Learning Theory Sample Complexity Examples

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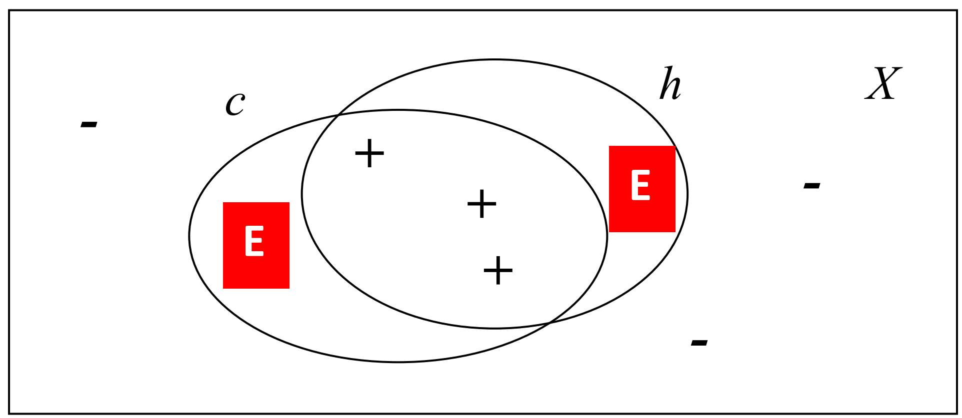
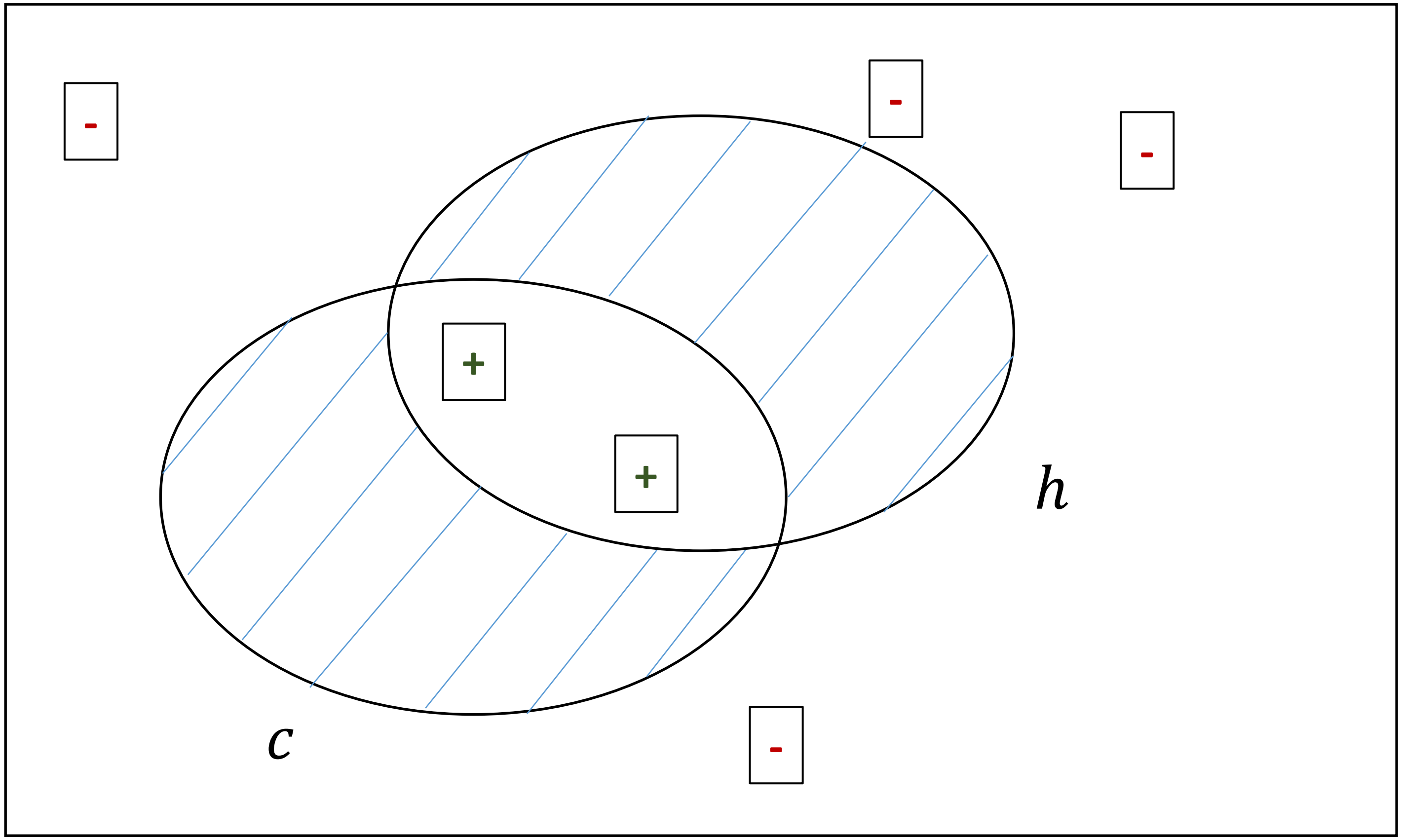


Outline

- The true error
- Consistent hypotheses, consistent learners
- Complexity of learning
- Sample complexity
- Bound on the sample complexity for finite H
- Examples:
  - Finite spaces of Boolean vectors
  - Circles, rectangles
- VC dimension

General setting

- Instances come from $\Omega = (X, Y, \pi)$
- The learning algorithm L takes training data $D \in \Omega^m$
- It works with some set of hypotheses, H
- It returns a hypothesis (or a model) $L(D) = h \in H$



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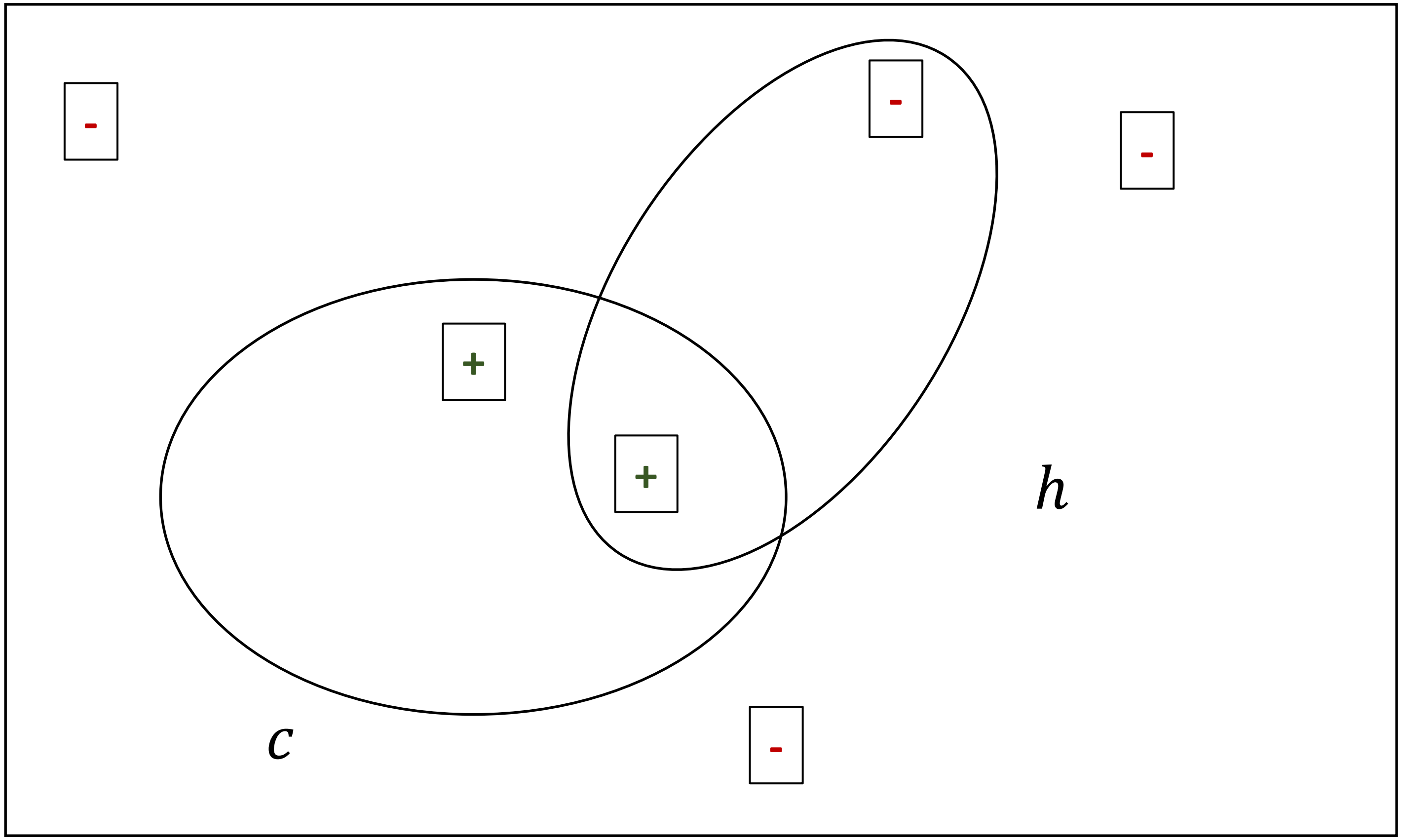
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Example from last class (disjnxns and Bool vectors)

$$m \geq \frac{1}{\varepsilon}\ln\frac{|H|}{\delta} = \frac{1}{\varepsilon}\left(\ln|H| + \ln\frac{1}{\delta}\right)$$

- $n = 20$ attributes in the same setting
- We get $|H| = 3^{20} \sim 3.5 * 10^9$
- In this case, we can obtain 95% certainty that our hypothesis will have error < 10% when using

$$m > \frac{1}{0.1}(\ln 3.5*10^9 + \ln\frac{1}{0.05}) = 10(22+3) = 250 \text{ instances}$$

- Note that here $|X| = 2^{20} \approx 10^6$

When |H| increases exponentially with the number of features then sample complexity increases linearly. The required fraction of the full population decreases.

PAC Learnability

- Consider a class **C** of possible target concepts defined over a space of instances **X**, and a learning algorithm **L** using hypothesis space **H**.

- Definition

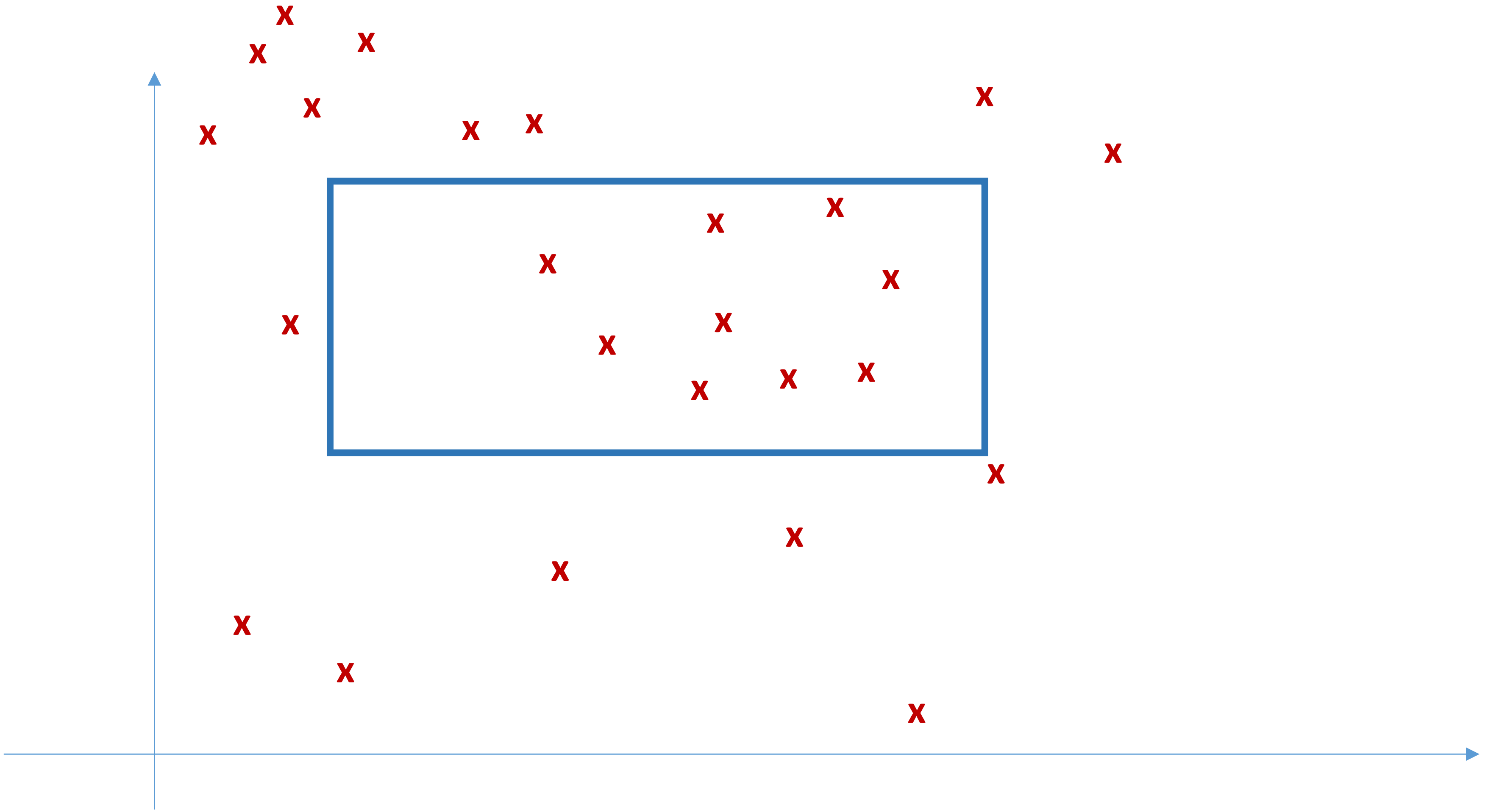
C is PAC-learnable by L using H

if for all $0 < \varepsilon < ½$, $0 < \delta < ½$, and for all $c \in$ **C** and distributions π over **X**, the following holds:

with data drawn independently according to π, **L** will output, with probability at least $(1-\delta)$, a hypothesis $h \in$ **H** such that $\text{error}_\pi(h) \leq \varepsilon$,

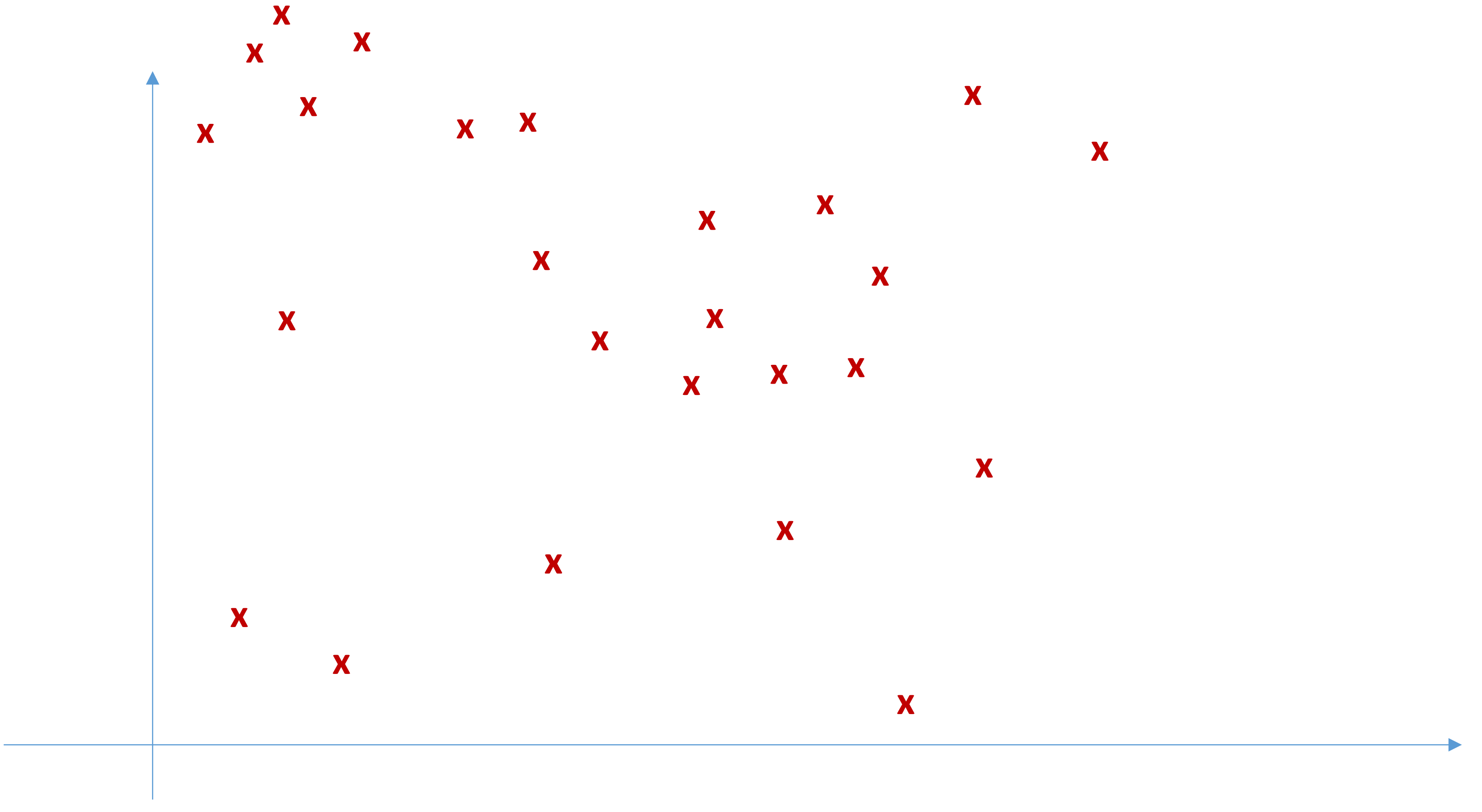
L operates in time (and sample) complexity that is polynomial in $1/\varepsilon$, $1/\delta$ (and in other possible parameters).

Learning axes aligned rectangles



$$X = \mathbb{R}^2$$

Learning axes aligned rectangles



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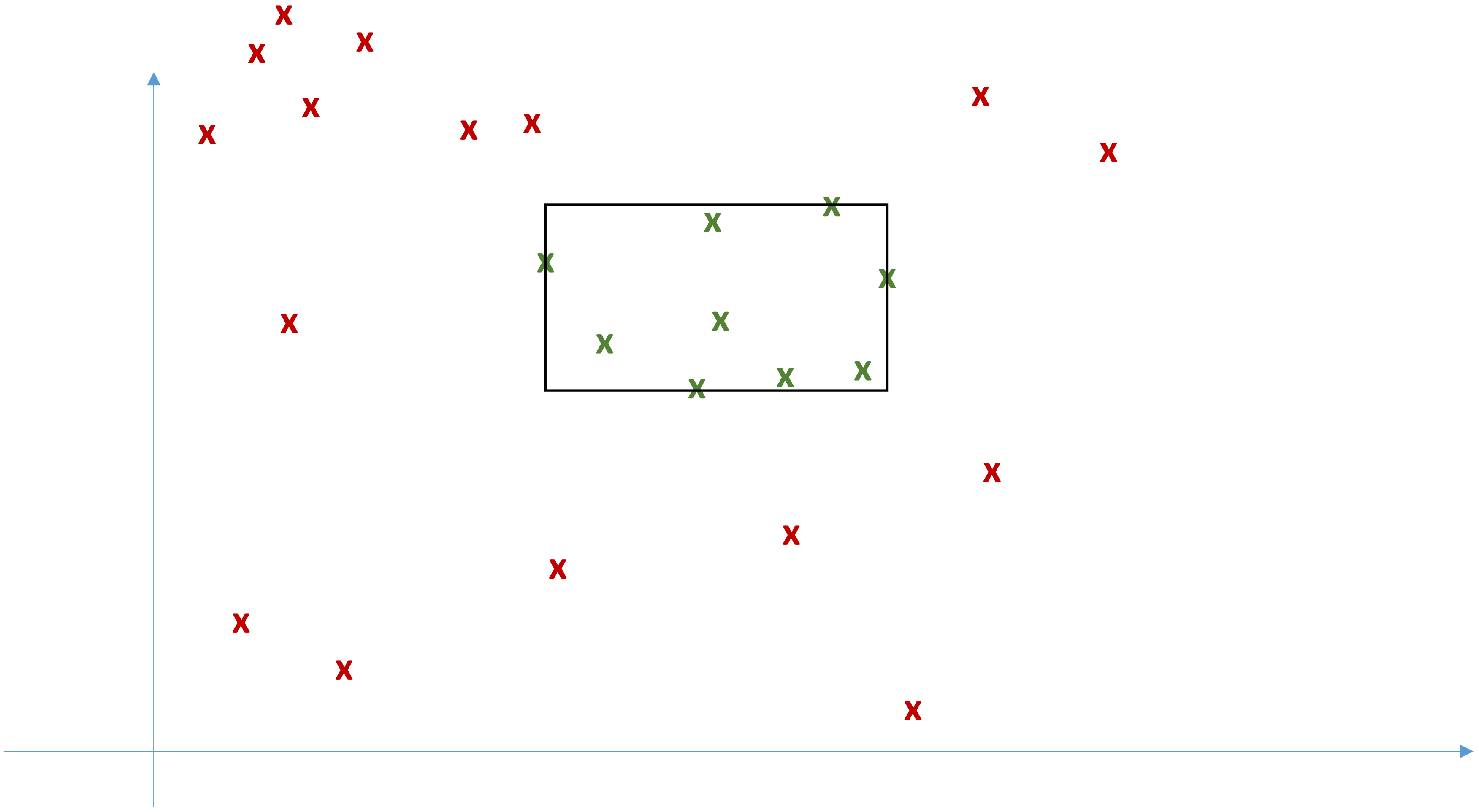
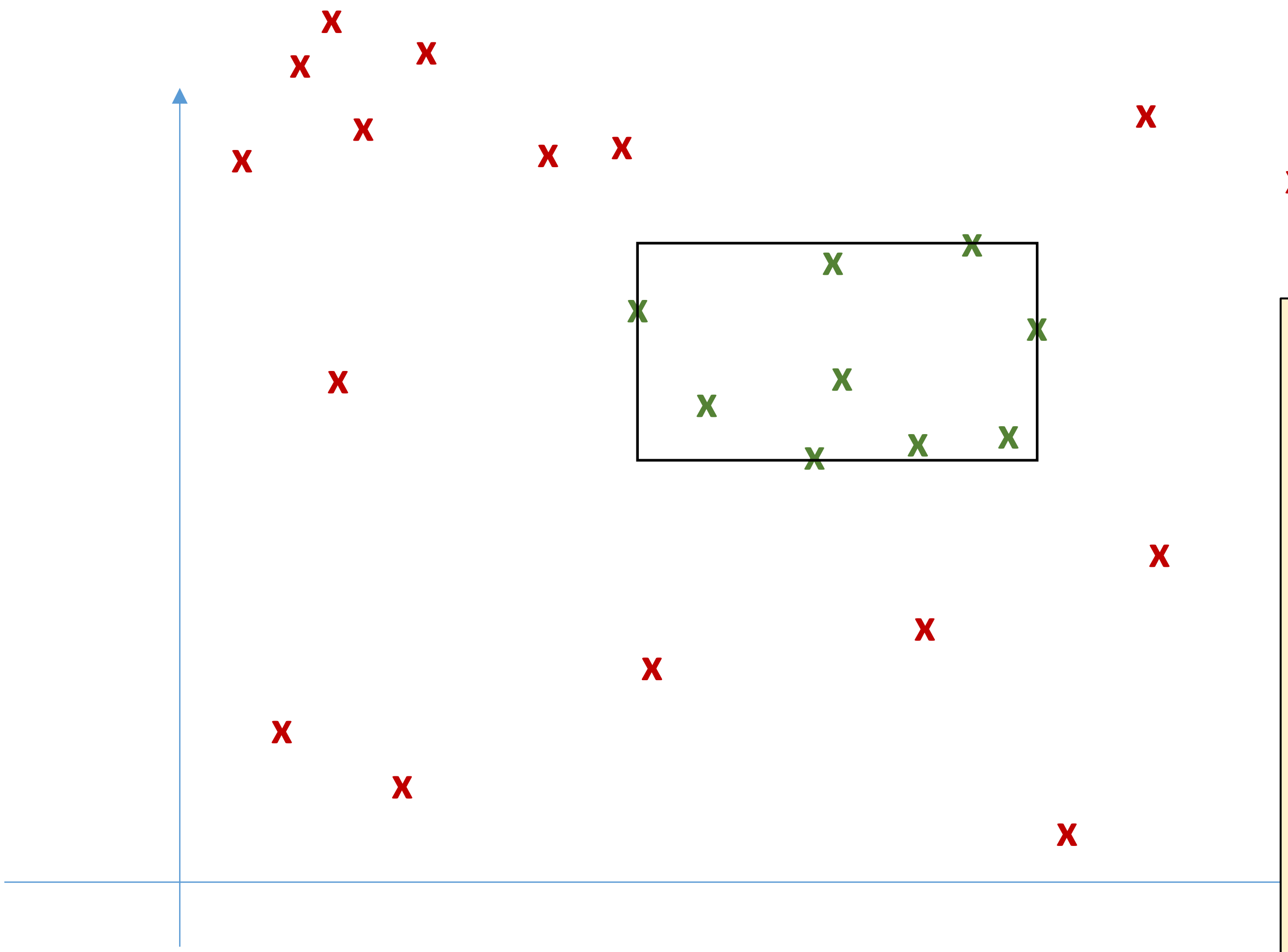
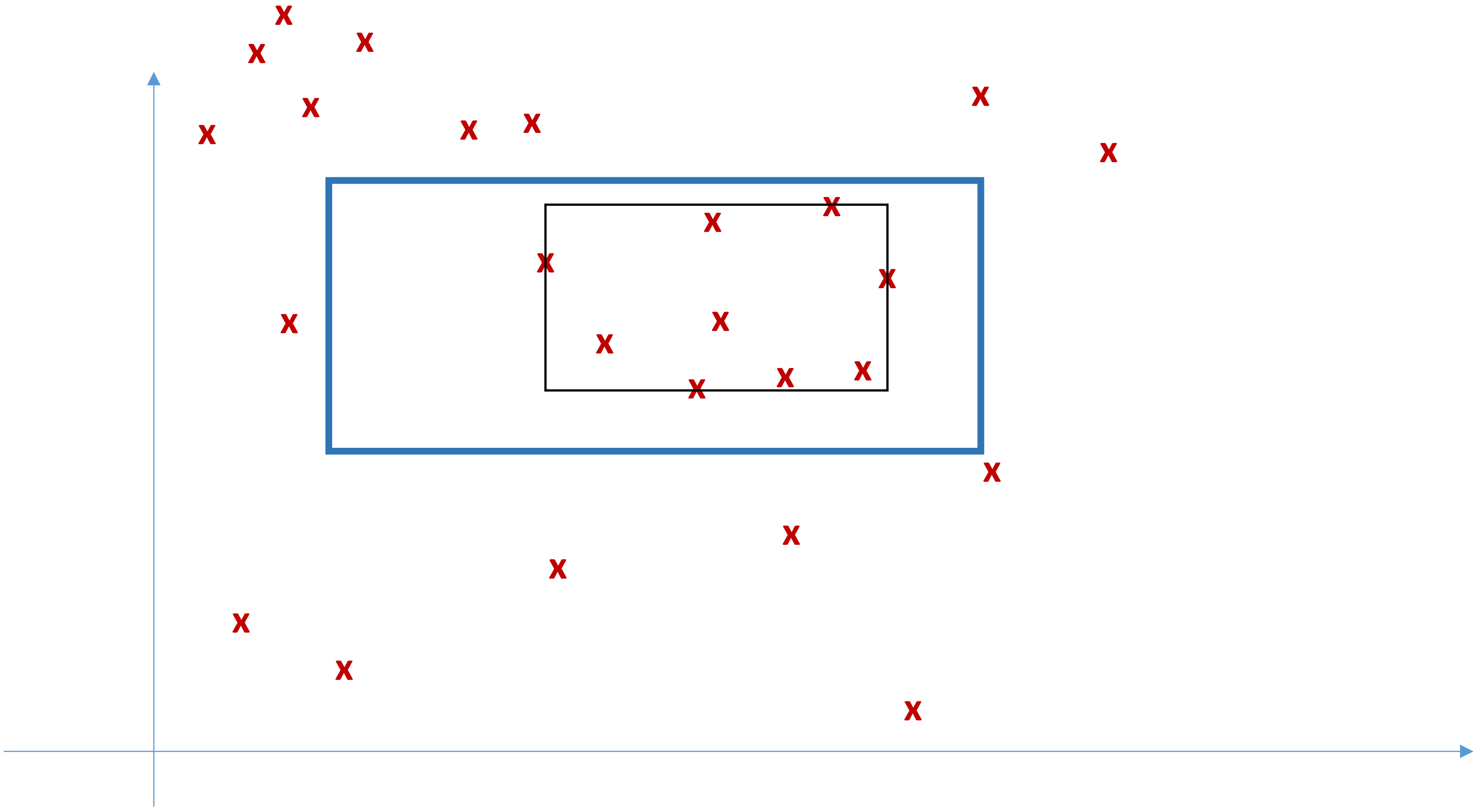


Learning axes aligned rectangles

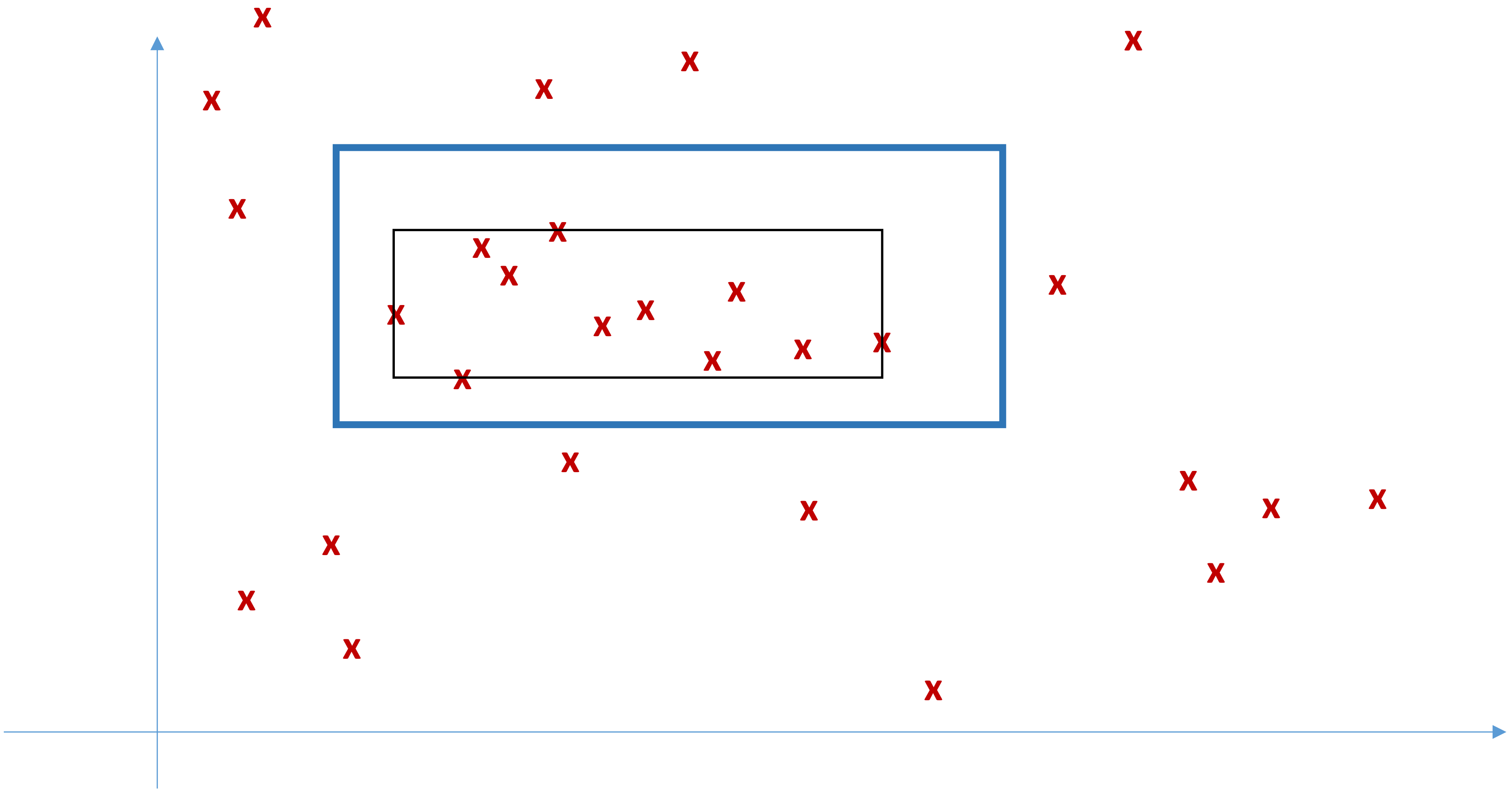


<u>Consistent learner</u>:
Find points with
max and min x,
max and min y.
Draw edges accordingly.
Linear in m.
Returns $h = L(D)$ such that
$\forall x \in X$
$h(x) = 1 \Longrightarrow c(x) = 1$
but not necessarily the opposite.
How about on D?

Learning axes aligned rectangles



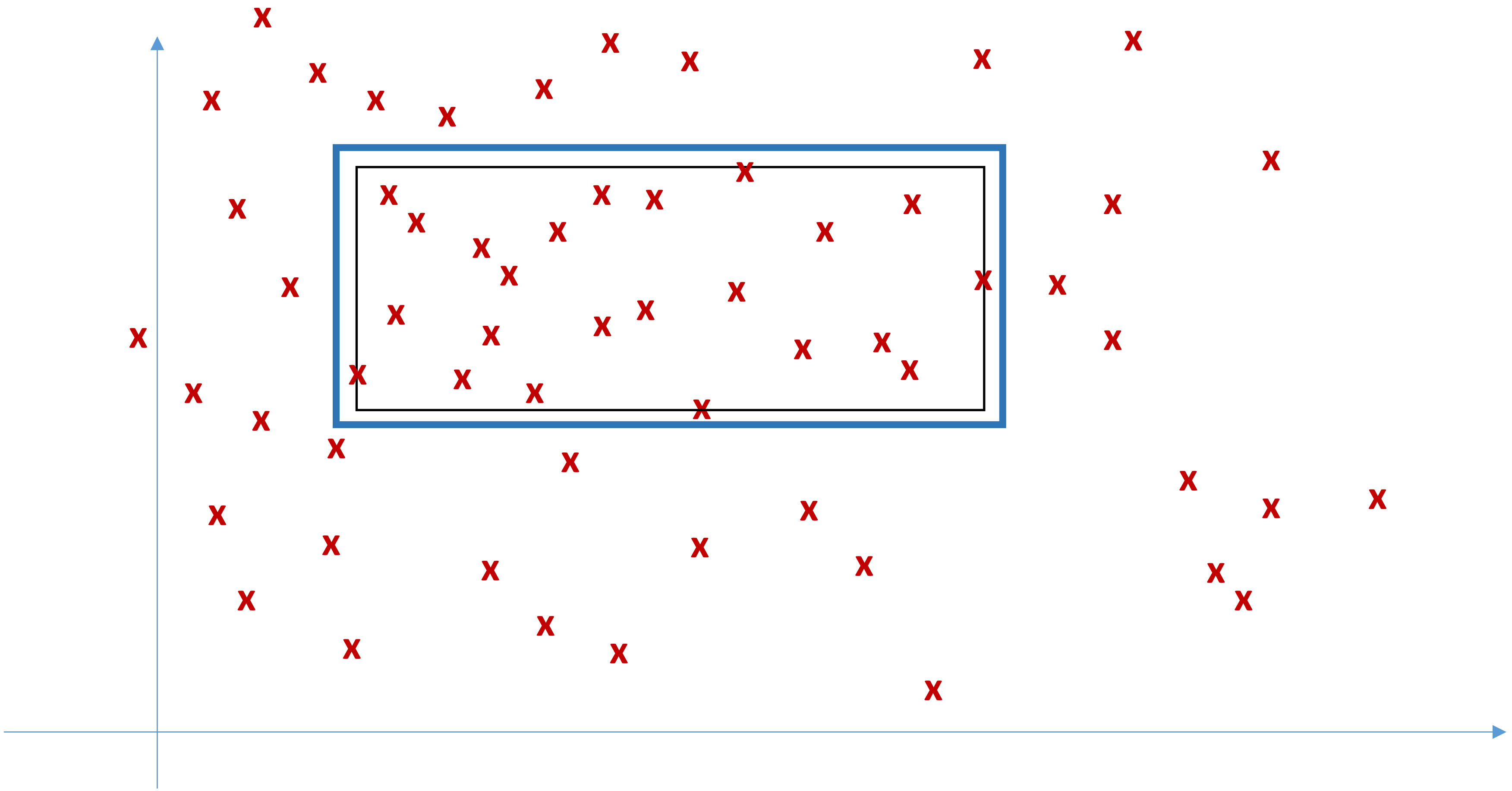
Learning axes aligned rectangles



Different training dataset lead to a different output hypotheses.

$$h = L(D), D \in (X, P)^m$$

Learning axes aligned rectangles



Different training dataset lead to a different output hypotheses.

$$h = L(D), D \in (X, P)^m$$

A bound on sample complexity

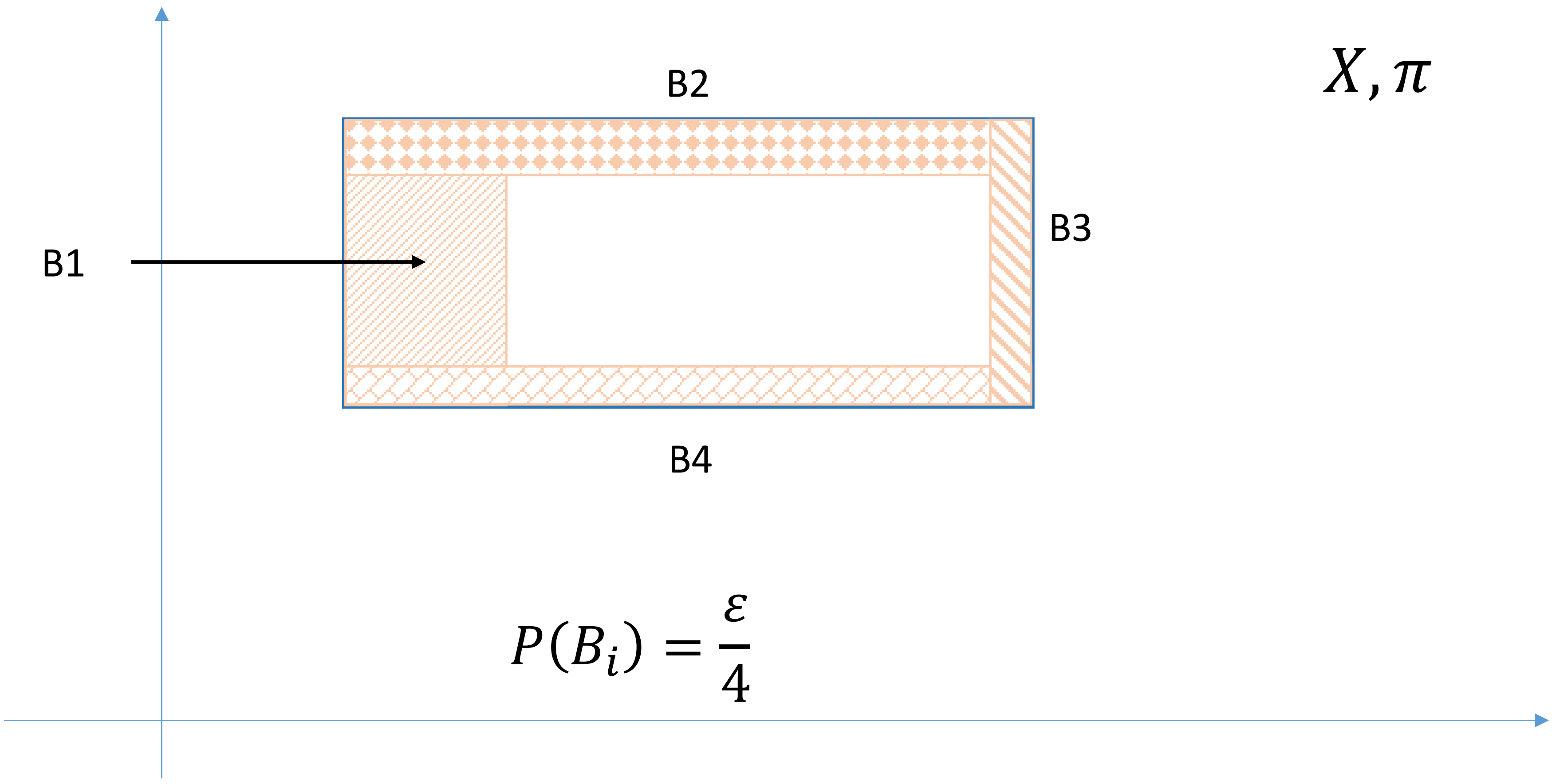
- For every $c \in C$
- We will now characterize (bound) the collection of all training datasets , D, that can conceivably lead to $h = L(D)$ with $\mathrm{Err}(h, c) > \varepsilon$
- We will show that this collection (a subset of X^m) is contained in a union of a small number of sets (B_i , subsets of X^m) characterizable by regions that their elements (points in X^m) do not visit.

A bound on the sufficient sample complexity, cont

- We will then estimate the probability (π^m) of each such set of datasets as well as that of their union.
- From here we will infer an upper bound on sufficient sample complexity, as a function of ε and δ, of the form

$$m(\varepsilon, \delta) \in O\big(f(\varepsilon), g(\delta)\big)$$

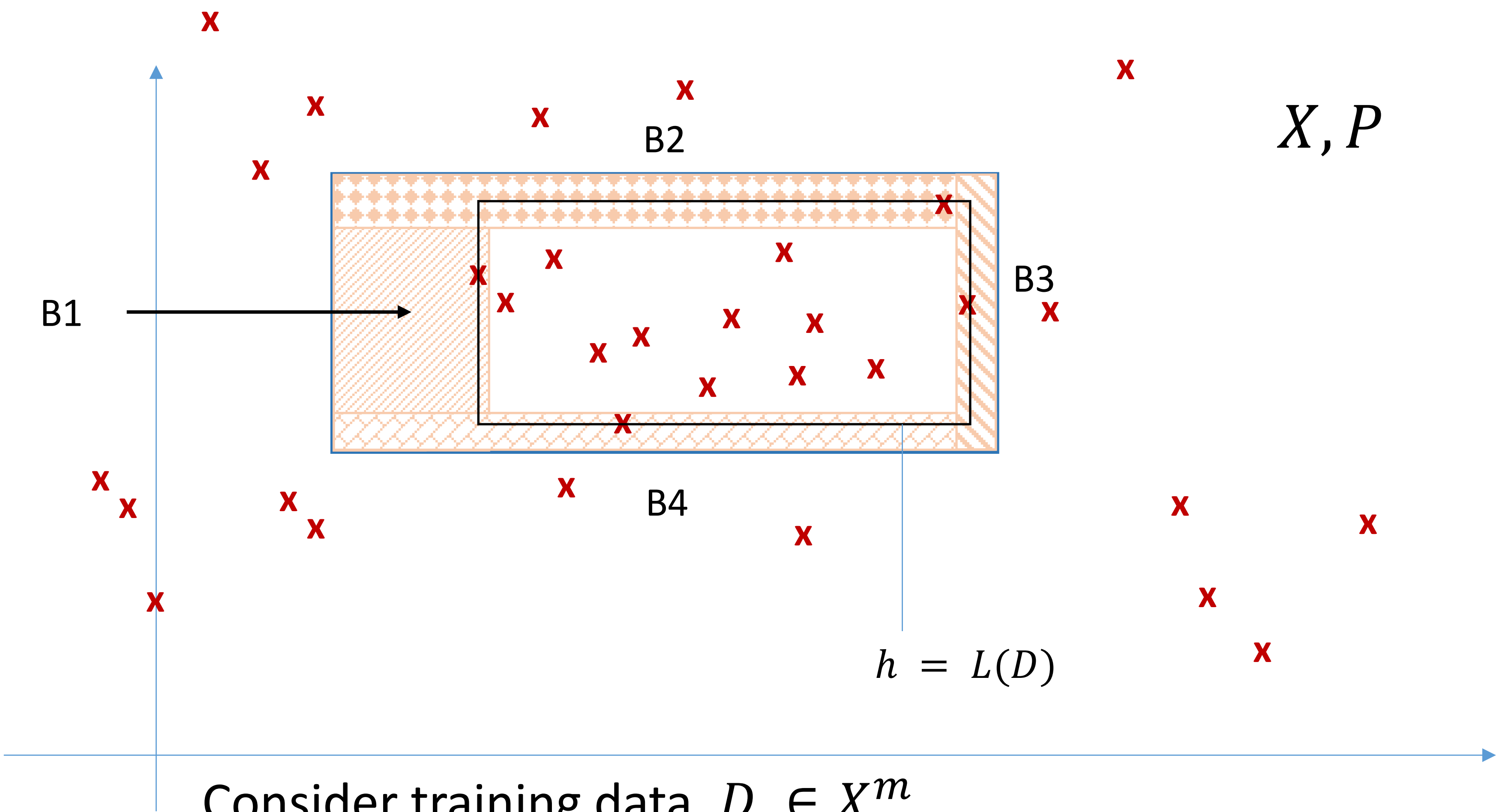
Learning axes aligned rectangles



X, π

$$P(B_i) = \frac{\varepsilon}{4}$$

Learning axes aligned rectangles

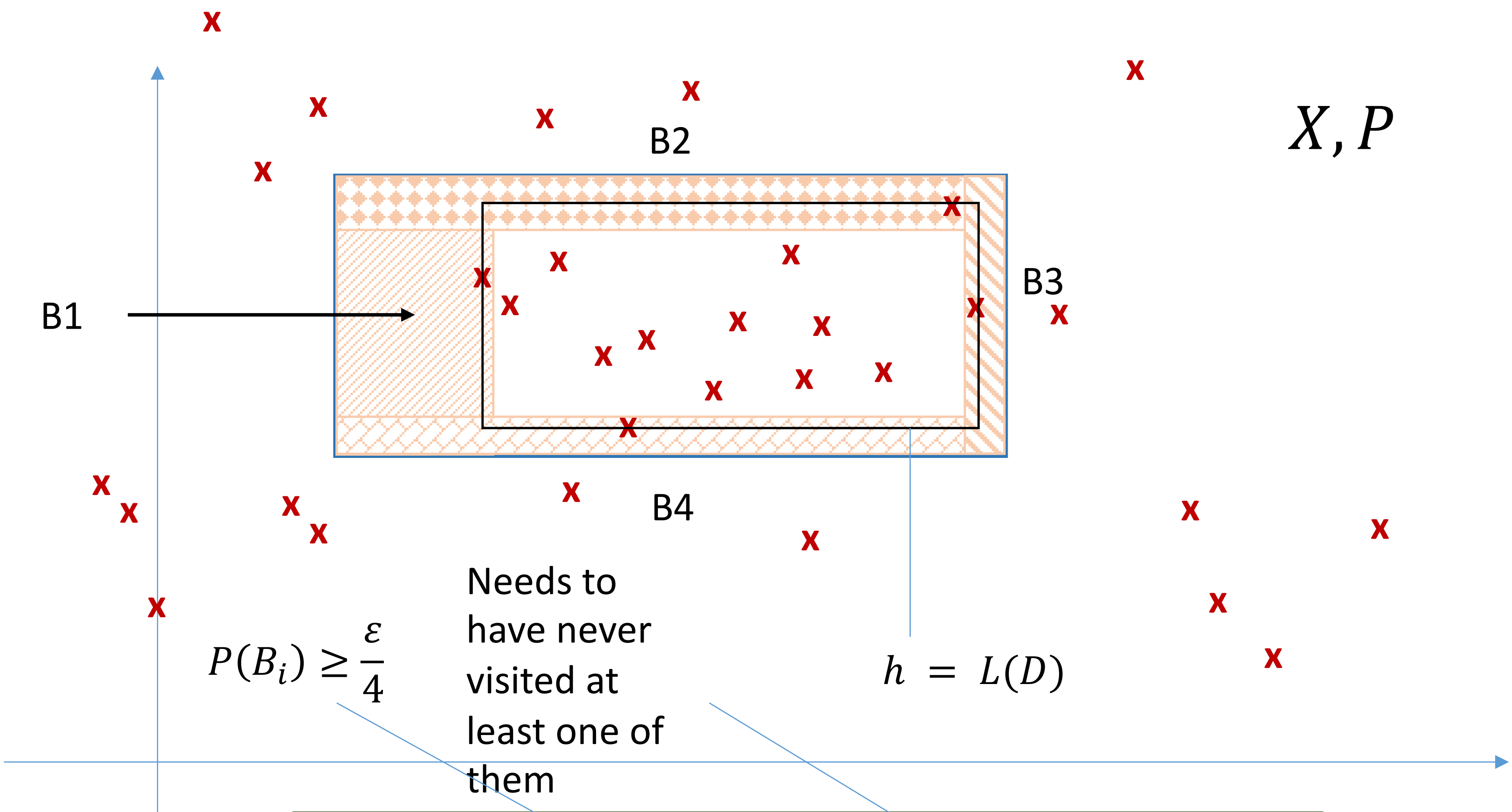


Consider training data, $D \in X^m$.

Assume that D visits <u>each one</u> of the 4 sets B_i defined above.

What can we say about $\mathrm{Err}(h, c)$? $(P(B_i) \leq \frac{\varepsilon}{4})$

Not visiting one of the regions B_i



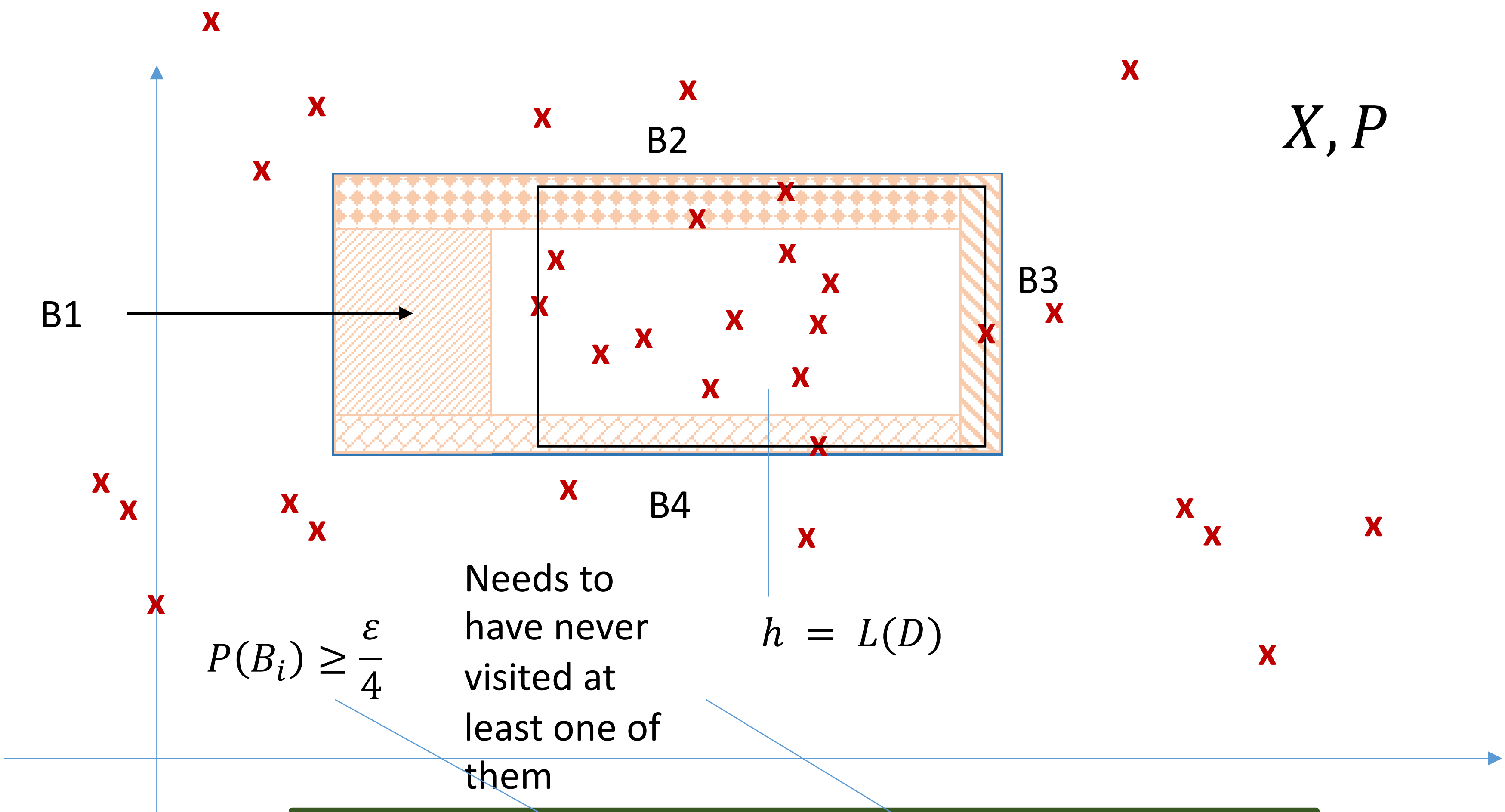
$P(B_i) \geq \frac{\varepsilon}{4}$

Needs to have never visited at least one of them

$h = L(D)$

$$P(\{D \in X^m : \mathrm{Err}(L(D), c) > \varepsilon\}) \leq$$
$$\sum_{i=1}^{4} \left(P(X - B_i)\right)^m \leq 4\left(1 - \frac{\varepsilon}{4}\right)^m \leq 4\exp\left(-\frac{m\varepsilon}{4}\right)$$

Learning axes aligned rectangles



$$P(\{D \in X^m : \mathrm{Err}(L(D), c) > \varepsilon\}) \leq \sum_{i=1}^{4} \left(P(X - B_i)\right)^m \leq 4\left(1 - \frac{\varepsilon}{4}\right)^m \leq 4\exp\left(-\frac{m\varepsilon}{4}\right)$$

Sufficient sample size

$$m(\varepsilon, \delta) = \frac{4}{\varepsilon} \cdot \ln\frac{4}{\delta}$$

Summary

- Sample complexity
- Consistent learners for finite hypotheses spaces
- Directly calculating bounds on the sample complexity of consistent learners
- Concepts in $\mathbb{R}^n$ - use the geometry!
- VC dimension - a more general formula
- In the recitation
  - More sample complexity examples
  - VC examples