Unit 2: Rational Functions
2.1 Rational Functions and Their Essential Characteristics

A **rational function** is a function that can be expressed in the form $f(x)=\dfrac{P(x)}{Q(x)}$ where both P(x) and Q(x) are polynomial functions and the denominator Q(x) is of degree 1 or higher. Although polynomial functions are defined for all real values of x, rational functions are **not defined** for those values of x for which the denominator, Q(x), is 0.

Examples of rational functions:

$$y=\frac{1}{x-2} \qquad f(x)=\frac{2x}{3-x} \qquad g(x)=\frac{x^2-4}{x^2-2x}$$

Q1. Explain why each of the following function is not a rational function?

a) $y=\dfrac{x+20}{4}$

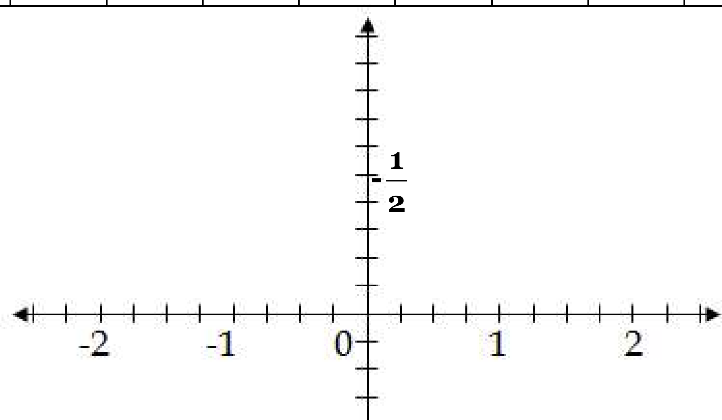
b) $f(x)=\dfrac{2\sqrt{x}-1}{x+3}$

Investigation: Properties of the simplest rational function $f(x)=\dfrac{1}{x}$

Graph the rational function $f(x)=\dfrac{1}{x}$ manually by completing a partial table of values.

Plot the *(x ,y)* points and join them with a smooth curve.

x	**-3**	**-2**	**-1**	**-1/2**	**-1/3**	**-1/4**	**0**	**1/4**	**1/3**	**1/2**	**1**	**2**	**3**
y													



Note that the function $f(x)=\dfrac{1}{x}$ is not defined for x = 0. The tables below show the behavior of f(x) near zero.

$$\frac{1}{\text{small number}} = \text{BIG NUMBER}$$

x	f(x)
-0.1	
-0.01	
-0.00001	

x	f(x)
0.1	
0.01	
0.00001	

This behavior can be described in the following analytical way:

____

The next two tables show how f (x) changes as |x| becomes large.

$$\frac{1}{\text{BIG NUMBER}} = \text{small number}$$

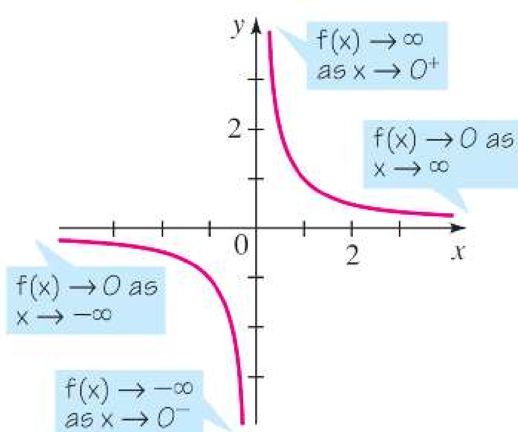
x	f(x)
-10	
-100	
-100 000	

x	f(x)
10	
100	
100 000	

This behavior can be described in the following analytical way:

____

The most important feature that distinguishes the graphs of rational functions is the presence of **asymptotes.**

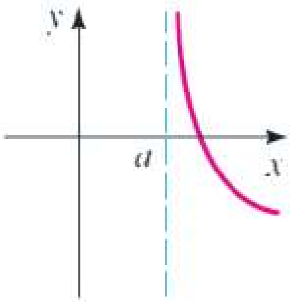


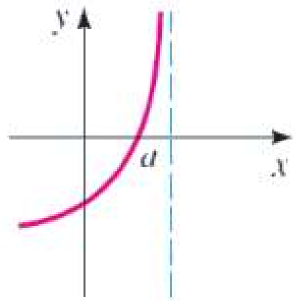
Definitions:

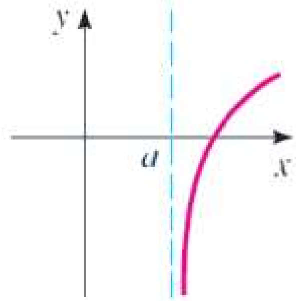
i) Vertical asymptote

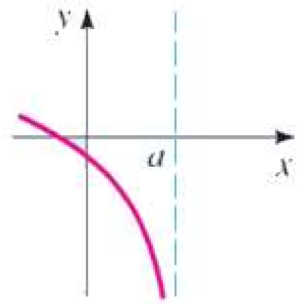
The line x=a is a vertical asymptote of the graph of function $\mathbf{f}(\mathbf{x})$ *,if y approaches* $\pm\infty$ *as x approaches a from the left or right.*

The following graphs illustrate each of the limit statements.





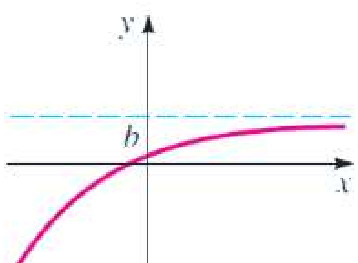


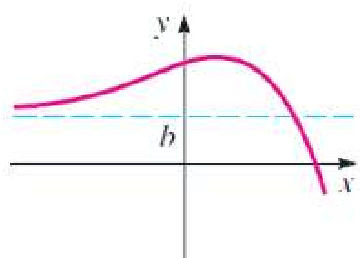


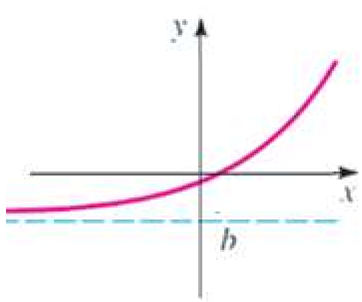
ii) Horizontal Asymptotes

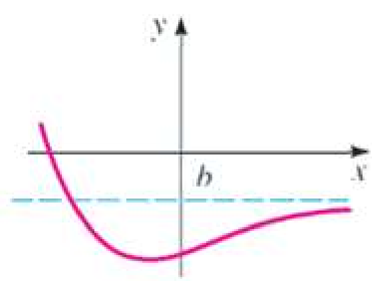
The line y = ***b*** *is a horizontal asymptote for the graph of a function* $\boldsymbol{f}(\boldsymbol{x})$ *if y approaches* ***b*** *(from above or below) as x approaches* $\pm\infty$.

The following graphs illustrate some typical ways that a curve may approach a horizontal asymptote:

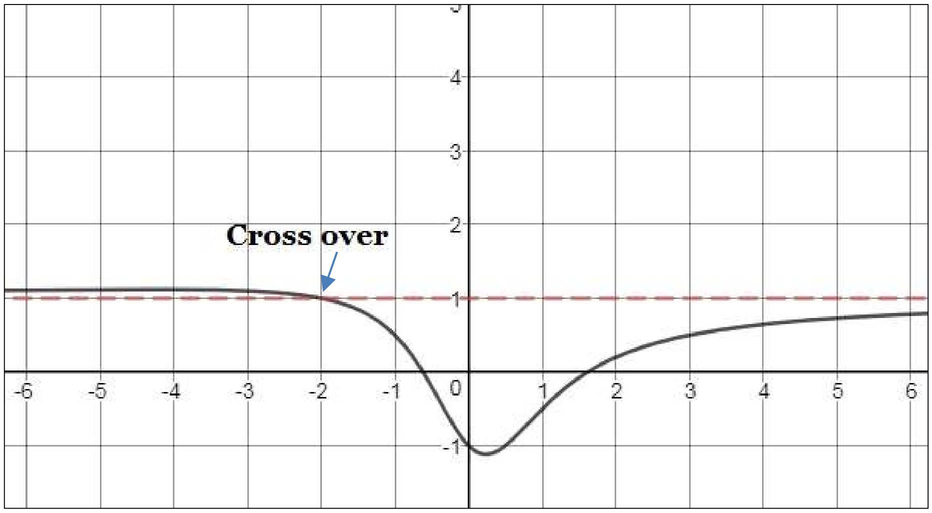








Note: A function can cross a horizontal asymptote for values of x that are "close" to the origin, it's called the **cross over**, but it can never cross a vertical asymptote.



General Rules on Finding the Horizontal and Vertical Asymptotes

Let *f* be the rational function

$$f(x)=\frac{a_n x^n+\ldots+a_1 x+a_0}{b_m x^m+\ldots+b_1 x+b_0}$$

- The vertical asymptotes of f(x) are the lines x=**a**, where **a** is the zero of denominator only.
- If n<m , then *f* has horizontal asymptote y = 0
- If n=m , then *f* has horizontal asymptote y = $\frac{a_n}{b_m}$
- If n>m , then *f* has no horizontal asymptote .

Example: Find the horizontal and vertical asymptotes for the following functions.

a. $f(x)=\frac{2x(x+1)(x-1)}{(x+2)(x-3)}$

b. $y=\frac{x^2-4x+5}{x^3-8}$

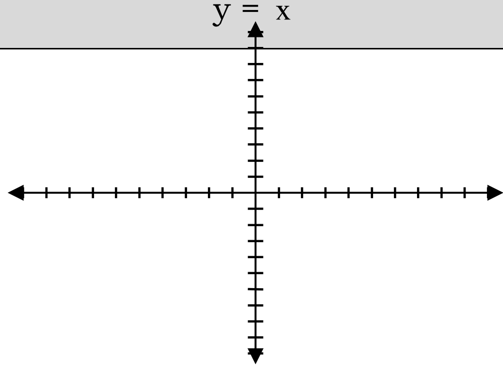
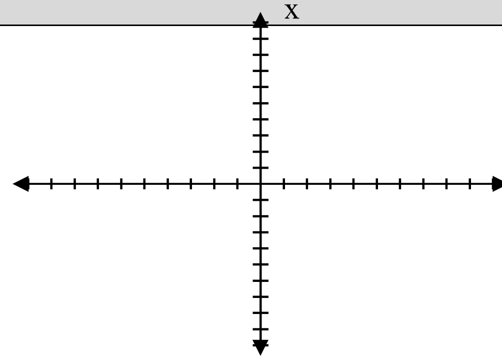
c. $f(x)=\frac{3x+1}{2-5x}$

d. $f(x)=\frac{(x-1)(x+1)(x+3)}{(x-4)(x-1)(2x+5)}$

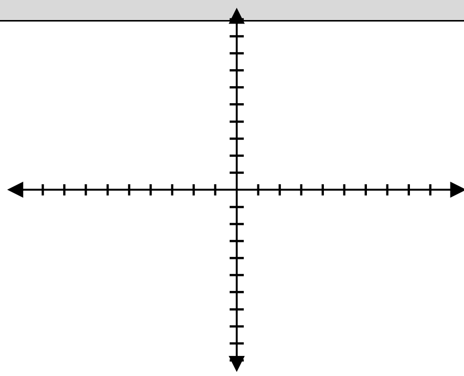
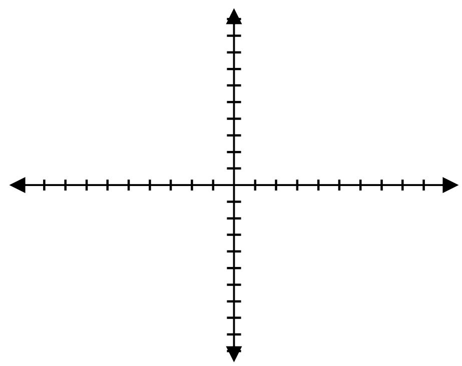
Reciprocal of Linear Functions

1. Use a graphing calculator to compare each of the following functions. Include a sketch of each.

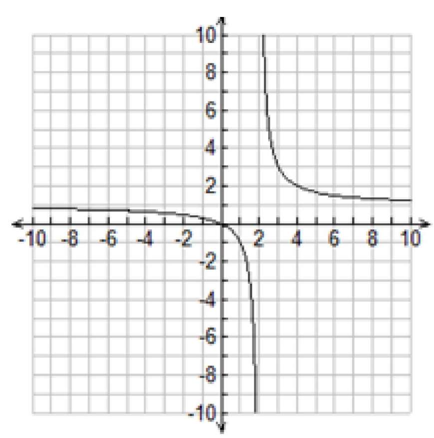
a)

$y = x$	$y = \frac{1}{x}$
	
Comparison	x-int: V.A: Interval that f >0: Interval that f <0:

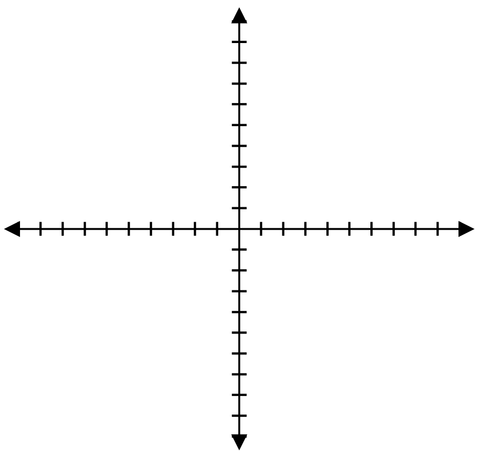
b)

$y = x+3$	$y = \frac{1}{x+3}$
	
Comparison	x-int: V.A: Interval that f >0: Interval that f <0:

2.Determine the equation of the following graph:



3. Determine the following information and sketch graph.

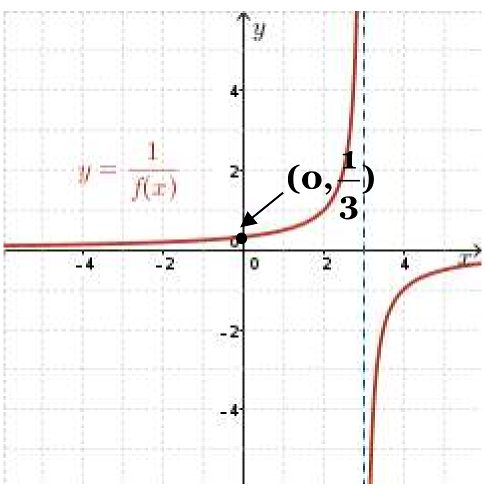


Equation	$y = \dfrac{1}{-2(x-2)}$		
Domain			
Range			
x-int		y-int	
H. Asymptote		V. Asymptote	

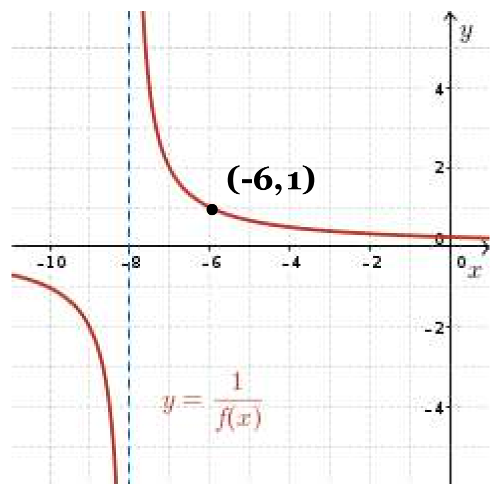
As $x \to$	**$f(x) \to$**
2^+	
2^-	
$+\infty$	
$-\infty$	

4. Given the graph of the reciprocal function $y = \dfrac{1}{f(x)}$, sketch the graph the function y=f(x). Determine an equation for each function.

a)



b)

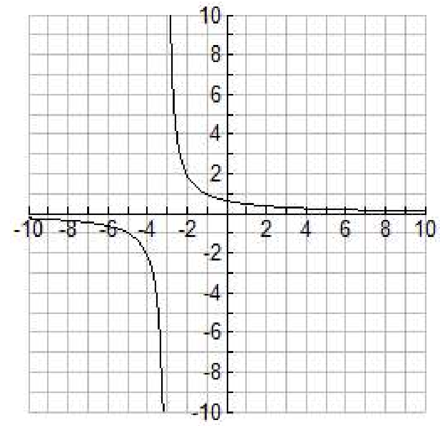


2.1-Practice:

1. Find the horizontal and vertical asymptotes for the following functions.

a. $f(x)=\dfrac{2x^2(x^2-1)}{(x+2)^2(x^2-4)}$ b. $f(x)=\dfrac{2(x-3)(x+2)(x+5)}{(x-1)(x+3)(x+5)}$

2. Determine the equation of the following graph:

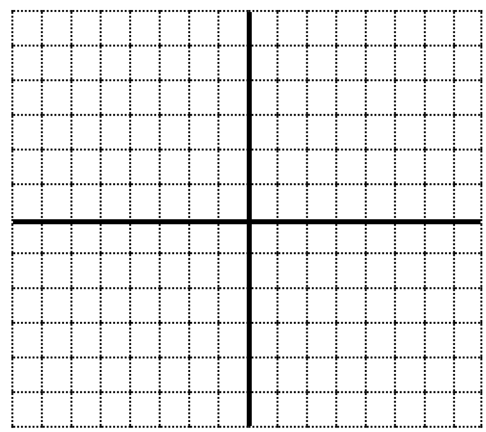


3. Graph $f(x)=-(4-x)$ and $g(x)=\dfrac{-1}{4-x}$ on the grid provided and for the g(x) identify:

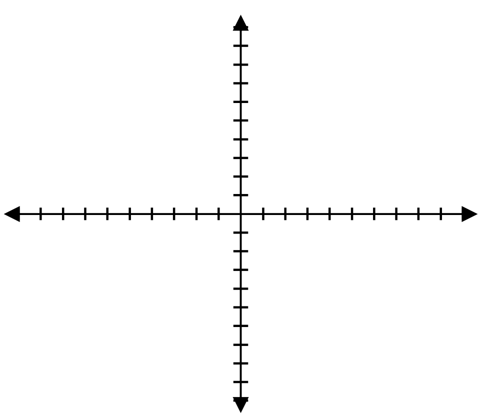
a) the domain

b) the range

c) the equation of the V.A. ____
d) the equation of the H.A. ____



4. Determine the following information and sketch graph.



Equation	$y=\dfrac{1}{2x-5}$		
Domain			
Range			
x-int		y-int	
H. Asymptote		V. Asymptote	

As $x\rightarrow$	$f(x)\rightarrow$
$\frac{5}{2}^{+}$	
$\frac{5}{2}^{-}$	
$+\infty$	
$-\infty$	

Warm Up

1. Which of the following are vertical asymptotes of $f(x)=\dfrac{(ax-b)^2}{(ax+b)(ax-b)}, a,b \neq 0 \text{ and } a,b \in \square$?

A) $x=\dfrac{b}{a}$ B) $x=\dfrac{-b}{a}$ C) $x=\pm\dfrac{b}{a}$ D) $y=1$

2. Which of the following is true regarding the function $f(x)=\dfrac{x+3}{x^2-5}$.

A) $f(x)$ has no vertical asymptotes
B) $f(x)$ has an x intercept at $x = 3$
C) As $x \to \infty, f(x) \to 0$ from above
D) $f(x)$ has a horizontal asymptote at $y = 1$

3. Which of the following functions has vertical asymptotes at $x=1$ and $x=-3$ and a horizontal asymptote at $y=0$?

A) $y=\dfrac{x^2-6x+9}{x^2-2x-3}$ B) $y=\dfrac{x^2}{x^2+2x-3}$ C) $y=\dfrac{x-1}{x+3}$ D) $y=\dfrac{x-9}{x^2+2x-3}$

4. Which of the following is true about the function $f(x)=\dfrac{-2}{x-6}$ as $x \to 6^+$?

A) $f(x) \to 0$ (from above)
B) $f(x) \to -\infty$
C) $f(x) \to \infty$
D) $f(x) \to 0$ (from below)

5. Write the equation of a rational function, in the form $f(x)=\dfrac{g(x)}{h(x)}$, with vertical asymptotes at $x=-\dfrac{3}{4}$ and $x=-5$, $x-$intercepts at ± 2, a horizontal asymptote at $y=-4$.