

Velocity and Acceleration in Two Dimensions



Figure 1 An object's velocity changes whenever there is a change in the velocity's magnitude (speed) or direction, such as when these cars turn with the track.

Suppose you are driving south along a straight side road that has a speed limit of 60 km/h. You stay on this road for 1 h and then reach a highway that has a speed limit of 100 km/h. You turn and travel southwest on the highway for 2 h.

This simple example gives you an idea of why you must carefully consider how to determine average velocity in a two-dimensional situation. Velocity is a vector, and like displacement, can be described in more than one dimension. A change in velocity occurs when there is a change in the velocity's magnitude (speed) or direction, such as the race cars taking a curve in **Figure 1**. Acceleration depends on the change in velocity, so acceleration in two dimensions also depends on a change in the velocity's magnitude, direction, or both.

Now that you are familiar with the component method for adding vectors, you can use this method to calculate two-dimensional average velocity and average acceleration. First, we look at velocity and speed in two dimensions and then subtracting vectors.

Velocity and Speed in Two Dimensions

In general, average velocity is the change in total displacement over time and is described by the equation

$$\vec{v}_{av} = \frac{\Delta \vec{d}}{\Delta t}$$

If displacement is in two dimensions, then you must first determine the total displacement using components (or a similar method) before determining the average velocity.

The notation for describing a velocity vector is the same as that for displacement, except that average velocity has units of length divided by time (for example, metres per second). Suppose a car has a displacement of 200 m [E 30° N], and travels this distance and in this direction in 10 s. The average velocity is therefore

$$\begin{aligned}\vec{v}_{av} &= \frac{200 \text{ m [E } 30^\circ \text{ N]}}{10 \text{ s}} \\ \vec{v}_{av} &= 20 \text{ m/s [E } 30^\circ \text{ N]}\end{aligned}$$

What happens when there are several displacements, each with a different direction? The average velocity for the entire trip is based on the total displacement. Therefore, this average velocity will always have the same direction as the total displacement. In the equation

$$\vec{v}_{av} = \frac{\Delta \vec{d}}{\Delta t},$$

$\Delta \vec{d} = \Delta \vec{d}_1 + \Delta \vec{d}_2 + \Delta \vec{d}_3 + \cdots$. To calculate total displacement, add the horizontal and vertical components of the individual displacements, and combine them to obtain the magnitude and direction of the total displacement as in Section 1.3.

Average speed, on the other hand, is a scalar property based on the length of time travelled and the total distance travelled, regardless of the direction. Therefore, when an object returns to its starting point, the distance it has travelled is the sum of all displacement magnitudes, and is thus not zero. Average speed is simply this total distance divided by the time of travel and is greater than zero. In the following Tutorial, you will review how to calculate average velocity and average speed in two dimensions.

Tutorial 1 Calculating Average Velocity and Average Speed in Two Dimensions

This Tutorial reviews how to calculate average velocity and average speed in two dimensions.

Sample Problem 1: Calculating Average Velocity and Average Speed

A family drives from Saint John, New Brunswick, to Moncton. Assuming a straight highway, this part of the drive has a displacement of 135.7 km [E 32.1° N]. From Moncton, they drive to Amherst, Nova Scotia. The second displacement is 51.9 km [E 25.9° S]. The total drive takes 2.5 h to complete.

(a) Calculate the average velocity of the family's vehicle.

(b) Calculate the average speed of the family's vehicle.

Solution

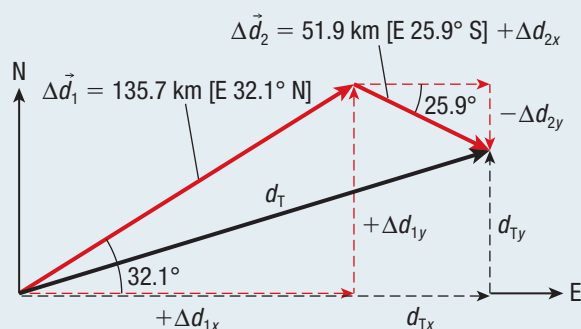
(a) **Given:** $\Delta \vec{d}_1 = 135.7 \text{ km [E } 32.1^\circ \text{ N]}$;
 $\Delta \vec{d}_2 = 51.9 \text{ km [E } 25.9^\circ \text{ S]}$; $\Delta t = 2.5 \text{ h}$

Required: \vec{v}_{av}

Analysis: Make a scale diagram to show the situation.

Determine the components of the two vectors. Then determine the total horizontal displacement, $\Delta d_{Tx} = \Delta d_{1x} + \Delta d_{2x}$, and the total vertical displacement, $\Delta d_{Ty} = \Delta d_{1y} + \Delta d_{2y}$. Calculate the magnitude of the total displacement using the Pythagorean theorem, and use the inverse tangent equation to calculate the angle of orientation for the total displacement. The average velocity is then the total displacement divided by the time of travel.

Solution:



$$\begin{aligned}\Delta d_{Tx} &= \Delta d_{1x} + \Delta d_{2x} \\ &= (+\Delta \vec{d}_1 \cos \theta_1) + (+\Delta \vec{d}_2 \cos \theta_2) \\ &= (135.7 \text{ km})(\cos 32.1^\circ) + (51.9 \text{ km})(\cos 25.9^\circ) \\ &= 161.6 \text{ km}\end{aligned}$$

$$\Delta d_{Tx} = 161.6 \text{ km [E]} \text{ (two extra digits carried)}$$

$$\begin{aligned}\Delta d_{Ty} &= \Delta d_{1y} + \Delta d_{2y} \\ &= (+\Delta \vec{d}_1 \sin \theta_1) + (-\Delta \vec{d}_2 \sin \theta_2) \\ &= (135.7 \text{ km})(\sin 32.1^\circ) - (51.9 \text{ km})(\sin 25.9^\circ) \\ &= 49.44 \text{ km}\end{aligned}$$

$$\Delta d_{Ty} = 49.44 \text{ km [N]} \text{ (two extra digits carried)}$$

$$\begin{aligned}|\Delta \vec{d}_T| &= \sqrt{(\Delta d_{Tx})^2 + (\Delta d_{Ty})^2} \\ &= \sqrt{(161.6 \text{ km})^2 + (49.44 \text{ km})^2} \\ |\Delta \vec{d}_T| &= 169.0 \text{ km (two extra digits carried)} \\ \theta_T &= \tan^{-1}\left(\frac{|\Delta d_{Ty}|}{|\Delta d_{Tx}|}\right) \\ &= \tan^{-1}\left(\frac{49.44 \text{ km}}{161.6 \text{ km}}\right) \\ \theta_T &= 17^\circ\end{aligned}$$

The total displacement is 170 km [E 17° N]. The average velocity is therefore

$$\begin{aligned}\vec{v}_{av} &= \frac{\Delta \vec{d}_T}{\Delta t} \\ &= \frac{169.0 \text{ km [E } 17^\circ \text{ N]}}{2.5 \text{ h}} \\ \vec{v}_{av} &= 68 \text{ km/h [E } 17^\circ \text{ N]}\end{aligned}$$

Statement: The average velocity of the vehicle is 68 km/h [E 17° N].

(b) **Given:** $\Delta \vec{d}_1 = 135.7 \text{ km [E } 32.1^\circ \text{ N]}$;
 $\Delta \vec{d}_2 = 51.9 \text{ km [E } 25.9^\circ \text{ S]}$; $\Delta t = 2.5 \text{ h}$

Required: v_{av}

Analysis: To calculate the average speed of the vehicle, determine the total distance travelled. Distance is not a vector sum, but a scalar addition of the separate displacement magnitudes. Therefore,

$$\begin{aligned}\Delta d_T &= |\Delta \vec{d}_1| + |\Delta \vec{d}_2| \\ \text{and } v_{av} &= \frac{\Delta d_T}{\Delta t}\end{aligned}$$

$$\begin{aligned}\text{Solution: } d_T &= |\Delta \vec{d}_1| + |\Delta \vec{d}_2| \\ &= 135.7 \text{ km} + 51.9 \text{ km} \\ d_T &= 187.6 \text{ km} \\ v_{av} &= \frac{\Delta d_T}{\Delta t} \\ &= \frac{187.6 \text{ km}}{2.5 \text{ h}} \\ v_{av} &= 75 \text{ km/h}\end{aligned}$$

Statement: The average speed of the vehicle is 75 km/h.

Practice

- A plane makes the following displacements: $\Delta \vec{d}_1 = 72.0 \text{ km [W } 30.0^\circ \text{ S]}$, $\Delta \vec{d}_2 = 48.0 \text{ km [S]}$, and $\Delta \vec{d}_3 = 150.0 \text{ km [W]}$. The entire flight takes 2.5 h. T/I
 - Calculate the total displacement of the plane. [ans: 230 km [W 22° S]]
 - Calculate the average velocity of the plane. [ans: 91 km/h [W 22° S]]
 - Calculate the average speed of the plane. [ans: 110 km/h]
- An elk walks 25.0 km [E 53.13° N], then walks 20.0 km [S], and then runs 15.0 km [W]. The journey takes 12 h. T/I
 - Calculate the elk's average velocity. [ans: 0 km/h]
 - Calculate the elk's average speed. [ans: 5.0 km/h]
 - Explain the difference in the two answers for the elk's average velocity and average speed.

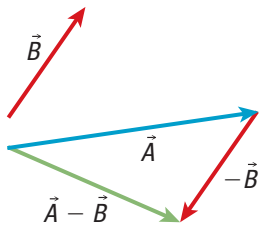


Figure 2 Vector subtraction is equivalent to adding a positive vector and a negative vector.

Subtracting Vectors in Two Dimensions

Up to now, you have been working with vector addition. In some cases, though, you need to multiply vectors by scalars and subtract vectors. Before dealing with vector subtraction, first consider the multiplication of a vector by a scalar. Scalars are simply numbers, such as 2 and 5.7. Multiplication of a vector by a scalar changes the vector's length, or magnitude. If a scalar k is greater than 1 ($k > 1$), then the product of k and vector \vec{A} is longer than \vec{A} . Similarly, if $0 < k < 1$, then the product of k and vector \vec{A} is shorter than \vec{A} .

Now suppose the scalar k is negative ($k < 0$). If you multiply vector \vec{B} by k , then \vec{B} and $k\vec{B}$ point in opposite directions. Now you can see how vector subtraction arises from scalar multiplication and vector addition. Subtracting vector \vec{B} from vector \vec{A} is equivalent to adding the vectors \vec{A} and $-\vec{B}$, for $k = -1$. See **Figure 2**. Expressing this as a vector equation yields

$$\begin{aligned}\vec{A} + (-1)(\vec{B}) &= \vec{A} + (-\vec{B}) \\ &= \vec{A} - \vec{B}\end{aligned}$$

You can use this same approach with the components of vectors. By multiplying one component by an appropriate negative scalar, you can subtract two vector components in one dimension by adding the positive component of one vector and the negative component of the other vector.

What does subtracting vectors mean physically? When a vector changes over an interval of time, the physical quantity of interest is the measure of the change, or the difference between the vectors in that time interval. For example, consider a car following a curve on a level road (**Figure 3**). Even if the driver keeps the speed of the car constant, the direction of the car changes. This change equals the difference between the velocity at one point in time and the velocity at any earlier point in time. In other words, the change in velocity is the subtraction of the final and initial velocities:

$$\Delta \vec{v} = \vec{v}_f - \vec{v}_i$$

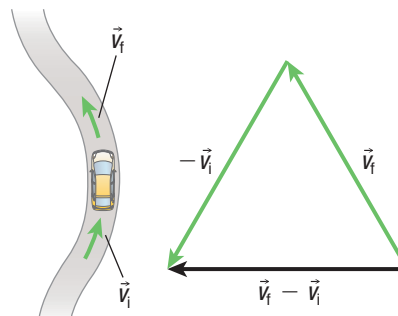


Figure 3 A change in a vector from \vec{v}_i to \vec{v}_f can be determined by vector subtraction, $\vec{v}_f - \vec{v}_i$.

Acceleration in Two Dimensions

We can now apply the principle of vector subtraction in two dimensions to determine the average acceleration in two dimensions. Recall that acceleration in one dimension is the change in velocity with time:

$$\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t}$$

$$\vec{a}_{av} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t}$$

Average acceleration occurs when the velocity vector of an object changes in magnitude, direction, or both.

As with two-dimensional displacement vectors, you can break down the two velocity vectors into horizontal and vertical components. By subtracting each dimension's components, you obtain the net horizontal and vertical components:

$$\Delta v_x = v_{fx} - v_{ix}$$

$$\Delta v_y = v_{fy} - v_{iy}$$

From these, you can calculate the magnitude and direction of the net velocity using the Pythagorean theorem and the inverse tangent equation, respectively:

$$|\Delta \vec{v}| = \sqrt{\Delta v_x^2 + \Delta v_y^2}$$

$$\theta = \tan^{-1} \left(\frac{|\Delta v_y|}{|\Delta v_x|} \right)$$

Finally, the change in velocity divided by the time interval yields the average acceleration.

In Tutorial 2, you will learn how to calculate acceleration in two dimensions by vector subtraction of velocity components.

UNIT TASK BOOKMARK

Calculate the acceleration during your extreme sport for the Unit Task on page 146.

Tutorial 2 Calculating Acceleration in Two Dimensions

The following Sample Problem models how to determine acceleration in two dimensions by vector subtraction of velocity components.

Sample Problem 1: Calculating Acceleration in Two Dimensions

A car turns from a road into a parking lot and into an available parking space. The car's initial velocity is 4.0 m/s [E 45.0° N]. The car's velocity just before the driver decreases speed is 4.0 m/s [E 10.0° N]. The turn takes 3.0 s. Calculate the average acceleration of the car during the turn.

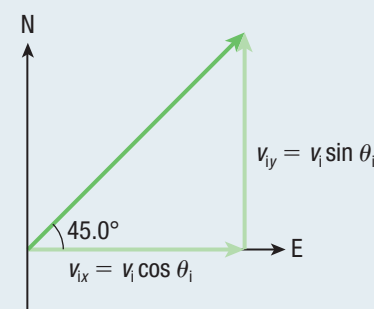
Given: $\vec{v}_i = 4.0 \text{ m/s [E } 45.0^\circ \text{ N]}$; $\vec{v}_f = 4.0 \text{ m/s [E } 10.0^\circ \text{ N]}$;
 $\Delta t = 3.0 \text{ s}$

Required: \vec{a}_{av}

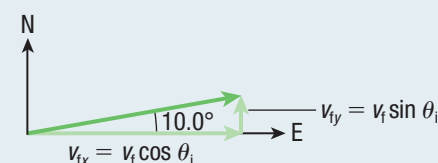
Analysis: Draw a vector diagram of the situation. Determine the components for each velocity vector, and then subtract the initial vector components from the final vector components, $\Delta v_x = \Delta v_{fx} - \Delta v_{ix}$ and $\Delta v_y = \Delta v_{fy} - \Delta v_{iy}$. Calculate the magnitude of the change in velocity using the Pythagorean theorem, and use the inverse tangent equation to calculate the angle of orientation for the net velocity. The average acceleration is then the change in velocity divided by the time interval.

Solution:

Components for the initial velocity vector:



Components for the final velocity vector:



For the initial vector:

$$\begin{aligned}v_{ix} &= \vec{v}_i \cos \theta_i \\&= (4.0 \text{ m/s [E]}) (\cos 45.0^\circ) \\v_{ix} &= 2.83 \text{ m/s [E]} \\v_{iy} &= \vec{v}_i \sin \theta_i \\&= (4.0 \text{ m/s [N]}) (\sin 45.0^\circ) \\v_{iy} &= 2.83 \text{ m/s [N]}\end{aligned}$$

For the final vector:

$$\begin{aligned}v_{fx} &= \vec{v}_f \cos \theta_f \\&= (4.0 \text{ m/s [E]}) (\cos 10.0^\circ) \\v_{fx} &= 3.94 \text{ m/s [E]} \\v_{fy} &= \vec{v}_f \sin \theta_f \\&= (4.0 \text{ m/s [N]}) (\sin 10.0^\circ) \\v_{fy} &= 0.695 \text{ m/s [N]}\end{aligned}$$

Subtract the horizontal components:

$$\begin{aligned}\Delta v_x &= v_{fx} - v_{ix} \\&= 3.94 \text{ m/s [E]} - 2.83 \text{ m/s [E]} \\\Delta v_x &= 1.11 \text{ m/s [E]}\end{aligned}$$

Subtract the vertical components:

$$\begin{aligned}\Delta v_y &= v_{fy} - v_{iy} \\&= 0.695 \text{ m/s [N]} - 2.83 \text{ m/s [N]} \\&= -2.14 \text{ m/s [N]} \\\Delta v_y &= 2.14 \text{ m/s [S]}\end{aligned}$$

Combine the net velocity components to determine the change in velocity:

$$\begin{aligned}|\Delta \vec{v}| &= \sqrt{\Delta v_x^2 + \Delta v_y^2} \\&= \sqrt{(1.11 \text{ m/s})^2 + (-2.14 \text{ m/s})^2} \\|\Delta \vec{v}| &= 2.4 \text{ m/s} \\\theta &= \tan^{-1} \left(\frac{|\Delta v_y|}{|\Delta v_x|} \right) \\&= \tan^{-1} \left(\frac{2.14 \text{ m/s}}{1.11 \text{ m/s}} \right) \\\theta &= 63^\circ\end{aligned}$$

The change in velocity is 2.4 m/s [E 63° S]. The average acceleration is therefore

$$\begin{aligned}\vec{a}_{av} &= \frac{\Delta \vec{v}}{\Delta t} \\&= \frac{2.4 \text{ m/s [E 63° S]}}{3.0 \text{ s}} \\\vec{a}_{av} &= 0.80 \text{ m/s}^2 \text{ [E 63° S]}\end{aligned}$$

Statement: The car's average acceleration is 0.80 m/s² [E 63° S].

Practice

1. A car heading east turns right at a corner. The car turns at a constant speed of 20.0 m/s. After 12 s, the car completes the turn, so that it is heading due south at 20.0 m/s. Calculate the car's average acceleration. **T/I A** [ans: 2.4 m/s² [W 45° S]]
2. Over a 15.0 min period, a truck travels on a road with many turns. The truck's initial velocity is 50.0 km/h [W 60.0° N]. The truck's final velocity is 80.0 km/h [E 60.0° N]. Calculate the truck's average acceleration, in kilometres per hour squared. **T/I** [ans: 2.80 × 10² km/h² [E 21.8° N]]
3. A bird flies from Lesser Slave Lake in northern Alberta to Dore Lake in northern Saskatchewan. The bird's displacement is 800.0 km [E 7.5° S]. The bird then flies from Dore Lake to Big Quill Lake, Saskatchewan. This displacement is 400.0 km [E 51° S]. The total time of flight is 18.0 h. Determine the bird's
 - (a) total distance travelled [ans: 1.2 × 10³ km]
 - (b) total displacement [ans: 1.1 × 10³ km [E 22° S]]
 - (c) average speed [ans: 62 km/h]
 - (d) average velocity **T/I** [ans: 62 km/h [E 22° S]]

1.4 Review

Summary

- Average velocity, in two dimensions, is the total displacement in two dimensions divided by the time interval during which the displacement occurs: $\vec{v}_{av} = \frac{\Delta \vec{d}}{\Delta t}$.
- You can determine the change in velocity in two dimensions by separating the velocity vectors into components and subtracting them using the vector property $\vec{v}_f - \vec{v}_i = \vec{v}_f + (-\vec{v}_i)$.
- Average acceleration in two dimensions is the change in velocity divided by the time interval between the two velocities: $\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t}$.

Questions

- Explain why the average speed is always greater than or equal to the magnitude of the average velocity for an object moving in two dimensions. Give an example in your answer. K/U T/I C
- During 4.0 min on a lake, a loon moves 25.0 m [E 30.0° N] and then 75.0 m [E 45.0° S]. Determine the loon's
 - total distance travelled
 - total displacement
 - average speed
 - average velocity T/I
- A car driver in northern Ontario makes the following displacements:

$$\Delta \vec{d}_1 = 15.0 \text{ km [W } 30.0^\circ \text{ N]},$$

$$\Delta \vec{d}_2 = 10.0 \text{ km [W } 75.0^\circ \text{ N]},$$
 and $\Delta \vec{d}_3 = 10.0 \text{ km [E } 70.0^\circ \text{ N]}.$
 The trip takes 0.50 h. Calculate the average velocity of the car and driver. T/I
- Explain how there can be average acceleration when there is no change in speed. K/U T/I C
- In your own words, explain how to subtract vectors in two dimensions. K/U C
- A pilot in a seaplane flies for a total of 3.0 h with an average velocity of 130 km/h [N 32° E]. In the first part of the trip, the pilot flies for 1.0 h through a displacement of 150 km [E 12° N]. She then flies directly to her final destination. Determine the displacement for the second part of the flight. T/I A
- A student goes for a jog at an average speed of 3.5 m/s. Starting from home, he first runs 1.8 km [E] and then runs 2.6 km [N 35° E]. Then he heads directly home. How long will the entire trip take? T/I A
- In **Figure 4**, a bird changes direction in 3.8 s while flying from point 1 to point 2. Determine the bird's average acceleration. T/I

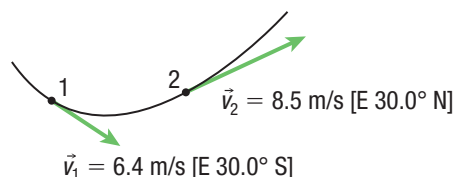


Figure 4

- A helicopter travelling horizontally at 50.0 m/s [W] turns steadily, so that after 45.0 s, its velocity is 35.0 m/s [S]. Calculate the average acceleration of the helicopter. T/I A
- A ball on a pool table bounces off the rail (side), as shown in **Figure 5**. The ball is in contact with the rail for 3.2 ms. Determine the average acceleration of the ball. T/I A

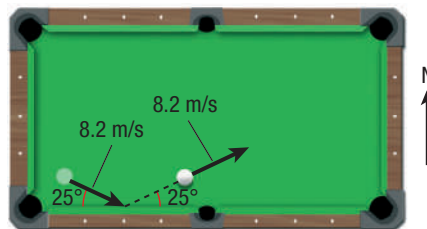


Figure 5

- A speed boat is moving at 6.4 m/s [W 35° N] when it starts accelerating at 2.2 m/s² [S] for 4.0 s. Calculate the final velocity of the boat. T/I A
- An airplane turns slowly for 9.2 s horizontally. The final velocity of the plane is $3.6 \times 10^2 \text{ km/h [N]}$; the average acceleration during the turn is $5.0 \text{ m/s}^2 \text{ [W]}$. What was the initial velocity of the plane? T/I A