Numerical Optimization

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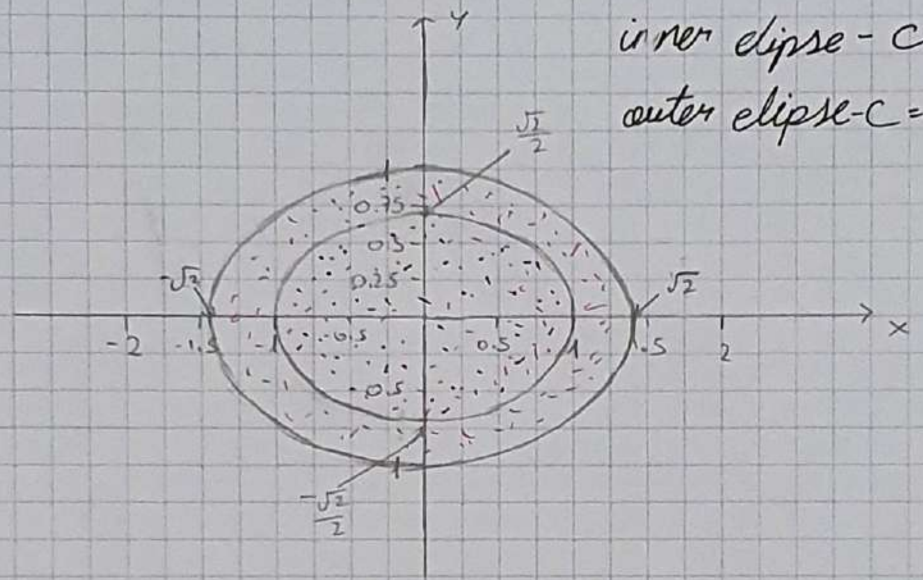
1. Let $F: X \to Y$, $F(x) = Ax + b$, X a convex set.
Given X is convex, $\forall x, y \in X$: $\alpha x + (1-\alpha)y \in X$ for $\alpha \in [0,1]$.
$F(\alpha x + (1-\alpha)y) = A(\alpha x + (1-\alpha)y) + b = \alpha \cdot Ax + (1-\alpha)Ay + (\alpha + (1-\alpha))b =$
$= \alpha \cdot (Ax + b) + (1-\alpha) \cdot (Ay + b) = \alpha \cdot F(x) + (1-\alpha) \cdot F(y) \in Y$
since $\alpha x + (1-\alpha)y \in X \Rightarrow Y$ is convex

2. a.



inner elipse - C = 1 (purple)
outer elipse - C = 2 (pink)

b. Let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ convex function $\Rightarrow \forall t \in [0,1]$, $x, y \in D$:
$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$
Let $x, y \in D$ s.t $f(x), f(y) \leq C$ where C is a constant.
Without ~~by~~ loss of generallity, assume $f(x) \leq f(y) \leq C$.
Let $t \in [0,1]$, x, y from above:
$f(tx + (1-t)y) \underset{\text{convex function}}{\leq} t f(x) + (1-t) f(y) \underset{\text{WLOG}}{\leq} t f(y) + (1-t) f(y) =$
$= f(y) \leq C \Rightarrow tx + (1-t)y$ belongs to the sub-level set of $C \Rightarrow$ sub-level sets of convex functions are convex.

3\. a. Let $g: D \subset \mathbb{R}^n \to \mathbb{R}$ be convex function

Let $h: \mathbb{R} \to \mathbb{R}$ be convex, monotonically increasing function.

from convexity: $g(tx+(1-t)y) \leq tg(x)+(1-t)g(y)$ for $0<t<1$, $x,y \in D$.

$$\Rightarrow h(g(tx+(1-t)y)) \underset{\text{monotonicity}}{\leq} h(tg(x)+(1-t)g(y)) \underset{\text{convexity}}{\leq} t\,h(g(x))+(1-t)\,h(g(y))$$

b. Let $g(x)=x^2$, $h(x)=-x$

Let $x=2$, $y=4$ $t=0.5$

$g(tx+(1-t)y) = g(0.5\cdot 2+0.5\cdot 4) = g(3) = 9$

$h(g(tx+(1-t)y)) = h(9) = -9$

$0.5\cdot h(g(2))+0.5\cdot h(g(4)) = 0.5\cdot h(4)+0.5\cdot h(16) = -2-8 = -10$

$h(g(tx+(1-t)y)) = -9 \not\leq -10 = t\,h(g(x))+(1-t)\,h(g(y))$

c. Let $g(x)=x$, $h(x)=\log_2(x)$

Let $x=2$. $y=4$ $t=0.5$

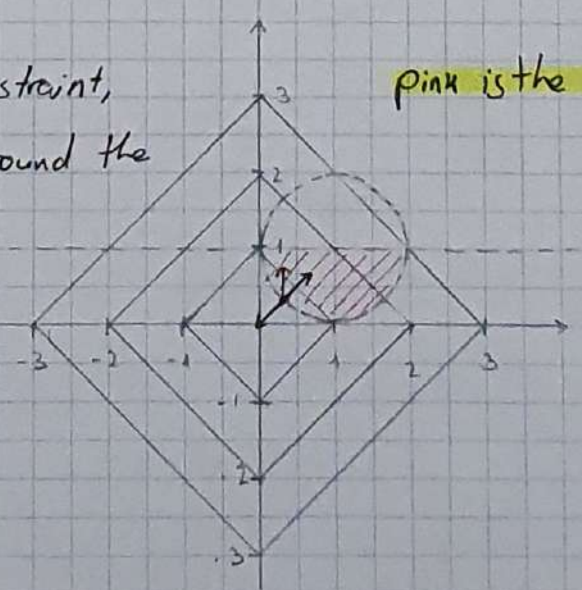
$h(g(tx+(1-t)y)) = h(g(0.5\cdot 2+0.5\cdot 4)) = h(g(3)) = \log_2 3 = 1.58$

$t\cdot h(g(x))+(1-t)\cdot h(g(y)) = 0.5\cdot h(g(2))+0.5\cdot h(g(4)) = 0.5\cdot h(2)+0.5\cdot h(4) =$
$= 0.5\cdot \log_2 2+0.5\cdot\log_2 4 = 0.5\cdot 1+0.5\cdot 2 = 1.5$

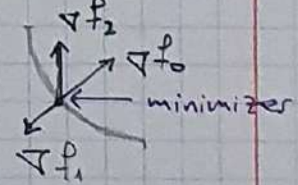
$h(g(tx+(1-t)y)) = 1.58 \not\leq 1.5 = t\cdot h(g(x))+(1-t)\,h(g(y))$

4\.

The first constraint, unit circle around the point (1,1) is an active constraint at the minimizer



pink is the feasible set



$f_0 = |x|+|y|$

$f_1 = (x-1)^2+(y-1)^2 \leq 1$

$f_2 = y-1 \leq 0$

5. $\min \frac{1}{2}(x_1^2 + x_2^2)$

s.t $x_1 \geq 1 \Rightarrow -x_1 \leq -1 \Rightarrow -x_1 + 1 \leq 0$

1. $\mathcal{L}(x, \lambda) = \frac{1}{2}(x_1^2 + x_2^2) + \lambda(-x_1 + 1)$

2. $g(\lambda) = \min\limits_{x} \frac{1}{2}(x_1^2 + x_2^2) + \lambda(-x_1 + 1)$

$$\nabla_x \mathcal{L}(x, \lambda) = \begin{pmatrix} x_1 - \lambda \\ x_2 \end{pmatrix}$$

$\nabla_x \mathcal{L}(x, \lambda) = \vec{0} \Rightarrow x_1 - \lambda = 0 \Rightarrow x_1 = \lambda$

$x_2 = 0$

$g(\lambda) = \frac{1}{2} \cdot \lambda^2 + \lambda \cdot (-\lambda + 1) = -\frac{1}{2}\lambda^2 + \lambda$

Dual problem: $\max\limits_{\lambda} -\frac{1}{2}\lambda^2 + \lambda$

3. The problem is convex, and Slater's condition holds (e.g exists x^* that is strictly feasible) hence, strong duality holds.

4. 1. $-x_1 + 1 \leq 0$

3. $\lambda \geq 0$

4. $\lambda(-x_1 + 1) = 0$

5. $x_1 - \lambda = 0$

$x_2 = 0$

5. For the given problem KKT conditions are both neccessry and sufficient.

from 5.3 strong duality holds $\Rightarrow$ KKT conditions are neccecary.
Given the strong duality yielded from convexity and slaters condition, KKT conditions are sufficient

6. KKT No 4: $\lambda(-x_1 + 1) = 0$, $\lambda = 0 \Rightarrow x_1 = 0 \Rightarrow (0,0), \lambda = 0$

$-x_1 + 1 = 0 \Rightarrow x_1 = 1 \underset{\text{from KKT No 5}}{\Rightarrow} \lambda = 1 \Rightarrow (1,0), \lambda = 1$

for $(0,0)$ exists:

$-x_1 + 1 = 0 + 1 \neq 0$, KKT No 1 does not hold.

for $(1,0)$ exists:

The minimizer of the problem is $x^* = (1,0)$, $\lambda^* = 1$

$-x_1 + 1 = -1 + 1 = 0$ KKT No 1 ✓

$\lambda \cdot (-x_1 + 1) = 1 \cdot (-1 + 1) = 0$ KKT No 4 ✓

$\lambda = 1 \geq 0$ KKT No 2 ✓

$x_1 - \lambda = 1 - 1 = 0$, $x_2 = 0$ KKT No 5 ✓