Module - IV

1) Basic advantage of Laplace Transform over Fourier Transform.

→ Incorporation of initial conditions: The laplace transform is particularly useful for solving LTI systems with initial conditions. It allows for a smooth transition from the time domain to the laplace domain where initial conditions can be directly incorporated into the analysis. But fourier transform is primarily concerned with steady-state frequency components.

Analysis of transient behaviour: The laplace transform is well-suited for analyzing transient behaviour in dynamic systems.

Frequency and time-domain analysis: While the fourier transform is mainly focused on frequency domain analysis, the laplace transform provides a broader perspective.

Convolution: The laplace transform simplifies convolution operations. In the laplace domain, convolution corresponds to algebraic multiplication, making it easier to analyze linear time-invariant systems.

★ Laplace Transform:

The laplace transform of a function $f(t)$ is given by the integral of $f(t)$ multiplied by e^{-st}, where s is a complex number in the laplace transform.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$$ [$F(s)$ is the laplace transform of $f(t)$]

Inverse Laplace Transform :

The inverse laplace transform is used to transform a function from the laplace domain back to the time domain.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\gamma - j\infty}^{\gamma + j\infty} F(s)\, e^{st}\, ds$$

$f(t)$ = time domain function

$F(s)$ = laplace domain function

s = complex variable.

2) Region of Convergence (ROC) :-

The region of convergence is a concept in the field of signal processing and the theory of LTI (linear time invariant) systems. It is a set of values in the complex plane for which a given discrete-time signal or system's Z-transform converges, meaning that the series or integral representing the Z-transform is mathematically well and does not diverge to infinity.

Various properties of ROC :

Uniqueness : The ROC is unique for a given signal or system. It's a specific region in the complex plane that characterizes the convergence behaviour of the Z-transform.

Stability : A system is stable if the ROC includes the unit circle. In this case, the Z-transform is bounded for all values within the unit circle, which ensures that the system's response remains bounded for bounded inputs.

Extendibility: The ROC can often be extended infinitely in some directions in the complex plane. This is particularly relevant when dealing with infinite-duration sequences.

ROC dependence on casualty: Casual systems have ROCs that extend outward ~~with~~ from the pole with a finite extent while non-casual systems have ROCs that extend inward from the pole with a finite extent.

3. i) $x(t) = e^{-at}u(t)$ $a > 0$

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)e^{-st}\,dt$$

Substituting $x(t) = e^{-at}u(t)$

$$X(s) = \int_0^\infty e^{-at}u(t)e^{-st}\,dt$$

Simplifying,

$$X(s) = \int_0^\infty e^{-(a+s)t}\,dt$$

Integrating with respect to t,

$$X(s) = \frac{-1}{a+s}e^{-(a+s)t}\Big|_0^\infty$$

Taking limits as t approaches infinity and zero:

$$X(s) = \lim_{t\to\infty}\left(\frac{-1}{a+s}e^{-(a+s)t}\right) - \lim_{t\to 0}\left(\frac{-1}{a+s}e^{-(a+s)t}\right)$$

So, the laplace transform of $x(t) = e^{-at}u(t)$ is

$$X(s) = \frac{1}{a+s}$$

The ROC is a range of values for s for which the laplace transform converges. In this case, ROC for $\frac{1}{s+a}$ is the right half of the complex plane, $Re(s) > -a$. This is because the laplace transform involves the exponential term $e^{-(a+s)t}$ and for convergence, the real part of s must be greater than $-a$. The boundary $Re(s) = -a$ is typically included in the ROC.

So, the ROC for $X(s) = \frac{1}{s+a}$ is $Re(s) > -a$, which can be represented in the complex s-plane as the right-half plane.

ii) $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$ where $a > b$.

→ $e^{-at}u(t)$, the laplace transform of $e^{-at}u(t)$ can be found using the standard transform for the exponential function

$$\mathcal{L}\{e^{-at}u(t)\} = \int_0^\infty e^{-at}e^{-st}\,dt = \int_0^\infty e^{-(a+s)t}\,dt$$

Using laplace transform, we get.

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$

Now, we can find the laplace transform of the entire expression by summing the transforms:

$$\mathcal{L}\{x(t)\} = \mathcal{L}\{e^{-at}u(t)\} + \mathcal{L}\{e^{-bt}u(t)\} = \frac{1}{s+a} + \frac{1}{s+b}$$

The region of convergence for the laplace transform depends on the poles of the transform in the complex plane. The laplace transform is valid for s such that the real part of s is greater than the maximum real part of the poles in the laplace transform. In this case, the poles are at $s = -a$ and $s - b$

So, the laplace transform is the region, s is greater than both a and b. $Re(s) > a$ and $Re(s) > b$. ROC is $Re(s) > \max(a, b)$

Scaling property of Laplace transform :-

If $F(s)$ is the laplace transform of a function $f(t)$, then the laplace transform of the scaled function $f(at)$, where a is a positive constant, is given by:

$$\mathcal{L}\{f(at)\} = 1/a\, F(s/a)$$

Proof:

Laplace transform of the scaled function.

$$\mathcal{L}\{f(at)\} = \int_0^\infty f(at)\, e^{-st}\, dt$$

Substituting $u = at$, so $du = a\,dt$,

$$\mathcal{L}\{f(at)\} = \int_0^\infty f(u)\, e^{-s(u/a)}\, 1/a\, du$$

Simplifying the integral,

$$\mathcal{L}\{f(at)\} = 1/a\, F(s/a)$$

The laplace transform of $f(at)$ is indeed $1/a\, F(s/a)$, where $F(s)$ is the laplace transform of the original function $f(t)$.

5) $x(t) = e^{-t} u(t)$

The laplace transform is defined as

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)\, e^{-st}\, dt$$

Substituting, $x(t) = e^{-t} u(t)$

$$X(s) = \int_0^\infty e^{-t} u(t)\, e^{-st}\, dt$$

Splitting the integral into two parts,

$$X(s) = \int_0^{\infty} e^{-t} e^{-st} \, dt$$

Now, integrating the expression with respect to t.

$$X(s) = \int_0^{\infty} e^{-(1+s)t} \, dt$$

$$X(s) = \frac{-1}{1+s} e^{-(1+s)t} \Big|_0^{\infty}$$

Evaluating the limits of integration:

$$X(s) = \left(0 - \frac{-1}{1+s}\right) = \frac{1}{1+s}$$

So, the laplace transform of $x(t) = e^{-t} u(t)$ is

$$X(s) = \frac{1}{1+s}$$

★ Using $X(s)$ the inverse laplace transform of the expression $e^{-3s} X(2s)$

The inverse laplace transform of $F(as)$ is $f(t/a)$

Here, $X(s) = \frac{1}{s+1}$, Now applying the scalling property $X(2s)$.

$$X(2s) = \frac{1}{2} \cdot \frac{1}{s/2+1} = \frac{1}{2} \cdot \frac{2}{s+2} = \frac{1}{s+2}$$

Now we have,

$$\mathcal{L}^{-1}\{e^{-3s} X(2s)\} = \mathcal{L}^{-1}\left\{e^{-3s} \cdot \frac{1}{s+2}\right\}$$

we can use the inverse laplace transform of $e^{-as} F(s)$, which is $u_c(t-a) \cdot \mathcal{L}^{-1}\{F(s)\}$. where $u_c(t)$ is the unit step function.

n this case, $a = 3$ and $F(s) = \frac{1}{s+2}$

So,

$$\mathcal{L}^{-1}\{e^{-3s}X(2s)\} = u_3(t-3) * \mathcal{L}^{-1}\{\frac{1}{s+2}\}$$

The laplace transform of $\frac{1}{s+2}$ is e^{-2t}, so

$$\mathcal{L}^{-1}\{\frac{1}{s+2}\} = e^{-2t}$$

Now, we have.

$$\mathcal{L}^{-1}\{e^{-3s}X(2s)\} = u_3(t-3) * e^{-2t}$$

which is the inverse laplace transform of $e^{-3s}X(2s)$

6) i) $x(t) = -te^{-2t}u(t)$

We can use the laplace transform.

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)e^{-st}\,dt$$

Substituting $x(t) = -te^{-2t}u(t))$

$$X(s) = \int_0^\infty (-te^{-2t}u(t))e^{-st}\,dt$$

integrating the expression with respect to t

$$X(s) = \int_0^\infty te^{-(2+s)t}\,dt$$

$$\int u\,dv = uv - \int v\,du$$ (Integration by parts)

$u = t \Rightarrow du = dt$

$dv = e^{-(2+s)t}\,dt \Rightarrow v = -\frac{1}{2+s}e^{-(2+s)t}$

Now applying integration by parts.

$$X(s) = -\left[t\left(-\frac{1}{2+s}e^{-(2+s)t}\right) - \int\left(-\frac{1}{2+s}e^{-(2+s)t}\right)dt\right]_0^\infty$$

Evaluating this expression at the limits

$$X(s) = \left[0 - \left(0 - \frac{1}{2+s}\right)\right] \cdot \frac{1}{2+s}$$

So, the laplace transform of $x(t) = -te^{-2t}u(t)$ is

$$X(s) = \frac{1}{2+s}$$

ii) $Y(s) = x_1(t-2) \cdot x_2(-t+3)$

we can first express $y(t)$ in terms of the given functions $x_1(t)$ and $x_2(t)$.

Given in the question,

$$x_1(t) = e^{-2t}u(t)$$
$$x_2(t) = e^{-3t}u(t)$$

We need to compute $y(t) = x_1(t-2) * x_2(-t+3)$

$$y(t) = e^{-2(t-2)}u(t-2) \cdot e^{-3(-t+3)}u(-t+3)$$

Simplifying the expression,

$$y(t) = e^{-2t+4}u(t-2) \cdot e^{3t-9}u(t-3)$$

Using unit step function, $y(t) = e^{-2t+4}u(t-2)u(t-3)$

$$Y(s) = \mathcal{L}\{e^{-2t+4}u(t-2)u(t-3)\}$$

Laplace transform e^{-2t+4}

$$\mathcal{L}\{e^{-2t+4}\} = \frac{1}{s+2}e^{4/s}$$

Laplace transform of $u(t-2)$

This is simply $\frac{e^{2s}}{s}$ as time-delayed step function.

$$Y(s) = \frac{1}{s+2}e^{4/s} \cdot \frac{e^{2s}}{s} \cdot \frac{e^{3s}}{s}$$

Simplifying and combining terms.

$$Y(s) = \frac{e^{4/s}}{s^3(s+2)}$$

This is the laplace transform of $y(t)$.