CS 156:Introduction to Artificial Intelligence

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Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(t) = \sum_s P(t, s)$$

$$P(s) = \sum_t P(t, s)$$

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

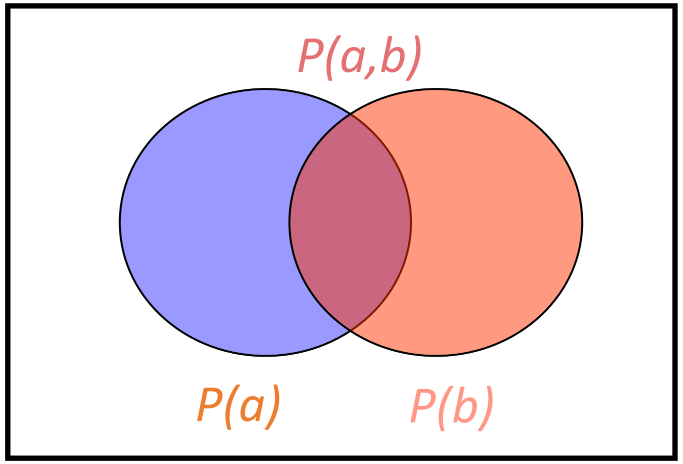
W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Conditional Probabilities

- A simple relation between joint and conditional probabilities
  - In fact, this is taken as the *definition* of a conditional probability

$$P(a|b) = \frac{P(a,b)}{P(b)}$$



$$P(T,W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

$$= P(W=s,T=c) + P(W=r,T=c)$$

$$= 0.2 + 0.3 = 0.5$$

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$P(W|T)$

$P(W|T = hot)$

W	P
sun	0.8
rain	0.2

$P(W|T = cold)$

W	P
sun	0.4
rain	0.6

Joint Distribution

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Normalization Trick

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$\begin{aligned} P(W = s | T = c) &= \frac{P(W = s, T = c)}{P(T = c)} \\ &= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\ &= \frac{0.2}{0.2 + 0.3} = 0.4 \end{aligned}$$

$$\begin{aligned} P(W = r | T = c) &= \frac{P(W = r, T = c)}{P(T = c)} \\ &= \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\ &= \frac{0.3}{0.2 + 0.3} = 0.6 \end{aligned}$$

$$P(W | T = c)$$

W	P
sun	0.4
rain	0.6

Normalization Trick

$$
\begin{aligned}
P(W=s|T=c) &= \frac{P(W=s,T=c)}{P(T=c)} \\
&= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\
&= \frac{0.2}{0.2+0.3} = 0.4
\end{aligned}
$$

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence

$P(c,W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)

$P(W|T=c)$

W	P
sun	0.4
rain	0.6

$$
\begin{aligned}
P(W=r|T=c) &= \frac{P(W=r,T=c)}{P(T=c)} \\
&= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\
&= \frac{0.3}{0.2+0.3} = 0.6
\end{aligned}
$$

To Normalize

- (Dictionary) To bring or restore to a normal condition

All entries sum to ONE

- Procedure:
  - Step 1: Compute Z = sum over all entries
  - Step 2: Divide every entry by Z

- Example 1

W	P
sun	0.2
rain	0.3

Normalize

Z = 0.5

W	P
sun	0.4
rain	0.6

- Example 2

T	W	P
hot	sun	20
hot	rain	5
cold	sun	10
cold	rain	15

Normalize

Z = 50

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Back to the dentist example ...

- We now represent the world of the dentist D using three propositions – Cavity, Toothache, and PCatch
- D's belief state consists of $2^3 = 8$ states each with some probability:

  $\{Cavity \wedge Toothache \wedge PCatch,$
  $\neg Cavity \wedge Toothache \wedge PCatch,$
  $Cavity \wedge \neg Toothache \wedge PCatch, \ldots\}$

The belief state is defined by the full joint probability of the propositions

	Toothache		¬Toothache	
	PCatch	¬PCatch	PCatch	¬PCatch
Cavity	0.108	0.012	0.072	0.008
¬Cavity	0.016	0.064	0.144	0.576

Probabilistic Inference

	Toothache		¬Toothache	
	PCatch	¬PCatch	PCatch	¬PCatch
Cavity	0.108	0.012	0.072	0.008
¬Cavity	0.016	0.064	0.144	0.576

$$P(Cavity \lor Toothache) = 0.108 + 0.012 + \ldots$$
$$= 0.28$$

Probabilistic Inference

	Toothache		¬Toothache	
	PCatch	¬PCatch	PCatch	¬PCatch
Cavity	0.108	0.012	0.072	0.008
¬Cavity	0.016	0.064	0.144	0.576

$$P(Cavity) = 0.108 + 0.012 + 0.072 + 0.008$$
$$= 0.2$$

Conditional Probability

$$P(a|b) = \frac{P(a, b)}{P(b)}$$

- $P(A \wedge B) = P(A|B)\,P(B)$
  $= P(B|A)\,P(A)$

	Toothache		¬Toothache	
	PCatch	¬PCatch	PCatch	¬PCatch
Cavity	0.108	0.012	0.072	0.008
¬Cavity	0.016	0.064	0.144	0.576

$$P(Cavity|Toothache) = P(Cavity \wedge Toothache)/P(Toothache)$$

$$= (0.108+0.012)/(0.108+0.012+0.016+0.064) = 0.6$$

Conditional Probability

- $P(A \wedge B) = P(A|B)\,P(B)$
  $= P(B|A)\,P(A)$
- $P(A \wedge B \wedge C) = P(A|B,C)\,P(B \wedge C)$
  $= P(A|B,C)\,P(B|C)\,P(C)$

Independence

- Two random variables A and B are independent if

  $$P(A \wedge B) = P(A)\, P(B)$$

  hence if $P(A|B) = P(A)$

- Two random variables A and B are independent given C, if

  $$P(A \wedge B|C) = P(A|C)\, P(B|C)$$

  hence if $P(A|B,C) = P(A|C)$

Bayes' Rule

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

where A and B are events and $P(B) \neq 0$.

- $P(A|B)$ is a conditional probability: the probability of event A occurring given that B is true. It is also called the posterior probability of A given B.
- $P(B|A)$ is also a conditional probability: the probability of event B occurring given that A is true. It can also be interpreted as the likelihood of A given a fixed B because $P(B|A) = L(A|B)$.
- $P(A)$ and $P(B)$ are the probabilities of observing A and B respectively without any given conditions; they are known as the prior probability and marginal probability.

Two ways to factor a joint distribution over two variables:

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- Example:
  - M: meningitis, S: stiff neck

$$\left.\begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned}\right\} \text{Example givens}$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m) + P(+s|-m)P(-m)} = \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.999}$$