L4 – 7.1/7.2 – Solving Exponential Equations
MHF4U

Part 1: Changing the Base of Powers

Exponential functions can be written in many different ways. It is often useful to express an exponential expression using a different base than the one that is given.

Example 1: Express each of the following in terms of a power with a base of 2.

a) 8

$= 2^3$

b) 4^3

$= (2^2)^3$

$= 2^6$

c) $\sqrt{16}\times\left(\sqrt[5]{32}\right)^3$

$= 16^{\frac{1}{2}}\times 32^{\frac{3}{5}}$

$= (2^4)^{\frac{1}{2}}\times(2^5)^{\frac{3}{5}}$

$= 2^2\times 2^3$

$= 2^5$

d) 12

$2^x = 12$

$\log 2^x = \log 12$

$x\log 2 = \log 12$

$x = \frac{\log 12}{\log 2}$

$\therefore 12 = 2^{\frac{\log 12}{\log 2}}$

Part d) shows that any positive number can be expressed as a power of any other positive number.

Example 2: Solve each equation by getting a common base

Remember: if $x^a = x^b$, then $a = b$

a) $4^{x+5} = 64^x$

$4^{x+5} = (4^3)^x$

$4^{x+5} = 4^{3x}$

$x + 5 = 3x$

$5 = 2x$

$x = \frac{5}{2}$

b) $4^{2x} = 8^{x-3}$

$(2^2)^{2x} = (2^3)^{x-3}$

$2^{4x} = 2^{3x-9}$

$4x = 3x - 9$

$x = -9$

Part 2: Solving Exponential Equations

When you have powers in your equation with different bases and it is difficult to write with the same base, it may be easier to solve by taking the **logarithm** of both sides and applying the **power law** of logarithms to remove the variable from the **exponent**.

Example 3: Solve each equation

a) $4^{2x-1} = 3^{x+2}$

$\log 4^{2x-1} = \log 3^{x+2}$	Take log of both sides
$(2x-1)\log 4 = (x+2)\log 3$	Use power law of logarithms
$2x\log 4 - \log 4 = x\log 3 + 2\log 3$	Use distributive property to expand
$2x\log 4 - x\log 3 = 2\log 3 + \log 4$	Move variable terms to one side
$x(2\log 4 - \log 3) = 2\log 3 + \log 4$	Common factor
$x = \dfrac{2\log 3 + \log 4}{2\log 4 - \log 3}$	Isolate the variable

$x \cong 2.14$

b) $2^{x+1} = 3^{x-1}$

$\log 2^{x+1} = \log 3^{x-1}$

$(x+1)\log 2 = (x-1)\log 3$

$x\log 2 + \log 2 = x\log 3 - \log 3$

$x\log 2 - x\log 3 = -\log 3 - \log 2$

$x(\log 2 - \log 3) = -\log 3 - \log 2$

$$x = \frac{-\log 3 - \log 2}{\log 2 - \log 3}$$

$x \cong 4.419$

Part 3: Applying the Quadratic Formula

Sometimes there is no obvious method of solving an exponential equation. If you notice two powers with the same base and an exponent of x, there may be a hidden quadratic.

Example 4: Solve the following equation

$2^x - 2^{-x} = 4$

Multiply both sides by 2^x

$2^x(2^x - 2^{-x}) = 2^x(4)$

Distribute

$2^{2x} - 2^0 = 4(2^x)$

Rearrange in to standard form $ax^2 + bx + c = 0$

$2^{2x} - 4(2^x) - 1 = 0$

$(2^x)^2 - 4(2^x) - 1 = 0$

Let $k = 2^x$ to see the quadratic

$k^2 - 4k - 1 = 0$

Solve using quadratic formula

$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$k = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$

$k = \dfrac{4 \pm \sqrt{20}}{2}$

Don't forget to simplify the radical expression

$k = \dfrac{4 \pm 2\sqrt{5}}{2}$

$k = \dfrac{2(2 \pm \sqrt{5})}{2}$

$k = 2 \pm \sqrt{5}$

Now substitute 2^x back in for k and solve

Case 1

$2^x = 2 + \sqrt{5}$

$\log 2^x = \log(2 + \sqrt{5})$

$x = \dfrac{\log(2 + \sqrt{5})}{\log 2}$

$x \cong 2.08$

Case 2

$2^x = 2 - \sqrt{5}$

$\log 2^x = \log(2 - \sqrt{5})$

Can't take the log of a negative number, therefore this is an extraneous root (No solution).

Part 4: Application Question

Remember:

Equation: $\boldsymbol{y = a(b)^x}$

$a =$ initial amount

$b =$ growth $(b > 1)$ or decay $(0 < b < 1)$ factor

$y =$ future amount

$x =$ number of times a has increased or decreased

To calculate x, use the equation: $x = \frac{total\ time}{time\ it\ takes\ for\ one\ growth\ or\ decay\ period}$

Example 5: A bacteria culture doubles every 15 minutes. How long will it take for a culture of 20 bacteria to grow to a population of 163 840?

$$163\ 840 = 20(2)^{\frac{t}{15}}$$

$$8192 = 2^{\frac{t}{15}}$$

$$\log 8192 = \log 2^{\frac{t}{15}}$$

$$\log 8192 = \frac{t}{15}\log 2$$

$$\frac{\log 8192}{\log 2} = \frac{t}{15}$$

$$13 = \frac{t}{15}$$

$t = 195$ minutes

Example 6: One minute after a 100-mg sample of Polonium-218 is placed into a nuclear chamber, only 80-mg remains. What is the half-life of polonium-218?

$$80 = 100\left(\frac{1}{2}\right)^{\frac{1}{h}}$$

$$0.8 = 0.5^{\frac{1}{h}}$$

$$\log 0.8 = \log 0.5^{\frac{1}{h}}$$

$$\log 0.8 = \frac{1}{h}\log 0.5$$

$$\frac{\log 0.8}{\log 0.5} = \frac{1}{h}$$

$$h = \frac{\log 0.5}{\log 0.8}$$

$h \cong 3.1$ minutes