MATH 125
Practice questions for the midterm exam

This is not an exhaustive list of the type of questions that might appear on the midterm exam, so you should review all the material covered previously, including all the assignments and all the learning activities in Blocks 1,2,3 to prepare for the midterm exam.

Show all your work, including all your computations.

1. Consider the following homogeneous system of linear equations:

$$\begin{cases} -3x_1 - 9x_2 + x_3 + 5x_4 = 0 \\ x_1 + 3x_2 - 2x_3 - 5x_4 = 0 \\ 4x_1 + 12x_2 - 3x_3 - 10x_4 = 0 \end{cases}$$

Let A be the matrix of coefficients of this linear system. Put A in reduced row echelon form.

2. Consider a linear system of 4 equations in 4 variables with matrix of coefficients denoted A and vector of constants denoted $\mathbf{b}$.

(a) Suppose that the augmented matrix $[A \,|\, \mathbf{b}]$ of that linear system of equations is row equivalent to the following matrix:

$$\left[\begin{array}{cccc|c} 2 & 1 & 6 & -1 & 4 \\ 0 & 0 & 3 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Write down all the solutions $\mathbf{x}$ of that system in vector form and in terms of free variables (or parameters).

(b) Do the columns of A span $\mathbb{R}^4$? Answer 'yes' or 'no' and justify your answer.

3. Find the equation in general form of the plane $\mathscr{P}$ in $\mathbb{R}^3$ which contains the point $(3, 1, -1)$ and is orthogonal to the line ℓ given by the vector equation $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$.

4. Consider the following four vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -4 \end{bmatrix}.$$

Is the vector $\mathbf{v}$ in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$? Answer 'yes' or 'no' and justify your answer. If it is, you do not have to express it in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

5. Give only the answer, you don't have to provide any justification.

(a) If A is a 4×6 matrix of rank 3, $\mathbf{b}$ is a 4×1 column vector and the linear system $A\mathbf{x} = \mathbf{b}$ is consistent (that is, it admits a solution), what is the number of free variables?

(b) Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are 3 non-zero vectors in $\mathbb{R}^5$. Let A be the 5×3 matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$, so the columns of A are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Give an example of a row echelon form of A that implies that those vectors are linearly independent.

6. Answer only true or false. You don't have to justify your answer.

(a) The line in $\mathbb{R}^3$ given by the vector equation $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$ is parallel to the plane given by the equation $2x - 3y + 4z = 1$.

(b) Let A be an $m \times n$ matrix and suppose that the linear system $A\mathbf{x} = \mathbf{b}$ always has at least one solution for any $m \times 1$ vector $\mathbf{b}$. Then the span of the columns of A must always equal $\mathbb{R}^m$.

(c) Consider a homogeneous system of linear equations with matrix of coefficients denoted A. If this system has more than one solution, then the columns of A must be linearly dependent.

(d) 3 non-zero vectors in $\mathbb{R}^3$ which are all orthogonal to each other must always be linearly independent.

(e) If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are 3 linearly independent vectors in $\mathbb{R}^n$ and the vector $\mathbf{v}$ is in the span of those vectors, then the scalars c_1, c_2, c_3 such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

are always unique.

7. Let $\mathscr{P}$ be the plane in $\mathbb{R}^3$ given by the equation $x - 2y - z = 2$. Let P be the point $(3, 1, -1)$ (which is on $\mathscr{P}$) and let Q be the point $(5, 0, -3)$. Determine $\cos(\theta)$ where θ is the angle between $\overrightarrow{PQ}$ and a normal vector $\mathbf{n}$ to the plane $\mathscr{P}$.

8. Solve the following system of linear equations by first forming its augmented matrix and then row reducing it to reduced row echelon form. Give the general solution in vector form.

$$\left\{ \begin{array}{rcrcrcrcr} x_1 & + & x_2 & & & + & 5x_4 & = & -4 \\ & & 2x_2 & + & 4x_3 & + & 2x_4 & = & -2 \\ -x_1 & & & + & 2x_3 & - & 4x_4 & = & 3 \end{array} \right.$$

9. (a) Let $\mathbf{u}_1 = [1,3]$, $\mathbf{u}_2 = [2,1]$, and $\mathbf{u}_3 = [-2,9]$ be vectors in $\mathbb{R}^2$. Write $\mathbf{u}_3$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

(b) Let $\mathbf{u}, \mathbf{v}$ be orthogonal unit vectors in $\mathbb{R}^n$. Compute $\| \mathbf{u} + \mathbf{v} \|$, the length of $\mathbf{u} + \mathbf{v}$.

(c) Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^3$. Consider the following statement:

If $\mathbf{x}$ is parallel to $\mathbf{y}$ and $\| \mathbf{x} \| = \| \mathbf{y} \|$, then $\mathbf{x} = \mathbf{y}$.

If the statement is true, provide a proof. If the statement is false, give a counterexample.

10. Consider the plane $\mathcal{P}$ in $\mathbb{R}^3$ with general equation $2x + y - 3z = 1$.

(a) Find a point in $\mathbb{R}^3$ that does not lie on $\mathcal{P}$. Justify your answer.

(b) Verify that the vector $\mathbf{d}$ is **parallel** to $\mathcal{P}$, where

$$\mathbf{d} = \begin{bmatrix} -5 \\ 4 \\ -2 \end{bmatrix}.$$

(c) Find a vector equation of a line ℓ in $\mathbb{R}^3$ which is parallel to $\mathcal{P}$, but not contained in $\mathcal{P}$.

11. (a) Give the definition of the **span** of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in $\mathbb{R}^n$.

(b) Give an example of a set of 3 distinct vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathbb{R}^3$ that do **not** span $\mathbb{R}^3$, i.e. $\text{Span}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \neq \mathbb{R}^3$. Justify your answer.

12. (a) State what it means for vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in $\mathbb{R}^n$ to be linearly independent.

(b) For what value(s) of k, if any, is the following set of vectors linearly independent? Justify your answer.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ k \end{bmatrix} \right\}$$

(c) Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are (fixed, but unknown) linearly independent vectors in $\mathbb{R}^3$. Do $\mathbf{u}, \mathbf{v}, \mathbf{w}$ span $\mathbb{R}^3$, that is, is $\text{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbb{R}^3$? Justify your answer.

13. (a) Let $A = \begin{bmatrix} 7 & 3 & 10 \\ 13 & 6 & 18 \\ 3 & 2 & 3 \end{bmatrix}$. Find the inverse of A. Show your calculations.

(b) Let $A = \begin{bmatrix} 12 & -3 & 4 \\ 8 & -3 & 3 \\ -2 & 2 & -1 \end{bmatrix}$. Find the inverse of A. Show your calculations.

14. Let $A = \begin{bmatrix} 3 & -3 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & -4 \end{bmatrix}$.

Do the columns of A span $\mathbb{R}^3$? Answer yes or no and justify your answer.

15. Answer only T (true) or F (false). You don't have to justify your answer.

15.1 If A is a square matrix for which the linear system with augmented matrix $[\, A \,|\, \mathbf{0} \,]$ has no free variables, then A is invertible.

15.2 If A and B are two $m \times n$ matrices, then $\operatorname{rank}(A + B) = \operatorname{rank}(A) + \operatorname{rank}(B)$.

15.3 If the matrix A has more columns than rows, then its columns are linearly dependent.

15.4 If an $n \times n$ matrix A has zeros on its main diagonal, then A is not invertible.

16. Suppose that the two $n \times n$ matrices A and B are invertible. Express the inverse of the matrix AB in terms of the inverse of A and the inverse of B. Give only your answer, no justification needed.