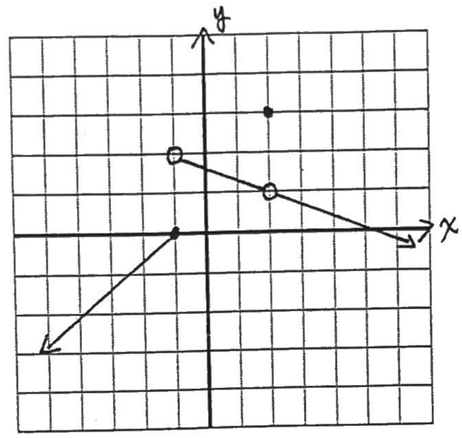
One-Sided Limits

Example 1: Suppose that the graph of the function $y = f(x)$ is as shown below:



a. Find: $f(-1)$ ________ $f(2)$ ________

b. Consider the behaviour of $f(x)$ near $x = 2$

As x approaches 2 <u>from the left</u> ($x < 2$) we see that $f(x)$ approaches 1.

We write this using the following notation: $\lim\limits_{x\to 2^-} f(x) = 1$

and call this the left hand limit.

As x approaches 2 <u>from the right</u> ($x > 2$) we see that $f(x)$ approaches 1.

We write this using the following notation: $\lim\limits_{x\to 2^+} f(x) = 1$

and call this the right hand limit.

In this case $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$

c. Consider the behaviour of $f(x)$ near $x = -1$ and complete the following:

$\lim\limits_{x\to -1^-} f(x) =$ ________ $\lim\limits_{x\to -1^+} f(x) =$ ________

Thus, in this case $\lim\limits_{x\to -1^-} f(x) \neq \lim\limits_{x\to -1^+} f(x)$

To test whether or not $\lim_{x \to a} f(x)$ exists, use the following criteria:

If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

If $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x) = L$.

Therefore for Example 1 we may say that:

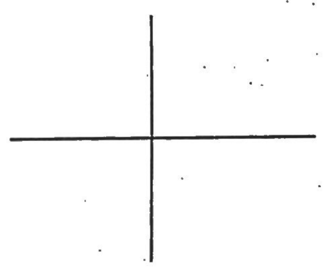
$\lim_{x \to 2} f(x)$ ________ and $\lim_{x \to -1} f(x)$ ____________

Example 2: The Heaviside function H is defined by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

It is named after the electrical engineer Oliver Heaviside and can be used to describe an electric current that is switched on at time $t = 0$.

a. Graph the function:

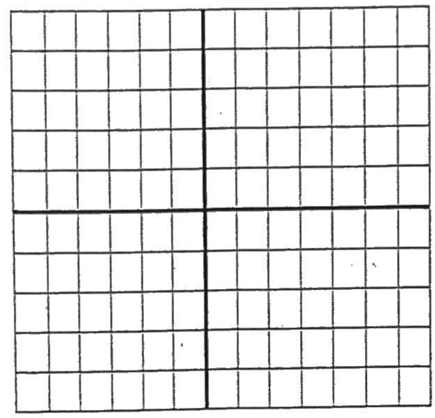


b. Evaluate the following if possible:

$\lim_{t \to 0^-} H(t)$ ______ $\lim_{t \to 0^+} H(t)$ ______ $\lim_{t \to 0} H(t)$ ______

Example 3: Draw the graph of the function f defined as follows:

$$f(x)=\begin{cases} x^2, & x\in(-\infty,1] \\ 3-x, & x\in(1,\infty) \end{cases}$$



Evaluate the following if possible:

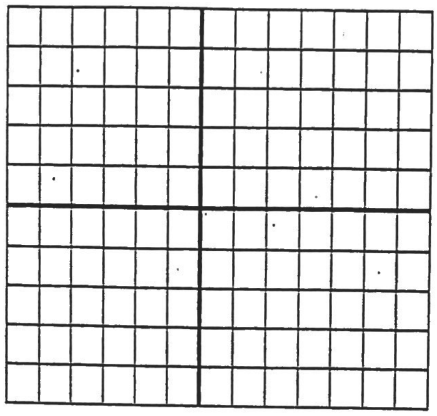
$\lim\limits_{x\to 1^+} f(x)$ ______ $\lim\limits_{x\to 1^-} f(x)$ ______ $\lim\limits_{x\to 1} f(x)$ ______ $\lim\limits_{x\to 2} f(x)$ ______

<u>Evaluating One-Sided Limits Algebraically</u>

Example: If $f(x)=\begin{cases} -x-2 & ,x\in(-\infty,-1] \\ x & ,x\in(-1,1) \\ x^2-2x & ,x\in[1,\infty) \end{cases}$

Determine whether or not $\lim\limits_{x\to -1} f(x)$ and $\lim\limits_{x\to 1} f(x)$ exist.

Verify your results by graphing the function:



CONTINUITY

A **continuous function** is one in which there are no "gaps" or "breaks". (ie. It is smooth)

A function is continuous at a point if *the limit of the function as you approach the point* equals *the value of the function at this point.*

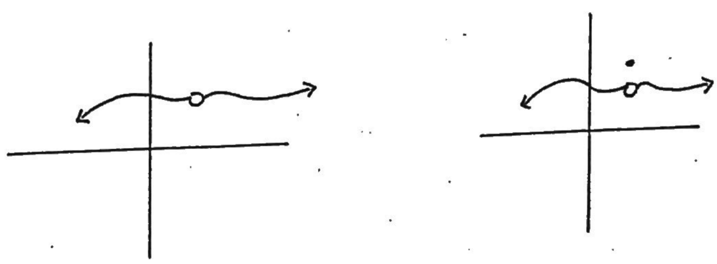
ie. A function is continuous at $x = a$ if:

i) $f(a)$ is defined
ii) $\lim_{x \to a} f(x)$ exists, and
iii) $\lim_{x \to a} f(x) = f(a)$

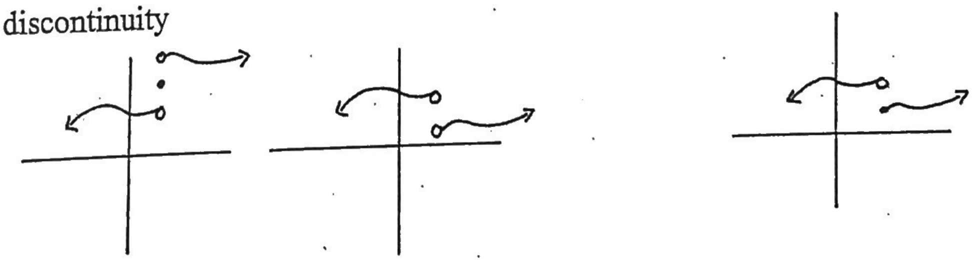
A function is continuous over an interval if it is continuous at every point in the interval.
Note: Polynomial functions are continuous at every number.

<u>Types of Discontinuities</u>:

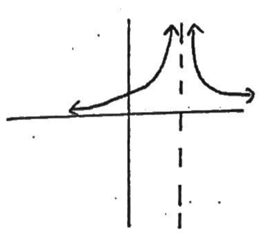
1. Removable discontinuity



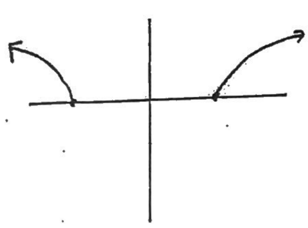
2. Jump discontinuity



3. Infinite discontinuity (Vertical Asymptote)

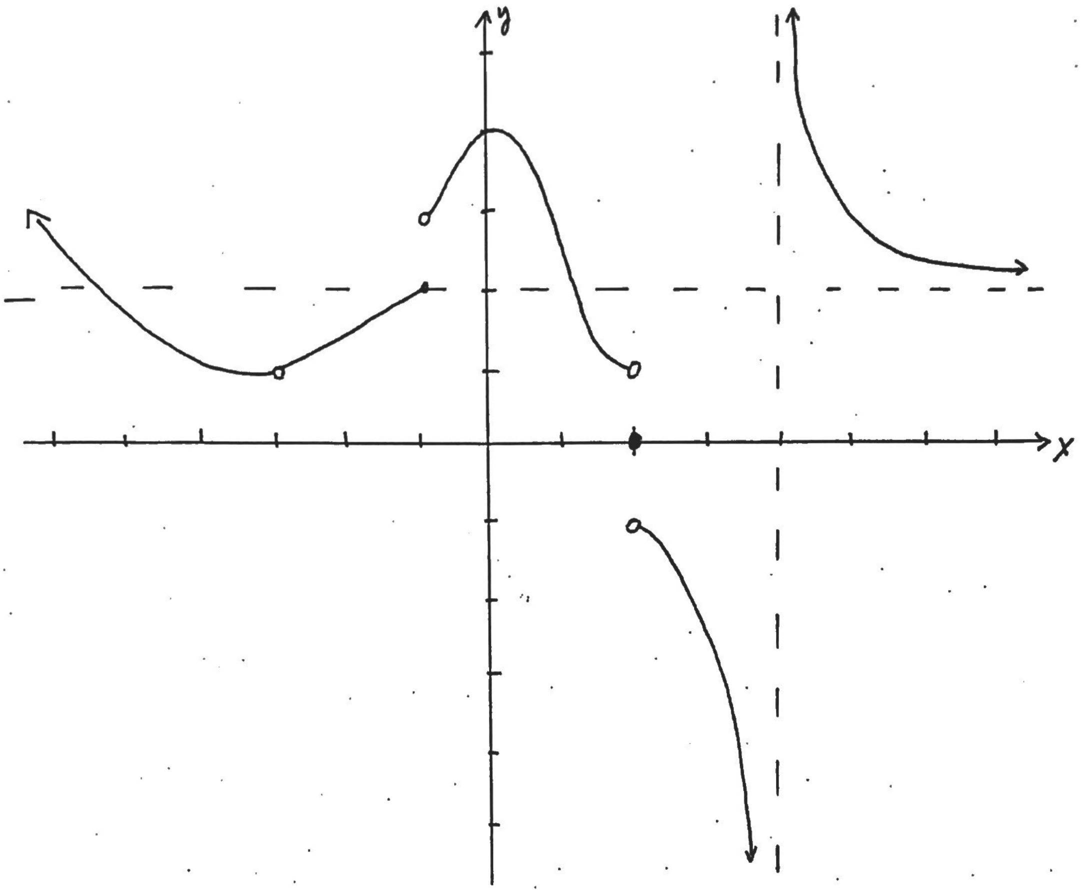


4. Gap



Limits

From the graph of $y = f(x)$ below, state the following:



$\lim_{x \to -3} f(x) =$ ________ $\lim_{x \to 0} f(x) =$ ________

$\lim_{x \to -1^-} f(x) =$ ________ $\lim_{x \to -1^+} f(x) =$ ________

$\lim_{x \to 2^-} f(x) =$ ________ $\lim_{x \to 2^+} f(x) =$ ________

$\lim_{x \to 4^-} f(x) =$ ________ $f(2) =$ ________

$\lim_{x \to +\infty} f(x) =$ ________ $\lim_{x \to -\infty} f(x) =$ ________

Limits do not exist at: ____________________

Discontinuities exist at: ____________________

Continuity

Example: Determine whether or not $f(x)$ is continuous on the interval $(-\infty, \infty)$.

$$f(x)=\begin{cases}(x+1)^2 & , \ x\in(-\infty,-1] \\ x+1 & , \ x\in(-1,1] \\ \sqrt{x}+1 & , \ x\in(1,\infty)\end{cases}$$