Limits at Infinity

Given $\lim_{x\to\infty} f(x)$ there are four (4) possible values for this limit.

$$\lim_{x\to\infty} f(x) = \begin{cases} 0 \\ +\infty \\ -\infty \\ \dfrac{a}{b}, & a \,\&\, b \text{ constants} \end{cases}$$

Let $f(x) = \dfrac{P_m(x)}{Q_n(x)}$, where m and n are the degrees of the polynomials $P(x)$ and $Q(x)$.

Note: To determine the value of $\lim_{x\to\infty} f(x)$, divide the numerator and the denominator by the highest degree of the denominator.

CASE 1: $m < n$

ex: $\lim_{x\to\infty} \dfrac{x+1}{x^2+4}$

$$= \lim_{x\to\infty} \frac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{4}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{4}{x^2}} = \frac{0+0}{1+0} = 0$$

Summary: If $m < n$, $\lim_{x\to\pm\infty} f(x) = 0$

* H.A. @ $y = 0$

CASE 2: $m > n$

ex: $\lim_{x\to\infty} \frac{x^3+x^2+4}{3x^2+1}$

$$= \lim_{x\to\infty} \frac{\frac{x^3}{x^2} + \frac{x^2}{x^2} + \frac{4}{x^2}}{\frac{3x^2}{x^2} + \frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{x + 1 + \frac{4}{x^2}}{3 + \frac{1}{x^2}}$$

$$= \frac{\infty + 1 + 0}{3 + 0} = \infty$$

ex: $\lim_{x\to-\infty} \frac{x^2+2}{x-1}$

$$\lim_{x\to-\infty} \frac{\frac{x^2}{x} + \frac{2}{x}}{\frac{x}{x} - \frac{1}{x}}$$

$$= \lim_{x\to-\infty} \frac{x + \frac{2}{x}}{1 - \frac{1}{x}}$$

$$= \frac{-\infty - 0}{1 + 0} = -\infty$$

Summary: If $m > n$ $\lim_{x\to\pm\infty} f(x) = \pm\infty$

CASE 3: $m = n$

ex: $\lim_{x \to \infty} \frac{x^3 + 2x + 3}{4x^3 + x^2 + x + 1}$

$$= \lim_{x \to \infty} \frac{\frac{x^3}{x^3} + \frac{2x}{x^3} + \frac{3}{x^3}}{\frac{4x^3}{x^3} + \frac{x^2}{x^3} + \frac{x}{x^3} + \frac{1}{x^3}}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{2}{x^2} + \frac{3}{x^3}}{4 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}}$$

$$= \frac{1 + 0 + 0}{4 + 0 + 0 + 0}$$

$$= \frac{1}{4}$$

* H.A. @ $y = \frac{a}{b}$,

where $\frac{a}{b}$ is the ratio of the leading coefficients.

Summary: If $m = n$ $\lim_{x \to \pm\infty} f(x) = \frac{a}{b}$