Pricing and Hedging in the Black-Scholes Framework

Vanna- Volga Pricing and Hedging

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Vana-Volga Pricing and Hedging

Illustrations

Interpolation of a Volatility Curve

Pricing Exotic Derivatives

Vanna-Volga Dynamic Hedging

- A practical method for pricing and hedging derivatives, taking into account an uncertain volatility.
- Popular for Foreign Exchange derivatives
- Relates the price of a complex derivative to the known price of simpler, liquid instruments.

Second order Greeks

- Vega: $\nu = \frac{\partial O}{\partial \sigma}$
- Vanna: $\frac{\partial \nu}{\partial S}$
- Volga: $\frac{\partial \nu}{\partial \sigma}$

Vanna-Volga Dynamic Hedging

An extended version of Ito's lemma with random volatility:

$$\begin{aligned} dO(t,K) &= \frac{\partial O}{\partial t}dt + \frac{\partial O}{\partial S}dS_t + \frac{\partial O}{\partial \sigma}d\sigma_t \\ &+\frac{1}{2}\frac{\partial^2 O}{\partial S^2}(dS_t)^2 + \frac{1}{2}\frac{\partial^2 O}{\partial \sigma^2}(d\sigma_t)^2 + \frac{\partial^2 O}{\partial S \partial \sigma}dS_t d\sigma_t \end{aligned}$$

Vanna-Volga Dynamic Hedging

$$
\begin{aligned}
& dO(t,K) - \Delta_t dS_t - \sum_{i=1}^{3} x_i dC_i(t,K_i) = \\
& \left[\frac{\partial O}{\partial t} - \sum_i x_i \frac{\partial C_i}{\partial t} \right] dt + \left[\frac{\partial O}{\partial S} - \Delta_t - \sum_i x_i \frac{\partial C_i}{\partial S} \right] dS_t \\
& + \left[\frac{\partial O}{\partial \sigma} - \sum_i x_i \frac{\partial C_i}{\partial \sigma} \right] d\sigma_t + \left[\frac{\partial^2 O}{\partial S^2} - \sum_i x_i \frac{\partial^2 C_i}{\partial S^2} \right] (dS_t)^2 \\
& + \left[\frac{\partial^2 O}{\partial \sigma^2} - \sum_i x_i \frac{\partial^2 C_i}{\partial \sigma^2} \right] (d\sigma_t)^2 + \left[\frac{\partial^2 O}{\partial S \partial \sigma} - \sum_i x_i \frac{\partial^2 C_i}{\partial S \partial \sigma} \right] dS_t d\sigma_t
\end{aligned}
$$

Vanna-Volga Hedging

$$dO(t,K) - \Delta_t dS_t - \sum_{i=1}^{3} x_i dC_i(t,K_i) = r\left[O(t,K) - \sum_i x_i C_i(t,K_i)\right] dt$$

Vanna-Volga Dynamic Hedging

The weights x_i are obtained by solving the system of linear equations:

$$
\begin{aligned}
\frac{\partial O}{\partial \sigma} &= \sum_i x_i \frac{\partial C_i}{\partial \sigma} \\
\frac{\partial^2 O}{\partial \sigma^2} &= \sum_i x_i \frac{\partial^2 C_i}{\partial \sigma^2} \\
\frac{\partial^2 O}{\partial S \partial \sigma} &= \sum_i x_i \frac{\partial^2 C_i}{\partial S \partial \sigma}
\end{aligned}
$$

or,

$$b = Ax$$

Vanna-Volga Dynamic Hedging

Choice of hedge instruments

- An at-the-money straddle:

$$C_1 = C(S) + P(S)$$

- A "risk reversal", traditionally defined as

$$C_2 = P(K_1) - C(K_2)$$

  with K_1 and K_2 chosen so that the options have a delta of .25.

- A "butterfly", defined as

$$C_3 = \beta(P(K_1) + C(K_2)) - (P(S) + C(S))$$

  with β determined to set the vega of the butterfly to 0.

Vanna-Volga Dynamic Hedging: Calculation Steps

- compute the risk indicators for the option to be priced:

$$b = \begin{pmatrix} \frac{\partial O}{\partial \sigma} \\ \frac{\partial^2 O}{\partial \sigma^2} \\ \frac{\partial^2 O}{\partial \sigma \partial S} \end{pmatrix} \tag{1}$$

- compute A matrix

$$A = \begin{pmatrix} \frac{\partial C_1}{\partial \sigma} & \dots & \frac{\partial C_3}{\partial \sigma} \\ \frac{\partial^2 C_1}{\partial \sigma^2} & \dots & \frac{\partial^2 C_3}{\partial \sigma^2} \\ \frac{\partial^2 C_1}{\partial \sigma \partial S} & \dots & \frac{\partial^2 C_3}{\partial \sigma \partial S} \end{pmatrix} \tag{2}$$

Vanna-Volga Dynamic Hedging: Calculation Steps

- solve for x:

$$b = Ax$$

- the corrected price for O is:

$$O^M(t,K) = O^{BS}(t,K) + \sum_{i=2}^{3} x_i \left(C_i^M(t) - C_i^{BS}(t) \right) \tag{3}$$

where $C_i^M(t)$ is the market price and $C_i^{BS}(t)$ the Black-Scholes price (i.e. with flat volatility).

Vanna-Volga Dynamic Hedging

Neglecting the off diagonal terms in A, a simplified procedure is to estimate x_i by:

$$
\begin{aligned}
x_2 &= \frac{\frac{\partial^2 O}{\partial \sigma^2}}{\frac{\partial^2 C_2}{\partial \sigma^2}} \\
x_3 &= \frac{\frac{\partial^2 O}{\partial \sigma \partial S}}{\frac{\partial^2 C_3}{\partial \sigma \partial S}}
\end{aligned}
$$

In practice, the weights x_i are scaled to better fit market prices.

Volatility Interpolation

Given the ATM volatility and at two other strikes, determine the volatility at an arbitrary strike K.

Vana-Volga Smile Interpolation

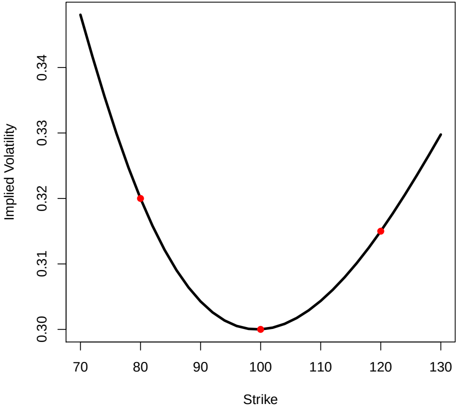


Figure: Interpolated Volatility Curve with Vanna-Volga algorithm

Vanna-Volga Pricing of a Binary Option

Consider a one-year binary call, struck at the money. Assume that the smile is quadratic.

Use the traditional benchmark instruments of the FX market:

- Straddle,
- Risk-reversal
- Butterfly

Vanna-Volga Pricing of a Binary Option

Steps:

- Compute the strikes corresponding to a 25Δ call and put,
- Compute the value of each benchmark instrument. The butterfly must be vega-neutral,
- Compute the risk indicators (vega, vanna, volga) for the binary option,
- ... and for the benchmark instruments,
- Compute the smile cost of each benchmark: the price with the smile effect less price at the ATM volatility.
- Compute the price correction for the binary option.

Pricing a Binary Option

In summary, we get:

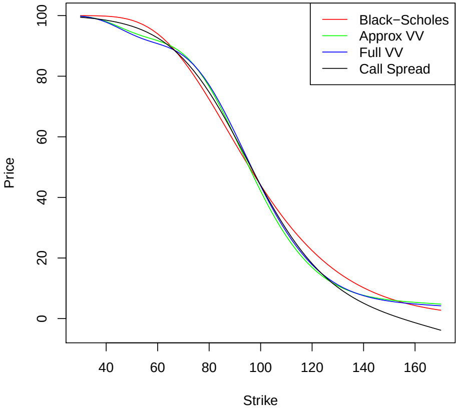
- Black-Scholes price: 32
- With approximate Vanna-Volga correction: $32 + -4.86 = 27.14$
- With acurate Vanna-Volga correction: $32 + -3.39 = 28.61$
- the approximation by a call spread is: 29.44

Pricing Comparison

comparison of binary option value for a range of strikes, computed with four methods:

1. the regular Black-Scholes method, assuming a flat volatility
2. the Black-Scholes price with approximate Volga-Vanna correction
3. same a above, but with an accurate calculation of the Vanna-Volga correction
4. the value of a call spread centered at the strike

Vana-Volga Pricing of a Binary Option



References