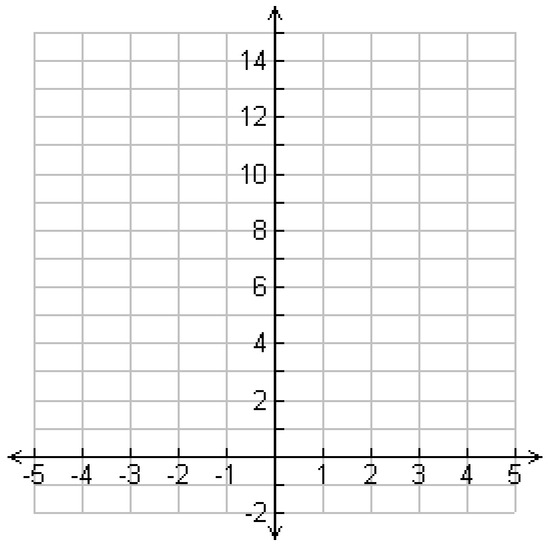
# Unit 5: Exponential and Logarithmic Functions

# Unit 5: Exponential and Logarithmic Functions
## 5.1 The Exponential Function and its Inverse

In this section, you will be investigating the exponential function $f(x) = b^x$. Since you will be drawing several curves on each grid, remember to label each curve with its equation.

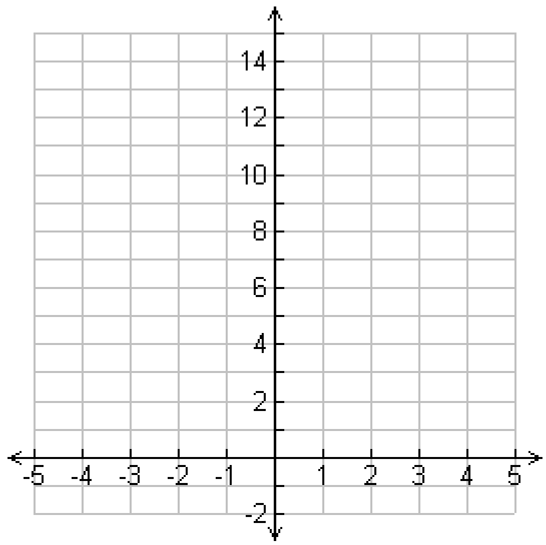
1. Use your graphing calculator to draw the graphs of $y = 2^x$, $y = 5^x$ and $y = 10^x$. Sketch the curves on the grid below. Be sure to label the y-intercept and any asymptotes.




| | $y = 2^x$ | $y = 5^x$ | $y = 10^x$ |
|---|---|---|---|
| Domain | | | |
| Range | | | |
| y-intercept | | | |
| Asymptotes | | | |
| Increasing/decreasing | | | |

What are the common characteristics of these curves?

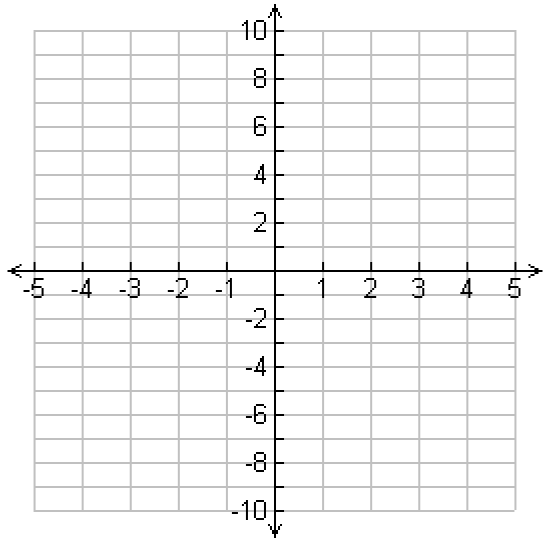
2. Use your graphing calculator to draw the graphs of $y=\left(\frac{1}{3}\right)^x$, $y=\left(\frac{1}{5}\right)^x$ and $y=\left(\frac{1}{10}\right)^x$. Note that we can express these functions as $y=3^{-x}$, $y=5^{-x}$, and $y=10^{-x}$. Sketch the curves on the grid below, labeling fully.




| | $y=3^{-x}$ | $y=5^{-x}$ | $y=10^{-x}$ |
|---|---|---|---|
| Domain | | | |
| Range | | | |
| y-intercept | | | |
| Asymptotes | | | |
| Increasing/decreasing | | | |

What are the common characteristics of these curves?

3. Graph $y = 3^x$, $y = \left(\frac{1}{3}\right)^x$ and $y = -3^x$. Sketch, labeling the functions carefully.




(a) What transformation on $y = 3^x$ will give $y = \left(\frac{1}{3}\right)^x$ as its image?

(b) What transformation on $y = 3^x$ will give $y = -3^x$ as its image?

(c) The inverse of a function is obtained by ______________________________________.

(d) The inverse of $y = 2^x$ is ________________________.

(e) The graph of the inverse is obtained_____________________________________.

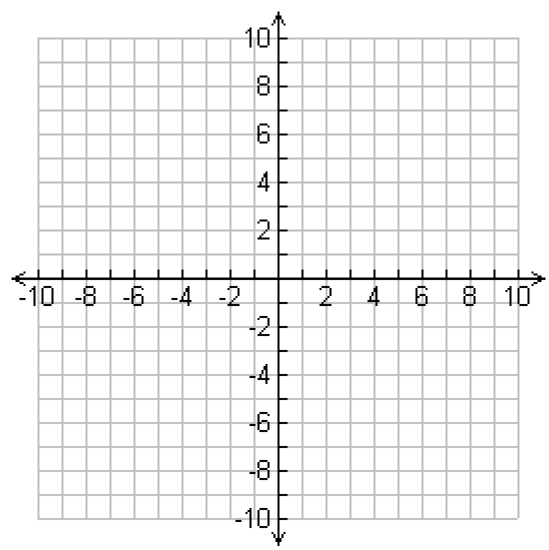
Ex. 1: Graph $y = 2^x$ and its inverse, on the same set of axes.

$y = 2^x$

| x | y |
|---|---|
| -1 | |
| 0 | |
| 1 | |
| 2 | |
| 3 | |

inverse

| x | y |
|---|---|
| | |
| | |
| | |
| | |
| | |



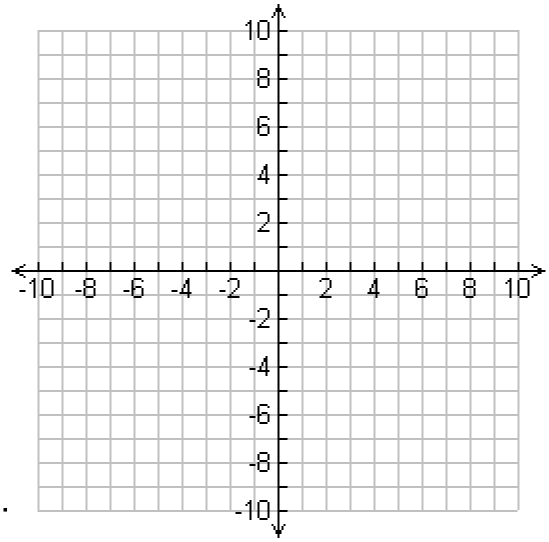
Ex. 2: Graph $y = \left(\frac{1}{2}\right)^x$ and its inverse, on the same set of axes.

$y = \left(\frac{1}{2}\right)^x$

| x | y |
|---|---|
| -3 | |
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |

inverse

| x | y |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |




Ex. 3: Write an equation to fit the data in the table of values.

| x | y | | | |
|---|---|---|---|---|
| -3 | $\frac{1}{64}$ | | | |
| -2 | $\frac{1}{16}$ | | | |
| -1 | $\frac{1}{4}$ | | | |
| 0 | 1 | | | |
| 1 | 4 | | | |
| 2 | 16 | | | |
| 3 | 64 | | | |

# Logarithmic Functions

Logarithms were first introduced by John Napier in the $17^{th}$ century for the purpose of simplifying calculations. This was accomplished with the development of logarithmic tables and, soon after, with logarithmic scales on a slide rule. With the introduction of the scientific calculator in the mid-1970s, this application of logarithms for computations became somewhat obsolete; however, logarithms are still used today in many areas such as

- scientific formulas and scales (the pH scale in chemistry and the Richter scale for measuring and comparing the intensity of earthquakes),
- astronomy (order of magnitude calculations comparing relative size of massive bodies),
- modelling and solving problems involving exponential growth and decay, and many areas of calculus

## Introduction

We will begin our study of logarithms by introducing and exploring the **logarithmic function**. The logarithmic function is simply the inverse of the exponential function.

**Exponential Form** ←→ **Logarithmic Form**

$x = b^y$ $y = \log_b x$ ($b > 0$ and $b \neq 1$)

The logarithm of a number x with a given base is the exponent to which that base must be raised to yield x.

**What is a Logarithm?**

Logarithms can be set to any base. The LOG key on your calculator represents $\log_{10}$. Record the results in the space provide. The first example is done for you.

| Logarithm | Value |
|---|---|
| $\log 100 = 2$ | $10^2 = 100$ |
| $\log 10$ | |
| $\log 1000$ | |
| $\log 0.01 =$ | |
| $\log 0.0001 =$ | |
| $\log \sqrt{10} =$ | |
| $\log \sqrt{10000} =$ | |
| $\log 0$ | |
| $\log(-3)$ | |
| $\log_3(81)$ | |
| $\log_2(16)$ | |
| $\log_6(216)$ | |
| $\log_{25}\left(\frac{1}{625}\right)$ | |
| $\log_4(64)$ | |

Ex. 1: Change to exponential form.

a. $\log_2 8 = 3$

b. $\log_2 32 = 5$

Ex. 2: Change to logarithmic form.

a. $4^3 = 64$

b. $\left(\frac{1}{2}\right)^{-4} = 16$

Ex. 3: Evaluate the following.

a. $\log_5 25$

b. $\log_3 27$

c. $\log_2\left(\frac{1}{4}\right)$

d. $\log_{\frac{1}{3}} 27$

e. $\log_2\left(\frac{1}{4}\right)+\log_{\frac{1}{2}} 4$

f. $\log_3\left(27\times\sqrt{27}\right)$

Ex. 4: For each function, $y=g(x)$, determine the equation of $y=g^{-1}(x)$.

a. $g(x)=3^{2(x-1)}+5$

b. $g(x)=2\log_5(x+4)-1$

c. $g(x)=-\log_3\left(\frac{1}{2}(x-3)\right)+4$

Ex.5. Determine the point of intersection of $f(x)=2\log_2(x-2)$ and $g(x)=2-\log_3(x-3)$.

To conclude…

1. An exponential function of the form $y = b^x$, $b > 0$, $b \neq 1$, has
   - a repeating pattern of finite differences
   - a rate of change that is increasing proportional to the function for $b > 1$
   - a rate of change that is decreasing proportional to the function of $0 < b < 1$
   - has domain $\{x \mid x \in R\}$
   - has range $\{y \mid y > 0, y \in R\}$
   - has y-intercept 1
   - has horizontal asymptote with equation $y = 0$

2. The inverse of $y = b^x$ is a function that can be written as $x = b^y$. This function
   - has domain $\{x \mid x > 0, x \in R\}$
   - has range $\{y \mid y \in R\}$
   - has x-intercept 1
   - has vertical asymptote at $x = 0$
   - is a reflection of $y = b^x$ about the line $y = x$

# Practice

## Part A - Multiple Choice.

_____1. The range of the function $f(x) = 4(2)^x + 1$ is:

A. $y \in R$ B. $y > 4,\ y \in R$ C. $y < 1,\ y \in R$ D. $y > 1,\ y \in R$

_____2. Another way to write $2^{-3} = \frac{1}{8}$ is

A. $\log_2(-3) = 8$ B. $\log_2(-3) = \frac{1}{8}$ C. $\log_{\frac{1}{8}}(2) = -3$ D. $\log_2\left(\frac{1}{8}\right) = -3$

_____3. The domain of the logarithmic function is:

A. $x \in R$ B. $x < 0,\ x \in R$ C. $x > 0,\ x \in R$ D. $x > 1,\ x \in R$

_____ 4. The graph of $y = 5\log_2(3x + 12) - 5$ has a vertical asymptote at

A. $x = -5$ B. $x = -12$ C. $x = 4$ D. $x = -4$

## Part B - Short Answer

1. Find the value(s) of $x$ such that $\log_x(19x - 30) = 3$.

2. Find the inverse of the following functions:

a) $f(x) = \dfrac{2^x}{1 - 2^x}$ b) $f(x) = -2\log_3\left(\frac{2}{3}(x+1)\right) - 2$

# Unit 5: Exponential and Logarithmic Functions
## 5.2 Laws of Logarithms

There are some basic properties of logarithms:

**Basic Property** **Proof**

#1: $\boxed{\log_b 1 = 0}$

#2: $\boxed{\log_b b = 1}$

#3: $\boxed{\log_b b^x = x}$

#4: $\boxed{b^{\log_b x} = x}$

Logarithms are exponents and the Exponent Laws give rise to corresponding Laws of Logarithms.
**Recall**: the power law of exponents

$$\left(b^x\right)^y = b^{xy}$$

We can use the power law of exponents to develop a power law of logarithms.

**The Power Law**: the logarithm of a power of a number is equal to the exponent multiplied by the logarithm of the number. $\boxed{\log_a\left(b^n\right) = n\log_a b}$

Proof:

**Ex. 1**: Rewrite the following:

a. $\log_6 36^4$

b. $3\log_5 25$

c. $\log_3 \sqrt{27}$

**Ex. 2**: Determine $\log_3 81^{\frac{51003}{2}}$.

**Ex. 3**: **Change of base Formula**: Show that $\log_b x = \dfrac{\log_a x}{\log_a b}$, where a > 0.

**Ex. 4:** Find the value of $\log_3 23$, correct to two decimal places.

**Ex. 5:** Prove that $\log_t b = \dfrac{1}{\log_b t}$.

**The Product Law**

The logarithm of a product is equal to the sum of the logarithms of the factors.

$$\boxed{\log_a (mn) = \log_a m + \log_a n}$$

Proof:

**Ex. 6:** Evaluate using properties of logarithms.

a. $\log_6 12 + \log_6 3$

b. $\log 25 + \log 4$

c. $\log_{\frac{1}{9}} \left(\sqrt[3]{81}\right)^2$

**Ex. 7:** Write $\log_a x^3y^4$ in terms of $\log_a x$ and $\log_a y$.

**The Quotient Law**

The logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator.

$$\boxed{\log_a\left(\frac{m}{n}\right)=\log_a m-\log_a n}$$

Proof:

**Ex. 8:** Evaluate using properties of logarithms.

a. $\log_2\left(\frac{25.6}{6.4}\right)$

b. $\log_4 48-\log_4 3$

c. $\log_5\left(\frac{1}{25}\right)$

**Ex. 9:** Write the following expressions as a single logarithm.

a) $3\log(x+3)-2\log(x-1)$

b) $2\log(x+5)-\log(x^2-25)+3\log x$

c) $\log\left(\dfrac{\sqrt[6]{x^5}}{x^2}\right)-\log\sqrt[3]{x}$

**Ex. 10:** Fully expand and simplify the following expressions.

$$log_k\left(\sqrt{\frac{x^3y^2}{w}}\right)$$

**Practice**

**1.** Evaluate using properties of logarithms.

a) $4\log_4(\log 100)$

b) $\dfrac{3}{\log_8 2}-\dfrac{5}{\log_2 2}-\dfrac{2}{\log_4 2}$

c) $\sqrt{25^{\frac{1}{\log_6(5)}}+49^{\frac{1}{\log_8(7)}}}$

d) $\dfrac{\log_3\sqrt{243\sqrt{81\sqrt[3]{3}}}}{\log_2\sqrt[4]{64}+\log_5 5^{-10}}$

e) $\log_9(243)^{-4}-\log_6\left(\dfrac{1}{1296}\right)+\log_5\sqrt[3]{78125}$

**2.** If $(x+y)^2=4xy$, prove that $\log_a\left(\dfrac{x+y}{2}\right)=\dfrac{1}{2}(\log_a x+\log_a y)$.

**3.** If $\log_8 5=m$ and $\log_4 3=n$ find an expression for $\log_2 15$ in terms of *m* and *n*.

4. If $\log_a b=\dfrac{2}{x^3}$ and $\log_b\sqrt{a}=4x^5$, determine x.

## Exit Card!

1. Write $\log(x)-4\log(x-5)+\frac{2}{3}\log\sqrt{x+1}$ as a single logarithm.

2. Evaluate $\left(\frac{1}{5}\right)^{-2+\log_{\sqrt{5}}10}$.

# Warm Up- Exponential and Logarithmic Functions

Multiple Choice: Write the capital letter corresponding to the most correct answer on the line provided

_____1. The inverse of $f(x)=1+3^x$ is :

A) $f^{-1}(x)=\log_{\frac{1}{3}}(x-1)$ B) $f^{-1}(x)=\left(\frac{1}{3}\right)^x-1$

C) $f^{-1}(x)=\log_3(x-1)$ D) $f^{-1}(x)=3\log(x-1)$

_____2. If $\log(2)=\text{m}$, and $\log(3)=\text{n}$, $\log_6(5)$ in terms of m and n is equivalent to ……

A) $\frac{m}{n}$ B) $1-\frac{\text{m}}{\text{n}+\text{m}}$ C) $\frac{1-\text{m}}{\text{n}+\text{m}}$ D) $\frac{\text{m}-1}{\text{n}+\text{m}}$

_____ 3. $\log_8\left(\frac{\sqrt{2}}{4}\right)$ is equal to:

A) $-\frac{1}{2}$ B) $-\frac{8}{6}$ C) $\frac{1}{2}$ D) $-\frac{3}{5}$

4. Evaluate. Show steps. Leave answers in exact form.

a) $\log_3\left(\log_3\left(\sqrt[3]{\sqrt[3]{3}}\right)\right)$ b) $2^{\log_2 5+2\log_2 3}$ c) $\frac{\log_{0.2}(125)^2}{\log_8\sqrt{512}}$

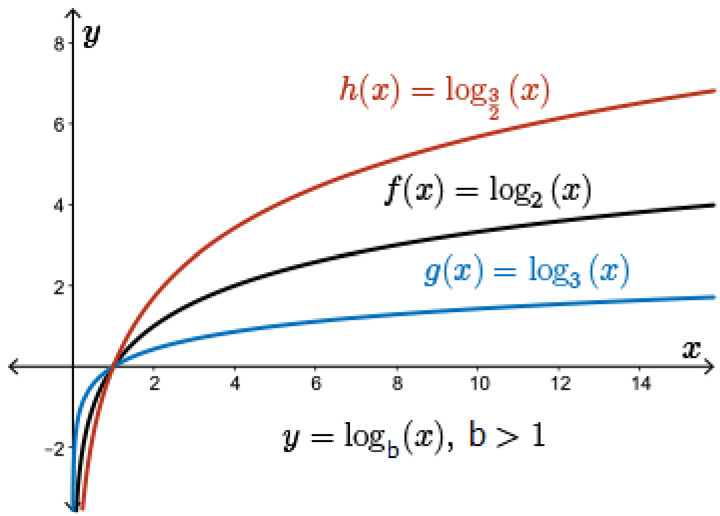
5. Let $\text{f}(\text{x})=\log_5(\text{x}^2-\text{x}-2)-2\log_5(\text{x}+1)$ .Determine the domain of the function and express f(x) as a single logarithm.

6. If $\log_{12}(3)=\text{m}$, find the value of $\log_{\sqrt{3}}(16)$ in terms of $m$.

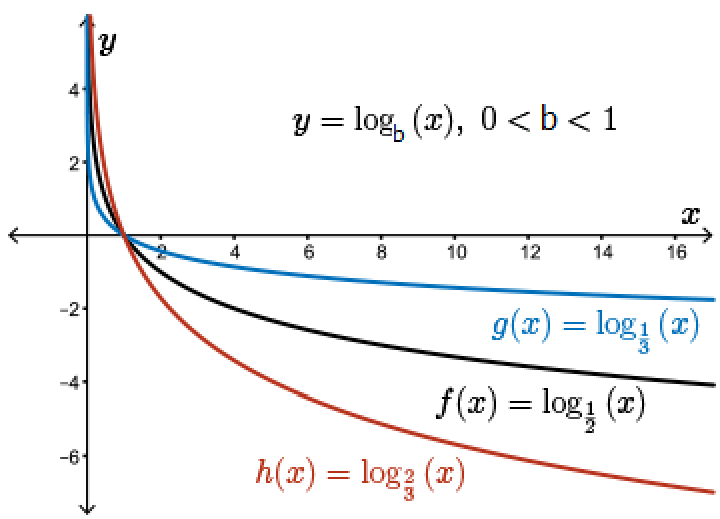
# Unit 5: Exponential and Logarithmic Functions
# 5.3 Transformations of Logarithmic Functions

## General Observations

The graph of a logarithmic function where $b > 1$ is always increasing. The greater the value of the base, $b$, the slower the curve increases as $x$ increases.




The graph of a logarithmic function where $0 < b < 1$ is always decreasing. The smaller the value of the base, $b$, the slower the curve decreases as $x$ increases.



# Transformations of Logarithmic Functions

The parameters $a$, $b$, $d$, and $c$ in the equation $y = a\log_b(k(x-d))+c$ correspond to the following transformations:

- If $a<0, y=log_b(x)$ is reflected in the $x$-axis.
- $y=log_b(x)$ is stretched vertically about the $x$-axis by a factor of $|a|$
- If $k<0, y=log_b(x)$ is reflected in the $y$-axis.
- $y=log_b(x)$ is stretched horizontally about the $y$-axis by a factor of $\frac{1}{|k|}$
- $y=log_b(x)$ is translated horizontally $d$ units.
  - If $d>0$, $then\ y=log_b(x)$ is translated right.
  - If $d<0$, $then\ y=log_b(x)$ is translated left.
- $y=log_b(x)$ is translated vertically $c$ units.
  - If $c>0$, $then\ y=log_b(x)$ is translated up.
  - If $c<0$, $then\ y=log_b(x)$ is translated down.

The transformation of each point is defined by the mapping $(x,y)\rightarrow\left(\frac{1}{k}x+d, ay+c\right)$

### Example 1

Graph the function $f(x)=2\log_5(3-x)+1$ . Identify the domain, range, and any asymptote of the function. Is the function increasing or decreasing?

**Solution:**

Start with the graph of $y=\log_5(x)$.

Mapping rule: $(x,y)\rightarrow($________,________$)$

| $x$ | $y=\log_5(x)$. |
|---|---|
| $\frac{1}{5}$ | |
| 1 | |
| 5 | |
| 25 | |

| $x$ | $f(x)$ |
|---|---|
| | |
| | |
| | |
| | |

**Practice-** Using a mapping rule, sketch the following functions on the grids provided.

1. a) $f(x) = 2\log_2(x+3)$

b) $f(x) = -\log_5\left(\sqrt{x}\right) - 3$

c) $f(x) = 3\log_2(-x) + 1$

d) $f(x) = -\log\left(\frac{1}{3}x\right) + 4$

e)* $f(x) = 1 - \log_2\left(x^2\right)$

f) $y = 2\log_4\left(3x + 12\right) - 4$

**Exit Card!**

Graph the function $y = \dfrac{6}{\log_{(5x-15)}(4)}$ on the grid below.

Mapping statement: $(x, y) \rightarrow$

| | |
|---|---|
| | |
| | |
| | |
| | |

→

| | |
|---|---|
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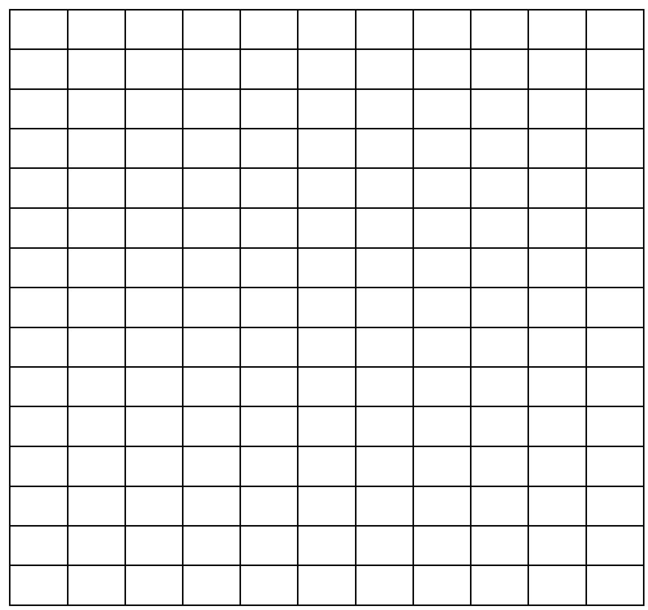


# Unit 5: Exponential and Logarithmic Functions
## 5.4 Techniques for Solving Exponential & Logarithmic Equations

**Part A: Solving Exponential Equations:**
Exponential equations have variables in the exponents or bases.

**Ex1.** $8^x = 4^{2x+1}$

**Ex2.** $5^{3x} = 63$

**Ex3.** $64^{3x-5} = \left(5^{6x+8}\right)$

**Ex4.** $4(2^x) = 3^{x+1}$

**Ex5.** Solve a) $5^{2x} - 5^x - 20 = 0$

b) $3^{2x} - 6(3)^x - 7 = 0$

**Ex.6:** All radioactive substances decrease in mass over time. Jamie works in a laboratory that uses radioactive substances. The laboratory received a shipment of 200 g of radioactive radon, and 16 days later, 12.5 g of the radon remained. What is the half-life of radon?

**Ex.7**: Solve $2(5^{6x})-9(5^{4x})+10(5^{2x})-3=0$.

**Ex.8:** Solve $2^{x+1}=3^{x-1}$ to three decimal places.

**Ex.9:** Express $\dfrac{2^6 \times \left(\dfrac{1}{4}\right)^5}{\left(\sqrt[4]{16}\right)^3}$ as a power with a base of 4.

## Exit Card!

Solve for x.

a) $4\left(\sqrt{2^x}\right) - \dfrac{5}{\sqrt{2^x}} = -1$

b) $\log_{(m-1)}(m^2 - 1) = 3$

c) $5^{4x} = 7(4^{x-2})$ (Round to 2dp)

d) $\left(3^{2x}\right) + 2\left(3^{x+1}\right) - 27 = 0$

**Practice**

Solve each of the following equations

a) $2^{2x}-8\left(2^{x}\right)+16=0$ b) $5^{2x}-26\left(5^{x}\right)+25=0$ c) $\left(27\times 3^{x}\right)=27^{x}\times 3^{0.25}$

d) $2^{2x}-8\left(2^{x}\right)+16=0$ e) $2^{2x+3}-3\left(2^{x+1}\right)+1=0$ f) $x^{\frac{4}{3}}-13x^{\frac{2}{3}}=-36$

g) $2^{2+x}-2^{2-x}=15$ h) $2^{x}+2^{2-x}=5$ i) $\left(x-5\right)^{\frac{2}{3}}=\left(27\right)^{-\frac{1}{9}}$

Answer

a) 2 b) 0 or 2 c) $\frac{11}{8}$ d) -1 or 3 e) -1 or -2

f) ±8 or ±27 g) 2 h) 0 or 2 i) $\frac{15\pm\sqrt{3}}{3}$

**Part B: Solving Logarithmic Equations**

The properties of logarithms we learned in the last sections can help us solve equations involving logarithmic expressions. We must remember that $y = \log_a x$ is defined only for x > 0. Some of the logarithmic equations we solve will appear to have a root that is less that zero. Such a root is inadmissible. This means that every time we solve a logarithmic equation, we must check that the roots obtained are admissible.

**Ex.1 Solve for x.**

a) $\log_x 0.01 = -2$

b) $\log_5(2x-4) = \log_5 36$

c) $\log_6 x + \log_6(x+1) = 1$

d) $\log_8\left(x^3\right) + 6\log_4(x) = -1$

e) $\log_3(x) - 4\log_x(9) = 2$

f) $\log_2\left(\log_4(x)\right) = 2$

g) $\left[\log_4(x)\right]^2 + 18\log_{x^3}(4) = 7$

h) $\log\sqrt[3]{x^2+48x} = \frac{2}{3}$

i) $\log_3\left(x^2+5x-36\right) - \log_3\left(x^2-2x-8\right) + \log_3\left(x+2\right) = 3$

j) $\log_{\sqrt{x}} 5^{\log_x 5} + 3(\log_x 5) - 2 = 0$

**Ex2**. Solve the system:

$$\begin{cases} y = 2\log_3(x) \\ y+1 = \log_3(9x) \end{cases}$$

**Ex.3** If the population of a colony of bacteria doubles every 30 minutes, how long will it take for the population to triple?

**Ex.4*** Determine the values of $x$ and $y$ given the following information.

- $\log_x\left(-\frac{1}{4}\log_y\left(\log_{x^y} x\right)^2\right)=1$
- $(\log_2 x)(\log_2 y)-3\log_2(4y)-\log_2(8x)=-16$

Warm Up

Solve for x.

a) $2\log_{x}(x)\log_{2}(x-6) = \log_{(x-6)}16 \cdot \log_{(x-6)}\left(x^{2} - 12x + 36\right)$

b) $\log_{3}(x) + \log_{2}(x) = 5$ (Round to 3dp)

## Mid-Review: Logarithmic Functions

1. Evaluate each of the following exactly.

a) $\log_a \frac{1}{\sqrt[5]{a}}$

b) $7^{-4\log_7 x^3}$

c) $\log_3 81 - 3\log_3 27$

d) $\log_9 81^{2x}$

e) $\log_2\left(\sin\left(\frac{\pi}{4}\right)\right)$

f) $\left(\frac{1}{5}\right)^{\log_{\sqrt{5}} 100}$

g) $\log_{64}(4096) - \frac{1}{2}\log_6(46656)$

h) $\log_{36} 2 - \frac{1}{2}\log_{\frac{1}{6}} 3$

i) $125^{\log_5 125}$

2. State

a) the domain and range of $f(x) = \log\sqrt{x^2 - 9}$ .

b) the $\frac{1}{2}\log_4 x^2 + \frac{3}{2}\log_4 y^4 - \frac{\log xy}{\log 4}$ as a single logarithm.

c) the $3^3 \times \left(\sqrt{729}\right)^5$ as a single power of 9.

3. Express $\log_3(x-2) + \log_3 y - \log_3(x^2 - 4)$ as a single logarithm. State your final answer in the simplest form possible.

4. Evaluate using properties of logarithms.

a) $\left(\sqrt{10}\right)^{4\log\sqrt[8]{6} - \log 4}$

b) $(0.2)^{-2 + \log_{\sqrt{5}} 10}$

c) $\log_6 3 + \left(\frac{1}{2}\right)\log_6 5 - \log_6 2$

d) $\log_8\left(\frac{\sqrt{2}}{4}\right)$

5. Using $\log_a\left(\frac{x+y}{5}\right) = \frac{1}{2}(\log_a x + \log_a y)$ , show that $x^2 + y^2 = 23xy$.

6. Solve for x $\log_b x = 2\log_b(1-a) + \frac{2}{\log_{(1+a)} b} - \log_b\left(\frac{1}{a} - a\right)^2$.

7. Explain why $\log_{-2}\left(-\frac{1}{8}\right) = -3$ is not a valid logarithmic equation, but it does make algebraic sense exponentially.

8. Explain the steps used to solve the equation $\sqrt[3]{256^2} \times 16^x = 64^{x-3}$.

9. If $\log_a 2 = x$ and $\log_a 3 = y$ find the value of $\log_{\sqrt{6}} 12$ in terms of x and y.

10. If $\log_{12} 3 = m$, find the value of $\log_{\sqrt{3}} 16$ in terms of $m$ .

11. How many digits are there in the number $3^{2015}$?

# Unit 5: Exponential and Logarithmic Functions
# 5.5 Applications of Logarithmic Functions

The most common examples of logarithmic scales are the Richter, decibel, and pH scales. The Richter scale measures the magnitude of earthquakes. The decibel scale measures the loudness of sound. The pH scale measures the acidity of liquids.

**The Richter Scale**
The Richter scale was developed in 1935 by seismologist Charles F. Richter. It measures the magnitude of an earthquake by comparing the intensity of the earthquake to some reference earthquake. The formula developed by Richter is

***Richter Scale*:** to measure earthquakes

$$M = \log\left(\frac{I}{I_o}\right),$$

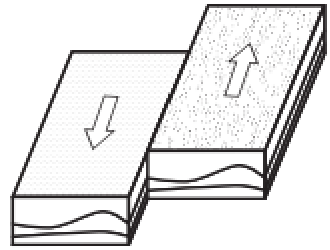
where, $M$ = magnitude of earthquake (measures on the Richter Scale)
$I$ = intensity of the earthquake
$I_o$ = intensity of a standard earthquake

**Ex. 1:** The San Francisco earthquake of 1989 measured 6.9 on the Richter scale. The Alaska earthquake of 1964 measured 8.5.

a) How many times as intense as the San Francisco earthquake was the Alaska earthquake?

b) Calculate the magnitude of an earthquake that is twice as intense as the 1989 San Francisco earthquake.

**GENERALISING THE PROCEDURE**



Earthquakes can be analyzed by the formula:

$$\boxed{M_2 - M_1 = \log\left(\frac{I_2}{I_1}\right)}$$

Where $M_2$ is the magnitude of the earthquake with intensity of $I_2$ and $M_1$ is the magnitude of the earthquake with intensity of $I_1$ .

**Ex.2:** How much more intense is an earthquake measuring 6.5 on the Richter scale than one measuring 6.4?

**The Decibel Scale**

The loudness of any sound is measured relative to the loudness of sound at the threshold of hearing. Sounds at this level are the softest that can still be heard.

| Sound | Loudness (dB) |
|---|---|
| soft whisper | 30 |
| normal conversation | 60 |
| shouting | 80 |
| subway | 90 |
| screaming | 100 |
| rock concert | 120 |
| jet engine | 140 |
| space-shuttle launch | 180 |

The formula used to compare sounds is

***Decibels:***

$$L = 10\log\left(\frac{I}{I_o}\right)$$

where, L = loudness measured in decibels (dB)
I = intensity of the sound (measured in $W/m^2$)
$I_o$ = intensity of sound at the threshold of hearing. $\left(I_o = 10^{-12}\,W/m^2\right)$

**Ex. 3:** A sound is 1000 times more intense than a sound you can just hear. What is the measure of its loudness in decibels?

**GENERALISING THE PROCEDURE**

Sound intensity can be analyzed by the formula:

$$\boxed{L_2 - L_1 = 10\log\left(\frac{I_2}{I_1}\right)}$$

Where $L_2$ is the loudness of the sound with intensity of $I_2$ and $L_1$ is the loudness of the sound with intensity of $I_1$ .

**Ex. 4:** The loudness level of a heavy snore is 69 dB. The loudness level of a conversation is 60 dB. The loudness level of a whisper is 30 dB.

a) How many times as loud as a conversation is a heavy snore?

b) How many times as loud as a whisper is a conversation?

## The pH Scale

The pH scale allows chemists to determine the concentration of hydrogen ion in a liquid. It ranges from values of 1 to 14. The higher the pH, the more basic, or less acidic the liquid. The lower the pH, the more acidic or less basic the liquid.

- A liquid with a pH of less than 7.0 is considered *acidic*
- A liquid with a pH of greater than 7.0 is considered *basic*
- A liquid with pH = 7.0 is considered to be *neutral*. Pure water has a pH of 7.0.

The relationship between pH and $H^+$ ion concentration is inversely proportional and can be summarized as:

Low pH = High $H^+$ ion concentration
High pH = Low $H^+$ ion concentration

The relationship between pH and hydrogen ion concentration is given by the formula

***pH Scale***:

$$pH = -\log[H^+]$$

where, pH = measure of acidity level of a substance
$[H^+]$ = concentration of hydrogen ion in moles/litre

**Ex. 5:** Find the pH of a swimming pool with a hydrogen ion concentration of $6.1 \times 10^{-8}$ mol/L.

**Ex. 6:** The pH of a fruit juice is 3.10. What is the hydrogen ion concentration of the fruit juice?

**GENERALISING THE PROCEDURE**

Formula can be used to compare two pH is

$$\boxed{pH_2 - pH_1 = -\log\left(\frac{[H^+]_2}{[H^+]_1}\right)}$$

**Ex. 7:** Refer to the table at the right to answer the following questions:

| **Solution** | **pH** |
|---|---|
| Lemon juice | 2 |
| Tomato juice | 4 |
| Pure water | 7 |
| Baking soda | 9 |
| Oven cleaner | 13 |

a) How many times as acidic as tomato juice is lemon juice?

b) How many times as acidic as pure water is lemon juice?

c) How many times as acidic as pure water is baking soda?

d) How many times as acidic as baking soda is oven cleaner?

# Practice

1. A sample of 500 cells in a medical research lab doubles every 20 min.
   a) Determine a formula for the number of cells at time t, where t is measured in minutes.
   b) How long will it take for the population to reach 18 000? Answer correct to 2 decimal places.

2. In 1987 there were about 130 000 cell phone users in Canada. In 1999, there were about 10 million cell phone users. What is the percent increase per year? Answer correct to 2 decimal places.

3. A sample of 700 cells in a medical research lab triples every 30 min.
   a) Determine a formula for the number of cells at time t.
   b) How long will it take for the population to reach 18 000? Answer correct to 2 decimal places

4. A sample of radioactive iodine-131 atoms has a half-life of about 8 days. Suppose that one million iodine-131 atoms are initially present.
   a) Determine a formula for the number of atoms at time t, where t represents number of days
   b) How long will it take for the sample to reach 180 000 atoms? Answer correct to 2 decimal places.

5. A new car costs $23 000. In 5 years it will be worth $9500. What is the rate of depreciation per year? Answer in percent, correct to 2 decimal places.

6. Most of Canada's earthquakes occur along the west coast. In 1949, there was an earthquake in the Queen Charlotte Islands that had a magnitude of 8.1 on the Richter Scale. In 1997 there was an earthquake in south-western B.C. with a magnitude of 4.6 on the Richter Scale. How many times as intense as the 1997 earthquake was the 1949 earthquake? Answer correct to 2 decimal places.

7. (a) The loudness level of a heavy snore is 69 dB. How many times is this more intense than conversational speech at 60 dB? Answer correct to 2 decimal places. (b) Sound is 316 times less intense if earplugs are worn. What would the decibel level of snoring be if earplugs were worn? Answer correct to the nearest dB.

8. Given the function $y = 4^x$, determine:
   (a) the average rate of change from t=5 seconds to t=6 seconds.
   (b) the instantaneous rate of change at t=5 seconds.

Answers:

1.(a) $N(t) = 500(2)^{\frac{t}{20}}$ (b) 103.40 min

2. 43.61%

3. (a) $N(t) = 700(3)^{\frac{t}{30}}$ (b) 88.67 min

4. (a) $N(t) = 1000000\left(\frac{1}{2}\right)^{\frac{t}{8}}$ (b) 19.79 days

5. 16.21%

6. 3162.28

7. (a) 7.94 (b) 44 dB

8. (a) 3072 (b) 1419.57

**Unit 5: Exponential & Logarithmic Functions-Review**

1. Solve each of the following. Show a proper solution.

a. $5^{2x+3} + 40 = 70$

b. $3\left(2^{x}\right) = 18^{x-1}$

c. $6^{x} - 15(6^{-x}) - 2 = 0$

d. $15 \times 3^{x+1} - 243 \times 5^{x-2} = 0$

e. $4^{2x} - 9(2^{2x}) + 14 = 0$

f. $4 - \log x = 3\sqrt{\log x}$

g. $\log_2 x + \log_4 x = \log_8 x + 10^{\log\frac{7}{6}}$

h. $\log(\log x) = \log(7 - 2\log x) - \log 5$

i. $\log_3(\log_2(\log_4(x^2 - 6x))) = 0$

j. $\log_2(x+1) - \log_2(x-1) = 1$

k. $\log_2 x + \log_4 x + \log_8 x + \log_{16} x = 25$

l. $9^{x} = 2 \times 3^{x+2} - 45$

m. $\log_2 x + \log_{16} x = 5$

2. The average annual salary, S, in dollars, of employees at a particular job in a manufacturing company is modeled by the equation $S = 25000(1.05)^n$, where \$25000 is the initial salary, which increases at 5% per year.
   (a) How long will it take the salary to increase by 50%?
   (b) If the starting salary is \$35 000, how long will it take the salary to increase by 50%? Explain your answer.
3. The speed, v, in kilometres per hour, of a water skier who drops the tow rope, can be given by the formula $v = v_0\left(10\right)^{-0.23t}$, where $v_0$ is the skier's speed at the time she drops the rope, and t is the time, in seconds, after she drops the rope. If the skier drops the rope when traveling at a speed of 65 km/h, how long will it take her to slow to a speed of 13 km/h?

4. On average, number of items, N, per day, on an assembly line, that a quality assurance trainee can inspect is $N = 40 - 24(0.74)^t$, where t is the number of days worked.
   (a) After how many days of training employee be able to inspect 32 items?
   (b) The company expects an experienced assurance employee to inspect 45 items per day. After the training period of 15 days is complete, how close will the trainee be to the experienced employee's quota?
5. Graph the function $y = -2\left(\dfrac{1}{\log_{2x+6} 3}\right) + 4$, State the domain, range and asymptote of the function.

6. Solve for x .

a) $5^{3x-2} = \left(2 \times 3^{4-2x}\right)^2$

b) $\log_2 \sqrt{x^2 + 12x} = 3$

c) $\log_{\sqrt[3]{5}}\left(\log_{\sqrt[5]{2}}\left(\log_{\sqrt{5}} x\right)\right) = 3$

e) $\dfrac{\log\left(35 - x^2\right)}{\log\left(5 - x\right)} = 2$

7. Noise in the school cafeteria is recorded at 78 dB at lunchtime. Exposure to sound at 90 dB can cause mild hearing loss. If students in the cafeteria yell 10 times louder, will Bayview start to have deaf kids?

8. Earthquakes of magnitude 7.0 or greater can cause metal buildings to collapse. On December 23, 1985, in Mackenzie Region, Northwest Territories, an earthquake of magnitude 6.9 occurred. On Vancouver Island, on June 23, 1946, an earthquake about 2.5 times as intense occurred. Was the Vancouver bland earthquake strong enough to cause metal bui1dings to collapse?

9. Lemon juice has a pH of 2.6 and coffee has a pH of 5.1. How many more times as great is the concentration of hydrogen ions in lemon juice to that of coffee? (Round final answer to 2 decimals)?

10. A 250 g sample of a radioactive material decays to 100 g in 2 weeks. What is its half-life in days?

11. A crate of apples is accidentally exposed to a particular radioactive element. The radioactive element is deemed to be safe when the amount present in a crate of apples is 0.1% of its original amount. If the half-life of the radioactive element is 6 hours, how long do consumers have to wait before the apples are safe to eat? (Note: would you even ***want*** to eat these apples?

12. For his dream car, James invested $24 000 at 5.4% interest, compounded monthly, for 7 years. After 7 years, he was still short. How much longer will he have to invest the money at 5% interest, compounded quarterly to have a total of $ 40 0000?

13. The SARS virus and the flu virus infect a city at the same time. At the start, the SARS virus had infected 60 people and the number of people infected was tripling every 8 days. The flu virus had infected 30 people and was doubling every 5 days. Predict in how many days the same number of people will be infected.

14. State the transformation needed to transform the graph of $y = \log_2 x$ to graph the following:

a) $y = \log_2\left(x - 3\right)$ ______________________________

b) $y = \log_2\left(\dfrac{1}{2}x\right)$ (two different ways)

i) ____________________________

ii) ____________________________