Module - III

1. Convergence Condition on Fourier Series :-

→ In general, there are several convergance conditions and theorems that ensure the convergance of a Fourier series for a given function

1. Periodic Function: The function must be periodic with a period T.
2. Piecewise Continuity: The function $f(t)$ should be piecewise continuous on the interval $[a, a+T]$. This means that $f(t)$ can have a finite number of discontinuities and a finite number of removable discontinuities.
3. Finite number of Extrema: The function should have a finite number of extrema in each period.
4. Finite variation: The function's total variation over one period should be finite.

If a function satisfies these conditions, then its Fourier series converges to the function in the mean at every point where the function is continuous and converges to the average of the left-hand and right-hand limits at each point of discontinuity. The fourier series can also converge to the function at points of discontinuity if the function has finite jumps.

* Dirichlet's condition for Fourier Series :-

→ Dirichlet's condition also known as Dirichlet's test or Dirichlet's condition for the convergence of Fourier series, Dirichlet's condition are a set of mathematical conditions that are used to determine when a Fourier series converges to the original function.

1. Periodicity: The function $f(x)$ must be piecewise continuous on the closed interval $[0, T]$ except for a finite number of discontinuities.

2. Bounded variation: The function $f(x)$ must be of bounded variation on the interval $[0, T]$. In other words, it should have a finite total variation on that interval.

3. Finite Number of Discontinuities: The function $f(x)$ can only have a finite number of discontinuities within the interval $[0, T]$. The discontinuities should be of the first kind, which means that they are jump discontinuities and the jumps should be finite.

When these conditions are satisfied, the Fourier series of the function $f(x)$ converges to $f(x)$ at every point of continuity of $f(x)$ and it converges to the midpoint of the jump discontinuities. Dirichlet's condition ensure that the Fourier series provides a good approximation to the original function in a piecewise continuous and bounded variation sense.

2. Fourier Series:-

The expression for a Fourier series represents a periodic function as a sum of sines and cosines with different frequencies and amplitudes. If we have a periodic function with a period T, The Fourier series representation is typically written

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)\right]$$

where,

$f(x)$ = periodic function

a_0 = average value of $f(x)$ over one period

a_n, b_n - Fourier coefficients

These coefficient represents the amplitudes and phases of the sinusoidal components in the Fourier series. The summation extends to infinity, and it includes all the harmonics with different frequencies, each weighted by its ~~repsusen~~ respective coefficient.

★ Coefficient of Fourier Series :-

→ Evaluating the coefficients of a Fourier series involves finding the specific values of the coefficients a_0, a_n and b_n that ~~repeat~~ represents the function $f(x)$ as a series of sines and cosines. over a given interval.

1. a_0 (Constant Term):

The a_0 coefficient represents the average value of the function $f(x)$ over one period. To calculate a_0,

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx \qquad T = \text{period of function.}$$

2. a_n (Cosine coefficient):

The a_n coefficients are associated with the cosine terms in the Fourier series. To find a_n,

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi n x}{T}\right)dx$$

3. b_n (Sine coefficient):

The b_n coefficients are associated with the sine terms in the Fourier series. $b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi n x}{T}\right)dx$

3 The trigonometric Fourier series for a periodic function $f(x)$ is given

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)\right)$$

We can use Euler's formula to express the cosine term as a combination of complex exponential:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Applying this to the trigonometric series, we get,

$$a_n \cos\left(\frac{2\pi n x}{T}\right) = a_n \cdot \frac{e^{i\frac{2\pi n x}{T}} + e^{-i\frac{2\pi n x}{T}}}{2}$$

Similarly, we can express the sine term as a combination of complex exponentials:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Applying this to the trigonometric series, we got

$$b_n \sin\left(\frac{2\pi n x}{T}\right) = b_n \cdot \frac{e^{i\frac{2\pi n x}{T}} - e^{-i\frac{2\pi n x}{T}}}{2i}$$

Combining the terms we get,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \frac{e^{i\frac{2\pi n x}{T}} + e^{-i\frac{2\pi n x}{T}}}{2} + b_n \cdot \frac{e^{i\frac{2\pi n x}{T}} - e^{-i\frac{2\pi n x}{T}}}{2i}\right)$$

Simplifying the terms,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{2\pi n x}{T}}$$

$$\begin{cases} \frac{a_n - i b_n}{2} & \text{for } n > 0 \\ \frac{a_0}{2} & \text{for } n = 0 \\ \frac{a_n + i b_{-n}}{2} & \text{for } n < 0 \end{cases}$$

Fourier transform Pair)

A Fourier transform pair is a pair of two functions in the domain of time and frequency that are related to each other by Fourier transform and its inverse.

The transform pair is defined as follows:

The Fourier transform of the original function $f(t)$ is given by $F(\omega)$ and is calculated as –

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

The inverse Fourier transform of $F(\omega)$ recovers the original function $f(t)$

$$f(t) = 1/2\pi \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega.$$

★ **Necessary Conditions for Fourier transform:**–

Function must be absolutely integrable) The function $f(x)$ must be absolutely integrable meaning its integral over the entire real line is finite –

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty$$

Sufficient conditions for Fourier transform:–

1. **Function of Finite total variation:** A function with finite total variation is a stronger condition than absolute integrability. It ensure that the function doesnot oscillate too rapidly. In mathematical terms:

$$\int_{-L}^{L} |f(x)| dx < \infty \text{ for all } L > 0$$

2. Piecewise Continuity: The function $f(x)$ should be piecewise continuous on real line, which means it can have a finite number of jump discontinuities and removable singularities.

3. Boundedness: Although not strictly necessary for the existance of the Fourier transform boundedness of the function can simplify the analysis and ensure that the Fourier transform is also bounded.

★ Merits & Demerits of Fourier Transform:-

Merits)

1) Frequency Analysis) The fourier transform provides a way to analyze a signal or function in the frequency domain. This is valuable for ~~domain~~ understanding the underlying frequency components of a signal, which is crucial in fields in such as signal processing.

2) Linear Transform) The fourier transform is a linear operation, which simplifies the analysis of linear systems. The response of a linear system to a signal can be calculated more easily in the frequency domain.

3) Convolution theorem) The fourier transform simplifies convolution operations, making them equivalent to multiplication in the frequency domain. This simplifies filtering and signal processing tasks.

Demerits

1) Limited applicability) The fourier transform is not suitable for all types of signals or functions. It

works best for signals that are well-behaved, such as those with finite energy or finite total variation. Some functions may not have Fourier transforms or may have highly complex transforms.

2) Temporal and Spatial Resolution> In the time-frequency duality increasing the temporal resolution decreases the frequency resolution. This can limit the precision of simultaneous time and frequency analysis.

3) Boundary effects> When analyzing finite-duration signals using the fourier transform, there can be boundary effects or spectral ~~knowledge~~ leakage that affect the accuracy of frequency components, especially if the signal is not periodic.

11) Duality property :-

If $f(t)$ is a function in the time domain and its Fourier transform is $F(\omega)$, then the Fourier transform of the function $F(\omega)$ is 2π times the function $f(-t)$.

$$\mathcal{F}\{F(\omega)\} = 2\pi f(-t)$$

In mathematical notation, this can be expressed as

$$\mathcal{F}\{\mathcal{F}\{f(t)\}\} = 2\pi f(-t)$$

Proof:

The fourier transform of $f(t)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$F\{F(\omega)\} = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega$$

Replacing $F(\omega)$ with its expression from original transform

$$F\{F(\omega)\} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-i\omega' t} dt \right) e^{-i\omega t} d\omega$$

$$F\{F(\omega)\} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{i\omega' t} \cancel{dt} e^{-i\omega t} dt \right) d\omega$$

$$F\{F(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega - \omega') d\omega$$

The result ~~is~~ of this integral is 2π when $\omega = \omega'$ and 0 otherwise –

$$F\{F(\omega)\} = 2\pi\delta(0) = 2\pi$$

$$F\{F(\omega)\} = 2\pi\delta(0) = 2\pi\delta(\omega)$$

Finally, replacing ω with $-t$ to obtain the desired result:

$$F\{F(\omega)\} = 2\pi\delta(-t) = 2\pi\delta(t)$$

This proves the duality property.

13. Fourier Transform of $u(t)$ using signum function.

→ The fourier transform of the unit step function $u(t)$ can be evaluated using the signum function,

$$\begin{cases} 1, & \text{for } t > 0 \\ 0, & \text{for } t < 0 \end{cases}$$

The signum function $sgn(t)$ is defined as :

$$\begin{cases} 1, & \text{for } t > 0 \\ 0, & \text{for } t = 0 \\ -1, & \text{for } t < 0 \end{cases}$$

To find the fourier transform of $u(t)$, ~~you~~ we can see that $u(t)$ is related to the signum function $sgn(t)$. Specifically the fourier transform of $u(t)$ denoted as $U(\omega)$ is given by,

$$U(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

$\left[\begin{array}{l} j = \text{imaginary unit } (j = -1) \\ \omega = \text{angular frequency} \\ \delta(\omega) = \text{dirac delta function} \end{array}\right]$

So, the fourier transform of $u(t)$ using the signum function

$$U(\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

15. The fourier transform of $u(t)$ is

$$U(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

Using the duality property, we can find the fourier transform of $x(t) = 1/t$ as follows.

$$X(\omega) = 2\pi U(-\omega)$$

we know that $U(\omega)$ is the dual of $u(t)$, it is related to $U(\omega)$ to $x(t)$ as $X(\omega) = 2\pi U(-\omega)$

$$X(\omega) = 2\pi\left(\frac{1}{j(-\omega)} + \pi\delta(-\omega)\right)$$

Simplifying it,

$$X(\omega) = -2\pi\left(\frac{1}{j\omega} + \pi\delta(-\omega)\right)$$

$$X(\omega) = -2\pi\left(\frac{1}{j\omega} - \pi\delta(\omega)\right)$$

So, the fourier transform of the signal $x(t) = \frac{1}{t}$ using the fourier transform of the signum function $U(\omega)$ is

$$X(\omega) = -2\pi\left(\frac{1}{j\omega} - \pi\delta(\omega)\right)$$

16. $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

Substituting it to fourier transform equation,

$$X(\omega) = \int_{-\infty}^{\infty} te^{-|t|} e^{-j\omega t} dt$$

Spliting into the integral into two parts because $e^{-|t|}$ is defined differently $t \geq 0$ and $t < 0$

1. For $t \geq 0$; $e^{-|t|} = e^{-t}$
2. For $t < 0$, $e^{-|t|} = e^{t}$.

Case 1 For $t \geq 0$

$$X_1(\omega) = \int_{0}^{\infty} te^{-t} e^{-j\omega t} dt$$

Simplifying this integral,

$$X_1(\omega) = \int_{0}^{\infty} te^{-(1+j\omega)t} dt$$

Case 2 For $t < 0$

$$X_2(\omega) = \int_{-\infty}^{0} te^{t} e^{-j\omega t} dt$$

Simplifying the integral,

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$$X_2(\omega) = \int_{-\infty}^{0} te^{-(1-j\omega)t} dt$$

Finally, the fourier transform of the signal $x(t) = te^{-|t|}$ is

$$X(\omega) = X_1(\omega) + X_2(\omega)$$

★ Using the fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Substituting $f(t) = \frac{2t}{(1+t^2)^2}$

$$F(\omega) = \int_{-\infty}^{\infty} \frac{2t}{(1+t^2)^2} e^{-j\omega t}\, dt$$

Applying the duality property:

The duality property states that the fourier transform of $F(\omega)$ is $2\pi f(-t)$

$$F(\omega) \longleftrightarrow 2\pi f(-t)$$

So, we need to find $2\pi f(-t)$. we can use the duality relationship

$$2\pi f(-t) = 2\pi \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

Now, we can substitute the expression for $F(\omega)$,

$$2\pi f(-t) = 2\pi \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{2t'}{(1+t'^2)^2} e^{-j\omega t}\, dt' \right) e^{j\omega t}\, d\omega$$

$$2\pi f(-t) = 2\pi \int_{-\infty}^{\infty} \frac{2t'}{(1+t'^2)^2}\, \delta(t-t')\, dt'$$

we evaluate the integral, recognizing that the delta function sets $t' = t$,

$$2\pi f(-t) = 2\pi \frac{2t}{(1+t^2)^2}$$

Simplifying,

$$2\pi f(-t) = \frac{4\pi t}{(1+t^2)^2}$$

So, the fourier transform of the function $f(t) = \frac{2t}{(1+t^2)^2}$ is given by,

$$F(\omega) = \frac{4\pi}{(1+\omega^2)^2}$$

8) The fourier transform of a signal $x(t)$ is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Substitute $x(t) = e^{-at} u(t)$ into the equation:

$$X(\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

Evaluating the integral,

$$X(\omega) = \frac{-1}{a+j\omega}(0-1) = \frac{1}{a+j\omega}$$

The magnitude and phase spectra of a fourier transform X(ω are given by.

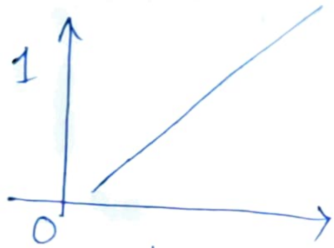
Magnitude Spectrum ($|X(\omega)|$):

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

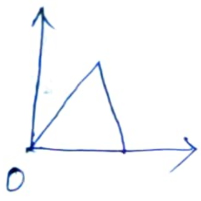
Phase spectrum ($\angle X(\omega)$):

$$\angle X(\omega) = \arctan\left(\frac{-\omega}{a}\right)$$

Plotting the magnitude spectrum.



Plotting the phase spectrum, ϕ



9) i) The fourier transform of the delta function $\delta(t)$ is a constan function:

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

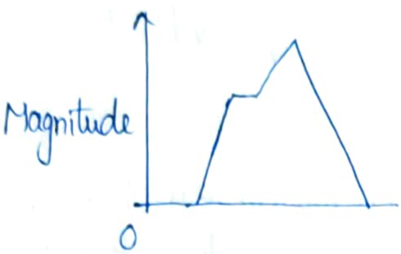
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

So, the fourier transform of $\delta(t)$ is

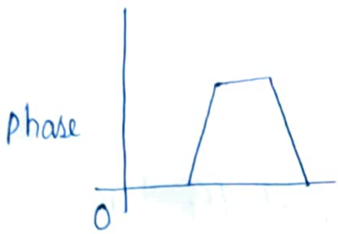
$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{0} = 1$$

The fourier transform of $\delta(t)$ is simply a constant function equal to 1

Magnitude spectrum:



Phase Spectrum:



ii) The inverse fourier transform of the delta function $\delta(\omega)$ in the frequency domain is the delta function $\delta(t)$ in the time domain. In other word's, it's a unit impulse located at $t=0$. The inverse fourier transform is given by.

$$f(t) = F^{-1}\{\delta(\omega)\} = \delta(t)$$

Here, F^{-1} denotes the inverse fourier transform.

So, the inverse fourier transform of $\delta(\omega)$ is a delta function $\delta(t)$ located at $t=0$.

10) a) Time Shifting :-

~~If X(ω) is the fourier~~

The time shifting property also known as the time domain shifting property is one of the fundamental properties of fourier transform.

If $X(\omega)$ is the fourier transform of the signal $x(t)$, then the fourier transform of a time-shifted signal $x(t-t_0)$ is given by,

$$F\{x(t-t_0)\} = X(\omega)e^{-j\omega t_0}$$

$F\{\cdot\}$ = fourier transform op.
$X(\omega)$ = fourier transform of the original signal $x(t)$.
t_0 = time shift

Proof :

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Now, we want to find the fourier transform of the time-shifted signal $x(t-t_0)$:

$$F\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\,e^{-j\omega t}\,dt$$

Substituting, $u = t - t_0$ which implies $t = u + t_0$

$$F\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(u)\,e^{-j\omega(u+t_0)}\,du$$

Now,
$$F\{x(t-t_0)\} = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(u)\,e^{-j\omega u}\,du$$

The final form is,

$$F\{x(t-t_0)\} = X(\omega)\,e^{-j\omega t_0}$$

Frequency Shifting:

The frequency shifting property also known as the modulation property ~~of~~ or frequency modulation property is a fundamental property of the fourier transform.

If $x(t)$ is a signal with fourier transform $X(\omega)$, and $X(\omega-\omega_0)$ is the fourier transform of $x(t)e^{j\omega_0 t}$ then,

$$X(\omega-\omega_0) = X(\omega)e^{-j\omega_0 t}$$

Proof

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Now, consider the signal $x(t)e^{j\omega_0 t}$,

$$X_1(\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}\,dt$$

Using the property of exponents, we can write the expression,

$$X_1(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}\,dt$$

Now, the fourier transform $X_1(\omega)$ is ~~the~~ related to the fourier transform $X(\omega)$ by a frequency shift of ω_0:

$$X_1(\omega) = X(\omega-\omega_0)$$

c. Time & Frequency Scaling:

The time and frequency scaling properties of the fourier transform are fundamental properties that describe how the transformation of a function is affected when the time domain signal is either stretched or compressed and how this relates to the frequency domain representation.

Time Scaling property

If $F(\omega)$ is the fourier transform of $f(t)$, then the fourier transform of $f(at)$, where a is a positive constant is given by:

$F(\omega/a)$

Proof >

$$F(at) = \int_{-\infty}^{\infty} f(at) e^{-j\omega t} dt$$

Substituting in the integral, $u = at$, so $du = a dt$

$$F(at) = \frac{1}{|a|} \int_{-\infty}^{\infty} f(u) e^{-j(\omega/a)u} du$$

This is the same as the fourier transform of $f(u)$ but with ω replaced by ω/a.

Frequency Scaling property:

If $F(\omega)$ is the fourier transform of $f(t)$, then the fourier transform of $f(t/a)$, where a is a positive constant is given by,

$|a| F(a\omega)$

Proof >

$$F(t/a) = \int_{-\infty}^{\infty} f(t/a) e^{-j\omega t} dt$$

Substituting in the integral, $v = t/a$, so $dv = (1/a) dt$

$$F(t/a) = \frac{1}{|a|} \int_{-\infty}^{\infty} f(v) e^{-j(a\omega)v} dv$$