L3 – 1.3 – Factored Form Polynomial Functions Lesson
MHF4U

In this section, you will investigate the relationship between the factored form of a polynomial function and the x-intercepts of the corresponding graph, and you will examine the effect of repeated factor on the graph of a polynomial function.

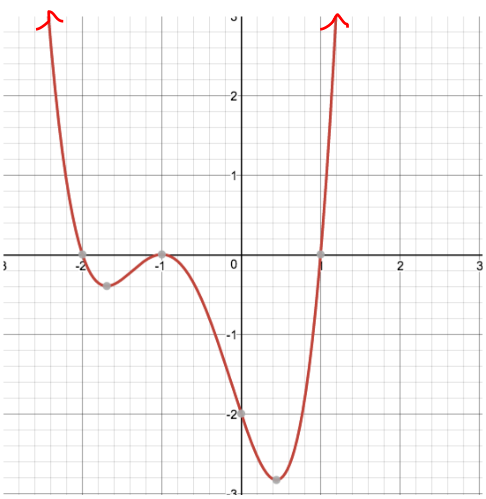
Factored Form Investigation

If we want to graph the polynomial function $f(x) = x^4 + 3x^3 + x^2 - 3x - 2$ accurately, it would be most useful to look at the factored form version of the function:

$f(x) = (x+1)^2(x+2)(x-1)$

Lets start by looking at the graph of the function and making connections to the factored form equation.

Graph of $f(x)$:



From the graph, answer the following questions…

a) What is the degree of the function?

The highest degree term is x^4, therefore the function is degree 4 (quartic)

b) What is the sign of the leading coefficient?

The leading coefficient is 1, therefore the leading coefficient is POSITIVE

c) What are the x-intercepts?

The x-intercepts are $(-2,0)$ of order 1, $(-1,0)$ of order 2, and $(1,0)$ of order 1

d) What is the y-intercept?

The y-intercept is the point $(0,-2)$

e) The x-intercepts divide the graph in to into four intervals. Write the intervals in the first row of the table. In the second row, choose a test point within the interval. In the third row, indicate whether the function is positive (above the x-axis) or negative (below the y-axis).

Interval	$(-\infty,-2)$	$(-2,-1)$	$(-1,1)$	$(1,\infty)$
Test Point	$f(-3)$ $=(-3+1)^2(-3+2)(-3-1)$ $=(-2)^2(-1)(-4)$ $=16$	$f(-1.5)$ $=(-1.5+1)^2(-1.5+2)(-1.5-1)$ $=(-0.5)^2(0.5)(-2.5)$ $=-0.3125$	$f(0)$ $=(0+1)^2(0+2)(0-1)$ $=(1)^2(2)(-1)$ $=-2$	$f(3)$ $=(3+1)^2(3+2)(3-1)$ $=(4)^2(5)(2)$ $=160$
Sign of $f(x)$	$+$	$-$	$-$	$+$

f) What happens to the sign of the of $f(x)$ near each x-intercept?

At (-2, 0) which is order 1, it changes signs
At (-1, 0) which is order 2, the sign does NOT change
At (1, 0) which is order 1, it changes signs

Conclusions from investigation:

The x-intercepts of the graph of the function correspond to the roots (zeros) of the corresponding equation. For example, the function $f(x) = (x-2)(x+1)$ has x-intercepts at **2** and **-1**. These are the roots of the equation $(x-2)(x+1) = 0$.

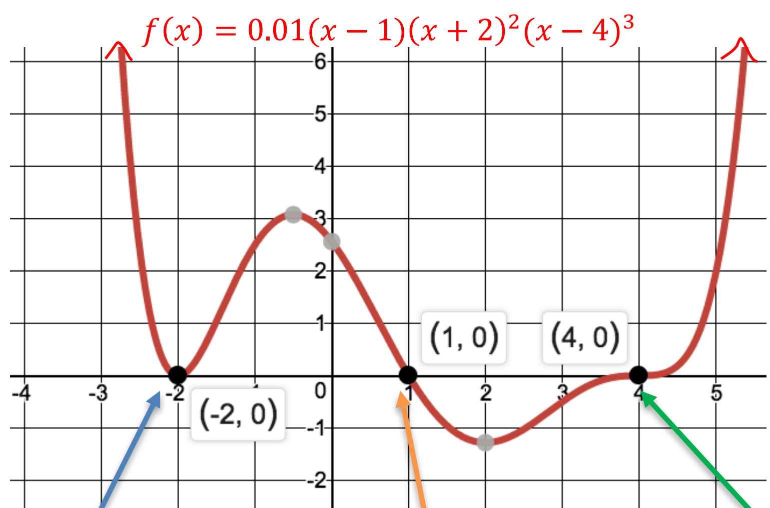
If a polynomial function has a factor $(x-a)$ that is repeated n times, then $x = a$ is a zero of **ORDER** n.

Order – the exponent to which each factor in an algebraic expression is raised.

For example, the function $f(x) = (x-3)^2(x-1)$ has a zero of order **two** at $x = 3$ and a zero of order **one** at $x = 1$.

The graph of a polynomial function changes sign at zeros of **odd** order but does not change sign at zeros of **even** order.

Shapes based on order of zero:



ORDER 2

(-2, 0) is an x-intercept of order 2. Therefore, it doesn't change sign.

"Bounces off" x-axis.

Parabolic shape.

ORDER 1

(1, 0) is an x-intercept of order 1. Therefore, it changes sign.

"Goes straight through" x-axis.

Linear Shape

ORDER 3

(4, 0) is an x-intercept of order 3. Therefore, it changes sign.

"S-shape" through x-axis.

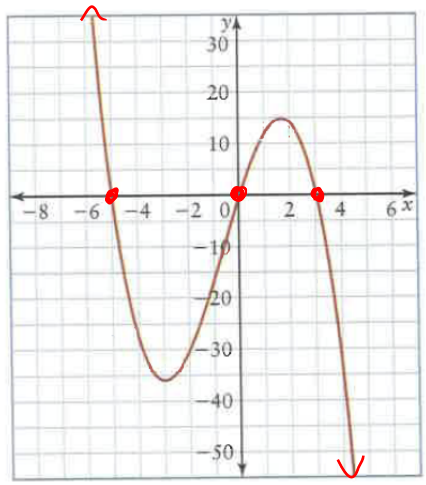
Cubic shape.

Example 1: Analyzing Graphs of Polynomial Functions

For each graph,

i) the least possible degree and the sign of the leading coefficient
ii) the x-intercepts and the factors of the function
iii) the intervals where the function is positive/negative

a)



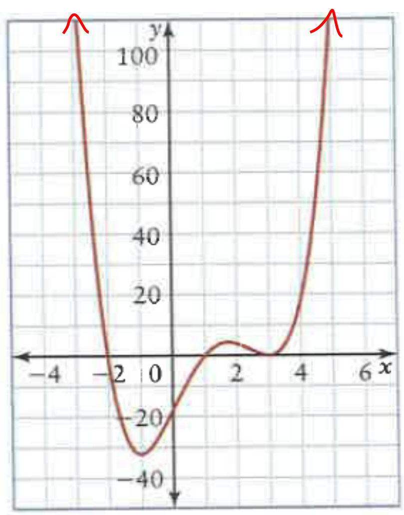
i) Three x-intercepts of order 1, so the least possible degree is 3. The graph goes from Q2 to Q4 so the leading coefficient is negative.

ii) The x-intercepts are -5, 0, and 3.
The factors are $(x+5)$, x, and $(x-3)$

iii)

Interval	$(-\infty, -5)$	$(-5, 0)$	$(0, 3)$	$(3, \infty)$
Sign of $f(x)$	+	−	+	−

b)



i) Two x-intercepts of order 1, and one x-intercept of order 2, so the least possible degree is 4. The graph goes from Q2 to Q1 so the leading coefficient is positive.

ii) The x-intercepts are -2, 1, and 3.
The factors are $(x+2)$, $(x-1)$, and $(x-3)^2$

iii)

Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, 3)$	$(3, \infty)$
Sign of $f(x)$	+	−	+	+

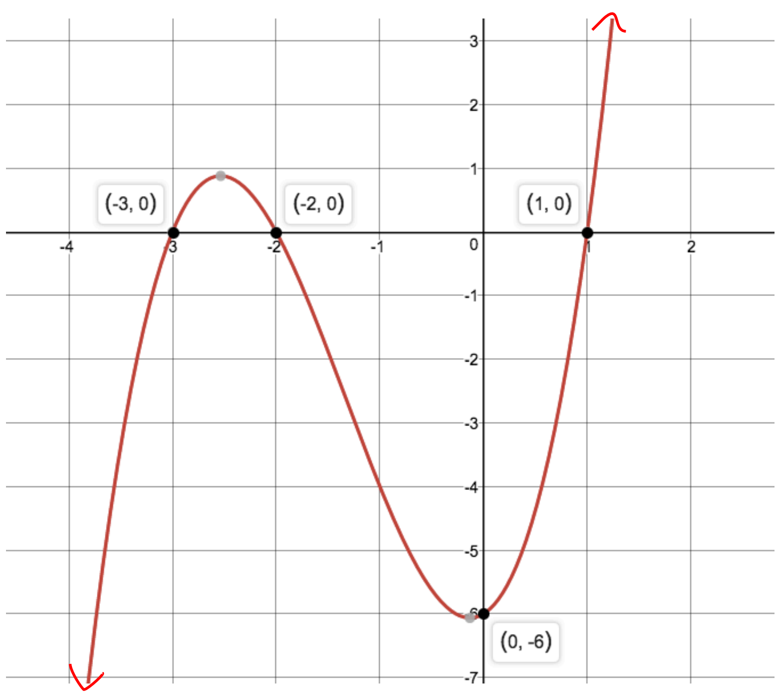
Example 2: Analyze Factored Form Equations to Sketch Graphs

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
The exponent on x when all factors of x are multiplied together OR Add the exponents on the factors that include an x.	The product of all the x coefficients	Use degree and sign of leading coefficient to determine this	Set each factor equal to zero and solve for x	Set $x = 0$ and solve for y

Sketch a graph of each polynomial function:

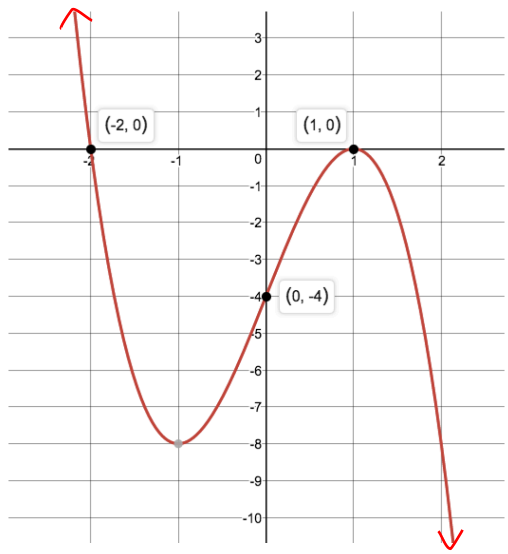
a) $f(x) = (x-1)(x+2)(x+3)$

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
The product of all factors of x is: $(x)(x)(x) = x^3$ The function is cubic. **DEGREE 3**	The product of all the x coefficients is: $(1)(1)(1) = 1$ **Leading Coefficient is 1**	Cubic with a positive leading coefficient extends from: **Q3 to Q1**	**The x-intercepts are 1, -2, and -3** (1, 0) (-2, 0) (-3, 0)	Set x equal to 0 and solve: $y = (0-1)(0+2)(0+3)$ $y = (-1)(2)(3)$ $y = -6$ **The y-intercept is at (0, -6)**



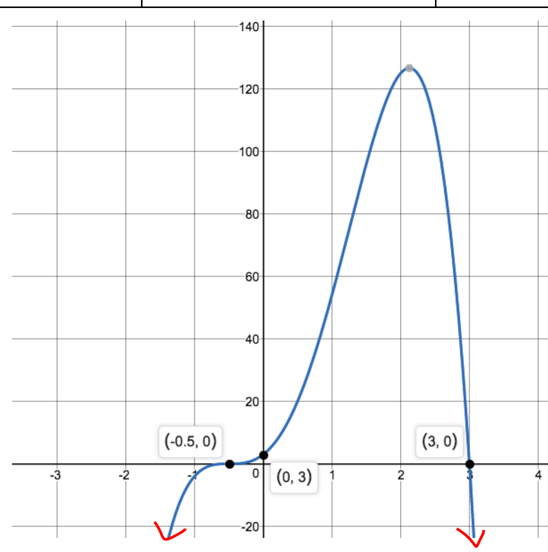
b) $g(x) = -2(x-1)^2(x+2)$

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
The product of all factors of x is: $(x^2)(x) = x^3$ The function is cubic. **DEGREE 3**	The product of all the x coefficients is: $(-2)(1)^2(1) = -2$ **Leading Coefficient is -2**	Cubic with a negative leading coefficient extends from: **Q2 to Q4**	**The x-intercepts are 1 (order 2), and -2.** (1, 0) (-2, 0)	Set x equal to 0 and solve: $y = -2(0-1)^2(0+2)$ $y = (-2)(1)(2)$ $y = -4$ **The y-intercept is at (0, -4)**



c) $h(x) = -(2x+1)^3(x-3)$

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
The product of all factors of x is: $(x^3)(x) = x^4$ The function is quartic. **DEGREE 4**	The product of all the x coefficients is: $(-1)(2)^3(1) = -8$ **Leading Coefficient is -8**	A quartic with a negative leading coefficient extends from: **Q3 to Q4**	**The x-intercepts are $-\frac{1}{2}$ (order 3), and 3.** $\left(-\frac{1}{2}, 0\right)$ $(3, 0)$	Set x equal to 0 and solve: $y = -[2(0)+1]^3[0-3]$ $y = (-1)(1)(-3)$ $y = 3$ **The y-intercept is at (0, 3)**

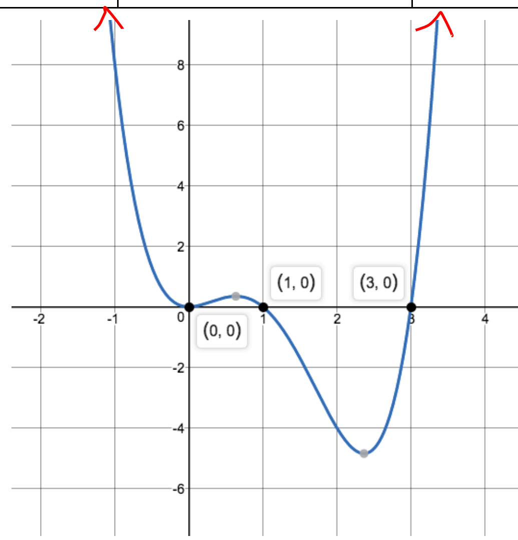


d) $j(x) = x^4 - 4x^3 + 3x^2$

Note: *must put in to factored form to find x-intercepts*

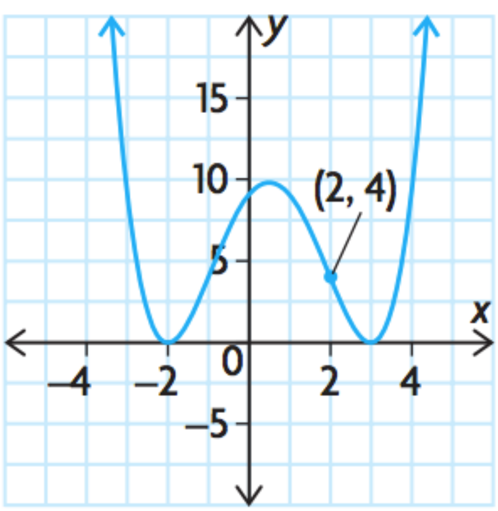
$j(x) = x^2(x^2 - 4x + 3)$
$j(x) = x^2(x - 3)(x - 1)$

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
The product of all factors of x is: $(x^2)(x)(x) = x^4$ The function is quartic. **DEGREE 4**	The product of all the x coefficients is: $(1)^2(1)(1) = 1$ **Leading Coefficient is 1**	A quartic with a positive leading coefficient extends from: **Q2 to Q1**	**The x-intercepts are 0 (order 2), 3, and 1.** $(0, 0)$ $(3, 0)$ $(1, 0)$	Set x equal to 0 and solve: $y = (0)^2(0-3)(0-1)$ $y = (0)(-3)(-1)$ $y = 0$ **The y-intercept is at (0, 0)**



<u>Example 3: Representing the Graph of a Polynomial Function with its Equation</u>

a) Write the equation of the function shown below:



The function has x-intercepts at -2 and 3. Both are of order 2.

$f(x) = k(x + 2)^2(x - 3)^2$

$4 = k(2 + 2)^2(2 - 3)^2$

$4 = k(4)^2(-1)^2$

$4 = 16k$

$k = \frac{1}{4}$

$\boldsymbol{f(x) = \frac{1}{4}(x + 2)^2(x - 3)^2}$

Steps:

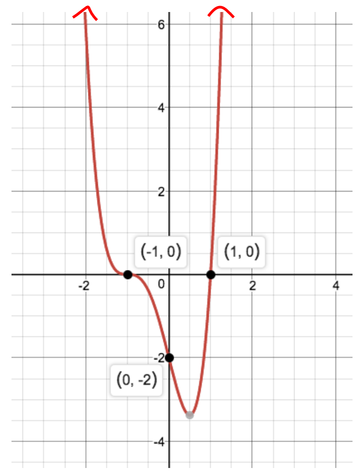
1) Write the equation of the family of polynomials using factors created from x-intercepts

2) Substitute the coordinates of another point (x, y) into the equation.

3) Solve for a

4) Write the equation in factored form

b) Find the equation of a polynomial function that is degree 4 with zeros -1 (order 3) and 1, and with a y-intercept of -2.



$f(x) = k(x+1)^3(x-1)$

$-2 = k(0+1)^3(0-1)$

$-2 = k(1)^3(-1)$

$-2 = -1k$

$k = 2$

$\boldsymbol{f(x) = 2(x+1)^3(x-1)}$