Calculus

- Used to compare quantities which vary in a non-linear way
- 2 main branches: -Differentiation-helps us to find a rate of change of one quantity compared to another
  -Integration-(reverse of differentiation) given a rate of change, find the original relationship between the 2 quantities
- The idea of zooming in to get a better approximation of the slope of the curve was the breakthrough that led to the development of differentiation
- Methods: ① Numerical approach to finding slopes (limits & slope of a tangent)
  ② Algebraic approach to finding slopes (differentiation from first principles & derivatives)

1-1 Exploring the concept of a limit

Recall: A sequence is a list of numbers written in a definite order: $t_1, t_2, t_3, t_4, \ldots, t_n, \ldots$ where t_1 is the first term and in general t_n is the nth term.

Ex 1: List the first 5 terms of the sequence defined by $t_n = \dfrac{n}{n+1}$ and draw the graph of the sequence.

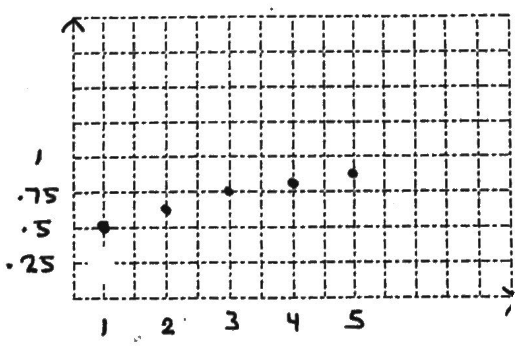
$t_1 = \frac{1}{2}$

$t_2 = \frac{2}{3}$

$t_3 = \frac{3}{4}$

$t_4 = \frac{4}{5}$

$t_5 = \frac{5}{6}$



$\therefore \lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$

In general, we say that the sequence with general term t_n has the limit L, and we write:

$$\boxed{\lim_{n\to\infty} t_n = L}$$

Ex 2: Find $\lim\limits_{n\to\infty} \dfrac{1}{n}$

$= 0$

Ex 3: Find $\lim\limits_{n\to\infty} (-1)^n$

= Does not Exist ("DNE")

Exercise 1.6

A 1. State the limits of the following sequences, or state that the limit does not exist.

(a) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \ldots, \left(\frac{1}{3}\right)^n, \ldots$

(b) $5, 4\frac{1}{2}, 4\frac{1}{3}, 4\frac{1}{4}, 4\frac{1}{5}, \ldots, 4 + \frac{1}{n}, \ldots$

(c) $1, 2, 3, 4, 5, \ldots, n, \ldots$

(d) $3, 3, 3, 3, 3, \ldots, 3, \ldots$

(e) $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots$

(f) $5, 6\frac{1}{2}, 5\frac{2}{3}, 6\frac{1}{4}, 5\frac{4}{5}, 6\frac{1}{6}, \ldots, 6 + \frac{(-1)^n}{n}, \ldots$

(g) $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \ldots$

(k) $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$ rational

(l) $\lim_{n\to\infty} (-1)^{n-1} n$

(m) $\lim_{n\to\infty} 5^{-n}$

(n) $\lim_{n\to\infty} (n^3 + n^2)$

(o) $\lim_{n\to\infty} \frac{1 + n - 2n^2}{1 - n + n^2} = -2$

(p) $\lim_{n\to\infty} \frac{1}{\sqrt{n}}$

(q) $\lim_{n\to\infty} \frac{1}{n^5}$

(r) $\lim_{n\to\infty} \frac{1 - n^3}{1 + 2n^3}$

(s) $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

(t) $\lim_{n\to\infty} \left(\frac{4}{3}\right)^n$

B 2. List the first six terms of the following sequences.

(a) $t_n = \frac{n-1}{2n-1}$

(b) $t_n = \frac{2n}{n^2+1}$

(c) $t_n = n2^n$

(d) $t_n = \frac{(-1)^{n-1}}{n}$

(e) $t_1 = 1, \; t_n = \frac{1}{1 + t_{n-1}} \; (n \geq 2)$

(f) $t_1 = 1, \; t_2 = 2, \; t_n = t_{n-1} - t_{n-2} \; (n \geq 3)$

3. Find the following limits or state that the limit does not exist.

(a) $\lim_{n\to\infty} \frac{1}{n^2}$

(b) $\lim_{n\to\infty} \frac{1}{5+n}$

(c) $\lim_{n\to\infty} \left(6 + \frac{1}{n^3}\right)$

(d) $\lim_{n\to\infty} \frac{n}{3n-1}$

(e) $\lim_{n\to\infty} \frac{6n+9}{3n-2}$

(f) $\lim_{n\to\infty} 5n$

(g) $\lim_{n\to\infty} \frac{n^2+1}{2n^2-1}$

(h) $\lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)}$

(i) $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n}$

(j) $\lim_{n\to\infty} \left(-\frac{1}{4}\right)^n$

Exercise 1.7

B 1. Find the sum of each of the following series or state that the series is divergent.

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

(b) $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \ldots$

(c) $\frac{1}{4} - \frac{5}{16} + \frac{25}{64} - \frac{125}{256} + \ldots$

(d) $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \ldots$

(e) $1 - 2 + 4 - 8 + \ldots$

(f) $60 + 40 + \frac{80}{3} + \frac{160}{9} + \ldots$

(g) $0.1 + 0.05 + 0.025 + 0.0125 + \ldots$

(h) $-3 + 3 - 3 + 3 - 3 + \ldots$

EXERCISE 1.6

1. (a) 0 (b) 4 (c) does not exist (d) 3 (e) 0 (f) 6 (g) does not exist

2. (a) $0, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}$ (b) $1, \frac{4}{5}, \frac{6}{10}, \frac{8}{17}, \frac{10}{26}, \frac{12}{37}$ (c) 2, 8, 24, 64, 160, 384 (d) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}$ (e) $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}$ (f) 1, 2, 1, −1, −2, −1

3. (a) 0 (b) 0 (c) 6 (d) $\frac{1}{3}$ (e) 2 (f) does not exist (g) $\frac{1}{2}$ (h) 1 (i) 0 (j) 0 (k) 0 (l) does not exist (m) 0 (n) does not exist (o) −2 (p) 0 (q) 0 (r) $-\frac{1}{2}$ (s) 0 (t) does not exist 4. $\frac{1}{3}$

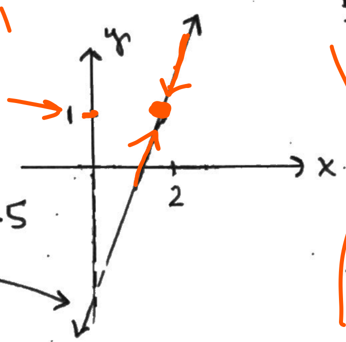
EXERCISE 1.7

1. (a) $\frac{3}{2}$ (b) $\frac{3}{5}$ (c) divergent (d) $\frac{15}{4}$ (e) divergent (f) 180 (g) $\frac{1}{5}$ (h) divergent

Introduction to Limits

Example 1: Consider the graph of the function $f(x)=3x-5$.
As the value of x "approaches" 2 (on the graph), what does the value of the function approach?

$\lim_{x\to 2} 3x-5=1$



$f(x)=3x-5$

Definition:

Limit → the notation $\lim_{x\to a} f(x)=L$ implies that as "x" approaches closer and closer to the value "a", the value of the function approaches a limiting value, L.

We say that "the *limit* of 3x-5 as x approaches 2 equals 1," and we write

$$\lim_{x\to 2}(3x-5)=1$$

$\lim_{x\to a} f(x) \neq f(a)$ ← not always true.

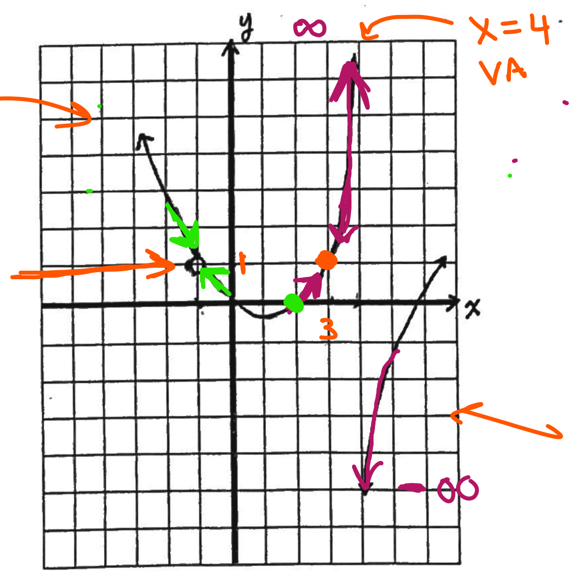
Notes:

- In this case $f(2)=1$ and $\therefore \lim_{x\to 2} f(x)=f(2)$.
- In retrospect it follows that $\lim_{x\to 2} f(x)$ could have been evaluated simply by substituting $x=2$ into the formula for $f(x)$.
- However, NOT all limits can be evaluated by substitution.

Example 2: Use the given graph of f to state the value of the limit, if it exists.

x=4
VA

hole



a. $\lim_{x\to 3} f(x) = 1$

b. $\lim_{x\to 2} f(x) = 0$

c. $\lim_{x\to -1} f(x) = 1$

d. $\lim_{x\to 4} f(x) = \text{DNE}$ (infinitely discontinuous)

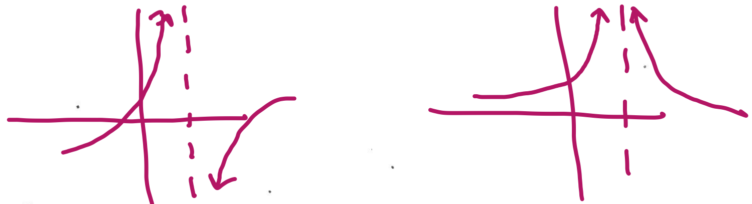
$f(4)$ is undefined VA.

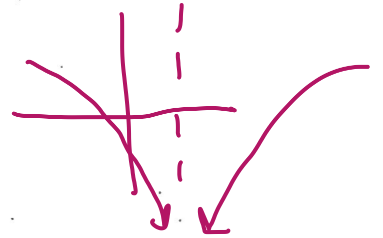
NOTE:
(*) $f(-1)$ is undefined.

$\lim_{x\to 2} f(x) = \text{DNE}$ VA $x=2$

$\lim_{x\to 2} f(x) = \infty$ (DNE) VA $x=2$

$\lim_{x\to 2} f(x) = -\infty$ (DNE) VA $x=2$





Example 3: Evaluate the following limits:

a. $\lim_{x\to 3} x^2 = 9$ ✓

$= f(3)$

b. $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$ ✓

$= f(2)$

c. $\lim_{x\to 1} \sqrt{x+3} = 2$

$= f(1)$

Notes: For the following types of functions

- polynomial functions
- rational functions
- algebraic functions

you can evaluate $\lim_{x\to a} f(x)$ by substitution $\left(\lim_{x\to a} f(x) = f(a) \right)$, provided that $\boxed{f(a) \text{ is defined.}}$

Example 4: Evaluate the following limit: $\lim_{x\to 3} \frac{x^2-9}{x-3} = \frac{0}{0}$ ← indeterminate form (need to do something algebraically)

$$\lim_{x\to 3} \frac{x^2-9}{x-3}$$

$$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}}$$

removed hole. $(3, 6)$

$$= \lim_{x\to 3} (x+3)$$

$$= 6$$

$f(3) = \text{undefined}.$

Example 5: Evaluate the following limits:

$$\lim_{x\to 2}\frac{(x-3)^2-1}{x-2} \quad \frac{0}{0}.$$

$$= \lim_{x\to 2}\frac{(x-3-1)(x-3+1)}{x-2}$$

$$= \lim_{x\to 2}\frac{(x-4)\cancel{(x-2)}}{\cancel{x-2}} \qquad \text{hole}(2,-2)$$

$$= \lim_{x\to 2}(x-4) = -2$$

$$\lim_{h\to 0}\frac{\sqrt{3+h}-\sqrt{3}}{h} \quad \frac{0}{0}$$

$$= \lim_{h\to 0}\left(\frac{\sqrt{3+h}-\sqrt{3}}{h}\right)\cdot\frac{(\sqrt{3+h}+\sqrt{3})}{(\sqrt{3+h}+\sqrt{3})} \qquad * \text{ conjugate}$$

$$= \lim_{h\to 0}\frac{\cancel{3+h-3}}{h(\sqrt{3+h}+\sqrt{3})}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{3+h}+\sqrt{3}} = \frac{1}{2\sqrt{3}} \quad \left(\text{or } \frac{\sqrt{3}}{6}\right)$$

$$\lim_{h\to 0}\frac{\frac{4}{2+h}-2}{h} \quad \frac{0}{0} \quad \text{common denom.} \qquad \text{hole }(0,-1)$$

$$= \lim_{h\to 0}\frac{\frac{4}{2+h}-\frac{2(2+h)}{2+h}}{h} \qquad = \lim_{h\to 0}\frac{-2}{2+h}$$

$$= \lim_{h\to 0}\frac{\frac{\cancel{4-4}-2h}{2+h}}{h} \qquad = -1$$

$$= \lim_{h\to 0}\frac{-2\cancel{h}}{\cancel{h}(2+h)} \qquad \frac{\frac{a}{b}}{c} = \frac{a}{b}\times\frac{1}{c} = \frac{a}{bc}$$

Limits – Change of Variable

1. $\lim_{x\to 8} \frac{x-8}{\sqrt[3]{x}-2}$ $\frac{0}{0}$ Let $t = \sqrt[3]{x}$ $\Rightarrow x = t^3$ | since $x \to 8$ then $t \to 2$

$\lim_{t\to 2} \frac{t^3-8}{t-2}$

$= \lim_{t\to 2} \frac{\cancel{(t-2)}(t^2+2t+4)}{\cancel{t-2}}$

$= \lim_{t\to 2} t^2+2t+4$

$= 12$

2. $\lim_{x\to 27} \frac{27-x}{x^{\frac{1}{3}}-3}$ Let $t = x^{\frac{1}{3}}$ $\Rightarrow x = t^3$ | Since $x \to 27$ then $t \to 3$

$\lim_{t\to 3} \frac{27-t^3}{t-3}$

$= \lim_{t\to 3} \frac{-\cancel{(3-t)}(9+3t+t^2)}{\cancel{t-3}}$

$= \lim_{t\to 3} -(9+3t+t^2)$

$= -27$

3. $\lim_{x \to 0} \frac{(x+8)^{\frac{1}{3}} - 2}{x}$

Let $t = (x+8)^{\frac{1}{3}}$

$x = t^3 - 8$

since $x \to 0$

then $t \to 2$

$$\lim_{t \to 2} \frac{t-2}{t^3-8}$$

$$= \lim_{t \to 2} \frac{\cancel{t-2}}{\cancel{(t-2)}(t^2+2t+4)}$$

$$= \lim_{t \to 2} \frac{1}{t^2+2t+4}$$

$$= \frac{1}{12}$$

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{\sqrt[3]{x} - 1}$$

$t = x^{\frac{1}{6}}$

$t^2 = x^{\frac{1}{3}}$

$t^3 = x^{\frac{1}{2}}$