MATH 125
Practice questions for the midterm exam - Solutions

This is not an exhaustive list of the type of questions that might appear on the midterm exam, so you should review all the material covered previously, including all the assignments and all the learning activities in Blocks 1,2,3, to prepare for the midterm exam.

Show all your work, including all your computations.

1. Consider the following homogeneous system of linear equations:

$$\begin{cases} -3x_1 - 9x_2 + x_3 + 5x_4 = 0 \\ x_1 + 3x_2 - 2x_3 - 5x_4 = 0 \\ 4x_1 + 12x_2 - 3x_3 - 10x_4 = 0 \end{cases}$$

Let A be the matrix of coefficients of this linear system. Put A in reduced row echelon form.

Solution:

$$\begin{bmatrix} -3 & -9 & 1 & 5 \\ 1 & 3 & -2 & -5 \\ 4 & 12 & -3 & -10 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 3 & -2 & -5 \\ -3 & -9 & 1 & 5 \\ 4 & 12 & -3 & -10 \end{bmatrix}$$

$$R_2 + 3R_1, \ R_3 - 4R_1$$

$$\begin{bmatrix} 1 & 3 & -2 & -5 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

$$R_3 + R_2$$

$$\begin{bmatrix} 1 & 3 & -2 & -5 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\frac{1}{5}R_2$$

$$\begin{bmatrix} 1 & 3 & -2 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 + 2R_2$$

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in reduced row echelon form. □

2. Consider a linear system of 4 equations in 4 variables with matrix of coefficients denoted A and vector of constants denoted $\mathbf{b}$.

(a) Suppose that the augmented matrix $[A\,|\,\mathbf{b}]$ of that linear system of equations is row equivalent to the following matrix:

$$\left[\begin{array}{cccc|c} 2 & 1 & 6 & -1 & 4 \\ 0 & 0 & 3 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Write down all the solutions $\mathbf{x}$ of that system in vector form and in terms of free variables (or parameters).

Solution: Let's put that augmented matrix in reduced row echelon form:

$$\left[\begin{array}{cccc|c} 2 & 1 & 6 & -1 & 4 \\ 0 & 0 & 3 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_1 - 2R_2 \left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & -8 \\ 0 & 0 & 3 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \frac{1}{2}R_1,\ \frac{1}{3}R_2 \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 0 & \frac{3}{2} & -4 \\ 0 & 0 & 1 & -\frac{2}{3} & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The corresponding system of equations is

$$\begin{cases} x_1 + \frac{1}{2}x_2 + \frac{3}{2}x_4 = -4 \\ \phantom{x_1 + \frac{1}{2}x_2 +} x_3 - \frac{2}{3}x_4 = 2 \end{cases}$$

x_2 and x_4 are free variables, so set $x_2 = s$ and $x_4 = t$. Then the solutions $\mathbf{x}$ can be written in vector form as:

$$\mathbf{x} = \begin{bmatrix} -4 - \frac{1}{2}s - \frac{3}{2}t \\ s \\ 2 + \frac{2}{3}t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{3}{2} \\ 0 \\ \frac{2}{3} \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

□

(b) Do the columns of A span $\mathbb{R}^4$? Answer 'yes' or 'no' and justify your answer.

Solution: No. From part (a), we observe that there is no pivot in row 3 or row 4 of a row echelon form of A (or $\operatorname{rank}(A) = 2 < 4 = \#$ of rows of A). Consequently, the columns of A do not span $\mathbb{R}^4$. □

3. Find the equation in general form of the plane $\mathscr{P}$ in $\mathbb{R}^3$ which contains the point $(3, 1, -1)$ and is orthogonal to the line ℓ given by the vector equation $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$.

Solution: The direction vector $\mathbf{d}$ of the line ℓ is $\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$. Since ℓ is orthogonal to the plane P, $\mathbf{d}$ is also a normal vector for P, so $\mathbf{n} = \mathbf{d} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$.

Let $\mathbf{p} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$. The normal equation of P is $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$, that is,

$$\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

Computing both dot products yields the equation of the plane in general form:

$$3x + 4y + 6z = 3 \cdot 3 + 4 \cdot 1 + 6 \cdot (-1), \text{ that is, } 3x + 4y + 6z = 7.$$

□

4. Consider the following four vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -4 \end{bmatrix}.$$

Is the vector $\mathbf{v}$ in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$? Answer 'yes' or 'no' and justify your answer. If it is, you do not have to express it in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution: Yes. Let A be the matrix $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 1 \\ 2 & 2 & 0 \\ -1 & 0 & -1 \end{bmatrix}$. Then $\mathbf{v}$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{v}$ has a solution. This vector equation is equivalent to a system of linear equations with augmented matrix $[A \mid \mathbf{v}]$, so let's find a row echelon form of this augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ -1 & -2 & 1 & 2 \\ 2 & 2 & 0 & 2 \\ -1 & 0 & -1 & -4 \end{array}\right]$$

$$R_2 + R_1,\ R_3 - 2R_1,\ R_4 + R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & -2 \end{array}\right]$$

$$R_3 + R_2,\ -\frac{1}{2}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$$R_4 - 2R_3,\ -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This last matrix is in row echelon form with no row of the form $\left[\begin{array}{ccc|c} 0 & 0 & 0 & b \end{array}\right]$ with $b \neq 0$, so the equation $A\mathbf{x} = \mathbf{v}$ has a solution. (This solution can be found to be $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.) Therefore, $\mathbf{v}$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. (More precisely, $\mathbf{v} = 3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$.) □

5. Give only the answer, you don't have to provide any justification.

(a) If A is a 4×6 matrix of rank 3, $\mathbf{b}$ is a 4×1 column vector and the linear system $A\mathbf{x} = \mathbf{b}$ is consistent (that is, it admits a solution), what is the number of free variables?

Solution: 3. By the Rank Theorem (see the third learning activity of Block 2), $\text{rank}(A) + k =$ number of columns of A, where k is the number of free variables. Therefore, $3 + k = 6$ and $k = 3$. □

(b) Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are 3 non-zero vectors in $\mathbb{R}^5$. Let A be the 5×3 matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$, so the columns of A are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Give an example of a row echelon form of A that implies that those vectors are linearly independent.

Solution: It should be a row echelon form with a pivot in every column (so $\text{rank}(A) = 3 = \#$ of columns of A), for instance: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ □

6. Answer only true or false. You don't have to justify your answer.

(a) The line in $\mathbb{R}^3$ given by the vector equation $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$ is parallel to the plane given by the equation $2x - 3y + 4z = 1$.

Solution: False. That line is parallel to the plane when it is orthogonal to a normal vector of the plane. A direction vector for the line is $\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$ and a normal vector for the plane is $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$. The dot product $\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ is non-zero since $\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = (-4)\cdot 2 + 1 \cdot (-3) + 1 \cdot 4 = -7$, so those vectors are not orthogonal to each other and therefore that line is not parallel to that plane. $\square$

(b) Let A be an $m \times n$ matrix and suppose that the linear system $A\mathbf{x} = \mathbf{b}$ always has at least one solution for any $m \times 1$ vector $\mathbf{b}$. Then the span of the columns of A must always equal $\mathbb{R}^m$.

Solution: True. The notation $A\mathbf{x} = \mathbf{b}$ means that A is the matrix of coefficients of the linear system and $\mathbf{b}$ is the vector of constants. The linear system admits a solution exactly when $\mathbf{b}$ is in the span of the columns of A. If this happens for any vector in $\mathbb{R}^m$, then the span of the columns of A must equal $\mathbb{R}^m$. $\square$

(c) Consider a homogeneous system of linear equations with matrix of coefficients denoted A. If this system has more than one solution, then the columns of A must be linearly dependent.

Solution: True. Any non-zero solution $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ of the system gives a linear dependence relation $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$ where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of A. If the system has more than one solution, than there must be a non-zero solution, that is, a solution for which not all the c_i are equal to 0. This implies that the columns of A must be linearly dependent. $\square$

(d) 3 non-zero vectors in $\mathbb{R}^3$ which are all orthogonal to each other must always be linearly independent.

Solution: True. See the Block 2 Extra Exercises Q11 solutions for a proof. $\square$

(e) If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are 3 linearly independent vectors in $\mathbb{R}^n$ and the vector $\mathbf{v}$ is in the span of those vectors, then the scalars c_1, c_2, c_3 such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

are always unique.

Solution: True.

Suppose that

$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3$$

for some $d_1, d_2, d_3 \in \mathbb{R}$, is another way to write $\mathbf{v}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Then, since both expressions are equal to $\mathbf{v}$, we have

$$\begin{aligned} & c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 \\ \implies \quad & c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 - d_1\mathbf{v}_1 - d_2\mathbf{v}_2 - d_3\mathbf{v}_3 = \mathbf{0} \\ \implies \quad & (c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + (c_3 - d_3)\mathbf{v}_3 = \mathbf{0} \end{aligned}$$

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, this implies $c_1 - d_1 = 0$, $c_2 - d_2 = 0$, and $c_3 - d_3 = 0$ and hence $c_1 = d_1$, $c_2 = d_2$, $c_3 = d_3$. □

7. Let $\mathscr{P}$ be the plane in $\mathbb{R}^3$ given by the equation $x - 2y - z = 2$. Let P be the point $(3, 1, -1)$ (which is on $\mathscr{P}$) and let Q be the point $(5, 0, -3)$. Determine $\cos(\theta)$ where θ is the angle between $\overrightarrow{PQ}$ and a normal vector $\mathbf{n}$ to the plane $\mathscr{P}$.

Solution: A normal vector $\mathbf{n}$ for the plane $\mathcal{P}$ is $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

$\| \mathbf{n} \| = \sqrt{1^2 + (-2)^2 + (-1)^2} = \sqrt{6}$.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$.

$\| \overrightarrow{PQ} \| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$.

$$\cos(\theta) = \frac{\overrightarrow{PQ} \cdot \mathbf{n}}{\| \overrightarrow{PQ} \| \| \mathbf{n} \|} = \frac{2 \cdot 1 + (-1) \cdot (-2) + (-2) \cdot (-1)}{3\sqrt{6}} = \frac{6}{3\sqrt{6}} = \frac{\sqrt{6}}{3}.$$

□

8. Solve the following system of linear equations by first forming its augmented matrix and then row reducing it to reduced row echelon form. Give the general solution in vector form.

$$\begin{cases} x_1 + x_2 + 5x_4 = -4 \\ 2x_2 + 4x_3 + 2x_4 = -2 \\ -x_1 + 2x_3 - 4x_4 = 3 \end{cases}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & -4 \\ 0 & 2 & 4 & 2 & -2 \\ -1 & 0 & 2 & -4 & 3 \end{array}\right] \xrightarrow[R_3+R_1]{} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & -4 \\ 0 & 2 & 4 & 2 & -2 \\ 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}R_2]{} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & -4 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 1 & 2 & 1 & -1 \end{array}\right] \xrightarrow[R_3-R_2]{R_1-R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 4 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The resulting system is:

$$\begin{aligned} x_1 - 2x_3 + 4x_4 &= -3 \\ x_2 + 2x_3 + x_4 &= -1 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{aligned}$$

So we have:

$$\begin{aligned} x_1 &= -3 + 2s - 4t \\ x_2 &= -1 - 2s - t \\ x_3 &= s \\ x_4 &= t \qquad\qquad s, t \in \mathbb{R} \end{aligned}$$

General solution in vector form:

$$\mathbf{x} = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad s, t \in \mathbb{R}$$

□

9. (a) Let $\mathbf{u}_1 = [1,3]$, $\mathbf{u}_2 = [2,1]$, and $\mathbf{u}_3 = [-2,9]$ be vectors in $\mathbb{R}^2$. Write $\mathbf{u}_3$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

Solution: We need to find scalars a and b such that $\mathbf{u}_3 = a\mathbf{u}_1 + b\mathbf{u}_2$.

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 3 & 1 & 9 \end{array}\right] \xrightarrow[R_2-3R_1]{} \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -5 & 15 \end{array}\right] \xrightarrow[-\frac{1}{5}R_2]{} \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -3 \end{array}\right] \xrightarrow[R_1-2R_2]{} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \end{array}\right]$$

Thus, $a = 4$ and $b = -3$, so that $\mathbf{u}_3 = 4\mathbf{u}_1 - 3\mathbf{u}_2$. □

(b) Let $\mathbf{u}, \mathbf{v}$ be orthogonal unit vectors in $\mathbb{R}^n$. Compute $\| \mathbf{u} + \mathbf{v} \|$, the length of $\mathbf{u} + \mathbf{v}$.

Solution: Since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, we have $\mathbf{u} \cdot \mathbf{v} = 0$. Since $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, we have $||\mathbf{u}|| = 1$ and $||\mathbf{v}|| = 1$.

$$||\mathbf{u}+\mathbf{v}||^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}+2(\mathbf{u}\cdot\mathbf{v})+\mathbf{v}\cdot\mathbf{v} = ||\mathbf{u}||^2+2(\mathbf{u}\cdot\mathbf{v})+||\mathbf{v}||^2 = 1^2+2(0)+1^2 = 2$$

Thus $||\mathbf{u} + \mathbf{v}|| = \sqrt{2}$. □

(c) Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^3$. Consider the following statement:

If $\mathbf{x}$ is parallel to $\mathbf{y}$ and $\| \mathbf{x} \| = \| \mathbf{y} \|$, then $\mathbf{x} = \mathbf{y}$.

If the statement is true, provide a proof. If the statement is false, give a counterexample.

Solution: This is false. Let $\mathbf{x} = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and let $\mathbf{y} = -\mathbf{e}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$. Then $\mathbf{y} = -\mathbf{x}$, so that $\mathbf{x}$ and $\mathbf{y}$ are parallel. Also

$$||\mathbf{x}|| = \sqrt{1^2 + 0^2 + 0^2} = \sqrt{1} = 1 \quad \text{and} \quad ||\mathbf{y}|| = \sqrt{(-1)^2 + 0^2 + 0^2} = \sqrt{1} = 1$$

Thus, in this case, $\mathbf{x}$ and $\mathbf{y}$ are parallel vectors with $||\mathbf{x}|| = ||\mathbf{y}||$, but $\mathbf{x} \neq \mathbf{y}$ and therefore the statement is false in general. □

10. Consider the plane $\mathcal{P}$ in $\mathbb{R}^3$ with general equation $2x + y - 3z = 1$.

(a) Find a point in $\mathbb{R}^3$ that does not lie on $\mathcal{P}$. Justify your answer.

Solution: There are many possible solutions here. For example, the point $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does not lie on $\mathcal{P}$ since

$$2(0) + 0 - 3(0) = 0 \neq 1.$$

□

(b) Verify that the vector $\mathbf{d}$ is **parallel** to $\mathcal{P}$, where

$$\mathbf{d} = \begin{bmatrix} -5 \\ 4 \\ -2 \end{bmatrix}.$$

Solution: A normal vector for $\mathcal{P}$ is $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$. The vector $\mathbf{d}$ is parallel to $\mathcal{P}$ if and only if $\mathbf{d}$ is orthogonal to $\mathbf{n}$:

$$\mathbf{d} \cdot \mathbf{n} = \begin{bmatrix} -5 \\ 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = -5(2) + 4(1) + -3(-2) = 0$$

Therefore, $\mathbf{d}$ is orthogonal to $\mathbf{n}$ and hence $\mathbf{d}$ is parallel to $\mathcal{P}$. □

(c) Find a vector equation of a line ℓ in $\mathbb{R}^3$ which is parallel to $\mathcal{P}$, but not contained in $\mathcal{P}$.

Solution: There are many possible solutions here. A line is parallel to $\mathcal{P}$ if and only if its direction vector is parallel to $\mathcal{P}$. Thus, we may choose the vector $\mathbf{d}$ from part b) to be a direction vector for ℓ. To find an equation of a line which is parallel to $\mathcal{P}$, but not contained in $\mathcal{P}$, we may choose any point that does not lie on $\mathcal{P}$ to form the equation. Thus, we may choose ℓ to be a line that passes through the point $\mathbf{0}$. Therefore, a vector equation of a line ℓ in $\mathbb{R}^3$ which is parallel to $\mathcal{P}$, but not contained in $\mathcal{P}$ is:

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 4 \\ -2 \end{bmatrix}, \quad t \in \mathbb{R}$$

or simply

$$\mathbf{x} = t \begin{bmatrix} -5 \\ 4 \\ -2 \end{bmatrix}, \quad t \in \mathbb{R}$$

□

11. (a) Give the definition of the **span** of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in $\mathbb{R}^n$.

Solution: The span of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in $\mathbb{R}^n$ is the set of all linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$. □

(b) Give an example of a set of 3 distinct vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathbb{R}^3$ that do **not** span $\mathbb{R}^3$, i.e. $\text{Span}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \neq \mathbb{R}^3$. Justify your answer.

Solution: Take for example, $\mathbf{x} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Then, $\text{Span}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \text{Span}(\mathbf{0}, \mathbf{y}, \mathbf{z}) = \text{span}(\mathbf{y}, \mathbf{z})$. Since fewer than three vectors (in this case two) cannot span $\mathbb{R}^3$, we have that $\text{Span}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \neq \mathbb{R}^3$.

Alternatively, let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Method 1: Then $\mathbf{x} + \mathbf{y} = \mathbf{z}$ (that is, $\mathbf{z}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$), so that $\text{Span}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \text{Span}(\mathbf{x}, \mathbf{y})$. Since fewer than three vectors (in this case two) cannot span $\mathbb{R}^3$, we have that $\text{Span}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \neq \mathbb{R}^3$.

Method 2: Write these vectors as the columns of a matrix.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in reduced row echelon form. Since there is no leading 1 in row 3 (so that $\text{rank}(A) = 2 < 3 = \#$ of rows of A), we have that $\text{Span}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \neq \mathbb{R}^3$.

There are many other examples. □

12. (a) State what it means for vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ in $\mathbb{R}^n$ to be linearly independent.

Solution: Vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ in $\mathbb{R}^n$ are said to be linearly independent if the following condition holds:

$$\text{If } c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}, \text{ then } c_1 = 0, c_2 = 0, \cdots, c_k = 0.$$

□

(b) For what value(s) of k, if any, is the following set of vectors linearly independent? Justify your answer.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ k \end{bmatrix} \right\}$$

Solution: The given vectors are linearly independent if and only if the following system has only the trivial solution (a unique solution):

$$[\,A\,|\,\mathbf{0}\,] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & k & 0 \end{array}\right] \xrightarrow[R_3+R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & k-1 & 0 \end{array}\right] \xrightarrow[R_3-R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & k-2 & 0 \end{array}\right]$$

To obtain a unique solution, there needs to be a leading entry in every column of an ref of A (so that $\text{rank}(A) = 3 = \#$ of columns of A). Thus, we require $k - 2 \neq 0$ and hence $k \neq 2$. Therefore, the vectors are linearly independent if and only if $k \neq 2$. □

(c) Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are (fixed, but unknown) linearly independent vectors in $\mathbb{R}^3$. Do $\mathbf{u}, \mathbf{v}, \mathbf{w}$ span $\mathbb{R}^3$, that is, is $\text{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbb{R}^3$? Justify your answer.

Solution: Yes! Let $A = [\mathbf{u}\,\mathbf{v}\,\mathbf{w}]$ be the 3×3 matrix with the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ as its columns. Since $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent, the linear system with augmented matrix $[\,A\,|\,\mathbf{0}\,]$ has only the trivial solution (a unique solution). Thus $\text{rank}(A) = 3 = \#$ of columns of A, that is, the rref(A) has a leading 1 in every column (and hence a 0 in all other entries). Therefore,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence there is also a leading 1 in every row of rref(A), i.e. $\text{rank}(A) = 3 = \#$ of rows of A. Thus $\text{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbb{R}^3$.

(Alternatively, once we have deduced that

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we can show that $\mathrm{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbb{R}^3$ as follows: let $\mathbf{b}$ be any vector in $\mathbb{R}^3$. Then the rref of the augmented matrix $[\,A\,|\,\mathbf{b}\,]$ is

$$\mathrm{rref}([\,A\,|\,\mathbf{b}\,]) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right], \quad \text{for some } \ a, b, c \in \mathbb{R}$$

Thus, the underlying system is consistent for all possible values of a, b, c (since there is a leading 1 in every row of $\mathrm{rref}(A)$), i.e. the linear system with augmented matrix $[\,A\,|\,\mathbf{b}\,]$ is consistent for all $\mathbf{b} \in \mathbb{R}^3$. Thus every $\mathbf{b} \in \mathbb{R}^3$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and hence $\mathrm{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbb{R}^3$.) $\square$

13. (a) Let $A = \begin{bmatrix} 7 & 3 & 10 \\ 13 & 6 & 18 \\ 3 & 2 & 3 \end{bmatrix}$. Find the inverse of A. Show your calculations.

Solution:

$$\left[\begin{array}{ccc|ccc} 7 & 3 & 10 & 1 & 0 & 0 \\ 13 & 6 & 18 & 0 & 1 & 0 \\ 3 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 3 & 0 & 0 & 1 \\ 13 & 6 & 18 & 0 & 1 & 0 \\ 7 & 3 & 10 & 1 & 0 & 0 \end{array}\right]$$

$$R_2 - 4R_1, \ R_3 - 2R_1$$

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 3 & 0 & 0 & 1 \\ 1 & -2 & 6 & 0 & 1 & -4 \\ 1 & -1 & 4 & 1 & 0 & -2 \end{array}\right]$$

$$R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 6 & 0 & 1 & -4 \\ 3 & 2 & 3 & 0 & 0 & 1 \\ 1 & -1 & 4 & 1 & 0 & -2 \end{array}\right]$$

$$R_2 - 3R_1, \ R_3 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 6 & 0 & 1 & -4 \\ 0 & 8 & -15 & 0 & -3 & 13 \\ 0 & 1 & -2 & 1 & -1 & 2 \end{array}\right]$$

$$R_2 \leftrightarrow R_3$$
$$\left[\begin{array}{ccc|ccc} 1 & -2 & 6 & 0 & 1 & -4 \\ 0 & 1 & -2 & 1 & -1 & 2 \\ 0 & 8 & -15 & 0 & -3 & 13 \end{array}\right]$$
$$R_3 - 8R_2$$
$$\left[\begin{array}{ccc|ccc} 1 & -2 & 6 & 0 & 1 & -4 \\ 0 & 1 & -2 & 1 & -1 & 2 \\ 0 & 0 & 1 & -8 & 5 & -3 \end{array}\right]$$
$$R_2 + 2R_3,\ R_1 - 6R_3$$
$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 48 & -29 & 14 \\ 0 & 1 & 0 & -15 & 9 & -4 \\ 0 & 0 & 1 & -8 & 5 & -3 \end{array}\right]$$
$$R_1 + 2R_2$$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 18 & -11 & 6 \\ 0 & 1 & 0 & -15 & 9 & -4 \\ 0 & 0 & 1 & -8 & 5 & -3 \end{array}\right]$$

Therefore, $A^{-1} = \begin{bmatrix} 18 & -11 & 6 \\ -15 & 9 & -4 \\ -8 & 5 & -3 \end{bmatrix}$ □

(b) Let $A = \begin{bmatrix} 12 & -3 & 4 \\ 8 & -3 & 3 \\ -2 & 2 & -1 \end{bmatrix}$. Find the inverse of A. Show your calculations.

Solution:

$$\left[\begin{array}{ccc|ccc} 12 & -3 & 4 & 1 & 0 & 0 \\ 8 & -3 & 3 & 0 & 1 & 0 \\ -2 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$
$$R_1 \leftrightarrow R_3$$
$$\left[\begin{array}{ccc|ccc} -2 & 2 & -1 & 0 & 0 & 1 \\ 8 & -3 & 3 & 0 & 1 & 0 \\ 12 & -3 & 4 & 1 & 0 & 0 \end{array}\right]$$
$$R_2 + 4R_1,\ R_3 + 6R_1$$
$$\left[\begin{array}{ccc|ccc} -2 & 2 & -1 & 0 & 0 & 1 \\ 0 & 5 & -1 & 0 & 1 & 4 \\ 0 & 9 & -2 & 1 & 0 & 6 \end{array}\right]$$

$$R_3 - 2R_2$$
$$\left[\begin{array}{ccc|ccc} -2 & 2 & -1 & 0 & 0 & 1 \\ 0 & 5 & -1 & 0 & 1 & 4 \\ 0 & -1 & 0 & 1 & -2 & -2 \end{array}\right]$$
$$R_2 \leftrightarrow R_3$$
$$\left[\begin{array}{ccc|ccc} -2 & 2 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & -2 & -2 \\ 0 & 5 & -1 & 0 & 1 & 4 \end{array}\right]$$
$$R_3 + 5R_2$$
$$\left[\begin{array}{ccc|ccc} -2 & 2 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & -2 & -2 \\ 0 & 0 & -1 & 5 & -9 & -6 \end{array}\right]$$
$$R_1 - R_3$$
$$\left[\begin{array}{ccc|ccc} -2 & 2 & 0 & -5 & 9 & 7 \\ 0 & -1 & 0 & 1 & -2 & -2 \\ 0 & 0 & -1 & 5 & -9 & -6 \end{array}\right]$$
$$R_1 + 2R_2$$
$$\left[\begin{array}{ccc|ccc} -2 & 0 & 0 & -3 & 5 & 3 \\ 0 & -1 & 0 & 1 & -2 & -2 \\ 0 & 0 & -1 & 5 & -9 & -6 \end{array}\right]$$
$$-\frac{1}{2}R_1,\ -R_2,\ -R_3$$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{5}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & -1 & 2 & 2 \\ 0 & 0 & 1 & -5 & 9 & 6 \end{array}\right]$$

Therefore, $A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} & -\frac{3}{2} \\ -1 & 2 & 2 \\ -5 & 9 & 6 \end{bmatrix}$ □

14. Let $A = \begin{bmatrix} 3 & -3 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & -4 \end{bmatrix}$.

Do the columns of A span $\mathbb{R}^3$? Answer yes or no and justify your answer.

Solution: Yes. Let's put A in row echelon form:

$$A = \begin{bmatrix} 3 & -3 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & -4 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 2 & -4 \\ 2 & 1 & 0 \\ 3 & -3 & 2 \end{bmatrix} \xrightarrow[\substack{R_2 - 2R_1 \\ R_3 - 3R_1}]{} \begin{bmatrix} 1 & 2 & -4 \\ 0 & -3 & 8 \\ 0 & -9 & 14 \end{bmatrix} \xrightarrow[R_3 - 3R_2]{} \begin{bmatrix} 1 & 2 & -4 \\ 0 & -3 & 8 \\ 0 & 0 & -10 \end{bmatrix}$$

This last matrix is a row echelon form of A and it has a pivot in every row, so A has rank 3 ($= \#$ of rows of A) and thus its columns span $\mathbb{R}^3$. $\square$

15. Answer only T (true) or F (false). You don't have to justify your answer.

15.1 If A is a square matrix for which the linear system with augmented matrix $[\, A \,|\, \mathbf{0} \,]$ has no free variables, then A is invertible.

Solution: True. If A is an $n \times n$ matrix for which the linear system with augmented matrix $[\, A \,|\, \mathbf{0} \,]$ has no free variables, then its rank must equal n by the Rank Theorem, hence it is invertible by the Invertible Matrix Theorem. $\square$

15.2 If A and B are two $m \times n$ matrices, then $\operatorname{rank}(A + B) = \operatorname{rank}(A) + \operatorname{rank}(B)$.

Solution: False. For a counterexample, let $A = I_2$ (the 2×2 identity matrix) and $B = -I_2$, so $A + B$ is the zero matrix of rank 0 but $\operatorname{rank}(A) = 2$ and $\operatorname{rank}(B) = 2$. $\square$

15.3 If the matrix A has more columns than rows, then its columns are linearly dependent.

Solution: True. If A is an $m \times n$ matrix and $m < n$, then its columns are n vectors in $\mathbb{R}^m$, hence they must be linearly dependent. $\square$

15.4 If an $n \times n$ matrix A has zeros on its main diagonal, then A is not invertible.

Solution: False. Here is a counterexample: the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is invertible and equal to its own inverse. $\square$

16. Suppose that the two $n \times n$ matrices A and B are invertible. Express the inverse of the matrix AB in terms of the inverse of A and the inverse of B. Give only your answer, no justification needed.

Solution: $(AB)^{-1} = B^{-1}A^{-1}$. $\square$