Final Review For MHF4U Advanced Functions SOLUTIONS

1. a. $x^3 + 3x^2 - 25x - 75$

$= x^2(x+3) - 25(x+3)$

$= (x^2-25)(x+3)$

$= (x+5)(x-5)(x+3)$

b. $64x^3 + 27y^3$ (sum of cubes)

$= (4x)^3 + (3y)^3$

$= (4x+3y)[(4x)^2 - (4x)(3y) + (3y)^2]$

$= (4x+3y)(16x^2 - 12xy + 9y^2)$

c. $2x^4 + 3x^3 - 11x^2 - 6x$

$= x(2x^3 + 3x^2 - 11x - 6)$

$f(2) = 0 \Rightarrow x-2$ is a factor

2	2	3	-11	-6
		4	14	6
	2	7	3	0

$= x(x-2)(2x^2+7x+3)$

$= x(x-2)(2x+1)(x+3)$

2. Graph $y = (x-3)^2\underbrace{(x^2-4x-7)}_{\text{Find roots}}$

$x = \frac{4 \pm \sqrt{16-4(1)(-7)}}{2}$

$= \frac{4 \pm \sqrt{44}}{2}$

$= \frac{4 \pm 2\sqrt{11}}{2}$

$= 2 \pm \sqrt{11}$

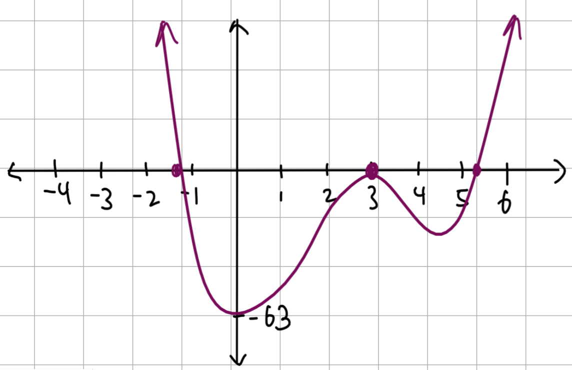
$x = 2+\sqrt{11} \doteq 5.32$

$x = 2-\sqrt{11} \doteq -1.32$

Roots: 3 (double), 5.32, -1.32

y-intercept: $y = (-3)^2(-7) = -63$

Degree 4 and leading coefficient is (+) ⇒ Q2 to Q1



3. Solve the following

a. $x^3 - 5x = 5x^2 - 1$

$\underbrace{x^3 - 5x^2 - 5x + 1}_{f(x)} = 0$

$f(-1) = 0 \Rightarrow x+1$ is a factor

-1	1	-5	-5	1
		-1	6	-1
	1	-6	1	0

$(x+1)(x^2-6x+1) = 0$

$x = \frac{6 \pm \sqrt{36-4(1)(1)}}{2}$

$= \frac{6 \pm \sqrt{32}}{2}$

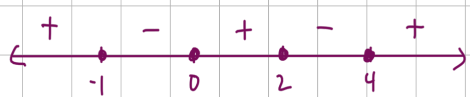
$= \frac{6 \pm 4\sqrt{2}}{2}$

$= 3 \pm 2\sqrt{2}$

$\boxed{\therefore x = \{-1, 3+2\sqrt{2}, 3-2\sqrt{2}\}}$

b. $x(x+1)(x-2)(x-4) > 0$

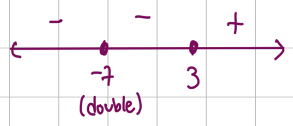
Roots: $x = 0, -1, 2, 4$



$\boxed{\therefore x \in (-\infty, -1) \cup (0, 2) \cup (4, \infty)}$

c. $(x+7)^2(x-3)^3 < 0$

Roots: $x = -7$ (double), 3 (triple)



$\therefore x \in (-\infty, -7) \cup (-7, 3)$

d. $2x^3 + 3x^2 - 11x \geq 6$

$\underbrace{2x^3 + 3x^2 - 11x - 6}_{f(x)} \geq 0$

$f(2) = 0 \Rightarrow x - 2$ is a factor

2	2	3	−11	−6
		4	14	6
	2	7	3	0

$(x-2)(2x^2+7x+3) \geq 0$

$(x-2)(2x+1)(x+3) \geq 0$

Roots: $x = 2, -\frac{1}{2}, -3$

$\boxed{\therefore x \in [-3, -\tfrac{1}{2}] \cup [2, \infty)}$

4. Determine if each of the following functions are even, odd, or neither

a. $f(x) = \frac{1}{x^3+1}$ [N]

$f(-x) = \frac{1}{(-x)^3+1}$

$= \frac{1}{-x^3+1}$

$\neq f(x) \neq -f(x)$

$\therefore$ It's neither

b. $f(x) = 2x^4 + 3x^2$ [E]

$f(-x) = 2(-x)^4 + 3(-x)^2$

$= 2x^4 + 3x^2$

$= f(x)$

$\therefore$ It's even

c. $f(x) = \left(\frac{1}{x^3+x}\right)^5$ [O]

$f(-x) = \left(\frac{1}{(-x)^3+(-x)}\right)^5$

$= \left(\frac{1}{-x^3-x}\right)^5$

$= -\left(\frac{1}{x^3+x}\right)^5$

$= -f(x)$

$\therefore$ It's odd

5. When $x^4 - 4x^3 + ax^2 + bx + 1$ is divided by $(x-1)$, the remainder is 7. When it is divided by $(x+1)$, the remainder is 3. Determine the values of a and b. $[a = 3, b = 6]$

Let $f(x) = x^4 - 4x^3 + ax^2 + bx + 1$

$f(1) = 7 \Rightarrow 1 - 4 + a + b + 1 = 7$

$a + b - 2 = 7$

$a + b = 9$ — ①

$f(-1) = 3 \Rightarrow 1 + 4 + a - b + 1 = 3$

$a - b + 6 = 3$

$a - b = -3$ — ②

① + ②: $a + b = 9$

$+)\ a - b = -3$

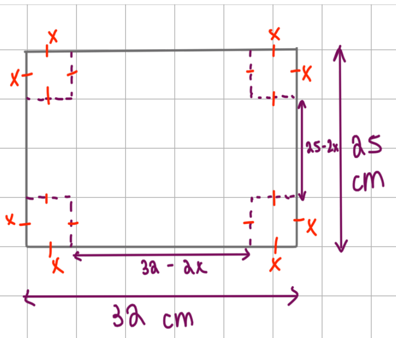
$2a = 6$

$\boxed{a = 3}$

Sub $a = 3$ into ①: $3 + b = 9$

$\boxed{b = 6}$

6. An open box, no more than 5 cm in height, is to be formed by cutting four identical squares from the corners of a sheet of metal 25 cm by 32 cm, and folding up the metal to form sides. The volume of the box must be 1650 cm^3. What is the side length of the squares removed? $\left[\frac{1}{4}(47-\sqrt{889})\ cm\right]$



$h < 5$ cm $\quad V = 1650$ cm^2

$V = lwh$

$1650 = (32-2x)(25-2x)(x)$

$1650 = x(800 - 64x - 50x + 4x^2)$

$1650 = x(4x^2 - 114x + 800)$

$0 = 4x^3 - 114x^2 + 800x - 1650$

$0 = (x-5)(4x^2 - 94x + 330)$

5	4	-114	800	1650
		20	-470	1650
	4	-94	330	0

$x = \dfrac{94 \pm \sqrt{(94)^2 - 4(4)(330)}}{2(4)}$

$= \dfrac{94 \pm \sqrt{3556}}{8}$

$= \dfrac{94 \pm 2\sqrt{889}}{8}$

$= \dfrac{47 \pm \sqrt{889}}{4}$

$x = 5$, $\quad x = \dfrac{47+\sqrt{889}}{4}$, $\quad x = \dfrac{47-\sqrt{889}}{4}$

inadmissible since $h < 5$

inadmissible since $\frac{47+\sqrt{889}}{4} > 32$

$\therefore$ The side length of the square is $\dfrac{47-\sqrt{889}}{4}$ cm.

7. The population of a town is modelled by $p(t) = 6t^2 + 110t + 3000$, where P is the population and t is the number of years since 1990. Find the average rate of change in population between 1995 and 2005. $\left[230\,\frac{people}{year}\right]$

$p(15) = 6(15)^2 + 110(15) + 3000$

$= 6000$

$p(5) = 6(5)^2 + 110(5) + 3000$

$= 3700$

$A.R.O.C = \dfrac{p(15) - p(5)}{15 - 5}$

$= \dfrac{6000 - 3700}{10}$

$= 230$

$\therefore$ The average rate of change is 230 people/year

8. Determine the instantaneous rate of change of $f(x) = x^3 + x^2$ at $x = 2$ [16]

$I.R.O.C = \dfrac{f(2.001) - f(2)}{0.001} = \dfrac{12.016007 - 12}{0.001} = \dfrac{0.016007}{0.001} \doteq 16$

$\therefore$ The instantaneous rate of change is 16.

9. The graph of $f(x) = x^4$ is horizontally stretched by a factor of 2, reflected in the y-axis, and shifted up 5 units. Find the equation of the transformed function. $\left[f(x) = \left(-\frac{1}{2}x\right)^4 + 5\right]$

$f(x) = \left(-\frac{1}{2}x\right)^4 + 5$

10. What other set of transformations applied to $f(x) = x^4$ would yield the same graph as that in #9?

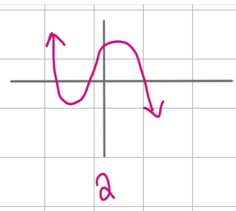
$f(x) = \left(-\frac{1}{2}x\right)^4 + 5$

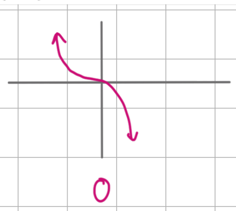
$= \frac{1}{16}x^4 + 5$

- vertical compression by a factor of 1/16
- 5 units up

11. A polynomial of degree 5 has a negative leading coefficient

a. How many turning points // local max/mins could the polynomial have? $[0, 2, or\ 4]$

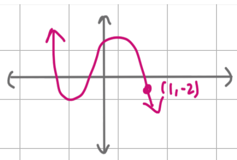


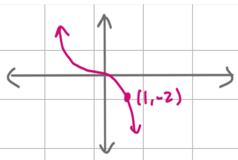


b. How many zeros could the function have? $[at\ most\ 5]$

c. Describe the end behaviour. $[Q2\ to\ Q4]$

d. Sketch two possible graphs, each passing through the point $(1, -2)$





12. A polynomial has a bounce at $x = -1$ and two x-intercepts of order 1 at $x = 2$ and $x = 4$.

a. Determine the equation of the polynomial if it goes through the point $(1, -24)$. $[a = -2]$

$f(x) = a(x+1)^2(x-2)(x-4)$

Sub. $(1,-24)$: $-24 = a(1+1)^2(1-2)(1-4)$

$-24 = a(2)^2(-1)(-3)$

$-24 = 12a$

$-2 = a$

$\boxed{\therefore f(x) = -2(x+1)^2(x-2)(x-4)}$

b. Solve for p if $(3, p)$ is a point on the graph of the function $[p = 32]$

Sub. $(3,p)$: $p = -2(3+1)^2(3-2)(3-4)$

$p = -2(16)(1)(-1)$

$p = 32$

Unit 2 Questions

13. Sketch the graph of $f(x) = \frac{2x^2+x-3}{x^2-4x+3} = \frac{(2x+3)\cancel{(x-1)}}{(x-3)\cancel{(x-1)}} = \frac{2x+3}{x-3}, x \neq 1$

Hole: $f(1) = \frac{2(1)+3}{1-3} = \frac{5}{-2}$

$\boxed{\therefore \text{Hole at } (1, -\frac{5}{2})}$

$\boxed{\text{y-intercept: } (0,-1)}$

$\boxed{\text{Root: } (-3/2, 0)}$

$\boxed{V.A: x = 3}$

As $x \to 3^-$, $f(x) = -\infty$

As $x \to 3^+$, $f(x) = \infty$

$\boxed{H.A: y = 2}$

As $x \to \infty$, $f(x) \to 2$ (above)

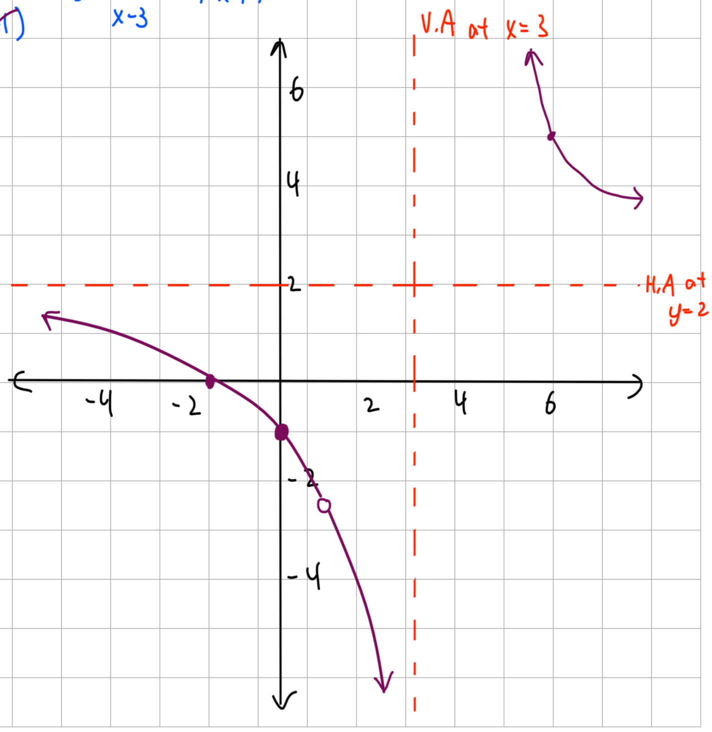
As $x \to -\infty$, $f(x) \to 2$ (below

Cross-over: $\frac{2x+3}{x-3} = 2$

$2x+3 = 2x-6$

$3 \neq -6$

$\therefore$ No cross-overs



14. Sketch the graph of $f(x) = \frac{x^2+4}{x^2-4} = \frac{x^2+4}{(x+2)(x-2)}$

$\boxed{\text{y-intercept: } (0,-1)}$

$\boxed{\text{Root: None}}$

$\boxed{\text{V.A: } x=-2}$

As $x \to -2^+$, $f(x) = -\infty$

As $x \to -2^-$, $f(x) = \infty$

$\boxed{\text{V.A: } x=2}$

As $x \to 2^+$, $f(x) = \infty$

As $x \to 2^-$, $f(x) = -\infty$

$\boxed{\text{H.A: } y=1}$

As $x \to \infty$, $f(x) \to 1$ (above)

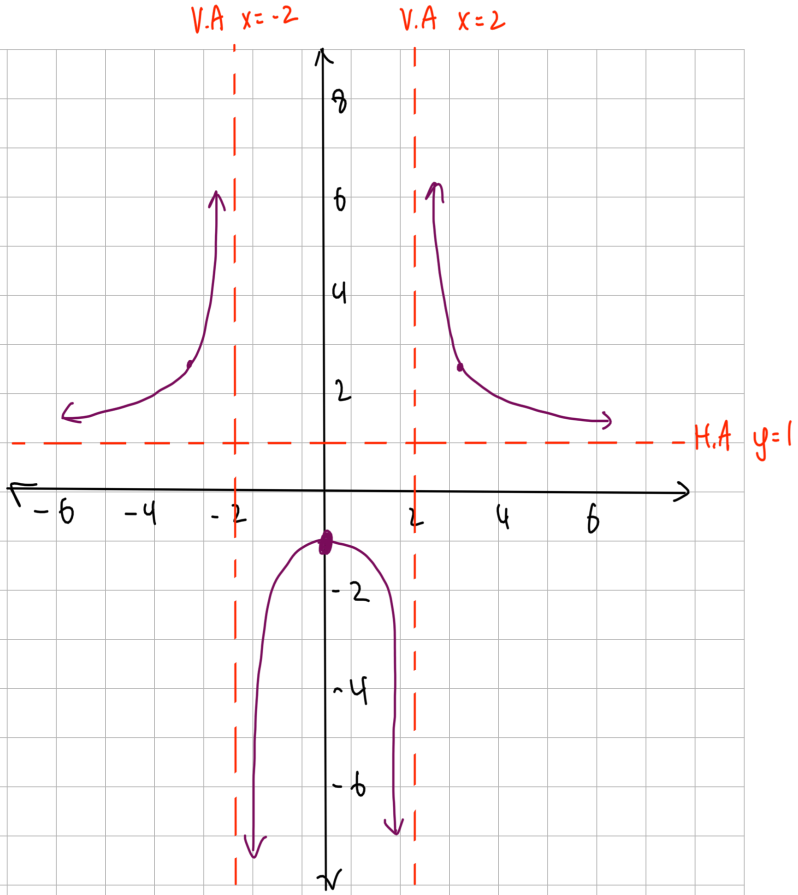
As $x \to -\infty$, $f(x) \to 1$ (above)

Cross-over: $\frac{x^2+4}{x^2-4} = 1$

$x^2+4 = x^2-4$

$4 \neq -4$

$\therefore$ No cross-overs



15. Solve the following using any method you want.

a. $x + \frac{1}{x-4} = 0$ $\quad [2 \pm \sqrt{3}]$

$\frac{x(x-4)+1}{x-4} = 0$

$\frac{x^2-4x+1}{x-4} = 0$

$x^2-4x+1 = 0$, $x \neq 4$

$x = \frac{4 \pm \sqrt{16-4(1)(1)}}{2}$

$x = \frac{4 \pm \sqrt{12}}{2}$

$x = \frac{4 \pm 2\sqrt{3}}{2}$

$\boxed{x = 2 \pm \sqrt{3}}$

b. $\frac{2x}{x-1} + \frac{1}{x-3} = \frac{2}{x^2-4x+3}$ $\quad \left[-\frac{1}{2}\right]$

$\frac{2x(x-3)+(x-1)}{(x-1)(x-3)} = \frac{2}{(x-1)(x-3)}$

$2x^2-6x+x-1 = 2$, $x \neq 1,3$

$2x^2-5x-3 = 0$

$(2x+1)(x-3) = 0$

$\boxed{x = -\frac{1}{2}}$ $\quad x=3$ ↳ inadmissible

c. $\frac{2x}{x-1} + \frac{1}{x-3} \geq \frac{2}{x^2-4x+3}$

simplified in #15b)

$\frac{(2x+1)\cancel{(x-3)}}{(x-1)\cancel{(x-3)}} \geq 0$

$\frac{2x+1}{x-1} \geq 0$

Hole at $x=3$

Root: $x=-1/2$

V.A: $x=1$

$\boxed{\therefore x \in (-\infty, -\tfrac{1}{2}] \cup (1,3) \cup (3,\infty)}$

d. $\frac{5}{x+3} < -\frac{3}{x-1}$

$\frac{5}{x+3} + \frac{3}{x-1} < 0$

$\frac{5(x-1)+3(x+3)}{(x+3)(x-1)} < 0$

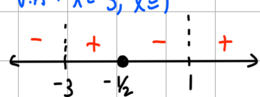
$\frac{5x-5+3x+9}{(x+3)(x-1)} < 0$

$\frac{8x+4}{(x+3)(x-1)} < 0$

$\frac{4(2x+1)}{(x+3)(x-1)} < 0$

Root: $x=-1/2$

V.A: $x=-3$, $x=1$



$\boxed{\therefore x \in (-\infty,-3) \cup (-\tfrac{1}{2}, 1)}$

Unit 3 and 4 Questions

16. Convert the following

a. $\frac{11\pi}{15}$ to degrees [132°]

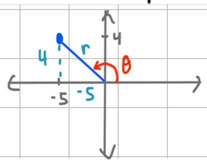
b. 420° to radians $\left[\frac{7\pi}{3}\right]$

$\frac{11\pi}{15} \cdot \frac{180}{\pi} = 132°$

$420° \cdot \frac{\pi}{180°} = \frac{7\pi}{3}$

17. The point P(-5, 4) is on the terminal arm of an angle measure of θ in standard position.

a. Sketch the principal angle



b. Determine the exact value of $\sin(\theta)$ $\left[\frac{4}{\sqrt{41}}\right]$

$r^2 = (4)^2 + (-5)^2$

$r = \sqrt{16 + 25}$ $\quad \therefore \sin\theta = \frac{4}{\sqrt{41}}$

$r = \sqrt{41}$

c. Determine the exact value of $\cos\left(\theta - \frac{\pi}{6}\right)$ $\left[\frac{-5\sqrt{3}+4}{2\sqrt{41}}\right]$

$\cos\left(\theta - \frac{\pi}{6}\right) = \cos\theta\cos\left(\frac{\pi}{6}\right) + \sin\theta\sin\left(\frac{\pi}{6}\right)$

$= \left(\frac{-5}{\sqrt{41}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{4}{\sqrt{41}}\right)\left(\frac{1}{2}\right)$

$= \frac{-5\sqrt{3}+4}{2\sqrt{41}} \cdot \frac{\sqrt{41}}{\sqrt{41}}$

$= \frac{-5\sqrt{123} + 4\sqrt{41}}{82}$

d. Determine the value of θ in degrees and radians. $[\theta = 141.34° \text{ or } 2.4669]$

R.A.A $= \sin^{-1}\left(\frac{4}{\sqrt{41}}\right)$

$\theta = \pi - \sin^{-1}\left(\frac{4}{\sqrt{41}}\right) \doteq 2.4669$ rads or $141.34°$

18. Find the exact value of

a. $\cos\left(\frac{3\pi}{4}\right)$ $\left[-\frac{1}{\sqrt{2}}\right]$

$= \cos\left(\pi - \frac{\pi}{4}\right)$

$= -\cos\left(\frac{\pi}{4}\right)$

$= -1/\sqrt{2}$

$= -\sqrt{2}/2$

b. $\csc\left(-\frac{3\pi}{2}\right)$ [1]

$= \frac{1}{\sin(-3\pi/2)}$

$= \frac{1}{-\sin(3\pi/2)}$ sine is an odd function

$= \frac{1}{-(-1)}$

$= 1$

c. $\sin\left(\frac{7\pi}{12}\right)$ $\left[\frac{\sqrt{6}+\sqrt{2}}{4}\right]$

$= \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$

$= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$

$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{2}+\sqrt{6}}{4}$

d. $\sec\left(\frac{5\pi}{6}\right)\cos\left(\frac{7\pi}{4}\right) - \cot\left(-\frac{\pi}{3}\right)$ $\left[\frac{\sqrt{3}-\sqrt{6}}{3}\right]$

$= \frac{\cos\left(2\pi - \frac{\pi}{4}\right)}{\cos\left(\pi - \frac{\pi}{6}\right)} + \cot\left(\frac{\pi}{3}\right)$ cot is an odd function

$= \frac{\cos\left(\frac{\pi}{4}\right)}{-\cos\left(\frac{\pi}{6}\right)} + \frac{1}{\tan\left(\frac{\pi}{3}\right)}$

$= \frac{\sqrt{2}/2}{-\sqrt{3}/2} + \frac{1}{\sqrt{3}}$

$= \frac{\sqrt{2}}{\cancel{2}} \cdot \frac{-\cancel{2}}{\sqrt{3}} + \frac{1}{\sqrt{3}}$

$= \frac{-\sqrt{2}+1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{-\sqrt{6}+\sqrt{3}}{3}$

e. $\cos\left(\frac{\pi}{8}\right)$ $\left[\frac{\sqrt{2+\sqrt{2}}}{2}\right]$

$\cos\left(\frac{\pi}{4}\right) = 2\cos^2\left(\frac{\pi}{8}\right) - 1$

$\frac{\sqrt{2}}{2} = 2\cos^2\left(\frac{\pi}{8}\right) - 1$

$\frac{\sqrt{2}}{2} + 1 = 2\cos^2\left(\frac{\pi}{8}\right)$

$\frac{\sqrt{2}+2}{4} = \cos^2\left(\frac{\pi}{8}\right)$

$\frac{\sqrt{\sqrt{2}+2}}{2} = \cos\left(\frac{\pi}{8}\right)$

f. $\csc\left(\frac{7\pi}{12}\right) = \dfrac{1}{\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)}$

$= \dfrac{1}{\frac{\sqrt{2}+\sqrt{6}}{4}}$ calculated in #18c)

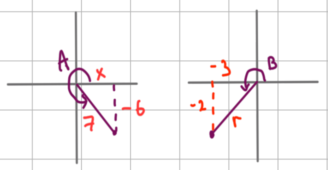
$= \dfrac{4}{\sqrt{2}+\sqrt{6}} \cdot \dfrac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}}$

$= \dfrac{4\sqrt{2}-4\sqrt{6}}{-4}$

$= \sqrt{6}-\sqrt{2}$

19. Given $\sin(A) = -\frac{6}{7}$ where $\frac{3\pi}{2} \le A \le 2\pi$ and $\tan(B) = \frac{2}{3}$ where $\pi \le B \le \frac{3\pi}{2}$. Determine the exact value of

a. $Sin(A+B)$ $\left[\frac{18\sqrt{13}-26}{91}\right]$



$x^2+(-6)^2=7^2$

$x^2 = 13$

$x = \sqrt{13}$

$(3)^2+(2)^2=r^2$

$9+4=r^2$

$\sqrt{13}=r$

$\sin(A+B) = \sin A\cos B + \cos A \sin B$

$= \left(-\frac{6}{7}\right)\left(\frac{-3}{\sqrt{13}}\right) + \left(\frac{\sqrt{13}}{7}\right)\left(-\frac{2}{\sqrt{13}}\right)$

$= \dfrac{18}{7\sqrt{13}} - \dfrac{2}{7}$

$= \dfrac{18-2\sqrt{13}}{7\sqrt{13}} \cdot \dfrac{\sqrt{13}}{\sqrt{13}}$

$= \dfrac{18\sqrt{13}-26}{91}$

b. $Cos(2B)$ $\left[\frac{5}{13}\right]$

$\cos(2B) = 2\cos^2(B)-1$

$= 2\left(\frac{-3}{\sqrt{13}}\right)^2-1$

$= 2\left(\frac{9}{13}\right)-1$

$= \frac{18}{13}-1$

$= 5/13$

20. Solve the following

a. $\cos(2x) = \cos(x), 0 \le x \le 2\pi$ $\left[0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi\right]$

$\cos(2x) - \cos(x) = 0$

$2\cos^2(x) - 1 - \cos(x) = 0$

$2\cos^2(x) - \cos(x) - 1 = 0$

Let $A = \cos(x)$: $2A^2 - A - 1 = 0$

$(2A+1)(A-1) = 0$

$2A+1=0$

$\cos(x) = -\frac{1}{2}$

R.A.A $= \frac{\pi}{3}$

$\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\theta_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

$A-1=0$

$\cos(x)=1$

$\theta_3 = 0$

$\theta_4 = 2\pi$

$\boxed{\therefore \theta \in \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi\right\}}$

b. $\sqrt{2}\tan(x)\cos(x) = \tan(x), 0 \le x \le 2\pi$ $\left[0, \frac{\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi\right]$

$\sqrt{2}\tan(x)\cos(x) - \tan(x) = 0$

$\tan(x)\left[\sqrt{2}\cos(x) - 1\right] = 0$

$\tan(x) = 0$

$x_1 = 0$

$x_2 = \pi$

$x_2 = 2\pi$

$\sqrt{2}\cos(x) - 1 = 0$

$\cos(x) = \frac{1}{\sqrt{2}}$

$R.A.A = \frac{\pi}{4}$

$x_3 = \frac{\pi}{4}$

$x_4 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$\boxed{\therefore x \in \left\{0, \frac{\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi\right\}}$

c. $2\cos(2x) = 1, 0 \le x \le 2\pi$ $\left[\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right]$

$\cos(2x) = \frac{1}{2}$, $0 \le 2x \le 4\pi$

$R.A.A = \frac{\pi}{3}$

$2x_1 = \frac{\pi}{3}$

$x_1 = \frac{\pi}{6}$

$2x_2 = 2\pi - \frac{\pi}{3}$

$2x_2 = \frac{5\pi}{3}$

$x_2 = 5\pi/6$

$2x_3 = 2\pi + \frac{\pi}{3}$

$2x_3 = \frac{7\pi}{3}$

$x_3 = 7\pi/6$

$2x_4 = 4\pi - \frac{\pi}{3}$

$2x_4 = \frac{11\pi}{3}$

$x_4 = 11\pi/6$

$\boxed{\therefore x \in \left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}}$

d. $\cos(2x) = -0.9541$ $[x = 1.4187 + k\pi, k\epsilon I, and\ x = 1.7229 + k\pi, k\epsilon I]$

$R.A.A \doteq 0.30416$

$2x_1 \doteq \pi - 0.30416$

$2x_1 \doteq 2.83744$

$x_1 = 1.4187$

$2x_2 \doteq \pi + 0.30416$

$2x_2 \doteq 3.44575$

$x_2 \doteq 1.7229$

$\boxed{\therefore x \in \{1.4187 + k\pi, 1.7229 + k\pi, k \in I\}}$

e. $6\sin^2(x) - 5\cos(x) - 2 = 0, 0 \le x \le 2\pi$ $\left[x = \frac{\pi}{3}, \frac{5\pi}{3}\right]$

$6(1 - \cos^2(x)) - 5\cos(x) - 2 = 0$

$6 - 6\cos^2(x) - 5\cos(x) - 2 = 0$

$-6\cos^2(x) - 5\cos(x) + 4 = 0$

$6\cos^2(x) + 5\cos(x) - 4 = 0$

Let $A = \cos(x)$: $6A^2 + 5A - 4 = 0$

$(3A + 4)(2A - 1) = 0$

$3A + 4 = 0$

$\cos(x) = -4/3$

↳ No solutions

$2A - 1 = 0$

$\cos(x) = \frac{1}{2}$

$R.A.A = \frac{\pi}{3}$

$x_1 = \frac{\pi}{3}$

$x_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\boxed{\therefore x \in \left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}}$

21. Prove that $f(x) = \tan(x)$ is an odd function.

$$f(-x) = \tan(-x) = \frac{\sin(-x)}{\cos(-x)}$$
$$= \frac{\sin(x)}{-\cos(x)}$$
$$= -\tan(x)$$
$$= -f(x)$$

$\therefore f(x)$ is an odd function

22. Prove the following

a. $\cos(x) + \sin(x) = \frac{1+\tan(x)}{\sec(x)}$

$$RS = \left(1 + \frac{\sin x}{\cos x}\right) \div \frac{1}{\cos(x)}$$
$$= \frac{\cos x + \sin x}{\cancel{\cos(x)}} \cdot \frac{\cancel{\cos(x)}}{1}$$
$$= \cos(x) + \sin(x)$$
$$= LS$$

b. $\frac{1}{1-\sec(x)} + \frac{1}{1+\sec(x)} = -2\cot^2(x)$

$$LS = \frac{1+\sec(x) + 1-\sec(x)}{(1-\sec(x))(1+\sec(x))}$$
$$= \frac{2}{1-\sec^2(x)}$$

$\tan^2(x) + 1 = \sec^2(x)$

$1 - \sec^2(x) = -\tan^2(x)$

$$= \frac{2}{-\tan^2(x)}$$
$$= -2\cot^2(x)$$
$$= RS$$

c. $\cos^2(2\theta) - \cos^2(\theta) = \sin^2(\theta) - \sin^2(2\theta)$

$$LS = [\cos(2\theta)]^2 - \cos^2\theta$$
$$= [1 - 2\sin^2\theta]^2 - (1 - \sin^2\theta)$$
$$= 1 - 4\sin^2\theta + 4\sin^4\theta - 1 + \sin^2\theta$$
$$= \sin^2\theta - 4\sin^2\theta + 4\sin^4\theta$$
$$= \sin^2\theta - 4\sin^2\theta[1 - \sin^2\theta]$$
$$= \sin^2\theta - 4\sin^2\theta[\cos^2\theta]$$
$$= \sin^2\theta - [2\sin\theta\cos\theta]^2$$
$$= \sin^2\theta - \sin^2(2\theta)$$
$$= RS$$

d. $\cos(x+y)\cos(x-y) = \cos^2(x) + \cos^2(y) - 1$

$$LS = [\cos(x)\cos(y) - \sin(x)\sin(y)][\cos(x)\cos(y) + \sin(x)\sin(y)]$$
$$= [\cos(x)\cos(y)]^2 - [\sin(x)\sin(y)]^2$$
$$= \cos^2(x)\cos^2(y) - \sin^2(x)\sin^2(y)$$
$$= \cos^2(x)\cos^2(y) - [1-\cos^2(x)][1-\cos^2(y)]$$
$$= \cancel{\cos^2(x)\cos^2(y)} - [1 - \cos^2(y) - \cos^2(x) + \cancel{\cos^2(x)\cos^2(y)}]$$
$$= \cos^2(x) + \cos^2(y) - 1$$
$$= RS$$

e. $\frac{\cos(x)}{1+\sin(x)} + \frac{\cos(x)}{1-\sin(x)} = \frac{2}{\cos(x)}$

$$LS = \frac{\cos(x)[1-\sin(x)] + \cos(x)[1+\sin(x)]}{(1+\sin x)(1-\sin x)}$$
$$= \frac{\cos(x) - \cancel{\cos(x)\sin(x)} + \cos(x) + \cancel{\cos(x)\sin(x)}}{(1+\sin x)(1-\sin x)}$$
$$= \frac{2\cos(x)}{(1+\sin x)(1-\sin x)}$$
$$= \frac{2\cos(x)}{1-\sin^2(x)}$$
$$= \frac{2\cos(x)}{\cos^2(x)}$$
$$= \frac{2}{\cos(x)}$$
$$= RS$$

f. $\sin(\pi + x) + \cos\left(\frac{\pi}{2} - x\right) + \tan\left(\frac{\pi}{2} + x\right) = -\cot(x)$

$$LS = -\sin(x) + \sin(x) + (-\cot(x))$$
$$= -\cot(x)$$
$$= RS$$

9. $\dfrac{\sin(\pi-x)\cos(\pi+x)\tan(2\pi-x)}{\sec\left(\frac{\pi}{2}+x\right)\csc\left(\frac{3\pi}{2}-x\right)\cot\left(\frac{3\pi}{2}+x\right)} = \sin^4(x) - \sin^2(x)$

$$LS = \frac{\sin(x)[\cos(x)][-\tan(x)]}{[-\csc(x)][-\sec(x)][-\tan(x)]}$$

$$= \frac{\sin(x)\cancel{\cos(x)} \cdot \frac{-\sin(x)}{\cancel{\cos(x)}}}{-\frac{1}{\cancel{\sin(x)}} \cdot \frac{1}{\cos(x)} \cdot -\frac{\cancel{\sin(x)}}{\cos(x)}}$$

$$= \frac{-\sin^2(x)}{1/\cos^2(x)}$$

$$= -\sin^2(x)\cos^2(x)$$

$$= -\sin^2(x)[1-\sin^2(x)]$$

$$= -\sin^2(x) + \sin^4(x)$$

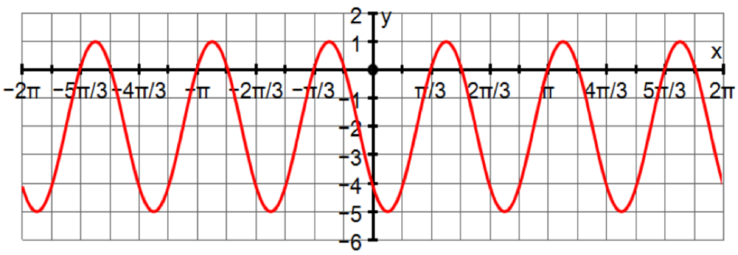
$$= \sin^4(x) - \sin^2(x)$$

$$= RS$$

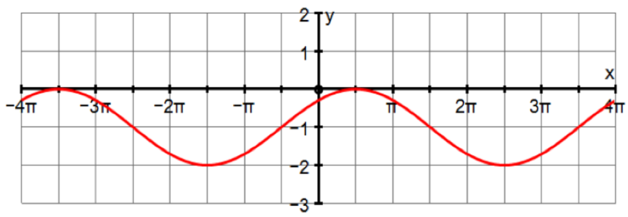
23. How does $y = -2\cos\left[\frac{1}{3}\left(x-\frac{\pi}{2}\right)\right] + 1$ compare to $f(x) = \cos(x)$?

- Reflected on the x-axis
- vertically stretched by a factor of 2 $\Rightarrow$ amplitude is 2
- horizontally stretched by a factor of 3 $\Rightarrow$ period is $2\pi/3$
- phase shift $\frac{\pi}{2}$ units right
- vertical displacement 1 unit up

24. Graph $y = -3\cos\left(3x - \frac{\pi}{4}\right) - 2$



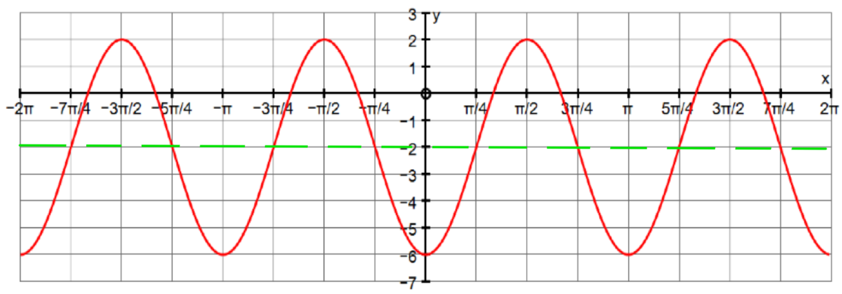
25. Graph $y = \sin\left(\frac{1}{2}x + \frac{\pi}{4}\right) - 1$



26. Determine the equation of $f(x)$ given the following graph

a. in terms of sine $f(x) = 4\sin\left[2\left(x-\frac{\pi}{4}\right)\right] - 2$

b. in terms of cosine $f(x) = -4\cos(2x) - 2$



$a = \frac{2-(-6)}{2} = 4$

$k = \frac{2\pi}{P} = \frac{2\pi}{\pi} = 2$

$c = \frac{2+(-6)}{2} = -2$

$d = 0 \rightarrow$ for $-$cosine

$d = \frac{\pi}{4} \rightarrow$ for sine

Unit 5 Questions

27. Write as a single logarithm

a. $alog_5(x-7) - \frac{2}{3}\log_5(w) + 2$ $\left[\log_5\left(\frac{25(x-7)^a}{w^{\frac{2}{3}}}\right)\right]$

$= \log_5(x-7)^a - \log_5(w)^{2/3} + \log_5 25$

$= \log_5\left[\frac{25(x-7)^a}{w^{2/3}}\right]$

b. $2\log_3\left(\frac{x^2-25}{x-5}\right) + \frac{7}{\log_{x+5}(9)}$ $\left[\log_3\left((x+5)^{\frac{11}{2}}\right)\right]$

$= 2\log_3\left[\frac{(x+5)\cancel{(x-5)}}{\cancel{x-5}}\right] + \frac{7}{\frac{1}{\log_9(x+5)}}$

$= 2\log_3(x+5) + 7\log_{3^2}(x+5)$

$= 2\log_3(x+5) + \frac{7}{2}\log_3(x+5)$

$= \frac{11}{2}\log_3(x+5)$

$= \log_3(x+5)^{11/2}$, $x \neq 5$

28. Determine the exact value of

a. $\log_8(6) - \log_8(3) + \log_8(4)$ [1]

$= \log_8\left[\frac{6(4)}{3}\right]$

$= \log_8(8)$

$= 1$

b. $\log_9(3^7\sqrt[5]{81})$ $\left[\frac{39}{10}\right]$

$= \log_{3^2}\left(3^7 \cdot (3^4)^{\frac{1}{5}}\right)$

$= \frac{1}{2}\log_3\left(3^{39/5}\right)$

$= \frac{39}{5}\left(\frac{1}{2}\right)\log_3 3$

$= 39/10$

29. Solve the following

a. $\log_4(x+3) = 2$ [13]

$x+3 = 4^2$

$\boxed{x = 13}$

b. $\log_7(x+2) = 1 - \log_7(x-4)$ [5]

$\log_7(x+2) + \log(x-4) = 1$

$\log_7\left((x+2)(x-4)\right) = 1$

$(x+2)(x-4) = 7^1$

$x^2 - 2x - 8 = 7$

$x^2 - 2x - 15 = 0$

$(x-5)(x+3) = 0$

$\boxed{\therefore x = 5}$ $x = -3 \rightarrow$ inadmissible

c. $\log_9(x-5) + \log_9(x+3) = 1$ [6]

$\log_9[(x-5)(x+3)] = 1$

$(x-5)(x+3) = 9^1$

$x^2 - 2x - 15 = 9$

$x^2 - 2x - 24 = 0$

$(x-6)(x+4) = 0$

$\boxed{x = 6}$ $x = -4 \rightarrow$ inadmissible

d. $\log_5(x+1) + \log_5(2) - \log_5(x+3) = \log_5(x-1)$ $[\sqrt{5}]$

$$\log_5(x+1) + \log_5(2) - \log_5(x+3) - \log_5(x-1) = 0$$

$$\log_5\left[\frac{2(x+1)}{(x+3)(x-1)}\right] = 0$$

$$\frac{2x+2}{x^2+2x-3} = 5^0$$

$$2x+2 = x^2+2x-3$$

$$5 = x^2$$

$$\boxed{\sqrt{5} = x}$$

($x = -\sqrt{5}$ is inadmissible)

e. $5 \cdot 8^{x+2} = 5^{7x}$ $[approx\ 0.627]$

$$8^{x+2} = \frac{5^{7x}}{5}$$

$$\log(8^{x+2}) = \log(5^{7x-1})$$

$$(x+2)(\log 8) = (7x-1)(\log 5)$$

$$x(\log 8) + 2(\log 8) = 7x(\log 5) - \log 5$$

$$x[\log 8 - 7\log 5] = -\log 5 - 2\log 8$$

$$\boxed{x = \frac{-\log 5 - 2\log 8}{\log 8 - 7\log 5} \qquad x \doteq 0.6279}$$

f. $(4^2)(2^{2x-3}) = (16^{x-2})\left(\frac{1}{\sqrt{2}}\right)$ $\left[\frac{19}{4}\right]$

$$(2^4)(2^{2x-3}) = 2^{4(x-2)} \cdot 2^{-1/2}$$

$$2^{2x-3+4} = 2^{4x-8-\frac{1}{2}}$$

$$2x+1 = 4x - \frac{17}{2}$$

$$19/2 = 2x$$

$$\boxed{19/4 = x}$$

g. $3^{2x} - 2(3^x) - 15 = 0$ $\left[\frac{log5}{log3}\right]$

Let $A = 3^x$: $A^2 - 2A - 15 = 0$

$$(A-5)(A+3) = 0$$

$A = 5$

$3^x = 5$

$\log 3^x = \log 5$

$\boxed{x = \frac{\log 5}{\log 3}}$

$A = -3$

$3^x = -3$

↳ no solutions

30. If $\log_b(a) = \frac{1}{x}$ and $\log_a(\sqrt{b}) = 3x^2$, show that $x = \frac{1}{6}$.

$$\frac{1}{\log_a(b)} = \frac{1}{x}$$

$$x = \log_a(b)$$

$$\log_a(b^{\frac{1}{2}}) = 3x^2$$

$$\frac{1}{2}\log_a(b) = 3x^2$$

$$\frac{1}{2}x = 3x^2$$

$$x = 6x^2$$

$$0 = 6x^2 - x$$

$$0 = x(6x-1)$$

$x = 0$ (inadmissible)

$\boxed{\therefore x = \frac{1}{6}}$

31. Graph $\log_2(8(x-3)^2)$.

$= \log_2(8) + \log_2(x-3)^2$

$= \log_2(2^3) + 2\log_2(x-3)$

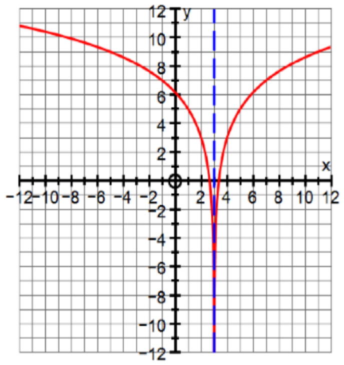
$= 2\log_2(x-3) + 3$

*Use mapping rule and parent function $y = \log_2(x)$

*Don't forget: · V.A at $x = 3$
· reflect along V.A due to the exponent in the logarithm.

$(x, y) \rightarrow (x+3, 2y+3)$

Q31) Answer:



32. In the year 1980, both towns had an earthquake. Springfield's earthquake measured 7.5 on the Richter Scale. The magnitude of the earthquake was 7.5. The earthquake in Shelbyville measured 6.4. How many times as intense was Springfield's earthquake when compared to Shelbyville's earthquake? [12.59 *times as intense*]

$M_2 - M_1 = \log\left(\frac{I_2}{I_1}\right)$

$7.5 - 6.4 = \log\left(\frac{I_2}{I_1}\right)$

$1.1 = \log(I_2/I_1)$

$10^{1.1} = I_2/I_1$

$12.59 \doteq I_2/I_1$

∴ The earthquake in Springfield was about 12.59 times as intense as the one in Shelbyville.

33. Let $f(x) = \{(3,2), (5,1), (7,4), (9,3), (11,5)\}$ and $g(x) = \{(1,3), (2,5), (3,7), (4,9), (5,11)\}$. Determine

a. $f(g(3))$ [4]

$= f(7)$

$= 4$

b. $g(f(9))$ [7]

$= g(3)$

$= 7$

c. $(f-g)(x)$ $[\{(3,-5), (5,-10)\}]$

$= \{(3,-5), (5,-10)\}$

34. Given $f(x) = \frac{1}{x-5}$ and $g(x) = x^2 + 8$. Find

a. $(f-g)(x)$

$= \frac{1}{x-5} - (x^2+8)$

$= \frac{1}{x-5} - \frac{(x^2+8)(x-5)}{x-5}$

$= \frac{1 - (x^3 - 5x^2 + 8x - 40)}{x-5}$

$= \frac{-x^3 + 5x^2 - 8x + 41}{x-5}$

b. $g(g(x))$

$= g(x^2+8)$

$= (x^2+8)^2 + 8$

$= x^4 + 16x^2 + 64 + 8$

$= x^4 + 16x^2 + 72$

c. $f^{-1}(x)$

$x = \frac{1}{y-5}$

$x(y-5) = 1$

$xy - 5x = 1$

$y = \frac{1+5x}{x}$

$\therefore f^{-1}(x) = \frac{1+5x}{x}$

d. $f(g(x))$

$= f(x^2+8)$

$= \frac{1}{x^2+8-5}$

$= \frac{1}{x^2+3}$

e. $f(f^{-1}(x))$

$= x$

35. Prove whether each of the following are Even, Odd, or Neither.

a. $f(x) = 2^x + 2^{-x}$ [E]

$f(-x) = 2^{-x} + 2^x$

$= 2^x + 2^{-x}$

$= f(x)$

∴ $f(x)$ is even

b. $f(x) = \frac{\sin(x)}{x^2-4}$ [O]

$f(-x) = \frac{\sin(-x)}{(-x)^2 - 4}$

since $\sin(x)$ is odd

$= \frac{-\sin(x)}{x^2-4}$

$= -f(x)$

∴ $f(x)$ is odd

c. $f(x) = \log(x^x)$ [*N*]

$f(-x) = \log[(-x)^{-x}]$

$= -x\log(-x)$

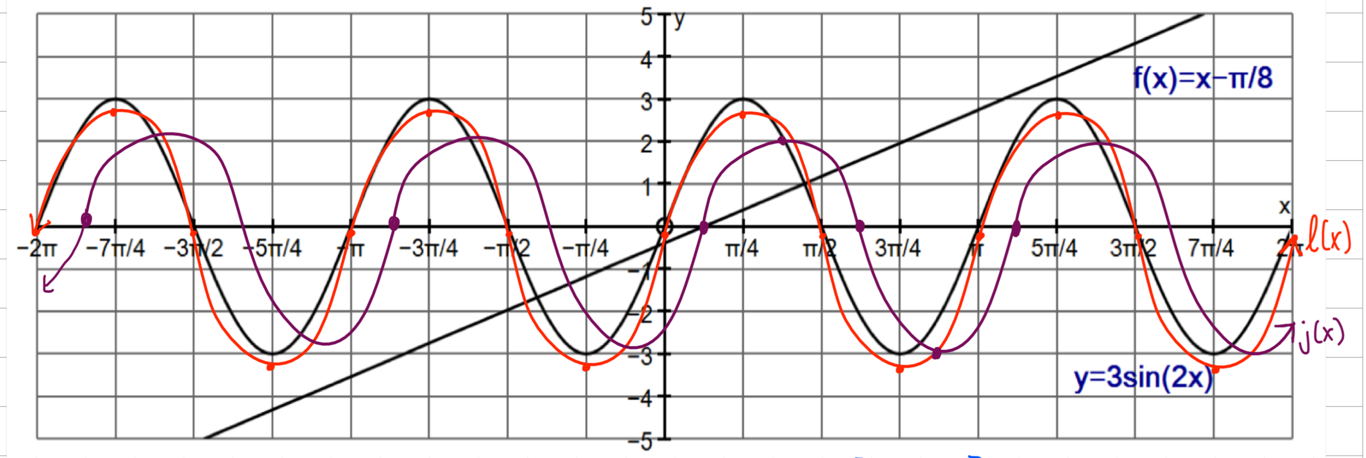
$f(-x) \neq f(x) \neq -f(x)$

∴ $f(x)$ is neither.

36. The graphs of $f(x)$ and $g(x)$ are given below. On the same grid (in different colours), sketch

a. $l(x) = f(g(x))$

b. $j(x) = g(f(x))$



$\ell(x) = 3\sin(2x) - \frac{\pi}{8}$

$(x,y) \to \left(\frac{1}{2}x,\ 3y - \frac{\pi}{8}\right)$

x	$y=\sin x$	
0	0	$\longrightarrow (0, -0.39)$
$\frac{\pi}{2}$	1	$\longrightarrow (\frac{\pi}{4}, 2.61)$
π	0	$\longrightarrow (\frac{\pi}{2}, -0.39)$
$\frac{3\pi}{2}$	-1	$\longrightarrow (\frac{3\pi}{4}, -3.39)$
2π	0	$\longrightarrow (\pi, -0.39)$

$j(x) = 3\sin\left[2\left(x - \frac{\pi}{8}\right)\right]$

$(x,y) \to \left(\frac{1}{2}x + \frac{\pi}{8},\ 3y\right)$

x	$y=\sin x$	
0	0	$\longrightarrow (\frac{\pi}{8}, 0)$
$\frac{\pi}{2}$	1	$\longrightarrow (\frac{3\pi}{8}, 3)$
π	0	$\longrightarrow (\frac{5\pi}{8}, 0)$
$\frac{3\pi}{2}$	-1	$\longrightarrow (\frac{7\pi}{8}, -3)$
2π	0	$\longrightarrow (\frac{9\pi}{8}, 0)$

37. What are the zeros of $f(x) = (x+2)(x+3)(x-4)\log(x)$?

$x+2=0$ $\quad$ $x+3=0$ $\quad$ $x-4=0$ $\quad$ $\log(x)=0$

$x=-2$ $\quad$ $x=-3$ $\quad$ $x=4$ $\quad$ $x=1$

inadmissible since they're not in the domain of log(x)

$\boxed{\therefore\ x \in \{1, 4\}}$

38. Determine the domain of $f(g(x))$ if $f(x) = \frac{1}{x-5}$ and $g(x) = \frac{2}{(x+3)(x-4)}$ $\left[\left\{x \epsilon R \middle| x \neq -3, 4, \frac{5 \pm \sqrt{1265}}{10}\right\}\right]$

$D_{f\circ g} = \{x \in D_g,\ g(x) \in D_f\}$

$D_g = \{x \in \mathbb{R} \mid x \neq -3, 4\}$

$D_f = \{x \in \mathbb{R} \mid x \neq 5\}$

$\hookrightarrow g(x) \neq 5$

$\frac{2}{(x+3)(x-4)} \neq 5$

$2 \neq 5(x+3)(x-4)$

$2 \neq 5(x^2 - x - 12)$

$2 \neq 5x^2 - 5x - 60$

$0 \neq 5x^2 - 5x - 62$

$x \neq \frac{5 \pm \sqrt{25 - 4(5)(-62)}}{2(5)}$

$x \neq \frac{5 \pm \sqrt{1265}}{10}$

$\boxed{\therefore D_{f\circ g} = \left\{x \in \mathbb{R} \mid x \neq -3, 4, \frac{5 \pm \sqrt{1265}}{10}\right\}}$