Combinatorial Analysis

$P(A) = \frac{\#\text{favorable}}{\#\text{possible}}$

Combinatorics is the science of counting. Usually, we count in how many ways something specific can happen to compute probabilities. Since combinatorics were introduced quite thoroughly in your introductory probability course, we will only briefly review key concepts and then move on to more interesting and challenging problems that you may not yet have seen in Math 161A.

The basic principle of counting: Suppose two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

The generalized principle of counting: If r experiments are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if ..., then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

Example: What is the probability that is a class of n students no two students have the same birthday?

There are two specific scenarios that occur quite frequently in combinatorics: They have to do with reordering distinct objects (permutations) or selecting subsets of distinct objects (combinations).

Permutation Rule: There are

$$n(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

ways in which to permute (rearrange) n different objects.

Example: In how many (distinguishable) ways can the letters in the word "PEPPER" be rearranged?

Choosing stuff when order doesn't matter

Combination Rule: There are

$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

different ways in which r objects can be selected from a set of n distinct objects. Note, that in this scenario, the order in which the r objects are selected is considered irrelevant.

Proof:

Note: The binomial coefficient $\binom{n}{r}$ is defined for $0 \leq r \leq n$ as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example: Consider a set of 12 antennas of which 4 are defective and 8 are functional. Assume that all the defective antennas and all the functional antennas are indistinguishable from each other. How many linear orderings are there in which no two defective antennas are consecutive?

Example: Consider n-digit numbers where each digit is one of the integers $0, 1, \ldots, 9$. How much such numbers are there for which

(a) no two consecutive digits are equal?

(b) 0 appears as a digit a total of i times, $i = 0, \ldots, n$

Often, what's hard in counting problems is to first understand very clearly what the question is (props or drawings often help to clarify your own understanding) and then deciding whether or not order matters.

Example: An urn contains balls labeled $0, \ldots, 9$. Three balls are drawn one after the other with replacement.

(a) What is the probability that the numbers 1,2,3 are drawn (in any order)?

(b) What is the probability that the three balls have different numbers on them?

Fact: A useful combinatorial identity is

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \qquad 1 \leq r \leq n$$

Proof:

Theorem: The Binomial Theorem

Let $x, y \in \mathbb{R}$ and let $n \geq 0$ be an integer. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

The Binomial Theorem can be proven by induction or though a combinatorial argument.

Proof: (Combinatorial version)

Example: How many subsets are there of a set of n distinct elements?

Multinomial Coefficients

In how many ways can n objects be divided into r groups of sizes $n_1, n_2, \ldots, n_r$ (with $\sum_{i=1}^{r} n_i = n$, of course)?

Example: In how many different ways can n distinct objects (think balls labeled $1, 2, \ldots, n$) be placed into r baskets such that the first basket receives n_1 balls, the second n_2 balls etc.? We would count an arrangement as different only if two balls from *different* baskets were to be exchanged.

Definition: Let $n_1, n_2, \ldots, n_r$ be integers and $n = n_1 + \cdots + n_r$. Then

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

is called the MULTINOMIAL COEFFICIENT.

Example: You have twelve different toys which you want to give to three children so that the oldest child receives five toys, the middle child four and the youngest child three. In how many ways can that be done? What if the toys were indistinguishable (think 12 chocolate chip cookies)?

n=12,
n_1=5,
n_2=4,
n_3=3

Theorem: Multinomial Theorem
Let $x_1, x_2, \ldots, x_r \in \mathbb{R}$ and let $n_1, \ldots, n_r$ be non-negative integers with $n_1 + \cdots + n_r = n$. Then

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1,\ldots,n_r):\\ n_1+\cdots+n_r=n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

This theorem follows as a generalization of the binomial theorem.

Example: Expand $(x_1 + x_2 + x_3)^3$.

Number of Integer Solutions of Equations

There are r^n ways to distribute n distinguishable balls into r distinguishable urns. Why?

For each of the n object , you have r choices of where to put it

What if the urns are distinguishable but the balls all look the same (and you would not be able to tell if two balls were interchanged)?

Example: In how many ways can n identical balls be distributed among r distinguishable urns?

(a) If it's ok for urns to remain empty?

(b) If every urn needs to receive at least one ball (assuming that $n \geq r$).

Fact: There are $\binom{n+r-1}{r-1}$ distinct non-negative integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying the equation

$$x_1 + x_2 + \cdots + x_r = n$$

Fact: There are $\binom{n-1}{r-1}$ distinct positive integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying the equation

$$x_1 + x_2 + \cdots + x_r = n, \qquad x_i > 0, \quad i = 1, \ldots, r$$

Stirling's Formula

Large factorial coefficients are hard to compute (for a calculator or computer) and in many practical applications an approximation would be useful. Stirling's approximation or Stirling's formula provides this approximation.

Theorem: For large n,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n = \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}$$

in the sense that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

Proof: