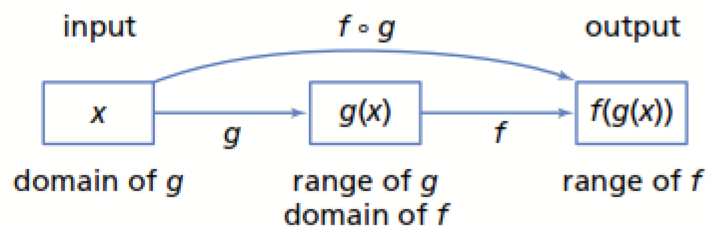
6.4 Compositions of Functions

Let $f(x)$ and $g(x)$ represent two functions. We have learned how to

- **Add** two functions $(f+g)(x)$
- **Subtract** two functions $(f-g)(x)$
- **Multiply** two functions $(fg)(x)$
- **Divide** two functions $\frac{f}{g}(x)$
- Find the **Inverse** of a function $f^{-1}(x)$

Now, we need to learn how to put a function into another function. Say we want to put $g(x)$ into $f(x)$.
This is written as $f(g(x))$ or $(f \circ g)(x)$.
$f(g(x))$ is an example of a **<u>Composite Function</u>**



Always remember to put the inner function in the outer function.

- If we want to put $f(x)$ into $g(x)$ the composition function is written as ____________

Example. Let $f(x) = 4 + \sqrt{x^2 + 4x}$ and $g(x) = x - 1$. Determine the equation for each composition Function.

a) $f(g(x))$

b) $g(f(x))$

c) $g(g(x))$

Find $g^{-1}(x)$

d) $g(g^{-1}(x))$

e) $g^{-1}(g(x))$

Method for obtaining inverses

This method utilizes the fact that

$$f\left(f^{-1}(x)\right)=x\ ,\quad x\in D_{f^{-1}}$$
$$f^{-1}\left(f(x)\right)=x\ ,\quad x\in D_{f}$$

This may seem confusing, but it is a result of the most basic principles of function inverses. Think of a function as some sort of process that we put x through, and it outputs some term. A functions inverse is simply the reverse process. So if we put x through a process, f, then put it through the reverse process, f^{-1}, we end up with just x again.

The methodical approach can be summarized in the following steps:

1. Replace x with $f^{-1}(x)$ on both sides of the equation.

At this point, you should have a $f(f^{-1}(x))$on one side of your equation and then a function of $f^{-1}(x)$ on the other.

2. Substitute in $f(f^{-1}(x)) = x$.

If we take the inverse of a function on the output of the function, f(x), we are left with the input. Therefore, if we take $f(f^{-1}(x))$, we are left with the original input, which is x. Therefore we know $f(f^{-1}(x)) = x$. Remember that, since the two expressions are equal, we can just replace $f(f^{-1}(x))$ with x.

3. Solve for $f^{-1}(x)$ **in terms of x.**

At this point, we have an x on one side of the equation, and then a function of $f^{-1}(x)$ on the other side. Try to solve for $f^{-1}(x)$ by getting it by itself on one side of the equation. This will tell you what the inverse function is.

Examples: Find the inverses for the following functions.

a) $f(x) = x + 4$

b) $f(x) = (x + 4)^2 - 5$

Example. If $(f\ o\ g)(x) = 8x^2 - 10x + 15$ and $f(x) - 2x + 9$ find $\mathrm{g}(x)$.

Example. Use the graphs of y = f(x) and y = g(x) to find each of the following compositions.

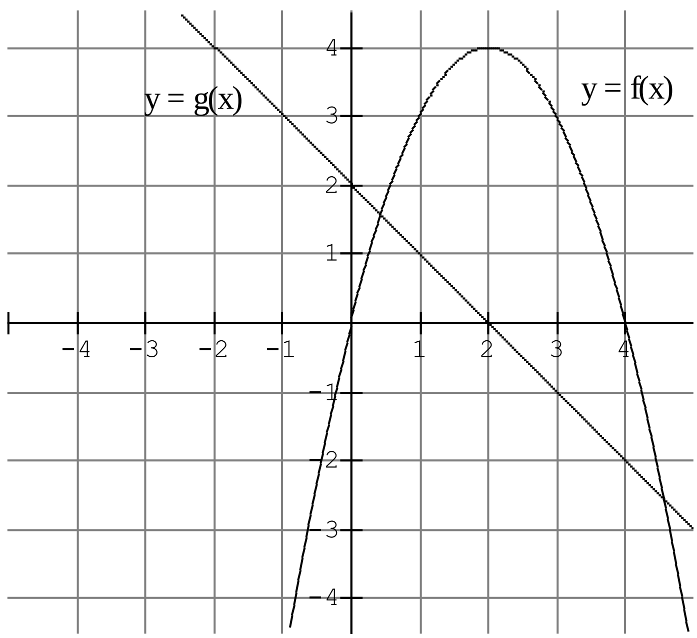
a. $f(g(3))$

b. $g(f(3))$

c. $f(g(0))$

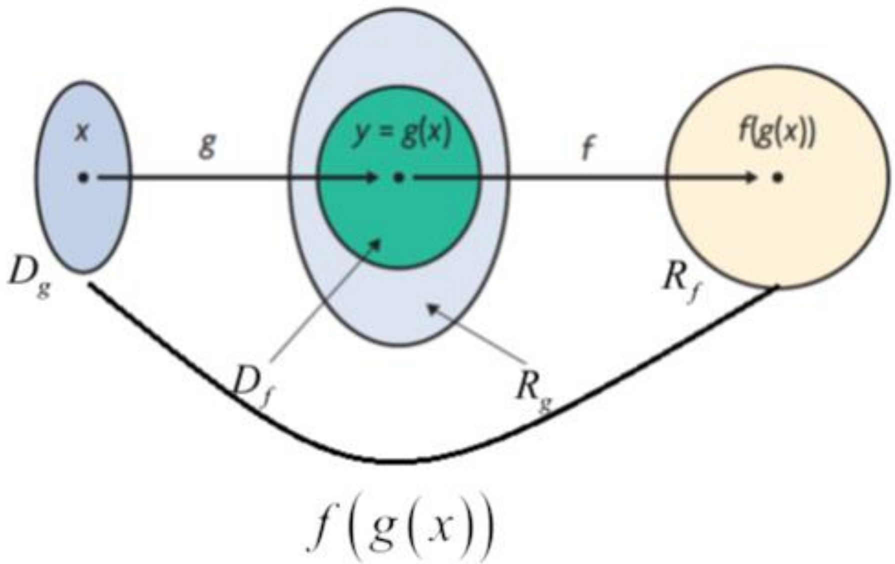
d. $g(f(0))$

e. $g(g(3))$



Domain of Composite Function Diagram

Let $(a, b)\epsilon g$ and $(b, c)\epsilon f$. Determine $f(g(a))$



NOTE: To define $(fg)(x)$,

$$\boxed{D_f \cap R_g \neq \varnothing}$$

Then

$$\boxed{D_{fog} = \{x \in D_g, g(x) \in D_f\}}$$

Example . Given that $f = \{(0,1), (1,3), (2,5), (3,7), (4,9)\}$ and $g = \{(0,0), (1,2), (2,4), (3,6), (4,8)\}$. Determine

a) $(f \circ g)(x)$

b) $(f \circ g)^{-1}(x)$

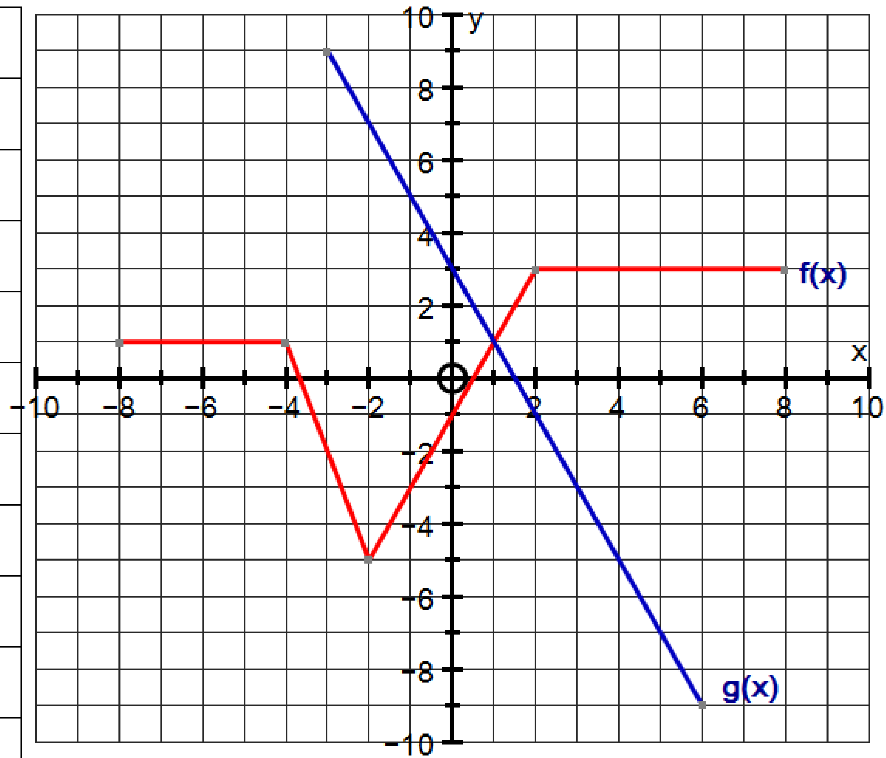
c) $g(f(x))$

d) $f^{-1}(x)$

e) $(g \circ f^{-1})(x)$

Example. Given the following graph of $f(x)$ and $g(x)$. Graph $(f \circ g)(x)$.

x	$g(x)$	$f(g(x))$

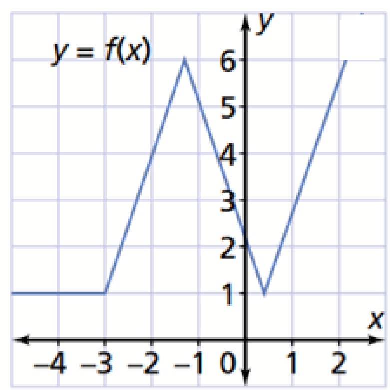


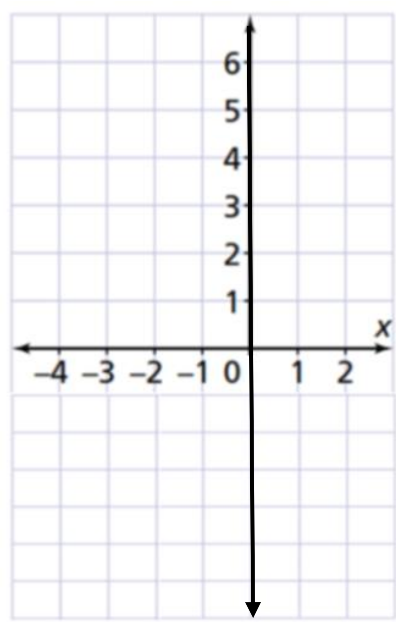
Example. Given the graph of y = f (x) on the right and the functions g(x) = 2x +1, h(x) =-x+3, and $k(x) = (g \circ f \circ h)(x)$,

(a) evaluate k(2)

(b) graph $(g \circ h \circ f)(x)$.

x	(g o h of)(x)





Example. Express h as the composition of two functions f and g, such that $h(x) = f(g(x))$.

a. $h(x) = 3^{2x^2-1}$

b. $h(x) = x^4 + 5x^2 + 6$

c. $h(x) = \frac{2x^2 - 1}{x^2}$

d. $h(x) = \frac{1}{x-4}$

Composite Functions

Exercise 6.4

Part A

1. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, find the following:

   a. $f(g(1))$ b. $g(f(1))$ c. $g(f(0))$

   d. $f(g(-4))$ e. $f(g(x))$ f. $g(f(x))$

2. For each of the following pairs of functions, find the composite functions $f \circ g$ and $g \circ f$. What is the domain of each composite function? Are the composite functions equal?

   a. $f(x) = x^2$, $g(x) = \sqrt{x}$

   b. $f(x) = \frac{1}{x}$, $g(x) = x^2 + 1$

   c. $f(x) = \frac{1}{x}$, $g(x) = \sqrt{x + 2}$

Part B

3. Use the functions $f(x) = 3x + 1$, $g(x) = x^3$, $h(x) = \frac{1}{x + 1}$, and $u(x) = \sqrt{x}$ to find expressions for the indicated composite function.

   a. $f \circ u$ b. $u \circ h$ c. $g \circ f$

   d. $u \circ g$ e. $h \circ u$ f. $f \circ g$

   g. $h \circ (f \circ u)$ h. $(f \circ g) \circ u$ i. $g \circ (h \circ u)$

4. Express h as the composition of two functions f and g, such that $h(x) = f(g(x))$.

   a. $h(x) = (2x^2 - 1)^4$ b. $h(x) = \sqrt{5x - 1}$

   c. $h(x) = \frac{1}{x - 4}$ d. $h(x) = (2 - 3x)^{\frac{5}{2}}$

   e. $h(x) = x^4 + 5x^2 + 6$ f. $h(x) = (x + 1)^2 - 9(x + 1)$

5. If $f(x) = \sqrt{2 - x}$ and $f(g(x)) = \sqrt{2 - x^3}$, then what is $g(x)$?

6. If $g(x) = \sqrt{x}$ and $f(g(x)) = (\sqrt{x} + 7)^2$, then what is $f(x)$?

7. Let $g(x) = x - 3$. Find a function f so that $f(g(x)) = x^2$.

8. Let $f(x) = x^2$. Find a function g so that $f(g(x)) = x^2 + 8x + 16$.

9. Let $f(x) = x + 4$ and $g(x) = (x - 2)^2$. Find a function u so that $f(g(u(x))) = 4x^2 - 8x + 8$.

10. If $f(x) = \frac{1}{1 - x}$ and $g(x) = 1 - x$, determine

    a. $g(f(x))$ b. $f(g(x))$

11. If $f(x) = 3x + 5$ and $g(x) = x^2 + 2x - 3$, determine x such that $f(g(x)) = g(f(x))$.

12. If $f(x) = 2x - 7$ and $g(x) = 5 - 2x$,

    a. determine $f \circ f^{-1}$ and $f^{-1} \circ f$.

    b. show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Exercise

1. a. 0 **b.** 0 **c.** -1 **d.** $\sqrt{15}$ **e.** $\sqrt{x^2 - 1}$ **f.** $x - 1$

2. a. $f(g(x)) = x, x \geq 0$; $g(f(x)) = |x|, x \in R$; $f \circ g \neq g \circ f$

b. $f(g(x)) = \frac{1}{x^2 + 1}, x \in R$; $g(f(x)) = \frac{1}{x^2} + 1, x \neq 0$; $f \circ g \neq g \circ f$

c. $f(g(x)) = \frac{1}{\sqrt{x + 2}}, x > -2$; $g(f(x)) = \sqrt{\frac{1 + 2x}{x}}, x < -\frac{1}{2}$ or $x > 0$; $f \circ g \neq g \circ f$

3. a. $3\sqrt{x} + 1$ **b.** $\frac{1}{\sqrt{x + 1}}$ **c.** $(3x + 1)^3$

d. $\sqrt{x^3}$ **e.** $\frac{1}{\sqrt{x} + 1}$ **f.** $3x^3 + 1$ **g.** $\frac{1}{3\sqrt{x} + 2}$ **h.** $3x\sqrt{x} + 1$

i. $\frac{1}{(\sqrt{x} + 1)^3}$

4. a. $f(x) = x^4$, $g(x) = 2x^2 - 1$ **b.** $f(x) = \sqrt{x}$, $g(x) = 5x - 1$

c. $f(x) = \frac{1}{x}$, $g(x) = x - 4$ **d.** $f(x) = x^{\frac{5}{2}}$, $g(x) = 2 - 3x$

e. $f(x) = x(x + 1)$, $g(x) = x^2 + 2$

f. $f(x) = x^2 - 9x$, $g(x) = x + 1$

5. $g(x) = x^3$

6. $f(x) = (x + 7)^2$

7. $f(x) = (x + 3)^2$

8. $g(x) = x + 4$ or $g(x) = -x - 4$

9. $u(x) = 2x$ or $u(x) = -2x + 4$

10. a. $\frac{x}{x - 1}$ **b.** $\frac{1}{x}$ **11.** $-2, -3$ **12. a.** x

1. Given the following functions, determine the following in simplest **exact** form.

$f(x) = \{(1,2),(2,3),(3,5),(5,7)\}$ $\quad$ $g(x) = \{(1,4),(3,7),(5,9),(9,2)\}$

$h(x) = -2\cos(2x+\pi)+1$ $\quad$ $m(x) = x^2 + 1$

a. $\left(f^{-1}+g\right)(x)$

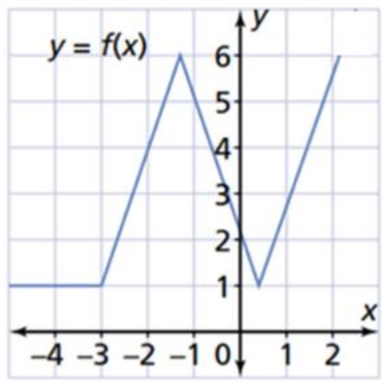
b. $(f \times g)(3)$

c. $(g \circ h)(\pi)$

d. $\left(\dfrac{f}{m}\right)(x)$

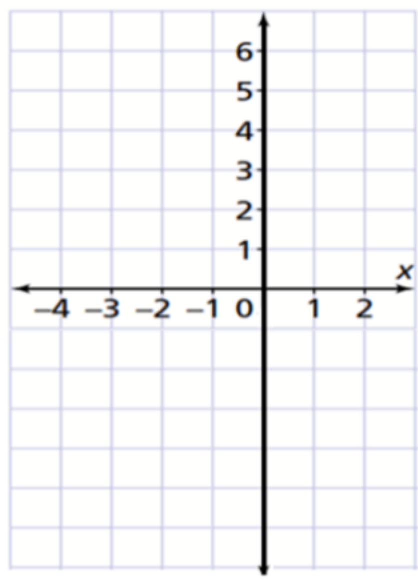
2. Given the graph of y = f (x) on the right and the functions $\boldsymbol{g(x) = 2x + 1}$, $\boldsymbol{h(x) = -x + 3}$, and $\boldsymbol{k(x) = (g \circ f \circ h)(x)}$

(a) evaluate $k(2)$

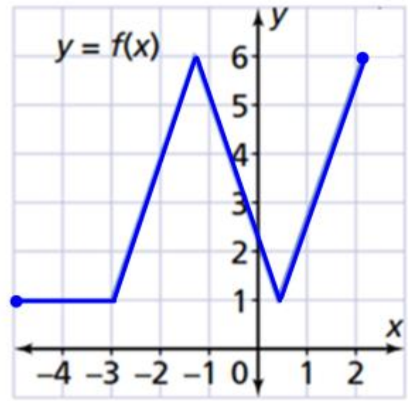


(b) graph $(g \circ h \circ f)(x)$.

x	$(g \circ h \circ f)(x)$



3. Given the graph $y = k(x)$ below and the functions $g(x) = 2^{x-1}$, $h(x) = \log_2(x^2+1)$, $m(x) = \sqrt{16-x^2}$,graph of $y = f(x)$ below, $k(x) = \dfrac{(g \circ f \circ h)(x)}{h(x)}$ and $n(x) = (f \circ m \circ f)(x)$ determine the following:



a. $k(2)$

b. $n(-2)$