Unit #1

Rates of Change

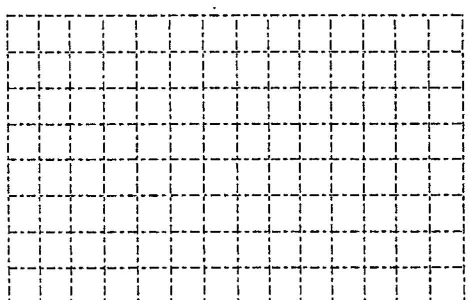
Calculus

- Used to compare quantities which vary in a non-linear way
- 2 main branches: -<u>Differentiation</u>-helps us to find a rate of change of one quantity compared to another
  -<u>Integration</u>-(reverse of differentiation) given a rate of change, find the original relationship between the 2 quantities
- The idea of zooming in to get a better approximation of the slope of the curve was the breakthrough that led to the development of differentiation
- <u>Methods</u>: ① Numerical approach to finding slopes (limits & slope of a tangent)
  ② Algebraic approach to finding slopes (differentiation from first principles & derivatives)

1-1 Exploring the concept of a limit

Recall: A sequence is a list of numbers written in a definite order: $t_1, t_2, t_3, t_4, \ldots, t_n, \ldots$
where t_1 is the first term and in general t_n is the nth term.

Ex 1: List the first 5 terms of the sequence defined by $t_n = \dfrac{n}{n+1}$ and draw the graph of the sequence.



In general, we say that the sequence with general term t_n has the limit L, and we write:

$$\boxed{\lim_{n\to\infty} t_n = L}$$

Ex 2: Find $\lim\limits_{n\to\infty} \dfrac{1}{n}$

Ex 3: Find $\lim\limits_{n\to\infty} (-1)^n$

Exercise 1.6

A **1.** State the limits of the following sequences, or state that the limit does not exist.

(a) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \ldots, \left(\frac{1}{3}\right)^n, \ldots$

(b) $5, 4\frac{1}{2}, 4\frac{1}{3}, 4\frac{1}{4}, 4\frac{1}{5}, \ldots, 4 + \frac{1}{n}, \ldots$

(c) $1, 2, 3, 4, 5, \ldots, n, \ldots$

(d) $3, 3, 3, 3, 3, \ldots, 3, \ldots$

(e) $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots$

(f) $5, 6\frac{1}{2}, 5\frac{2}{3}, 6\frac{1}{4}, 5\frac{4}{5}, 6\frac{1}{6}, \ldots, 6 + \frac{(-1)^n}{n}, \ldots$

(g) $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \ldots$

(k) $\lim_{n\to\infty} \frac{n}{n^2+1}$

(l) $\lim_{n\to\infty} (-1)^{n-1} n$

(m) $\lim_{n\to\infty} 5^{-n}$

(n) $\lim_{n\to\infty} (n^3 + n^2)$

(o) $\lim_{n\to\infty} \frac{1 + n - 2n^2}{1 - n + n^2}$

(p) $\lim_{n\to\infty} \frac{1}{\sqrt{n}}$

(q) $\lim_{n\to\infty} \frac{1}{n^5}$

(r) $\lim_{n\to\infty} \frac{1 - n^3}{1 + 2n^3}$

(s) $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n$

(t) $\lim_{n\to\infty} \left(\frac{4}{3}\right)^n$

B **2.** List the first six terms of the following sequences.

(a) $t_n = \frac{n-1}{2n-1}$

(b) $t_n = \frac{2n}{n^2+1}$

(c) $t_n = n2^n$

(d) $t_n = \frac{(-1)^{n-1}}{n}$

(e) $t_1 = 1,\ t_n = \frac{1}{1 + t_{n-1}}\ (n \geqslant 2)$

(f) $t_1 = 1,\ t_2 = 2,\ t_n = t_{n-1} - t_{n-2}\ (n \geqslant 3)$

3. Find the following limits or state that the limit does not exist.

(a) $\lim_{n\to\infty} \frac{1}{n^2}$

(b) $\lim_{n\to\infty} \frac{1}{5+n}$

(c) $\lim_{n\to\infty} \left(6 + \frac{1}{n^3}\right)$

(d) $\lim_{n\to\infty} \frac{n}{3n-1}$

(e) $\lim_{n\to\infty} \frac{6n+9}{3n-2}$

(f) $\lim_{n\to\infty} 5n$

(g) $\lim_{n\to\infty} \frac{n^2+1}{2n^2-1}$

(h) $\lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)}$

(i) $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n}$

(j) $\lim_{n\to\infty} \left(-\frac{1}{4}\right)^n$

Exercise 1.7

B ~~1.~~ Find the sum of each of the following series or state that the series is divergent.

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

(b) $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \ldots$

(c) $\frac{1}{4} - \frac{5}{16} + \frac{25}{64} - \frac{125}{256} + \ldots$

(d) $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \ldots$

(e) $1 - 2 + 4 - 8 + \ldots$

(f) $60 + 40 + \frac{80}{3} + \frac{160}{9} + \ldots$

(g) $0.1 + 0.05 + 0.025 + 0.0125 + \ldots$

(h) $-3 + 3 - 3 + 3 - 3 + \ldots$

EXERCISE 1.6

1. (a) 0 (b) 4 (c) does not exist (d) 3 (e) 0 (f) 6 (g) does not exist

2. (a) $0, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}$ (b) $1, \frac{4}{5}, \frac{6}{10}, \frac{8}{17}, \frac{10}{26}, \frac{12}{37}$ (c) 2, 8, 24, 64, 160, 384 (d) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}$ (e) $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}$ (f) $1, 2, 1, -1, -2, -1$

3. (a) 0 (b) 0 (c) 6 (d) $\frac{1}{3}$ (e) 2 (f) does not exist (g) $\frac{1}{2}$ (h) 1 (i) 0 (j) 0 (k) 0 (l) does not exist (m) 0 (n) does not exist (o) -2 (p) 0 (q) 0 (r) $-\frac{1}{2}$ (s) 0 (t) does not exist **4.** $\frac{1}{3}$

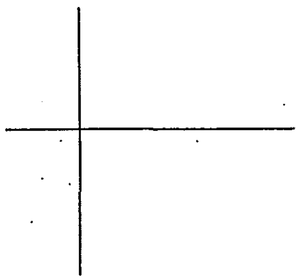
EXERCISE 1.7

1. (a) $\frac{3}{2}$ (b) $\frac{3}{5}$ (c) divergent (d) $\frac{15}{4}$ (e) divergent (f) 180 (g) $\frac{1}{5}$ (h) divergent

Introduction to Limits

Example 1: Consider the graph of the function $f(x) = 3x - 5$.

As the value of x "approaches" 2 (on the graph), what does the value of the function approach?



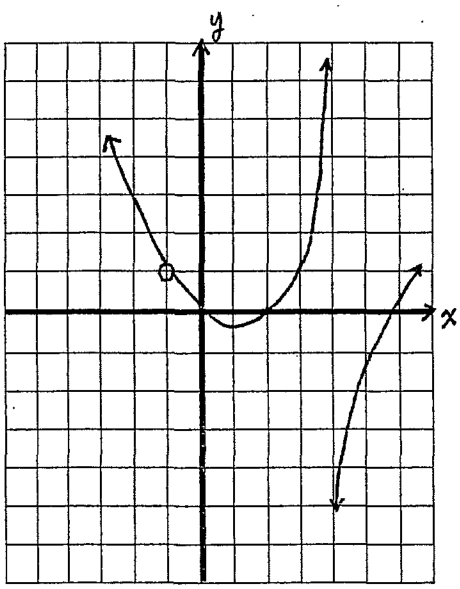
We say that "the *limit* of $3x$-5 as x approaches 2 equals 1," and we write

$$\lim_{x \to 2} (3x - 5) = 1$$

Notes:

- In this case $f(2) = 1$ and $\therefore \lim_{x \to 2} f(x) = f(2)$.
- In retrospect it follows that $\lim_{x \to 2} f(x)$ could have been evaluated simply by substituting $x = 2$ into the formula for $f(x)$.
- However, NOT all limits can be evaluated by substitution.

Example 2: Use the given graph of f to state the value of the limit, if it exists.



a. $\lim_{x \to 3} f(x)$

b. $\lim_{x \to 2} f(x)$

c. $\lim_{x \to -1} f(x)$

d. $\lim_{x \to 4} f(x)$

Example 3: Evaluate the following limits:

a. $\lim\limits_{x \to 3} x^2$

b. $\lim\limits_{x \to 2} \frac{1}{x}$

c. $\lim\limits_{x \to 1} \sqrt{x+3}$

Notes: For the following types of functions

- polynomial functions
- rational functions
- algebraic functions

you can evaluate $\lim\limits_{x \to a} f(x)$ by substitution $\left(\lim\limits_{x \to a} f(x) = f(a) \right)$, provided that $f(a)$ is defined.

Example 4: Evaluate the following limit: $\lim\limits_{x \to 3} \frac{x^2 - 9}{x - 3}$

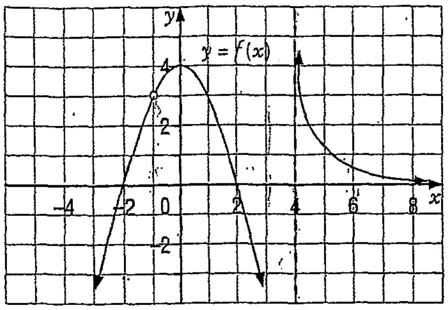
Example 5: Evaluate the following limits:

$$\lim_{x \to 2} \frac{(x-3)^2 - 1}{x-2}$$

$$\lim_{h \to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h}$$

$$\lim_{h \to 0} \frac{\frac{4}{2+h} - 2}{h}$$

1. Use the graph of $f(x)$ to state the value of each limit, if it exists. If it does not exist, explain why.



a) $\lim_{x \to -1} f(x)$ b) $\lim_{x \to 0} f(x)$

c) $\lim_{x \to 2} f(x)$ d) $\lim_{x \to 4} f(x)$

e) $\lim_{x \to 6} f(x)$

10. Determine the value of each limit.

a) $\lim_{x \to 2} 4$ b) $\lim_{x \to 6} \pi$

c) $\lim_{x \to 5} x$ d) $\lim_{x \to 4} x^2$

e) $\lim_{x \to 2} (3x - 4)$ f) $\lim_{x \to -1} (3x^2 - 4x + 10)$

g) $\lim_{x \to 1} (x^3 - x^2 + x - 1)$ h) $\lim_{x \to 0} \dfrac{x - 4}{x + 2}$

i) $\lim_{x \to 3} \dfrac{x^2 + 8x + 12}{x^2 + 2x}$ j) $\lim_{t \to -2} \sqrt{3t^4 + 4t^2}$

11. Find each limit.

a) $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$ b) $\lim_{x \to 2} \dfrac{x - 2}{x^2 - 4}$

c) $\lim_{x \to 4} \dfrac{x^2 - 6x + 8}{x - 4}$ d) $\lim_{x \to -2} \dfrac{x + 2}{x^2 - 3x - 10}$

e) $\lim_{x \to -3} \dfrac{2x^2 + 7x + 3}{x^2 - 9}$ f) $\lim_{x \to 4} \dfrac{x^2 + 2x - 24}{x^2 - 6x + 8}$

g) $\lim_{x \to 1} \dfrac{5x^2 - 3x - 2}{3x^2 - 7x + 4}$ h) $\lim_{x \to 2} \dfrac{x^3 - 8}{x^2 - 4}$

i) $\lim_{x \to 3} \dfrac{x - 3}{x^3 - 27}$ j) $\lim_{x \to -2} \dfrac{x^3 + 8}{x + 2}$

k) $\lim_{x \to 4} \dfrac{x - 4}{\sqrt{x} - 2}$ l) $\lim_{x \to 3} \dfrac{\frac{1}{x} - \frac{1}{3}}{x - 3}$

1. a) 3 b) 4 c) 0 d) does not exist; infinite discontinuity e) $\frac{1}{2}$

10. a) 4 b) π c) 5 d) 16 e) 2 f) 17 g) 0 h) -2 i) 3 j) 8 **11.** a) 6 b) $\frac{1}{4}$ c) 2 d) $-\frac{1}{7}$ e) $\frac{5}{6}$ f) 5 g) -7 h) 3 i) $\frac{1}{27}$ j) 12 k) 4 l) $-\frac{1}{9}$

(#1) Evaluate the following.

(a) $\lim_{h\to 0} \frac{(4+h)^3-64}{h}$ (b) $\lim_{h\to 0} \frac{(h-2)^2-4}{h}$

(c) $\lim_{h\to 0} \frac{\frac{1}{1+h}-1}{h}$ (d) $\lim_{h\to 0} \frac{(2+h)^4-16}{h}$

(2.) Evaluate the limit of each indeterminate quotient.

a. $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$ b. $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$

c. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ d. $\lim_{x\to 0} \frac{\sqrt{4-x}-\sqrt{4+x}}{x}$

e. $\lim_{x\to 1} \frac{\sqrt{5-x}-\sqrt{3+x}}{x-1}$

(#3) Let $f(x) = Ax + B$, where A and B are constants. If $\lim_{x\to 1} f(x) = -2$ and $\lim_{x\to -1} f(x) = 4$, find A and B.

Answers

(1) (a) 48 (b) -4 (c) -1 (d) 32

(2) a. $\frac{1}{2}$ b. $\frac{1}{4}$ c. $\frac{1}{4}$ d. $-\frac{1}{2}$ e. $-\frac{1}{2}$

(3) $A = -3$, $B = 1$

A Limited Number of Limit Questions

7. Evaluate the limit of each indeterminate quotient.

a. $\lim\limits_{x\to2}\dfrac{4-x^2}{2-x}$ b. $\lim\limits_{x\to-2}\dfrac{4-x^2}{2+x}$ c. $\lim\limits_{x\to0}\dfrac{7x-x^2}{x}$

d. $\lim\limits_{x\to-1}\dfrac{2x^2+5x+3}{x+1}$ X $\lim\dfrac{3x^2+x-4}{3x+4}$ f. $\lim\limits_{x\to3}\dfrac{x^3-27}{x-3}$

g. $\lim\limits_{x\to-2}\dfrac{x^3+2x^2-4x-8}{x+2}$ h. $\lim\limits_{x\to2}\dfrac{2x^3-5x^2+3x-2}{2x-4}$ i. $\lim\limits_{x\to0}\dfrac{\sqrt{x+1}-1}{x}$

j. $\lim\limits_{x\to0}\dfrac{2-\sqrt{4+x}}{x}$ k. $\lim\limits_{x\to4}\dfrac{\sqrt{x}-2}{x-4}$ l. $\lim\limits_{x\to0}\dfrac{\sqrt{7-x}-\sqrt{7+x}}{x}$

m. $\lim\limits_{x\to1}\dfrac{\sqrt{5-x}-\sqrt{3+x}}{x-1}$ n. $\lim\limits_{x\to4}\dfrac{2-\sqrt{x}}{3-\sqrt{2x+1}}$ o. $\lim\limits_{x\to0}\dfrac{2^{2x}-2^x}{2^x-1}$

8. Evaluate the limit by change of variable.

a. $\lim\limits_{x\to8}\dfrac{\sqrt[3]{x}-2}{x-8}$ b. $\lim\limits_{x\to27}\dfrac{27-x}{x^{\frac{1}{3}}-3}$ c. $\lim\limits_{x\to1}\dfrac{x^{\frac{1}{6}}-1}{x-1}$

d. $\lim\limits_{x\to1}\dfrac{x^{\frac{1}{6}}-1}{x^{\frac{1}{3}}-1}$ e. $\lim\limits_{x\to4}\dfrac{\sqrt{x}-2}{\sqrt{x^3}-8}$ f. $\lim\limits_{x\to0}\dfrac{(x+8)^{\frac{1}{3}}-2}{x}$

9. Evaluate each limit, if it exists, using any appropriate technique.

a. $\lim\limits_{x\to4}\dfrac{16-x^2}{x^3+64}$ b. $\lim\limits_{x\to4}\dfrac{x^2-16}{x^2-5x+6}$ c. $\lim\limits_{x\to1}\dfrac{x^3+x^2-5x+3}{x^2-2x+1}$

d. $\lim\limits_{x\to-1}\dfrac{x^2+x}{x+1}$ X $\lim\limits_{x\to6^+}\dfrac{\sqrt{x^2-5x-6}}{x-3}$ f. $\lim\limits_{x\to0}\dfrac{(2x+1)^{\frac{1}{3}}-1}{x}$

g. $\lim\limits_{x\to2}\dfrac{x^2-4}{\left(\frac{1}{x}\right)-\frac{1}{2}}$ h. $\lim\limits_{x\to3}\dfrac{\sqrt{x+1}-2}{x-3}$ i. $\lim\limits_{x\to0}\dfrac{x^2-9x}{5x^3+6x}$

j. $\lim\limits_{x\to0}\dfrac{\sqrt{x+1}-1}{x}$ k. $\lim\limits_{h\to0}\dfrac{(x+h)^2-x^2}{h}$

l. $\lim\limits_{x\to1}\left(\dfrac{1}{x-1}\right)\left(\dfrac{1}{x+3}-\dfrac{2}{3x+5}\right)$

ANSWERS

7. a. 4 b. 4 c. 7 d. 1 e. $\frac{-7}{3}$ f. 27 g. 0 h. $\frac{7}{2}$ i. $\frac{1}{2}$ j. $-\frac{1}{4}$ k. $\frac{1}{4}$ l. $-\frac{1}{\sqrt{7}}$ m. $\frac{-1}{2}$ n. $\frac{3}{4}$ o. 1

8. a. $\frac{1}{12}$ b. -27 c. $\frac{1}{6}$ d. $\frac{1}{2}$ e. $\frac{1}{12}$ f. $\frac{1}{12}$

9. a. 0 b. 0 c. 4 d. -1 e. 0 f. $\frac{2}{3}$ g. -16 h. $\frac{1}{4}$ i. $\frac{-3}{2}$ j. $\frac{1}{2}$ k. $2x$ l. $\frac{1}{32}$

Properties of Limits

Definition: $\lim_{x \to a} f(x)$ refers to the limit of f(x) as x approaches a.

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, and c is a contant, then:

1. $\lim_{x \to a} c = c$.

2. $\lim_{x \to a} x = a$.

3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$.

4. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$.

5. $\lim_{x \to a} [f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)]$.

6. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, $\lim_{x \to a} g(x) \neq 0$.

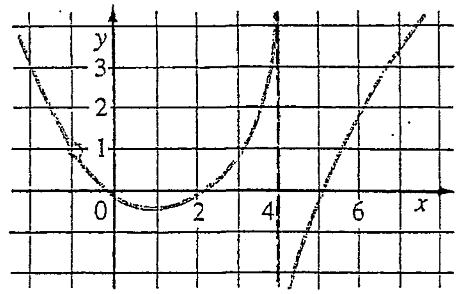
7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$, $n \in Q$.

Functions whose limit as x $\to$ a is equal to the function evaluated at a are **continuous at a.**

$$\lim_{x \to a} f(x) = f(a).$$

EXERCISE 1.2

A 1. Use the given graph of f to state the value of the limit, if it exists.



(a) $\lim_{x\to 3} f(x)$ (b) $\lim_{x\to 2} f(x)$ (c) $\lim_{x\to -1} f(x)$ (d) $\lim_{x\to 4} f(x)$

2. State the value of each limit.

(a) $\lim_{x\to 2} x^3$ (b) $\lim_{x\to \pi} x$

(c) $\lim_{x\to 8} 3$ (d) $\lim_{x\to 4} \sqrt{x}$

(e) $\lim_{x\to k} x^6$ (f) $\lim_{x\to 0} \pi$

B 3. Use the properties of limits to evaluate the following.

(a) $\lim_{x\to 1} (3x - 7)$ (b) $\lim_{x\to -1} (2x^2 - 5x + 3)$

(c) $\lim_{x\to 2} (x^3 + x^2 - 2x - 8)$ (d) $\lim_{x\to -2} (x^2 + 5x + 3)^6$

(e) $\lim_{x\to 0} \dfrac{x-1}{x+1}$ (f) $\lim_{x\to 4} \dfrac{x^2 + 2x - 3}{x^2 + 2}$

(g) $\lim_{t\to 2} \dfrac{t^4 - 3t + 1}{t^2(t-1)^3}$ (h) $\lim_{u\to -4} \sqrt{u^4 + 2u^2}$

(i) $\lim_{x\to 5} \sqrt[3]{x^2 + 2x - 8}$ (j) $\lim_{t\to 3} \left(2t^2 + \sqrt{\dfrac{6+t}{4-t}}\right)$

4. Find the following limits.

(a) $\lim_{x\to -2} \dfrac{x+2}{x^2 - 4}$ (b) $\lim_{x\to 1} \dfrac{x^2 - 3x + 2}{x - 1}$

(c) $\lim_{x\to 3} \dfrac{x^2 - 2x - 3}{x^2 - 4x + 3}$ (d) $\lim_{x\to -2} \dfrac{2x^2 + 5x + 2}{x^2 - 2x - 8}$

(e) $\lim_{x\to 1} \dfrac{x^3 - 1}{x^2 - 1}$ (f) $\lim_{x\to -3} \dfrac{x+3}{x^3 + 27}$

(g) $\lim_{x\to 9} \dfrac{x - 9}{\sqrt{x} - 3}$ (h) $\lim_{x\to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x - 2}$

5. Evaluate the following.

(a) $\lim_{h\to 0} \dfrac{(4+h)^3 - 64}{h}$ (b) $\lim_{h\to 0} \dfrac{(h-2)^2 - 4}{h}$

(c) $\lim_{h\to 0} \dfrac{\dfrac{1}{1+h} - 1}{h}$ (d) $\lim_{h\to 0} \dfrac{(2+h)^4 - 16}{h}$

(e) $\lim_{h\to 0} \dfrac{\sqrt{9+h} - 3}{h}$ (f) $\lim_{h\to 0} \dfrac{\dfrac{1}{(2+h)^2} - \dfrac{1}{4}}{h}$

6. Find the following limits, if they exist.

(a) $\lim_{x\to 3} \dfrac{1}{(x-3)^2}$ (b) $\lim_{x\to -8} \dfrac{x^2 + 16x + 64}{x + 8}$

(c) $\lim_{x\to 1} \dfrac{x^4 - 1}{x - 1}$ (d) $\lim_{x\to -1} \dfrac{x - 1}{x^2 - 1}$

(e) $\lim_{x\to 1} \dfrac{x^2 + x - 2}{x^2 - 2x + 1}$ (f) $\lim_{x\to -2} \dfrac{x^2 - x - 2}{x^2 + 3x + 2}$

Answers

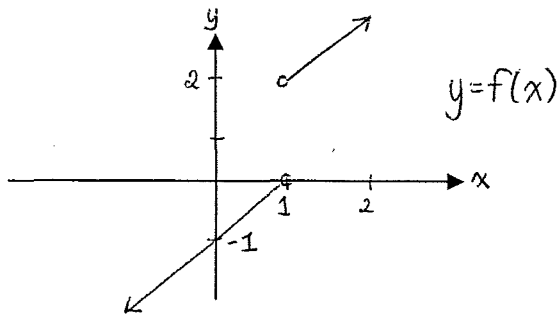
EXERCISE 1.2

1. (a) 1 (b) 0 (c) 1 (d) does not exist
2. (a) 8 (b) π (c) 3 (d) 2 (d) 2 (e) k^6 (f) π
3. (a) -4 (b) 10 (c) 0 (d) 729 (e) -1 (f) $\frac{7}{6}$ (g) $\frac{11}{4}$ (h) $12\sqrt{2}$ (i) 3 (j) 21
4. (a) $-\frac{1}{4}$ (b) -1 (c) 2 (d) $\frac{1}{2}$ (e) $\frac{3}{2}$ (f) $\frac{1}{27}$ (g) 6 (h) $-\frac{1}{4}$
5. (a) 48 (b) -4 (c) -1 (d) 32 (e) $\frac{1}{6}$ (f) $-\frac{1}{4}$
6. (a) does not exist (b) 0 (c) 4 (d) does not exist (e) does not exist (f) does not exist (g) $-\frac{2}{27}$ (h) $-\frac{1}{16}$ (i) -1 (j) -2
7. (a) 2.000 000, 2.593 742, 2.704 814, 2.716 924, 2.718 146, 2.718 268, 2.718 280. 2.718 282 (b) 2.718 28
8. (a) 1.0000, 0.7177, 0.6956, 0.6934, 0.6932 (b) 0.693
9. (a) 12 (b) $\frac{1}{2}$ **11.** within 0.000 25

13. $f(x) = \dfrac{1}{x}, g(x) = -\dfrac{1}{x}$

1-3 One-Sided Limits

Recall:



- The limit of $f(x)$ as x approaches 1 from the right equals 2
- The limit of $f(x)$ as x approaches 1 from the left equals 0

$\because \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$ One sided limits are **unequal.**

$\therefore \lim_{x\to 1} f(x)$ does not exits (*f(x)* does not approach a single value as $x \to 1$)

> In general:
>
> If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ **does not exist.**
>
> If $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x) = L$ where L is the limiting value.

Continuous vs. Discontinuous functions

A function $f(x)$ is continuous at $x = a$ if

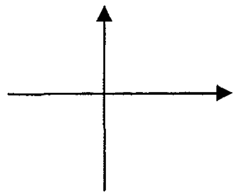
- $f(a)$ is defined (a is in the domain of $f(x)$)
- $\lim_{x\to a} f(x)$ exists
- $\lim_{x\to a} f(x) = f(a)$

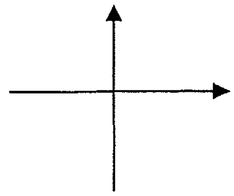
If $f(x)$ is not continuous at $x = a$, then $f(x)$ is discontinuous at $x = a$, or $f(x)$ has a discontinuity at 'a'.

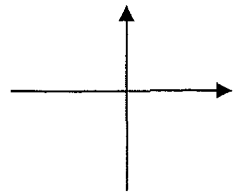
Note: The limit of a function at $x = a$ may exist even though the function is discontinuous at $x = a$.

Functions can have 3 types of discontinuities:

1. Jump discontinuity
2. Infinite discontinuity
3. Removable discontinuity

1. 

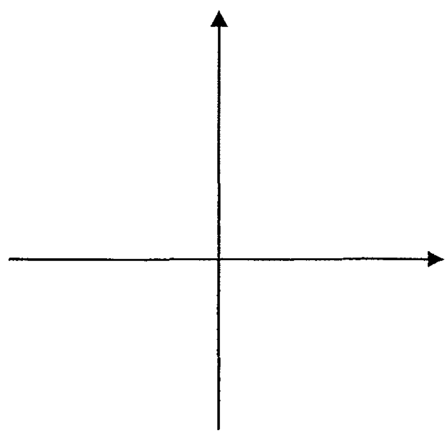
2. 

3. 

Ex. 1 Consider the function:

$$f(x)=\begin{cases} x^2-3 & \text{if } x\le -1 \\ x-1 & \text{if } x>-1 \end{cases}$$

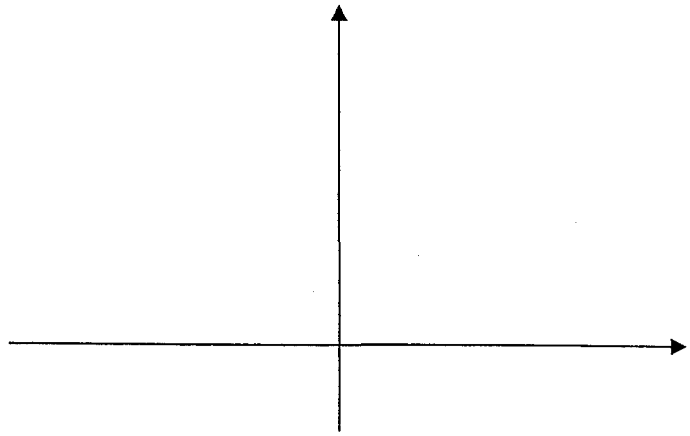
By evaluating one-sided limits, determine whether $\lim\limits_{x\to -1} f(x)$ exists. Sketch the graph of $f(x)$.



- Note that $f(-1)=-2$ and hence $\lim\limits_{x\to -1} f(x)=f(-1)$
- This equality describes the fact that the separate parts of the graph meet at $x=-1$ without a break, so that $f(x)$ is continuous at $x=-1$.

Ex. 2 By sketching the graph of the function, find all x values at which the function is discontinuous.

$$f(x)=\begin{cases} x^2 & if \quad x<1 \\ 2 & if \quad x\geq 1 \end{cases}$$



From the graph:

Algebraically:

Ex. 3 Let $f(x)=\begin{cases} \frac{1}{x} & if \quad x\leq -1 \\ \frac{x-1}{2} & if \quad -1<x<+1 \\ \sqrt{x} & if \quad x\geq 1 \end{cases}$

a) Find the following limits.

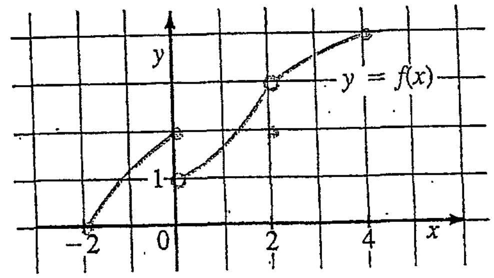
i) $\lim\limits_{x\to -1^-} f(x)$ ii) $\lim\limits_{x\to -1^+} f(x)$ iii) $\lim\limits_{x\to 1^-} f(x)$ iv) $\lim\limits_{x\to 1^+} f(x)$

b) Sketch the graph of $f(x)$.

c) Where is $f(x)$ discontinuous?

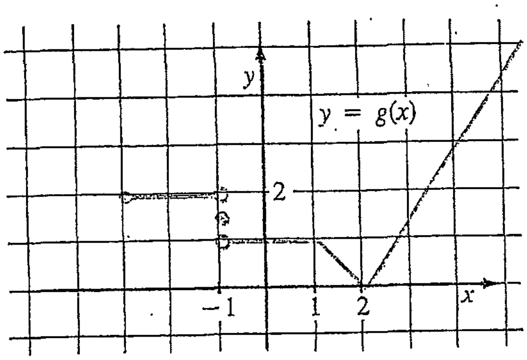
:CISE 1.3

A **1.** Use the given graph of f to state the value of the limit, if it exists.



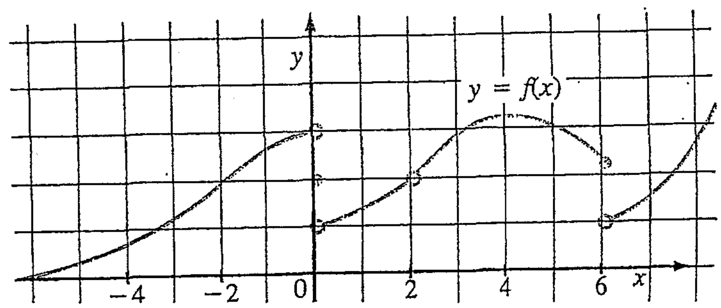
(a) $\lim_{x \to -2^+} f(x)$ (b) $\lim_{x \to 0^-} f(x)$ (c) $\lim_{x \to 0^+} f(x)$ (d) $\lim_{x \to 0} f(x)$
(e) $\lim_{x \to 2^-} f(x)$ (f) $\lim_{x \to 2^+} f(x)$ (g) $\lim_{x \to 2} f(x)$ (h) $\lim_{x \to 4^-} f(x)$

2. Use the given graph of g to state the value of the limit, if it exists.

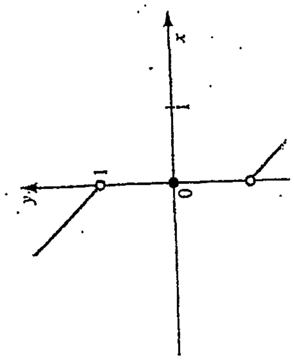


(a) $\lim_{x \to -3^+} g(x)$ (b) $\lim_{x \to -1^-} g(x)$ (c) $\lim_{x \to -1^+} g(x)$ (d) $\lim_{x \to -1} g(x)$
(e) $\lim_{x \to 2^-} g(x)$ (f) $\lim_{x \to 2^+} g(x)$ (g) $\lim_{x \to 2} g(x)$ (h) $\lim_{x \to 1} g(x)$

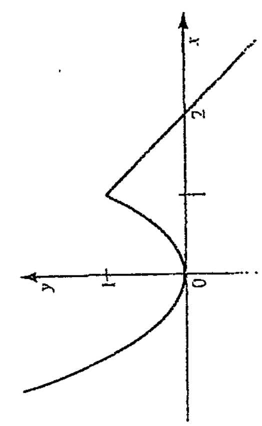
3. The graph of f is given. State whether f is continuous or discontinuous at each of the following numbers.
(a) -2 (b) 0 (c) 2 (d) 4 (e) 6



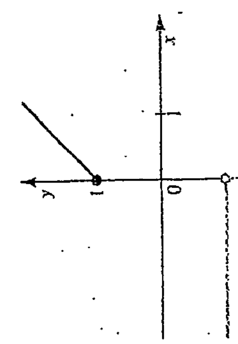
7. (a) 1 (b) −1 (c) does not exist



6. (a) 1 (b) 1 (c) 1



5. (a) −1 (b) 1 (c) does not exist



Answers

EXERCISE 1.3

1. (a) 0 (b) 2 (c) 1 (d) does not exist (e) 3
(f) 3 (g) 3 (h) 4

2. (a) 2 (b) 2 (c) 1 (d) does not exist (e) 0
(f) 0 (g) 0 (h) 1

3. (a) continuous (b) discontinuous
(c) discontinuous (d) continuous
(e) discontinuous

4. (a) 0 (b) 0 (c) 0 (d) 0 (e) 0 (f) 0 (g) 0
(h) 1 (i) −1 (j) does not exist

B **4.** Find the following limits, if they exist.

(a) $\lim_{x\to 0^+} \sqrt[4]{x}$ (b) $\lim_{x\to 3^+} \sqrt{x-3}$

(c) $\lim_{x\to 1^-} \sqrt{1-x}$ (d) $\lim_{x\to \frac{1}{2}^-} \sqrt[4]{1-2x}$

(e) $\lim_{x\to 6^+} |x-6|$ (f) $\lim_{x\to 6^-} |x-6|$

(g) $\lim_{x\to 6} |x-6|$ (h) $\lim_{x\to 0^+} \frac{|x|}{x}$

(i) $\lim_{x\to 0^-} \frac{|x|}{x}$ (j) $\lim_{x\to 0} \frac{|x|}{x}$

5. Let

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ x+1 & \text{if } x \geqslant 0 \end{cases}$$

Find the following limits, if they exist. Then sketch the graph of f.

(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$

6. Let

$$g(x) = \begin{cases} x^2 & \text{if } x \leqslant 1 \\ 2-x & \text{if } x > 1 \end{cases}$$

Find the following limits, if they exist. Then sketch the graph of g.

(a) $\lim_{x\to 1^-} g(x)$ (b) $\lim_{x\to 1^+} g(x)$ (c) $\lim_{x\to 1} g(x)$

7. Let

$$h(x) = \begin{cases} 1-x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ -x-1 & \text{if } x > 0 \end{cases}$$

Find the following limits, if they exist. Then sketch the graph of h.

(a) $\lim_{x\to 0^-} h(x)$ (b) $\lim_{x\to 0^+} h(x)$ (c) $\lim_{x\to 0} h(x)$

8. Let

$$f(x) = \begin{cases} -1 & \text{if } x \leqslant -2 \\ \frac{1}{2}x & \text{if } -2 < x < 2 \\ 1 & \text{if } x \geqslant 2 \end{cases}$$

(a) Find the following limits.

(i) $\lim_{x\to -2^-} f(x)$ (ii) $\lim_{x\to -2^+} f(x)$

(iii) $\lim_{x\to 2^-} f(x)$ (iv) $\lim_{x\to 2^+} f(x)$

(b) Sketch the graph of f.

(c) Where is f discontinuous?

9. Let

$$f(x) = \begin{cases} (x+1)^2 & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x \leqslant 1 \\ 2x - x^2 & \text{if } x > 1 \end{cases}$$

(a) Find the following limits, if they exist.

(i) $\lim_{x\to -1^-} f(x)$ (ii) $\lim_{x\to -1^+} f(x)$ (iii) $\lim_{x\to -1} f(x)$

(iv) $\lim_{x\to 1^-} f(x)$ (v) $\lim_{x\to 1^+} f(x)$ (vi) $\lim_{x\to 1} f(x)$

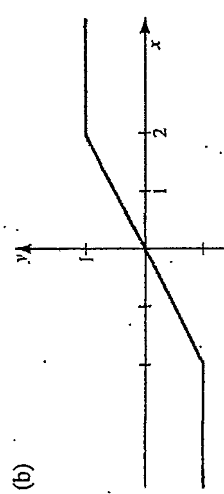
(b) Sketch the graph of f.

(c) Where is f discontinuous?

Answers Cont'd:

8. (a) (i) -1 (ii) -1 (iii) 1 (iv) 1

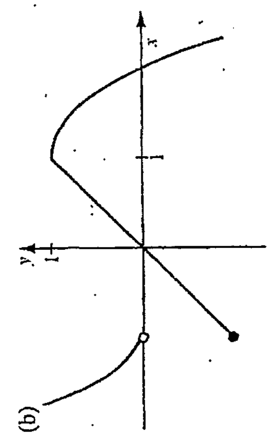
(b)



c) nowhere

9. (a) (i) 0 (ii) -1 (iii) does not exist (iv) 1 (v) 1 (vi) 1

(b)



(c) -1

Date:________ Name:________________

Intuitive Approach to Limits

1. Based on the graph below, determine:

a) $\lim_{x\to -12^-} f(x) =$ ____ b) $\lim_{x\to -9^+} f(x) =$ ____ c) $\lim_{x\to 5^-} f(x) =$ ____

d) $\lim_{x\to -9^-} f(x) =$ ____ e) $\lim_{x\to -6^+} f(x) =$ ____ f) $\lim_{x\to 0^+} f(x) =$ ____

g) $\lim_{x\to -6^-} f(x) =$ ____ h) $\lim_{x\to -3^+} f(x) =$ ____ i) $\lim_{x\to 2^+} f(x) =$ ____

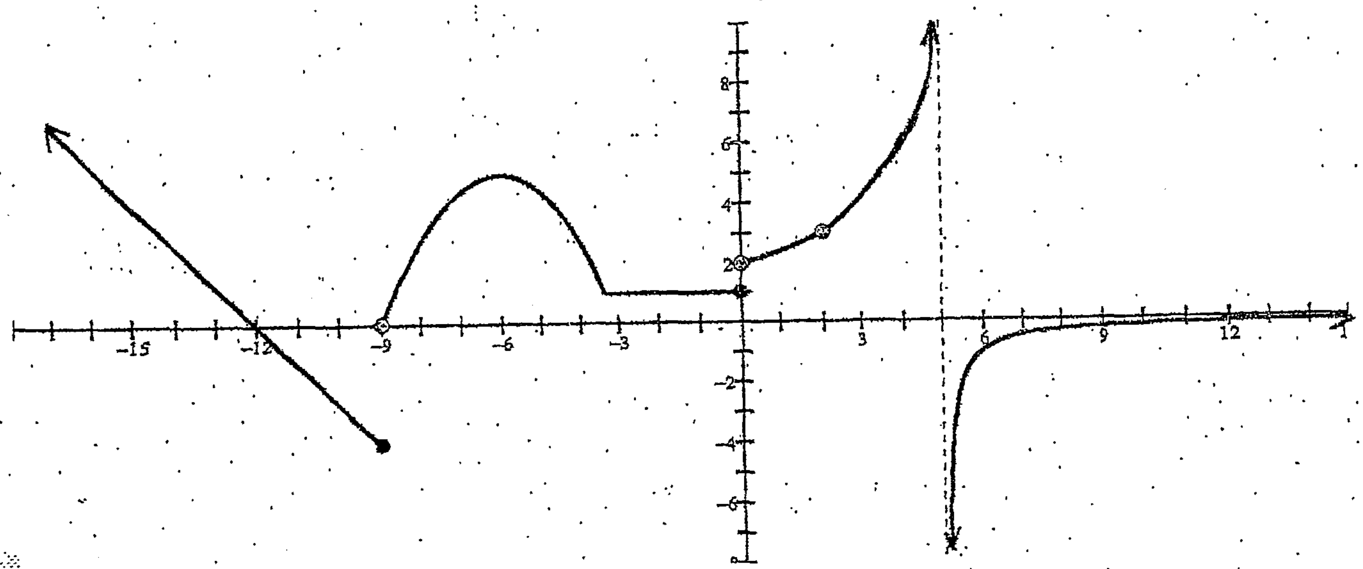
j) $\lim_{x\to -3^-} f(x) =$ ____ k) $\lim_{x\to 0^-} f(x) =$ ____ l) $\lim_{x\to 5^+} f(x) =$ ____

m) $\lim_{x\to -12^+} f(x) =$ ____ n) $\lim_{x\to 2^-} f(x) =$ ____

State whether or not the following limits exist. State the values of those limits that do exist.

a) $\lim_{x\to -12} f(x) =$ ____ b) $\lim_{x\to 5} f(x) =$ ____ c) $\lim_{x\to -3} f(x) =$ ____ d) $\lim_{x\to -6} f(x) =$ ____

e) $\lim_{x\to -9} f(x) =$ ____ f) $\lim_{x\to 2} f(x) =$ ____ g) $\lim_{x\to 0} f(x) =$ ____ h) $\lim_{x\to \infty} f(x) =$ ____

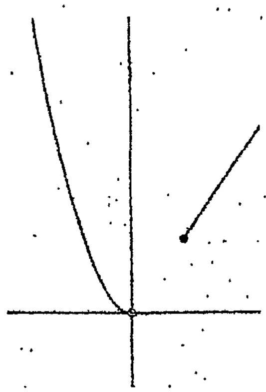


2. a) Given the function defined by:

$$f(x)=\begin{cases} x \text{ if } x \geq 1 \\ x^2 \text{ if } x < 0 \end{cases}$$

explain why:

$$\lim_{x\to 1} f(x) \neq 1$$



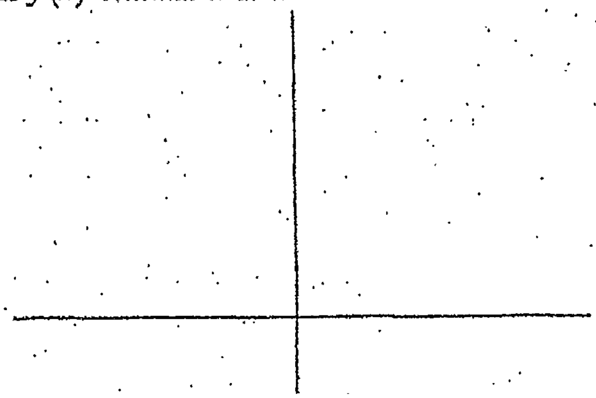
b) Let the function f be defined by:

$$f(x)=\begin{cases} x-2 \text{ if } x \neq 4 \\ 1 \text{ if } x=4 \end{cases}$$

i) Draw the graph of this function.

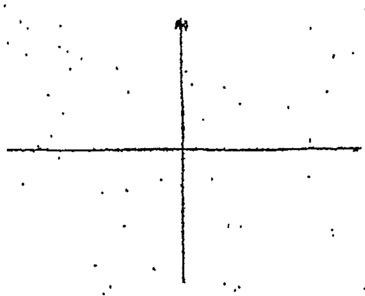
ii) $\lim_{x\to 4} f(x) = ?$

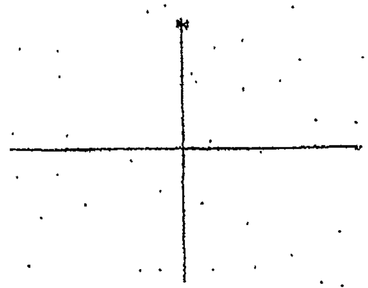
iii) Is $f(x)$ continuous at $x=4$?

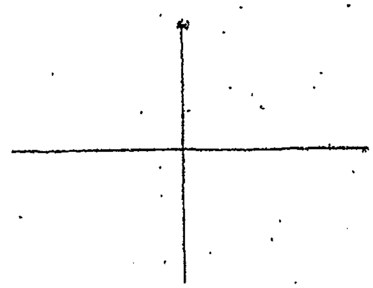


3. By drawing the graph for each of the following, find the limits:

a) $\lim_{x\to 0^+} |x-2| =$ ______ b) $\lim_{x\to 0^+} \sqrt{x} =$ ______ c) $\lim_{x\to 1}(x^2-2) =$ ______







Exercise 3.1

Part A

1. Find the slope of the line through each pair of points.
   a. (2, 7), (−3, −8) b. $\left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, -\frac{7}{2}\right)$ c. (6.3, −2.6), (1.5, −1)

2. What is the slope of a line perpendicular to the following?
   a. $y = 3x - 5$ b. $13x - 7y - 11 = 0$

3. State the equation and sketch the graph of the following straight lines:
   a. passing through (−4, −4) and $\left(-\frac{5}{3}, -\frac{5}{3}\right)$
   b. having slope 8 and y-intercept 6
   c. having x-intercept 5 and y-intercept −3
   d. passing through (5, 6) and (5, −9)

4. Simplify each of the following:
   a. $\frac{(2+h)^2-4}{h}$ b. $\frac{(5+h)^3-125}{h}$ c. $\frac{(3+h)^4-81}{h}$ d. $\frac{\frac{1}{1+h}-1}{h}$
   e. $\frac{3(1+h)^2-3}{h}$ f. $\frac{(2+h)^3-8}{h}$ g. $\frac{\frac{3}{4+h}-\frac{3}{4}}{h}$ h. $\frac{\frac{-1}{2+h}+\frac{1}{2}}{h}$

7. Consider the function $f(x) = x^3$.
   a. Copy and complete the following table of values; P and Q are points on the graph of $f(x)$.

P	Q	Slope of Line PQ	P	Q	Slope of Line PQ
(2,)	(3,)		(2,)	(1,)	
(2,)	(2.5,)		(2,)	(1.5,)	
(2,)	(2.1,)		(2,)	(1.9,)	
(2,)	(2.01,)		(2,)	(1.99,)	

   b. Use the results of part a to approximate the slope of the tangent to the graph of $f(x)$ at point P.
   c. Calculate the slope of the secant PR, where the x-coordinate of R is $2 + h$.
   d. Use the result of part c to calculate the slope of the tangent to the graph of $f(x)$ at point P.
   e. Compare your answers for parts b and d.
   f. Sketch the graph of $f(x)$ and the tangent to the graph at point P.

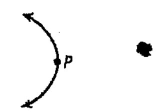
8. Find the slope of the tangent to each curve at the point whose x-value is given.
   a. $y = 3x^2$; (−2, 12) b. $y = x^2 - x$ at $x = 3$ c. $y = x^3$ at $x = -2$

9. Find the slope of the tangent to each curve at the point whose x-value is given.
   a. $y = \sqrt{x-2}$; (3, 1) b. $y = \sqrt{x-5}$ at $x = 9$ c. $y = \sqrt{5x-1}$ at $x = 2$

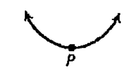
10. Find the slope of the tangent to each curve at the point whose x-value is given.
   a. $y = \frac{8}{x}$; (2, 4) b. $y = \frac{8}{3+x}$ at $x = 1$ c. $y = \frac{1}{x+2}$ at $x = 3$

15. Copy the following figures. Draw an approximate tangent for each curve at point P.

a.



b.

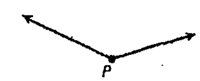


c.

d.

e.

f.



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Exercise 3.1

1. a. 3 **b.** $\frac{-5}{3}$ **c.** $\frac{-1}{3}$
2. a. $\frac{-1}{3}$ **b.** $\frac{-7}{13}$
3. a. $x - y = 0$ **b.** $y = 8x + 6$ **c.** $3x - 5y - 15 = 0$ **d.** $x - 5 = 0$
4. a. $4 + h$ **b.** $75 + 15h + h^2$ **c.** $108 + 54h + 12h^2 + h^3$ **d.** $\frac{-1}{1+h}$ **e.** $6 + 3h$ **f.** $12 + 6h + h^2$ **g.** $\frac{-3}{4(4+h)}$ **h.** $\frac{1}{4+2h}$
5. a. $\frac{1}{\sqrt{16+h}+4}$ **b.** $\frac{h+5}{\sqrt{h^2+5h+4}+2}$ **c.** $\frac{1}{\sqrt{5+h}+\sqrt{5}}$
6. a. $6 + 3h$ **b.** $3 + 3h + h^2$ **c.** $\frac{1}{\sqrt{9+h}+3}$
7. a. $P(2, 8)$

Q	Slope of PQ
(3, 27)	19
(2.5, 15.625)	15.25
(2.1, 9.261)	12.61
(2.01, 8.120601)	12.0601
(1, 1)	7
(1.5, 3.375)	9.25
(1.9, 6.859)	11.41
(1.99, 7.880599)	11.9401

b. 12 **c.** $12 + 6h + h^2$ **d.** 12
8. a. −12 **b.** 5 **c.** 12
9. a. $\frac{1}{2}$ **b.** $\frac{1}{4}$ **c.** $\frac{5}{6}$
10. a. −2 **b.** $\frac{-1}{2}$ **c.** $\frac{-1}{25}$
11. a. 1 **b.** −1 **c.** 9 **d.** $\frac{1}{4}$ **e.** $\frac{-1}{10}$ **f.** $\frac{-3}{4}$ **g.** $\frac{-1}{6}$ **h.** $\frac{-1}{16}$
16. $\frac{-5}{4}$
17. 1600 papers/month
18. (2, 4)
19. $\left(-2, \frac{28}{3}\right), \left(-1, \frac{26}{3}\right), \left(1, \frac{-26}{3}\right), \left(2, \frac{-28}{3}\right)$

1. Find the slope of the secant line to the curve $f(x) = 3x^2 + 4x - 10$ when $x_1 = -2$ and $x_2 = 3$.

2. Find the slope of the tangent line to the curve $f(x) = 3x^2 + 4x - 10$ when $x = 2$.

3. Find the equation of the tangent line to the curve $f(x) = 3x^2 + 4x - 10$ when $x = 2$.
   - Hint: use the point on the curve that the tangent line touches to solve $y = mx + b$

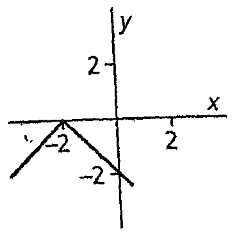
4. Find the slope of the tangent line to the curve $f(x) = \frac{2}{x+1}$ when $x = 3$.

Exercise 4.1

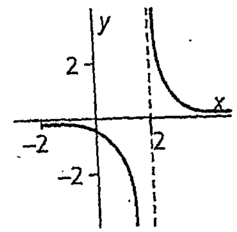
Part A

1. State the domain on which f is differentiable.

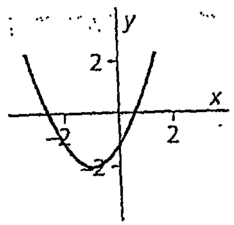
a.



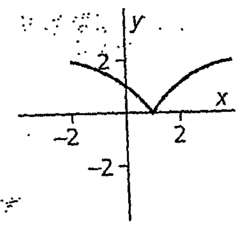
b.



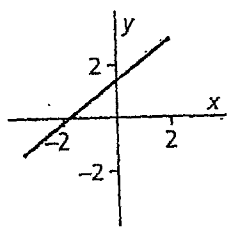
c.



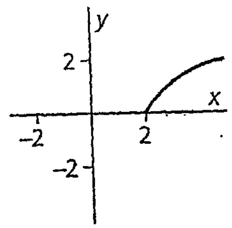
d.



e.



f.



2. Explain what the derivative of a function represents.

3. Illustrate two situations in which a function does not have a derivative at $x = 1$.

Part B

4. For each function, find the value of the derivative $f'(a)$ for the given value of a.
   a. $f(x) = x^2; a = 1$
   b. $f(x) = x^2 + 3x + 1; a = 3$
   c. $f(x) = \sqrt{x+1}; a = 0$

5. Use the definition of the derivative to find $f'(x)$ for each function.
   a. $f(x) = x^2 + 3x$ b. $f(x) = \frac{3}{x+2}$ c. $f(x) = \sqrt{3x+2}$ d. $f(x) = \frac{1}{x^2}$

6. In each case, find the derivative $\frac{dy}{dx}$.
   a. $y = 6 - 7x$ b. $y = \frac{x+1}{x-1}$ c. $y = 3x^2$

7. Find the slope of the tangents to $y = 2x^2 - 4x$ when $x = 0$, $x = 1$, and $x = 2$. Sketch the graph, showing these tangents.

8. An object moves in a straight line with its position at time t seconds given by $s(t) = -t^2 + 8t$ where s is measured in metres. Find the velocity when $t = 0$, $t = 4$, and $t = 6$.

9. Find an equation of the straight line that is tangent to the graph of $f(x) = \sqrt{x+1}$ and parallel to $x - 6y + 4 = 0$.

Exercise 4.1

1. a. $x \in R, x \neq -2$ b. $x \in R, x \neq 2$ c. $x \in R$ d. $x \in R, x \neq 1$ e. $x \in R$ f. $x > 2, x \in R$
4. a. 2 b. 9 c. $\frac{1}{2}$
5. a. $2x + 3$ b. $\frac{-3}{(x+2)^2}$ c. $\frac{3}{2\sqrt{3x+2}}$ d. $\frac{-2}{x^3}$
6. a. -7 b. $\frac{-2}{(x-1)^2}$ c. $6x$
7. $-4, 0, 4$
8. 8 m/s, 0 m/s, -4 m/s
9. $x - 6y + 10 = 0$
10. a. 0 b. 1 c. m d. $2ax + b$
12. a and e, b and f, c and d
13. -1
14. $f'(0) = 0$
15. 3
16. $f(x) = (x-3)^{\frac{1}{3}}$, answers will vary

21

13. Determine where each function is discontinuous.

a) $f(x)=\begin{cases} 3x+1 & \text{if } x\neq 2 \\ 8 & \text{if } x=2 \end{cases}$

b) $f(x)=\begin{cases} x+2 & \text{if } x\in(-\infty,\ 0] \\ x^2+2 & \text{if } x\in(0,\ 3] \\ 2x+4 & \text{if } x\in(3,\ \infty) \end{cases}$

c) $f(x)=\begin{cases} x & \text{if } x\in(-\infty,\ -1] \\ -x^3 & \text{if } x\in(-1,\ 1) \\ -x & \text{if } x\in[1,\ \infty) \end{cases}$

d) $f(x)=\begin{cases} x^2-1 & \text{if } x\in(-\infty,\ -1) \\ x^2+2x+1 & \text{if } x\in[-1,\ 1] \\ x^2-2x+1 & \text{if } x\in(1,\ \infty) \end{cases}$

20. Consider the function $f(x)$ described by

$f(x)=\begin{cases} 1 & \text{if } x \text{ is an integer} \\ 0 & \text{if } x \text{ is not an integer} \end{cases}$

Find each of the following, if it exists.

a) $\lim\limits_{x\to\frac{1}{2}} f(x)$ b) $\lim\limits_{x\to 2} f(x)$

21. Find each limit, if it exists.

a) $\lim\limits_{x\to\sqrt{3}} \dfrac{x^2-3}{3-\sqrt{6+x^2}}$

b) $\lim\limits_{x\to 8} \dfrac{x-8}{\sqrt[3]{x}-2}$

c) $\lim\limits_{x\to 3} \dfrac{\sqrt{7-x}-2}{1-\sqrt{4-x}}$

23. Evaluate $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$.

24. Let

$f(x)=\begin{cases} 1-|x| & \text{if } x\in[-1,\ 1] \\ |x|-1 & \text{if } x\in[-2,\ -1) \text{ or } x\in(1,\ 2] \\ (x-3)^2 & \text{if } x\in(2,\ \infty) \\ (x+3)^2 & \text{if } x\in(-\infty,\ -2) \end{cases}$

Sketch the graph of $f(x)$ and determine the values of x at which $f(x)$ is discontinuous.

25. For what values of c is the function

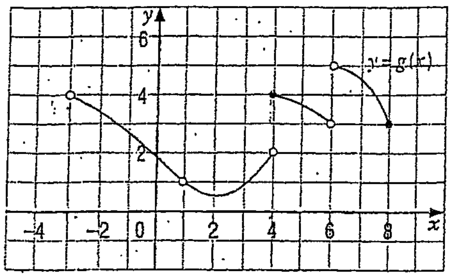
$f(x)=\begin{cases} (cx-1)^3 & \text{if } x\in(-\infty,\ 2) \\ c^2x^2-1 & \text{if } x\in[2,\ \infty) \end{cases}$

continuous at every number?

Answers:

13. a) $x=2$ b) $x=3$ c) $x=-1$ d) $x=1$ **14.** b) i) 800 ii) 900 iii) does not exist iv) 900 v) 1000 vi) does not exist **15.** b) i) 0.94 ii) 1.55 iii) does not exist iv) 1.55 v) 2.05 vi) does not exist **16.** b) any whole number values of a **17.** At 0.25 km, 0.35 km, 0.45 km, ... **18.** b) $t=0,\ 1,\ 2,\ 3$ **19.** a) $I(R)=\dfrac{2(5+R)}{R}(1-H(2-R));\ R\geq 0$ c) i) 0 ii) 7 iii) 0 iv) does not exist **20.** a) 0 b) 0 **21.** a) -6 b) 12 c) $-\dfrac{1}{2}$ **22.** Answers will vary. For example, $f(x)=\dfrac{1}{x},\ g(x)=-\dfrac{1}{x}$. **23.** $\dfrac{3}{2}$ **24.** $f(x)$ is not discontinuous anywhere. **25.** $c=0,\ \dfrac{1}{2}$, or $\dfrac{3}{2}$

2. Use the graph of $g(x)$ to state the value of each limit, if it exists. If it does not exist, explain why.



a) $\lim_{x \to -3^+} g(x)$ b) $\lim_{x \to 1^-} g(x)$

c) $\lim_{x \to 1^+} g(x)$ d) $\lim_{x \to 1} g(x)$

e) $\lim_{x \to 2^-} g(x)$ f) $\lim_{x \to 2^+} g(x)$

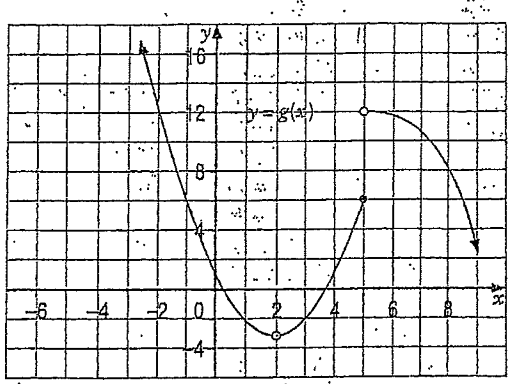
g) $\lim_{x \to 2} g(x)$ h) $\lim_{x \to 4^-} g(x)$

i) $\lim_{x \to 4^+} g(x)$ j) $\lim_{x \to 4} g(x)$

k) $\lim_{x \to 6^-} g(x)$ l) $\lim_{x \to 6^+} g(x)$

m) $\lim_{x \to 6} g(x)$ n) $\lim_{x \to 8^-} g(x)$

3. Use the graph of $g(x)$ to state the value of each quantity, if it exists. If it does not exist, explain why.



a) $\lim_{x \to -2} g(x)$ b) $\lim_{x \to 0} g(x)$

c) $\lim_{x \to 2^-} g(x)$ d) $g(2)$

e) $\lim_{x \to 2^+} g(x)$ f) $\lim_{x \to 2} g(x)$

g) $\lim_{x \to 5^-} g(x)$ h) $g(5)$

i) $\lim_{x \to 5^+} g(x)$ j) $\lim_{x \to 5} g(x)$

5. Let

$$f(x) = \begin{cases} -2 & \text{if } x \in (-\infty, 1) \\ x - 3 & \text{if } x \in [1, \infty). \end{cases}$$

Sketch the graph of $f(x)$. Then, find each limit, if it exists. If the limit does not exist, explain why.

a) $\lim_{x \to 1^-} f(x)$ b) $\lim_{x \to 1^+} f(x)$ c) $\lim_{x \to 1} f(x)$

6. Let

$$g(x) = \begin{cases} x^2 & \text{if } x \in (-\infty, 2] \\ -x + 1 & \text{if } x \in (2, \infty) \end{cases}$$

Sketch the graph of $g(x)$. Then, find each limit, if it exists. If the limit does not exist, explain why.

a) $\lim_{x \to 2^-} g(x)$ b) $\lim_{x \to 2^+} g(x)$ c) $\lim_{x \to 2} g(x)$

7. Let

$$h(x) = \begin{cases} x + 2 & \text{if } x \in (-\infty, 0) \\ 0 & \text{if } x = 0 \\ -x - 2 & \text{if } x \in (0, \infty) \end{cases}$$

Sketch the graph of $h(x)$. Then, find each limit, if it exists. If the limit does not exist, explain why.

a) $\lim_{x \to 0^-} h(x)$ b) $\lim_{x \to 0^+} h(x)$ c) $\lim_{x \to 0} h(x)$

8. Let

$$f(x) = \begin{cases} -x^2 & \text{if } x \in (-\infty, -1) \\ 1 & \text{if } x = -1 \\ x & \text{if } x \in (-1, \infty) \end{cases}$$

Sketch the graph of $f(x)$. Then, find each limit, if it exists. If the limit does not exist, explain why.

a) $\lim_{x \to -1^-} f(x)$ b) $\lim_{x \to -1^+} f(x)$ c) $\lim_{x \to -1} f(x)$

Answers :)

2. a) 4 b) 1 c) 1 d) 1 e) $\frac{1}{2}$ f) $\frac{1}{2}$ g) $\frac{1}{2}$ h) 2 i) 4 j) does not exist; jump discontinuity k) 3 l) 5 m) does not exist; jump discontinuity n) 3 3. a) 12 b) 1 c) −3 d) undefined e) −3 f) −3 g) 6 h) 6 i) 12 j) does not exist; jump discontinuity 4. Question 1: removable discontinuity at $x = -1$; infinite discontinuity at $x = 4$. Question 2: removable discontinuity at $x = 1$; jump discontinuity at $x = 4$; jump discontinuity at $x = 6$. Question 3: removable discontinuity at $x = 2$; jump discontinuity at $x = 5$. 5. a) −2 b) −2 c) −2 6. a) 4 b) −1 c) does not exist; one-sided limits are different (jump discontinuity) 7. a) 2 b) −2 c) does not exist; one-sided limits are different (jump discontinuity) 8. a) −1 b) −1 c) −1

One-Sided Limits - Exercise

1. By evaluating one-sided limits, find the indicated limit (if it exists).

a. $f(x) = \begin{cases} x+2, & \text{if } x < -1 \\ -x+2, & \text{if } x \geq -1 \end{cases}; \lim_{x \to -1} f(x)$

b. $f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ -x+3, & \text{if } x > 1 \end{cases}; \lim_{x \to 1} f(x)$

c. $f(x) = \begin{cases} -x+4, & \text{if } x \leq 2 \\ -2x+6, & \text{if } x > 2 \end{cases}; \lim_{x \to 2} f(x)$

d. $f(x) = \begin{cases} 4-x^2, & \text{if } x < 1 \\ x, & \text{if } x \geq 1 \end{cases}; \lim_{x \to 1} f(x)$

e. $f(x) = \begin{cases} 4x, & \text{if } x \geq \frac{1}{2} \\ \frac{1}{x}, & \text{if } x < \frac{1}{2} \end{cases}; \lim_{x \to \frac{1}{2}} f(x)$

f. $f(x) = \begin{cases} x+3, & \text{if } x < 2 \\ \frac{4}{x}, & \text{if } x \geq 2 \end{cases}; \lim_{x \to 2} f(x)$

g. $f(x) = \begin{cases} 1, & \text{if } x < -\frac{1}{2} \\ x^2 - \frac{1}{4}, & \text{if } x \geq -\frac{1}{2} \end{cases}; \lim_{x \to -\frac{1}{2}} f(x)$

h. $f(x) = \begin{cases} 2+x^2, & \text{if } x < 0 \\ 2-x^2, & \text{if } x \geq 0 \end{cases}; \lim_{x \to 0} f(x)$

2. Sketch the graph of any function that satisfies the given conditions in each case.

a. $\lim_{x \to 1^+} f(x) = 3$, $\lim_{x \to 1^-} f(x) = 2$, $f(1) = 1$

b. $\lim_{x \to 2} f(x) = 0$, $f(2) = 1$

c. $\lim_{x \to 0^-} f(x) = 1$, $\lim_{x \to 0^+} f(x) = -1$, $f(0) = 1$

d. $f(x) = 1$, if $x < 1$ and $\lim_{x \to 1^+} f(x) = 2$

Answers 1. a. dne; b. dne; c. 2; d. dne; e. 2; f. dne; g. dne; h. 2;

LIMITS AS X APPROACHES INFINITY

Find the following limits:

1. $\lim_{x\to\infty}\frac{1}{x^2}$
2. $\lim_{x\to\infty}\frac{1}{5+x}$
3. $\lim_{x\to\infty}\left(6+\frac{1}{x^3}\right)$
4. $\lim_{n\to\infty}\frac{n}{3n-1}$
5. $\lim_{n\to\infty}\frac{6n+9}{3n-2}$
6. $\lim_{n\to\infty}5n$
7. $\lim_{n\to\infty}\frac{n^2+1}{2n^2-1}$
8. $\lim_{n\to\infty}\frac{(n+1)^2}{n(n+2)}$
9. $\lim_{n\to\infty}\frac{n}{n^2+1}$
10. $\lim_{n\to\infty}5^{-n}$
11. $\lim_{n\to\infty}(n^3+n^2)$
12. $\lim_{n\to\infty}\frac{1+n-2n^2}{1-n+n^2}$
13. $\lim_{n\to\infty}\frac{1}{\sqrt{n}}$
14. $\lim_{n\to\infty}\frac{1}{n^5}$
15. $\lim_{n\to\infty}\frac{1-n^3}{1+2n^3}$
16. $\lim_{n\to\infty}\left(\frac{2}{3}\right)^n$
17. $\lim_{n\to\infty}\left(\frac{4}{3}\right)^n$
18. $\lim_{n\to-\infty}\frac{2x+3}{\sqrt{x^2+7}}$
19. $\lim_{n\to\infty}\left(\sqrt{x+8}-\sqrt{x}\right)$
20. $\lim_{n\to\infty}\left(\sqrt{x^2-8}-\sqrt{x}\right)$
21. $\lim_{n\to\infty}\left(\frac{x}{x+1}-\frac{x}{x-1}\right)$
22. $\lim_{b\to\infty}(-1)^b$
23. $\lim_{n\to\infty}\frac{1-n^3}{5+n^2}$
24. $\lim_{n\to\infty}\frac{1+n+2n^3}{1-n+n^2}$
25. $\lim_{n\to\infty}\frac{n^3}{n-3}$
26. $\lim_{x\to\infty}\frac{3^{2x}-3^{-2x}}{3^{2x}+3^{-2x}}$
27. $\lim_{p\to\infty}\left(-\frac{3}{4}\right)^p$
28. $\lim_{p\to\infty}-\left(\frac{3}{4}\right)^p$

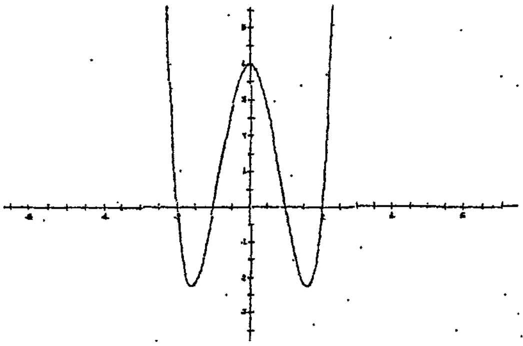
ANSWERS

1. 0	2. 0	3. 6	4. 1/3	5. 2	6. ∞
7. $\frac{1}{2}$	8. 1	9. 0	10. 0	11. ∞	12. -2
13. 0	14. 0	15. -1/2	16. 0	17. ∞	18. 2
19. 0	20. ∞	21. 0	22. DNE	23. $-\infty$	24. ∞
25. ∞	26. 1	27. DNE	28. 0		

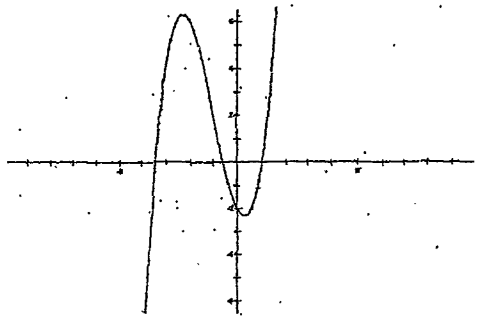
Name________________

GRAPH OF THE DERIVATIVE FUNCTION

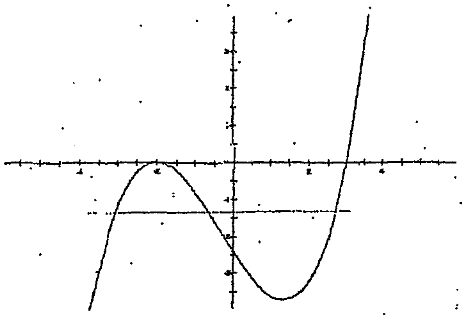
Given graph of function, sketch graph of derivative on the same grid:



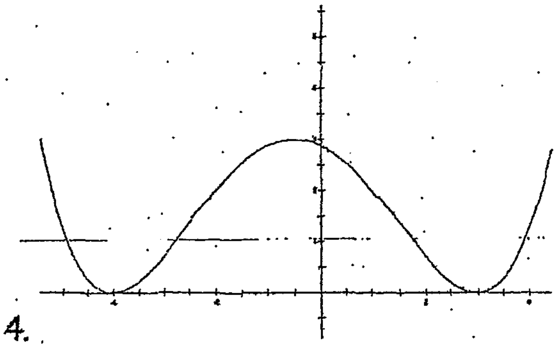
1.



2.



3.



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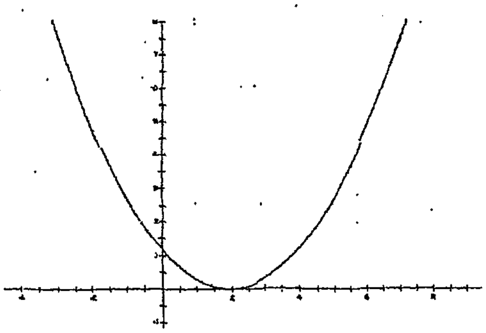
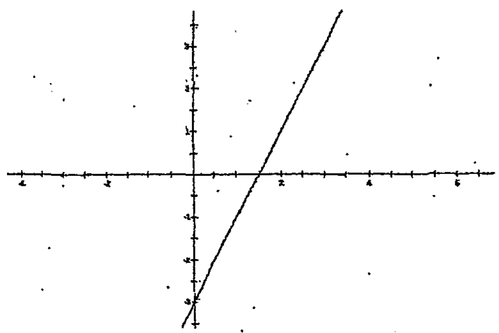


5.



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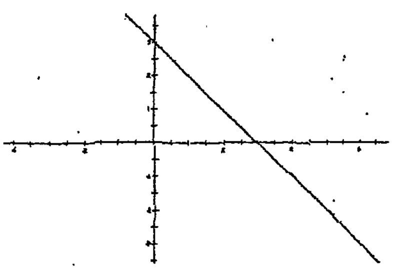

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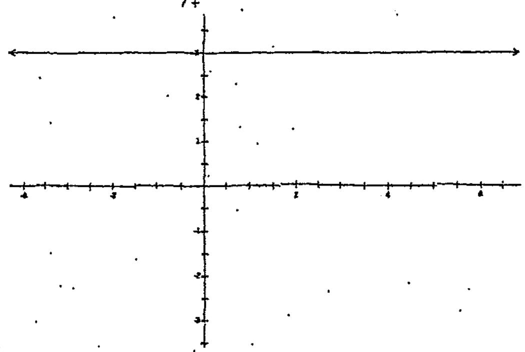
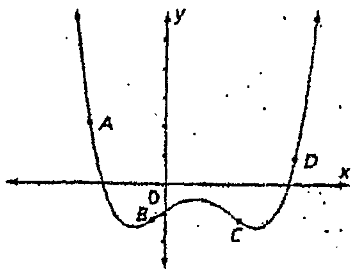


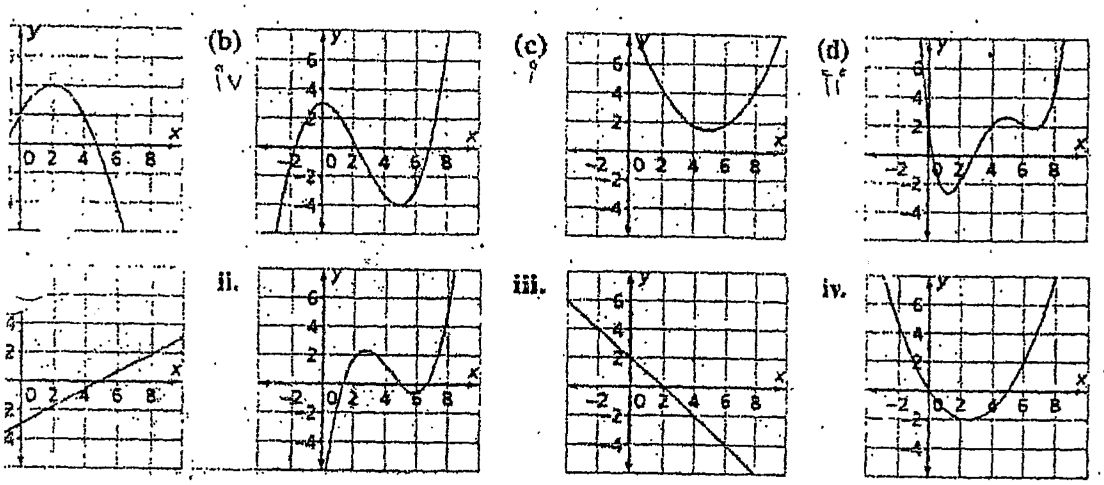
8.

Relating the Graph of a function and its Derivative.

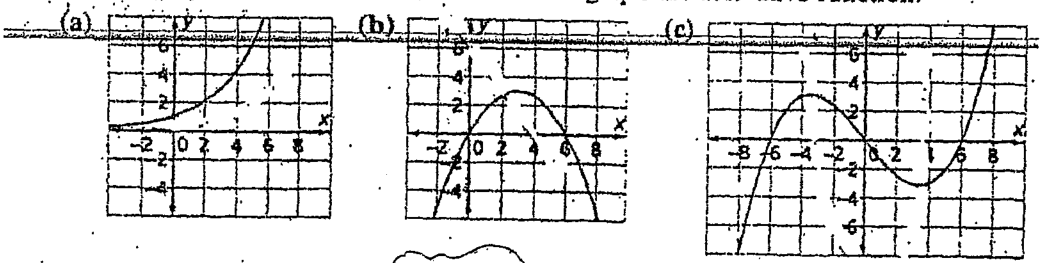
3. Determine the sign of $\frac{dy}{dx}$ at points A, B, C and D.



1. Match each graph in the top row with the graph of its derivative function in the bottom row.



2. For the graph of each function, estimate and graph the derivative function.



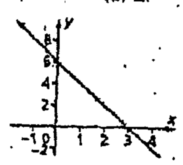
Answers:

3. A: neg.; B: pos ; C: neg., D: pos.

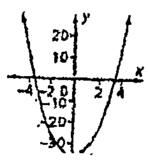
1. (a) iii. (b) iv. (c) i. (d) ii.

2. (a)

(b)



(c)

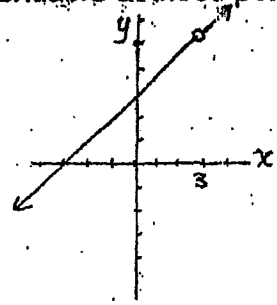


Non-differentiable Points

There are three situations in which a function may have point(s) that are not differentiable, or in other words, at which the derivative does not exist.

1) Functions that are not continuous are also not differentiable at those points.

Ex: $f(x) = \dfrac{x^2-9}{x-3} = \dfrac{(x+3)(x-3)}{(x-3)}$



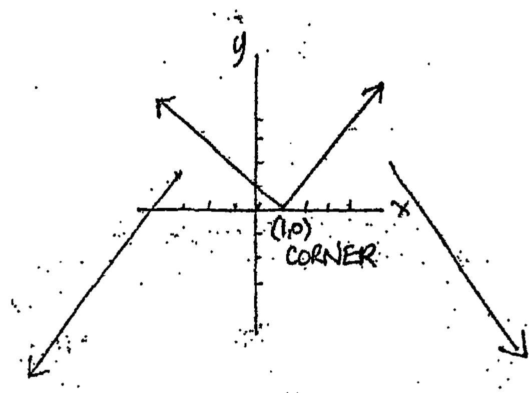
- Function is not defined at x = 3
- No tangent can be drawn at x = 3
- No derivative exists at x = 3

2) Functions that contain 'corners' or 'cusps'

Ex: $f(x) = |x-1|$

Consider:

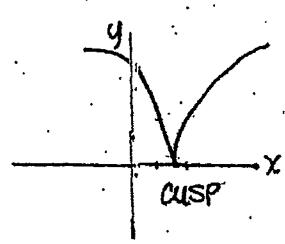
x	f(x)
-3	4
-2	3
-1	2
0	1
1	0
2	1
3	2



$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \frac{-1}{1} = -1 \qquad \lim_{h\to 0+}\frac{f(1+h)-f(1)}{h} = \frac{1}{1} = 1$$

- Since $\lim\limits_{h\to 0^-} \neq \lim\limits_{h\to 0+}$, the limit does not exist at x=1
- Since the limit does not exist, the derivative f' (1) does not exist
- The slope changes abruptly from –1 to +1 at (1, 0) (a corner)
- No unique tangent line can be drawn at this point

Ex: $f(x) = (3x-6)^{\frac{2}{3}}$ which gives a curve like



$$\lim_{h\to 0-}\frac{f(2+h)-f(2)}{h} = -\infty \qquad \lim_{h\to 0+}\frac{f(2+h)-f(2)}{h} = \infty$$

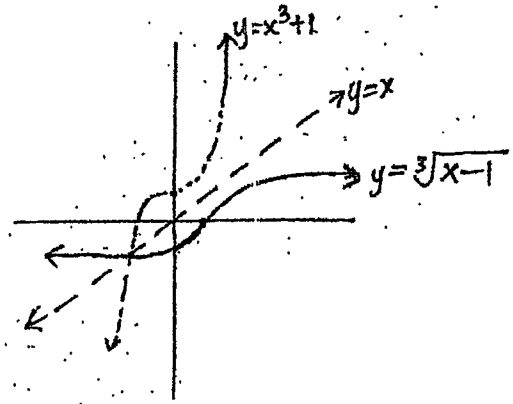
- The slope changes from $-\infty$ to ∞ at x = 2, therefore f(x) is not differentiable at x = 2.

Non-differentiable Points

3) Functions that contain a vertical tangent :

Ex: $f(x) = (x-1)^{\frac{1}{3}} = \sqrt[3]{x-1}$

{Inverse of $f(x) = x^3 + 1$}



$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \frac{-1}{1} = -1$$

$\lim_{x \to 1}$ does not exist.

$$\lim_{h \to 0+} \frac{f(1+h) - f(1)}{h} = \frac{1}{1} = 1$$

- A tangent can be drawn at (1,0), but this tangent is vertical, thus its slope is undefined.
- In turn the limit is undefined at x = 1 and the function is not differentiable at this point.

Conclusions:

1. Discontinuous functions are not differentiable at the points of discont nuity. (rational and square root functions)

2. Absolute value functions are not differentiable at their vertex.

3. Cube root functions are not differentiable at their points of inflection.

Quiz: Limits Name: ____________________

1. Find the following limits.

a) $\lim_{x\to 0} \pi$ ✓

b) $\lim_{x\to -1} 2x^2 - 5x + 3$ ✓

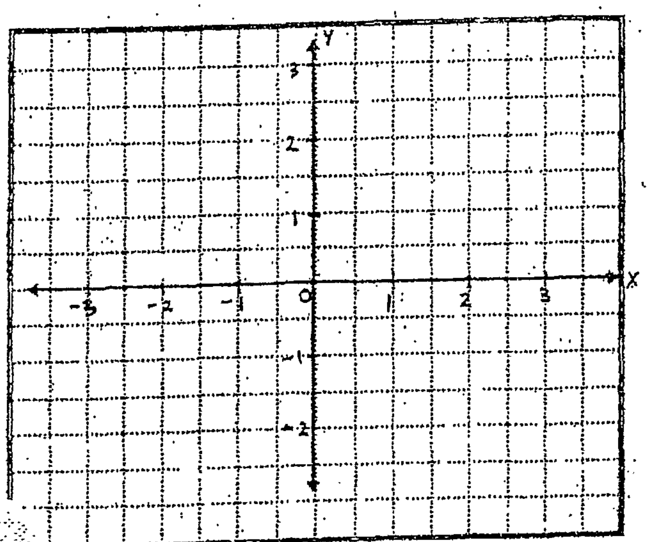
c) $\lim_{x\to 4} \frac{x^2+2x-6}{x-4}$ ✓✓

d) $\lim_{x\to 4} \frac{x^2-16}{3x-12}$ ✓✓

e) $\lim_{h\to 0} \frac{h}{\sqrt{3+h}-\sqrt{3}}$ ✓✓✓

2. Sketch the graph of the function $f(x) = \begin{cases} -x-2, & \text{if } x \le -1 \\ x, & \text{if } -1 < x < 1 \\ x^2 - 2x, & \text{if } x \ge 1 \end{cases}$. ✓✓✓

Determine $\lim_{x\to -1} f(x)$ and $\lim_{x\to 1} f(x)$. ✓✓✓✓



Review

1. Evaluate the following limits

a) $\lim_{x\to-4}\frac{x^2+5x+4}{x^2+3x-4}$ b) $\lim_{x\to3}\frac{2x^2-x-15}{3x^2-13x+12}$ c) $\lim_{x\to-4}\frac{\sqrt{x^2+9}-5}{x+4}$ d) $\lim_{x\to16}\frac{4-\sqrt{x}}{16x-x^2}$

e) $\lim_{x\to-4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x}$ f) $\lim_{x\to4}\frac{x^3-64}{x-4}$ g) $\lim_{x\to-\infty}\frac{-2x^3+5x-7}{x^2+5x+4}$ h) $\lim_{x\to\infty}\frac{2x^3-1}{5x^3+2x^2-3x}$

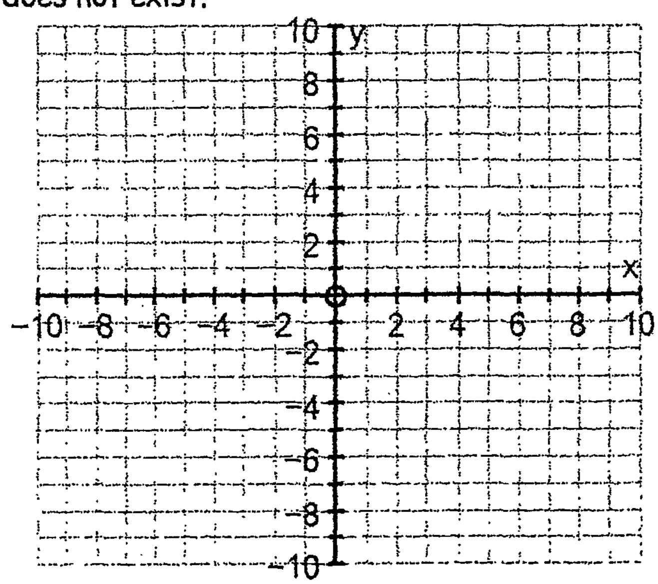
2. Using **first principles**, determine the derivative of the following functions
   a) $f(x) = 2x^2 - 3x$
   b) $f(x) = \frac{1}{3x+2}$

3. Find the equation of the tangent line and normal line to the curve $y = x^2 - 6x + 9$ at $x = 2$.

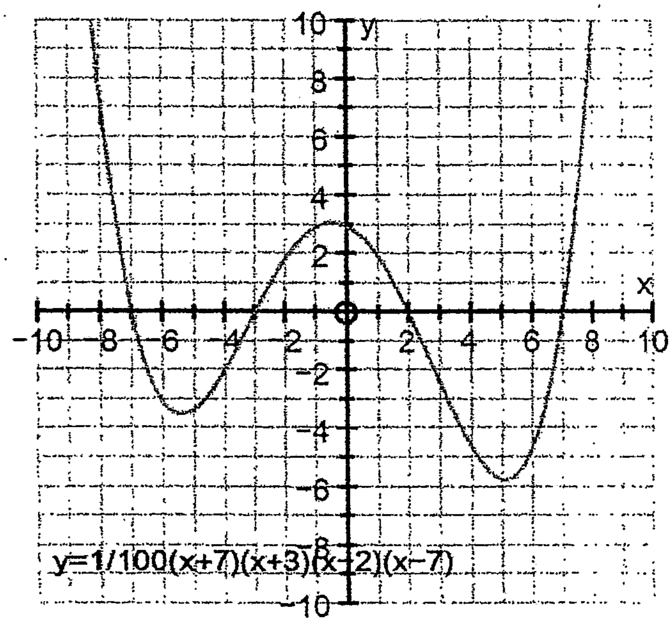
4. Find an equation of the tangent line to the parabola $y = x^2 - 5x + 4$ that is perpendicular to the line $y = -\frac{1}{3}x + 5$

5. If a stone is dropped off a 500 m high cliff, then its height after t seconds, and before it hits the ground, is $h(t) = 500 - 5t^2$.
   a) Find the average velocity of the stone in the time interval $1 \le t \le 3$.
   b) How would you find the instantaneous velocity at $t = 2$?

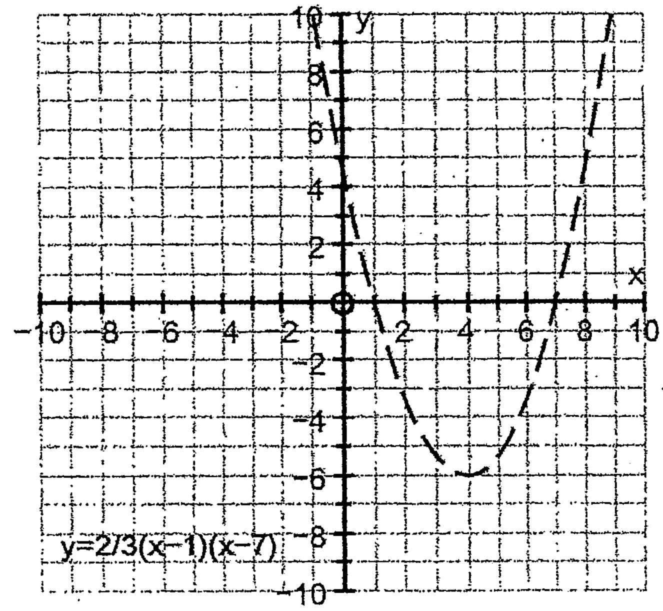
6. Sketch the graph of $f(x) = \begin{cases} -2x-8, & x \le -2 \\ \frac{3}{2}x - 1, & -2 < x \le 2 \\ (x-4)^2 - 6, & x > 2 \end{cases}$. Determine where $f(x)$ is continuous and state the points where the derivative of $f(x)$ does not exist.



7. Given the graph of $f(x)$, sketch $f'(x)$



8. Given the graph of $f'(x)$, sketch $f(x)$



Challenge Question

1. Find all a and all b for which f is continuous at $x = 0$ *and* $x = 1$. $f(x) = \begin{cases} x^2 - a, & x \le 0 \\ \frac{-2}{x+b}, & 0 < x \le 1 \\ ax - 3, & x > 1 \end{cases}$

$\left[a = -3 \text{ or } 2, b = -\frac{2}{3} \text{ or } 1\right]$