Learning Theory Sample Complexity PAC Learning

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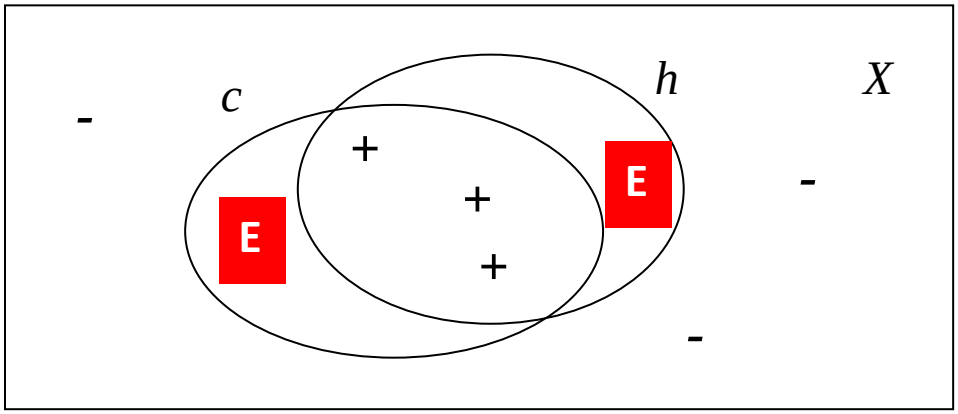
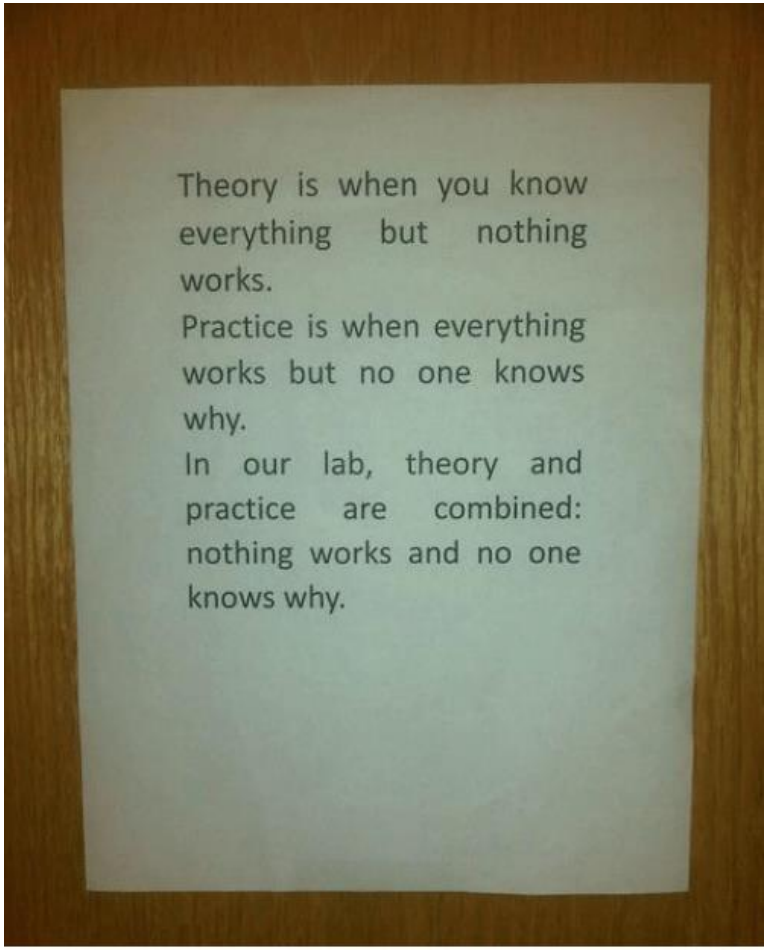


In theory, theory and practice are the same. In practice, they are not.

Albert Einstein

meetville.com



Outline

- Generalization and evaluating the true error (again …)
- Complexity of learning
- Sample complexity
- Examples:
  - Finite spaces and Boolean expressions
  - Circles, rectangles
- VC dimension

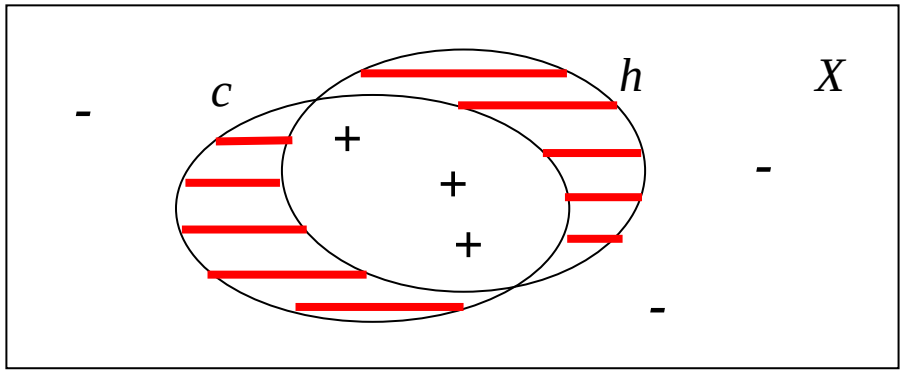
General setting

- Instances come from $\Omega = (X, Y, \pi)$
- Consider a concept $c \subseteq X$, which we are trying to learn. That is – provide a model (hypothesis) h that will be positive on c and negative otherwise
- The learning algorithm L takes training data $D \in \Omega^m$
- It works with some set of hypotheses, H
- It returns a hypothesis $L(D) = h \in H$

The True Error of h

Assume that we have a probability distribution over X (called π in the previous slide)
Then we can define:

$$TrueErr(h) = \ error_{\pi}(h) = \pi\left(x \in X \ : \ c(X) \neq h(X)\right)$$



Statistical Estimation – a detour into practice

- In practice, we use a test set to estimate the true error of a candidate hypothesis. It is a representative sample of data we have not used for learning.
- If the test set is all the rest of X then we know the true error!
  (of course this is unrealistic and we use sampling)

Statistical Estimation of the Classification Error

- Using a test set of size n, assume that we counted r errors.
- We estimate the generalization error by $\hat{p} = \frac{r}{n}$.

- From statistical sampling theory it follows that a 95% confidence interval for the generalization error is

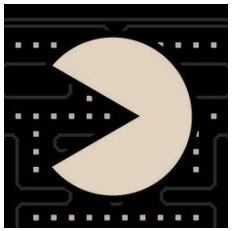
$$(\hat{p} - 2se, \hat{p} + 2se)$$

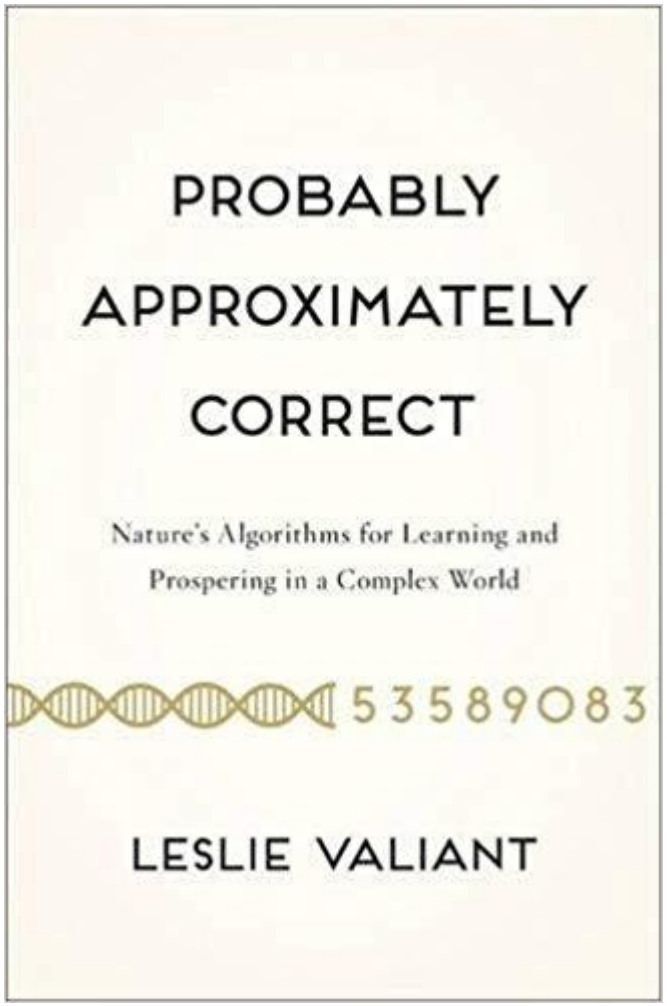
where

$$se = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Back to the sample complexity theory

PAC Learning





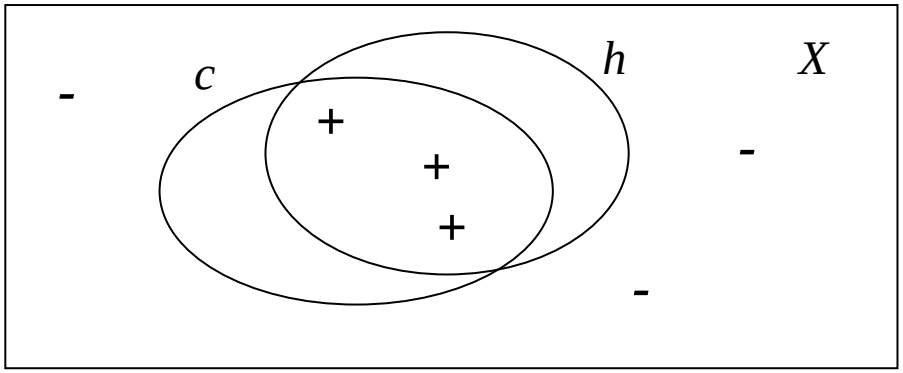
Leslie Valiant
Harvard Univ
Born 1949
Turing Award Laureate in 2010
American computer scientist

Probably Approximately Correct Framework

1. **Probability**
   (guarantees given with $1 - \delta < 1$ certainty)
2. **Approximation**
   (a desired bound, $\varepsilon > 0$, on the true error will be specified)
3. We will study the use of **resources**
   - Size of training set (sample complexity)
   - Time/space of learning
     (learnability in polynomial time per training instance)

PAC framework, cont

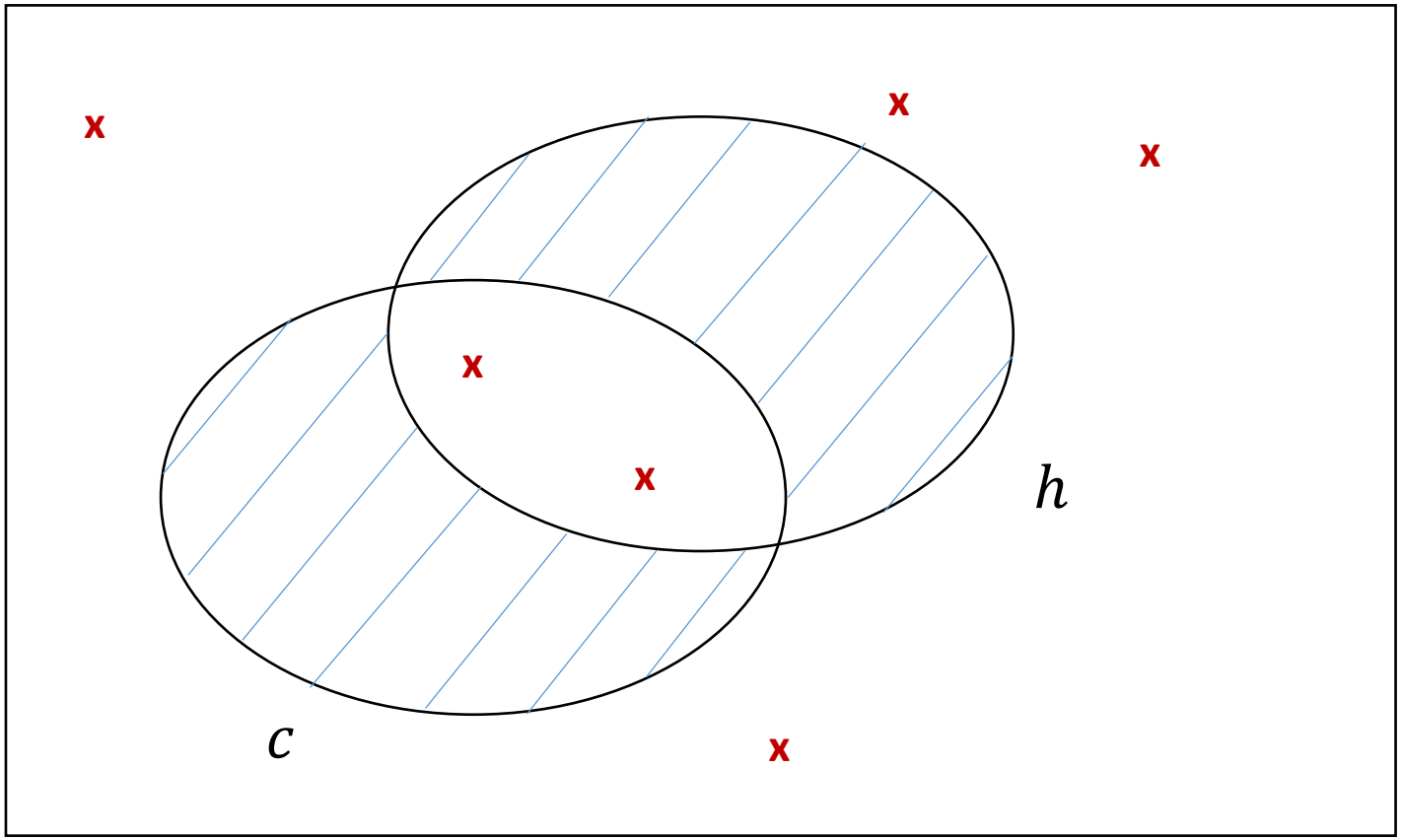
- We are trying to learn a concept c (in this framework, a subset of X)
- We have a training set in the instance space X, drawn at random according to Ω^m and labeled without errors
- Working with a hypothesis space H we seek a hypothesis h that is learned based on our data
- We want to guarantee that with probability $1-\delta$ it will not have a true error of more than ε



Consistent hypotheses

A hypothesis h is *D-consistent* with respect to a concept c and training data D if

$$\forall d \in D \;\; h(d) = c(d)$$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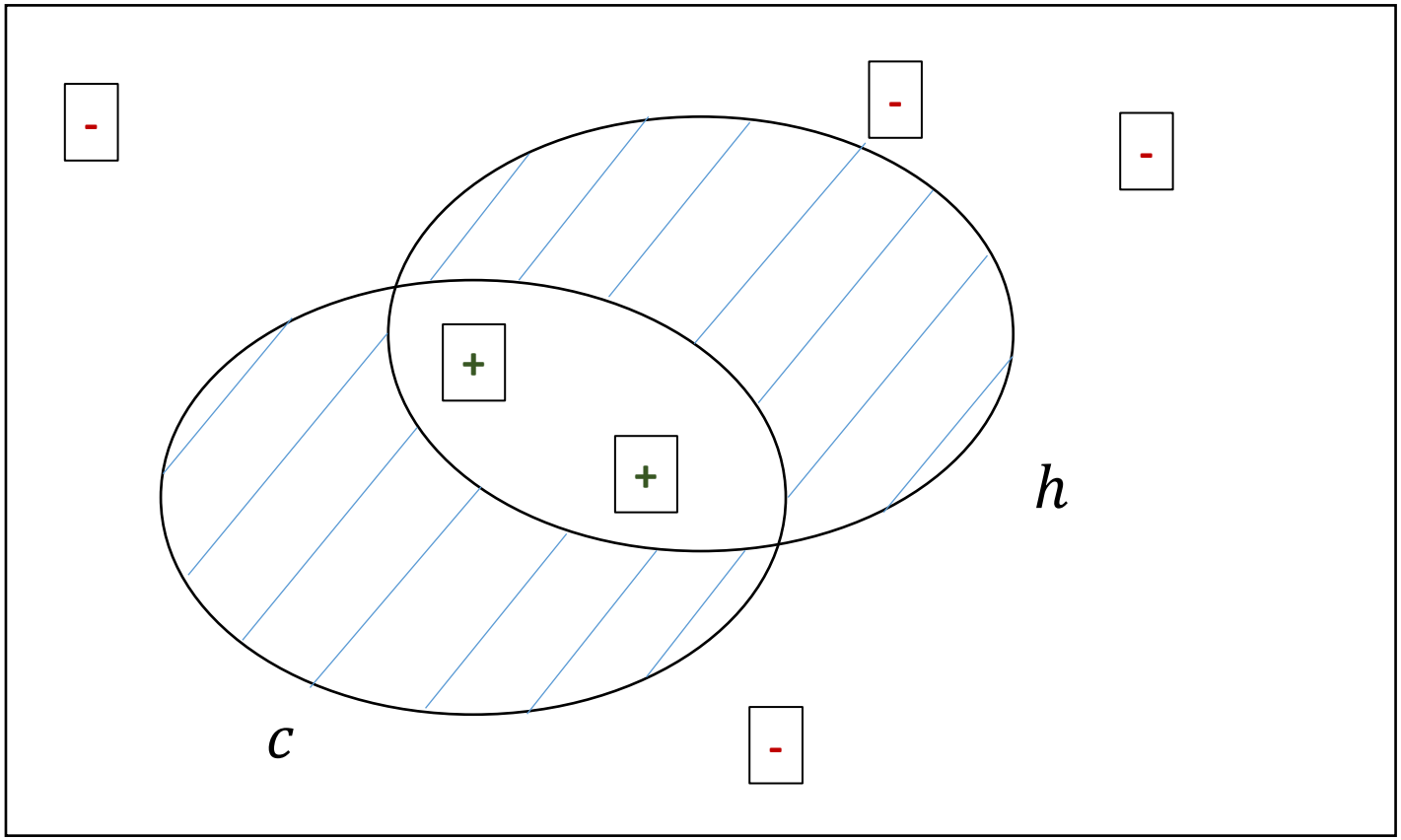
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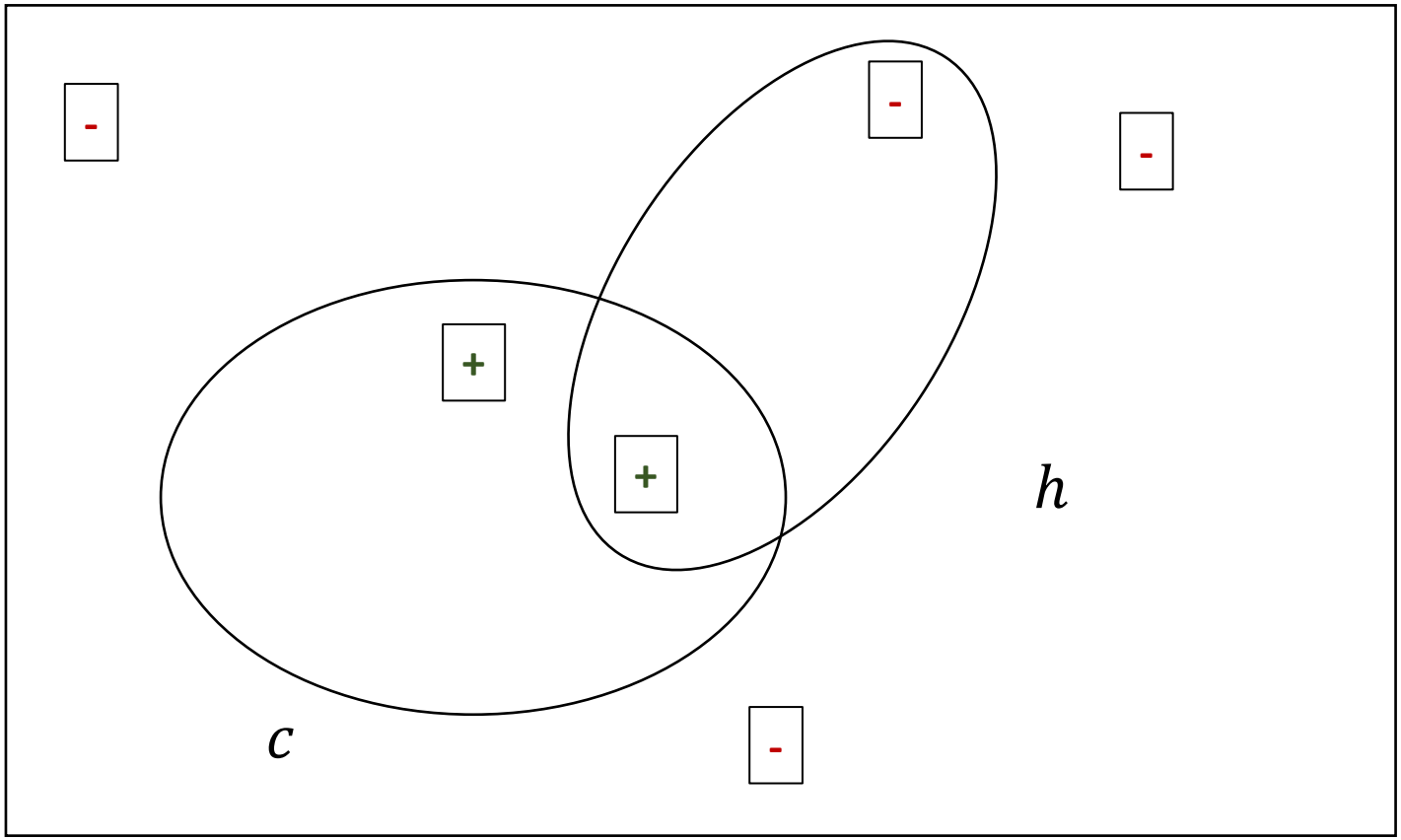
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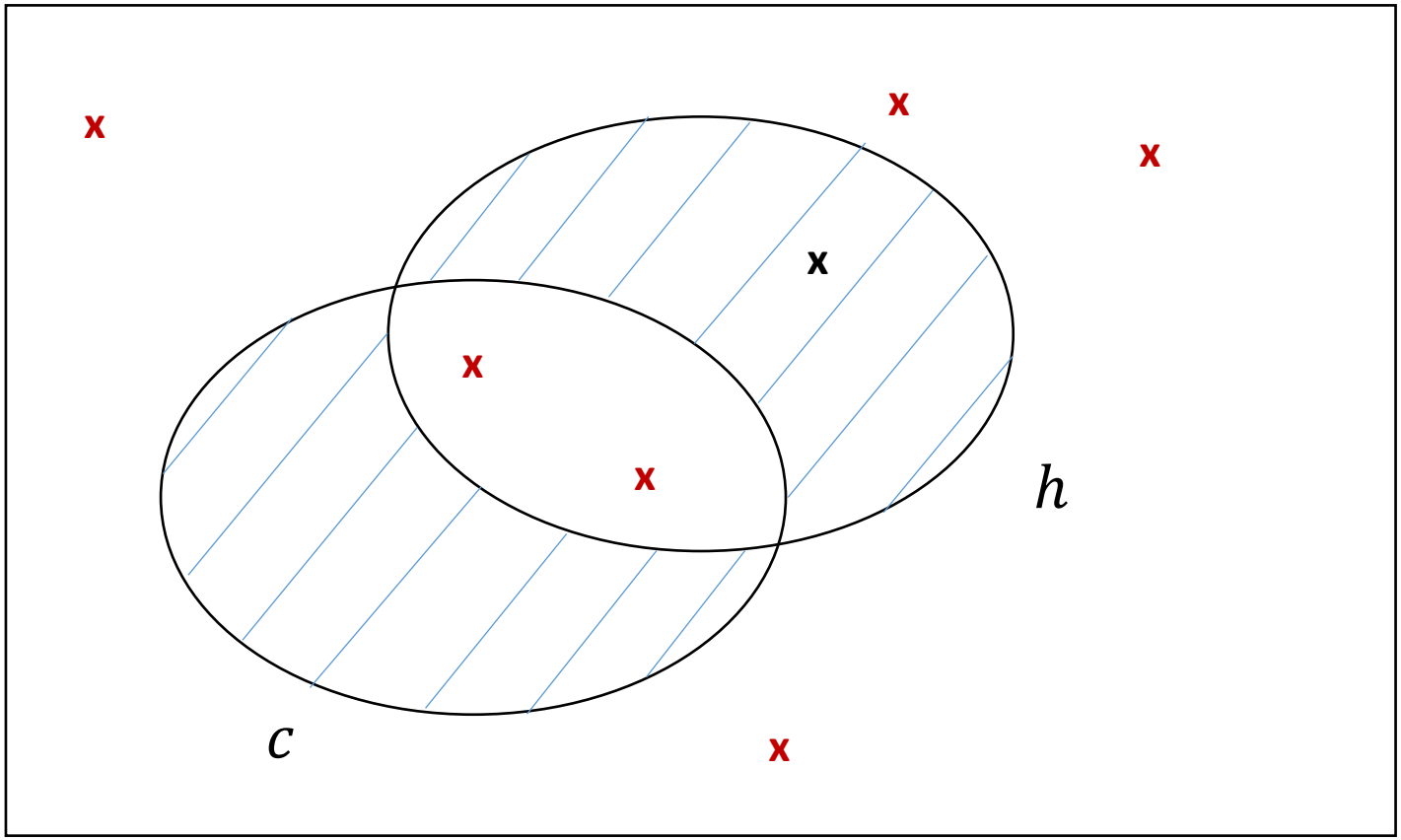
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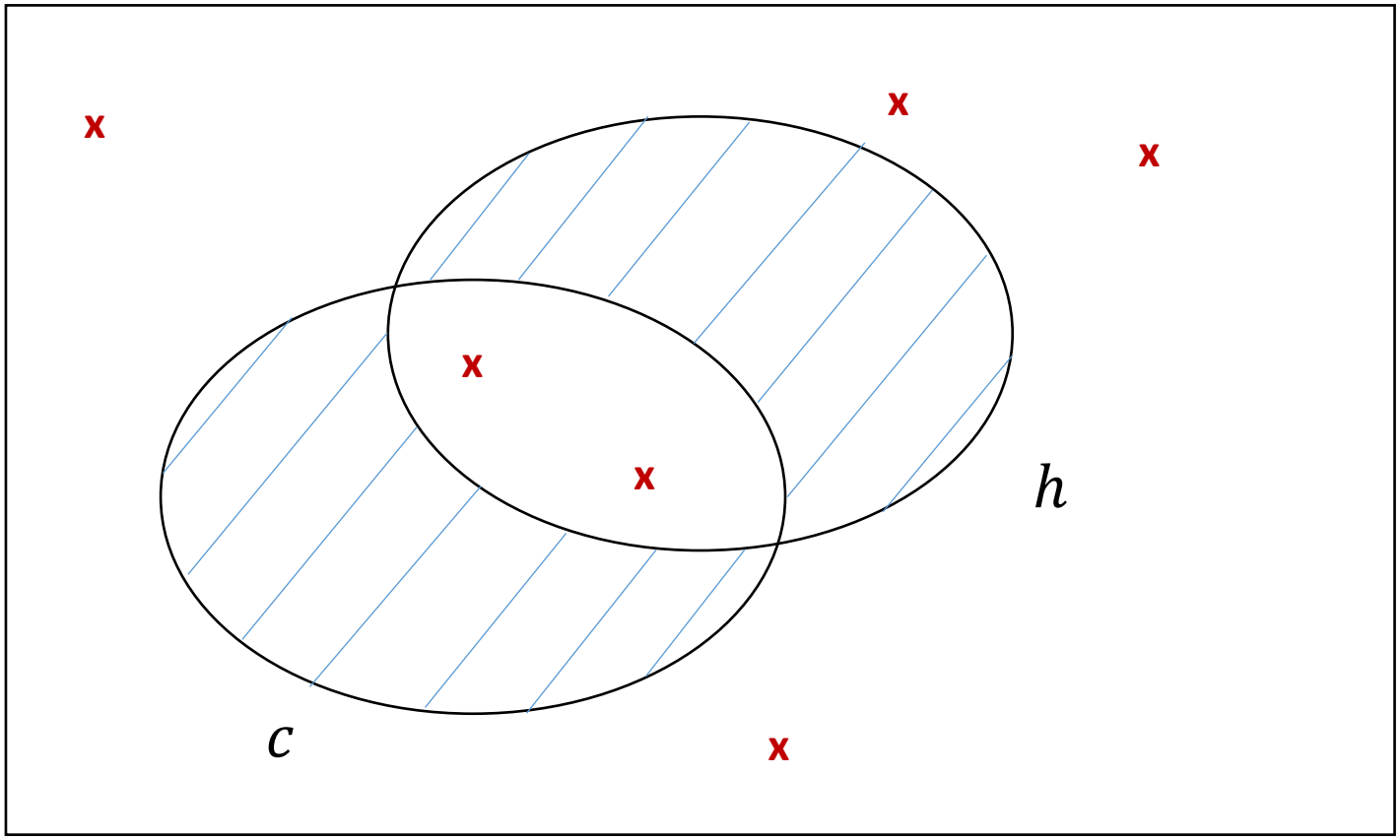
Consistent learners

A learning algorithm L , operating on training data produced by concepts from C and using a hypothesis space H is said to be a consistent learner if for any training data D and for any $c \in C$ the output $h = L(D)$ is D-*consistent* with respect to c .

Sample Complexity: Finite Hypothesis Spaces

- A hypothesis h is ε-bad if $error_{\pi}(h) > \varepsilon$ (e.g. 5%)
- A **consistent learner** using H must output some consistent h for any m samples
- Question: what is the probability that such a hypothesis h (an output of a consistent learner) will be ε-bad?
- In other words:
  what is the probability of obtaining a training dataset that may lead to such h, when our learner is consistent?



If h is ε bad then $\pi(\text{err}) \geq \varepsilon$

To be the output of a consistent learning algorithm, all m training data points had to have avoided the blue region.

Otherwise h would not have been consistent

A bound on $Prob(h$ is ε-bad)

- Consider some ε bad hypothesis $h \in H$
- For h to be the output of a consistent learner the training data has to have avoided a region $E \subset X$ with $\pi(E) \geq \varepsilon$.
- Since all m samples are independent, the probability (in $\Omega^m = (X, \pi)^m$) of an ε-bad h to be the output of a consistent learner is therefore $\leq (1 - \varepsilon)^m$

That is:

If L is a consistent learner for C
then,
for any $c \in C$ and any $\varepsilon - \text{bad}$ hypothesis $h \in H$,
we have:

$$\pi^m(D: L(D) = h) \leq (1 - \varepsilon)^m$$

Denote this set of datasets by $T(h)$

Cont …

Assume that H is finite.
Considering m sampled data points, $D \in X^m$, the probability that $L(D)$ is $\varepsilon -$ bad can now be bounded as follows:

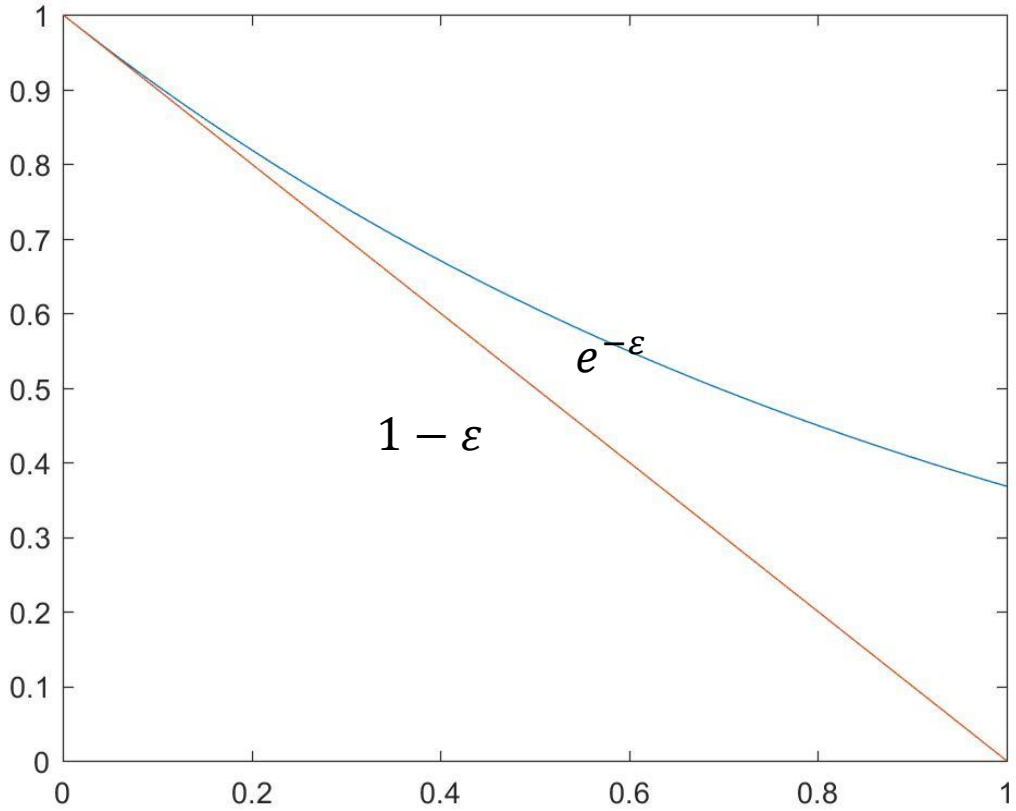
$$Prob(L(D) \text{ is } \varepsilon - \text{bad }) = \pi^m(D: L(D) \text{ is } \varepsilon - \text{bad})$$

$$= \pi^m\left(\bigcup_{h \text{ is } \varepsilon-\text{bad}} T(h)\right) \leq \sum_{h \text{ is } \varepsilon-\text{bad}} \pi^m(T(h))$$

$$\leq \sum_{h \text{ is } \varepsilon-\text{bad}} (1 - \varepsilon)^m \leq |H|(1 - \varepsilon)^m$$

$$\leq |H| \cdot \exp(-\varepsilon m)$$

$1 - \varepsilon < e^{-\varepsilon}$



For a formal proof try the Taylor/MacLaurin expansion of $\exp(-x)$ around 0.

First Bound on Sample Complexity

- Note that linear increase in the sample size reduces the chance of error in an exponential rate!
- In order to reduce the failure probability below some desired level δ we can require:

$$|H|e^{-\varepsilon m} \leq \delta \text{ or}$$

$$m \geq \frac{1}{\varepsilon}\ln\frac{|H|}{\delta} = \frac{1}{\varepsilon}\left(\ln|H| + \ln\frac{1}{\delta}\right)$$

- This is not a tight bound (mainly due to replacing the number of ε-bad hypotheses by the size of H)

Example: Disjunctions of Boolean Literals

- The instance space X is n dimensional Boolean vectors.
- The hypotheses space, H, and the concept space, C, both consist of disjunctions of n Boolean literals of the form:

$$x_1 \vee \overline{x}_5 \vee \overline{x}_{22}$$

- No further limitations on the hypotheses/concept space:

  For each Boolean variable x_i our hypothesis may contain either x_i or $\overline{x_i}$ (but not both) or none of them. The latter will mean that we get F for x_i

- Can we construct a consistent learner?

A Consistent Learner

Begin with $h = x_1 \vee \bar{x}_1 \vee x_2 \vee \bar{x}_2 \vee \cdots \vee x_n \vee \bar{x}_n$

For each negative training instance $\vec{x}$ ($c(\vec{x}) = \text{false}$):

Remove all literals l_i from h, which are consistent with $\vec{x}$

We end up with $h = l_{i_1} \vee l_{i_2} \vee \cdots \vee l_{i_k}$
that is consistent also with the positive instances,

since one of their literals must be in h .

Thus we end up with h that is consistent with all training examples

Example

1. Start: $x_1 \vee \overline{x_1} \vee x_2 \vee \overline{x_2} \vee x_3 \vee \overline{x_3} \vee x_4 \vee \overline{x_4}$
2. Instance 1: $x_1 \vee x_2 \vee \overline{x_3} \vee x_4$
3. Instance 2: $x_1 \vee x_4$

- Consistent with Instances 3 & 4

Example	x_1	x_2	x_3	x_4	y
1	0	0	1	0	0
2	0	1	0	0	0
3	0	0	1	1	1
4	1	0	0	1	1

0 x x x
x 0 x x
x x 1 x
x x x 1

How many instances are sufficient?

- Assume we have 10 attributes
- In our example, using conjunctions of features (or empty conjunctions at some is) we get $|H| = 3^{10} = 59{,}049$
- We want to ensure with 95% certainty that our hypothesis will have error < 10% .
- We then need $m > \frac{1}{0.1}(\ln 59049 + \ln \frac{1}{0.05}) = 10(11+3) = 140$ instances
- Note that we have 1024 instances in total (all of X)

Another example

- 20 attributes in the same setting
- We get $|H| = 3^{20} \sim 3.5 * 10^9$
- In this case, to have 95% certainty that our hypothesis will have error < 10% , we need $m > \frac{1}{0.1}(\ln 3.5*10^9 + \ln \frac{1}{0.05}) = 10(22+3) = 250$ instances
- In this case we have about 10^6 possible instances

When |H| increases exponentially with the number of features then sample complexity increases linearly.
The required fraction of the full population decreases.

PAC Learnability

- Consider a class **C** of possible target concepts defined over a space of instances **X**, and a learning algorithm **L** using hypothesis space **H**.
- Definition

  C is **PAC-learnable** by **L** using **H**

  if for all $0 < \varepsilon < ½$, $0 < \delta < ½$, and for all $c \in$ **C** and distributions π over **X**, the following holds:

  with data drawn independently according to π, **L** will output, with probability at least (1-δ), a hypothesis $h \in$ **H** such that $\text{error}_{\pi}(h) \leq \varepsilon$,

  L operates in time (and sample) complexity that is polynomial in $1/\varepsilon$, $1/\delta$ (and in other possible parameters).

Summary and next steps

- Generalization errors
- PAC learning framework
- Consistent learners for finite hypotheses spaces
- Directly calculating bounds on the sample complexity of consistent learners
- Next week – complete our theory section:
  - Concepts in $\mathbb{R}^n$
  - Agnostic learning
  - VC dimension
- After theory – unsupervised learning, special topics