**L5 – 3.4 Solve Rational Equations and Inequalities**
**MHF4U**

## Part 1: Rational Expressions

**Rational Expression:** the quotient of two polynomials, $\frac{p(x)}{q(x)}$, where $q(x) \neq 0$

**Example 1:** Simplify and state the restrictions of each rational expression

a) $\frac{2x^2-8}{x^2+3x+2}$

$$\frac{2x^2-8}{x^2+3x+2}$$

$$= \frac{2(x^2-4)}{(x+2)(x+1)} \text{ ← DOS}$$

$$= \frac{2(x-2)\cancel{(x+2)}}{\cancel{(x+2)}(x+1)}$$

$$= \frac{2(x-2)}{x+1} ;\ x \neq -2, -1$$

b) $\frac{x^3-x^2-x+1}{3x^3-3}$

$$\frac{x^3-x^2-x+1}{3x^3-3}$$

$$= \frac{(x^3-x^2)+(-x+1)}{3(x^3-1)} \text{ ← DOC}$$

$$= \frac{x^2(x-1)-1(x-1)}{3(x-1)(x^2+x+1)}$$

$$= \frac{\cancel{(x-1)}(x^2-1)}{3\cancel{(x-1)}(x^2+x+1)} \text{ ← DOS}$$

$$= \frac{(x-1)(x+1)}{3(x^2+x+1)} ;\ x \neq 1$$

## Part 2: Solve Rational Equations

**Steps for solving rational equations:**

1) Fully factor both sides of the equation
2) Multiply both sides by a common denominator (cross multiply if appropriate)
3) Continue to solve as you would a normal polynomial equation
4) State restrictions throughout (values of $x$ that would make denominator equal zero)

**Example 2:** Solve each equation

**a)** $\frac{4}{3x-5} = 4$

$$\frac{4}{3x-5} = 4$$

$$(\cancel{3x-5})\left(\frac{4}{\cancel{3x-5}}\right) = 4(3x-5) \quad ; x \neq \frac{5}{3}$$

$$4 = 4(3x-5)$$

$$4 = 12x - 20$$

$$24 = 12x$$

$$\boxed{x = 2}$$

**b)** $\frac{6}{x-2} = x - 1$

$$\frac{6}{x-2} = x - 1$$

$$(\cancel{x-2})\left(\frac{6}{\cancel{x-2}}\right) = (x-2)(x-1) \quad ; x \neq 2$$

$$6 = x^2 - 1x - 2x + 2$$

$$0 = x^2 - 3x - 4$$

$$0 = (x-4)(x+1)$$

$$x - 4 = 0 \qquad x + 1 = 0$$

$$\boxed{x_1 = 4} \qquad \boxed{x_2 = -1}$$

**c)** $\frac{x-5}{x^2-3x-4} = \frac{3x+2}{x^2-1}$

$$\frac{x-5}{x^2-3x-4} = \frac{3x+2}{x^2-1}$$

$$\frac{x-5}{(x-4)(x+1)} = \frac{3x+2}{(x-1)(x+1)} \quad ; x \neq -1, 1, 4$$

$$\frac{(x-1)(x+1)(x-5)}{x+1} = \frac{(x-4)(x+1)(3x+2)}{x+1}$$

$$(x-1)(x-5) = (x-4)(3x+2)$$

$$x^2 - 6x + 5 = 3x^2 - 10x - 8$$

$$0 = 2x^2 - 4x - 13$$

NOT Factorable
use QF

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(-13)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{120}}{4}$$

$$x = \frac{4 \pm 2\sqrt{30}}{4}$$

$$x = \frac{2(2 \pm \sqrt{30})}{4}$$

$$x = \frac{2 \pm \sqrt{30}}{2}$$

## Part 3: Solve Rational Inequalities

**REMEMBER:** Solving inequalities is the same as solving equations. However, when both sides of an inequality are multiplied or divided by a negative number, the inequality sign must be reversed.

**Steps for solving rational inequalities algebraically:**

1) Use inverse operations to move all terms to one side of the inequality
2) Combine the expressions by using a common denominator
3) Factor the polynomial
4) Find the interval(s) where the function is positive or negative by making a factor table

**To make a factor table:**

- Use $x$-intercepts and vertical asymptotes to divide in to intervals
- Use a test point within each interval to find the sign of each factor
- Determine the overall sign of the product by multiplying signs of each factor within each interval.

**Example 3:** Solve each inequality algebraically

a) $\frac{x^2+6x+5}{2x^2-7x+3} < 0$

P: 6
S: -7
$\frac{-3}{1} = \frac{-6}{2}$, $\frac{-1}{2}$

$$\frac{x^2+6x+5}{2x^2-7x+3} < 0$$

$$\frac{(x+5)(x+1)}{(2x-1)(x-3)} < 0$$

zeros: $x = -5, -1$

restrictions: $x \neq \frac{1}{2}, 3$

| | $-\infty$ to $-5$: $-6$ | $-5$ to $-1$: $-2$ | $-1$ to $\frac{1}{2}$: $0$ | $\frac{1}{2}$ to $3$: $1$ | $3$ to $\infty$: $4$ |
|---|---|---|---|---|---|
| $x+5$ | − | + | + | + | + |
| $x+1$ | − | − | + | + | + |
| $2x-1$ | − | − | − | + | + |
| $x-3$ | − | − | − | − | + |
| overall | + | − | + | − | + |

The inequality is true when $-5 < x < -1$ OR $0.5 < x < 3$

The inequality is true when $x \in (-5,-1) \cup (0.5,3)$

**b)** $x - 2 < \frac{8}{x}$

$$x - 2 < \frac{8}{x}$$

$$x - 2 - \frac{8}{x} < 0$$

$$\frac{x^2}{x} - \frac{2x}{x} - \frac{8}{x} < 0$$

$$\frac{x^2 - 2x - 8}{x} < 0$$

$f(x)$

$$\frac{(x-4)(x+2)}{x} < 0$$

zeros: $x = -2, 4$

restriction: $x \neq 0$

$-\infty \quad -2 \quad 0 \quad 4 \quad \infty$

| | -3 | -1 | 1 | 5 |
|---|---|---|---|---|
| $x-4$ | − | − | − | + |
| $x+2$ | − | + | + | + |
| $x$ | − | − | + | + |
| overall | − | + | − | + |

The inequality is true when $x < -2$ OR $0 < x < 4$

The inequality is true when $x \in (-\infty, -2) \cup (0, 4)$

c) $\frac{x+3}{x+1} \geq \frac{x-2}{x-3}$

$$\frac{x+3}{x+1} \geq \frac{x-2}{x-3}$$

$$\frac{x+3}{x+1}\cdot\frac{(x-3)}{(x-3)} - \frac{x-2}{x-3}\cdot\frac{(x+1)}{(x+1)} \geq 0$$

$$\frac{(x+3)(x-3)-(x-2)(x+1)}{(x+1)(x-3)} \geq 0$$

$$\frac{x^2-9-(x^2-x-2)}{(x+1)(x-3)} \geq 0$$

$$\frac{x^2-9-x^2+x+2}{(x+1)(x-3)} \geq 0$$

$$\frac{x-7}{(x+1)(x-3)} \geq 0$$

zero: $x=7$

restrictions: $x \neq -1, 3$

| | $-\infty$ to $-1$ | $-1$ to $3$ | $3$ to $7$ | $7$ to $\infty$ |
|---|---|---|---|---|
| | $-2$ | $0$ | $4$ | $8$ |
| $x-7$ | $-$ | $-$ | $-$ | $+$ |
| $x+1$ | $-$ | $+$ | $+$ | $+$ |
| $x-3$ | $-$ | $-$ | $+$ | $+$ |
| overall | $-$ | $+$ | $-$ | $+$ |

The inequality is true when $-1 < x < 3$ OR $x \geq 7$

The inequality is true when $x \in (-1, 3) \cup [7, \infty)$

**d)** $\frac{x^3+6x^2-2x}{x^2+4} \geq 2$

$$\frac{x^3+6x^2-2x}{x^2+4} \geq 2$$

$$\frac{x^3+6x^2-2x}{x^2+4} - 2 \geq 0$$

$$\frac{x^3+6x^2-2x}{x^2+4} - \frac{2(x^2+4)}{x^2+4} \geq 0$$

$$\frac{x^3+6x^2-2x-2(x^2+4)}{x^2+4} \geq 0$$

$$\frac{x^3+6x^2-2x-2x^2-8}{x^2+4} \geq 0$$

$$\frac{x^3+4x^2-2x-8}{x^2+4} \geq 0$$

$$\frac{x^2(x+4)-2(x+4)}{x^2+4} \geq 0$$

$$\frac{(x+4)(x^2-2)}{x^2+4} \geq 0$$

zeros: $x+4=0$, $x=-4$; $x^2-2=0$, $x^2=2$, $x=\pm\sqrt{2}$

no restrictions

| | $-\infty$ to $-4$ | $-4$ to $-\sqrt{2}$ | $-\sqrt{2}$ to $\sqrt{2}$ | $\sqrt{2}$ to $\infty$ |
|---|---|---|---|---|
| | $-5$ | $-3$ | $0$ | $2$ |
| $x+4$ | $-$ | $+$ | $+$ | $+$ |
| $x^2-2$ | $+$ | $+$ | $-$ | $+$ |
| $x^2+4$ | $+$ | $+$ | $+$ | $+$ |
| overall | $-$ | $+$ | $-$ | $+$ |

The inequality is true when $-4 \leq x \leq -\sqrt{2}$ OR $x \geq \sqrt{2}$

The inequality is true when $x \in [-4, -\sqrt{2}] \cup [\sqrt{2}, \infty)$