Counting Techniques

If you have a sample space $\mathcal{S}$ in which each outcome is equally likely, then computing probabilities of events reduces to counting the number of ways an event can occur. Let N denote the number of events in the sample space and $N(A)$ the number of outcomes contained in event A, then

$$P(A) = \frac{N(A)}{N}.$$

This means we count in how many ways A can occur (numerator, "favorables") and divide by the number of possible outcomes of the experiments (denominator, "possibles").

Example: The experiment "rolling a die" has $N = 6$ possible outcomes. The event "rolling an even number" has three favorable outcomes. Hence

$$P(\text{even}) =$$

Example: A box contains three marbles, one blue, one green, and one red. We draw two marbles

(a) with replacement (that means, we draw one, look at it, put it back and draw the next one)

(b) without replacement (that means we draw two at the same time)

Write down the sample space for each case. What is the probability of the event E "one is red and one is green"?

Unfortunately, even seemingly simple experiments can have *very* many possible outcomes. That makes listing all outcomes impractical. In this leture we will take a closer look at COMBINATORICS which is the mathematical theory of "counting".

Examples:

- There are 2,598,960 different ways to deal a poker hand (5 cards out of a 52 card deck).
- There are 1024 ways to flip a coin ten times (if order of flips matters)
- There are 13,983,816 ways to select 6 out of 49 lottery numbers (and only one Jackpot winning combination!)

Example: Mr. Smith owns 3 different suit jackets, 4 different ties, and 3 different pairs of pants. How many different outfits can he wear?

If one jacket, 2 of the ties and 2 pairs of pants are blue, what is the probability that he ends up with an all-blue outfit?

$$p = \frac{\#\text{favorate}}{\#\text{Possible}}$$

THEOREM: **General Product Rule**

Suppose a set consists of ordered collections of k elements (k-tuples) and that there are n_1 possible choices for the first element, n_2 choices for the second element, etc., and n_k choices for the k^{th} element. Then there are $n_1 n_2 \cdots n_k$ possible k-tuples.

Example: The Birthday problem revisited.

What is the probability that in a group of n people at least two people share a Birthday?

FACT: A set of n elements has 2^n subsets. (Each element can either be included or excluded in the subset $\rightarrow$ n actions with 2 possible outcomes each.)

Poll Question 3.1

Permutations and Combinations

In many applications it makes a difference if the order of elements matters or not. Here are some examples:

Order matters in:

- License plates, Codes, Phone numbers.
- People sitting down at a table, etc.

Order does not matter in

- Poker hands, committees of people.
- Drawing coins out of a purse when all you are interested in is the sum of money you get, etc.

Poll Question 3.2

DEFINITION: An ordered subset is called a PERMUTATION. The number of permutations of size k that can be formed from the n different objects or individuals in a group will be denoted by $P_{k,n}$.

Example: Find $P_{k,n}$ for a password made up out of four different upper-case letters.

$P_{4,26}$

DEFINITION: An unordered subset is called a COMBINATION. One way to denote the number of combinations is $C_{k,n}$, but we will use the notation $\binom{n}{k}$ (read: "n choose k") instead.

Example: Find $C_{k,n}$ for a poker hand. A poker hand is five cards selected at random, and without replacement from a deck of 52 different cards.

Order matters → Permutations
Order does not matter → Combinations

Example: You have a box with 10 different cookies and 3 hungry kids. In how many different ways can you give each kid a cookie?

$P_{3,10}$

PROPOSITION: **Permutation Rule**

The number of possible permutations of k objects chosen from a set of n objects is

$$P_{k,n} = \frac{n!}{(n-k)!}.$$

FACT: $P_{n,n} = n!$ is the number of ways to order n distinct objects.
FACT: $0! = 1$ by definition.

Example: Five kids will randomly sit down on 5 chairs in a row. In how many ways is that possible?

Jim and Joe are friends. What is the probability that they'll sit next to each other?

Example: Eight ballots in a ballot box are numbered $1, \ldots, 8$. In how many ways can we draw three ballots without replacement? We don't care about the order in which the ballots are drawn.

PROPOSITION: **Combination Rule**

The number of possible combinations of k objects chosen from a set of n possible objects is

$$C_{k,n} = \binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{(n-k)!k!}.$$

Note: There are "!", $P_{k,n}$, and $C_{k,n}$ buttons on most calculators. If you cannot locate yours, make sure to ask somebody (your instructor, for instance).

Practice Problems

Example: How many different letter arrangements can be formed using the letters

$$P\ E\ P\ P\ E\ R$$ 3 P, 2E, 1 R

One possibility would be e.g. $REPPPE$ (it does not need to be an english word).

total: P_{6,6}=6!

PROPOSITION: There are

$$\frac{m!}{m_1! \cdots m_k!}$$

different permutations of m objects of which $m_1, \ldots, m_k$ are alike, respectively.

Example: In how many different ways can the letters

$$C\ A\ L\ I\ F\ O\ R\ N\ I\ A$$

2A,2I

be arranged?

What is the probability that in a random arrangement the two I's end up next to each other?

Total: P_{10,10}/2!2!

N(A)=P_{9,9}/2!

Poll Question 3.3

Example: In how many ways can a committee of 2 teachers and 4 students be chosen from 10 teachers and 20 students?

Example: What is the probability to have at least one ace in a poker hand? A poker hand consists of 5 cards randomly drawn from a standard 52-card deck.

Remark: There is often more than one way of looking at (and solving) a particular problem. If you compare your solution with that of a friend you may both have used different reasoning, and different theorems. But you should end up with the same probability! The same is true for the homework or review problems. Just because the way you did the problem is different from the one printed in the solutions does not mean you did it the wrong way (as long as your answer is correct).

Mixed Examples on Counting problems

Remark: For students who are not used to this new way of thinking, it can sometimes be hard to distinguish between the different rules. Which one applies to which problem? There is no definite answer to this question. A good approach is to imagine yourself having to do the "experiment". When you want to count the possible outcomes, keep very close track of **all** the decisions you have to make and apply the product rule.

Another rule of thumb:

Choosing with replacement $\rightarrow$ Order matters $\rightarrow$ Permutations

Choosing without replacement $\rightarrow$ Order does not matter $\rightarrow$ Combinations

Example: A fair coin is flipped ten times. What is the probability to see exactly five heads in the ten tosses?

Example: What is the probability to be dealt a poker hand that has exactly two kings?

Poll Question 3.4

Example: A team of four bridge players is to be chosen from 10 players. Two of the players hate each other and refuse to serve on the team together. How many different team formations are possible?

Example: What is the probability to win the jackpot in the lottery "6 out of 49"?

A couple harder counting problems

Example: You have eight pennies and four dimes. If you randomly put them all in a row, what is the probability that no two dimes are next to each other?

Remark: Other than practice, what helps with counting problems is relating new problems to old ones that you have previously solved. Which previous problem could be useful here?

Solution to previous example:

Example: A poker hand is dealt at random. What is the probability, that the hand contains at least one card from each suit?

Solution attempt:

$$P(\text{at least one card from each suit}) = \frac{13 \cdot 13 \cdot 13 \cdot 13 \cdot 48}{\binom{52}{5}}$$

Reasoning: Pick one card from each suit. Then pick any one of the remaining cards for your favorables.

Note: That solution is WRONG! Explain what the mistake is. Then find the correct solution.