L1 – 2.1 – Long Division of Polynomials and The Remainder Theorem Lesson
MHF4U

In this section you will apply the method of long division to divide a polynomial by a binomial. You will also learn to use the remainder theorem to determine the remainder of a division without dividing.

Part 1: Do You Remember Long Division (divide, multiply, subtract, repeat)?

$107 \div 4$ can be completed using long division as follows:

$$\begin{array}{r} 26 \\ 4\overline{)107} \\ \underline{8} \\ 27 \\ \underline{24} \\ R = 3 \end{array}$$

4 does not go in to 1, so we start by determining how many times 4 goes in to 10. It goes in 2 times. So we put a 2 in the quotient above the 10. This is the division step.

Then, multiply the 2 by 4 (the divisor) and put the product below the 10 in the dividend. This is the multiply step.

Now, subtract 8 from the 10 in the dividend. Then bring down the next digit in the dividend and put it beside the difference you calculated. This is the subtract step.

You then repeat these steps until there are no more digits in the dividend to bring down.

Every division statement that involves numbers can be rewritten using multiplication and addition.

We can express the results of our example in two different ways:

$107 = (4)(26) + 3$ OR $\frac{107}{4} = 26 + \frac{3}{4}$

=26.75

=26 + 0.75
=26.75

Example 1: Use long division to calculate $753 \div 22$

$$\begin{array}{r} 34 \\ 22\overline{)753} \\ \underline{66} \\ 93 \\ \underline{88} \\ R = 5 \end{array}$$

$753 = (22)(34) + 5$

OR

divide both sides by 22

$\frac{753}{22} = 34 + \frac{5}{22}$

34.227

$34 + 0.227$
$= 34.227$

Part 2: Using Long Division to Divide a Polynomial by a Binomial

The quotient of $(3x^3 - 5x^2 - 7x - 1) \div (x - 3)$ can be found using long division as well…

$$
\begin{array}{r}
3x^2 + 4x + 5 \\
x - 3 \overline{)\, 3x^3 - 5x^2 - 7x - 1} \\
\underline{3x^3 - 9x^2} \qquad\qquad \\
4x^2 - 7x \qquad \\
\underline{4x^2 - 12x} \qquad \\
5x - 1 \\
\underline{5x - 15} \\
R = 14 \qquad
\end{array}
$$

Focus only on the first terms of the dividend and the divisor. Find the quotient of these terms.

Since $3x^3 \div x = 3x^2$, this becomes the first term of the quotient. Place $3x^2$ above the term of the dividend with the same degree.

Multiply $3x^2$ by the divisor, and write the answer below the dividend. Make sure to line up 'like terms'. $3x^2(x-3) = 3x^3 - 9x^2$. Subtract this product from the dividend and then bring down the next term in the dividend.

Now, once again, find the quotient of the first terms of the divisor and the new expression you have in the dividend. Since $4x^2 \div x = 4x$, this becomes the next term in the quotient.

Multiply $4x$ by the divisor, and write the answer below the last line in the dividend. Make sure to line up 'like terms'. $4x(x-3) = 4x^2 - 12x$. Subtract this product from the dividend and then bring down the next term.

Now, find the quotient of the first terms of the divisor and the new expression in the dividend. Since $5x \div x = 5$, this becomes the next term in the quotient. Multiply $5(x-3) = 5x - 15$. Subtract this product from the dividend.

The process is stopped once the degree of the remainder is less than the degree of the divisor. The divisor is degree 1 and the remainder is now degree 0, so we stop.

The result in quotient form is:

$$\frac{(3x^3 - 5x^2 - 7x - 1)}{(x-3)} = 3x^2 + 4x + 5 + \frac{14}{x-3}$$

The expression that can be used to check the division is:

$$(3x^3 - 5x^2 - 7x - 1) = (x-3)(3x^2 + 4x + 5) + 14$$

Note: you could check this answer by FOILing the product and collecting like terms.

The result of the division of $P(x)$ by a binomial of the form $x - b$ is:

$$\frac{P(x)}{x-b} = Q(x) + \frac{R}{x-b}$$

Where R is the remainder. The statement that can be used to check the division is:

$$P(x) = (x-b)Q(x) + R$$

Example 2: Find the following quotients using long division. Express the result in quotient form. Also, write the statement that can be used to check the division (then check it!).

a) $x^2 + 5x + 7$ divided by $x + 2$

$$\begin{array}{r|l} & x + 3 \\ \hline x+2 & x^2 + 5x + 7 \\ & \underline{x^2 + 2x} \\ & \quad 3x + 7 \\ & \quad \underline{3x + 6} \\ & \quad R = 1 \end{array}$$

The result in quotient form is:

$$\frac{x^2 + 5x + 7}{x + 2} = x + 3 + \frac{1}{x + 2}$$

The expression that can be used to check the division is:

$$x^2 + 5x + 7 = (x + 2)(x + 3) + 1$$

b) $2x^3 - 3x^2 + 8x - 12$ divided by $x - 1$

$$\begin{array}{r|l} & 2x^2 - x + 7 \\ \hline x-1 & 2x^3 - 3x^2 + 8x - 12 \\ & \underline{2x^3 - 2x^2} \\ & \quad -1x^2 + 8x \\ & \quad \underline{-1x^2 + 1x} \\ & \qquad 7x - 12 \\ & \qquad \underline{7x - 7} \\ & \qquad R = -5 \end{array}$$

The result in quotient form is:

$$\frac{2x^3 - 3x^2 + 8x - 12}{x - 1} = 2x^2 - x + 7 + \frac{-5}{x - 1}$$

The expression that can be used to check the division is:

$$2x^3 - 3x^2 + 8x - 12 = (x - 1)(2x^2 - x + 7) - 5$$

c) $4x^3 + 9x - 12$ divided by $2x + 1$

$$\begin{array}{r|l} & 2x^2 - x + 5 \\ \hline 2x+1 & 4x^3 + 0x^2 + 9x - 12 \\ & \underline{4x^3 + 2x^2} \\ & \quad -2x^2 + 9x \\ & \quad \underline{-2x^2 - 1x} \\ & \qquad 10x - 12 \\ & \qquad \underline{10x + 5} \\ & \qquad R = -17 \end{array}$$

The result in quotient form is:

$$\frac{4x^3 + 9x - 12}{2x + 1} = 2x^2 - x + 5 + \frac{-17}{2x + 1}$$

The expression that can be used to check the division is:

$$4x^3 + 9x - 12 = (2x + 1)(2x^2 - x + 5) - 17$$

Example 3: The volume, in cubic cm, of a rectangular box is given by $V(x) = x^3 + 7x^2 + 14x + 8$. Determine expressions for possible dimensions of the box if the height is given by $x + 2$.

$$
\begin{array}{rl}
 & x^2 + 5x + 4 \\
x+2 & \overline{)\, x^3 + 7x^2 + 14x + 8} \\
 & \underline{x^3 + 2x^2} \\
 & \quad 5x^2 + 14x \\
 & \quad \underline{5x^2 + 10x} \\
 & \qquad 4x + 8 \\
 & \qquad \underline{4x + 8} \\
 & \qquad R = 0
\end{array}
$$

Dividing the volume by the height will give an expression for the area of the base of the box.

$$\underbrace{x^3 + 7x^2 + 14x + 8}_{\text{Volume}} = \underbrace{(x+2)}_{\text{height}}\underbrace{(x^2+5x+4)}_{\text{area of base}}$$

$$= (x+2)(x+4)(x+1)$$

Factor the area of the base to get possible dimensions for the length and width of the box.

Expressions for the possible dimensions of the box are $x + 1$, $x + 2$, and $x + 4$.

Part 3: Remainder Theorem

> When a polynomial function $P(x)$ is divided by $x - b$, the remainder is $P(b)$; and when it is divided by $ax - b$, the remainder is $P\left(\frac{b}{a}\right)$, where a and b are integers, and $a \neq 0$.

Example 3: Apply the remainder theorem

a) Use the remainder theorem to determine the remainder when $P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $x + 1$

Since $x + 1$ is $x - (-1)$, the remainder is $P(-1)$.

$P(-1) = 2(-1)^3 + (-1)^2 - 3(-1) - 6$

$\quad = -2 + 1 + 3 - 6$

$\quad = -4$

Therefore, the remainder is -4

b) Verify your answer using long division

$$
\begin{array}{r|l}
 & 2x^2 - x - 2 \\
\hline
x+1 & 2x^3 + x^2 - 3x - 6 \\
 & \underline{2x^3 + 2x^2} \\
 & \quad -1x^2 - 3x \\
 & \quad \underline{-1x^2 - 1x} \\
 & \qquad -2x - 6 \\
 & \qquad \underline{-2x - 2} \\
 & \qquad R = -4
\end{array}
$$

Example 4: Use the remainder theorem to determine the remainder when $P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $2x - 3$

The remainder is $P\left(\frac{3}{2}\right)$

$$P\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) - 6$$

$$= 2\left(\frac{27}{8}\right) + \left(\frac{9}{4}\right) - 3\left(\frac{3}{2}\right) - 6$$

$$= \frac{54}{8} + \frac{9}{4} - \frac{9}{2} - 6$$

$$= \frac{27}{4} + \frac{9}{4} - \frac{18}{4} - \frac{24}{4}$$

$$= -\frac{3}{2}$$

Therefore, the remainder is $-\frac{3}{2}$

Example 5: Determine the value of k such that when $3x^4 + kx^3 - 7x - 10$ is divided by $x - 2$, the remainder is 8.

The remainder is $P(2)$. Solve for k when $P(2)$ is set to equal 8.

$P(2) = 3(2)^4 + k(2)^3 - 7(2) - 10$

$8 = 3(2)^4 + k(2)^3 - 7(2) - 10$

$8 = 3(16) + 8k - 14 - 10$

$8 = 48 + 8k - 24$

$8 = 24 + 8k$

$-16 = 8k$

$-2 = k$

Therefore, the value of k is -2.