L6 – 6.5 – Applications of Logarithms AND Exponentials in Physical Sciences
MHF4U

Exponential Formulas

$A(t) = A_0(1+i)^t$ — general, where i is percent growth(+) or decay(-)

$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$ — half-life, H is the half-life period

$A(t) = A_0(2)^{\frac{t}{D}}$ — doubling, D is the doubling period

Logarithmic Formulas

$pH = -\log[H^+]$ — Where pH is acidity and [H+] is concentration of hydronium ions mol/L

$\beta_2 - \beta_1 = 10\log\left(\frac{I_2}{I_1}\right)$ — Where β is loudness in dB and I is intensity of sound in W/m^2

$M = \log(\frac{I}{I_0})$ — Where M is magnitude measure by richters, I is intensity

1) The half-life of a radioactive form of tritium is about 2 years. How much of a 5-kg sample of this material would remain after …

a) 8 years

$A(8) = 5\left(\frac{1}{2}\right)^{8/2}$

$A(8) = 5\left(\frac{1}{2}\right)^4$

$A(8) = \frac{5}{16}$

$A(8) = 0.3125\text{ kg}$

b) 12 months = 1 year

$A(1) = 5\left(\frac{1}{2}\right)^{1/2}$

$\approx 3.536\text{ kg}$

2) The population of Littleton is currently (2014) 23000, and is increasing exponentially with a growth rate of 2% per year. Estimate when Littleton will have a population of 30000.

$30\,000 = 23\,000(1.02)^t$

$\left(\frac{30}{23}\right) = (1.02)^t$

$\log_{1.02}\left(\frac{30}{23}\right) = t$

$\frac{\log\left(\frac{30}{23}\right)}{\log 1.02} = t$

$t = 13.42\text{ years}$

3) The population of purple martins in Algonquin park was estimated to be 35000 in 1992. Ten years later, in 2002, the population had risen to 44400.

a) What is the annual growth rate for the purple martin population?

$44\,400 = 35000(b)^{10}$

$\frac{222}{175} = (b)^{10}$

$\sqrt[10]{\frac{222}{175}} = b$

$1.024 = b$

∴ growth rate is 2.4% per year.

b) Estimate the population for 2010 to the nearest hundred.

$A(8) = 44\,400\,(1+0.024)^8$

$A(8) = 53\,676.3$

$A(8) \simeq 53\,700$.

4) After an accident at a nuclear plant, the radiation level in the plant was 950 R (roentgens). Five hours later the level was 800 R. How long will it take before safe levels of radiation are reached, which is less than 0.01 R?

$800 = 950(b)^5$

$\sqrt[5]{\frac{800}{950}} = b$

$b = 0.9662$

$0.01 = 950(0.9662)^t$

$\frac{0.01}{950} = 0.9662^t$

$\frac{\log\left(\frac{0.01}{950}\right)}{\log(0.9662)} = t$

$t = 333.3$ hours

5) The value of a new minivan drops 40% after the first year, and then decreases exponentially at a rate of 12% per year after that. When will a minivan that cost $35000 new be worth less than $10000?

Value after 1st year $= 35000(0.6) = 21000$

$A(t) = A_0(1-i)^t$

$10\,000 = 21\,000(1-0.12)^t$

$\frac{10}{21} = (0.88)^t$

$\log_{0.88}\left(\frac{10}{21}\right) = t$

$\frac{\log\left(\frac{10}{21}\right)}{\log(0.88)} = t$

$t \simeq 5.8$ years

1 year at 40% and then 5.8 years at 12%; this means it would take about 6.8 years to be worth less than $10000

6) A crab fossil contains 38.6% of its original Carbon14 isotope, which has a half life of 5370 years. Approximately how old is the crab fossil?

$A(t) = A_0\left(\frac{1}{2}\right)^{t/H}$

$0.386(A_0) = A_0\left(\frac{1}{2}\right)^{t/5370}$

$\frac{0.386(A_0)}{A_0} = \frac{A_0\left(\frac{1}{2}\right)^{t/5370}}{A_0}$

$0.386 = \left(\frac{1}{2}\right)^{t/5370}$

$\log(0.386) = \log\left(\frac{1}{2}\right)^{t/5370}$

$\log(0.386) = \frac{t}{5370}\log\left(\frac{1}{2}\right)$

$\frac{\log(0.386)}{\log(0.5)} = \frac{t}{5370}$

$\frac{5370\log(0.386)}{\log(0.5)} = t$

$t = 7374.8$ years old

7) A Trimark mutual fund has track record of 4.2% growth per year. What is the doubling period for this investment?

$2 = 1(1.042)^t$

$\log 2 = \log(1.042)^t$

$\log 2 = t\log(1.042)$

$t = \frac{\log(2)}{\log(1.042)}$

$t = 16.85$ years

8) A treatment to help a patient stop smoking involves chewing nicotine gum. Each gum introduces 1.5 mg of nicotine into the patient's system. Nicotine has a half life of 3 hours. The patient will feel the urge to smoke when the level of nicotine drops below 0.45 mg in her system. If she chewed a gum at 8:00 a.m., and another at 10:00 a.m., at what time will she next feel the urge to smoke?

Concentration at 10:00 am

$A(2) = 1.5\left(\frac{1}{2}\right)^{2/3}$

$A(2) = 0.94494$

then +1.5 mg for the new piece

$= 2.44494$ mg

when will 2.44494 mg reduce to 0.45 mg?

$0.45 = 2.44494\left(\frac{1}{2}\right)^{t/3}$

$0.184053596 4 = \left(\frac{1}{2}\right)^{t/3}$

$\frac{\log 0.1840535964}{\log\left(\frac{1}{2}\right)} = \frac{t}{3}$

$\frac{3\log(0.1840535964)}{\log(0.5)} = t$

$t = 7.3$ hours or about 5:18 pm

9) Determine the pH of a solution with hydronium ion concentration:

a) 0.01

$pH = -\log(0.01)$

$pH = -(-2)$

$pH = 2$

b) 1.5×10^{-10}

$pH = -\log(1.5 \times 10^{-10})$

$pH \simeq 9.8$

$pH = -\log[H^+]$

10) Determine the hydronium concentration, in moles per litre, of a solution with pH:

a) 8.5

$8.5 = -\log[H^+]$

$-8.5 = \log[H^+]$

$[H^+] = 10^{-8.5}$

$[H^+] \simeq 3.16 \times 10^{-9}$ mol/L

b) 3

$3 = -\log[H^+]$

$-3 = \log[H^+]$

$[H^+] = 10^{-3}$

$[H^+] = 0.001$ mol/L

11) How many times as intense is the sound of a shout as the sound of a whisper?

$\beta_2 - \beta_1 = 10 \log\left(\frac{I_2}{I_1}\right)$

$80 - 30 = 10 \log\left(\frac{I_2}{I_1}\right)$

$5 = \log\left(\frac{I_2}{I_1}\right)$

$\frac{I_2}{I_1} = 10^5$

$\frac{I_2}{I_1} = 100\,000$

∴ A shout is 100 000 as intense as a whisper

12) A loud car stereo has a decibel level of 110 dB. How many times as intense as the sound of a loud car stereo is the sound of a rock concert speaker?

$150 - 110 = 10 \log\left(\frac{I_2}{I_1}\right)$

$4 = \log\left(\frac{I_2}{I_1}\right)$

$10^4 = \frac{I_2}{I_1}$

$10\,000 = \frac{I_2}{I_1}$

∴ Rock concert is 10 000 times as intense as a car stereo.

β_2 $\frac{I_2}{I_1}$ $= 60dB$

13) The sound intensity of a pin drop is about 1/30 000 of the sound intensity of a normal conversation. What is the decibel level of a pin drop? β_1

$$\beta_2 - 60 = 10 \log\left(\frac{1}{30000}\right)$$

$$\beta_2 = 10 \log\left(\frac{1}{30000}\right) + 60$$

$$\beta_2 \simeq 15.23 \text{ dB}$$

Decibel level of a pin drop is about 15.23 dB

14) On September 26, 2001, an earthquake in North Bay, Ontario, occurred that was 10 000 times as intense as I_0. What was the measure of this earthquake on the Richter scale?

$$M = \log\left(\frac{I}{I_0}\right)$$

$$M = \log(10000)$$

$M = 4$

15) On February 10, 2000, an earthquake happen in Welland, Ontario, that measured 2.3 on the Richter scale.

a) How many times as intense was this as a standard low-level earthquake?

$$2.3 = \log\left(\frac{I}{I_0}\right)$$

$$10^{2.3} = \left(\frac{I}{I_0}\right)$$

$$\frac{I}{I_0} \simeq 199.53$$

About 200 times as intense

b) On July 22, 2001, an earthquake in St. Catharines measured 1.1 on the Richter scale. How many times as intense as the St.Catharines earthquake was the Welland earthquake?

$$2.3 - 1.1 = 1.2$$

$$1.2 = \log\left(\frac{I}{I_0}\right)$$

$$10^{1.2} = \frac{I}{I_0}$$

$$\frac{I}{I_0} \simeq 15.85$$

About 16 times as intense

16) The stellar magnitude scale compares the brightness of stars using the equation $m_2 - m_1 = \log(\frac{b_1}{b_2})$, where m_2 *and* m_1 are the apparent magnitude of the two stars being compared (how bright they appear in the sky) and b_1 and b_2 are their brightness (how much light they actually emit). This relationship does not factor in how far om Earth the stars are.

a) Sirius is the brightest-appearing star in our sky, with an apparent magnitude of -1.5. How much brighter does Sirius appear than Betelgeuse, whose apparent magnitude in 0.12?

m_2 m_1

$0.12 - (-1.5) = \log(\frac{b_1}{b_2})$

$1.62 = \log(\frac{b_1}{b_2})$

$10^{1.62} = \frac{b_1}{b_2}$

$\frac{b_1}{b_2} \approx 41.7$

About 41.7 times brighter

b) The Sun appears about 1.3 x 10^{10} times as brightly in our sky as does Sirius. What is the apparent magnitude of the Sun?

m_2 m_1

$-1.5 - m_1 = \log(1.3 \times 10^{10})$

$-1.5 - \log(1.3 \times 10^{10}) = m_1$

$m_1 \approx -11.61$

ANSWER KEY – Exponentials

1. a) 8 years **b)** 12 months
0.3125 kg 3.536 kg

2. In approx. 13.418 years

3. a) Growth rate of 2.4% per year
3. b) Approx. 53 700 birds

4. Approx. 333 hours

5. Approx. 7 years

6. Approx. 7375 years

7. Approx. 16.85 years

8. $t \sim 7.3$ hours, so about 5:20 p.m

ANSWER KEY: Logarithms

9. a) 2 **b)** 9.8

10. a) 3.2×10^{-9} mol/L
b) 0.001 mol/L

11. A shout is 100,000 times more intense than a whisper

12. A rock concert speaker is 10,000 times more intense than a loud car speaker

13. A pin drop is 15dB

14. level 4 on the Richter scale

15. a) about 200 times more intense
b) 15.85 times more intense

16. a) 41.69 times brighter
b) apparent magnitude is -11.61