## Conditional Probability and Independence

The concept of conditional probability is immensely useful. It allows to connect events that may have already happend with events that have not yet happened and makes it possible to measure how much events influence each other. Conditional probabilities are also often easier to compute than straightforward probabilities of complex events.

**Note:** So far, we have used the notation $P(A \cap B)$ to indicate the probability that events $A$ *and* $B$ both occur. From now on, we will abbreviate this to $P(AB)$, that is, omit the $\cap$ symbol.

**Definition:** If $P(F) > 0$, then the conditional probability of event $E$, given event $F$ is defined as

$$P(E|F) = \frac{P(EF)}{P(F)}$$

**Example 20.** A fair coin was flipped twice. At least one of the flips resulted in "heads". What is the probability the other flip also resulted in "heads"?

**Example 21.** * A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease.

Calculate the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease.

**Theorem:** Multiplication rule

It follows from the definition of conditional probability that

$$P(EF) = P(E|F)P(F)$$

This can be generalized to any number of events:

$$P(E_1E_2\cdots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1})$$

**Proof:**

**Example 22.** Cards are drawn one by one from a standard 52-card deck without replacement. What is the probability that the fourth heart is drawn on the tenth draw?

**Example 23.** Matching problem (cont.)

In the matching problem in which $N$ individuals toss their cell phones in a pile and then randomly each pick one back out we showed that the probability that nobody gets his or her own phone back is

$$P_N = \sum_{i=0}^{N}(-1)^i/i!$$

What is the probability that exactly $k$ people have matches?

**Theorem:** Law of total probability

Let $E$ and $F$ be any events. Then

$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$$

**Proof:**

**Definition:** A collection of sets $F_1, \ldots, F_n$ are called a partition of a sample space $S$ if they are mutually exclusive and exhaustive. That means that no two events $F_i$ and $F_j$ can occur together

$$P(F_iF_j) = 0 \text{ for all } i, j$$

and that one of the events has to occur

$$P\left(\bigcup_{i=1}^{n} F_i\right) = 1$$

**Theorem:** Law of total probability (general version)

Let $F_1, \ldots, F_n$ be a partition of the sample space. Then for any event $E$

$$P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i)$$

**Example 24.** * An insurance company estimates that 40% of policyholders who have only an auto policy will renew next year and 60% of policyholders who have only a homeowners policy will renew next year. The company estimates that 80% of policyholders who have both an auto policy and a homeowners policy will renew at least one of those policies next year.

Company records show that 65% of policyholders have an auto policy, 50% of policyholders have a homeowners policy, and 15% of policyholders have both an auto policy and a homeowners policy. Using the company's estimates, calculate the percentage of policyholders that will renew at least one policy next year.

**Example 25.** Twins can be either identical or fraternal. Identical, also called monozygotic, twins share the same set of genes and consequently always have the same gender. Fraternal twins develop when two eggs are fertilized and implant in the uterus. The genetic connection of fraternal twins is not different from siblings born at different times. A scientist in LA wanted to find the proportion of twins born in the county that are identical twins. The problem is, that for newborn babies it is not obvious whether twins are identical or fraternal. Only a DNA test, which is costly, would be able to tell for sure. Instead of asking for DNA tests, the scientist asked the hospitals for data listing all twin births with an indication of the gender of both babies. The hospitals reported that approximately 64% of twin births were same-sexed. How could this information be used to estimate the proportion of identical twin births?

**Theorem:** Bayes' Theorem
Let $F_1, \ldots, F_n$ be a partition of the sample space and let $E$ be any event. Then

$$P(F_j|E) = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}$$

**Proof:**

**Corollary:** For any two events $E$ and $F$

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)} = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)}$$

**Example 26.** At a certain stage of a criminal investigation, the inspector in charge is 60% convinced of the guilt of a certain suspect. Suppose, however, that a new piece of evidence which shows that the criminal has a certain characteristic (such as left-handedness, baldness or brown hair) is uncovered. If 20% of the population possess this characteristic, how certain of the guilt of the suspect should be inspector be now if it turns out that the suspect has the characteristic?

The change in the probability of a hypothesis when new evidence is introduced can be expressed compactly in terms of the change in the *odds* of that hypothesis.

**Definition:** The odds of an event $A$ are defined as

$$\frac{P(A)}{P(A^c)} = \frac{P(A)}{1 - P(A)}$$

That is, the odds tell us how much more likely it is that the event $A$ occurs than it is that $A$ does not occur. If the odds are $\alpha$, then one sometimes says that the odds are $\alpha$ to 1 in favor of event $A$.

**Example 27.** If $P(A) = \frac{2}{3}$, what are the odds of event $A$?

If a hypothesis $H$ is true with probability $P(H)$ before new evidence is introduced and true with probability $P(H|E)$ after the new evidence is introduced then the new odds of the hypothesis (after the introduction of the new evidence) are:

$$\frac{P(H|E)}{P(H^c|E)} = \frac{P(H)}{P(H^c)} \frac{P(E|H)}{P(E|H^c)}$$

That means that the new value of odds is equal to the old value of the odds multiplied by the ratio of the probabilities of the evidence given that the hypothesis is true or not true, respectively. That means that if the evidence is more likely if the hypothesis is true than if not, the odds for the hypothesis increase after introduction of the evidence.

**Example 28.** According to the Arizona Chapter of the American Lung Association, 7% of the population has lung disease. Of those people having lung disease, 90% are smokers; and of those not having lung disease, 74.7% are non-smokers. What are the chances that a smoker has lung disease?

**Example 29.** Betty is a mother of two. She has just been seen walking with one of her children and that child is a girl. Suppose mothers of two who have both a girl and a boy are twice as likely to ask their daughters out for a walk than their sons. Suppose further that children are equally likely to be boys or girls. What is the probability that Betty's other child is also a girl?

**Example 30.** * An insurance company issues life insurance policies in three separate categories: standard, preferred, and ultra-preferred. Of the company's policyholders, 50% are standard, 40% are preferred, and 10% are ultra-preferred. Each standard policyholder has probability 0.010 of dying in the next year, each preferred policyholder has probability 0.005 of dying in the next year, and each ultra-preferred policyholder has probability 0.001 of dying in the next year. A policyholder dies in the next year. Calculate the probability that the deceased policyholder was ultra-preferred.

## Independent Events

In several of the previous examples we have seen that the probability $P(E)$ of an event $E$ can change after another is event $F$ is known to have occurred. That is the conditional probability $P(E|F)$ does not have to equal the unconditional probability $P(E)$. If that is the case, then the occurrence of $F$ influences the chances of the occurrence of $E$. If the occurrence of event $F$ does not affect the chances of the occurrence of event $E$, then we say that events $E$ and $F$ are independent.

$$P(E|F) = P(E) \qquad \Leftrightarrow \qquad P(EF) = P(E)P(F)$$

**Definition:** Two events $E$ and $F$ are said to be independent, if

$$P(EF) = P(E)P(F)$$

or, equivalently if

$$P(E|F) = P(E)$$

or, equivalently if

$$P(F|E) = P(F)$$

Two events that are not independent are said to be dependent.

**Example 31.** A card is drawn at random from a standard 52-card deck. Define the events $A$ = card is an ace and $C$ = card is a club ($\clubsuit$). Are the events $A$ and $C$ independent?

**Proposition:** If $E$ is independent of $F$, then $E$ is also independent of $F^c$.

**Proof:**

**Example 32.** Coupon collector problem

Suppose there are $n$ types of coupons and that each new coupon collected is, independent of previous selections, a type $i$ coupon with probability $p_i, \sum_{i=1}^n p_i = 1$. Suppose $k$ coupons are to be collected. If $A_j$ is the event that there is at least one type $j$ coupon among those collected, then, for $i \neq j$ find

(a) $P(A_i)$

(b) $P(A_i \cup A_j)$

(c) $P(A_i | A_j)$

**Historical Note:** Suppose two players both put up stakes and play some game of chance with the stakes to go to the winner of the game. Suppose they are interrupted before the game ends with some "partial score". How should the stakes be divided based on this score?

This question was posed to the French mathematician Blaise Pascal (left) by the Chevalier de Méré who was a professional gambler in 1654. Pascal discussed it in his letters with Pierre de Fermat (right) who had a reputation as a great mathematician. This discussion gave rise to the first definition of an expected value and is regarded by many as the birth date of probability theory.

**Example 33.** The problem of the points

Independent trials resulting in a success with probability $p$ and a failure with probability $1 - p$ are performed. What is the probability that $n$ successes occur before $m$ failures. Think of $A$ and $B$ playing a game in which $A$ gains a point every time a success occurs and otherwise $B$ gains a point. What is the probability that $A$ wins the game if currently $A$ would need $n$ more points to win and $B$ would need $m$ more points to win.

**Example 34.** The gambler's ruin problem

Independent trials are performed which result in a success with probability $p$ and a failure with probability $1-p$. Two players bet on each trial. If the trial is a success, $A$ collects one unit from $B$. If the trial is a failure $B$ collects one unit from $A$. They continue to do this until one of them runs out of money. What is the probability that $A$ will end up with all the money if he starts with $i$ units and $B$ starts with $N - i$ units?

**Note:** Before we find a solution to the gambler's ruin problem, describe how this problem differs from the problem of the points.

Proof of gambler's ruin problem (cont.):

## $P(\cdot|F)$ is a Probability

Conditional probabilities satisfy all the properties of ordinary probabilities. That is

(a) $0 \leq P(E|F) \leq 1$

(b) $P(S|F) = 1$

(c) If $E_1, E_2, \ldots$ are mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{\infty} E_i|F\right) = \sum_{i=1}^{\infty} P(E_i|F)$$

**Example 35.** An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. Assume that 30% of people are accident prone. During any given year, an accident prone person will have an accident with probability 0.4, whereas the corresponding figure for a person who is not accident prone is 0.2. What is the conditional probability that a new policy holder who had an accident in the first year will have another accident in the second year?