Probability Theory

Probability is the study of randomness. In this course you will learn how to build mathematical models that can take uncertainty into account. The origins of probability theory lie in the 16^{th} and 17^{th} century (P. de Fermat, B. Pascal) and were based mostly on the analysis of games of chance. Modern probability theory which is based on the mathematical foundation of measure theory began in the 19^{th} century with A.N. Kolmogorov and P.S. Laplace.

Vocabulary:

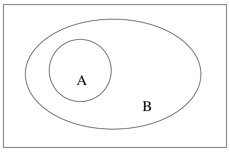
EXPERIMENT: an action or process whose outcome is subject to uncertainty.
Example: Rolling a die.

SAMPLE SPACE: denoted $\mathcal{S}$ is the set of all possible outcomes of the experiment.
Example: Write down the sample spaces for the experiments "rolling a die" and "tossing a coin".

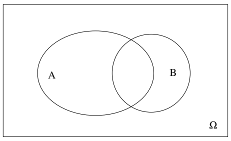
EVENT: a subset of outcomes from the sample space. A SIMPLE EVENT consists of only one outcome from the sample space $\mathcal{S}$.
Example: Write down the outcome "rolling an even number" in set notation. Write down the simple event "tossing heads".

Set Theory

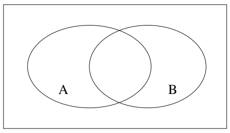
Remark: You can draw pictures of sets, so called Venn Diagrams, that show the relationship between the sets. The following picture shows sets A and B such that A is (literally) contained in B.



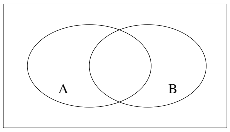
Definition: The COMPLEMENT of an event A, denoted by A', is the set of all outcomes in $\mathcal{S}$ that are not in A.



Definition: The UNION of two events A and B, denoted by $A \cup B$ (read: "A or B"), is the set of all outcomes contained in either A or B (or both).



Definition: The INTERSECTION of two events A and B, denoted $A \cap B$ (read "A and B"), is the set of all outcomes that are contained in both A and B.



Definition: Two events A and B are called MUTUALLY EXCLUSIVE or DISJOINT, if their intersection is empty. The empty set is denoty by $\emptyset$.

Important: The set notation that will be used most in this course is the notation for intersection, union, and complement.

Notation	Symbol	Meaning
Intersection	$\cap$	"and"
Union	$\cup$	"or"
Complement	$(\cdot)'$	"not"

Exercise: Draw Venn diagrams with two intersecting sets A and B and shade the area that represents:

(a) A

(b) $(A \cap B)'$

(c) $A' \cap B'$

What would the Venn diagrams above (and the shaded sets) look like if B were contained in A?

PollEverywhere Poll

Please log into your PollEverywhere account on either your phone, tablet or computer. Use the same email that is on your Canvas account (which is not always your sjsu email). Respond to the quiz that will pop up in a moment. Any response earns points, but a correct response earns more points. Your PollEverywhere account needs to be synched to your Canvas account in order to receive credit for your responses. My instructor ID is martinabremer513.

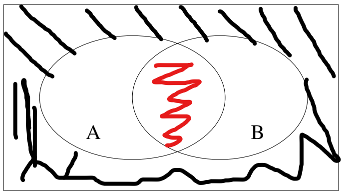
There are laws for working with sets.

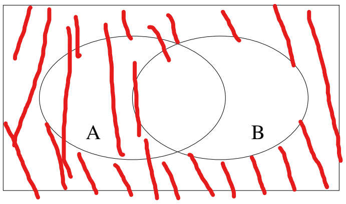
De Morgan's Laws:
$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

How can you see if a law such as the above is true? Draw two pictures of the sets A and B. In one picture shade the set described on the left side of the equation and in the other picture shade the set in the right side of the equation. If the shaded sets are the same, then the above equality holds.

Exercise: Convince yourself that the first De Morgan's law is true.

A'





$(A \cap B)'$

It should be obvious that the following always hold:

$$A \cup A' = \mathcal{S},$$

$$A \cap A' = \emptyset \text{ where } \emptyset \text{ is the "empty" set or "null event".}$$

Example: If F is the event that a randomly chosen person is female, and S is the event that the person is single, then how would you describe the events that the randomly chosen person is

(a) Married? S'

(b) A married female? $S' \cap F$

Poll Question

Axioms of Probability

An axiom is a primary assumption about something. Building up on axioms, a theory can be derived. The following are the underlying assumptions we make on probabilities.

Axiom 1: For any event A, its probability is non-negative

$$P(A) \geq 0;$$

Axiom 2: The probability of the whole sample space is equal to one

$$P(\mathcal{S}) = 1;$$

Axiom 3: If $A_1, A_2, A_3, \ldots$ is a (finite or infinite) collection of disjoint events, then

$$P\left(A_1 \cup A_2 \cup A_3 \cup \cdots\right) = \sum P(A_i).$$

From these axioms we can make some immediate conclusions:

1 PROPOSITION: Null event $P(\emptyset) = 0$.

PROOF: $1 \overset{(2)}{=} P(S) = P(S \cup \emptyset) \overset{(3)}{=} P(S) + P(\emptyset) = 1 + P(\emptyset)$

2 PROPOSITION: It is sometimes easier to compute the probability that something will *not* happen

$$P(A') = 1 - P(A)$$

PROOF: $1 = P(S) = P(A+\bar{A}) = P(A) + P(\bar{A})$

3 PROPOSITION: If the events A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B)$$

PROOF:

PROPOSITION: If $A \subset B$, then $P(A) \leq P(B)$. In particular, $P(A) \leq 1$ for every event A.

PROOF:

4 PROPOSITION: For any two events A and B (not necessarily mututally exclusive) we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

PROOF:

5 PROPOSITION: For three events A, B, C, the general addition rule looks like this

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

PROOF:

Example: According to *Current Population Reports*, published by the U.S. Bureau of Census, 51.0% of U.S. adults are female, 7.1% are divorced, and 4.1% are divorced females. Determine the probability that a randomly selected adult in the U.S. is

(a) Male

$1 - 0.51$

(b) Female, but not divorced

$0.51 - 0.041$

(c) A divorced male

$0.071 - 0.041$

Example: Consider randomly selecting a student at San Jose State and define the events I = the student owns an iPod, and C = the student owns a cell phone. Suppose that $P(I) = 0.6$, $P(C) = 0.8$ and $P(I \cap C) = 0.55$.

(a) Determine the probability that a randomly chosen student owns either an iPod or a cell phone.

(b) Determine the probability that a randomly chosen student owns an iPod, but not a cell phone.

Poll Question