UNIT 2

Chapter 2- Factor Theorem and Inequalities

Lesson Package

MHF4U

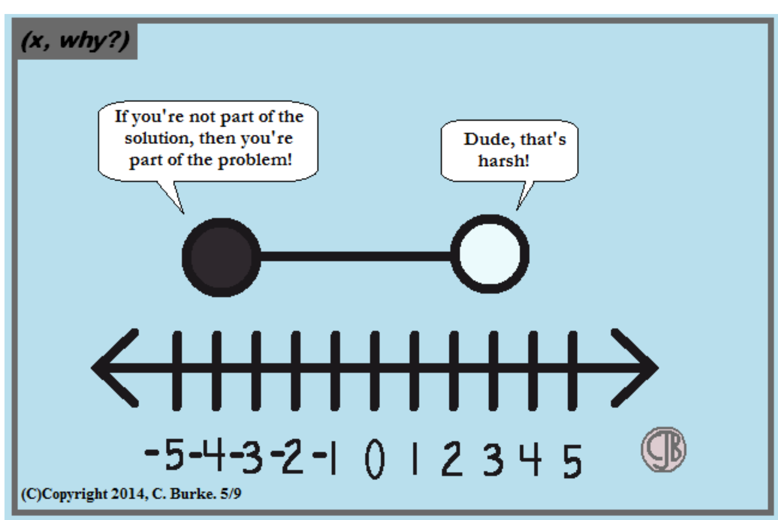
"I don't like long division; I always feel bad for the remainders."

Chapter 2 Outline

Unit Goal: By the end of this unit, you will be able to factor and solve polynomials up to degree 4 using the factor theorem, long division, and synthetic division. You will also learn how to solve factorable polynomial inequalities.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Long Division	- divide polynomial expressions using long division - understand the remainder theorem	C3.1
L2	Synthetic Division	- divide polynomial expressions using synthetic division	3.1
L3	Factor Theorem	- be able to determine factors of polynomial expressions by testing values	C3.2
L4	Solving Polynomial Equations	- solve polynomial equations up to degree 4 by factoring - make connections between solutions and x-intercepts of the graph	C3.2, C3.3, C3.4, C3.7
L5	Families of Polynomial Functions	- determine the equation of a family of polynomial functions given the x-intercepts	C1.8
L5	Solving Polynomial Inequalities	- Solve factorable polynomial inequalities	C4.1, C4.2, C4.3

Assessments	**F/A/O**	**Ministry Code**	**P/O/C**	**KTAC**
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
Quiz – Factor Theorem	F		P	
PreTest Review	F/A		P	
Test – Factor Theorem and Inequalities	O	C1.8 C3.1, 3.2, 3.3, 3.3, 3.7 C4.1, 4.2, 4.3	P	K(21%), T(34%), A(10%), C(34%)



L1 – 2.1 – Long Division of Polynomials and The Remainder Theorem Lesson
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In this section you will apply the method of long division to divide a polynomial by a binomial. You will also learn to use the remainder theorem to determine the remainder of a division without dividing.

Part 1: Do You Remember Long Division (divide, multiply, subtract, repeat)?

$107 \div 4$ can be completed using long division as follows:

$$\begin{array}{r} 26 \\ 4\enclose{longdiv}{107} \\ \underline{8}\downarrow \\ 27 \\ \underline{24} \\ R = 3 \end{array}$$

4 does not go in to 1, so we start by determining how many times 4 goes in to 10. It goes in 2 times. So we put a 2 in the quotient above the 10. This is the division step.

Then, multiply the 2 by 4 (the divisor) and put the product below the 10 in the dividend. This is the multiply step.

Now, subtract 8 from the 10 in the dividend. Then bring down the next digit in the dividend and put it beside the difference you calculated. This is the subtract step.

You then repeat these steps until there are no more digits in the dividend to bring down.

Every division statement that involves numbers can be rewritten using multiplication and addition.

We can express the results of our example in two different ways:

$107 = (4)(26) + 3$ OR $\frac{107}{4} = 26 + \frac{3}{4}$

$=26.75$

$=26 + 0.75$
$=26.75$

Example 1: Use long division to calculate $753 \div 22$

$$\begin{array}{r} 34 \\ 22\enclose{longdiv}{753} \\ \underline{66}\downarrow \\ 93 \\ \underline{88} \\ R = 5 \end{array}$$

$753 = (22)(34) + 5$

OR

Divide both sides by 22

$\frac{753}{22} = 34 + \frac{5}{22}$

34.227

$34 + 0.227$
$= 34.227$

Part 2: Using Long Division to Divide a Polynomial by a Binomial

The quotient of $(3x^3 - 5x^2 - 7x - 1) \div (x - 3)$ can be found using long division as well…

$$
\begin{array}{r}
3x^2 + 4x + 5 \\
x - 3 \overline{\big)\, 3x^3 - 5x^2 - 7x - 1} \\
\underline{3x^3 - 9x^2 } \\
4x^2 - 7x \\
\underline{4x^2 - 12x } \\
5x - 1 \\
\underline{5x - 15} \\
R = 14
\end{array}
$$

Focus only on the first terms of the dividend and the divisor. Find the quotient of these terms.

Since $3x^3 \div x = 3x^2$, this becomes the first term of the quotient. Place $3x^2$ above the term of the dividend with the same degree.

Multiply $3x^2$ by the divisor, and write the answer below the dividend. Make sure to line up 'like terms'. $3x^2(x - 3) = 3x^3 - 9x^2$. Subtract this product from the dividend and then bring down the next term in the dividend.

Now, once again, find the quotient of the first terms of the divisor and the new expression you have in the dividend. Since $4x^2 \div x = 4x$, this becomes the next term in the quotient.

Multiply $4x$ by the divisor, and write the answer below the last line in the dividend. Make sure to line up 'like terms'. $4x(x - 3) = 4x^2 - 12x$. Subtract this product from the dividend and then bring down the next term.

Now, find the quotient of the first terms of the divisor and the new expression in the dividend. Since $5x \div x = 5$, this becomes the next term in the quotient. Multiply $5(x - 3) = 5x - 15$. Subtract this product from the dividend.

The process is stopped once the degree of the remainder is less than the degree of the divisor. The divisor is degree 1 and the remainder is now degree 0, so we stop.

The result in quotient form is:

$$\frac{(3x^3 - 5x^2 - 7x - 1)}{(x - 3)} = 3x^2 + 4x + 5 + \frac{14}{x - 3}$$

The expression that can be used to check the division is:

$$(3x^3 - 5x^2 - 7x - 1) = (x - 3)(3x^2 + 4x + 5) + 14$$

Note: you could check this answer by FOILing the product and collecting like terms.

The result of the division of $P(x)$ by a binomial of the form $x - b$ is:

$$\frac{P(x)}{x - b} = Q(x) + \frac{R}{x - b}$$

Where R is the remainder. The statement that can be used to check the division is:

$$P(x) = (x - b)Q(x) + R$$

Example 2: Find the following quotients using long division. Express the result in quotient form. Also, write the statement that can be used to check the division (then check it!).

a) $x^2 + 5x + 7$ divided by $x + 2$

$$\begin{array}{r|l} & x + 3 \\ \hline x+2 & x^2 + 5x + 7 \\ & \underline{x^2 + 2x} \\ & \quad 3x + 7 \\ & \quad \underline{3x + 6} \\ & \quad R = 1 \end{array}$$

The result in quotient form is:

$$\frac{x^2 + 5x + 7}{x + 2} = x + 3 + \frac{1}{x + 2}$$

The expression that can be used to check the division is:

$$x^2 + 5x + 7 = (x + 2)(x + 3) + 1$$

b) $2x^3 - 3x^2 + 8x - 12$ divided by $x - 1$

$$\begin{array}{r|l} & 2x^2 - x + 7 \\ \hline x-1 & 2x^3 - 3x^2 + 8x - 12 \\ & \underline{2x^3 - 2x^2} \\ & \quad -1x^2 + 8x \\ & \quad \underline{-1x^2 + 1x} \\ & \qquad 7x - 12 \\ & \qquad \underline{7x - 7} \\ & \qquad R = -5 \end{array}$$

The result in quotient form is:

$$\frac{2x^3 - 3x^2 + 8x - 12}{x - 1} = 2x^2 - x + 7 + \frac{-5}{x - 1}$$

The expression that can be used to check the division is:

$$2x^3 - 3x^2 + 8x - 12 = (x - 1)(2x^2 - x + 7) - 5$$

c) $4x^3 + 9x - 12$ divided by $2x + 1$

$$\begin{array}{r|l} & 2x^2 - x + 5 \\ \hline 2x+1 & 4x^3 + 0x^2 + 9x - 12 \\ & \underline{4x^3 + 2x^2} \\ & \quad -2x^2 + 9x \\ & \quad \underline{-2x^2 - 1x} \\ & \qquad 10x - 12 \\ & \qquad \underline{10x + 5} \\ & \qquad R = -17 \end{array}$$

The result in quotient form is:

$$\frac{4x^3 + 9x - 12}{2x + 1} = 2x^2 - x + 5 + \frac{-17}{2x + 1}$$

The expression that can be used to check the division is:

$$4x^3 + 9x - 12 = (2x + 1)(2x^2 - x + 5) - 17$$

Example 3: The volume, in cubic cm, of a rectangular box is given by $V(x) = x^3 + 7x^2 + 14x + 8$. Determine expressions for possible dimensions of the box if the height is given by $x + 2$.

$$
\begin{array}{r|l}
 & x^2 + 5x + 4 \\
\hline
x+2 & x^3 + 7x^2 + 14x + 8 \\
 & \underline{x^3 + 2x^2} \\
 & \quad 5x^2 + 14x \\
 & \quad \underline{5x^2 + 10x} \\
 & \qquad 4x + 8 \\
 & \qquad \underline{4x + 8} \\
 & \qquad R = 0
\end{array}
$$

Dividing the volume by the height will give an expression for the area of the base of the box.

$$\overbrace{x^3 + 7x^2 + 14x + 8}^{\text{Volume}} = \overbrace{(x+2)}^{\text{height}}\overbrace{(x^2 + 5x + 4)}^{\text{area of base}}$$

$$= (x+2)(x+4)(x+1)$$

Factor the area of the base to get possible dimensions for the length and width of the box.

Expressions for the possible dimensions of the box are $x + 1$, $x + 2$, and $x + 4$.

Part 3: Remainder Theorem

When a polynomial function $P(x)$ is divided by $x - b$, the remainder is $P(b)$; and when it is divided by $ax - b$, the remainder is $P\left(\frac{b}{a}\right)$, where a and b are integers, and $a \neq 0$.

Example 3: Apply the remainder theorem

a) Use the remainder theorem to determine the remainder when $P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $x + 1$

Since $x + 1$ is $x - (-1)$, the remainder is $P(-1)$.

$P(-1) = 2(-1)^3 + (-1)^2 - 3(-1) - 6$

$= -2 + 1 + 3 - 6$

$= -4$

Therefore, the remainder is -4

b) Verify your answer using long division

$$
\begin{array}{r}
2x^2 - x - 2 \\
x+1 \overline{\smash{)}\,2x^3 + x^2 - 3x - 6} \\
\underline{2x^3 + 2x^2 } \\
-1x^2 - 3x \\
\underline{-1x^2 - 1x } \\
-2x - 6 \\
\underline{-2x - 2} \\
R = -4
\end{array}
$$

Example 4: Use the remainder theorem to determine the remainder when $P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $2x - 3$

The remainder is $P\left(\frac{3}{2}\right)$

$$P\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) - 6$$

$$= 2\left(\frac{27}{8}\right) + \left(\frac{9}{4}\right) - 3\left(\frac{3}{2}\right) - 6$$

$$= \frac{54}{8} + \frac{9}{4} - \frac{9}{2} - 6$$

$$= \frac{27}{4} + \frac{9}{4} - \frac{18}{4} - \frac{24}{4}$$

$$= -\frac{3}{2}$$

Therefore, the remainder is $-\frac{3}{2}$

Example 5: Determine the value of k such that when $3x^4 + kx^3 - 7x - 10$ is divided by $x - 2$, the remainder is 8.

The remainder is $P(2)$. Solve for k when $P(2)$ is set to equal 8.

$P(2) = 3(2)^4 + k(2)^3 - 7(2) - 10$

$8 = 3(2)^4 + k(2)^3 - 7(2) - 10$

$8 = 3(16) + 8k - 14 - 10$

$8 = 48 + 8k - 24$

$8 = 24 + 8k$

$-16 = 8k$

$-2 = k$

Therefore, the value of k is -2.

L2 – 2.1 – Synthetic Division Lesson
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In this section you will learn how to use synthetic division as an alternate method to dividing a polynomial by a binomial. Synthetic division is an efficient way to divide a polynomial by a binomial of the form $x - b$.

> **IMPORTANT:** When using Polynomial OR Synthetic division…
>
> - Terms must be arranged in descending order of degree, in both the divisor and the dividend.
> - Zero must be used as the coefficient of any missing powers of the variable in both the divisor and the dividend.

Part 1: Synthetic division when the binomial is of the form $x - b$

Divide $3x^3 - 5x^2 - 7x - 1$ by $x - 3$. In this question, $b = 3$.

coefficients of dividend

zero of divisor

$$\begin{array}{c|cccc} 3 & 3 & -5 & -7 & -1 \\ & & 9 & 12 & 15 \quad + \\ \hline \times & 3 & 4 & 5 & 14 \\ & x^2 & x & \# & R \end{array}$$

List the coefficients of the dividend in the first row. To the left, write the b value (the zero of the divisor). Place a + sign above the horizontal line to represent addition and a × sign below the horizontal line to indicate multiplication of the divisor and the terms of the quotient.

Bring the first term down, this is the coefficient of the first term of the quotient. Multiply it by the b value and write this product below the second term of the dividend.

Now add the terms together.

Multiply this sum by the b value and write the product below the third term of the dividend. Repeat this process until you have a completed the chart.

The last number below the chart is the remainder. The first numbers are the coefficients of the quotient, starting with degree that is one less than the dividend.

Don't forget that the answer can be written in two ways…

$$\frac{3x^3 - 5x^2 - 7x - 1}{x - 3} = \mathbf{3x^2 + 4x + 5 + \frac{14}{x - 3}}$$

OR

$$3x^3 - 5x^2 - 7x - 1 = \mathbf{(x - 3)(3x^2 + 4x + 5) + 14}$$

Example 1: Use synthetic division to divide. Then write the multiplication statement that could be used to check the division.

a) $(x^4 - 2x^3 + 13x - 6) \div (x + 2)$

$$\boxed{= x^4 - 2x^3 + 0x^2 + 13x - 6}$$

$$\begin{array}{r|rrrrr}
-2 & 1 & -2 & 0 & 13 & -6 \\
 & \downarrow & -2 & 8 & -16 & 6 \quad + \\
\hline
\times & 1 & -4 & 8 & -3 & 0 \\
 & x^3 & x^2 & x & \# & R
\end{array}$$

$$x^4 - 2x^3 + 13x - 6 = (x+2)(x^3 - 4x^2 + 8x - 3)$$

> ***Note:** since the remainder is zero, both the quotient and divisor are __factors__ of the dividend.*

b) $(2x^3 - 5x^2 + 8x + 4) \div (x - 3)$

$$\begin{array}{r|rrrr}
3 & 2 & -5 & 8 & 4 \\
 & \downarrow & 6 & 3 & 33 \quad + \\
\hline
\times & 2 & 1 & 11 & 37 \\
 & x^2 & x & \# & R
\end{array}$$

$$2x^3 - 5x^2 + 8x + 4 = (x-3)(2x^2 + x + 11) + 37$$

Part 2: Synthetic division when the binomial is of the form $ax - b$

Divide $6x^3 + 5x^2 - 16x - 15$ by $2x + 3$

$(2x + 3) = 2\left(x + \frac{3}{2}\right) \rightarrow b = -\frac{3}{2}$

To use synthetic division, the divisor must be in the form $x - b$. Re-write the divisor by factoring out the coefficient of the x.

We can now divide $6x^3 + 5x^2 - 16x - 15$ by $\left(x + \frac{3}{2}\right)$ using synthetic division as long as you remember to divide the quotient by 2 after.

$$
\begin{array}{r|rrrrl}
-3/2 & 6 & 5 & -16 & -15 & \\
 & \downarrow & -9 & 6 & 15 & + \\
\hline
\times & 6 & -4 & -10 & 0 & \\
 & \multicolumn{3}{c}{\overline{\quad 2 \quad}} & R &
\end{array}
$$

$$
\begin{array}{rrrr}
\div & 3 & -2 & 5 \\
 & x^2 & x & \#
\end{array}
$$

$6x^3 + 5x^2 - 16x - 15 = (2x+3)(3x^2 - 2x - 5)$

Check answer using long division

$$
\begin{array}{r l}
 & \quad 3x^2 - 2x - 5 \\
2x + 3 & \overline{\big) \, 6x^3 + 5x^2 - 16x - 15} \\
 & \underline{6x^3 + 9x^2} \quad \downarrow \\
 & \quad -4x^2 - 16x \\
 & \quad \underline{-4x^2 - 6x} \quad \downarrow \\
 & \qquad -10x - 15 \\
 & \qquad \underline{-10x - 15} \\
 & \qquad R = 0
\end{array}
$$

$6x^3 + 5x^2 - 16x - 15 = (2x+3)(3x^2 - 2x - 5)$

Note: Synthetic division can only be used with a linear divisor. It is most useful with a divisor of the form $x - b$. If the divisor is $ax - b$, it can be used but long division may be easier.

Example 2: Find each quotient by choosing an appropriate strategy.

a) Divide $x^3 - 4x^2 + 2x + 3$ by $x - 3$

$$\begin{array}{c|cccc} 3 & 1 & -4 & 2 & 3 \\ & \downarrow & 3 & -3 & -3 \\ \hline & 1 & -1 & -1 & 0 \\ & x^2 & x & \# & R \end{array}$$

use synthetic division because we have a linear divisor of the form $x - b$

$$x^3 - 4x^2 + 2x + 3 = (x-3)(x^2 - x - 1)$$

b) Divide $12x^4 - 56x^3 + 59x^2 + 9x - 18$ by $2x + 1$

$$\begin{array}{r} 6x^3 - 31x^2 + 45x - 18 \\ 2x+1 \enclose{longdiv}{12x^4 - 56x^3 + 59x^2 + 9x - 18} \\ \underline{12x^4 + 6x^3} \qquad\qquad\qquad\quad \\ -62x^3 + 59x^2 \qquad\qquad \\ \underline{-62x^3 - 31x^2} \qquad\qquad \\ 90x^2 + 9x \qquad \\ \underline{90x^2 + 45x} \qquad \\ -36x - 18 \\ \underline{-36x - 18} \\ R = 0 \end{array}$$

$$12x^4 - 56x^3 + 59x^2 + 9x - 18 = (2x+1)(6x^3 - 31x^2 + 45x - 18)$$

c) Divide $x^4 - 2x^3 + 5x + 3$ by $x^2 + 2x + 1$

$$
\begin{array}{r|l}
 & x^2 - 4x + 7 \\
\hline
x^2+2x+1 & x^4 - 2x^3 + 0x^2 + 5x + 3 \\
 & \underline{x^4 + 2x^3 + x^2} \\
 & -4x^3 - 1x^2 + 5x \\
 & \underline{-4x^3 - 8x^2 - 4x} \\
 & 7x^2 + 9x + 3 \\
 & \underline{7x^2 + 14x + 7} \\
 & R = -5x - 4
\end{array}
$$

use long division since it is a non-linear divisor

$$x^4 - 2x^3 + 5x + 3 = (x^2+2x+1)(x^2-4x+7) - 5x - 4$$

d) Divide $x^4 - x^3 - x^2 + 2x + 1$ by $x^2 + 2$

$$
\begin{array}{r|l}
 & x^2 - 1x - 3 \\
\hline
x^2+0x+2 & x^4 - x^3 - x^2 + 2x + 1 \\
 & \underline{x^4 + 0x^3 + 2x^2} \\
 & -1x^3 - 3x^2 + 2x \\
 & \underline{-1x^3 + 0x^2 - 2x} \\
 & -3x^2 + 4x + 1 \\
 & \underline{-3x^2 + 0x - 6} \\
 & 4x + 7
\end{array}
$$

use long division since it is a non-linear divisor

$$x^4 - x^3 - x^2 + 2x + 1 = (x^2+2)(x^2 - x - 3) + 4x + 7$$

L3 – 2.2 – Factor Theorem Lesson
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In this section, you will learn how to determine the factors of a polynomial function of degree 3 or greater.

Part 1: Remainder Theorem Refresher

a) Use the remainder theorem to determine the remainder when $f(x) = x^3 + 4x^2 + x - 6$ is divided by $x + 2$

> **Remainder Theorem:** When a polynomial function $P(x)$ is divided by $x - b$, the remainder is $P(b)$; and when it is divided by $ax - b$, the remainder is $P\left(\frac{b}{a}\right)$, where a and b are integers, and $a \neq 0$.

$f(-2) = (-2)^3 + 4(-2)^2 + (-2) - 6$

$f(-2) = -8 + 16 - 2 - 6$

$f(-2) = 0$

The remainder when divided by $x + 2$ is 0. This means that $x + 2$ is a factor of the dividend.

b) Verify your answer to part a) by completing the division using long division or synthetic division.

> **Note:** I chose synthetic since it is a linear divisor of the form $x - b$.

$$
\begin{array}{r|rrrr}
-2 & 1 & 4 & 1 & -6 \\
 & \downarrow & -2 & -4 & 6 \quad + \\
\hline
\times & 1 & 2 & -3 & 0 \\
 & x^2 & x & \# & R
\end{array}
$$

$x^3 + 4x^2 + x - 6 = (x+2)(x^2 + 2x - 3)$

> **Factor Theorem:**
>
> $x - b$ is a factor of a polynomial $P(x)$ if and only if $P(b) = 0$. Similarly, $ax - b$ is a factor of $P(x)$ if and only if $P\left(\frac{b}{a}\right) = 0$.

Example 1: Determine if $x - 3$ and $x + 2$ are factors of $P(x) = x^3 - x^2 - 14x + 24$

$P(3) = (3)^3 - (3)^2 - 14(3) + 24$

$P(3) = 27 - 9 - 42 + 24$

$P(3) = 0$

Since the remainder is **0**, $x - 3$ divides evenly into $P(x)$; that means $x - 3$ **is a factor** of $P(x)$.

$P(-2) = (-2)^3 - (-2)^2 - 14(-2) + 24$

$P(-2) = -8 - 4 + 28 + 24$

$P(-2) = 40$

Since the remainder is not **0**, $x + 2$ does not divide evenly into $P(x)$; that means $x + 2$ **is not a factor** of $P(x)$.

Part 2: How to determine a factor of a Polynomial With Leading Coefficient 1

You could guess and check values of b that make $P(b) = 0$ until you find one that works…

Or you can use the Integral Zero Theorem to help.

Integral Zero Theorem

If $x - b$ is a factor of a polynomial function $P(x)$ with leading coefficient 1 and remaining coefficients that are integers, then ***b* is a factor of the constant term** of $P(x)$.

Note: Once one of the factors of a polynomial is found, division is used to determine the other factors.

Example 2: Factor $x^3 + 2x^2 - 5x - 6$ fully.

Let $P(x) = x^3 + 2x^2 - 5x - 6$

Find a value of b such that $P(b) = 0$. Based on the factor theorem, if $P(b) = 0$, then we know that $x - b$ is a factor. We can then divide $P(x)$ by that factor.

The integral zero theorem tells us to test factors of **-6.**

Test $\pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}$, ***and*** $\pm\,\mathbf{6}$. Once one factor is found, you can stop testing and use that factor to divide $P(x)$.

$P(1) = (1)^3 + 2(1)^2 - 5(1) - 6$

$P(1) = 1 + 2 - 5 - 6$

$P(1) = -8$

Since $P(1) \neq 0$, we know that $x - 1$ is NOT a factor of $P(x)$.

$P(2) = (2)^3 + 2(2)^2 - 5(2) - 6$

$P(2) = 8 + 8 - 10 - 6$

$P(2) = 0$

Since $P(2) = 0$, we know that $x - 2$ is a factor of $P(x)$.

You can now use either long division or synthetic division to find the other factors

Method 1: Long division

$$
\begin{array}{r|l}
 & x^2 + 4x + 3 \\
\hline
x - 2 & x^3 + 2x^2 - 5x - 6 \\
 & \underline{x^3 - 2x^2} \downarrow \\
 & \quad 4x^2 - 5x \\
 & \quad \underline{4x^2 - 8x} \downarrow \\
 & \qquad 3x - 6 \\
 & \qquad \underline{3x - 6} \\
 & \qquad R = 0
\end{array}
$$

$x^3 + 2x^2 - 5x - 6 = (x-2)(x^2+4x+3)$ factor further if possible.

$= (x-2)(x+3)(x+1)$

Method 2: Synthetic Division

$$
\begin{array}{c|cccc}
2 & 1 & 2 & -5 & -6 \\
 & \downarrow & 2 & 8 & 6 \quad + \\
\hline
x & 1 & 4 & 3 & 0 \\
 & x^2 & x & \# & R
\end{array}
$$

factor further if possible

$x^3 + 2x^2 - 5x - 6 = (x-2)(x^2+4x+3)$

$= (x-2)(x+3)(x+1)$

Example 3: Factor $x^4 + 3x^3 - 7x^2 - 27x - 18$ completely.

Let $P(x) = x^4 + 3x^3 - 7x^2 - 27x - 18$

Find a value of b such that $P(b) = 0$. Based on the factor theorem, if $P(b) = 0$, then we know that $x - b$ is a factor. We can then divide $P(x)$ by that factor.

The integral zero theorem tells us to test factors of **-18.**

Test **$\pm 1, \pm 2, \pm 3, \pm 6, \pm 9$ *and* ± 18**. Once one factor is found, you can stop testing and use that factor to divide $P(x)$.

$P(1) = (1)^4 + 3(1)^3 - 7(1)^2 - 27(1) - 18$

$P(1) = -48$

$x - 1$ is NOT a factor of $P(x)$.

$P(-1) = (-1)^4 + 3(-1)^3 - 7(-1)^2 - 27(-1) - 18$

$P(-1) = 0$

$x + 1$ IS a factor of $P(x)$.

Since **$P(-1) = 0$**, this tell us that **$x + 1$** is a factor. Use division to determine the other factor.

$$
\begin{array}{r|rrrrr}
-1 & 1 & 3 & -7 & -27 & -18 \\
 & \downarrow & -1 & -2 & 9 & 18 \quad + \\
\hline
\times & 1 & 2 & -9 & -18 & 0 \\
 & x^3 & x^2 & x & \# & R
\end{array}
$$

$$x^4 + 3x^3 - 7x^2 - 27x - 18 = (x+1)(x^3 + 2x^2 - 9x - 18)$$

We can now further divide $x^3 + 2x^2 - 9x - 18$ using division again or by factoring by grouping.

Method 1: Division

Test factors of -18

$f(-2) = (-2)^3 + 2(-2)^2 - 9(-2) - 18$

$f(-2) = 0$

$x + 2$ is a factor

$$
\begin{array}{r|rrrr}
-2 & 1 & 2 & -9 & -18 \\
 & \downarrow & -2 & 0 & 18 \quad + \\
\hline
\times & 1 & 0 & -9 & 0 \\
 & x^2 & x & \# & R
\end{array}
$$

DOS

$$x^4 + 3x^3 - 7x^2 - 27x - 18 = (x+1)(x+2)(x^2 - 9)$$
$$= (x+1)(x+2)(x-3)(x+3)$$

Method 2: Factoring by Grouping

$$f(x) = x^3 + 2x^2 - 9x - 18$$

Group the first 2 terms and the last 2 terms and separate with an addition sign.

$$f(x) = (x^3 + 2x^2) + (-9x - 18)$$

Common factor within each group

$$f(x) = x^2(x + 2) - 9(x + 2)$$

Factor out the common binomial

$$f(x) = (x + 2)(x^2 - 9)$$

Therefore,

$$\begin{aligned} x^4 + 3x^3 - 7x^2 - 27x - 18 &= (x + 1)(x^3 + 2x^2 - 9x - 18) \\ &= (x + 1)(x + 2)(x^2 - 9) \\ &= (x + 1)(x + 2)(x - 3)(x + 3) \end{aligned}$$

Example 4: Try Factoring by Grouping Again

$x^4 - 6x^3 + 2x^2 - 12x$

Note: Factoring by grouping does not always work…but when it does, it saves you time!

$= \boldsymbol{(x^4 - 6x^3) + (2x^2 - 12x)}$

$= \boldsymbol{x^3(x - 6) + 2x(x - 6)}$

$= \boldsymbol{(x - 6)(x^3 + 2x)}$

$= \boldsymbol{(x - 6)(x)(x^2 + 2)}$

Part 3: How to determine a factor of a Polynomial With Leading Coefficient NOT 1

The integral zero theorem can be extended to include polynomials with leading coefficients that are not 1. This extension is known as the rational zero theorem.

Rational Zero Theorem:

Suppose $P(x)$ is a polynomial function with integer coefficients and $x = \frac{b}{a}$ is a zero of $P(x)$, where a and b are integers and $a \neq 0$. Then,

- b is a factor of the constant term of $P(x)$
- a is a factor of the leading coefficient of $P(x)$
- $(ax - b)$ is a factor of $P(x)$

Example 5: Factor $P(x) = 3x^3 + 2x^2 - 7x + 2$

We must start by finding a value of $\frac{b}{a}$ where $P\left(\frac{b}{a}\right) = 0$.

b must be a factor of the constant term. Possible values for b are: $\pm\mathbf{1}, \pm\mathbf{2}$

a must be a factor of the leading coefficient. Possible values of a are: $\pm\mathbf{1}, \pm\mathbf{3}$

Therefore, possible values for $\frac{b}{a}$ are: $\pm\mathbf{1}, \pm\frac{\mathbf{1}}{\mathbf{3}}, \pm\mathbf{2}, \pm\frac{\mathbf{2}}{\mathbf{3}}$

Test values of $\frac{b}{a}$ for x in $P(x)$ to find a zero.

$P(1) = 3(1)^3 + 2(1)^2 - 7(1) + 2 = 0$

Since ***$P(1) = 0$*, $x - 1$ is a factor** of $P(x)$. Use division to find the other factors.

$$\begin{array}{c|cccc}
1 & 3 & 2 & -7 & 2 \\
 & \downarrow & 3 & 5 & -2 \quad + \\
\hline
x & 3 & 5 & -2 & 0 \\
 & x^2 & x & \# & R
\end{array}$$

P: −6
S: 5
6 and −1

$$\begin{aligned}
3x^3+2x^2-7x+2 &= (x-1)(3x^2+5x-2) \\
&= (x-1)[3x^2+6x-1x-2] \\
&= (x-1)[(3x^2+6x)+(-1x-2)] \\
&= (x-1)[3x(x+2)-1(x+2)] \\
&= (x-1)(x+2)(3x-1)
\end{aligned}$$

Example 6: Factor $P(x) = 2x^3 + x^2 - 7x - 6$

Possible values for b are: $\pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}, \pm\mathbf{6}$

Possible values of a are: $\pm\mathbf{1}, \pm\mathbf{2}$

Therefore, possible values for $\frac{b}{a}$ are: $\pm\mathbf{1}, \pm\frac{\mathbf{1}}{\mathbf{2}}, \pm\mathbf{2}, \pm\mathbf{3}, \pm\frac{\mathbf{3}}{\mathbf{2}}, \pm\mathbf{6}$

$f(-1) = 2(-1)^3 + (-1)^2 - 7(-1) - 6 = 0$

Therefore, $x + 1$ is a factor of $P(x)$

$$
\begin{array}{r|rrrr}
-1 & 2 & 1 & -7 & -6 \\
 & \downarrow & -2 & 1 & 6 \quad + \\
\hline
\times & 2 & -1 & -6 & 0 \\
 & x^2 & x & \# & R
\end{array}
$$

P: -12

S: -1

-4 and 3

$$2x^3 + x^2 - 7x - 6 = (x+1)(2x^2 - x - 6)$$
$$= (x+1)\left[(2x^2 - 4x) + (3x - 6)\right]$$
$$= (x+1)\left[2x(x-2) + 3(x-2)\right]$$
$$= (x+1)(x-2)(2x+3)$$

Part 4: Application Question

Example 7: When $f(x) = 2x^3 - mx^2 + nx - 2$ is divided by $x + 1$, the remainder is -12 and $x - 2$ is a factor. Determine the values of m and n.

Hint: Use the information given to create 2 equations and then use substitution or elimination to solve.

$$f(-1) = 2(-1)^3 - m(-1)^2 + n(-1) - 2$$
$$-12 = -2 - m - n - 2$$
$$① \quad -8 = -m - n$$

$$f(2) = 2(2)^3 - m(2)^2 + n(2) - 2$$
$$0 = 16 - 4m + 2n - 2$$
$$② \quad -14 = -4m + 2n$$

$$① \quad -8 = -m - n \xrightarrow{\times 2} -16 = -2m - 2n$$
$$② \quad -14 = -4m + 2n \longrightarrow \underline{-14 = -4m + 2n} \; +$$
$$-30 = -6m$$
$$5 = m$$

sub $m = 5$ into ① or ②

$$-8 = -5 - n$$
$$n = 3$$

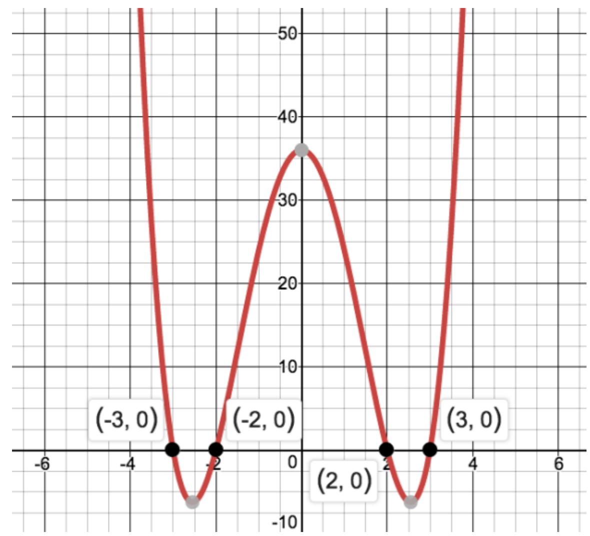
$\therefore m = 5$ and $n = 3$

L4 – 2.3 – Solving Polynomial Equations Lesson
MHF4U

In this section, you will learn methods of solving polynomial equations of degree higher than two by factoring (using the factor theorem). You will also identify the connection between the roots of polynomial equations, the x-intercepts of the graph of a polynomial function, and the zeros of the function.

Part 1: Investigation

a) Use technology to graph the function $f(x) = x^4 - 13x^2 + 36$



b) Determine the x-intercepts from the graph

The x-intercepts are:

(-3, 0), (-2, 0), (2, 0), and (3, 0)

c) Factor $f(x)$. Then, use the factors to determine the zeros (roots) of $f(x)$.

$f(x) = x^4 - 13x^2 + 36$ ← can factor like a quadratic

$0 = (x^2)^2 - 13(x^2) + 36$ P: 36 S: -13 -9 and -4

$0 = (x^2 - 9)(x^2 - 4)$ Both are DOS

$0 = (x-3)(x+3)(x-2)(x+2)$

$x - 3 = 0$ $x + 3 = 0$ $x - 2 = 0$ $x + 2 = 0$
$x_1 = 3$ $x_2 = -3$ $x_3 = 2$ $x_4 = -2$

> **Remember:** The zeros of the function are the values of x that make $f(x) = 0$. If the polynomial equation is factorable, then the values of the zeros (roots) can be determined algebraically by solving each linear or quadratic factor.

d) How are the x-intercepts from the graph related to the roots (zeros) of the equation?

The zeros of the equation ARE the x-intercepts of the graph of the function.

Example 1: State the solutions to the following polynomials that are already in factored form

a) $x(2x+3)(x-5)=0$

$\boxed{x_1=0}$

$2x+3=0$

$\boxed{x_2=-\frac{3}{2}}$

$x-5=0$

$\boxed{x_3=5}$

b) $(2x^2-3)(3x^2+1)$

$2x^2-3=0$

$x^2=\frac{3}{2}$

$x=\pm\sqrt{\frac{3}{2}}$

$\boxed{x_1=\sqrt{\frac{3}{2}}}$ $\boxed{x_2=-\sqrt{\frac{3}{2}}}$

$3x^2+1=0$

$x^2=-\frac{1}{3}$

$x=\pm\sqrt{-\frac{1}{3}}$

Not a REAL solution

Methods of factoring:

- Long division and synthetic division
- Factor by grouping
- Difference of squares $a^2-b^2=(a-b)(a+b)$
- Common Factoring
- Trinomial factoring (sum and product)
- Sum and difference of cubes $a^3+b^3=(a+b)(a^2-ab+b^2)$
  $a^3-b^3=(a-b)(a^2+ab+b^2)$

Example 2: Solve each polynomial equation by factoring

a) $x^3-x^2-2x=0$

$x^3-x^2-2x=0$

common factor

$x(x^2-x-2)=0$

P: -2
S: -1
-2 and 1

$x(x-2)(x+1)=0$

$\boxed{x_1=0}$

$x-2=0$

$\boxed{x_2=2}$

$x+1=0$

$\boxed{x_3=-1}$

Solution(s):

(0, 0), (2, 0), and (-1, 0)

b) $3x^3+x^2-12x-4=0$

$x^2(3x+1)-4(3x+1)=0$

$(3x+1)(x^2-4)=0$

$(3x+1)(x-2)(x+2)=0$

$x_1=-\frac{1}{3}$ $\quad x_2=2$ $\quad x_3=-2$

Solution(s):

$\left(-\frac{1}{3},0\right)$, (2, 0), and (-2, 0)

Example 3:

a) Use the factor theorem to solve $f(x) = 2x^3 + 3x^2 - 11x - 6$

Possible values of b are: $\pm 1, \pm 2, \pm 3, \pm 6$

Possible values for are: $\pm 1, \pm 2$

Possible values for $\frac{b}{a} = \pm 1, \pm\frac{1}{2}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm 6$

$f(2) = 2(2)^3 + 3(2)^2 - 11(2) - 6 = 0, \therefore x - 2$ is a factor

$$
\begin{array}{c|cccc}
2 & 2 & 3 & -11 & -6 \\
 & \downarrow & 4 & 14 & 6 \quad + \\
\hline
\times & 2 & 7 & 3 & 0 \\
 & x^2 & x & \# & R
\end{array}
$$

$f(x) = (x-2)(2x^2 + 7x + 3)$ p: 6 s: 7 6 and 1

$0 = (x-2)[(2x^2+6x) + (1x+3)]$

$0 = (x-2)[2x(x+3) + 1(x+3)]$

$0 = (x-2)(x+3)(2x+1)$

$x - 2 = 0$ $\quad$ $x + 3 = 0$ $\quad$ $2x + 1 = 0$

$\boxed{x_1 = 2}$ $\quad$ $\boxed{x_2 = -3}$ $\quad$ $\boxed{x_3 = -1/2}$

Solution(s):

(2, 0), (-3, 0), and $\left(-\frac{1}{2}, 0\right)$

b) What do your answers to part a) represent?

The values of $2, -\frac{1}{2}$, and -3 are the roots of the equation $2x^3 + 3x^2 - 11x - 6 = 0$ which means they are the x-intercepts of the graph of the function $f(x) = 2x^3 + 3x^2 - 11x - 6$.

Example 4: Find the zeros of the polynomial function $f(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$

Possible values for b are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$

$f(-1) = (-1)^4 - 2(-1)^3 - 7(-1)^2 + 8(-1) + 12 = 0, \therefore x + 1$ is a factor

$$\begin{array}{c|ccccc}
-1 & 1 & -2 & -7 & 8 & 12 \\
 & \downarrow & -1 & 3 & 4 & -12 \quad + \\
\hline
\times & 1 & -3 & -4 & 12 & 0 \\
 & x^3 & x^2 & x & \# & R
\end{array}$$

$$f(x) = (x+1)(x^3 - 3x^2 - 4x + 12)$$

$$= (x+1)\left[x^2(x-3) - 4(x-3)\right]$$

$$= (x+1)(x-3)(x^2-4)$$

$$= (x+1)(x-3)(x-2)(x+2)$$

$x+1=0$ $\quad$ $x-3=0$ $\quad$ $x-2=0$ $\quad$ $x+2=0$

$\boxed{x_1 = -1}$ $\quad$ $\boxed{x_2 = 3}$ $\quad$ $\boxed{x_3 = 2}$ $\quad$ $\boxed{x_4 = -2}$

Solution(s):

(-1, 0), (3, 0), (2, 0), and (-2, 0)

Example 5:

a) Find the roots of the polynomial function $f(x) = x^3 + x - 3x^2 - 3$

Start by rearranging in descending order of degree: $f(x) = x^3 - 3x^2 + x - 3$

$$f(x) = x^3 - 3x^2 + x - 3$$
$$0 = (x^3 - 3x^2) + (x - 3)$$
$$0 = x^2(x-3) + 1(x-3)$$
$$0 = (x-3)(x^2+1)$$

$$x - 3 = 0$$
$$\boxed{x = 3}$$

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$\cancel{x = \pm\sqrt{-1}}$$

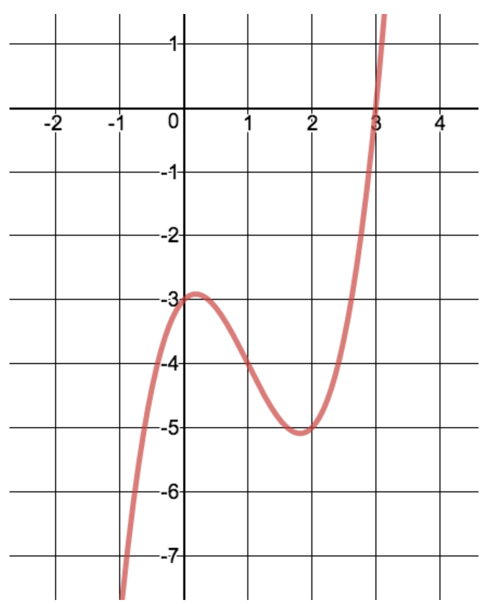
Not a real root.

Solution(s):
$(3, 0)$

Note: Since the square root of a negative number is not a real number, the only REAL root is $x = 3$. $x = \pm\sqrt{-1}$ is considered a NON-REAL root.

b) Use technology to look at the graph of the function $f(x)$. Comment on how x-intercept(s) of the graph are related to the REAL and NON-REAL roots of the equation.

The x-intercepts of the graph of a polynomial function correspond to only the REAL roots of the related polynomial equation. There are no x-intercepts on the graph that correspond to the NON-REAL roots of the equation.



Example 6: Find all real roots for each polynomial equation

a) $f(x) = 2x^3 - 3x^2 - x - 2$

Possible values of b are: $\pm 1, \pm 2$

Possible values for are: $\pm 1, \pm 2$

Possible values for $\frac{b}{a} = \pm 1, \pm \frac{1}{2}, \pm 2$

$f(2) = 2(2)^3 - 3(2)^2 - (2) - 2 = 0, \therefore x - 2$ is a factor of $f(x)$

$$\begin{array}{c|cccc} 2 & 2 & -3 & -1 & -2 \\ & \downarrow & 4 & 2 & 2 \\ \hline \times & 2 & 1 & 1 & 0 \\ & x^2 & x & \# & R \end{array} \quad +$$

$f(x) = 2x^3 - 3x^2 - x - 2$

$0 = (x-2)(2x^2 + x + 1)$

$x - 2 = 0$

$x = 2$

no real roots

P: 2 ? NOTHING!
S: 1

check discriminant to see if there are other real roots

$b^2 - 4ac = (1)^2 - 4(2)(1)$

$= -7$

$\therefore$ no real roots

Solution(s):

$(2, 0)$

b) $g(x) = 8x^3 + 125$

Hint: *This is a difference of cubes* → $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$g(x) = (2x)^3 + (5)^3$

$0 = (2x+5)(4x^2 - 10x + 25)$

$2x + 5 = 0$

$x = -5/2$

NO REAL ROOTS

P: 100 NOTHING!
S: −10

Check discriminant to see if there are other real roots.

$b^2 - 4ac = (-10)^2 - 4(4)(25)$

$= -300$

NO REAL ROOTS

Solution(s):

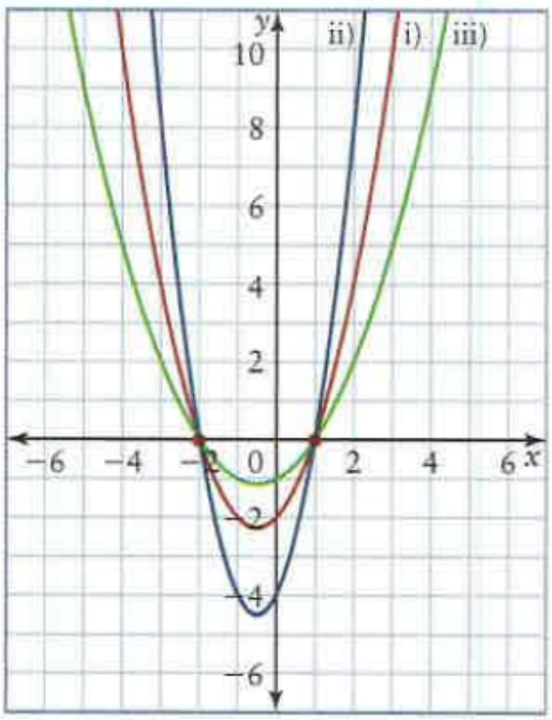
$\left(-\frac{5}{2}, 0\right)$

L5 – 2.4 – Families of Polynomial Functions Lesson
MHF4U

In this section, you will determine equations for a family of polynomial functions from a set of zeros. Given additional information, you will determine an equation for a particular member of the family.

Part 1: Investigation

1) Set A

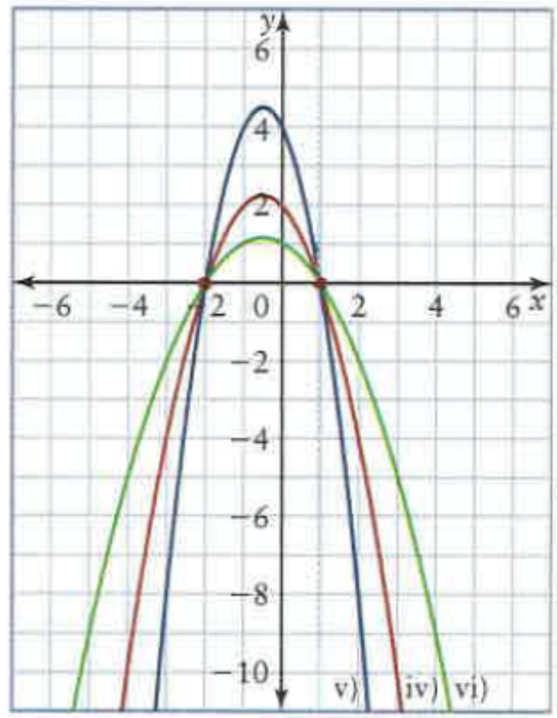


i) $y = (x - 1)(x + 2)$

ii) $y = 2(x - 1)(x + 2)$

iii) $y = \frac{1}{2}(x - 1)(x + 2)$

Set B



iv) $y = -(x - 1)(x + 2)$

v) $y = -2(x - 1)(x + 2)$

vi) $y = -\frac{1}{2}(x - 1)(x + 2)$

a) How are the graphs of the functions similar and how are they different?

Same	Different
• x-intercepts (zeros) • equations have same degree	• y-intercepts • stretch or compression factors • vertices

b) Describe the relationship between the graphs of functions of the form $y = k(x - 1)(x + 2)$, where $k \in \mathbb{R}$

They have the same x-intercepts.

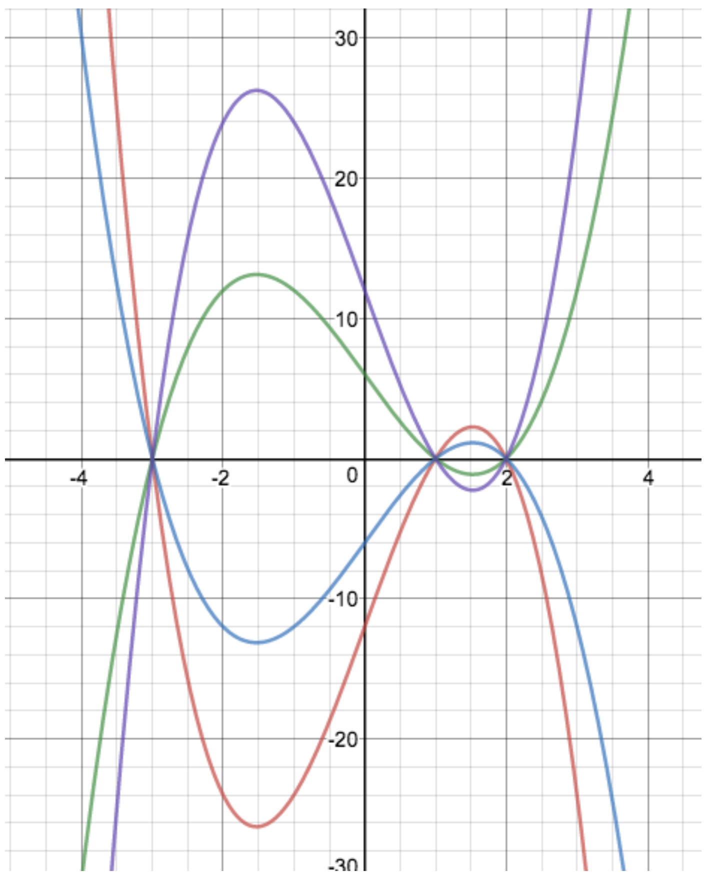
2) a) Examine the following functions. How are they similar? How are they different?

i) $y = -2(x-1)(x+3)(x-2)$
ii) $y = -(x-1)(x+3)(x-2)$
iii) $y = (x-1)(x+3)(x-2)$
iv) $y = 2(x-1)(x+3)(x-2)$

b) Predict how the graphs of the functions will be similar and how they will be different.

They will have the same x-intercepts but their shape and direction will be different due to the sign and value of the leading coefficient.

c) Use technology to help you sketch the graphs of all four functions on the same set of axes.



A **family** of functions is a set of functions that have the same characteristics. Polynomial functions with the same **zeros** are said to belong to the same family. The graphs of polynomial functions that belong to the same family have the same x-intercepts but have different y-intercepts (unless 0 is one of the x-intercepts).

An equation for the family of polynomial functions with zeros $a_1, a_2, a_3, \ldots, a_n$ is:

$$y = k(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n), \text{ where } k \in \mathbb{R}, k \neq 0$$

Part 2: Represent a Family of Functions Algebraically

1) The zeros of a family of quadratic functions are 2 and -3.

a) Determine an equation for this family of functions.

$y = k(x - 2)(x + 3)$

b) Write equations for two functions that belong to this family

$y = 8(x - 2)(x + 3)$
$y = -3(x - 2)(x + 3)$

c) Determine an equation for the member of the family that passes through the point (1, 4).

$y = k(x - 2)(x + 3)$

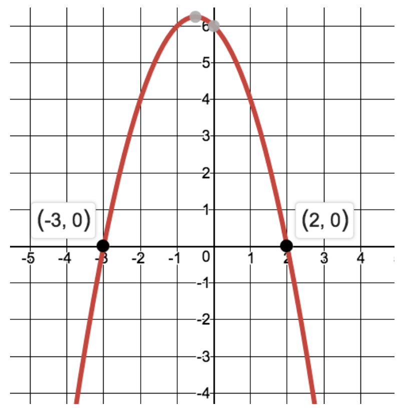
$4 = k(1 - 2)(1 + 3)$

$4 = k(-1)(4)$

$4 = -4k$

$-1 = k$

$y = -(x - 2)(x + 3)$



2) The zeros of a family of cubic functions are -2, 1, and 3.

a) Determine an equation for this family.

$y = k(x + 2)(x - 1)(x - 3)$

b) Determine an equation for the member of the family whose graph has a y-intercept of -15.

$-15 = k(0 + 2)(0 - 1)(0 - 3)$

$-15 = k(2)(-1)(-3)$

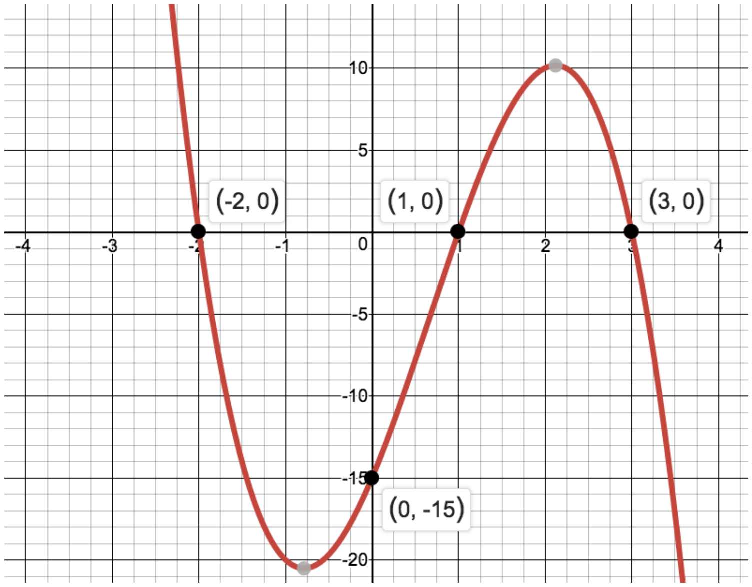
$-15 = 6k$

$k = -2.5$

$y = -2.5(x + 2)(x - 1)(x - 3)$

d) Sketch a graph of the function

Negative leading coefficient and odd degree so it will extend from Q2 to Q4



To sketch a graph:

- Plot y-intercept
- Plot x-intercepts
- Use degree and leading coefficient to determine end behaviour

3) Determine an equation for the family of cubic functions with zeros $3 \pm \sqrt{5}$ and $-\frac{1}{2}$

Factors:

$$x = 3 \pm \sqrt{5}$$

$$x - 3 = \pm\sqrt{5}$$

$$(x-3)^2 = 5$$

$$x^2 - 6x + 9 = 5$$

$$x^2 - 6x + 4 = 0$$

$$x = -\frac{1}{2}$$

$$2x = -1$$

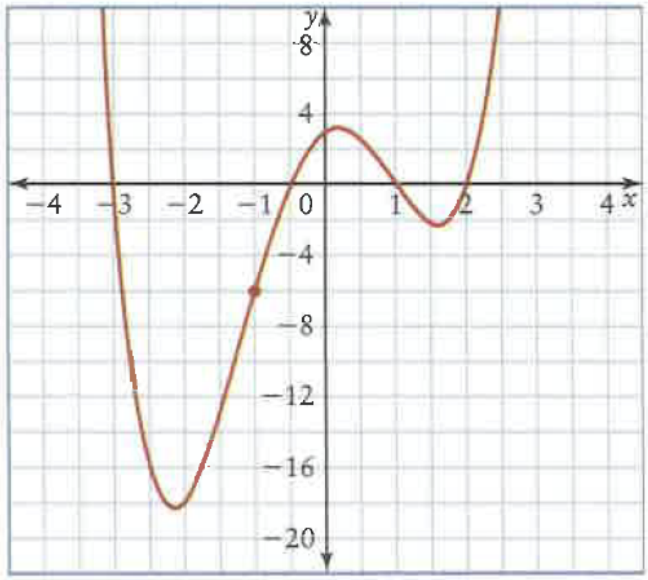
$$2x + 1 = 0$$

Equation:

$$P(x) = k(2x+1)(x^2 - 6x + 4)$$

Part 3: Determine an Equation for a Function From a Graph

3) Determine an equation for the quartic function represented by this graph.



The x-intercepts are $-3, -\frac{1}{2}, 1$, and 2

$y = k(x+3)(2x+1)(x-1)(x-2)$

The graph passes through the point (-1, -6)

$-6 = k(-1+3)(2(-1)+1)(-1-1)(-1-2)$

$-6 = k(2)(-1)(-2)(-3)$

$-6 = -12k$

$k = 0.5$

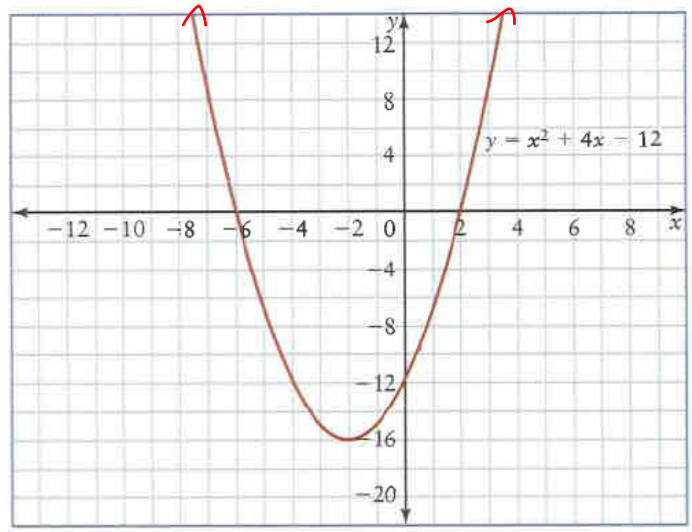
$y = 0.5(x+3)(2x+1)(x-1)(x-2)$

L6 – 2.5 – Solving Inequalities Lesson
MHF4U

In this section, you will learn the meaning of a polynomial inequality and examine methods for solving polynomial inequalities.

Part 1: Intro to Inequalities

Task: Read the following on your own



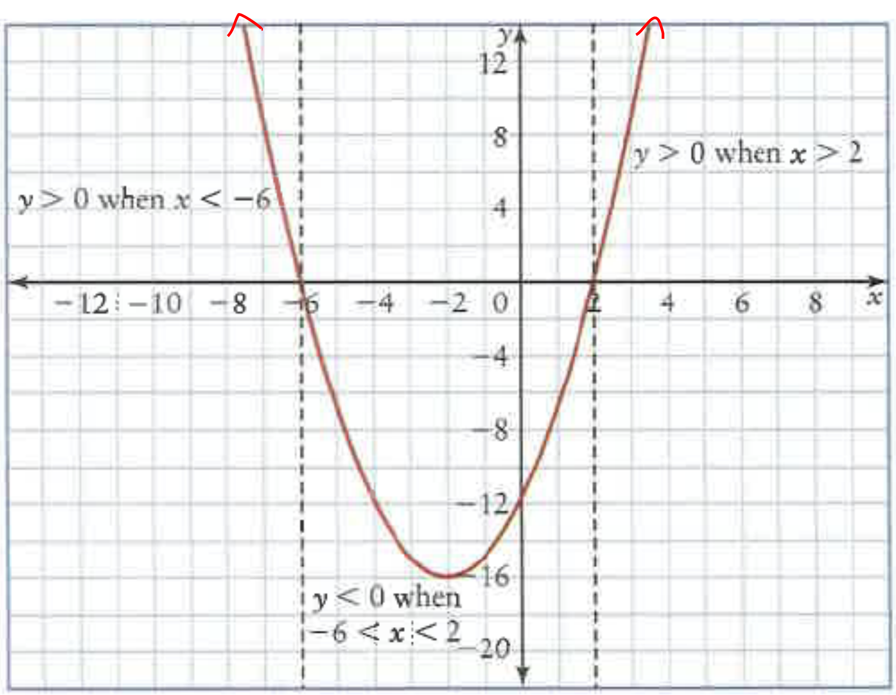
Examine the graph of $y = x^2 + 4x - 12$.

The x-intercepts are 6 and -2. These correspond to the zeros of the function $y = x^2 + 4x - 12$. Note that the factored form version of the function is $y = (x + 6)(x - 2)$. By moving from left to right along the x-axis, we can make the following observations:

- The function is positive when $x < -6$ since the y-values are positive
- The function is negative when $-6 < x < 2$ since the y-values are negative
- The function is positive when $x > 2$ since the y-values are positive.

The zeros -6 and 2 divide the x-axis into three intervals. In each interval, the function is either positive or negative. The information can be summarized in a table:

Interval	$x < -6$	$-6 < x < 2$	$x > 2$
Sign of Function	$+$	$-$	$+$



Polynomial Inequalities

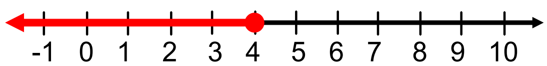
A polynomial inequality results when the equal sign in a polynomial equation is replaced with an inequality symbol.

The real zeros of a polynomial function, or x-intercepts of the corresponding graph, divide the x-axis into intervals that can be used to solve a polynomial inequality.

Part 1: Inequalities and Number Lines

Example 1: Write an inequality that corresponds to the values of x shown on each number line

a)

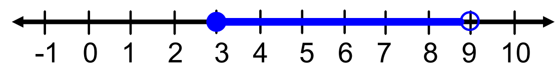


$x \leq 4$

OR

$(-\infty, 4]$

b)



$3 \leq x < 9$

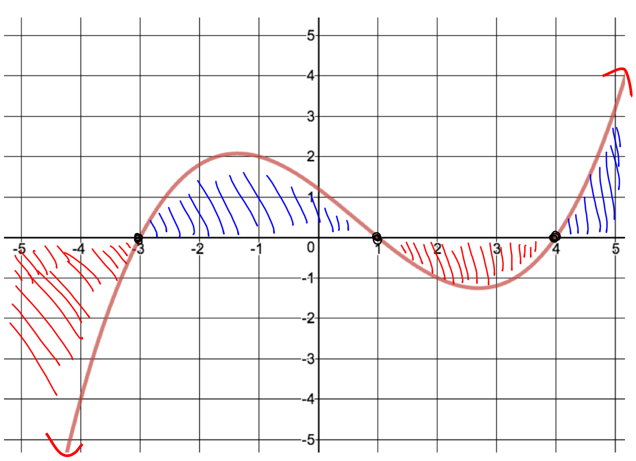
OR

$[3, 9)$

Part 2: Solve an Inequality given the Graph

Example 2: Use the graph of the function $f(x)$ to answer the following inequalities…

$f(x) = 0.1(x-1)(x+3)(x-4)$



a) $f(x) < 0$

$f(x) < 0$ when: $x < -3$ or $1 < x < 4$

$(-\infty, -3) \cup (1, 4)$

b) $f(x) \geq 0$

$f(x) \geq 0$ when: $-3 \leq x \leq 1$ or $x \geq 4$

$[-3, 1] \cup [4, \infty)$

Part 2: Solve Linear Inequalities

Note: Solving linear inequalities is the same as solving linear equations. However, when both sides of an inequality are multiplied or divided by a negative number, the inequality sign must be reversed.

Example 3: Solve each inequality

a) $x - 8 \geq 3$

$x - 8 \geq 3$

$x \geq 3 + 8$

$x \geq 11$

b) $-4 - 2x < 12$

$-4 - 2x < 12$

$\frac{-2x}{-2} < \frac{16}{-2}$

Reverse inequality when dividing by a negative

$x > -8$

Part 2: Solve Inequalities of Degree 2 and Higher

Steps for solving polynomial inequalities algebraically:

1) Use inverse operations to move all terms to one side of the inequality
2) Factor the polynomial to determine the zeros of the corresponding equation
3) Find the interval(s) where the function is positive or negative by either:
   a. Graphing the function using the zeros, leading coefficient, and degree
   b. Make a factor table and test values in each interval

Example 4: Solve each polynomial inequality algebraically

a) $2x^2 + 3x - 9 > 0$ ← when is it above the x-axis?

Method 1: Graph the inequality

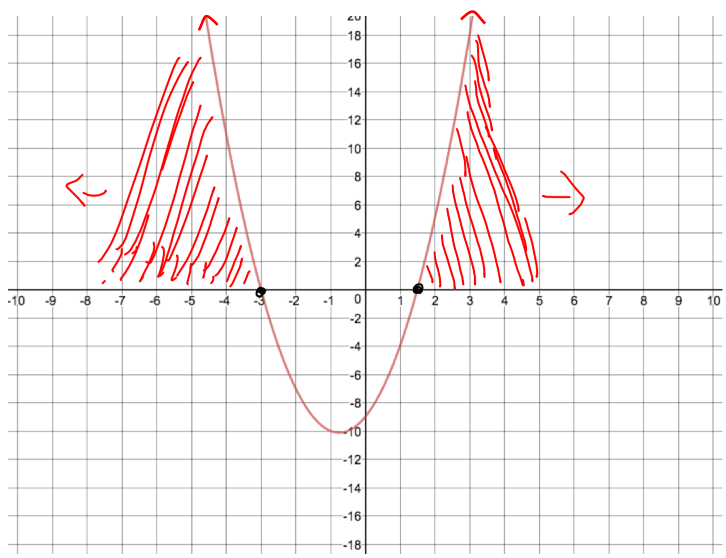
$2x^2 + 3x - 9 > 0$ P: -18 S: 3 6 and -3

$(2x^2 + 6x) + (-3x - 9) > 0$

$2x(x + 3) - 3(x + 3) > 0$

$(x + 3)(2x - 3) > 0$

x-int at -3 and 1.5



$2x^2 + 3x - 9 > 0$ when... $x < -3$ or $x > 1.5$

$(-\infty, -3) \cup (1.5, \infty)$

Method 2: Factor Table (sign chart)

To make a factor table:

- Use x-intercepts and vertical asymptotes to divide in to intervals
- Use a test point within each interval to find the sign of each factor
- Determine the overall sign of the product by multiplying signs of each factor within each interval.

	$-\infty$ to -3	-3 to 1.5	1.5 to ∞
	-4	0	2
$x+3$	$-$	$+$	$+$
$2x-3$	$-$	$-$	$+$
overall sign	$+$ *	$-$	$+$ *

$\therefore\ 2x^2+3x-9>0$ when $x<-3$ or $x>1.5$

$(-\infty,-3)\cup(1.5,\infty)$

b) $-2x^3 - 6x^2 + 12x \leq -16$

Method 1: Graph the inequality

$$-2x^3 - 6x^2 + 12x + 16 \leq 0$$

common factor

$$\frac{-2(x^3 + 3x^2 - 6x - 8)}{-2} \leq \frac{0}{-2}$$

reverse inequality

$$x^3 + 3x^2 - 6x - 8 \geq 0$$

factors of -8 are: $\pm 1, \pm 2$, and ± 3

$f(-1) = 0$, $\therefore$ $x+1$ is a factor.

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & \downarrow & -1 & -2 & 8 \\ \hline \times & 1 & 2 & -8 & 0 \\ & x^2 & x & \# & R \end{array}$$

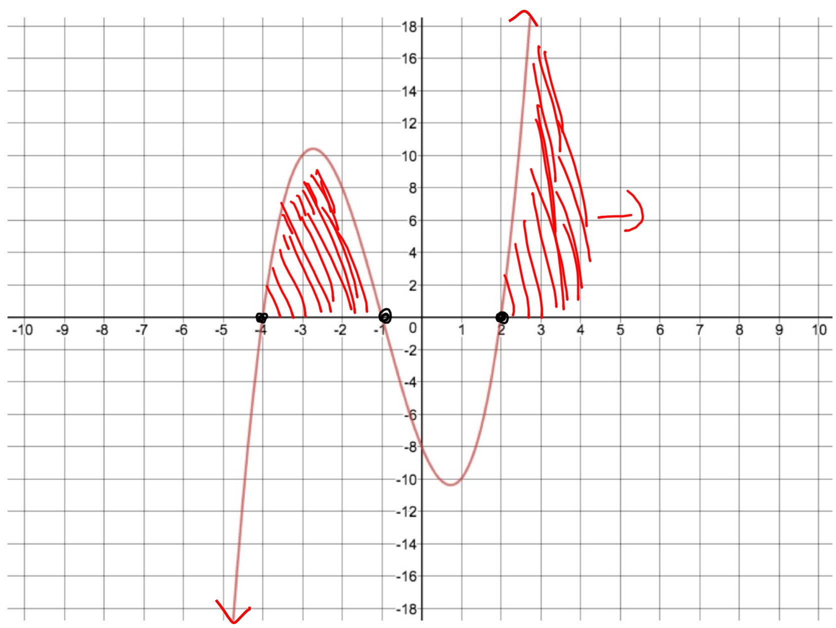
$$(x+1)(x^2 + 2x - 8) \geq 0$$

$$(x+1)(x+4)(x-2) \geq 0$$

x-int at $-1, -4$, and 2

Positive leading coefficient

Degree 3



Solution: $-4 \leq x \leq -1$ or $x \geq 2$

$[-4, -1] \cup [2, \infty)$

Method 2: Factor Table (sign chart)

	$-\infty$ to -4	-4 to -1	-1 to 2	2 to ∞
	-5	-2	0	3
$x+1$	$-$	$-$	$+$	$+$
$x+4$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
overall sign	$-$	$+$ *	$-$	$+$ *

Solution: $[-4, -1] \cup [2, \infty)$

c) $x^3 + 4x^2 + 6x < -24$

$x^3 + 4x^2 + 6x + 24 < 0$

factor by grouping

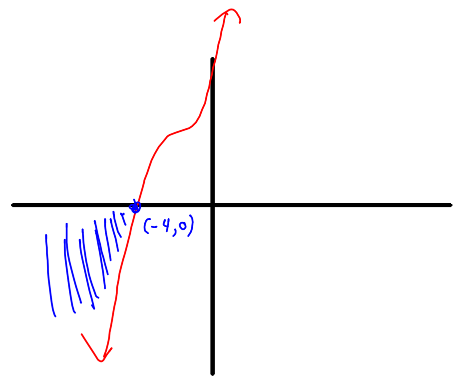
$x^2(x+4) + 6(x+4) < 0$

$(x+4)(x^2+6) < 0$

x-int at -4

no real solutions

	$-\infty$ -5	-4 0	∞
$x+4$	$-$	$+$	
x^2+6	$+$	$+$	
sign	$-$	$+$	



Solution: $x < -4$ OR $(-\infty, -4)$

Part 2: Applications of Inequalities

3) The price, p, in dollars, of a stock t years after 1999 can be modeled by the function $p(t) = 0.5t^3 - 5.5t^2 + 14t$. When will the price of the stock be more than \$90?

$0.5t^3 - 5.5t^2 + 14t > 90$

$0.5t^3 - 5.5t^2 + 14t - 90 > 0$

$0.5(t^3 - 11t^2 + 28t - 180) > 0$

$t^3 - 11t^2 + 28t - 180 > 0$

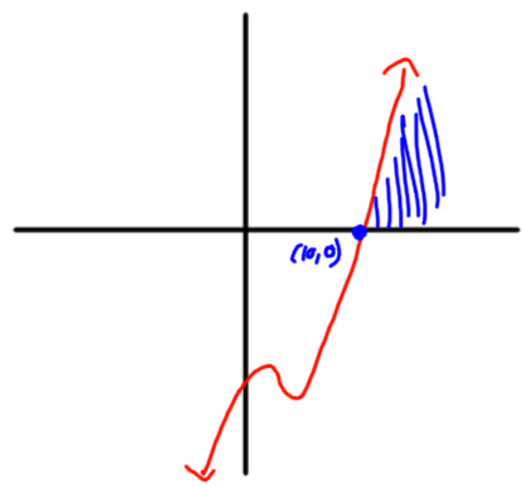
$f(10) = 0$; ∴ $x - 10$ is a factor

10	1	-11	28	-180
	↓	10	-10	180
	1	-1	18	0
	t^2	t	#	R

$(t-10)(t^2 - t + 18) > 0$

x-int at (10,0)

$b^2 - 4ac = (-1)^2 - 4(1)(18)$
$= -71$
∴ no solutions



Solution: $t > 10$

∴ the price of the stock will be above \$90 after year 2009