Adding Fractions

In arithmetic we know that the sum or difference of fractions with same denominator is given by the sum or difference of the numerators divided by the common denominator. If fractions do not have a common denominator one must be obtained before addition or subtraction can take place.

Example: Find the sum: $\frac{1}{2}+\frac{2}{3}$

$$\frac{1}{2}+\frac{2}{3}=\frac{1}{2}\cdot\left(\frac{3}{3}\right)+\frac{2}{3}\cdot\left(\frac{2}{2}\right) \quad \text{get denomintor of 6 for both fractions}$$

$$=\frac{3}{6}+\frac{4}{6}=\frac{7}{6} \quad \text{add numerators of fractions together}$$

When we add or subtract algebraic fractions the method is exactly the same. First a common denominator must be obtained. To find the common denominator:

1. Factor completely each denominator.
2. The least common denominator is the product of all the different factors with each factor raised to the highest power to which it appears in any one factorization.

Example: Find the difference: $\frac{x+h+1}{x+h}-\frac{x+1}{x}$

$$\frac{x+h+1}{x+h}-\frac{x+1}{x} \quad \text{the common denominator is } (x+h)x$$

$$\frac{x+h+1}{x+h}\cdot\left(\frac{x}{x}\right)-\frac{x+1}{x}\cdot\left(\frac{x+h}{x+h}\right) \quad \text{get denominator of } (x+h)x \text{ for both fractions}$$

$$\frac{(x+h+1)x-(x+1)(x+h)}{(x+h)x} \quad \text{subtract the numerators}$$

$$\frac{x^2+xh+x-x^2-xh-x-h}{x(x+h)}=\frac{-h}{x(x+h)} \quad \text{simplify and distribute the negative sign}$$

Practice Problems

Find the sum or difference.

1. $\frac{1}{4x}+\frac{1}{3x}$

2. $\frac{3x+4}{x+2}-\frac{2x+5}{x+2}$

3. $z+\frac{1}{z}$

4. $\frac{5}{8y}-\frac{2}{12y}$

5. $\frac{2-y}{9y+6}+\frac{y-2}{6y+4}$

6. $\frac{3}{2x-3}-\frac{2}{9-4x^2}$

7. $\frac{7}{n^2}-\frac{5n-2}{n}$

8. $\frac{3}{a-3}-\frac{3}{a}$

9. $\frac{2x+3}{2x^3-4x^2}-\frac{1}{x-2}$

Functions

Functions are most commonly written as $y = f(x)$, where x is the input value and y is the output value. In other words, the value for y is completely determined by the value of x. Sometimes x is referred to as the independent variable and y as the dependent variable.

The **domain of a function** is the set of all the values that can be plugged into the function.

The **range of a function** is the set of all the possible outputs of the function.

Consider the function $f(x) = x^2 - 2$. Find $f(a+2)$.
To do this, you will need to plug $a+2$ into the function wherever there is an x.
So, $f(a+2) = (a+2)^2 - 2 = a^2 + 4a + 4 - 2 = a^2 + 4a + 2$.
No matter what the input value is, it will always be plugged in wherever there is an x.
For example, find $f($☺$)$.
$f($☺$) =$ ☺$^2 - 2$; the ☺ is plugged in for the x values.

Examples:

a. Let $f(x) = x^2 - 2$. Find $f(2)$.

$$\begin{aligned} f(2) &= (2)^2 - 2 \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

b. Let $f(x) = 4x^2 - x$. Find $f(a+3)$.

$$\begin{aligned} f(a+3) &= 4(a+3)^2 - (a+3) \\ &= 4(a+3)(a+3) - (a+3) \\ &= 4(a^2 + 6a + 9) - (a+3) \\ &= 4a^2 + 24a + 36 - (a+3) \\ &= 4a^2 + 23a + 33 \end{aligned}$$

c. Let $g(x) = \dfrac{1}{6x}$. Find $g(x+h)$.

$$\begin{aligned} g(x+h) &= \frac{1}{6(x+h)} \\ &= \frac{1}{6x + 6h} \end{aligned}$$

d. Let $g(x) = x^2$. Find $\dfrac{g(a+h) - g(a)}{h}$.

$$\begin{aligned} \frac{g(a+h) - g(a)}{h} &= \frac{(a+h)^2 - a^2}{h} \\ &= \frac{a^2 + 2ah + h^2 - a^2}{h} \\ &= \frac{2ah + h^2}{h} \end{aligned}$$

Equations of Lines

The equation of a line is a function that can be written in the form $ax + by = c$.
For a line that passes through the points (x_1, y_1) and (x_2, y_2), the slope of the line, m, is given by
$m = \frac{y_2 - y_1}{x_2 - x_1}$.

Note that horizontal lines have a slope of 0, and vertical lines have an undefined slope.

The slope-intercept form of a line is

$$y = mx + b$$

where m is the slope of the line and b is the y-intercept, or the y-value at the point where the line crosses the y-axis.

The point-slope form of the line passing through the point (x_1, y_1) with slope m is

$$y - y_1 = m(x - x_1)$$

Practice Problems

Find the slope of the line that goes through the following points.

1. $(2,5)$ and $(-4,7)$ **2.** $(0,6)$ and $(5,-2)$ **3.** $(3,4)$ and $(3,12)$

Write an equation of the line using either point-slope or slope-intercept form…

4. with slope $m = -\frac{1}{2}$ and through the point $(3,-4)$

5. through the points $(0,7)$ and $(-5,2)$

6. through the points $(-2,6)$ and $(9,6)$

7. with slope $m = \frac{2}{3}$ and through the point $(0,-2)$

Factoring

Being able to factor polynomials is an essential skill needed in calculus. Below are some of the techniques used to factor polynomials.

Factoring out the Greatest Common Factor (GCF)

Examples:

$4x+12=4(x+3)$
GCF of $4x$ and 12 is 4

$6x^2y+9xy^2+3xy=3xy(2x+3y+1)$
GCF is $3xy$

Perfect Square Trinomials

$$x^2+2xy+y^2=(x+y)(x+y)=(x+y)^2$$
$$x^2-2xy+y^2=(x-y)(x-y)=(x-y)^2$$

Difference of Two Squares

$$x^2-y^2=(x+y)(x-y)$$

General Trinomials

$$(x^2+(a+b)x+ab)=(x+a)(x+b)$$

Difference of Two Cubes

$$x^3-y^3=(x-y)(x^2+xy+y^2)$$

Sum of Two Cubes

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$

The Greatest Common Factor (GCF) is the largest factor that divides into every term in a given polynomial.

Perfect Square Trinomial Example

$$x^2+6x+9=(x+3)^2$$
$$x^2-12x+36=(x-6)^2$$

Difference of two Squares Example

$$4x^2-9y^2=(2x+3y)(2x-3y)$$

Trinomial Factoring Examples

$$x^2+7x+12=(x+3)(x+4)$$
$$x^2-5x+6=(x-2)(x-3)$$
$$x^2-4x+21=(x-7)(x+3)$$

Difference of Two Cubes Example

$$8x^3-y^3=(2x-y)(4x^2-2xy+y^2)$$

Sum of Two Cubes Example

$$x^3+27y^3=(x+3y)(x^2-3xy+9y^2)$$

Remember!

Always try to factor out the greatest common factor first! A polynomial may look like it is not factorable, but by taking out a common factor you may be able to factor it with ease.

Solving Quadratic Equations

Give a quadratic equation, $ax^2+bx+c=0$, there are two basic methods that one can use to solve for the value of x: factoring or using the quadratic formula.

Factoring

Example:

$x^2-3x-10=0$

$(x-5)(x+2)=0$ factor as much as possible

$x-5=0$ or $x+2=0$ set each factor equal to zero

$x=5$ or $x=-2$ solve each equation

Thus, the solutions are $x=5$ or $x=-2$.

Quadratic Formula

Example: $x^2-3x-10$

Recall the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}, \text{ so here}$$

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-10)}}{2(1)}=\frac{3\pm\sqrt{9+40}}{2}$$

$$x=\frac{3\pm\sqrt{49}}{2}$$

$$x=\frac{3+7}{2}=5 \text{ or } x=\frac{3-7}{2}=-2$$

The solutions are $x=5$ or $x=-2$.

Often note that the quadratic formula is used to solve an equation that cannot easily be solved by the method of factoring. When a factorable quadratic is given (as in this example), so the quadratic formula is not necessary, but the solutions do end up the same whichever method is used.

Practice Problems

Solve for x.

1. $2x^2+5x-7=0$

2. $5x^2-15x-10=0$

3. $8x^3-32x=0$

4. $x^2-7x+12=0$

5. $6x^2+18x+2=0$

6. $x^3+3x^2+2x=0$

7. $x^2-5x+4=0$

8. $5x^2-3x-6=0$

9. $9x^2-6x^3+x^4=0$

10. $5x^3-20x=0$

11. $\frac{3}{2}x^3-6x=0$

12. $-3x^2+9x+10=0$

Laws of Exponents

Assume a and b are real numbers and m and n are integers.

1. $a^m \cdot a^n = a^{m+n}$

2. $(ab)^n = a^n b^n$

3. $\dfrac{a^m}{a^n} = a^{m-n}$

4. $\left(a^m\right)^n = a^{mn}$

5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n} \quad (b \neq 0)$

Zero Exponent

If a is a nonzero real number, then

6. $a^0 = 1$

Negative Exponent

If a is a nonzero real number and n is a nonzero integer, then

7. $a^{-n} = \dfrac{1}{a^n}$

Examples using the Laws of Exponents

1. a. $x^7 \cdot x^5 = x^{12}$ **b.** $2^3 \cdot 2^2 = 2^5 = 32$

2. a. $(xy)^4 = x^4 y^4$ **b.** $(2 \cdot 5)^3 = 2^3 \cdot 5^3 = 8 \cdot 125 = 1000$

3. a. $\dfrac{x^{10}}{x^7} = x^3$ **b.** $\dfrac{4^5}{4^7} = 4^{-2} = \dfrac{1}{16}$

4. a. $\left(z^3\right)^2 = z^6$ **b.** $\left(5^2\right)^3 = 5^6 = 15625$

5. a. $\left(\dfrac{x}{y}\right)^4 = \dfrac{x^4}{y^4}$ **b.** $\left(\dfrac{2}{5}\right)^3 = \dfrac{2^3}{5^3} = \dfrac{8}{125}$

Examples of Zero Exponents

6. a. $x^0 = 1$ **b.** $12^0 = 1$

Examples of Negative Exponents

7. a. $x^{-3} = \dfrac{1}{x^3}$ **b.** $4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$

Watch out for the following common exponent mistakes!

1. Exponents applied to polynomials: remember, these need to be multiplied out!

$(a+b)^2 \neq a^2 + b^2$. The correct way is $(a+b)^2 = a^2 + 2ab + b^2$

2. Parentheses and negative signs:

a. $(-2)^4 = 16$ **b.** $-2^4 = -16$ **c.** $(-2)^3 = -8$

In part (a), the exponent is applied to the number -2; notice the even exponent makes the outcome positive. In part (b), the exponent is applied only to the number 2, so the outcome is negative. In part (c) the exponent is applied to the number -2, but since the exponent is odd the outcome is negative.

3. Negative exponents: remember, a negative exponent requires a reciprocal to make it positive !

Note: $3^{-2} \neq -3^2$. The correct way is $3^{-2} = \dfrac{1}{3^2}$.

Radicals and Rational Exponents

Definitions of $a^{1/n}$ and $\sqrt[n]{a}$

For any positive integer n,
$a^{1/n} = \sqrt[n]{a}$

Note that $\sqrt[n]{a}$ is not a real number if $a < 0$ and n is even.

Rational Exponents

Assume m and n are integers with $n > 0$

$$a^{m/n} = \left(a^{1/n}\right)^m = \left(a^m\right)^{1/n}$$

or equivalently

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

Properties of Radicals

Assume a, b, $\sqrt[n]{a}$, and $\sqrt[n]{b}$ are real numbers.

1. $\left(\sqrt[n]{a}\right)^n = \sqrt[n]{a^n} = a$

This is a special case of $a^{m/n} = \left(\sqrt[n]{a}\right)^m$.

2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{1/n} \cdot b^{1/n} = \left(a \cdot b\right)^{1/n} = \sqrt[n]{a \cdot b}$

This is a special case of the rule $\left(ab\right)^m = a^m b^m$ from the exponents section (p.1).

3. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \dfrac{a^{1/n}}{b^{1/b}} = \left(\dfrac{a}{b}\right)^{1/n} = \sqrt[n]{\dfrac{a}{b}}$ where $b \neq 0$

Examples using the Properties of Radicals

1. a. $\left(\sqrt{4}\right)^2 = \left(2\right)^2 = 4$ **b.** $\sqrt[2]{8} = \sqrt[3]{2^3} = 2$

2. $\sqrt[4]{8} \cdot \sqrt[4]{2} = \sqrt[4]{8 \cdot 2} = \sqrt[4]{16} = 2$

3. $\dfrac{\sqrt[3]{54}}{\sqrt[3]{2}} = \sqrt[3]{\dfrac{54}{2}} = \sqrt[3]{27} = 3$

Watch out for the following.

1. $\sqrt{8} = \sqrt[2]{8} = 8^{1/2}$; remember, if there is no index given it means square root.

2. $\sqrt[4]{x^3} = x^{3/4}$;when switching from radical notation to exponential notation, remember that the root goes on the bottom of the fractional exponent. Think of it like a tree; the roots are always at the bottom!

Logarithms

Logarithms are closely related to exponents. In general: $\log_b x = y$ is equivalent to $b^y = x$.

For example the equation $10^2 = 100$ can be written as $\log_{10} 100 = 2$. This is read as "log base 10 of 100 equals 2."

To solve $\log_4 16 = x$, we want to think: what power do we raise 4 to in order to give us 16? Or, if $4^x = 16$, what is x? We know $4^2 = 16$, so $x = 2$.

Logarithmic functions

For each $b > 0$, there is a function called "log-base-b" defined by $f(x) = \log_b x$ for all $x > 0$.

Properties of Logarithms

1. $\log_b (xy) = \log_b x + \log_b y$

2. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

3. $\log_b (x)^k = k \log_b x$

4. $\log_b b^x = x$

5. $b^{\log_b x} = x$

6. $\log_b 1 = 0$

Examples Using Logarithm Properties

1. $\log_{10} 20 = \log_{10}(4 \cdot 5) = \log_{10} 4 + \log_{10} 5$

2. $\log_3 \left(\frac{2}{3}\right) = \log_3 2 - \log_3 3$

3. $\log_4 \sqrt{5} = \log_4 5^{1/2} = \frac{1}{2} \log_4 5$

4. $\log_2 2^5 = 5$

5. $3^{\log_3 4} = 4$

6. $\log_5 1 = 0$

Note: The properties of logarithms can be performed in either direction. In certain cases it might be necessary to write expressions as a single logarithm.

$$\textbf{Ex: } \log_b 4 + \log_b x - 2\log_b y = \log_b \left(\frac{4x}{y^2}\right)$$

In other cases it might be necessary write single logarithm as the sum/difference of logs.

$$\textbf{Ex: } \log_b \frac{x^2}{4} = 2\log_b x - \log_b 4$$

Natural Log and Base *e*

A common base that pops up in many applications is base e, where $e \approx 2.718...$

We call $\log_e$ the natural log, which is often written as $\ln$.

The two functions $y = \ln x$ and $y = e^x$ are commonly used in the study of calculus, and it is a good idea to familiarize yourself with them. The function $y = \ln x$ has domain $x > 0$ and the function $y = e^x$ has domain all real numbers.

Trigonometry

Radians

Like degrees, radians give us a way to measure angles. One radian is the measure of the angle on the unit circle, where the arc it intercepts is equal to 1.

The circumference of a circle is given by $2\pi r$, where r is the length of the radius. By using this definition it is easy to see that there are 2π radians in a complete circle.

2π radians $= 360°$ and π radians $= 180°$

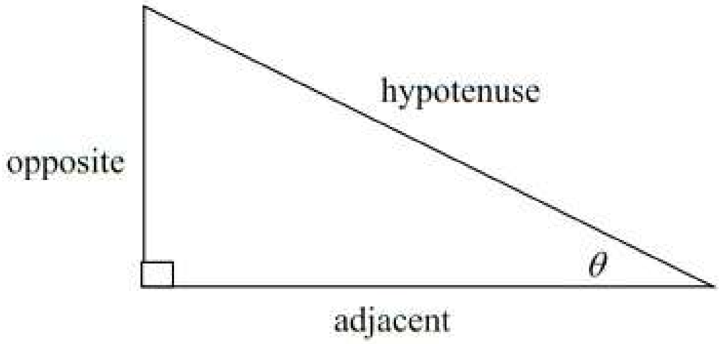
Therefore,

$1° = \dfrac{\pi}{180}$ radians and $1 \text{ radian} = \left(\dfrac{180}{\pi}\right)^{\circ}$

In calculus radians will ALWAYS be used for measuring angles.

Right Triangle Definition

For this definition we assume that $0 < \theta < \frac{\pi}{2}$.



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

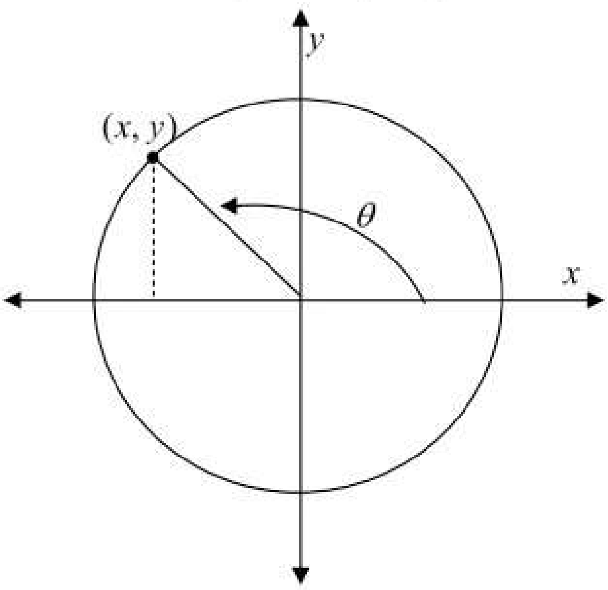
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Trick: A way to remember this is SOH CAH TOA, where SOH refers to Sine Opp Hyp, CAH refers to Cosine Adj Hyp, and TOA refers to Tangent Opp Adj.

Unit Circle Definition

For this definition, θ is any angle and the circle has a radius of 1 unit.



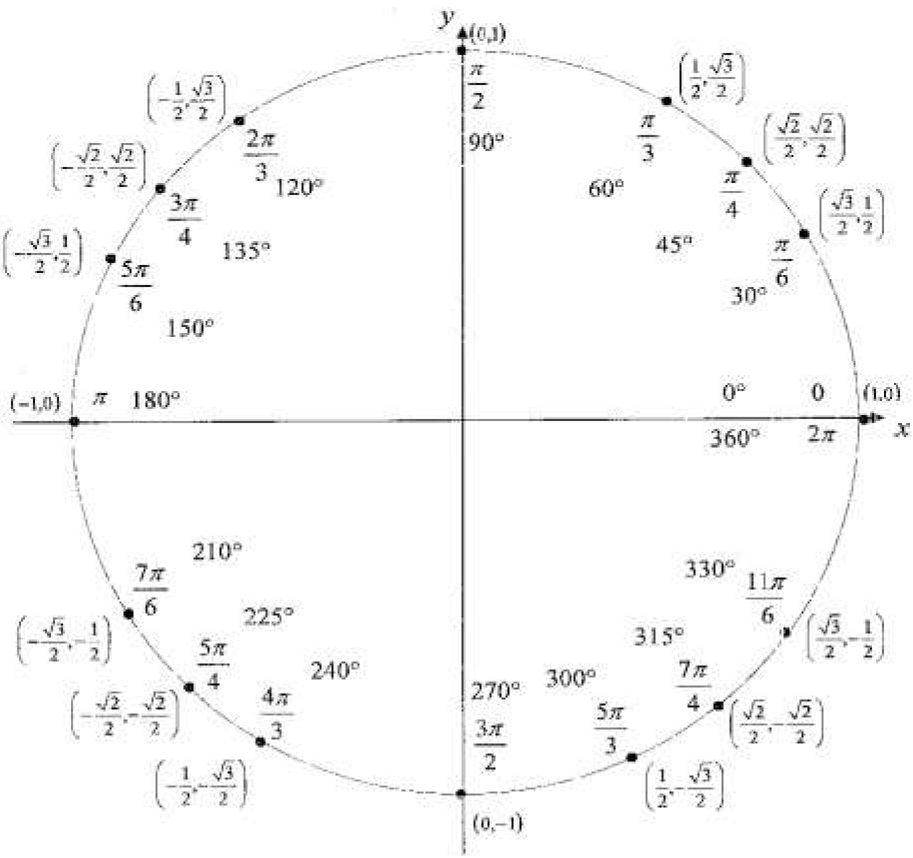
$\sin\theta = y$	$\csc\theta = \frac{1}{y}$
$\cos\theta = x$	$\sec\theta = \frac{1}{x}$
$\tan\theta = \frac{y}{x}$	$\cot\theta = \frac{x}{y}$

Note: The Right Triangle Definition and the Unit Circle Definition are equivalent for $0 \le \theta \le \frac{\pi}{2}$. Both definitions will be useful to know when studying calculus.

The Unit Circle

The unit circle can be helpful in remembering certain values of $\sin\theta$ and $\cos\theta$. The image below depicts the important values along the unit circle.

Unit Circle



For any ordered pair on the unit circle (x, y) : $\cos\theta = x$ and $\sin\theta = y$

Example

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2} \qquad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

Note: It is most helpful to know the values of sine and cosine at each of the intercepts $\left(0, \frac{\pi}{2}, \pi. \frac{3\pi}{2}, \text{ and } 2\pi\right)$ and in the first quadrant. From these values all the other values of sine and cosine can be determined using knowledge of the signs of x and y in other quadrants. For example, if you need the value of $\sin\frac{5\pi}{4}$, it is determined by the value of $\sin\frac{\pi}{4}$, with a sign adjustment, since $\frac{5\pi}{4}$ is in the 2nd quadrant, where y is negative.

Formulas and Identities

Tangent and Cotangent Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Reciprocal Identities

$\csc\theta = \frac{1}{\sin\theta}$ $\quad$ $\sin\theta = \frac{1}{\csc\theta}$

$\sec\theta = \frac{1}{\cos\theta}$ $\quad$ $\cos\theta = \frac{1}{\sec\theta}$

$\cot\theta = \frac{1}{\tan\theta}$ $\quad$ $\tan\theta = \frac{1}{\cot\theta}$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Double Angle Formulas

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2 - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

Other Identities

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Practice Problems

Use trigonometric identities to rewrite each of the following.

1. $\sqrt{1 - \sin^2\theta}$

2. $\tan^2\theta + 1$

3. $\cos x \cdot \tan x$

4. $\sin^2 x + \cos^2 x$

5. $\cos x + \sin x \tan x$

6. $\sec\theta\cos\theta$

7. $\frac{\cos\theta}{1 - \sin\theta}$

8. $\frac{\tan\theta}{1 - \sec^2\theta}$

9. $\frac{2}{\sqrt{1 + \tan^2 x}}$

Inverse Trigonometric Functions

In general the inverse function "undoes" whatever the function does. For example, if $f^{-1}(x)$ is the inverse function of $f(x)$ and $f(1)=7$ then $f^{-1}(7)=1$.

As another example, consider the function $f(x)=2x+4$. This function takes the input value x, multiplies it by 2, and then adds 4 to get a y-value $f(x)$. The inverse of this function, $f^{-1}(x)=\frac{x-4}{2}$, will do just the opposite to the input value, subtracting 4 from it and then dividing it by 2.

The easiest way to find the inverse of a given function is to switch the variables and solve for y.
Ex. $y=f(x)=2x+4$. Find $f^{-1}(x)$.

$$\begin{aligned} x &= 2y+4 && \text{(switching } x \text{ and } y\text{)} \\ x-4 &= 2y && \\ \frac{x-4}{2} &= y && \text{(solving for y)} \\ f^{-1}(x) &= \frac{x-4}{2} && \text{is the inverse function} \end{aligned}$$

Inverse functions have the property that $f\left(f^{-1}(x)\right)=f^{-1}\left(f(x)\right)=x$; is, that composing a function with its inverse results in simply x. Note in our example:

$$f\left(f^{-1}(x)\right)=2\left(\frac{x-4}{2}\right)+4=x-4+4=x$$

and

$$f^{-1}\left(f(x)\right)=\frac{(2x+4)-4}{2}=\frac{2x}{2}=x$$

Inverse Trigonometric Functions (Continued)

Finding inverse trigonometric functions using the method above would be a bit difficult. For example consider the function $y = \sin x$, which means x is an angle whose sine is y. If we switch the variables we get $x = \sin y$, but we cannot use algebra to solve for y as we did in the example above. This is why we define special names for inverse trigonometric functions.

$y = \arcsin x$ is the inverse function of $y = \sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. In other words, $\arcsin x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.

Ex 1: Find the value of $\arccos\left(\frac{\sqrt{2}}{2}\right)$.

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad \text{(using the values from the unit circle)}$$

$$\text{so, } \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

Ex 2: Find the value of $\arctan\left(\sqrt{3}\right)$.

$$\tan x = \frac{\sin x}{\cos x} \qquad \text{(using identities)}$$

$$\frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \qquad \text{(using values from the unit circle)}$$

$$\text{so, } \tan\frac{\pi}{3} = \sqrt{3}$$

$$\text{so, } \arctan\sqrt{3} = \frac{\pi}{3}$$

We can define inverse functions for all the trigonometric functions. Here are the 3 most commonly used inverse trigonometric functions.

Function	$y = \arcsin x$	$y = \arccos x$	$y = \arctan x$
Domain	$-1 \le x \le 1$	$-1 \le x \le 1$	$-\infty < x < \infty$
Range	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	$0 \le y \le \pi$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$