Axioms of Probability

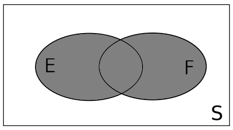
Definition: The set of all possible outcomes of an experiment is called the SAMPLE SPACE. The possible outcomes themselves are called ELEMENTARY EVENTS. Any subset of the sample space is called an EVENT.

Example 11. Consider the experiment of tossing a coin twice. List the sample space and define an event. Under which condition are the elementary events all equally likely?

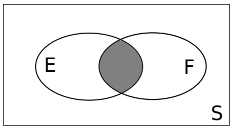
Set Theory

Sets and the relationships between them can be depicted in Venn Diagrams. Since events are subsets of sample spaces and thus sets, the same diagrams are useful in understanding complex events and relations between events.

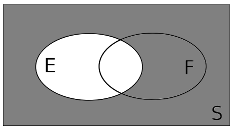
Definition: The UNION of two events E and F, denoted by $E \cup F$ (read: E *or* F), is the set of all outcomes contained in either E or F (or both). For events, this means that either E *or* F occurs.



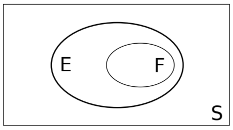
Definition: The INTERSECTION of two events E and F, denoted $E \cap F$ or EF (read: E *and* F), is the set of all outcomes that are contained in both E and F. For events, this means that both E *and* F occur simultaneously.



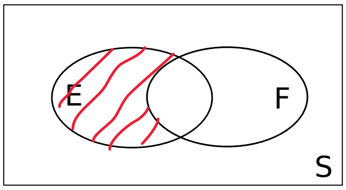
Definition: The COMPLEMENT of an event E, denoted by E^c, is the set of all outcomes in $\mathcal{S}$ that are not in E. For events, this means that the event E does *not* occur.



Definition: A set F is said to be contained in a another set E (or a subset of E, $F \subset E$) if every element that is in F is also in E. In terms of events, this means that if F occurs then E occurs.



Example 12. Shade the area that represents $E \cap F^c$ and describe in words what this area represents.



Theorem: De Morgan's laws
Let $E_1, \ldots, E_n$ be events in a sample space S. Then

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$$

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Proof:

Axioms of Probability

Probabilities are often thought of in terms of long-run frequencies. If you toss a (possibly biased) coin very many times, then the proportion of heads in those many tosses represents the probability with which the coin tosses heads. But how can we be sure that if the experiment were repeated with the same coin that the proportion of heads tossed would converge to the same value? We really don't. This is an assumption on which the frequency interpretation of probability is based. This assumption is already fairly complex. It is possible to make a small number of simpler assumptions (called the axioms of probability) based on which the notion of probability can be defined.

Axiom 1: Probabilities are always between 0 and 1

$$0 \leq P(E) \leq 1$$

Axiom 2: The probability of the whole sample space (the certain event) is 1.

$$P(S) = 1$$

Axiom 3: For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is, events for which $E_i E_j = \varnothing$ when $i \neq j$),

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Example 13. Show how these three axioms together imply that the probability of the empty set (impossible event) is zero.

Some Propositions

Since these propositions likely have been proved in your Math 161A classes, proofs are omitted here. They may, however, appear on your homework assignments.

Proposition: $P(E^c) = 1 - P(E)$

Proposition: If $E \subset F$, the $P(E) \leq P(F)$.

Proposition: Addition rule for two events.

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Proposition: General addition rule.

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r})$$

Example 14. Write out the general addition rule explicitly for three events E, F, G and convince yourself that it is true with the help of a Venn Diagram.

Example 15. Four married couples are arranged in a row. What is the probability that no husband will sit next to his wife?

Example 16. * Suppose $P(A \cup B) = 0.7$ and $P(A \cup B^c) = 0.9$. Find $P(A)$.

Note: All "starred" examples are taken directly from old actuarial exams.

Example 17. * An auto insurance company has 10,000 policyholders. Each policyholder is classified as either young or old; male or female; married or single.

Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males. How many are young single females?

Sample Spaces With Equally Likely Outcomes

In many (but not all!) experiments it is possible to create sample spaces in which each outcome is equally likely. In this case, the probability of an event E can be computed through counting:

$$P(E) = \frac{\#\text{ of outcomes in } E}{\#\text{ of outcomes in } S}$$

Example 18. A regular card deck contains 52 different cards with 13 cards from each suit ($\heartsuit, \spadesuit, \clubsuit, \diamondsuit$). In the game of bridge, each of four players receives 13 cards from the shuffled deck. What is the probability that

(a) One player receives all 13 spades?

(b) Each player receives one ace?

Example 19. The matching problem

Suppose that N students each throw their cell phones into a bag. The phones are mixed up and each student draws one phone back out of the bag. What is the probability that none of the students get their own phone back?