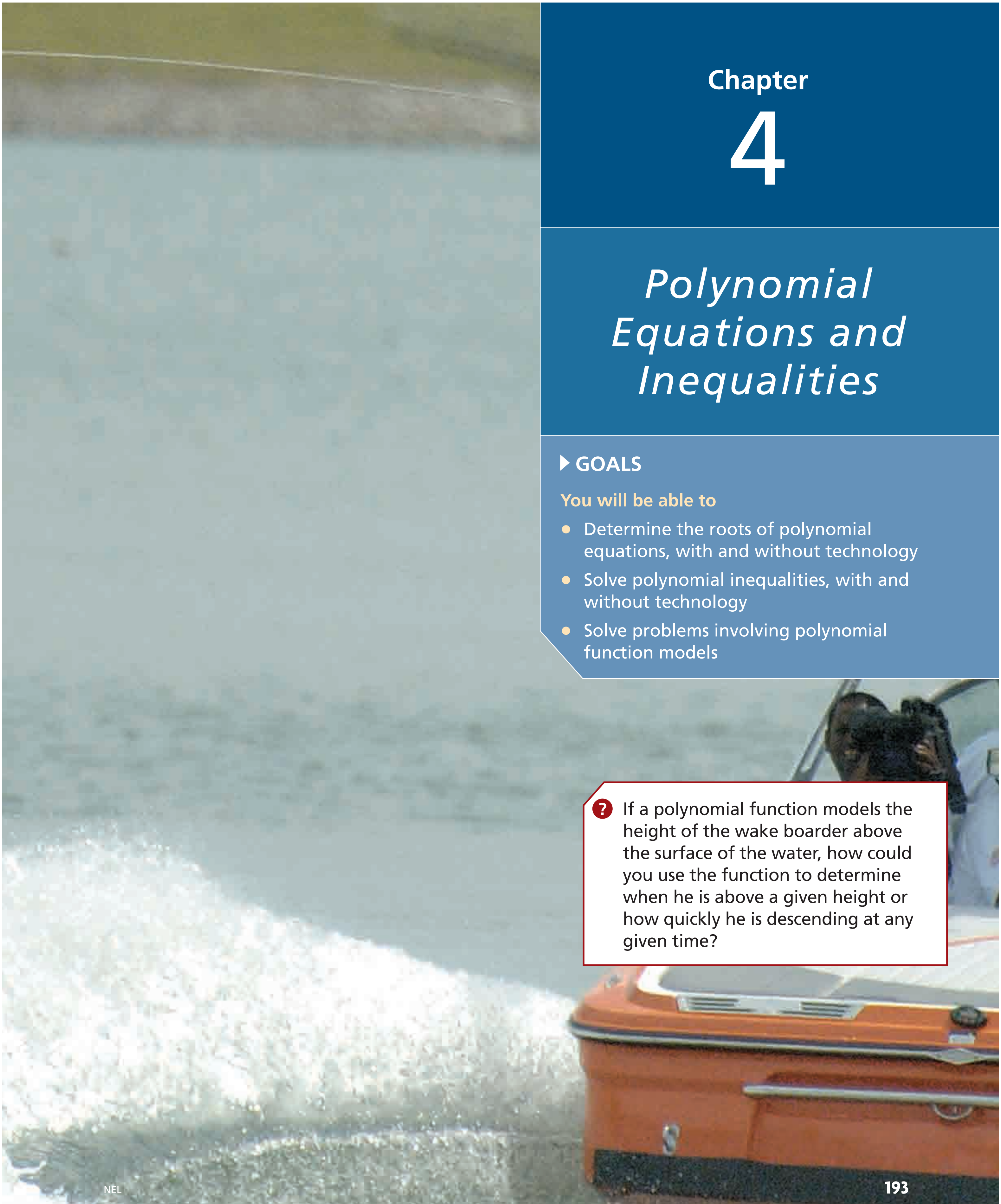


Chapter

4

Polynomial Equations and Inequalities

GOALS

You will be able to

- Determine the roots of polynomial equations, with and without technology
- Solve polynomial inequalities, with and without technology
- Solve problems involving polynomial function models

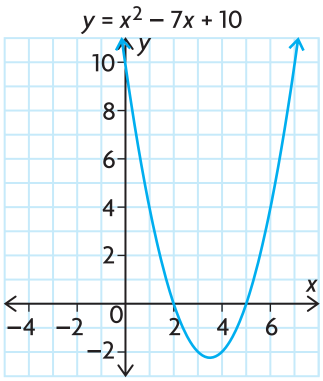
? If a polynomial function models the height of the wake boarder above the surface of the water, how could you use the function to determine when he is above a given height or how quickly he is descending at any given time?

4 Getting Started

Study *Aid*

- For help, see the Review of Essential Skills found at the Nelson Advanced Functions website.

Question	Appendix/Lesson
1, 4, 5	R-6
2	R-3, 3.6
3	3.3
6, 7	2.5



SKILLS AND CONCEPTS You Need

1. Solve the following linear equations.

a) $5x - 7 = -3x + 17$

b) $12x - 9 - 6x = 5 + 3x + 1$

c) $2(3x - 5) = -4(3x - 2)$

d) $\dfrac{2x + 5}{3} = 7 - \dfrac{x}{4}$

2. Factor the following expressions.

a) $x^3 + x^2 - 30x$

b) $x^3 - 64$

c) $24x^4 + 81x$

d) $2x^3 + 7x^2 - 18x - 63$

3. Sketch a graph of each of the following functions.

a) $y = (x - 2)(x + 3)(x - 4)$

b) $y = 2(x + 6)^3 - 10$

4. Given the graph of the function shown, determine the roots of the equation $x^2 - 7x + 10 = 0$.

5. Determine the roots of each of the following quadratic equations.

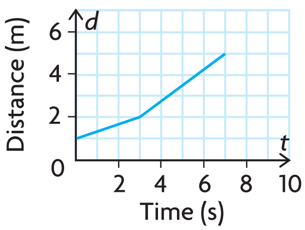
a) $2x^2 = 18$

b) $x^2 + 8x - 20 = 0$

c) $6x^2 = 11x + 10$

d) $x(x + 3) = 3 - 5x - x^2$

6. The graph below shows Erika's walk in front of a motion sensor.



a) In which time interval is she walking the fastest? Explain.

b) Calculate the speeds at which she walks on the intervals $t \in (0, 3)$ and $t \in (3, 7)$.

c) Is she moving away or toward the motion sensor? How do you know?

7. A T-ball player hits a baseball from a tee that is 0.5 m tall. The height of the ball is modelled by $h(t) = -5t^2 + 9.75t + 0.5$, where h is the height in metres at t seconds.

a) How long is the ball in the air?

b) Determine the average rate of change in the ball's height during the first second of flight.

c) Estimate the instantaneous rate of change in the ball's height when it hits the ground.

8. Copy and complete the anticipation guide in your notes.

Statement	Agree	Disagree	Justification
The quadratic formula can only be used when solving a quadratic equation.			
Cubic equations always have three real roots.			
The graph of a cubic function always passes through all four quadrants.			
The graphs of all polynomial functions must pass through at least two quadrants.			
The expression $x^2 > 4$ is only true if $x > 2$.			
If you know the instantaneous rates of change for a function at $x = 2$ and $x = 3$, you can predict fairly well what the function looks like in between.			

APPLYING *What You Know*

Modelling a Situation with a Polynomial Function

Shown is a picture of the Gateway Arch located in St. Louis, Missouri, U.S.A. The arch is about 192 m wide and 192 m tall.

The city of St. Louis would like to hang a banner from the arch for their New Year celebrations. They have determined that the banner should be suspended from a horizontal cable that spans the arch 175 m off the ground to ensure optimal viewing around the city.

? Assuming that the arch is parabolic in shape, how long should the cable be?

A. Draw a sketch showing the arch on a coordinate grid using an appropriate scale.

B. Determine a quadratic function that could model the inside of the arch using vertex or factored form.

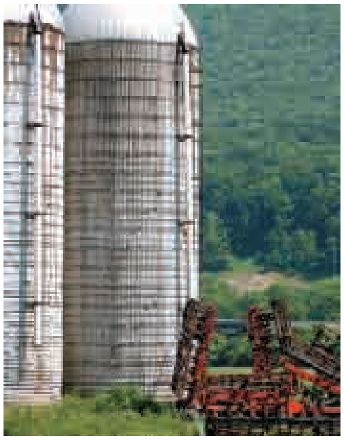
C. How did you predict what the sign of the leading coefficient in the function would be? Explain.

D. Use your model to determine the length of the cable needed to support the banner.

4.1 Solving Polynomial Equations

YOU WILL NEED

- graphing calculator or graphing software



GOAL

Solve polynomial equations using a variety of strategies.

LEARN ABOUT the Math

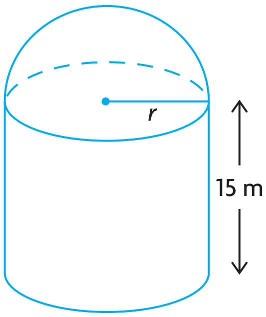
Amelia's family is planning to build another silo for grain storage, identical to those they have on their farm. The cylindrical portion of those they currently have is 15 m tall, and the silo's total volume is 684π m^3.

? What are the possible values for the radius of the new silo?

EXAMPLE 1 Representing a problem with a polynomial model

Determine possible values for the radius of the silo.

Solution



Draw a diagram to represent the silo. In this case, the height must be 15 m.

$$V = V_{cylinder} + V_{hemisphere}$$

$$V = \pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$$

Determine a **polynomial equation** for the volume of the silo using the formula for the volume of a cylinder and a hemisphere.

polynomial equation
an equation in which one polynomial expression is set equal to another (e.g., $x^3 - 5x^2 = 4x - 3$, or $5x^4 - 3x^3 + x^2 - 6x = 9$)

$$684\pi = \pi r^2(15) + \frac{2}{3}\pi r^3$$

$$684\pi = 15\pi r^2 + \frac{2}{3}\pi r^3$$

Substitute the given values for the volume and the height into the formula and simplify the equation.

$$0 = 15\pi r^2 + \frac{2}{3}\pi r^3 - 684\pi$$

$$0 = \frac{\pi}{3}(45r^2 + 2r^3 - 2052)$$

Divide out the common factor of $\frac{\pi}{3}$, then divide both sides of the equation by this value.

$$0 = 45r^2 + 2r^3 - 2052$$
$$0 = 2r^3 + 45r^2 - 2052$$

Rewrite the polynomial in descending order of degree.

$$\text{Let } f(r) = 2r^3 + 45r^2 - 2052$$

Solve the equation by factoring the polynomial.

$$f(2) = 2(2)^3 + 45(2)^2 - 2052$$
$$= -1856$$
$$f(3) = 2(3)^3 + 45(3)^2 - 2052$$
$$= -1593$$
$$f(6) = 2(6)^3 + 45(6)^2 - 2052$$
$$= 0$$

Since the roots of the equation are the *x*-intercepts of the related function, use the factor theorem to determine one factor.

By the factor theorem, $(r - 6)$ is a factor of $f(r)$.

$$(2r^3 + 45r^2 - 2052) \div (r - 6)$$

$$\begin{array}{r|rrrr} 6 & 2 & 45 & 0 & -2052 \\ & \downarrow & 12 & 342 & 2052 \\ \hline & 2 & 57 & 342 & 0 \end{array}$$

Use synthetic division to divide $f(r)$ by $(r - 6)$ to determine the other factor.

$$(r - 6)(2r^2 + 57r + 342) = 0$$

$r - 6 = 0$

$r = 6$ is one solution

Set each factor equal to zero and solve.

$$2r^2 + 57r + 342 = 0$$

$$a = 2, b = 57, c = 342$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-57 \pm \sqrt{(57)^2 - 4(2)(342)}}{2(2)}$$

$$r = \frac{-57 \pm \sqrt{513}}{4}$$

$$r = \frac{-57 + \sqrt{513}}{4} \quad \text{or} \quad r = \frac{-57 - \sqrt{513}}{4}$$

$$r \doteq -8.6 \qquad\qquad r \doteq -19.9$$

The second factor does not appear to be factorable, so use the quadratic formula to solve for the other roots.

The silo must have a radius of 6 m.

The radius cannot be negative, and so only the positive root can be the radius of the silo.

Reflecting

A. How could you verify the solutions you found, with and without using a graphing calculator?

B. What restriction was placed on the variable in the polynomial equation? Explain why this was necessary.

C. Do you think it is possible to solve all cubic and quartic equations using an algebraic strategy involving factoring? Explain.

APPLY the Math

EXAMPLE 2 Selecting a strategy to solve a cubic equation

Solve $4x^3 - 12x^2 - x + 3 = 0$.

Solution A: Using the factor theorem

Let $f(x) = 4x^3 - 12x^2 - x + 3$

Possible values for x where $f(x) = 0$:

$\pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}$

Use the factor theorem and the related polynomial function to determine one factor of the equation. Numbers that could make $f(x) = 0$ are of the form $\frac{p}{q}$, where p is a factor of the constant term 3 and q is a factor of the leading coefficient 4.

$$\begin{aligned} f(1) &= 4(1)^3 - 12(1)^2 - (1) + 3 \\ &= -6 \end{aligned}$$

$$\begin{aligned} f(-1) &= 4(-1)^3 - 12(-1)^2 - (-1) + 3 \\ &= -12 \end{aligned}$$

$$\begin{aligned} f(3) &= 4(3)^3 - 12(3)^2 - (3) + 3 \\ &= 0 \end{aligned}$$

By the factor theorem, $(x - 3)$ is a factor of $f(x)$.

$(4x^3 - 12x^2 - x + 3) \div (x - 3)$

$$\begin{array}{r} 4x^2 + 0x - 1 \\ x - 3 \overline{)\,4x^3 - 12x^2 - x + 3} \\ \underline{4x^3 - 12x^2} \qquad\qquad \\ 0x^2 - x \qquad \\ \underline{0x^2 - 0x} \qquad \\ -x + 3 \\ \underline{-x + 3} \\ 0 \end{array}$$

Divide $f(x)$ by $(x - 3)$ to find the second factor using either long or synthetic division.

$(x - 3)(4x^2 - 1) = 0$

The quotient $4x^2 - 1$ is a difference of squares. Factor this.

$$(x - 3)(2x - 1)(2x + 1) = 0$$

$$x - 3 = 0 \text{ or } 2x - 1 = 0 \text{ or } 2x + 1 = 0$$

Set each of the factors equal to zero to solve.

$$x = 3 \qquad 2x = 1 \qquad 2x = -1$$

$$x = \frac{1}{2} \qquad x = -\frac{1}{2}$$

Check:

Verify the solutions by substitution.

You only need to check $x = \frac{1}{2}$ and $-\frac{1}{2}$ since $x = 3$ was obtained using substitution.

$x = \frac{1}{2}$

LS	RS
$4x^3 - 12x^2 - x + 3$	0
$= 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) + 3$	
$= \frac{1}{2} - 3 - \frac{1}{2} + 3$	
$= 0$	

LS = RS ✓

$x = -\frac{1}{2}$

LS	RS
$4x^3 - 12x^2 - x + 3$	0
$= 4\left(-\frac{1}{2}\right)^3 - 12\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 3$	
$= -\frac{1}{2} - 3 + \frac{1}{2} + 3$	
$= 0$	

LS = RS ✓

The solutions to $4x^3 - 12x^2 - x + 3 = 0$ are $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 3$.

Solution B: Factoring by grouping

$$4x^3 - 12x^2 - x + 3 = 0$$

$$4x^2(x - 3) - 1(x - 3) = 0$$

The first two terms and the last two terms have a common factor, so you can factor by grouping.

$$(x - 3)(4x^2 - 1) = 0$$

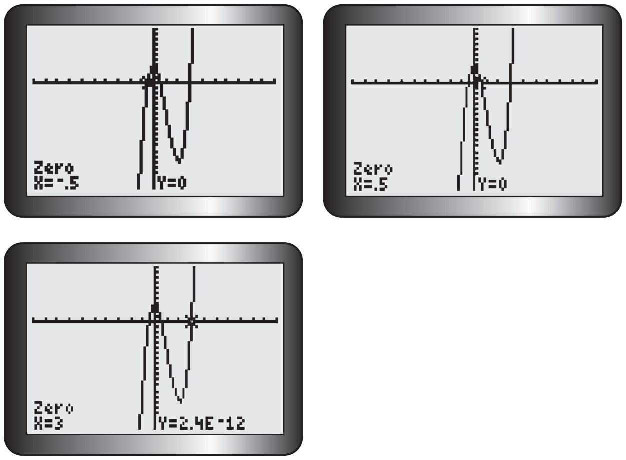
$$(x - 3)(2x - 1)(2x + 1) = 0$$

$$x - 3 = 0 \text{ or } 2x - 1 = 0 \text{ or } 2x + 1 = 0$$

Set each of the factors equal to zero to solve.

$$x = 3 \qquad 2x = 1 \text{ or } \qquad 2x = -1$$

$$x = \frac{1}{2} \qquad x = -\frac{1}{2}$$



Verify the solutions, by graphing the corresponding polynomial function and determining its zeros.

The solutions to $4x^3 - 12x^2 - x + 3 = 0$ are $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 3$.

EXAMPLE 3 Selecting tools to solve a question involving modelling

The paths of two orcas playing in the ocean were recorded by some oceanographers. The first orca's path could be modelled by the equation $h(t) = 2t^4 - 17t^3 + 27t^2 - 252t + 232$, and the second by $h(t) = 20t^3 - 200t^2 + 300t - 200$, where h is their height above/below the water's surface in centimetres and t is the time during the first 8 s of play. Over this 8-second period, at what times were the two orcas at the same height or depth?

Solution

$$2t^4 - 17t^3 + 27t^2 - 252t + 232 = 20t^3 - 200t^2 + 300t - 200$$
$$2t^4 - 37t^3 + 227t^2 - 552t + 432 = 0$$

Since you are solving for the time when the heights or depths are the same, set the two equations equal to each other and use inverse operations to make the right side of the equation equal to zero.

Let $f(t) = 2t^4 - 37t^3 + 227t^2 - 552t + 432$

Solve the equation by factoring.

Some possible values for t where $f(t) = 0$:

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9$

$f(1) = 72$

$f(2) = -28$

$f(3) = -18$

$f(4) = 0$

Use the factor theorem to determine one factor of $f(t)$. Since the question specifies that the time is within the first 10 s, you only need to consider values of $\frac{p}{q}$ between 0 and 10. In this case, consider just the factors of 432.

By the factor theorem, $(t - 4)$ is a factor of $f(t)$.

$(2t^4 - 37t^3 + 227t^2 - 552t + 432) \div (t - 4)$

$$\begin{array}{r|rrrrr} 4 & 2 & -37 & 227 & -552 & 432 \\ & \downarrow & 8 & -116 & 444 & -432 \\ \hline & 2 & -29 & 111 & -108 & 0 \end{array}$$

Divide $f(t)$ by $(t - 4)$ to determine the second factor.

$f(t) = (t - 4)(2t^3 - 29t^2 + 111t - 108)$

$f(6) = -108$

$f(8) = -208$

$f(9) = 0$

Since the second factor is cubic, you must continue looking for more zeros using the factor theorem.
It is not necessary to recheck the values that were used in an earlier step, so carry on with the other possibilities.

By the factor theorem, $(t - 9)$ is a factor of $f(t)$.

$(2t^3 - 29t^2 + 111t - 108) \div (t - 9)$

$$\begin{array}{r|rrrr} 9 & 2 & -29 & 111 & -108 \\ & \downarrow & 18 & -99 & 108 \\ \hline & 2 & -11 & 12 & 0 \end{array}$$

Divide the cubic polynomial by $(t - 9)$ to determine the other factor.

$$\begin{aligned} f(t) &= (t - 4)(t - 9)(2t^2 - 11t + 12) \\ &= (t - 4)(t - 9)(2t - 3)(t - 4) \\ &= (t - 4)^2(t - 9)(2t - 3) \end{aligned}$$

Factor the quadratic.

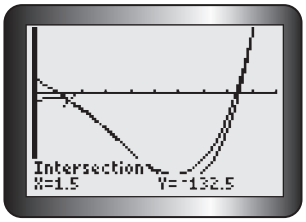
$(t - 4)^2 = 0,\ t - 9 = 0,$ or $2t - 3 = 0$

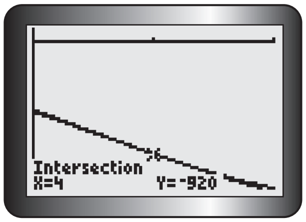
$t = 4 \qquad t = 9 \qquad t = 1.5$

Set each factor equal to zero and solve.

The solutions that are valid on the given domain are $t = 1.5$ and $t = 4$.

The polynomial functions given only model the orca's movement between 0 s and 8 s, so the solution $t = 9$ is inadmissable.





Graph both functions and adjust the window settings to determine the points of intersection to verify the solutions.

The orcas were at the same depth after 1.5 s and 4 s on the interval between 0 s and 8 s.

EXAMPLE 4 Selecting a strategy to solve a polynomial equation that is unfactorable

Solve each of the following.

a) $x^4 + 5x^2 = -1$ **b)** $x^3 - 2x + 3 = 0$

Solution

a) $x^4 + 5x^2 = -1$

$x^4 + 5x^2 + 1 = 0$

Add 1 to both sides of the equation to make the right side of the equation equal to zero.

Let $f(x) = x^4 + 5x^2 + 1$

$f(1) = (1)^4 + 5(1)^2 + 1 = 7$

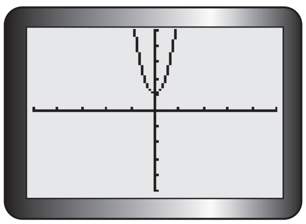
$f(-1) = (-1)^4 + 5(-1)^2 + 1 = 7$

If the equation is factorable, then either $f(1)$ or $f(-1)$ should give a value of 0.

The polynomial in this equation cannot be factored.

The function $f(x) = x^4 + 5x^2 + 1$ has an even degree and a positive leading coefficient, so its end behaviours are the same. In this case, as $x \to \pm\infty$, $y \to \infty$. This function has a degree of 4, so it could have 4, 3, 2, 1, or 0 x-intercepts.

Use the corresponding polynomial function to visualize the graph and determine the possible number of zeros.



Graph the function on a graphing calculator to determine its zeros.

Based on the graph and the function's end behaviours, it never crosses the x-axis, so it has no zeros. As a result, the equation $x^4 + 5x = -1$ has no solutions.

b) $x^3 - 2x + 3 = 0$

Let $f(x) = x^3 - 2x + 3$

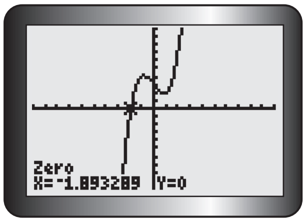
If the equation is factorable, then either $f(1)$, $f(-1)$, $f(3)$, or $f(-3)$ should equal 0.

$$f(1) = (1)^3 - 2(1) + 3 = 2$$
$$f(-1) = (-1)^3 - 2(-1) + 3 = 4$$
$$f(3) = (3)^3 - 2(3) + 3 = 24$$
$$f(-3) = (-3)^3 - 2(-3) + 3 = -18$$

The polynomial in this equation cannot be factored.

The function $f(x) = x^3 - 2x^2 + 3$ has an odd degree and a positive leading coefficient, so its end behaviours are opposite. In this case, as $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$. This function has a degree of 3, so it could have 3, 2, or 1 x-intercepts.

Use the corresponding polynomial function to visualize the graph and determine the possible number of zeros.



Graph the function on a graphing calculator to determine its zeros.

Based on the graph and the function's end behaviours, it crosses the x-axis only once. The solution is $x \doteq -1.89$.

In Summary

Key Idea

- The solutions to a polynomial equation $f(x) = 0$ are the zeros of the corresponding polynomial function, $y = f(x)$.

Need to Know

- Polynomial equations can be solved using a variety of strategies:
  - algebraically using a factoring strategy
  - graphically using a table of values, transformations, or a graphing calculator
- Only some polynomial equations can be solved by factoring, since not all polynomials are factorable. In these cases, graphing technology must be used.
- When solving problems using polynomial models, it may be necessary to ignore the solutions that are outside the domain defined by the conditions of the problem.

CHECK Your Understanding

1. State the zeros of the following functions.

a) $y = 2x(x - 1)(x + 2)(x - 2)$
b) $y = 5(2x + 3)(4x - 5)(x + 7)$
c) $y = 2(x - 3)^2(x + 5)(x - 4)$
d) $y = (x + 6)^3(2x - 5)$
e) $y = -5x(x^2 - 9)$
f) $y = (x + 5)(x^2 - 4x - 12)$

2. Solve each of the following equations by factoring. Verify your solutions using graphing technology.

a) $3x^3 = 27x$
b) $4x^4 = 24x^2 + 108$
c) $3x^4 + 5x^3 - 12x^2 - 20x = 0$
d) $10x^3 + 26x^2 - 12x = 0$
e) $2x^3 + 162 = 0$
f) $2x^4 = 48x^2$

3. a) Determine the zeros of the function $y = 2x^3 - 17x^2 + 23x + 42$.
b) Write the polynomial equation whose roots are the zeros of the function in part a).

4. Explain how you can solve
$x^3 + 12x^2 + 21x - 4 = x^4 - 2x^3 - 13x^2 - 4$
using two different strategies.

5. Determine the zeros of the function
$f(x) = 2x^4 - 11x^3 - 37x^2 + 156x$ algebraically.
Verify your solution using graphing technology.

PRACTISING

6. State the zeros of the following functions.

a) $f(x) = x(x - 2)^2(x + 5)$
b) $f(x) = (x^3 + 1)(x - 17)$
c) $f(x) = (x^2 + 36)(8x - 16)$
d) $f(x) = -3x^3(2x + 4)(x^2 - 25)$
e) $f(x) = (x^2 - x - 12)(3x)$
f) $f(x) = (x + 1)(x^2 + 2x + 1)$

7. Determine the roots algebraically by factoring.

a) $x^3 - 8x^2 - 3x + 90 = 0$
b) $x^4 + 9x^3 + 21x^2 - x - 30 = 0$
c) $2x^3 - 5x^2 - 4x + 3 = 0$
d) $2x^3 + 3x^2 = 5x + 6$
e) $4x^4 - 4x^3 - 51x^2 + 106x = 40$
f) $12x^3 - 44x^2 = -49x + 15$

8. Use graphing technology to find the real roots to two decimal places.

a) $x^3 - 7x + 6 = 0$

b) $x^4 - 5x^3 - 17x^2 + 3x + 18 = 0$

c) $3x^3 - 2x^2 + 16 = x^4 + 16x$

d) $x^5 + x^4 = 5x^3 - x^2 + 6x$

e) $105x^3 = 344x^2 - 69x - 378$

f) $21x^3 - 58x^2 + 10 = -18x^4 - 51x$

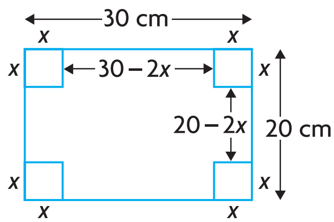
9. Solve each of the following equations.
K
a) $x^3 - 6x^2 - x + 30 = 0$

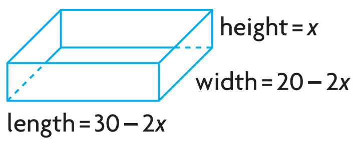
b) $9x^4 - 42x^3 + 64x^2 - 32x = 0$

c) $6x^4 - 13x^3 - 29x^2 + 52x = -20$

d) $x^4 - 6x^3 + 10x^2 - 2x = x^2 - 2x$

10. An open-topped box can be created by cutting congruent squares from each of the four corners of a piece of cardboard that has dimensions of 20 cm by 30 cm and folding up the sides. Determine the dimensions of the squares that must be cut to create a box with a volume of 1008 cm^3.





11. The Sickle-Lichti family members are very competitive card players.
A
They keep score using a complicated system that incorporates positives and negatives. Maya's score for the last game night could be modelled by the function $S(x) = x(x - 4)(x - 6)$, $x < 10$, $x \in \mathbf{W}$, where x represents the game number.

a) After which game was Maya's score equal to zero?

b) After which game was Maya's score -5?

c) After which game was Maya's score 16?

d) Draw a sketch of the graph of $S(x)$ if $x \in \mathbf{R}$. Explain why this graph is *not* a good model to represent Maya's score during this game night.

12. The function $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$ can be used to calculate s, the height above a planet's surface in metres, where g is the acceleration due to gravity, t is the time in seconds, v_0 is the initial velocity in metres per second, and s_0 is the initial height in metres. The acceleration due to gravity on Mars is $g = -3.92$ m/s^2. Find, to two decimal places, how long it takes an object to hit the surface of Mars if the object is dropped from 1000 m above the surface.

13. The distance of a ship from its harbour is modelled by the function $d(t) = -3t^3 + 3t^2 + 18t$, where t is the time elapsed in hours since departure from the harbour.
   a) Factor the time function.
   b) When does the ship return to the harbour?
   c) There is another zero of $d(t)$. What is it, and why is it not relevant to the problem?
   d) Draw a sketch of the function where $0 \leq t \leq 3$.
   e) Estimate the time that the ship begins its return trip back to the harbour.

14. During a normal 5 s respiratory cycle in which a person inhales and then exhales, the volume of air in a person's lungs can be modelled by $V(t) = 0.027t^3 - 0.27t^2 + 0.675t$, where the volume, V, is measured in litres at t seconds.
   a) What restriction(s) must be placed on t?
   b) If asked, "How many seconds have passed if the volume of air in a person's lungs is 0.25 L?" would you answer this question algebraically or by using graphing technology? Justify your decision.
   c) Solve the problem in part b).

15. Explain why the following polynomial equation has no real solutions: $0 = 5x^8 + 10x^6 + 7x^4 + 18x^2 + 132$

16. **T** Determine algebraically where the cubic polynomal function that has zeros at 2, 3, and -5 and passes through the point (4, 36) has a value of 120.

17. **C** For each strategy below, create a cubic or quartic equation you might solve by using that strategy (the same equation could be used more than once). Explain why you picked the equation you did.
   a) factor theorem
   b) common factor
   c) factor by grouping
   d) quadratic formula
   e) difference or sum of cubes
   f) graphing technology

Extending

18. a) It is possible that a polynomial equation of degree 4 can have no real roots. Create such a polynomial equation and explain why it cannot have any real roots.
   b) Explain why a degree 5 polynomial equation must have at least one real root.

19. The factor theorem only deals with rational zeros. Create a polynomial of degree 5 that has no rational zeros. Explain why your polynomial has no rational zero but has at least one irrational zero.

4.2 Solving Linear Inequalities

GOAL

Solve linear inequalities.

YOU WILL NEED

- graphing calculator or graphing software

LEARN ABOUT the Math

In mathematics, you must be able to represent intervals and identify smaller sections of a relation or a set of numbers. You have used the following inequality symbols:

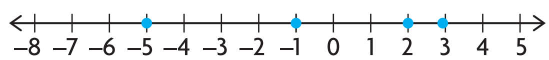
$>$ greater than $\quad$ $<$ less than
$\geq$ greater than or equal to $\quad$ $\leq$ less than or equal to

When you write one of these symbols between two or more linear expressions, the result is called a **linear inequality**.

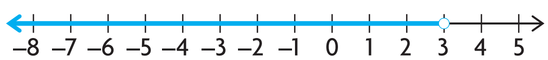
linear inequality
an inequality that contains an algebraic expression of degree 1 (e.g., $5x + 3 > 6x - 2$)

To solve an inequality, you have to find all the possible values of the variable that satisfy the inequality.

For example, $x = 2$ satisfies $3x - 1 < 8$, but so do $x = 2.9$, $x = -1$, and $x = -5$.



In fact, every real number less than 3 results in a number smaller than 8. So all real numbers less than 3 satisfy this inequality. The thicker part of the number line below represents this solution.



Communication *Tip*

To show that a number is not included in the solution set, use an open dot at this value. A solid dot shows that this value is included in the solution set.

The solution to $3x - 1 < 8$ can be written in set notation as $\{x \in \mathbf{R} \mid x < 3\}$ or in interval notation $x \in (-\infty, 3)$.

? How can you determine algebraically the solution set to a linear inequality like $3x - 1 < 8$?

EXAMPLE 1 Selecting an inverse operation strategy to solve a linear inequality

Solve the linear inequality $3x - 1 < 8$.

Solution

$$3x - 1 < 8$$

$$3x - 1 + 1 < 8 + 1$$

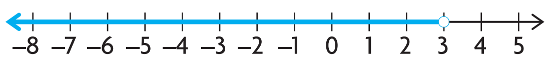
Treat the inequality like a linear equation and use inverse operations to isolate *x*. Add 1 to both sides of the inequality and simplify.

$$3x < 9$$

$$\frac{3x}{3} < \frac{9}{3}$$

Divide both sides of the inequality by 3.

$$x < 3$$



A number line helps visualize the solution.

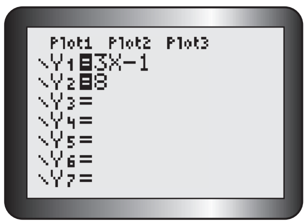
Check $x = 0$.

LS	RS
$3x - 1$	8
$= 3(0) - 1$	
$= -1$	

$LS < RS$

Choose a value for *x* that is less than 3 to verify that any number in the solution set satisfies the original inequality.

You can also verify the solution set using a graphing calculator. Graph each side of the inequality as a function.



The *y*-values on the line $y = 3x - 1$ that are less than 8 are found on all points that lie on the line below the horizontal line $y = 8$. This happens when *x* is smaller than 3.

The solution set is $\{x \in \mathbf{R} \mid x < 3\}$ or in interval notation $x \in (-\infty, 3)$.

Reflecting

A. How was solving a linear inequality like solving a linear equation? How was it different?

B. When checking the solution to an inequality, why is it not necessary for the left side to equal the right side?

C. Why do most linear equations have only one solution, but linear inequalities have many?

APPLY the Math

EXAMPLE 2 Using reasoning to determine which operations preserve the truth of a linear inequality

Can you add, subtract, multiply, or divide both sides of an inequality by a non-zero value and still have a valid inequality?

Solution

$4 < 8$

$4 + 5 < 8 + 5$

$9 < 13$

Write a true inequality using two numbers. Add the same positive quantity to both sides.

The result is still true.

$4 + (-5) < 8 + (-5)$

$-1 < 3$

Add the same negative quantity to both sides of the initial inequality.

The result is still true.

$4 - 10 < 8 - 10$

$-6 < -2$

Subtract the same positive quantity from both sides of the initial inequality.

The result is still true.

$4 - (-3) < 8 - (-3)$

$7 < 11$

Subtract the same negative quantity from both sides of the initial inequality.

The result is still true.

$4(6) < 8(6)$

$24 < 48$

Multiply by the same positive quantity on both sides of the initial inequality.

The result is still true.

$$4(-2) < 8(-2)$$
$$-8 < -16$$

Multiply by the same negative quantity on both sides of the initial inequality.

The result is false.
In this case, $-8 > -16$.

$$4 \div 2 < 8 \div 2$$
$$2 < 4$$

Divide by the same positive quantity on both sides of the initial inequality.

The result is still true.

$$4 \div (-2) < 8 \div (-2)$$
$$-2 < -4$$

Divide by the same negative quantity on both sides of the initial inequality.

The result is false.
In this case, $-2 > -4$.

Most of the operations preserve the validity of the inequality. The exception occurs when both sides are multiplied or divided by a negative number. In these two cases, reversing the inequality sign preserves the validity.

Since algebraic expressions represent numbers, this conclusion applies to linear inequalities that contain variables.

EXAMPLE 3 Reflecting to verify a solution

Solve the inequality $35 - 2x \geq 20$.

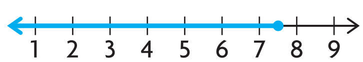
Solution

$$35 - 2x \geq 20$$
$$-2x \geq 20 - 35$$
$$-2x \geq -15$$

Use inverse operations to isolate *x*. Subtract 35 from both sides and simplify.

$$\frac{-2x}{-2} \leq \frac{-15}{-2}$$
$$x \leq 7.5$$

Divide both sides by -2. Since the division involves a negative number, reverse the inequality sign.



Represent the solution on a number line.
A solid dot is placed on 7.5 since this number is included in the solution set.

If $x = 5$, then

LS	RS
$35 - 2x$	20
$= 35 - 2(5)$	
$= 35 - 10$	
$= 25$	

LS $\geq$ RS: This is the desired outcome.

Test a value less than 7.5 and a value greater than 7.5 to verify the solution. Since $x = 5$ makes the inequality true and $x = 8$ makes the inequality false, the solution is correct.

If $x = 8$, then

LS	RS
$35 - 2x$	20
$= 35 - 2(8)$	
$= 35 - 16$	
$= 19$	

LS $\leq$ RS: This is <u>not</u> the desired outcome.

The solution set is $\{x \in \mathbf{R} | x \leq 7.5\}$ or in interval notation $x \in (-\infty, 7.5]$.

The value of 7.5 makes both sides equal. Since the inequality sign includes an equal sign, 7.5 must be part of the solution set.

EXAMPLE 4 Connecting the process of solving a double inequality to solving a linear inequality

Solve the inequality $30 \leq 3(2x + 4) - 2(x + 1) \leq 46$.

Solution

$$30 \leq 3(2x + 4) - 2(x + 1) \leq 46$$

This is a combination of two inequalities:
$30 \leq 3(2x + 4) - 2(x + 1)$ and $3(2x + 4) - 2(x + 1) \leq 46$.
A valid solution must satisfy both inequalities.
Expand using the distributive property and simplify.

$$30 \leq 6x + 12 - 2x - 2 \leq 46$$

$$30 \leq 4x + 10 \leq 46$$

$$30 - 10 \leq 4x + 10 - 10 \leq 46 - 10$$

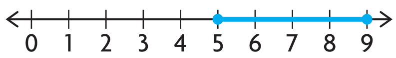
Subtract 10 from all three parts of the inequality.

$$20 \leq 4x \leq 36$$

$$\frac{20}{4} \leq \frac{4x}{4} \leq \frac{36}{4}$$

Divide all parts of the inequality by 4.

$$5 \leq x \leq 9$$

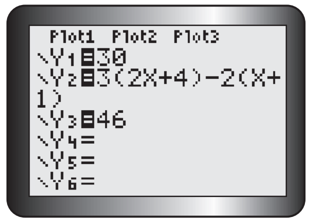


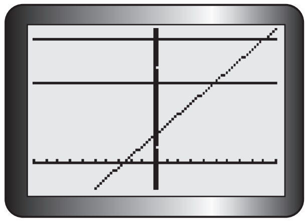
A number line helps to visualize the solution. Solid dots are placed on 5 and 9 since these numbers are included in the solution set.

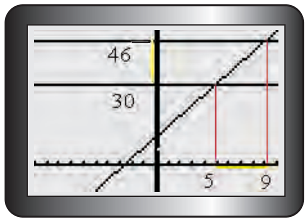
The solution using interval notation is

$x \in [5, 9]$

To verify the solution, graph the functions that correspond to all three parts of the inequality.







The x-values that satisfy the inequality are the x-coordinates of points on the diagonal line defined by $y = 3(2x + 4) - 2(x + 1)$ whose y-values are bounded by 30 and 46.

In Summary

Key Idea

- You can solve a linear inequality using inverse operations in much the same way you solve linear equations.

Need to Know

- If you multiply or divide an inequality by a negative number, you must reverse the inequality sign.
- Most linear equations have only one solution, whereas linear inequalities have many solutions.
- A number line can help you visualize the solution set to an inequality. A solid dot is used to indicate that a number is included in the solution set, whereas an open dot indicates that a number is excluded.

CHECK Your Understanding

1. Solve the following inequalities graphically. Express your answer using set notation.
   a) $3x - 1 \leq 11$
   b) $-x + 5 > -2$
   c) $x - 2 > 3x + 8$
   d) $3(2x + 4) \geq 2x$
   e) $-2(1 - 2x) < 5x + 8$
   f) $\dfrac{6x + 8}{5} \leq 2x - 4$

2. Solve the following inequalities algebraically. Express your answer using interval notation.
   a) $2x - 5 \leq 4x + 1$
   b) $2(x + 3) < -(x - 4)$
   c) $\dfrac{2x + 3}{3} \leq x - 5$
   d) $2x + 1 \leq 5x - 2$
   e) $-x + 1 > x + 1$
   f) $\dfrac{x + 4}{2} \geq \dfrac{x - 2}{4}$

3. Solve the double inequality $3 \leq 2x + 5 < 17$ algebraically and illustrate your solution on a number line.

4. For each of the following inequalities, determine whether $x = 2$ is contained in the solution set.
   a) $x > -1$
   b) $5x - 4 > 3x + 2$
   c) $4(3x - 5) \geq 6x$
   d) $5x + 3 \leq -3x + 1$
   e) $x - 2 \leq 3x + 4 \leq x + 14$
   f) $33 < -10x + 3 < 54$

PRACTISING

5. Solve the following algebraically. Verify your results graphically.
   K
   a) $2x - 1 \leq 13$
   b) $-2x - 1 > -1$
   c) $2x - 8 > 4x + 12$
   d) $5(x - 3) \geq 2x$
   e) $-4(5 - 3x) < 2(3x + 8)$
   f) $\dfrac{x - 2}{3} \leq 2x - 3$

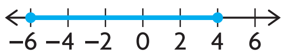
6. For the following inequalities, determine if 0 is a number in the solution set.
   a) $3x \leq 4x + 1$
   b) $-6x < x + 4 < 12$
   c) $-x + 1 > x + 12$
   d) $3x \leq x + 1 \leq x - 1$
   e) $x(2x - 1) \leq x + 7$
   f) $x + 6 < (x + 2)(5x + 3)$

7. Solve the following inequalities algebraically.
   a) $-5 < 2x + 7 < 11$
   b) $11 < 3x - 1 < 23$
   c) $-1 \leq -x + 9 \leq 13$
   d) $0 \leq -2(x + 4) \leq 6$
   e) $59 < 7x + 10 < 73$
   f) $18 \leq -12(x - 1) \leq 48$

8. **a)** Create a linear inequality, with both constant and linear terms on each side, for which the solution is $x > 4$.

b) Create a linear inequality, with both constant and linear terms on each side, for which the solution is $x \leq \frac{3}{2}$.

9. The following number line shows the solution to a double inequality.

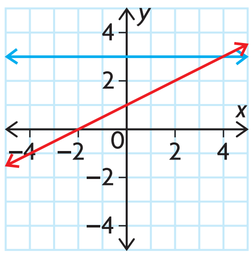


a) Write the solution using set notation.

b) Create a double inequality for which this is the solution set.

10. Which of the following inequalities has a solution. Explain.
$x - 3 < 3 - x < x - 5$ or $x - 3 > 3 - x > x - 5$

11. Consider the following graph.



a) Write an inequality that is modelled by the graph.

b) Find the solution by examining the graph.

c) Confirm the solution by solving your inequality algebraically.

12. A The relationship between Celsius and Fahrenheit is represented by $C = \frac{5}{9}(F - 32)$. In order to be comfortable, but also economical, the temperature in your house should be between 18 °C and 22 °C.

a) Write this statement as a double linear inequality.

b) Solve the inequality to determine the temperature range in degrees Fahrenheit.

13. Some volunteers are making long distance phone calls to raise money for a charity. The calls are billed at the rate of \$0.50 for the first 3 min and \$0.10/min for each additional minute or part thereof. If each call cannot cost more that \$2.00, how long can each volunteer talk to a prospective donor?

14. **a)** Find the equation that allows for the conversion of Celsius to Fahrenheit by solving the relation given in question 12 for F.

b) For what values of C is the Fahrenheit temperature greater than the equivalent Celsius temperature?

15. The inequality $|2x - 1| < 7$ can be expressed as a double inequality.

T **a)** Depict the inequality graphically.

b) Use your graph to solve the inequality.

16. Will the solution to a double inequality always have an upper and

C lower limit? Explain.

Extending

17. Some inequalities are very difficult to solve algebraically. Other methods, however, can be very helpful in solving such problems. Consider the inequality $2^x - 3 < x + 1$.

a) Explain why solving the inequality might be very difficult to do algebraically.

b) Describe an alternative method that could work, and use it to solve the inequality.

18. Some operations result in switching the direction of the inequality when done to both sides, but others result in maintaining the direction. For instance, if you add a constant to both sides, the direction is maintained, whereas multiplying both sides by a negative constant causes the sign to switch. For each of the following, determine if the inequality direction should be maintained, should switch, or if it sometimes switches and sometimes is maintained.

a) cubing both sides

b) squaring both sides

c) making each side the exponent with 2 as the base, i.e., $3 < 5$, so $2^3 < 2^5$

d) making each side the exponent with 0.5 as the base

e) taking the reciprocal of both sides

f) rounding both sides up to the nearest integer

g) taking the square root of both sides

19. Solve each of the following, $x \in \mathbf{R}$. Express your answers using both set and interval notation and graph the solution set on a number line.

a) $x^2 < 4$

b) $4x^2 + 5 \geq 41$

c) $|2x + 2| < 8$

d) $-3x^3 \geq 81$

4 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study *Aid*

- See Lesson 4.1, Examples 1 to 4.
- Try Mid-Chapter Review Questions 1, 2, and 3.

Q: How can you solve a polynomial equation?

A1: You can use an algebraic strategy using the corresponding polynomial function, the factor theorem, and division to factor the polynomial. Set each factor equal to zero and solve for the independent variable. You will need to use the quadratic formula if one of the factors is a nonfactorable quadratic.

For example, to solve the equation

$2x^4 + x^3 - 19x^2 - 14x + 24 = 0$, let $f(x) = 2x^4 + x^3 - 19x^2 - 14x + 24$. Possible values of x that make $f(x) = 0$ are numbers of the form $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient. Some possible values in this case are:

$$\frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24}{\pm 1, \pm 2}$$

Since $f(-2) = 0$, by the factor theorem, $(x + 2)$ is a factor of $f(x)$. Determine $f(x) \div (x + 2)$ to find the other factor.

$$\begin{array}{r|rrrrr} -2 & 2 & 1 & -19 & -14 & 24 \\ & \downarrow & -4 & 6 & 26 & -24 \\ \hline & 2 & -3 & -13 & 12 & 0 \end{array}$$

Now, $f(x) = (x + 2)(2x^3 - 3x^2 - 13x + 12)$.

Possible values of x that make the cubic polynomial 0 are numbers of the form:

$$\frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12,}{\pm 2}$$

Since $f(3) = 0$, $(x - 3)$ is also a factor of $f(x)$. Divide the cubic polynomial by $(x - 3)$ to determine the other factor.

$$\begin{array}{r|rrrr} 3 & 2 & -3 & -13 & 12 \\ & \downarrow & 6 & 9 & -12 \\ \hline & 2 & 3 & -4 & 0 \end{array}$$

So, $f(x) = (x + 2)(x - 3)(2x^2 + 3x - 4)$.

$x + 2 = 0$ or $x - 3 = 0$ or $2x^2 + 3x - 4 = 0$

$x = -2$ or $x = 3$

Since $2x^2 + 3x - 4$ is not factorable, use the quadratic formula to determine the other zeros.

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-4)}}{2(2)}$$

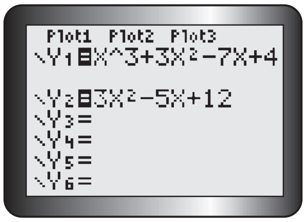
$$x = \frac{-3 \pm \sqrt{41}}{4}$$

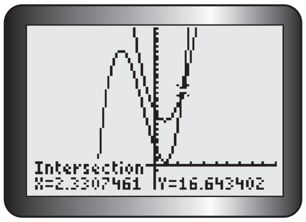
$x \doteq -2.35$ or 0.85

The equation has four roots: $x = -2$, $x \doteq -2.35$, $x \doteq 0.85$, and $x = 3$.

A2: You can use a graphing strategy to find the zeros of a rearranged equation, or graph both sides of the equation separately and then determine the point(s) of intersection.

For example, to solve the equation $x^3 + 3x^2 - 7x + 4 = 3x^2 - 5x + 12$, enter both polynomials in the equation editor of the graphing calculator and then graph both corresponding functions. Use the intersect operation to determine the point of intersection of the two graphs.





The solution is $x \doteq 2.33$.

Q: How can you solve a linear inequality?

A: You solve a linear inequality using inverse operations in much the same way you would solve a linear equation. If at any time you multiply or divide the inequality by a negative number, you must reverse the inequality sign.

Study Aid

- See Lesson 4.2, Examples 1, 3, and 4.
- Try Mid-Chapter Review Questions 4 to 8.

PRACTICE Questions

Lesson 4.1

1. Determine the solutions for each of the following.

a) $0 = -2x^3(2x - 5)(x - 4)^2$
b) $0 = (x^2 + 1)(2x + 4)(x + 2)$
c) $x^3 - 4x^2 = 7x - 10$
d) $0 = (x^2 - 2x - 24)(x^2 - 25)$
e) $0 = (x^3 + 2x^2)(x + 9)$
f) $-x^4 = -13x^2 + 36$

2. Jude is diving from a cliff into the ocean. His height above sea level in metres is represented by the function $h(t) = -5(t - 0.3)^2 + 25$, where t is measured in seconds.

a) Expand the height function.
b) How high is the cliff?
c) When does Jude hit the water?
d) Determine where the function is negative. What is the significance of the negative values?

3. Chris makes an open-topped box from a 30 cm by 30 cm piece of cardboard by cutting out equal squares from the corners and folding up the flaps to make the sides. What are the dimensions of each square, to the nearest hundredth of a centimetre, so that the volume of the resulting box is 1000 cm^3?

Lesson 4.2

4. Solve the following inequalities algebraically and plot the solution on a number line.

a) $2x - 4 < 3x + 7$
b) $-x - 4 \leq x + 4$
c) $-2(x - 4) \geq 16$
d) $2(3x - 7) > 3(7x - 3)$

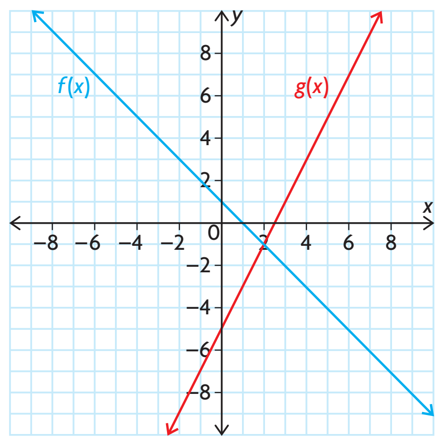
5. Solve and state your solution using inverval notation.

$$2x < \frac{3x + 6}{2} \leq 4 + 2x$$

6. Create a linear inequality with both a constant and a linear term on each side and that has each of the following as a solution.

a) $x > 7$
b) $x \in (-\infty, -8)$
c) $-1 \leq x \leq 7$
d) $x \in [3, \infty)$

7. Consider the following functions.



a) Find the equations of the lines depicted.
b) Solve the inequality $f(x) < g(x)$ by examining the graph.
c) Confirm your solution by solving the inequality algebraically.

8. The New Network cell phone company charges \$20 a month for service and \$0.02 per minute of talking time. The My Mobile company charges \$15 a month for service and \$0.03 per minute of talking time.

a) Write expressions for the total bill of each company.
b) Set up an inequality that can be used to determine for what amount of time (in minutes) My Mobile is the better plan.
c) Solve your inequality.
d) Why did you have to put a restriction on the algebraic solution from part c)?

4.3 Solving Polynomial Inequalities

GOAL

Solve polynomial inequalities.

YOU WILL NEED

- graphing calculator or graphing software

LEARN ABOUT the Math

The elevation of a hiking trail is modelled by the function $h(x) = 2x^3 + 3x^2 - 17x + 12$, where h is the height measured in metres above sea level and x is the horizontal position from a ranger station measured in kilometres. If x is negative, the position is to the west of the station, and if x is positive, the position is to the east. Since the trail extends 4.2 km to the west of the ranger station and 4 km to the east, the model is accurate where $x \in [-4.2, 4]$.

? How can you determine which sections of the trail are above sea level?

EXAMPLE 1 Selecting a strategy to solve the problem

At what distances from the ranger station is the trail above sea level?

Solution A: Using an algebraic strategy and a number line

$$2x^3 + 3x^2 - 17x + 12 > 0$$

The trail is above sea level when the height is positive, i.e., $h(x) > 0$. Write the mathematical model using a **polynomial inequality**.

polynomial inequality
an inequality that contains a polynomial expression (e.g., $5x^3 + 3x^2 - 6x \le 2$)

$$h(1) = 2(1)^3 + 3(1)^2 - 17(1) + 12$$
$$= 0 \text{ so } (x - 1) \text{ is a factor of } h(x)$$

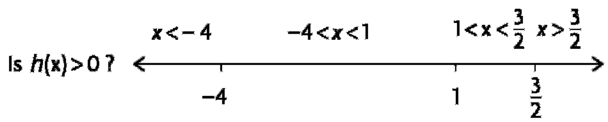
Factor the corresponding function $y = h(x)$ to locate the x-intercepts. Use the factor theorem to determine the first factor.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -17 & 12 \\ & \downarrow & 2 & 5 & -12 \\ \hline & 2 & 5 & -12 & 0 \end{array}$$

$$h(x) = (x - 1)(2x^2 + 5x - 12)$$
$$0 = (x - 1)(2x - 3)(x + 4)$$
$$x = 1,\ x = \frac{3}{2}, \text{ or } x = -4$$

Set $h(x) = 0$. Set each factor equal to 0 and solve.

The x-intercepts are at -4, 1, and $\frac{3}{2}$. These numbers divide the domain of real numbers into four intervals: $x < -4$, $-4 < x < 1$, $1 < x < \frac{3}{2}$, $x > \frac{3}{2}$



Draw a number line and test points in each interval to see whether the function has a positive or negative value.

Interval	$x < -4$	$-4 < x < 1$	$1 < x < \frac{3}{2}$	$x > \frac{3}{2}$
Value of $h(x)$	$h(-5) = -78$	$h(0) = 12$	$h(1.2) \doteq -0.6$	$h(2) = 6$
Is $h(x) > 0$?	no	yes	no	yes

$h(x) > 0$ when $-4 < x < 1$ and $x > \frac{3}{2}$.

Identify the intervals where $h(x)$ is positive.

The hiking trail is above sea level from 4 km west of the ranger station to 1 km east, and for distances more than 1.5 km east.

Write a concluding statement.

Solution B: Using a graphing strategy

$$2x^3 + 3x^2 - 17x + 12 > 0$$

The trail is above sea level when the height is positive, i.e., $h(x) > 0$.

$$h(1) = 2(1)^3 + 3(1)^2 - 17(1) + 12$$
$$= 0 \text{ so } (x - 1) \text{ is a factor of } h(x)$$

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -17 & 12 \\ & \downarrow & 2 & 5 & -12 \\ \hline & 2 & 5 & -12 & 0 \end{array}$$

Factor the corresponding function $y = h(x)$ to locate the x-intercepts. Use the factor theorem to determine the first factor.

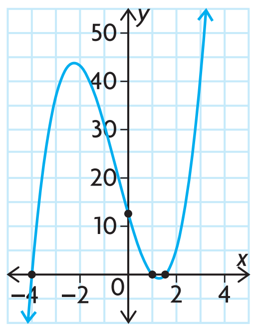
$$h(x) = (x - 1)(2x^2 + 5x - 12)$$
$$0 = (x - 1)(2x - 3)(x + 4)$$
$$x = 1, x = \frac{3}{2}, \text{ or } x = -4$$

Set $h(x) = 0$.
Set each factor equal to 0 and solve.

The x-intercepts are at -4, 1, and $\frac{3}{2}$.

$$h(0) = 12$$

The y-intercept occurs when $x = 0$.



Analyze the function and draw a sketch of $h(x)$. Plot the x- and y-intercepts. Because the leading coefficient of the function is positive and the degree of the function is odd, the graph has opposite end behaviours. The graph must start in the third quadrant and proceed to the first quadrant. Estimate the location of the turning points.

The graph lies above the x-axis on the intervals $-4 < x < 1$ and $x > \frac{3}{2}$.

Determine the intervals where $h(x) > 0$.

The hiking trail is above sea level from 4 km west of the ranger station to 1 km east, and for distances beyond 1.5 km to the east of the ranger station.

Write a concluding statement that answers the question.

Reflecting

A. When solving a polynomial inequality, which steps are the same as those used when solving a polynomial equation?

B. What additional steps must be taken when solving a polynomial inequality?

C. The zeros of $y = h(x)$ were used to identify the intervals where $h(x)$ was positive and negative but were not included in the solution set of $h(x) > 0$. Explain why.

D. How could you verify the solution set to the polynomial inequality using graphing technology?

APPLY the Math

EXAMPLE 2 Selecting tools and strategies to solve a factorable polynomial inequality

Solve the inequality $x^3 - 2x^2 + 5x + 20 \geq 2x^2 + 14x - 16$.

Solution A: Using algebra and a factor table

$x^3 - 2x^2 + 5x + 20 \geq 2x^2 + 14x - 16$

$x^3 - 4x^2 - 9x + 36 \geq 0$

Use inverse operations to make the right side of the inequality equal to zero.

$x^2(x - 4) - 9(x - 4) \geq 0$

Factor the polynomial on the left by grouping.

$(x - 4)(x^2 - 9) \geq 0$

$(x - 4)(x - 3)(x + 3) \geq 0$

$(x - 4)(x - 3)(x + 3) = 0$

Determine the roots of the corresponding polynomial equation.

The roots are -3, 3, and 4. These numbers divide the real numbers into four intervals:

$x < -3, -3 < x < 3, 3 < x < 4, x > 4$

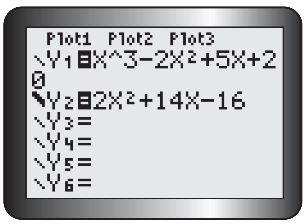
	$x < -3$	$-3 < x < 3$	$3 < x < 4$	$x > 4$
$(x - 4)$	$-$	$-$	$-$	$+$
$(x - 3)$	$-$	$-$	$+$	$+$
$(x + 3)$	$-$	$+$	$+$	$+$
their product	$(-)(-)(-) = -$	$(-)(-)(+) = +$	$(-)(+)(+) = -$	$(+)(+)(+) = +$

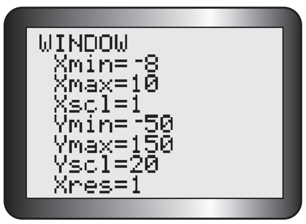
Create a table to consider the sign of each factor in each of the intervals and examine the sign of their product. In this case, the intervals that correspond to a positive product are the solutions to the polynomial inequality.

$x^3 - 2x^2 + 5x + 20 \geq 2x^2 + 14x - 16$ when $-3 \leq x \leq 3$ or $x \geq 4$.

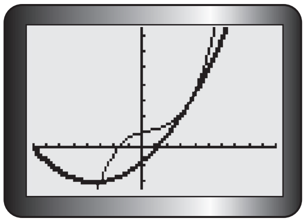
Write a concluding statement.

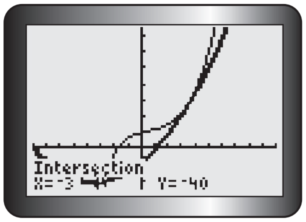
Solution B: Using graphing technology



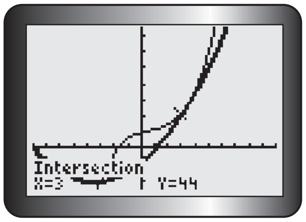


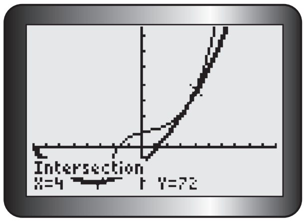
Graph each side of the inequality as a separate function. Bold the graph of the second function (the quadratic) so you can distinguish one from the other. Experiment with different window settings to make the intersecting parts of the graph visible.





The two graphs intersect somewhere to the left of the y-axis and intersect twice to the right of the y-axis. Use the intersect operation to determine all points of intersection.





You can see on the graph that the cubic function lies above the quadratic function in the interval $-3 \le x \le 3$ or $x \ge 4$.

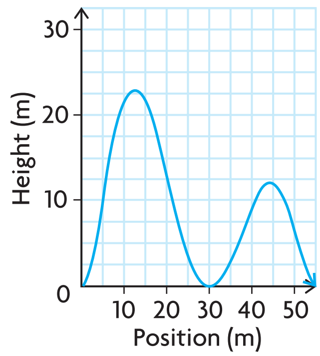
The two functions intersect at $(-3, -40)$, $(3, 44)$, and $(4, 72)$.

$x^3 - 10x^2 + 15x + 11 \ge -x^2 - 8x + 26$ when $x \in [-3, 3]$ or $x \in [4, \infty)$.

Write a concluding statement.

EXAMPLE 3 Selecting a strategy to solve a polynomial inequality that is unfactorable

The height of one section of the roller coaster can be described by the polynomial function $h(x) = \frac{1}{4\,000\,000}x^2(x - 30)^2(x - 55)^2$, where h is the height, measured in metres, and x is the position from the start, measured in metres along the ground.



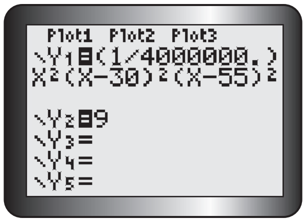
When will the roller coaster car be more than 9 m above the ground?

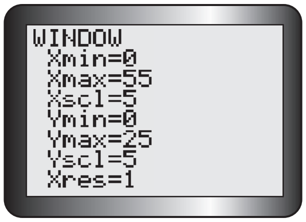
Solution

Solve

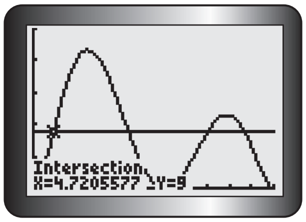
$$\frac{1}{4\,000\,000}x^2(x - 30)^2(x - 55)^2 > 9$$

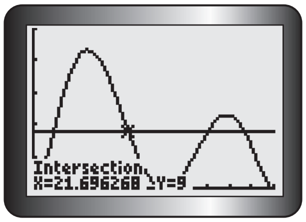
In this case, the solution set corresponds to all values of x where $h(x) > 9$. Using an algebraic approach involving factoring would be tedious, so use a graphing strategy.

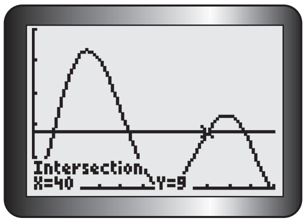


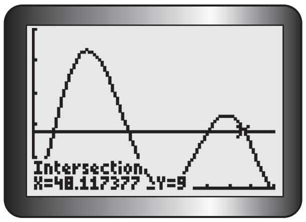


Graph the function $h(x) = \frac{1}{4\,000\,000}x^2(x - 30)^2(x - 55)^2$ and the line $y = 9$ on the graphing calculator and locate intervals where the roller coaster is higher than 9 m. On the graph, this will correspond to when $Y1 > Y2$.









Determine the four points of intersection of the height function and the horizontal line.

The four points where the height function and the horizontal line intersect are approximately $(4.7, 9)$, $(21.7, 9)$, $(40, 9)$, and $(48.1, 9)$.

The roller coaster will be more than 9 m above the ground when it is between 4.7 m and 21.7 m from the starting point and between 40 m and 48.1 m from the starting point, as measured along the ground.

$Y1 > Y2$ when $4.7 < x < 21.7$ or $40 < x < 48.1$.

In Summary

Key Idea

- To solve a polynomial inequality algebraically, you must first determine the roots of the corresponding polynomial equation. Then you must consider the sign of the polynomial in each of the intervals created by these roots. The solution set is determined by the interval(s) that satisfy the given inequality.

Need to Know

- Some polynomial inequalities can be solved algebraically by
  - using inverse operations to move all terms to one side of the inequality
  - factoring the polynomial to determine the zeros of the corresponding polynomial equation
  - using a number line, a graph, or a factor table to determine the intervals on which the polynomial is positive or negative
- All polynomial inequalities can be solved using graphing technology by
  - graphing each side of the inequality as a separate function
  - determining the intersection point(s) of the functions
  - examining the graph to determine the intervals where one function is above or below the other, as required

    or
  - creating an equivalent inequality with zero on one side
  - identifying the intervals created by the zeros of the graph of the new function
  - finding where the graph lies above the x-axis (where $f(x) > 0$) or below (where $f(x) < 0$), as required

CHECK Your Understanding

1. Solve each of the following using a number line strategy. Express your answers using set notation.

a) $(x + 2)(x - 3)(x + 1) \geq 0$

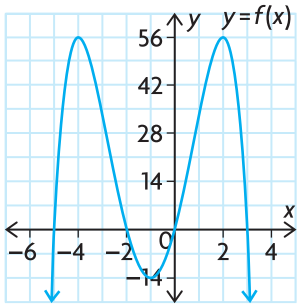
b) $-2(x - 2)(x - 4)(x + 3) < 0$

c) $(x - 3)(5x + 2)(4x - 3) < 0$

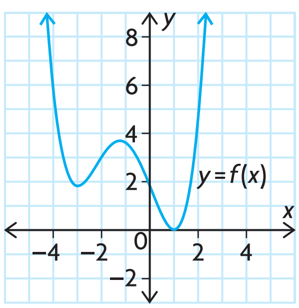
d) $(x - 5)(4x + 1)(2x - 5) \geq 0$

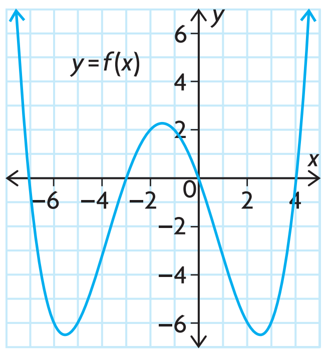
2. For each graph shown, determine where $f(x) \leq 0$. Express your answers using interval notation.

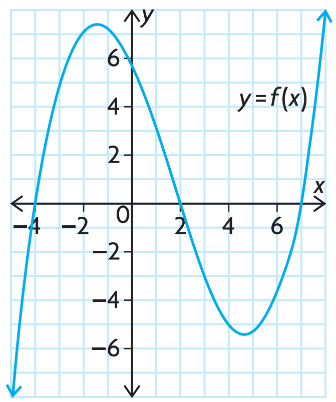
a)



b)



c) 

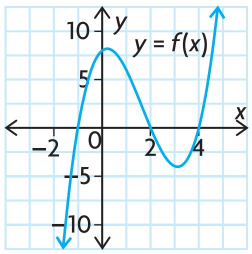
d) 

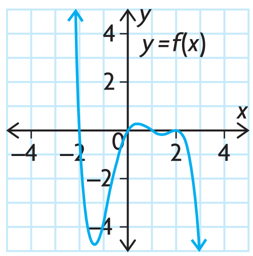
3. If $f(x) = 2x^3 - x^2 + 3x + 10$ and $g(x) = x^3 + 3x^2 + 2x + 4$, determine when $f(x) > g(x)$ using a factor table strategy.

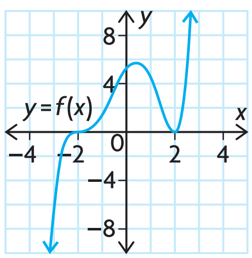
4. Solve the inequality $x^3 - 7x^2 + 4x + 12 > x^2 - 4x - 9$ using a graphing calculator.

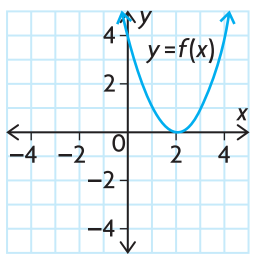
PRACTISING

5. For each of the following polynomial functions, state the intervals where $f(x) > 0$.
K

a) 

c) 

b) 

d) 

6. Solve the following inequalities.

a) $(x - 1)(x + 1) > 0$
b) $(x + 3)(x - 4) < 0$
c) $(2x + 1)(x - 5) \geq 0$
d) $-3x(x + 7)(x - 2) < 0$
e) $(x - 3)(x + 1) + (x - 3)(x + 2) \geq 0$
f) $2x(x + 4) - 3(x + 4) \leq 0$

7. Solve the following inequalities algebraically. Confirm your answer with a graph.

a) $x^2 - 6x + 9 \geq 16$

b) $x^4 - 8x < 0$

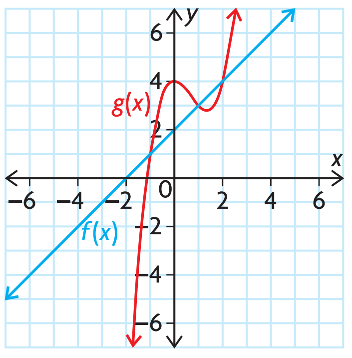
c) $x^3 + 4x^2 + x \leq 6$

d) $x^4 - 5x^2 + 4 > 0$

e) $3x^3 - 3x^2 - 2x \leq 2x^3 - x^2 + x$

f) $x^3 - x^2 - 3x + 3 > -x^3 + 2x + 5$

8. For the following pair of functions, determine when $f(x) < g(x)$.



9. Consider $x^3 + 11x^2 + 18x + 10 > 10$.

a) What is the equation of the corresponding function that could be graphed and used to solve this inequality?

b) Explain how the graph of the corresponding function can be used in this case to solve the inequality.

c) Solve this inequality algebraically.

10. Determine an expression for $f(x)$ in which $f(x)$ is a quartic function, $f(x) > 0$ when $-2 < x < 1$, $f(x) \leq 0$ when $x < -2$ or $x > 1$, $f(x)$ has a double root when $x = 3$, and $f(-1) = 96$.

11. The viscosity, v, of oil used in cars is related to its temperature, t, by the formula $v = -t^3 - 6t^2 + 12t + 50$, where each unit of t is equivalent to 50 °C.

a) Graph the function on your graphing calculator.

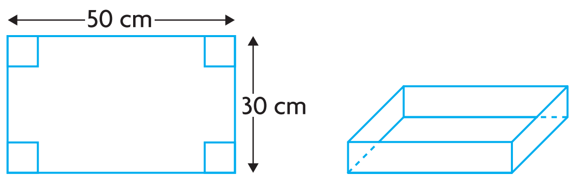
b) Determine the temperature range for which $v > 0$ to two decimal places.

c) Determine the temperature ranges for which $15 < v < 20$ to two decimal places.

12. A rock is tossed from a platform and follows a parabolic path through the air. The height of the rock in metres is given by $h(t) = -5t^2 + 12t + 14$, where t is measured in seconds.

a) How high is the rock off the ground when it is thrown?
b) How long is the rock in the air?
c) For what times is the height of the rock greater than 17 m?
d) How long is the rock above a height of 17 m?

13. **A** An open-topped box can be made from a sheet of aluminium measuring 50 cm by 30 cm by cutting congruent squares from the four corners and folding up the sides. Write a polynomial function to represent the volume of such a box. Determine the range of side lengths that are possible for each square that is cut out and removed that result in a volume greater than 4000 cm^3.



14. **T** **a)** Without a calculator, explain why the inequality $2x^{24} + x^4 + 15x^2 + 80 < 0$ has no solution.
b) Without a calculator, explain why $-4x^{12} - 7x^6 + 9x^2 + 20 < 30 + 11x^2$ has a solution of $-\infty < x < \infty$.

15. Explain why the following solution strategy fails, and then solve the inequality correctly.

Solve: $(x + 1)(x - 2) > (x + 1)(-x + 6)$.
Divide both sides by $x + 1$ and get $x - 2 > -x + 6$.
Add x to both sides: $2x - 2 > 6$.
Add 2 to both sides: $2x > 8$.
Divide both sides by 2: $x > 4$.

16. **C** Create a concept web that illustrates all of the different methods you could use to solve a polynomial inequality.

Extending

17. Use what you know about the factoring method to solve the following inequalities.

a) $\dfrac{x^2 + x - 12}{x^2 + 5x + 6} < 0$ **b)** $\dfrac{x^2 - 25}{x^3 + 6x^2 + 5x} > 0$

18. Solve the inequality $(x + 1)(x - 2)(2^x) \geq 0$ algebraically.

4.4 Rates of Change in Polynomial Functions

GOAL

Determine average and instantaneous rates of change in polynomial functions.

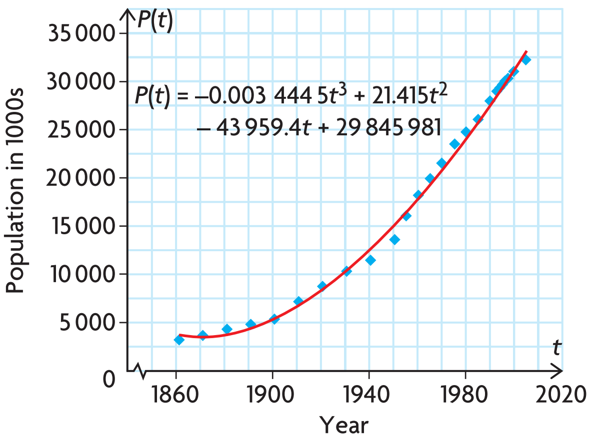
YOU WILL NEED

- graphing calculator or graphing software

LEARN ABOUT the Math

Emile is researching Canada's population growth. He obtained the data online and used graphing software to create the following graph and fit a curve to the data.

Canada's Population Since 1861



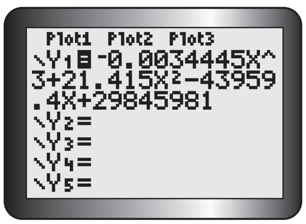
Year	Population (1000s)
1861	3 230
1871	3 689
1881	4 325
1891	4 833
1901	5 371
1911	7 207
1921	8 788
1931	10 377
1941	11 507
1951	13 648
1956	16 081
1961	18 238
1966	20 015
1971	21 568
1976	23 550
1981	24 820
1986	26 101
1991	28 031
1994	29 036
1995	29 354
1996	29 672
1997	30 011
1998	30 301
2001	31 051
2006	32 249

? Was Canada's population growing faster in 1997 or 2005?

EXAMPLE 1 Selecting tools and strategies to determine the instantaneous rate of change

Estimate the instantaneous rates of change in Canada's population in 1997 and 2005, and compare them.

Solution A: Using an algebraic strategy



Enter the equation into the graphing calculator.

Tech Support

For help using the graphing calculator to evaluate a function at a given point, see Technical Appendix, T–3.

Average rate of change

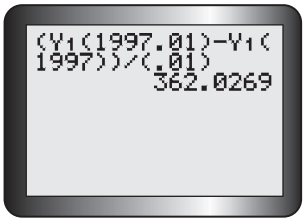
$$= \frac{P(a + h) - P(a)}{h}$$

$$h = 0.01$$

$$= \frac{P(1997 + 0.01) - P(1997)}{0.01}$$

$$= \frac{P(1997.01) - P(1997)}{0.01}$$

Use the difference quotient and a very small value for h where $a = 1997$ to estimate the instantaneous rate of change in 1997.



Enter the rate of change expression into the graphing calculator to determine its value using the equation entered into Y1.

Average rate of change

$$= \frac{P(a + h) - P(a)}{h}$$

$$h = 0.01$$

$$= \frac{P(2005 + 0.01) - P(2005)}{0.01}$$

$$= \frac{P(2005.01) - P(2005)}{0.01}$$

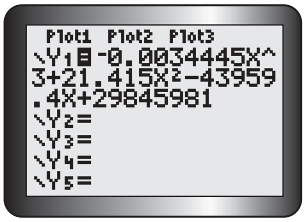
Use the difference quotient and a very small value for h where $a = 2005$ to estimate the instantaneous rate of change in 2005.

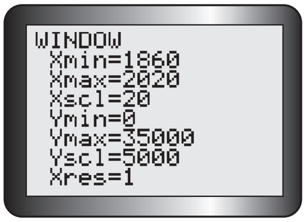
Enter the rate of change expression into the graphing calculator to determine its value using the equation entered into Y1.

The population was increasing by approximately 362 000 people/year in 1997 and 374 000 people/year in 2005. Canada's population was growing faster in 2005.

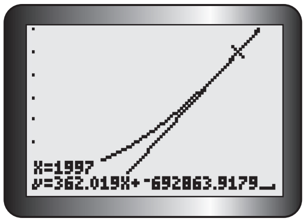
Round off and multiply the rates of change by 1000 since the population is given in thousands.

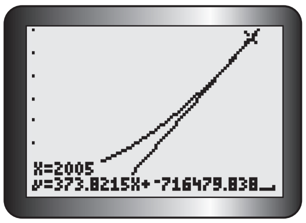
Solution B: Using a graphing strategy





Enter the equation into Y1 on the graphing calculator and adjust the window setting to match the data given.





The instantaneous rate of change at any given point is equal to the slope of the **tangent** line to the curve at that point.
Use the draw tangent operation to draw tangents to the curve at $x = 1997$ and $x = 2005$.
The graphing calculator gives the equation of the tangent in $y = mx + b$ form; thus, the slope of the tangent is the coefficient of x.

The population was increasing by approximately 362 000 people/year in 1997 and 374 000 people/year in 2005. Canada's population was growing faster in 2005.

Round off and multiply the rate of change by 1000 since the population is given in thousands.

Tech Support

For help to draw tangent lines using the graphing calculator's draw operation, see Technical Appendix, T–17.

Reflecting

A. The estimates for the instantaneous rates of change in population for 1997 and 2005 were both positive. Why does this make sense? Explain.

B. Explain how you could determine whether Canada's population was growing faster in 1880 or 1920 by just using the graph that was given.

C. Was Canada's population growing at a constant rate between 1860 and 2006? Explain.

APPLY the Math

EXAMPLE 2 Selecting tools and strategies to determine the slope of a secant

Determine the average rate of change from $x = 2$ to $x = 5$ on the function $f(x) = (x - 3)^3 - 1$.

Solution A: Using an algebraic strategy

Average rate of change

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Use the average rate of change formula for the interval $2 \le x \le 5$.

$$= \frac{f(5) - f(2)}{5 - 2}$$

$$= \frac{[(5-3)^3 - 1] - [(2-3)^3 - 1]}{3}$$

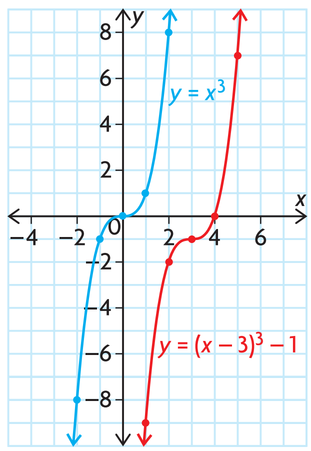
$$= \frac{7 - (-2)}{3}$$

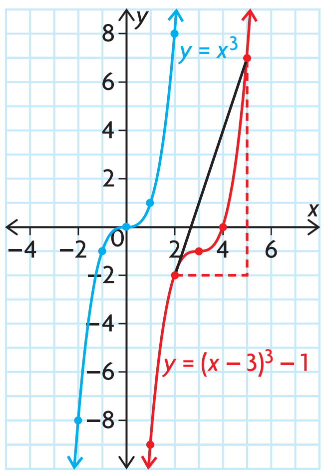
$$= 3$$

Solution B: Using a graphing strategy

$f(x) = (x - 3)^3 - 1$ is a translation right 3 units and down 1 unit of the graph of $y = x^3$.

Use transformations to sketch the graph of the function.



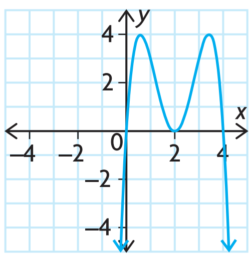


Draw the secant through the points $(2, f(2))$ and $(5, f(5))$, and calculate its slope since the slope of this secant line equals the average rate of change in $f(x)$ on this interval.

$$\begin{aligned} m_{\text{secant}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{7 - (-2)}{5 - 2} \\ &= \frac{9}{3} \\ &= 3 \end{aligned}$$

EXAMPLE 3 Selecting tools and strategies to determine the slope of a tangent

The graph of a polynomial function is shown. Estimate the instantaneous rate of change in $f(x)$ at the point $(2, 0)$.



Solution A: Using an algebraic strategy

$$f(x) = ax(x - 2)^2(x - 4)$$

Determine the equation of the polynomial function. The graph has zeros at $x = 0$, 2, and 4. Since the graph is parabolic at $x = 2$, the factor $(x - 2)$ is squared.

$$3 = a(1)(1 - 2)^2(1 - 4)$$
$$3 = a(-3)$$
$$-1 = a$$

Substitute the point $(1, 3)$ into the equation and solve for a.

$$f(x) = -x(x - 2)^2(x - 4)$$

State the equation that represents the function.

$$\text{Slope} = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{f(2 + h) - f(2)}{h}$$

Use the difference quotient to estimate the slope of the tangent line at $(2, 0)$. In this case, $a = 2$ and $f(a) = f(2) = 0$.

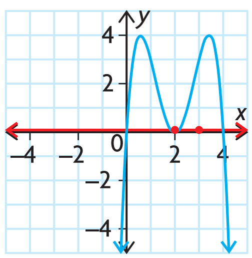
Let $h = 0.001$

$$h = \frac{f(2.001) - f(2)}{0.001}$$
$$h = \frac{[-2.001(2.001 - 2)^2(2.001 - 4)] - 0}{0.001}$$
$$h \doteq \frac{0.000\,004}{0.001}$$
$$h = 0.004$$

The instantaneous rate of change at $(2, 0)$ is approximately 0.

The slope of the tangent line at a turning point on a polynomial function is 0.

Solution B: Using a graphing strategy



Draw the graph on graph paper and sketch the tangent line at the point $A(2, 0)$. Estimate the coordinates of a second point that lies on the tangent line. In this case, use the point $B(3, 0)$.

$$m_{\text{tangent}} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{0 - 0}{2 - 3}$$
$$= 0$$

Calculate the slope of line AB using the slope formula.

The instantaneous rate of change at $(2, 0)$ is 0.

In Summary

Key Idea

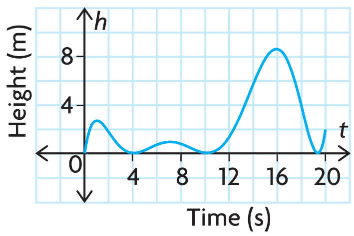
- The methods used previously to calculate average rate of change and estimate instantaneous rate of change can be used with polynomial functions.

Need to Know

- The average rate of change of a polynomial function $y = f(x)$ on the interval from $x_1 \leq x \leq x_2$ is $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$. Graphically, this is equivalent to the slope of the secant line that passes through the points (x_1, y_1) and (x_2, y_2) on the graph of $y = f(x)$.
- The instantaneous rate of change of a polynomial function $y = f(x)$ at $x = a$ can be approximated by using the difference quotient $\frac{f(a + h) - f(a)}{h}$, where h is a very small value. Graphically, this is equivalent to estimating the slope of the tangent line by calculating the slope of the secant line that passes through the points $(a, f(a))$ and $(a + h, f(a + h))$.
- The instantaneous rate of change of a polynomial function $y = f(x)$ at any of its turning points is 0.

CHECK Your Understanding

1. Consider the graph showing a bicyclist's elevation relative to his elevation above sea level at the start of the race. The first 20 s of the race are shown.



a) On which intervals will the tangent slope be positive? negative? zero?

b) What do these slopes tell you about the elevation of the bicyclist?

2. Consider the function $f(x) = 3(x - 2)^2 - 2$.

a) Determine the average rate of change in $f(x)$ on each of the following intervals.

i) $2 \leq x \leq 4$ **ii)** $2 \leq x \leq 6$ **iii)** $4 \leq x \leq 6$

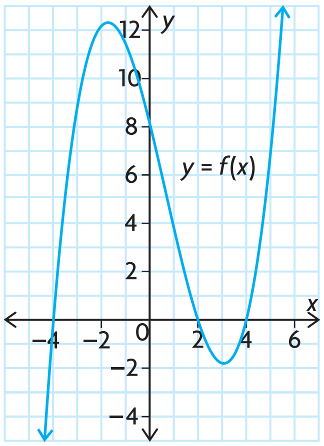
b) Estimate the instantaneous rate of change at $x = 4$.

c) Explain why all the rates of change in $f(x)$ calculated in parts a) and b) are positive.

d) State an interval on which the average rate of change in $f(x)$ will be negative.

e) State the coordinates of a point where the instantaneous rate of change in $f(x)$ will be negative.

3. Consider the function $f(x) = x^3 - 4x^2 + 4x$.
   a) Estimate the instantaneous rate of change in $f(x)$ at $x = 2$.
   b) What does your answer to part a) tell you about the graph of the function at $x = 2$?
   c) Sketch a graph of $f(x)$ by first finding the zeros of $f(x)$ to verify your answer to part b).



4. You are given the following graph of $y = f(x)$.
   a) Calculate the average rate of change in $f(x)$ on the interval $4 \leq x \leq 5$.
   b) Estimate the coordinates of the point on the graph of $f(x)$ whose instantaneous rate of change in $f(x)$ is the same as that found in part a).

PRACTISING

5. **K** For each of the following functions, calculate the average rate of change on the interval $x \in [2, 5]$.
   a) $f(x) = 3x + 1$
   b) $t(x) = 3x^2 - 4x + 1$
   c) $g(x) = \frac{1}{x}$
   d) $d(x) = -x^2 + 7$
   e) $h(x) = 2^x$
   f) $v(x) = 9$

6. For each of the functions in question 5, estimate the instantaneous rate of change at $x = 3$.

7. Graph the function $f(x) = x^3 - 2x^2 + x$ by finding its zeros. Use the graph to estimate where the instantaneous rate of change is positive, negative, and zero.

8. A construction worker drops a bolt while working on a high-rise building 320 m above the ground. After t seconds, the bolt's height above the ground is s metres, where $s(t) = 320 - 5t^2$, $0 \leq t \leq 8$.
   a) Find the average velocity for the interval $3 \leq t \leq 8$.
   b) Find the bolt's velocity at $t = 2$.

9. Consider the function $f(x) = 3x^2 - 4x - 1$.
   a) Estimate the slope of the tangent line at $x = 1$.
   b) Find the y-coordinate of the point of tangency.
   c) Use the coordinates of the point of tangency and the slope to find the equation of the tangent line at $x = 1$.

10. **A** The height, h, in metres of a toy rocket above the ground can be modelled by the function $h(t) = -5t^2 + 50t$, where t represents time in seconds.
   a) Use an average speed to approximate the instantaneous speed at $t = 4$.
   b) Use an average speed to approximate the instantaneous speed at $t = 10$.
   c) What is the average speed over the interval from $t = 0$ s to $t = 10$?

11. The distance in kilometres of a boat from its dock can be modelled by the function $d(t) = \left(\frac{1}{200}\right)t^2(t - 8)^2$, where t is in minutes and $t \in [0, 8]$. Sketch a graph that models this situation.

a) Estimate when the instantaneous rate of change in distance to the dock is positive, negative, and zero.

b) What happens to the boat when the instantaneous rate of change in distance is zero? What does it mean when the boat's rate of change in distance is negative?

12. Approximate the instantaneous rate of change at the zeros of the following function: $y = x^4 - 2x^3 - 8x^2 + 18x - 9$.

13. Consider the function $f(x) = x^2 + 3x - 5$.

T **a)** Estimate the instantaneous rate of change at $x = 1$.

b) Simplify the expression $\frac{f(x + h) - f(x)}{h}$.

c) Examine the expression in part b) and discuss what happens as h becomes very close to 0.

d) Use your result from part c) to come up with an expression for the instantaneous rate of change at the point x, and check your result from part a) using the expression.

14. Explain how instantaneous rates of change could be used to locate the

C local maxima and local minima for a polynomial function.

Extending

15. Consider the function $f(x) = e^x$ (e is called Euler's Number where $e \doteq 2.7183$).

a) Estimate the instantaneous rate of change at $x = 5$. Find $f(5)$.

b) Repeat part a) with three more x-values.

c) Generalize your findings.

16. Consider the function $f(x) = x^3 - 4x$.

a) Estimate the slope of the tangent line at $x = 1$.

b) Using the slope and the point of tangency, find the equation of the tangent line.

c) The tangent line intersects the original graph one more time. Where? Graph both the original function and the tangent line to illustrate this.

17. Determine, to two decimal places, where the slope of a tangent line and the slope of the secant line that passes through $A(2, -4)$ and $B(3, 0)$ are equal on the graph of $f(x) = x^3 - 3x^2$.

4 Chapter Review

FREQUENTLY ASKED Questions

Study | *Aid*

- See Lesson 4.3, Examples 1, 2, and 3.
- Try Chapter Review Questions 10 to 13.

Q: How do you solve a polynomial inequality?

A1: Sometimes you can use an algebraic strategy if the polynomial is factorable. Use inverse operations to make one side of the inequality equal to zero, factor the polynomial to determine its zeros, then test values to the left, between, and to the right of the zeros to determine which intervals will satisfy the inequality. This can be done using a number line or a factor table.

For example, to solve $3x^3 - 4x^2 - 3x - 10 > 2x^3 - 6x^2 + 6x + 8$

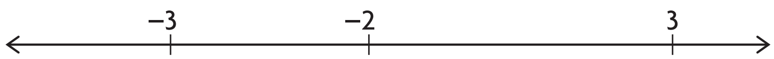
$$\begin{aligned} x^3 + 2x^2 - 9x - 18 &> 0 \\ x^2(x + 2) - 9(x + 2) &> 0 \\ (x^2 - 9)(x + 2) &> 0 \\ (x + 3)(x - 3)(x + 2) &> 0 \end{aligned}$$

The equation $(x + 3)(x - 3)(x + 2) = 0$ has solutions $x = -3$, $x = 3$, or $x = -2$. These numbers divide the domain of real numbers into the following intervals:

$x < -3$, $-3 < x < -2$, $-2 < x < 3$, and $x > 3$

Substitute values that lie in each interval into the original inequality, $3x^3 - 4x^2 - 3x - 10 > 2x^3 - 6x^2 + 6x + 8$.

Let $f(x) = 3x^3 - 4x^2 - 3x - 10$ and let $g(x) = 2x^3 - 6x^2 + 6x + 8$.



$x < -3$	$-3 < x < -2$	$-2 < x < 3$	$x > 3$
$f(-4) = -254$	$f(-2.5) = -74.375$	$f(1) = -14$	$f(4) = 106$
$g(-4) = -240$	$g(-2.5) = -75.75$	$g(1) = 10$	$g(4) = 64$
$f(x) < g(x)$	$f(x) > g(x)$	$f(x) < g(x)$	$f(x) > g(x)$

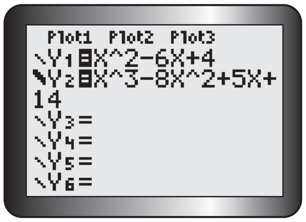
$3x^3 - 4x^2 - 3x - 10 > 2x^3 - 6x^2 + 6x + 8$ when $-3 < x < -2$ and $x > 3$.

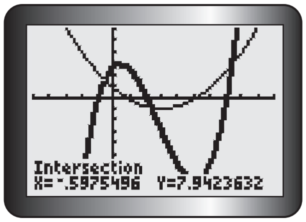
A2: You can always use a graphing strategy using one of the following methods.

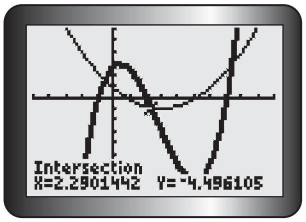
1. Treat each side of the inequality as two separate functions and graph them. Then determine their intersection points and identify the intervals for which one function is above or below the other, as required.

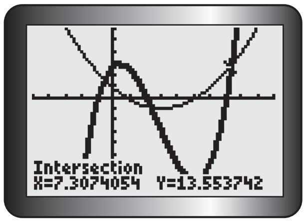
2. Create an equivalent inequality with zero on one side, and then identify the intervals created by the zeros of the graph of the corresponding function. Find where the graph lies above the x-axis (where $f(x) > 0$) or below (where $f(x) < 0$), as required.

For example, to solve $x^2 - 6x + 4 \geq x^3 - 8x^2 + 5x + 14$, use the graphing calculator to determine the intersection points for the functions.









The two functions intersect when $x \doteq -0.598$, 2.290, and 7.307.

Refer to the graph to see where Y1 is above Y2 on the intervals defined by these three points. For example, for points to the left of $x = -0.598$, Y1 is above Y2.

So, $x^2 - 6x + 4 \geq x^3 - 8x^2 + 5x + 14$ when $x \leq -0.598$ and when $2.290 \leq x \leq 7.307$.

Q: How do you calculate an average rate of change for a polynomial function?

A: The average rate of change is the slope of a secant that connects two points on the function. To calculate the average rate of change on the interval $x_1 \leq x \leq x_2$ for a function, $f(x)$, calculate the average rate of change, $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Study *Aid*

- See Lesson 4.4, Example 2.
- Try Chapter Review Questions 14, 17, and 18.

Q: How do you approximate the instantaneous rate of change for a polynomial function?

A1: You can calculate the average rate of change for a very small interval on either side of the point at which you wish to calculate the instantaneous rate of change using the difference quotient $\dfrac{f(a + h) - f(a)}{h}$, where h is a very small value.

A2: You can graph the function, either by hand or by using a graphing calculator, and draw a tangent line at the required point and estimate its slope.

Study *Aid*

- See Lesson 4.4, Examples 1 and 3.
- Try Chapter Review Questions 15, 16, and 18.

PRACTICE Questions

Lesson 4.1

1. Solve each of the following equations by factoring.

a) $x^4 - 16x^2 + 75 = 2x^2 - 6$
b) $2x^2 + 4x - 1 = x + 1$
c) $4x^3 - x^2 - 2x + 2 = 3x^3 - 2(x^2 - 1)$
d) $-2x^2 + x - 6 = -x^3 + 2x - 8$

2. Solve the equation algebraically, and check your solution graphically:
$18x^4 - 53x^3 + 52x^2 - 14x - 8 =$
$3x^4 - x^3 + 2x - 8$

3. a) Write the equation of a polynomial $f(x)$ that has a degree of 4, zeros at $x = 1, 2, -2,$ and -1, and has a y-intercept of 4.
b) Determine the values of x where $f(x) = 48$.

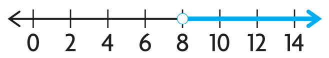
4. An open-topped box is made from a rectangular piece of cardboard, with dimensions of 24 cm by 30 cm, by cutting congruent squares from each corner and folding up the sides. Determine the dimensions of the squares to be cut to create a box with a volume of 1040 cm^3.

5. Between 1985 through 1995, the number of home computers, in thousands, sold in Canada is estimated by
$C(t) = 0.92(t^3 + 8t^2 + 40t + 400)$,
where t is in years and $t = 0$ for 1985.

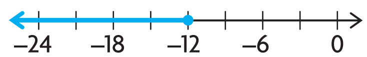
a) Explain why you can use this model to predict the number of home computers sold in 1993, but not to predict sales in 2005.
b) Explain how to find when the number of home computer sales in Canada reached 1.5 million, using this model.
c) In what year did home computer sales reach 1.5 million?

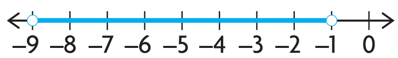
Lesson 4.2

6. For each number line given, write an inequality with both constant and linear terms on each side that has the corresponding solution.

a) 

b) 

c) 

d) 

7. Solve the following inequalities algebraically. State your answers using interval notation.

a) $2(4x - 7) > 4(x + 9)$
b) $\dfrac{x - 4}{5} \geq \dfrac{2x + 3}{2}$
c) $-x + 2 > x - 2$
d) $5x - 7 \leq 2x + 2$

8. Solve the following inequalities. State your answers using set notation.

a) $-3 < 2x + 1 < 9$
b) $8 \leq -x + 8 \leq 9$
c) $6 + 2x \geq 0 \geq -10 + 2x$
d) $x + 1 < 2x + 7 < x + 5$

9. A phone company offers two options. The first plan is an unlimited calling plan for \$34.95 a month. The second plan is a \$20.95 monthly fee plus \$0.04 a minute for call time.

a) When is the unlimited plan a better deal?
b) Graph the situation to confirm your answer from part a).

Lesson 4.3

10. Select a strategy and determine the interval(s) for which each inequality is true.

a) $(x + 1)(x - 2)(x + 3)^2 < 0$

b) $\frac{(x - 4)(2x + 3)}{5} \geq \frac{2x + 3}{5}$

c) $-2(x - 1)(2x + 5)(x - 7) > 0$

d) $x^3 + x^2 - 21x + 21 \leq 3x^2 - 2x + 1$

11. Determine algebraically where the intervals of the function are positive and negative.
$f(x) = 2x^4 - 2x^3 - 32x^2 - 40x$

12. Solve the following inequality using graphing technology:
$x^3 - 2x^2 + x - 3 \geq 2x^3 + x^2 - x + 1$

13. In Canada, hundreds of thousands of cubic metres of wood are harvested each year. The function
$f(x) = 1135x^4 - 8197x^3 + 15\,868x^2 - 2157x + 176\,608,\ 0 \leq x \leq 4$, models the volume harvested, in cubic metres, from 1993 to 1997. Estimate the intervals (in years and months) when less than 185 000 m^3 were harvested.

Lesson 4.4

14. For each of the following functions, determine the average rate of change in $f(x)$ from $x = 2$ to $x = 7$, and estimate the instantaneous rate of change at $x = 5$.

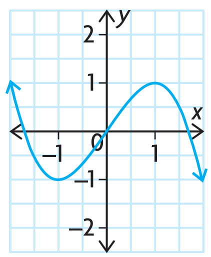
a) $f(x) = x^2 - 2x + 3$

b) $h(x) = (x - 3)(2x + 1)$

c) $g(x) = 2x^3 - 5x$

d) $v(x) = -x^4 + 2x^2 - 5x + 1$

15. Given the following graph, determine the intervals of x where the instantaneous rate of change is positive, negative, and zero.



16. The height in metres of a projectile is modelled by the function $h(t) = -5t^2 + 25$, where t is the time in seconds.

a) Find the point when the object hits the ground.

b) Find the average rate of change from the point when the projectile is launched $(t = 0)$ to the point in which it hits the ground.

c) Estimate the object's speed at the point of impact.

17. Consider the function $f(x) = 2x^3 + 3x - 1$.

a) Find the average rate of change from $x = 3$ to $x = 3.0001$.

b) Find the average rate of change from $x = 2.9999$ to $x = 3$.

c) Why are your answers so similar? Estimate the instantaneous rate of change at $x = 3$.

18. The incidence of lung cancer in Canadians per 100 000 people is shown below.

Year	Males	Females
1975	73.1	14.7
1980	83.2	21.7
1985	93.2	30.9
1990	92.7	36.5
1995	84.7	40.8
2000	78.6	46.4

Source: Cancer Bureau, Health Canada

a) Use regression to determine a cubic function to represent the curve of best fit for both the male and female data.

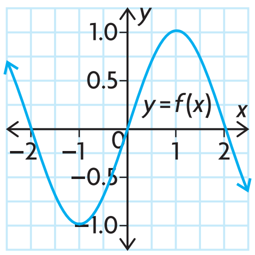
b) According to your models, when will more females have lung cancer than males?

c) Was the incidence of lung cancer changing at a faster rate in the male or female population during the period fom 1975 to 2000? Justify your answer.

d) Was the incidence of lung cancer changing at a faster rate in the male or female population in 1998? Justify your answer.

4 Chapter Self-Test

1. Solve for x in $3x^3 - 3x^2 - 7x + 5 = x^3 - 2x^2 - 1$.



2. Consider the graph shown of the function $y = f(x)$.

a) Determine where $f(x)$ is positive, negative, and zero.

b) Determine where the instantaneous rate of change in $f(x)$ is positive, negative, and zero. Find the average rate of change in $f(x)$ from $x = 1$ to $x = 2$.

3. A pizza company is advertising a special card. The card costs \$50, but allows the owner to purchase pizzas for \$5 each for one full year. Pizzas are normally \$12 each.

a) Write expressions that represent the cost of n pizzas with and without the card.

b) How many pizzas would you have to purchase in a year to make the card worthwhile?

4. Solve the following inequalities.

a) $4x - 5 < -2(x + 1)$

b) $-4 \leq -(3x + 1) \leq 5$

c) $(x + 1)(x - 5)(x + 2) > 0$

d) $(2x - 4)^2(x + 3) \geq 0$

5. The height in metres of a projectile launched from the top of a building is given by $h(t) = -5t^2 + 20t + 15$, where t is the time in seconds since it was launched.

a) How high was the projectile at the moment of launch?

b) At what time does the projectile hit the ground?

c) What is the average rate of change in height from the time the object was launched until the time it hit the ground?

6. Consider the following function: $f(x) = x^3 + x^2 + 1$.

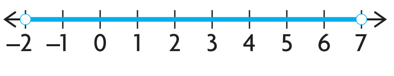
a) Estimate the slope of the tangent line at $x = 1$.

b) What are the coordinates of the point of tangency?

c) Determine the equation of the tangent line.

7. Explain why the polynomial $f(x) = 4x^{2008} + 2008x^4 + 4$ has no zeros.

8. The following number line shows the solution to a double inequality.



a) Write the solution using set notation.

b) Create a double inequality that has both a linear and a constant term for which this is the solution set.

9. A box that holds an expensive pen has square ends, and its length is 13 cm longer than its width. The volume of the box is 60 cm^3. Determine the dimensions of the box.

4 Chapter Task

Flight of an Osprey

An observer in a fishing boat watched as an osprey dove under water and re-emerged with a fish in its talons. The following table shows the bird's estimated height above the water as given by that observer.

Time (s)	Height (m)
0	7
2	10
4	5
6	0
7	0
8	3

? Is the osprey travelling at its greatest speed when it hits the water?

A. Plot the given data on graph paper. What type of function best models these data?

B. Without using graphing technology, determine an equation to model the data and state a suitable domain.

C. Describe the osprey's flight, making reference to your graph and equation. Include information about the time, its height, direction of flight, and relative rate of ascent and descent (faster/slower).

D. According to your model, how long was the osprey under water? Does this seem reasonable? Explain.

E. According to your model, when was the osprey more than 6 m above the water?

F. Use your model to estimate the rate at which the osprey's height is changing at the time it hits the water.

G. Using tangent lines on your graph, do you think the rate you calculated in part F is the greatest at this point? Explain.

H. Check your result for part F using graphing technology by creating a scatter plot, determining the equation of the curve of best fit, and using it to find the slope of the appropriate tangent line.

I. Use the graphing calculator and the graph you created in part H to help you determine when the osprey's rate of change in height was greatest between 0 s and 8 s.

Task *Checklist*

- ✔ Did you explain your thinking clearly?
- ✔ Did you justify your answers mathematically?
- ✔ Did you show all work and calculations?
- ✔ Did you check your calculations?
- ✔ Did you label your work properly?