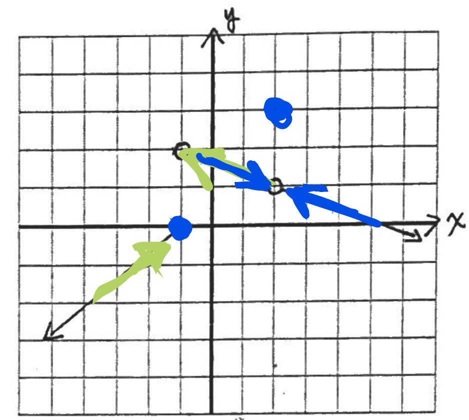
One-Sided Limits

Example 1: Suppose that the graph of the function $y = f(x)$ is as shown below:



a. Find: $f(-1) = $ 0 $\qquad f(2) = $ 3

b. Consider the behaviour of $f(x)$ near $x = 2$

As x approaches 2 from the left ($x < 2$) we see that $f(x)$ approaches 1.

We write this using the following notation: $\boxed{\lim_{x\to 2^-} f(x) = 1}$

and call this the left hand limit.

As x approaches 2 from the right ($x > 2$) we see that $f(x)$ approaches 1.

We write this using the following notation: $\boxed{\lim_{x\to 2^+} f(x) = 1}$

and call this the right hand limit.

In this case $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

c. Consider the behaviour of $f(x)$ near $x = -1$ and complete the following:

$\lim_{x\to -1^-} f(x) = $ 0 $\qquad \lim_{x\to -1^+} f(x) = $ 2

Thus, in this case $\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$

$\boxed{\therefore \lim_{x\to -1} f(x) \text{ DNE}}$

To test whether or not $\lim\limits_{x\to a} f(x)$ exists, use the following criteria:

If $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$, then $\lim\limits_{x\to a} f(x)$ does not exist. (DNE)

If $\lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$, then $\lim\limits_{x\to a} f(x) = L$.

Therefore for Example 1 we may say that:

$\lim\limits_{x\to 2} f(x)$ = 1 and $\lim\limits_{x\to -1} f(x)$ does not exist (DNE)

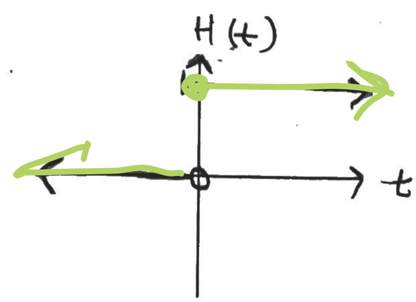
Example 2: The Heaviside function H is defined by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

It is named after the electrical engineer Oliver Heaviside and can be used to describe an electric current that is switched on at time $t = 0$.

a. Graph the function:

$H(0) = 1$



b. Evaluate the following if possible:

$\lim\limits_{t\to 0^-} H(t)$ 0 $\quad$ $\lim\limits_{t\to 0^+} H(t)$ 1 $\quad$ $\lim\limits_{t\to 0} H(t)$ DNE

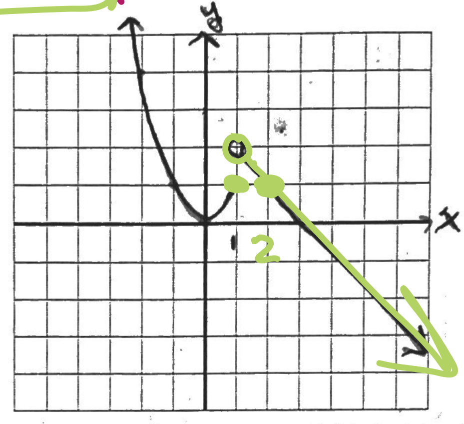
* Since $\lim\limits_{t\to 0^-} H(t) \neq \lim\limits_{t\to 0^+} H(t)$ *

$\Rightarrow \lim\limits_{t\to 0} H(t)$ DNE

Example 3: Draw the graph of the function f defined as follows:

$$f(x)=\begin{cases} x^2, & x\in(-\infty,1] \\ 3-x, & x\in(1,\infty) \end{cases}$$

$f(1)=1$



Evaluate the following if possible:

$\lim_{x\to 1^+} f(x) = 2$ $\lim_{x\to 1^-} f(x) = 1$ $\lim_{x\to 1} f(x)$ DNE $\lim_{x\to 2} f(x) = 1$

Evaluating One-Sided Limits Algebraically

Example: If $f(x)=\begin{cases} -x-2 & ,x\in(-\infty,-1] \\ x & ,x\in(-1,1) \\ x^2-2x & ,x\in[1,\infty) \end{cases}$

Determine whether or not $\lim_{x\to -1} f(x)$ and $\lim_{x\to 1} f(x)$ exist.

$\lim_{x\to -1^-} f(x)$

$=\lim_{x\to -1^-} -x-2$

$=-1$

$\lim_{x\to -1^+} f(x)$

$=\lim_{x\to -1^+} x$

$=-1$

$\boxed{\therefore \lim_{x\to -1} f(x) = -1}$

$\lim_{x\to 1^-} f(x)$

$=\lim_{x\to 1^-} x$

$=1$

$\lim_{x\to 1^+} f(x)$

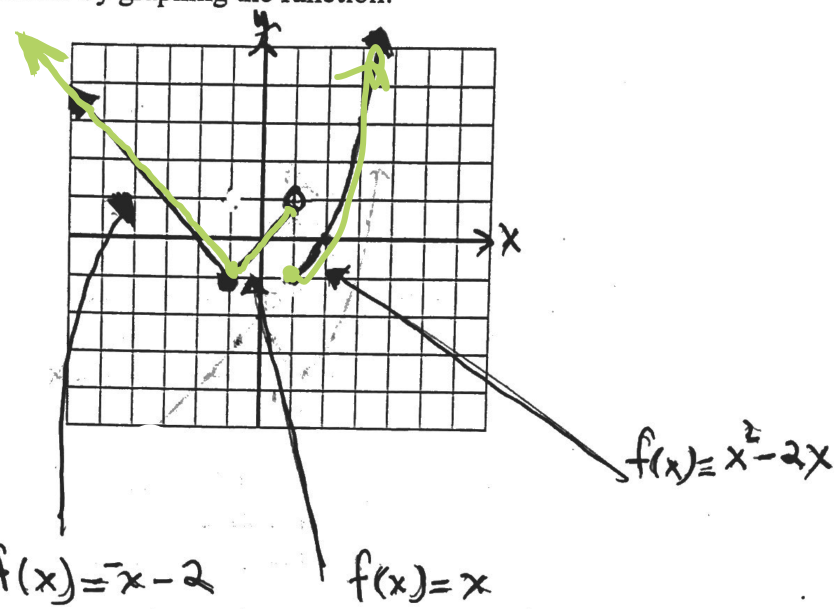
$=\lim_{x\to 1^+} x^2-2x$

$=1-2$

$=-1$

$\boxed{\therefore \lim_{x\to 1} f(x) \text{ DNE!}}$

Verify your results by graphing the function:



CONTINUITY

A **continuous function** is one in which there are no “gaps” or “breaks”. (ie. It is smooth)

A function is continuous at a point if *the limit of the function as you approach the point* equals *the value of the function at this point.*

$\frac{1}{x^2+2}$

ie. A function is continuous at $x = a$ if:

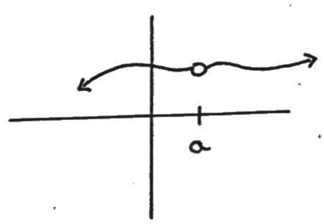
i) $f(a)$ is defined
ii) $\lim_{x \to a} f(x)$ exists, and
iii) $\lim_{x \to a} f(x) = f(a)$

A function is continuous over an interval if it is continuous at every point in the interval.
Note: Polynomial functions are continuous at every number.

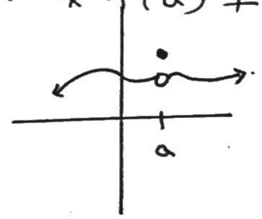
Types of Discontinuities:

1. Removable discontinuity
"hole"

* $f(a)$ is not defined
* $\lim_{x \to a} f(x)$ exists

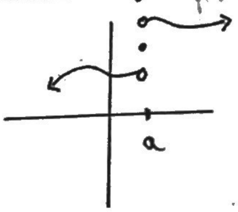


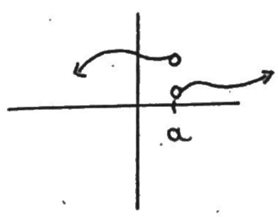
* $f(a)$ is defined
* $\lim_{x \to a} f(x)$ exists
* $f(a) \neq \lim_{x \to a} f(x)$

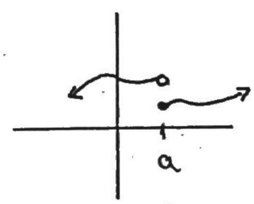


2. Jump discontinuity

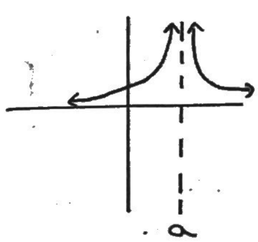
$\lim_{x \to a} f(x)$ DNE





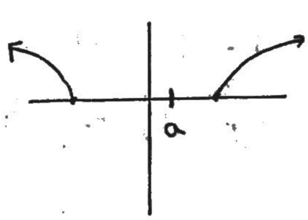


3. Infinite discontinuity (Vertical Asymptote)



* $f(a)$ is not defined
* $\lim_{x \to a} f(x)$ DNE $(+\infty)$

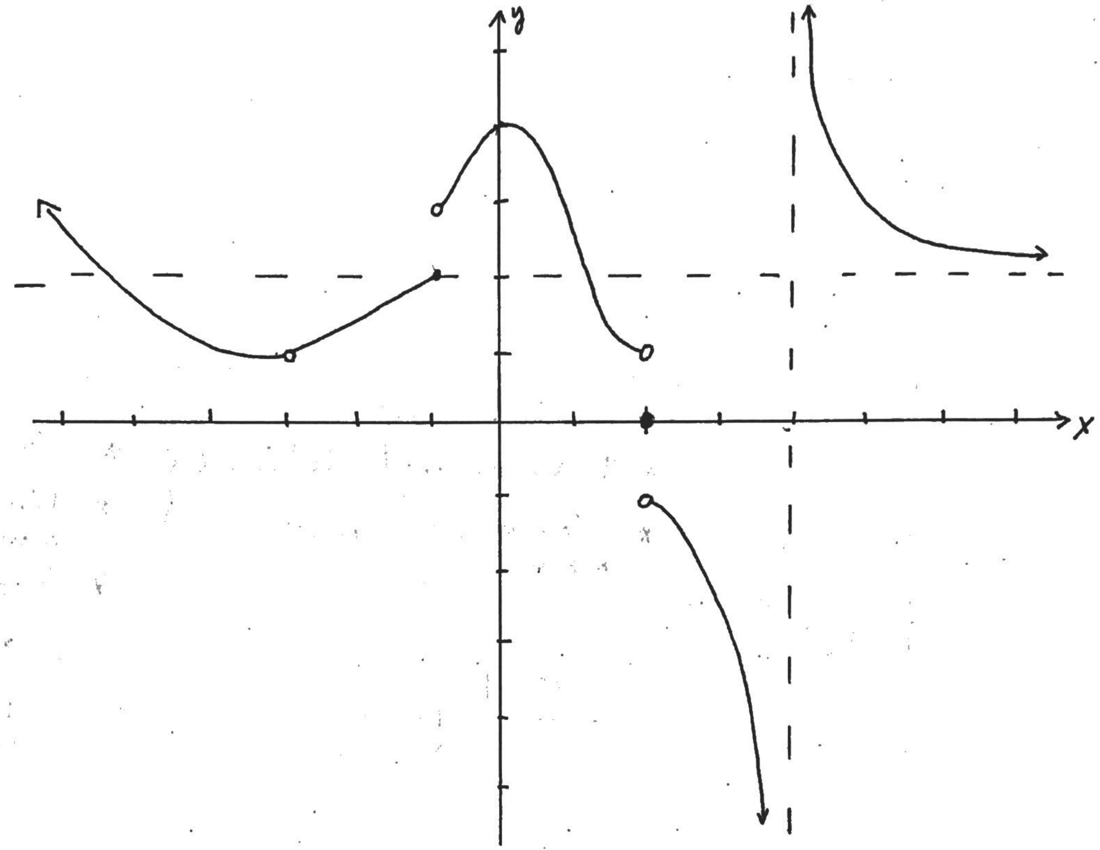
4. Gap



* $f(a)$ is not defined
* $\lim_{x \to a} f(x)$ DNE

Limits

From the graph of $y = f(x)$ below, state the following:



$\lim_{x \to -3} f(x) =$ 1

$\lim_{x \to 0} f(x) =$ 4

$\lim_{x \to -1^-} f(x) =$ 2

$\lim_{x \to -1^+} f(x) =$ 3

$\lim_{x \to 2^-} f(x) =$ 1

$\lim_{x \to 2^+} f(x) =$ −1

$\lim_{x \to 4^-} f(x) =$ $-\infty$

$f(2) =$ 0

$\lim_{x \to +\infty} f(x) =$ 2

$\lim_{x \to -\infty} f(x) =$ ∞

Limits do not exist at: −1, 2, 4

Discontinuities exist at: −3, −1, 2, 4

Continuity

Example: Determine whether or not $f(x)$ is continuous on the interval $(-\infty, \infty)$.

$$f(x) = \begin{cases} (x+1)^2 & , x \in (-\infty, -1] \\ x+1 & , x \in (-1, 1] \\ \sqrt{x}+1 & , x \in (1, \infty) \end{cases}$$

1st point @ $x = -1$

① $f(-1) = 0 \Rightarrow$ Defined

② $\lim_{x \to -1^-} f(x)$ $\qquad$ $\lim_{x \to -1^+} f(x)$ $\qquad$ $\lim_{x \to -1} f(x)$

$= \lim_{x \to -1^-} (x+1)^2$ $\qquad$ $= \lim_{x \to -1^+} x+1$ $\qquad \Rightarrow$ $= 0$

$= 0$ $\qquad$ $= 0$

③ $f(-1) = \lim_{x \to -1} f(x) = 0 \Rightarrow f(x)$ is continuous at $x = -1$.

2nd point @ $x = 1$

① $f(1) = 2 \Rightarrow$ Defined

② $\lim_{x \to 1^-} f(x)$ $\qquad$ $\lim_{x \to 1^+} f(x)$ $\qquad \Rightarrow \lim_{x \to 1} f(x)$

$= \lim_{x \to 1^-} x+1$ $\qquad$ $= \lim_{x \to 1^+} \sqrt{x}+1$ $\qquad$ $= 2$

$= 2$ $\qquad$ $= 2$

③ $f(1) = \lim_{x \to 1} f(x) = 2 \Rightarrow f(x)$ is continuous at $x = 1$.

$\therefore f(x)$ is continuous on $(-\infty, \infty)$