Math 125 – Fall 2025
Practice Midterm Exam: Solutions

Question 1. State the augmented matrix associated with the linear system below, and fully reduce that augmented matrix to reduced row echelon form as if to solve the system. Show and label all row reduction steps!

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 3x_3 & - & 2x_4 & = & -2 \\ 2x_1 & - & x_2 & + & 8x_3 & - & x_4 & = & 3 \\ 4x_1 & - & 2x_2 & + & 16x_3 & & & = & 4 \end{array}$$

(You do not need to provide the general solution to this system.)

Solution. The augmented matrix associated to the linear system is

$$[A \,|\, \mathbf{b}] = \left[\begin{array}{cccc|c} 1 & -1 & 3 & -2 & -2 \\ 2 & -1 & 8 & -1 & 3 \\ 4 & -2 & 16 & 0 & 4 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & -2 & -2 \\ 2 & -1 & 8 & -1 & 3 \\ 4 & -2 & 16 & 0 & 4 \end{array}\right] \xrightarrow[R_3-4R_1]{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & -1 & 3 & -2 & -2 \\ 0 & 1 & 2 & 3 & 7 \\ 0 & 2 & 4 & 8 & 12 \end{array}\right] \xrightarrow[R_3-2R_2]{R_1+R_2} \left[\begin{array}{cccc|c} 1 & 0 & 5 & 1 & 5 \\ 0 & 1 & 2 & 3 & 7 \\ 0 & 0 & 0 & 2 & -2 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}R_3]{} \left[\begin{array}{cccc|c} 1 & 0 & 5 & 1 & 5 \\ 0 & 1 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[R_2-3R_3]{R_1-R_3} \left[\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 6 \\ 0 & 1 & 2 & 0 & 10 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Thus $\mathrm{rref}([A \,|\, \mathbf{b}]) = \left[\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 6 \\ 0 & 1 & 2 & 0 & 10 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$.

Question 2.

Suppose that a linear system has an augmented matrix that is row equivalent to the matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & -3 & 0 & -1 & -8 \\ 0 & 1 & -5 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Provide the general solution to the linear system in vector form. Show your work.

Solution.

The matrix above is almost in reduced row echelon form. Applying the row operation $R_1 - R_2$ gives the reduced row echelon form

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -5 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The resulting system is:

$$\begin{aligned} x_1 + 2x_3 - x_5 &= 0 \\ x_2 - 5x_3 &= -8 \\ x_3 &\text{ is free} \\ x_4 + x_5 &= 3 \\ x_5 &\text{ is free} \end{aligned}$$

So we have:

$$\begin{aligned} x_1 &= -2s + t \\ x_2 &= -8 + 5s \\ x_3 &= s \\ x_4 &= 3 - t \\ x_5 &= t, \qquad s, t \in \mathbb{R} \end{aligned}$$

Therefore, the general solution of the system is:

$$\mathbf{x} = \begin{bmatrix} 0 \\ -8 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

Question 3. Give a counterexample to show that the following statement is false:

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero parallel vectors in $\mathbb{R}^3$, then $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$.

Solution.

There are infinitely many possible counterexamples. Take $\mathbf{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}-2\\0\\0\end{bmatrix}$ in $\mathbb{R}^3$. Then

$$\mathbf{v} = \begin{bmatrix}-2\\0\\0\end{bmatrix} = -2\begin{bmatrix}1\\0\\0\end{bmatrix} = -2\mathbf{u}$$

$$\|\mathbf{u}+\mathbf{v}\| = \left\|\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}-2\\0\\0\end{bmatrix}\right\| = \left\|\begin{bmatrix}-1\\0\\0\end{bmatrix}\right\| = \sqrt{(-1)^2+0^2+0^2} = \sqrt{1} = 1$$

$$\|\mathbf{u}\| + \|\mathbf{v}\| = \left\|\begin{bmatrix}1\\0\\0\end{bmatrix}\right\| + \left\|\begin{bmatrix}-2\\0\\0\end{bmatrix}\right\| = \sqrt{1^2+0^2+0^2} + \sqrt{(-2)^2+0^2+0^2} = \sqrt{1}+\sqrt{4} = 1+2 = 3$$

Thus, $\mathbf{u}$ and $\mathbf{v}$ are nonzero parallel vectors in $\mathbb{R}^3$, but $\|\mathbf{u}+\mathbf{v}\| \neq \|\mathbf{u}\| + \|\mathbf{v}\|$. Hence, the statement is false in this case, and so false in general.

Question 4. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ be vectors in $\mathbb{R}^n$. Prove the following statement:

If $\mathbf{u} \in \operatorname{Span}(\mathbf{v},\, \mathbf{w})$ and $\mathbf{w} \in \operatorname{Span}(\mathbf{v},\, \mathbf{x})$, then $\mathbf{u} \in \operatorname{Span}(3\mathbf{v},\, -5\mathbf{x})$.

Solution.

Assume: $\mathbf{u} \in \operatorname{Span}(\mathbf{v},\, \mathbf{w})$ and $\mathbf{w} \in \operatorname{Span}(\mathbf{v},\, \mathbf{x})$

Show: $\mathbf{u} \in \operatorname{Span}(3\mathbf{v},\, -5\mathbf{x})$

Since $\mathbf{u} \in \operatorname{Span}(\mathbf{v},\, \mathbf{w})$,

$$\mathbf{u} = a\mathbf{v} + b\mathbf{w}$$

for some $a, b \in \mathbb{R}$ and since $\mathbf{w} \in \operatorname{Span}(\mathbf{v},\, \mathbf{x})$,

$$\mathbf{w} = c\mathbf{v} + d\mathbf{x}$$

for some $c, d \in \mathbb{R}$. Then

$$\begin{aligned}\mathbf{u} &= a\mathbf{v} + b\mathbf{w}\\ &= a\mathbf{v} + b(c\mathbf{v} + d\mathbf{x})\\ &= a\mathbf{v} + bc\mathbf{v} + bd\mathbf{x}\\ &= (a+bc)\mathbf{v} + bd\mathbf{x}\\ &= \left(\frac{a+bc}{3}\right)(3\mathbf{v}) + \left(\frac{-bd}{5}\right)(-5\mathbf{x})\end{aligned}$$

Thus, $\mathbf{u} \in \operatorname{Span}(3\mathbf{v},\, -5\mathbf{x})$, as required.

Question 5. Let

$$\mathbf{u} = \begin{bmatrix} -1 \\ -5 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}.$$

(a) If $(\mathbf{u} \cdot \mathbf{v})\mathbf{x} = \mathbf{u} - \mathbf{v}$ then $\mathbf{x} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$.

(b) The vector $\mathbf{w}$ with second entry 1 that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{w} = \begin{bmatrix} ? \\ 1 \\ ? \end{bmatrix}$.

(c) Suppose that $\mathbf{v} = \overrightarrow{AB}$, where A and B are points in $\mathbb{R}^3$. If $B = (9, -8, 4)$ what is the point A?

Solution.

(a) $\mathbf{u} \cdot \mathbf{v} = (-1)(1) + (-5)(-5) + 0(-2) = 24$, so

$$\mathbf{x} = \frac{1}{\mathbf{u} \cdot \mathbf{v}}(\mathbf{u} - \mathbf{v}) = \frac{1}{24}\left(\begin{bmatrix} -1 \\ -5 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}\right) = \frac{1}{24}\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} \\ 0 \\ \frac{1}{12} \end{bmatrix}$$

(b) If $\mathbf{w} = \begin{bmatrix} a \\ 1 \\ c \end{bmatrix}$, then

$$\begin{aligned} \mathbf{u} \cdot \mathbf{w} &= (-1)a + (-5)(1) + 0c = -a - 5 \quad \text{and} \\ \mathbf{v} \cdot \mathbf{w} &= a + (-5)(1) - 2c = a - 5 - 2c. \end{aligned}$$

$\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ exactly when $\mathbf{u} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$, that is, exactly when $-a - 5 = 0$ and $a - 5 - 2c = 0$. It follows that $a = -5$ and $-5 - 5 - 2c = 0$, hence $c = -5$ also. Therefore,

$$\mathbf{w} = \begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix}$$

(c) We note that

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

so that

$$\overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{AB}$$

Thus,

$$\overrightarrow{OA} = \begin{bmatrix} 9 \\ -8 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 6 \end{bmatrix}$$

and so $A = (8, -3, 6)$.

Question 6. For $\mathbf{v}$ and $\mathbf{w} \in \mathbb{R}^{15}$, if $\|\mathbf{v}\| = 3$, $\|\mathbf{w}\| = 2$ and $\mathbf{v} \cdot \mathbf{w} = -2$, then

$$(2\mathbf{v} - 5\mathbf{w}) \cdot (3\mathbf{v} + 2\mathbf{w}) =$$

Solution. 36

$$\begin{aligned}
(2\mathbf{v} - 5\mathbf{w}) \cdot (3\mathbf{v} + 2\mathbf{w}) &= (2\mathbf{v} - 5\mathbf{w}) \cdot (3\mathbf{v}) + (2\mathbf{v} - 5\mathbf{w}) \cdot (2\mathbf{w}) \\
&= (2\mathbf{v}) \cdot (3\mathbf{v}) - (5\mathbf{w}) \cdot (3\mathbf{v}) + (2\mathbf{v}) \cdot (2\mathbf{w}) - (5\mathbf{w}) \cdot (2\mathbf{w}) \\
&= 6(\mathbf{v} \cdot \mathbf{v}) - 15(\mathbf{w} \cdot \mathbf{v}) + 4(\mathbf{v} \cdot \mathbf{w}) - 10(\mathbf{w} \cdot \mathbf{w}) \\
&= 6(\mathbf{v} \cdot \mathbf{v}) - 15(\mathbf{v} \cdot \mathbf{w}) + 4(\mathbf{v} \cdot \mathbf{w}) - 10(\mathbf{w} \cdot \mathbf{w}) \\
&= 6\|\mathbf{v}\|^2 - 11(\mathbf{v} \cdot \mathbf{w}) - 10\|\mathbf{w}\|^2 \\
&= 6(3)^2 - 11(-2) - 10(2)^2 \\
&= 36
\end{aligned}$$

Question 7. If $\mathbf{u}$ is a non-zero vector in $\mathbb{R}^n$ and $\mathbf{v} = 3\mathbf{u}$, is $\text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{u}, \left(\frac{1}{3}\right)\mathbf{u}, 3\mathbf{u}$, or $\left(\frac{1}{9}\right)\mathbf{u}$?

Solution. u

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{u} \cdot (3\mathbf{u})}{(3\mathbf{u}) \cdot (3\mathbf{u})}\right)(3\mathbf{u}) = \left(\frac{3(\mathbf{u} \cdot \mathbf{u})}{9(\mathbf{u} \cdot \mathbf{u})}\right)(3\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u} = \mathbf{u}$$

Question 8.

Find a general equation of the plane $\mathcal{P}$ in $\mathbb{R}^3$ which contains the point $(4, 2, -1)$ and is parallel to the plane in $\mathbb{R}^3$ with general equation $2x + 5y - 2z = 1$.

Solution. $2x + 5y - 2z = 20$.

Since the planes are parallel, they have the same normal vectors. A normal vector for the plane with general equation $2x + 5y - 2z = 1$ is

$$\begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix},$$

and so this is also a normal vector for the parallel plane $\mathcal{P}$ which passes through the point $(4, 2, -1)$. Thus, a normal equation for $\mathcal{P}$ is

$$\begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

and hence a general equation is

$$2x + 5y - 2z = 20$$

Question 9. Find a vector equation of the plane $\mathcal{P}$ in $\mathbb{R}^3$ given by the general equation $x+2y+3z=6$.

Solution. There are many different vector equations that all describe the plane $\mathcal{P}$. To find a vector equation for $\mathcal{P}$, we require a point on $\mathcal{P}$ and a pair of direction vectors $\mathbf{v}, \mathbf{w}$ for $\mathcal{P}$.

We first note that $(1,1,1)$ is a point on $\mathcal{P}$ since

$$1+2(1)+3(1)=6$$

From the given general equation of $\mathcal{P}$, we obtain that

$$\mathbf{n}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

is a normal vector for $\mathcal{P}$. To find a pair of direction vectors for $\mathcal{P}$, we can choose two non-parallel vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that

$$\mathbf{n}\cdot\mathbf{v}=0 \quad \text{and} \quad \mathbf{n}\cdot\mathbf{w}=0.$$

For example, let

$$\mathbf{v}=\begin{bmatrix}-2\\1\\0\end{bmatrix} \quad \text{and} \quad \mathbf{w}=\begin{bmatrix}-3\\0\\1\end{bmatrix}$$

Since $\mathbf{v}$ and $\mathbf{w}$ are not scalar multiples of each other, they are not parallel. Furthermore, we have

$$\mathbf{n}\cdot\mathbf{v}=\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}-2\\1\\0\end{bmatrix}=1(-2)+2(1)+3(0)=0$$

and

$$\mathbf{n}\cdot\mathbf{w}=\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}-3\\0\\1\end{bmatrix}=1(-3)+2(0)+3(1)=0$$

Thus, $\mathbf{v}$ and $\mathbf{w}$ are both orthogonal to $\mathbf{n}$ and hence they are parallel to $\mathcal{P}$. Therefore, $\mathbf{v}$ and $\mathbf{w}$ serve as a pair of direction vectors for $\mathcal{P}$.

Putting everything together, we obtain that a vector equation for $\mathcal{P}$ is

$$\mathbf{x}=\begin{bmatrix}1\\1\\1\end{bmatrix}+s\begin{bmatrix}-2\\1\\0\end{bmatrix}+t\begin{bmatrix}-3\\0\\1\end{bmatrix}, \quad s,t\in\mathbb{R}$$

See the solution for Q4 (a) on Assignment #1 for alternative methods to answer this question.

Question 10. Suppose that the matrix below is a row echelon form of the augmented matrix of a system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 2 & a+1 & 2a+3 \\ 0 & a-4 & a-2 & a-6 \\ 0 & 0 & a^2+2a-3 & a^2+4a+3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & a+1 & 2a+3 \\ 0 & a-4 & a-2 & a-6 \\ 0 & 0 & (a+3)(a-1) & (a+3)(a+1) \end{array}\right]$$

(a) For all $a \neq \boxed{}$, the linear system has a unique solution.

(b) List all $a \in \mathbb{R}$ so that the linear system has infinitely many solutions.

(c) List all $a \in \mathbb{R}$ so that the system has no solution.

Solution.

(a) $a \neq -3, 1, 4$

To get a unique solution, the system has to be consistent and the rank of the coefficient matrix must be 3, i.e., we must have a leading entry in every column of a row echelon form of the coefficient matrix. Thus, we require

$$a-4 \neq 0 \quad \text{and} \quad (a+3)(a-1) \neq 0$$

that is, we require that $a \neq 4, -3, 1$ (or in ascending order $a \neq -3, 1, 4$).

(b) $a = -3$

For the system to have infinitely many solutions, it must be consistent with at least one free variable. When $a = -3$ the matrix above becomes

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -3 \\ 0 & -7 & -5 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is consistent with one free variable (namely x_3). So when $a = -3$ the system has infinitely many solutions.

(c) $a = 1, 4$

For the system to be an inconsistent system, we need to have a leading entry in the last column of an ref of the augmented matrix. When $a = 1$ the matrix above becomes

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 0 & -3 & -1 & -5 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

which is inconsistent, since there is a leading entry in the augmented column. When $a = 4$ the matrix above becomes

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 11 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 21 & 35 \end{array}\right] \xrightarrow[\frac{1}{2}R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 5 & 11 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 21 & 35 \end{array}\right] \xrightarrow[R_3-21R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 5 & 11 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 56 \end{array}\right]$$

which is inconsistent, since there is a leading entry in the augmented column. So when $a = 1$ or $a = 4$ the system has no solution.

Question 11. Let

$$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ -4 \\ -4 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} -3 \\ 6 \\ 6 \end{bmatrix}.$$

Is the set $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ linearly independent?

Is $\mathrm{Span}(\mathbf{u},\ \mathbf{v},\ \mathbf{w})$ equal to a line that passes through the origin, a plane that passes through the origin, or all of $\mathbb{R}^3$?

Solution. Observe that $\mathbf{v} = -2\mathbf{u}$ and $\mathbf{w} = 3\mathbf{u}$, so these vectors are linearly dependent. Since

$$\mathrm{Span}(\mathbf{u},\ \mathbf{v},\ \mathbf{w}) = \mathrm{Span}(\mathbf{u},\ -2\mathbf{u},\ 3\mathbf{u}) = \mathrm{Span}(\mathbf{u})$$

their span is a line in $\mathbb{R}^3$ which passes through the origin.

Alternatively, following the standard procedure to answer this type of question, let's form the matrix $A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ having these vectors as columns and let's put it in row echelon form.

$$A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \begin{bmatrix} -1 & 2 & -3 \\ 2 & -4 & 6 \\ 2 & -4 & 6 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2+2R_1} \begin{bmatrix} -1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\mathrm{rank}(A) = 1 < 3\ = \#$ of columns of A, the vectors are linearly dependent. Since the rank of A is 1, the span of those vectors must be a line in $\mathbb{R}^3$ (which passes through the origin).

Question 12. Suppose that the matrix $[\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\ \mathbf{v_4}\ \mathbf{v_5}]$ has a reduced row echelon form of

$$\begin{bmatrix} 1 & 3 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

(a) Then $\mathbf{v_5} = \boxed{}\mathbf{v_1} + \boxed{}\mathbf{v_2} + \boxed{}\mathbf{v_3} + \boxed{}\mathbf{v_4}$.

(b) True or False? $\quad \mathrm{Span}(\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3},\ \mathbf{v_4},\ \mathbf{v_5}) = \mathbb{R}^3$

Solution. Let $A = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\ \mathbf{v_4}\ \mathbf{v_5}]$.

(a) $\mathbf{v_5} = -6\mathbf{v_1} + 0\mathbf{v_2} + 0\mathbf{v_3} + 4\mathbf{v_4}$

Let $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \mathbf{u}_4$ and $\mathbf{u}_5$ be the columns of the $\mathrm{rref}(A)$. Linear relations between column vectors are preserved by row operations. So a matrix and its rref have the same linear relations holding between their column vectors. From the $\mathrm{rref}(A)$, we see that

$$\mathbf{u_5} = -6\mathbf{u_1} + 0\mathbf{u_2} + 0\mathbf{u_3} + 4\mathbf{u_4}$$

and it follows that the same relation holds between $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4$ and $\mathbf{v}_5$.

There is more than one correct answer: for instance, it can be seen also directly from the $\mathrm{rref}(A)$ that $\mathbf{u_5} = 0\mathbf{u_1} - 2\mathbf{u_2} + 0\mathbf{u_3} + 4\mathbf{u_4}$, hence $\mathbf{v_5} = 0\mathbf{v_1} - 2\mathbf{v_2} + 0\mathbf{v_3} + 4\mathbf{v_4}$.

(b) True. Since there is a leading entry in every row of the $\mathrm{rref}(A)$ (or since $\mathrm{rank}(A) = 3 = \#$ of rows of A), these vectors span $\mathbb{R}^3$.

Question 13.

If $A = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$ and $X = 2A + B^{-1}$, then $BA = \begin{bmatrix} & \\ & \end{bmatrix}$ and $X = \begin{bmatrix} & \\ & \end{bmatrix}$?

Solution.

$$BA = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 7 & -6 \end{bmatrix}$$

Using the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

for the inverse of a 2×2 matrix with non-zero determinant, we can compute directly that

$$B^{-1} = \frac{1}{2(3) - 2(4)} \begin{bmatrix} 3 & -2 \\ -4 & 2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 3 & -2 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 1 \\ 2 & -1 \end{bmatrix}$$

It follows that

$$X = 2A + B^{-1} = \begin{bmatrix} 2 & -6 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -5 \\ 4 & 3 \end{bmatrix}.$$

Question 14. Suppose that V, W, and X are invertible square matrices of the same size such that

$$(V^2X^TW^T)^{-1} = V^3W^{-1}V^{-2}$$

Solve for X in terms of V and W. Which one of the following matrices is equal to X?

(A) $(W^{-1})^T(V^T)^3W$

(B) $(V^{-3})^TW^TW^{-1}$

(C) $W^{-1}(V^T)^{-3}W^T$

(D) $WV^{-3}(W^{-1})^T$

(E) None of these.

Solution. (C)

$$\begin{aligned}
& & (V^2X^TW^T)^{-1} &= V^3W^{-1}V^{-2} \\
&\iff & ((V^2X^TW^T)^{-1})^{-1} &= (V^3W^{-1}V^{-2})^{-1} \\
&\iff & V^2X^TW^T &= (V^{-2})^{-1}(W^{-1})^{-1}(V^3)^{-1} \\
&\iff & V^2X^TW^T &= V^2WV^{-3} \\
&\iff & V^{-2}\left(V^2X^TW^T\right)(W^T)^{-1} &= V^{-2}\left(V^2WV^{-3}\right)(W^T)^{-1} \\
&\iff & IX^TI &= IWV^{-3}(W^T)^{-1} \\
&\iff & X^T &= WV^{-3}(W^T)^{-1} \\
&\iff & (X^T)^T &= (WV^{-3}(W^T)^{-1})^T \\
&\iff & X &= ((W^T)^{-1})^T(V^{-3})^TW^T \\
&\iff & X &= ((W^{-1})^T)^T(V^T)^{-3}W^T \\
&\iff & X &= W^{-1}(V^T)^{-3}W^T
\end{aligned}$$

Question 15. True or false?

Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^9$. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, then $\mathbf{x}$ and $\mathbf{x}+\mathbf{y}$ are linearly independent.

Solution. True.

Assume: $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

Show: $\mathbf{x}$ and $\mathbf{x}+\mathbf{y}$ are linearly independent.

Suppose that

$$a\mathbf{x} + b(\mathbf{x}+\mathbf{y}) = \mathbf{0},$$

for some $a, b \in \mathbb{R}$.

Thus, collecting together like terms, we find that

$$(a+b)\mathbf{x} + b\mathbf{y} = \mathbf{0}$$

Since $\mathbf{x}, \mathbf{y}$ are linearly independent, we have $a+b=0$ and $b=0$, and hence $a+0=0$, so that $a=0$. Therefore, $a=0$ and $b=0$, and hence $\mathbf{x}$ and $\mathbf{x}+\mathbf{y}$ are linearly independent.

Question 16. True or false?

The line in $\mathbb{R}^3$ with vector equation

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, t \in \mathbb{R}$$

is parallel to the plane in $\mathbb{R}^3$ with general equation

$$2x - y + 4z = 5$$

Solution. False.

The direction vector $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ of the line is the same as the normal vector $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ of the plane with general equation $2x - y + 4z = 5$. This means that the line is perpendicular to the plane and not parallel to it.

Question 17. True or false?

For vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, if $\mathbf{u} \neq 0$ and $\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{v}$, then $\mathbf{w} = \mathbf{v}$.

Solution. False.

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then

$$\mathbf{u} \cdot \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1(0) + 0(0) = 0$$

and

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1(0) + 0(1) = 0$$

Therefore, $\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \neq \mathbf{0}$, but $\mathbf{w} \neq \mathbf{v}$. Hence the statement is false in this case, and so false in general.

Question 18. True or false?

Let A be a 3×3 matrix. Suppose that the system of linear equations with augmented matrix $[\, A \mid \mathbf{b} \,]$ is inconsistent for some vector $\mathbf{b} \in \mathbb{R}^3$. Then the columns of A are linearly independent.

Solution. False.

Since the system of linear equations with augmented matrix $[\, A \mid \mathbf{b} \,]$ is inconsistent for some vector $\mathbf{b} \in \mathbb{R}^3$, there is a leading entry in the last column of an ref of $[\, A \mid \mathbf{b} \,]$ for that $\mathbf{b} \in \mathbb{R}^3$. Thus, at least one of the three rows of an ref of A does not contain a leading entry, that is, there are at most two leading entries in an ref of A, and so $\operatorname{rank}(A) \leq 2$. Therefore, at least one of the three columns of an ref of A does not contain a leading entry, or equivalently, $\operatorname{rank}(A) \leq 2 \neq 3 = \#$ of columns of A, and hence the columns of A are linearly dependent, not linearly independent.

Question 19. True or false?

Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. If $A^8 = I$, then A is invertible.

Solution. True.

Since $A^8 = I$, we obtain that

$$A(A^7) = I,$$

where A and A^7 are both $n \times n$ matrices. Thus, A is invertible and $A^{-1} = A^7$.