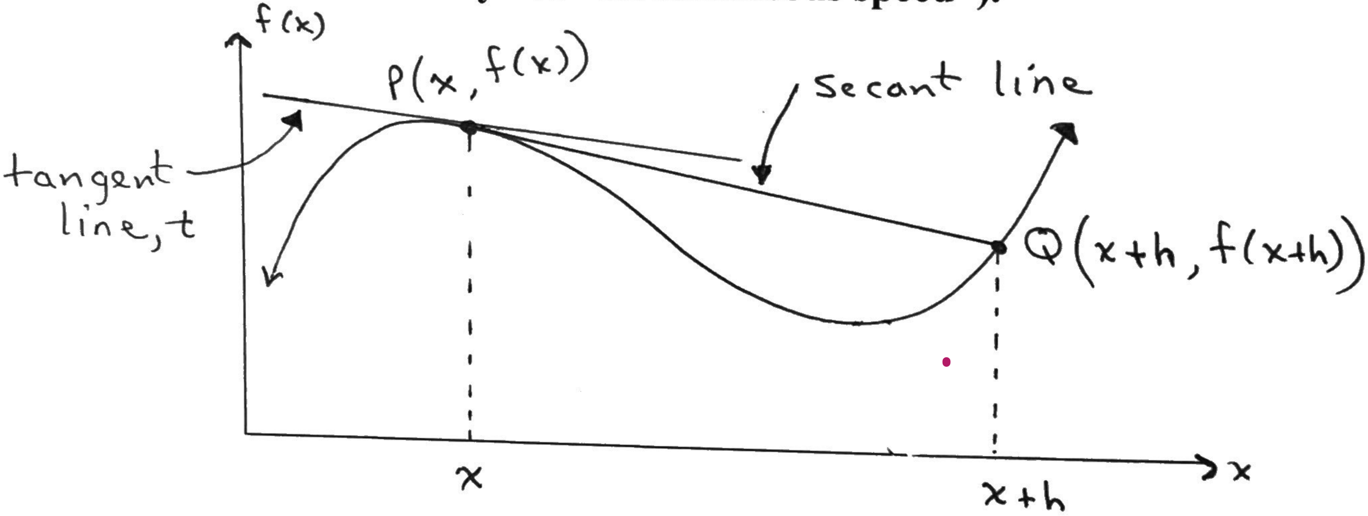
Slopes of Secants and Tangents

- **The slope of the secant is also the average rate of change (AROC). (From an application point of view it is often referred to "average velocity" or "average speed").**

- **The slope of the tangent is also the instantaneous rate of change (IROC). (From an application point of view it is often referred to "instantaneous velocity" or "instantaneous speed").**



Slope of the secant line PQ is given by $$m_{PQ} = \frac{f(x+h) - f(x)}{h}.$$

Sometimes called the "Newton Quotient" or the "difference quotient".

Slope of the tangent line is given by $$m_t = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Example 1: **Determine the slope of the tangent to the curve $f(x) = x^2 + 1$ at the point where $x = 2$.**

$$m_t = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$m_t = \lim_{h \to 0} \frac{(x+h)^2 + 1 - (x^2+1)}{h}$$

$$m_t = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{1} - \cancel{x^2} - \cancel{1}}{h}$$

$$m_t = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$m_t = 2x$$

$$\therefore m_t = 2x$$

$$m_t\Big|_{x=2} = 4$$

Example 2: **Determine the equation of the tangent to the curve $f(x) = \sqrt{x+3}$ at the point where $x = 1$.**

$$m_t = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$m_t = \lim_{h \to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h}$$

$$m_t = \lim_{h \to 0} \frac{(\sqrt{x+h+3} - \sqrt{x+3})}{h} \cdot \frac{(\sqrt{x+h+3} + \sqrt{x+3})}{(\sqrt{x+h+3} + \sqrt{x+3})}$$

$$m_t = \lim_{h \to 0} \frac{\cancel{x} + \cancel{h} + \cancel{3} - (\cancel{x} + \cancel{3})}{\cancel{h}(\sqrt{x+h+3} + \sqrt{x+3})}$$

$$m_t = \lim_{h \to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}}$$

$$m_t = \frac{1}{2\sqrt{x+3}}$$

$$\therefore m_t\Big|_{x=1} = \frac{1}{4}$$

Equation:

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - 2 = \frac{1}{4}(x-1)}$$

Example 3: **At what point on the curve $f(x) = -2x^4$ is the tangent line perpendicular to the line $x - y + 1 = 0$?**

Note: the line that is perpendicular to the tangent line at the point of tangency, P is called the "normal line".

* $x - y + 1 = 0 \Rightarrow y = x + 1 \quad \therefore m = 1 \;;\; m_t = -1$ (negative reciprocal)

* $m_t = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$m_t = \lim_{h \to 0} \dfrac{-2(x+h)^4 + 2x^4}{h}$

$m_t = \lim_{h \to 0} \dfrac{-2\left[(x+h)^4 - x^4\right]}{h}$

$m_t = \lim_{h \to 0} \dfrac{-2\left[(x+h)^2 - x^2\right]\left[(x+h)^2 + x^2\right]}{h}$

$m_t = \lim_{h \to 0} \dfrac{-2\left[\cancel{x^2} + 2xh + h^2 - \cancel{x^2}\right]\left[(x+h)^2 + x^2\right]}{h}$

$m_t = \lim_{h \to 0} \dfrac{-2\cancel{h}\left[2x + h\right]\left[(x+h)^2 + x^2\right]}{\cancel{h}}$

$m_t = \lim_{h \to 0} -2\left[2x + h\right]\left[(x+h)^2 + x^2\right]$

$m_t = -8x^3$

But $m_t = -1 \Rightarrow -8x^3 = -1$

$x^3 = \frac{1}{8}$

$x = \frac{1}{2}$

$\therefore$ point is $\left(\frac{1}{2}, \frac{-1}{8}\right)$