## Calculus

> Used to compare quantities which vary in a non-linear way
> 2 main branches: -<u>Differentiation</u>-helps us to find a rate of change of one quantity compared to another
>       -<u>Integration</u>-(reverse of differentiation) given a rate of change, find the original relationship between the 2 quantities
> The idea of zooming in to get a better approximation of the slope of the curve was the breakthrough that led to the development of differentiation
> <u>Methods:</u> ① Numerical approach to finding slopes (limits & slope of a tangent)
>       ② Algebraic approach to finding slopes (differentiation from first principles & derivatives)

## 1-1 Exploring the concept of a limit

Recall: A sequence is a list of numbers written in a definite order: $t_1, t_2, t_3, t_4, \ldots, t_n, \ldots$
where $t_1$ is the first term and in general $t_n$ is the nth term.

Ex 1: List the first 5 terms of the sequence defined by $t_n = \dfrac{n}{n+1}$ and draw the graph of the sequence.
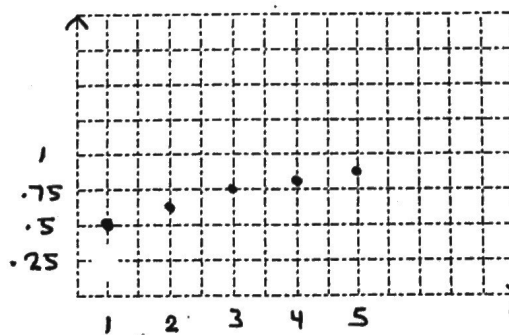
$t_1 = \dfrac{1}{2}$

$t_2 = \dfrac{2}{3}$

$t_3 = \dfrac{3}{4}$

$t_4 = \dfrac{4}{5}$

$t_5 = \dfrac{5}{6}$

$\therefore \lim_{n \to \infty} \dfrac{n}{n+1} = 1$

*In general, we say that the sequence with general term $t_n$ has the limit L, and we write:*

$$\lim_{n \to \infty} t_n = L$$

Ex 2: Find $\lim_{n \to \infty} \dfrac{1}{n}$

$= 0$

Ex 3: Find $\lim_{n \to \infty} (-1)^n$

$= \text{Does not Exist ("DNE")}$

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## Exercise 1.6

**A 1.** State the limits of the following sequences, or state that the limit does not exist.

(a) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \ldots, \left(\frac{1}{3}\right)^n, \ldots$

(b) $5, 4\frac{1}{2}, 4\frac{1}{3}, 4\frac{1}{4}, 4\frac{1}{5}, \ldots, 4 + \frac{1}{n}, \ldots$

(c) $1, 2, 3, 4, 5, \ldots, n, \ldots$

(d) $3, 3, 3, 3, 3, \ldots, 3, \ldots$

(e) $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots$

(f) $5, 6\frac{1}{2}, 5\frac{2}{3}, 6\frac{1}{4}, 5\frac{4}{5}, 6\frac{1}{6}, \ldots, 6 + \frac{(-1)^n}{n}, \ldots$

(g) $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \ldots$

(k) $\lim\limits_{n \to \infty} \dfrac{n}{n^2 + 1}$

(m) $\lim\limits_{n \to \infty} 5^{-n}$

(o) $\lim\limits_{n \to \infty} \dfrac{1 + n - 2n^2}{1 - n + n^2}$

(q) $\lim\limits_{n \to \infty} \dfrac{1}{n^5}$

(s) $\lim\limits_{n \to \infty} \left(\frac{2}{3}\right)^n$

(l) $\lim\limits_{n \to \infty} (-1)^{n-1} n$

(n) $\lim\limits_{n \to \infty} (n^3 + n^2)$

(p) $\lim\limits_{n \to \infty} \dfrac{1}{\sqrt{n}}$

(r) $\lim\limits_{n \to \infty} \dfrac{1 - n^3}{1 + 2n^3}$

(t) $\lim\limits_{n \to \infty} \left(\frac{4}{3}\right)^n$

**B 2.** List the first six terms of the following sequences.

(a) $t_n = \dfrac{n - 1}{2n - 1}$

(b) $t_n = \dfrac{2n}{n^2 + 1}$

(c) $t_n = n2^n$

(d) $t_n = \dfrac{(-1)^{n-1}}{n}$

(e) $t_1 = 1, \ t_n = \dfrac{1}{1 + t_{n-1}} \ (n \geqslant 2)$

(f) $t_1 = 1, \ t_2 = 2, \ t_n = t_{n-1} - t_{n-2} \ (n \geqslant 3)$

**3.** Find the following limits or state that the limit does not exist.

(a) $\lim\limits_{n \to \infty} \dfrac{1}{n^2}$

(b) $\lim\limits_{n \to \infty} \dfrac{1}{5 + n}$

(c) $\lim\limits_{n \to \infty} \left(6 + \dfrac{1}{n^3}\right)$

(d) $\lim\limits_{n \to \infty} \dfrac{n}{3n - 1}$

(e) $\lim\limits_{n \to \infty} \dfrac{6n + 9}{3n - 2}$

(f) $\lim\limits_{n \to \infty} 5n$

(g) $\lim\limits_{n \to \infty} \dfrac{n^2 + 1}{2n^2 - 1}$

(h) $\lim\limits_{n \to \infty} \dfrac{(n + 1)^2}{n(n + 2)}$

(i) $\lim\limits_{n \to \infty} \dfrac{(-1)^{n+1}}{n}$

(j) $\lim\limits_{n \to \infty} \left(-\frac{1}{4}\right)^n$

## Exercise 1.7

**B 1.** Find the sum of each of the following series or state that the series is divergent.

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

(b) $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \ldots$

(c) $\frac{1}{4} - \frac{5}{16} + \frac{25}{64} - \frac{125}{256} + \ldots$

(d) $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \ldots$

(e) $1 - 2 + 4 - 8 + \ldots$

(f) $60 + 40 + \frac{80}{3} + \frac{160}{9} + \ldots$

(g) $0.1 + 0.05 + 0.025 + 0.0125 + \ldots$

(h) $-3 + 3 - 3 + 3 - 3 + \ldots$

**EXERCISE 1.6**

1. (a) 0  (b) 4  (c) does not exist  (d) 3  (c) 0
(f) 6  (g) does not exist

2. (a) $0, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}$  (b) $1, \frac{4}{5}, \frac{6}{10}, \frac{8}{17}, \frac{10}{26}, \frac{12}{37}$
(c) 2, 8, 24, 64, 160, 384
(d) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}$  (e) $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}$
(f) $1, 2, 1, -1, -1, -2, -1$

3. (a) 0  (b) 0  (c) 6  (d) $\frac{1}{3}$  (c) 2  (f) does not
exist  (g) $\frac{1}{2}$  (h) 1  (i) 0  (j) 0  (k) 0
(l) does not exist  (m) 0  (n) does not exist
(o) $-2$  (p) 0  (q) 0  (r) $-\frac{1}{2}$  (s) 0  (t) does
not exist        4. $\frac{4}{3}$

**EXERCISE 1.7**

1. (a) $\frac{3}{2}$  (b) $\frac{3}{5}$  (c) divergent  (d) $\frac{15}{4}$
(e) divergent  (f) 180  (g) $\frac{1}{5}$  (h) divergent

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Example 1:  Consider the graph of the function $f(x) = 3x - 5$.

As the value of $x$ "approaches" 2 (on the graph), what does the value of the function approach?
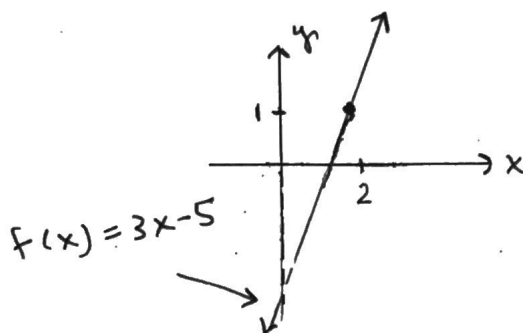
**Definition:**

Limit → the notation $\lim\limits_{x \to a} f(x) = L$ implies that as "$x$" approaches closer and closer to the value "$a$", the value of the function approaches a limiting value, $L$.
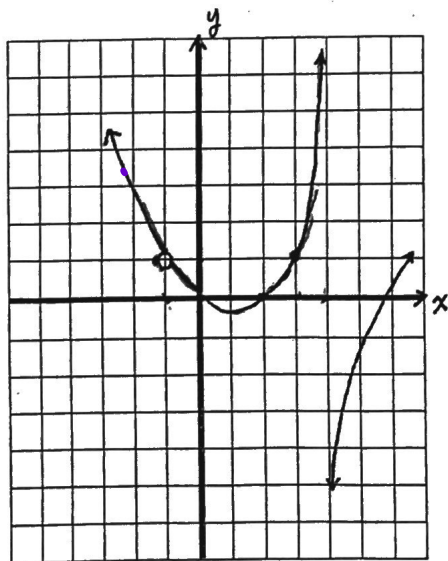
$f(x) = 3x - 5$

We say that "the *limit* of $3x-5$ as $x$ approaches 2 equals 1," and we write

$$\lim_{x \to 2} (3x - 5) = 1$$

Notes:

- In this case $f(2) = 1$ and $\therefore \lim\limits_{x \to 2} f(x) = f(2)$.
- In retrospect it follows that $\lim\limits_{x \to 2} f(x)$ could have been evaluated simply by substituting $x = 2$ into the formula for $f(x)$.
- However, NOT all limits can be evaluated by substitution.

Example 2:  Use the given graph of $f$ to state the value of the limit, if it exists.

a. $\lim\limits_{x \to 3} f(x) = 1$

b. $\lim\limits_{x \to 2} f(x) = 0$

c. $\lim\limits_{x \to -1} f(x) = 1$

d. $\lim\limits_{x \to 4} f(x) = DNE \quad \left( \begin{array}{c} \text{infinitely} \\ \text{discontinuous} \end{array} \right)$

NOTE:

$f(-1)$ is undefined.

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Example 3:     Evaluate the following limits:

a.    $\lim\limits_{x \to 3} x^2 = 9$                                 b.    $\lim\limits_{x \to 2} \dfrac{1}{x} = \dfrac{1}{2}$

c.    $\lim\limits_{x \to 1} \sqrt{x+3} = 2$

Notes: For the following types of functions

- polynomial functions
- rational functions
- algebraic functions

you can evaluate $\lim\limits_{x \to a} f(x)$ by substitution $\left( \lim\limits_{x \to a} f(x) = f(a) \right)$, provided that $f(a)$ is defined.

Example 4:     Evaluate the following limit: $\lim\limits_{x \to 3} \dfrac{x^2-9}{x-3} = \dfrac{0}{0}$ ← indeterminate form

$\left( \text{need to do something algebraically} \right)$

$\lim\limits_{x \to 3} \dfrac{x^2-9}{x-3}$

$= \lim\limits_{x \to 3} \dfrac{(x-3)(x+3)}{x-3}$

$= \lim\limits_{x \to 3} (x+3)$

$= 6$

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Example 5:   Evaluate the following limits:

$$\lim_{x \to 2} \frac{(x-3)^2 - 1}{x-2}$$

$$= \lim_{x \to 2} \frac{(x-3-1)(x-3+1)}{x-2}$$

$$= \lim_{x \to 2} \frac{(x-4)(x-2)}{x-2}$$

$$= \lim_{x \to 2} (x-4) = -2$$

$$\lim_{h \to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h}$$

$$= \lim_{h \to 0} \left( \frac{\sqrt{3+h} - \sqrt{3}}{h} \right) \cdot \frac{(\sqrt{3+h} + \sqrt{3})}{(\sqrt{3+h} + \sqrt{3})} \qquad * \text{ conjugate}$$

$$= \lim_{h \to 0} \frac{3+h-3}{h(\sqrt{3+h} + \sqrt{3})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{3+h} + \sqrt{3}} = \frac{1}{2\sqrt{3}} \quad \left( \text{or } \frac{\sqrt{3}}{6} \right)$$

$$\lim_{h \to 0} \frac{\frac{4}{2+h} - 2}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4}{2+h} - \frac{2(2+h)}{2+h}}{h} \qquad = \lim_{h \to 0} \frac{-2}{2+h}$$

$$= \lim_{h \to 0} \frac{\frac{4-4-2h}{2+h}}{h} \qquad = -1$$

$$= \lim_{h \to 0} \frac{-2h}{h(2+h)}$$

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# Limits – Change of Variable

1. $\displaystyle\lim_{x \to 8} \frac{x-8}{\sqrt[3]{x}-2}$  Let $t = \sqrt[3]{x}$  since $x \to 8$ then $t \to 2$

$\Rightarrow x = t^3$

$\displaystyle\lim_{t \to 2} \frac{t^3 - 8}{t - 2}$

$= \displaystyle\lim_{t \to 2} \frac{(t-2)(t^2 + 2t + 4)}{t-2}$

$= \displaystyle\lim_{t \to 2} t^2 + 2t + 4$

$= 12$

2. $\displaystyle\lim_{x \to 27} \frac{27 - x}{x^{\frac{1}{3}} - 3}$  Let $t = x^{\frac{1}{3}}$  since $x \to 27$ then $t \to 3$

$\Rightarrow x = t^3$

$\displaystyle\lim_{t \to 3} \frac{27 - t^3}{t - 3}$

$= \displaystyle\lim_{t \to 3} \frac{(3-t)(9 + 3t + t^2)}{t - 3}$

$= \displaystyle\lim_{t \to 3} -(9 + 3t + t^2)$

$= -27$

3. $\displaystyle\lim_{x\to 0}\frac{(x+8)^{\frac{1}{3}}-2}{x}$

Let $t = (x+8)^{\frac{1}{3}}$

$x = t^3 - 8$

since $x \to 0$

then $t \to 2$

$\displaystyle\lim_{t\to 2}\frac{t-2}{t^3-8}$

$= \displaystyle\lim_{t\to 2}\frac{\cancel{t-2}}{\cancel{(t-2)}(t^2+2t+4)}$

$= \displaystyle\lim_{t\to 2}\frac{1}{t^2+2t+4}$

$= \dfrac{1}{12}$