

# ***Rates of Change***

## *Lesson Package*

*MHF4U*

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

## Unit 6 Outline

**Unit Goal:** By the end of this unit, you will be able to demonstrate an understanding of average and instantaneous rates of change.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Average Rates of Change	- Calculate and interpret average rates of change of functions	D1.1, 1.2, 1.3, 1.4
L2	Instantaneous Rates of Change	- understand what an instantaneous rate of change is and be able to use average rates of change to estimate instantaneous rates	D1.5, 1.6, 1.7, 1.8, 1.9
L3	Newton Quotient	- Calculate the equation of the derivative	D1.8
L4	Limits	- Be able to calculate limits from a graph and from an equation	D1.8

Assessments	F/A/O	Ministry Code	P/O/C	KTAC
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
Quiz – Solving Rational Equations and Inequalities	F		P	
PreTest Review	F/A		P	
Test –Rates of Change	O	C3.5, 3.6 D1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9 D4.1, 4.2, 4.3	P	K(21%), T(34%), A(10%), C(34%)



## L1 – 1.5 Average Rates of Change

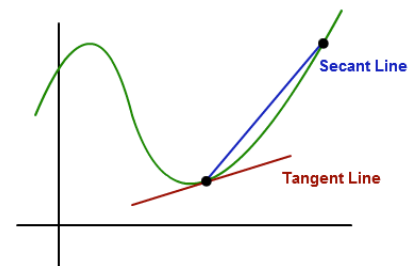
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### Part 1: Terminology

**Rate of Change:** a measure of the change in one quantity (the dependent variable) with respect to a change in another quantity (the independent variable).

**Secant Line:** a line that passes through two points on the graph of a relation

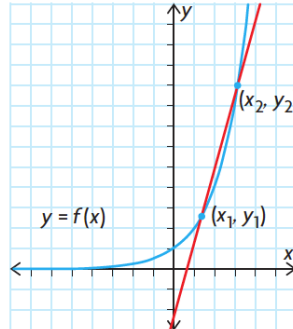
**Tangent Line:** a line that touches the graph of a relation at only one point within a small interval



An **average rate of change** is a change that takes place over an **interval**, while an **instantaneous rate of change** is a change that takes place in an **instant**. We will focus on average rates of change in this section.

An average rate of change corresponds to the slope of a **SECANT** between two points on a curve.

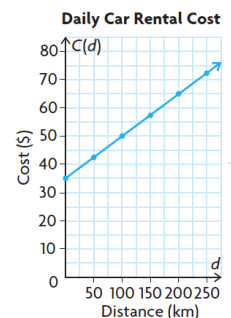
$$\text{Average rate of change} = \text{slope of secant} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$



### Part 2: Average Rates of Change from a Table or Graph

**Note:** All **linear** relationships have a constant rate of change. Average rate of change calculations over different intervals of the independent variable give the **SAME** result.

We will be focusing on **non-linear** relationships. Non-linear relationships do not have a constant rate of change. Average rate of change calculations over different intervals of the independent interval give **DIFFERENT** results.



**Example 1:** Andrew drains water from a hot tub. The tub holds 1600 L of water. It takes 2 hours for the water to drain completely. The volume  $V$ , in Liters, of water remaining in the tub at various times  $t$ , in minutes, is shown in the table and graph.

**a)** Calculate the average rate of change in volume during each of the following time intervals.

**i)**  $30 \leq t \leq 90$

$$m = \frac{\Delta V}{\Delta t}$$

$$= \frac{V(90) - V(30)}{90 - 30}$$

$$= \frac{100 - 900}{60}$$

$$= -13.3 \text{ L/min}$$

The volume of water is decreasing by 13.3 L/min.

**ii)**  $60 \leq t \leq 90$

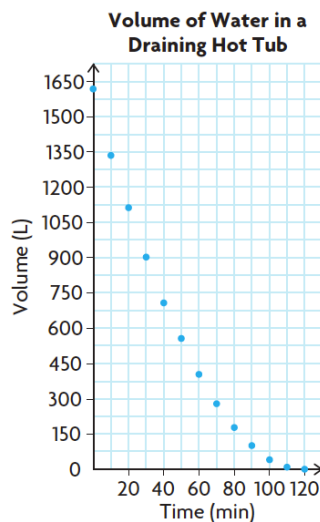
$$m = \frac{\Delta V}{\Delta t}$$

$$= \frac{V(90) - V(60)}{90 - 60}$$

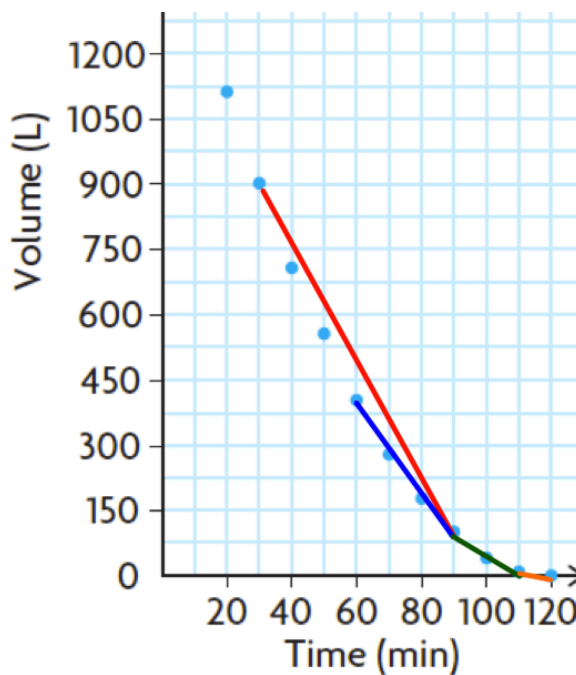
$$= \frac{100 - 400}{30}$$

$$= -10 \text{ L/min}$$

The volume of water is decreasing by 10 L/min.



Time (min)	Volume (L)
0	1600
10	1344
20	1111
30	900
40	711
50	544
60	400
70	278
80	178
90	100
100	44
110	10
120	0



iii)  $90 \leq 110$

$$m = \frac{\Delta V}{\Delta t}$$

$$= \frac{V(110) - V(90)}{110 - 90}$$

$$= \frac{10 - 100}{20}$$

$$= -4.5 \text{ L/min}$$

The volume of water is decreasing by 4.5 L/min.

iv)  $110 \leq 120$

$$m = \frac{\Delta V}{\Delta t}$$

$$= \frac{V(120) - V(110)}{120 - 110}$$

$$= \frac{0 - 10}{10}$$

$$= -1 \text{ L/min}$$

The volume of water is decreasing by 1 L/min.

b) Does the tub drain at a constant rate?

*The tub does not drain at a constant rate. This can be seen from the graph which shows a non-linear relationship. The rate slows as the water empties because of the pressure decrease.*

A **negative** rate of change indicates the quantity of the dependent variable is decreasing over the interval. The secant line has a negative slope.

A **positive** rate of change indicates the quantity of the dependent variable is increasing over the interval. The secant line has a positive slope.

## Part 2: Average Rate of Change from an Equation

**Example 2:** A rock is tossed upward from a cliff that is 120 meters above the water. The height of the rock above the water is modelled by  $h(t) = -5t^2 + 10t + 120$ , where  $h$  is the height in meters and  $t$  is the time in seconds. Calculate the average rate of change in height during each time intervals.

a)  $0 \leq t \leq 1$

$$\begin{aligned} m &= \frac{\Delta h}{\Delta t} \\ &= \frac{h(1) - h(0)}{1 - 0} \\ &= \frac{125 - 120}{1} \\ &= 5 \text{ m/s} \end{aligned}$$

b)  $1 \leq t \leq 2$

$$\begin{aligned} m &= \frac{\Delta h}{\Delta t} \\ &= \frac{h(2) - h(1)}{2 - 1} \\ &= \frac{120 - 125}{1} \\ &= -5 \text{ m/s} \end{aligned}$$

c)  $2 \leq t \leq 3$

$$\begin{aligned} m &= \frac{\Delta h}{\Delta t} \\ &= \frac{h(3) - h(2)}{3 - 2} \\ &= \frac{105 - 120}{1} \\ &= -15 \text{ m/s} \end{aligned}$$

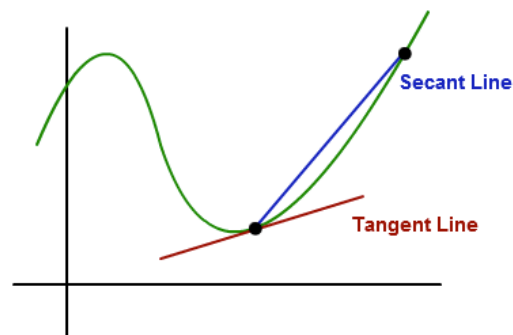
## L2 – 1.6 Instantaneous Rates of Change

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### Part 1: Review of Average Rates of Change

An average rate of change corresponds to the slope of a **SECANT** between two points on a curve.

An instantaneous rate of change corresponds to the slope of a **TANGENT** at one specific point on a curve.



**Example 1:** The height,  $h$ , in meters, after  $t$  seconds is modelled by the equation  $h(t) = -4.9t^2 + 14t + 1$ . Determine the average rate of change of the height of the ball for each time interval.

a)  $[0, 0.5]$

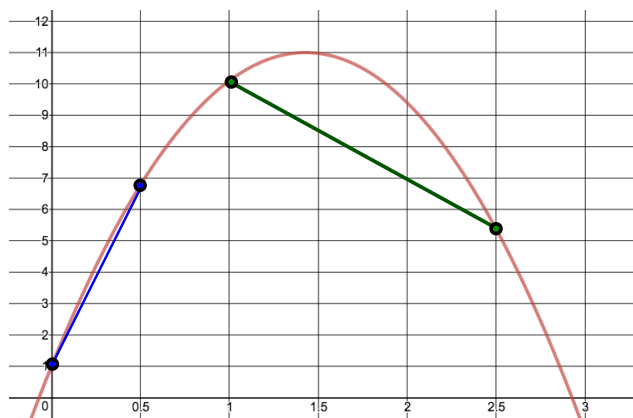
$$h(0) = -4.9(0)^2 + 14(0) + 1$$

$$h(0) = 1$$

$$h(0.5) = -4.9(0.5)^2 + 14(0.5) + 1$$

$$h(0.5) = 6.775$$

$$m = \frac{\Delta h}{\Delta t} = \frac{h(0.5) - h(0)}{0.5 - 0} = \frac{6.775 - 1}{0.5} = 11.55 \text{ m/s}$$



b)  $[1, 2.5]$

$$h(1) = -4.9(1)^2 + 14(1) + 1$$

$$h(1) = 10.1$$

$$h(2.5) = -4.9(2.5)^2 + 14(2.5) + 1$$

$$h(2.5) = 5.375$$

$$m = \frac{\Delta h}{\Delta t} = \frac{h(2.5) - h(1)}{2.5 - 1} = \frac{5.375 - 10.1}{1.5} = -3.15 \text{ m/s}$$

## Part 2: Find Instantaneous Rate of Change from a Graph or Table

**Instantaneous Rate of Change:** The exact rate of change of a function at one specific value of the independent variable. Can be estimated using average rates of change for small intervals of the independent variable.

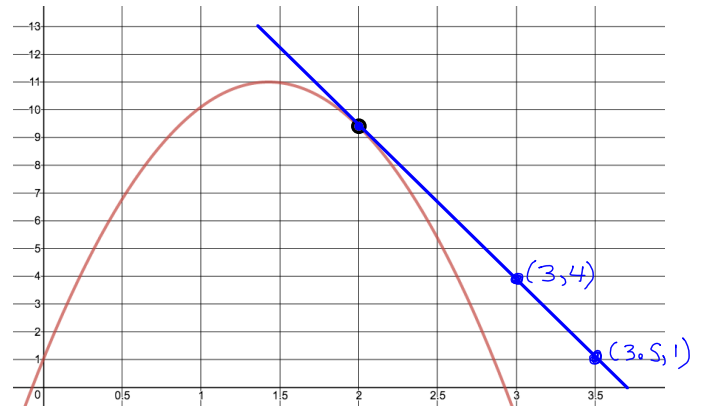
**Note:** the best estimate for the instantaneous rate of change occurs when the interval used to calculate the average rate of change is made as small as possible.

**Example 2:** Find the instantaneous rate of change of the height of the ball at 2 seconds. The equation for the height of the ball is  $h(t) = -4.9t^2 + 14t + 1$ .

### Method 1: Draw Tangent Line

The slope of a tangent at a point on a graph is equivalent to the instantaneous rate of change of a function at this point.

$$\left. \frac{dh}{dt} \right|_{t=2} \approx \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 4}{3.5 - 3} = -6 \text{ m/s}$$

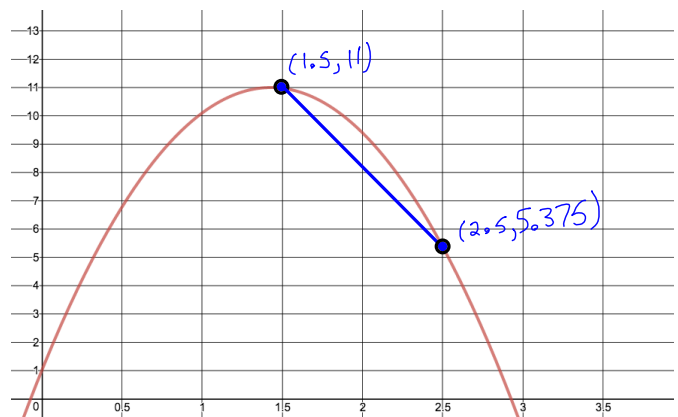


### Method 2: Choose a Surrounding Interval

Choose an interval that has the instantaneous rate of change location as its midpoint if possible.

For this example, choose the interval  $[1.5, 2.5]$ . Use the equation of the function to determine the corresponding  $y$  values.

$$\left. \frac{dh}{dt} \right|_{t=2} \approx \frac{h(2.5) - h(1.5)}{2.5 - 1.5} = \frac{5.375 - 11}{1} = -5.625 \text{ m/s}$$





### Method 3: Averaging a Preceding and Following Interval

Calculate the average rate of change on either side of  $t = 2$  and then average them.

For this example, find the average rate of change for the intervals  $[1.5, 2]$  and  $[2, 2.5]$  and then average them.

Average Rate for interval  $[1.5, 2]$

$$m = \frac{h(2) - h(1.5)}{2 - 1.5}$$

$$= \frac{9.4 - 11}{0.5}$$

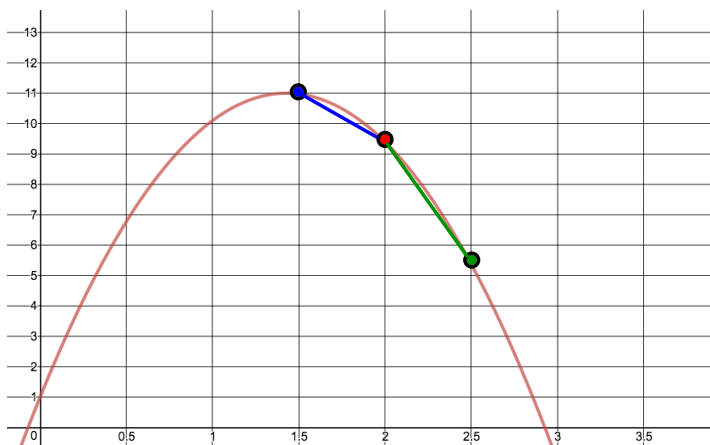
$$= -3.2 \text{ m/s}$$

Average Rate for interval  $[2, 2.5]$

$$m = \frac{h(2.5) - h(2)}{2.5 - 2}$$

$$= \frac{5.375 - 9.4}{0.5}$$

$$= -8.05 \text{ m/s}$$



$$\left. \frac{dh}{dt} \right|_{t=2} \approx \frac{-3.2 + (-8.05)}{2} = -5.625 \text{ m/s}$$

**Example 3:** Consider the data from the Bank of Canada for the beginning of November in 2007. Estimate the instantaneous rate of change of the value of the Canadian Dollar on November 7<sup>th</sup>.

Day	1	2	5	7	8	13	14
Value of Canadian \$ in US Cents	94.99	93.59	93.39	91.70	93.06	96.43	96.07

### Method 1: Choose a Surrounding Interval

Use the interval  $[5, 8]$

$$\left. \frac{dv}{dt} \right|_{t=7} \approx \frac{V(8) - V(5)}{8 - 5} = \frac{93.06 - 93.39}{3} = -0.11 \text{ cents/day}$$

## Method 2: Averaging a Preceding and Following Interval (more accurate)

Average Rate for interval [5, 7]

$$\begin{aligned}m &= \frac{V(7)-V(5)}{7-5} \\&= \frac{91.70-93.39}{2} \\&= -0.845 \text{ cents/day}\end{aligned}$$

Average Rate for interval [7, 8]

$$\begin{aligned}m &= \frac{V(8)-V(7)}{8-7} \\&= \frac{93.06-91.70}{1} \\&= 1.36 \text{ cents/day}\end{aligned}$$

$$\left. \frac{dV}{dt} \right|_{t=2} \approx \frac{-0.845+1.36}{2} = 0.258 \text{ cents/day}$$

## Part 3: Find Instantaneous Rate of Change from an Equation

Squeezing a preceding or following interval is a technique for estimating instantaneous rates of change. By finding average rates of change that get progressively smaller, you can make more accurate estimates at to what the rate of change is **APPROACHING** at an exact point.

**Example 4:** Use the chart below to estimate the slope of the tangent to the curve  $f(x) = 2x^3 - x$  at  $x = 2$ .

Interval	$\Delta y$	$\Delta x$	Slope of secant = $\frac{\Delta y}{\Delta x}$
$2 \leq x \leq 3$	$\begin{aligned} &= f(3) - f(2) \\ &= [2(3)^3 - 3] - [2(2)^3 - 2] \\ &= 51 - 14 \\ &= 37 \end{aligned}$	$\begin{aligned} &= 3 - 2 \\ &= 1 \end{aligned}$	$\begin{aligned} &= \frac{37}{1} \\ &= 37 \end{aligned}$
$2 \leq x \leq 2.5$	$\begin{aligned} &= f(2.5) - f(2) \\ &= 28.75 - 14 \\ &= 14.75 \end{aligned}$	$\begin{aligned} &= 2.5 - 2 \\ &= 0.5 \end{aligned}$	$\begin{aligned} &= \frac{14.75}{0.5} \\ &= 29.5 \end{aligned}$
$2 \leq x \leq 2.1$	$\begin{aligned} &= f(2.1) - f(2) \\ &= 16.422 - 14 \\ &= 2.422 \end{aligned}$	$\begin{aligned} &= 2.1 - 2 \\ &= 0.1 \end{aligned}$	$\begin{aligned} &= \frac{2.422}{0.1} \\ &= 24.22 \end{aligned}$
$2 \leq x \leq 2.01$	$\begin{aligned} &= f(2.01) - f(2) \\ &= 14.231202 - 14 \\ &= 0.231202 \end{aligned}$	$\begin{aligned} &= 2.01 - 2 \\ &= 0.01 \end{aligned}$	$\begin{aligned} &= \frac{0.231202}{0.01} \\ &= 23.1202 \end{aligned}$
$2 \leq x \leq 2.001$	$\begin{aligned} &= f(2.001) - f(2) \\ &= 14.023012 - 14 \\ &= 0.023012 \end{aligned}$	$\begin{aligned} &= 2.001 - 2 \\ &= 0.001 \end{aligned}$	$\begin{aligned} &= \frac{0.023012}{0.001} \\ &= 23.012 \end{aligned}$

By looking at the limit of the secant slopes as we approach  $x = 2$ , we can estimate the slope of the tangent to be:

$$\left. \frac{dy}{dx} \right|_{x=2} \approx 23$$

### L3 – Newton Quotient

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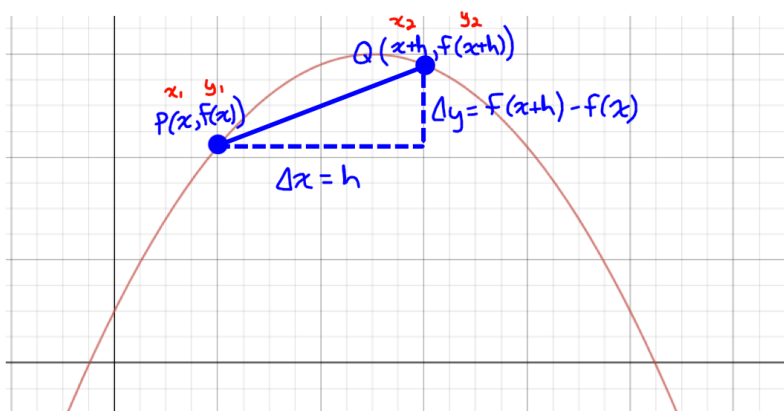
#### Part 1: Difference between Average and Instantaneous Rates of Change

Average Rate of Change	Instantaneous Rate of Change
Over an interval	At one exact point
Slope of secant	Slope of tangent
Calculated	Estimated
$m$	$\frac{dy}{dx}$

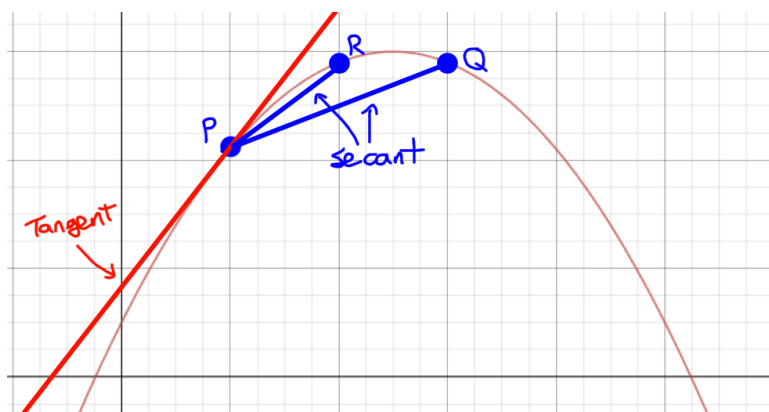
#### Part 2: Estimating Instantaneous Rate of Change Using Difference Quotient

Looking at the graph of  $f(x) = -0.2(x - 5)^2 + 6$ , points  $P(x, f(x))$  and  $Q(x + h, f(x + h))$  are on the graph of the function. By connecting the points we have a secant line.

Finding the slope of the secant line will tell us the average rate of change between the two points.



The closer we move point Q towards point P, the closer the average rate of change will get to the instantaneous rate of change at P. In other words, the slope of the secant will get closer to the slope of the tangent line at P. Notice that the slope of PR is much closer to the tangent's slope than the slope of PQ is.



The smaller the difference in the  $x$  values (smaller the secant line), the closer we get to the actual instantaneous rate of change. What if we put the difference in the  $x$  values at 0? This would give us the instantaneous rate of change exactly, but how do we divide by 0?

We have to find the LIMIT of the secant slopes as the  $\Delta x$  approaches 0.

It is often useful to find the general equation for instantaneous rate of change of  $f(x)$  that can be used to find the slope of the tangent at any point throughout the domain. This equation is called the 'derivative' and is written as  $f'(x)$ .

### Newton Quotient:

The instantaneous rate of change at any  $x$  for  $y = f(x)$  is

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

This is called "Derivative from First Principles."

**Example 1:** Find an equation for the instantaneous rate of change of  $f(x) = \frac{1}{4}x^2 - x + 1$ . Then use the equation to find  $f'(4)$ .

**Note:** To find a general formula for instantaneous rate of change, leave the  $x$ 's in the equation.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{4}(x+h)^2 - (x+h) + 1 - \left(\frac{1}{4}x^2 - x + 1\right)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{4}(x^2 + 2xh + h^2) - x - h + 1 - \frac{1}{4}x^2 + x - 1}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{4}x^2 + \frac{1}{2}xh + \frac{1}{4}h^2 - x - h + 1 - \frac{1}{4}x^2 + x - 1}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{2}xh + \frac{1}{4}h^2 - h}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{h\left(\frac{1}{2}x + \frac{1}{4}h - 1\right)}{h}$$

$$f'(x) = \frac{1}{2}x + \frac{1}{4}(0) - 1$$

$$f'(x) = \frac{1}{2}x - 1$$

$$f'(4) = \frac{1}{2}(4) - 1$$

$$f'(4) = 1$$

**Example 2:** Find a formula for the instantaneous rate of change for the function  $f(x) = x^2 - 5x + 4$

$$f(x+h) = (x+h)^2 - 5(x+h) + 4$$

$$= x^2 + 2xh + h^2 - 5x - 5h + 4$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 5x - 5h + 4 - (x^2 - 5x + 4)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 5x - 5h + 4 - x^2 + 5x - 4}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2xh + h^2 - 5h}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{h(2x + h - 5)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} 2x + h - 5$$

$$f'(x) = 2x + 0 - 5$$

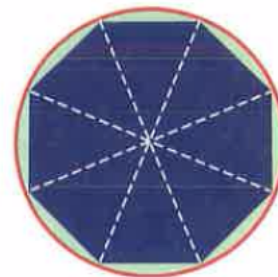
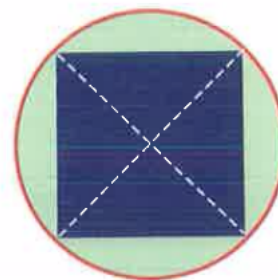
$$f'(x) = 2x - 5$$

## L4 – Limits

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The Greek mathematician Archimedes developed a proof of the formula for the area of a circle,  $A = \pi r^2$ . His method, known as the “method of exhaustion,” involved calculating the area of regular polygons inscribed in the circle. The area of the polygon provides an estimate of the area of the circle. As Archimedes increased the number of sides of the polygon, its shape came closer and closer to the shape of a circle. An octagon provides a much better estimate of the area of a circle than a square does. What about a myriagon, a polygon with 10 000 sides? What happens to the estimate as the number of sides approaches infinity?

Archimedes’ method of finding the area of a circle is based on the concept of a LIMIT. The circle is the limiting shape of the polygon. As the number of sides gets larger, the area of the polygon approaches its limit, the shape of a circle, without ever becoming an actual circle. You used a similar strategy to estimate the instantaneous rate of change of a function. Your estimate became increasingly accurate as the interval between two points was made smaller. Using limits, the interval can be made infinitely small, approaching zero. As this happens, the slope of the secant approaches its limiting value, the slope of the tangent.



### Section 1: Intro to Limits

**Example 1:** Complete the following table:

$x$	2	2.5	2.9	2.99	3	3.01	3.1	3.5	4
$y = x^2 - 2$	2	4.25	6.41	6.9401	7	7.0601	7.61	10.25	14

Find the limit as  $x$  approaches 3 from the left

$$\lim_{x \rightarrow 3^-} (x^2 - 2) = 7$$

Find the limit as  $x$  approaches 3 from the right

$$\lim_{x \rightarrow 3^+} (x^2 - 2) = 7$$

$$\text{Therefore, } \lim_{x \rightarrow 3} (x^2 - 2) = 7$$

$\lim_{x \rightarrow a} f(x)$  exists if the following 3 criteria are met

1.  $\lim_{x \rightarrow a^-} f(x)$  exists
2.  $\lim_{x \rightarrow a^+} f(x)$  exists
3.  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$

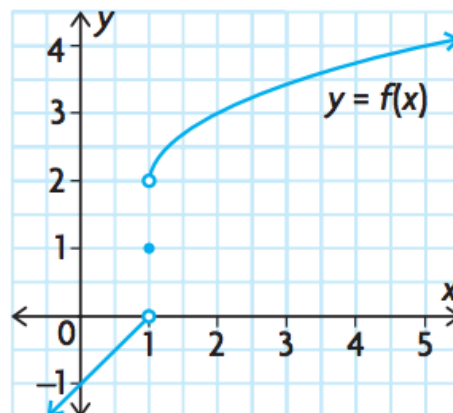
## Section 2: Find limits from a graph

**Example 2:** Given the following graph of the piecewise function  $f(x) = \begin{cases} x - 1, & \text{if } x < 1 \\ 1, & \text{if } x = 1 \\ 2 + \sqrt{x - 1}, & \text{if } x > 1 \end{cases}$

$$\lim_{x \rightarrow 1^-} f(x) = 0$$

$$\lim_{x \rightarrow 1^+} f(x) = 2$$

$\therefore \lim_{x \rightarrow 1} f(x)$  does not exist since  $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$

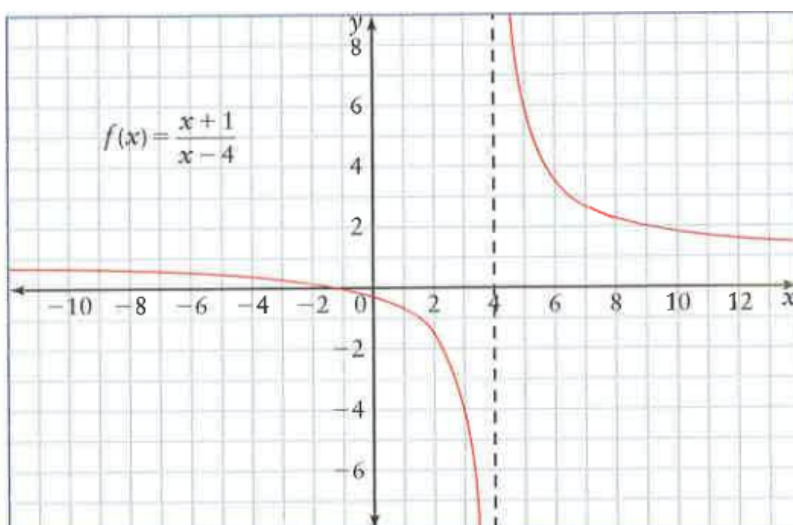


**Example 3:** Use the graph of  $f(x) = \frac{x+1}{x-4}$  to find:

$$\lim_{x \rightarrow 4^-} f(x) = -\infty$$

$$\lim_{x \rightarrow 4^+} f(x) = \infty$$

$\therefore \lim_{x \rightarrow 4} f(x)$  does not exist



$$\lim_{x \rightarrow \infty} f(x) = 1$$

$$\lim_{x \rightarrow -\infty} f(x) = 1$$

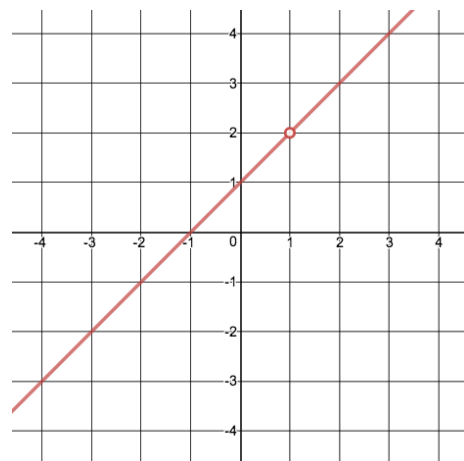
**Note:** If numerator and denominator are same degree, quotient of leading coefficients indicate where the horizontal asymptote will be.



### Section 3: Limit of a Function from Graph (indeterminate form)

**Example 4:** Analyze the equation, table, and graph of  $f(x) = \frac{x^2-1}{x-1}$  to find  $\lim_{x \rightarrow 1} f(x)$

x approaches 1 from the left $\rightarrow$						$\leftarrow$ x approaches 1 from the right					
x	0	0.5	0.9	0.99	0.999	1	1.001	1.01	1.1	1.5	2
$\frac{x^2-1}{x-1}$	1	1.5	1.9	1.99	1.999	undefined	2.001	2.01	2.1	2.5	3
$f(x) = \frac{x^2-1}{x-1}$ approaches 2 from below $\rightarrow$						$\leftarrow f(x) = \frac{x^2-1}{x-1}$ approaches 2 from above					



This table and graph suggests that

$$\lim_{x \rightarrow 1^-} f(x) = 2$$

$$\lim_{x \rightarrow 1^+} f(x) = 2$$

$$\therefore \lim_{x \rightarrow 1} f(x) = 2$$

However, when plugging in  $x = 1$  in to the equation, we get  $\frac{0}{0}$  which is called the indeterminate form. The function is not defined at this point. However, it is not necessary for the function to be defined at a point for its limit to exist. Try factoring to see if it has a removable discontinuity:

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \rightarrow 1} x + 1$$

$$= 1 + 1$$

$$= 2$$

#### Section 4: Limit of a Function from an Equation (indeterminate form)

**Example 5:** Find the following limits.

$$\text{a) } \lim_{x \rightarrow -1} \frac{x^2 - 4}{x^2 + 3}$$

$$= \frac{(-1)^2 - 4}{(-1)^2 + 3}$$

$$= \frac{-3}{4}$$

$$\text{b) } \lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x - 3}$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(x+1)}{x-3}$$

$$= \lim_{x \rightarrow 3} x + 1$$

$$= 4$$