Unit 2: Rational Functions

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2.1 Rational Functions and Their Essential Characteristics

A **rational function** is a function that can be expressed in the form $f(x)=\dfrac{P(x)}{Q(x)}$ where both P(x) and Q(x) are polynomial functions and the denominator Q(x) is of degree 1 or higher. Although polynomial functions are defined for all real values of x, rational functions are **not defined** for those values of x for which the denominator, Q(x), is 0.

Examples of rational functions:

$$y=\frac{1}{x-2} \qquad f(x)=\frac{2x}{3-x} \qquad g(x)=\frac{x^2-4}{x^2-2x}$$

Q1. Explain why each of the following function is not a rational function?

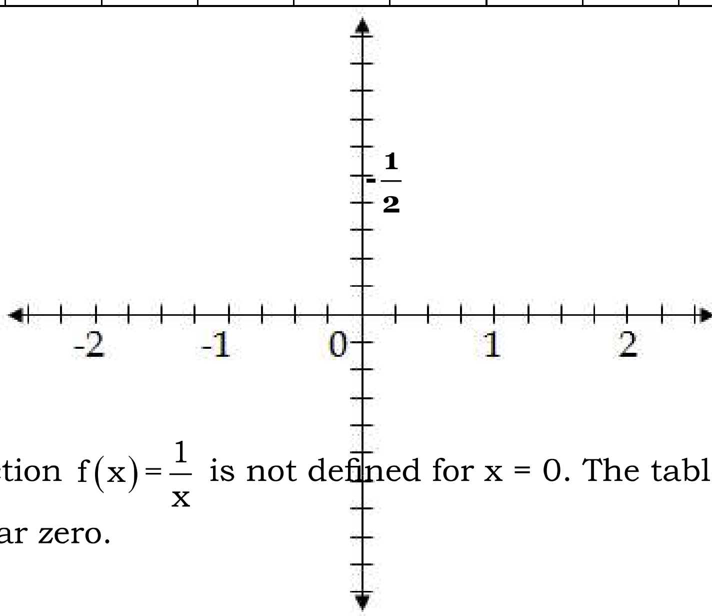
a) $y=\dfrac{x+20}{4}$

b) $f(x)=\dfrac{2\sqrt{x}-1}{x+3}$

Investigation: Properties of the simplest rational function $f(x)=\dfrac{1}{x}$

Graph the rational function $f(x)=\dfrac{1}{x}$ manually by completing a partial table of values. Plot the (*x* ,*y*) points and join them with a smooth curve.

x	**-3**	**-2**	**-1**	**-1/2**	**-1/3**	**-1/4**	**0**	**1/4**	**1/3**	**1/2**	**1**	**2**	**3**
y													



Note that the function $f(x)=\dfrac{1}{x}$ is not defined for x = 0. The tables below show the behavior of f(x) near zero.

$$\frac{1}{\text{small number}} = \text{BIG NUMBER}$$

x	f(x)
-0.1	
-0.01	
-0.00001	

x	f(x)
0.1	
0.01	
0.00001	

This behavior can be described in the following analytical way:

The next two tables show how f (x) changes as |x| becomes large.

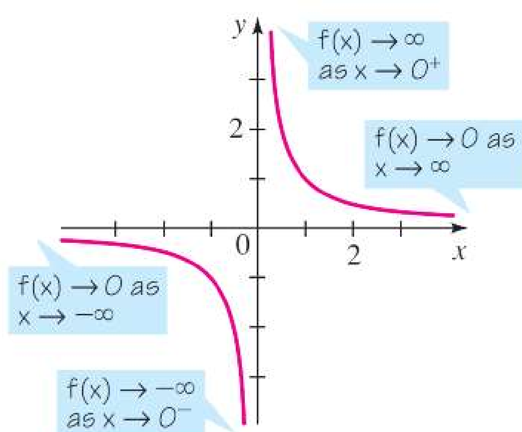
$$\frac{1}{\text{BIG NUMBER}} = \text{small number}$$

x	f(x)
-10	
-100	
-100 000	

x	f(x)
10	
100	
100 000	

This behavior can be described in the following analytical way:

The most important feature that distinguishes the graphs of rational functions is the presence of **asymptotes.**

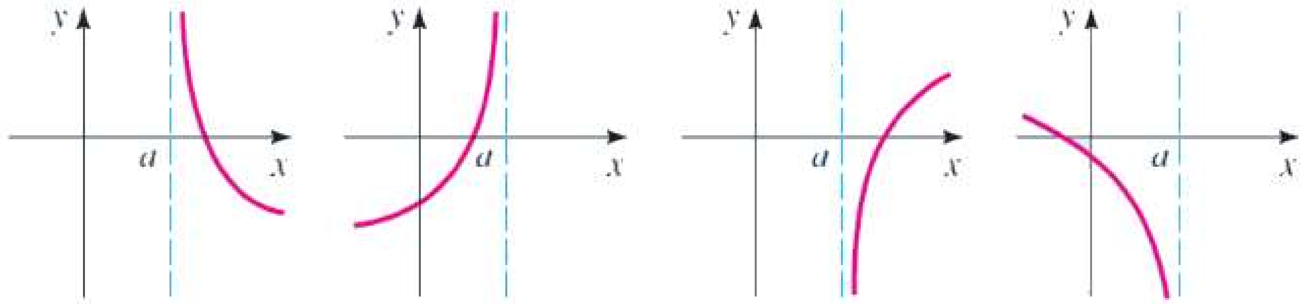


Definitions:

i) Vertical asymptote

The line x=a is a vertical asymptote of the graph of function $\mathbf{f}(\mathbf{x})$ *,if y approaches* $\pm\infty$ *as x approaches a from the left or right.*

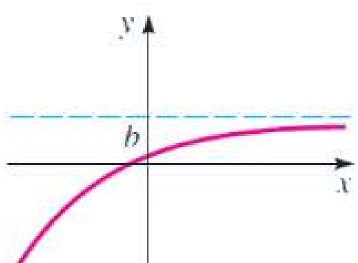
The following graphs illustrate each of the limit statements.

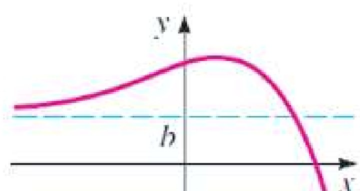


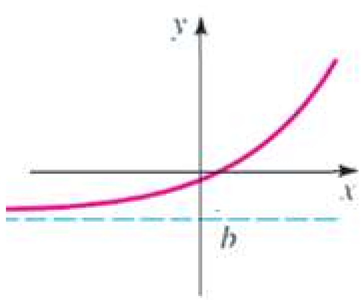
ii) Horizontal Asymptotes

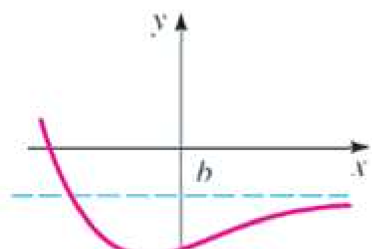
The line y = ***b*** *is a horizontal asymptote for the graph of a function* $\boldsymbol{f}(\boldsymbol{x})$ *if y approaches* ***b*** *(from above or below) as x approaches* $\pm\infty$*.*

The following graphs illustrate some typical ways that a curve may approach a horizontal asymptote:

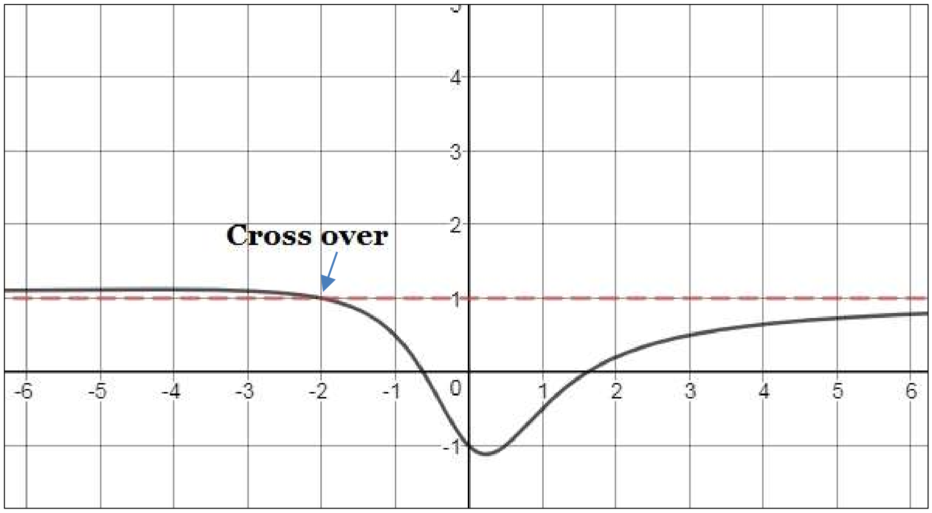








Note: A function can cross a horizontal asymptote for values of x that are "close" to the origin, it's called the **cross over**, but it can never cross a vertical asymptote.



General Rules on Finding the Horizontal and Vertical Asymptotes
Let *f* be the rational function

$$f(x)=\frac{a_n x^n+\ldots+a_1 x+a_0}{b_m x^m+\ldots+b_1 x+b_0}$$

- The vertical asymptotes of f(x) are the lines x=**a**, where **a** is the zero of denominator only.
- If n<m , then *f* has horizontal asymptote y = 0
- If n=m , then *f* has horizontal asymptote y = $\frac{a_n}{b_m}$
- If n>m , then *f* has no horizontal asymptote .

Example: Find the horizontal and vertical asymptotes for the following functions.

a. $f(x)=\frac{2x(x+1)(x-1)}{(x+2)(x-3)}$

b. $y=\frac{x^2-4x+5}{x^3-8}$

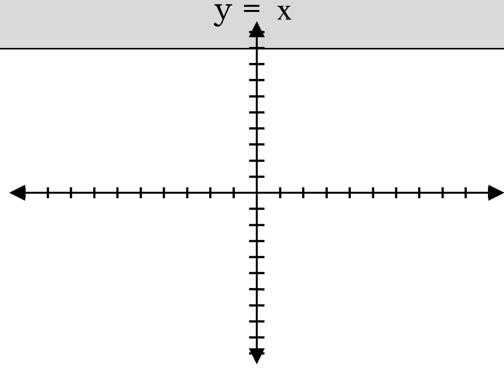
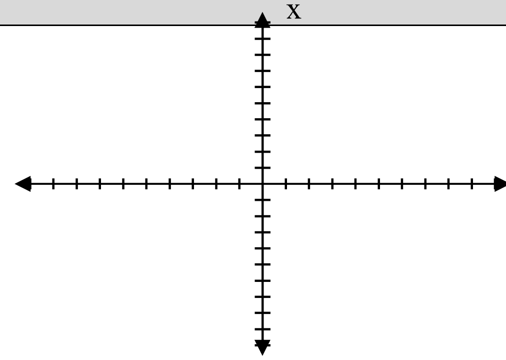
c. $f(x)=\frac{3x+1}{2-5x}$

d. $f(x)=\frac{(x-1)(x+1)(x+3)}{(x-4)(x-1)(2x+5)}$

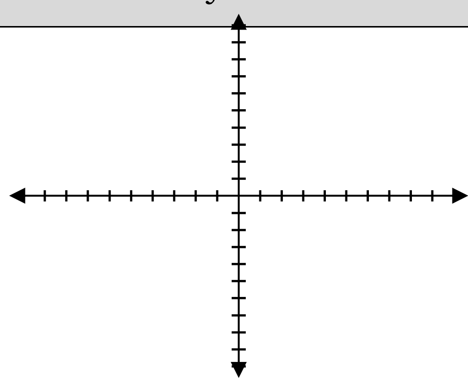
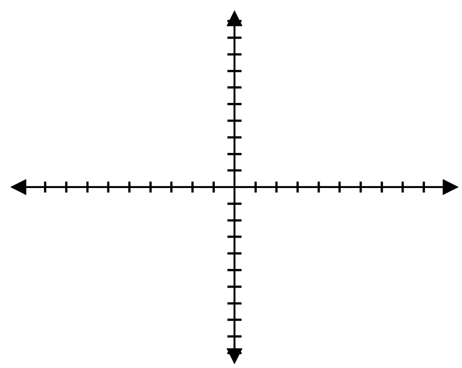
Reciprocal of Linear Functions

1. Use a graphing calculator to compare each of the following functions. Include a sketch of each.

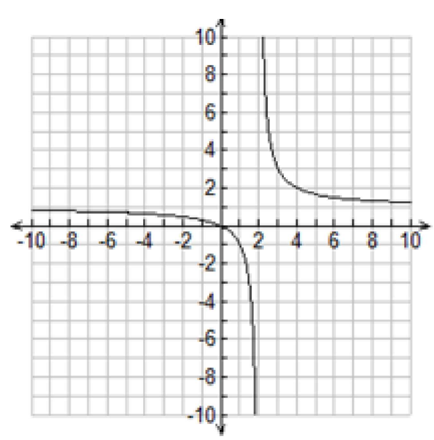
a)

$y = x$	$y = \frac{1}{x}$
	
Comparison	x-int: V.A: Interval that f >0: Interval that f <0:

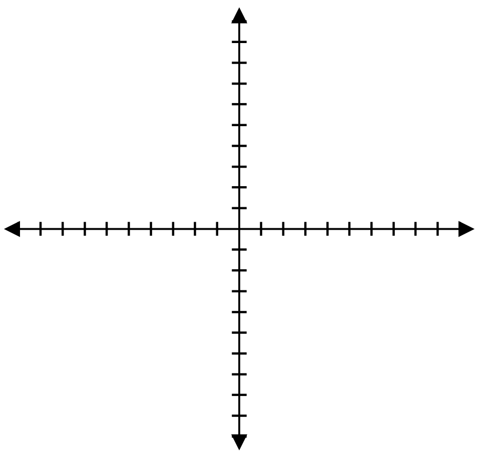
b)

$y = x+3$	$y = \frac{1}{x+3}$
	
Comparison	x-int: V.A: Interval that f >0: Interval that f <0:

2. Determine the equation of the following graph:



3. Determine the following information and sketch graph.

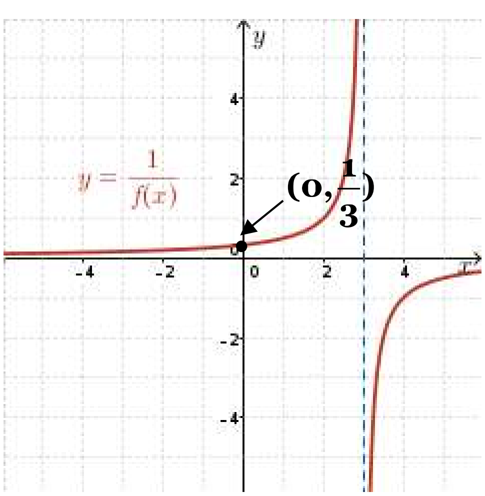


Equation	$y = \dfrac{1}{-2(x-2)}$		
Domain			
Range			
x-int		y-int	
H. Asymptote		V. Asymptote	

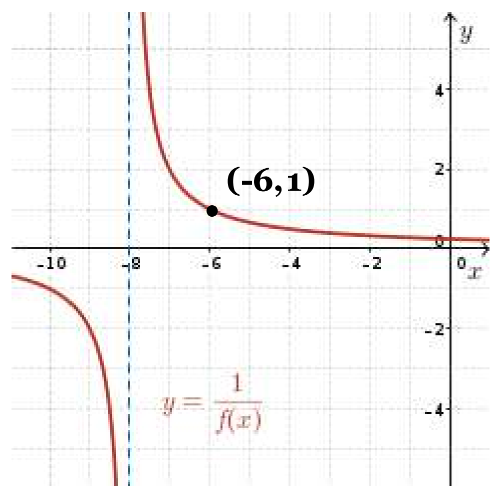
As $x \to$	**$f(x) \to$**
2^+	
2^-	
$+\infty$	
$-\infty$	

4. Given the graph of the reciprocal function $y = \dfrac{1}{f(x)}$, sketch the graph the function y=f(x). Determine an equation for each function.

a)



b)

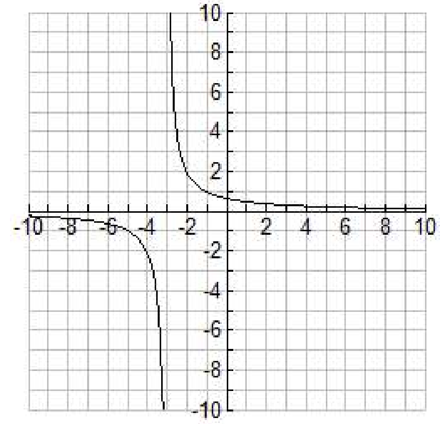


2.1-Practice:

1. Find the horizontal and vertical asymptotes for the following functions.

a. $f(x)=\dfrac{2x^2(x^2-1)}{(x+2)^2(x^2-4)}$ b. $f(x)=\dfrac{2(x-3)(x+2)(x+5)}{(x-1)(x+3)(x+5)}$

2. Determine the equation of the following graph:

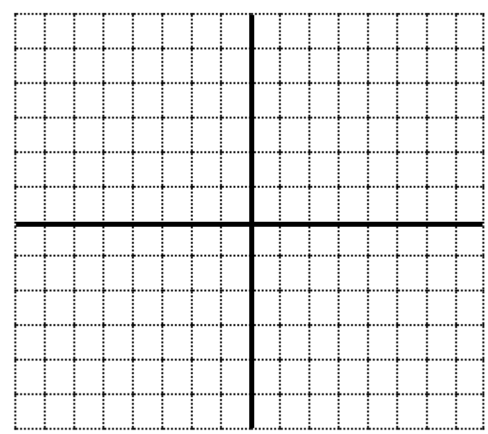


3. Graph $f(x)=-(4-x)$ and $g(x)=\dfrac{-1}{4-x}$ on the grid provided and for the g(x) identify:

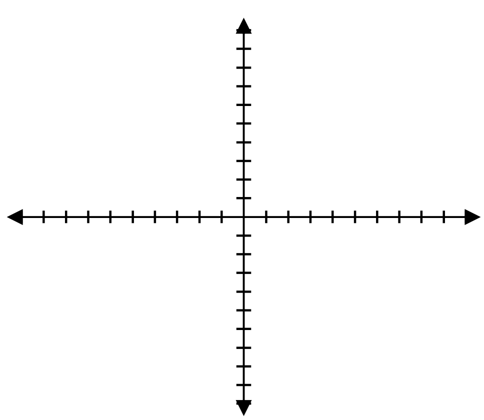
a) the domain

b) the range

c) the equation of the V.A. ____
d) the equation of the H.A. ____



4. Determine the following information and sketch graph.



Equation	$y=\dfrac{1}{2x-5}$		
Domain			
Range			
x-int		y-int	
H. Asymptote		V. Asymptote	

As $x\rightarrow$	$f(x)\rightarrow$
$\dfrac{5}{2}^{+}$	
$\dfrac{5}{2}^{-}$	
$+\infty$	
$-\infty$	

Warm Up

1. Which of the following are vertical asymptotes of $f(x)=\dfrac{(ax-b)^2}{(ax+b)(ax-b)}, a, b \neq 0 \text{ and } a, b \in \mathbb{R}$?

   A) $x=\dfrac{b}{a}$ B) $x=\dfrac{-b}{a}$ C) $x=\pm\dfrac{b}{a}$ D) $y=1$

2. Which of the following is true regarding the function $f(x)=\dfrac{x+3}{x^2-5}$.

   A) $f(x)$ has no vertical asymptotes
   B) $f(x)$ has an x intercept at $x=3$
   C) As $x \to \infty, f(x) \to 0$ from above
   D) $f(x)$ has a horizontal asymptote at $y=1$

3. Which of the following functions has vertical asymptotes at $x=1$ and $x=-3$ and a horizontal asymptote at $y=0$?

   A) $y=\dfrac{x^2-6x+9}{x^2-2x-3}$ B) $y=\dfrac{x^2}{x^2+2x-3}$ C) $y=\dfrac{x-1}{x+3}$ D) $y=\dfrac{x-9}{x^2+2x-3}$

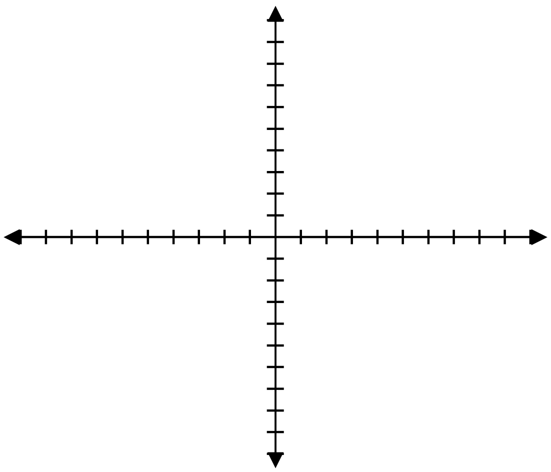
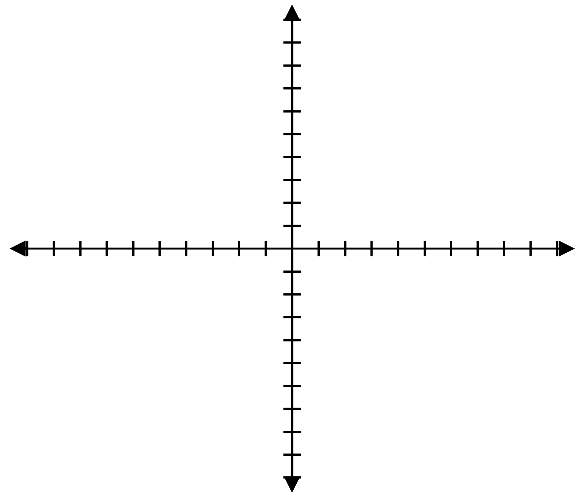
4. Which of the following is true about the function $f(x)=\dfrac{-2}{x-6}$ as $x \to 6^+$?

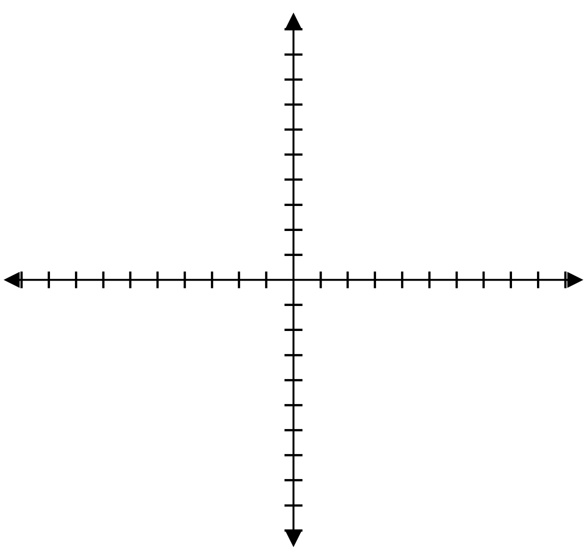
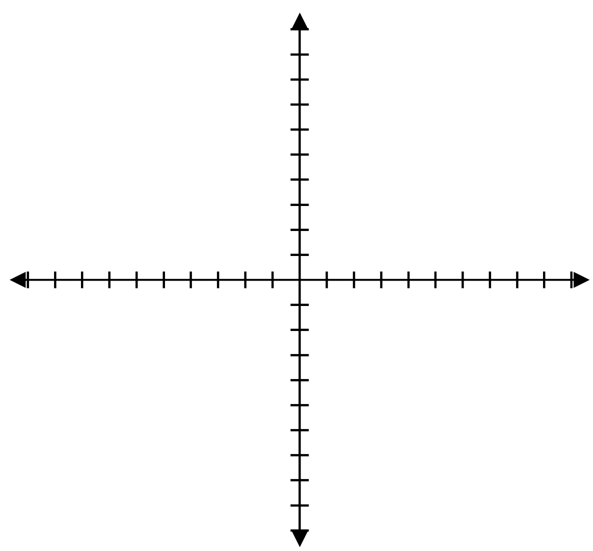
   A) $f(x) \to 0$ (from above) B) $f(x) \to -\infty$
   C) $f(x) \to \infty$ D) $f(x) \to 0$ (from below)

5. Write the equation of a rational function, in the form $f(x)=\dfrac{g(x)}{h(x)}$, with vertical asymptotes at $x=-\dfrac{3}{4}$ and $x=-5$, $x-$intercepts at ± 2, a horizontal asymptote at $y=-4$.

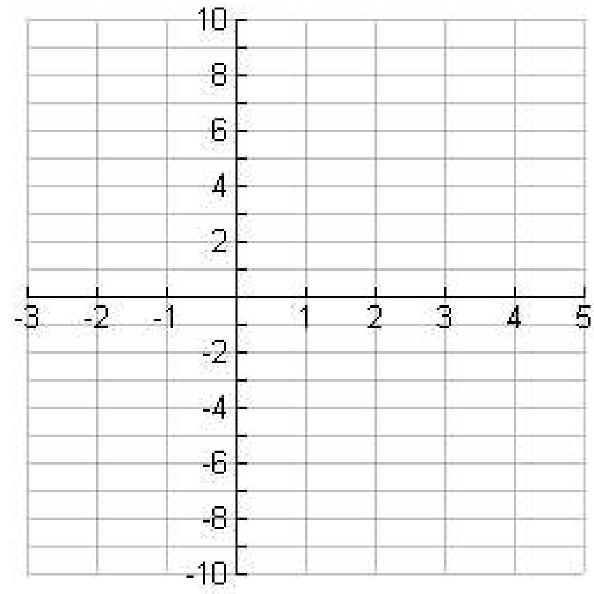
2.2 Reciprocal of Quadratic Functions

1. Use a graphing calculator to compare each of the following functions. Include a sketch of each.

$y = x^2$	$y = \dfrac{1}{x^2}$
	
comparison	

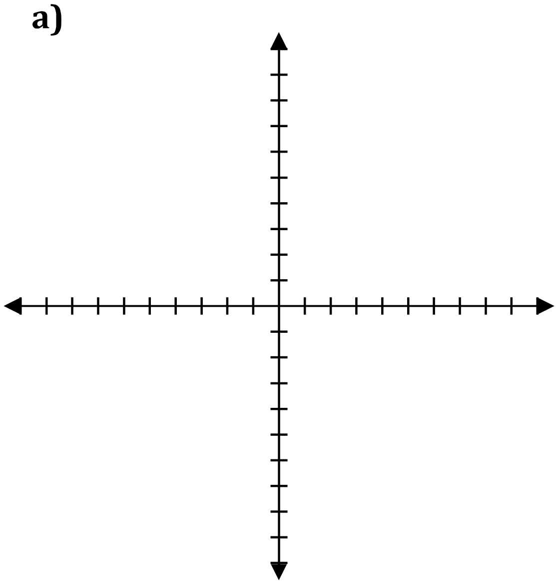
$y = (x-3)(x+1)$	$y = \dfrac{1}{(x-3)(x+1)}$
	
comparison	

2. Sketch the graph of $f(x) = -x^2 + 5x - 6$ and its reciprocal on the same axis. Clearly identify the intersection(s) between f(x) and its reciprocal.



3. Determine the following information about each of the following graphs

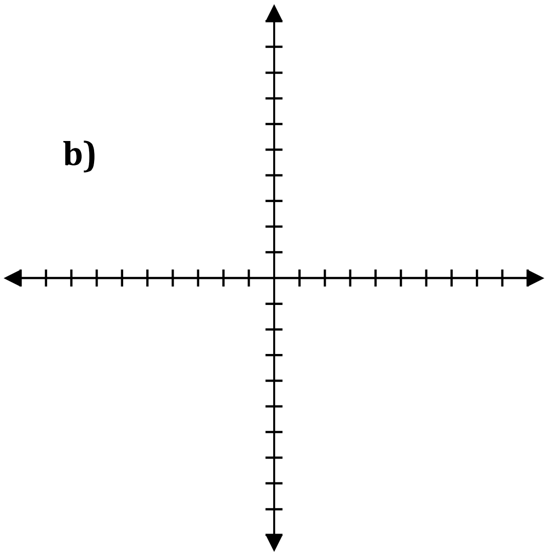
a)



Equation	$y = \dfrac{1}{(x+3)(x-3)}$		
Domain			
Range			
x-int		y-int	
Max/Min			
H. Asymptote:		V. Asymptote(s):	

As $x \rightarrow$	**$f(x) \rightarrow$**
3^+	
3^-	
-3^+	
-3^-	
$+\infty$	
$-\infty$	

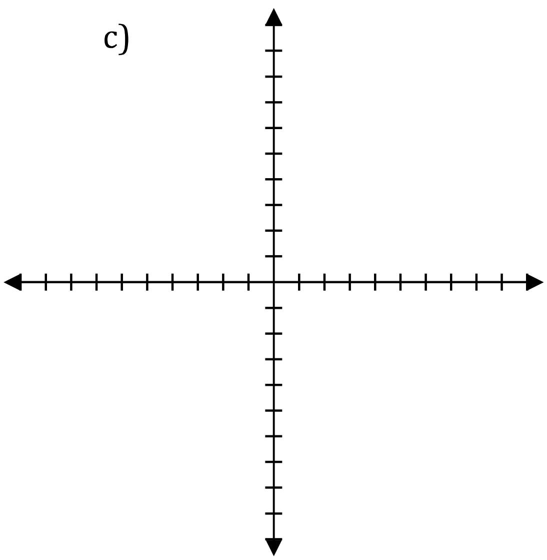
b)



Equation	$y = \dfrac{-1}{(x-2)^2}$		
Domain			
Range			
x-int		y-int	
Max/Min			
H. Asymptote:		V. Asymptote(s):	

As $x\to$	$f(x)\to$
2^+	
2^-	
$+\infty$	
$-\infty$	

c)

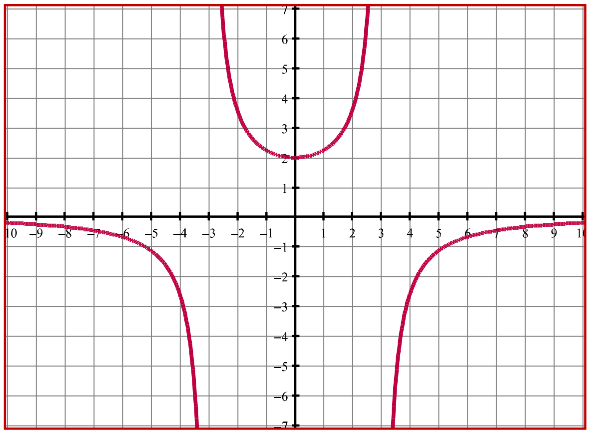


Equation	$y = \dfrac{1}{x^2+4}$		
Domain			
Range			
x-int		y-int	
Max/Min			
H. Asymptote:		V. Asymptote(s):	

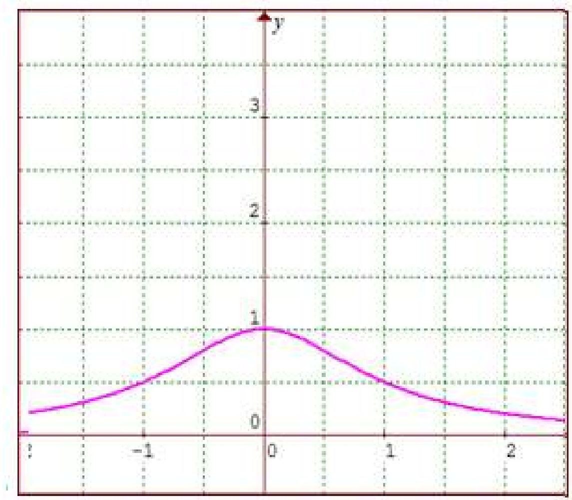
As $x\to$	$f(x)\to$
$+\infty$	
$-\infty$	

4. Determine the equation of each of the following graphs:

a)



b)



Summary

Reciprocals of quadratics can be classified into 3 different types:

i. $f(x) = \frac{1}{(x-a)^2}$

_______ vertical asymptotes and

_______ "branches"

ii. $f(x) = \frac{k}{(x-a)(x-b)}$

_______ vertical asymptotes and

_______ "branches"

iii. $f(x) = \frac{1}{x^2+a}$, $a > 0$

_______ vertical asymptotes (hat shaped)

- All the y-coordinates of a reciprocal function are the reciprocals of the y-coordinates of the original function.
- The graph of a reciprocal function has a vertical asymptote at each zero of the original function.
- A reciprocal function will always have as a horizontal asymptote if the original function is linear or quadratic.
- A reciprocal function has the same positive/negative intervals as the original function.
- Intervals of increase on the original function are intervals of decrease on the reciprocal function. Intervals of decrease on the original function are intervals of increase on the reciprocal function.
- If the range of the original function includes 1 and/or-1 the reciprocal function will intersect the original function at a point (or points) where the y-coordinate is 1 or -1 .
- If the original function has a local minimum point, the reciprocal function will have a local maximum point at the same x-value (and vice versa).

2.2 Practice

1) Find constants a and b that guarantee that the graph of the function defined by $h(x)=\dfrac{ax^2+7}{9-bx^2}$ will have a vertical asymptote at $x=\pm\dfrac{3}{5}$ and a horizontal asymptote at $y=-2$.

2) Sketch the graph of following functions .

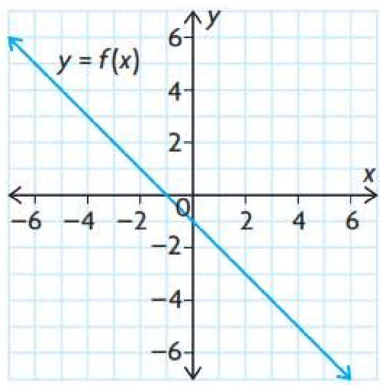
   a) $y=\dfrac{-1}{x(x-5)}$

   b) $y=\dfrac{-3}{(x+3)^2}$

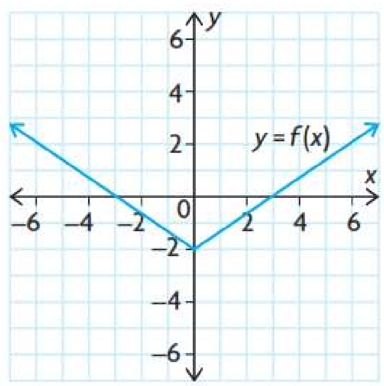
3) For each case, create a function that has a graph with the given features.
   (a) a vertical asymptote x = 1 and a horizontal asymptote y = 0
   (b) two vertical asymptotes x = -1 and x = 3, horizontal asymptote y = -1, and x-intercepts – 2 and 4.

4) Sketch the graph of the reciprocal of each function.

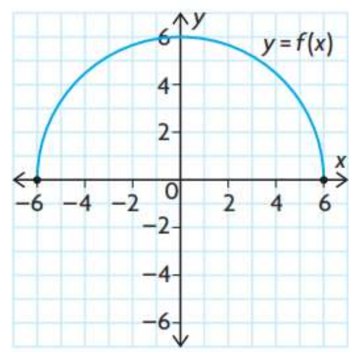
a)



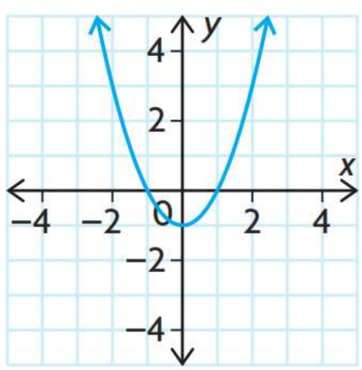
b)



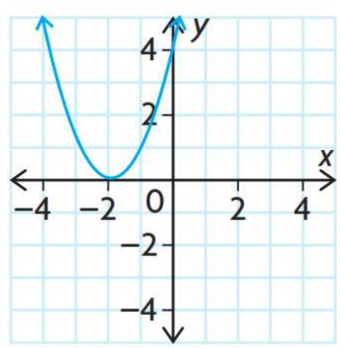
c)



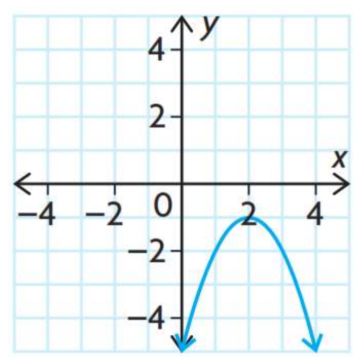
d)



e)



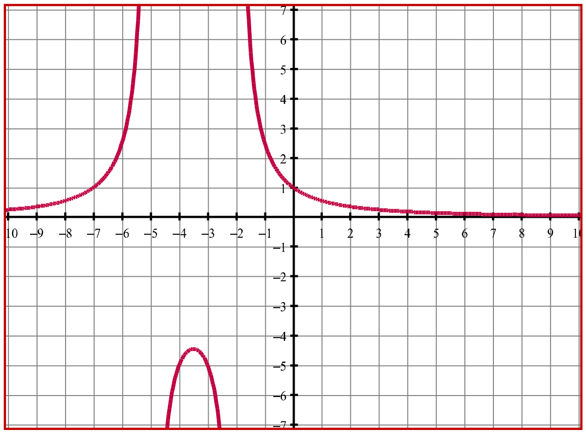
f)



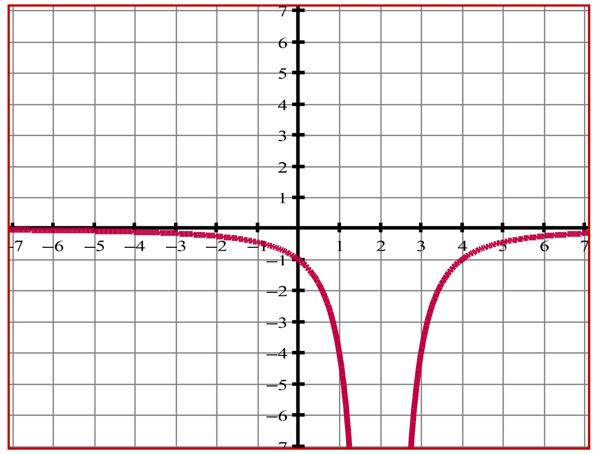
5) Sketch the graph of $f(x)=-2x^2+10x-12$ and its reciprocal on the same axis. Clearly identify the intersection(s) between f(x) and its reciprocal.

6) Determine the equation of each of the following graphs:

a)



b)



Warm up

1. Which of the following functions does **not** have a vertical asymptote?

A) $f(x)=\dfrac{x}{x^2-x}$ B) $f(x)=\dfrac{x^2-1}{x}$

C) $f(x)=\dfrac{x-1}{x^2-x}$ D) $f(x)=\dfrac{x^2-1}{x-1}$

2. The function $f(x)=\dfrac{1}{x^2-6x-16}$ has a local maximum at

A) $f(x)=\pm 1$ B) $(3,-25)$

C) $\left(3,\dfrac{-1}{25}\right)$ D) $\left(0,\dfrac{-1}{16}\right)$

3. Which of the following has a horizontal asymptote of $y=1$?

A) $f(x)=\dfrac{x^2+2x-24}{x^3-64}$ B) $f(x)=\dfrac{2x^2-2x-24}{4x^2-64}$

C) $f(x)=\dfrac{x^4+3x^2-40}{(x^2+1)(x+8)}$ D) $f(x)=\dfrac{(x^2-4)(2x-9)}{(x)(2x-3)(x-1)}$

4. Consider the function $f(x)=\dfrac{1}{x^2+6x+8}$.

a) Determine the point at which the slope of the tangent is 0.

b) Determine the equation of the tangent line at this point. ________________

2.3 Rational Functions of the Form $f(x)=\dfrac{ax+b}{cx+d}$

1. Consider the following:

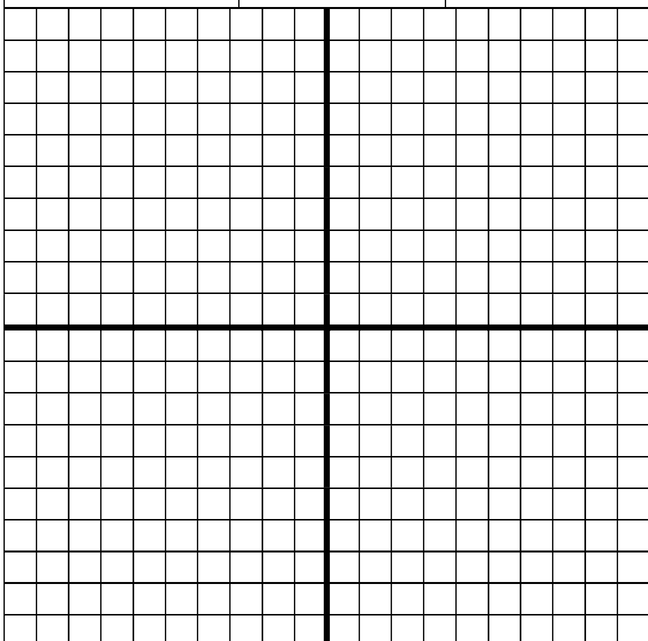
Function	H.A	$y =$	V.A	$x =$
$f(x)=\dfrac{1}{x}$	*as x* $\to -\infty$, *f(x)* $\to$ --------- *as x* $\to \infty$, *f(x)* $\to$ ---------		*as x* $\to$ ---------- , *f(x)* $\to$ ---------- *as x* $\to$ --------- , *f(x)* $\to$ ---------	
Function	**H.A**	$y =$	**V.A**	$x =$
$f(x)=\dfrac{1}{3x+2}$	*as x* $\to -\infty$, *f(x)* $\to$ --------- *as x* $\to \infty$, *f(x)* $\to$ ---------		*as x* $\to$ --------- *f(x)* $\to$ --------- *as x* $\to$ ---------, *f(x)* $\to$ ---------	
Function	**H.A**	$y =$	**V.A**	$x =$
$f(x)=\dfrac{1}{x-2}$	*as x* $\to -\infty$, *f(x)* $\to$ --------- *as x* $\to \infty$, *f(x)* $\to$ ---------		*as x* $\to$ --------- *f(x)* $\to$ --------- *as x* $\to$ ---------, *f(x)* $\to$ ---------	

2. What happens if we change the numerator so that the number is not a constant?

Function	H.A	$y =$	V.A	$x =$
$f(x)=\dfrac{x}{x+5}$	*as x* $\to -\infty$, *f(x)* $\to$ --------- *as x* $\to \infty$, *f(x)* $\to$ ---------		*as x* $\to$ ---------- , *f(x)* $\to$ -------- *as x* $\to$ --------- , *f(x)* $\to$ -------	
Function	**H.A**	$y =$	**V.A**	$x =$
$f(x)=\dfrac{x+2}{4x-5}$	*as x* $\to -\infty$, *f(x)* $\to$ --------- *as x* $\to \infty$, *f(x)* $\to$ ---------		*as x* $\to$ ---------- , *f(x)* $\to$ -------- *as x* $\to$ --------- , *f(x)* $\to$ -------	
Function	**H.A**	$y =$	**V.A**	$x =$
$f(x)=\dfrac{3-2x}{4x-1}$	*as x* $\to -\infty$, *f(x)* $\to$ --------- *as x* $\to \infty$, *f(x)* $\to$ ---------		*as x* $\to$ ---------, *f(x)* $\to$ --------- *as x* $\to$ ---------, *f(x)* $\to$ --------	

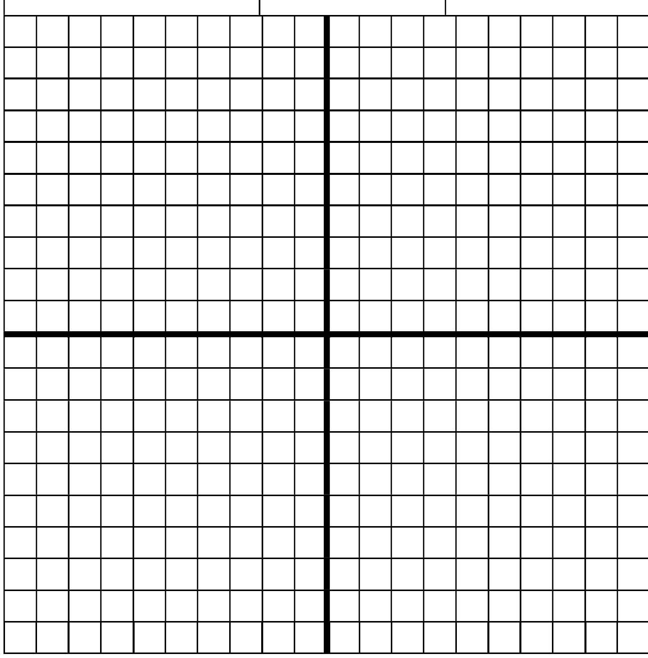
Example#1: Fill in the chart for each of the following, and then sketch the graph.

Function	x-intercept	y-intercept	Equation of the V.A.	Equation of the H.A.
$f(x)=\dfrac{x}{x-3}$				



As $\boldsymbol{x}\to$	$\boldsymbol{f(x)}\to$
3^{+}	
3^{-}	
$+\infty$	
$-\infty$	

Function	x-intercept	y-intercept	Equation of the V.A.	Equation of the H.A.
$f(x)=\dfrac{-4x-3}{2x+1}$				



As $\boldsymbol{x}\to$	$\boldsymbol{f(x)}\to$
$-\frac{1}{2}^{+}$	
$-\frac{1}{2}^{-}$	
$+\infty$	
$-\infty$	

Example#2. Determine the value of constants *a* and *b* that guarantee the graph of the function defined by $f(x) = \frac{7 - ax}{bx + 2}$ will have a vertical asymptote of $x = -3$ and a horizontal asymptote of $y = -2$.

Example#3. Find the linear over linear function , $f(x) = \frac{ax+b}{cx + d}$, satisfying the given requirements: $f(10) = 20$, $f(30) = 25$, and the graph of $f(x)$ has $y = 30$ as its horizontal asymptote.

Exit Card!

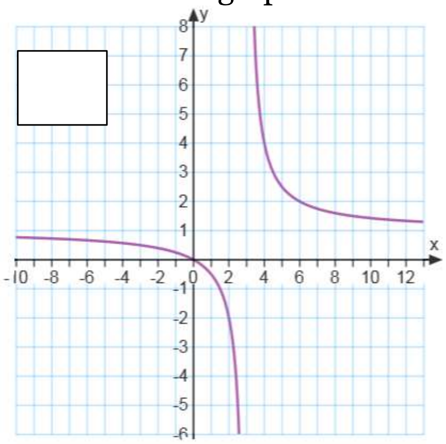
1. State the equations for all asymptotes of $y = \dfrac{1-2x}{3x-1}$.

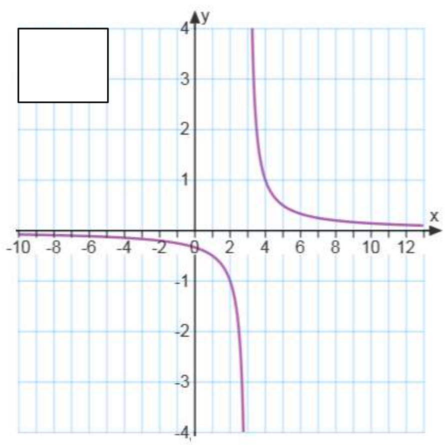
   H.A. ___________ , V.A. _____________

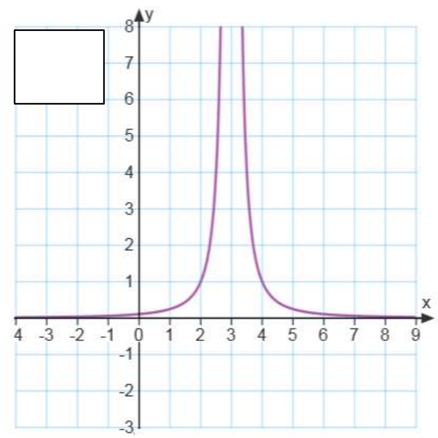
2. The following graphs represent the functions:

   a) $y = \dfrac{x}{x-3}$ b) $y = \dfrac{1}{x-3}$ c) $y = \dfrac{1}{(x-3)^2}$

   Match each graph with the appropriate equation.







3. Find constants **a** and **b** that guarantee that the graph of the function defined by $f(x) = \dfrac{ax+3}{4-bx}$ will have a vertical asymptote at x = 2 and a horizontal asymptote at y = -2.

2.3 Practice

1. The more you study for a certain exam, the better your performance on it. If you study for 10 hours, your score will be 55%. If you study for 26 hours, your score will be 95%. You can get as close as you want to a perfect score just by studying long enough. Assume your percentage score is a linear-to-linear function of the number of hours that you study. If you want a score of 80%, how long do you need to study?

2. Clyde makes extra money selling tickets in front of the Safeco Field. The amount he charges for a ticket depends on how many he has. If he only has one ticket, he charges $100 for it. If he has 10 tickets, he charges $80 a piece. But if he has a large number of tickets, he will sell them for $50 each. How much will he charge for a ticket if he holds 20 tickets?

3. Sketch he graph of the following functions.
   a. $y = \dfrac{2x}{x-1}$
   b. $y = \dfrac{4x+3}{x+2}$
   c. $y = \dfrac{-1}{x+3} + 2$
   d. $y = \dfrac{-3x+5}{2x-4}$
   e. $y = \dfrac{4}{2x+3} - 1$

4. Rosetta is growing a bamboo plant in her apartment. The height of the plant is a linear-over-linear function of time. Thirty days ago, the plant was 14 cm high. Today, the plant is 18 cm high. The plant always increases in height, and will approach (but never exceed) a height of 32 cm. Find a function representing the height of the plant as a function of time.

5. Sketch a possible function with the given features:
   i. Vertical asymptotes at $x = -1$ and $x = 3$
   ii. x-intercepts at $x = -2$ and $x = 4$
   iii. y-intercept at y = -4
   iv. the function is not defined at $x = 1$
   v. As $x \to 3^-, f(x) \to -\infty$
   vi. As $x \to -\infty, f(x) \to -2$ from above
   vii. As $x \to +\infty, f(x) \to -2$ from above

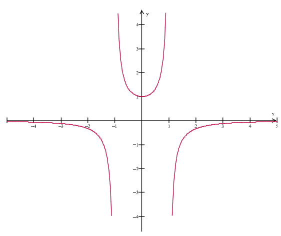
Self Assessment

Part A:

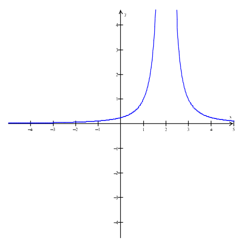
Multiple Choice: Write the CAPITAL letter corresponding to the correct answer on the line provided. One mark per question.

1. Which graph represents $f(x)=\dfrac{1}{(x-2)^2}$?

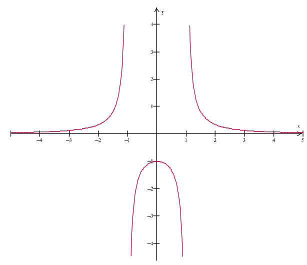
A)



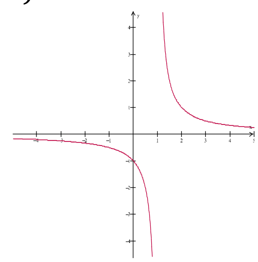
B)



C)



D)



2. State the interval(s) for which the function $h(x)=\dfrac{1}{(x-1)(x-3)}$ is above the x-axis

A) $x<3$ B) $1<x<3$

C) $x>1$ D) $x<1\ ,\ x>3$

3. Which of the following functions does not have a horizontal asymptote?

A) $y=\dfrac{x^3-3x^2+3x-1}{x^2-5}$ B) $y=\dfrac{x+1}{x-1}$

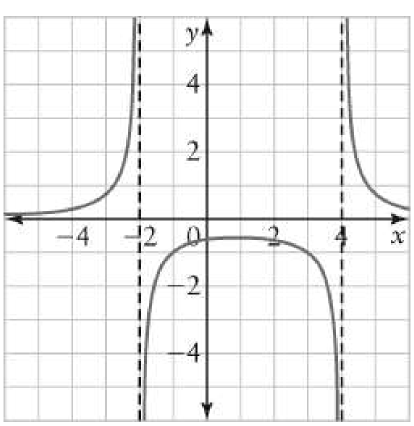
C) $y=\dfrac{x^2-1}{x^3+8}$ D) $y=\dfrac{3x^2-5x+2}{2x^2-5}$

4. What is true about the function $f(x)=\dfrac{2-x}{3x+5}$ as $x\to\pm\infty$?

A) $f(x)\to\dfrac{2}{3}$ B) $f(x)\to-\dfrac{1}{3}$

C) $f(x)\to 0$ D) $f(x)\to\dfrac{1}{3}$

5. Over what interval(s) is the graph of the rational function decreasing?



A) $x\in(-2,4)$
B) $x\in(1,4)\cup(4,\infty)$
C) $x\in(-\infty,-2)\cup(-2,1)$
D) $x\in(-\infty,-2)\cup(4,\infty)$

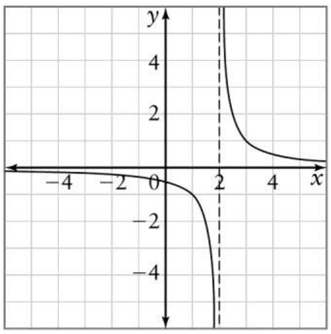
Part B: Full Solutions

1. Copy and complete the table to describe the behaviour of the function $f(x) = -\frac{1}{x+4}$.

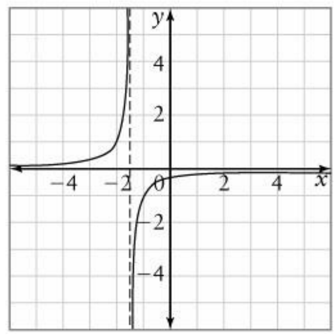
As *x*→	***f*(*x*)→**
-4^+	
-4^-	
$+\infty$	
$-\infty$	

2. Determine a possible equation to represent each function shown.

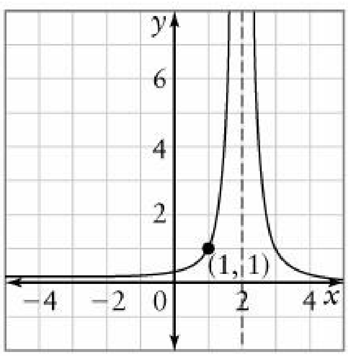
a)



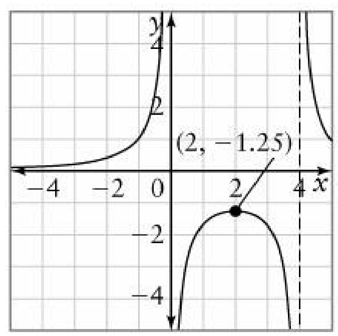
b)



c)



d)



3. Sketch a graph of each function. Label the *y*-intercept. State the domain, the range and the equations of asymptotes.

a) $f(x) = -\frac{x+2}{x-5}$ **b)** $h(x) = \frac{2x}{5-x}$

4. Copy and complete the table to describe the behaviour of the function $f(x) = \frac{1}{(x+2)(x+5)}$.

As *x*→	***f*(*x*)→**
-2^+	
-2^-	
-5^+	
-5^-	
$+\infty$	
$-\infty$	

5. Determine the equations for the vertical asymptotes, if they exist, for each function. Then, state the domain.

a) $f(x) = -\frac{1}{x^2 - 7x + 6}$

b) $f(x) = \frac{1}{x^2 + 4x + 6}$

6. For each function,
   i) determine the equations for the asymptotes, if they exist
   ii) give the domain
   iii) determine *y*-intercept
   iv) sketch a graph of the function

a) $y = \frac{1}{(x-2)(x+4)}$

b) $y = \frac{-1}{(x+4)^2}$

c) $y = \frac{1}{x^2 + x + 4}$

7. Determine the **point(s)** of intersection between $f(x) = \frac{1}{2x^2 - 16x + 33}$ and its reciprocal, if any.

2.4 Oblique (Slant) Asymptotes

If $f(x)=\dfrac{P(x)}{Q(x)}$ is a rational function in which the degree of the numerator is one more than the degree of the denominator, we can use the Division Algorithm to express the function in the form

$$f(x)=(mx+b)+\frac{R(x)}{Q(x)}$$

where the degree of R is less than the degree of Q and m ≠ 0. This means that as $x \to \pm\infty$, $\dfrac{R(x)}{Q(x)} \to 0$, so for large values of |x|, the graph of y = f(x) approaches the graph of the line y = mx + b. In this situation we say that y = mx + b is a **slant asymptote**, or an **oblique asymptote**.

To test the behaviour, we must find $\left[f(x)-(mx+b)\right]$ which is $\dfrac{remainder}{Q(x)}$.

- Let $x \to \infty$, sub x = 100 into $\dfrac{remainder}{Q(x)}$. If $\dfrac{remainder}{Q(x)}$>0, the approach is from above otherwise the approach is from below.

- Let $x \to -\infty$, sub x = – 100 into $\dfrac{remainder}{Q(x)}$ .If $\dfrac{remainder}{Q(x)}$>0, the approach is from above otherwise the approach is from below.

To conclude, we must write:

as $x \to \infty, \left[f(x)-(mx+b)\right] \to 0$ from above/below or

as $x \to -\infty, \left[f(x)-(mx+b)\right] \to 0$ from above/below

Ex.1 Find the oblique asymptote of the following rational functions.

a. $f(x)=\dfrac{x^2-4x-5}{x-3}$

$$\begin{array}{r|l} & x-1 \\ x-3 & x^2-4x-5 \\ & \underline{-x^2+3x} \\ & -x-5 \\ & \underline{x-3} \\ & -8 \end{array}$$

O.A. $y=x-1$

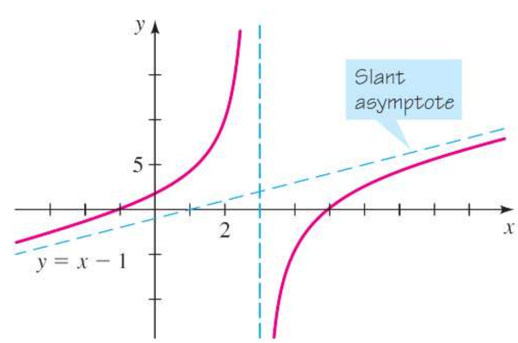
Since $f(x)=x-1+\left(\dfrac{-8}{x-3}\right)$

As $x \to \infty$, sub x = 100 into $\dfrac{-8}{x-3}$. Since $\dfrac{-8}{100-3}<0$ the approach is from below.

As $x \to -\infty$, sub x =- 100 into $\dfrac{-8}{x-3}$. Since $\dfrac{-8}{-100-3}>0$ the approach is from above.

Conclusion: As $x \to \infty$, $\left[f(x)-(x-1)\right] \to 0$ (below)

As $x \to -\infty$,$\left[f(x)-(x-1)\right] \to 0$ (above)



b. $g(x)=\dfrac{2x^3-3x^2+2x-7}{x^2-4x+2}$

O.A. ___________

As $x\to\infty$, (above/below)

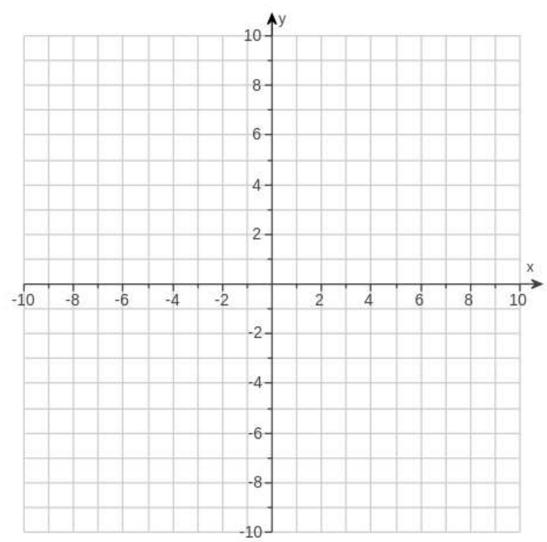
Recall: sub x = 100 into $\dfrac{remainder}{Q(x)}$. If $\dfrac{remainder}{Q(x)}$>0, the approach is from above otherwise the approach is from below

As $x\to-\infty$, (above/below)

Recall: sub x = – 100 into $\dfrac{remainder}{Q(x)}$ .If $\dfrac{remainder}{Q(x)}$>0, the approach is from above otherwise the approach is from below.

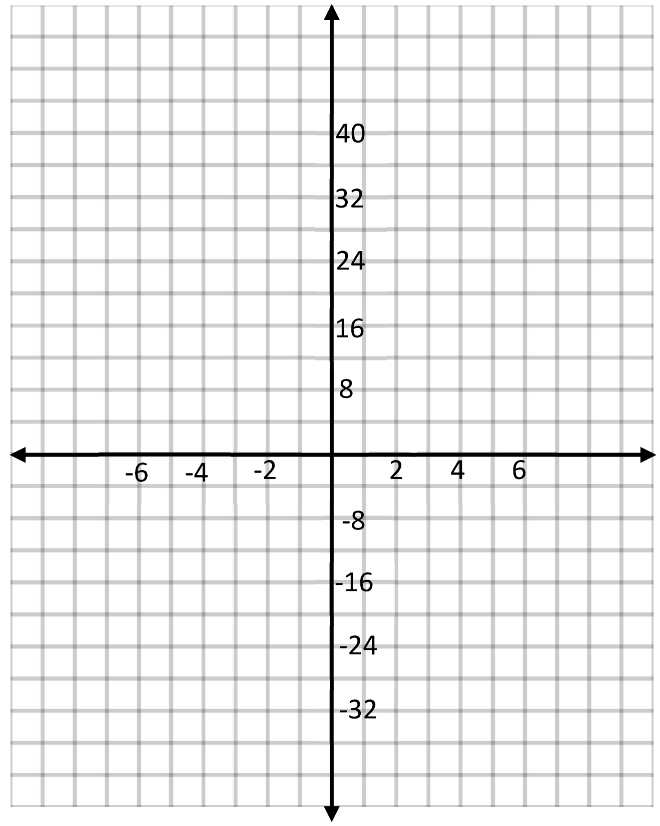
Conclusion:

c. Determine the vertical asymptote, oblique asymptote, x-intercept(s) and y-intercept of $f(x)=\dfrac{x^2-3x-4}{x-1}$. Then graph $f(x)$.



Does f(x) cross the oblique asymptote? How do you know?

d. Graph $f(x) = \frac{2x^3 - 4x^2 - 9}{4 - x^2}$. Be sure to check for any cross-overs.



2.4 Practice

1) Which of the following has an oblique asymptote?

A) $f\left(x\right)=\dfrac{x^2-49}{x+7}$ B) $f\left(x\right)=\dfrac{x^2+49}{x+7}$

C) $f\left(x\right)=\dfrac{x^2-49}{x-7}$ D) both A) and C)

2) Which of the following functions crosses its horizontal asymptote?

A) $f\left(x\right)=\dfrac{x^2-2x-8}{(x+1)(x^2-16)}$ B) $f\left(x\right)=\dfrac{-5}{x^2-7x+17}$

C) $f\left(x\right)=\dfrac{x^2-49}{x+7}$ D) Horizontal asymptotes can never be crossed

3) Find all asymptotes, then analyze behavior of function near asymptotes:

a) $f(x)=\dfrac{x^2+3x+2}{x-2}$ b) $f(x)=\dfrac{2x^3-4x^2-9}{4-x^2}$ c) $f(x)=\dfrac{-x^2}{x-3}$

4) a. Given $f(x)=\dfrac{3x^3+x^2+9}{x^2+2x+1}$. Determine the equation of the oblique asymptote.

b. Determine the **exact** point(s) where $f(x)$ crosses the oblique asymptote.

5) Find constants a,b,c and n that guarantee that the graph of the function defined by $f(x)=\dfrac{ax^2+bx-3}{x^n+c}$ will have a vertical asymptote of x=−3 and an oblique asymptote of y=2x+1.

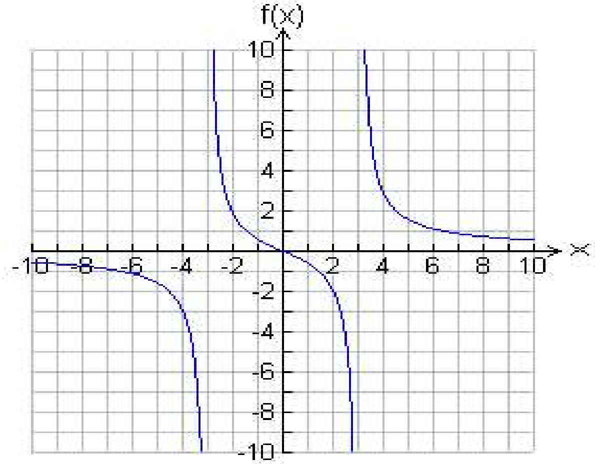
2.5 Solving Rational Function Inequalities

Solving rational inequalities is very similar to solving polynomial inequalities.
We need to consider rational functions are not defined at the zeros of the denominator.

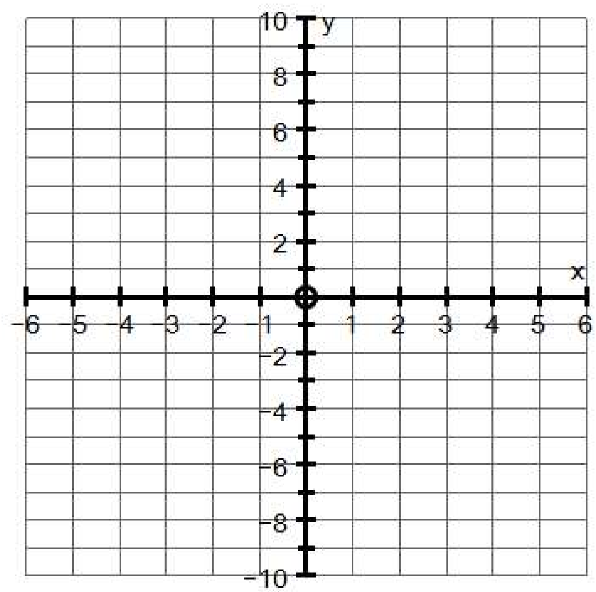
Example 1: For the rational function, f(x), solve when

i) $f(x) > 0$

ii) $f(x) \leq 0$



Example 2: Solve $\frac{2}{x} \geq x-1$ graphically.



Solving Rational Inequalities Algebraically
Rational inequalities can be solved algebraically, graphically, or comparing values in a table.

- Move everything to the left side and set equal to zero.
- Rewrite the left side as a single fraction.
- Factor the numerator and denominator.
- Use the numerator to find the zero points.

  < 0 or > 0 open

  ≤ 0 or ≥ 0 closed
- Use the denominator to find discontinuity points.
- Locate the points on a number line.
- Label intervals formed as positive (+) or negative (–) by testing points.
- Shade appropriate intervals.

  < 0 or ≤ 0 negative

  > 0 or ≥ 0 positive
- Give a numeric solution.

Example 3 : Solve each of the following, $x \in R$

a) $\frac{2}{x-1} > 0$

b) $\frac{x-2}{x^2+x} \geq 0$

c) $\frac{x^2+4x-5}{1-x} < 0$

d) $\frac{x}{x+1} \le \frac{4x-3}{x-7}$

e) $\frac{2x}{x-1} \le \frac{x}{x+2}$

f) $\frac{x^2+3x}{(x^2+10x+21)(x^2+4)} \ge \frac{-x}{x^3-2x^2+4x-8}$

Exit Card!

Solve $\frac{2x}{x+1} \geq \frac{x+6}{x+3}$.

Collect all terms to the left side of the inequality and 0to the right side.

Simplify the expression on the left side by finding a common denominator and adding the terms.

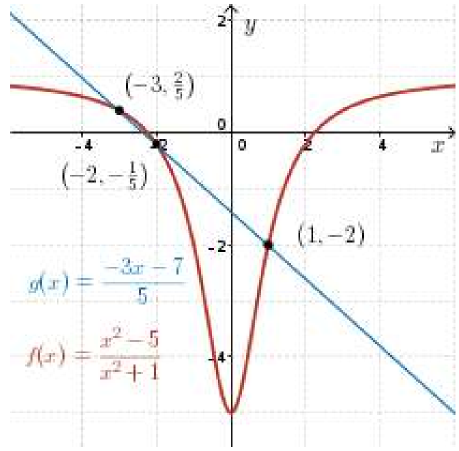
Once simplified, factor the numerator, if possible, always leaving the denominator in factored form

Create an interval table and identify the sign of each factor in the rational expression within each interval.

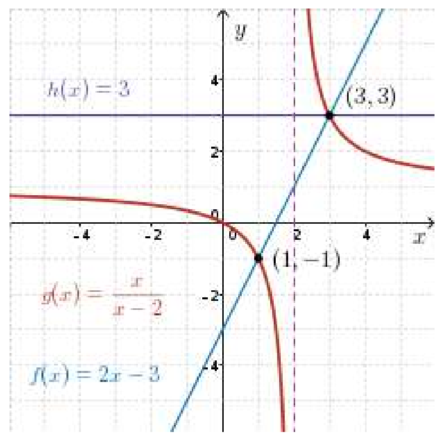
The zero value of each factor in the numerator and denominator indicates when a sign change may occur for that factor.

2.5 Practice

1. a. Given the graphs of $f(x)=\dfrac{x^2-5}{x^2+1}$ and $g(x)=\dfrac{-3x-7}{5}$, determine the solution of $f(x)<g(x)$.



b. Given the graphs of $f(x)=2x-3$, $g(x)=\dfrac{x}{x-2}$ and $h(x)=3$, determine the solution of



i. $\dfrac{x}{x-2}\geq 2x-3$

ii. $\dfrac{x}{x-2}\leq 3$

iii. $2x-3\leq\dfrac{x}{x-2}\leq 3$

2. Solve each inequality graphically, where $x\in\mathbb{R}$

a. $\dfrac{3}{x+2}\geq x$

b. $\dfrac{x}{x+7}<\dfrac{-x}{x-2}$

3. Solve each inequality algebraically. State the solution using interval notation, where $x\in\mathbb{R}$.

a. $\dfrac{3x+4}{2x-1}>0$ b. $\dfrac{12x^2+11x+2}{2x^2-7x+3}\leqslant 0$ c. $\dfrac{3-x}{2x+2}>\dfrac{x}{2}$

d. $\dfrac{1}{-x^2-1}<-\dfrac{1}{4}$ e. $\dfrac{2x}{x-1}\leqslant\dfrac{x}{x+2}$ f. $\dfrac{3}{x-2}-\dfrac{x-3}{x+1}>\dfrac{x}{x-2}$

g. $\dfrac{x}{x^2-4}\geqslant\dfrac{-1}{x+1}$

4. Given $f(x)=\dfrac{2x}{1-x}$ determine the values of x for which $f(f(x))<-\dfrac{3}{2}, x\in\mathbb{R}$.

Solving Rational Inequalities

Answers

1. a. $x \in (-\infty, -3) \cup (-2, 1), x \in \mathbb{R}$

   b. i. $x \in (-\infty, 1] \cup (2, 3], x \in \mathbb{R}$

      ii. $x \in (-\infty, 2) \cup [3, \infty), x \in \mathbb{R}$

      iii. $x \in (-\infty, 1], x \in \mathbb{R}$

2. a. $x \leq -3$ or $-2 < x \leq 1, x \in \mathbb{R}$

   b. $-7 < x < -2.5$ or $0 < x < 2, x \in \mathbb{R}$

3. a. $x \in \left(-\infty, -\frac{4}{3}\right) \cup \left(\frac{1}{2}, \infty\right), x \in \mathbb{R}$

   b. $x \in \left[-\frac{2}{3}, -\frac{1}{4}\right] \cup \left(\frac{1}{2}, 3\right), x \in \mathbb{R}$

   c. $x \in (-\infty, -3) \cup (-1, 1), x \in \mathbb{R}$

   d. $x \in \left(-\sqrt{3}, \sqrt{3}\right), x \in \mathbb{R}$

   e. $x \in \left[-5, -2\right) \cup \left[0, 1\right), x \in \mathbb{R}$

   f. $x \in \left(-1, \frac{1}{2}\right) \cup (2, 3), x \in \mathbb{R}$

   g. $x \in \left(-2, \frac{-1-\sqrt{33}}{4}\right] \cup \left(-1, \frac{-1+\sqrt{33}}{4}\right] \cup (2, \infty), x \in \mathbb{R}$

4. $\frac{1}{3} < x < 1$ or $1 < x \leq 3, x \in \mathbb{R}$

Warm up

Solve $\frac{3}{x-2} - \frac{x-3}{x+1} > \frac{x}{x-2}$.

2.6 Making Connections with Rational Functions and Equations

A Quick Review... Steps to Graph a Rational Function:

1. Factor the numerator and the denominator and simplify if possible.
2. Find the x-intercept(s) by setting the numerator equal to zero and solving.
3. Find the y-intercept [Let $x = 0$ and evaluate]
4. Find the vertical asymptotes by setting the denominator equal to zero and solving.

NOTE THAT THERE ARE CASES THAT THERE ARE NO VERTICAL ASYMPTOTE FOR RATIONAL FUNCTION, eg. $f(x) = \dfrac{2}{x^2 + 2}$.
Determine and illustrate the behaviour of the graph near the vertical asymptote(s).
Also, consider any **"holes"** in the graph.

The rational function will have a **"HOLE"** in the graph instead of a vertical asymptote for the zero(s) in the denominator equal to the zero(s) in the numerator.

Examples:

a) $$\begin{aligned} f(x) &= \frac{2x - 2}{x - 1} \\ &= \frac{2(x - 1)}{x - 1} \\ &= 2 ,\ x \neq 1 \end{aligned}$$

b) $$f(x) = \frac{x^2 - x - 2}{x^2 + 4x + 3}$$

Let's look at what will happen in each of these cases.

- There are more (x-a) factors in the denominator. After dividing out all duplicate factors, the (x-a) is still in the denominator. Factors in the denominator result in vertical asymptotes. Therefore, there will be a vertical asymptote at x=a.

- There are more (x-a) factors in the numerator. After dividing out all the duplicate factors, the (x-a) is still in the numerator. Factors in the numerator result in x-intercepts. But, because you can't use x=a, there will be a hole in the graph on the x-axis.

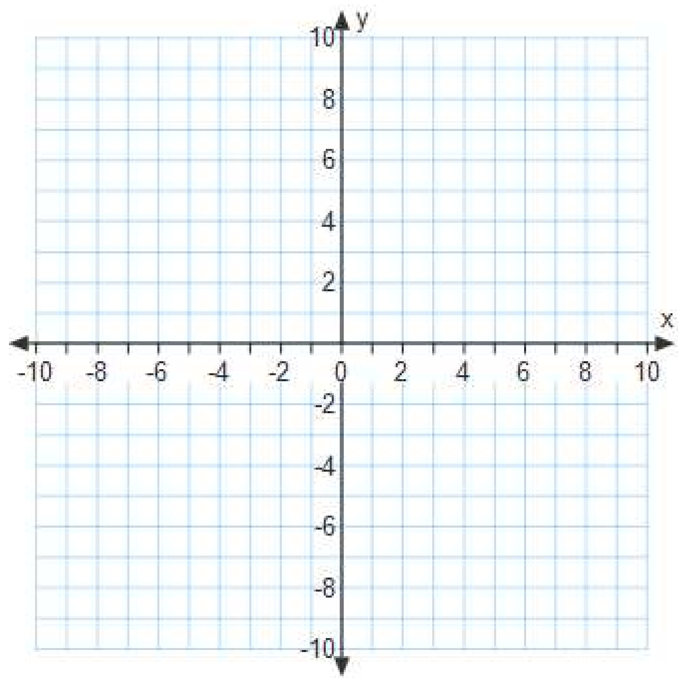
5. Find the horizontal asymptote, if it exists, using the rules. Determine and illustrate the end behaviour of the graph near the horizontal asymptote.

6. Check for any points that cross the horizontal asymptote. [If $y = L$ then solve $f(x) = \dfrac{P(x)}{Q(x)} = L$ or if $y = mx + b$ then solve $f(x) = \dfrac{P(x)}{Q(x)} = mx + b$

7. The vertical asymptote will divide the x number line into regions. If necessary, graph at least one point in each region. This point will tell us whether the graph will be above or below the horizontal asymptote and if you should need several points to determine the general shape of the graph.

GRAPHING RATIONAL FUNCTIONS WITH SPECIAL CASES OF "HOLE"

Graph $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}$



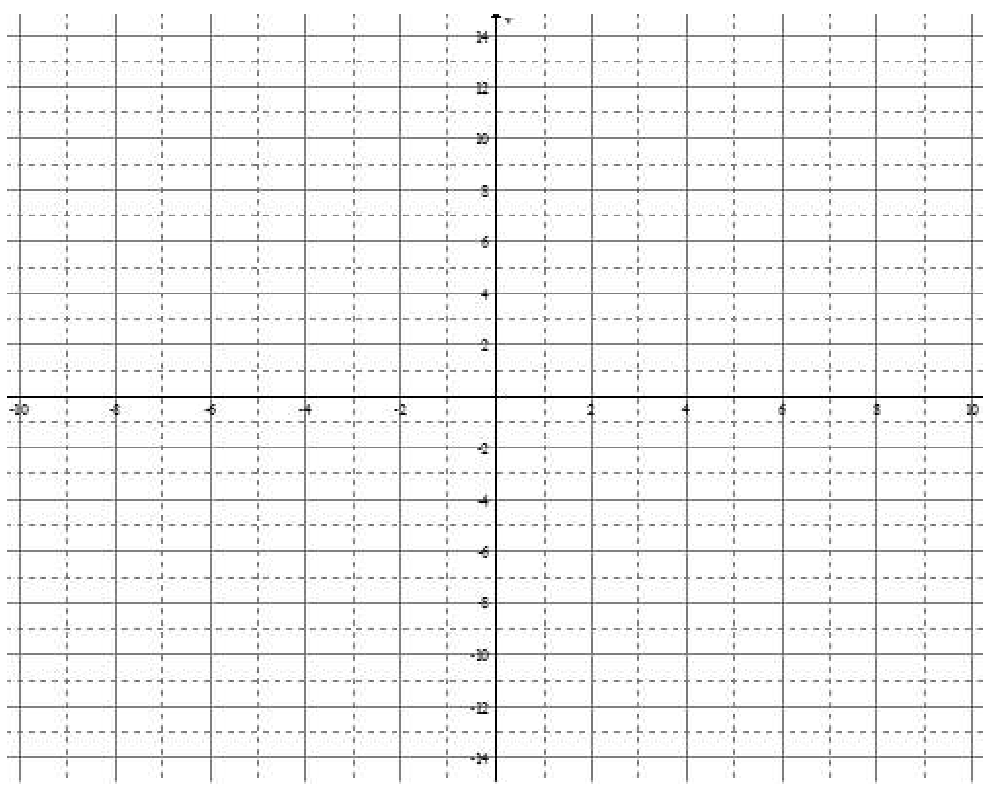
As x→	***f*(*x*)→**
$+\infty$	
$-\infty$	

Putting it all together

Ex1. Given the following functions, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function

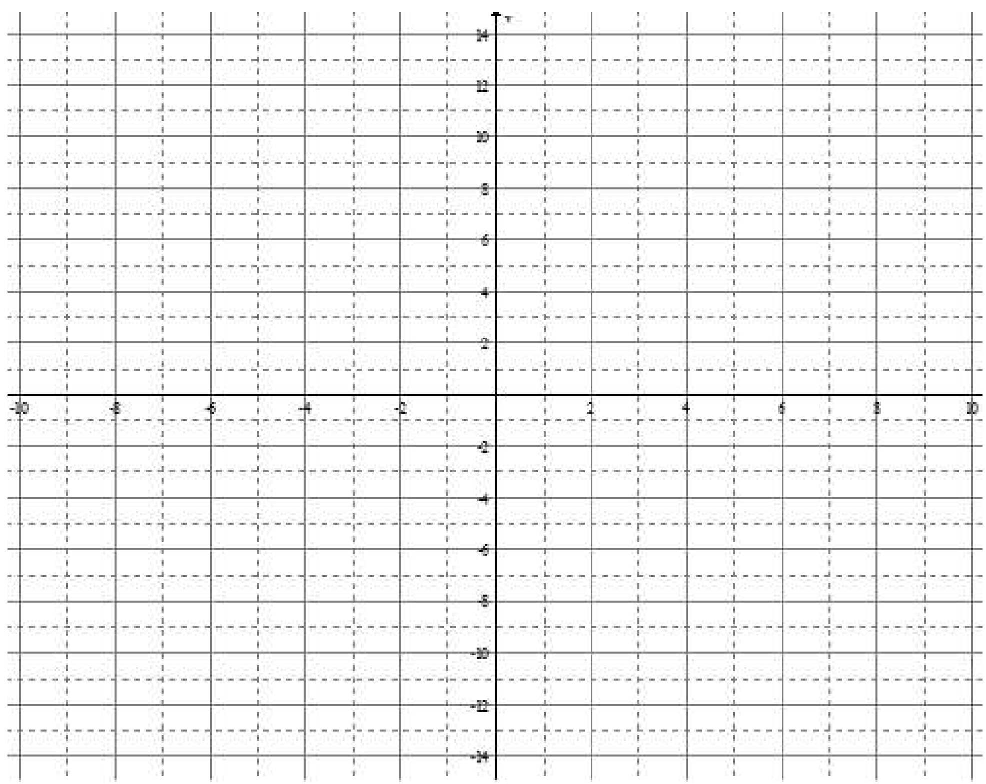
a) $f(x)=\dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$

As $x\to$	$f(x)\to$
-3^+	
-3^-	
2^+	
2^-	
$+\infty$	
$-\infty$	



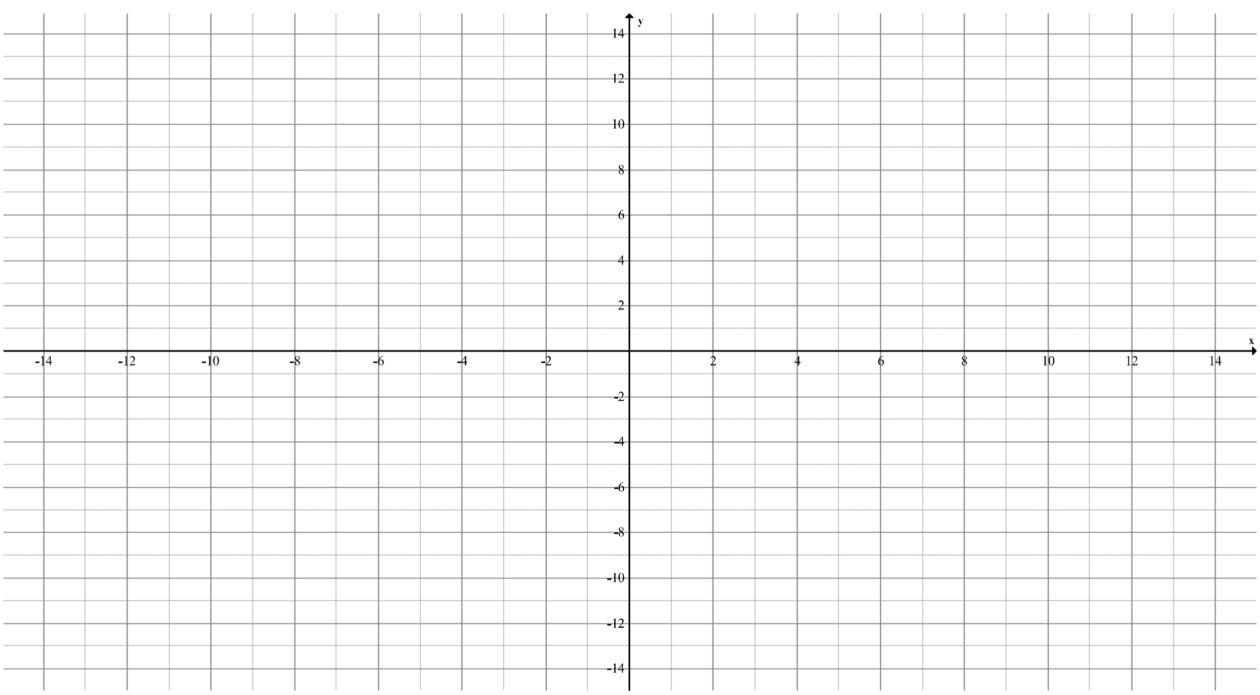
b) $f(x)=\dfrac{(x+2)(x-3)}{(x+1)^2(x-2)}$

As $x \to$	$f(x) \to$
-1^+	
-1^-	
2^+	
2^-	
$+\infty$	
$-\infty$	

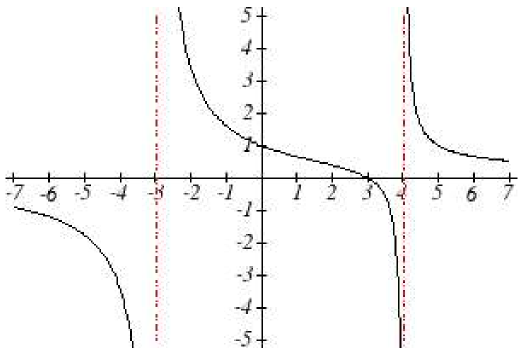
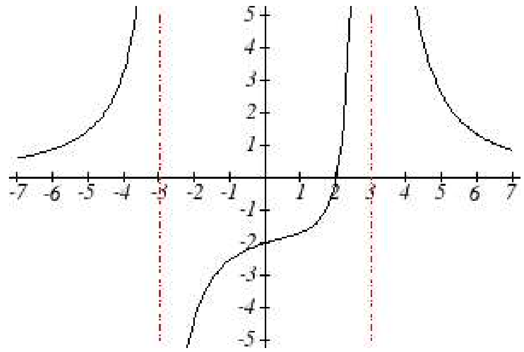


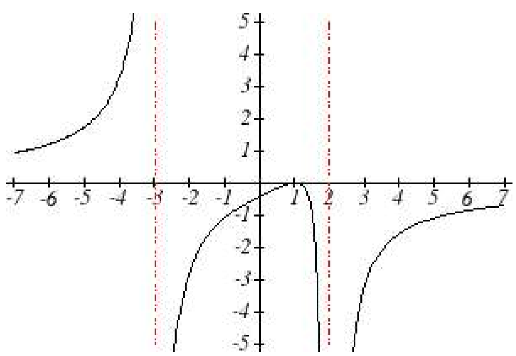
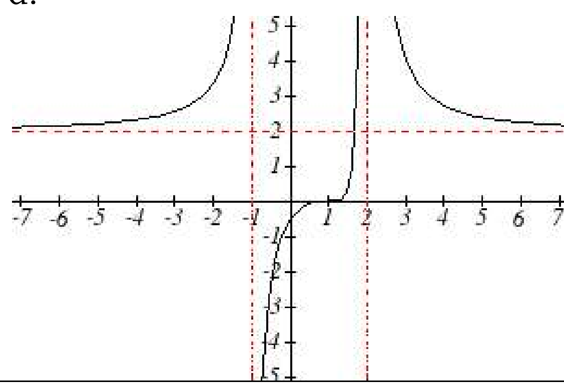
c) $f(x)=\dfrac{x^2-4x+4}{x^2+x-20}$

As $x\to$	**$f(x)\to$**
$+\infty$	
$-\infty$	



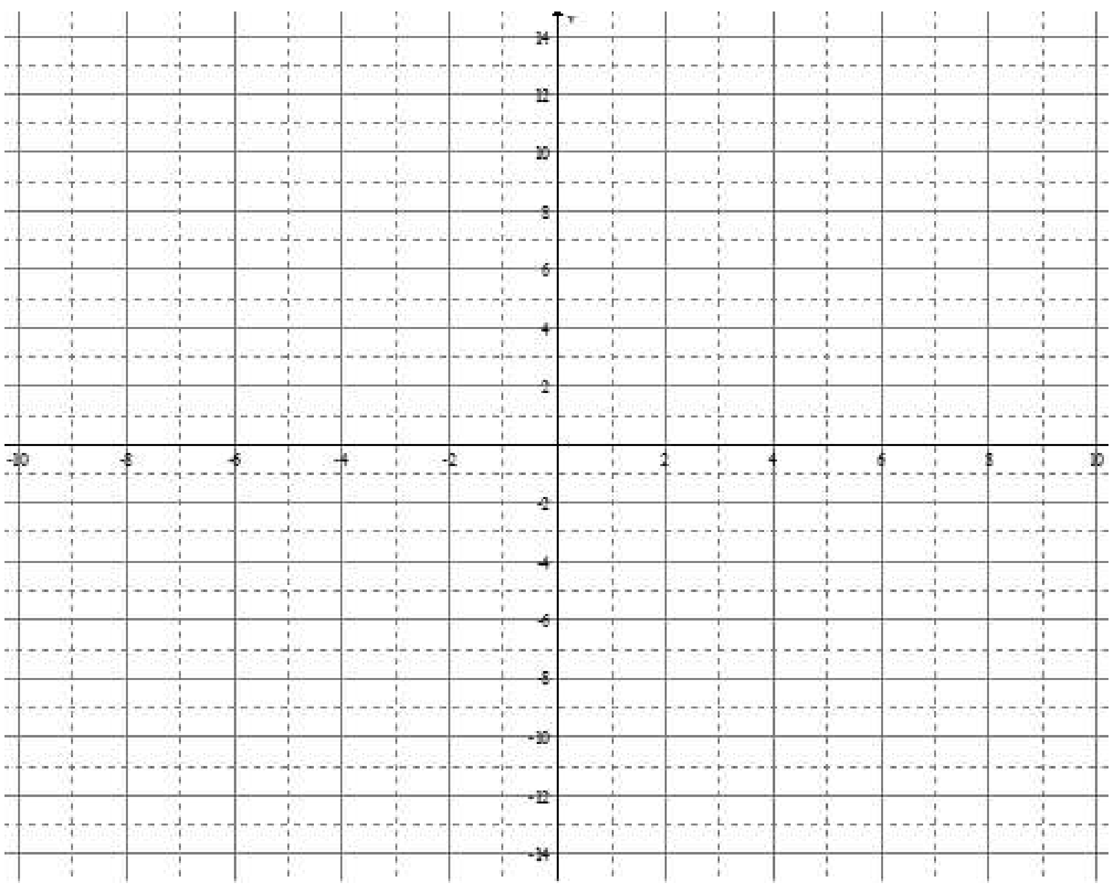
Ex.2 Write an equation for the function graphed

a. 	b. 

c. 	d. 

Ex. 3 Use the information below to sketch the function.

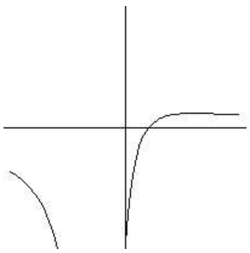
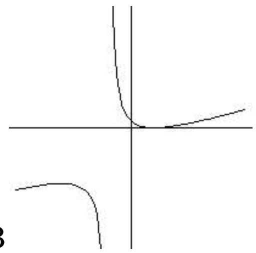
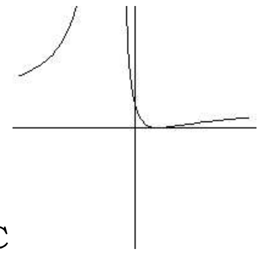
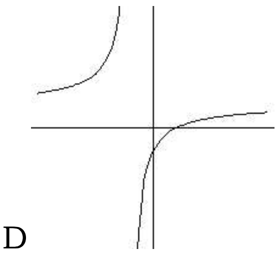
- There is a horizontal asymptote at $f(x) = 0$.
- The Domain of the function is D: $\{x \neq 4,\ x \neq 3,\ x \in \mathbb{R}\}$
- $f(2) = \frac{3}{2}$
- $f(1) = f(5) = 0$, $f(3) =$ undefined
- $f(x) < 0$ when $x < 1,\ 3 < x < 4,\ 4 < x < 5$
- $f(x) > 0$ when $1 < x < 3, x > 5$



2.6 Practice

Match each equation form with one of the graphs

1. $f(x)=\dfrac{x-A}{x-B}$ 2. $g(x)=\dfrac{(x-A)^2}{x-B}$ 3. $h(x)=\dfrac{x-A}{(x-B)^2}$ 4. $k(x)=\dfrac{(x-A)^2}{(x-B)^2}$

A  B  C  D 

Answer questions 5-12 in separate sheet of paper.

For each function, find the intercepts, the asymptotes and end behaviour of function. Use that information to sketch a graph.

5. $p(x)=\dfrac{2x-3}{x+4}$

6. $s(x)=\dfrac{4}{(x-2)^2}$

7. $f(x)=\dfrac{3x^2-14x-5}{3x^2+8x-16}$

8. $a(x)=\dfrac{x^2+2x-3}{x^2-1}$

9. $h(x)=\dfrac{2x^2+x-1}{x-4}$

10. $n(x)=\dfrac{3x^2+4x-4}{x^3-4x^2}$

11. $w(x)=\dfrac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

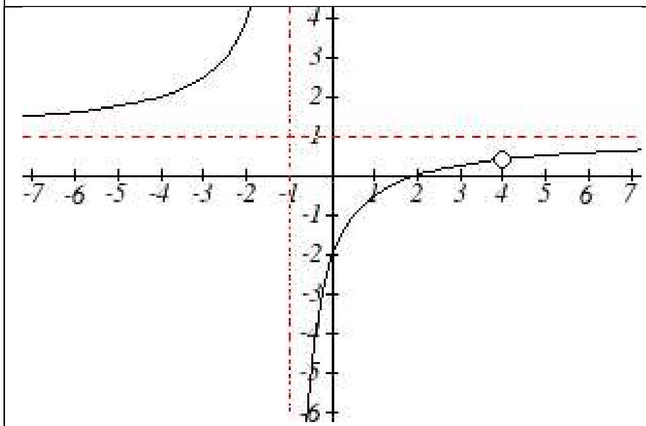
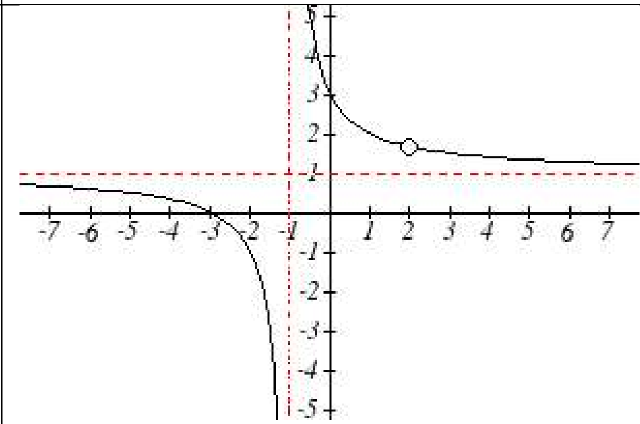
12. $m(x)=\dfrac{5-x}{2x^2+7x+3}$

Write an equation for a rational function with the given characteristics.

13. Vertical asymptotes at $x=5$ and $x=-5$
 x- intercepts at $(2,0)$ and $(-1,0)$
 y- intercept at $(0,4)$
 hole at (1,25/6)

14. Vertical asymptotes at $x=-4$ and $x=-5$
 x -intercepts at $(4,0)$ and $(-6,0)$
 Horizontal asymptote at $y=7$

15. Vertical asymptote at $x=-1$ Oblique asymptote : y=x-5
 Double zero at $x=2$
 y-intercept (0,4)

16.	17.
	

18.Estimate the slope of the tangent to the graph of $f(x)=\dfrac{x}{x^2-4}$ at the point where x=3. Explain why there cannot be a tangent line at x=2.

19. The concentration, *C*, of a drug in the bloodstream *t* hours after the drug was taken orally is given by $C(t)=\dfrac{5t}{7+t^2}$ where *c* is measured in milligrams per litre.

(a) Calculate the average rate of change in the drug's concentration during the first 2 h since ingestion.

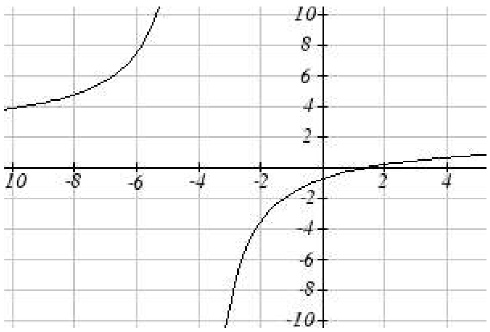
(b) Estimate the rate at which the concentration of the drug is changing after exactly 3 h.

Answer to Practice Questions

1. D 2.B 3. A 4.C

	Vertical Asymptotes	Horizontal Asymptote	*y*-Intercept	*x*-intercept
5.	$x=-4$	$y=2$	(0,-3/4)	(3/2, 0)
6.	$x=2$	$y=0$	(0,1)	DNE
7.	$x=-4,1\frac{1}{3}$	$y=1$	(0, 5/16)	(-1/3, 0), (5,0)
8.	$x=-1$, hole at $x=1$	$y=1$	(0,3)	(-3, 0)
9.	$x=4$	none	(0, ¼)	(-1, 0), (1/2, 0)

		y=2x (oblique)		
10.	$x=0,\ 4$	$y=0$	DNE	(-2, 0), (2/3, 0)
11.	$x=-2,\ 4$	$y=1$	(0, 15/16)	(1, 0), (-3, 0), (5, 0)
12.	$x=-3,\ \frac{-1}{2}$	$y=0$	(0,5/3)	(5,0)

5.


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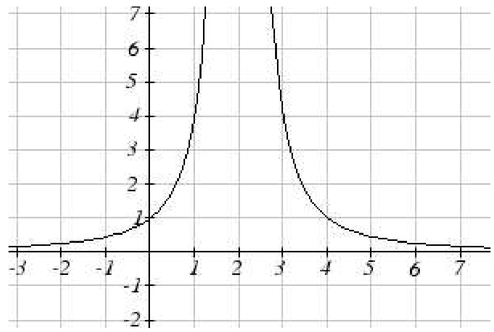

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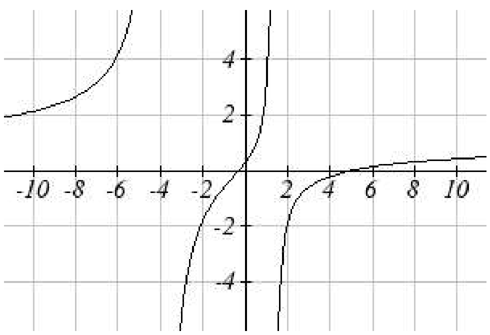

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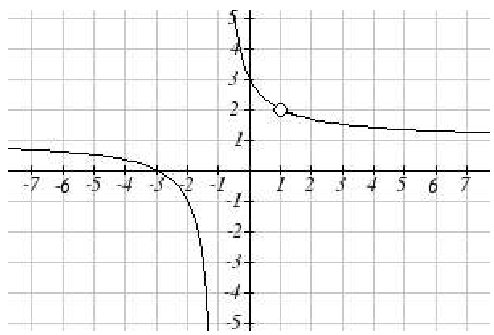

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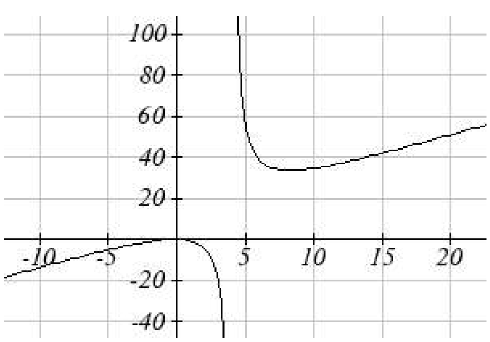

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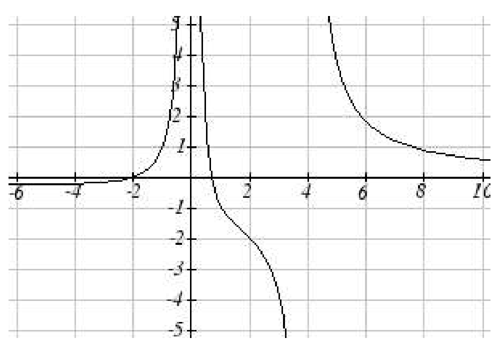
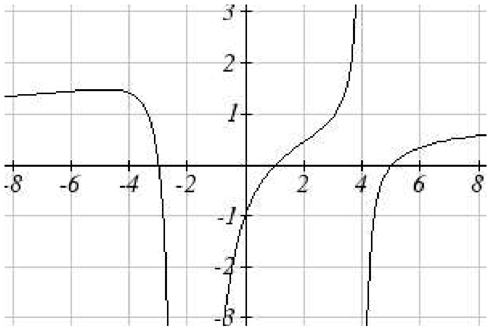
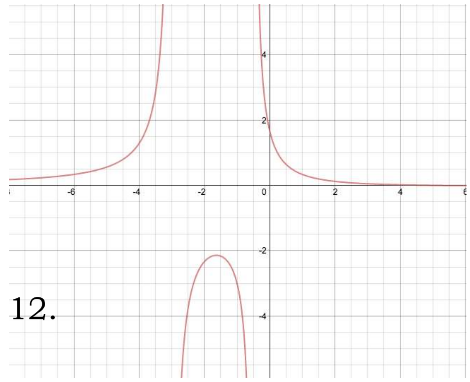


11. 

12. 

13. $y=\dfrac{50(x-2)(x+1)(x-1)}{(x+5)(x-5)(x-1)}$

14. $y=\dfrac{7(x-4)(x+6)}{(x+4)(x+5)}$

15. $y=\dfrac{(x-2)^2}{x+1}$

16. $y=\dfrac{(x-4)(x-2)}{(x-4)(x+1)}$

17. $y=\dfrac{(x+3)(x-2)}{(x+1)(x-2)}$

18 . slope of tangent at x=3 is -0.52 , slope of tangent at x=2 is undefined.

19. a) AROC when $t \in [0,2]$ is 5/11.

b) IROC when t=3 is approximately -0.04.

Warm up

1. Solve for $x \in R$.

a) $\dfrac{x^2+2x-15}{x^2+7x} > 0$

b) $\dfrac{x^2+5x-10}{1-x^2} \le 2$

2. A closed-topped cylindrical tin can is to be made with a volume of 25π cm^3. Determine the values of the radius that will produce a surface area no more than the area of a letter sized piece of paper measuring 190π cm^2.

Unit 2-Review

Multiple Choice: Write the CAPITAL letter corresponding to the correct answer on the line provided.

1. Given the function $f(x)=\dfrac{x^a+k}{x^b+m}$, a linear oblique asymptote will occur when: _________

A) $a\geq b$ B) $b>a$ C) $a-b=1$ D) $a-b=2$ E) none of the above

2. Which of the following statements is **true** if $f(x)$ is the reciprocal of a quadratic with x intercepts at $x=\pm4$ and a vertex of $(0,8)$? ________

A) $f(x)$ has two vertical asymptotes

B) $\frac{1}{f(x)}$ has a local maximum at $\left(0,\frac{1}{8}\right)$

C) $\frac{1}{f(x)}$ has a local minimum at $\left(0,\frac{-1}{8}\right)$

D) $\frac{1}{f(x)}>0$ when $x\in(-\infty,\infty)$

3. Given $f(x)=\frac{(6-2x)}{(x^2-4)(x-3)}$ which of the following is **true**? ________

A) $f(x)$ crosses at least one of its asymptotes

B) $f(x)$ has a hole at $\left(3,\frac{-2}{5}\right)$

C) $f(x)$ has a horizontal asymptote at $y=-2$

D) $f(x)$ has 3 vertical asymptotes

4. Which of the following functions has an asymptote that passes through the origin? ________

A) $f(x)=\frac{x(x-4)(x-9)}{(x-8)(x-3)^2}$

B) $f(x)=\frac{(x^2-4)(2x-9)}{(x)(2x-3)}$, $x\neq-1$

C) $f(x)=\frac{(x-6)(x+4)}{x^3-8}$

D) Both B and C

5. Which of the following function(s) cross at least one of their asymptotes? _________

A) $f(x)=\frac{(x+1)(x+5)(2x-9)}{2x^2-7x-9}$

B) $f(x)=\frac{1}{x^2+16}$

C) $f(x)=\frac{x(x-4)(x-9)}{x^3-14x^2+57x-72}$

D) All of the above

6. Complete the table below given the following function $f(x)=\frac{-x(x-3)(x-4)}{(2x-8)(x+2)(x+5)}$

x-intercept(s), if any.	
y-intercept, if any.	
Equation of vertical asymptote(s), if any.	
Equation of horizontal or oblique asymptote, if any.	

7. Determine the equation of the oblique asymptote given $f(x) = \frac{2x^2+9x-12}{x+4}$

8. Create the equation of a function $g(x)$ with the following properties:

B) x-intercept of $\frac{1}{4}$, y-intercept of $\frac{-1}{2}$, vertical asymptote of $x = \frac{-2}{3}$ and horizontal asymptote of $y = \frac{4}{3}$.

9. For the function, $g(x) = \frac{mx-3}{4-nx}$, find the values of m and n such that $g(x)$ has a vertical asymptote when $x = 6$ and a horizontal asymptote at $y = -3$.

10. Sketch the graph of $f(x) = \frac{x^3 - x^2 - 4x + 4}{x^2 + x - 20}$

11. Determine any points of intersection for the function $f(x) = -x^2 - 5x - 6$ and its reciprocal function. Leave answer(s) in exact form where necessary.

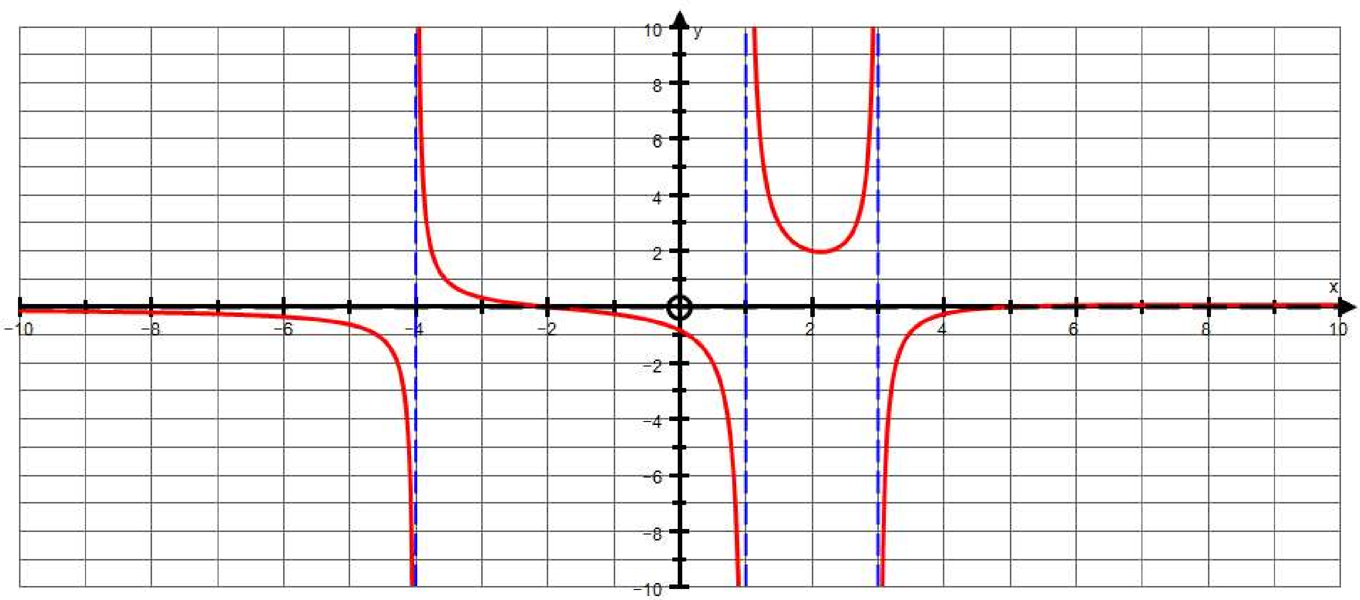
12. Given the graph of the rational function $f(x)$ below solve $\frac{4x^2-20x}{f(x)} \geq 0$

Please note two things regarding $f(x)$:

a) The y intercept is $\left(0, \frac{-5}{6}\right)$

b) $f(x)$ crosses its horizontal asymptote at $x = -2$ and $x = 5$

c) The degree of the numerator of $f(x)$ is 2 and the degree of the denominator of $f(x)$ is 3.



13. Solve

a) $\frac{x^3 + 7x^2 + 12x}{x^2 + 9x + 20} \geq \frac{x+1}{x^2 - 3x - 4} - \frac{2x+5}{2x^2 - 3x - 20}$

b) $\frac{x^3 + 2x^2 - 46x - 125}{(x^2 + 5x - 6)(x+3)} \geq \frac{-1}{(x+6)(x+3)}$

14. Determine the perimeter of the quadrilateral, rounded to the nearest tenth of a unit, created by the intersections of $h(x)$ and its reciprocal graph $f(x)$. Given: $h(x) = -2(x+4)(x-6)$ and

$f(x) = \frac{1}{-2x^2 + 4x + 48}$.

15. Describe what is known about the equation of a rational function with vertical asymptotes at $x = 5$ and

$x = -3$ and a horizontal asymptote of $y = 0$.

16. The concentration of a toxic chemical in a spring-fed lake is given by the equation $C(x) = \dfrac{60x}{x^2 + 3x + 6}$, where C is given in grams per litre and x is the time in days. Find the instantaneous rate of change at 4 days.

17. Sketch the graph of $y = \dfrac{x^3 - 6x^2 + 32}{x^3 - x^2 - 4x + 4}$.

18. Use the information below to sketch the function.
    - There is a horizontal asymptote at $f(x) = -2$.
    - The Domain of the function is $D: \left\{x \neq -3, x \neq 4, x \in \Re\right\}$
    - The range of the function is $R: \left\{y > -8, y \in \Re\right\}$
    - $f(0) = \dfrac{28}{81}$
    - There is a hole at $(4, \dfrac{9}{40})$
    - One of the factors of the numerator is $(x^2 + 5x - 6)$
    - $f(x) > 0 \quad when \quad -6 < x < -3, \; -3 < x < 1, \; 1 < x < 4, \; 4 < x < 7$
    - $f(7) = 0$