L2 – 4.4 Compound Angle Formulas
MHF4U

Compound angle: *an angle that is created by adding or subtracting two or more angles.*

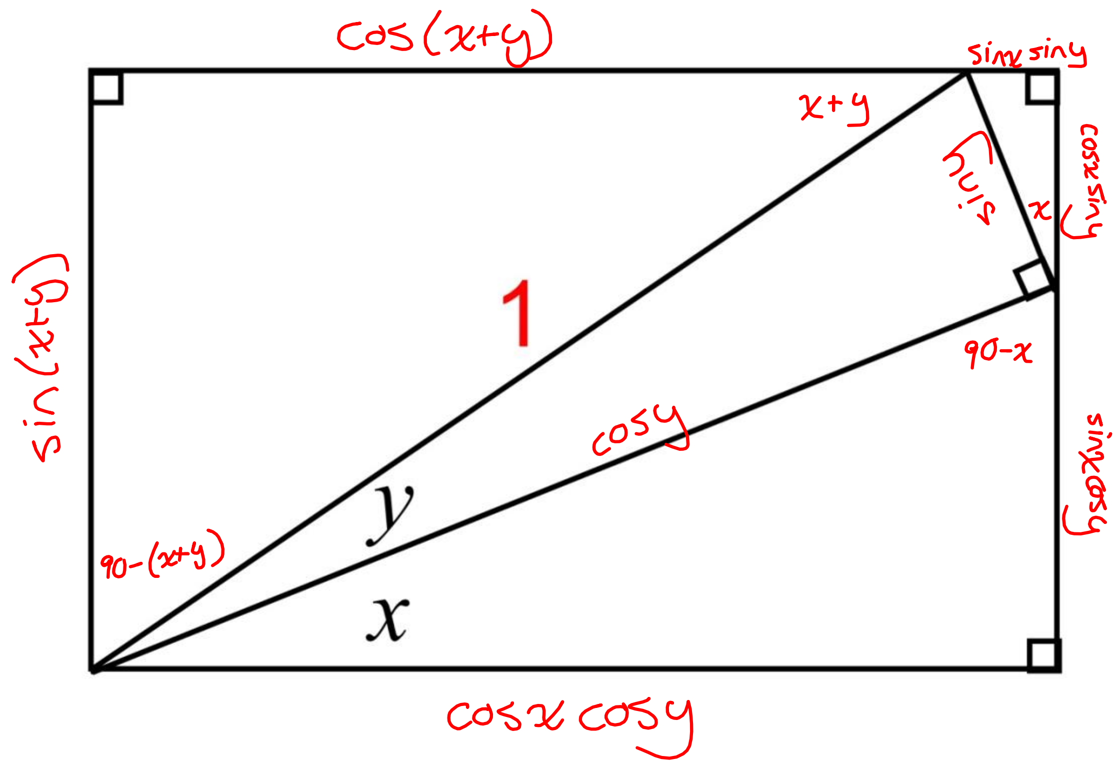
Part 1: Proof of $\cos(x-y)$

Normal algebra rules do not apply:

$$\cos(x-y) \neq \cos x - \cos y$$

So what does $\cos(x-y) = ?$

Using the diagram below, label all angles and sides:



$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

Part 2: Proofs of other compound angle formulas

Even/Odd Properties	
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$

Example 1: Prove $\cos(x - y) = \cos x \cos y + \sin x \sin y$

LS	RS
$= \cos(x - y)$	$= \cos x \cos y + \sin x \sin y$
$= \cos[x + (-y)]$	
$= \cos x \cos(-y) - \sin x \sin(-y)$	
$= \cos x \cos y - \sin x\,(-\sin y)$	
$= \cos x \cos y + \sin x \sin y$	

LS = RS

Example 2:

a) Prove $\sin(x - y) = \sin x \cos y - \cos x \sin y$

LS	RS
$= \sin(x - y)$	$= \sin x \cos y - \cos x \sin y$
$= \sin x \cos(-y) + \cos x \sin(-y)$	
$= \sin x \cos y + \cos x\,(-\sin y)$	
$= \sin x \cos y - \cos x \sin y$	

LS = RS

Compound Angle Formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

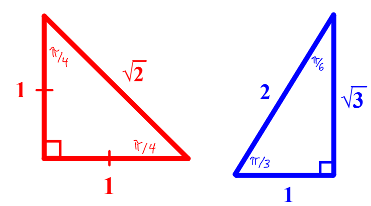
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Part 3: Determine Exact Trig Ratios for Angles other than Special Angles



By expressing an angle as a sum or difference of angles in the special triangles, exact values of other angles can be determined.

Example 3: Use compound angle formulas to determine exact values for

a) $\sin\frac{\pi}{12}$

$$\sin\frac{\pi}{12} = \sin\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$$

$$= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

b) $\tan\left(-\frac{5\pi}{12}\right)$

$$\tan\left(-\frac{5\pi}{12}\right) = -\tan\left(\frac{5\pi}{12}\right)$$

$$= -\tan\left(\frac{2\pi}{12} + \frac{3\pi}{12}\right)$$

$$= -\frac{\tan\left(\frac{2\pi}{12}\right) + \tan\left(\frac{3\pi}{12}\right)}{1 - \tan\left(\frac{2\pi}{12}\right)\tan\left(\frac{3\pi}{12}\right)}$$

$$= -\frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{4}\right)}$$

$$= -\frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}}$$

$$= -\frac{\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}}}$$

$$= -\frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$$= -\frac{1+\sqrt{3}}{\sqrt{3}-1}$$

Part 4: Use Compound Angle Formulas to Simplify Trig Expressions

Example 4: Simplify the following expression

$$\cos\frac{7\pi}{12}\cos\frac{5\pi}{12}+\sin\frac{7\pi}{12}\sin\frac{5\pi}{12}$$

$$=\cos\left(\frac{7\pi}{12}-\frac{5\pi}{12}\right)$$

$$=\cos\frac{2\pi}{12}$$

$$=\cos\frac{\pi}{6}$$

$$=\frac{\sqrt{3}}{2}$$

Part 5: Application

Example 5: Evaluate $\sin(a+b)$, where a and b are both angles in the second quadrant; given $\sin a=\frac{3}{5}$ and $\sin b=\frac{5}{13}$

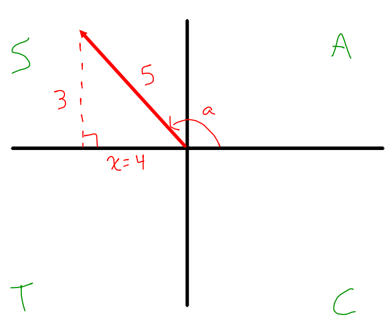
Start by drawing both terminal arms in the second quadrant and solving for the third side.

$$\sin(a+b)=\sin a\cos b+\cos a\sin b$$

$$=\left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\left(\frac{5}{13}\right)$$

$$=-\frac{36}{65}-\frac{20}{65}$$

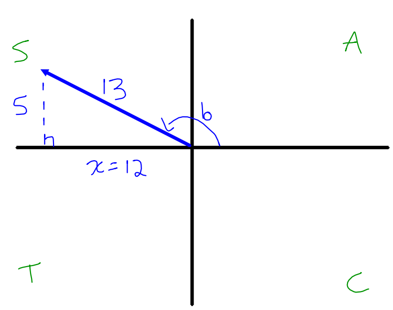
$$=-\frac{56}{65}$$



$x^2+3^2=5^2$

$x^2=16$

$x=4$



$x^2+5^2=13^2$

$x^2=144$

$x=12$