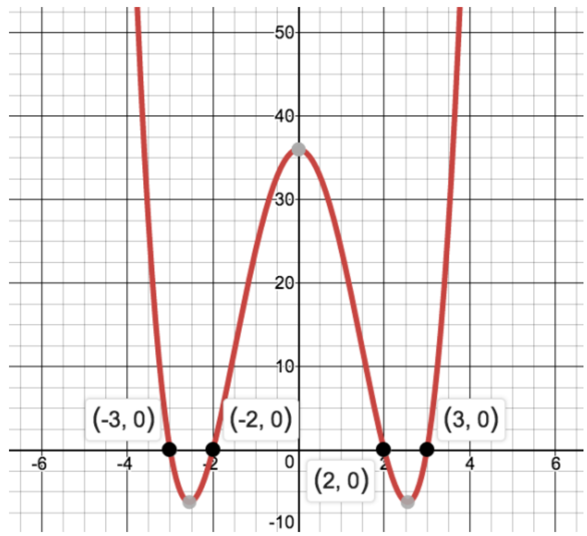
L4 – 2.3 – Solving Polynomial Equations Lesson
MHF4U

In this section, you will learn methods of solving polynomial equations of degree higher than two by factoring (using the factor theorem). You will also identify the connection between the roots of polynomial equations, the x-intercepts of the graph of a polynomial function, and the zeros of the function.

Part 1: Investigation

a) Use technology to graph the function $f(x) = x^4 - 13x^2 + 36$



b) Determine the x-intercepts from the graph

The x-intercepts are:

(-3, 0), (-2, 0), (2, 0), and (3, 0)

c) Factor $f(x)$. Then, use the factors to determine the zeros (roots) of $f(x)$.

$f(x) = x^4 - 13x^2 + 36$ ← can factor like a quadratic

$0 = (x^2)^2 - 13(x^2) + 36$ P: 36 S: -13 -9 and -4

$0 = (x^2 - 9)(x^2 - 4)$ Both are DOS

$0 = (x-3)(x+3)(x-2)(x+2)$

$x - 3 = 0$ $x+3=0$ $x-2=0$ $x+2=0$

$x_1 = 3$ $x_2 = -3$ $x_3 = 2$ $x_4 = -2$

Remember: The zeros of the function are the values of x that make $f(x) = 0$. If the polynomial equation is factorable, then the values of the zeros (roots) can be determined algebraically by solving each linear or quadratic factor.

d) How are the x-intercepts from the graph related to the roots (zeros) of the equation?

The zeros of the equation ARE the x-intercepts of the graph of the function.

Example 1: State the solutions to the following polynomials that are already in factored form

a) $x(2x+3)(x-5)=0$

$\boxed{x_1=0}$

$2x+3=0$

$\boxed{x_2=-3/2}$

$x-5=0$

$\boxed{x_3=5}$

b) $(2x^2-3)(3x^2+1)$

$2x^2-3=0$

$x^2=3/2$

$x=\pm\sqrt{3/2}$

$\boxed{x_1=\sqrt{3/2}}$ $\boxed{x_2=-\sqrt{3/2}}$

$3x^2+1=0$

$x^2=-1/3$

$x=\pm\sqrt{-1/3}$

Not a REAL solution

Methods of factoring:

- Long division and synthetic division
- Factor by grouping
- Difference of squares $a^2-b^2=(a-b)(a+b)$
- Common Factoring
- Trinomial factoring (sum and product)
- Sum and difference of cubes $a^3+b^3=(a+b)(a^2-ab+b^2)$
  $a^3-b^3=(a-b)(a^2+ab+b^2)$

Example 2: Solve each polynomial equation by factoring

a) $x^3-x^2-2x=0$

$x^3-x^2-2x=0$

common factor

$x(x^2-x-2)=0$

P: -2
S: -1

-2 and 1

$x(x-2)(x+1)=0$

$\boxed{x_1=0}$

$x-2=0$

$\boxed{x_2=2}$

$x+1=0$

$\boxed{x_3=-1}$

Solution(s):

(0, 0), (2, 0), and (-1, 0)

b) $3x^3+x^2-12x-4=0$

$x^2(3x+1)-4(3x+1)=0$

$(3x+1)(x^2-4)=0$

$(3x+1)(x-2)(x+2)=0$

$x_1=-\frac{1}{3}$ $\quad x_2=2$ $\quad x_3=-2$

Solution(s):

$\left(-\frac{1}{3},0\right)$, (2, 0), and (-2, 0)

Example 3:

a) Use the factor theorem to solve $f(x) = 2x^3 + 3x^2 - 11x - 6$

Possible values of b are: $\pm1, \pm2, \pm3, \pm6$

Possible values for are: $\pm1, \pm2$

Possible values for $\frac{b}{a} = \pm1, \pm\frac{1}{2}, \pm2, \pm3, \pm\frac{3}{2}, \pm6$

$f(2) = 2(2)^3 + 3(2)^2 - 11(2) - 6 = 0, \therefore x - 2$ is a factor

$$\begin{array}{c|cccc}
2 & 2 & 3 & -11 & -6 \\
 & \downarrow & 4 & 14 & 6 \\
\hline
x & 2 & 7 & 3 & 0 \\
 & x^2 & x & \# & R
\end{array} \quad +$$

$f(x) = (x-2)(2x^2+7x+3)$ p: 6 s: 7 6 and 1

$0 = (x-2)[(2x^2+6x)+(1x+3)]$

$0 = (x-2)[2x(x+3)+1(x+3)]$

$0 = (x-2)(x+3)(2x+1)$

$x-2=0$ $\boxed{x_1=2}$

$x+3=0$ $\boxed{x_2=-3}$

$2x+1=0$ $\boxed{x_3=-1/2}$

Solution(s):

(2, 0), (-3, 0), and $\left(-\frac{1}{2}, 0\right)$

b) What do your answers to part a) represent?

The values of $2, -\frac{1}{2}$, and -3 are the roots of the equation $2x^3 + 3x^2 - 11x - 6 = 0$ which means they are the x-intercepts of the graph of the function $f(x) = 2x^3 + 3x^2 - 11x - 6$.

Example 4: Find the zeros of the polynomial function $f(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$

Possible values for b are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$

$f(-1) = (-1)^4 - 2(-1)^3 - 7(-1)^2 + 8(-1) + 12 = 0, \therefore x + 1$ is a factor

$$
\begin{array}{r|rrrrr}
-1 & 1 & -2 & -7 & 8 & 12 \\
 & \downarrow & -1 & 3 & 4 & -12 \\
\hline
x & 1 & -3 & -4 & 12 & 0 \\
 & x^3 & x^2 & x & \# & R
\end{array}
\quad +
$$

$$f(x) = (x+1)(x^3 - 3x^2 - 4x + 12)$$

$$= (x+1)\left[x^2(x-3) - 4(x-3)\right]$$

$$= (x+1)(x-3)(x^2-4)$$

$$= (x+1)(x-3)(x-2)(x+2)$$

$x+1=0$ $\boxed{x_1 = -1}$

$x-3=0$ $\boxed{x_2 = 3}$

$x-2=0$ $\boxed{x_3 = 2}$

$x+2=0$ $\boxed{x_4 = -2}$

Solution(s):

(-1, 0), (3, 0), (2, 0), and (-2, 0)

Example 5:

a) Find the roots of the polynomial function $f(x) = x^3 + x - 3x^2 - 3$

Start by rearranging in descending order of degree: $f(x) = x^3 - 3x^2 + x - 3$

$$f(x) = x^3 - 3x^2 + x - 3$$
$$0 = (x^3 - 3x^2) + (x - 3)$$
$$0 = x^2(x-3) + 1(x-3)$$
$$0 = (x-3)(x^2+1)$$

$x - 3 = 0$

$\boxed{x = 3}$

$x^2 + 1 = 0$

$x^2 = -1$

$\cancel{x = \pm\sqrt{-1}}$

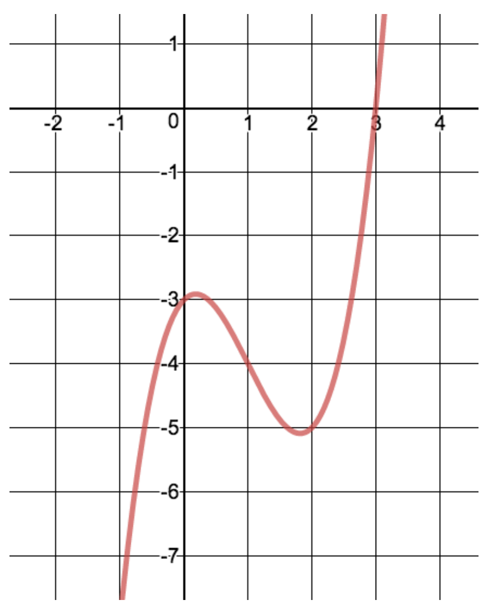
Not a real root.

Solution(s):

(3, 0)

Note: Since the square root of a negative number is not a real number, the only REAL root is $x = 3$. $x = \pm\sqrt{-1}$ is considered a NON-REAL root.

b) Use technology to look at the graph of the function $f(x)$. Comment on how x-intercept(s) of the graph are related to the REAL and NON-REAL roots of the equation.



The x-intercepts of the graph of a polynomial function correspond to only the REAL roots of the related polynomial equation. There are no x-intercepts on the graph that correspond to the NON-REAL roots of the equation.

Example 6: Find all real roots for each polynomial equation

a) $f(x) = 2x^3 - 3x^2 - x - 2$

Possible values of b are: $\pm 1, \pm 2$

Possible values for are: $\pm 1, \pm 2$

Possible values for $\frac{b}{a} = \pm 1, \pm \frac{1}{2}, \pm 2$

$f(2) = 2(2)^3 - 3(2)^2 - (2) - 2 = 0, \therefore x - 2$ is a factor of $f(x)$

2	2	-3	-1	-2	
	↓	4	2	2	+
x	2	1	1	0	
	x^2	x	#	R	

$f(x) = 2x^3 - 3x^2 - x - 2$

$0 = (x-2)(2x^2 + x + 1)$

$x - 2 = 0$

$x = 2$

no real roots

P: 2 ? NOTHING!
S: 1

check discriminant to see if there are other real roots

$b^2 - 4ac = (1)^2 - 4(2)(1)$

$= -7$

$\therefore$ no real roots

Solution(s):

(2, 0)

b) $g(x) = 8x^3 + 125$

Hint: *This is a difference of cubes* $\rightarrow a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$g(x) = (2x)^3 + (5)^3$

$0 = (2x+5)(4x^2 - 10x + 25)$

$2x + 5 = 0$

$x = -5/2$

NO REAL ROOTS

P: 100 NOTHING!
S: -10

Check discriminant to see if there are other real roots.

$b^2 - 4ac = (-10)^2 - 4(4)(25)$

$= -300$

NO REAL ROOTS

Solution(s):

$\left(-\frac{5}{2}, 0\right)$