Unit 1: Polynomial Functions
1.7 The Factor Theorem

Review

- Division of Polynomials $\frac{f(x)}{x-a}$ - may use long division or synthetic division

- Division Statement: f(x) = (x-a) Q(x) +r(x)
- Remainder Theorem: When a polynomial f(x) is divided by (x - a), the remainder, r, is f(a)

Investigation:
Find the remainder when $x^3 + 2x^2 - 11x - 12$ is divided by x + 1 and write the division statement.
Solution:

$\therefore x^3+2x^2 - 11x - 12 =$ - Factor the quotient if possible

$=$

- Notice that the products of the constant terms in the factors is (1)(4)(-3) = -12 . This is also the constant term of the polynomial.

Since division gives zero as a remainder, both the **divisor** and **quotient** are factors of the polynomial function. This special case of the remainder theorem where the remainder is **zero** is called the **factor theorem**.

Factor Theorem:
A polynomial function f(x) has x - a as a factor if and only if f(a) = 0.

USE THE FACTOR THEOREM FOR FACTORING POLYNOMIALS WITH DEGREE HIGHER THAN 2.

Example: Is $x - 3$ a factor of $x^3 - 2x^2 - 2x - 3$?

Example: Use the Factor Theorem to factor fully each of the following polynomials

a) $x^3 - 4x^2 + x + 6$

SOLUTION

step 1: Let $f(x)=x^3 - 4x^2 + x + 6$

step 2: Find all factors of constant: $\{\pm 1, \pm 2, \pm 3, \pm 6\}$. This is a set of possible roots for $f(x)=0$.

step 3: Observe that $f(-1) = 0 \quad \therefore x + 1$ is a factor of $f(x)$

step 4: Dividing $f(x)$ by $x + 1$ determines a quotient of $x^2 - 5x + 6$

step 5: $f(x) = x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6) = (x + 1)(x - 2)(x - 3)$

b) $x^4 - 3x^3 - 13x^2 + 3x + 12$

The Rational Root Theorem (Extended Factor Theorem)

The factor theorem can be extended over the set of Rationals, Q, so that more test values can be used to determine a factor of the polynomial.

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ can also be written as

$$f(x) = a_n[x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \frac{a_{n-2}}{a_n}x^{n-2} + \ldots + \frac{a_2}{a_n}x^2 + \frac{a_1}{a_n}x + \frac{a_0}{a_n}]$$

$\therefore \dfrac{\text{all of the factors of } a_0}{\text{all of the factors of } a_n}$ should be considered when determining possible factors of f(x)

Example: Factor fully each of the following polynomials

a) $4x^3 + 16x^2 + 9x - 9$

Now you can consider $\pm$ combination of $\dfrac{\text{all the factors of } 9}{\text{all the factors of } 4} = \pm\dfrac{1, 3, 9}{1, 2, 4}$.

That means $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{9}{1}, \pm\frac{9}{2}, \pm\frac{9}{4}\right\}$ is a **set of possible roots** for f(x)=0.

b) $30x^3 + 13x^2 - 30x + 8$

Example: If x + 1 is a factor of $x^3 + x^2 + kx + 2$, what is the value of k?

Example: Find the value of k for which a-3b is a factor of $a^4-7a^2b^2+kb^4$. Hence, for this value of k, factorize $a^4-7a^2b^2+kb^4$ccompletely.

Example: The function $h(x) = 3x^2 - x^3$ has been shifted to the left 1 and vertically stretched by 2.

a) Determine the equation for the transformed function, g(x) in factored form.

b) Determine the zeroes of the transformed function and state the order of each zero.

Factoring a Sum or Difference of Cubes

Recall:

- Factoring a difference of squares

$$\begin{aligned} &x^2 - 9 \\ &= (x-3)(x+3) \end{aligned} \qquad \begin{aligned} &4x^2 - 16 \\ &= (2x-4)(2x+4) \end{aligned}$$

$$\begin{aligned} &a^2 - b^2 \\ &= (a-b)(a+b) \end{aligned}$$

IS THERE A WAY TO FACTOR A SUM OF CUBES OR A DIFFERENCE OF CUBES IN ONE STEP???????

$$\begin{aligned} &x^3 - 27 \\ &= \ ? \end{aligned} \qquad \begin{aligned} &8a^3 - 27b^3 \\ &= \ ? \end{aligned}$$

Factoring a Sum or Difference of Cubes

- **Sum of Cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- **Difference of Cubes:** $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Factor the following expressions **completely**.

a) $x^3 - 64$

b) $27x^3 + 125$

c) $7x^4 - 448x$

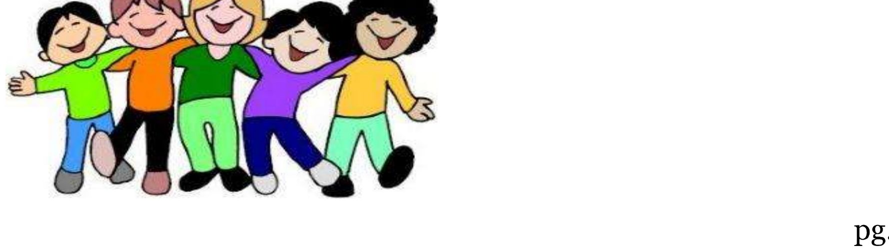
d) $64x^3 + \frac{8}{125}y^{12}$

e) $\frac{1}{8}a^3 - \frac{27}{125}b^{18}$

f) $x^9 - 512$

g) $(x-2)^3 - (3x-2)^3$

h) $3(x-2)^3 - 24(x+2)^3$



Practice:

1. Factor the following polynomials using the factor theorem.

   (a) x^3-4x^2+x+6
   (b) $x^3+8x^2+21x+18$
   (c) $x^4-x^3-3x^2+x+2$

2. Factor each expression

   (a) x^3-8
   (b) $27x^3+1$
   (c) $625x^3-40$
   (d) $125-64x^3$

3. Factor fully: $abx^3+(a+b-ab)x^2+(1-a-b)x-1$ [note $P(1)=0$]

4. a) Factor $x^{12}-1$ fully.
   b) List all polynomials of the form $x^4+bx^3+cx^2+dx+e$ with rational coefficients that are factors of the polynomial, $x^{12}-1$.

Answer

1. a) $(x+1)(x-2)(x-3)$ b) $(x+2)(x+3)^2$ c) $(x-2)(x-1)(x+1)^2$

2. a) $(x-2)(x^2+2x+4)$ b) $(3x+1)(9x^2-3x+1)$

   c) $5(5x-2)(25x^2+10x+4)$ d) $-(4x-5)(16x^2+20x+25)$

3. $abx^3+(a+b-ab)x^2+(1-a-b)x-1=(ax+1)(bx+1)(x-1)$; note $P(1)=0$

4. a) $x^{12}-1=(x-1)(x+1)(x^2+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$

   b) There are seven such $4th$ degree polynomial factors:

   - x^4-x^2+1
   - $(x^2+x+1)(x^2-x+1)=x^4+x^2+1$
   - $(x-1)(x+1)(x^2+1)=x^4-1$
   - $(x-1)(x+1)(x^2+x+1)=x^4+x^3-x-1$
   - $(x-1)(x+1)(x^2-x+1)=x^4-x^3+x-1$
   - $(x^2+1)(x^2+x+1)=x^4+x^3+2x^2+x+1$
   - $(x^2+1)(x^2-x+1)=x^4-x^3+2x^2-x+1$

Warm Up

1. Completely factor and fully simplify the following expressions.

   a) $(2+x)^3 - (2-x)^3$

   b) $(x-1)^6 - 1$

   c) $x^4 - 5x^3 - 21x^2 + 125x - 100$

   d) $6x^3 - 5x^2 - 49x + 60$