Numerical Optimization with Python

Dry HW 02

1. The set Y is *an affine transformation of a set* X, if it is simply the set of all affine transformations of elements of X, namely: $Y = \{Ax + b: x \in X\}$ where A, b are a constant matrix and vector of the appropriate dimensions. Prove that affine transformations of convex sets are convex sets.
2. A sub-level set of a function $f: \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is defined as follows: $\{x \in \mathcal{D}: f(x) \leq c\}$ where c is a constant scalar.
    a. Sketch the sublevel sets of $f(x, y) = x^2 + 2y^2$ for $c = 1, 2$.
    b. Prove that sub-level sets of convex functions are convex sets
3. Let $f: \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $h: \mathbb{R} \to \mathbb{R}$ be convex and monotone increasing.
    a. Prove that the composition $h \circ g: \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is convex.
    b. Provide a counter example if we drop only the monotonic requirement on h.
    c. Provide another counter example if we drop only the convexity requirement on h.
4. Consider the following problem:

$$\min[|x| + |y|]$$

Subject to:

$$(x - 1)^2 + (y - 1)^2 \leq 1$$

$$y \leq 1$$

4.1. Draw a sketch of the contour lines of the objective function and the constraints, show the feasible region

4.2. Only by inspection of the picture (no calculation or solution needed), find the minimizer. Which constraints are active at the minimizer?

4.3. At the minimizer, draw arrows that denote the direction of the gradient of the objective function and of each of the constraints.

5. Consider the problem:

$$\min \frac{1}{2}(x_1^2 + x_2^2)$$

$$\text{s.t. } x_1 \geq 1$$

5.1. Write the Lagrangian function for the problem

5.2. Find the dual function $g(\lambda)$, and formulate the dual problem

5.3. For this problem, does strong duality hold? Justify your answer.

5.4. For the given primal problem, write the KKT optimality conditions

5.5. For this problem, are the KKT conditions necessary? Sufficient? Justify your answer

5.6. Solve the KKT system and conclude the minimizer of the problem.