Named Continuous Distributions

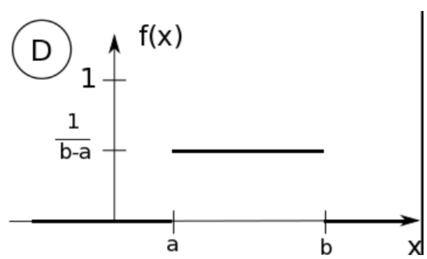
Similarly to the discrete case, we will now take a closer look at a few especially important continuous distibutions. In each case we will (where possible) derive the PDF, CDF and formulas for the expected value and variance.

The Uniform Distribution

In the beginning of this course we have considered only sample spaces in which every outcome is equally likely. If the outcomes are all numbers (e.g., $1, 2, \ldots, 6$ for rolling a fair six-sided die), then this distribution has a name - it is called the discrete Uniform distribution.

The analogous continuous distribution is called the (continuous) Uniform distribution. We have to consider all possible values in an interval (a, b). The probability that X falls into a subinterval should only depend on the length (and not the position) of the subinterval (a, b).

PDF: What should the probability density function of such a random variable look like?



Poll Question 8.1

DEFINITION: A random variable is called a continuous Uniform with parameters a and b, if its PDF can be written as

$$f(x) = \begin{cases} 1/b-a & a<=x<=b \\ 0 & \text{else} \end{cases}$$

Notation: $X \sim \text{Uniform}(a, b)$.

CDF: The CDF for a Uniform random variable can be computed in closed form

$$F(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \le x \le b \\ 1 & x > b \end{cases}$$

EXPECTED VALUE AND VARIANCE:

$E(X) =$ (a+b)/2

$E(X^2) = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a}\left[\frac{1}{3}x^3\right]_a^b = \frac{1}{3}\frac{b^3-a^3}{b-a} = \frac{1}{3}(a^2+ab+b^2)$.

$V(X) = E(X^2) - E(X)^2 = \cdots = \frac{(b-a)^2}{12}$.

Example: Suppose you arrive at a bus stop at 10:00am. The bus will arrive at a time T uniformly distributed between 10:00am and 10:30am.

(a) What is the probability that you'll have to wait more than 10 minutes for the bus?

T=arrive time of the bus(in min after 10)

T~uniform(0,30)

(b) If at 10:15am the bus has not yet arrived, what is the probability that you have to wait at least 10 more minutes?

P(T>25 | T>15)=P(T>25)/P(T>15)

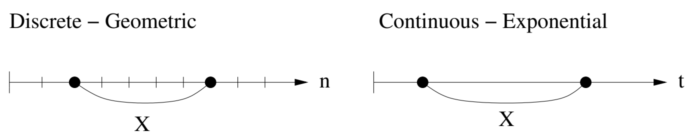
(c) What is the probability that the bus will arrive exactly at 10:15am?

0

Poll Question 8.2

The Exponential Distribution

Recall: The Geometric distribution was used to model the number of independent trials until (and including) the first success.



SITUATION: The Exponential distribution is the continuous waiting time distribution.

Examples:

- X is the time until the next customer arrives at a bank.
- X is the time until a lightbulb burns out (lifetime).
- X is the mileage you get out of one tank of gas.

DEFINITION: A continuous random variable is said to have an Exponential distribution with parameter $\lambda > 0$ if it has the PDF

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

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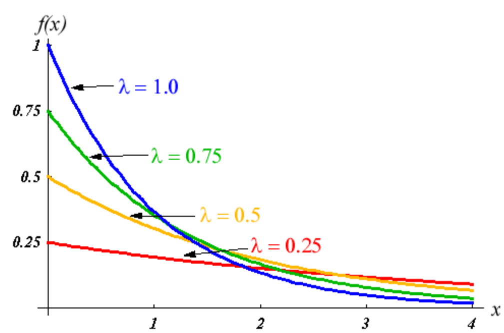

NOTATION: $X \sim \text{Exponential}(\lambda)$ or $X \sim \mathcal{E}(\lambda)$.

CDF: Using Calculus, we can find the CDF of X:

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Survival Function: If X is used to model a lifetime, then the probability of survival until time t is

$$P(X > t) = 1 - P(X \leq t) = 1 - F(t) = e^{-\lambda t}.$$

This function is called the survival function for X.

Expectation and Variance:

The expected value and variance of X can be found by using integration by parts.

$E(X) = \frac{1}{\lambda}$

$V(X) = \left(\frac{1}{\lambda}\right)^2$.

Note: Like the discrete waiting time distribution (Geometric) the Exponential distribution also has the "lack-of-memory" property.

$$P(X > s + t | X > s) = P(X > t) \text{ for } s, t \geq 0.$$

Example: Suppose the time X is takes a customer representative to help a customer at a phone hot-line has an exponential distribution with mean 2.5 minutes.

(a) What is the probability that it takes the customer representative more than three minutes to help the next customer?

(b) What is the probability that it takes the customer representative less than three minutes to help each of the next five customers?

Poll Question 8.3

The Gamma Distribution

SITUATION: The Gamma distribution is used to model continuous waiting times until the r^{th} occurrence of an event. The distribution is named after the Gamma function which appears in the PDF.

Recall:

	Discrete	Continuous
Waiting for the first event	Geometric	Exponential
Waiting for the r^{th} event	Negative Binomial	**Gamma**

Examples:

- Time until the r^{th} customer enters a bank
- Mileage that you get out of r tanks of gas - note that r here does not need to be an integer.

DEFINITION: A continuous random variable with PDF

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} & x \geq 0 \end{cases}$$

is said to have a Gamma distribution with parameters λ and r.

DEFINITION: Here, $\Gamma(\alpha)$ is the Gamma function defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \qquad \alpha > 0$$

For a positive integer n it is $\Gamma(n) = (n-1)!$.

NOTATION: $X \sim \text{Gamma}(\lambda, r)$

THEOREM: If $X_1, \ldots, X_r$ are independent Exponential(λ) random variables, then

$$Y = \sum_{i=1}^r X_i \sim \text{ Gamma } (\lambda, r).$$

EXPECTATION AND VARIANCE:

If r is an integer, then we can use the above theorem to find

$$E(Y) = E\left(\sum_{i=1}^r X_i\right) = r \cdot \frac{1}{\lambda} \qquad V(Y) = V\left(\sum_{i=1}^r X_i\right) = r \cdot \left(\frac{1}{\lambda}\right)^2.$$

Remark: The above formulas are also true if r is not an integer (but the derivation is much more complicated). Your textbook calls $\frac{1}{\lambda} = \beta$ and $r = \alpha$.

Example: Suppose that accidents happen at a certain intersection at an average rate of two every day.
In each part of the problem, define the random variable you are working with and state its distribution along with all relevant parameters.

(a) What is the probability that we will have to wait more than a day for the next two accidents to happen?

X = time until 2nd accident(in days)
X~Gamma(r=2,lambda=2)
P(X>1)

(b) How long do you expect to have to wait until the fifth accident occurs?

T=days until 5th accident
T~Gamma(r=5,lambda=2)
E(T)=5/2

Poll Question 8.4

The Normal Distribution

Remark: The Normal Distribution is by far the most important distribution that we will study in this course. It is the distribution most often applied to "real life" (or normal) data by scientists in all disciplines. It was first used in 1733 by Abraham DeMoivre and used by Carl Friedrich Gauß to predict the location of astronomical bodies.

DEFINITION: A continuous random variable X is said to have a Normal distribution with mean μ and variance $\sigma^2 > 0$, if it has the PDF

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \text{for } -\infty < x < \infty.$$

NOTATION: $X \sim \text{Normal}(\mu, \sigma^2)$ or $X \sim N(\mu, \sigma^2)$.

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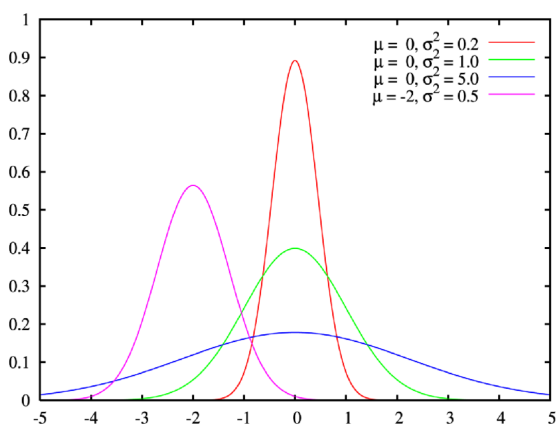


Remarks:

- This distribution is sometimes also called the "Gaussian" distribution.
- $E(X) = \mu$ and $V(X) = \sigma^2$.
- The PDF has the characteristic "bell-shape" and is symmetric about the mean.
- Because of this symmetry, the median is equal to the mean.

DEFINITION: A normal random variable with mean $\mu = 0$ and variance $\sigma^2 = 1$ is called a STANDARD NORMAL random variable.

$$f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Notation: $Z \sim$ Normal(0,1). The CDF of Z is $P(Z \leq z)$ and will be denoted by $\Phi(z)$.

Remark: The standard normal distribution does not often arise naturally, but it rather serves as a standard against which other normal distributions are measured.

Example: Computing probabilities for Normal random variables by integrating the PDF is not straightforward. Instead, tables are used which contain values of the standard normal CDF $\Phi(z)$ for many different values of z.

Let Z be a standard normal random variable. Use the table in the back of the textbook (table A3 in Devore) to find

(a) $P(Z \leq 1.17)$

(b) $P(Z > -2.73)$ 0.9968

(c) $P(-0.5 \leq Z \leq 0.64)$ 0.4304

Remark: Of course, you can also use software (Excel, or your calculator) to find values of the standard normal CDF. In Excel, the command for $\Phi(z)$ is Normsdist(z). Read the manual for your calculator, to find the command for $\Phi(z)$. For example, on a TI-84, you would hit DISTR and `normalcdf` and enter the lower, and upper bounds and μ and σ (in that order).

Poll Question 8.5

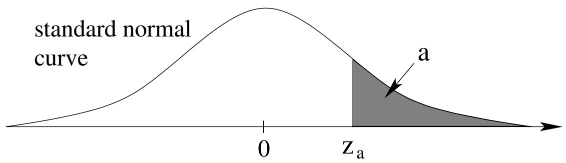
Normal Percentiles

The table in the back of the book can be read backwards to find standard normal percentiles.

Example: Use the table in the back of the book to find the 95^{th} percentile of the standard normal distribution.

Later in this course we will frequently work with *right* tail areas of the standard normal distribution.

DEFINITION: Let z_a denote the value on the measurement axis, for which the area under the standard normal curve to the right of z_a is equal to a.



Example: Find $z_{0.01}$.

Poll Question 8.6

Probabilities for General Normal Random Variables

Probabilities for Normal random variables with any mean μ and variance σ^2 can be computed by comparing them to standard normal distributions. This process is referred to as STANDARDIZING the random variable.

PROPOSITION: Let X be a Normal random variable with mean μ and standard deviation σ. Then

$$Z = \frac{X-\mu}{\sigma} \sim \text{ Normal}(0,1)$$

Thus

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$
$$P(X \le a) = \Phi\left(\frac{a-\mu}{\sigma}\right), \qquad P(X \ge b) = 1 - \Phi\left(\frac{b-\mu}{\sigma}\right)$$

Example: LSAT scores have approximately a Normal distribution with mean $\mu = 150$ and standard deviation $\sigma = 10$.

(a) If you have a score of 167, then you did better than how many percent of the people taking the test with you?

(b) How high would your score have to be so that you are in the top 90%?

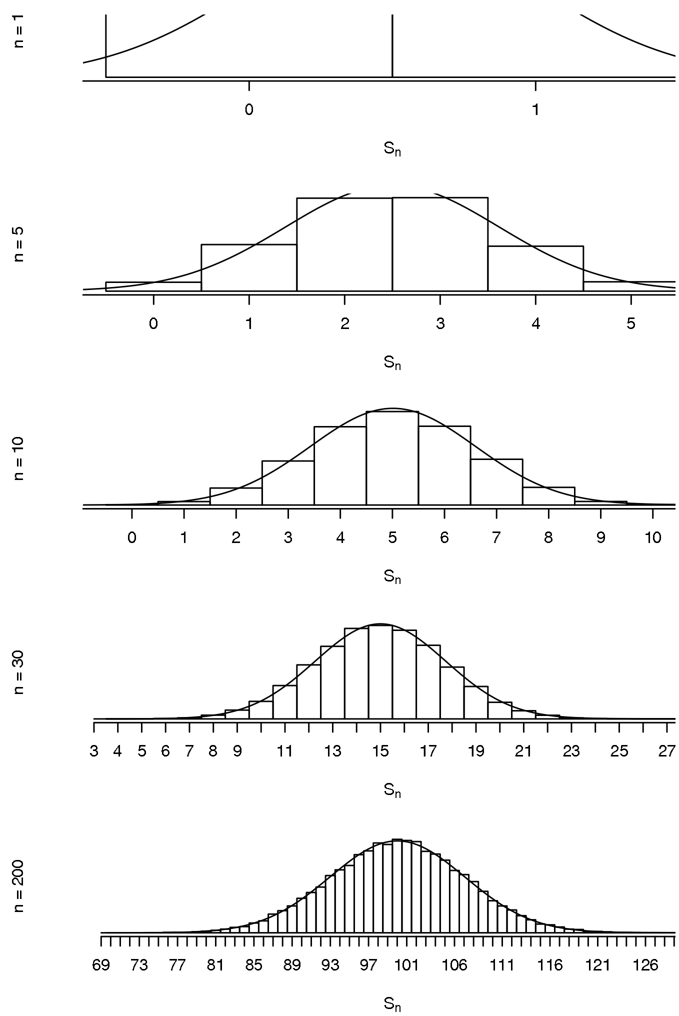
P(X<=x)=0.1
P(Z<=(x-150)/10)=0.9
(x-150)/10=1.28

Poll Question 8.7

Normal Approximation to the Binomial

For certain parameter values n and p, the Binomial(n, p) PMF becomes very similar to the Normal PDF. We want to use this similarity to approximate Binomial probabilities that are tedious to compute by Normal probabilities.

The following graphs show the Binomial PMF of X for $p = 0.5$ and selected values of n. Superimposed is the continuous curve of a Normal PDF.



For large values of n and moderate (close to 0.5) values of p the PMF of X is very close to a Normal PDF with $\mu = np$ and $\sigma^2 = np(1-p)$. Use $np > 5$ and $n(1-p) > 5$ as a rule of thumb for when to use this approximation.

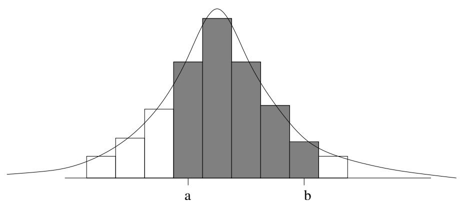
Probabilities for the Binomial random variable X can therefore be approximated by corresponding probabilities for a Normal$(\mu = np, \sigma^2 = np(1-p))$ random variable. The larger n is, the more precise this approximation will be.

Remark: There is one important detail that we have to take care of when computing such probabilities.

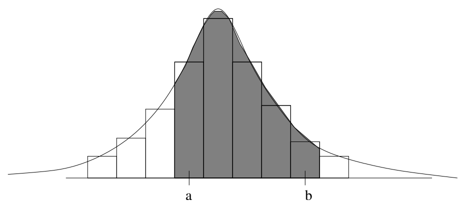
$X \sim \text{Binomial}(n, p)$ is a discrete random variable. It does therefore make a difference if we compute $P(X \leq a)$ or $P(X < a)$, for example.

This means that we have to either include or exclude the "a"-bar in our computation of the probability. We can do that by going a step of 1/2 to the left or right depending on whether we want to include or exclude the bar.

This procedure is called a "Continuity Correction".



$P(a \leq X \leq b)$



$P(a - \frac{1}{2} < Z < b + \frac{1}{2})$

Continuity Correction: For limits a and b we have, for example,

$$P(a \leq X \leq b) \approxeq \Phi\left(\frac{b + \frac{1}{2} - \mu}{\sigma}\right) - \Phi\left(\frac{a - \frac{1}{2} - \mu}{\sigma}\right)$$

The Continuity Correction can also be used to compute probabilities like $P(a < X \leq b)$, but then the direction of the "step" needs to be changed.

Poll Question 8.8

Example: A multiple choice test has 150 questions. Each question has three possible answers.

(a) Find the probability that a student will get at least 56 answers right simply by guessing.

X=# of correct
X~Binomial(n=150,p=1/3)
X≈Y~Normal(μ =np=50, σ =np(1-p)=100/3)
P(X>=56)≈P(Y>55.5)

(b) Find the probability that the student will get a "B" on the test (i.e. at least 120 questions correct but less than 135).

P(120<=X<135)=P(119.5<Y<134.5)