Unit 5: Exponential & Logarithmic Functions-Review

SOLUTIONS

1. Solve each of the following. Show a proper solution.

a. $5^{2x+3}+40=70$

$\log_5\left[5^{2x+3}\right]=\log_5(30)$

$2x+3=\log_5 30$

$\boxed{x=\frac{\log_5(30)-3}{2}}$

b. $3\left(2^x\right)=18^{x-1}$

$3(2^x)=\frac{18^x}{18}$

$54(2^x)=18^x$

$54=\frac{18^x}{2^x}$

$54=\left(\frac{18}{2}\right)^x$

$\log_9(54)=\log_9(9^x)$

$\boxed{\log_9(54)=x}$

c. $6^x\left(6^x-15(6^{-x})-2\right)=(0)6^x$

$6^{2x}-15-2(6^x)=0$

$6^{2x}-2(6^x)-15=0$

Let $A=6^x$: $A^2-2A-15=0$

$(A-5)(A+3)=0$

$A-5=0$

$6^x=5$

$\log_6(6^x)=\log_6(5)$

$\boxed{x=\log_6(5)}$

$A+3=0$

$6^x=-3$

↳ no solutions

d. $15\times3^{x+1}-243\times5^{x-2}=0$

$15(3^{x+1})=243(5^{x-2})$

$5\cdot3(3^{x+1})=3^5(5^{x-2})$

$\frac{3^{x+2}}{3^5}=\frac{5^{x-2}}{5}$

$3^{x-3}=5^{x-3}$

$1=\frac{5^{x-3}}{3^{x-3}}$

$\left(\frac{5}{3}\right)^0=\left(\frac{5}{3}\right)^{x-3}$

$0=x-3$

$\boxed{3=x}$

e. $4^{2x}-9(2^{2x})+14=0$

$(2^2)^{2x}-9(2^{2x})+14=0$

$2^{4x}-9(2^{2x})+14=0$

Let $A=2^{2x}$: $A^2-9A+14=0$

$(A-7)(A-2)=0$

$A-7=0$

$2^{2x}=7$

$\log_2(2^{2x})=\log_2(7)$

$2x=\log_2(7)$

$\boxed{x=\frac{\log_2(7)}{2}}$

$A-2=0$

$2^{2x}=2^1$

$2x=1$

$\boxed{x=\frac{1}{2}}$

f. $\left(4-\log x\right)^2=\left(3\sqrt{\log x}\right)^2$

Restriction:
$x>0$
$\log x>0 \Rightarrow \therefore x>1$

$16-8(\log x)+(\log x)^2=9(\log x)$

$(\log x)^2-17(\log x)+16=0$

Let $A=\log x$: $A^2-17A+16=0$

$(A-16)(A-1)=0$

$A-16=0$

$\log(x)=16$

$\boxed{x=10^{16}}$

↳ extraneous

$A-1=0$

$\log(x)=1$

$\boxed{x=10}$

g. $\log_2 x + \log_4 x = \log_8 x + 10^{\log\frac{7}{6}}$

Restriction: $x>0$

$$\log_2 x + \log_{2^2} x = \log_{2^3} x + \frac{7}{6}$$
$$\log_2 x + \frac{1}{2}\log_2 x = \frac{1}{3}\log_2 x + \frac{7}{6}$$
$$\frac{7}{6}\log_2 x = \frac{7}{6}$$
$$\log_2 x = 1$$
$$\boxed{x = 2}$$

h. $\log(\log x) = \log(7 - 2\log x) - \log 5$

Restriction: $\log x > 0 \Rightarrow x > 1$

$7 - 2\log x > 0 \Rightarrow \frac{7}{2} > \log x \Rightarrow 10^{7/2} > x$

$\therefore 1 < x < 10^{3.5}$

$$\log(\log x) = \log\left[\frac{7-2\log x}{5}\right]$$
$$\log x = \frac{7-2\log x}{5}$$
$$5\log x = 7 - 2\log x$$
$$7\log x = 7$$
$$\log x = 1$$
$$10^{\log x} = 10^1$$
$$\boxed{x = 10}$$

i. $\log_3(\log_2(\log_4(x^2 - 6x))) = 0$

Restriction: $x^2 - 6x > 0 \Rightarrow x(x-6) > 0$

$+\;\;-\;\;+$ on number line with 0, 6 $\;\therefore x<0$ or $x>6$

$$3^{\log_3(\log_2(\log_4(x^2-6x)))} = 3^0$$
$$\log_2(\log_4(x^2-6x)) = 1$$
$$2^{\log_2(\log_4(x^2-6x))} = 2^1$$
$$\log_4(x^2-6x) = 2$$
$$4^{\log_4(x^2-6x)} = 4^2$$
$$x^2 - 6x = 16$$
$$x^2 - 6x - 16 = 0$$
$$(x-8)(x+2) = 0$$
$$\boxed{\therefore x = 8 \text{ and } x = -2}$$

j. $\log_2(x+1) - \log_2(x-1) = 1$

Restrictions: $x+1>0 \Rightarrow x>-1$

$x-1>0 \Rightarrow x>1$

$\therefore x>1$

$$\log_2\left[\frac{x+1}{x-1}\right] = 1$$
$$2^{\log_2\left[\frac{x+1}{x-1}\right]} = 2^1$$
$$\frac{x+1}{x-1} = 2$$
$$x+1 = 2(x-1)$$
$$x+1 = 2x-2$$
$$\boxed{3 = x}$$

k. $\log_2 x + \log_4 x + \log_8 x + \log_{16} x = 25$

Restriction: $x>0$

$$\log_2 x + \log_{2^2} x + \log_{2^3} x + \log_{2^4} x = 25$$
$$\log_2 x + \frac{1}{2}\log_2 x + \frac{1}{3}\log_2 x + \frac{1}{4}\log_2 x = 25$$
$$\frac{25}{12}\log_2 x = 25$$
$$\log_2 x = 12$$
$$2^{\log_2 x} = 2^{12}$$
$$\boxed{x = 4096}$$

l. $9^x = 2\times 3^{x+2} - 45$

$$3^{2x} - 2(3^{x+2}) + 45 = 0$$
$$3^{2x} - 2(3^2)3^x + 45 = 0$$
$$3^{2x} - 18(3^x) + 45 = 0$$

Let $A = 3^x$: $A^2 - 18A + 45 = 0$

$$(A-15)(A-3) = 0$$

$A - 15 = 0$

$3^x = 15$

$\log_3(3^x) = \log_3(15)$

$\boxed{x = \log_3(15)}$

$A - 3 = 0$

$3^x = 3$

$\boxed{x = 1}$

m. $\log_2 x + \log_{16} x = 5$

Restriction: $x > 0$

$\log_2 x + \log_{2^4} x = 5$

$\log_2 x + \frac{1}{4}\log_2 x = 5$

$\frac{5}{4}\log_2 x = 5$

$\log_2 x = 4$

$2^{\log_2 x} = 2^4$

$\boxed{x = 16}$

2. The average annual salary, S, in dollars, of employees at a particular job in a manufacturing company is modeled by the equation S = 25000(1.05)ⁿ, where $25000 is the initial salary, which increases at 5% per year.
 (a)How long will it take the salary to increase by 50%?
 (b) If the starting salary is $35 000, how long will it take the salary to increase by 50%? Explain your answer.

a) Salary increased by 50% = 25000(1.5) = 37500

$37500 = 25000(1.05)^n$

$1.5 = (1.05)^n$

$\log(1.5) = \log(1.05)^n$

$\frac{\log(1.5)}{\log(1.05)} = n$

$8.3 \doteq n$

$\boxed{\therefore \text{It will take about 8.3 years}}$

b) It will also take about 8.3 years a 50% increase is always represented by $1.5 = (1.05)^n$ no matter what the initial value is.

3. The speed, v, in kilometres per hour, of a water skier who drops the tow rope, can be given by the formula $v = v_0(10)^{-0.23t}$, where v_0 is the skier's speed at the time she drops the rope, and t is the time, in seconds, after she drops the rope. If the skier drops the rope when traveling at a speed of 65 km/h, how long will it take her to slow to a speed of 13 km/h?

$v = 13$

$v_0 = 65$

$13 = 65(10)^{-0.23t}$

$\frac{1}{5} = 10^{-0.23t}$

$\log\left(\frac{1}{5}\right) = \log\left(10^{-0.23t}\right)$

$\log\left(\frac{1}{5}\right) = -0.23t$

$\frac{\log\left(\frac{1}{5}\right)}{-0.23} = t$

$3 \doteq t$

$\boxed{\therefore \text{It will take about 3 hours.}}$

4. On average, number of items, N, per day, on an assembly line, that a quality assurance trainee can inspect is $N = 40 - 24(0.74)^t$, where t is the number of days worked.

(a) After how many days of training employee be able to inspect 32 items?

(b) The company expects an experienced assurance employee to inspect 45 items per day. After the training period of 15 days is complete, how close will the trainee be to the experienced employee's quota?

a) $32 = 40 - 24(0.74)^t$

$\frac{1}{3} = 0.74^t$

$\log\left(\frac{1}{3}\right) = \log(0.74)^t$

$\frac{\log\left(\frac{1}{3}\right)}{\log(0.74)} = t$

$3.6 \doteq t$

$\boxed{\therefore \text{After about } 3.6 \text{ days of training.}}$

b) $N = 40 - 24(0.74)^{15}$

$\doteq 39.7$

As $t \to \infty$, $(0.74)^t \to 0$.

Hence, $N \to 40$ but never reaches it.

$\boxed{\therefore \text{An employee will never be able to inspect 45 items per day.}}$

5. Graph the function $y = -2\left(\frac{1}{\log_{2x+6} 3}\right) + 4$, State the domain, range and asymptote of the function.

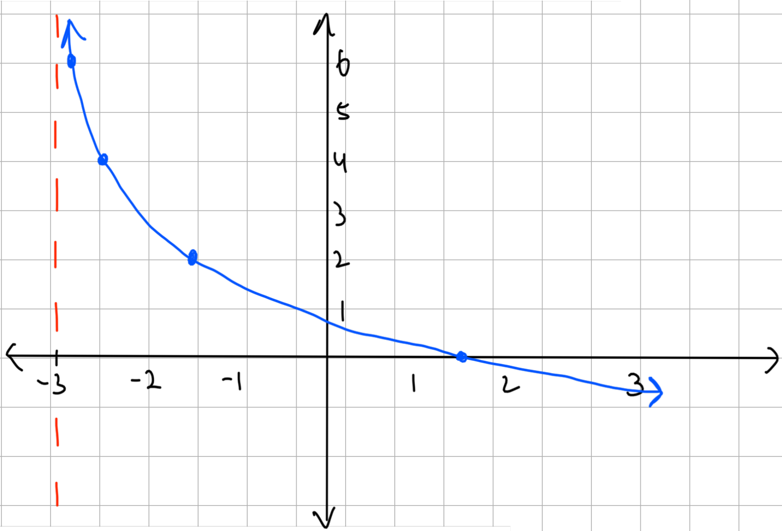
$y = -2\log_3(2x+6) + 4$

$y = -2\log_3[2(x+3)] + 4$

$(x, y) \to \left(\frac{1}{2}x - 3, -2y + 4\right)$

x	$y = \log_3 x$	
$\frac{1}{3}$	-1	$\Rightarrow \left(-\frac{17}{6}, 6\right) \doteq (-2.8, 6)$
1	0	$\Rightarrow \left(-\frac{5}{2}, 4\right) = (-2.5, 4)$
3	1	$\Rightarrow \left(-\frac{3}{2}, 2\right) = (-1.5, 2)$
9	2	$\Rightarrow \left(\frac{3}{2}, 0\right) = (1.5, 0)$

V.A at $x = 0 \Rightarrow$ V.A at $x = -3$



6. Solve for x .

a) $5^{3x-2} = \left(2 \times 3^{4-2x}\right)^2$

$5^{3x-2} = 4\left(3^{8-4x}\right)$

$\log 5^{3x-2} = \log\left[4\left(3^{8-4x}\right)\right]$

$(3x-2)\log 5 = \log 4 + \log 3^{8-4x}$

$(3x-2)\log 5 = \log 4 + (8-4x)\log 3$

$3x(\log 5) - 2(\log 5) = \log 4 + 8(\log 3) - 4x(\log 3)$

$3x(\log 5) + 4x(\log 3) = \log 4 + 8(\log 3) + 2(\log 5)$

$x\left[3(\log 5) + 4(\log 3)\right] = \log 4 + 8(\log 3) + 2(\log 5)$

$$\boxed{x = \frac{\log 4 + 8(\log 3) + 2(\log 5)}{3(\log 5) + 4(\log 3)} \qquad x \doteq 1.5}$$

b) $\log_2 \sqrt{x^2 + 12x} = 3$

Restrictions: $x^2 + 12x > 0$

$x(x+12) > 0$

$+ \quad - \quad +$

$-12 \quad 0$

$\therefore x < -12, x > 0$

$\log_2\left(x^2 + 12x\right)^{\frac{1}{2}} = 3$

$\frac{1}{2}\log_2\left(x^2 + 12x\right) = 3$

$\log_2\left(x^2 + 12x\right) = 6$

$2^{\log_2\left(x^2 + 12x\right)} = 2^6$

$x^2 + 12x = 64$

$x^2 + 12x - 64 = 0$

$(x + 16)(x - 4) = 0$

$\boxed{\therefore x = -16, x = 4}$

c) $\log_{\sqrt[3]{5}}\left(\log_{\sqrt[5]{2}}\left(\log_{\sqrt{5}} x\right)\right)=3$

$\log_{5^{\frac{1}{3}}}\left[\log_{2^{\frac{1}{5}}}\left(\log_{5^{\frac{1}{2}}}(x)\right)\right]=3$

$\left(5^{\frac{1}{3}}\right)^{\log_{5^{\frac{1}{3}}}\left[\log_{2^{\frac{1}{5}}}\left(\log_{5^{\frac{1}{2}}}(x)\right)\right]}=\left(5^{\frac{1}{3}}\right)^{3}$

$\log_{2^{\frac{1}{5}}}\left(\log_{5^{\frac{1}{2}}}(x)\right)=5$

$\left(2^{\frac{1}{5}}\right)^{\log_{2^{\frac{1}{5}}}\left(\log_{5^{\frac{1}{2}}}(x)\right)}=\left(2^{\frac{1}{5}}\right)^{5}$

$\log_{5^{\frac{1}{2}}}(x)=2$

$\left(5^{\frac{1}{2}}\right)^{\log_{5^{\frac{1}{2}}}(x)}=\left(5^{\frac{1}{2}}\right)^{2}$

$\boxed{x=5}$

e) $\dfrac{\log\left(35-x^2\right)}{\log(5-x)}=2$

Restrictions:

$35-x^2>0$

$35>x^2 \Rightarrow x<\sqrt{35},\ x>-\sqrt{35}$

$5-x>0 \Rightarrow 5>x$

$5-x\neq 1 \Rightarrow 4\neq x$

$\log(35-x^2)=2\log(5-x)$

$\log(35-x^2)=\log(5-x)^2$

$10^{\log(35-x^2)}=10^{\log(5-x)^2}$

$35-x^2=(5-x)^2$

$35-x^2=25-10x+x^2$

$0=2x^2-10x-10$

$0=x^2-5x-5$

$x=\dfrac{5\pm\sqrt{25-4(1)(-5)}}{2}=\dfrac{5\pm\sqrt{45}}{2}=\dfrac{5\pm3\sqrt5}{2}$

$x=\dfrac{5+3\sqrt5}{2}>5 \Rightarrow$ inadmissible

$\boxed{\therefore\ x=\dfrac{5-3\sqrt5}{2}}$

7. Noise in the school cafeteria is recorded at 78 dB at lunchtime. Exposure to sound at 90 dB can cause mild hearing loss. If students in the cafeteria yell 10 times louder, will Bayview start to have deaf kids?

$L_2-L_1=10\log\left(\dfrac{I_2}{I_1}\right)$

$L_2-78=10\log\left(\dfrac{10I_1}{I_1}\right)$

$L_2-78=10\log(10)$

$L_2=10+78$

$L_2=88\text{ dB}$

$\therefore$ No, the students at Bayview won't go deaf.

8. Earthquakes of magnitude 7.0 or greater can cause metal buildings to collapse. On December 23, 1985, in Mackenzie Region, Northwest Territories, an earthquake of magnitude 6.9 occurred. On Vancouver Island, on June 23, 1946, an earthquake about 2.5 times as intense occurred. Was the Vancouver bland earthquake strong enough to cause metal buildings to collapse?

$M_2-M_1=\log\left(\dfrac{I_2}{I_1}\right)$

$M_2-6.9=\log\left(\dfrac{2.5I_1}{I_1}\right)$

$M_2=\log(2.5)+6.9$

$M_2\doteq 7.3$

$\therefore$ Yes, it was strong enough to cause metal buildings to collapse.

9. Lemon juice has a pH of 2.6 and coffee has a pH of 5.1. How many more times as great is the concentration of hydrogen ions in lemon juice to that of coffee? (Round final answer to 2 decimals)?

$$2.6 - 5.1 = -\log\left(\frac{H^+_2}{H^+_1}\right)$$

$$-2.5 = -\log\left(\frac{H^+_2}{H^+_1}\right)$$

$$2.5 = \log\left(\frac{H^+_2}{H^+_1}\right)$$

$$10^{2.5} = \frac{H^+_2}{H^+_1}$$

$$316.22 = \frac{H^+_2}{H^+_1}$$

∴ The hydrogen ions concentration is about 316.22 times as great.

10. A 250 g sample of a radioactive material decays to 100 g in 2 weeks. What is its half-life in days?

$$A = A_0\left(\frac{1}{2}\right)^{t/d}$$

$$100 = 250\left(\frac{1}{2}\right)^{14/d}$$

$$\frac{2}{5} = \left(\frac{1}{2}\right)^{14/d}$$

$$\log\left(\frac{2}{5}\right) = \log\left(\frac{1}{2}\right)^{14/d}$$

$$\log\left(\frac{2}{5}\right) = \frac{14}{d}\log\left(\frac{1}{2}\right)$$

$$\frac{\log(2/5)}{\log(1/2)} = \frac{14}{d}$$

$$d = \frac{14\log(1/2)}{\log(2/5)}$$

$$d \doteq 10.6$$

∴ The half life is about 10.5 days.

11. A crate of apples is accidentally exposed to a particular radioactive element. The radioactive element is deemed to be safe when the amount present in a crate of apples is 0.1% of its original amount. If the half-life of the radioactive element is 6 hours, how long do consumers have to wait before the apples are safe to eat? (Note: would you even ***want*** to eat these apples?

$$A = A_0\left(\frac{1}{2}\right)^{t/d}$$

$$0.001A_0 = A_0\left(\frac{1}{2}\right)^{t/6}$$

$$\log(0.001) = \log\left(\frac{1}{2}\right)^{t/6}$$

$$-3 = \frac{t}{6}\log\left(\frac{1}{2}\right)$$

$$\frac{-3}{\log(\frac{1}{2})} = \frac{t}{6}$$

$$\frac{-18}{\log(\frac{1}{2})} = t$$

$$59.8 = t$$

∴ They would need to wait about 59.8 hours for the apples to be safe to eat.
(But I wouldn't want to eat it anyways... I'd rather not grow another head ☺)

12. For his dream car, James invested $24 000 at 5.4% interest, compounded monthly, for 7 years. After 7 years, he was still short. How much longer will he have to invest the money at 5% interest, compounded quarterly to have a total of $ 40 0000?

$$A = A_0\left(1+\frac{r}{n}\right)^{t \times n}$$

$$A = 24000\left(1+\frac{0.054}{12}\right)^{7\times12}$$
$$= 24000(1.0045)^{84}$$
$$= \$34995.02$$

$$40\,0000 = 34995.02\left(1+\frac{0.05}{4}\right)^{4n}$$
$$11.43 \doteq (1.0125)^{4n}$$
$$\log(11.43) = \log(1.0125)^{4n}$$
$$\frac{\log(11.43)}{\log(1.0125)} = 4n$$
$$\frac{\log(11.43)}{4\log(1.0125)} = n$$
$$49 \doteq n$$

$\therefore$ He will need to invest it for about 49 more years.

13. The SARS virus and the flu virus infect a city at the same time. At the start, the SARS virus had infected 60 people and the number of people infected was tripling every 8 days. The flu virus had infected 30 people and was doubling every 5 days. Predict in how many days the same number of people will be infected.

Let y represent the number of people infected and x the number of days.

SARS virus: $y = 60(3)^{x/8}$

Flu virus: $y = 30(2)^{x/5}$

Solve:
$$60(3)^{x/8} = 30(2)^{x/5}$$
$$(3)^{x/8} = \frac{1}{2}(2)^{x/5}$$
$$(3)^{x/8} = 2^{-1}(2)^{x/5}$$
$$(3)^{x/8} = (2)^{\frac{x}{5}-\frac{5}{5}}$$
$$\log(3)^{x/8} = \log(2)^{\frac{x-5}{5}}$$
$$\frac{x}{8}\log(3) = \left(\frac{x-5}{5}\right)\log(2)$$
$$5x(\log 3) = 8(x-5)(\log 2)$$
$$5x(\log 3) = 8x(\log 2) - 40(\log 2)$$
$$40(\log 2) = 8x(\log 2) - 5x(\log 3)$$
$$40(\log 2) = x\,[8(\log 2) - 5(\log 3)]$$
$$\frac{40(\log 2)}{[8(\log 2) - 5(\log 3)]} = x$$
$$532 \doteq x$$

$\therefore$ The same number of people will be infected in about 532 days.

14. State the transformation needed to transform the graph of $y = \log_2 x$ to graph the following:

a) $y = \log_2(x-3)$ Horizontal shift 3 units right

b) $y = \log_2\left(\frac{1}{2}x\right)$ (two different ways)

i) Horizontal stretch by a factor of 2.

ii) vertical shift 1 unit down.

$y = \log_2\left(\frac{1}{2}\right) + \log_2(x)$

$y = -1 + \log_2(x)$

$y = \log_2(x) - 1$