# Unit 1: Polynomial Functions

**Unit 1: Polynomial Functions**
**1.1 Power Functions**

## POLYNOMIAL EXPRESSIONS

*A polynomial expression is one or more terms where each term is the product of a constant and a variable raised to a non-negative integral exponent only.*

Example 1: Which of the following is a polynomial expression? Explain.

| | Yes/No | Reason |
|---|---|---|
| $5x$ | | |
| $2x^{-3} + 3\sqrt{x} - 4$ | | |
| $t^2 + 3.5t$ | | |
| $3xy + 4x^2$ | | |
| $\frac{1}{2}x^5 - 3x^2 - x + 1$ | | |

## POLYNOMIAL FUNCTIONS

*A polynomial function is a function defined by a polynomial in one variable written in the form* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$.

To be a polynomial function, the following conditions must be met:

1. $a_n \neq 0$ (This means the lead coefficient cannot equal zero) if n is the highest term
2. The coefficients ($a_n$, $a_{n-1}$, ...$a_0$) are all real numbers
3. The exponents are all whole numbers
   - The degree of a polynomial in one variable *x* is the highest power of *x*.
   - The leading coefficient of a polynomial function is the constant belonging to the power with the highest exponent.
   - The domain of a polynomial function is all real numbers.
   - There are n+1 terms in a polynomial function of degree n.

## Types of Polynomial Functions

| Type | Degree | Standard Form a,b,c,d,e are Real numbers | Example |
|---|---|---|---|
| **Constant** | 0 | $f(x) = a$ | |
| **Linear** | 1 | $f(x) = ax+b$ , $a\neq 0$ | |
| **Quadratic** | 2 | $f(x) = ax^2+bx+c$ , $a\neq 0$ | |
| **Cubic** | 3 | $f(x) = ax^3+bx^2+cx+d$ , $a\neq 0$ | |
| **Quartic** | 4 | $f(x) = ax^4+bx^3+cx^2+dx+e$ , $a\neq 0$ | |
| **Quintic** | 5 | $f(x) = ax^5+bx^4+cx^3+dx^2+ex+f$ , $a\neq 0$ | |

**Example 2:** Complete the table below:

| | Type | Degree | Leading Coefficient |
|---|---|---|---|
| $f(x) = 10 + 7x$ | | | |
| $f(x) = 3.7x^4 - 2x^2 + 7.4$ | | | |
| $g(x) = 2.5$ | | | |
| $g(x) = -\frac{1}{2}x^2 + \sqrt{2}$ | | | |
| $s(t) = t^3 - 3t$ | | | |
| $f(x) = x(x+1)(x-2)$ | | | |
| $h(x) = x(2x+1)^2(2-x)^3$ | | | |

**Example 3:** Explain why each of the following are not polynomial functions.

a) $h(x) = 3x^2 + 2x - 5x^{-1}$

b) $f(x) = \frac{1}{x}$

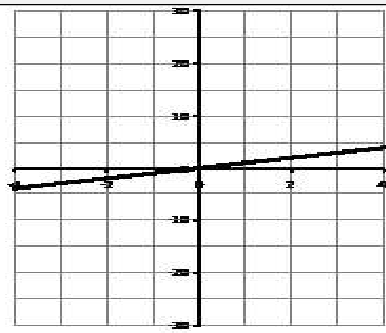
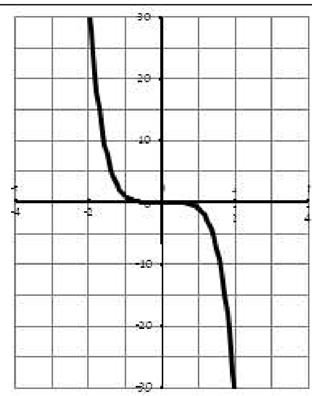
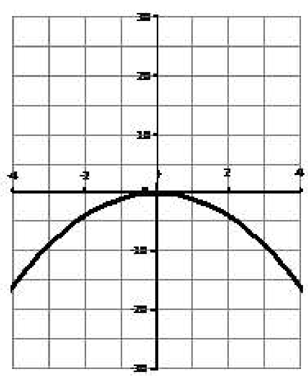
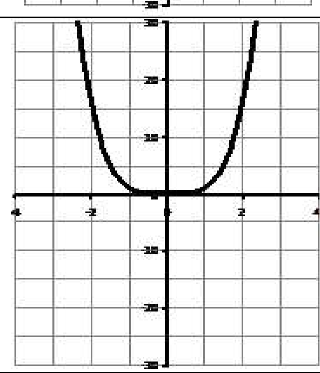
c) $g(x) = 2^x$

**POWER FUNCTIONS**

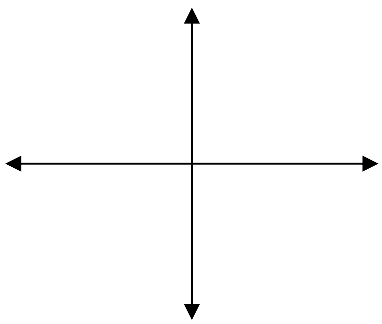
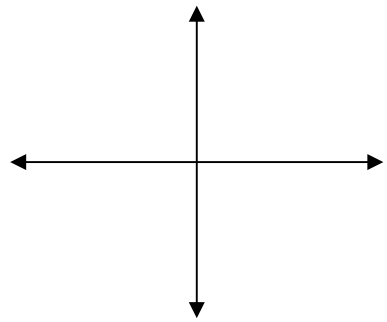
A power function is a polynomial of the form $f(x) = ax^n$, where n is a whole number.
Power functions have similar characteristics depending on whether their degree is even or odd.

| |
|---|
| ________________________________________________ |
| ________________________________________________ |
| ________________________________________________ |

| |
|---|
| ________________________________________________ |
| ________________________________________________ |
| ________________________________________________ |

| Function | Degree | Graph | Sign of Leading coefficient | End behavior |
|---|---|---|---|---|
| $y = \frac{1}{3}x$ | |  | | As $x \to -\infty$, $y \to$ ______<br>As $x \to \infty$, $y \to$ ______ |
| $y = -2x^3$ | |  | | As $x \to -\infty$, $y \to$ ______<br>As $x \to \infty$, $y \to$ ______ |
| $y = -x^2$ | |  | | As $x \to -\infty$, $y \to$ ______<br>As $x \to \infty$, $y \to$ ______ |
| $y = x^4$ | |  | | As $x \to -\infty$, $y \to$ ______<br>As $x \to \infty$, $y \to$ ______ |

## Summary

<table>
<tr>
<td>If a polynomial function has an odd degree and its lead coefficient is positive, then, the function extends from the _______ quadrant to the _______ quadrant.<br>Therefore:<br><br>$as\ \ x \to -\infty$, ____________<br>$as\ \ x \to \infty$, ____________ .<br><br></td>
<td>If a polynomial function has an odd degree and its lead coefficient is negative, then, the function extends from the _____ quadrant to the _______ quadrant.<br>Therefore:<br><br>$as\ \ x \to -\infty$, ____________<br>$as\ \ x \to \infty$, ____________ .<br><br></td>
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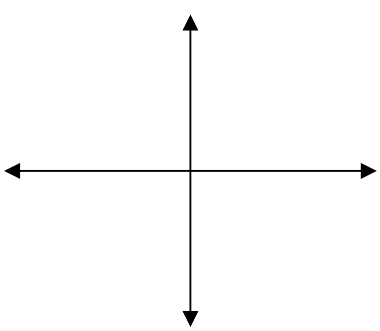
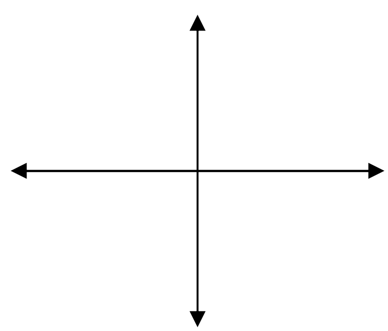
<tr>
<td>If a polynomial function has an even degree and its lead coefficient is positive, then, the function extends from the _______ quadrant to the _______ quadrant.<br>Therefore:<br><br>$as\ \ x \to -\infty$, ____________<br>$as\ \ x \to \infty$, ____________ .<br><br></td>
<td>If a polynomial function has an even degree and its lead coefficient is negative, then, the function extends from the _____ quadrant to the _______ quadrant.<br>Therefore:<br><br>$as\ \ x \to -\infty$, ____________<br>$as\ \ x \to \infty$, ____________ .<br><br></td>
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**Polynomial Identities**

An equation shows that two mathematical expressions are equal.

$$2x-5=6x-1$$

It can be solved to give the ***roots*** or ***solutions*** of the equation. An *identity* is an equation that is true for **all values of *x*.**

If an equation is an identity, the symbol $"="$ in the equation can be replaced by $"\equiv"$ which means "*is identical to*".

For example: $(x+1)^3 \equiv x^3+3x^2+3x+1$ is an **identity** since the equation is satisfied for all values of *x*.

Ex#1.Given that $x^3-2x^2+4x+3 \equiv (x-1)(x^2-x+\mathrm{a})+\mathrm{b}$, find the value of *a* and *b*.

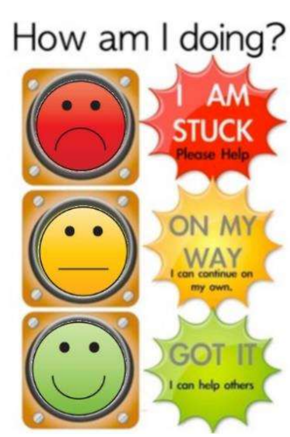
Ex#2. Given that $15x^3+\mathrm{C}x^2-x+2 \equiv (3x+1)(\mathrm{A}x+\mathrm{B})(x-1)$, find the values of A,B and C.

Ex#3.Given $2x^2+x+\mathrm{C} \equiv \mathrm{A}(x+1)^2+\mathrm{B}(x-1)+4$ for all values of x, find the values of A,B and C.

## Exit Card!

Complete the chart below.

| **Function** | **Type** | **degree** | **leading coefficient** | **End behavior** |
|---|---|---|---|---|
| $y = x^2 - 2x$ | | | | As $x \to -\infty$, $y \to$ ________<br>As $x \to \infty$, $y \to$ ________ |
| $y = -(x-1)(1-x)^2$ | | | | As $x \to -\infty$, $y \to$ ________<br>As $x \to \infty$, $y \to$ ________ |
| $y = -2x^5 + 7x^4 - 3x^3 - 18x^2 + 5$ | | | | As $x \to -\infty$, $y \to$ ________<br>As $x \to \infty$, $y \to$ ________ |
| $y = -(1-2x)^3(x+1)^2$ | | | | As $x \to -\infty$, $y \to$ ________<br>As $x \to \infty$, $y \to$ ________ |
| $y = -3x^2(2-x)^3(2x-1)^2$ | | | | As $x \to -\infty$, $y \to$ ________<br>As $x \to \infty$, $y \to$ ________ |

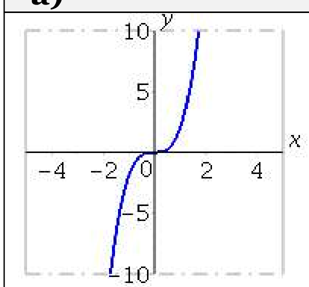
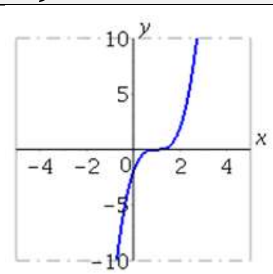
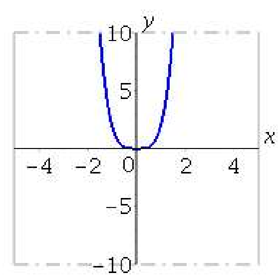
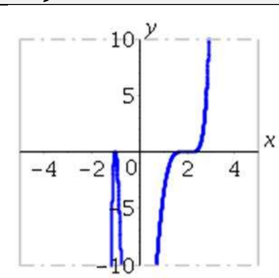


## Warm Up

1. Which of the following relations are polynomial functions? For each polynomial function name the type of polynomial, its degree, the leading coefficient and end behaviour. If a relation is not a polynomial function, provide at least one reason why it is not.

| | |
|---|---|
| a. $y = 3x^5 - 2x + 17$ | b. $y = 3x^4 + \sqrt{5x}$ |
| Type of polynomial: __________<br>Degree : __________<br>Leading coefficient:__________<br>$x \to -\infty, y \to$<br>$x \to \infty\ , y \to$ | Reason: |
| c. $y = -\frac{1}{2}x(x-4)^2(x+4)^2$ | d. $y = (x-2)^2(4-2x)(x+5)$ |
| Type of polynomial: __________<br>Degree : __________<br>Leading coefficient:__________<br>$x \to -\infty, y \to$<br>$x \to \infty\ , y \to$ | Type of polynomial: __________<br>Degree : __________<br>Leading coefficient:__________<br>$x \to -\infty, y \to$<br>$x \to \infty\ , y \to$ |
| e. $y = 4$ | f. $y = \left(-\left(x-4\right)\right)^3 + 1$ |
| Type of polynomial: __________<br>Degree : __________<br>Leading coefficient:__________<br>$x \to -\infty, y \to$<br>$x \to \infty\ , y \to$ | Type of polynomial: __________<br>Degree : __________<br>Leading coefficient:__________<br>$x \to -\infty, y \to$<br>$x \to \infty\ , y \to$ |

2. From the graphs given, select all graphs that represent power functions of the form $y=ax^n$.

| a) | b) | c) | d) |
|---|---|---|---|
|  |  |  |  |
| | | | |

## 1.2 Characteristics of Polynomial Functions (Part 1)
INVESTIGATING FINITE DIFFERENCES

A **first difference** or **finite difference** is the difference between **consecutive y-coordinates** for **evenly spaced integral x-coordinates**. To calculate a first difference, **subtract consecutive y-values**.

1. Complete the following table for the relation $y = 2x + 1$:

| x | f(x) | Δf(x) |
|---|---|---|
| -2 | | |
| -1 | | |
| 0 | | |
| 1 | | |
| 2 | | |

**Observations**

How can you identify a linear relation?

In general, function in the form
$y = mx + b$, is a ________________.

For a linear relation, all of the first differences are ______________________.

**The constant differences** = _____× _____

2. Complete the following table for the relation $y = 3x^2 – 1$ :

| x | f(x) | Δf(x) | Δ²f(x) |
|---|---|---|---|
| -2 | | | |
| -1 | | | |
| 0 | | | |
| 1 | | | |
| 2 | | | |

**Observations**
In general, function in the form
$f(x) = ax^2 + bx + c$ is a _________________.

Its first differences form an arithmetic sequence.

Its second differences are _______________.

**The constant differences** = _____× ____

3. Complete the following table for the relation $y = 2x^3 + 1$ :

| x | f(x) | Δf(x) | Δ²f(x) | Δ³f(x) |
|---|---|---|---|---|
| -2 | | | | |
| -1 | | | | |
| 0 | | | | |
| 1 | | | | |
| 2 | | | | |

**Observations**
In general, a function in the form $f(x) = ax^3 + bx^2 + cx + d$ is a _________________.
Its third differences are ______________.
**The constant differences** = _____× _____

**Summary and Extension:**

1. If the first differences are equal:
   - ➔ The function is a ______ degree function;
   - ➔ The function is called a _________________;
   - ➔ The constant differences = _____× _____
2. If the second differences are equal:
   - ➔ The function is a ______ degree function;
   - ➔ The function is called a _________________;
   - ➔ The constant differences = _____× _____
3. If the third differences are equal:
   - ➔ The function is a ______ degree function;
   - ➔ The function is called a _________________;
   - ➔ The constant differences = _____× _____

**GENERAL RULE:**

- Finite differences can be used to determine the degree of a polynomial function.
- For example, the fourth differences of a quartic function are constant.
- The constant finite differences have the same sign as the leading coefficient.
- The constant finite differences are equal to **a[n x (n – 1) x … x 2 x 1]=an!** where **a** is the value of the leading coefficient. (**The constant differences= an!**)

**Example 1:**
The table of values represents a polynomial function.

| x | f(x) | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| -2 | -54 | | | | |
| -1 | -8 | | | | |
| 0 | 0 | | | | |
| 1 | 6 | | | | |
| 2 | 22 | | | | |
| 3 | 36 | | | | |
| 4 | 12 | | | | |
| 5 | -110 | | | | |

Use finite differences to determine:

(a) The degree of the polynomial function. ___________________

(b) The sign of the leading coefficient ______________________

(c) The value of the leading coefficient ______________________

**Example 2:** The points (1,-4), (2,0), (3,30), (4, 98) (5,216) (6,396) are on a function. Find the equation of the function.

| x | f(x) | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| 1 | -4 | | | | |
| 2 | 0 | | | | |
| 3 | 30 | | | | |
| 4 | 98 | | | | |
| 5 | 216 | | | | |
| 6 | 396 | | | | |

**Method:**

Step 1: Complete a table of values to determine the type of function and its general equation.
Step 2: Find the equations for consecutive values of y.
Step 3: Create a system of equations and solve for the variables ‘a’, ‘b’, and ‘c’.
Step 4: Write the equation.

**Example 3:** The points (1,0), (2,-2), (3,-2), (4, 0) (5,4) (6,10) are on a function. Find the equation of the function.

| x | f(x) | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | | | | |
| 6 | | | | | |

## Exit Card!

1. For the polynomial function $f(x) = (9x^3 - 9x^2 - 9x + 9)(x^2 - x + 2)$. State:
   (a) the degree of the function:________________

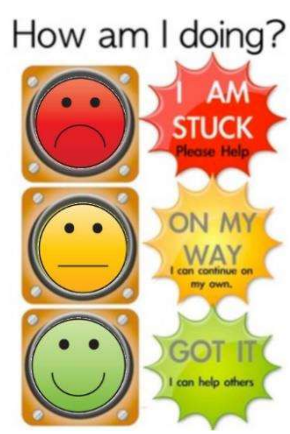
   (b) the leading coefficient:______________________

   (c) the value of the constant finite differences :________________

2. A polynomial function has a constant fourth difference of -132. Determine
   (a) the type of the function: ____________

   (b) the degree of the function ____________

   (c) the value of the leading coefficient: _____________



## Practice

1. In each of the following, the points given lie on the graph of a polynomial f unction. Determine the equation of the function using the algebraic method developed in class:

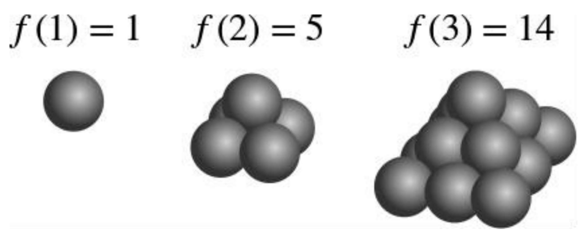
a) (1,4), (2,15), (3,30), (4,49), (5,72), (6,99)

b) (1, -34), (2,-42), (3,-38), (4,-16), (5,30), (6,106)

c) (1,12), (2,-10), (3,-18), (4,0), (5,56), (6,162)

d) (1,-2), (2,-4), (3,-6), (4,-8), (5,14), (6,108), (7,346)

2. The first three square pyramidal numbers are 1,5, and 14, as shown in the diagram. Find the next three pyramidal numbers and determine the equation of a polynomial function that gives the $x^{th}$ square pyramidal number.



Warm-Up

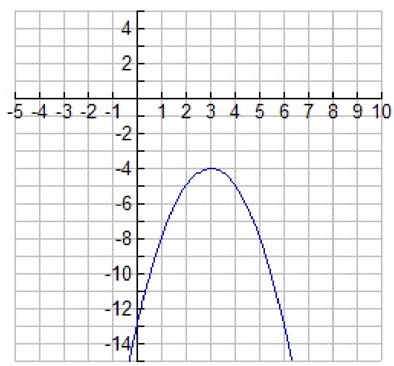
Determine the equation of the function that has the following points on its curve:

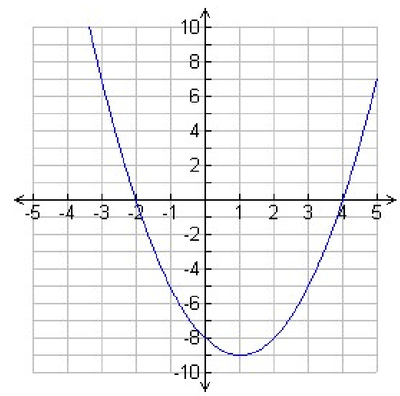
(-2,-24), (-1, -7), (0, -2), (1, -3), (2, -4), (3, 1) .

| x | f(x) | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| -2 | -24 | | | | |
| -1 | -7 | | | | |
| 0 | -2 | | | | |
| 1 | -3 | | | | |
| 2 | -4 | | | | |
| 3 | 1 | | | | |

# 1.2 PROPERTIES OF POLYNOMIAL FUNCTIONS (Part 2)

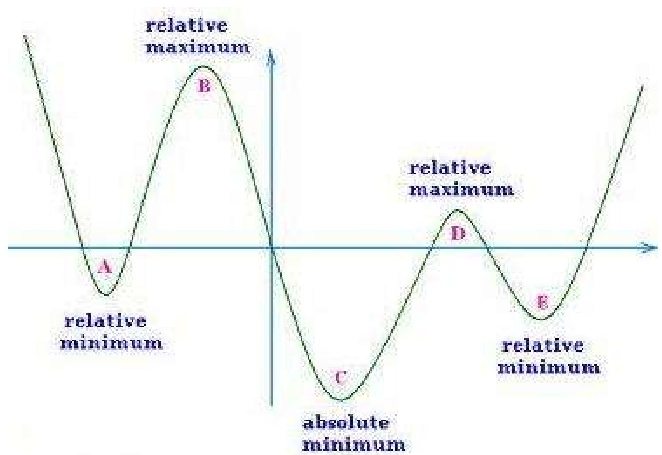
**ABSOLUTE (GLOBAL) MAXIMUM AND MINIMUM VALUES:**
An **absolute maximum** is **the highest y-value** on the graph.
An **absolute minimum** is **the lowest y-value** on the graph.



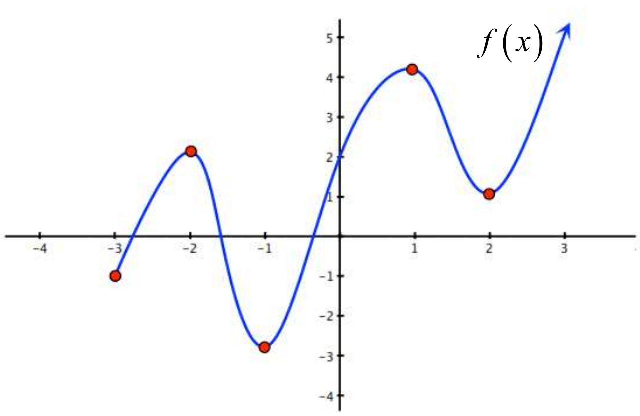




**RELATIVE (LOCAL) MAXIMUM AND MINIMUM VALUES:**

A **relative maximum** is the greatest value of a *function in **its neighborhood***.
A **relative minimum** is the least value of a *function in **its neighborhood***.




Ex1. Graph of function f(x) is given .Identify global and local max/min values.



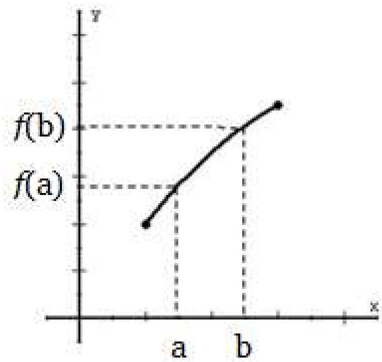

Local max:___________________
Local min:___________________
Absolute max: ___________________
Absolute min: ___________________

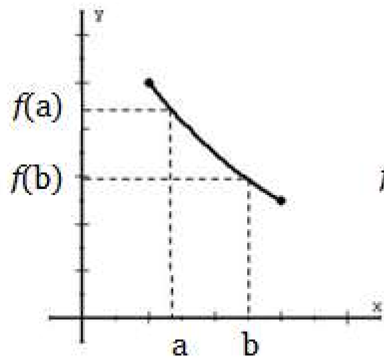
<u>**INTERVALS OF INCREASE/DECREASE**</u>

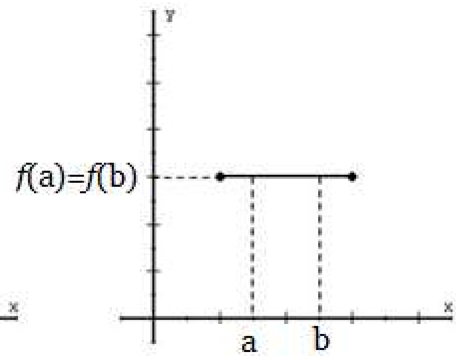
Suppose S is an interval in the domain of $f(x)$ ,so $f(x)$ is defined for all x in S.

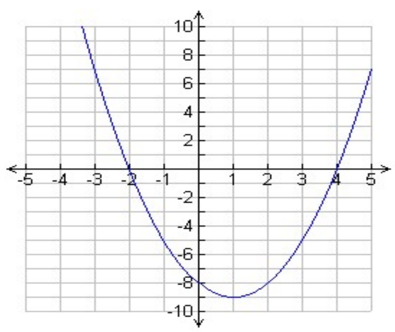
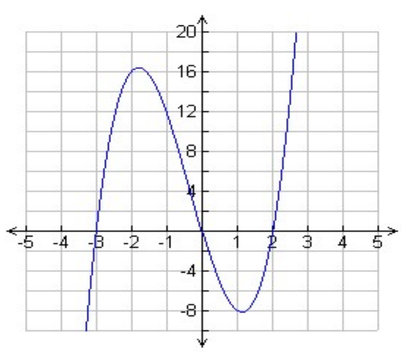
$f(x)$ is **increasing** on S , $\Leftrightarrow f(a) < f(b)$ for all $a, b \in S$ such that $a < b$

$f(x)$ is **decreasing** on S $\Leftrightarrow f(a) > f(b)$ for all $a, b \in S$ such that $a < b$



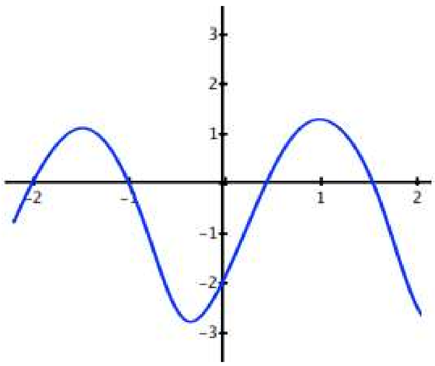




| FUNCTION GRAPH | INTERVALS OF INCREASE/DECREASE | TYPE OF MAX OR MIN |
|---|---|---|
|  | | |
|  | | |

**Conclusion:**

## ZEROS OF A FUNCTION:

The **zeros** of a function are known by two other names: ***x-intercepts*** and **roots**.
A zero/x-intercept/root of a function is the value of the x-coordinate where the function cuts or just touches the x-axis. The x-intercept is the **value** of the x-coordinate from the point (x, 0). The Y-INTERCEPT is the value of the y-coordinate of the point where the graph crosses the y axis. It is the value of y coordinate of the point (0, y).




x-intercepts :____________________
y-intercept :______________________

### Investigation of the Properties of Polynomial Functions:

Using a graphing calculator or Desmos[1], fill-in the following charts and draw appropriate conclusions:

**a) Quadratic Functions:**

| **Function** | **Degree** | **Number of zeroes/ x-intercepts/ roots** | **Number of Turning Points (MAX or Min)** |
|---|---|---|---|
| $y = x^2$ | | | |
| $y = x^2 + 1$ | | | |
| $y = 3x^2 - 4x - 1$ | | | |

**Conclusions:**

- Quadratic functions have a degree of ______.
- The maximum number of roots that a quadratic function can have is ______
- The least number of roots that a quadratic function can have is ______
- The maximum number of turning points (max.min) a quadratic function can have is ______

**b) Cubic Functions:**

| **Function** | **Degree** | **Number of zeroes/ x-intercepts/ roots** | **Number of Turning Points (MAX or Min)** |
|---|---|---|---|
| $y = x^3$ | | | |
| $y = x^3 + 2x^2 - x - 2$ | | | |
| $y = -4x^3 + 16x^2 - 13x + 3$ | | | |

**Conclusions:**

- Cubic functions have a degree of ______.
- The maximum number of roots that a cubic function can have is ______
- The least number of roots that a cubic function can have is ______
- The maximum number of turning points (max/min) a cubic function can have is ______

(1) https://www.desmos.com/calculator/kxbhcq6bix

**c) Quartic Functions:**

| **Function** | **Degree** | **Number of zeroes/ x-intercepts/ roots** | **Number of Turning Points (MAX or Min)** |
|---|---|---|---|
| $y = x^4$ | | | |
| $y = -x^4 - 5$ | | | |
| $y = x^4 + 3x^3 + x^2 - 3x - 2$ | | | |
| $y = -x^4 + 5x^2 - 4$ | | | |

**<u>Conclusions:</u>**

- Quartic functions have a degree of ______.
- The maximum number of roots that a quartic function can have is ______
- The least number of roots that a quartic function can have is ______
- The maximum number of turning points a quartic function can have is ______

**d) Quintic Functions:**

| **Function** | **Degree** | **Number of zeroes/ x-intercepts/ roots** | **Number of Turning Points (MAX or Min)** |
|---|---|---|---|
| $y = x^5 + 7$ | | | |
| $y = 2x^5 + 7x^4 - 3x^3 - 18x^2 + 5$ | | | |
| $y = 5x^5 + 5x^4 - 2x^3 + 4x^2 - 3x$ | | | |

**<u>Conclusions:</u>**

- Quintic functions have a degree of ______.
- The maximum number of roots/x-intercepts that a quintic function can have is ______
- The least number of roots/x-intercepts that a quintic function can have is ______
- The maximum number of turning points a quintic function can have is ______

**e) 6th Degree Functions:**

| **Function** | **Degree** | **Number of zeroes/ x-intercepts/ roots** | **Number of Turning Points (MAX or Min)** |
|---|---|---|---|
| $y = x^6$ | | | |
| $y = 2x^6 - 12x^4 + 18x^2 + x - 5$ | | | |
| $y = -x^6 - 3$ | | | |

**<u>Conclusions:</u>**

- The maximum number of roots/x-intercepts that a 6th degree function can have is ______
- The least number of roots/x-intercepts that a 6th degree function can have is ______
- The maximum number of turning points a 6th degree function can have is ______

# Overall Conclusions

**Number of Zeros:**

The maximum number of zeros/x-intercepts that a polynomial function can have is the ________as its ______________________.

The minimum number of zeros/x-intercepts that an ***odd degree*** polynomial can have is _______.

However, an ***even degree*** polynomial function can have ______ zeros/x-ints at all.

**Turning Points:**

The maximum number of turning points that a polynomial function can have is _________.

An ***even degree*** function must have at least ___________ turning point.

An ***odd degree*** function could have _________ turning points at all.

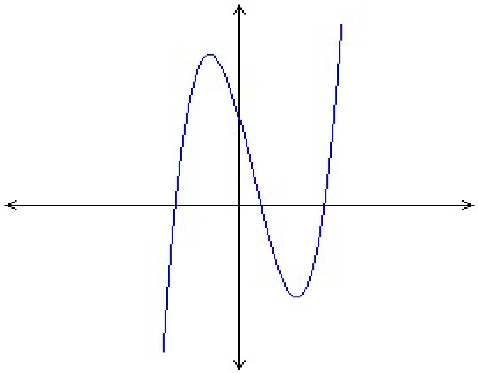
Complete the chart:

| Type of Polynomial | Degree | Maximum Number of zeros / x-intercepts / roots | Minimum Number of Zeros/ x-intercepts/roots | Maximum Number of Turning Points |
|---|---|---|---|---|
| **Linear** | | | | |
| **Quadratic** | | | | |
| **Cubic** | | | | |
| **Quartic** | | | | |
| **Quintic** | | | | |
| | 6 | | | |
| | 7 | | | |
| | n | | If n is even:_____<br>If n is odd:______ | |

# Exit Card!

Identify number of zeros/x-intercepts/roots, the sign of the leading coefficient and describe the end behaviour. Using this information, decide if each function is cubic or quartic.

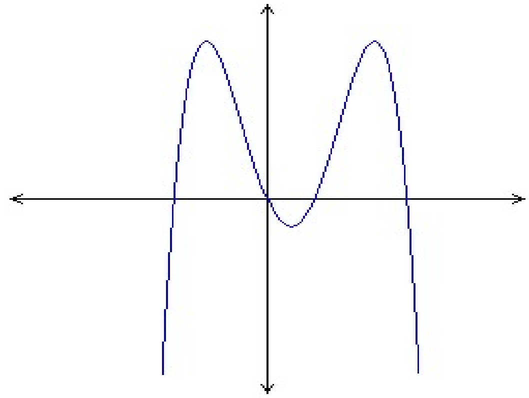
a)



# of zeros/x-int/roots: ______
Sign of Leading Coefficient: _______
End Behaviour:

Type of function:

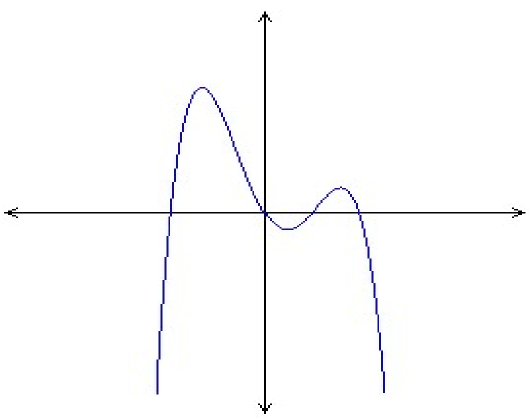
b)



# of zeros/x-int/roots: ______
Sign of Leading Coefficient: _______
End Behaviour:

Type of function:

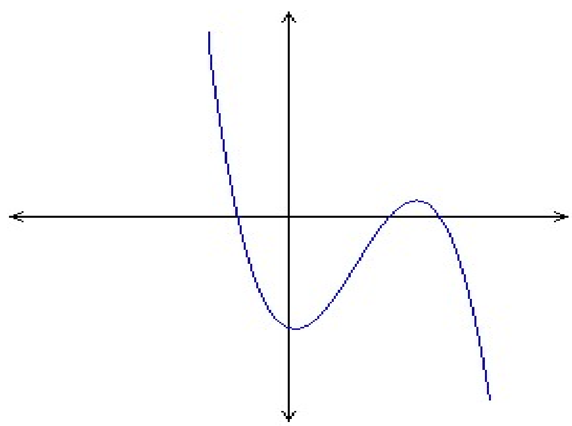
c)



# of zeros/x-int/roots: ______
Sign of Leading Coefficient: _______
End Behaviour:

Type of function:

d)



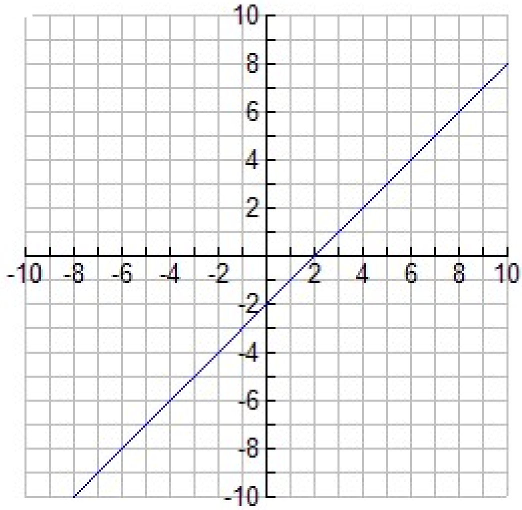
# of zeros/x-int/roots: ______
Sign of Leading Coefficient: _______
End Behaviour:

Type of function:

## Practice:

Take a look at the following graphs and answer the questions.

a)




Domain:

Range:

Number of roots:

Roots:

End Behaviour:

Degree:

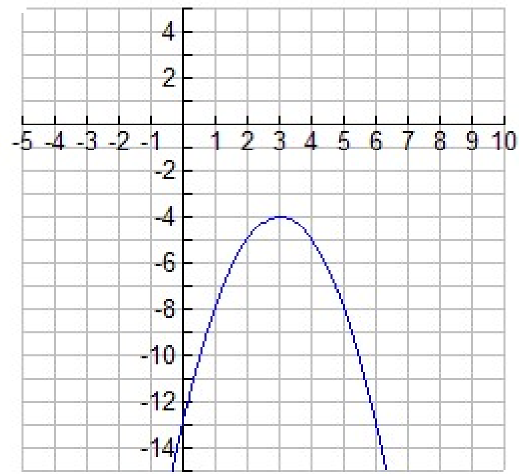
Intervals of positive:

Intervals of negative:

Intervals of increasing:

Intervals of decreasing:

b)




Domain:

Range:

Number of roots:

Roots:

End Behaviour:

Degree:

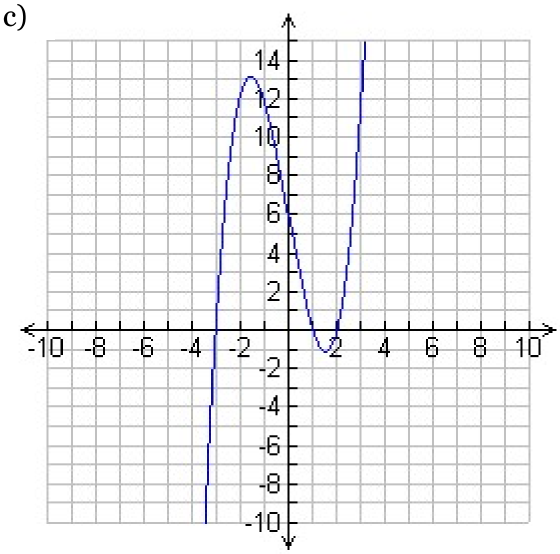
Intervals of positive:

Intervals of negative:

Intervals of increasing:

Intervals of decreasing

c)




Domain:

Range:

Number of roots:

Roots:

End Behaviour:

Degree: Name:

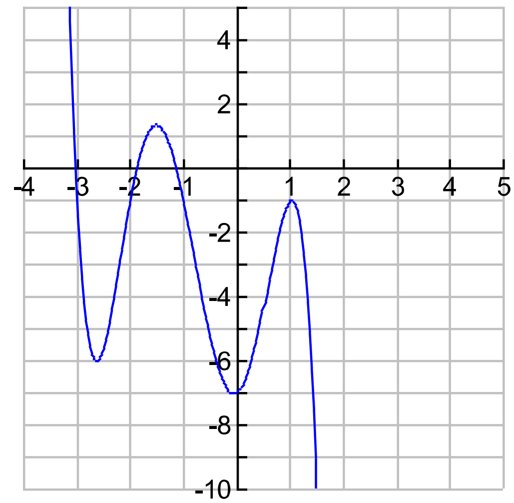
Intervals of positive:

Intervals of negative:

Intervals of increasing:

Intervals of decreasing

d)




Domain:

Range:

Number of roots:

Roots:

End Behaviour:

Degree: Name:

Intervals of positive:

Intervals of negative:

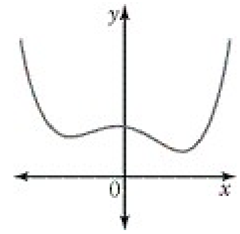
Intervals of increasing:

Intervals of decreasing

# Warm up

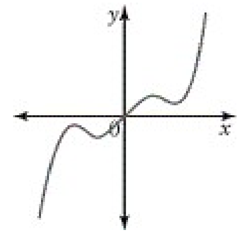
## Part I. Multiple Choice

1. The least possible degree of the polynomial function represented by the graph shown is




a) 2 c) 4
b) 3 d) 5

2. The least possible degree of the polynomial function represented by the graph shown is




a) 3 c) 5
b) 4 d) 7

3. If y = f(x) is a quartic function with a constant difference of -48, then the following statement is **false**:

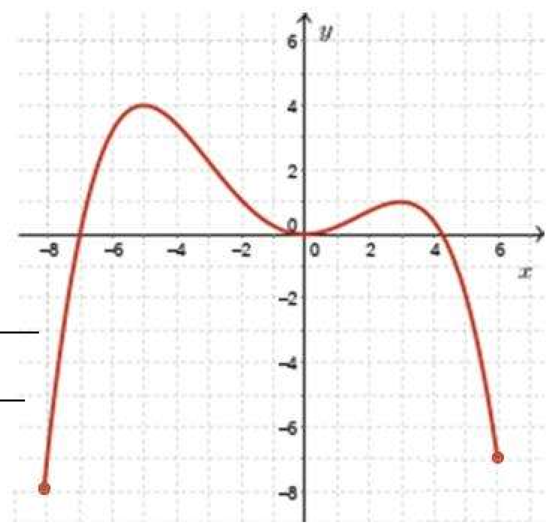
 a) the function starts in Q3 and ends in Q4
 b) the sign of the leading coefficient is negative
 c) the function might not have any roots
 d) the sign of the leading coefficient is positive

## Part II. True/False

a) The function $y = -3x^4 + 1$ extends from quadrant 3 to quadrant 4. _______________
b) Odd-degree polynomials have at least one $x$-intercept. _______________
c) Even-degree polynomial functions always begin and end on the same side of the $x$-axis. ________
d) The graph of a quartic function cannot have exactly three $x$-intercepts. _____________
e) The function $y = x^4 + 2x^2 + 1$ never crosses the $x$-axis. _____________
f) All quartic polynomial equations have at least one real solution. _____________

## Part III. Short Answers

The following is the graph of $y = f(x)$. Answer the following questions.



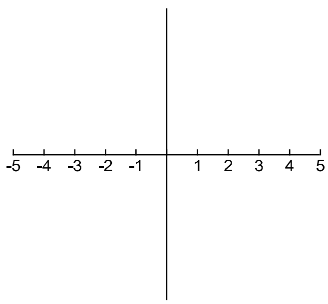

a) absolute max:___________________________
b) absolute min: ___________________________
c) local max:_______________________________
d) local min.: _____________________________
e) interval(s) of increasing : ______________________________
f) interval(s) of decreasing : ______________________________

## 1.3 Equations and Graphs of Polynomial Functions

Use your graphing calculator to complete the following charts:

**<u>Cubic Functions:</u>**

**1. $y = x^3$**




Degree:

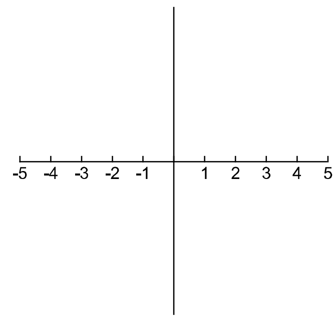
Sign of the lead coefficient:

Quadrants:

Roots/x-intercepts:

y-intercept:

**2. $y = -x^3$**




Degree:

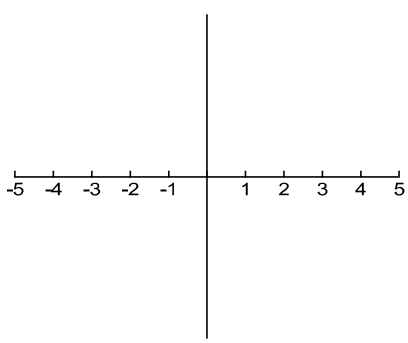
Sign of the lead coefficient:

Quadrants:

Roots/x-intercepts:

y-intercept:

**3. $y = x(x + 2)(x-1)$**




Degree:

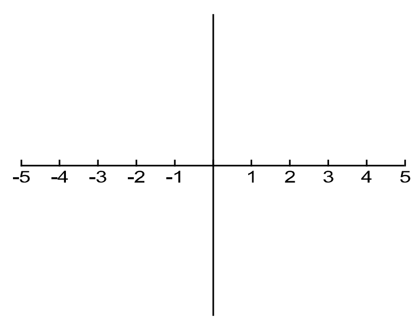
Sign of the lead coefficient:

Quadrants:

Roots/x-intercepts:

y-intercept:

**4. $y = -(x - 3)(x + 2)(x - 1)$**




Degree:

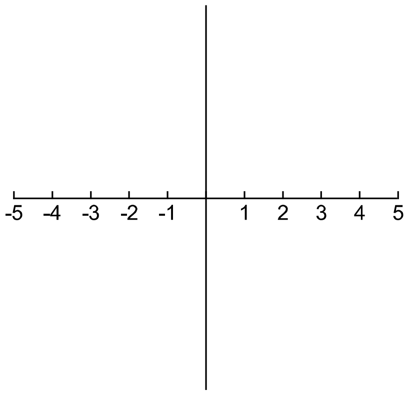
Sign of the lead coefficient:

Quadrants:

Roots/x-intercepts:

y-intercept:

**5. $y = (x + 2)(x-4)^2$**




Degree:

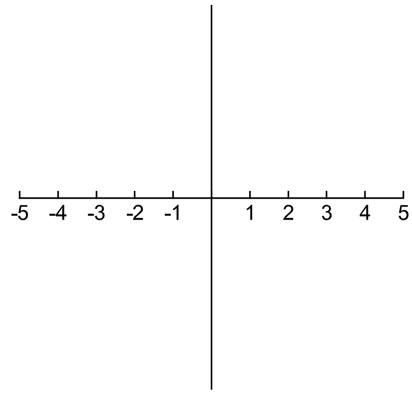
Sign of the lead coefficient:

Quadrants:

Roots/x-intercepts:

y-intercept:

**6. $y = -(x + 3)^3$**




Degree:

Sign of the lead coefficient:

Quadrants :

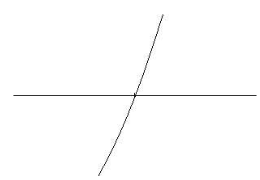
Roots/x-intercepts:

y-intercept:

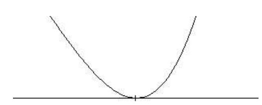
<u>Cubic functions</u>

- Have a hill and a valley in the middle.
- Their degree is ________________
- The maximum number of zeros or x-intercepts is ________
- If the lead coefficient is positive, the graph starts in the ______ quadrant and ends in the _______ quadrant.

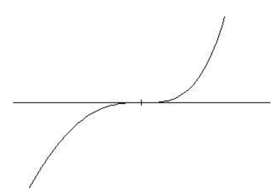
- If the lead coefficient is negative, the graph starts in the ______quadrant and ends in the ________ quadrant.



- If the exponent on the variable or bracket is 1 [ i.e. x or (x+2)], the curve ____________________________ the x-intercept.

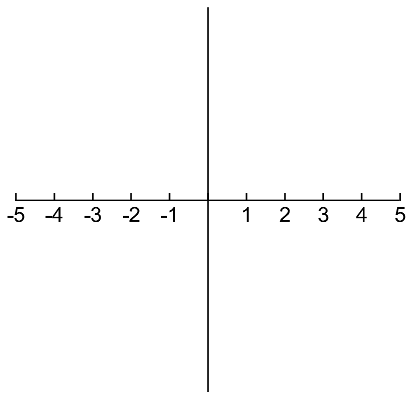


- If the exponent on the variable or bracket is squared [i.e. $x^2$ or $(x+2)^2$], the curve will_____________________ the x-intercept.
- If the exponent on the variable or bracket is cubed [i.e. $x^3$ or $(x+2)^3$], the curve will appear flatter and have a "**point of inflection**" as it passes through the x-intercept.



## Quartic Functions

**1. $y = x^4$**




Degree:

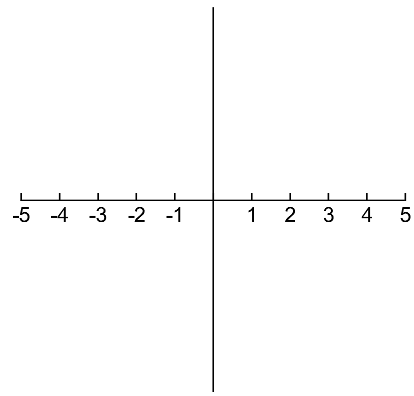
Sign of the lead coefficient:

Quadrants :

Roots/x-intercepts:

y-intercept:

**2. $y = -x^4$**




Degree:

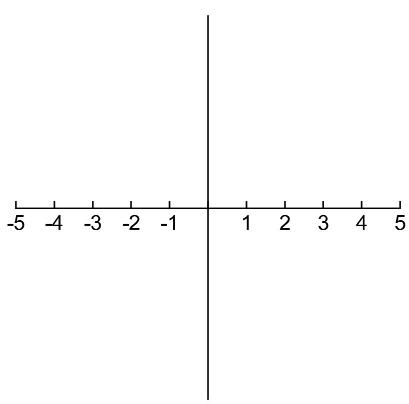
Sign of the lead coefficient:

Quadrants :

Roots/x-intercepts:

y-intercept:

**3. $y = (x-2)(x-1)(x+3)(x-3)$**




Degree:

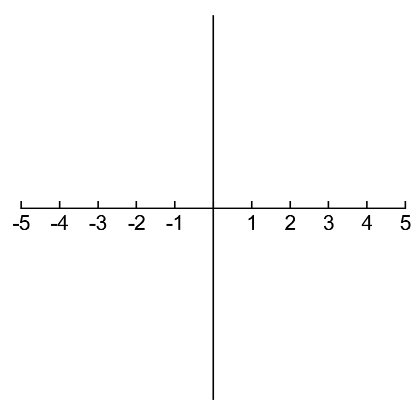
Sign of the lead coefficient:

Quadrants :

Roots/x-intercepts:

y-intercept:

**4. $y = -(x + 2)^2(x-3)^2$**




Degree:

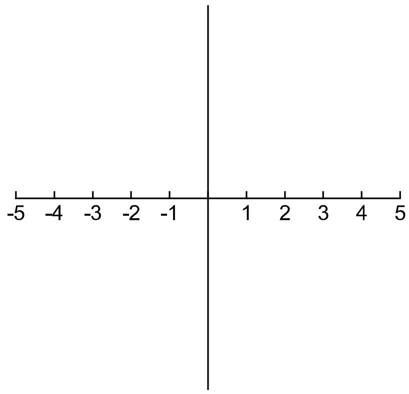
Sign of the lead coefficient:

Quadrants :

Roots/x-intercepts:

y-intercept :

**5. $y = (x+2)^3(x-1)$**




Degree:

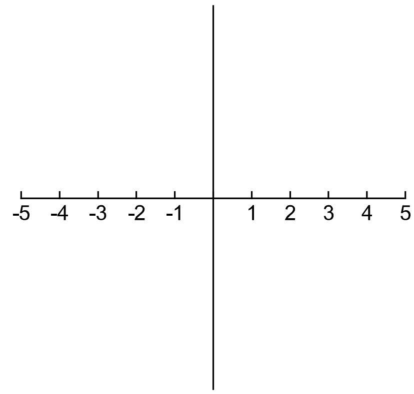
Sign of the lead coefficient:

Quadrants :

Roots/x-intercepts:

y-intercept

**6. $y = (x+1)^2(x-2)(x+3)$**




Degree:

Sign of the lead coefficient:

Quadrants :

Roots/x-intercepts:

y-intercept

## Quartic Functions

There degree is ______Their shape is similar to a ____________________, or U-shaped. The maximum number of x-intercepts is ________.
If the lead coefficient /value in front of the bracket is positive, the curve opens __________ and it begins and ends in quadrants ________ ←→_______.
If the lead coefficient /value in front of the bracket is negative, the curve opens ___________. and it begins and ends in quadrants ________ ←→______.

- Note that if the variable or bracket has an exponent of 1, the curve__________________.
- If the variable or bracket has an even exponent, the curve_____________________ at the x-intercept.
- If the variable or bracket has an odd exponent greater that 1, the curve passes through the x-intercept with a slight____________________________________________.

**Example 1: Do not use a graphing calculator.** Sketch the following graphs**:**

**a) $y = (x^2 - 4)(x+2)$ (BE SURE TO FACTOR FULLY BEFORE GRAPHING!)**

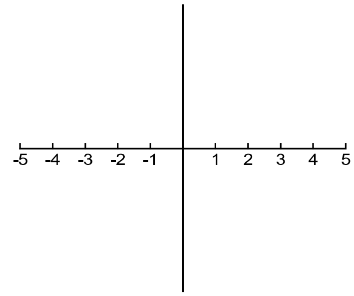
Degree:

Sign of the lead coefficient:

Quadrants :

Roots/x-intercepts:

y-intercept




**b) $y = -x^3 + x$**

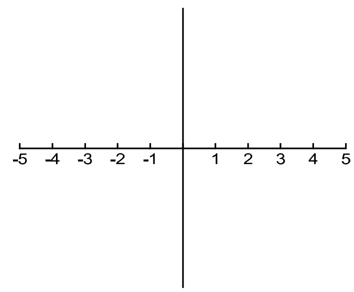
Degree:

Sign of the lead coefficient:

Quadrants :

Roots/x-intercepts:

y-intercept




**c) $y = x(x - 3)^3$**

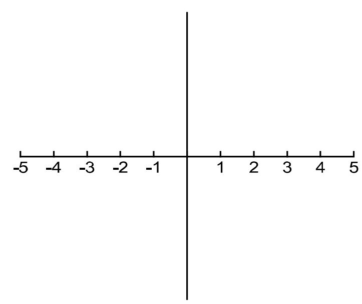
Degree:

Sign of the lead coefficient:

Quadrants :

Roots/x-intercepts:

y-intercept




**d) $y = x(x-2)^2(x+3)^3$**

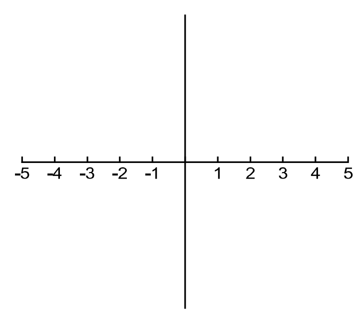
Degree:

Sign of the lead coefficient:

Quadrants :

Roots/x-intercepts:

y-intercept :



## Sketching Polynomial Functions Summary

To sketch a polynomial function,

- Factor the polynomial fully, if it is not in factored form.
- Identify the degree. This will indicate the general shape of the curve.
- Look to see if the lead coefficient is positive or negative. This will help peg down the shape and quadrants.
- Find the x-intercepts. Let y = 0 solve for x.
- Find the y-intercept. Let x = 0 solve for y.
- Plot the intercepts and use the shape to sketch the curve.
- **Remember that if the variable or bracket has an even exponent, the curve "bounces" off the intercept**.
  However, if the variable or bracket has an odd exponent, the curve passes through the intercept in one of two ways:
  - If the odd exponent is 1, then the curve passes straight through the curve.
  - If the odd exponent is greater than 1, (i.e. 3, 5, 7. . . ) then, the curve bends creating a slight shelf( saddle) at the x-intercept.

**HOMEWORK:**

By writing down the degree, the sign of the lead coefficient, quadrants/shape, the roots/x-intercepts, and the y-intercepts, Sketch each of the following in separate sheet of paper:

a) $y = x(x-2)(x+3)$

b) $y = -(x-1)(x+3)^2$

c) $y = -x(x-3)(x+2)(x+4)$

d) $y = (x+2)^3(x-3)$

e) $y = x(x+2)^2(x-2)$

f) $y = x^2(x+2)^3$

g) $y = (x-1)^2(x+1)^3(x-2)(x+2)$

h) $y = x^3 - x$

i) $y = -x^3 + 4x^2 - 3x$

j) $y = -(x^2 - 9)$

**Exit card!**

Sketch the graph of the following function: $y = (x-1)^2(x+1)^3(x-2)(x+2)$

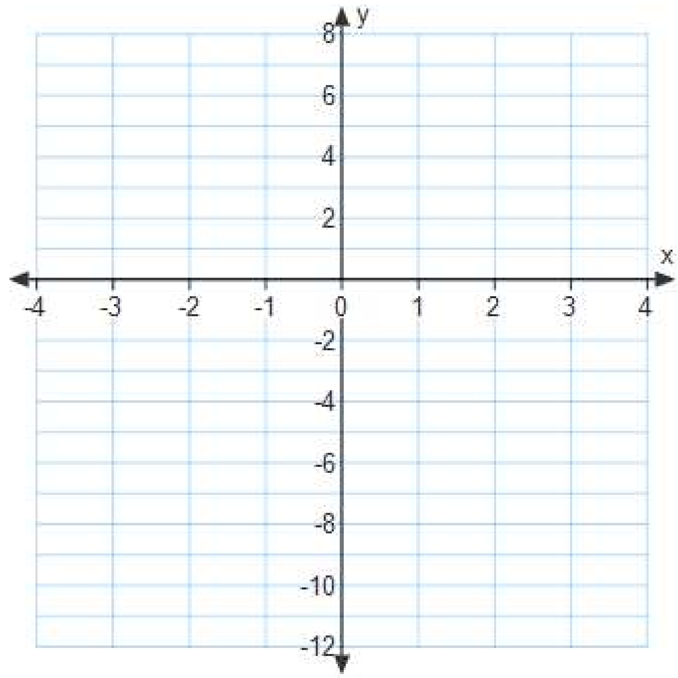
Degree:

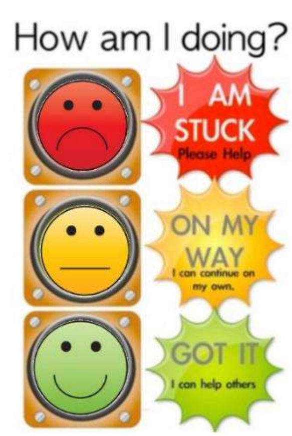
Sign of the lead coefficient:

Quadrants:

Roots/x-intercepts:

y-intercept :





**Group Activity**

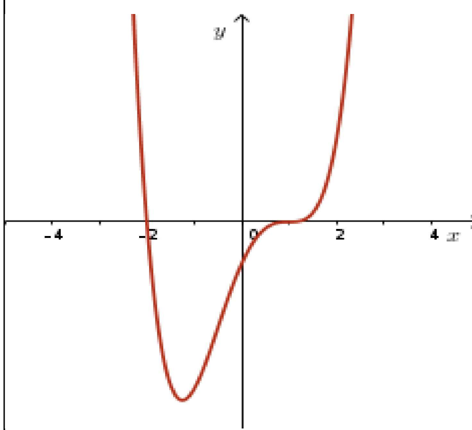
**Names:** ________________________________________

1. Match each equation to the most appropriate graph (There are more graphs than equations)

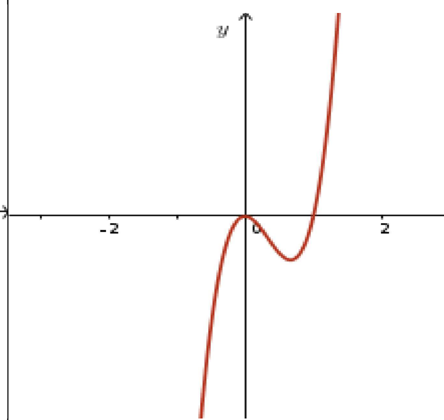
a. $f(x) = 2x(x-1)^2$ _____ b. $y = -(x-1)^2(x+2)^2$ _____ c. $y = -x^3(x+1)$ _____

d. $g(x) = (x+2)(x-1)^3$ _____ e. $g(x) = -2(x+1)^2(x-2)^3$ _____

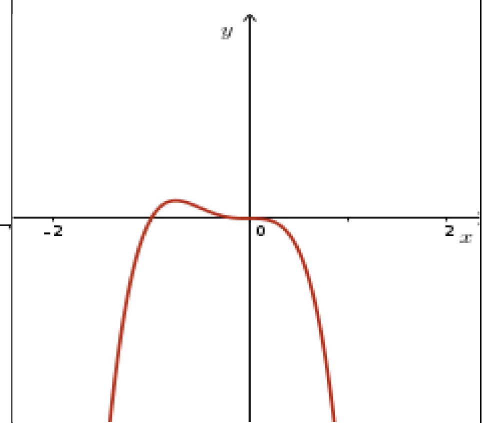
I.



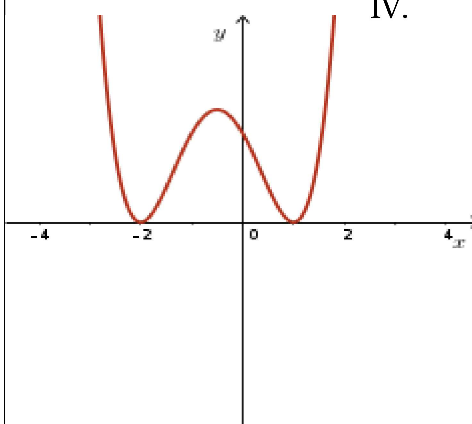

II.

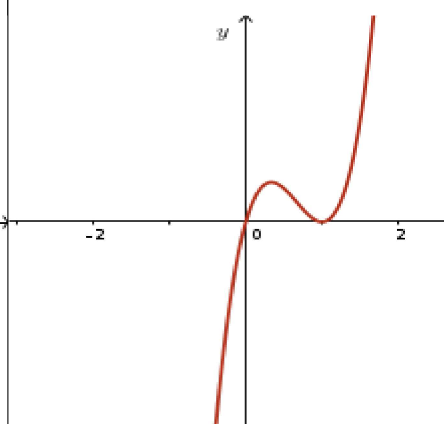



III.

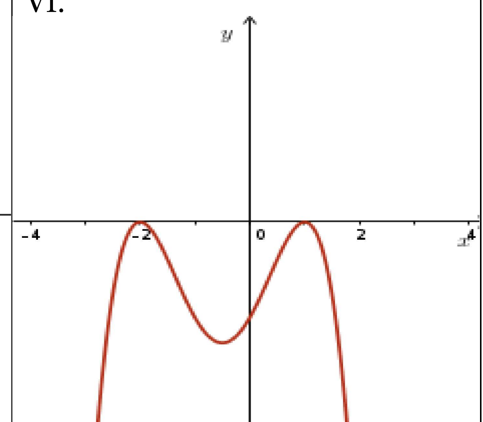



IV.

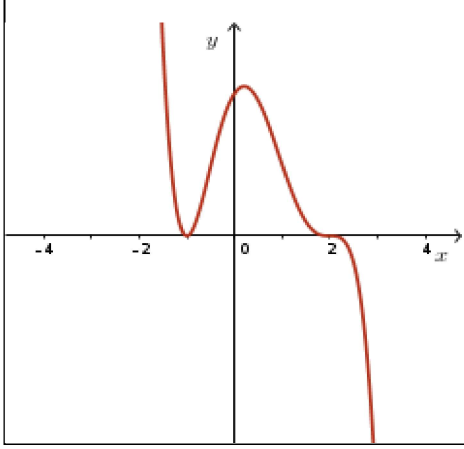



VI.

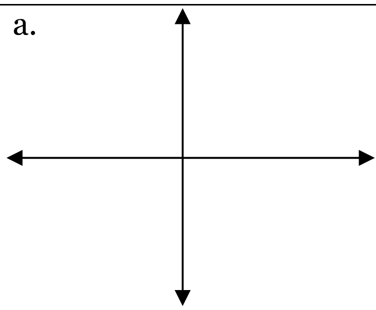
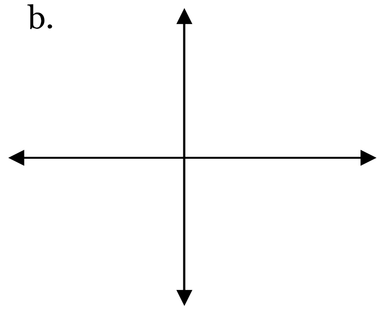
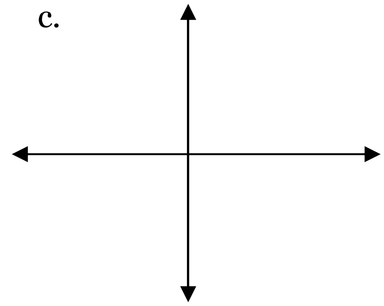



VII.



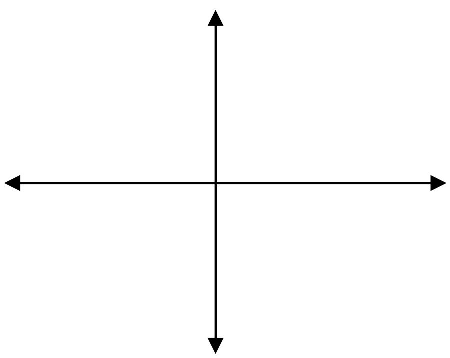
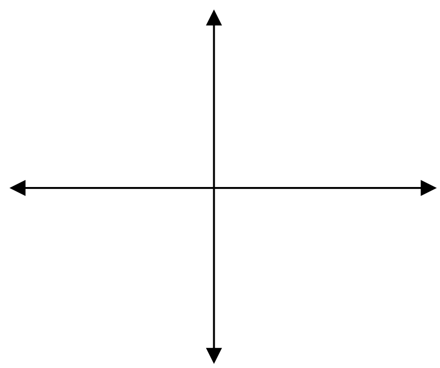
2. Sketch a possible graph of a polynomial function that satisfies the following conditions.

   (a) A quadratic function with a negative leading coefficient and a zero at $x=-5$ of order 2.
   (b) A $5^{th}$ degree function with a positive leading coefficient, a zero at the origin of order 2, and a zero at $x=3$ of order 3.
   (c) A quartic function with a negative leading coefficient and two real zeros, $x=0$ and $x=3$ of order 2.

| | | |
|---|---|---|
| a.  | b.  | c.  |

3. Sketch a possible graph for each of the following functions.

   (a) $y=-0.5(3-x)(x+1)^3$ (b) $f(x)=-x(2x+3)(x-2)^2$

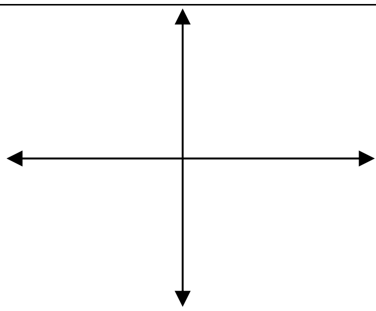
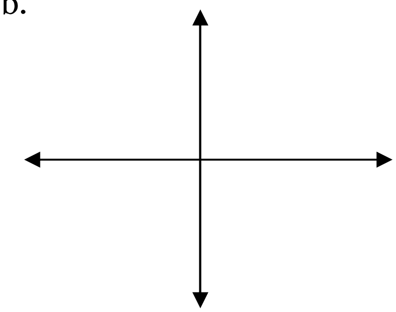
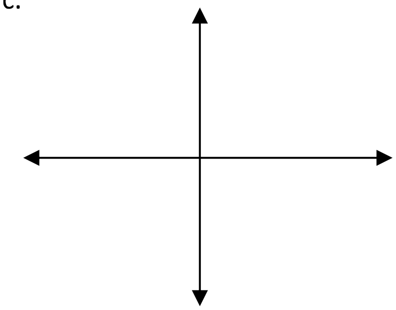
| | |
|---|---|
| a.  | b.  |

4. State the x-intercepts of each function and identify at which zeros the value of the function, f(x), changes sign.

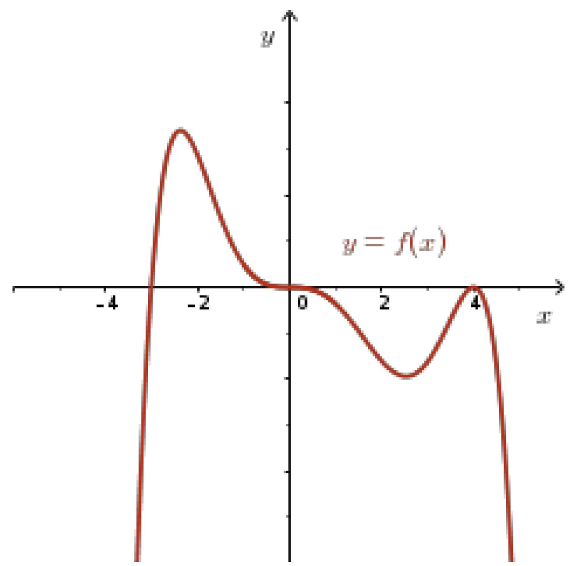
   (a) $f(x)=-2(x-1)^3(x+4)^2$ (b) $f(x)=-2(x+3)(x-4)^2$

5. Identify the intervals in which the following polynomial functions are positive and the intervals in which they are negative. (Hint: sketch the graph of each function)

(a) $f(x)=(x-2)(x+1)(x+4)$  (b) $f(x)=-2x(x-4)(x+3)^2$  (c) $f(x)=-x^2(x-3)^3$

| | | |
|---|---|---|
| a.  | b.  | c.  |
| | | |

6. Determine a possible equation for the polynomial function y=f(x) shown below.



# 1.4 Part A: Transformations

Transformation of functions in the form $y = x^n$ is $y = \boldsymbol{a}\,[\,\boldsymbol{k}\,(x - \boldsymbol{d})]^n + \boldsymbol{c}$

$$y = \boldsymbol{a}\,[\,\boldsymbol{k}\,(x - \boldsymbol{d})]^n + \boldsymbol{c}$$

**Example 1:** For $f(x) = x^3$, describe $y = -2f(-3x+6) - 1$.

Equation: ______________________________

- ______________________________
- ______________________________
- ______________________________
- ______________________________

**Example 2:** For $f(x) = x^4$, describe $y = \frac{1}{2}f\left(\frac{3}{4}x - 3\right) + 1$.

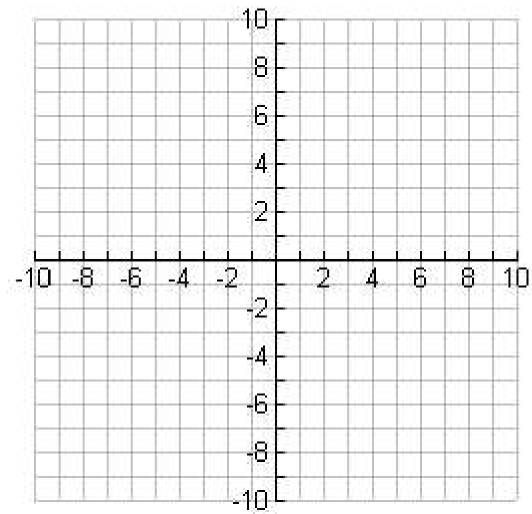
Equation: ______________________________

- ______________________________
- ______________________________
- ______________________________
- ______________________________

The **MAPPING NOTATION** for graphing $y = a\left[k\left(x-d\right)\right]^n + c$ is :

$$(x, y) \rightarrow \left(\frac{x}{k} + d, ay + c\right)$$

**Example 3:** Complete the chart and graph the function.

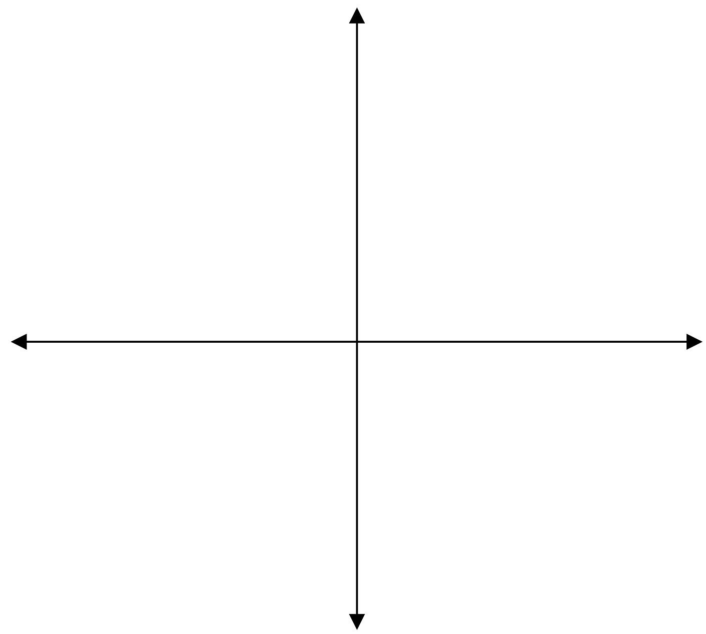
| Base Graph: | **Describe Transformation** | **Graph of the function** |
|---|---|---|
| $f(x) = -2\left(\frac{1}{3}x+1\right)^3 + 2$ | • ________________<br>• ________________<br>• ________________<br>• ________________<br>• ________________ |  |
| Mapping Notation: | $(x, y) \rightarrow (\qquad , \qquad)$ | |

| $(x, x^3)$ | |
|---|---|
| (-2,-8) | |
| (-1,-1) | |
| (0,0) | |
| (1,1) | |
| (2,8) | |

**Example 4:** Graph $y = 3(x+6)^4 - 48$ using mapping rule.

$(x, y) \rightarrow ( \qquad , \qquad )$

| $(x, x^4)$ | |
|---|---|
| (-2,16) | |
| (-1,1) | |
| (0,0) | |
| (1,1) | |
| (2,16) | |



**Example 5:**

**(a)** The function $h(x) = 2(x-4)(x+2)(x-3)$ is reflected in the x-axis, vertically stretched about the x-axis by a factor of $\frac{5}{2}$, and translated 4 units left, 5 units down. Write an equation for the transformed function.

**(b)** What transformations are applied to the function $p(x) = 3(2x-4)(x+2)(x-3)$ to obtain the function $q(x) = (x-2)(x-1)(x+3)$?

**Example 6:** Describe the transformations for each of the following functions compared to $y = x^4$ in two different ways.

a) $y = -\left(\frac{1}{2}x - 5\right)^4$

- ______________________________
- ______________________________
- ______________________________
- ______________________________
- ______________________________

- ______________________________
- ______________________________
- ______________________________
- ______________________________
- ______________________________

b) $y = -7(-x+6)^4 + 9$

- ______________________________
- ______________________________
- ______________________________
- ______________________________
- ______________________________

- ______________________________
- ______________________________
- ______________________________
- ______________________________
- ______________________________

## 1.4 Part B : EVEN AND ODD FUNCTIONS

**EVEN FUNCTIONS**

- A function is an even function if it shows symmetry in the y-axis.
- This means the left half of the function looks like a mirror image of the right half of the function on either side of the y-axis.
- To test if a function is even, we check to see if f(x) = f(-x).
- Alternately, if (x, y) maps onto (-x,y), the function is even.

**ODD FUNCTIONS**

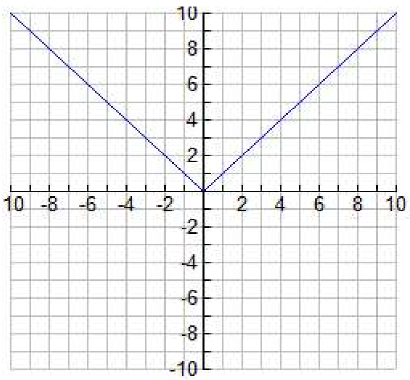
- A function is an odd function if it shows symmetry in the origin.
- This means that any line drawn across the plane through the origin that crosses the curve will cross pairs of points on the graph that are the same distance from the origin.
- To test if a function is odd, we check to see if f(-x) = -f(x).
- Alternately, if (x, y) maps onto (-x, -y), the function is odd.

**NOTE:**

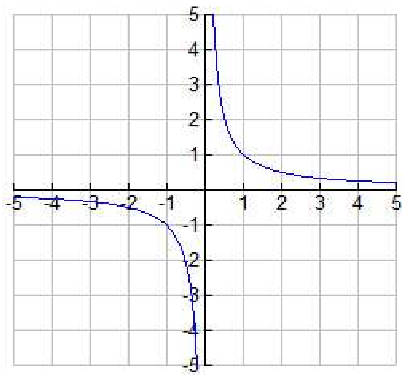
- Not all functions can be classified as even or odd. Some functions are neither.

EXAMPLE 1: Determine if the following functions are even, odd, or neither. Justify your reasoning.

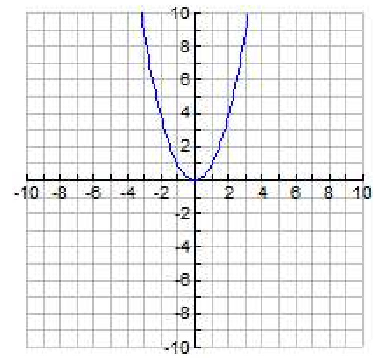
1.



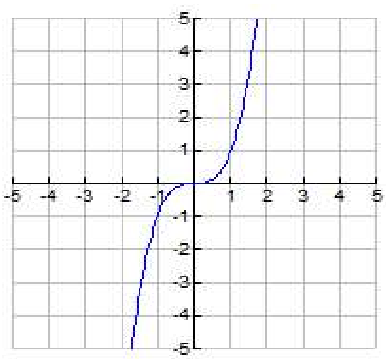

2.



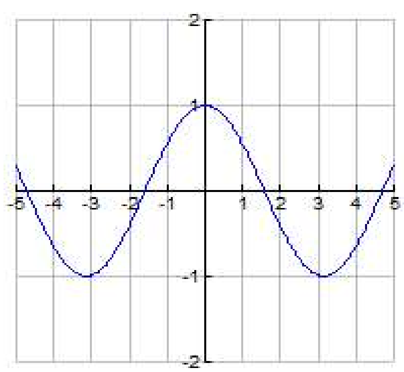

3.



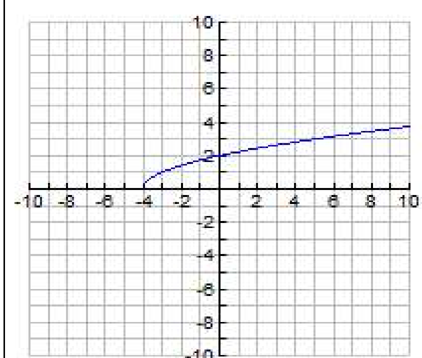

4.




5.




6.




EXAMPLE 2: Determine if the following functions are even, odd or neither. Justify your reasoning algebraically.

a) $f(x) = 3x^5 - 2x^3 + x$

b) $f(x) = \dfrac{6x^4 + x^2}{(x^3 - 2x)^2}$

c) $h(x) = \dfrac{-5x^3 + x}{(3x - 5)^2}$

## Practice:

1. Review your notes and the examples below. Then, for each of the practice questions prove algebraically that the function is even, odd, or neither. Use good form!

| Function | ***Algebraic Proof:*** | **Odd or even** |
| --- | --- | --- |
| $f(x) = x^2$ | **$f(x) = x^2$**<br>$f(-x) = (-x)^2$<br>**$= x^2$**<br>$= f(x)$<br>*Therefore $f(x)$ is even.* | Even since $f(x)$ is symmetrical about the y-axis. |
| $f(x) = x^3$ | *If $f(x)$ is odd then $f(-x)=-f(x)$*<br>$f(-x) = (-x^3)$<br>$= -x^3$<br>$= -f(x)$<br>$\therefore f(x)$ *is odd* | Odd since $f(x)$ is symmetrical through the origin. |
| $f(x)=x\sqrt{(5-x)}$ | $f(x) = x\sqrt{(5-x)}$ :<br>$f(-x) = (-x)\sqrt{(5-(-x)}$<br>$= -x\sqrt{(5+x)}$<br>$\neq f(x)$<br>$\neq -f(x)$<br>*$f(x)$ is neither even nor odd*<br>*Hint: Always check if the function is even first. Then, check if it is odd.* | Neither even nor odd since the function is not symmetrical through the origin or about the y-axis. |
| $f(x) = x^2 - x$ | | |
| $f(x) = (x + x^3)^5$ | | |

| | | |
|---|---|---|
| $f(x)=2^x$ | | |
| $f(x)=\frac{1}{x^4+1}$ | | |
| $f(x)=\frac{x-1}{x+1}$ | | |
| $f(x)=\frac{x}{x+2}$ | | |

2 Determine an equation, in factored form, for an **even** polynomial function with a turning point at (2,0). Show algebraically that the function is even.

3. Determine an equation, in factored form, for an **odd** polynomial function with one of its zeros at x=2. Show algebraically that the function is odd.

4. Sketch the graph of a polynomial function with the following characteristics:

   i. the function is an even function
   ii. as $x\rightarrow\infty, y\rightarrow-\infty$
   iii. the function has exactly 3 x-intercepts

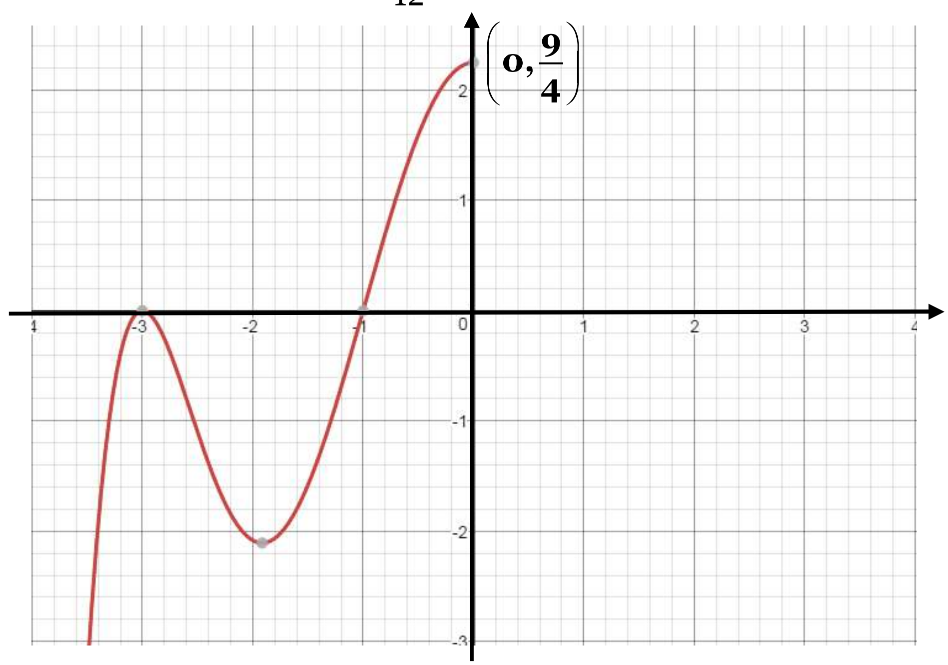
5. Sketch the graph of two polynomial functions of different degree with the following characteristics:
   i. the function is an odd function
   ii. as $x \to \infty, y \to \infty$
   iii. the function has exactly 3 x-intercepts

6. Determine, algebraically, if $f(x) = -2x(x-4)^2(x+4)^2$ is even, odd, or neither.

7. Determine, algebraically, if $f(x) = (4x+8)(x-2)^2(2+x)$ is even, odd, or neither.

8. A quintic polynomial function that is classified as an **odd** function passes through the points $(2,0), (3,210), (4,0)$. Determine the equation of $f(x)$ in factored form.

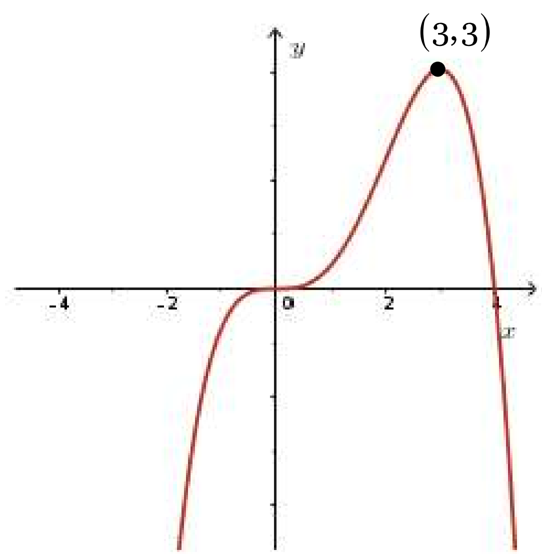
9. Given the function, $f(x)$, below is classified as an **even** polynomial function. Determine the exact value of $f(2)$. [Answer: $\frac{-25}{12}$ ]



Warm- up

1. Determine algebraically if the function is an even function, odd function, or neither.
   $f(x) = x^2(2x^5 - 7x^3 + 9x)$.

2. Given the graph of $f(x)$, determine the equation of $f(x)$.



# 1.5 Dividing Polynomials

Dividing a polynomial by another polynomial is similar to performing a division of numbers using long division. For example, divide the polynomial $x^3+13x^2+39x+46$ by $x+9$

**Solution**:

1)
$$\begin{array}{rl}
 & x^2 \\ \cline{2-2}
x+9\big) & x^3+13x^2+39x+46
\end{array}$$

first divide x into $x^3$ to get $x^2$

---

2)
$$\begin{array}{rl}
 & x^2 \\ \cline{2-2}
x+9\big) & x^3+13x^2+39x+46 \\
 & \underline{x^3+9x^2} \\
 & \phantom{x^3+}4x^2
\end{array}$$

now multiply $x^2$ by $x + 9$ to get $x^3 + 9x^2$
then subtract $x^3 + 9x^2$ from $x^3 + 13x^2$ to get $4x^2$

---

3)
$$\begin{array}{rl}
 & x^2+4x \\ \cline{2-2}
x+9\big) & x^3+13x^2+39x+46 \\
 & \underline{x^3+9x^2}\ \downarrow \\
 & \phantom{x^3+}4x^2+39x
\end{array}$$

bring down the + 39x
divide $4x^2$ by x to get 4x

---

4)
$$\begin{array}{rl}
 & x^2+4x \\ \cline{2-2}
x+9\big) & x^3+13x^2+39x+46 \\
 & \underline{x^3+9x^2}\ \downarrow \\
 & \phantom{x^3+}4x^2+39x \\
 & \phantom{x^3+}\underline{4x^2+36x} \\
 & \phantom{x^3+4x^2+}3x
\end{array}$$

now multiply 4x by $x + 9$ to get $4x^2 + 36x$
then subtract $4x^2 + 36x$ from $4x^2 + 39x$ to get 3x

---

5)
$$\begin{array}{rl}
 & x^2+4x+3 \\ \cline{2-2}
x+9\big) & x^3+13x^2+39x+46 \\
 & \underline{x^3+9x^2}\ \downarrow\ \ \downarrow \\
 & \phantom{x^3+}4x^2+39x\ \downarrow \\
 & \phantom{x^3+}\underline{4x^2+36x}\ \downarrow \\
 & \phantom{x^3+4x^2+}3x+46 \\
 & \phantom{x^3+4x^2+}\underline{3x+27} \\
 & \phantom{x^3+4x^2+3x+}19
\end{array}$$

bring down the + 46
divide 3x by x to get 3

multiply 3 by $x + 9$ to get $3x + 27$
then subtract $3x + 27$ from $3x + 46$ to get 19

Since the remainder has a lower degree than the divisor, the division is now complete. The result can be written as:

$$x^3+13x^2+39x+46=\underbrace{(x+9)}_{\text{divisor}}\underbrace{(x^2+4x+3)}_{\text{quotient}}+\underbrace{19}_{\text{remainder}}$$

The result of the division of a polynomial $P(x)$ by a binomial of the form $x - b$ is $\frac{P(x)}{x - b} = Q(x) + \frac{R}{x - b}$, where $Q(x)$ is the quotient and $R$ is the remainder. The corresponding statement that can be used to check the division, is $P(x) = (x - b)Q(x) + R$.

## Dividing Polynomials

Using the previous example, complete the polynomial division questions below:

**1.** $x^3 - 5x^2 - x - 10$ by $x - 2$

**2.** $y^4 + 2y^2 - 28y - 36$ by $2y^2 + 4y - 2$

# Remainder Theorem

**Remainder Theorem:**
When a polynomial $f(x)$ is divided by $x-a$, the remainder is $f(a)$.

*Proof of the Remainder Theorem:*

**Ex. 1.** Find the remainder when $2x^3+3x^2-17x-30$ is divided by each of the following:

(a) $x-1$

(b) $x-2$

(c) $x-3$

(d) $x+1$

**Similarly, when a polynomial $f(x)$ is divided by $ax+b$, its remainder is given by $f\left(-\frac{b}{a}\right)$.**

**Ex. 2.** When $8x^3+4kx^2-2x+3$ is divided by $2x+1$ the remainder is 6, find *k*.

**Ex. 3.** When $f(x)=2x^3-px^2+qx-1$ is divided by $x+2$ the remainder is -3; $f(x)$ is divisible by $x-1$. Find the values of p and q.

**Ex. 4.** Polynomial $f(x)$ has a remainder of 3 when divided by $x-2$ and a remainder of $-5$ when it is divided by $x+2$. Determine the remainder when the polynomial is divided by $x^2-4$.

# Practice

1. Without using long division, find each remainder:
   (a) $(2x^2+6x+8)\div(x+1)$ (b) $(x^2+4x+12)\div(3x-1)$
2. When the polynomial $x^n+x-8$ is divided by $x-2$ the remainder is 10. What is the value of n ?
3. Given that $g(x)=(x+2)(3x^2+4)+5$ *and* $h(x)=(6x+1)(3x^3-2x^2+x)+8$ . Find the remainder when $g(x)+h(x)$ is divided by $x+1$ .
4. The remainders when a polynomial is divided by $x-1$ and $x+3$ are 2 and -6 respectively. Find the remainder when the same polynomial is divided by $(x-1)(x+3)$.
5. Find the remainder when $x^{2012}+x-1$ is divided by $x+1$.
6. Find the value of k for which $x^2+(k-1)x+k^2-16$ is exactly divisible by $x-3$ but not divisible by $x+4$.
7. Given that the expression $2x^3+px^2-8x+q$ is exactly divisible by $2x^2-7x+6$,evaluate p and q.
8. The polynomial $2x^3-3ax^2+ax+b$ has a factor $x-1$ and a remainder of -10 when divided by $x+1$.Find the values of a and b.
9. Find each quotient and remainder:

(a) $(2x^2+6x+15)\div(x+3)$ (b) $(x^2-4x+13)\div(2x-1)$
(c) $(x^2-x+3)\div(x+2)$ (d) $(2x^4+x^3-24x^2-3x+2)\div(x^2+x-4)$

10. When a certain polynomial is divided by $x+3$, the quotient is $x^2-3x+5$ and the remainder is 6. What is the polynomial?

## Warm- up

1. Divide $8x^4 - 30x^2 + 6x - 3$ by $1 + x + 2x^2$ using long **division** and write the division statement.

2. Consider the function $f(x) = ax^3 + 3x + b + 5$, where a and b are constants and $a \neq 0$ & $b \neq 0$. $f(x)$ has a remainder of 2a when divided by $x$ and a remainder of 2b when divided by $x - 1$. Determine the values of a and b.

# 1.6 Synthetic Division (Optional)

Synthetic division is another way to divide a polynomial by the binomial $x - b$, where $b$ is a constant.

**Example**: Divide $2x^3 - 3x^2 + 4x - 1$ by $x + 1$ use synthetic division.

**<u>Step 1</u>**

The root of divisor (b value) goes on the outside of the box. The dividend coefficients go on the inside of the box.

b of divisor x - b    coefficients of dividend

$$\begin{array}{r|rrrr} -1 & 2 & -3 & 4 & -1 \\ & & & & \\ \hline & & & & \end{array}$$

***When you write out the dividend make sure that you write it in descending powers and you insert 0's for any missing terms.***

**<u>Step 2</u>**

Bring down the leading coefficient to the bottom row

- **Bring down the 2**

$$\begin{array}{r|rrrr} -1 & 2 & -3 & 4 & -1 \\ & \downarrow & & & \\ \hline & 2 & & & \end{array}$$

**<u>Step 3</u>**

Multiply this by the ***b*** value ( in this case -1), and carry the result up into the next column:

- **(-1)(2) = -2**
- **Place -2 in next column**

$$\begin{array}{r|rrrr} -1 & 2 & -3 & 4 & -1 \\ & & -2 & & \\ \hline & 2 & & & \end{array}$$

**<u>Step 4</u>**

Add down the column

- **-3 + (-2) = -5**

$$\begin{array}{r|rrrr} -1 & 2 & -3 & 4 & -1 \\ & & -2 & & \\ \hline & 2 & -5 & & \end{array}$$

**<u>Step 5</u>**

Multiply the previous result by the ***b*** value, and carry the new resul

- **(-1)(-5) = 5**
- **Place 5 in next column**

$$\begin{array}{r|rrrr} -1 & 2 & -3 & 4 & -1 \\ & & -2 & & \\ \hline & 2 & -5 & & \end{array}$$

**<u>Step 6</u>** Repeat until down

The numbers in the last row make up your coefficients of the quotient as well as the remainder.

$$\begin{array}{r|rrr|r} -1 & 2 & -3 & 4 & -1 \\ & & -2 & 5 & -9 \\ \hline & 2 & -5 & 9 & -10 \end{array}$$

The final value on the right is the remainder. Working right to left, the next number is your constant, the next is the coefficient for $x$, the next is the coefficient for $x^2$ , etc.

**In this example, the remainder is -10 and the quotient is $x^2$-5x+9.**

## Practice

1. Complete the indicated division.

$$
\begin{array}{r|rrrrr}
2 & 2 & -3 & -5 & 3 & 8 \\
\hline
\end{array}
$$

2. Divide $3x^3 - 2x^2 + 3x - 4$ by $x - 3$ using synthetic division. Write the answer in the form $\frac{P(x)}{x-b} = Q(x) + \frac{R}{x-b}$

3. Divide $2x^3 + x - 12$ by $x - 2$ using synthetic division.

# Extended Synthetic Division

1. Divide $2x^4 + 4x^3 - 5x^2 + 3x - 2$ by $x^2 + 2x - 3$

2. Divide $x^4 + 3x^2 + 1$ by $x^2 - 2x + 3$.

**You Try!** Divide $3x^3 - 5x^2 + 6x + 10$ by $x^2 + x + 2$

# Unit 1: Polynomial Functions
## 1.7 The Factor Theorem

Review

- Division of Polynomials $\frac{f(x)}{x-a}$ - may use long division or synthetic division

- Division Statement: f(x) = (x-a) Q(x) +r(x)
- Remainder Theorem: When a polynomial f(x) is divided by (x - a), the remainder, r, is f(a)

Investigation:
Find the remainder when $x^3 + 2x^2 - 11x - 12$ is divided by x + 1 and write the division statement.
Solution:

$\therefore x^3+2x^2 - 11x - 12 =$
$=$

- Factor the quotient if possible

- Notice that the products of the constant terms in the factors is (1)(4)(-3) = -12 . This is also the constant term of the polynomial.

Since division gives zero as a remainder, both the **divisor** and **quotient** are factors of the polynomial function. This special case of the remainder theorem where the remainder is **zero** is called the **factor theorem**.

**Factor Theorem:**
A polynomial function f(x) has x - a as a factor if and only if f(a) = 0.

USE THE FACTOR THEOREM FOR FACTORING POLYNOMIALS WITH DEGREE HIGHER THAN 2.

Example: Is $x - 3$ a factor of $x^3 - 2x^2 - 2x - 3$ ?

Example: Use the Factor Theorem to factor fully each of the following polynomials

a) $x^3 - 4x^2 + x + 6$

**SOLUTION**

**step 1:** Let $f(x)=x^3 - 4x^2 + x + 6$

**step 2:** Find all factors of constant: $\{\pm1,\pm2,\pm3,\pm6\}$ . This is a set of possible roots for $f(x)=0$.

**step 3:** Observe that $f(-1) = 0$ $\therefore$ $x+1$ is a factor of $f(x)$

**step 4:** Dividing $f(x)$ by $x+1$ determines a quotient of $x^2 - 5x + 6$

**step 5:** $f(x) = x^3 - 4x^2 + x + 6 = (x+1)(x^2 - 5x + 6) = (x+1)(x-2)(x-3)$

b) $x^4 - 3x^3 - 13x^2 + 3x + 12$

**The Rational Root Theorem (Extended Factor Theorem)**
The factor theorem can be extended over the set of Rationals, Q, so that more test values can be used to determine a factor of the polynomial.

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ can also be written as

$$f(x) = a_n[x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \frac{a_{n-2}}{a_n}x^{n-2} + \ldots + \frac{a_2}{a_n}x^2 + \frac{a_1}{a_n}x + \frac{a_0}{a_n}]$$

$\therefore \dfrac{\text{all of the factors of } a_0}{\text{all of the factors of } a_n}$ should be considered when determining possible factors of f(x)

**Example:** Factor fully each of the following polynomials

a) $4x^3 + 16x^2 + 9x - 9$

Now you can consider $\pm$ combination of $\dfrac{\text{all the factors of } 9}{\text{all the factors of } 4} = \pm\dfrac{1,3,9}{1,2,4}$.

That means $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{9}{1}, \pm\frac{9}{2}, \pm\frac{9}{4}\right\}$ is a **set of possible roots** for f(x)=0.

b) $30x^3 + 13x^2 - 30x + 8$

**Example:** If x + 1 is a factor of $x^3 + x^2 + kx + 2$, what is the value of k?

**Example:** Find the value of k for which a-3b is a factor of $a^4 - 7a^2b^2 + kb^4$. Hence, for this value of k, factorize $a^4 - 7a^2b^2 + kb^4$ccompletely.

**Example:** The function $h(x) = 3x^2 - x^3$ has been shifted to the left 1 and vertically stretched by 2.

**a)** Determine the equation for the transformed function, g(x) in factored form.

**b)** Determine the zeroes of the transformed function and state the order of each zero.

# Factoring a Sum or Difference of Cubes

**Recall:**

- Factoring a difference of squares

$$x^2 - 9 = (x-3)(x+3)$$

$$4x^2 - 16 = (2x-4)(2x+4)$$

$$a^2 - b^2 = (a-b)(a+b)$$

**IS THERE A WAY TO FACTOR A SUM OF CUBES OR A DIFFERENCE OF CUBES IN ONE STEP???????**

$$x^3 - 27 = ?$$

$$8a^3 - 27b^3 = ?$$

## Factoring a Sum or Difference of Cubes

- **Sum of Cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- **Difference of Cubes:** $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Factor the following expressions **completely**.

a) $x^3 - 64$

b) $27x^3 + 125$

c) $7x^4 - 448x$

d) $64x^3 + \frac{8}{125}y^{12}$

e) $\frac{1}{8}a^3 - \frac{27}{125}b^{18}$

f) $x^9 - 512$

g) $(x-2)^3 - (3x-2)^3$

h) $3(x-2)^3 - 24(x+2)^3$

# Practice:

**1.** Factor the following polynomials using the factor theorem.

(a) $x^3-4x^2+x+6$
(b) $x^3+8x^2+21x+18$
(c) $x^4-x^3-3x^2+x+2$

**2.** Factor each expression

(a) $x^3-8$
(b) $27x^3+1$
(c) $625x^3-40$
(d) $125-64x^3$

**3.** Factor fully: $abx^3+(a+b-ab)x^2+(1-a-b)x-1$ [note $P(1)=0$ ]

**4.** a) Factor $x^{12}-1$ fully.
b) List all polynomials of the form $x^4+bx^3+cx^2+dx+e$ with rational coefficients that are factors of the polynomial, $x^{12}-1$.

Answer

1. a) $(x+1)(x-2)(x-3)$ b) $(x+2)(x+3)^2$ c) $(x-2)(x-1)(x+1)^2$

2. a) $(x-2)(x^2+2x+4)$ b) $(3x+1)(9x^2-3x+1)$
c) $5(5x-2)(25x^2+10x+4)$ d) $-(4x-5)(16x^2+20x+25)$

3. $abx^3+(a+b-ab)x^2+(1-a-b)x-1=(ax+1)(bx+1)(x-1)$; note $P(1)=0$

4. a) $x^{12}-1=(x-1)(x+1)(x^2+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$

b) There are seven such $4th$ degree polynomial factors:

- $x^4-x^2+1$
- $(x^2+x+1)(x^2-x+1)=x^4+x^2+1$
- $(x-1)(x+1)(x^2+1)=x^4-1$
- $(x-1)(x+1)(x^2+x+1)=x^4+x^3-x-1$
- $(x-1)(x+1)(x^2-x+1)=x^4-x^3+x-1$
- $(x^2+1)(x^2+x+1)=x^4+x^3+2x^2+x+1$
- $(x^2+1)(x^2-x+1)=x^4-x^3+2x^2-x+1$

## Warm Up

1. Completely factor and fully simplify the following expressions.

    a) $(2+x)^3-(2-x)^3$

    b) $(x-1)^6-1$

    c) $x^4-5x^3-21x^2+125x-100$

    d) $6x^3-5x^2-49x+60$

# Unit 1: Polynomial Functions
# 1.8 Solving Polynomial Equations

If a polynomial equation, P(x)=0, is factorable, the roots of the equation are determined by factoring the polynomial, setting each factor to zero, and then solving each of these equations individually.

- **An $n^{th}$ degree polynomial equation has at most n distinct roots.**
- **The solutions to a polynomial equation P(x)=0 are the zeros of the corresponding** polynomial function y=P(x).

The x-intercepts of the graph of y=P(x) are the real zeros of the polynomial function.

- **If a polynomial equation of degree 3 or greater cannot be factored, the roots of the equation must be determined using technology or higher-level mathematical procedures.**

**Steps to Solve Polynomial Equations:**

**Example1:**

$$f(x) = 2x^3 + 3x^2 - 8x + 3$$

1st Find all the factors of the constant and of the leading coefficient.

2nd Form all rational numbers by making fractions with the numerator a factor of the constant and the denominator a factor of the leading coefficient

3rd Using synthetic substitution, find one solution to the problem

4th Rewrite the function as two factors.

5th Repeat this process as needed to get all your factors of the polynomial.

6th Set each factor equal to zero and solve the resulting equations.

**Example 2:** Solve for x, $x \llcorner\ \Sigma$

a. $x^3 - x = 0$

b. $8x^3 + 26x^2 + 17x - 6 = 0$

c. $x^3 - 3x^2 - 4x + 12 = 0$

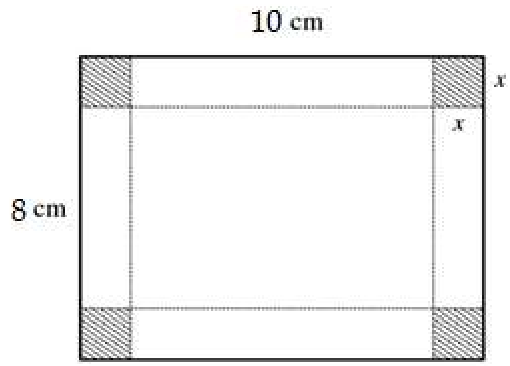
d. $3x^4 - 10x^3 = 24x^2 + 6x - 5$

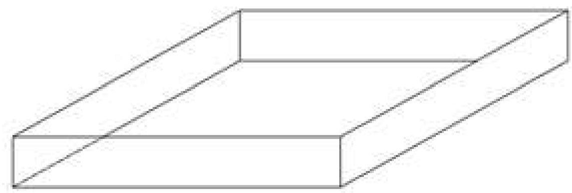
e. $6x^3 + x^2 - 5x - 2 = 0$

f. $x^4 - 15x^2 - 16 = 0$

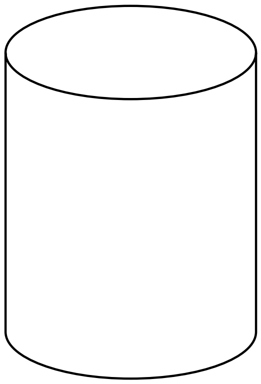
g. $(x^2 - 5x - 5)(x^2 - 5x + 3) = 9$

**Example 3:** A rectangular piece of cardboard measuring 10 cm by 8 cm is made into an open box by cutting squares from the corners and turning up the sides. If the box is to hold a volume of 48 $cm^3$ , what size of square must be removed?



**Example 4:** A cylindrical chemical storage tank must have a height 4 meters greater than radius of the top of the tank. Determine the radius of the top and the height of the tank if the tank must have a volume of $5\pi$ cubic meters.



**Example 5:** The length, width and height of a small box are three consecutive **odd** integers, where the width is the least and the length is the greatest integer. If the width is double and the length and height are increased by 2 cm each, then the volume is increased by 273 $cm^3$. Find the dimensions of the original box

# Practice

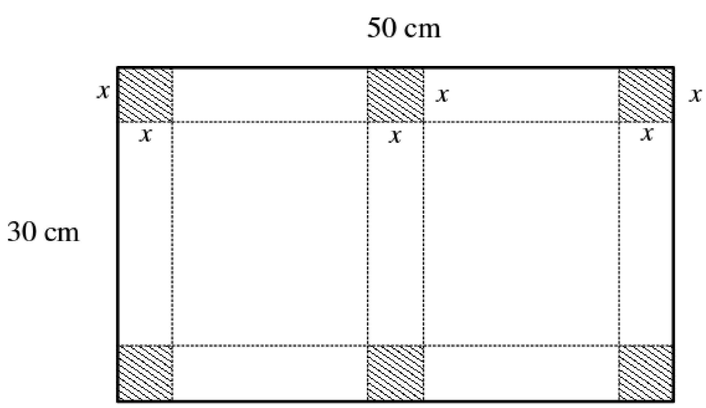
1. Solve the following polynomial equations by factoring where $x \in \mathbb{R}$.
   a) $x^3 - 5x^2 - 4x + 20 = 0$
   b) $2x^3 + 3x^2 = 11x + 6$
   c) $4x^2 = x^3 + 2x + 3$
   d) $x^4 - 7x^2 + 12 = 0$
   e) $2x^3 + 15 = 6x^2 + 5x$
   f) $2x(x^3 + 1) = x^2(4x + 1)$
   g) $2x^4 + 8x + 12 = 3x^2(x + 3)$

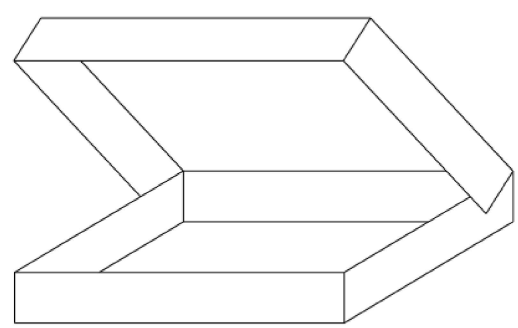
2. Explain why
   a) $15x^5 + 4x^4 + 9x^2 + 7x + 380 = 0$ has at least one real root.
   b) $5x^6 + 3x^4 + 8x^2 + 120 = 0$ has no real roots.

3. A rectangular holding tank is x metres deep, (6x–8) metres long, and (6x–16) metres wide. Find the dimensions of the tank with a volume of 512 $m^3$.

4. The product of the squares of two consecutive integers is 1764. Find all possible values for the integers.

5. A box with a lid is to be created from a 50 cm by 30 cm piece of cardboard by cutting x by x squares from the four corners of the cardboard, and at the centre of the two sides, as shown in the diagram. Determine the function that represents the volume of the box in terms of x, and state the restrictions on x. If the box is to have a volume of 1750 $cm^3$, determine the side length of the squares that need to be cut



6*. Solve $x^2\left(x^2 + 6\right) = 5x^3 - x + 1, x \in \mathbb{R}$.

# Unit 1: Polynomial Functions
# 1.9 Families of Polynomial Functions

**Families of Polynomial Functions**

A **family of** $n^{th}$ **degree polynomial functions** that share the same x-intercepts and differ only in vertical scale factors can be defined by $f(x)=k(x-a_1)(x-a_2)\cdots(x-a_n)$ where k is the leading coefficient, $k\in\mathbb{R}$, $k\neq 0$ and $a_1,a_2,a_3,\ldots,a_n$ are the zeros of the function.

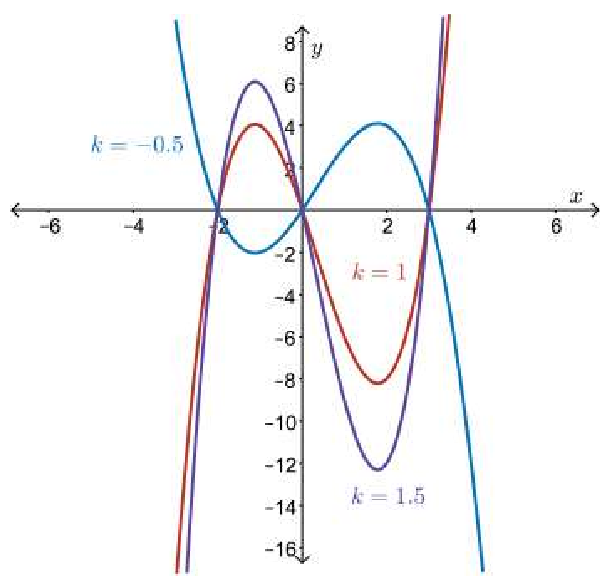
- **General Forms in factored form:**
  quadratic: $f(x) = k(x-s)(x-u)$ s and u are zeros, $k\in R$
  cubic: $f(x) = k(x-s)(x-t)(x-u)$ s, u, and t are zeros ,$k\in R$ e.g. $f(x) = 3(x-1)(x+2)(x+5)$
  $f(x) = -5(x-1)(x+2)(x+5)$
  quartic: $f(x) = k(x-s)(x-t)(x-u)(x-v)$ s,u,t, and v are zeros, $k\in R$

**Example 1:**The function $f(x) = -3x^3 + kx^2 - 5x + k$ has a zero/x-intercept at $x = 1$. Determine the value of 'k' and the equation of the polynomial function.

**Example 2:**
a) Find the family of cubic functions whose x-intercepts are -2,0, and 3.

Factors: ___________________________

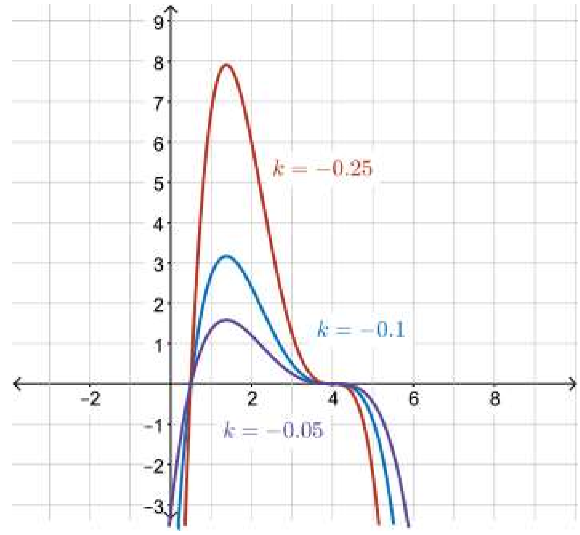
Family:___________________________




b) Find the specific member of this family that has remainder of 12 when it's divided by $x+1$.

**Example 3:**

Determine the general equation of a quartic function with end behavior f(x)→−∞ as x→±∞, a zero at $x = \frac{1}{2}$, and a point of inflection at $x = 4$ .




**Example 4:**

a) Give an example of polynomial function that has single roots at $2 \pm \sqrt{3}$ and a double root at 4.

b) How many other relations share the same zeros? How do you know?

c) Find specific member of family that has y-intercept of -8.

# Practice

1. Match each graph with the corresponding equation.

(a) $y=-x(x-2)^2$

(b) $y=-(x-2)(x^2+2x+3)$

(c) $y=x(x-2)(x+2)$

(d) $y=12x^3(x-2)$

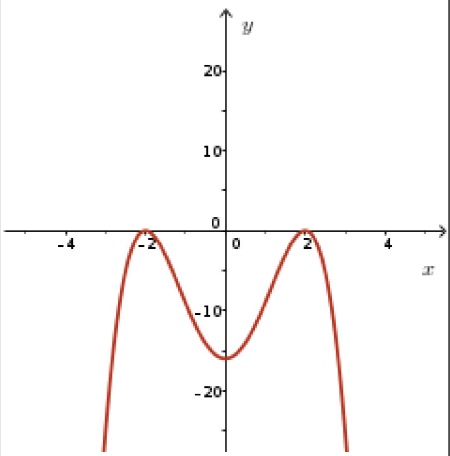
(e) $y=x(x+2)^2$

(f) $y=-2x(x+2)(x-2)$

(g) $y=-(x-2)^2(x+2)^2$

(h) $y=-(x-2)(x^2+2x-5)$

I.

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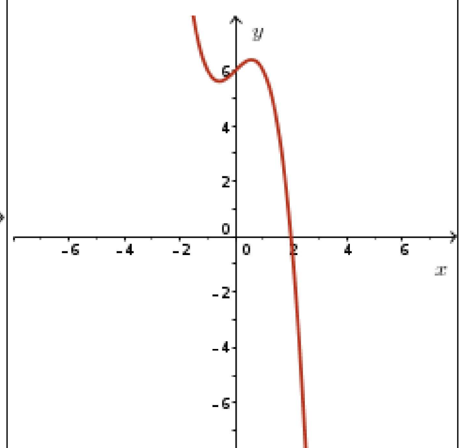


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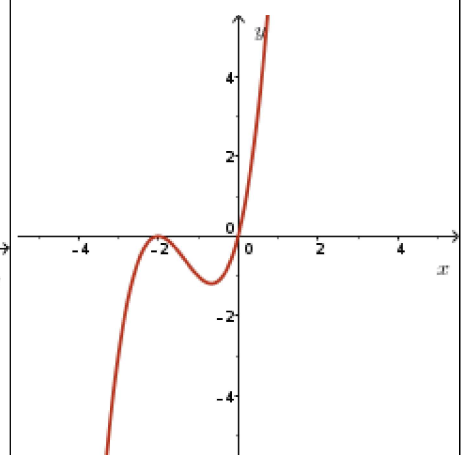
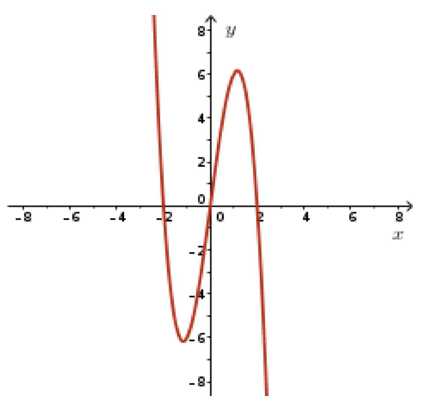
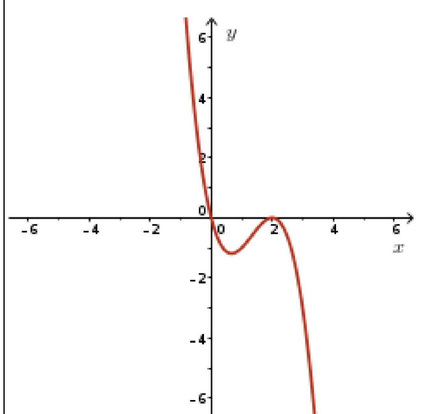


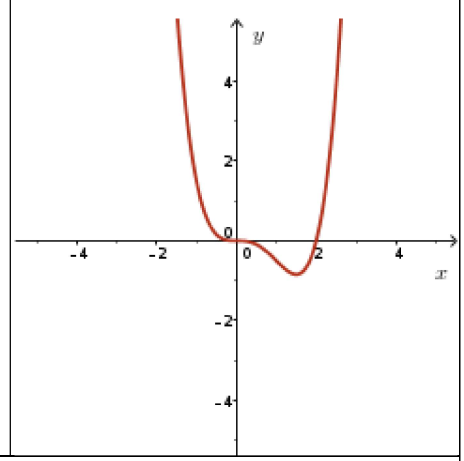

IV.



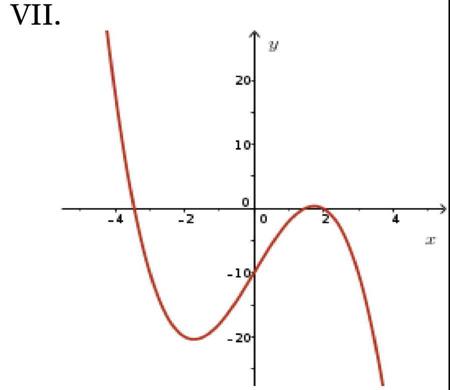

V.



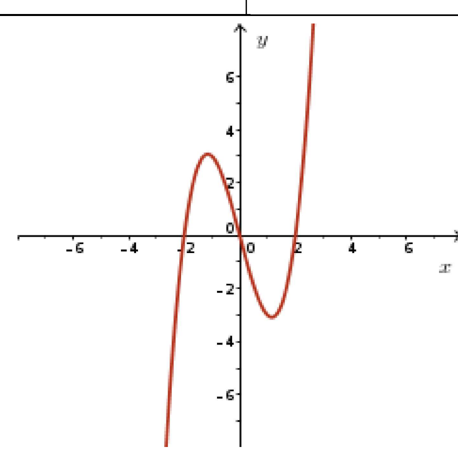

VI.




VII.

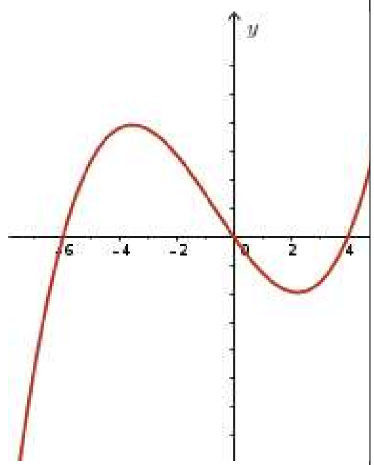



VIII.

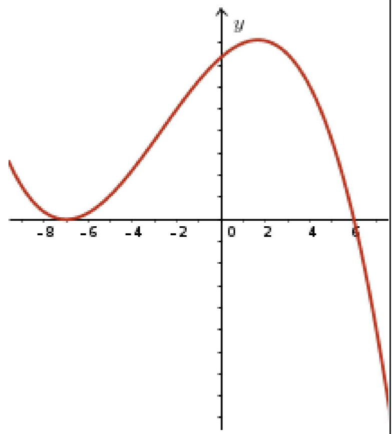


2. Given the graph of y=f(x), determine **a general equation** for a family of polynomials with the same end behavior and zeros of f(x) (note: all zeros are integer in value).

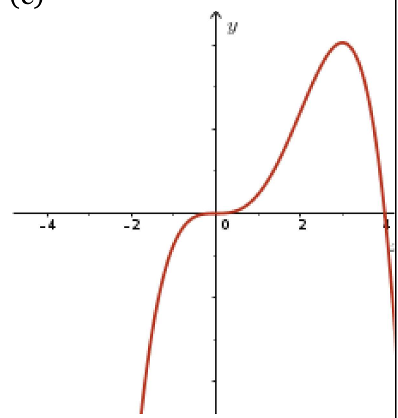
(a)



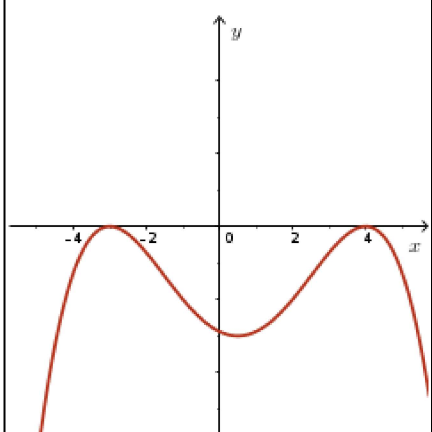

(b)



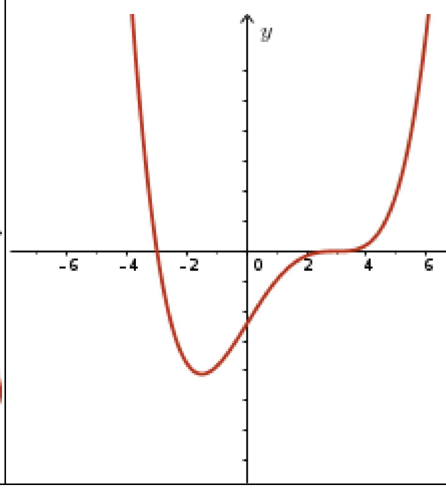

(c)



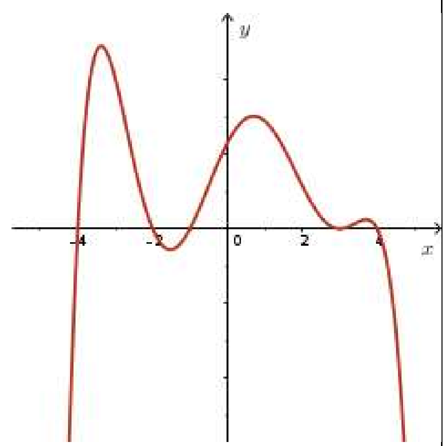

(d)




(e)

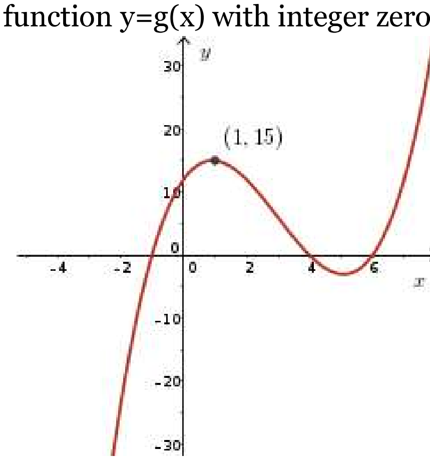



(f)



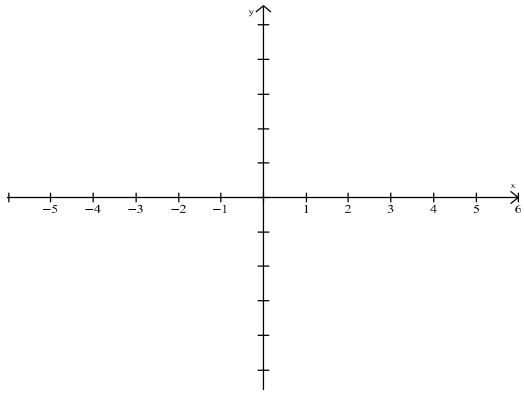
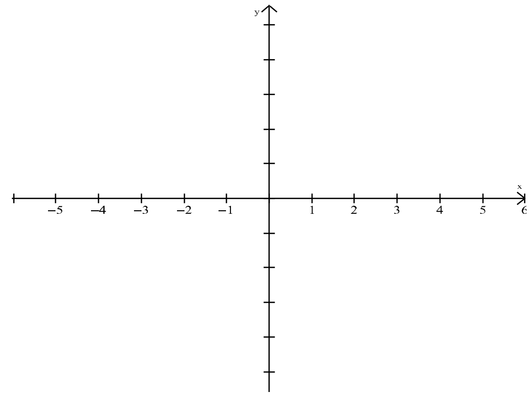

3. State the equation of the quartic function with zeros $x = \frac{-1}{2}$ and 5 (both of multiplicity 1) and x=2 (multiplicity 2), having a y-intercept of 4.

4. Determine the equation given the graph of the polynomial function y=g(x) with integer zeros.

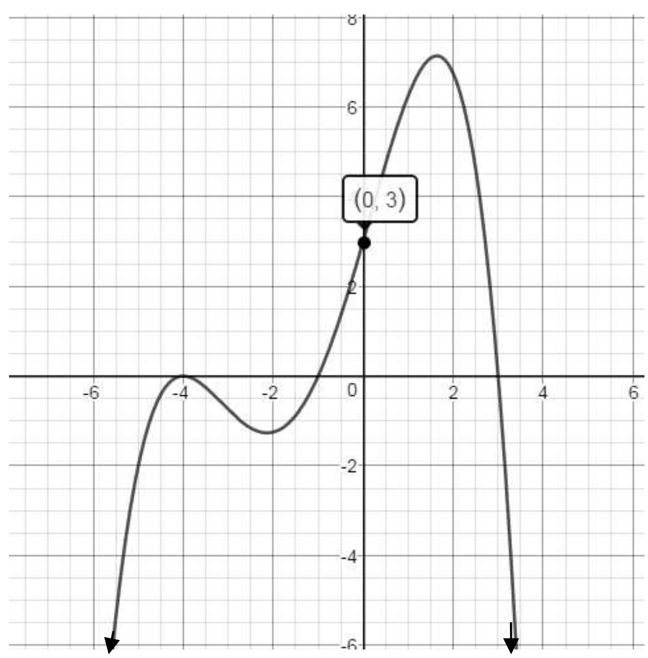


5. Determine the equation of the
   a. quadratic functions with zeros $-3 \pm \sqrt{5}$ ,passing through (-1,2).
   b. cubic functions with zeros 0 and $1 \pm 2\sqrt{3}$ , passing through (2,22).
   c. quartic functions with zeros –2,1, and $-1 \pm \sqrt{2}$ , and y-intercept -36.
   d. Cubic function with zeros –1,2/3, and 3, passing through (4,5).
6. Determine the equation of the quartic function with rational coefficients, zeros $4 - \sqrt{2}$ and $-3 + \sqrt{6}$ and a y-intercept of –21.

1. Sketch the graph of the following functions using in the properties of functions discussed in class.

| a) $f(x) = -(2-x)(x^2 - 4)$ | b) $f(x) = -(x - 2)^2(x + 1)^3(x+3)$ |
|---|---|
|  |  |
| Degree of the function: _______<br>End behaviour:<br>$x \to \infty$ ,<br>$x \to -\infty$ , | Degree of the function: _______<br>End behaviour:<br>$x \to \infty$ ,<br>$x \to -\infty$ , |

2. Use the graph of the polynomial function to answer the following questions.




a. The least possible degree of the function is _____________.

b. The sign of the leading coefficient is ____________.

c. The x-intercepts of the function are ___________________.

d. The intervals where the function is increasing are ____________________.

e. The intervals where the function is negative are _________________________.

f. Determine an equation in factored form.

# Unit 1: Polynomial Functions
## 1.10 Solve Inequalities

PART A: Solve Factorable Polynomial Inequalities Graphically

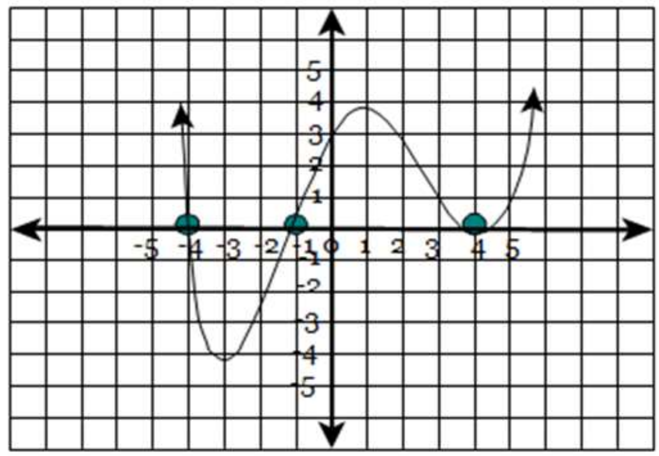
The solutions to polynomial inequalities are intervals or sets of numbers that are subset of the domain of the corresponding function.

Examples:

1. For the function on the right state when

   i) $f(x) > 0$

   ii) $f(x) < 0$




2. Use a **graphing calculator** to solve each of the following polynomial inequalities.

   a) $f(x) = x^3 - x^2 + 3x - 9$ , solve $f(x) \geq 0$ ____________________________

   b) $f(x) = x^3 + 5x^2 + 3x - 9$, solve $f(x) < 0$ ____________________________

   $f(x) > 0$ ____________________________

   c) $f(x) = x^4 - 1$ , solve $f(x) \leq 0$ ____________________________________

   d) $f(x) = x(x+2)^2(x-3)$, solve $f(x) > 0$ ______________________________

   $f(x) \geq 0$ ______________________________

## PART B: Solve Factorable Polynomial Inequalities Algebraically

**Method:**

1. Rearrange inequality so that the right side is 0.
2. Find the zeros( or x-intercepts) of the polynomial.
3. Draw a number line representing the x-axis and label the zero(s) ( or x-intercepts).

$\longleftrightarrow$

4. Pick a test value( your choice!) between the zero(s) to determine if the interval or region is positive(+) or negative(-) .
   - positive(+) : y-values are positive for all the x in the interval (Graph is above x-axis)
   - negative(-) : y-values are negative for all the x in the interval (Graph is above x-axis)

**NOTE:** May sketch function to determine positive or negative intervals.

5. State the solution to the inequality given.

   If $f(x) > 0$ , positive interval(s) are required only

   If $f(x) \geq 0$ , positive(including zeros) interval(s) are required only

   If $f(x) < 0$ , negative interval(s) are required only

   If $f(x) \leq 0$ , negative(including zeros) interval(s) are required only

**Example#1:** Solve each of the following, $x \in R$

a) $x^2 - 3x > 10$

b) $x^3 + 4x^2 + x - 6 < 0$

c) $125 - 8x^3 \leqslant 0$

d) $(3x-1)^5 (x+5)^7 - (3x-1)^4 (x+5)^8 > 0$

e) $(4-x^2)(x^2-3x+2)<0$

f) $x^2+1>0$

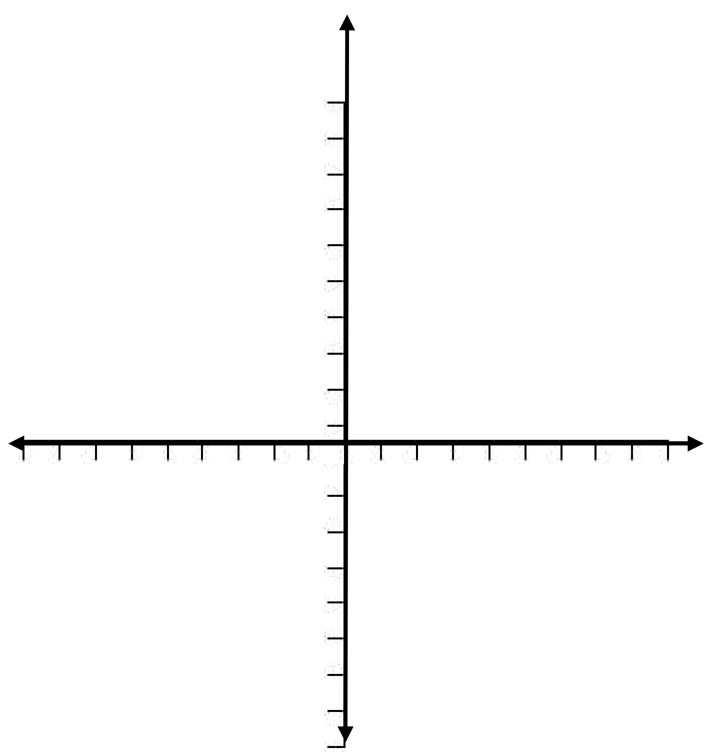
g) $x^4+2x^3-4x^2>8x$

h) $x^3+3x^2+x+3\leq 0$

**Example #2**: Laurie and Dave play on an Ultimate Frisbee team. On a windy day, and throwing against the wind, the height, in metres, of the Frisbee, t seconds after it leaves Laurie's hand, is determined by the function $h(t) = -t^3 + 2t^2 + t - 2$. How many seconds after it is thrown must Dave catch the Frisbee to ensure that it does not hit the ground?

**Example#3** : Determine the equation of a quartic function $f(x)$ that satisfies the following conditions:

- $f(x) \geq 0$ when $x \in (-2, 5)$
- $f(x) < 0$ when $x \in (-\infty, -2) \cup (5, \infty)$
- $f(x)$ has a root of order 2 at x= 2
- $f(x)$ has a maximum point at ( 4 ,10).



## Practice

1. Solve the following polynomial inequalities, $x \in \mathbb{R}$.

   a) $x^2 - 4x + 3 < 0$
   b) $x^3 - 3x - 2 \geqslant 0$
   c) $x^4 - 1 \geqslant 0$
   d) $-x^2 + 3x + 1 < 0$
   e) $-2x^4 - 2x^3 + 16x^2 + 24x < 0$
   f) $2(x+3)(x-1)^2(x-5) \leqslant 0$
   g) $-3(x+4)(x-3)^3 > 0$
   h) $x^4 < 22x^2 + 75$
   i) $2x^2 - 2x \geqslant 2 - x$

2. Let $f(x) = -2x + 1, g(x) = x^2 - 2x + 1$ and $h(x) = x^3 - 1$ . Determine all values of x such that $f(x) < g(x) < h(x)$ and illustrate the situation graphically.

3. The number *n* (**in hundreds**), of mosquitoes in a camping area after *t* weeks can be modelled by the equation $n(t) = 2t^4 - 5t^3 - 16t^2 + 45t$ .According to this model, when will the population of mosquitoes be greater than 1800?

4. A zoo wishes to construct an aquarium in the shape of a rectangular prism such that the length is twice the width and 5 m greater than the height. If the aquarium must have a volume strictly between 1125 $m^3$ and 3000 $m^3$, determine the restrictions on the length of the aquarium.

5. Determine the equation of a quintic function *f*(*x*) that satisfies the following conditions:

   - $f(-3) = f(0) = f(4) = 0$
   - $f(1) = -9$
   - $f(x) > 0$ when $x < -3$ or $-3 < x < 0$
   - $f(x) < 0$ when $0 < x < 4$ or $x > 4$

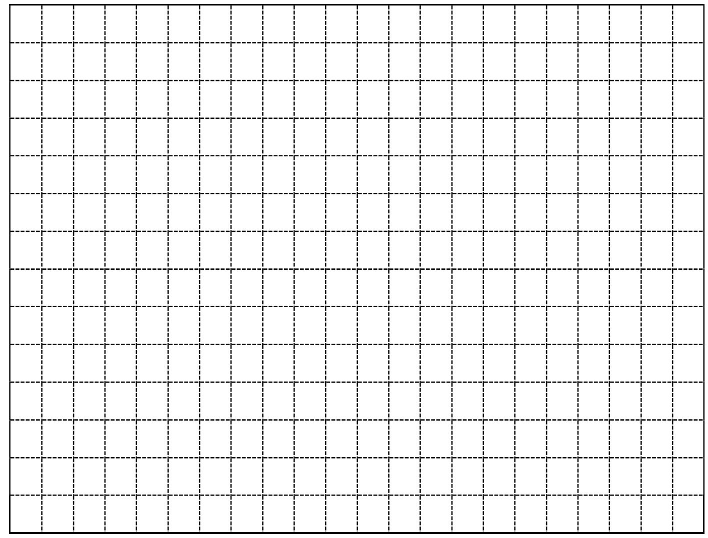
   Illustrate the situation graphically.

6. The solution to $x^2 + bx + 24 < 0$ is the set of all values of *x* such that $k < x < k + 2$ for some real value of *k*. Determine all possible values of *b*, $\mathbf{b} \in \mathbb{R}$. Justify your answer.

7. A quartic function has turning points at $(-3,0)$,$(1,0)$, and $(-1,-16)$. Determine all values of *x* such that $-9 < f(x) < 0$.

**Warm up**

1. Solve the inequality $x^3 - x^2 - 4x + 4 > 0$ graphically.



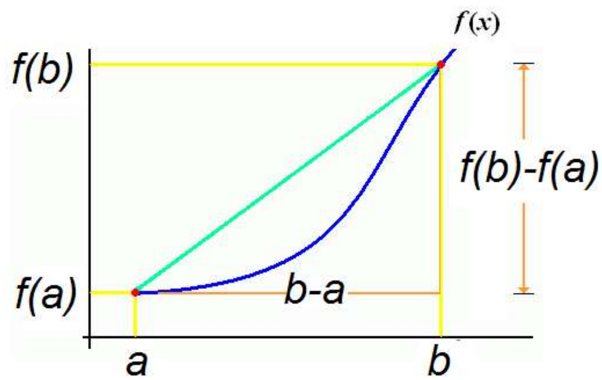
2. Solve the inequality $-x^3 + 7x^2 - 48 \le 0$ algebraically.

# 1.11 Average Rate of Change & Instantaneous Rate of Change

The average rate of change of f(x) on the interval [a,b] is defined as the slope of the secant drawn to the graph over the interval [a,b].

$$\boxed{\text{A.R.O.C} = m_s = \frac{f(b) - f(a)}{b - a}}$$

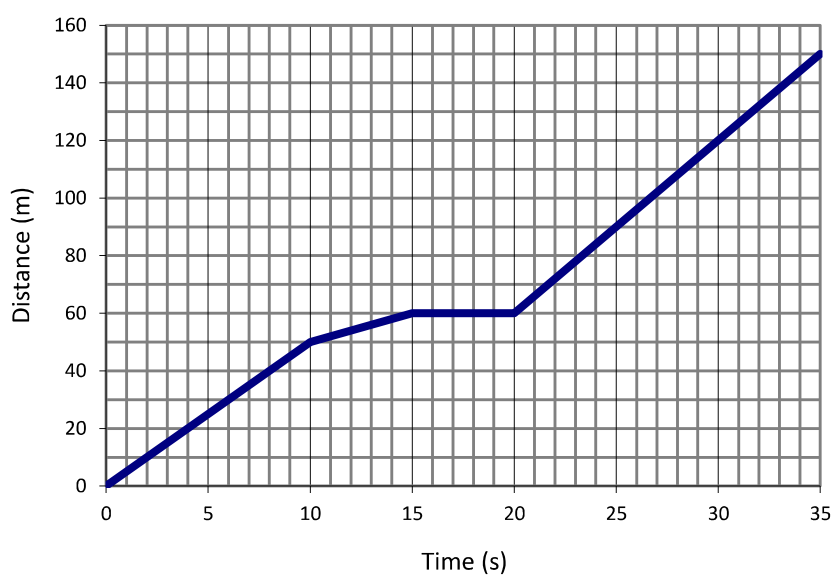
Recall: A secant line is a straight line that joins two points on the function.




Example#1: Calculate the average rate of change

i. between 10s and 15 s.

ii. between 4s and 30s.

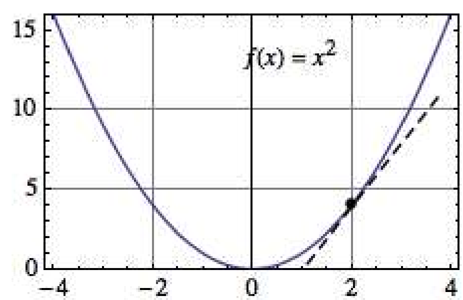


**Example#2** Determine the average rate of change of the secant line on the graph of $g(x)=x^2-4x$ in interval [1,5].

**Instantaneous Rate of Change**

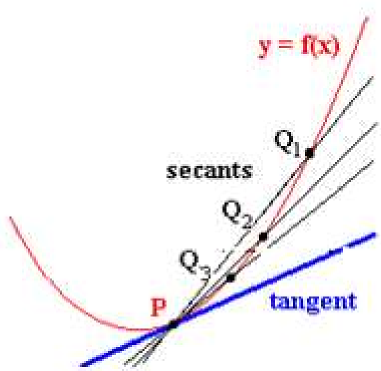
The exact rate of change of a function $y = f(x)$ at a specific value of the independent variable $x = a$

- ***Instantaneous rate of change*** = slope of tangent
  Recall: **Tangent Line** is a line which only touches the curve at one point




**Example #3:** Estimate the instantaneous rate of change of $f(x) = x^2$ at the point $x = 2$. We cannot find the slope of the tangent because we only know one point on the tangent line, P.

We can estimate the slope of the tangent at P (i.e. the instantaneous rate of change at P) by finding the slopes of secants for smaller and smaller intervals around P.As the point Q on the function approaches the point P, the slope of the secant approaches the slope of the tangent.



**Mathematically:**

(1) select points closer and closer to the point P
(2) calculate the slope of each secant
(3) the slopes of the secants will approach a value, which is the slope of the tangent

**Method 1**

Choose a point whose x-value is very close to the one given and calculate the slope using a table of values until a pattern emerges

| Point 1 | |
|---|---|
| X1 | Y1 |
| 2 | 4 |
| 2 | 4 |
| 2 | 4 |
| 2 | 4 |

| Point 2 | | $\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$ |
|---|---|---|
| X2 | Y2 | |
| 2.1 | 4.41 | 4.1 |
| 2.01 | | |
| 2.001 | | |
| 2.0001 | | |

**The slope of tangent is ______.**

**Method 2**

If we want to know the instantaneous rate of change of *f(x)* at $x = a$, consider the intervals between $x = a$ and $x = a \pm h$, where $h$ is a really small number.(We can always consider $h$=0.001)

$$\textbf{Instantaneous Rate of Change} = \frac{f(a+h)-f(a)}{(a+h)-a}$$

$$\boxed{I.R.O.C = \frac{f(a+h)-f(a)}{h}, as\ h \to 0}$$

(This is known as the **difference quotient**.)

$$I.R.O.C = \frac{f(2+0.001)-f(2)}{0.001}$$

$$= \underline{\hspace{4cm}}$$

**<u>Example #4:</u>** An emergency flare is shot into the air. Its height, in metres, above the ground at various times in its flight is given by the following table.

| Time (s) | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|---|---|---|---|
| Height (m) | 2.0 | 15.75 | 27.0 | 37.1 | 42.0 | 46.8 | 47.0 | 45.75 | 42.0 |

Estimate the instantaneous rate of change in height at exactly t = 2.0 s.

**Example #5**: The height of a soccer ball above the ground at time t after it is kicked into the air, is given by the formula $h(t) = -4.9t^2 + 3.5t + 1$ where h is the height in metres, t is the time in seconds.

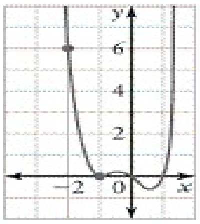
(a) Calculate the average velocity of the ball between t=0.1s and t =0. 3s

(b) Calculate the instantaneous velocity at t=0.6 s.

**Practice**

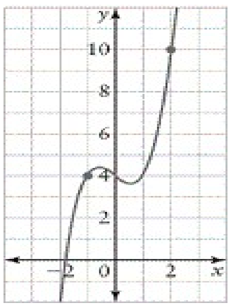
**Part A** –*Identify the choice that best completes the statement or answers the question.*

_____ 1. A secant drawn through the points shown on the graph has a slope of



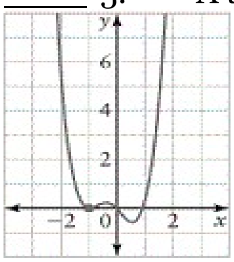

a. −1
b. −2
c. −3
d. −6

_____ 2. A secant drawn through the points shown on the graph has a slope of



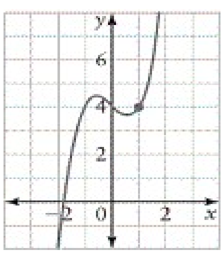

a. $\frac{1}{2}$
b. 2
c. 5
d. $\frac{10}{3}$

_____ 3. A tangent to the graph of the function at the point shown has a slope of



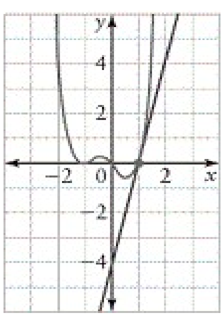

a. −2
b. −1
c. 0
d. 1

_____ 4. A tangent to the graph of the function at the point shown has a slope of approximately




a. $\frac{1}{2}$
b. 2
c. −2
d. 0

_____ 5. The slope of the tangent at the point indicated on the graph is




a. $\frac{1}{2}$
b. 1
c. 2
d. 4

_____ 6. The area, $A$, in square metres, of a circular oil spill after $t$ min is given by $A = kt^2$. The average rate of change of the area of the oil spill over the first 5 min is approximately 15 $m^2/min$, the value of k is :

a. 3
c. 2
b. 6
d. 5

**Part B**

1. Suppose the revenue, R, in dollars, from the sales of x (in hundred) MP3 players is given by $R(x) = x(350 - 0.1x^2)$. Find the average rate of change of revenue from selling from 100 to 200 MP3 players.

2. A medical researcher establishes that a patient's reaction time, r, in minutes, to a dose of a particular drug is $r(d) = -0.8d^3 + d^2$, where d is the amount of the drug, in millilitres, that is absorbed into the patient's blood. Determine the instantaneous rate of change of the time reaction with respect to the amount of the drug at 1 millilitre.

3. The following table represents the growth of a bacteria population over a 4.5 h period.

| **Time (h)** | 0 | 0.75 | 1.5 | 2.25 | 3 | 3.75 |
|---|---|---|---|---|---|---|
| **Number of bacteria** | 850 | 1122 | 1481 | 1954 | 2577 | 3400 |

Estimate the instantaneous rate of change in number of bacteria at exactly t = 3.

4. For the function $f(x) = -2x^3 + 3x - 1$ , determine the slope of a tangent at x = 1
5. Find the equation of tangent line to the curve $f(x) = -x^4 + 1$ at x = -1 on the curve.

## Unit 1 Review – Polynomial Functions

1. Fill in the blanks.

a) State the remainder when $-4x^3+3x^2+2x-1$ is divided by $x-2$. _______________

b) State the roots and the order of each root of $g(x)=2x^2(2x+3)^3$. ______________

c) When a function is divided by $2x-1$, the remainder is -2; Determine the remainder when the same function is divided by $x-\frac{1}{2}$. _____________

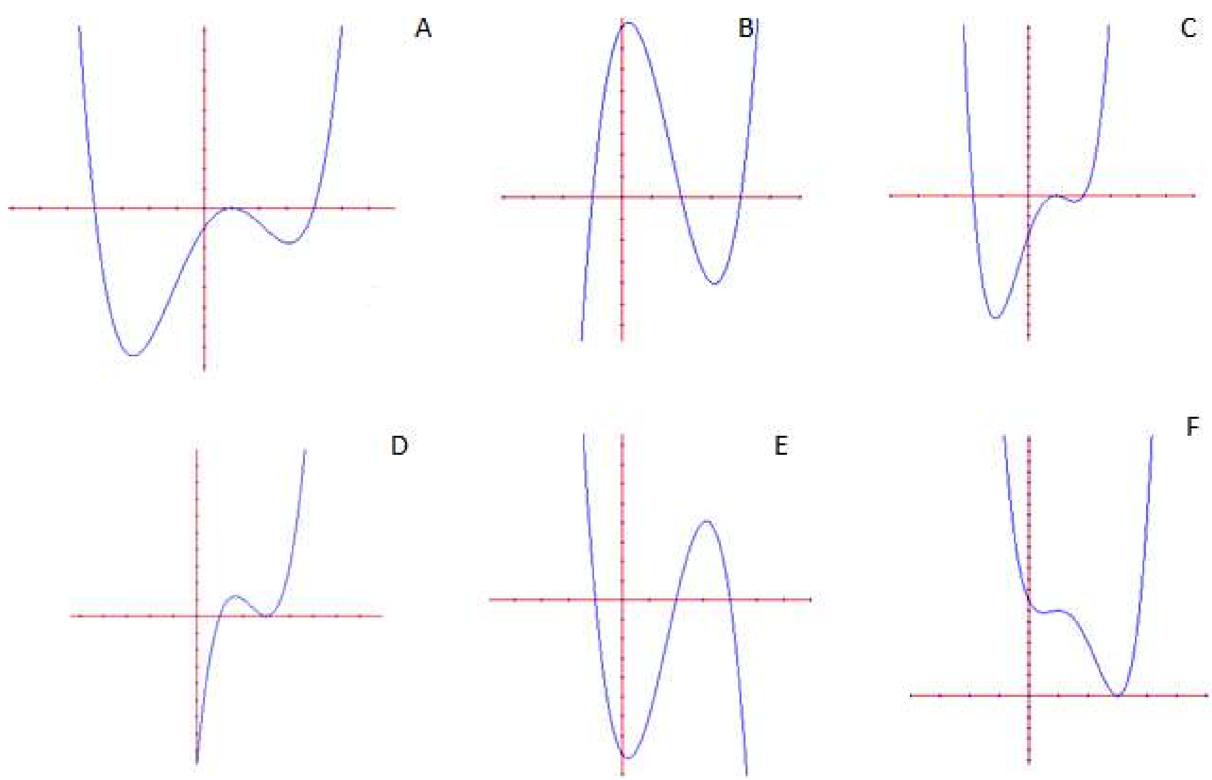
d) Values that could be zeros for the polynomial $f(x)=4x^3+2x^2-7x-8$ are___________

e) State if $y=-2x^4+3x^2+1$ is odd, even or neither. _____________________________

f) Beside each equation below, put the letter of the graph that best describes the equation:

i) $y=(x^2-16)(x-1)^2$ ______ ______

ii) $y=(2-x)(x-4)(x+1)$ ______ ______




2. Write the equation in factored form of any quartic function with following characteristics. Sketch the graph of function:

- $f(0)=0$
- $f(x)<0$, when $x<-2$
- $f(x)\geq 0$, when $-2\leq x\leq 3$
- $f(x)<0$, when $x>3$

3. Fully factor

a) $2x^5 - 2x^4 - 4x^3 + 4x^2 + 2x - 2$ b) $64y^6x^3 - 125$

c) $6(x+2)^{-5} + 2x^2(x+2)^{-4}$ d) $4x^4 - 13x^3 - 13x^2 + 28x - 6 = 0$

4. Divide $8x^4 - 30x^2 + 6x - 3$ by $1 + x + 2x^2$ using **long division** and write the division statement.

5. When $f(x) = x^4 - 4x^3 + mx^2 + nx + 1$ is divided by $x-1$, the remainder is 7. When it is divided by $x+1$, the remainder is 3. Determine the values of $m$ and $n$.

6. Solve each of the following, $x \in R$.

a) $x(x-1)(3-x)(x+3) < 0$ b) $x^3 - x^2 < 5x + 3$

7. Find the value of a and b and the remaining factor if the expression $ax^3 - 11x^2 + bx + 3$ is divisible by $x^2 - 4x + 3$.

8. Graph the following function $f(x) = (x^2+x+1)(2x+5)^2(x-3)^3$.Show all your work.

9. The passenger section of a train has a width $2x - 7$, length $2x + 3$, and height $x - 2$, with all dimensions in metres. Solve a polynomial equation to determine the dimensions of the section of the train if the volume is $117 m^3$.

10. Determine algebraically, whether each function is even, odd, or neither.

a) $f(x) = 4x^3$ b) $f(x) = 2x^4 - x^2$ c) $g(x) = \sqrt[3]{2x^2 + 1}$

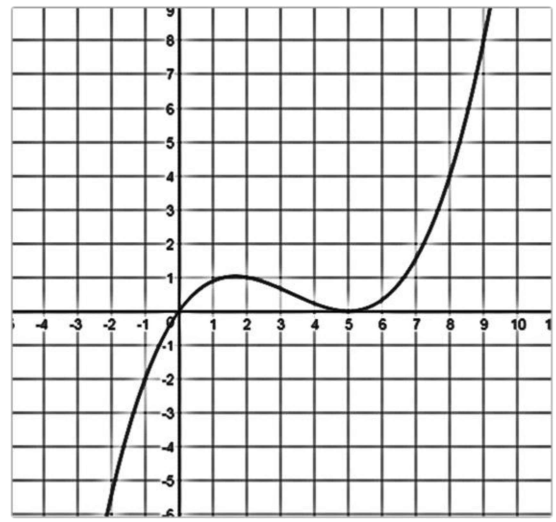
d) $h(x) = \dfrac{-x^3}{(3x^3 - 9x)^2}$ e) $f(x) = x + |x|$ f) $g(x) = \dfrac{2x}{|x|}$

11. The table of values below represents a polynomial function. Determine the equation of this function.

| x | y | | | |
|---|---|---|---|---|
| -2 | -19 | | | |
| -1 | -3 | | | |
| 0 | 1 | | | |
| 1 | -1 | | | |
| 2 | -3 | | | |
| 3 | 1 | | | |

12. Find the general equation of quartic functions that has negative leading coefficient, two equal roots at 2, and roots at $3 \pm 2\sqrt{2}$.

13. Given the graph of a polynomial function g(x), answer the following:



a) Is the function even-degree or odd-degree?

b) Is the function even or odd or neither?

c) State the zeroes and the lowest possible order of each zero ____________________

d) State the interval where the function is positive ____________________________________

e) State the interval where the function is negative ____________________________________

f) Determine the value of the remainder when f(x) is divided by x+2. _______________

14. Water is draining from a container. The height, in millimeters, of the water as a function of time, in seconds, can be modeled by the function

$$h(t) = 0.00185(250 - t)^2.$$

a) Calculate the average rate of change of height with respect to time from 50s to 100s.

b) Calculate the instantaneous rate of change of height with respect to time at t=60s.

c) Create a sketch of the function indicating the secant line and tangent line from part a.

15. When polynomial $x^3$-**a**x+21 is divided by x +**b**, the quotient is $x^2$ - 3x +5 and the remainder is 6. Determine values of **a** and **b**.

16. Is x+b a factor of $x^9+5b^2x^7+5bx^8- b^9$?