L3 – 2.2 – Factor Theorem Lesson
MHF4U

In this section, you will learn how to determine the factors of a polynomial function of degree 3 or greater.

Part 1: Remainder Theorem Refresher

a) Use the remainder theorem to determine the remainder when $f(x) = x^3 + 4x^2 + x - 6$ is divided by $x + 2$

> **Remainder Theorem:** When a polynomial function $P(x)$ is divided by $x - b$, the remainder is $P(b)$; and when it is divided by $ax - b$, the remainder is $P\left(\frac{b}{a}\right)$, where a and b are integers, and $a \neq 0$.

$f(-2) = (-2)^3 + 4(-2)^2 + (-2) - 6$

$f(-2) = -8 + 16 - 2 - 6$

$f(-2) = 0$

The remainder when divided by $x + 2$ is 0. This means that $x + 2$ is a factor of the dividend.

b) Verify your answer to part a) by completing the division using long division or synthetic division.

> **Note:** I chose synthetic since it is a linear divisor of the form $x - b$.

$$
\begin{array}{r|rrrrl}
-2 & 1 & 4 & 1 & -6 & \\
 & \downarrow & -2 & -4 & 6 & + \\
\hline
\times & 1 & 2 & -3 & 0 & \\
 & x^2 & x & \# & R &
\end{array}
$$

$x^3 + 4x^2 + x - 6 = (x+2)(x^2 + 2x - 3)$

> **Factor Theorem:**
>
> $x - b$ is a factor of a polynomial $P(x)$ if and only if $P(b) = 0$. Similarly, $ax - b$ is a factor of $P(x)$ if and only if $P\left(\frac{b}{a}\right) = 0$.

Example 1: Determine if $x - 3$ and $x + 2$ are factors of $P(x) = x^3 - x^2 - 14x + 24$

$P(3) = (3)^3 - (3)^2 - 14(3) + 24$

$P(3) = 27 - 9 - 42 + 24$

$P(3) = 0$

Since the remainder is **0**, $x - 3$ divides evenly into $P(x)$; that means $x - 3$ **is a factor** of $P(x)$.

$P(-2) = (-2)^3 - (-2)^2 - 14(-2) + 24$

$P(-2) = -8 - 4 + 28 + 24$

$P(-2) = 40$

Since the remainder is not **0**, $x + 2$ does not divide evenly into $P(x)$; that means $x + 2$ **is not a factor** of $P(x)$.

Part 2: How to determine a factor of a Polynomial With Leading Coefficient 1

You could guess and check values of b that make $P(b) = 0$ until you find one that works...

Or you can use the Integral Zero Theorem to help.

Integral Zero Theorem

If $x - b$ is a factor of a polynomial function $P(x)$ with leading coefficient 1 and remaining coefficients that are integers, then ***b* is a factor of the constant term** of $P(x)$.

Note: Once one of the factors of a polynomial is found, division is used to determine the other factors.

Example 2: Factor $x^3 + 2x^2 - 5x - 6$ fully.

Let $P(x) = x^3 + 2x^2 - 5x - 6$

Find a value of b such that $P(b) = 0$. Based on the factor theorem, if $P(b) = 0$, then we know that $x - b$ is a factor. We can then divide $P(x)$ by that factor.

The integral zero theorem tells us to test factors of **-6.**

Test $\pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}$, ***and*** $\pm\ \mathbf{6}$. Once one factor is found, you can stop testing and use that factor to divide $P(x)$.

$P(1) = (1)^3 + 2(1)^2 - 5(1) - 6$

$P(1) = 1 + 2 - 5 - 6$

$P(1) = -8$

Since $P(1) \neq 0$, we know that $x - 1$ is NOT a factor of $P(x)$.

$P(2) = (2)^3 + 2(2)^2 - 5(2) - 6$

$P(2) = 8 + 8 - 10 - 6$

$P(2) = 0$

Since $P(2) = 0$, we know that $x - 2$ is a factor of $P(x)$.

You can now use either long division or synthetic division to find the other factors

Method 1: Long division

$$
\begin{array}{r l}
 & x^2 + 4x + 3 \\
x - 2 \,\big) & x^3 + 2x^2 - 5x - 6 \\
 & \underline{x^3 - 2x^2} \\
 & \quad 4x^2 - 5x \\
 & \quad \underline{4x^2 - 8x} \\
 & \qquad 3x - 6 \\
 & \qquad \underline{3x - 6} \\
 & \qquad R = 0
\end{array}
$$

$x^3 + 2x^2 - 5x - 6 = (x-2)(x^2+4x+3)$ factor further if possible.

$= (x-2)(x+3)(x+1)$

Method 2: Synthetic Division

$$
\begin{array}{c|cccc}
2 & 1 & 2 & -5 & -6 \\
 & \downarrow & 2 & 8 & 6 \quad + \\
\hline
x & 1 & 4 & 3 & 0 \\
 & x^2 & x & \# & R
\end{array}
$$

factor further if possible

$x^3 + 2x^2 - 5x - 6 = (x-2)(x^2+4x+3)$

$= (x-2)(x+3)(x+1)$

Example 3: Factor $x^4 + 3x^3 - 7x^2 - 27x - 18$ completely.

Let $P(x) = x^4 + 3x^3 - 7x^2 - 27x - 18$

Find a value of b such that $P(b) = 0$. Based on the factor theorem, if $P(b) = 0$, then we know that $x - b$ is a factor. We can then divide $P(x)$ by that factor.

The integral zero theorem tells us to test factors of **$\underline{-18}$.**

Test $\pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}, \pm\mathbf{6}, \pm\mathbf{9}\ \boldsymbol{and} \pm \mathbf{18}$. Once one factor is found, you can stop testing and use that factor to divide $P(x)$.

$P(1) = (1)^4 + 3(1)^3 - 7(1)^2 - 27(1) - 18$

$P(1) = -48$

$x - 1$ is NOT a factor of $P(x)$.

$P(-1) = (-1)^4 + 3(-1)^3 - 7(-1)^2 - 27(-1) - 18$

$P(-1) = 0$

$x + 1$ IS a factor of $P(x)$.

Since $\underline{\boldsymbol{P(-1) = 0}}$, this tell us that $\underline{\boldsymbol{x + 1}}$ is a factor. Use division to determine the other factor.

$$
\begin{array}{c|ccccc}
-1 & 1 & 3 & -7 & -27 & -18 \\
 & \downarrow & -1 & -2 & 9 & 18 \quad + \\
\hline
\times & 1 & 2 & -9 & -18 & 0 \\
 & x^3 & x^2 & x & \# & R
\end{array}
$$

$x^4 + 3x^3 - 7x^2 - 27x - 18 = (x+1)(x^3 + 2x^2 - 9x - 18)$

We can now further divide $x^3 + 2x^2 - 9x - 18$ using division again or by factoring by grouping.

Method 1: Division

Test factors of -18

$f(-2) = (-2)^3 + 2(-2)^2 - 9(-2) - 18$

$f(-2) = 0$

$x + 2$ is a factor

$$
\begin{array}{c|cccc}
-2 & 1 & 2 & -9 & -18 \\
 & \downarrow & -2 & 0 & 18 \quad + \\
\hline
\times & 1 & 0 & -9 & 0 \\
 & x^2 & x & \# & R
\end{array}
$$

DOS

$x^4 + 3x^3 - 7x^2 - 27x - 18 = (x+1)(x+2)(x^2 - 9)$

$= (x+1)(x+2)(x-3)(x+3)$

Method 2: Factoring by Grouping

$$f(x) = x^3 + 2x^2 - 9x - 18$$

Group the first 2 terms and the last 2 terms and separate with an addition sign.

$$f(x) = (x^3 + 2x^2) + (-9x - 18)$$

Common factor within each group

$$f(x) = x^2(x + 2) - 9(x + 2)$$

Factor out the common binomial

$$f(x) = (x + 2)(x^2 - 9)$$

Therefore,

$$\begin{aligned} x^4 + 3x^3 - 7x^2 - 27x - 18 &= (x + 1)(x^3 + 2x^2 - 9x - 18) \\ &= (x + 1)(x + 2)(x^2 - 9) \\ &= (x + 1)(x + 2)(x - 3)(x + 3) \end{aligned}$$

Example 4: Try Factoring by Grouping Again

$x^4 - 6x^3 + 2x^2 - 12x$

Note: Factoring by grouping does not always work…but when it does, it saves you time!

$= \boldsymbol{(x^4 - 6x^3) + (2x^2 - 12x)}$

$= \boldsymbol{x^3(x - 6) + 2x(x - 6)}$

$= \boldsymbol{(x - 6)(x^3 + 2x)}$

$= \boldsymbol{(x - 6)(x)(x^2 + 2)}$

Part 3: How to determine a factor of a Polynomial With Leading Coefficient NOT 1

The integral zero theorem can be extended to include polynomials with leading coefficients that are not 1. This extension is known as the rational zero theorem.

Rational Zero Theorem:

Suppose $P(x)$ is a polynomial function with integer coefficients and $x = \frac{b}{a}$ is a zero of $P(x)$, where a and b are integers and $a \neq 0$. Then,

- b is a factor of the constant term of $P(x)$
- a is a factor of the leading coefficient of $P(x)$
- $(ax - b)$ is a factor of $P(x)$

Example 5: Factor $P(x) = 3x^3 + 2x^2 - 7x + 2$

We must start by finding a value of $\frac{b}{a}$ where $P\left(\frac{b}{a}\right) = 0$.

b must be a factor of the constant term. Possible values for b are: $\pm 1, \pm 2$

a must be a factor of the leading coefficient. Possible values of a are: $\pm 1, \pm 3$

Therefore, possible values for $\frac{b}{a}$ are: $\pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}$

Test values of $\frac{b}{a}$ for x in $P(x)$ to find a zero.

$P(1) = 3(1)^3 + 2(1)^2 - 7(1) + 2 = 0$

Since **$P(1) = 0$, $x - 1$ is a factor** of $P(x)$. Use division to find the other factors.

1	3	2	-7	2	
	↓	3	5	-2	+
x	3	5	-2	0	
	x^2	x	#	R	

P: -6
S: 5
6 and -1

$$3x^3 + 2x^2 - 7x + 2 = (x-1)(3x^2 + 5x - 2)$$
$$= (x-1)[3x^2 + 6x - 1x - 2]$$
$$= (x-1)[(3x^2 + 6x) + (-1x - 2)]$$
$$= (x-1)[3x(x+2) - 1(x+2)]$$
$$= (x-1)(x+2)(3x-1)$$

Example 6: Factor $P(x) = 2x^3 + x^2 - 7x - 6$

Possible values for b are: $\pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}, \pm\mathbf{6}$

Possible values of a are: $\pm\mathbf{1}, \pm\mathbf{2}$

Therefore, possible values for $\frac{b}{a}$ are: $\pm\mathbf{1}, \pm\frac{\mathbf{1}}{\mathbf{2}}, \pm\mathbf{2}, \pm\mathbf{3}, \pm\frac{\mathbf{3}}{\mathbf{2}}, \pm\mathbf{6}$

$f(-1) = 2(-1)^3 + (-1)^2 - 7(-1) - 6 = 0$

Therefore, $x + 1$ is a factor of $P(x)$

$$\begin{array}{c|cccc}
-1 & 2 & 1 & -7 & -6 \\
 & \downarrow & -2 & 1 & 6 \quad + \\
\hline
\times & 2 & -1 & -6 & 0 \\
 & x^2 & x & \# & R
\end{array}$$

P: -12
S: -1
-4 and 3

$$2x^3 + x^2 - 7x - 6 = (x+1)(2x^2 - x - 6)$$
$$= (x+1)\left[(2x^2 - 4x) + (3x - 6)\right]$$
$$= (x+1)\left[2x(x-2) + 3(x-2)\right]$$
$$= (x+1)(x-2)(2x+3)$$

Part 4: Application Question

Example 7: When $f(x) = 2x^3 - mx^2 + nx - 2$ is divided by $x + 1$, the remainder is -12 and $x - 2$ is a factor. Determine the values of m and n.

Hint: Use the information given to create 2 equations and then use substitution or elimination to solve.

$f(-1) = 2(-1)^3 - m(-1)^2 + n(-1) - 2$

$-12 = -2 - m - n - 2$

① $-8 = -m - n$

$f(2) = 2(2)^3 - m(2)^2 + n(2) - 2$

$0 = 16 - 4m + 2n - 2$

② $-14 = -4m + 2n$

① $-8 = -m - n \xrightarrow{\times 2} -16 = -2m - 2n$

② $-14 = -4m + 2n \longrightarrow -14 = -4m + 2n$ $+$

$-30 = -6m$

$5 = m$

sub $m = 5$ into ① or ②

$-8 = -5 - n$

$n = 3$

$\therefore m = 5$ and $n = 3$