Unit 1 Review – Polynomial Functions

1. Fill in the blanks.

a) State the remainder when $-4x^3+3x^2+2x-1$ is divided by $x-2$. $f(2)=-17$

b) State the roots and the order of each root of $g(x)=2x^2(2x+3)^3$. 0 (order 2), $-\frac{3}{2}$ (order 3)

c) When a function is divided by $2x-1$, the remainder is -2; Determine the remainder when the same function is divided by $x-\frac{1}{2}$. -2

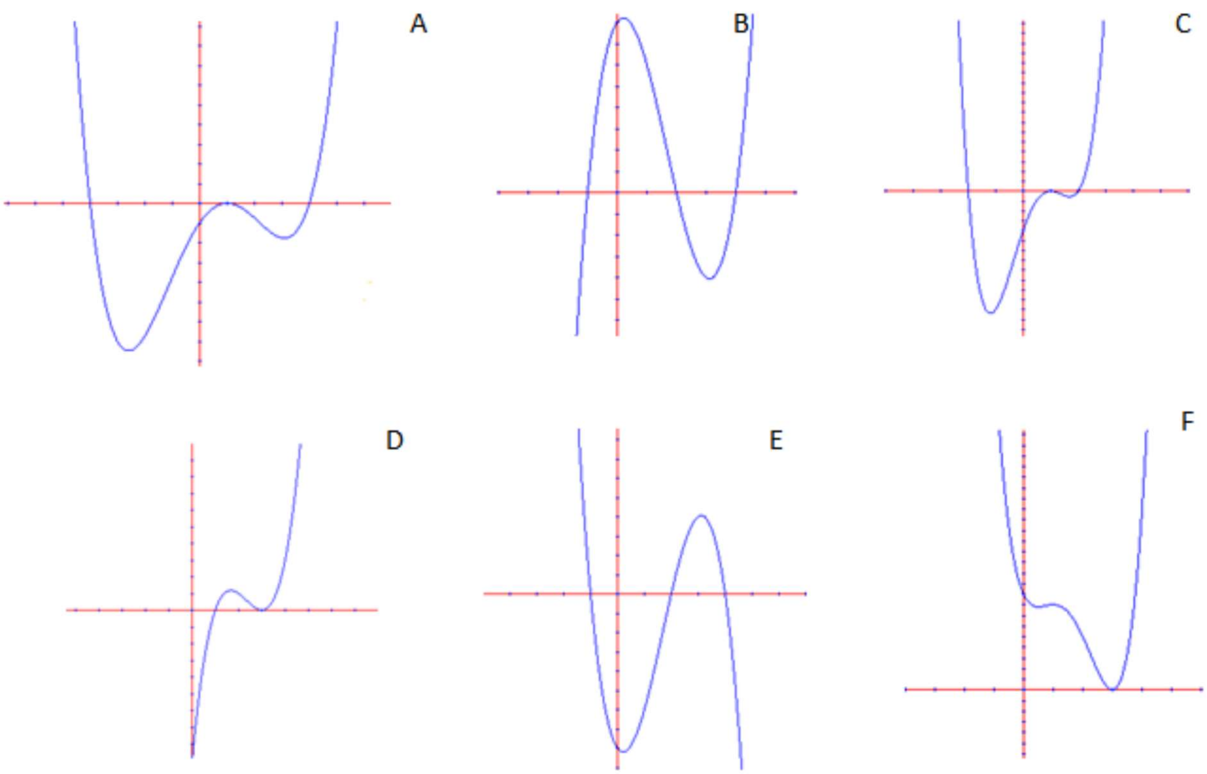
d) Values that could be zeros for the polynomial $f(x)=4x^3+2x^2-7x-8$ are $\pm\{1,2,4,8,\frac{1}{2},\frac{1}{4}\}$

e) State if $y=-2x^4+3x^2+1$ is odd, even or neither. even

f) Beside each equation below, put the letter of the graph that best describes the equation:

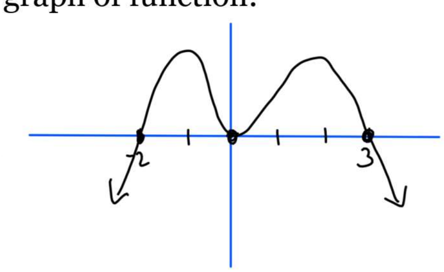
i) $y=(x^2-16)(x-1)^2$ A

ii) $y=(2-x)(x-4)(x+1)$ E



2. Write the equation in factored form of any quartic function with following characteristics. Sketch the graph of function:

- $f(0)=0$
- $f(x)<0$, when $x<-2$
- $f(x)\geq 0$, when $-2\leq x\leq 3$
- $f(x)<0$, when $x>3$



$f(x)=-x^2(x+2)(x-3)$

3. Fully factor

a) $2x^5 - 2x^4 - 4x^3 + 4x^2 + 2x - 2$

$f(1) = 0 \Rightarrow x-1$ is a factor

$$\begin{array}{r|rrrrrr} 1 & 2 & -2 & -4 & 4 & 2 & -2 \\ & & 2 & 0 & -4 & 0 & 2 \\ \hline & 2 & 0 & -4 & 0 & 2 & 0 \end{array}$$

$$\begin{aligned} f(x) &= (x-1)(2x^4-4x^2+2) \\ &= 2(x-1)(x^4-2x^2+1) \\ &= 2(x-1)(x^2-1)^2 \\ &= 2(x-1)(x-1)^2(x+1)^2 \\ &= 2(x-1)^3(x+1)^2 \end{aligned}$$

b) $64y^6x^3 - 125$

$= (4y^2x-5)(16y^4x^2+20y^2x+25)$

c) $6(x+2)^{-5} + 2x^2(x+2)^{-4}$

$$\begin{aligned} &= 2(x+2)^{-5}[3 + x^2(x+2)] \\ &= 2(x+2)^{-5}[3 + x^3 + 2x^2] \\ &= 2(x+2)^{-5}(x^3+2x^2+3) \\ &= \frac{2(x^3+2x^2+3)}{(x+2)^5} \end{aligned}$$

d) $4x^4 - 13x^3 - 13x^2 + 28x - 6 = 0$

possible factors: $\frac{\pm 1, 2, 3, 6}{\pm 1, 2, 4}$

$f(1) = 0 \Rightarrow x-1$ is a factor

$$\begin{array}{r|rrrrr} 1 & 4 & -13 & -13 & 28 & -6 \\ & & 4 & -9 & -22 & 6 \\ \hline & 4 & -9 & -22 & 6 & 0 \end{array}$$

$f(x) = (x-1)(4x^3-9x^2-22x+6)$

$f(\frac{1}{4}) = 0 \Rightarrow x - \frac{1}{4}$ is a factor

$$\begin{array}{r|rrrr} \frac{1}{4} & 4 & -9 & -22 & 6 \\ & & 1 & -2 & -6 \\ \hline & 4 & -8 & -24 & 0 \end{array}$$

$$\begin{aligned} \therefore f(x) &= (x-1)(x-\tfrac{1}{4})(4x^2-8x-24) \\ &= (x-1)(4x-1)(x^2-2x-6) \end{aligned}$$

4. Divide $8x^4 - 30x^2 + 6x - 3$ by $1 + x + 2x^2$ using **long division** and write the division statement.

$$\begin{array}{r l} & 4x^2 - 2x - 16 \\ 2x^2+x+1 \, \big) & 8x^4 + 0x^3 - 30x^2 + 6x - 3 \\ -) & 8x^4 + 4x^3 + 4x^2 \\ \hline & -4x^3 - 34x^2 + 6x \\ -) & -4x^3 - 2x^2 - 2x \\ \hline & -32x^2 + 8x - 3 \\ -) & -32x^2 - 16x - 16 \\ \hline & 24x + 13 \end{array}$$

$$\therefore \frac{8x^4-30x^2+6x-3}{1+x+2x^2} = 4x^2-2x-16 + \frac{24x+13}{1+x+2x^2}$$

5. When $f(x) = x^4 - 4x^3 + mx^2 + nx + 1$ is divided by $x-1$, the remainder is 7. When it is divided by $x+1$, the remainder is 3. Determine the values of m and n.

$f(1) = 7 \Rightarrow 1 - 4 + m + n + 1 = 7$

$m + n = 9$ — ①

$f(-1) = 3 \Rightarrow 1 + 4 + m - n + 1 = 3$

$m - n = -3$ — ②

①+②:

$$\begin{array}{rl} & m + n = 9 \\ + & m - n = -3 \\ \hline & 2m = 6 \end{array}$$

$\boxed{m = 3}$

Sub. $m = 3$ into ①:

$3 + n = 9$

$\boxed{n = 6}$

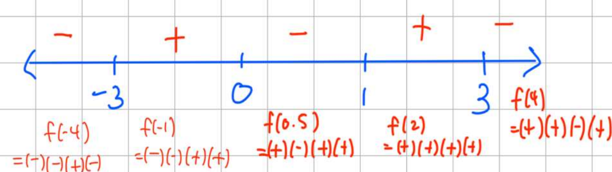
6. Solve each of the following, $x \in \mathbb{R}$.

a) $x(x-1)(3-x)(x+3) < 0$

b) $x^3 - x^2 < 5x + 3$

a)

Roots: $0, 1, 3, -3$



$\therefore x \in (-\infty, -3) \cup (0, 1) \cup (3, \infty)$

b)

$x^3 - x^2 - 5x - 3 < 0$

possible factors: $\pm 1, 3$

$f(-1) = 0 \Rightarrow x+1$ is a factor

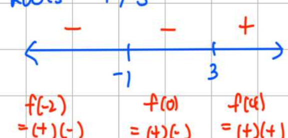
$$\begin{array}{r|rrrr} -1 & 1 & -1 & -5 & -3 \\ & & -1 & 2 & 3 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

$(x+1)(x^2 - 2x - 3) < 0$

$(x+1)(x-3)(x+1) < 0$

$(x+1)^2(x-3) < 0$

Roots: $-1, 3$



$\therefore x \in (-\infty, -1) \cup (-1, 3)$

7. Find the value of a and b and the remaining factor if the expression $ax^3 - 11x^2 + bx + 3$ is divisible by $x^2 - 4x + 3$.

$x^2 - 4x + 3 = (x-3)(x-1)$

$f(3) = 0 \Rightarrow 27a - 99 + 3b + 3 = 0$

$27a + 3b = 96$

$9a + b = 32$ — ①

$f(1) = 0 \Rightarrow a - 11 + b + 3 = 0$

$a + b = 8$ — ②

① − ②:

$$\begin{array}{rl} & 9a + b = 32 \\ - & a + b = 8 \\ \hline & 8a = 24 \end{array}$$

$\boxed{a = 3}$

Sub. $a = 3$ into ②:

$3 + b = 8$

$\boxed{b = 5}$

$(3x^3 - 11x^2 + 5x + 3) \div (x^2 - 4x + 3)$

$$\begin{array}{rr|rrrr} 4 & -3 & 3 & -11 & 5 & 3 \\ & & & & -9 & -3 \\ & & & -11 & -4 & 0 \\ & & & 12 & 4 & \\ \hline & & 3 & 1 & 0 & 0 \end{array}$$

$3x + 1$

$\therefore$ The remaining factor is $3x+1$

8. Graph the following function $f(x) = (x^2+x+1)(2x+5)^2(x-3)^3$. Show all your work.

Roots: $-\frac{5}{2}$ (double)

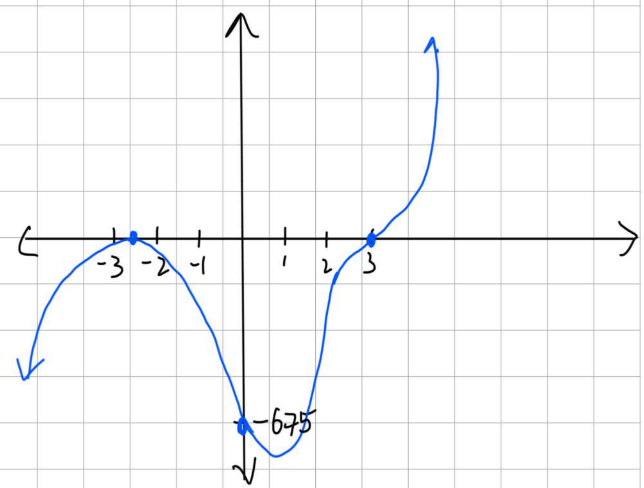
3 (triple)

degree: 7

leading coefficient: +2

y-intercept: $y = (1)(5)^2(-3)^3$

$= -675$



9. The passenger section of a train has a width $2x - 7$, length $2x + 3$, and height $x - 2$, with all dimensions in metres. Solve a polynomial equation to determine the dimensions of the section of the train if the volume is $117 m^3$.

$$(2x-7)(2x+3)(x-2) = 117$$
$$4x^3 - 16x^2 - 5x + 42 = 117$$
$$4x^3 - 16x^2 - 5x - 75 = 0$$

possible factors: $\frac{\pm 1, 3, 5, 15, 25, 75}{\pm 1, 2, 4}$

$f(5) = 0 \Rightarrow x-5$ is a factor

5	4	-16	-5	-75
		20	20	75
	4	4	15	0

$$(x-5)\underbrace{(4x^2+4x+15)}_{\text{no real solutions}} = 0$$

$x = 5$

$2x - 7 = 2(5) - 7 = 3$

$2x + 3 = 2(5) + 3 = 13$

$x - 2 = 5 - 3 = 2$

∴ The dimensions are 3 m by 13 m by 2m

10. Determine algebraically, whether each function is even, odd, or neither.

a) $f(x) = 4x^3$

$f(-x) = 4(-x)^3$
$= -4x^3$
$= -f(x)$
$\therefore$ Odd

b) $f(x) = 2x^4 - x^2$

$f(-x) = 2(-x)^4 - (-x)^2$
$= 2x^4 - x^2$
$= f(x)$
$\therefore$ Even

c) $g(x) = \sqrt[3]{2x^2 + 1}$

$g(-x) = \sqrt[3]{2(-x)^2 + 1}$
$= \sqrt[3]{2x^2 + 1}$
$= f(x)$
$\therefore$ Even

d) $h(x) = \dfrac{-x^3}{(3x^3 - 9x)^2}$

$h(-x) = \dfrac{-(-x)^3}{[3(-x)^3 - 9(-x)]^2}$
$= \dfrac{x^3}{(-3x^3 + 9x)^2}$
$= \dfrac{x^3}{[-(3x^3 - 9x)]^2}$
$= \dfrac{x^3}{(3x^3 - 9x)^2}$
$= -f(x)$
$\therefore$ Odd

e) $f(x) = x + |x|$

$f(-x) = -x + |-x|$
$= -x + |x|$
$\neq -f(x) \neq f(x)$
$\therefore$ Neither

f) $g(x) = \dfrac{2x}{|x|}$

$g(-x) = \dfrac{2(-x)}{|-x|}$
$= \dfrac{-2x}{|x|}$
$= -g(x)$
$\therefore$ Odd

11. The table of values below represents a polynomial function. Determine the equation of this function.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-2	-19			
-1	-3	16		
0	1	4	-12	
1	-1	-2	-6	6
2	-3	-2	0	6
3	1	4	6	6

$6 = a \cdot 3!$
$6 = 6a$
$1 = a$

$y = ax^3 + bx^2 + cx + d$
$y = x^3 + bx^2 + cx + d$

Sub. $(0,1)$: $1 = d$

Sub. $(1,-1)$: $-1 = 1 + b + c + 1$
$-3 = b + c$ — ①

Sub. $(-1,-3)$: $-3 = -1 + b - c + 1$
$-3 = b - c$ — ②

① + ②:
$-3 = b + c$
$+)\ -3 = b - c$
$-6 = 2b$
$-3 = b$

Sub. $b = -3$ into ①:
$-3 = -3 + c$
$0 = c$

$\therefore y = x^3 - 3x^2 + 1$

12. Find the general equation of quartic functions that has negative leading coefficient, two equal roots at 2, and roots at $3 \pm 2\sqrt{2}$.

$x = 3 \pm 2\sqrt{2}$

$(x-3)^2 = (\pm 2\sqrt{2})^2$

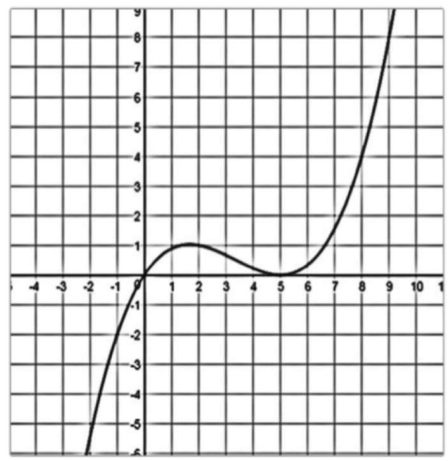
$x^2 - 6x + 9 = 4(2)$

$x^2 - 6x + 9 = 8$

$x^2 - 6x + 1 = 0$

$\therefore f(x) = k(x-2)^2(x^2-6x+1), \; k<0$

13. Given the graph of a polynomial function g(x), answer the following:



a) Is the function even-degree or odd-degree? Odd

b) Is the function even or odd or neither? Neither

c) State the zeroes and the lowest possible order of each zero Ø (order 1), 5 (order 2)

d) State the interval where the function is positive

$x \in (0,5) \cup (5,\infty)$

e) State the interval where the function is negative

$x \in (-\infty, 0)$

f) Determine the value of the remainder when f(x) is divided by x+2. $f(-2) = -5$

14. Water is draining from a container. The height, in millimeters, of the water as a function of time, in seconds, can be modeled by the function

$$h(t) = 0.00185(250-t)^2.$$

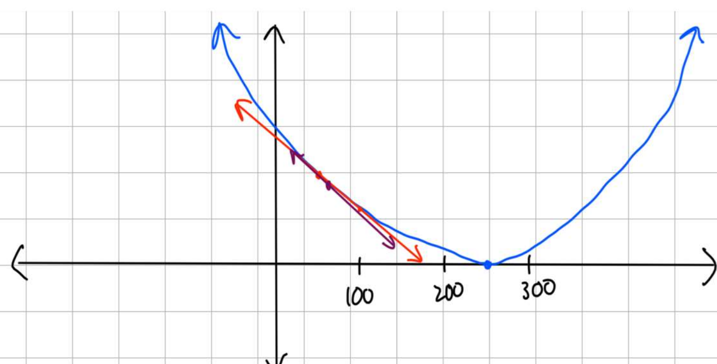
a) Calculate the average rate of change of height with respect to time from 50s to 100s.

$$A.R.O.C = \frac{h(100)-h(50)}{100-50} = \frac{41.625-74}{50} = -0.6475 \text{ mL/s}$$

b) Calculate the instantaneous rate of change of height with respect to time at t=60s.

$$I.R.O.C = \frac{h(60.001)-h(60)}{0.001} \approx \frac{66.784297-66.785}{0.001} = -0.703 \text{ mL/s}$$

c) Create a sketch of the function indicating the secant line and tangent line from part a.



15. When polynomial x^3-**a**x+21 is divided by x +**b**, the quotient is $x^2 - 3x + 5$ and the remainder is 6. Determine values of **a** and **b**.

$$f(x) = x^3 - ax + 21 = (x^2 - 3x + 5)(x + b) + 6$$
$$= x^3 + bx^2 - 3x^2 - 3xb + 5x + 5b + 6$$
$$= x^3 + (b-3)x^2 + (-3b+5)x + (5b+6)$$

$0 = b - 3$

$\boxed{3 = b}$

$-a = -3b + 5$

$-a = -9 + 5$

$\boxed{a = 4}$

16. Is x+b a factor of $x^9 + 5b^2x^7 + 5bx^8 - b^9$?

$$f(-b) = (-b)^9 + 5b^2(-b)^7 + 5b(-b)^8 - b^9$$
$$= -b^9 - 5b^9 + 5b^9 - b^9$$
$$= -2b^9$$
$$\neq 0$$

$\therefore$ x+b is not a factor.