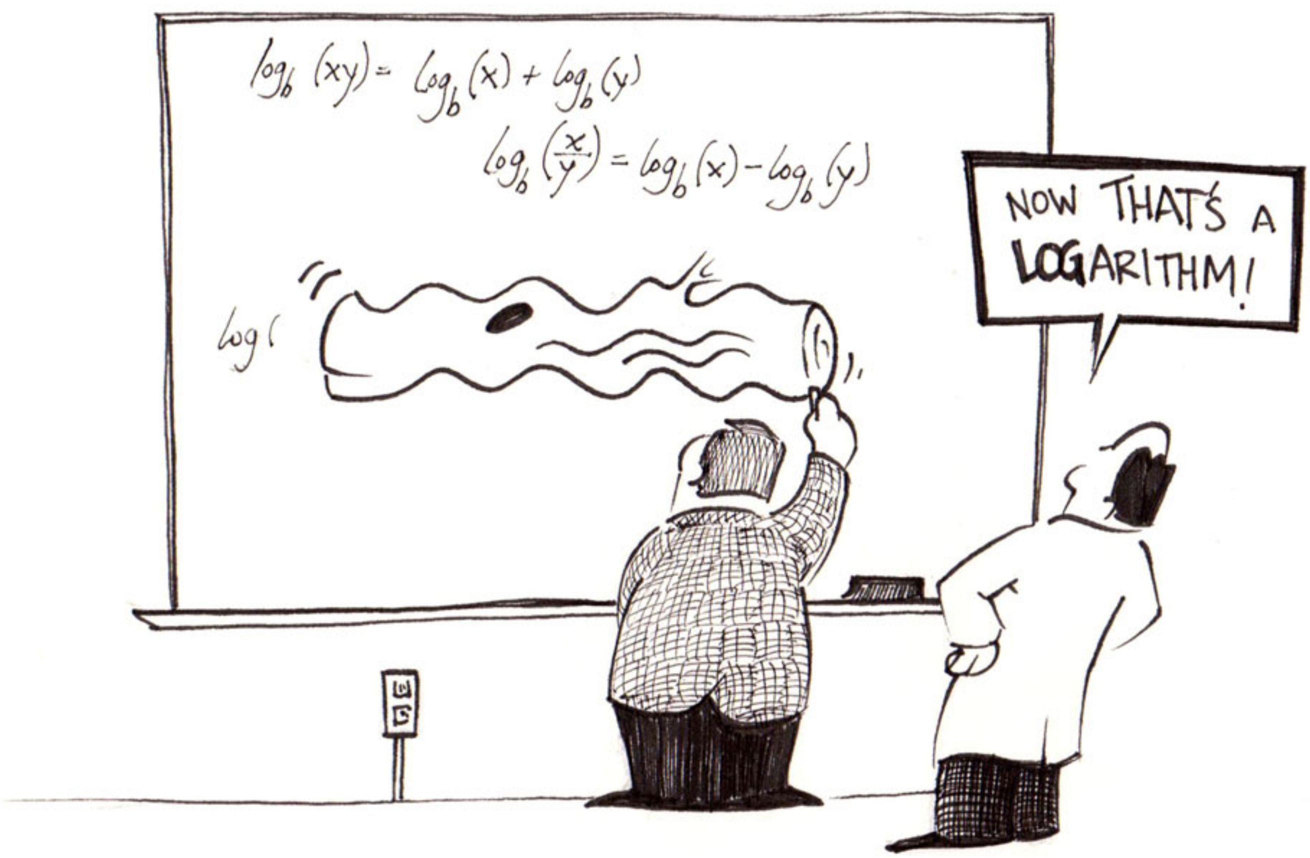
UNIT 3

Chapter 6/7- Logarithmic and Exponential Functions

Lesson Package

MHF4U



Chapter 6/7 Outline

Unit Goal: By the end of this unit, you will be able to demonstrate an understanding of the relationship between exponential and logarithmic expressions. You will also be able to solve exponential and logarithmic equations.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Log as Inverse	- recognize the operation of finding the logarithm to be the inverse operation of exponentiation - evaluate simple logarithmic expressions - understand that the logarithm of a number to a given base is the exponent to which the base must be raised to get the number	A1.1, 1.2, 1.3, 2.1, 2.2
L2	Power Law of Logarithms	- use laws of logarithms to simplify expressions - understand change of base formula	A1.4
L3	Product and Quotient Laws of Logarithms	- use laws of logarithms to simplify expressions	A1.4
L4	Solving Exponential Equations	- recognize equivalent algebraic expressions - solve exponential equations	A3.1, 3.2
L5	Solving Logarithmic Equations	- solve logarithmic equations	A3.3
L5	Applications of Logarithms	- Solve problems arising from real world applications involving exponential and logarithmic equations	A3.4

Assessments	F/A/O	Ministry Code	P/O/C	KTAC
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
Quiz – Log Rules	F		P	
PreTest Review	F/A		P	
Test – Log and Exponential Funcitons	O	A1.1, 1.2, 1.3, 1.4 A2.1, 2.2 A3.1, 3.2, 3.3, 3.4	P	K(21%), T(34%), A(10%), C(34%)

L1 – 6.1/6.2 – Intro to Logarithms and Review of Exponentials
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In this section you will learn about how a logarithmic function is the inverse of an exponential function. You will also learn how to express exponential equations in logarithmic form.

Part 1: Review of Exponential Functions

Equation: $\boldsymbol{y = a(b)^x}$

$a =$ initial amount

$b =$ growth $(b > 1)$ or decay $(0 < b < 1)$ factor

$y =$ future amount

$x =$ number of times a has increased or decreased

To calculate x, use the equation: $x = \frac{total\ time}{time\ it\ takes\ for\ one\ growth\ or\ decay\ period}$

Example 1: An insect colony has a current population of 50 insects. Its population doubles every 3 days.

a) What is the population after 12 days?

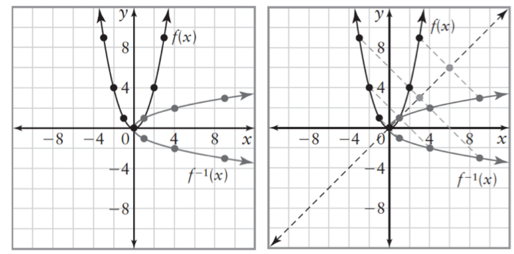
b) How long until the population reaches 25 600?

Part 2: Review of Inverse Functions

Inverse of a function:

· The inverse of a function *f* is denoted as f^{-1}
· The function and its inverse have the property that if f(a) = b, then $f^{-1}(b) = a$
· So if $f(5) = 13$, then $f^{-1}(13) = 5$
· More simply put: The inverse of a function has all the same points as the original function, except that the *x*'s and *y*'s have been reversed.

The **graph** of $f^{-1}(x)$ is the graph of $f(x)$ reflected in the line $y = x$. This is true for all functions and their inverses.



Example 2: Determine the equation of the inverse of the function $f(x) = 3(x-5)^2 + 1$

Algebraic Method for finding the inverse:

1. Replace $f(x)$ with "y"
2. Switch the x and y variables
3. Isolate for y
4. replace y with $f^{-1}(x)$

Equation of inverse:

Part 3: Review of Exponent Laws

Name	Rule
Product Rule	$x^a \cdot x^b =$
Quotient Rule	$\frac{x^a}{x^b} =$
Power of a Power Rule	$(x^a)^b =$
Negative Exponent Rule	$x^{-a} =$
Exponent of Zero	$x^0 =$

Part 4: Inverse of an Exponential Function

Example 3:

a) Find the equation of the inverse of $f(x) = 2^x$.

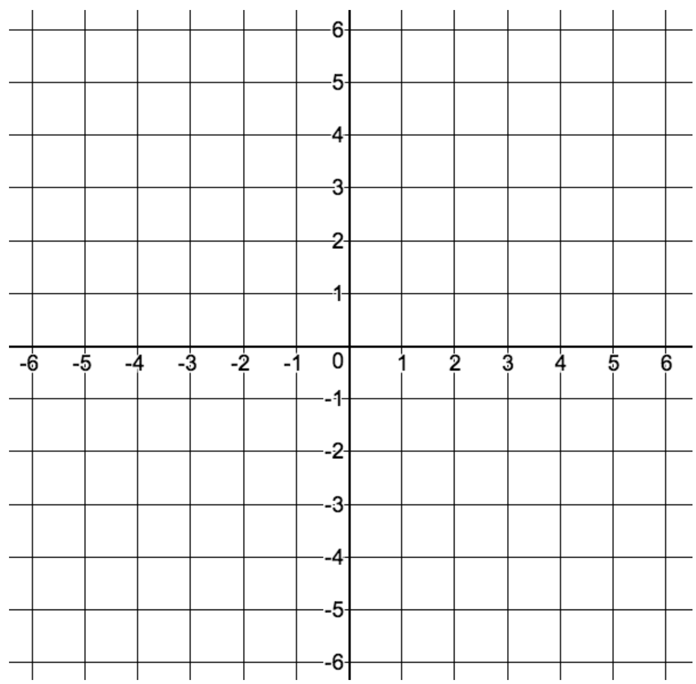
This step uses the 'change of base' formula that we will cover later in the unit.

$$\log_b m = \frac{\log m}{\log b}$$

b) Graph the both $f(x)$ and $f^{-1}(x)$.

$\boldsymbol{f(x) = 2^x}$	
$\boldsymbol{x}$	$\boldsymbol{y}$

$\boldsymbol{f^{-1}(x) = log_2 x}$	
$\boldsymbol{x}$	$\boldsymbol{y}$



Note: just swap x and y coordinates to get key points for the inverse of a function. The graph should appear to be a reflection across the line $y = x$.

c) Complete the chart of key properties for both functions

$\boldsymbol{y = 2^x}$	$\boldsymbol{y = \log_2 x}$
x-int:	x-int:
y-int:	y-int:
Domain:	Domain:
Range:	Range:
Asymptote:	Asymptote:

Part 5: What is a Logarithmic Function?

The logarithmic function is the _______________ of the exponential function with the same base.

> The **logarithmic function** is defined as $y = \log_b x$, or y equals the logarithm of x to the base b.
>
> The function is defined only for ____________________
>
> In this notation, ____ is the exponent to which the base, ____, must be raised to give the value of ____.

In other words, the solution to a logarithm is always an _______________.

The logarithmic function is most useful for solving for unknown ________________

________________________________ are logarithms with a base of 10. It is not necessary to write the base for common logarithms: $\log x$ means the same as $\log_{10} x$

Part 6: Writing Equivalent Exponential and Logarithmic Expressions

Exponential equations can be written in logarithmic form, and vice versa

$y = b^x \rightarrow$

$y = \log_b x \rightarrow$

Example 4: Rewrite each equation in logarithmic form

a) $16 = 2^4$ **b)** $m = n^3$ **c)** $3^{-2} = \frac{1}{9}$

Example 5: Write each logarithmic equation in exponential form

a) $\log_4 64 = 3$ **b)** $y = \log x$

Note: because there is no base written, this is understood to be the common logarithm of x.

Part 7: Evaluate a Logarithm

Example 6: Evaluate each logarithm without a calculator

Rule: if $x^a = x^b$, then $a = b$

a) $y = \log_3 81$

Rule: $\log_a(a^b) = b$

a) $y = \log_4 64$

Note: *either of the rules presented above are appropriate to use for evaluating logarithmic expressions*

b) $y = \log\left(\frac{1}{100}\right)$

c) $y = \log_2\left(\frac{1}{8}\right)$

L2 – 6.4 – Power Law of Logarithms
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Part 1: Solving for an Unknown Exponent

Example 1: Suppose you invest \$100 in an account that pays 5% interest, compounded annually. The amount, A, in dollars, in the account after any given time, t, in years, is given by $A = 100(1.05)^t$. How long will it take for the amount in this account to double?

In this example, we used the power law of logarithms to help solve for an unknown exponent.

Power Law of Logarithms:

$$\boldsymbol{\log_b x^n = n \log_b x}\,,\, b > 0, b \neq 1, x > 0$$

Proof of Power Law of Logarithms:

Let $w = \log_b x$

Write in exponential form

Raise both sides to the exponent of n

Apply power law of exponents

Write as a logarithmic expression

Substitute $w = \log_b x$

Part 2: Practice the Power Law of Logarithms

Example 2: Evaluate each of the following

a) $\log_3 9^4$

Method 1: Simplify and Evaluate using rules from last lesson

Rule: $\log_a(a^b) = b$

Method 2: Use Power Law of Logarithms

Rule: $\log_b x^n = n \log_b x$

b) $\log_2 8^5$

c) $\log_5 \sqrt{125}$

Part 3: Change of Base Formula

Thinking back to example 1, we had the equation:

$2 = 1.05^t$

We could have written this in logarithmic form as $\log_{1.05} 2 = t$, but unfortunately, there is no easy way to change 2 to a power with base 1.05 and you can't just type on your calculator to evaluate because most scientific calculators can only evaluate logarithms in base 10. So we used the power law of logarithms instead.

Any time you want to evaluate a logarithm that is not base 10, such as $\log_{1.05} 2$, you can use the ___________________________________:

To calculate a logarithm with any base, express in terms of common logarithms use the **change of base formula:**

Using this formula, we could determine that $\log_{1.05} 2 =$, which is exactly what we ended up with by using the power law of logarithms.

Part 4: Evaluate Logarithms with Various Bases

Example 3: Evaluate, correct to three decimal places

a) $\log_5 17$

b) $\log_{\frac{1}{2}} 10$

Example 4: Solve for y in the equation $100 = 2^y$

OR

L3 – 7.3 – Product and Quotient Laws of Logarithms
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Part 1: Proof of Product Law of Logarithms

Let $x = \log_b m$ and $y = \log_b n$

Written in exponential form:

Part 2: Summary of Log Rules

Power Law of Logarithms	for $b > 0, b \neq 1, x > 0$
Product Law of Logarithms	for $b > 0, b \neq 1, m > 0, n > 0$
Quotient Law of Logarithms	for $b > 0, b \neq 1, m > 0, n > 0$
Change of Base Formula	for $m > 0, b > 0, b \neq 1$
Exponential to Logarithmic	
Logarithmic to Exponential	
Other useful tips	

Part 3: Practice Using Log Rules

Example 1: Write as a single logarithm

a) $\log_5 6 + \log_5 8 - \log_5 16$

b) $\log x + \log y + \log(3x) - \log y$

Started by collecting like terms. Must have same base and argument.

Can't use power law because the exponent 2 applies only to x, not to $3x$.

c) $\frac{\log_2 7}{\log_2 5}$

Used change of base formula.

d) $\log 12 - 3\log 2 + 2\log 3$

Example 2: Write as a single logarithm and then evaluate

a) $\log_8 4 + \log_8 16$

b) $\log_3 405 - \log_3 5$

c) $2\log 5 + \frac{1}{2}\log 16$

Example 3: Write the Logarithm as a Sum or Difference of Logarithms

a) $\log_3(xy)$ **b)** $\log 20$ **c)** $\log(ab^2c)$

Example 4: Simplify the following algebraic expressions

a) $\log\left(\frac{\sqrt{x}}{x^2}\right)$ **b)** $\log\left(\sqrt{x}\right)^3 + \log x^2 - \log\sqrt{x}$ **c)** $\log(2x-2) - \log(x^2-1)$

L4 – 7.1/7.2 – Solving Exponential Equations
MHF4U

Part 1: Changing the Base of Powers

Exponential functions can be written in many different ways. It is often useful to express an exponential expression using a different base than the one that is given.

Example 1: Express each of the following in terms of a power with a base of 2.

a) 8 **b)** 4^3 **c)** $\sqrt{16}\times\left(\sqrt[5]{32}\right)^3$ **d)** 12

Part d) shows that any positive number can be expressed as a power of any other positive number.

Example 2: Solve each equation by getting a common base

Remember: if $x^a = x^b$, then $a = b$

a) $4^{x+5} = 64^x$ **b)** $4^{2x} = 8^{x-3}$

Part 2: Solving Exponential Equations

When you have powers in your equation with different bases and it is difficult to write with the same base, it may be easier to solve by taking the _______________ of both sides and applying the ______________ of logarithms to remove the variable from the __________________.

Example 3: Solve each equation

a) $4^{2x-1} = 3^{x+2}$

Take log of both sides
Use power law of logarithms
Use distributive property to expand
Move variable terms to one side
Common factor
Isolate the variable

b) $2^{x+1} = 3^{x-1}$

Part 3: Applying the Quadratic Formula

Sometimes there is no obvious method of solving an exponential equation. If you notice two powers with the same base and an exponent of x, there may be a hidden quadratic.

Example 4: Solve the following equation

Multiply both sides by 2^x

Distribute

Rearrange in to standard form $ax^2 + bx + c = 0$

Solve using quadratic formula

Don't forget to simplify the radical expression

Now substitute 2^x back in for k and solve

Case 1

Case 2

<u>Part 4: Application Question</u>

Remember:

Equation: $\boldsymbol{y = a(b)^x}$

$a =$ initial amount

$b =$ growth $(b > 1)$ or decay $(0 < b < 1)$ factor

$y =$ future amount

$x =$ number of times a has increased or decreased

To calculate x, use the equation: $x = \frac{total\ time}{time\ it\ takes\ for\ one\ growth\ or\ decay\ period}$

Example 5: A bacteria culture doubles every 15 minutes. How long will it take for a culture of 20 bacteria to grow to a population of 163 840?

Example 6: One minute after a 100-mg sample of Polonium-218 is placed into a nuclear chamber, only 80-mg remains. What is the half-life of polonium-218?

L5 – 7.4 – Solving Logarithmic Equations
MHF4U

Part 1: Try and Solve a Logarithmic Equation

Solve the equation $\log(x+5) = 2\log(x-1)$

Hint: apply the power law of logarithms to the right side of the equation

Note:

If $\log_m a = \log_m b$, then $a = b$.

Part 1: Solve Simple Logarithmic Equations

To complete this lesson, you will need to remember how to change from logarithmic to exponential:

$y = \log_b x$ →

Example 2: Solve each of the following equations

a) $\log(x+4) = 1$

Method 1: re-write in exponential form

Method 1: express both sides as a logarithm of the same base

b) $\log_5(2x-3)=2$

Part 2: Apply Factoring Strategies to Solve Equations

Example 3: Solve each equation and reject any extraneous roots

a) $\log(x-1)-1=-\log(x+2)$

b) $\log \sqrt[3]{x^2 + 48x} = \frac{2}{3}$

c) $\log_3 x - \log_3(x - 4) = 2$

Example 4: If $\log_a b = 3$, then use log rules to find the value of…

a) $\log_a ab^2$

b) $\log_b a$

Hint: need to change the base

$\boldsymbol{\log_b m =}$

L6 – 6.5 – Applications of Logarithms in Physical Sciences
MHF4U

Part 1: Review of Solving Logarithmic Equations

Example 1: Solve for x in the following equation

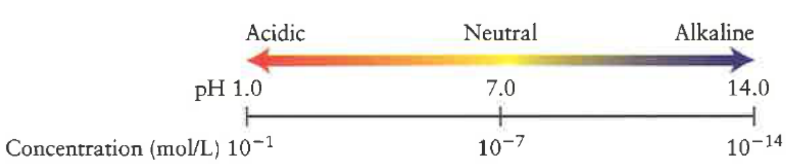
$$\log_2(x-6) = 4 - \log_2 x$$

Part 2: pH Scale

The pH scale is used to measure the acidity or alkalinity of a chemical solution. It is defined as:

$$\boldsymbol{pH = -\log[H^+]}$$

where $[H^+]$ is the concentration of hydronium ions, measured in moles per liter.



pH = 0	battery acid, strong hydrofluoric acid
pH = 1	hydrochloric acid secreted by stomach lining
pH = 2	lemon juice, gastric acid, vinegar
pH = 3	grapefruit, orange juice, soda
pH = 4	tomato juice, acid rain
pH = 5	soft drinking water, black coffee
pH = 6	urine, saliva
pH = 7	"pure" water
pH = 8	seawater
pH = 9	baking soda
pH = 10	Great Salt Lake, milk of magnesia
pH = 11	ammonia solution
pH = 12	soapy water
pH = 13	bleaches, oven cleaner
pH = 14	liquid drain cleaner

Example 2: Answer the following pH scale questions

a) Tomato juice has a hydronium ion concentration of approximately 0.0001 mol/L. What is its pH?

b) Blood has a hydronium ion concentration of approximately 4×10^{-7} mol/L. Is blood acidic or alkaline?

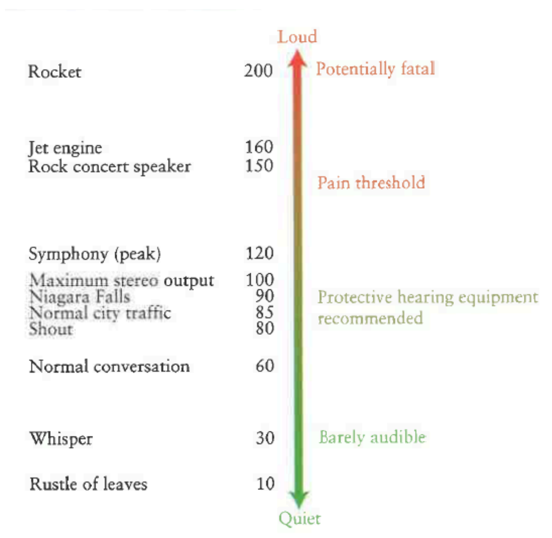
c) Orange juice has a pH of approximately 3. What is the concentration of hydronium ions in orange juice?

Part 3: Decibel Scale

Some common sound levels are indicated on the decibel scale shown. The difference in sound levels, in decibels, can be found using the equation:

$$\boldsymbol{\beta_2 - \beta_1 = 10 \log\left(\frac{I_2}{I_1}\right)}$$

where, $\beta_2 - \beta_1$ is the difference in sound levels, in decibels, and $\frac{I_2}{I_1}$ is the ratio of their sound intensities, where I is measured in watts per square meter (W/m^2)



Example 3: Answer the following questions about decibels

a) How many times as intense as a whisper is the sound of a normal conversation

b) The sound level in normal city traffic is approximately 85 dB. The sound level while riding a snowmobile is about 32 times as intense. What is the sound level while riding a snowmobile, in decibels?

Part 4: Richter Scale

The magnitude, M, of an earthquake is measured using the Richter scale, which is defined as:

$$\boldsymbol{M = \log\left(\frac{I}{I_0}\right)}$$

where I is the intensity of the earthquake being measured and I_0 is the intensity of a standard, low-level earthquake.

Example 4: Answer the following questions about the Richter Scale

a) How many times as intense as a standard earthquake is an earthquake measuring 2.4 on the Richter scale?

b) What is the magnitude of an earthquake 1000 times as intense as a standard earthquake?

L7 – 6.3 Transformations of Exponential and Logarithmic Functions
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Part 1: Properties of Exponential Functions

General Equation: $y = a(b)^{k(x-d)} + c$ where the base function is $y = b^x$

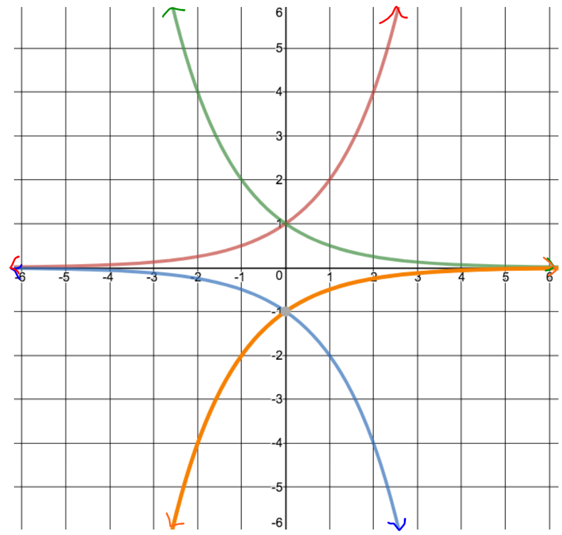
There are 4 possible shapes for an exponential function

1) $a > 0$ and $b > 1$ (ex. $y = 2^x$)

2) $a > 0$ and $0 < b < 1$ (ex. $y = \left(\frac{1}{2}\right)^x$)

3) $a < 0$ and $b > 1$ (ex. $y = -1(2)^x$)

4) $a < 0$ and $0 < b < 1$ (ex. $y = -1\left(\frac{1}{2}\right)^x$)



To graph the base function $y = b^x$, Find the following key features:

- Horizontal asymptote
  - Starts at $y = 0$ and can be shifted by c
- $y - intercept$
  - set $x = 0$ and solve
- At least one other point to be sure of shape
  - Common to choose $x = 1$ and solve for y

You can then use transformational properties of a, k, d, and c to graph a transformed function

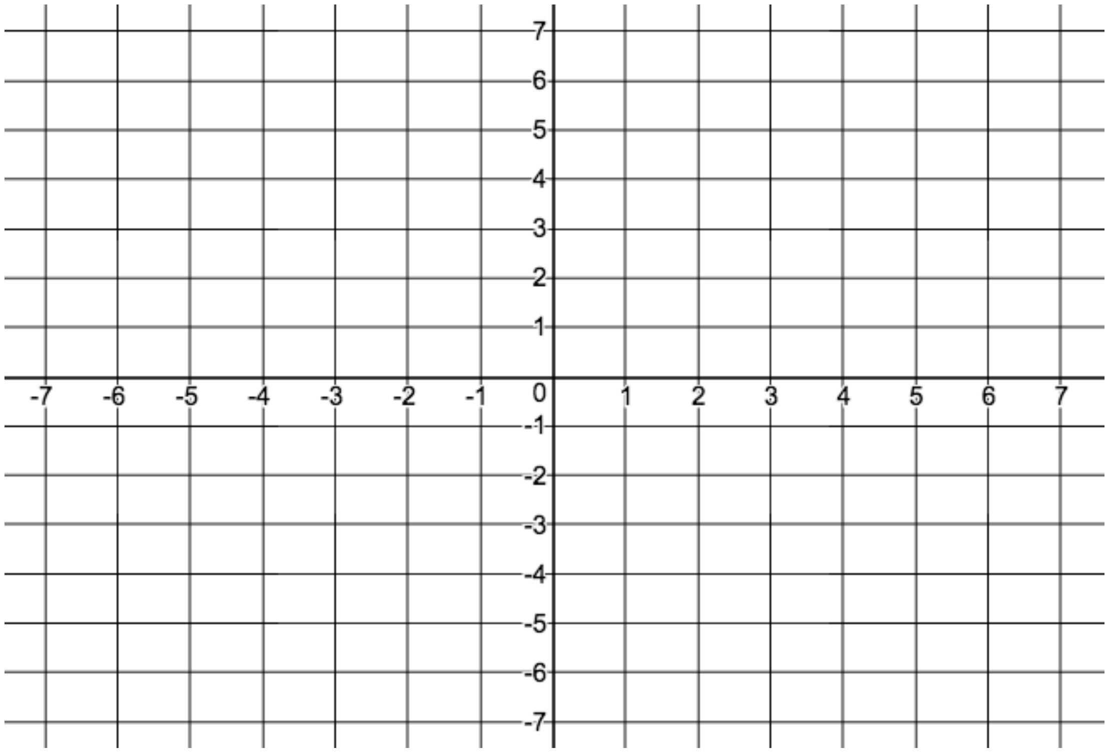
Part 2: Transformations of Exponential Functions

Example 1: Sketch the graph of $f(x) = 2(3)^{x+4} - 5$ and $g(x) = -3^{\frac{1}{2}x} + 4$ using transformations

$y = 3^x$	
x	y

$f(x) = 2(3)^{x+4} - 5$	

$g(x) = -3^{\frac{1}{2}x} + 4$	

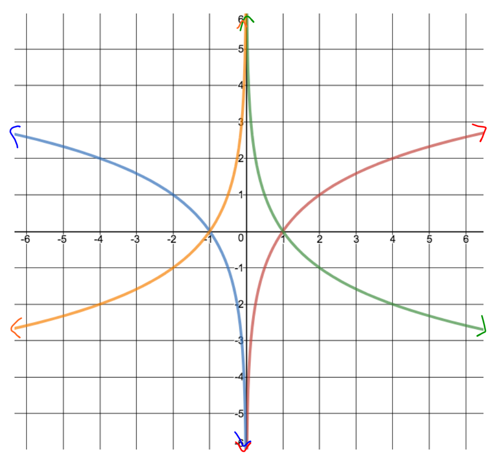


<u>Part 3: Properties of Logarithmic Functions</u>

General Equation: $y = a\log_b[k(x-d)] + c$ where the base function is $y = \log_b x$

Remember that $y = \log_b x$ is the inverse of the exponential function $y = b^x$

There are 4 possible shapes for a logarithmic function



1) $k > 0$ and $b > 1$ (ex. $y = \log_2(x)$)
2) $k > 0$ and $0 < b < 1$ (ex. $y = \log_{0.5}(x)$)
3) $k < 0$ and $b > 1$ (ex. $y = \log_2(-x)$)
4) $k < 0$ and $0 < b < 1$ (ex. $y = \log_{0.5}(-x)$)

To graph the base function $y = \log_b x$, Find the following key features:

- Vertical asymptote
  - Starts at $x = 0$ and can be shifted by d
- $x - intercept$
  - set $y = 0$ and solve
- At least one other point to be sure of shape
  - Common to choose $y = 1$ and solve for x

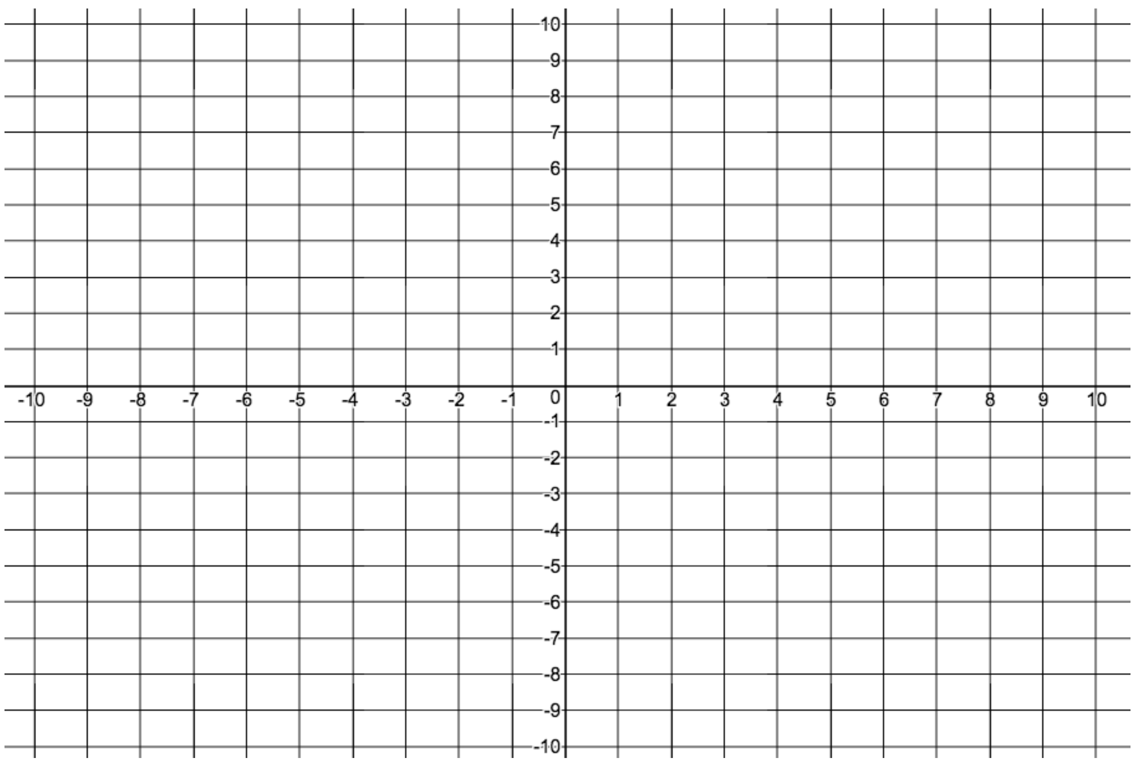
Part 4: Transformations of Logarithmic Functions

Example 2: Sketch the graph of $f(x) = -4\log_3(x) + 2$ and $g(x) = \log_3[-(x+2)] - 4$ using transformations

$y = \log_3(x)$	
x	y

$f(x) = -4\log_3(x) + 2$	

$g(x) = \log_3[-(x+2)] - 4$	



L8 – The Natural Logarithm
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Part 1: What is $'e'$?

Example 1: Suppose you invest $1 at 100% interest for 1 year at various compounding levels. What is the highest amount of money you can have after 1 year?

Note: the formula used for compound interest of $1 at 100% interest annually compounded n times during the year is:

$$A = 1\left(1 + \frac{1}{n}\right)^n$$

Compounding Level, n	Amount, A in dollars
Annualy (once a year)	
Semi-annually (2-times)	
Quarterly (4-times)	
Monthly (12-times)	
Daily (365-times)	
Secondly (31 536 000-times)	
Continuously (1 000 000 000-times)	

Properties of e:

- $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ ____________________________
- e is an _______________ number, similar to π. They are non-terminating and non-repeating.
- $\log_e x$ is known as the _________________________ and can be written as ______
- Many naturally occurring phenomena can be modelled using base-e exponential and logarithmic functions.
- $\log_e e = \ln e =$ ___

Part 2: Reminder of Log Rules

Power Law of Logarithms	$\log_b x^n = n \log_b x$ for $b > 0, b \neq 1, x > 0$
Product Law of Logarithms	$\log_b(mn) = \log_b m + \log_b n$ for $b > 0, b \neq 1, m > 0, n > 0$
Quotient Law of Logarithms	$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$ for $b > 0, b \neq 1, m > 0, n > 0$
Change of Base Formula	$\log_b m = \frac{\log m}{\log b}, m > 0, b > 0, b \neq 1$
Exponential to Logarithmic	$y = b^x \rightarrow \mathrm{x} = \log_b y$
Logarithmic to Exponential	$y = \log_b x \rightarrow x = b^y$
Other useful tips	$\log_a(a^b) = b$ $\quad\quad$ $\log a = \log_{10} a$ $\quad\quad$ $\log_b b = 1$

Part 2: Solving Problems Involving *e*

Example 2: Evaluate each of the following

a) e^3

b) $\ln 10$

c) $\ln e$

Example 3: Solve each of the following equations

a) $20 = 3e^x$

b) $e^{1-2x} = 55$

c) $2\ln(x-3)-7=3$

d) $\ln(4e^x)=2$

Part 3: Graphing Functions Involving *e*

Example 4: Graph the functions $y=e^x$ and $y=\ln x$

Note: $y=\ln x$ is the inverse of $y=e^x$

$y=e^x$	
x	y

$y=\ln x$	
x	y