Unit 3-Review SOLUTIONS

Multiple Choice: Select the best answer for each of the following

1. Sin(50°) cos(30°) + cos(50°)sin(30°) is equivalent to : ____ —

a. sin80° b. cos80° c. sin20° d. cos20°

2. $\dfrac{\tan\left(\frac{5\pi}{9}\right) - \tan\left(\frac{\pi}{6}\right)}{1+\tan\left(\frac{5\pi}{9}\right)\tan\left(\frac{\pi}{6}\right)}$ is equivalent to : ____

a. $\tan\left(\frac{13\pi}{18}\right)$ b. $\tan\left(\frac{\pi}{6}\right)$ c. $\tan\left(\frac{7\pi}{18}\right)$ d. $\tan\left(\frac{\pi}{6}\right)$ ____

$\tan\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$
$= \dfrac{\tan\left(\frac{\pi}{3}\right)-\tan\left(\frac{\pi}{4}\right)}{1+\tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)}$
$= \dfrac{\sqrt{3}-1}{1+\sqrt{3}} \cdot \dfrac{1-\sqrt{3}}{1-\sqrt{3}}$
$= \dfrac{\sqrt{3}-3-1+\sqrt{3}}{1-3}$
$= \dfrac{-4+2\sqrt{3}}{-2}$
$= 2-\sqrt{3}$

3. The exact value of $\tan\left(\frac{\pi}{12}\right)$ is :

a. $2-\sqrt{3}$ b. $2+\sqrt{3}$ c. $3-\sqrt{3}$ d. $3+\sqrt{3}$

$\cos(2\theta)$
$= 1-2\sin^2\theta$
$= 1-2\left(\frac{4}{5}\right)^2$
$= 1-2\left(\frac{16}{25}\right)$
$= \frac{25}{25}-\frac{32}{25}$
$= -7/25$

4. Given $\sin(\theta) = \frac{4}{5}$ and terminates in the second quadrant, the value of $\cos(2\theta)$ is ____

a. $\frac{-24}{25}$ b. $\frac{-7}{25}$ c. $\frac{-16}{25}$ d. $\frac{-3}{25}$

$\theta = \frac{\pi}{3}$

$\sin 2\theta = 2\sin\theta\cos\theta$
$= 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)$
$= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$
$= \frac{\sqrt{3}}{2}$

5. Given $\sin(\theta) = \frac{\sqrt{3}}{2}$ and terminates in quadrant one, the value of $\sin 2\theta$ is : ____

a. $\sqrt{3}$ b. $\frac{\sqrt{3}}{2}$ c. $\frac{\sqrt{3}}{4}$ d. $\frac{\sqrt{3}}{8}$

6. $\sin(-\theta)$ is equivalent to : ____

a. $-\sin(\theta)$ b. $\sin(\theta)$ c. $\cos(\theta)$ d. $-\cos(\theta)$

7. The exact value of $\sin\left(\frac{5\pi}{12}\right)$ is : ____

$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{2}-\frac{\pi}{12}\right)$
$= \cos\left(\frac{\pi}{12}\right)$
$= \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$
$= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$
$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$
$= \frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$

a. $\frac{\sqrt{6}+\sqrt{2}}{4}$ b. $\frac{\sqrt{6}+\sqrt{2}}{2}$ c. $\frac{\sqrt{6}-\sqrt{2}}{4}$ d. $\frac{\sqrt{6}-\sqrt{2}}{2}$

8. The expression $\cos(70°)$ is equivalent to : ____

$\cos(70°) = \cos(90° - 20°)$
$= \sin(20°)$

a. $\sin(70°)$ b. $\cos(20°)$ c. $\sec(20°)$ d. $\sin(20°)$

9. The expression $\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{5\pi}{18}\right) - \sin\left(\frac{5\pi}{18}\right)\cos\left(\frac{2\pi}{3}\right)$ is equal to : ____

$\sin\left(\frac{2\pi}{3}-\frac{5\pi}{18}\right) = \sin\left(\frac{7\pi}{18}\right)$

a. $\cos\left(\frac{7\pi}{18}\right)$ b. $\cos(\pi)$ c. $\sin\left(-\frac{7\pi}{18}\right)$ d. $\sin\left(\frac{7\pi}{18}\right)$

10. The expression $\dfrac{2\tan\left(\frac{\pi}{3}\right)}{1-\tan^2\left(\frac{\pi}{3}\right)}$ is equal to : ____

$\tan\left(2\cdot\frac{\pi}{3}\right) = \tan\left(\frac{2\pi}{3}\right) = \tan\left(\pi-\frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$

a. 1 b. $\sqrt{3}$ c. $-\sqrt{3}$ d. -1

11. The expression $\cot(A+B)$ is equivalent to : ____

$\frac{1}{\tan(A+B)}$

$= \dfrac{1}{\frac{\tan A+\tan B}{1-\tan A\tan B}}$

$= \dfrac{1-\tan A\tan B}{\tan A+\tan B}$

a. $\cot(A)+\cot(B)$ b. $\cot(A)-\cot(B)$

c. $\dfrac{1-\tan(A)\tan(B)}{\tan(A)+\tan(B)}$ d. $\dfrac{\cot(A)+\cot(B)}{1-\cot(A)\cot(B)}$

12. The expression $\cos\left(\frac{\pi}{12}\right)$ is equal to : ____

$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$

$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$

$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{2}}{4} + \frac{\sqrt{3}\cdot\sqrt{2}}{4}$

$= \frac{\sqrt{2}(1+\sqrt{3})}{4}\cdot\frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{2(1+\sqrt{3})}{4\sqrt{2}}$

$= \frac{1+\sqrt{3}}{2\sqrt{2}}$

a. $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ b. $\dfrac{\sqrt{3}+1}{-2\sqrt{2}}$ c. $\dfrac{\sqrt{3}-1}{-2\sqrt{2}}$

d. $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$

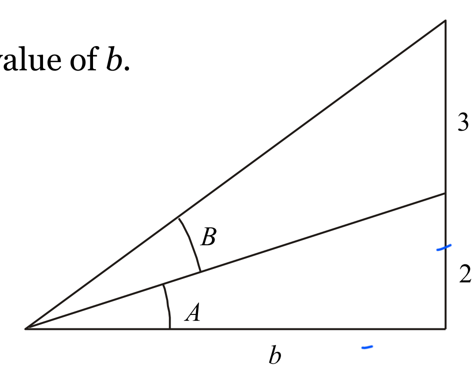
13. The expression $\cos(A+B)-\cos(A-B)$ is equivalent to : ____

$= \cancel{\cos A\cos B} - \sin A\sin B - (\cancel{\cos A\cos B} + \sin A\sin B) = -2\sin A\sin B$

a. 2 cos(A) cos (B) b. $-2\sin(A)\sin(B)$ c. $2\sin(A)\sin(B)$ d. 0

Full Solution

1. In the given diagram $\angle A = \angle B$,find the exact value of *b*.



$\tan(A+B) = \frac{5}{b}$ $\tan A = \frac{2}{b}$

$\tan(2A) = \frac{5}{b}$

$\dfrac{2\tan A}{1-\tan^2 A} = \frac{5}{b}$

$\dfrac{2\left(\frac{2}{b}\right)}{1-\left(\frac{2}{b}\right)^2} = \frac{5}{b}$

$\dfrac{\frac{4}{b}}{1-\frac{4}{b^2}} = \frac{5}{b}$

$\dfrac{4/b}{\frac{b^2-4}{b^2}} = \frac{5}{b}$

$\frac{4b}{b^2-4} = \frac{5}{b}$

$4b^2 = 5(b^2-4)$

$20 = b^2$

$2\sqrt{5} = b$

2. Express $12\cos\theta + 5\sin\theta = R\cos(\theta - \alpha)$, where $R > 0$ and α is acute.

$$R\cos(\theta-\alpha) = R[\cos\theta\cos\alpha + \sin\theta\sin\alpha]$$
$$= (R\cos\alpha)\cos\theta + (R\sin\alpha)\sin\theta$$

Since $12\cos\theta + 5\sin\theta = (R\cos\alpha)\cos\theta + (R\sin\alpha)\sin\theta$:

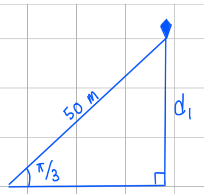
$$12 = R\cos\alpha \qquad 5 = R\sin\alpha$$
$$12^2 = (R\cos\alpha)^2 \qquad 5^2 = (R\sin\alpha)^2$$
$$144 = R^2\cos^2\alpha \text{ — ①} \qquad 25 = R^2\sin^2\alpha \text{ — ②}$$

① + ②:
$$144 = R^2\cos^2\alpha$$
$$+)\ 25 = R^2\sin^2\alpha$$
$$169 = R^2\cos^2\alpha + R^2\sin^2\alpha$$
$$169 = R^2(\cos^2\alpha + \sin^2\alpha)$$
$$169 = R^2$$
$$\pm 13 = R$$
$$13 = R$$
$(R \neq -13 \text{ since } R>0)$

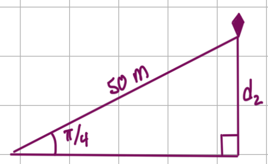
Sub. $R=13$: $5 = 13\sin\alpha$
$$\frac{5}{13} = \sin\alpha$$
$$\sin^{-1}\left(\frac{5}{13}\right) = \alpha$$
$$0.39 \text{ rads} \doteq \alpha$$

$\therefore 12\cos\theta + 5\sin\theta \doteq 13\cos(\theta - 0.39)$

3. Jimmy is flying his kite at the end of a 50 m string. The string makes an angle of $\frac{\pi}{3}$ with the ground. The wind increases, and the kite flies higher until the string makes an angle of $\frac{\pi}{4}$ with the ground. Determine the **vertical displacement** of the kite as an **exact** value.



$$\sin\left(\frac{\pi}{3}\right) = \frac{d_1}{50}$$
$$\frac{\sqrt{3}}{2} = \frac{d_1}{50}$$
$$25\sqrt{3} = d_1$$



$$\sin\left(\frac{\pi}{4}\right) = \frac{d_2}{50}$$
$$\frac{\sqrt{2}}{2} = \frac{d_2}{50}$$
$$25\sqrt{2} = d_2$$

Vertical displacement $= d_1 - d_2$
$$= 25\sqrt{3} - 25\sqrt{2}$$
$$= 25(\sqrt{3} - \sqrt{2})$$

$\therefore$ The vertical displacement is $25(\sqrt{3}-\sqrt{2})$ m.

4. Fill in the blanks.

a) Evaluate the exact value of $\cot\left(\frac{3\pi}{2}\right)$ undefined

b) Evaluate $\csc\left(\frac{\pi}{5}\right)$ to 4 decimal places 1.7013

c) Evaluate the exact value of $\sec\left(\frac{7\pi}{6}\right)$ $-\frac{2}{\sqrt{3}}$

$$\to = \sec\left(\pi + \frac{\pi}{6}\right)$$
$$= -\sec\left(\frac{\pi}{6}\right)$$
$$= -\frac{2}{\sqrt{3}}$$

5. Evaluate the exact value of the following expressions:

a) $\cos\left(\frac{7\pi}{12}\right)$.

$= \cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$

$= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$

$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$

$= \frac{\sqrt{2} - \sqrt{6}}{4}$

b) $\cos\left(\frac{5\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)$

$= \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= \left[\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \cancel{\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)}\right] + \left[\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cancel{\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)}\right]$

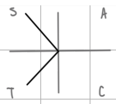
$= 2\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right)$

$= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{6}}{2}$

6. If $\cos\theta = -\frac{3}{8}$, $\pi \le \theta \le 3\pi$, determine the exact value of $\cos\left(\frac{\theta}{2}\right)$.

$\pi \le \theta \le 3\pi \implies \frac{\pi}{2} \le \frac{\theta}{2} \le \frac{3\pi}{2}$



double angle identity

$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

$-\frac{3}{8} = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

$\frac{5}{8} = 2\cos^2\left(\theta/2\right)$

$\frac{5}{16} = \cos^2\left(\theta/2\right)$

$\pm\frac{\sqrt{5}}{4} = \cos\left(\frac{\theta}{2}\right)$

($+\frac{\sqrt{5}}{4}$ is inadmissible since $\frac{\theta}{2}$ is in Q2 or Q3)

$\boxed{\therefore \cos\left(\frac{\theta}{2}\right) = -\frac{\sqrt{5}}{4}}$

7. Prove that $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi - x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{3\pi}{2} + x\right)} = \sin^4(x) - \sin^2(x)$.

$LS = \dfrac{\sin(x)[-\cos(x)][\cancel{-\tan(x)}]}{[-\csc(x)][-\sec(x)][\cancel{-\tan(x)}]}$

$= -\sin(x)\cos(x) \div \frac{1}{\sin(x)\cos(x)}$

$= -\sin^2(x)\cos^2(x)$

$= -\sin^2(x)\left[1 - \sin^2(x)\right]$

$= -\sin^2(x) + \sin^4(x)$

$= \sin^4(x) - \sin^2(x)$

$= RS$

8. Prove the identity

a) $\csc^4(x) - \cot^4(x) = \dfrac{1+\cos^2(x)}{1-\cos^2(x)}$.

$$LS = [\csc^2(x) + \cot^2(x)][\csc^2(x) - \cot^2(x)]$$
$$= \left[\frac{1}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)}\right]\left[\frac{1}{\sin^2(x)} - \frac{\cos^2(x)}{\sin^2(x)}\right]$$
$$= \frac{1+\cos^2(x)}{\sin^2(x)} \cdot \frac{1-\cos^2(x)}{\sin^2(x)}$$
$$= \frac{(1+\cos^2(x))\cancel{(1-\cos^2(x))}}{(1-\cos^2(x))^{\cancel{2}}}$$
$$= \frac{1+\cos^2(x)}{1-\cos^2(x)}$$
$$= RS$$

b) $\sin^4(x) + \cos^4(x) = \dfrac{3}{4} + \dfrac{1}{4}\cos(4x)$

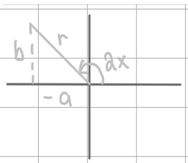
$(a+b)^2 = a^2 + 2ab + b^2$
$\Rightarrow (a+b)^2 - 2ab = a^2 + b^2$

$$LS = [\sin^2(x) + \cos^2(x)]^2 - 2\sin^2(x)\cos^2(x)$$
$$= 1 - 2\sin^2(x)\cos^2(x)$$
$$= 1 - 2(\sin(x)\cos(x))^2$$
$$= 1 - 2\left[\tfrac{1}{2}\sin(2x)\right]^2$$
$$= 1 - 2 \cdot \tfrac{1}{4}\sin^2(2x)$$
$$= 1 - \tfrac{1}{2}\sin^2(2x)$$

$\cos(2A) = 1 - 2\sin^2(A)$
$2\sin^2(A) = 1 - \cos(2A)$
$\sin^2(A) = \frac{1}{2} - \frac{1}{2}\cos(2A)$

$$= 1 - \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 4x\right)$$
$$= 1 - \tfrac{1}{4} + \tfrac{1}{4}\cos(4x)$$
$$= \tfrac{3}{4} + \tfrac{1}{4}\cos(4x)$$
$$= RS$$

10. If $\tan(2x) = -\dfrac{b}{a}$, $\dfrac{\pi}{2} \le 2x \le \pi$, then determine an expression for $\sin x \cos x$ in terms of a and b.

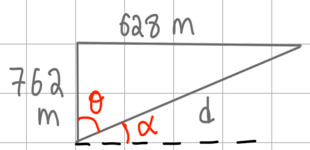


$r^2 = a^2 + b^2$
$r = \sqrt{a^2+b^2}$

$$\sin(2x) = \frac{b}{\sqrt{a^2+b^2}}$$
$$2\sin(x)\cos(x) = \frac{b}{\sqrt{a^2+b^2}}$$
$$\sin(x)\cos(x) = \frac{b}{2\sqrt{a^2+b^2}}$$

11 . A cable car rises 762 m as it moves a horizontal distance of 628 m.

a) How long is the ride?



$$d^2 = (628)^2 + (762)^2$$
$$d^2 = 975028$$
$$d \doteq 987.4\text{ m}$$

$\therefore$ The ride is 987.4 m long

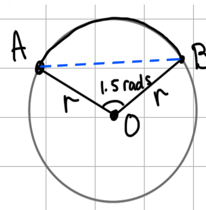
b) What is the angle of inclination of the cable to the nearest degree?

$$\tan\theta = \frac{628}{762}$$
$$\theta \doteq 0.689 \text{ rads}$$

$$\alpha \doteq \frac{\pi}{2} - 0.689 \doteq 0.88 \text{ rads}$$

$\therefore$ The angle of inclination is about 0.88 rads

12. An arc of a circle, centre O, subtends an angle of 1.5 radians at the centre. Determine the ratio of the length of arc AB to the length of line segment AB.



By cosine law:

$(AB)^2 = r^2 + r^2 - 2r^2[\cos(1.5)]$

$(AB)^2 = 2r^2 - 2r^2[\cos(1.5)]$

$(AB)^2 = 2r^2[1 - \cos(1.5)]$

$AB = \sqrt{2r^2[1-\cos(1.5)]}$

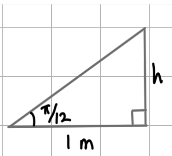
$AB \doteq 1.36r$

$\text{Arc } AB = \theta r$

$= 1.5r$

$\therefore \frac{\text{Arc } AB}{AB} \doteq \frac{1.5r}{1.36r} \doteq 1.1$

13. A skateboard ramp is built with an incline angle of $\frac{\pi}{12}$. If the base of the ramp is 1m in length, determine the exact height of the ramp.



$\tan\left(\frac{\pi}{12}\right) = \frac{h}{1}$

$\tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = h$

$\frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)} = h$

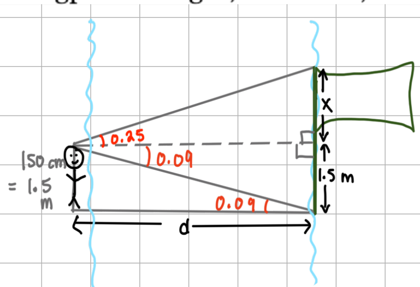
$\frac{\sqrt{3} - 1}{1 + \sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}} = h$

$\frac{\sqrt{3} - 3 - 1 + \sqrt{3}}{1 - 3} = h$

$\frac{-4 + 2\sqrt{3}}{-2} = h$

$2 - \sqrt{3} = h$ $\therefore$ The height of the ramp is $2-\sqrt{3}$ m.

14. A 150-cm tall person stands on the bank of a narrow river and observes a flagpole on the opposite bank. If the angle of elevation to the top of the flagpole is 0.25 and the angle of depression to the bottom of the flagpole is 0.09, determine the flagpole's height, in metres, to two decimal places.



$\tan(0.09) = \frac{1.5}{d}$

$d = \frac{1.5}{\tan(0.09)}$

$d \doteq 16.621$ m

$\tan(0.25) \doteq \frac{x}{16.621}$

$4.24 \doteq x$

$\therefore$ The height is $1.5 + 4.24 \doteq 5.74$ m

15. If $\cos(x)+\sin(x)=k$, for what value(s) of k does $\sin(x)\cos(x)=\frac{1}{2}$?

$[\cos(x) + \sin(x)]^2 = (k)^2$

$\cos^2(x) + 2\sin(x)\cos(x) + \sin^2(x) = k^2$

$\cos^2(x) + \sin^2(x) + 2\left(\frac{1}{2}\right) = k^2$

$1 + 1 = k^2$

$2 = k^2$

$\boxed{\pm\sqrt{2} = k}$