

Calculus

- Used to compare quantities which vary in a non-linear way
- 2 main branches:
 - Differentiation-helps us to find a rate of change of one quantity compared to another
 - Integration-(reverse of differentiation) given a rate of change, find the original relationship between the 2 quantities
- The idea of zooming in to get a better approximation of the slope of the curve was the breakthrough that led to the development of differentiation
- Methods: ① Numerical approach to finding slopes (limits & slope of a tangent)
② Algebraic approach to finding slopes (differentiation from first principles & derivatives)

1-1 Exploring the concept of a limit

 Recall: A sequence is a list of numbers written in a definite order: $t_1, t_2, t_3, t_4, \dots, t_n, \dots$, where t_1 is the first term and in general t_n is the nth term.

Ex 1: List the first 5 terms of the sequence defined by $t_n = \frac{n}{n+1}$ and draw the graph of the sequence.

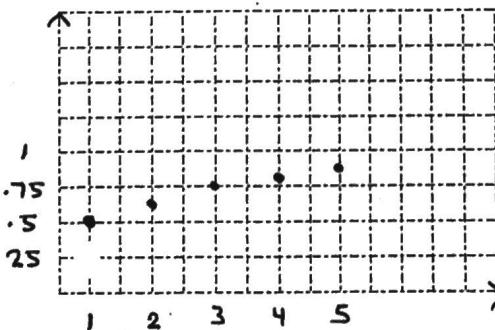
$$t_1 = \frac{1}{2}$$

$$t_2 = \frac{2}{3}$$

$$t_3 = \frac{3}{4}$$

$$t_4 = \frac{4}{5}$$

$$t_5 = \frac{5}{6}$$



$$\therefore \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

In general, we say that the sequence with general term t_n has the limit L, and we write:

$$\lim_{n \rightarrow \infty} t_n = L$$

Ex 2: Find $\lim_{n \rightarrow \infty} \frac{1}{n}$

$$= 0$$

Ex 3: Find $\lim_{n \rightarrow \infty} (-1)^n$

= Does not exist ("DNE")

Exercise 1.6

A 1. State the limits of the following sequences, or state that the limit does not exist.

(a) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \dots, \left(\frac{1}{3}\right)^n, \dots$

(b) $5, \frac{41}{2}, \frac{41}{3}, \frac{41}{4}, \frac{41}{5}, \dots, 4 + \frac{1}{n}, \dots$

(c) $1, 2, 3, 4, 5, \dots, n, \dots$

(d) $3, 3, 3, 3, 3, \dots, 3, \dots$

(e) $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \dots$

(f) $5, \frac{61}{2}, \frac{52}{3}, \frac{61}{4}, \frac{54}{5}, \frac{61}{6}, \dots, 6 + \frac{(-1)^n}{n}, \dots$

(g) $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \dots$

(k) $\lim_{n \rightarrow \infty} \frac{n}{n^2 + 1}$

(m) $\lim_{n \rightarrow \infty} 5^{-n}$

(o) $\lim_{n \rightarrow \infty} \frac{1+n-2n^2}{1-n+n^2}$

(q) $\lim_{n \rightarrow \infty} \frac{1}{n^5}$

(s) $\lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n$

(l) $\lim_{n \rightarrow \infty} (-1)^{n-1} n$

(n) $\lim_{n \rightarrow \infty} (n^3 + n^2)$

(p) $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}}$

(r) $\lim_{n \rightarrow \infty} \frac{1-n^3}{1+2n^3}$

(t) $\lim_{n \rightarrow \infty} \left(\frac{4}{3}\right)^n$

B 2. List the first six terms of the following sequences.

(a) $t_n = \frac{n-1}{2n-1}$

(b) $t_n = \frac{2n}{n^2 + 1}$

(c) $t_n = n2^n$

(d) $t_n = \frac{(-1)^{n-1}}{n}$

(e) $t_1 = 1, t_n = \frac{1}{1+t_{n-1}} (n \geq 2)$

(f) $t_1 = 1, t_2 = 2, t_n = t_{n-1} - t_{n-2} (n \geq 3)$

3. Find the following limits or state that the limit does not exist.

(a) $\lim_{n \rightarrow \infty} \frac{1}{n^2}$

(b) $\lim_{n \rightarrow \infty} \frac{1}{5+n}$

(c) $\lim_{n \rightarrow \infty} \left(6 + \frac{1}{n^3}\right)$

(d) $\lim_{n \rightarrow \infty} \frac{n}{3n-1}$

(e) $\lim_{n \rightarrow \infty} \frac{6n+9}{3n-2}$

(f) $\lim_{n \rightarrow \infty} 5n^2$

(g) $\lim_{n \rightarrow \infty} \frac{n^2+1}{2n^2-1}$

(h) $\lim_{n \rightarrow \infty} \frac{(n+1)^2}{n(n+2)}$

(i) $\lim_{n \rightarrow \infty} \frac{(-1)^{n+1}}{n}$

(j) $\lim_{n \rightarrow \infty} \left(-\frac{1}{4}\right)^n$

EXERCISE 1.6

1. (a) 0 (b) 4 (c) does not exist (d) 3 (e) 0
 (f) 6 (g) does not exist
 2. (a) 0, $\frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{5}{9}, \frac{1}{11}$ (b) 1, $\frac{4}{5}, \frac{6}{10}, \frac{8}{17}, \frac{10}{26}, \frac{12}{37}$
 (c) 2, 8, 24, 64, 160, 384
 (d) 1, $-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}$ (e) 1, $\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{3}{5}, \frac{4}{6}$
 (f) 1, 2, 1, -1, -2, -1
 3. (a) 0 (b) 0 (c) 6 (d) $\frac{1}{3}$ (e) 2 (f) does not exist
 (g) $\frac{1}{2}$ (h) 1 (i) 0 (j) 0 (k) 0
 (l) does not exist (m) 0 (n) does not exist
 (o) -2 (p) 0 (q) 0 (r) $-\frac{1}{2}$ (s) 0 (t) does not exist 4. $\frac{1}{3}$

Exercise 1.7

B Find the sum of each of the following series or state that the series is divergent.

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$

(b) $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \dots$

(c) $\frac{1}{4} - \frac{5}{16} + \frac{25}{64} - \frac{125}{256} + \dots$

(d) $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \dots$

(e) $1 - 2 + 4 - 8 + \dots$

(f) $60 + 40 + \frac{80}{3} + \frac{160}{9} + \dots$

(g) $0.1 + 0.05 + 0.025 + 0.0125 + \dots$

(h) $-3 + 3 - 3 + 3 - 3 + \dots$

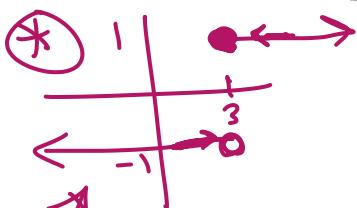
EXERCISE 1.7

1. (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) divergent (d) $\frac{15}{4}$
 (e) divergent (f) 180 (g) $\frac{1}{2}$ (h) divergent

Introduction to Limits

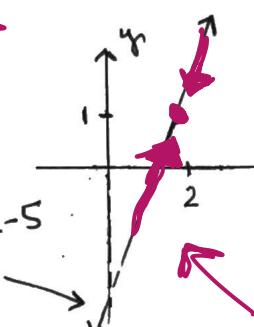
Example 1: Consider the graph of the function $f(x) = 3x - 5$.

As the value of x "approaches" 2 (on the graph), what does the value of the function approach?



$$\lim_{x \rightarrow 3} f(x) = \text{DNE}$$

$$\underline{f(3) = 1}$$



We say that "the limit of $3x-5$ as x approaches 2 equals 1," and we write

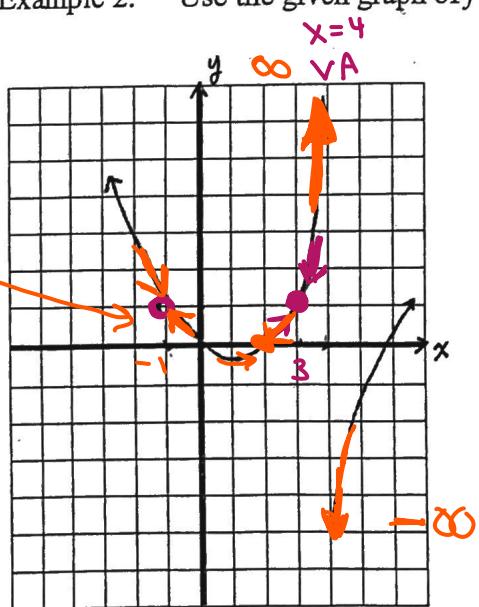
$$\lim_{x \rightarrow 2} (3x - 5) = 1$$

Notes:

- In this case $f(2) = 1$ and $\therefore \boxed{\lim_{x \rightarrow 2} f(x) = f(2)}$
- In retrospect it follows that $\lim_{x \rightarrow 2} f(x)$ could have been evaluated simply by substituting $x = 2$ into the formula for $f(x)$.
- However, NOT all limits can be evaluated by substitution.

$$3(2) - 5 = 1$$

Example 2: Use the given graph of f to state the value of the limit, if it exists.



$$\lim_{x \rightarrow 5} f(x) = \text{DNE} \quad VA \quad x = 5$$

$$\left| \begin{array}{l} \lim_{x \rightarrow s} f(x) \\ \text{VA } x = s \\ = \infty \end{array} \right| \quad \left| \begin{array}{l} \lim_{x \rightarrow s} f(x) \\ \text{VA } x = s \\ = 3 \end{array} \right|$$

$$a. \lim_{x \rightarrow 3} f(x) = 1$$

$$f(3) = 1$$

NOTE:

*(-1) is undefined.

$$b. \lim_{x \rightarrow 2} f(x) = 0$$

$$f(2) = 0$$

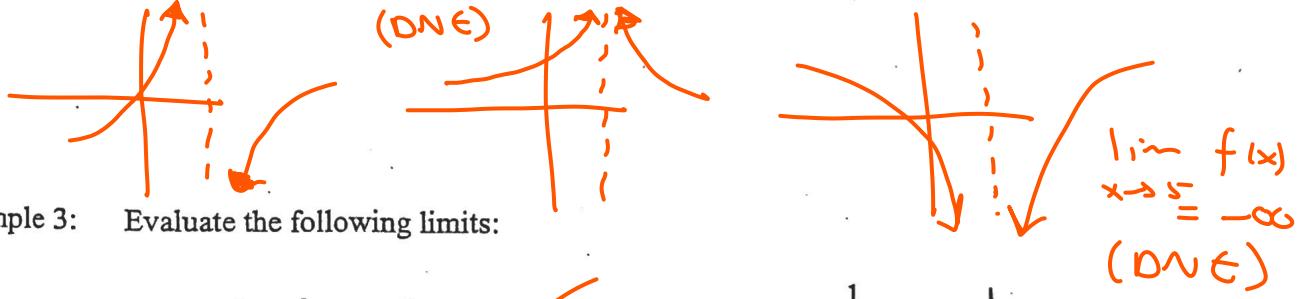
$$c. \lim_{x \rightarrow -1} f(x) = 1$$

$$d. \lim_{x \rightarrow 4} f(x) = \text{DNE}$$

(infinitely discontinuous)

$f(4)$ is undefined.

$$\sqrt{A} \quad x = 5$$



Example 3: Evaluate the following limits:

a. $\lim_{x \rightarrow 3} x^2 = 9$

b. $\lim_{x \rightarrow 2} \frac{1}{x} = \frac{1}{2}$

c. $\lim_{x \rightarrow 1} \sqrt{x+3} = 2$

Notes: For the following types of functions

- polynomial functions
- rational functions
- algebraic functions

you can evaluate $\lim_{x \rightarrow a} f(x)$ by substitution ($\lim_{x \rightarrow a} f(x) = f(a)$), provided that $f(a)$ is defined.

$$f(x) =$$

Example 4: Evaluate the following limit: $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

$$\frac{0}{0}$$

indeterminate form

(need to do something algebraically)

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} \quad \frac{0}{0}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{x-3}$$

hole (3, 6)

$$= \lim_{x \rightarrow 3} (x+3)$$

$f(3)$ is undefined

$$= 6$$

Example 5: Evaluate the following limits:

$$\lim_{x \rightarrow 2} \frac{(x-3)^2 - 1}{x-2} \quad \frac{0}{0}$$

$$= \lim_{x \rightarrow 2} \frac{(x-3-1)(x-3+1)}{x-2}$$

$$= \lim_{x \rightarrow 2} \frac{(x-4)(x-2)}{x-2}$$

$$= \lim_{x \rightarrow 2} (x-4) = -2$$

$$\lim_{h \rightarrow 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} \quad \frac{0}{0}$$

$$= \lim_{h \rightarrow 0} \left(\frac{\sqrt{3+h} - \sqrt{3}}{h} \right) \cdot \left(\frac{\sqrt{3+h} + \sqrt{3}}{\sqrt{3+h} + \sqrt{3}} \right)$$

$$= \lim_{h \rightarrow 0} \frac{3+h - 3}{h(\sqrt{3+h} + \sqrt{3})}$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{3+h} + \sqrt{3}} = \frac{1}{2\sqrt{3}} \quad \left(\text{or } \frac{\sqrt{3}}{6} \right)$$

$$\lim_{h \rightarrow 0} \frac{\frac{4}{2+h} - 2}{h} \quad \frac{0}{0}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{4}{2+h} - 2}{h} = \frac{2(2+h)}{2+h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{4-4-2h}{2+h}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-2h}{h(2+h)}$$

$$= \frac{-2h}{2+h} \cdot \frac{1}{h}$$

~~cancel~~

$$\frac{0}{0} \rightarrow \# \quad \text{DNE}$$

hole

(2, -2)

$f(2)$ is undefined.

Something

* conjugate
fact

X1

$$= \frac{1}{2\sqrt{3}} \quad \left(\text{or } \frac{\sqrt{3}}{6} \right)$$

common denominator
do not need to rationalize

+ to rationalize

$$\lim_{h \rightarrow 0} \frac{-2}{2+h}$$

$$= -1$$

hole
(0, -1)

$$\frac{a}{bc} = \frac{a}{b} \times \frac{1}{c}$$

$$= \frac{a}{bc}$$

Limits – Change of Variable

Equivalent

- $$\lim_{x \rightarrow 8} \frac{x-8}{\sqrt[3]{x}-2}$$

Let $t = \sqrt[3]{x}$

$\Rightarrow x = t^3$

$\left. \begin{array}{l} \text{since} \\ x \rightarrow 8 \\ \text{then} \\ t \rightarrow 2 \end{array} \right\}$

$$\lim_{t \rightarrow 2} \frac{t^3 - 8}{t - 2}$$

$$= \lim_{t \rightarrow 2} \frac{(t-2)(t^2 + 2t + 4)}{t-2}$$

$$= \lim_{t \rightarrow 2} t^2 + 2t + 4$$

$$= 12$$

- $$\lim_{x \rightarrow 27} \frac{27-x}{x^{\frac{1}{3}}-3}$$

Let $t = x^{\frac{1}{3}}$

$\Rightarrow x = t^3$

$\left. \begin{array}{l} \text{since} \\ x \rightarrow 27 \\ \text{then} \\ t \rightarrow 3 \end{array} \right\}$

$$\lim_{t \rightarrow 3} \frac{27-t^3}{t-3}$$

$$= \lim_{t \rightarrow 3} \frac{(3-t)(9+3t+t^2)}{t-3}$$

$$= \lim_{t \rightarrow 3} -(9+3t+t^2)$$

$$= -27$$

3. $\lim_{x \rightarrow 0} \frac{(x+8)^{\frac{1}{3}} - 2}{x}$

Let $t = (x+8)^{\frac{1}{3}}$
 $x = t^3 - 8$

since $x \rightarrow 0$

then $t \rightarrow 2$

$$\lim_{t \rightarrow 2} \frac{t-2}{t^3-8}$$

$$= \lim_{t \rightarrow 2} \frac{t-2}{(t-2)(t^2+2t+4)}$$

$$= \lim_{t \rightarrow 2} \frac{1}{t^2+2t+4}$$

$$= \frac{1}{12}$$

$$\lim_{t \rightarrow 1} \frac{t^3-1}{t^2-1}$$

$$\left. \begin{aligned} t &= x^{\frac{1}{3}} \\ t^2 &= x^{\frac{2}{3}} = x^{\frac{1}{3}} \\ t^3 &= x^{\frac{3}{6}} = x^{\frac{1}{2}} \end{aligned} \right\}$$

Also

$$\lim_{x \rightarrow 1} \frac{\sqrt[3]{x}-1}{x-1} = \lim_{x \rightarrow 1} \frac{x^{\frac{1}{3}}-1}{x^{\frac{1}{3}}-1}$$



$$\lim_{x \rightarrow 1} \frac{x^{\frac{1}{3}}-1}{x^{\frac{1}{3}}-1} = \lim_{x \rightarrow 1} \frac{\frac{1}{3}x^{-\frac{2}{3}}}{\frac{1}{3}x^{-\frac{2}{3}}} = 1$$