Unit 3: Trigonometry
3.4 Compound Angle Formulas

Activity 1: Investigation- Compound angles

Sara is studying for a trigonometry test and completes the following question:
Question: Evaluate the following:

$$\cos\left(\pi - \frac{\pi}{3}\right)$$

Sara's solution:

$$\begin{aligned}\cos\left(\pi - \frac{\pi}{3}\right) &= \cos(\pi) - \cos\left(\frac{\pi}{3}\right)\\ &= -1 - \frac{1}{2}\\ &= -\frac{3}{2}\end{aligned}$$

Use a calculator to determine if Sara's answer is right or wrong.

Describe in words the mistake(s) in her solution if there is any.

____
____

Is the following statement true or false?

"A trigonometric ratio can be distributed to the angles that lie within the brackets."

From the investigation above, we know that:

____
____

As you might expect, there are formulas for $\cos(\alpha \pm \beta)$, $\sin(\alpha \pm \beta)$ and $\tan(\alpha \pm \beta)$, but Activity 1 shows it is wrong to apply the distributive law to the trigonometric ratios of compound angles.

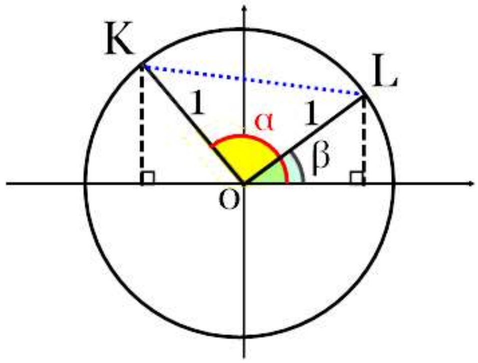
Derivation of cos(α−β)
Prove that, cos(α−β)=cos(α)cos(β)+sin(α)sin(β).

Figure on the next page shows angles α and β in standard position on the unit circle, determining points L(a,b) and K($-x,y$) ,respectively.
We use the distance formula and cosine law to find the length of LK.

Recall:

$$\text{Distance Formula : } d_{AB} = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

$$\text{Cosine Law: } a^2 = b^2 + c^2 - 2bc.\cos(A)$$



Step 1: Express the coordinates of L and K in terms of the angles α and β:

L(_________,________)

K(_________, ________)

Step 2: Use the distance formula to determine KL^2 :

$KL^2 = (x_K - x_L)^2 + (y_K - y_L)^2$

$KL^2 =$

$=$

$=$

$=$

Step 3: Now determine KL^2 using the cosine law for △KOL:

$KL^2 = KO^2 + LO^2 - 2.KO.LO.\cos(\alpha - \beta)$

$=$

$=$

Step 4: Equating the two expressions for KL^2, we have :

Example 1: Prove that, $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.

Example 2: Use the compound angle formula and co-functions to expand $\sin(\alpha-\beta)$.

Hint: Using co-functions, we know that $\sin(A) = \cos(\frac{\pi}{2} - A)$, so we can write $\sin(\alpha+\beta)$ in terms of the cosine function as:

$$\begin{aligned}\sin(\alpha-\beta) &= \cos(\frac{\pi}{2} - (\alpha-\beta)) \\ &= \cos(\frac{\pi}{2} - \alpha + \beta) \\ &= \cos[(\frac{\pi}{2} - \alpha) + \beta]\end{aligned}$$

Compound Angel Identities

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} \qquad (*)$$

$$\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)} \qquad (*)$$

(*) see Appendix 2 for proof

Example 3: Determine the **exact** value for each of the following:

a) $\cos\left(75^{\circ}\right)$

b) $\sin\left(\frac{\pi}{12}\right)$

c) $\tan\left(\frac{7\pi}{12}\right)$

d) $\cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{2\pi}{3}\right)+\sin\left(\frac{5\pi}{6}\right)\sin\left(\frac{2\pi}{3}\right)$

Example 4: Given $\sec(\alpha)=\frac{5}{3}$, $0<\alpha<\frac{\pi}{2}$ and $\sin(\beta)=\frac{1}{3}$,$0<\beta<\frac{\pi}{2}$**,** draw a sketch and determine the exact value of $\sin\left(\alpha+\beta\right)$ without the use of a calculator.

3.4 PRACTICE

1. Evaluate (provide the exact values).

a) $\sin\left(\frac{\pi}{12}\right)$ b) $\tan\left(\frac{\pi}{12}\right)$ c) $\tan\left(\frac{17\pi}{12}\right)$

d) $\sin\left(\frac{13\pi}{12}\right)$ e) $\cos\left(\frac{13\pi}{12}\right)$ f) $\cos(50^\circ)\sin(95^\circ)-\sin(50^\circ)\cos(95^\circ)$

g) $\dfrac{\tan\left(\frac{11\pi}{6}\right)+\tan\left(\frac{\pi}{3}\right)}{1-\tan\left(\frac{11\pi}{6}\right)\tan\left(\frac{\pi}{3}\right)}$ h*) $\dfrac{\cos^2\left(\frac{5\pi}{3}\right)\times\sin\left(\frac{5\pi}{6}\right)}{\tan\left(\frac{11\pi}{6}\right)\times\sec\left(\frac{\pi}{12}\right)}$

2. Given: $\sin(\alpha)=\frac{5}{13}, \alpha\in\left(\frac{\pi}{2},\pi\right)$ and $\cos(\beta)=\frac{-3}{5}, \beta\in\left(\pi,\frac{3\pi}{2}\right)$. Evaluate:

a) $\cos(\alpha+\beta)$ b) $\tan(\alpha+\beta)$

3. Given: $\cot(x)=\frac{4}{3}$, $\cot(y)=\frac{5}{12}$.Find: $\tan(x-y)$.

4. Solve for x, in the domain $0\le x\le 2\pi$.

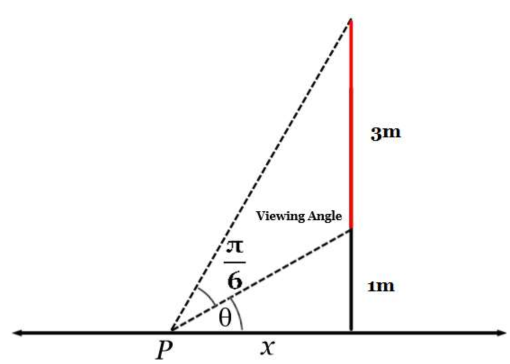
a) $\cos\left(\frac{\pi}{4}\right)\cos(x)-\sin\left(\frac{\pi}{4}\right)\sin(x)=1$ b) $\sin(x)\cos\left(\frac{\pi}{6}\right)-\cos(x)\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$

5*. Consider a triangle with angles A, B, C. Show that :

$$\tan(A)+\tan(B)+\tan(C)=\tan(A)\tan(B)\tan(C).$$

6*. Find the exact acute angle between the lines $x-\sqrt{3}y+\sqrt{3}=0$ and $x+y-5=0$.

7*. A room in an art gallery contains a picture you are interested in viewing. The picture is 3 meters high and is hanging so the bottom of the picture is one meter above your eye level. How far from the wall on which the picture is hanging should you stand so that the angle of vision occupied by the picture is $\frac{\pi}{6}$?



Answers

1. a) $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$ b) $2-\sqrt{3}$ c) $2+\sqrt{3}$ d) $-\frac{\sqrt{2}(\sqrt{3}-1)}{4}$

   e) $-\frac{\sqrt{2}(\sqrt{3}+1)}{4}$ f) $\frac{\sqrt{2}}{2}$ g) $\frac{\sqrt{3}}{3}$ h) $\frac{-\left(\sqrt{6}+3\sqrt{2}\right)}{32}$

2. a) $\frac{56}{65}$ b) $\frac{33}{56}$

3. $-\frac{33}{56}$

4. a) $\frac{7\pi}{4}$ b) $\frac{\pi}{3}$ or π

5. Hint : In any triangle , A+B+C=π , or A+C= π-C therefore tan(A+C)= tan(π-C).

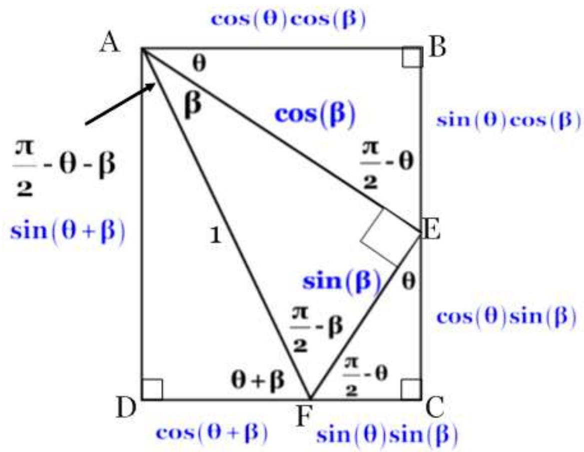
6. $\frac{5\pi}{12}$

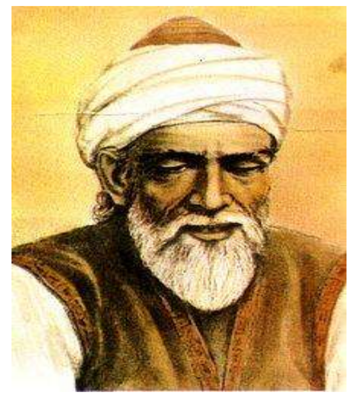
7. 4.2m or 0.94m

Appendix 1

The formula for cos (θ ±β) originally established by the 10th century Persian mathematician **Abu al-Wafa' Buzjani.** This appendix presents the proof of compound angle formula for sin(θ+β) and cos(θ+β).

Triangle AEF is constructed in rectangle ABCD such that ∠AEF=90∘, E lies on BC, and F lies on CD. AF is 1 unit in length, ∠BAE=θ, and ∠FAE=β, as shown in the diagram. Using the diagram, develop the compound angle formulas for sin(θ+β) and cos(θ+β).





$$\text{In}\,\Delta AEF, \angle EAF = \beta \ \text{so} \angle AFE = \frac{\pi}{2} - \beta.$$

$$\text{Since } \sin(\beta) = \frac{EF}{1}, \text{then } EF = \sin(\beta). \text{Similarly, } AE = \cos(\beta).$$

$$\text{In}\,\Delta ABE, \angle BAE = \theta \ \text{so} \angle AEB = \frac{\pi}{2} - \theta.$$

$$\text{Since } \sin(\theta) = \frac{BE}{AE}$$

$$= \frac{BE}{\cos(\beta)}, \text{then } BE = \sin(\theta)\cos(\beta). \text{Similarly, } AB = \cos(\theta)\cos(\beta).$$

$$\text{In}\,\Delta ADF, \angle DAF = \frac{\pi}{2} - \theta - \beta \ \text{and} \angle AFD = \pi - \frac{\pi}{2} - \left(\frac{\pi}{2} - \theta - \beta\right) = \theta + \beta.$$

$$\text{Since } \sin(\theta+\beta) = \frac{AD}{AF} = \frac{AD}{1}, \text{then } AD = \sin(\theta+\beta). \text{Similarly, we get } DF = \cos(\theta+\beta).$$

$$\text{In}\,\Delta CEF, \angle CEF = \pi - \frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right) = \theta. \text{Thus, } \angle CFE = \frac{\pi}{2} - \theta$$

$$\text{Since } \sin(\theta) = \frac{CF}{EF} = \frac{BE}{\sin(\beta)}, \text{then } CF = \sin(\theta)\sin(\beta). \text{Similarly, } CE = \cos(\theta)\sin(\beta).$$

In rectangle ABCD, $AD = BC = BE + CE$,

$\therefore \sin(\theta+\beta) = \sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta).$

Also, $AB = CD = CF + DF$, so

$\cos(\theta)\cos(\beta) = \cos(\theta+\beta) + \sin(\theta)\cos(\beta)$

Therefore, $\cos(\theta+\beta) = \cos(\theta)\cos(\beta) - \sin(\theta)\cos(\beta).$

Appendix 2

This appendix provides the mathematical proof of the the angle sum formula for tangent. We begin by using the quotient identity.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin(A)\cos(B)+\cos(A)\sin(B)}{\cos(A)\cos(B)-\sin(A)\sin(B)}$$

$$= \frac{\left[\sin(A)\cos(B)+\cos(A)\sin(B)\right] \div \mathbf{\cos(A)\cos(B)}}{\left[\cos(A)\cos(B)-\sin(A)\sin(B)\right] \div \mathbf{\cos(A)\cos(B)}}$$

$$= \frac{\dfrac{\sin(A)\cancel{\cos(B)}}{\mathbf{\cos(A)\cancel{\cos(B)}}} + \dfrac{\cancel{\cos(A)}\sin(B)}{\mathbf{\cancel{\cos(A)}\cos(B)}}}{\dfrac{\cancel{\cos(A)\cos(B)}}{\mathbf{\cancel{\cos(A)\cos(B)}}} - \dfrac{\sin(A)\sin(B)}{\mathbf{\cos(A)\cos(B)}}}$$

$$= \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

Warm up

Simplify the following without using a calculator. Provide the exact values.

a) $$\frac{\tan\left(\frac{5\pi}{6}\right) - \tan\left(\frac{2\pi}{3}\right)}{1 + \tan\left(\frac{5\pi}{6}\right)\tan\left(\frac{2\pi}{3}\right)}$$

b) $$\frac{\sin(x)\cos(\frac{\pi}{4} - x) + \cos(x)\sin(\frac{\pi}{4} - x)}{\cos(x)\cos(\frac{\pi}{3} - x) - \sin(x)\sin(\frac{\pi}{3} - x)}$$