Unit-5: Derivatives of Exponential and Trigonometric Functions

Day	Date	Topic	Textbook Reference Student Work
1		5.1 Derivatives of exponential functions $y = e^x$ 5.2 Derivatives of the general exponential function $y = b^x$	p. 232, #2cef, 3bcf, 5, 8, 10, 11c, 12, 16 p. 240, #1, 2bc, 4, 6
2		5.3 Optimization Problems Involving Exponential Functions	p. 245, #2, 3, 4, 6, 11, 13
3		5.4 The Derivative of $y = \sin(x)$ and $y = \cos(x)$ 5.5 The Derivative of $y = \tan(x)$	p. 256, #1cegi, 2cde, 3e, 5, 6d, 12, 13* Super challenging p. 260, #1cdef, 3bcf, 6
4		Lesson 5.6 More practice with trigonometric derivatives Lesson 5.7 Derivative rules and curve sketching	Handout #1 (yellow) # 30, 31, 33, 34, 37ac, 38 Handout #2 (blue) Section 7.2 # 3ace, 4a, 5a, 7, 11e Section 7.4 # 1d, 2a, 3 - 6
5		Review	p. 263, #1bdf, 2bc, 3def, 4, 8, 10, 12, 13ad, 14cf, 17, 19, 20, 21
6		Test	

5.1 Derivatives of e^x and
5.2 Derivatives of b^x

Learning Goal / Success Criteria:
Define e and the derivative of $y = e^x$ / p. 232, #3.
Determine the derivative of the general exponential function $y = b^x$ / p. 240, #1.
Compare the graph of an exponential function with the graph of its derivative / p4, notes.
Solve rate of change problems involving exponential and trigonometric function models using their derivatives / p. 233, #12.
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Activity 1:
Use graphing technology to graph $f(x) = b^x$ where the base, b, is a slider.

Define $g(x) = f'(x)$.

Which value for b is needed so that $f(x) = f'(x)$?

What is special about 2.17828?

Activity 2: Use spreadsheet technology to determine $\lim_{h \to 0} (1 + h)^{\frac{1}{h}}$

Activity 3: Use spreadsheet technology to determine $\lim_{h \to \infty} \left(1 + \frac{1}{h}\right)^h$

Activity 4:

Use spreadsheet technology to show $\lim_{h \to 0} \left(\frac{a^h - 1}{h}\right) = \ln(a)$

Use first principles to find the derivative of:

(a) $f(x) = 2^x$ b) $f(x) = 3^x$ c) $f(x) = 10^x$

Define e so that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$ so that $\frac{d}{dx} e^x = e^x$

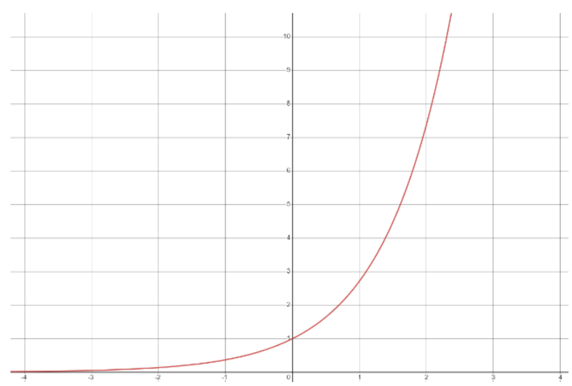
$$\lim_{h \to 0} e^h - 1 = h$$

$$\lim_{h \to 0} e^h = 1 + h$$

$$e = \lim_{h \to 0} (1 + h)^{\frac{1}{h}}$$

$$\cong 2.71828 \ldots$$

$f(x) = e^x = f'(x)$



The derivative of $f(x) = e^x$

$f'(x) = e^x$

The derivative of $f(x) = e^x$

$f'(x) = e^x$

Derivative of a composite function involving e^x

$$\frac{d}{dx}e^{g(x)} = e^{g(x)}g'(x)$$

1. Find the derivative of each of the following:

(a) $f(x) = x^2e^x$

(b) $f(x) = \frac{e^x}{x^2}$

(c) $f(x) = e^{3x}$

(d) $f(x) = e^{x^2-x}$

2. Find the equation of the line tangent to $f(x) = 3e^{x^2}$ at $x = -1$.

Use first principles and the result from Activity 4 to determine the derivative of:

(a) $f(x) = 2^x$

b) $f(x) = 2^{5x}$

Based on the result, we can generalize for the derivative of $f(x) = b^x$

$\frac{d}{dx} b^x = (\ln b) \times b^x$

Derivative $f(x) = b^{g(x)}$

$f'(x) = b^{g(x)} (\ln b)\big(g'(x)\big)$

4. Use these new rules to find each derivative:

(a) $f(x) = 5^x$

(b) $f(x) = x\, 3^x$

(c) $f(x) = 5^{3x-2}$

(d) $f(x) = e^x$

6. Restate $y = 5^x$ using base e, and then differentiate using $\frac{d}{dx}e^{g(x)} = e^{g(x)} \times g'(x)$.

7. The population of a town is modelled by the equation $P(t) = 50000(0.98)^t$ where t is the number of years since January 1, 1850.
(a) What was the population of the town on January 1, 1900?
(b) At what rate was the population changing on January 1, 1900?

5.3 Optimization Problems Involving Exponential Functions

Learning Goal / Success Criteria:
Solve optimization problems using exponential functions / p. 245, #3.
Solve rate of change problems involving exponential models using their derivatives / p. 246, #8.
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1. The effectiveness of studying for an exam depends on how many hours a student studies. (It depends on other things, too.) Some experiments show that if the effectiveness E is put on a scale from 0-10, then the relationship can be modelled as $E(t) = 0.5[10 + te^{-\frac{t}{20}}]$, where t is the number of hours spent studying for an exam. If a student has up to 30 hours for studying, how many hours are needed for maximum effectiveness?

2. A mathematical consultant determines that the proportion of people who will have responded to the advertisement of a new product after it has been marketed for *t* days is given by $f(t) = 0.7\left(1 - e^{-0.2t}\right)$. The area covered by the advertisement covers 10 million potential customers and each response to the advertisement results in revenue to the company $0.70 (on average), excluding the cost of advertising. The advertising costs $30 000 to produce plus $5000 per day to run.

(a) Determine $\lim_{t \to \infty} f(t)$ and interpret the result.

(b) What percent of potential customers have responded after seven days of advertising?

2. continued:
A mathematical consultant determines that the proportion of people who will have responded to the advertisement of a new product after it has been marketed for *t* days is given by $f(t) = 0.7\left(1 - e^{-0.2t}\right)$. The area covered by the advertisement covers 10 million potential customers and each response to the advertisement results in revenue to the company $0.70 (on average), excluding the cost of advertising. The advertising costs $30 000 to produce plus $5000 per day to run.

(c) Write the function for $P(t)$ that represents the average profit after t days of advertising.

(d) What is the average profit after seven days?

(e) For how many full days should the advertising campaign run to maximize the average profit? Assume the advertising budget is $200 000.

5.4 The Derivative of $y = sin(x), y = cos(x)$
5.5 The Derivative of $y = tan(x)$

Learning Goals:
Investigate and determine the derivatives of sinusoidal functions / p. 256, #1.
Determine the derivative of the tangent function / p. 260, #1.
Solve rate of change problems involving exponential and trigonometric function models using their derivatives / p.257, #7.
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On the same set of axes, graph: $f(x) = \sin(x)$ and $g(x) = \frac{\sin(x+0.001)-\sin(x)}{0.001}$

Sketch both graphs.

If $f(x) = \sin(x)$, then $f'(x) =$ * but only when ____________________[1]

Sketch the graph of $f(x) = \cos(x)$ and its derivative.

If $f(x) = \cos(x)$, then $f'(x) =$

Use a trigonometric identity to rewrite $f(x) = \tan(x)$, and use the quotient rule to find the derivative.

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[1] Use graphing tech to show $f(x) = \cos x$, $g(x) = f'(x)$, $g\left(\frac{\pi}{2}\right) = -1$ and the tangent $y = -1x + \frac{\pi}{2}$.
Repeat in degrees to show the slope of tangent at 90 is not -1.

$$\frac{d}{dx}\sin\left(g\left(x\right)\right)=\cos\left(g\left(x\right)\right)\times g'\left(x\right) \qquad \frac{d}{dx}\cos\left(g\left(x\right)\right)=-\sin\left(g\left(x\right)\right)\times g'\left(x\right)$$

$$\frac{d}{dx}\tan\left(g\left(x\right)\right)=\sec^2\left(g\left(x\right)\right)\times g'\left(x\right)$$

1. Differentiate each of the following.

(a) $y=\cos 3x$

(b) $f(x) = \tan\,(x^2+3x)$

(c) $y=x\sin x$

(d) $y=\sin x^2$

(e) $y=\sin^2 x$

(f) $y=\cos\left(1+x^3\right)$

(g) $y = 2\sin^3 x - 4\cos^2 x$

(h) $y = \cos(\sin x)$

(i) $y = \cos(4t)\sin^2 t$

(j) $y = \dfrac{1}{\sin x \cos x}$

(k) $f(x) = (sinx + tanx)^4$

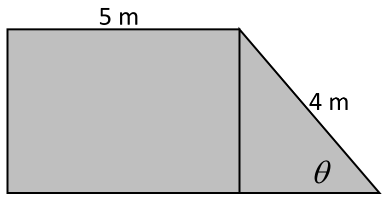
(l) $f(x) = x\tan(2x - 1)$

5.6 More Practice with Trigonometric Derivatives

1. Find and classify the critical points of $f(x) = sin^2(x) - \sin(x), 0 \leq x \leq 2\pi$

2. Find the maximum and minimum perimeter of a rectangle with a 50 cm diagonal.

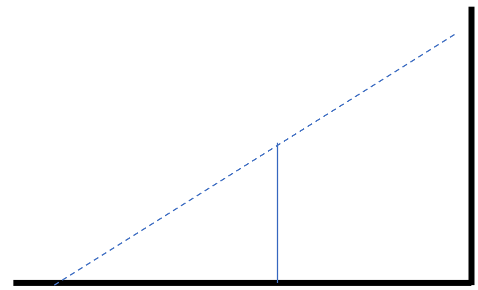
3. In the diagram, there is a blueprint for a car shelter. Find the angle θ that maximizes the area under the shelter.



4. The position of a vibrating string, measured in cm from its rest position, is given by $x(t) = 4\cos(30\pi t)$ with t measured in seconds. Find the maximum velocity and acceleration of the string.

5. Complete **one** of the following, A or B.
A. When a car hits a bump, t seconds later, the height of the car is modeled by the equation $h(t) = 10e^{-t}\sin(t)$ with h measured in cm. Find the maximum and the minimum vertical velocity of the car.

B. A fence is 2 m tall, 3 m from a wall. A ladder must touch the ground, the top of the fence and then reach the wall. Find the angle between the ladder and the ground that makes the ladder as short as possible.



5.7 Exponential and Trigonometric Derivative Rules Review and Curve Sketching

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Remember the rules of derivatives so far for trigonometric and exponential functions:

y	$\frac{dy}{dx}$
e^x	
a^x	
$sinx$	
$cosx$	
$tanx$	

1. Find the derivative of the following

(a) $f(x) = x^3 e^x$

(b) $h(x) = \sin\left(x^2\right)$

(c) $g(x) = x + 2^x$

(d) $j(x) = \dfrac{\cos\sqrt{x}}{x^2}$

(e) $y = e^{2x} \times \cos 4x$

(f) $y = xe^{-x^2}$

(g) $f(x) = \sqrt{\tan x}$

(h) $y = 100\left(2^{\frac{-t}{6}}\right)$

2. Find the equation of the tangent to $f(x) = x^2 \cos 4x$ at $x = \frac{\pi}{8}$

3. A spring hanging from the ceiling vibrates up and down. Its vertical position at time t is given by $d(t) = 4\sin(3t)$

(a) Find the velocity of the spring at time t

(b) Find the maximum velocity and the time(s) at which it occurs.

4. For the function $h(x) = x^2 e^{-\frac{x}{2}}$
(a) Determine the interval(s) of increase and decrease.
(b) Determine the interval(s) of concavity.
(c) Find and classify critical values.
(d) Find $\lim_{x \to \infty} h(x)$ and $\lim_{x \to -\infty} h(x)$, use a calculator.
(e) How do you know that h(x) is always positive?
(f) Sketch the graph.

5. For the function $f(x) = \frac{sinx}{2+cosx} \quad 0 \le x \le 2\pi$

(a) Determine the interval(s) of increase and decrease.
(b) Determine the interval(s) of concavity.
(c)Find and classify critical values.
(d) How do you know quickly that 2+cosx is never zero? Why is this important?
(e) Sketch the graph.

Trigonometry and Exponential Practice – Yellow

7.6 Derivatives of Exponential Functions

Refer to the Key Concepts on page 458.

29. Differentiate.

a) $y = e^x$ b) $y = x^3e^{-x}$

c) $f(x) = x^2e^x$ d) $y = \frac{e^x}{x}$

e) $h(x) = e^{2x}$ f) $g(x) = \frac{e^x}{1+e^{-3x}}$

g) $y = \frac{2}{e^x}$ h) $f(x) = (1 - e^{2x})^2$

i) $k(x) = \frac{e^{\sqrt{x}}}{e^{-1}}$ j) $y = xe^{\sqrt{x}}$

30. Find an equation of the tangent to the curve $y = e^x$ at the point where $x = 1$.

31. The atmospheric pressure, p, in kilopascals, for distances up to 10 km above sea level, is given by the formula $p = 101.3e^{-0.125x}$, where x is the altitude, in kilometres, above sea level. At what altitude will the atmospheric pressure be approximately 70 kPa?

32. Determine the local extrema, inflection points, and all asymptotes of each function. Then, sketch a graph of each function.

a) $y = x - e^x$ b) $y = \frac{e^x}{x+2}$

33. The power output, in watts, of a certain battery system decreases at a rate proportional to the power, and is defined by the equation $P(t) = 50e^{-0.004t}$, where t is the time in days.

a) Determine the equation representing the rate of change of the power output.

b) At what rate is the power output decreasing after 100 days?

34. The cost, in dollars, of producing x CDs is given by the equation $C(x) = 10x - 75x^2e^{-x} + 1500$.

a) Determine an equation for the marginal cost.

b) Determine the marginal cost of producing 100 CDs.

c) Determine the marginal cost of producing 1000 CDs.

7.7 Derivatives of Logarithmic Functions

Refer to the Key Concepts on page 468.

35. Differentiate.

a) $g(x) = \ln x^7$ b) $y = 8^x$

c) $h(x) = \ln(5x + 1)$ d) $g(x) = \log(4x + 15)$

e) $y = 2^x \log_2(x - 8)$ f) $y = \log_3(7x^2 + 2)$

g) $f(x) = \frac{4^x}{1+x}$ h) $y = \frac{1}{x \ln x}$

37. Find the equation of the tangent to each curve at the given point.

a) $y = 5^x$, at the point where $x = 5$

b) $g(x) = 3^x$, (3, 27)

c) $f(x) = \left(\frac{1}{2}\right)^x$, (−3, 8)

d) $y = 7^x$, (2, 49)

38. The population, P, of a certain kind of bacteria, after t hours, is represented by an equation of the form $P = P_0a^t$. There were 6400 bacteria after 90 min and 12 800 bacteria after 2 h. Find the rate of growth after 90 min.

40. When a particular medication is swallowed by a patient, the concentration of the active ingredient, in parts per million, in the bloodstream is given by the equation $C(t) = 150t(0.5)^t$, after t hours.

a) Estimate the highest concentration of the medicine.

b) How fast is the concentration decreasing after 2 h?

41. How long will it take \$7000 to triple if it is invested at

a) 8.5%, compounded semi-annually

b) 9%, compounded continuously

42. Over a span of 20 years, the population of Brampton increased from 51 003 to 149 030. The population of Brampton in 2000 was 310 792.

a) Assuming that the same population model applies for the entire relevant domain, when was the population 51 003?

b) Research population statistics for Brampton. When was its population 51 003?

44. A population of mould doubles every 3 h. If the initial population was 600, how long will it take to reach a population of 40 000?

45. The rate of growth of the world's urban population varies at a rate proportional to the existing population. In 1989, the world's urban population was approximately 2.3 billion. In 1999, it had reached 3 billion.

a) Determine an equation representing the world's urban population between 1989 and 1999.

b) Assuming the same model applies for a further 20 years, what would the world's urban population be in 2019?

Practise

B **1.** Determine the derivative of y with respect to x for each function.

a) $y = \sin(4x + 7)$ b) $y = \sin(x^2 + 3)$
c) $y = \cos(5x + 3)$ d) $y = 3\tan(2x + 3)$
e) $y = \cos(3x^2 - 1)$ f) $y = \tan(5x^2 + 1)$
g) $y = \sin(\cos^2 x)$ h) $y = \sin(\cos x)$

2. Differentiate.

a) $f(x) = \dfrac{\tan x}{x}$ b) $y = x\sin(3x - 2)$
c) $y = x^2\sin(2x^2 + 5)$ d) $f(x) = \dfrac{\cos x}{x}$
e) $g(x) = x\cos(8x - 17)$
f) $y = x^2\cos(0.4x^2 + 9)$
g) $y = 5x^2\tan^3(3x^2 - 1)$
h) $h(x) = 4x^2\sin^3(6x^2 - 2)$

3. Find the derivative.

a) $y = \sin^3(\cos^2 x)$ b) $f(x) = \cos^4(\sin x^3)$
c) $y = x\cos(\tan x)$ d) $y = \sin(e^x)$
e) $h(x) = \cos(\ln x)$ f) $g(x) = \ln(\sin(e^x))$
g) $y = \sin^2 x + \cos^2 x$ h) $f(x) = \dfrac{\sin x}{1 + \cos x}$

Apply, Solve, Communicate

4. Determine the equation of the tangent to the curve at the given point.

a) $y = x\sin 2x,\ \left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$
b) $y = \cos^2 x,\ \left(\dfrac{\pi}{3}, \dfrac{1}{4}\right)$
c) $y = \dfrac{1}{\tan^2 x},\ \left(\dfrac{\pi}{4}, 1\right)$

5. Determine any extrema.

a) $y = \cos x - \sin x,\ x \in [-\pi, \pi]$
b) $y = \sin^2 x - \sin x,\ x \in [-\pi, \pi]$
c) $y = 2\cos x - \cos 2x,\ x \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$
d) $y = \dfrac{1}{\cos x} + \tan x,\ x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

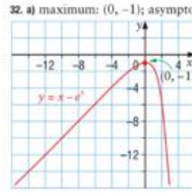
6. Determine the points of inflection.

a) $y = 2\cos x + \sin 2x,\ x \in [0, \pi]$
b) $y = 2\sin^2 x - 1,\ x \in [-\pi, \pi]$
c) $y = \sin x - \tan x,\ x \in \left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$

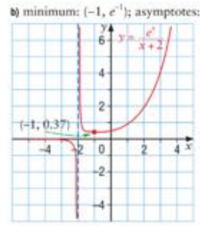
Answers – front

d) 3520 Hz **29. a)** e^x **b)** $x^2e^{-x}(3 - x)$ **c)** $xe^x(2 + x)$
d) $\dfrac{e^x(x-1)}{x^2}$ **e)** $2e^{2x}$ **f)** $\dfrac{e^x(1+4e^{-3x})}{(1+e^{-3x})^2}$ **g)** $-\dfrac{2}{e^x}$ **h)** $-4e^{2x}(1 - e^{2x})$
i) $\dfrac{e^{\sqrt{x}+1}}{2\sqrt{x}}$ **j)** $e^{\sqrt{x}}\left(1 + \dfrac{x}{2\sqrt{x}}\right)$ **30.** $y = ex$ **31.** 2.957 km

32. a) maximum: $(0, -1)$; asymptote: $y = x$

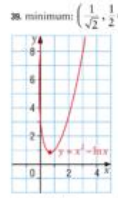


b) minimum: $(-1, e^{-1})$; asymptotes: $x = -2$, $y = 0$



33. a) $P'(t) = -0.2e^{-0.004t}$ **b)** 0.13 W/day
34. a) $C'(x) = 10 + 75xe^{-x}(x - 2)$ **b)** \$10 **c)** \$10 **35. a)** $\dfrac{7}{x}$
b) $8^x\ln 8$ **c)** $\dfrac{5}{5x+1}$ **d)** $\dfrac{4}{(4x+15)\ln 10}$
e) $\dfrac{2^x}{(x-8)\ln 2} + 2^x\ln(x - 8)$ **f)** $\dfrac{14x}{(7x^2+2)\ln 3}$
g) $\dfrac{4^x(x\ln 4 + \ln 4 - 1)}{(1+x)^2}$ **h)** $-\dfrac{\ln x + 1}{(x\ln x)^2}$

37. a) $y = (5^5\ln 5)x + 5^5 - 5^6\ln 5$
b) $y = (27\ln 3)x + 27 - 81\ln 3$
c) $y = (-8\ln 2)x + 8 - 24\ln 2$
d) $y = (49\ln 7)x + 49 - 98\ln 7$
38. $6400\ln 4$ bacteria/hour
39. minimum: $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}(1 + \ln 2)\right)$; asymptote: $x = 0$



40. a) 79.6 ppm **b)** 14.5 ppm/h **41. a)** 13.198 years
b) 12.207 years **42. a)** 1966 (33.71 years before 2000)
43. after 19.2 min **44.** 18.177 h **45. a)** $P = 2.3e^{0.026\,57t}$
b) 5.1 billion **46. a)** $P(x) = 100e^{-0.139x}$ **b)** 49.9 kPa
c) 24.9 kPa **47. a)** $I = 2e^{-0.1999x}$ **b)** $I = I_0e^{-0.1999x}$
c) 36.8% of the light compared to 5 mm.

Answers – back

1. a) $4\cos(4x + 7)$ **b)** $2x\cos(x^2 + 3)$ **c)** $-5\sin(5x + 3)$
d) $6\sec^2(2x + 3)$ **e)** $-6x\sin(3x^2 - 1)$ **f)** $10x\sec^2(5x^2 + 1)$
g) $-2\cos x\sin x\cos(\cos^2 x)$ **h)** $-\sin x\cos(\cos x)$
2. a) $\dfrac{x\sec^2 x - \tan x}{x^2}$ **b)** $\sin(3x - 2) + 3x\cos(3x - 2)$
c) $2x\sin(2x^2 + 5) + 4x^3\cos(2x^2 + 5)$ **d)** $-\dfrac{x\sin x + \cos x}{x^2}$
e) $\cos(8x - 17) - 8x\sin(8x - 17)$
f) $2x\cos(0.4x^2 + 9) - 0.8x^3\sin(0.4x^2 + 9)$
g) $10x\tan^3(3x^2 - 1) + 90x^3\tan^2(3x^2 - 1)\sec^2(3x^2 - 1)$
h) $8x\sin^3(6x^2 - 2) + 144x^3\sin^2(6x^2 - 2)\cos(6x^2 - 2)$
3. a) $-6\cos x\sin x\sin^2(\cos^2 x)\cos(\cos^2 x)$
b) $-12x^2\cos x^3\cos^3(\sin x^3)\sin(\sin x^3)$
c) $\cos(\tan x) - x\sin(\tan x)\sec^2 x$ **d)** $e^x\cos e^x$ **e)** $-\dfrac{\sin(\ln x)}{x}$
f) $\dfrac{e^x\cos e^x}{\sin e^x}$ **g)** 0 **h)** $\dfrac{1}{1+\cos x}$

4. a) $y = x$ **b)** $y = -\dfrac{\sqrt{3}}{2}x + \dfrac{\pi\sqrt{3}}{6} + \dfrac{1}{4}$ **c)** $y = -4x + \pi + 1$
5. a) minimum $-\sqrt{2}$ when $x = \dfrac{3\pi}{4}$, maximum $\sqrt{2}$ when $x = -\dfrac{\pi}{4}$ **b)** maxima: 0 when $x = \dfrac{\pi}{2}$, 2 when $x = -\dfrac{\pi}{2}$; minima: $-\dfrac{1}{4}$ when $x = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$ **c)** minimum -3 when $x = \pi$ **d)** none **6. a)** $\left(\dfrac{\pi}{2}, 0\right)$ **b)** $\left(-\dfrac{3\pi}{4}, 0\right), \left(-\dfrac{\pi}{4}, 0\right), \left(\dfrac{\pi}{4}, 0\right), \left(\dfrac{3\pi}{4}, 0\right)$ **c)** $(0, 0), (\pi, 0)$ **7. a)** Every fourth one

Trigonometric and Exponential Practice - Blue

Exercise 7.2

B 1. Find the de... ...ve of y with respect to x in each of the following.

(a) $y = \cos(-4x)$ (b) $y = \sin(3x + 2\pi)$

(c) $y = 4\sin(-2x^2 - 3)$ (d) $y = -\frac{1}{2}\cos(4 + 2x)$

(e) $y = \sin x^2$ (f) $y = -\cos x^2$

(g) $y = \sin^{-2}(x^3)$ (h) $y = \cos(x^2 - 2)^2$

(i) $y = 3\sin^4(2 - x)^{-1}$ (j) $y = x\cos x$

(k) $y = \dfrac{x}{\sin x}$ (l) $y = \dfrac{\sin x}{1 + \cos x}$

(m) $y = (1 + \cos^2 x)^6$ (n) $y = \sin\dfrac{1}{x}$

(o) $y = \sin(\cos x)$ (p) $y = \cos^3(\sin x)$

(q) $y = x\cos\dfrac{1}{x}$ (r) $y = \dfrac{\sin^2 x}{\cos x}$

(s) $y = \dfrac{1 + \sin x}{1 - \sin 2x}$ (t) $y = \sin^3 x + \cos^3 x$

(u) $y = \cos^2\left(\dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}\right)$

2. Find $\dfrac{dy}{dx}$ in each of the following.

(a) $\sin y = \cos 2x$ (b) $x\cos y = \sin(x + y)$

(c) $\sin y + y = \cos x + x$ (d) $\sin(\cos x) = \cos(\sin y)$

(e) $\sin x\cos y + \cos x\sin y = 1$

(f) $\sin x + \cos 2x = 2xy$

3. Find an equation of the tangent line to the given curve at the given point.

(a) $y = 2\sin x$ at $\left(\frac{\pi}{6}, 1\right)$ (b) $y = \dfrac{\sin x}{\cos x}$ at $\left(\frac{\pi}{4}, 1\right)$

(c) $y = \dfrac{1}{\cos x} - 2\cos x$ at $\left(\frac{\pi}{3}, 1\right)$

(d) $y = \dfrac{\cos^2 x}{\sin^2 x}$ at $\left(\frac{\pi}{4}, 1\right)$

(e) $y = \sin x + \cos 2x$ at $\left(\frac{\pi}{6}, 1\right)$

(f) $y = \cos(\cos x)$ at $x = \frac{\pi}{2}$

4. Find the critical numbers, the intervals of increase and decrease, and any maximum or minimum values.

(a) $y = \sin^2 x,\ -\pi \leqslant x \leqslant \pi$

(b) $y = \cos x - \sin x,\ -\pi \leqslant x \leqslant \pi$

5. Determine the concavity and find the points of inflection.

(a) $y = 2\cos x + \sin 2x,\ 0 \leqslant x \leqslant 2\pi$

(b) $y = 4\sin^2 x - 1,\ -\pi \leqslant x \leqslant \pi$

6. Use the procedure of Example 5 to sketch the graph of each of the following.

(a) $y = x + \sin x,\ 0 \leqslant x \leqslant 2\pi$ (b) $y = x\cos x,\ 0 \leqslant x \leqslant \pi$

7. If $f(x) = \sin x\cos 3x$, evaluate $f''\left(\frac{\pi}{3}\right)$.

8. Use Newton's method to find all roots of the given equation correct to 6 decimal places.

(a) $\cos x - x = 0$ (b) $2\sin x = 2 - x$ (c) $\sin x = \dfrac{x}{2}$

C 9. Use the results of this section to find the derivative of $y = \tan x$ and $y = \csc x$.

10. If $\sin y + \cos x = 1$ find $\dfrac{d^2y}{dx^2}$.

7.2 (continued)

11. Find $\frac{dy}{dx}$ in each of the following.

(a) $y = \frac{1}{\sin(x - \sin x)}$

(b) $y = \sqrt{\sin\sqrt{x}}$

(c) $y = \sqrt[3]{x \cos x}$

(d) $y = \cos^3(\cos x) + \sin^2(\cos x)$

(e) $y = \sqrt{\cos(\sin^2 x)}$

7.3

B **1.** Find the derivative of each of the following.

(a) $y = 3 \tan 2x$

(b) $y = \frac{1}{3} \cot 9x$

(c) $y = 12 \sec \frac{1}{4}x$

(d) $y = -\frac{1}{4} \csc(-8x)$

(e) $y = \tan x^2$

(f) $y = \tan^2 x$

(g) $y = \sec \sqrt[3]{x}$

(h) $y = x^2 \csc x$

(i) $y = \cot^3(1 - 2x)^2$

(j) $y = \sec^2 x - \tan^2 x$

(k) $y = \frac{1}{\sqrt{(\sec 2x - 1)^3}}$

(l) $y = \frac{x^2 \tan x}{\sec x}$

(m) $y = 2x(\sqrt{x} - \cot x)$

(n) $y = \sin(\tan x)$

(o) $y = \tan^2(\cos x)$

(p) $y = [\tan(x^2 - x)^{-2}]^{-3}$

7.4

B **1.** Find the local maxima and/or minima of each of the following functions.

(a) $f(x) = x - 2 \sin x,\ 0 \leq x \leq 2\pi$

(b) $f(x) = x + \cos x,\ 0 \leq x \leq 2\pi$

(c) $f(x) = \sin^4 x + \cos^4 x,\ 0 \leq x \leq 2\pi$

(d) $f(x) = x \sin x + \cos x,\ -\pi \leq x \leq \pi$

2. The position of a particle as it moves horizontally is described by the given equations. If s is the displacement in metres and t is the time in seconds find the absolute maximum and absolute minimum displacements.

(a) $s = 2 \sin t + \sin 2t,\ -\pi \leq t \leq \pi$

(b) $f(t) = \sin^2 t - 2 \cos^2 t,\ -\pi \leq t \leq \pi$

3. Triangle ABC is inscribed in a semicircle with diameter $BC = 10$ cm. Find the value of angle B that produces the triangle of maximum area.

4. Points A and B lie on a circle, centre O, radius 5 cm. Find the value of angle AOB that produces a maximum area for triangle AOB.

5. Triangle ABC has $AB = AC$. It is inscribed in a circle centre O, radius 10 cm. Find the value of angle BAC that produces a maximum area for triangle ABC.

6. Rectangle $ABCD$ has A and D on the equal sides of an isosceles triangle and B and C on its base. If $AB = 2$ cm and $BC = 6$ cm, find the value of the base angle that produces the triangle of minimum area.

Answers

EXERCISE 7.2

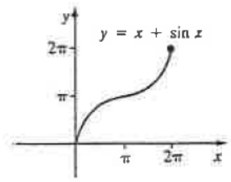
1. (a) $-4 \sin 4x$ (b) $3 \cos(3x + 2\pi)$
(c) $-16x \cos(-2x^2 - 3)$ (d) $\sin(4 + 2x)$
(e) $2x \cos x^2$ (f) $2x \sin x^2$ (g) $\dfrac{-6x^2 \cos x^3}{\sin^3 x^3}$
(h) $-4x(x^2 - 2) \sin(x^2 - 2)^2$
(i) $\dfrac{12 \cos (2 - x)^{-1} \sin^3(2 - x)^{-1}}{(2 - x)^2}$
(j) $\cos x - x \sin x$ (k) $\dfrac{\sin x - x \cos x}{\sin^2 x}$
(l) $\dfrac{1}{1 + \cos x}$ (m) $-6 \sin 2x(1 + \cos^2 x)^5$
(n) $-\dfrac{1}{x^2} \cos \dfrac{1}{x}$ (o) $-\sin x \cos(\cos x)$
(p) $-3 \cos x \sin(\sin x) \cos^2(\sin x)$
(q) $\dfrac{1}{x} \sin \dfrac{1}{x} + \cos \dfrac{1}{x}$
(r) $\dfrac{\sin x(2 \cos^2 x + \sin^2 x)}{\cos^2 x}$
(s) $\dfrac{\cos x - \cos x \sin 2x + 2 \cos 2x + 2 \sin x \cos 2x}{(1 - \sin 2x)^2}$
(t) $3 \sin x \cos x(\sin x - \cos x)$
(u) $\sin 2 \left(\dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}\right) \left(\dfrac{1}{\sqrt{x}(1 + \sqrt{x})^2}\right)$

3. (a) [illegible] $y - \pi\sqrt{3} + 6 = 0$
(b) $4x - 2y + 2 - \pi = 0$
(c) $3\sqrt{3}x - y + 1 - \pi\sqrt{3} = 0$
(d) $4x + y - 1 - \pi = 0$
(e) $6\sqrt{3}x + 12y - 12 - \pi\sqrt{3} = 0$
(f) $y = 1$

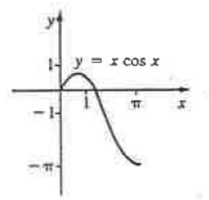
4. (a) critical numbers $\pm\pi, \pm\dfrac{\pi}{2}, 0$;
increasing on $\left(-\pi, -\dfrac{\pi}{2}\right)$ and $\left(0, \dfrac{\pi}{2}\right)$,
decreasing on $\left(-\dfrac{\pi}{2}, 0\right)$ and $\left(\dfrac{\pi}{2}, \pi\right)$;
local maximums $f\left(-\dfrac{\pi}{2}\right) = 1, f\left(\dfrac{\pi}{2}\right) = 1$;
local minimum $f(0) = 0$.
(b) critical numbers $-\dfrac{\pi}{4}, \dfrac{3\pi}{4}$;
increasing on $\left(-\pi, -\dfrac{\pi}{4}\right)$ or $\left(\dfrac{3\pi}{4}, \pi\right)$;
decreasing on $\left(-\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$;
local maximum $f\left(-\dfrac{\pi}{4}\right) = \dfrac{2}{\sqrt{2}}$;
local minimum $f\left(\dfrac{3\pi}{4}\right) = -\dfrac{2}{\sqrt{2}}$

5. (a) concave down on $\left(0, \dfrac{\pi}{2}\right)$ or $\left(3.394, \dfrac{3\pi}{2}\right)$ or $(6.030, 2\pi)$; concave up on $\left(\dfrac{\pi}{2}, 3.394\right)$ or $\left(\dfrac{3\pi}{2}, 6.030\right)$; points of inflection $\left(\dfrac{\pi}{2}, 0\right)$, $(3.394, -1.453)$, $\left(\dfrac{3\pi}{2}, 0\right)$, and $(6.030, 1.451)$
(b) concave down on $\left(-\dfrac{3\pi}{4}, -\dfrac{\pi}{4}\right)$ or $\left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$;
concave up on $\left(-\pi, -\dfrac{3\pi}{4}\right)$ or $\left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$
or $\left(\dfrac{3\pi}{4}, \pi\right)$;
points [illegible]tion $\left(-\dfrac{3\pi}{4}, 1\right)$, $\left(-\dfrac{\pi}{4}, 1\right)$,
$\left(\dfrac{\pi}{4}, 1\right)$ and $\left(\dfrac{5\pi}{4}, 1\right)$.

6. (a) A. $[0, 2\pi]$ B. y-intercept $= 0$, x-intercept $= 0$ C. none D. none E. always increasing F. minimum $f(0) = 0$, maximum $f(2\pi) = 2\pi$ G. CD on $(0, \pi)$ and CU on $(\pi, 2\pi)$; IP (π, π)



(b) A. $(0, \pi)$ B. y-intercept 0, x-intercepts 0, $\dfrac{\pi}{2}$ C. none D. none E. increasing on $(0, 0.86)$, decreasing on $(0.86, \pi)$ F. local maximum $f(0.86) = 0.56$, minimums $f(0) = 0$ and $f(\pi) = -\pi$ G. CD on $(0, 2.29)$ and CU on $(2.29, \pi)$; IP $(2.29, -1.51)$



7. $5\sqrt{3}$
8. (a) 0.739 085 (b) 0.704 576 (c) 1.895 494
9. $\dfrac{d}{dx} \tan x = \sec^2 x$, $\dfrac{d}{dx} \csc x = -\csc x \cot x$
10. $\dfrac{\cos^2 y \cos x + \sin^2 x \sin y}{\cos^2 y}$
11. (a) $(\cos x - 1) \cot(x - \sin x) \csc(x - \sin x)$
(b) $\dfrac{\cos \sqrt{x}}{4\sqrt{x} \sin \sqrt{x}}$ (c) $\dfrac{\cos x - x \sin x}{3\sqrt[3]{x^2 \cos^2 x}}$
(d) $\sin x[\sin(\cos x)][\cos(\cos x)][3 \cos (\cos x) - 2]$
(e) $\dfrac{-\sin x \cos x[\sin(\sin^2 x)]}{\sqrt{\cos(\sin^2 x)}}$
12. $2x - y = 0$ 13. $\dfrac{\sin x + y \sec^2(xy) + 1}{\cos y - x \sec^2(xy)}$

EXERCISE 7.3

1. (a) $6 \sec^2 2x$ (b) $-3 \csc^2 9x$

Answers continued.

(c) $3 \sec \frac{x}{4} \tan \frac{x}{4}$ (d) $-2 \csc 8x \cot 8x$
(e) $2x \sec^2 x^2$ (f) $2 \tan x \sec^2 x$
(g) $\frac{\sec \sqrt[3]{x} \tan \sqrt[3]{x}}{3\sqrt[3]{x^2}}$ (h) $x \csc x(2 - x \cot x)$
(i) $12(1 - 2x) \csc^2(1 - 2x)^2 \cot^2(1 - 2x)^2$
(j) 0 (k) $\frac{-3 \sec 2x \tan 2x}{\sqrt{(\sec 2x - 1)^3}}$ (l) $x(x \cos x + 2 \sin x)$ (m) $3\sqrt{x} + 2x \csc^2 x - 2 \cot x$
(n) $\sec^2 x[\cos(\tan x)]$ (o) $-2 \sin x[\tan(\cos x)][\sec^2(\cos x)]$
(p) $\frac{6(2x - 1) \sec^2(x^2 - x)^{-2}}{(x^2 - x)^3 \tan^4(x^2 - x)^{-2}}$

2. (a) $\frac{\sec^2 x}{1 - \sec y \tan y}$ (b) $-\frac{2 \sec^2 2x}{\sin 3y}$
(c) $-\frac{\csc^2 x + \csc^2(x + y)}{\csc^2(x + y) + \csc^2 y}$
(d) $-\frac{y \csc(xy) \cot(xy)}{2y + x \csc(xy)\cot(xy)}$
(e) $\frac{2xy^2 + y \sec\left(\frac{x}{y}\right) \tan\left(\frac{x}{y}\right)}{x \sec\left(\frac{x}{y}\right) \tan\left(\frac{x}{y}\right)}$
(f) $\frac{2x}{2y - \sec^2 y[\cos(\tan y)]}$

3. (a) $4x + y - 1 - \pi = 0$
(b) $3\sqrt{3}x - 6y - \sqrt{3}\pi + 3 = 0$
(c) $4y + 8\sqrt{2}x + 4\sqrt{2} + \pi\sqrt{2} = 0$
(d) $4\sqrt{2}x - 2y - 3\sqrt{2}\pi = 0$

4. No critical numbers exist and $y' > 0$

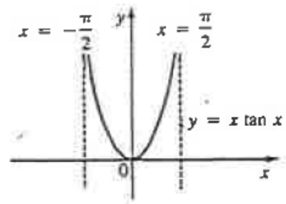
5. (a) $x = \pi$
(b) $x = -\frac{\pi}{2}, x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$

6. No critical numbers; always increasing

7. CU on $\left(-\frac{\pi}{2}, 0\right)$ or $\left(\frac{\pi}{2}, \pi\right)$;
CD on $\left(0, \frac{\pi}{2}\right)$ or $\left(\pi, \frac{3\pi}{2}\right)$

8. A. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ B. y-intercept is 0, x-intercept is 0 C. y-axis D. $x = \pm\frac{\pi}{2}$
E. increasing on $\left(0, \frac{\pi}{2}\right)$, decreasing on $\left(-\frac{\pi}{2}, 0\right)$
F. minimum $f(0) = 0$ G. CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

H.



10. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ 11. $\pm\frac{1}{2}$

12. $\frac{3 \sec^{\frac{3}{2}}(\sqrt[4]{x}) \tan(\sqrt[4]{x})}{8(\sqrt[4]{x^3})}$

13. $\frac{dy}{dx} = -2x, \frac{d^2y}{dx^2} = -2$

EXERCISE 7.4

1. (a) local minimum $f\left(\frac{\pi}{3}\right) = -0.685$; local maximum $f\left(\frac{5\pi}{3}\right) \doteq 6.968$ (b) none (c) local minima $f\left(\frac{\pi}{4}\right), f\left(\frac{3\pi}{4}\right), f\left(\frac{5\pi}{4}\right), f\left(\frac{7\pi}{4}\right) = \frac{1}{2}$; local maxima $f(0), f\left(\frac{\pi}{2}\right), f(\pi), f\left(\frac{3\pi}{2}\right), f(2\pi) = 1$
(d) local minimum $f(0) = 1$; local maxima $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) \doteq 1.571$

2. (a) abs min $= -\frac{3\sqrt{3}}{2}$; abs max $= \frac{3\sqrt{3}}{2}$
(b) abs min $= -2$; abs max $= 1$

3. $\frac{\pi}{4}$ 4. $\frac{\pi}{2}$ 5. $\frac{\pi}{3}$ 6. 0.588 00

7. 0.588 00 8. 25 m/h 9. 0.02 m/s

10. 33 708 m/min and 11 624 m/min

11. 0.727 m²/min 12. $\frac{\sqrt{2}}{5}$ rad/s

13. 0.05 rad/s 14. 0.128 rad/s 15. $\frac{\pi}{3}$

16. 24.5 m² 18. 125 m