Part 1: Review of Average Rates of Change

An average rate of change corresponds to the slope of a **SECANT** between two points on a curve.

An instantaneous rate of change corresponds to the slope of a **TANGENT** a one specific point on a curve.
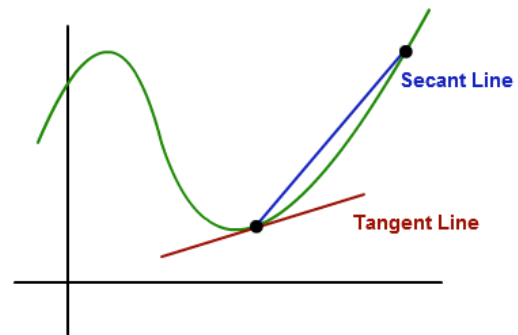
Example 1: The height, h, in meters, after t seconds is modelled by the equation $h(t) = -4.9t^2 + 14t + 1$. Determine the average rate of change of the height of the ball for each time interval.
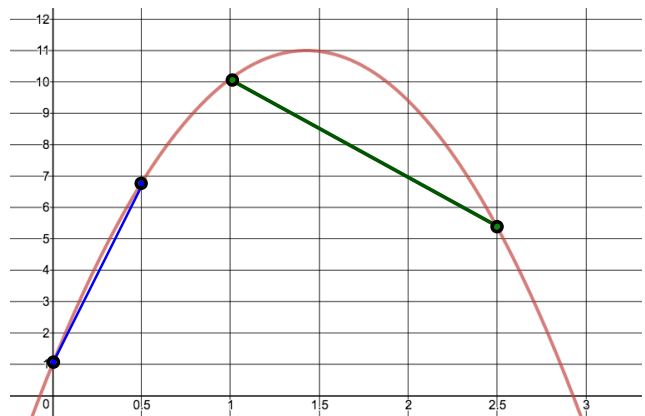
a) $[0,0.5]$

$h(0) = -4.9(0)^2 + 14(0) + 1$

$h(0) = 1$

$h(0.5) = -4.9(0.5)^2 + 14(0.5) + 1$

$h(0.5) = 6.775$

$m = \frac{\Delta h}{\Delta t} = \frac{h(0.5)-h(0)}{0.5-0} = \frac{6.775-1}{0.5} = 11.55$ m/s

b) $[1,2.5]$

$h(1) = -4.9(1)^2 + 14(1) + 1$

$h(1) = 10.1$

$h(2.5) = -4.9(2.5)^2 + 14(2.5) + 1$

$h(2.5) = 5.375$

$m = \frac{\Delta h}{\Delta t} = \frac{h(2.5)-h(1)}{2.5-1} = \frac{5.375-10.1}{1.5} = -3.15$ m/s

Part 2: Find Instantaneous Rate of Change from a Graph or Table

Instantaneous Rate of Change: The exact rate of change of a function at one specific value of the independent variable. Can be estimated using average rates of change for small intervals of the independent variable.

Note: the best estimate for the instantaneous rate of change occurs when the interval used to calculate the average rate of change is made as small as possible.

Example 2: Find the instantaneous rate of change of the height of the ball at 2 seconds. The equation for the height of the ball is $h(t) = -4.9t^2 + 14t + 1$.

Method 1: Draw Tangent Line

The slope of a tangent at a point on a graph is equivalent to the instantaneous rate of change of a function at this point.

$$\left.\frac{dh}{dt}\right|_{t=2} \approx \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-4}{3.5-3} = -6 \text{ m/s}$$
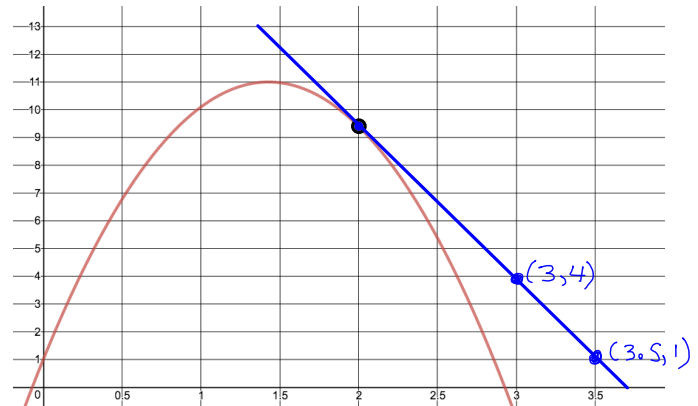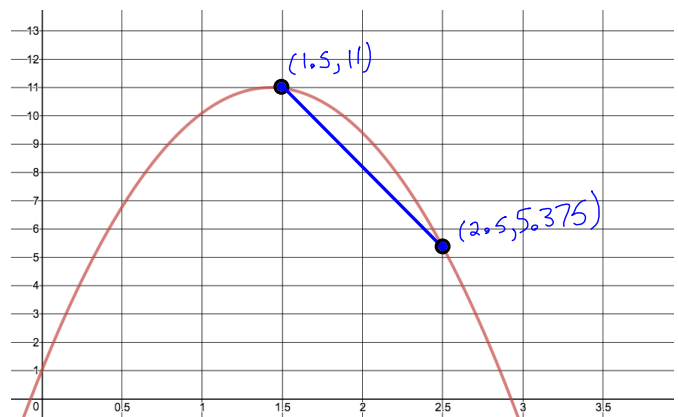

Method 2: Choose a Surrounding Interval

Choose an interval that has the instantaneous rate of change location as its midpoint if possible.

For this example, choose the interval $[1.5, 2.5]$. Use the equation of the function to determine the corresponding y values.

$$\left.\frac{dh}{dt}\right|_{t=2} \approx \frac{h(2.5) - h(1.5)}{2.5 - 1.5} = \frac{5.375 - 11}{1} = -5.625 \text{ m/s}$$

Method 3: Averaging a Preceding and Following Interval

Calculate the average rate of change on either side of $t = 2$ and then average them.

For this example, find the average rate of change for the intervals [1.5, 2] and [2, 2.5] and then average them.

Average Rate for interval [1.5, 2]
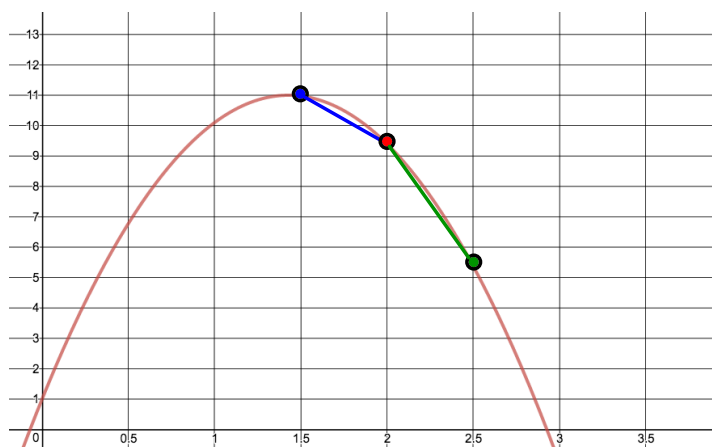
$$m = \frac{h(2)-h(1.5)}{2-1.5}$$

$$= \frac{9.4-11}{0.5}$$

$$= -3.2 \text{ m/s}$$

Average Rate for interval [2, 2.5]

$$m = \frac{h(2.5)-h(2)}{2.5-2}$$

$$= \frac{5.375-9.4}{0.5}$$

$$= -8.05 \text{ m/s}$$

$$\left.\frac{dh}{dt}\right|_{t=2} \approx \frac{-3.2+(-8.05)}{2} = -5.625 \text{ m/s}$$

Example 3: Consider the data from the Bank of Canada for the beginning of November in 2007. Estimate the instantaneous rate of change of the value of the Canadian Dollar on November 7th.

Day	1	2	5	7	8	13	14
Value of Canadian $ in US Cents	94.99	93.59	93.39	91.70	93.06	96.43	96.07

Method 1: Choose a Surrounding Interval

Use the interval [5, 8]

$$\left.\frac{dv}{dt}\right|_{t=7} \approx \frac{V(8)-V(5)}{8-5} = \frac{93.06-93.39}{3} = -0.11 \text{ cents/day}$$

Method 2: Averaging a Preceding and Following Interval (more accurate)

Average Rate for interval [5, 7]

$$m = \frac{V(7)-V(5)}{7-5}$$

$$= \frac{91.70-93.39}{2}$$

$$= -0.845 \text{ cents/day}$$

Average Rate for interval [7, 8]

$$m = \frac{V(8)-V(7)}{8-7}$$

$$= \frac{93.06-91.70}{1}$$

$$= 1.36 \text{ cents/day}$$

$$\frac{dV}{dt}\bigg|_{t=2} \approx \frac{-0.845+1.36}{2} = 0.258 \text{ cents/day}$$

Part 3: Find Instantaneous Rate of Change from an Equation

Squeezing a preceding or following interval is a technique for estimating instantaneous rates of change. By finding average rates of change that get progressively smaller, you can make more accurate estimates at to what the rate of change is **APPROACHING** at an exact point.

Example 4: Use the chart below to estimate the slope of the tangent to the curve $f(x) = 2x^3 - x$ at $x = 2$.

Interval	Δy	Δx	Slope of secant $= \frac{\Delta y}{\Delta x}$
$2 \le x \le 3$	$= f(3) - f(2)$ $= [2(3)^3 - 3] - [2(2)^3 - 2]$ $= 51 - 14$ $= 37$	$= 3 - 2$ $= 1$	$= \frac{37}{1}$ $= 37$
$2 \le x \le 2.5$	$= f(2.5) - f(2)$ $= 28.75 - 14$ $= 14.75$	$= 2.5 - 2$ $= 0.5$	$= \frac{14.75}{0.5}$ $= 29.5$
$2 \le x \le 2.1$	$= f(2.1) - f(2)$ $= 16.422 - 14$ $= 2.422$	$= 2.1 - 2$ $= 0.1$	$= \frac{2.422}{0.1}$ $= 24.22$
$2 \le x \le 2.01$	$= f(2.01) - f(2)$ $= 14.231202 - 14$ $= 0.231202$	$= 2.01 - 2$ $= 0.01$	$= \frac{0.231202}{0.01}$ $= 23.1202$
$2 \le x \le 2.001$	$= f(2.001) - f(2)$ $= 14.023012 - 14$ $= 0.023012$	$= 2.001 - 2$ $= 0.001$	$= \frac{0.023012}{0.001}$ $= 23.012$

By looking at the **limit** of the secant slopes as we **approach** $x = 2$, we can estimate the slope of the tangent to be:

$$\left.\frac{dy}{dx}\right|_{x=2} \approx 23$$