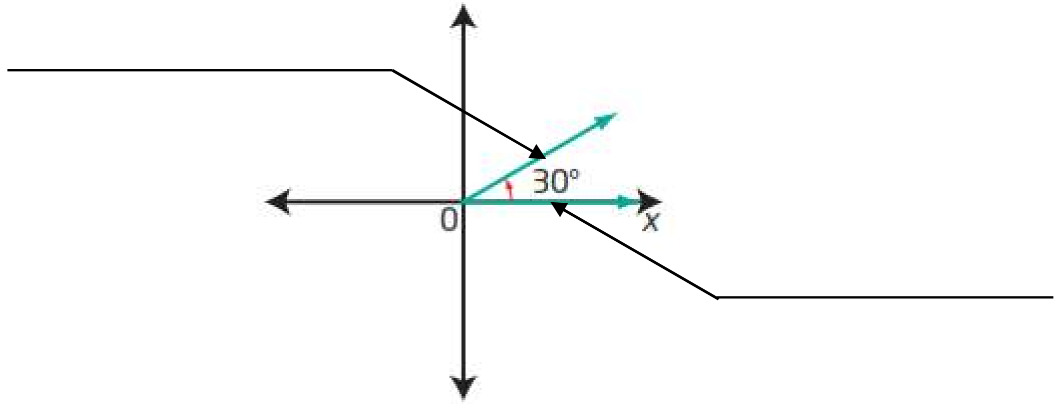
Unit 3: Trigonometric Function(part I)

Grade 11 Trigonometry -Review

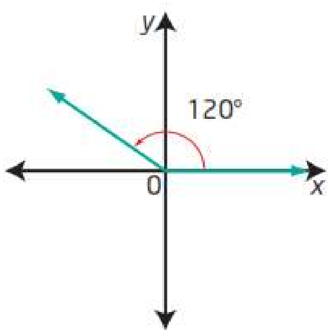
Part A: Standard Position and Co-terminal Angles

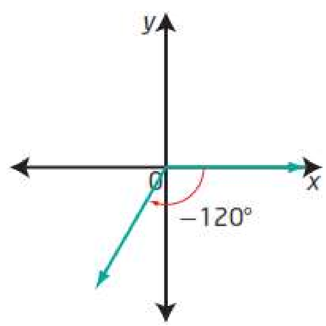
STANDARD POSITION An angle in standard position has its center at the origin and its initial arm along the positive *x*-axis.

Label the initial arm and terminal arm for the angle 30° below.



An angle can be either positive or negative. If the terminal arm of an angle opens in a counter-clockwise direction, the angle is positive. If the terminal arm opens in a clock-wise direction, the angle is negative.

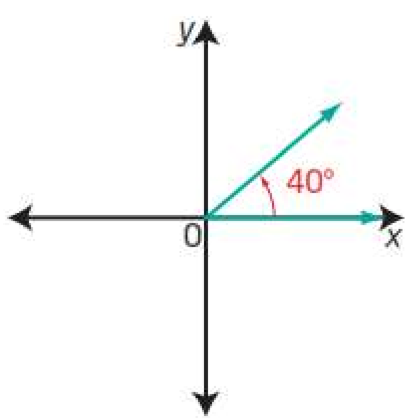




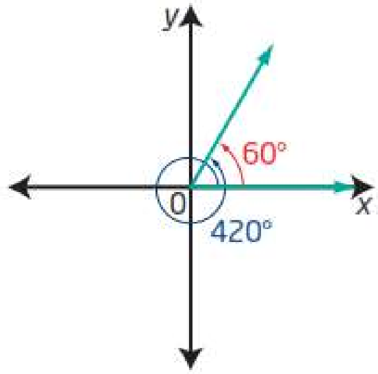
Exercises:

1. Draw the following angles in standard position. The first one has been done for you.

a) 40 °　　b) 120°　　c) -45°　　d) 430°



COTERMINAL ANGLES are angles in standard position with the same terminal arms. coterminal angles, as are 40° and -320°.



Any given angle has an infinite number of angles coterminal with it, since each time you make one full rotation from the terminal arm, you arrive back at the same terminal arm. Angles coterminal with any
angle θ can be described using the expression $\theta \pm (360°)n$, where n is a natural number. This way of expressing an answer is called the **general form**

Exercises:
2. For each angle below, find one positive coterminal angle and one negative coterminal angle.

a) 30° b) 310° c) -90°

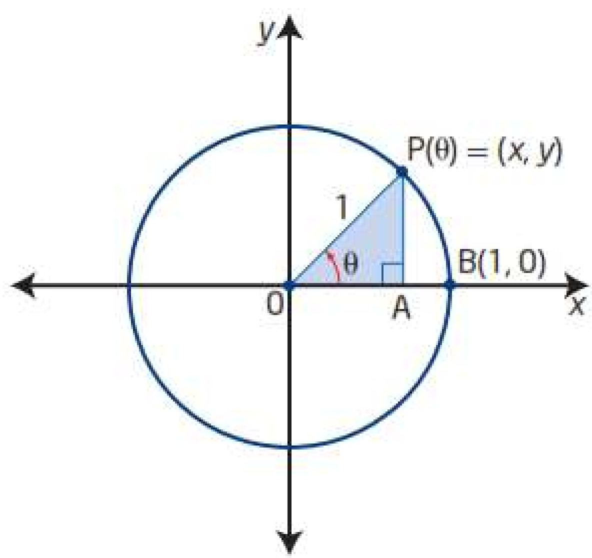
Part B: Definitions of the Primary & Reciprocal Trigonometric Ratios

The **Standard Position** of angles allows us to <u>**define**</u> trigonometric ratios for ANY angle, even angles bigger than 90°. To find the trig ratios for θ , pick a point P(x, y) on the terminal arm. Drop a <u>vertical</u> line to the x-axis to construct a right triangle.

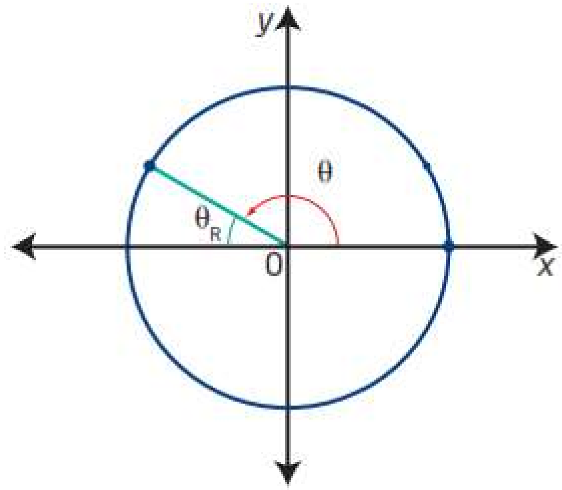
$\text{Sin}(\theta) = \frac{y}{r}$

$\cos(\theta) =$

$\tan(\theta) =$



When the terminal arm of θ lies in the 2^{nd}, 3^{rd}, or 4^{th} quadrant then θ will not be contained in the right triangle. So we *define* the sine, cosine and tangent for such angles to be **the related acute angle, θ_R ,** the acute angle formed between the terminal arm and the x-axis taking into account whether the trig ratio is positive or negative as indicated by the CAST rule (see Part D).



The sine, cosine and tangent ratios for an angle are called the PRIMARY TRIGONOMETRIC RATIOS. The RECIPROCAL TRIGONOMETRIC RATIOS are defined as follows:

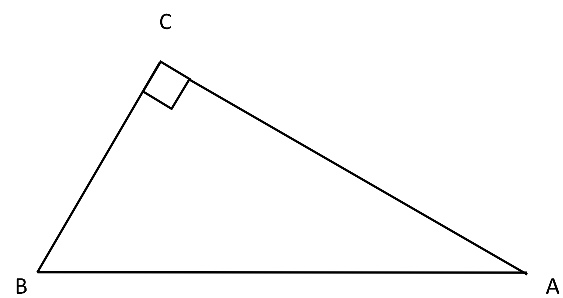
cosecant θ or csc(θ) $= \dfrac{1}{\sin(\theta)} = \dfrac{r}{y}$

secant θ or sec(θ) =

cotangent θ or cot(θ) =

Exercises:

3. In the triangle drawn at the right, ∠C = 90°, c = 13, and a = 5. State all six trig ratios for ∠B.



Part C: Special Angles

Certain angles are considered ‘special’ because finding their trig ratios is relatively easy as such angles are readily formed in ‘special’ triangles. Sketch the special triangles that allow you find the primary and reciprocal trig ratios for 45°, 30° & 60°, and 0° & 90°. For example, the special triangle for 45° is a right isosceles triangle with side lengths 1, 1, $\sqrt{2}$. Use your triangles to complete the chart below.

Special Angle	0°	30°	45°	60°	90°	180°
Sin(θ)						
cos(θ)						
tan(θ)						
sec(θ)						
csc(θ)						

Part D: The CAST Rule

For each diagram below, P is a point on the terminal arm of an angle **θ.**

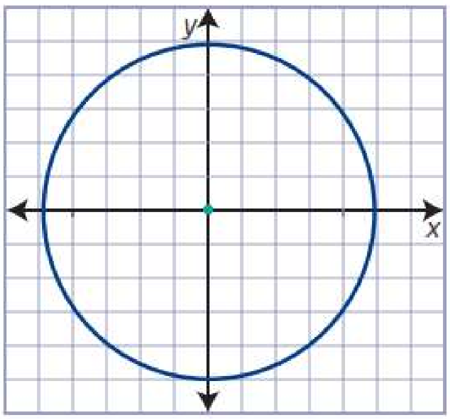
i) Construct a right triangle by dropping a perpendicular from P to the x-axis.

ii) Determine the 'lengths' of all three sides of the right triangle constructed, including whether the 'length' is positive or negative.

iii) Determine the three primary trig ratios, sine, cosine and tangent. (NOTE: some of these trig ratios WILL be negative)

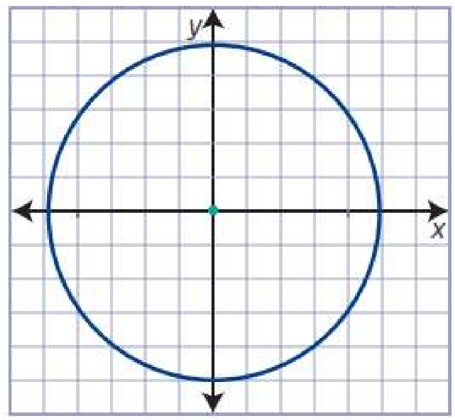
iv) Determine θ in degrees.

QUADRANT 1
P(4, 3)



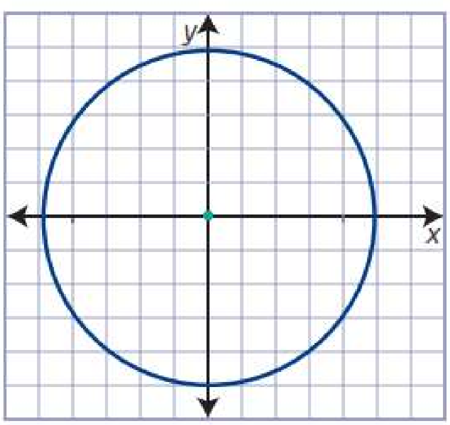
sin**(θ)** =
cos**(θ)** =
tan**(θ)** =

QUADRANT 2
P(-4, 3)



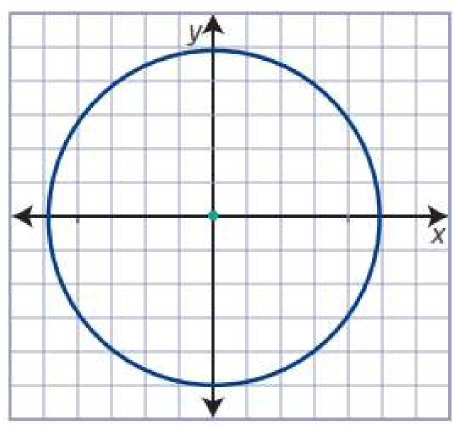
sin**(θ)** =
cos**(θ)** =
tan**(θ)** =

QUADRANT 3
P(-4, -3)



sin**(θ)** =
cos**(θ)** =
tan**(θ)** =

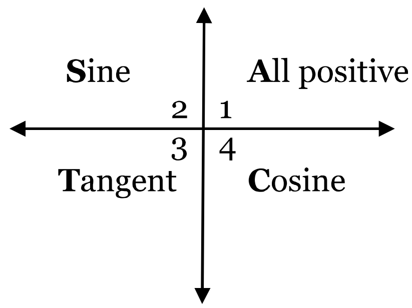
QUADRANT 4
P(4, -3)



sin**(θ)** =
cos**(θ)** =
tan**(θ)** =

CAST rule

Notice that exactly one trig ratio is positive in every quadrant except of the first quadrant in which all the ratios are positive. The CAST rule helps us remember in which quadrants a particular ratio is positive.

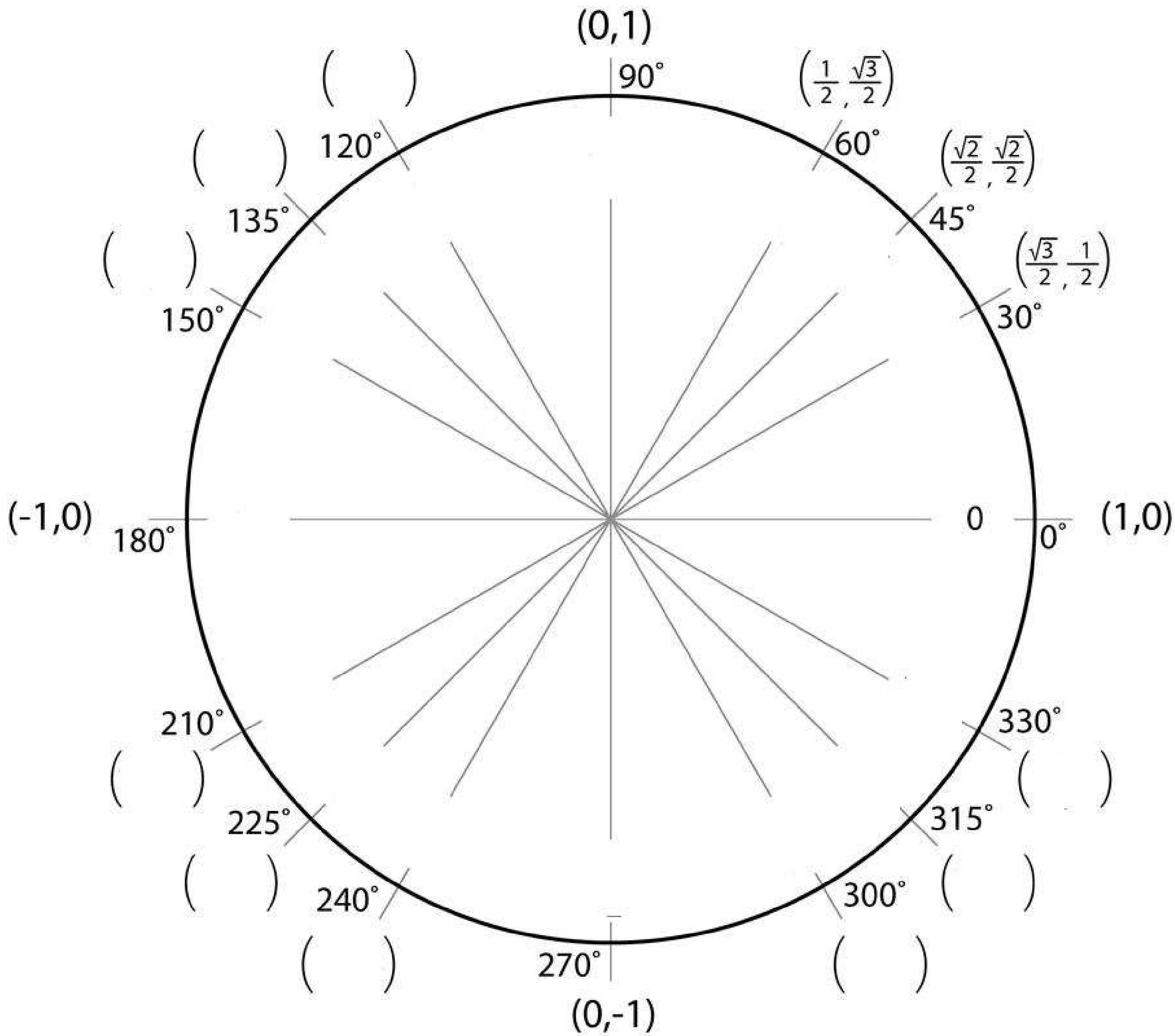


Exercise:

Complete the following chart for the trig ratios listed in the top row. You will have to use your knowledge of SPECIAL ANGLES. Do NOT use your calculator.

Trig Ratio	**a) $\cos 120°$**	**b) $\sin 150°$**	**c) $\tan 330°$**
Standard Position of Angle ⇒Construct right triangle ⇒Determine the reference angle $\boldsymbol{\theta_R}$			
Positive or Negative?			
Exact Value of Trig Ratio			

Unit Circle



(0,1)
90°
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
60°
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
45°
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
30°
( )
120°
( )
135°
( )
150°
(-1,0)
180°
0
0°
(1,0)
210°
( )
225°
( )
240°
( )
270°
(0,-1)
300°
( )
315°
( )
330°
( )

Grade 11 Trig Review

1. Predict whether each value will be positive or negative. Sketch each angle on a coordinate grid.
   a) $\tan 167°$ b) $\sin 99°$ c) $\cos 132°$

2. Given that ($0 \le \theta \le 180°$), determine the value(s) of θ *to* 1 decimal place.
   a) $\cos\theta = 0.4772$ b) $\tan\theta = -0.2272$ c) $\sin\theta = 0.5476$
   d) $\tan\theta = 1.6191$ e) $\sin\theta = 0.3486$ f) $\cos\theta = 0.5577$

3. Angle θ is obtuse.
   a) $\tan\theta = -0.4452$; calculate $\sin\theta$ to 4 decimal places.
   b) $\sin\theta = 0.9707$; calculate $\cos\theta$ to 4 decimal places.

4. Sketch each angle θ in standard position, then write a coterminal angle.
   a) $\theta = 170°$ b) $\theta = 293°$ c) $\theta = -30°$ d) $\theta = -320°$
   e) $\theta = 450°$ f) $\theta = 600°$ g) $\theta = -370°$ h) $\theta = 200°$

5. Determine two angles between 0° and 360° that have each trigonometric function value. Write the angle to the nearest degree.
   a) $\sin\theta = 0.42$ b) $\cos\theta = -0.31$ c) $\tan\theta = 3.46$

6. The point P(4, -15) lies on the terminal arm of an angle θ in standard position. Determine each trigonometric function value to 3 decimal places.
   a) $\sin\theta$ b) $\cos\theta$ c) $\tan\theta$

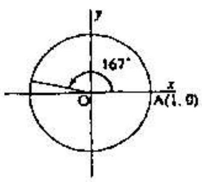
7. The terminal arm of an angle θ lies in Quadrant II on the line with equation $4x + 3y = 0$. Determine each trigonometric function value.
   a) $\sin\theta$ b) $\cos\theta$ c) $\tan\theta$

8. State each exact value. Do not use a calculator.
   a) $\cos 135°$ b) $\tan 225°$ c) $\sin 210°$ d) $\dfrac{1}{\tan 60°}$

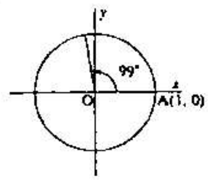
9. Simplify each expression. Do not use a calculator.
   a) $\sin 30° + \cos 60°$ b) $\tan 45° + \tan 225°$ c) $\sin 240° + \cos 300°$

Answers

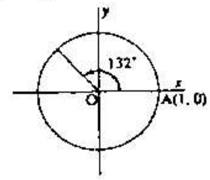
1. a) Negative



b) Positive



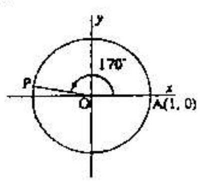
c) Negative



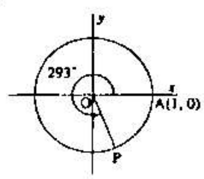
2. a) 61.5° b) 167.2° c) 33.2°, 146.8°
d) 58.3° e) 20.4°, 159.6° f) 123.9°

4. Coterminal angles may vary.

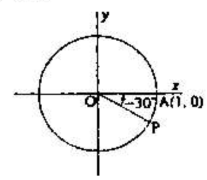
a) 530°



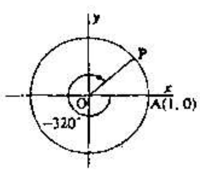
b) 653°



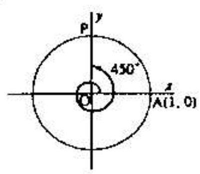
c) 330°



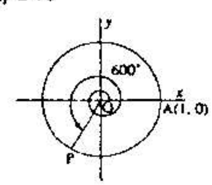
d) 40°



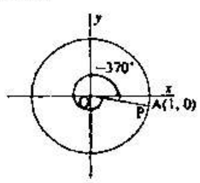
e) 90°



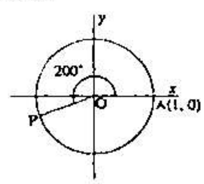
f) 240°



g) 350°



h) 560°



5. a) 25°, 155° b) 108°, 252° c) 74°, 254°

6. a) −0.966 b) 0.258 c) −3.750

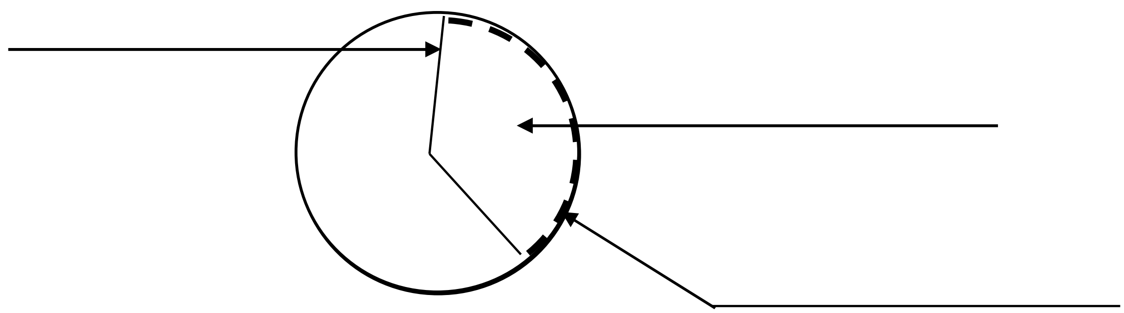
7. a) $\frac{4}{5}$ b) $-\frac{3}{5}$ c) $-\frac{4}{3}$

8. a) $-\frac{1}{\sqrt{2}}$ b) 1 c) $-\frac{1}{2}$ d) $\frac{1}{\sqrt{3}}$

9. a) 1 b) 2 c) $\frac{1-\sqrt{3}}{2}$

3.1 Investigation- RADIANS

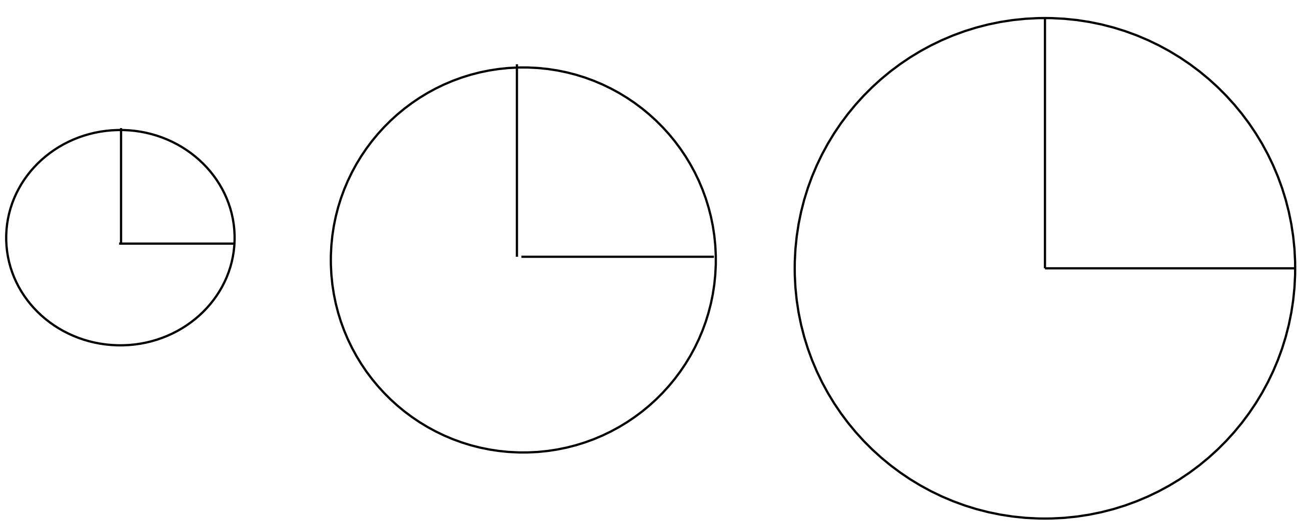
PART A: TERMINOLOGY
Fill in each blank with one of the following terms: arc length, radius, sector



PART B: Investigate Angle Measure
Materials

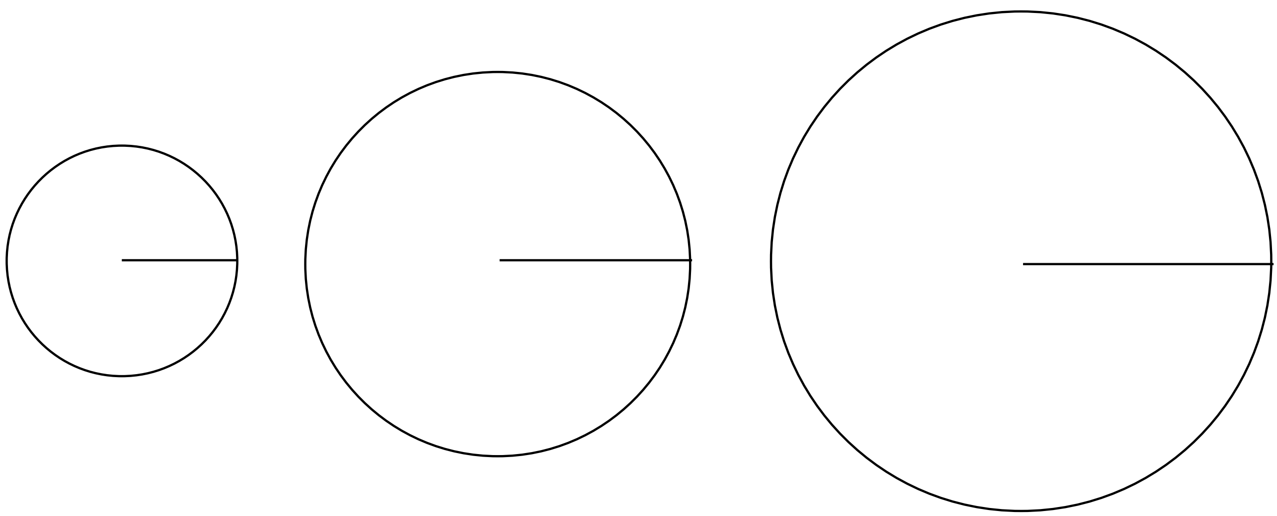
- protractor
- ruler
- string
- calculator

1. Use a protractor to confirm that in each of the 3 circles below the sector drawn has an angle measuring 90°.
2. Use a ruler to measure the radius of each circle and record in the chart on the next page.
3. Use a string to measure the arc length of each sector and record in the chart.
4. Complete the 5th column using a calculator. Provide an answer to **1 decimal place**.



Size of Angle	Size of Circle	arc length (**a**)	radius (**r**)	$\frac{a}{r}$
90°	small			
	medium			
	large			
120°	small			
	medium			
	large			
57°	small			
	medium			
	large			

5. In the following circles, use a protractor to construct sectors that have a sector angle of 120°. Measure the arc length and radius for each angle you constructed and record in the chart.
6. Repeat #5 but this time construct angles of 57° . Use the same circles.



PART C: SUMMARY
Answer the following questions on a separate sheet of paper.

1. What do you notice about the chart?

2. How does the size of the circle affect the ratio, $\frac{a}{r}$?

3. How does the size of the angle affect the ratio, $\frac{a}{r}$?

4. Use your answers to questions 2 and 3 to describe another way to give an angle's size other than by using degrees.

Part E: Definition of a Radian

Radians are an alternative way to measure angles, other than using degrees. The number of radians contained in an angle is equal to the ratio $\frac{a}{r}$, where a is the arc length of the sector containing the angle and r is the radius of that circle.

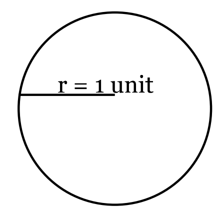
One radian is defined to be the measure of the angle where $\boldsymbol{\frac{a}{r} = 1}$***. In other words, one radian is the size an angle that has an arc length equal to the radius, which makes the angle about 57°.*** *(See the last column in the chart of the investigation.)*

Converting Between Degrees and Radians

Question:

How many radians are in a circle? (i.e. How many radians equal 360°?)

Solution:



Step 1: To find the number of radians, we need to know the arc length and radius. Since the size of a circle does not matter, consider a circle with a radius of 1 unit.
Radius equals _________.

Step 2: Now we need to find the arc length. Hmmmm…what is another name for the arc length of a sector whose angle measures 360°? ____________________.
Find the arc length. Do not use decimals. Leave your answer in terms of π.

Arc length equals ___________.

Step 3: Remember that: $\# \text{ of radians } = \frac{a}{r}$

$= \underline{\hspace{3cm}}$

$=$

Step 4: Therefore, 360° = _______ # of radians.

OR:

$$\boxed{180° = \underline{\hspace{3cm}} \text{ \# of radians}}$$

Rules for conversion between
DEGREES $\Leftrightarrow$ RADIANS

Since $180^{\circ} = \pi$ radians…

- Dividing both sides by 180 gives that 1° =_________radians.
- Dividing both sides by π gives that 1 radian =_______$^{\circ}$.

These deductions give us the following nice conversion shortcuts:

DEGREES=_________ RADIANS   **RADIANS =________DEGREES**

Example 1: Convert from degrees to radians.

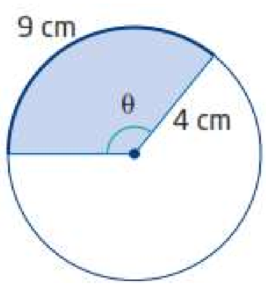
a) 90° b) 270° c) 138°

Example 2: Convert from radians to degrees.

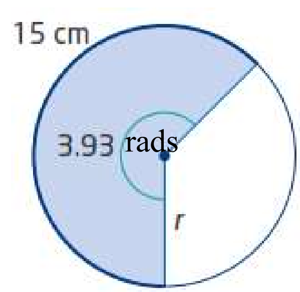
a) 3π b) $\frac{3}{4}\pi$ c) 1.45 rads.

Example 3 : Use the information in each diagram to determine the value of the variable. Give your answers to the nearest hundredth of a unit.

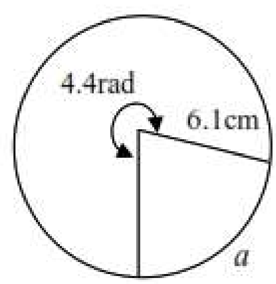
a)



b)



c)



Exit Card!

1. Change each degree to radian measure in terms of π and vice versa.

Degree	Radian	Radian	Degree
(a) - 315°		(c) $\frac{\pi}{12}$	
(b) 135°		(d) 6 rad	

2. Sara is taking a course in industrial engineering. For an assignment, she is designing the interface of a DVD player. In her plan, she includes a decorative arc below the on/off button. The arc has central angle 130° in a circle with radius 6.7 mm. Determine the length of the arc, to the nearest tenth of a millimeter.

3.1 Practice

1. Find the exact number of degrees in the angles with the following radian measures.

a) $\dfrac{\pi}{3}$ b) $\dfrac{\pi}{4}$ c) 2π d) $\dfrac{\pi}{2}$ e) $\dfrac{3}{4}\pi$ f) $\dfrac{3}{2}\pi$

g) 4π h) $\dfrac{5}{6}\pi$ i) $\dfrac{\pi}{18}$ j) $\dfrac{11}{3}\pi$ k) $\dfrac{7}{6}\pi$ l) 5π

2. Find the exact radian measure in terms of π for each of the following angles.

a) 40^o b) 75^o c) 10^o d) 120^o e) 225^o

f) 315^o g) 330^o h) 240^o i) 540^o j) 1080^o

3. Find the approximate number of degrees, to the nearest tenth, in the angles with the following radian measures.

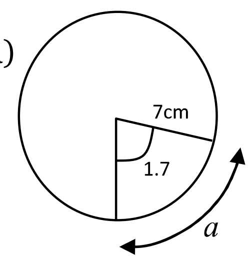
a) 2.5 b) 1.75 c) 0.35 d) $\dfrac{17}{13}\pi$ e) $\dfrac{5}{7}\pi$

4. Find the approximate number of radians (nearest hundredth), in the angles with the following degree measures.

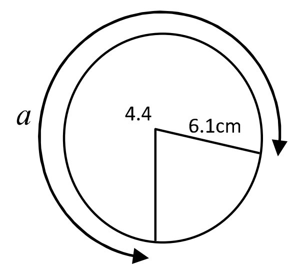
a) 60^o b) 150^o c) 310.5^o d) 145^o e) 230^o

5. Determine the length of each arc, a, to the nearest tenth.

a)

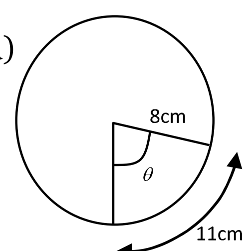


b)

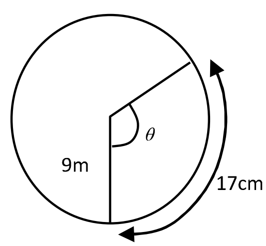


6. Determine the approximate measure of $\angle\theta$, (nearest hundredth) of a radian and to the tenth of a degree.

a)

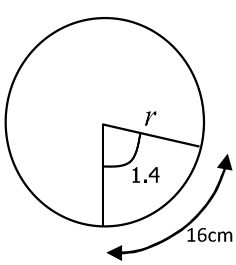


b)

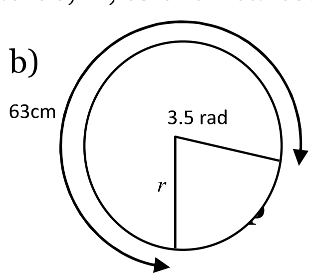


7. Determine the length of each radius, r, to the nearest tenth.

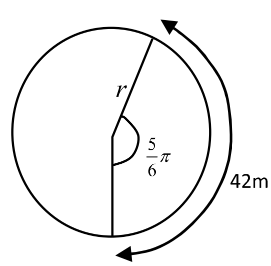
a)



b)



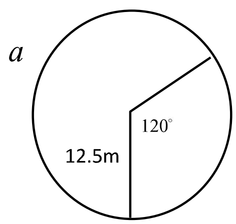
c)



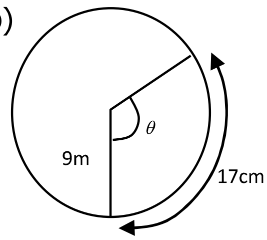
Warm up

1. Determine the missing value in each diagram

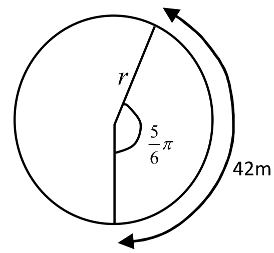
a)



b)



c)



2. State two co-terminal angles for each of the following. Your answers must include π.

a) $\frac{2\pi}{3}$

b) $-\frac{3\pi}{4}$

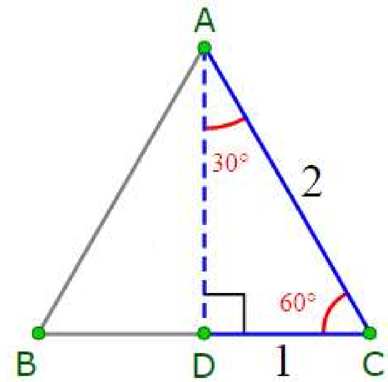
3. Change each degree to radian measure in terms of π and vise versa.

Degree	**Radian**	**Radian**	**Degree**
(c) - 315º		(d) $\frac{20\pi}{3}$	
(d) 135º		(e) 6 rad	
(e) 540º		(f) $\frac{3\pi}{4}$	

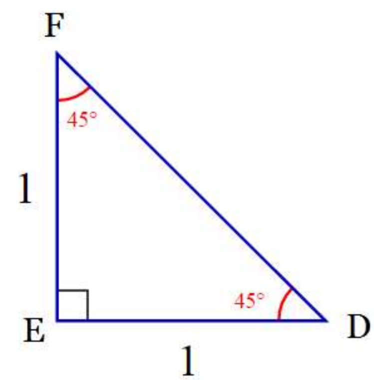
Unit 3: Trigonometry
3.2 Trigonometric Ratios and Special Angles

Angles that measure 30^o, 45^o, and 60^o occur often in trigonometry. They are sometimes called **special angles**.

- **30^o-60^o-90^o Triangle**
  ΔABC is an equilateral triangle with side lengths of 2 units. AD bisects $\angle BAC$ to form two congruent triangles. ΔABD and ΔACD are called 30^o-60^o-90^o triangles. What is the length of the side lengths of ΔACD? Express the length of AD in radical form.



- **45^o-45^o-90^o Triangle**
  ΔDEF is an isosceles right triangle whose equal sides, DE and EF, have a length of 1 unit. ΔDEF is called 45^o-45^o-90^o triangle. Determine the length of DF. Express the answer in radical form.



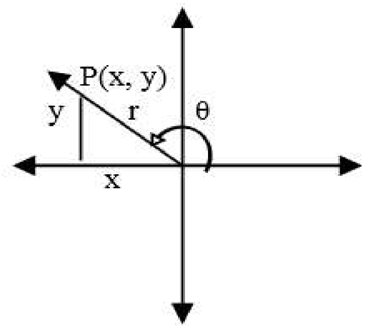
Use the 30^o-60^o-90^o and 45^o-45^o-90^o Triangle triangle to determine the exact values of the sine, cosine and tangent of 30^o, 45^o and 60^o.

θ in degrees	θ in radians	sin (θ)	cos(θ)	tan(θ)
30^o				
45^o				
60^o				

Angle Measures and the Coordinate Plane

Let P(x, y) be any point on the terminal arm of angle θ in standard position.

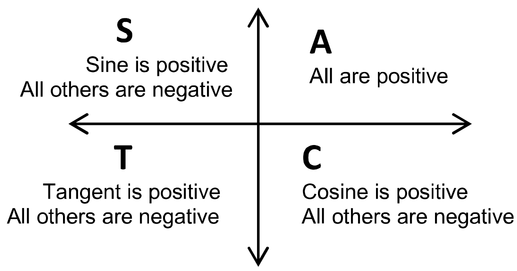
By the Pythagorean Theorem, r = $\sqrt{x^2 + y^2}$. State the three Primary Trigonometric Ratios in terms of x, y and r for angle θ.



Recall that supplementary angles have the same ratio.

Recall the CAST rule. We can use this to help us remember the signs of the trigonometric ratios in each of the quadrants.

CAST Rule:



Sine, Cosine and Tangent of Any Angle

Ex 1: The point P(-3, -6) lies on the terminal arm of an angle θ in standard position. Determine the exact values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.

Special Cases

Ex 2: Find the sine, cosine and tangent of any angle that measures 0, $\frac{\pi}{2}$, $\frac{3\pi}{2}$ and 2π.

Case 1: Angle of 0	**Case 2: Angle of $\frac{\pi}{2}$**	**Case 3: Angle of π**	**Case 4: Angle of $\frac{3\pi}{2}$**

Exact Trigonometric Ratios From Radian Measures

Ex 3: Determine the exact values of

a) $\sin\left(\frac{7}{4}\pi\right)$ b) $\cos\left(\frac{5}{6}\pi\right)$ c) $\tan\left(-\frac{11\pi}{6}\right)$

Recall: Reciprocal Trigonometric Functions

$\csc(\theta) = \frac{1}{\sin(\theta)}$ $\sec(\theta) = \frac{1}{\cos(\theta)}$ $\cot(\theta) = \frac{1}{\tan(\theta)}$

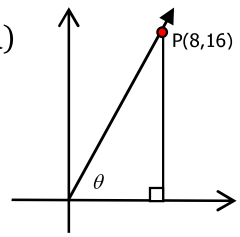
Ex 4: Determine the exact values of csc($\frac{2\pi}{3}$), sec($\frac{2\pi}{3}$) and cot($\frac{2\pi}{3}$).

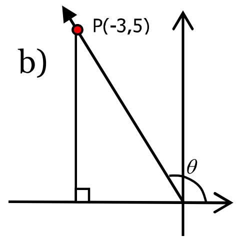
Ex 5: Determine the angle θ, where $0 \le \theta \le 2\pi$.

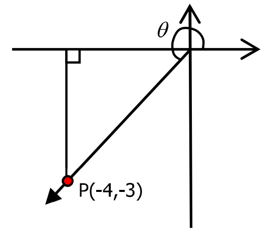
a) $\cos(\theta) = -\frac{\sqrt{3}}{2}$ b) $\sin(\theta) = -\frac{1}{\sqrt{2}}$

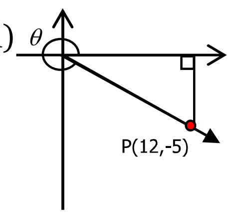
3.2 Practice

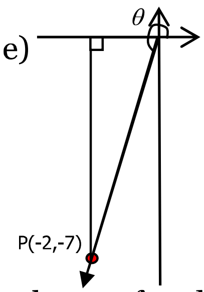
1. The co-ordinates of point P on the terminal arm of each $\angle\theta$ in standard position are shown,
where $0 \le \theta \le 2\pi$. Determine the exact values of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$.

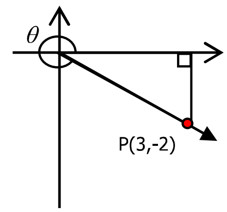
a) 

b) 

c) 

d) 

e) 

f) 

2. The co-ordinates of point P on a terminal arm of each $\angle\theta$ in standard position are shown, where $0 \le \theta \le 2\pi$. Determine the exact values of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$.

a) $P(6,5)$ b) $P(-1,8)$ c) $P(-2,-5)$ d) $P(6,-1)$

e) $P(2,-4)$ f) $P(-3,-9)$ g) $P(3,3)$ h) $P(-2,6)$

3. Find the exact value of each trigonometric ratio.

a) $\sin\left(\frac{5\pi}{4}\right)$ b) $\tan\left(\frac{11\pi}{6}\right)$ c) $\cos\left(\frac{\pi}{6}\right)$ d) $\cos\left(\frac{7\pi}{4}\right)$

e) $\tan\left(\frac{4\pi}{3}\right)$ f) $\cos\left(\frac{7\pi}{6}\right)$ g) $\sin\left(\frac{5\pi}{6}\right)$ h) $\cos\left(\frac{3\pi}{4}\right)$

4. $\angle\theta$ in standard position with its terminal arm in the stated quadrant, and $0 \le \theta \le 2\pi$.
A trigonometric ratio is given. Find the exact values of the other two trigonometric ratios.

a) $\sin(\theta) = \frac{4}{5}$, quadrant II b) $\cos(\theta) = -\frac{2}{3}$, quadrant III

c) $\tan(\theta) = -\frac{5}{2}$, quadrant IV d) $\sin(\theta) = -\frac{3}{7}$, quadrant III

5. Determine $\angle\theta$, given $0 \le \theta \le 2\pi$.

a) $\sin(\theta) = \frac{1}{2}$ b) $\cos(\theta) = \frac{\sqrt{3}}{2}$ c) $\tan(\theta) = -1$ d) $\sec(\theta) = -2$

e) $\cot(\theta) = -\sqrt{3}$ f) $\sin(\theta) = -\frac{5}{6}$

Warm up

1. For each function find the exact value of the given function.

(a) $\tan\left(\frac{-11\pi}{6}\right)$ (b) $\sec\left(-\frac{4\pi}{3}\right)$ (c) $\sin\left(\frac{13\pi}{6}\right)$

2. Given:

$$13\sin(\alpha)+5=0 \quad \left(0<\alpha<\frac{3\pi}{2}\right)$$

$$13\cos(\beta)-12=0 \quad \left(\frac{3\pi}{2}<\beta<2\pi\right)$$

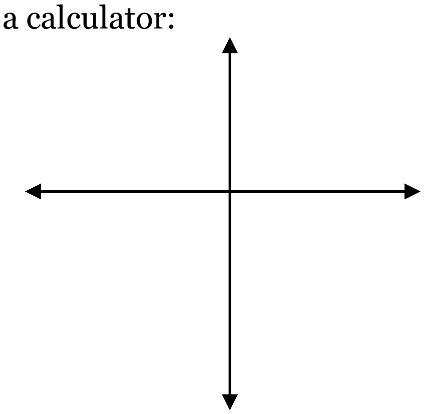
Draw a sketch and determine the following, without the use of a calculator:

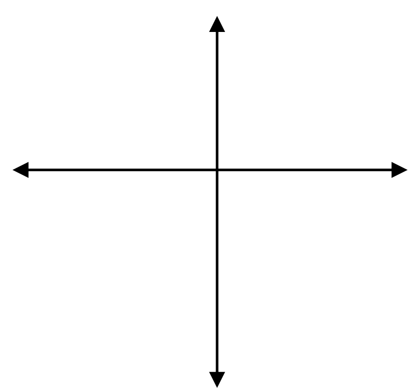
(a) $\tan(\alpha)$

(b) $\tan(\beta)$

(c) $\sin(\beta)$

(d) $\cos(\alpha)$





Unit 3: Trigonometry
3.3 Equivalent Trigonometric Expressions

Investigation

1. Sketch ΔABC such that $\angle B$ is a right angle. Mark $\angle B$ as $\frac{\pi}{2}$.

2. Mark $\angle A$ as x. Use the sum of the interior angles of a triangle to derive an expression for the measure of $\angle C$ in terms of x.

3. Determine an expression for $\sin(x)$ in terms of the sides of the triangle, a, b and c.

4. Determine expressions for the sine, cosine, and tangent of $\angle C$ in terms of the sides of the triangle. Which of these is equal to $\sin(x)$?

5. You have shown that $\sin x$ and $\cos\left(\frac{\pi}{2} - x\right)$ are equivalent trigonometric expressions. Does the relationship between these equivalent expressions depend on the value of x? Justify your answer.

6. What is the sum of x and $\left(\frac{\pi}{2} - x\right)$? What name is given to a pair of angles such as x and $\left(\frac{\pi}{2} - x\right)$?

7. Use ΔABC to determine equivalent trigonometric expressions for $\cos(x)$ and $\tan(x)$.Use ΔABC to determine equivalent trigonometric expressions for $\csc(x)$, $\sec(x)$, and $\cot(x)$.

8. Summarize the six relations among trigonometric functions in the table.

Co-Function Identities	
$\sin\left(\frac{\pi}{2} - x\right) =$	$\csc\left(\frac{\pi}{2} - x\right) =$
$\cos\left(\frac{\pi}{2} - x\right) =$	$\sec\left(\frac{\pi}{2} - x\right) =$
$\tan\left(\frac{\pi}{2} - x\right) =$	$\cot\left(\frac{\pi}{2} - x\right) =$

Other Co-Function Identities	
$\sin\left(\frac{\pi}{2} + x\right) =$	$\csc\left(\frac{\pi}{2} + x\right) =$
$\cos\left(\frac{\pi}{2} + x\right) =$	$\sec\left(\frac{\pi}{2} + x\right) =$
$\tan\left(\frac{\pi}{2} + x\right) =$	$\cot\left(\frac{\pi}{2} + x\right) =$

An identity involving trigonometric expressions is called a **trigonometric identity**. The trigonometric identities in the table in step 8 are known as the **co-function or co-related identities.**
Summary of the other identities are below.

More Co-Function Identities	
$\sin\left(\frac{3\pi}{2} - x\right) =$	$\csc\left(\frac{3\pi}{2} - x\right) =$
$\cos\left(\frac{3\pi}{2} - x\right) =$	$\sec\left(\frac{3\pi}{2} - x\right) =$
$\tan\left(\frac{3\pi}{2} - x\right) =$	$\cot\left(\frac{3\pi}{2} - x\right) =$

More Co-Function Identities	
$\sin\left(\frac{3\pi}{2} + x\right) =$	$\csc\left(\frac{3\pi}{2} + x\right) =$
$\cos\left(\frac{3\pi}{2} + x\right) =$	$\sec\left(\frac{3\pi}{2} + x\right) =$
$\tan\left(\frac{3\pi}{2} + x\right) =$	$\cot\left(\frac{3\pi}{2} + x\right) =$

Recall- Even and odd functions identity:

Reflection Identities	
$\sin(-x) = -\sin(x)$	$\csc(-x) = -\csc(x)$
$\cos(-x) = \cos(x)$	$\sec(-x) = \sec(x)$
$\tan(-x) = -\tan(x)$	$\cot(-x) = -\cot(x)$

Examples

1. State an equivalent trigonometric expression for $\cos\left(\frac{\pi}{12}\right)$ using co-function identity.

2. Given that $\sec\left(\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}$, use an equivalent trigonometric expression to show that $\csc\left(\frac{\pi}{3}\right)=\frac{2}{\sqrt{3}}$.

3. Simplify the following expressions

a) $\dfrac{\cos\left(\frac{\pi}{2}-2x\right)}{\tan(\pi-2x)}+\dfrac{\csc\left(\frac{\pi}{2}-2x\right)}{\sec(2x+2\pi)}$

b) $\dfrac{\sin(x-\pi)}{\cos(x+\pi)}-\dfrac{\csc(-x-\pi)}{\sec\left(\frac{\pi}{2}-x\right)}$

4. Determine a value of b such that $\csc\left(2b+\frac{\pi}{8}\right)=\sec\left(3b-\frac{\pi}{8}\right)$.

3.3 Practice

1. Write each of the following in terms of the co-function identity:

(a) $\sin\left(\frac{5\pi}{8}\right)$ (b) $\csc\left(\frac{5\pi}{18}\right)$

(c) $\cos\left(\frac{\pi}{9}\right)$ (d) $\cos\left(\frac{7\pi}{36}\right)$

2. Find the exact value of special angles using equivalent trigonometric expressions.

a) $\cot\left(\frac{-5\pi}{3}\right)$ b) $\csc\left(\frac{4\pi}{3}\right)$ c) $\sin\left(\frac{10\pi}{3}\right)$

d) $\sec\left(\frac{2\pi}{3}\right)$ e) $\sin\left(\frac{-7\pi}{6}\right)$ f) $\cot\left(\frac{\pi}{2}-x\right)\cot(x)$

3. Express each of the following as a single trigonometric ratio, <u>and then</u> evaluate the ratio.

a) $\sec^2(\frac{7\pi}{6})\cot^2(\frac{7\pi}{6})$ b) $\csc^2(\frac{3\pi}{4})\tan^2(\frac{3\pi}{4})$ c) $\cos\left(\frac{4\pi}{3}\right)-\csc\left(\frac{5\pi}{6}\right)$

4. Determine a value for c such that $\tan\left(2c+\frac{\pi}{3}\right)-\cot\left(3c-\frac{\pi}{4}\right)=0$.

5. Determine a value for θ that satisfies $\sin\left(2\theta-\frac{\pi}{3}\right)=\cos\left(\frac{5\pi}{4}-3\theta\right)$.

6. Given $\sin\left(x+\frac{3\pi}{2}\right)=\frac{2}{3}$ where $\pi<x<\frac{3\pi}{2}$, determine the exact value of $\sin(x)$.

7. Simplify.

(a) $\cos(-\theta)+\cos(180^\circ-\theta)-\cos(180^\circ+\theta)$

(b) $\dfrac{\cos(\pi+x)+\sin(\pi-x)}{\sin(2\pi-x)-\cos(\pi-x)}$

(c) $\sin(\pi+x)\cos\left(\frac{\pi}{2}-x\right)-\cos(\pi+x)\sin\left(x+\frac{3\pi}{2}\right)$

(d) $\dfrac{\cos(-x)\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{3\pi}{2}-x\right)\csc(x)}{\sin\left(\frac{\pi}{2}+x\right)-\cos\left(x-\frac{\pi}{2}\right)+\sin\left(\frac{3\pi}{2}-x\right)}$

8. Fill in the blanks with the appropriate trig function name:

(a) $\sin\frac{2\pi}{3}=$ __________ $\left(\frac{-\pi}{6}\right)$

(b) ____________ $\frac{11\pi}{60}=\sin\frac{19\pi}{60}$

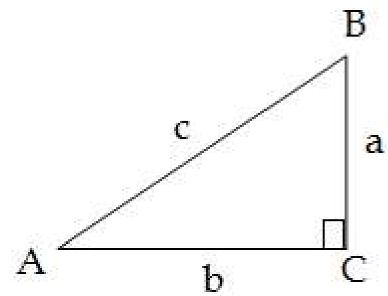
(c) $\cos\frac{7\pi}{18} = \frac{1}{\underline{\hspace{4em}}\,\frac{\pi}{9}}$

9. For right triangle ABC:

(a) If $\sin A = \frac{\sqrt{3}}{3}$, what is the value of cos B?

(b) If cos A=0.109, what is $\sin\left(\frac{\pi}{2} - A\right)$?

(c) If $\cos\frac{11\pi}{180} = 0.9816$, what is $\sin\frac{79\pi}{180}$?



Warm-up

1. Find without using calculators, the value of

$$\tan\left(\frac{5\pi}{4}\right)-\tan\left(\frac{\pi}{9}\right)\cot\left(\frac{\pi}{9}\right)-\sec\left(\frac{\pi}{6}\right)\csc\left(\frac{\pi}{3}\right)-\sin\left(\frac{11\pi}{6}\right)+\sin\left(\frac{\pi}{2}\right)$$

2. Given $A=\frac{\pi}{6}, B=\frac{\pi}{4}, C=\frac{\pi}{3}$, **find the value of** $\sin^2(A)-\cos^2(C)+2\tan(B)-\sec^2(B)$.

Unit 3: Trigonometry
3.4 Compound Angle Formulas

Activity 1: Investigation- Compound angles

Sara is studying for a trigonometry test and completes the following question:
Question: Evaluate the following:

$$\cos\left(\pi - \frac{\pi}{3}\right)$$

Sara's solution:

$$\begin{aligned}\cos\left(\pi - \frac{\pi}{3}\right) &= \cos(\pi) - \cos\left(\frac{\pi}{3}\right)\\ &= -1 - \frac{1}{2}\\ &= -\frac{3}{2}\end{aligned}$$

Use a calculator to determine if Sara's answer is right or wrong.

Describe in words the mistake(s) in her solution if there is any.

____

____

Is the following statement true or false?

"A trigonometric ratio can be distributed to the angles that lie within the brackets."

From the investigation above, we know that:

____

____

As you might expect, there are formulas for $\cos(\alpha \pm \beta)$, $\sin(\alpha \pm \beta)$ and $\tan(\alpha \pm \beta)$, but Activity 1 shows it is wrong to apply the distributive law to the trigonometric ratios of compound angles.

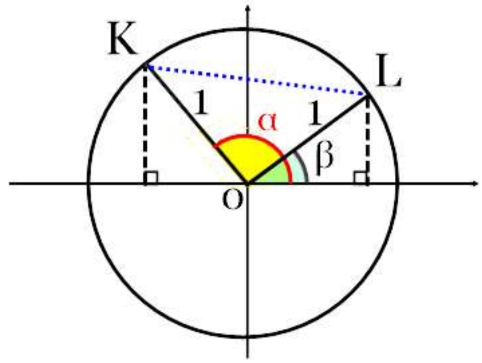
Derivation of cos(α−β)
Prove that, cos(α−β)=cos(α)cos(β)+sin(α)sin(β).

Figure on the next page shows angles α and β in standard position on the unit circle, determining points L(a,b) and K($-x$,y) ,respectively.
We use the distance formula and cosine law to find the length of LK.

Recall:

$$\text{Distance Formula : } d_{AB} = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

$$\text{Cosine Law: } a^2 = b^2 + c^2 - 2bc.\cos(A)$$



Step 1: Express the coordinates of L and K in terms of the angles α and β:

L(__________,_________)

K(__________, _________)

Step 2: Use the distance formula to determine KL^2:

$KL^2 = (x_K - x_L)^2 + (y_K - y_L)^2$

$KL^2 =$

$=$

$=$

$=$

Step 3: Now determine KL^2 using the cosine law for △KOL:

$KL^2 = KO^2 + LO^2 - 2.KO.LO.\cos(\alpha - \beta)$

$=$

$=$

Step 4: Equating the two expressions for KL^2, we have :

Example 1: Prove that, $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.

Example 2: Use the compound angle formula and co-functions to expand $\sin(\alpha-\beta)$.

Hint: Using co-functions, we know that $\sin(A) = \cos(\frac{\pi}{2} - A)$, so we can write $\sin(\alpha+\beta)$ in terms of the cosine function as:

$$\begin{aligned}\sin(\alpha-\beta) &= \cos(\frac{\pi}{2} - (\alpha-\beta)) \\ &= \cos(\frac{\pi}{2} - \alpha + \beta) \\ &= \cos[(\frac{\pi}{2} - \alpha) + \beta]\end{aligned}$$

Compound Angel Identities

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\tan(\alpha+\beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)} \qquad (*)$$

$$\tan(\alpha-\beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)} \qquad (*)$$

(*) see Appendix 2 for proof

Example 3: Determine the **exact** value for each of the following:

a) $\cos\left(75^{\circ}\right)$

b) $\sin\left(\frac{\pi}{12}\right)$

c) $\tan\left(\frac{7\pi}{12}\right)$

d) $\cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{2\pi}{3}\right)+\sin\left(\frac{5\pi}{6}\right)\sin\left(\frac{2\pi}{3}\right)$

Example 4: Given $\sec(\alpha)=\frac{5}{3}$, $0<\alpha<\frac{\pi}{2}$ and $\sin(\beta)=\frac{1}{3}$, $0<\beta<\frac{\pi}{2}$, draw a sketch and determine the exact value of $\sin\left(\alpha+\beta\right)$ without the use of a calculator.

3.4 PRACTICE

1. Evaluate (provide the exact values).

a) $\sin\left(\frac{\pi}{12}\right)$ b) $\tan\left(\frac{\pi}{12}\right)$ c) $\tan\left(\frac{17\pi}{12}\right)$

d) $\sin\left(\frac{13\pi}{12}\right)$ e) $\cos\left(\frac{13\pi}{12}\right)$ f) $\cos(50^\circ)\sin(95^\circ)-\sin(50^\circ)\cos(95^\circ)$

g) $\dfrac{\tan\left(\frac{11\pi}{6}\right)+\tan\left(\frac{\pi}{3}\right)}{1-\tan\left(\frac{11\pi}{6}\right)\tan\left(\frac{\pi}{3}\right)}$ h*) $\dfrac{\cos^2\left(\frac{5\pi}{3}\right)\times\sin\left(\frac{5\pi}{6}\right)}{\tan\left(\frac{11\pi}{6}\right)\times\sec\left(\frac{\pi}{12}\right)}$

2. Given: $\sin(\alpha)=\frac{5}{13}, \alpha\in\left(\frac{\pi}{2},\pi\right)$ and $\cos(\beta)=\frac{-3}{5}, \beta\in\left(\pi,\frac{3\pi}{2}\right)$. Evaluate:

a) $\cos(\alpha+\beta)$ b) $\tan(\alpha+\beta)$

3. Given: $\cot(x)=\frac{4}{3}$, $\cot(y)=\frac{5}{12}$. Find: $\tan(x-y)$.

4. Solve for x, in the domain $0\le x\le 2\pi$.

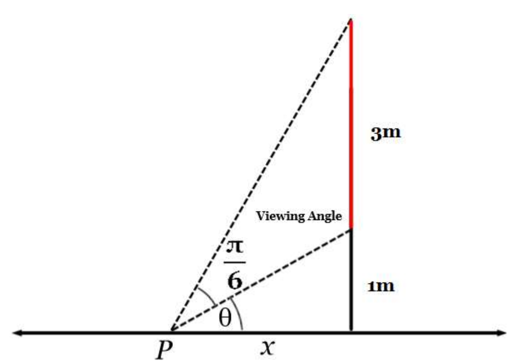
a) $\cos\left(\frac{\pi}{4}\right)\cos(x)-\sin\left(\frac{\pi}{4}\right)\sin(x)=1$ b) $\sin(x)\cos\left(\frac{\pi}{6}\right)-\cos(x)\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$

5*. Consider a triangle with angles A, B, C. Show that :

$$\tan(A)+\tan(B)+\tan(C)=\tan(A)\tan(B)\tan(C).$$

6*. Find the exact acute angle between the lines $x-\sqrt{3}y+\sqrt{3}=0$ and $x+y-5=0$.

7*. A room in an art gallery contains a picture you are interested in viewing. The picture is 3 meters high and is hanging so the bottom of the picture is one meter above your eye level. How far from the wall on which the picture is hanging should you stand so that the angle of vision occupied by the picture is $\frac{\pi}{6}$?



Answers

1. a) $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$ b) $2-\sqrt{3}$ c) $2+\sqrt{3}$ d) $-\frac{\sqrt{2}(\sqrt{3}-1)}{4}$

   e) $-\frac{\sqrt{2}(\sqrt{3}+1)}{4}$ f) $\frac{\sqrt{2}}{2}$ g) $\frac{\sqrt{3}}{3}$ h) $\frac{-\left(\sqrt{6}+3\sqrt{2}\right)}{32}$

2. a) $\frac{56}{65}$ b) $\frac{33}{56}$

3. $-\frac{33}{56}$

4. a) $\frac{7\pi}{4}$ b) $\frac{\pi}{3}$ or π

5. Hint : In any triangle , $A+B+C=\pi$, or $A+C=\pi-C$ therefore $\tan(A+C)=\tan(\pi-C)$.

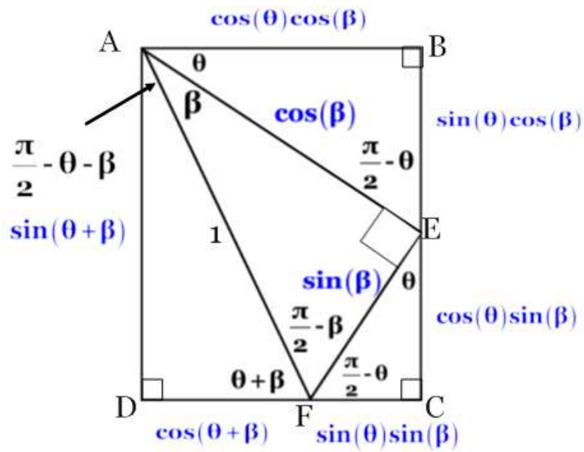
6. $\frac{5\pi}{12}$

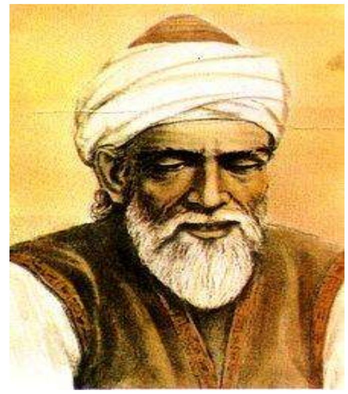
7. 4.2m or 0.94m

Appendix 1

The formula for cos (θ ±β) originally established by the 10th century Persian mathematician **Abu al-Wafa' Buzjani.** This appendix presents the proof of compound angle formula for sin(θ+β) and cos(θ+β).

Triangle AEF is constructed in rectangle ABCD such that ∠AEF=90°, E lies on BC, and F lies on CD. AF is 1 unit in length, ∠BAE=θ, and ∠FAE=β, as shown in the diagram. Using the diagram, develop the compound angle formulas for sin(θ+β) and cos(θ+β).





$$\text{In}\,\Delta\text{AEF}, \angle\text{EAF} = \beta \ \text{so}\ \angle\text{AFE} = \frac{\pi}{2} - \beta.$$

$$\text{Since } \sin(\beta) = \frac{\text{EF}}{1}, \text{then EF} = \sin(\beta). \text{Similarly, AE} = \cos(\beta).$$

$$\text{In}\,\Delta\text{ABE}, \angle\text{BAE} = \theta \ \text{so}\ \angle\text{AEB} = \frac{\pi}{2} - \theta.$$

$$\text{Since } \sin(\theta) = \frac{\text{BE}}{\text{AE}}$$

$$= \frac{\text{BE}}{\cos(\beta)}, \text{then BE} = \sin(\theta)\cos(\beta). \text{Similarly, AB} = \cos(\theta)\cos(\beta).$$

$$\text{In}\,\Delta\text{ADF}, \angle\text{DAF} = \frac{\pi}{2} - \theta - \beta \text{ and } \angle\text{AFD} = \pi - \frac{\pi}{2} - \left(\frac{\pi}{2} - \theta - \beta\right) = \theta + \beta.$$

$$\text{Since } \sin(\theta+\beta) = \frac{\text{AD}}{\text{AF}} = \frac{\text{AD}}{1}, \text{then AD} = \sin(\theta+\beta). \text{Similarly, we get DF} = \cos(\theta+\beta).$$

$$\text{In}\,\Delta\text{CEF}, \angle\text{CEF} = \pi - \frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right) = \theta. \text{Thus}, \angle\text{CFE} = \frac{\pi}{2} - \theta$$

$$\text{Since } \sin(\theta) = \frac{\text{CF}}{\text{EF}} = \frac{\text{BE}}{\sin(\beta)}, \text{then CF} = \sin(\theta)\sin(\beta). \text{Similarly, CE} = \cos(\theta)\sin(\beta).$$

In rectangle ABCD, AD = BC = BE + CE,

$$\therefore \sin(\theta+\beta) = \sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta).$$

Also, AB = CD = CF + DF, so

$$\cos(\theta)\cos(\beta) = \cos(\theta+\beta) + \sin(\theta)\cos(\beta)$$

$$\text{Therefore, } \cos(\theta+\beta) = \cos(\theta)\cos(\beta) - \sin(\theta)\cos(\beta).$$

Appendix 2

This appendix provides the mathematical proof of the the angle sum formula for tangent. We begin by using the quotient identity.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin(A)\cos(B) + \cos(A)\sin(B)}{\cos(A)\cos(B) - \sin(A)\sin(B)}$$

$$= \frac{[\sin(A)\cos(B) + \cos(A)\sin(B)] \div \mathbf{\cos(A)\cos(B)}}{[\cos(A)\cos(B) - \sin(A)\sin(B)] \div \mathbf{\cos(A)\cos(B)}}$$

$$= \frac{\dfrac{\sin(A)\cancel{\cos(B)}}{\mathbf{\cos(A)}\cancel{\mathbf{\cos(B)}}} + \dfrac{\cancel{\cos(A)}\sin(B)}{\cancel{\mathbf{\cos(A)}}\mathbf{\cos(B)}}}{\dfrac{\cancel{\cos(A)\cos(B)}}{\cancel{\mathbf{\cos(A)\cos(B)}}} - \dfrac{\sin(A)\sin(B)}{\mathbf{\cos(A)\cos(B)}}}$$

$$= \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

Warm up

Simplify the following without using a calculator. Provide the exact values.

a) $$\frac{\tan\left(\frac{5\pi}{6}\right) - \tan\left(\frac{2\pi}{3}\right)}{1+\tan\left(\frac{5\pi}{6}\right)\tan\left(\frac{2\pi}{3}\right)}$$

b) $$\frac{\sin(x)\cos(\frac{\pi}{4} - x) + \cos(x)\sin(\frac{\pi}{4} - x)}{\cos(x)\cos(\frac{\pi}{3} - x) - \sin(x)\sin(\frac{\pi}{3} - x)}$$

Unit 3: Trigonometry
3.4 Double Angle Formulas

We will extend our knowledge of compound angle formulas to include the double angle formulas. These formulas are special cases of the angle sum formulas studied in the previous lesson.

Double Angle Formula for Sine

$\sin(2A) = 2\sin(A)\cos(A)$

Proof:

Double Angle Formula for Cosine

$\cos(2A) = \cos^2(A) - \sin^2(A)$ $\quad$ $\cos(2A) = 2\cos^2(A) - 1$ $\quad$ $\cos(2A) = 1 - 2\sin^2(A)$

Proof:

Double Angle Formula for Tangent

$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

.

Proof:

Example 1: Express the following as a single trigonometric ratio.

a) $2\sin(5y)\cos(5y)$

b) $1-2\sin^2(3x)$

c) $4\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

d) $\sin(6x)\cos(6x)$

e) $2\cos^2(3\theta-2)-1$

f) $\cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)$

Example 2: Express the following as a single trigonometric ratio and then evaluate.

a) $\cos^2\left(\frac{\pi}{12}\right)-\sin^2\left(\frac{\pi}{12}\right)$

b) $\dfrac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$

c) $\sin^2(75°)-\cos^2(75°)$

d) $\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$

Example 3: Determine the exact values of $\sin(2\theta)$ and $\cos(2\theta)$ if $\sin(\theta)=\frac{4}{5}$ for $\frac{\pi}{2}\leq\theta\leq\pi$.

Example 4: If $\tan(\theta) = -\frac{\sqrt{5}}{2}$ for $\frac{\pi}{2} \le \theta \le \pi$, determine the exact value of sin(2θ).

Example 5: Use a double angle formula to rewrite each of the following:

a) sin(6x) =

b) sin(x)=

c) cos(x) =

Example 6*: Use the fact that $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ to determine the exact value of $\cos\left(\frac{\pi}{8}\right)$.

3.4 Practice

1. **Multiple Choice:** Select the best answer for each of the following

i. Given $\cos(\theta)=\frac{2}{3}$ in the first quadrant, the value of $\sin\left(\frac{\theta}{2}\right)$ is: ____

a. $\frac{\sqrt{3}}{2}$ b. $\frac{\sqrt{6}}{2}$ c. $\frac{\sqrt{6}}{6}$ d. $\frac{\sqrt{3}}{6}$

ii. $2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$ is equivalent to : ____

a. $\sin\left(\frac{\pi}{8}\right)$ b. $\cos\left(\frac{\pi}{8}\right)$ c. $\sin\left(\frac{\pi}{4}\right)$ d. $\cos\left(\frac{\pi}{4}\right)$

iii. The exact value of $1-2\sin^2\left(\frac{\pi}{8}\right)$ is: ____

a. $-\sqrt{2}$ b. $\frac{\sqrt{2}}{2}$ c. $-\frac{\sqrt{2}}{2}$ d. $\sqrt{2}$

2. Express as a single sine or cosine function.

a) $10\sin(x)\cos(x)$

b) $1-2\sin^2\left(\frac{2\theta}{3}\right)$

c) $5\sin(2x)\cos(2x)$

d) $2\cos^2(5\theta)-1$

3. Simplify each expression. State any restrictions on the variable in the domain $[0,2\pi]$

a) $\frac{\sin(2a)}{\cos(a)}=$

b) $2\tan(a)\cos^2(a)=$

c) $2\sin^2(a)+\cos(2a)=$

4. Expand using a double angle formula.

a) $3\sin(4x)=$

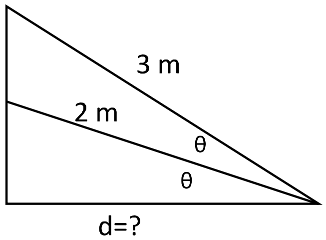
b) $6\cos(6x)=$

c) $1-\cos(8x)=$

d) $\tan(4x)=$

e) $\cos(2x)-\frac{\sin(2x)}{\sin(x)}=$

5. Two ropes (2m and 3m long) used to stabilize a pole for a volleyball net are anchored to the ground. The angle between the two ropes is equal to the angle between the ground and the lower rope. Determine the distance from the base of the pole to the point at which the ropes are anchored to the ground.



6. a) Express $\sec(2\theta)$ in terms of $\sec(\theta)$ and $\tan(\theta)$.

b) Express $\csc(2\theta)$ in terms of $\csc(\theta)$ and $\sec(\theta)$.

7. Determine formulas for

a) $\sin(3\theta)$ in terms of $\sin(\theta)$

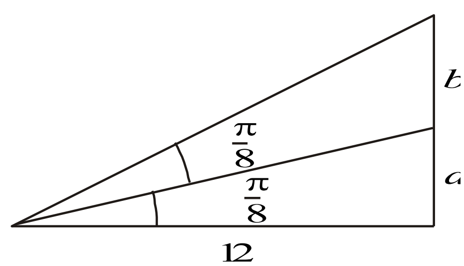
b) $\cos(3\theta)$ in terms of $\cos(\theta)$

d) $\tan(3\theta)$ in terms of $\tan(\theta)$

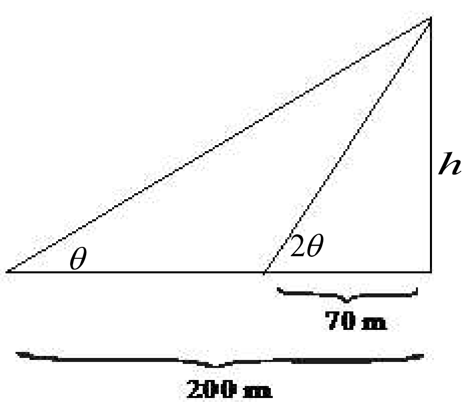
8. Express $\sin(2\theta)$ and $\cos(2\theta)$ in terms of $\tan(\theta)$.

9. If x and y are all first quadrant angles and $\sec(x) = \frac{5}{3}$ and $\sin(y) = \frac{1}{3}$, then determine the **exact** value of $\sin(2x + y)$.

10. Find the exact value of *b*.



11. Find the value of h in the below diagram.



Warm up

1. Express each of the following as a single trigonometric ratio, <u>and then</u> evaluate the exact value of the ratio.

b) $\sin^2\left(\frac{7\pi}{3}\right)-\cos^2\left(\frac{7\pi}{3}\right)$

c) $1-2\sin^2\left(\frac{5\pi}{12}\right)$

d) $\sin\left(\frac{5\pi}{6}\right)\cos\left(\frac{3\pi}{8}\right)-\cos\left(\frac{5\pi}{6}\right)\sin\left(\frac{3\pi}{8}\right)$

e) $2\cos^2(\frac{5\pi}{8})-1$

f) $\cos\frac{4\pi}{3}-\csc\frac{5\pi}{6}$

2. If $\cos\alpha = \frac{4}{5}$, and $\sin\beta = -\frac{12}{13}$, $0 < \alpha < \frac{\pi}{2}$, $\pi < \beta < \frac{3\pi}{2}$, evaluate $\cos(2\alpha + \beta)$.

3. If $\frac{\pi}{2} < x < \pi$ and $\cos^2 x = \frac{8}{9}$, determine the **exact** value of $\sin(4x)$.

Unit 3: Trigonometry
3.5 Proving Trigonometric Identities

Fill in the blanks with the words in the box.

Counter-example	**trig-identity**	**equal**
	identity	

- A statement of equality between two expressions that is true for all values of the variables for which the expressions are defined is called an ________________.
- An identity involving trigonometric expressions is called a ________________.
- Our goal is to prove that one side of an expression is __________ to the other side of the expression.
- A ________________ can be used to show that an equation is not an identity

Strategies for Proving Trig Identities:

- Write everything in terms of sine and cosine
- Be aware of equivalent forms of the fundamental identities, ie: $\sin^2\theta+\cos^2\theta=1$ has an alternative form: $\sin^2\theta = 1-\cos^2\theta$
- Try to rewrite the more complicated side of the equation so that it is identical to the simpler side.
- Usually any factoring or indicated algebraic operations should be performed, ie: $\sin^2 x + 2\sin x + 1 = (\sin x+1)^2$ or

$$\sin^3 x+\cos^3 x = (\sin x+\cos x)(\sin^2 x - \sin x\cos x+\cos^2 x)$$
$$= (\sin x+\cos x)\left(1-\frac{1}{2}\sin 2x\right)$$

- If an expression contains $1\pm\sin x$, $\sec x\pm\tan x$ or $\csc x\pm\cot x$ multiplying both numerator and denominator by $1\mp\sin x$, $\sec x\mp\tan x$ or $\csc x\mp\cot x$ would give $\cos^2 x$, 1 or -1.
- If there is more than one angle in the identity, consider using a Compound Identity

Reciprocal Identities

$$\csc A = \frac{1}{\sin A}$$
$$\sec(A) = \frac{1}{\cos(A)}$$
$$\cot(A) = \frac{1}{\tan(A)}$$
$$\tan(A) = \frac{1}{\cot(A)}$$
$$\cot(A) = \frac{1}{\tan(A)}$$

Quotient Identities

$$\tan(A) = \frac{\sin(A)}{\cos(A)}$$
$$\cot(A) = \frac{\cos(A)}{\sin(A)}$$

Pythagorean Identity

$$\sin^2(A)+\cos^2(A)=1$$
$$\tan^2(A)+1=\sec^2(A)$$
$$\cot^2(A)+1=\csc^2(A)$$

Reflection Identities

$$\sin(-A) = -\sin(A)$$
$$\cos(-A) = \cos(A)$$
$$\tan(-A) = -\tan(A)$$

COMPOUND ANGLE IDENTITIES

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

DOUBLE ANGLE IDENTITIES

$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$\cos(2A) = 2\cos^2(A) - 1$$
$$\cos(2A) = 1 - 2\sin^2(A)$$
$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

1. Prove the following identity:

a) $\dfrac{1+\sec(x)}{\tan(x)+\sin(x)} = \csc(x)$

b) $\cot^2(\theta)[\tan^2(\theta)+1] = \csc^2(\theta)$

c) $\dfrac{\tan^2(\theta)}{\sec^2(\theta)} = \left[1+\cos(\theta)\right]\left[1-\cos(\theta)\right]$

d) $\dfrac{\sec(x)+\tan(x)}{\sin(x)} = \dfrac{\csc(x)}{\sec(x)-\tan(x)}$

e) $\dfrac{\sin(x)+\sin(2x)}{1+\cos(x)+\cos(2x)} = \tan(x)$

f) $\dfrac{\sin(2x)}{1+\cos(2x)} = \tan(x)$

g) $\tan(2x) - \sin(2x) = 2\tan(2x)\sin^2(x)$

h) $\sin(7x) = \sin(x)\left[\cos^2(3x) - \sin^2(3x)\right] + 2\cos(x)\cos(3x)\sin(3x)$

2. If $2\cos^2(x) + 4\sin(x)\cos(x)$ is expressed in the form $A\sin(2x) + B\cos(2x) + C$ where $A, B, C \epsilon R$, determine the values of A, B, and C.

3. Write $2\sin(2x) + \sqrt{12}\cos(2x)$ in the form $y = A\cos(2x - \theta)$ by finding $A > 0$ and $\theta \in [0, 2\pi]$.

3.5 PRACTICE

1. Prove each identity.

a) $\dfrac{\sec(\theta)-1}{1-\cos(\theta)}=\sec(\theta)$

b) $\tan(x)+\cot(x)=\dfrac{2}{\sin(2x)}$

$\dfrac{\sin(t)-\cos(t)}{\cos(t)}+\dfrac{\sin(t)+\cos(t)}{\sin(t)}=\sec(t)\csc(t)$

c) $\dfrac{1+\tan^2(x)}{1+\cot^2(x)}=\dfrac{1-\cos^2(x)}{\cos^2(x)}$

d) $\dfrac{1}{1+\sec(\theta)}+\dfrac{1}{1-\sec(\theta)}=-2\cot^2(\theta)$

e) $\dfrac{1+\sec(x)}{\tan(x)+\sin(x)}=\csc(x)$

f) $\cos(a+b)\cos(a-b)=\cos^2(a)-\sin^2(b)$

g) $\dfrac{\cos(x)-\sin(y)}{\cos(y)-\sin(x)}=\dfrac{\cos(y)+\sin(x)}{\cos(x)+\sin(y)}$

h) $\dfrac{1+\tan(A)}{\sin(A)}-\sec(A)=\csc(A)$

i)

j) $\tan(A)+\cot(A)=\sec^2(A)\cot(A)$

k) $\dfrac{4-\sin^2(2x)}{4\cos^4(x)}=\tan^4(x)+\tan^2(x)+1$

l) $1-\sin(x)\cos(x)=\dfrac{\sin^2(x)}{1+\cot(x)}+\dfrac{\cos^2(x)}{1+\tan(x)}$

m) $\dfrac{\sin(x-y)}{\sin(x)\sin(y)}=\cot(y)-\cot(x)$

n) $\dfrac{\sin(5x)}{\sin(x)}-\dfrac{\cos(5x)}{\cos(x)}=4-8\sin^2(x)$

Unit 3-Review

Multiple Choice: Select the best answer for each of the following

1. Sin(50˚) cos(30˚) + cos(50˚)sin(30˚) is equivalent to : _____

a. sin80˚ b. cos80˚ c. sin20˚ d. cos20˚

2. $\dfrac{\tan\left(\frac{5\pi}{9}\right) - \tan\left(\frac{\pi}{6}\right)}{1+\tan\left(\frac{5\pi}{9}\right)\tan\left(\frac{\pi}{6}\right)}$ is equivalent to : _____

a. $\tan\left(\frac{13\pi}{18}\right)$ b. $\tan\left(\frac{\pi}{6}\right)$ c. $\tan\left(\frac{7\pi}{18}\right)$ d. $\tan\left(\frac{\pi}{6}\right)$ _____

3. The exact value of $\tan\left(\frac{\pi}{12}\right)$ is :

a. $2-\sqrt{3}$ b. $2+\sqrt{3}$ c. $3-\sqrt{3}$ d. $3+\sqrt{3}$

4. Given $\sin(\theta)=\frac{4}{5}$ and terminates in the second quadrant, the value of $\cos(2\theta)$ is _____

a. $\frac{-24}{25}$ b. $\frac{-7}{25}$ c. $\frac{-16}{25}$ d. $\frac{-3}{25}$

5. Given $\sin(\theta)=\frac{\sqrt{3}}{2}$ and terminates in quadrant one, the value of $\sin 2\theta$ is : _____

a. $\sqrt{3}$ b. $\frac{\sqrt{3}}{2}$ c. $\frac{\sqrt{3}}{4}$ d. $\frac{\sqrt{3}}{8}$

6. $\sin(-\theta)$ is equivalent to : _____

a. $-\sin(\theta)$ b. $\sin(\theta)$ c. $\cos(\theta)$ d. $-\cos(\theta)$

7. The exact value of $\sin\left(\frac{5\pi}{12}\right)$ is : _____

a. $\frac{\sqrt{6}+\sqrt{2}}{4}$ b. $\frac{\sqrt{6}+\sqrt{2}}{2}$ c. $\frac{\sqrt{6}-\sqrt{2}}{4}$ d. $\frac{\sqrt{6}-\sqrt{2}}{2}$

8. The expression $\cos(70°)$ is equivalent to : _____

a. $\sin(70°)$ b. $\cos(20°)$ c. $\sec(20°)$ d. $\sin(20°)$

9. The expression $\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{5\pi}{18}\right)-\sin\left(\frac{5\pi}{18}\right)\cos\left(\frac{2\pi}{3}\right)$ is equal to : ____

a. $\cos\left(\frac{7\pi}{18}\right)$ b. $\cos(\pi)$ c. $\sin\left(-\frac{7\pi}{18}\right)$ d. $\sin\left(\frac{7\pi}{18}\right)$

10. The expression $\dfrac{2\tan\left(\frac{\pi}{3}\right)}{1-\tan^2\left(\frac{\pi}{3}\right)}$ is equal to : ____

a. 1 b. $\sqrt{3}$ c. $-\sqrt{3}$ d. -1

11. The expression $\cot(A+B)$ is equivalent to : ____

a. $\cot(A)+\cot(B)$ b. $\cot(A)-\cot(B)$

c. $\dfrac{1-\tan(A)\tan(B)}{\tan(A)+\tan(B)}$ d. $\dfrac{\cot(A)+\cot(B)}{1-\cot(A)\cot(B)}$

12. The expression $\cos\left(\frac{\pi}{12}\right)$ is equal to : ____

a. $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ b. $\dfrac{\sqrt{3}+1}{-2\sqrt{2}}$ c. $\dfrac{\sqrt{3}-1}{-2\sqrt{2}}$

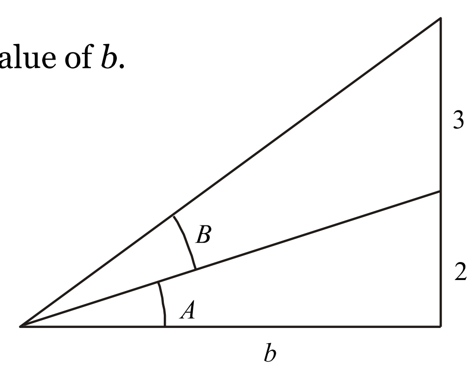
d. $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$

13. The expression $\cos(A+B)-\cos(A-B)$ is equivalent to : ____

a. $2\cos(A)\cos(B)$ b. $-2\sin(A)\sin(B)$ c. $2\sin(A)\sin(B)$ d. 0

Full Solution

1. In the given diagram $\angle A = \angle B$,find the exact value of b.



2. Express $12\cos\theta + 5\sin\theta = R\cos(\theta - \alpha)$,where $R > 0$ and α is acute.

3. Jimmy is flying his kite at the end of a 50 m string. The string makes an angle of $\frac{\pi}{3}$ with the ground. The wind increases, and the kite flies higher until the string makes an angle of $\frac{\pi}{4}$ with the ground. Determine the **vertical displacement** of the kite as an **exact** value.

4. Fill in the blanks.

a) Evaluate the exact value of $\cot\left(\frac{3\pi}{2}\right)$ ____________________

b) Evaluate $\csc\left(\frac{\pi}{5}\right)$ to 4 decimal places ____________________

c) Evaluate the exact value of $\sec\left(\frac{7\pi}{6}\right)$ ____________________

5. Evaluate the exact value of the following expressions:

a) $\cos\left(\frac{7\pi}{12}\right)$.

b) $\cos\left(\frac{5\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)$

6. If $\cos\theta = -\frac{3}{8}$, $\pi \le \theta \le 3\pi$,determine the exact value of $\cos\left(\frac{\theta}{2}\right)$.

7. Prove that $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi - x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{3\pi}{2} + x\right)} = \sin^4(x) - \sin^2(x)$.

8. Prove the identity

a) $\csc^4(x) - \cot^4(x) = \dfrac{1 + \cos^2(x)}{1 - \cos^2(x)}$.

b) $\sin^4(x) + \cos^4(x) = \frac{3}{4} + \frac{1}{4}\cos(4x)$

10. If $\tan(2x) = -\frac{b}{a}$, $\frac{\pi}{2} \le 2x \le \pi$, then determine an expression for $\sin x \cos x$ in terms of a and b.

11 . A cable car rises 762 m as it moves a horizontal distance of 628 m.
a) How long is the ride?
b) What is the angle of inclination of the cable to the nearest degree?

12. An arc of a circle, centre O, subtends an angle of 1.5 radians at the centre. Determine the ratio of the length of arc AB to the length of line segment AB.

13. A skateboard ramp is built with an incline angle of $\frac{\pi}{12}$. If the base of the ramp is 1m in length, determine the exact height of the ramp.

14. A 150-cm tall person stands on the bank of a narrow river and observes a flagpole on the opposite bank. If the angle of elevation to the top of the flagpole is 0.25 and the angle of depression to the bottom of the flagpole is 0.09, determine the flagpole's height, in metres, to two decimal places.

15. If $\cos(x)+\sin(x)=k$, for what value(s) of k does $\sin(x)\cos(x)=\frac{1}{2}$?