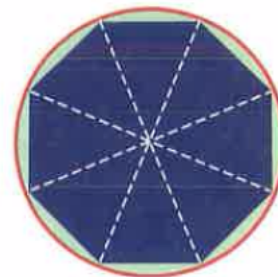
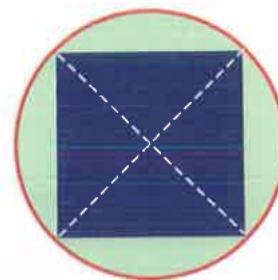


L4 – Limits

MHF4U

The Greek mathematician Archimedes developed a proof of the formula for the area of a circle, $A = \pi r^2$. His method, known as the “method of exhaustion,” involved calculating the area of regular polygons inscribed in the circle. The area of the polygon provides an estimate of the area of the circle. As Archimedes increased the number of sides of the polygon, its shape came closer and closer to the shape of a circle. An octagon provides a much better estimate of the area of a circle than a square does. What about a myriagon, a polygon with 10 000 sides? What happens to the estimate as the number of sides approaches infinity?

Archimedes’ method of finding the area of a circle is based on the concept of a LIMIT. The circle is the limiting shape of the polygon. As the number of sides gets larger, the area of the polygon approaches its limit, the shape of a circle, without ever becoming an actual circle. You used a similar strategy to estimate the instantaneous rate of change of a function. Your estimate became increasingly accurate as the interval between two points was made smaller. Using limits, the interval can be made infinitely small, approaching zero. As this happens, the slope of the secant approaches its limiting value, the slope of the tangent.



Section 1: Intro to Limits

Example 1: Complete the following table:

x	2	2.5	2.9	2.99	3	3.01	3.1	3.5	4
$y = x^2 - 2$	2	4.25	6.41	6.9401	7	7.0601	7.61	10.25	14

Find the limit as x approaches 3 from the left

$$\lim_{x \rightarrow 3^-} (x^2 - 2) = 7$$

Find the limit as x approaches 3 from the right

$$\lim_{x \rightarrow 3^+} (x^2 - 2) = 7$$

$$\text{Therefore, } \lim_{x \rightarrow 3} (x^2 - 2) = 7$$

$\lim_{x \rightarrow a} f(x)$ exists if the following 3 criteria are met

1. $\lim_{x \rightarrow a^-} f(x)$ exists
2. $\lim_{x \rightarrow a^+} f(x)$ exists
3. $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$

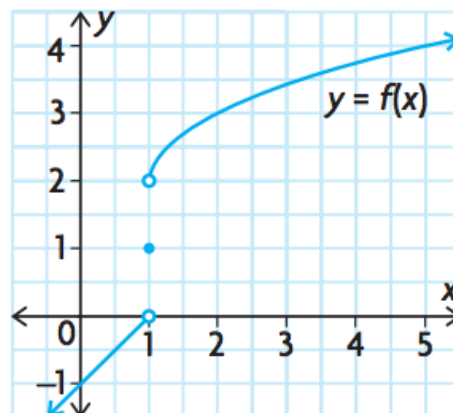
Section 2: Find limits from a graph

Example 2: Given the following graph of the piecewise function $f(x) = \begin{cases} x - 1, & \text{if } x < 1 \\ 1, & \text{if } x = 1 \\ 2 + \sqrt{x - 1}, & \text{if } x > 1 \end{cases}$

$$\lim_{x \rightarrow 1^-} f(x) = 0$$

$$\lim_{x \rightarrow 1^+} f(x) = 2$$

$\therefore \lim_{x \rightarrow 1} f(x)$ does not exist since $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$

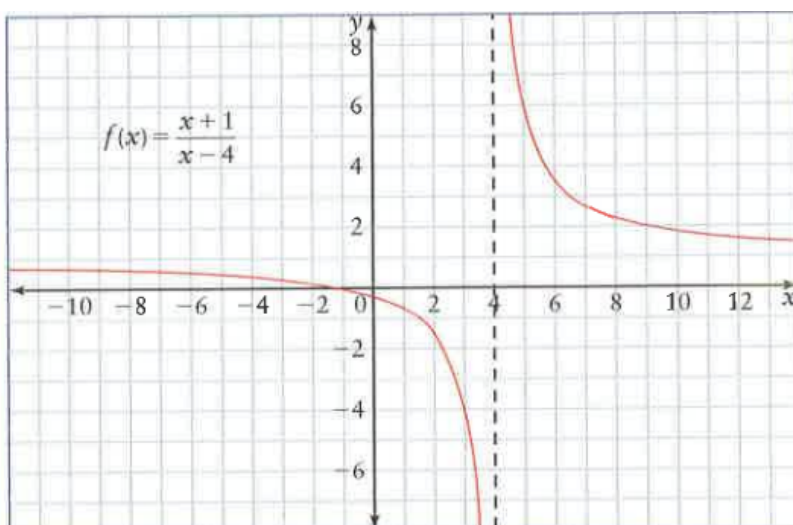


Example 3: Use the graph of $f(x) = \frac{x+1}{x-4}$ to find:

$$\lim_{x \rightarrow 4^-} f(x) = -\infty$$

$$\lim_{x \rightarrow 4^+} f(x) = \infty$$

$\therefore \lim_{x \rightarrow 4} f(x)$ does not exist



$$\lim_{x \rightarrow \infty} f(x) = 1$$

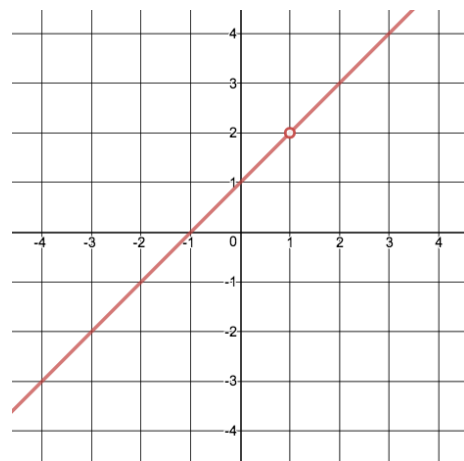
$$\lim_{x \rightarrow -\infty} f(x) = 1$$

Note: If numerator and denominator are same degree, quotient of leading coefficients indicate where the horizontal asymptote will be.

Section 3: Limit of a Function from Graph (indeterminate form)

Example 4: Analyze the equation, table, and graph of $f(x) = \frac{x^2-1}{x-1}$ to find $\lim_{x \rightarrow 1} f(x)$

x approaches 1 from the left \rightarrow						\leftarrow x approaches 1 from the right					
x	0	0.5	0.9	0.99	0.999	1	1.001	1.01	1.1	1.5	2
$\frac{x^2-1}{x-1}$	1	1.5	1.9	1.99	1.999	undefined	2.001	2.01	2.1	2.5	3
$f(x) = \frac{x^2-1}{x-1}$ approaches 2 from below \rightarrow						$\leftarrow f(x) = \frac{x^2-1}{x-1}$ approaches 2 from above					



This table and graph suggests that

$$\lim_{x \rightarrow 1^-} f(x) = 2$$

$$\lim_{x \rightarrow 1^+} f(x) = 2$$

$$\therefore \lim_{x \rightarrow 1} f(x) = 2$$

However, when plugging in $x = 1$ in to the equation, we get $\frac{0}{0}$ which is called the indeterminate form. The function is not defined at this point. However, it is not necessary for the function to be defined at a point for its limit to exist. Try factoring to see if it has a removable discontinuity:

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \rightarrow 1} x + 1$$

$$= 1 + 1$$

$$= 2$$

Section 4: Limit of a Function from an Equation (indeterminate form)

Example 5: Find the following limits.

$$\text{a) } \lim_{x \rightarrow -1} \frac{x^2 - 4}{x^2 + 3}$$

$$= \frac{(-1)^2 - 4}{(-1)^2 + 3}$$

$$= \frac{-3}{4}$$

$$\text{b) } \lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x - 3}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(x+1)}{x-3}$$

$$= \lim_{x \rightarrow 3} x + 1$$

$$= 4$$