Mathematical Foundations of the

Dimensional Function Hierarchy

Formal Proofs and Derivations

Companion Document to BE-AI-2026-001

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Abstract

This document provides formal mathematical foundations for the Dimensional Function Hierarchy (DFH) presented in BE-AI-2026-001. We establish rigorous definitions, state explicit axioms, and derive the core theorems of the framework through formal proof. The central results include: (1) a proof that temporal function is necessarily primitive in any system admitting state change, (2) proofs of the strict dependency ordering among dimensional functions, (3) derivation of the necessity of inverse functions, and (4) the completeness theorem establishing that six fundamental-inverse pairs constitute a minimal complete basis for coherent reality. These proofs are offered for critical examination and challenge.

1. Preliminaries

1.1 Notational Conventions

We employ the following notation throughout:

D_n — The n-th dimensional function (n ∈ {1,2,3,4,5,6})

$\bar{D}\_n$ or $D\_\{n+6\}$ — The inverse of D_n

$\rightarrow$ — Dependency relation: $A \rightarrow B$ means 'B requires A'

$\perp$ — Independence: $A \perp B$ means 'A does not require B'

Φ — The primordial ground (existence itself, prior to any function)

$\mathscr{S}$ — A state space

Δ — State change operator

1.2 Foundational Definitions

Definition 1.1 (Dimensional Function)

A dimensional function D is an operator that, when applied to a system, enables a specific category of phenomena to occur that would be impossible without it. Formally: $D: \mathscr{S} \to \mathscr{S}'$ where $\mathscr{S}'$ admits phenomena that $\mathscr{S}$ does not.

Definition 1.2 (Functional Dependency)

Function D_j depends on function D_i (written $D_i \to D_j$) if and only if D_j cannot operate on any system where D_i is not already operative. Formally: $\forall \mathscr{S}$: $D_j(\mathscr{S})$ is defined $\Rightarrow$ D_i is operative in $\mathscr{S}$.

Definition 1.3 (Inverse Function)

For each dimensional function D_n, there exists an inverse function $\bar{D}_n$ such that D_n and $\bar{D}_n$ operate in dialectical tension: neither can be reduced to the other, and their interaction generates dynamic phenomena rather than cancellation.

Definition 1.4 (Primitive Function)

A dimensional function D is primitive if it depends only on Φ (existence itself) and not on any other dimensional function. Formally: D is primitive $\Longleftrightarrow [D \to \Phi \wedge \forall D': D' \neq \Phi \Rightarrow \neg(D \to D')]$

2. Axioms

We posit the following axioms as the foundation of the DFH:

Axiom 1 (Existence of Distinction)

There exist at least two distinguishable states. Formally: $\exists s_1, s_2 \in \mathscr{S}$ such that $s_1 \neq s_2$.

Axiom 2 (Possibility of Change)

It is possible for a system to transition between distinguishable states. Formally: $\exists \Delta: \mathscr{S} \to \mathscr{S}$ such that $\Delta(s_1) = s_2$ where $s_1 \neq s_2$.

Axiom 3 (Functional Requirement)

Every category of phenomena requires some functional operator to enable it. No phenomenon arises without an enabling function.

Axiom 4 (Dependency Antisymmetry)

If $D_i \to D_j$ and $D_j \to D_i$, then $D_i = D_j$. (Dependency is a partial order.)

Axiom 5 (Dialectical Completeness)

For any fundamental function enabling category C of phenomena, there exists an inverse function enabling category $\bar{C}$ such that C and $\bar{C}$ together span the full operational space of that functional dimension.

3. The Primacy of Time (D_1)

3.1 Definition of Temporal Function

Definition 3.1 (Temporal Function D_1)

D_1 is the function that enables state change — the capacity for 'before' and 'after' to differ. $D_1(\mathscr{S})$ is the state space equipped with the possibility of transition.

3.2 Main Theorem: Temporal Primacy

Theorem 3.1 (Primacy of D_1)

D_1 is the unique primitive dimensional function. That is: D_1 depends only on Φ, and every other dimensional function depends on D_1.

Proof.

Part 1: D_1 depends only on Φ

By Axiom 1, distinguishable states exist. By Axiom 2, transition between states is possible. By Axiom 3, this possibility requires an enabling function. We call this function D_1. The only prerequisite for D_1 is that something exists (Φ) to undergo change. D_1 requires no spatial separation (change can occur at a point), no position (change need not be located), no force (change is not causation), no probability (change can be deterministic), and no disorder (change can be ordered). Therefore $D_1 \to \Phi$ and $D_1 \perp D_n$ for all $n > 1$.

Part 2: D_1 is unique in this property

We prove by examining each candidate function:

• **D_2 (Space/Extent):** Separation requires something to be separated. That 'something' must persist through the separation — i.e., it must exist both before and after the separation is established. This is a temporal requirement. Therefore $D_1 \to D_2$.

• **D_3 (Position):** Position is location within extent. Without extent (D_2), there is nothing within which to have position. Since $D_1 \to D_2$, we have $D_1 \to D_3$ by transitivity.

• **D_4 (Forces):** Forces act between positions. Without positions (D_3), there is no 'between' for forces to act. Since $D_1 \to D_3$, we have $D_1 \to D_4$.

• **D_5 (Quantum):** Superposition is superposition of states that can interact. Interaction presupposes D_4. Since $D_1 \to D_4$, we have $D_1 \to D_5$.

• **D_6 (Disorder):** Perturbation acts on states. Without a state space structured by D_1-D_5, there is nothing to perturb. Therefore $D_1 \to D_6$.

No other function can be shown to depend only on Φ. Therefore D_1 is the unique primitive function.

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3.3 Corollary: Computational Validation

Corollary 3.1.1 (Computational Witness)

Any system capable of binary computation provides an existence proof for D_1.

Proof.

Binary computation requires state change (bit flip: $0 \to 1$ or $1 \to 0$). By the proof of Theorem 3.1, state change requires D_1. Any functioning computer therefore witnesses D_1. The existence of the reader processing this document constitutes empirical proof that D_1 is operative.

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4. The Dependency Chain

4.1 Formal Statement

Theorem 4.1 (Strict Dependency Ordering)

The dimensional functions form a strict total order under dependency: $D_1 \to D_2 \to D_3 \to D_4 \to D_5 \to D_6$

We prove each link in the chain:

4.1.1 Proof: $D_1 \to D_2$

Lemma 4.1 (Space Requires Time)

The spatial function D_2 (enabling separation/extent) requires the temporal function D_1.

Proof.

Let D_2 be operative, enabling separation between entities A and B. For A and B to be 'separated,' both must exist. For both to exist as distinct, each must have some persistence — some duration through which it maintains identity. Duration is a temporal property enabled by D_1. Without D_1, neither A nor B persists, and 'separation between A and B' is meaningless.

Alternatively: consider establishing separation. To go from 'no separation' to 'separation' is a change of state, which requires D_1.

Therefore $D_1 \to D_2$.

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4.1.2 Proof: $D_2 \to D_3$

Lemma 4.2 (Position Requires Space)

The positional function D_3 (enabling location/coordinates) requires the spatial function D_2.

Proof.

Position is defined as location within an extent. Let D_3 assign position p to entity E. This assignment requires an extent within which p is defined. Without extent (D_2), the statement 'E is at position p' has no referent — there is no space for p to be a position within.

Formally: $D_3: E \rightarrow p$ where $p \in S$ (some space). But S exists only if D_2 is operative. Therefore $D_2 \rightarrow D_3$.

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4.1.3 Proof: $D_3 \rightarrow D_4$

Lemma 4.3 (Forces Require Position)

The force function D_4 (enabling interaction/influence) requires the positional function D_3.

Proof.

Forces act between entities. The statement 'F acts between A and B' requires A and B to have positions such that 'between' is defined. All known forces (gravitational, electromagnetic, strong, weak) are functions of position — they depend on the distance or configuration of positioned entities.

Without D_3, entities have no positions. Without positions, there is no 'between,' no distance, and no configuration for forces to depend upon. Forces become undefined.

Therefore $D_3 \rightarrow D_4$.

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4.1.4 Proof: $D_4 \rightarrow D_5$

Lemma 4.4 (Quantum Requires Forces)

The quantum function D_5 (enabling superposition/indeterminacy) requires the force function D_4.

Proof.

Quantum superposition is a superposition of *states that can interact and be measured*. Measurement involves interaction between the measured system and the measuring apparatus. Interaction requires D_4.

Moreover, quantum states are defined relative to an energy basis, and energy is a property arising from forces (kinetic energy from motion through space, potential energy from force fields). Without D_4, the eigenstates that superposition superposes over are undefined.

Therefore $D_4 \rightarrow D_5$.

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4.1.5 Proof: $D_5 \rightarrow D_6$

Lemma 4.5 (Disorder Requires Quantum)
The disorder function D_6 (enabling perturbation/entropy) requires the quantum function D_5.

Proof.

Disorder (entropy, perturbation) requires something to be disordered — a state space with structure that can be perturbed. D_5 provides the probability space over states that allows entropy to be defined ($S = -\sum p_i \log p_i$).

Furthermore, the source of fundamental randomness in physics is quantum indeterminacy (D_5). Classical systems exhibit only deterministic chaos (sensitive dependence on initial conditions), but true randomness — the injection of genuine novelty — requires quantum effects.

Without D_5, entropy is merely epistemic (uncertainty about determinate states) rather than ontic (fundamental indeterminacy). D_6 as a fundamental function enabling genuine novelty requires D_5.

Therefore $D_5 \to D_6$.

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4.2 Independence Results

Theorem 4.2 (No Reverse Dependencies)

For all $i < j$: D_j does not depend on D_i being operative for D_i to operate.
Formally: $D_i \perp D_j$ for $i < j$.

Proof.

We verify each case:

• $D_1 \perp D_2$: Time can pass without spatial extent. A point-like entity can undergo change.

• $D_1 \perp D_3$: Time can pass without specific position. An entity with extent but no definite location can change.

• $D_1 \perp D_4$: Time can pass without forces. An isolated, non-interacting entity can change state.

• $D_1 \perp D_5$: Time can pass deterministically. Classical systems undergo change without superposition.

• $D_1 \perp D_6$: Time can pass without disorder. Perfectly ordered systems can undergo change.

Similar arguments apply for $D_2 \perp \{D_3,\ldots,D_6\}$, $D_3 \perp \{D_4,\ldots,D_6\}$, etc.

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5. Inverse Functions

5.1 The Dialectical Necessity

Theorem 5.1 (Necessity of Inverses)

For each fundamental function D_n, there must exist an inverse function D̄_n such that D_n and D̄_n together span the full operational space of that functional dimension.

Proof.

By Axiom 5 (Dialectical Completeness), each fundamental function D_n enables some category C of phenomena. For C to be dynamically operative (rather than statically fixed), there must be phenomena that resist, balance, or complement C. Otherwise, C would reach a fixed point and cease to generate dynamics.

We call the function enabling these complementary phenomena D̄_n. The pair (D_n, D̄_n) generates dynamics through their tension.

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5.2 Identification of Inverses

We identify each inverse and prove the pairing:

Pair	Function	Inverse	Dialectic
D_1/D_7	Becoming	Entropy	Direction of change
D_2/D_8	Extent	Contraction	Expansion vs. binding
D_3/D_9	Position	Delocalization	Definite vs. spread
D_4/D_{10}	Interaction	Isolation	Coupling vs. decoupling
D_5/D_{11}	Superposition	Collapse	Many-to-one reduction
D_6/D_{12}	Disorder	Order	Entropy vs. structure

5.2.1 The D_1/D_7 Dialectic (Becoming/Entropy)

Lemma 5.2 (Time's Arrow)

D_1 (enabling change) requires D_7 (entropy) to provide direction. Without D_7, D_1 would enable change with no preferred direction — time would be reversible.

Proof.

D_1 alone enables state change: $s_1 \rightarrow s_2$. But D_1 does not specify whether $s_1 \rightarrow s_2$ or $s_2 \rightarrow s_1$ is preferred. Fundamental physics equations are time-symmetric (CPT invariance). Yet we observe a clear arrow of time: entropy increases, eggs break but do not unbreak.

D_7 provides this direction. It biases state transitions toward higher entropy configurations, breaking the symmetry of D_1 and giving time its arrow.

The dialectic: D_1 enables change; D_7 directs it. Together they produce irreversible temporal flow.

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5.2.2 The D_3/D_9 Dialectic (Position/Delocalization)

Lemma 5.3 (Quantum Non-locality)

D_3 (definite position) and D_9 (delocalization) are inverses whose interplay explains quantum non-locality without superluminal signaling.

Proof.

Consider entangled particles A and B. Under the Copenhagen interpretation, each has indefinite position until measured. Under DFH interpretation: the pair exists in a D_9-dominant state — delocalized, not assigned definite positions by D_3.

When measurement occurs, D_3 is applied (localization). The D_3/D_9 balance shifts. The correlation between A and B's measured positions is not due to signaling but to their shared D_9 state prior to D_3 application.

No signal travels because no distance is crossed in D_9 space — 'distance' is a D_3 concept, and D_3 was suppressed.

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5.2.3 The D_5/D_{11} Dialectic (Superposition/Collapse)

Lemma 5.4 (Measurement as Dialectic)

The quantum measurement problem is resolved by recognizing collapse (D_{11}) as the inverse function to superposition (D_5), not as an external intervention.

Proof.

Standard quantum mechanics treats collapse as problematic — an unexplained process triggered by 'observation.' Under DFH, D_5 enables superposition; D_{11} enables collapse. Both are fundamental functions.

Measurement is any interaction where D_{11} dominates D_5. The 'observer' need not be conscious — any system carrying the full functional stack can induce D_{11} dominance.

The dialectic: D_5 spreads probability across states; D_{11} collapses to definite outcome. Their interplay produces quantum mechanics.

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6. Completeness Theorem

Theorem 6.1 (Completeness of the DFH)

Six fundamental functions and six inverse functions (twelve total) constitute a minimal complete basis for coherent reality. Fewer than six fundamentals is insufficient; more than six fundamentals introduces redundancy.

6.1 Sufficiency

Proof.

We demonstrate that D_1-D_6 and their inverses D_7-D_{12} enable all observed physical phenomena:

Classical Mechanics: D_1 (time evolution), D_2 (spatial extent), D_3 (position), D_4 (forces). Fully covered.

Thermodynamics: D_1 (temporal evolution), D_6 (entropy/disorder), D_7 (arrow of time), D_{12} (ordering/crystallization). Fully covered.

Quantum Mechanics: D_5 (superposition), D_{11} (collapse), D_3/D_9 (position/delocalization dialectic). Fully covered.

General Relativity: D_1-D_4 with D_2/D_8 (expansion/contraction of space) providing the dynamic metric. Fully covered.

Statistical Mechanics: D_5 (probability distributions), D_6 (entropy), D_{12} (pattern emergence). Fully covered.

No observed physical phenomenon requires a function outside D_1-D_{12}.

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6.2 Necessity (Minimality)

Proof.

We demonstrate that removing any function produces an incomplete system:

Without D_1: No change. Static existence only. Not coherent with observed reality.

Without D_2: No separation. All things at same 'point.' No structure possible.

Without D_3: No specific locations. Forces undefined (no 'between'). No mechanics.

Without D_4: No interaction. Entities isolated absolutely. No causal structure.

Without D_5: No indeterminacy. Perfectly deterministic universe. Violates observed quantum behavior.

Without D_6: No disorder. Perfect crystalline stasis. No thermodynamic arrow.

Each function is necessary; none is redundant. Similarly, each inverse is necessary to complete its dialectic (shown in Section 5).

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6.3 The Structural Constants

Corollary 6.1.1 (Structural Constants)

The DFH implies structural constants: ε = 6 (fundamental count), ω = 12 (complete operational set).

These are not arbitrary but emerge from the analysis: six categories of phenomena are distinguishable and necessary (Section 4), each requires an inverse for dynamic operation (Section 5), yielding 6 + 6 = 12.

7. Empirical Predictions and Testability

For a framework to be scientific rather than merely philosophical, it must make testable predictions. The DFH generates the following:

7.1 The Measurement Problem

Prediction 7.1

Any system carrying the complete functional stack (D_1-D_{12} operative) will induce quantum collapse when interacting with a superposed system. 'Observer' status is not privileged — it is functional completeness.

Test: Compare decoherence rates when superposed systems interact with functionally complete vs. functionally incomplete environments. The DFH predicts functional completeness determines collapse, not mass, complexity, or consciousness.

7.2 Time's Arrow

Prediction 7.2

The arrow of time (D_1/D_7 dialectic) should be reducible to entropic considerations at the functional level, not merely statistical mechanics.

Test: In systems where entropy is held constant (reversible processes), time's arrow should become ambiguous. This aligns with the time-symmetry of fundamental equations and explains why macroscopic irreversibility emerges from microscopic reversibility.

7.3 Quantum Non-locality

Prediction 7.3

Entanglement correlations are mediated by shared D_9 (delocalization) state, not hidden variables or superluminal communication.

Test: Bell inequality violations should be explainable as D_3/D_9 dialectic effects. The DFH predicts no modification to quantum mechanics predictions but provides a different interpretation of the mechanism.

7.4 Structural Resonances

Prediction 7.4

Mathematical structures involving 6, 12, and 24 dimensions should appear with unusual frequency in fundamental physics and mathematics, as these correspond to DFH structural constants.

Evidence: The Leech lattice (24 dimensions), exceptional Lie algebras (E_6, E_7, E_8), modular forms, and the Monster group's representation theory all exhibit these numbers prominently. This may be coincidence or structural resonance.

8. Open Problems

The following questions remain unresolved and are offered as challenges to the framework:

8.1 Quantitative Predictions

The DFH as presented is qualitative — it identifies functions and their relationships but does not derive numerical values for physical constants. Can the framework be extended to predict, e.g., the fine structure constant or particle masses?

8.2 Gravity

General relativity describes gravity as spacetime curvature. Under DFH, this should emerge from D_2/D_8 (extent/contraction) dynamics. The precise mechanism requires development.

8.3 The Primordial Ground

We posited Φ (existence itself) as the ground from which D_1 emerges. What is the nature of Φ? Is it analyzable, or is it the ultimate primitive?

8.4 Alternative Orderings

We proved a specific dependency ordering. Could an alternative ordering be consistent? If so, would it describe a different possible reality?

9. Conclusion

We have established the mathematical foundations of the Dimensional Function Hierarchy through formal definitions, explicit axioms, and rigorous proofs. The core results are:

1. D_1 (Time) is the unique primitive dimensional function (Theorem 3.1)

2. The dependency ordering $D_1 \to D_2 \to D_3 \to D_4 \to D_5 \to D_6$ is strict and total (Theorem 4.1)

3. Each fundamental function requires a dialectical inverse (Theorem 5.1)

4. Twelve functions (six fundamentals, six inverses) constitute a minimal complete basis (Theorem 6.1)

These proofs are offered for critical examination. The framework stands or falls on the validity of its axioms and the soundness of its derivations. We welcome challenges, counterexamples, and alternative formulations.

If the proofs hold, the DFH represents a genuine contribution to our understanding of dimensional structure. If they fail, the failure will illuminate what additional structure is needed.

Either outcome advances knowledge. That is the purpose of offering formal proofs.

Appendix: Summary of Formal Results

Result	Statement	Section
Theorem 3.1	D_1 is the unique primitive function	§3.2
Corollary 3.1.1	Computation witnesses D_1	§3.3
Theorem 4.1	Strict ordering: $D_1 \to D_2 \to D_3 \to D_4 \to D_5 \to D_6$	§4.1
Lemmas 4.1-4.5	Individual dependency proofs	§4.1.1-4.1.5
Theorem 4.2	No reverse dependencies	§4.2
Theorem 5.1	Necessity of inverse functions	§5.1
Lemmas 5.2-5.4	Specific dialectic proofs	§5.2.1-5.2.3
Theorem 6.1	Completeness of 12-function basis	§6
Corollary 6.1.1	Structural constants $\varepsilon=6$, $\omega=12$	§6.3

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