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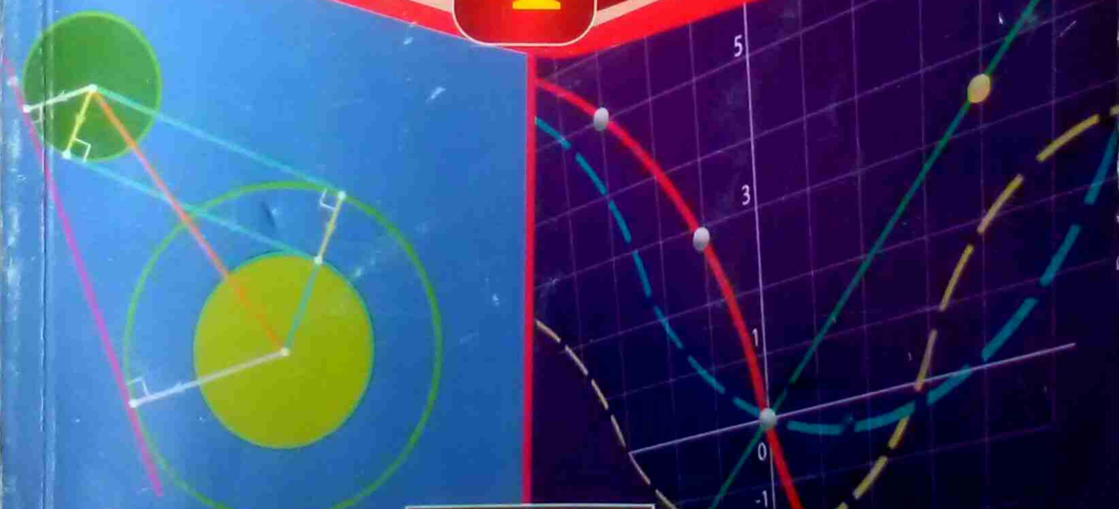
Senior Secondary

Mathematics

Student's Book

Form 4



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Contents

1 MATRICES

Success criteria

By the end of this topic, the student must be able to:
- *Describe matrices.*
- *Carry out basic operations involving matrices.*

Introduction to Matrices

There are many instances when large quantities of numeric information have to be stored. Very often this information is arranged in the form of tables, which is one of the most convenient ways of arranging information.

Table 1.1 shows how different types of packets of biscuits are packed in a certain factory.

Name of packet	Type of biscuit			
	A	B	C	D
Economy	14	14	10	10
Family	5	8	9	14
Standard	8	3	7	6

Table 1.1

This means, for example, that the "Economy" packet contains 14 type A, 14 type B, 10 type C and 10 type D biscuits.

With time, the packers get to know what each row and each column refers to, and they need to remember only the patterns:

$$\begin{pmatrix} 14 & 14 & 10 & 10 \\ 5 & 8 & 9 & 14 \\ 8 & 3 & 7 & 6 \end{pmatrix}$$

Such an arrangement is called a **matrix** (plural: **matrices**).

Thus:

A **matrix** is a rectangular array of numbers arrange in rows and columns, and whose value and position in the arrangement is significant.

A matrix is usually shown in curl or square brackets as:

$$\begin{pmatrix} 1 & 8 \\ 3 & 7 \end{pmatrix} \text{ or } \begin{bmatrix} 1 & 8 \\ 3 & 7 \end{bmatrix}$$

Normally, a capital letter in bold type e.g. **A**, **B**, is used to denote a matrix. However, it is difficult to bold in our normal hand writing, we use a wavy underline on the letter, e.g. $\underset{\sim}{A}$, $\underset{\sim}{B}$.

Each number in a matrix is called an **element** of the matrix. For example in the above matrix, 1,8 3 and 7 are elements.

Order of a matrix

A convenient way of describing the shape or size of a matrix is by using rows and columns. For example, the matrix from Table 1.1 has 3 rows and 4 columns. It is said to be a matrix of **order** 3×4 (read "three by four") or a "three by four matrix".

Thus:

The **order** of a matrix denotes the number of rows and columns in the matrix. A matrix of order $m \times n$ has m rows and n columns.

How many rows and columns does a matrix of order 4×3 have?

State the order of each of the following matrices.

1. $\begin{pmatrix} 7 & 0 & 0 \\ 2 & -1 & 5 \end{pmatrix}$
2. $\begin{pmatrix} 7 & 0 & 3 \\ 1 & -4 & 4 \\ -5 & 2 & 4 \end{pmatrix}$
3. $\begin{pmatrix} 5 & 5 & 9 \end{pmatrix}$
4. $\begin{pmatrix} 4 \\ 8 \\ 9 \end{pmatrix}$

Types of matrices

Matrices 1 to 4 on the previous page are rectangular, square, row and column matrices respectively.

- A matrix of order $1 \times n$ is called a **row matrix**. An example is matrix 3 on page 1.
- Matrix of order $m \times 1$ is called a **column matrix**. An example is matrix 4 on page 1.
- Matrix of order $n \times n$ is called a **square matrix**. An example is matrix 2 on page 1, which is a 3×3 square matrix. Other examples of square matrices are of orders 2×2, 4×4, 5×5 and so on.

Activity 1.1

Make up matrices having order

(a) 1×3 (b) 5×1 (c) 3×4 (d) 5×2 (e) 2×2

A matrix whose elements are zeros is called a **zero** or **null matrix**.

A matrix which has each elements in the leading diagonal as ones and all the other elements as zeros e.g $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called an **identity matrix**.

The **position of an element** inside a matrix is described using suffices (plural of suffix). Thus, if **A** is a matrix and $a_{m,n}$ is an element in it, then $a_{m,n}$ is the element in the m^{th} row and n^{th} column.

A **suffix** is also known as a **subscript.**

Exercise 1.1

1. Table 1.2 shows the number of times that three couples attended various types of entertainment in one year.

Type of entertainment	Couple		
	The Pambukas	The Umis	The Tsokas
Cinema	7	2	5
Dance	1	2	9
Play	5	8	1
Circus	0	3	2

Table 1.2

(a) Write down the information in the table in the form of a matrix and state the order of the matrix.

(b) Write the Umis' attendance as a column matrix. What is the order of this matrix?

(c) Write, as a row matrix, the number of times that plays have been attended, and state the order of the matrix.

2. Make up a matrix of order:

(a) 3×5 (b) 5×4 (c) 3×1

(d) 1×2 (e) 2×2 (f) 3×3

3. How many elements are there in a matrix of order:

(a) 2×4 (b) 4×2 (c) 4×4

(d) 1×3 (e) 1×1 (f) $m \times n$?

4. Given $\mathbf{A} = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 2 & 1 & {}^-1 & {}^-2 \\ {}^-4 & {}^-3 & 0 & 7 \end{pmatrix}$, what is the element: (a) $a_{1,3}$ (b) $a_{2,1}$ (c) $a_{3,4}$ (d) $a_{3,3}$?

5. Write down, as $a_{m,n}$, the following elements of **A** in Question 4.

(a) ${}^-1$ (b) ${}^-4$ (c) 3 (d) 1

6. Three salesgirls sold the following numbers of bottles of perfume on a certain day:

Ivy sold 9 bottles of *She*, 13 of *Rosy* and 6 of *Shield*.

Alinate sold 8 bottles of *Yu*, 7 of *Rosy* and 10 of *Shield*.

Pempho sold 15 bottles of *Yu*, 1 of *She* and 18 of *Rosy*.

Show this information in a 3×4 matrix.

Addition and subtraction of matrices

Over a period of two weeks, two families used the amounts of bread, milk and sugar shown in Table 1.3 below.

Item	Week 1		Week 2	
	Kasiya's family	Mlenga's family	Kasiya's family	Mlenga's family
Bread (loaves)	7	12	7	16
Milk (litres)	10	14	13	16
Sugar (kg)	2	3	2	4

Table 1.3

Regard the table as two 3 × 2 matrices. We will use these to complete the 3 × 2 matrix shown below, giving the total amounts of each item used by each family in the two weeks.

$$\begin{pmatrix} 7 & 12 \\ 10 & 14 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 7 & 16 \\ 13 & 16 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 7+7 & 12+16 \\ 10+13 & 14+16 \\ 2+2 & 3+4 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 28 \\ 23 & 30 \\ 4 & 7 \end{pmatrix}$$

To do this you must *add* together the elements in corresponding positions in the first two matrices. This is how matrices are added.

The method of *subtraction* follows the same pattern as that of addition. For, example, to find out *how much more* of each food item that the families in Table 1.3 used in the second week than in the first week, each quantity in the first matrix is subtracted from the corresponding quantity in the second matrix.

$$\text{i.e. } \begin{pmatrix} 7 & 16 \\ 13 & 16 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 7 & 12 \\ 10 & 14 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 7-7 & 16-12 \\ 13-10 & 16-14 \\ 2-2 & 4-3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 4 \\ 3 & 2 \\ 0 & 1 \end{pmatrix}$$

Compatibility in addition or subtraction

Matrices can be added or subtracted only if they are of the **same order**. Such matrices are said to be **compatible** for addition or for subtraction.

The resulting matrix is of the same order.

The operation is done as follows:

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \pm \begin{pmatrix} g & h & i \\ j & k & l \end{pmatrix} = \begin{pmatrix} a \pm g & b \pm h & c \pm i \\ d \pm j & e \pm k & f \pm l \end{pmatrix}$$

Example 1.1

If $A = \begin{pmatrix} 2 & -5 & 7 \\ -3 & 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 7 & -2 \\ 3 & -2 & -4 \end{pmatrix}$ *and* $C = \begin{pmatrix} 4 & 6 \\ 6 & 2 \\ 1 & 5 \end{pmatrix}$, *find:*

(a) $A + B$ (b) $B + A$

(c) $A - B$ (d) $B - A$

(e) $A + C$ (f) $C - B$

Solution

(a) $A + B = \begin{pmatrix} 2 & -5 & 7 \\ -3 & 2 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 7 & -2 \\ 3 & -2 & -4 \end{pmatrix}$

$= \begin{pmatrix} 2+3 & -5+7 & 7+-2 \\ -3+3 & 2+-2 & 4+-4 \end{pmatrix}$

$= \begin{pmatrix} 5 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $B + A = \begin{pmatrix} 3 & 7 & -2 \\ 3 & -2 & -4 \end{pmatrix} + \begin{pmatrix} 2 & -5 & 7 \\ -3 & 2 & 4 \end{pmatrix}$

$= \begin{pmatrix} 3+2 & 7+-5 & -2+7 \\ 3+-3 & -2+2 & -4+4 \end{pmatrix}$

$= \begin{pmatrix} 5 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

(c) $A - B = \begin{pmatrix} 2 & -5 & 7 \\ -3 & 2 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 7 & -2 \\ 3 & -2 & -4 \end{pmatrix}$

$= \begin{pmatrix} 2-3 & -5-7 & 7--2 \\ -3-3 & 2--2 & 4--4 \end{pmatrix}$

$= \begin{pmatrix} -1 & -12 & 9 \\ -6 & 4 & 8 \end{pmatrix}$

(d) $B - A = \begin{pmatrix} 3 & 7 & -2 \\ 3 & -2 & -4 \end{pmatrix} - \begin{pmatrix} 2 & -5 & 7 \\ -3 & 2 & 4 \end{pmatrix}$

$= \begin{pmatrix} 3-2 & 7-{}^-5 & {}^-2-7 \\ 3-{}^-3 & {}^-2-2 & {}^-4-4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 12 & {}^-9 \\ 6 & {}^-4 & {}^-8 \end{pmatrix}$

(e) $A + C = \begin{pmatrix} 2 & {}^-5 & 7 \\ {}^-3 & 2 & {}^-4 \end{pmatrix} + \begin{pmatrix} 4 & 6 \\ 6 & 2 \\ 1 & 5 \end{pmatrix}$

$= \begin{pmatrix} 2+4 & {}^-5+6 & 7+?? \\ {}^-3+6 & 2+2 & 4+?? \\ ??+1 & ??+5 & ??+?? \end{pmatrix}$

= *Not compatible*

(f) $C - B = \begin{pmatrix} 4 & 6 \\ 6 & 2 \\ 1 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 7 & {}^-2 \\ 3 & {}^-2 & {}^-4 \end{pmatrix}$

$= \begin{pmatrix} 4-3 & 6-7 & ??-2 \\ 6-3 & 2-2 & ??+4 \\ 1-?? & 5-?? & ??-4 \end{pmatrix}$

= *Not compatible*

From Example 1.1, we notice that:

Matrix addition is commutative, i.e

$A + B = B + A$, but

Matrix subtraction is non-commutative, i.e.

$A - B \neq B - A$.

Exercise 1.2

1. Add the following pairs of matrices where possible.

   (a) $\begin{pmatrix} -3 & 2 \\ 4 & 0 \end{pmatrix}$ and $\begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 5 \\ 5 & 4 \end{pmatrix}$ and $\begin{pmatrix} -5 & 4 & 1 \\ 0 & 5 & 2 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 1\frac{1}{2} \\ 2\frac{1}{2} & 3 \\ \frac{2}{3} & 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} & 1\frac{1}{2} \\ -3 & 2\frac{1}{2} \\ \frac{5}{6} & 2 \end{pmatrix}$

   (d) $(3 \;\; {}^-2 \;\; 5)$ and $\begin{pmatrix} -1 \\ 4 \\ {}^-5 \end{pmatrix}$

   Explain why in some cases it is not possible to add.

2. Work out the following, where possible.

   (a) $\begin{pmatrix} 3 & 2 & 5 & 6 \\ 4 & 6 & 2 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 4 & 2 & 1 \\ 4 & 3 & 5 & 5 \end{pmatrix}$

   (b) $\begin{pmatrix} 4 \\ 3 \\ {}^-2 \end{pmatrix} - \begin{pmatrix} 5 \\ {}^-2 \\ {}^-4 \end{pmatrix}$ (c) $\begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$

   (d) $\begin{pmatrix} 4 & 8 \\ 9 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} - \begin{pmatrix} 6 & 4 \\ 5 & 3 \end{pmatrix}$

   (e) $(3 \;\; 5 \;\; {}^-4) + (2 \;\; 5 \;\; 5) - \begin{pmatrix} 3 \\ 2 \\ {}^-1 \end{pmatrix}$

   (f) $(5 \;\; 4 \;\; 5) - (3 \;\; -2 \;\; -4) + (3 \;\; -5 \;\; -2)$

3. Write down any three matrices **A**, **B** and **C** which have the same order. Work out:

   (a) $A + (B + C)$ (b) $(A + B) + C$

   (c) Now complete the following statement about the implied property of matrix addition in (a) and (b). Matrix addition is ______.

4. Copy the following, replacing the stars with appropriate values.

   (a) $\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} * & * \\ * & * \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 1 \\ {}^-2 & 0 \\ 4 & {}^-7 \end{pmatrix} - \begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{pmatrix}$

5. If $\begin{pmatrix} 2 & 6 \\ 7 & {}^-7 \end{pmatrix} + \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ a & {}^-4 \end{pmatrix}$, find the values of a and b.

6. In relation to matrix addition or subtraction, what does it mean to say "the matrices are incompatible"?

Scalar multiplication of matrices

If $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$, $\mathbf{A} + \mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 8 \\ 4 & 10 \end{pmatrix}$$

But $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$.

Thus, we see that $2\mathbf{A} = 2\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 2 \times 3 & 2 \times 4 \\ 2 \times 2 & 2 \times 5 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 8 \\ 4 & 10 \end{pmatrix}$$

The number, for example 2, multiplying the matrix is called a **scalar**.

To multiply a matrix by a scalar, we multiply each element of the matrix by the scalar.

e.g. $k\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} ka & kb & kc \\ kd & ke & kf \end{pmatrix}$

Example 1.2

Work out $4\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

Solution

$$4\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 \times 1 & 4 \times 4 \\ 4 \times 3 & 4 \times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 16 \\ 12 & 8 \end{pmatrix}$$

Example 1.3

Find the values of m, n, p and q if

$$3\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

Solution

$$3\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 12 \\ 6 & 15 \end{pmatrix} - \begin{pmatrix} 4 & 8 \\ 12 & 16 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 \\ -6 & -1 \end{pmatrix}$$

Thus, $\begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -6 & -1 \end{pmatrix}$

Hence, $m = 5$, $n = 4$, $p = -6$ and $q = -1$.

Two matrices are equal if they are of the same order and their corresponding elements are equal.

Exercise 1.3

1. Given that $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix}$, find:

   (a) $3\mathbf{A}$ (b) $^-4\mathbf{C}$

   (c) $\mathbf{A} + 2\mathbf{B}$ (d) $\mathbf{B} - 2\mathbf{C}$

2. Find k if, $k\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} + 3\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 12 & -6 \end{pmatrix}$.

3. Find the matrix **M** for which:

   (a) $\begin{pmatrix} 3 & 4 \\ -2 & 0 \end{pmatrix} + 2\begin{pmatrix} -4 & 2 \\ 7 & -1 \end{pmatrix} = 4\mathbf{M}$

   (b) $\begin{pmatrix} 1 & 9 \\ 6 & 2 \end{pmatrix} - 2\mathbf{M} = \begin{pmatrix} 7 & 3 \\ 8 & 6 \end{pmatrix} + \mathbf{M}$

4. Find x and y if,

   $$5\begin{pmatrix} x & 3 \\ 7 & 3 \end{pmatrix} - 2\begin{pmatrix} x & 4 \\ y & 2 \end{pmatrix} = \begin{pmatrix} 18 & 7 \\ 29 & 11 \end{pmatrix}.$$

5. Find the unknowns in $\begin{pmatrix} p + q + r \\ p + 2q \\ 3p \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 6 \end{pmatrix}$.

Multiplication of matrices

Multiplication of row and column matrices

We have seen how matrices are added or subtracted. Sometimes, it is necessary to combine matrices in a different way. Now consider the following example.

Example 1.4

Mrs. Mandondo bought 2 kg of meat at K 600 per kilogram and 3 tins of milling at K 100 per tin. How much did she spend?

Solution

Two kinds of information are given: the quantities of food bought and their costs. This information can be shown in a matrix form as:

M represents milk and U represents milling

$$\text{Quantity}\ \begin{array}{cc} M & U \\ (2 & 3) \end{array} \qquad \text{Price}\ \begin{array}{cc} M & U \\ (600 & 100) \end{array}$$

To find the total cost, we calculate it as follows:

$$\begin{aligned} \text{Cost} &= K\,(2 \times 600 + 3 \times 100) \\ &= K\,(1\,200 + 300) \\ &= K\,1\,500 \end{aligned}$$

When using matrices, this calculation is written in the form

$$\begin{aligned} (2\ \ 3)\begin{pmatrix} 600 \\ 100 \end{pmatrix} &= (2 \times 600 + 3 \times 100) \\ &= (1\,500) \end{aligned}$$

The matrix (1 500) in Example 1.4 is known as the **product** of the two matrices.

Note: 1. The first matrix is written as a row matrix and the second as a column matrix but in the same order.

2. No multiplication symbol is placed between the two matrices.

Notice that:

> To combine a row matrix and a column matrix with the same number of elements, we multiply their corresponding elements and add, i.e. $(a\ \ b)\begin{pmatrix} x \\ y \end{pmatrix} = (ax + by)$.

Example 1.5

In a national soccer league, the result of two soccer clubs, Tigers FC and Juke Box FC, were as shown in Table 1.4

Club	*Won*	*Drawn*
Tigers FC	*8*	*2*
Juke Box FC	*7*	*4*

Table 1.4

If three points are awarded when a match is won and 1 point when it is drawn, use matrices to find the total number of points obtained by each club.

Solution

W D

$$\begin{aligned} (8\ \ 2)\begin{pmatrix} 3 \\ 1 \end{pmatrix} &= (8 \times 3 + 2 \times 1) \\ &= (26) \end{aligned}$$

Points per win

Points per draw

$$\begin{aligned} (7\ \ 4)\begin{pmatrix} 3 \\ 1 \end{pmatrix} &= (7 \times 3 + 4 \times 1) \\ &= (25) \end{aligned}$$

Thus, Tigers FC had 26 points and Juke Box FC has 25 points.

Give a reason why we cannot find the product

$$(8\ \ 2\ \ 2)\begin{pmatrix} 3 \\ 1 \end{pmatrix}?$$

Exercise 1.4

1. Where possible, work out the following:

(a) $(0\ \ 2\ \ 1)\begin{pmatrix} 12 \\ 5 \\ 2 \end{pmatrix}$ (b) $(-5\ \ 1\ \ -2)\begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$

(c) $(5\ \ 5)\begin{pmatrix} 6 \\ 5 \end{pmatrix}$ (d) $(3\ \ -6\ \ 5)\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

(e) $(0\ \ 3)\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ (f) $(6\ \ 2\ \ 3\ \ 4)\begin{pmatrix} -2 \\ 6 \\ -4 \\ 3 \end{pmatrix}$

2. Find the value (or values) of the unknown in each of the following cases.

(a) $(x\ \ 5)\begin{pmatrix} 2 \\ -4 \end{pmatrix} = (-24)$

(b) $(x\ \ -5)\begin{pmatrix} 2 \\ x \end{pmatrix} = (9)$

(c) $(2\ \ x\ \ 5)\begin{pmatrix} x \\ 3 \\ 4 \end{pmatrix} = (28)$

(d) $(3\ \ 0\ \ x)\begin{pmatrix} 5 \\ 1 \\ x \end{pmatrix} = (40)$

x

3. A wholesaler sells salt in packets of two sizes: small and large. The amounts of salt contained in these packets are 500 g and 1 kg. A retailer bought them at K 40 and K 85 respectively.
   (a) Write down two column matrices, one for the amount and another for the cost.
   (b) The retailer ordered 2 dozens of large and $3\frac{1}{2}$ dozens of small packets. Write this information as a row matrix.
   (c) By multiplying matrices, calculate the total amount of salt ordered.
   (d) How much did the order cost?

4. A farmer took the following to the market: 5 boxes of cassava and 5 sacks of cabbages. He sold the vegetable at the rates of K 700 per box and K 450 per sack respectively. Use a matrix to find how much the farmer got.

5. It costs an average K 24 to feed a goat per day and K 60 to feed a cow per day. A farmer has 5 cows and 15 goats and another has 10 cows and 10 goats. Find the difference in their expenditure per day on feeding of their animals.

Multiplying 2 × 2 matrices

A snack shop owner uses a computer to store data, in a matrix form. The owner can obtain information about the sales at one of his shops where he sells doughnuts and hotdogs.

$$(62,\ 27) \text{ and } \begin{pmatrix} 2.0 \\ 1.65 \end{pmatrix} \begin{array}{l} \to \text{cost of 1 doughnut in \$} \\ \to \text{cost of 1 hotdog in \$} \end{array}$$

62: number of doughnuts sold; 27: number of hot dogs sold

The expression 62 (2.0) + 27(1.65) is used to calculate

number of doughnuts (62), cost of 1 doughnuts (2.0), number of hotdogs (27), cost of 1 hotdog (1.65)

the total sales at that outlet.

If more than one shop was involved, the data can be recorded as a matrix

Number of doughnuts, number of hotdogs

$$\begin{array}{l} \text{Shop A} \\ \text{Shop B} \end{array} \begin{pmatrix} 62 & 27 \\ 49 & 83 \end{pmatrix}$$

To calculate the sales of each shop we multiply the matrices

$$\begin{pmatrix} 62 & 27 \\ 49 & 83 \end{pmatrix} \begin{pmatrix} 2.0 \\ 1.65 \end{pmatrix} = \begin{pmatrix} 62 \times 2 + 27 \times 1.65 \\ 49 \times 2 + 83 \times 1.65 \end{pmatrix}$$

$$= \begin{pmatrix} 124 + 44.55 \\ 98 + 136.95 \end{pmatrix}$$

$$= \begin{pmatrix} 168.55 \\ 234.95 \end{pmatrix} \begin{array}{l} \to \text{this component gives the total sales of shop A} \\ \to \text{total sales of shop B} \end{array}$$

The example above suggest a method for multiplying matrices.

For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

1st row, 1st column

i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} \square & \\ & \end{pmatrix}$ ← component in 1st row, 1st column

2nd row, 1st column

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} & \\ \square & \end{pmatrix}$ ← component in 2nd row, 1st column

The remaining elements can be found in a similar way.

1st row of the first matrix combined with 2nd column of the second matrix gives the component in 1st row and 2nd column.

Similarly, 2nd row of first matrix and 2nd column of second matrix gives the component in 2nd row, 2nd column.

Similar procedure is followed to find the product of matrices of other orders.

Example 1.6

Mrs. Pempho bought 2 kg of meat at K 600 per kilogram and 3 tins of milling at K 100 per tin.

At the same time and at the same store, Mrs. Ntambo bought 3 kg of meat and 2 tins of milling. On a different day, the two ladies bought the same quantities of food items at a store where the prices were K 650 per kilogram of meat and K 80 per tin of milling. Use matrix method to find how much each lady spent at each place.

Solution

Mrs. Pempho, 1st store:

$$(2\ \ 3)\begin{pmatrix}600\\100\end{pmatrix} = (2 \times 600 + 3 \times 100)$$
$$= (1\ 500) \text{ i.e. she spent K 1 500.}$$

Mrs. Pempho, 2nd store:

$$(2\ \ 3)\begin{pmatrix}650\\80\end{pmatrix} = (2 \times 650 + 3 \times 80)$$
$$= (1\ 540) \text{ i.e. she spent K 1 540.}$$

Mrs. Ntambo, 1st store:

$$(3\ \ 2)\begin{pmatrix}600\\100\end{pmatrix} = (3 \times 600 + 2 \times 100)$$
$$= (2\ 000) \text{ i.e. she spent K 2 000.}$$

Mrs. Ntambo, 2nd store:

$$(3\ \ 2)\begin{pmatrix}650\\80\end{pmatrix} = (3 \times 650 + 2 \times 80)$$
$$= (2\ 110) \text{ i.e. she spent K 2 110.}$$

Note that, in calculating Mrs. Pempho's expenditure, the first matrix is the same in each case. We can combine these multiplications and write for Mrs. Pempho.

$$(2\ \ 3)\begin{pmatrix}600 & 650\\100 & 80\end{pmatrix} = (1\ 500\ \ 1\ 540)$$

Likewise, we can write for Mrs. Ntambo:

$$(3\ \ 2)\begin{pmatrix}600 & 650\\100 & 80\end{pmatrix} = (2\ 000\ \ 2\ 110)$$

Finally, since the matrix giving the prices is the same in each case, we can now represent the whole calculation by a single product as:

$$\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}\begin{pmatrix}600 & 650\\100 & 80\end{pmatrix} = \begin{pmatrix}1\ 500 & 1\ 540\\2\ 000 & 2\ 110\end{pmatrix}$$

We note that:

In working out the product of two matrices, the **rows of the left-hand matrix** are combined, in turns, with the **columns of the right-hand matrix**.

Example 1.7

If $\mathbf{A} = \begin{pmatrix}3 & 2\\0 & ^-2\end{pmatrix}$ *and* $\mathbf{B} = \begin{pmatrix}2 & ^-1\\^-2 & 4\end{pmatrix}$, *work out the product* ***AB***.

Solution

$$\mathbf{AB} = \begin{pmatrix}3 & 2\\0 & ^-2\end{pmatrix}\begin{pmatrix}2 & ^-1\\^-2 & 4\end{pmatrix}.$$

To work out this product, we combine the first row of ***A*** *with the columns of* ***B*** *in turn to give*

$$\begin{pmatrix}3 & 2\\ * & *\end{pmatrix}\begin{pmatrix}2 & ^-1\\^-2 & 4\end{pmatrix} = \begin{pmatrix}2 & 5\\ * & *\end{pmatrix}.$$

We then combine the second row of ***A*** *with the columns of* ***B*** *to give*

$$\begin{pmatrix}* & *\\0 & -2\end{pmatrix}\begin{pmatrix}2 & -1\\-2 & 4\end{pmatrix} = \begin{pmatrix}* & *\\-4 & -8\end{pmatrix}.$$

Thus, $\begin{pmatrix}3 & 2\\0 & -2\end{pmatrix}\begin{pmatrix}2 & ^-1\\-2 & 4\end{pmatrix} = \begin{pmatrix}2 & 5\\-4 & -8\end{pmatrix}.$

In the product **AB**, we say that **B** has been pre-multiplied by **A**.

If the product matrix **AB**, which is in Example 1.7, is $\begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}$, each of its elements is obtained as shown in Table 1.5.

Elements of product matrix	Got by combining	
	Row of 1st matrix	Column of 2nd matrix
a_{11}	1st row	1st column
a_{12}	1st row	2nd column
a_{21}	2nd row	1st column
a_{22}	2nd row	2nd column

Table 1.5

This enables us to write down any required element in the product immediately when dealing with more complicated products.

Example 1.8

If $\mathbf{C} = \begin{pmatrix} 4 & 2 \\ -3 & 1 \end{pmatrix}$ *and* $\mathbf{D} = \begin{pmatrix} 2 & 6 \\ 3 & 1 \end{pmatrix}$ *, find* **CD.**

Solution

$$\mathbf{CD} = \begin{pmatrix} 4 & 2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 3 & 1 \end{pmatrix}.$$

$$= \begin{pmatrix} 4 \times 2 + 2 \times 3 & 4 \times 6 + 2 \times 1 \\ -3 \times 2 + 1 \times 3 & -3 \times 6 + 1 \times 1 \end{pmatrix}.$$

$$= \begin{pmatrix} 8 + 6 & 24 + 2 \\ ^-6 + 3 & -18 + 1 \end{pmatrix}.$$

$$= \begin{pmatrix} 14 & 26 \\ -3 & -17 \end{pmatrix}.$$

Example 1.9

Write down the values of c and d in the product

$$\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 7 & -2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Solution

Element c is obtained by combining the 2nd row of the 1st matrix with the 1st column of the 2nd matrix.

$\therefore\ c = 2 \times 3 + 7 \times 7 = 55.$

Element d is obtained by combining the 2nd row of the 1st matrix with the 2nd column of the 2nd matrix.

$\therefore\ d = 2 \times 1 + 7 \times {^-2} = {^-12}.$

Compatibility in multiplication

Activity 1.2

Given that $\mathbf{P} = (0\ \ 5)$ and $\mathbf{Q} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, which of the products **PQ** and **QP** is possible?

Work out the one which is possible.

Try **PQ**: $\mathbf{PQ} = (0\ \ 5) \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = ?$

Try **QP**: $\mathbf{QP} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} (0\ \ 5) = ?$

How many rows and columns are in the possible product?

You should have now noticed that:

Two matrices can be multiplied if, and only if, **the number of columns in the matrix to the left is the same as the number of rows in the matrix to the right**. When this is the case, the two matrices are said to be **compatible** for multiplication. For example, a $(p \times q)$ matrix and a $(q \times r)$ matrix are compatible in that order but a $(q \times r)$ matrix and a $(p \times q)$ matrix are not compatible.

You should also have noticed that:

The product of two matrices has the same number of rows as the first matrix and the same number of columns as the second matrix. Thus, a $(p \times q)$ matrix will pre-multiply a $(q \times r)$ matrix to give a $(p \times r)$ matrix,

i.e $(p \times q)(q \times r) \Rightarrow (p \times r)$

Example 1.10

If $\mathbf{P} = \begin{pmatrix} 1 & 3 \\ ^-2 & ^-1 \end{pmatrix}$ *and* $\mathbf{Q} = \begin{pmatrix} 2 & 0 \\ ^-1 & 2 \end{pmatrix}$*, find* **PQ** *and* **QP**. *What do you notice?*

Solution

$$\mathbf{PQ} = \begin{pmatrix} 1 & 3 \\ ^-2 & ^-1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ ^-1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 2 + 3 \times {^-1} & 1 \times 0 + 3 \times 2 \\ -2 \times 2 + -1 \times -1 & -2 \times 0 + -1 \times 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 6 \\ -3 & -2 \end{pmatrix}$$

$$\mathbf{QP} = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times 1 + 0 \times -2 & 2 \times 3 + 0 \times -1 \\ -1 \times 1 + 2 \times -2 & -1 \times 3 + 2 \times -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 6 \\ -5 & -5 \end{pmatrix}$$

We notice that:

PQ ≠ QP, i.e matrix multiplication is **not commutative.**

Exercise 1.5

1. In which of the following pairs of matrices is it possible to pre-multiply the second matrix by the first? Work out the product where possible.

   (a) $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $(3 \;\; 7)$ (d) $(3 \;\; 7)$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

   (e) $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

   (f) $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix}$

2. Given that $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix}$ find **AB** and **BA**.

   Do your results lead you to the same conclusion as that of Example 1.10?

3. Given that $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = k \begin{pmatrix} 2.5 \\ 1 \end{pmatrix}$,

   find the value of k.

4. Find the values of x and y if,

   (a) $\begin{pmatrix} x & 2 \\ -1 & y \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$

   (b) $(x \;\; y) \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} = (9 \;\; 6)$

5. A matrix **B** is such that $(2 \;\; 3)$ **B** $= (9 \;\; 2)$. What is the order of the matrix **B**?

6. Work out the following products.

   (a) $\begin{pmatrix} 4 & 3 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

   (c) $\begin{pmatrix} 0 & 2 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$

   (e) $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ (f) $\begin{pmatrix} p & q \\ r & q \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix}$

7. Given that $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$

   and $\mathbf{C} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$,

   (a) Work out **BC** and **A**(**BC**),

   (b) Work out **AB** and (**AB**)**C.**

8. If $\mathbf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

   calculate:

   (a) **PR** (b) **QR**

   (c) **PR** + **QR** and (d) (**P** + **Q**)**R**

   (e) Show that (3**P** + 5**Q**)**R** = 3**PR** + 5**QR**.

9. When shopping for Christmas, Penina bought 2 skirts and 3 blouses at her local urban centre where the prices were K 360 per skirt and K 300 per blouse. In the main town, a bargain shop was offering the same commodities at K 350 and K 270 respectively.

   (a) How much would she have saved by going to buy the items at the bargain shop if the fare was K 50 return?

   (b) Tessie's purchases were 3 skirts and 2 blouses.

      (i) Express Penina's and Tessies's purchases as a 2 × 2 matrix (**A**) and

      (ii) the prices in the urban centre and bargain shop as a 2 × 2 matrix (**B**).

   (c) Find the matrix product **P** = **AB**. What does **P** tell you?

      Find also the matrix product **BA**. What does this tell you?

10. In Form 4N, there are 5 candidates for Computer Studies and 12 for Agriculture. The numbers in Form 4S are 6 for Computer Studies and 10 for Agriculture.

Each Computer student is required to buy 4 textbooks and 3 exercise books while each Agriculture student is required to buy 3 textbooks and 3 exercise books.

Find, by matrix multiplication, the total number of each kind of book bought by each class.

Multiplying with identity and zero matrices

Activity 1.3

Look at the results that you obtained in Exercise 1.5, Question 6, parts (a), (b) and (e). What do you notice?

Multiply any other 2×2 matrix by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Does it matter which matrix is the pre-multiplier?

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the **identity** (or **unit**) matrix of order 2. It is denoted by $\mathbf{I}_2$.

Note that an identity matrix is a square matrix which has only 1's in the diagonal from the top left corner to the bottom right corner and 0's elsewhere. The 1's are said to be in the **leading** or **main diagonal**.

Note: The matrices $\mathbf{I}_2$ and $\mathbf{I}_3$ behave like the number 1 in the multiplication of numbers, e.g. $3 \times 1 = 1 \times 3 = 3$, $a \times 1 = 1 \times a = a$.

Thus:

If **M** is a square matrix, and **I** is the identity matrix of the same order as **M**, then

$$\mathbf{IM} = \mathbf{MI} = \mathbf{M}$$

I is more specifically, a **multiplicative identity** of the same order as **M**.

In Question 6(c) of Exercise 1.5, did you obtain a matrix with all its elements zeros as shown below?

$$\begin{pmatrix} 0 & 2 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \times 5 + 2 \times 0 & 0 \times 2 + 2 \times 0 \\ 0 \times 5 + 5 \times 0 & 0 \times 2 + 5 \times 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Such a matrix is called a **zero matrix**.

A **zero matrix** is a matrix of any order, not necessarily square, with all its elements zero. It is denoted by **0**, and has the property that **OM** = **MO** = **0**, provided that **O** and **M** are compatible both ways.

O is also an **additive identity**, i.e. **M** + **O** = **O** + **M** = **M**, provided that **O** and **M** are of the same order. Thus, **O** behaves like the number zero.

Exercise 1.6

1. Write down the additive identity of each of the following matrices.

   (a) $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 4 \\ 2 & -9 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$

2. For each of the following matrices, write down a multiplicative identity, making sure to specify the order in which it is an identity.

   (a) $\begin{pmatrix} 5 & 1 \\ 2 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$

3. If $\mathbf{P} = \begin{pmatrix} 6 & 1 \\ 2 & 3 \end{pmatrix}$ what is:

   (a) $\mathbf{I}_4\mathbf{P}$ (b) $\mathbf{PI}_4$?

4. Work out the product $\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$.

   What do you notice? Is it always true that if **XY** = **Y**, then **X** = **I**?

5. If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}$, work out the product **AB**.

Is it always true that if $\mathbf{AB} = \mathbf{0}$, then $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$?

Is it true that if $\mathbf{X} = \mathbf{0}$ and $\mathbf{Y} = \mathbf{0}$, then $\mathbf{XY} = \mathbf{0}$?

6. Given that $= \begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$,

find the possible values of a, b, c and d which are non-zero.

7. (a) Find the products of the following matrices

(i) $\begin{pmatrix} 3 & -6 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} -2 & 3 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Give a reason why the name **identity matrix** is a suitable description of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

2 TANGENTS TO CIRCLES

Success criteria

By the end of this topic, the student must be able to:

- *Describe the properties of tangents to circles.*
- *Apply the properties of tangents to circles in solving problems.*
- *Construct tangents.*

Tangent to a circle

In this chapter, we will concentrate on construction and properties of tangents to a circle. Let us first understand what a tangent to a circle is.

Consider Fig. 2.1.

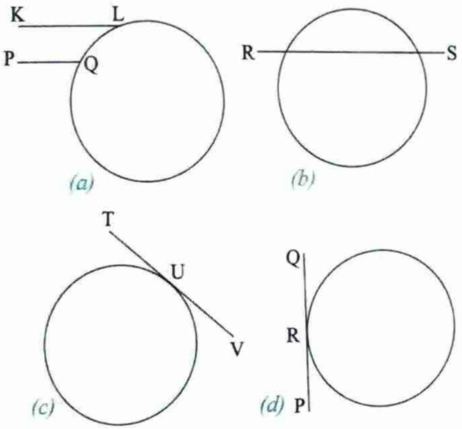


Fig. 2.1

In Fig. 2.1(a), lines KL and PQ have only one common point with the circle. A line with at least one point common with the circle is said to **meet** the circle at that point.

In Fig. 2.1(b), line RS has two distinct common points with the circle. Such a line is said to **meet and cut** the circle at the two points. Such a line is called a **secant**.

In Fig. 2.1(c), the line TV has one point of contact with the circle. Line TV is said to **meet and touch** the circle at that point of contact. Point U is called the **point of contact**. Such a line is called a **tangent to the circle**.

Thus, in Fig. 2.1 (c) TV is the tangent to the circle at point U.

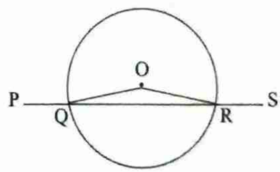
Similarly, in Fig. 2.1 (d) PQ is the tangent to the circle at point R.

1. A line which cuts a circle at two distinct points (as in Fig. 2.1(b)) is called a **secant** of the circle.
2. A line which has one, and only one point in contact with a circle (as in Fig. 2.1(c) and (d)), however far it is produced either way, is called a **tangent** to the circle.

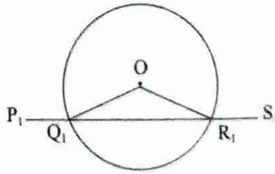
Relationship between tangent and radius of a circle at the point of contact.

Consider Fig. 2.2.

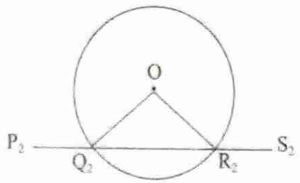
(a)



(b)



(c)



(d)

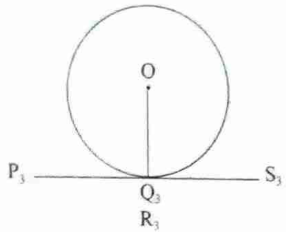


Fig. 2.2

Fig. 2.2(a) to (d) shows what happens when the secant PQRS moves away from the centre of the circle. As the secant moves further away, the points Q and R get closer to each other and the chord QR gets shorter each time. Eventually, Q and R coincide at one point [Fig. 2.2(d)].

On the other hand, angles OQP and ORS become smaller and smaller. Eventually when Q and R coincide, angles OQR and ORS each becomes 90°.

Note that in ΔOQR, since $OQ = OR$,

$\angle OQR = \angle ORQ$.

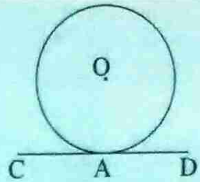
It follows that $\angle PQO = \angle SRO$.

Therefore, when Q and R coincide [Fig. 2.2(d)], $\angle PQO = \angle SRO = 90°$.

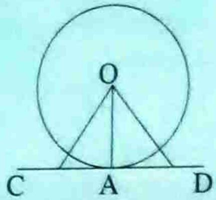
Hence the radius is perpendicular to the tangent PS.

Activity 2.1

1. Draw a circle of any radius centre O.
2. Mark a point A anywhere on the circumference of the circle and draw a line through A so that the line just touches the circle at A extending on both sides of A to points C and D.



3. Join OA and measure the two angles OAC and OAD.



4. What can you comment on the angles?

Note that:

1. A tangent to a circle is perpendicular to the radius drawn through the point of contact.
2. At any point on a circle, one, and only one, tangent can be drawn to the circle.
3. The perpendicular to a tangent at its point of contact passes through the centre of the circle.

Thus, Fig. 2.3 (a) to (d) shows the angles between the radius and tangent at various points on the circumference.

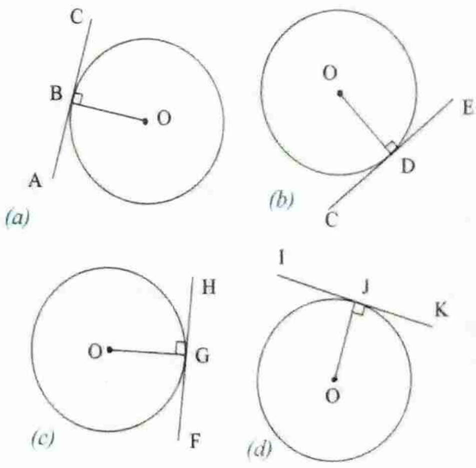
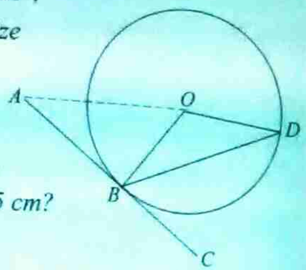


Fig. 2.3

Example 2.1

In Fig. 2.4, AC is a tangent to the circle, centre O. If $\angle ABD = 120°$,



(a) what is the size of $\angle ODB$?

(b) what is the length of OA if OB = 6 cm and AB = 7.5 cm?

Solution

Fig. 2.4

(a) Since AC is a tangent and OB is a radius, AC is perpendicular to OB.

$\therefore \angle ABO = 90°$.

$\angle OBD = \angle ABD - \angle ABO$

$= 120° - 90°$

$= 30°$

Since OB = OD (radii)

$\angle ODB = \angle OBD$ *(base angles of an isosceles Δ)*

$\therefore \angle ODB = 30°$

(b) Using ΔAOB $OA^2 = AB^2 + BO^2$ (right angled Δ; Pythagoras theorem)

$= 7.5^2 + 6^2$

$\sqrt{OA^2} = \sqrt{92.25}$

$\therefore OA = 9.605$ *cm* ≈ 9.6 *cm (1 d.p.)*

Exercise 2.1

1. Fig. 2.5 shows a circle, centre O. PR is a tangent to the circle, at P and PQ is a chord. Calculate

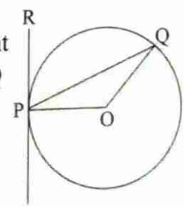


Fig. 2.5

   (a) $\angle RPQ$ given that $\angle POQ = 85°$.
   (b) $\angle RPQ$ given that $\angle PQO = 26°$.
   (c) $\angle POQ$ given that $\angle RPQ = 54°$.
   (d) $\angle POQ$ given that $\angle QPO = 17°$.

2. In Fig. 2.6, ABC is a tangent and BE is a diameter to the circle. Calculate:

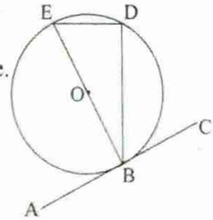


Fig. 2.6

   (a) $\angle EBD$ if $\angle CBD = 33°$.
   (b) $\angle BED$ if $\angle ABD = 150°$.
   (c) $\angle DBC$ if $\angle DEB = 65°$.
   (d) $\angle ABD$ if $\angle BED = 38°$.

3. PQ is a diameter of a circle. Let R be a point on the circumference of the circle. Show that PR is a tangent to a circle, centre O, radius OR.

4. AB is a chord of a circle, centre O. If BC is a perpendicular to the tangent at A, show that $\angle OBA = \angle ABC$.

5. Two circles have the same centre O, but different radii. PQ is a chord of the bigger circle but touches the smaller circle at A. Show that PA = AQ.

6. Two circles have the same centre O and radii of 13 cm and 10 cm. AB is a chord of the bigger circle, but a tangent to the small circle. What is the length of AB?

7. A tangent is drawn from a point 17 cm away from the centre of a circle of radius 8 cm. What is the length of the tangent?

Constructing a tangent at any given point on the circle

To construct a tangent to a circle, we use the fact that a tangent is perpendicular to the circle at the point of contact.

Procedure

1. Draw a circle, centre O, using any radius.
2. Draw a line OB through any point A on the circumference, with B outside the circle.

3. At A, construct a line PQ perpendicular to OB.

The line PQ (Fig. 2.7) is a tangent to the circle at A.

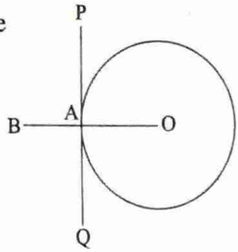


Fig. 2.7

Constructing tangents to a circle from a common point

Activity 2.2

Procedure

1. Draw a circle of any radius, centre O.
2. Mark a point T outside the circle.
3. Join OT. Construct the perpendicular bisector of TO to meet TO at P.
4. With centre P, radius PO, construct arcs to cut the circle at points A and B.
5. Join AT and BT. These are the required tangents from the external point T.
6. Join OA and OB to get Fig. 2.8 below.

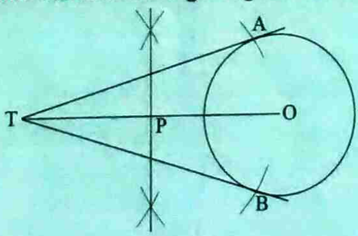


Fig. 2.8

7. Measure:
   (a) AT, BT.
   (b) ∠ATO, ∠BTO,
   (c) ∠AOT, ∠BOT
   (d) ∠TAO, ∠TBO

   What do you notice?
   Which points on a circle would have tangents that do not meet?

You should have observed that:

AT = BT.
∠ATO = ∠BTO,
∠AOT = ∠BOT
∠TAO = ∠TBO

Tangents at points on a circle which are diametrically opposite do not meet.

If two tangents are drawn to a circle from a common point outside the circle:

(a) the tangents are equal;
(b) the tangents subtend equal angles at the centre;
(c) the line joining the centre to the common point bisects the angles between the tangents.
(d) the angle between the tangent and the radius at the point of contact is 90°.

Theorem

Tangents from an external point to the same circle are equal in length.

Consider Fig. 2.9.

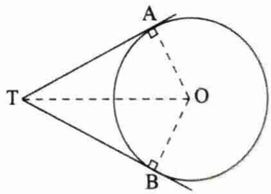


Fig. 2.9

O is the centre of the circle. AT and BT are the tangents to the circle from an external point T.

To prove AT = BT;
Join OT, OA, OB.

Proof

Using Δs AOT and BOT;

OA = OB .... radii of the same circle.

OT = OT .... OT is common to the two triangles

∠TAO =∠TBO = 90° .... radius meets tangent at right angles at the point of contact.

Δs AOT and BOT are right angled at A and B respectively.

Δs AOT and BOT are congruent .... RHS

∴ AT = BT .... corresponding sides of congruent triangles.

∴ Tangents from an external point to the same circle are equal.

Theorem

The line joining an external point to the centre of a circle bisects that angle between the tangents from that point.

Consider Fig 2.10

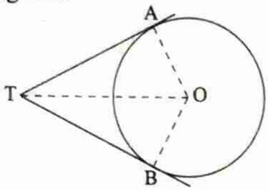


Fig. 2.10

O is the centre of the circle. AT and BT are tangents to the circle from a common external point T.

To prove ∠ATO = ∠BTO

Using ΔATO and BTO,

OA = OB .....radii

AT = BT .....tangents from an external point to the same circle are equal.

TO is common.

Δs ATO and BTO are congruent ..... SSS.

Corresponding angles of the triangles are equal.

<ATO = ∠BTO ...... corresponding angles of congruent Δs.

∴ Line OT bisects the angle between the two tangents.

Example 2.2

In Fig. 2.11, TA and TB are tangents to the circle, centre O.

If ∠ABO = 28°, what is the size of ∠ATO?

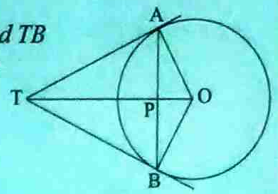


Fig. 2.11

Solution

In ΔABO, ∠ABO = ∠BAO = 28° (isosceles Δ)

∴ ∠AOB = 180° – 56° = 124°

∠AOT = $\frac{1}{2}$ ∠AOB (tangents subtend equal angles at the centre of circle)

= 62°

In ΔATO, ∠OAT = 90° (tangent is perpendicular to radius)

∴ ∠ATO = 90° – ∠AOT

= 90° – 62°

= 28°.

Exercise 2.2

1. In Fig. 2.12, O is the centre of the circle and PT, RT are tangents to the circle. Calculate

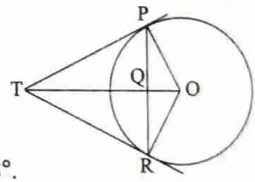


Fig. 2.12

   (a) ∠POT if ∠OTR = 34°.
   (b) ∠PRO if ∠PTR = 58°.
   (c) ∠TPR if ∠PRO = 15°.
   (d) ∠RTO if ∠POR = 148°.

2. Draw a circle, centre O, and radius 2.5 cm. Mark points A and B on the circle such that ∠AOB = 130°. Construct tangents at A and B. Measure:
   (a) the lengths of the tangents
   (b) the angle formed where the tangents meet.

3. In Fig. 2.13, O is the centre of the circle. If BO = 19.5 cm, BQ = 18 cm, QC = 8.8 cm and AO = 9.9 cm, what are the lengths of:
(a) AB (b) BC (c) AC?

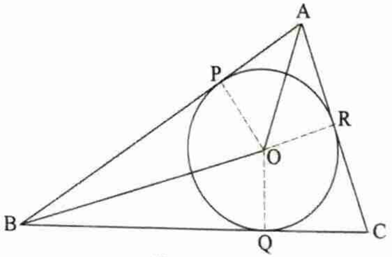


Fig. 2.13

4. A tangent is drawn to a circle of radius 5.8 cm from a point 14.6 cm from the centre of the circle. What is the length of the tangent?

5. Tangents are drawn from a point 10 cm away from the centre of a circle of radius 4 cm. What is the length of the chord joining the two points of contact?

6. Tangents TA and TB each of length 8 cm, are drawn to a circle of radius 6 cm. What is the length of the minor arc AB?

7. Construct two tangents from a point A which is 6 cm from the centre of a circle of radius 4 cm.
   (a) What is the length of the tangent?
   (b) Measure the angle subtended at the centre of the circle.

8. Draw a line KL = 6 cm long. Construct a circle centre K radius 3.9 cm such that the tangent LM from L to the circle is 4.5 cm. Measure ∠KLM.

Common tangents to two circles

Fig. 2.14 shows non-intersecting pairs of circles which have common tangents.

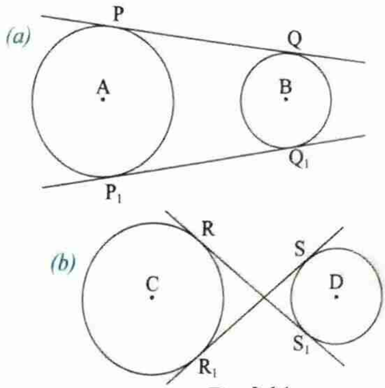


Fig. 2.14

The tangents in Fig. 2.14(a) are called **exterior** or **direct common tangents**, while those in Fig. 2.14(b) are called **transverse** or **interior common tangents.**

Properties derived from tangents to a circle from a common point apply to all types of common tangents. Examples 2.3 and 2.4 are used to illustrate further work or application of properties of tangents to circles.

Example 2.3

Two circles, of radii 4 cm and 9 cm, are positioned in such a way that the distance between their centres is 21 cm, as shown in Fig. 2.15.

Calculate:

(a) the length of the tangent MN.

(b) ∠PQN.

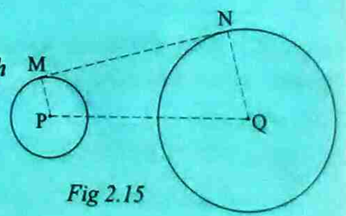


Fig 2.15

Solution

In Fig 2.16, line PR has been drawn such that PR ⊥ QN, hence forming rectangle MNRP.

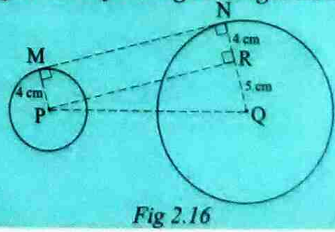


Fig 2.16

(a) *Since MNRP is a rectangle, MN = PR.*

Using ΔPQR,

$$PR^2 = PQ^2 - QR^2 \text{ (Pythagoras' theorem)}$$
$$= 21^2 - 5^2 = 416$$
$$\therefore \quad MN = PR = \sqrt{416} = 20.40 \text{ cm}$$

(b) *Again, using* ΔPQR,

$$\cos \angle PQR = \frac{5}{21} = 0.238\ 1$$

Hence, $\angle PQN = \angle PQR = 76.23°$ *(4 s.f.).*

Example 2.4

Find the length of a transverse common tangent to two circles of radii 7 cm and 3 cm given that the centres are 12 cm apart.

Solution

Refer to Fig. 2.17.

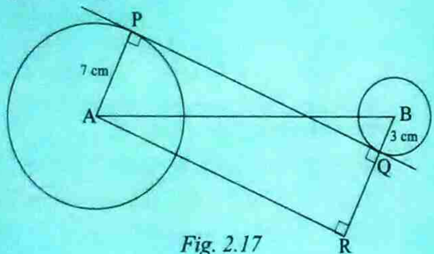


Fig. 2.17

Let A and B be centres of the circles.

Let PQ be a transverse common tangent.

Draw AR $\perp$ *B produced*

Since $\angle APQ = \angle PQR = 90°$,

APQR is a rectangle.

$$BR = BQ + QR = BQ + PA$$
$$= (3 + 7) \text{ cm} = 10 \text{ cm}$$
$$AR^2 + BR^2 = AB^2 \text{ (Pythagoras theorem)}$$
$$\Rightarrow AR^2 = AB^2 - BR^2$$
$$= 12^2 - 10^2$$
$$= 144 - 100$$
$$= 44$$
$$\Rightarrow AR = \sqrt{44} = 6.633 \text{ cm.}$$

But $PQ = AR$ *(opposite sides of rectangle)*

$$\therefore PQ = 6.633 \text{ cm.}$$

Exercise 2.3

1. The centres of two circles of radii 10 cm and 6 cm are 20 cm apart. Find the length of:
   (a) a direct common tangent to the circles.
   (b) a transverse common tangent of the circles.

2. Draw two circles, radii 5 cm and 2 cm, such that their centres are 8.5 cm apart. Construct a common tangent direct to the circles. Measure its length.

3. Draw two circles, radii 3.5 cm and 2.5 cm, such that their centres are 9 cm apart. Construct a transverse common tangent. Measure its length.

4. Two circles of radii 6.5 cm and 1.5 cm have their centres 10 cm apart. What angle does:
   (a) the direct common tangent make with the line joining the centres?
   (b) the transverse common tangent make with the line joining the centres?

5. The centres of two circles of radii R and r, are d units apart. What is the length of:
   (a) a direct common tangent to the two circles?
   (b) a transverse common tangent to the two circles?

6. Two circles with radii 3 cm and 8 cm are positioned in such a way that their centres are 13 cm apart. What is the length of their common direct tangent?

7. Two circles, with radii 12 cm and 4 cm, are placed such that the length of their direct common tangent is 15 cm. What is the distance between their centres?

Angles in alternate segment

In Fig. 2.18, ABC is a tangent to the circle at B. The chord BD divides the circle into two segments BED and BFD.

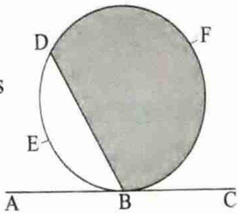


Fig. 2.18

We see that ∠ABD is in the minor segment (unshaded segment). The shaded segment BFD is the major segment. We say that segment BFD is the **alternate segment** to ∠ABD.

Similarly, segment BED is the alternate segment to ∠CBD.

Activity 2.3

1. Draw a circle of any radius.
2. Draw a tangent at any point B.
3. Draw a chord BD. Measure ∠ABD.
4. Mark points P, Q, R on the circumference in the same segment as in Fig. 2.19. Join BP, BQ, BR, DP, DQ and DR.

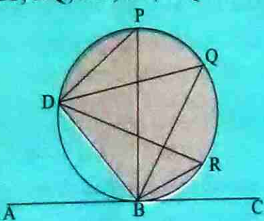


Fig. 2.19

5. Measure angles ABD, BPD, BQD and BRD; and compare them with ∠ABD. What do you notice?

You should have observed that

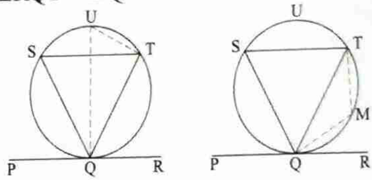
∠ABD = ∠BPD = ∠BQD = ∠BRD.

In general:

1. If a straight line touches a circle, and from the point of contact a chord is drawn, the angle which the chord makes with the tangent is equal to the angle the chord subtends in the alternate segment of the circle. This is called the alternate segment theorem.
2. If a straight line is drawn at the end of a chord of a circle making with the chord an angle equal to an angle in the alternate segment, the straight line touches the circle (i.e. it is a tangent to the circle).

Fig. 2.20 is used to explain further the properties of angles in alternate segments.

(a) We use Fig. 2.20(a) to show that ∠RQT = ∠QST.



(a) *(b)*

Fig. 2.20

Draw diameter QU. Join UT.

Since QU is a diameter and PR is a tangent,

∠RQT + ∠TQU = 90° (tangent ⊥ radius)

∠QTU = 90° (∠ in semi-circle)

∴ ∠QUT + ∠TQU = 90° (∠ sum of Δ)

∴ ∠RQT + ∠TQU = ∠QUT + ∠TQU

⇒ ∠RQT = ∠QUT.

But ∠QUT = ∠QST (∠s in same segment)

∴ ∠RQT = ∠QST.

(b) We use Fig. 2.20(b) to show that ∠PQT = ∠QMT

∠PQT + ∠RQT = 180° (adj. ∠s on straight line).

∠QMT + ∠QST = 180° (opp. ∠s of cyclic quadrilateral).

∴ ∠PQT + ∠RQT = ∠QMT + ∠QST.

But ∠RQT = ∠QST (shown in (a) above)

∴ ∠PQT = ∠QMT

Theorem:

The angle between a chord and a tangent at the point of contact is equal to the angle subtended by the same chord in the alternate segment.

Given: Circle centre O

Tangent at B

Chord intersecting tangent at the point of contact.

Let the tangent be ABC.

Chord: BD and E another point on the circle as in Fig. 2.21.

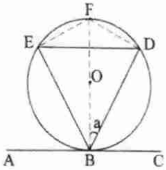


Fig. 2.21

To prove $\angle CBD = \angle BED$

Construction

(i) Draw in the diameter through B and O to intersect the circle at a point F.

(ii) Join DF and EF.

Proof

Let $\angle FED = a$, $\angle DBF = b$

$\angle CBF = 90^\circ$ .....radius meets tangent at the point of contact at right angles.

But $\angle CBF = \angle CBD + \angle FBD$

$= \angle CBD + a$

$\angle BEF = 90^\circ$ angle on a semi circle $= 90^\circ$

$\angle BEF = \angle BED + \angle FED$

$\angle CBD + \angle FBD = \angle BED + \angle FED$

But $\angle FBD = \angle FED$ ....angles subtended by the same chord in the same segment.

$\angle CBD = \angle BED$

i.e. $\angle CBD$ is between tangent at B and chord BD.

BED is subtended by chord BD in the alternate segment.

Angle between tangent and a chord at the point of contact is equal to the angle subtended by the same chord in the alternate segment.

Similarly, we can also prove that $\angle ABE = \angle BDE$

Example 2.5

In Fig. 2.22, PQR is a tangent to the circle at Q. QT is a chord and PST is a straight line. Given that $\Delta PQT = 110^\circ$, $\angle TPQ = 25^\circ$, find $\angle SQP$.

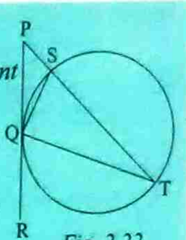


Fig. 2.22

Solution

In $\angle PQT$, $\angle PQT + \angle QTP + \angle TPQ = 180^\circ$ ($\angle$sum of Δ)

But $\angle PQT = 110^\circ$, $\angle TPQ = 25^\circ$

$\therefore \angle QTP = 180^\circ - 135^\circ = 45^\circ$

But $\angle SQP = \angle QTP$ ($\angle$s in alternate segment)

$\therefore \angle SQP = 45^\circ$.

Exercise 2.4

1. In Fig. 2.23, AC is a tangent to the circle and BE//CD.

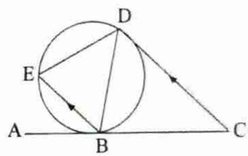


Fig. 2.23

(a) If $\angle ABE = 42^\circ$, $\angle BDC = 59^\circ$, find $\angle BED$

(b) If $\angle DBE = 62^\circ$, $\angle BCD = 56^\circ$, find $\angle BED$.

2. In Fig. 2.24, PR is a tangent to the circle.

(a) If $\angle PQT = 66^\circ$, find $\angle QST$.

(b) If $\angle QTS = 38^\circ$ and $\angle QRS = 30^\circ$, find $\angle QST$.

(c) If $\angle QTS = 35^\circ$ and $\angle TQS = 58^\circ$, find $\angle QRS$.

(d) If $\angle PQT = 50^\circ$ and $\angle PRS = 30^\circ$, find $\angle SQT$.

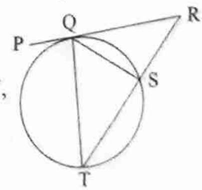


Fig. 2.24

3. In Fig. 2.25, AB, BC and AC are tangents to the circle. If $\angle BAC = 75°$ and $\angle ABC = 44°$, find $\angle EDF$, $\angle DEF$ and $\angle EFD$.

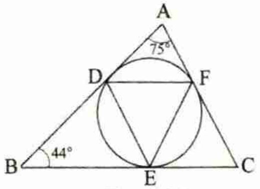


Fig. 2.25

4. In Fig. 2.26, KLM is a tangent to the circle. If $\angle LPN = 38°$ and $\angle KLP = 85°$, find $\angle PQN$.

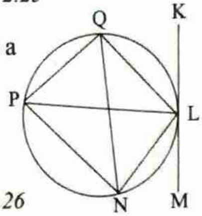


Fig. 2.26

5. In Fig. 2.27, DC is a tangent to the circle. Show that $\angle CBD = \angle ADC$.

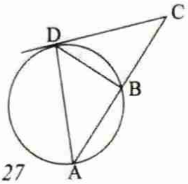


Fig. 2.27

6. In Fig. 2.28, AB and DE are tangents to the circle. $\angle ABC = 40°$ and $\angle BCD = 38°$. Find $\angle CDE$.

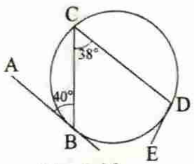


Fig. 2.28

7. In Fig. 2.29, ABC is a tangent to the circle at B and ADE is a straight line. If $\angle BAD = \angle DBE$, show that BE is a diameter.

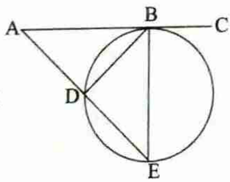


Fig. 2.29

8. In Fig. 2.30, AD is a tangent to the circle. BC is a diameter of the circle and $\angle BCD = 30°$. Find $\angle DAB$.

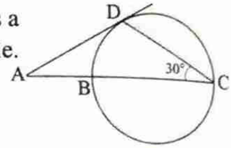


Fig. 2.30

9. In Fig. 2.31, AD is a tangent to the circle at D, $\angle DAB = 28°$ and $\angle ADC = 112°$. Find the angle subtended at the centre of the circle by the chord DC.

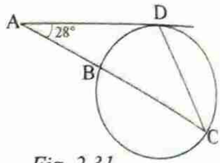


Fig. 2.31

10. Points A, B and C are on a circle such that $\angle ABC = 108°$. Find the angle between the tangents at A and C.

11. In Fig. 2.32, O is the centre of the circle. AB and CD are chords that meet at X. XT is a tangent to the circle.

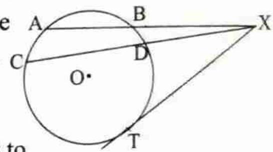


Fig. 2.32

Show that: (a) $XT^2 = XA \cdot XB$

(b) $XT^2 = XC \cdot XD$.

3 STATISTICS III

Success criteria

By the end of this topic, the student must be able to:

- *Calculate measures of dispersion.*
- *Interpret data using measures of dispersion.*

Introduction

In book 3, we learnt the three common measures of central tendency. These are the **mean, median** and **mode**. In this unit, we will learn about the measures of dispersion. How are the two measures related?

Measures of central tendency tell us where the middle of a set of data lies. It summarizes a set of data to its centre. Measures of dispersion denote how **stretched** or **squeezed** a distribution is. Measures of dispersion use measures of central tendency to show how data is spread out or dispersed. For example, a mean of data shows the average value of such a data, but a measure of dispersion shows how close data points are to a mean value.

Measures of dispersion

Consider the distributions in Table 3.1.

A	50	50	50	50	50	50	50
B	42	45	46	50	52	56	59
C	34	37	49	53	57	59	61

Table 3.1

$$\text{Mean } (\bar{x}) = \frac{\sum fx}{\sum f}$$

$$\text{Mean of A} = \frac{50+50+50+50+50+50+50}{7}$$

$$= \frac{350}{7} = 50$$

$$\text{Mean of B} = \frac{42+45+46+50+52+56+59}{7}$$

$$= \frac{350}{7} = 50$$

$$\text{Mean of C} = \frac{34+37+49+53+57+59+61}{7}$$

$$= \frac{350}{7} = 50$$

The mean of each distribution is 50.

In distribution A, the values do not vary, while in distributions B and C, they do. Some of the values in B and C are above the mean while others are below. The values show **variation** or **dispersion**. Those of distribution C are more **spread** out than those of distribution B. Those of distribution A have no spread.

It is useful, for statistical purposes, to have a way of measuring the dispersion (or spread) of a distribution. In this unit, we will look at such measures which include the **range, mean deviation, standard deviation** and **variance**.

Range

Range is the difference between the largest and smallest values in a distribution.

Example 3.1

Find the range of each of the distributions in Table 3.1.

Solution

Distribution A: Range = 50 – 50 = 0

Distribution B: Range = 59 – 42 = 17

Distribution C: Range = 61 – 34 = 27

Note: 1. The greater the variation of the values in a distribution, the greater the range.

2. The range is very easy to determine. However, it is disadvantageous in that it depends on only two extreme values.

Mean deviation (M.D.)

Table 3.2 shows the deviations from the mean of each of the values in distributions B and C in Table 3.1.

Deviations from the mean
B: –8, –5, –4, 0, 2, 6, 9
C: –16, –13, –1, 3, 7, 9, 11

Table 3.2

Sum of deviation from the mean in distribution B are:

$$= -8 + -5 + -4 + 0 + 2 + 6 + 9$$
$$= -17 + 17 = 0$$

Sum of deviation from mean in distribution C are:

$$= -16 + -13 + -1 + 3 + 7 + 9 + 11$$
$$= -30 + 30 = 0$$

In each case, the sum of the deviations is zero.

For any distribution, the sum of the deviations is zero. This does not reveal anything about the dispersion of the values.

Since we are interested only in how far above or below the mean that the values are, we may ignore the signs on the deviations and take the absolute values (i.e sizes of the deviations irrespective of the signs). For distributions B and C, the absolute deviations are as in Table 3.3.

Absolute deviations from the mean
B: 8, 5, 4, 0, 2, 6, 9
C: 16, 13, 1, 3, 7, 9, 11

Table 3.3

Absolute deviation from the mean are denoted as $|x - \bar{x}|$ where $\bar{x}$ is the mean of the set.

The mean of absolute deviations is called **mean absolute deviation** or simply **mean deviation (MD)**. It tells us how far, on average, the values are above or below the mean.

For distribution A (Table 3.1),

MD = 0.

This means that every value is equal to the mean.

For distribution B,

$$\text{MD} = \frac{8+5+4+0+2+6+9}{7}$$
$$= \frac{34}{7} \approx 4.857 \text{ (4 s,f,).}$$

This means that, on average, the values are 4.9 more or less than the mean.

For distribution C,

$$\text{MD} = \frac{16+13+1+3+7+9+11}{7}$$
$$= \frac{60}{7} \approx 8.571 \text{ (4 s.f.).}$$

The formula for finding mean deviation is

$$\text{MD} = \frac{\sum f\,|x - \bar{x}|}{\sum f}.$$

Note: 1. The greater the dispersion, the higher the value of MD.

2. The mean deviation may be calculated from any other average, e.g. from the median or mode. However, mean deviation about the mean is the one most commonly used and preferred.

3. When the data is grouped, we use the class mid-values to find MD.

Exercise 3.1

1. For the following distributions, determine the range and the mean absolute deviation.

(a) 65, 69, 70, 72, 76, 78, 80, 81, 84.

(b) 16, 23, 26, 38, 42, 47, 53, 58, 61, 64, 73, 75, 79, 83, 87.

2. Table 3.4 shows the distribution of shoe sizes of 100 students in a certain school.

Shoe size	4	5	6	7	8	9
No. of students	11	26	33	16	10	4

Table 3.4

Find:

(a) the range,

(b) the mean deviation of the distribution.

3. Table 3.5 is a frequency distribution of the mass of tobacco harvested on a single day by labourers working on a tobacco estate.

Mass (kg)	58	61	64	67	70	73	76	79
Frequency	5	6	8	12	13	8	6	2

Table 3.5

Find the mean deviation.

Variance and standard deviation

Standard deviation is used for quality control. In the field of manufacturing, a certain amount of deviation from the norm is considered acceptable.

To determine the amount of deviation, a representative measure of the data called the ***standard deviation*** is calculated.

The purpose of this measure is to determine how closely measurements cluster about the mean.

Although items that are mass produced can never be identical, they need to be similar.

For example, machine parts must match other machine parts that need replacing. Thus, they need to be manufactured to have measurements that are as close as possible.

In the production of merchandise, it is important to check quality constantly. The standard deviation of sample batches is the way to determine how closely measurements cluster about the mean.

Rather than ignore the signs of the deviations, we can square each deviation so that we get only positive values.

The mean of the squares of the deviations from the mean is called the **mean squared deviation** or **variance**, denoted as s^2.

Consider the following distribution.

B: 42, 45, 46, 50, 52, 56, 59.

The mean of this distribution is 50.

The variance of the distribution is worked out as follows (Table 3.6).

x	$d = x - x$	d^2
42	–8	64
45	–5	25
46	–4	16
50	0	0
52	2	4
56	6	36
59	9	81
		$\sum d^2 = 226$

Table 3.6

$$\text{Variance} = \frac{\sum d^2}{N} = \frac{226}{7} \approx 32.29.$$

For a frequency distribution, variance is given by the formula

$$s^2 = \frac{\sum f(x - \overline{x})^2}{\sum f} \quad \text{i.e. } s^2 = \frac{\sum fd^2}{\sum f}.$$

This is known as the basic formula for finding the variance.

If the units of the values in the distribution were centimetres, what would the units of the variance be?

To be useful, any measure of spread must have the following properties:

1. Translation along the number line (i.e. adding a constant A to each value in the distribution) should not affect it.
2. Enlargement with scale factor c (i.e. multiplying or dividing each value in the distribution by c) should multiply or divide the spread by the same factor c.
3. Multiplying the frequencies by any factor should not change the spread.
4. All members of the distribution should be taken into account, but extreme values must not influence the spread unduly.

The variance does not satisfy property 2! This is because we have squared the deviations. To restore this property, we take the square root of the variance.

The square root of the variance is known as **root mean squared deviation** or **standard deviation** (denoted by s). Thus,

$$s = \sqrt{\text{variance}}$$

$$\Rightarrow s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \text{ i.e. } s = \sqrt{\frac{\sum fd^2}{\sum f}}$$

Where $d = (x - \bar{x})$

This is the basic formula for finding the standard deviation.

Example 3.2

Calculate the mean, the variance and the standard deviation of the distribution in Table 3.7.

x	5	7	9	11	13
f	2	4	8	6	4

Table 3.7

Solution

The working for the mean may be tabulated as in Table 3.8(a).

x	f	fx
5	2	10
7	4	28
9	8	72
11	6	66
13	4	52
	$\sum f = 24$	$\sum fx = 228$

Table 3.8 (a)

$$\textit{Mean, } x = \frac{\sum fx}{\sum f} = \frac{228}{24} = 9.5$$

This value of x is used to complete Table 3.8(b).

x	f	$d = x - \bar{x}$	d^2	fd^2
5	2	–4.5	20.25	40.5
7	4	–2.5	6.25	25.0
9	8	–0.5	0.25	2.0
11	6	1.5	2.25	13.5
13	4	3.5	12.25	49.0
	$\sum f = 24$			$\sum fd^2 = 130$

Table 3.8 (b)

$$\textit{Variance, } s^2 = \frac{\sum fd^2}{\sum f} = \frac{130}{24}$$

$$= 5.417 \ (4 \text{ s.f})$$

$$\textit{Standard deviation, } s = \sqrt{\text{variance}}$$

$$= \sqrt{\frac{\sum fd^2}{\sum f}} = \sqrt{\frac{130}{24}}$$

$$= \sqrt{5.417}$$

$$= 2.327 \ (4 \text{ s.f.})$$

Example 3.3

Five students A, B, C, D and E obtained the marks 53, 41, 60, 80 and 56 respectively. Table 3.9 shows part of the work to find the standard deviation.

Students	*mark x*	$d = x - \bar{x}$	d^2
A	53	–5	25
B	41	–17	289
C	60	2	4
D	80	22	484
E	56	–2	4

Table 3.9

(a) Complete the table.

(b) Find the standard deviation of the marks.

Solution

(a) Table 3.10 shows the complete table.

Pupil	*A*	*B*	*C*	*D*	*E*	Efd^2
d^2	*25*	*289*	*4*	*484*	*4*	*806*

Table 3.10

(b) Variance, $s^2 = \frac{\sum fd^2}{\sum f} = \frac{806}{5}$

$$\text{Standard deviation} = \sqrt{\frac{\sum fd^2}{\sum f}} = \sqrt{\frac{806}{5}}$$
$$= \sqrt{161.2}$$
$$= 12.696$$
$$= 12.7$$

Note:

1. If s is small, the numbers are closely grouped about the mean.
2. When the data is grouped, we use the class mid-values to calculate standard deviation, *s*.

Exercise 3.2

Calculate the mean and standard deviation of each of the following distributions, giving your answer correct to 4 s.f. where appropriate.

1. 6, 8, 9, 10, 10, 12, 15
2. 34, 37, 49, 53, 57, 59, 61
3.

x	2	4	6	8	10
f	1	2	4	3	2

Table 3.11

4.

x	1	2	3	4	5	6	7
f	2	3	6	9	4	4	2

Table 3.12

5.

Quality	10.5	25.5	35.5	45.5	55.5	65.5
Frequency	5	11	16	9	5	4

Table 3.13

6.

Quality	12	17	22	27	32	37	42
Frequency	5	11	16	9	5	4	

Table 3.14

Computational formula

We have already seen that the variance is given by the formula

$$s^2 = \frac{\sum f(x - \bar{x})^2}{\sum f}.$$

Expanding this formula gives:

$$s^2 = \frac{\sum f(x^2 - 2\bar{x}x + \bar{x}^2)}{\sum f}$$
$$= \frac{\sum fx^2 - 2\bar{x}\sum fx + \bar{x}^2\sum f}{\sum f}$$
$$= \frac{\sum fx^2}{\sum f} - \frac{2\bar{x}\sum fx}{\sum f} + \frac{\bar{x}^2\sum f}{\sum f}$$
$$= \frac{\sum fx^2}{\sum f} - 2\bar{x}^2 + \bar{x}^2$$
$$= \frac{\sum fx^2}{\sum f} - \bar{x}^2$$

Thus:

The variance may be found using the formula

$$s^2 = \frac{\sum fx^2}{\sum f} - \bar{x}^2.$$

The standard deviation is thus given by

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

This method is very useful in cases where the mean is fractional, in which case the working would be more difficult if we tried to use the basic formula.

Example 3.4

Use the formula $s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$ *to find the standard deviation of this distribution.*

x	5	7	9	11	13
f	2	4	8	6	4

Table 3.15

Solution

The working may be tabulated as shown in Table 3.16.

x	f	fx	fx^2
5	2	10	50
7	4	28	196
9	8	72	648
11	6	66	726
13	4	52	676
	$\sum f = 24$	$\sum fx = 228$	$\sum fx^2 = 2\,296$

Table 3.16

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{228}{24} = 9.5.$$

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2} = \sqrt{\frac{2\,296}{24} - 9.5^2}$$

$$= \sqrt{5.417}$$

$$= 2.327 \ (4 \text{ s.f.})$$

Example 3.5

In an agricultural centre, the lengths of a sample of 50 maize cobs were measured and recorded as shown in Table 3.17.

Length (cm)	Number of cobs
8 - 10	4
11 - 13	7
14 - 16	11
17 - 19	15
20 - 22	8
23 - 25	5

Table 3.17

Calculate: *(a) the mean,*

(b) (i) the variance,

(ii) the standard deviation.

Solution

Table 3.17 shows the required working.

(a)

Length (cm)	f	x	fx	fx^2
8 - 10	4	9	36	324
11 - 13	7	12	84	1 008
14 - 16	11	15	165	2 475
17 - 19	15	18	270	4 860
20 - 22	8	21	168	3 528
23 - 25	5	24	120	2 880
	$\sum f = 50$		$\sum fx = 843$	$\sum fx^2 = 15\,075$

Table 3.18

$$\text{Mean,} = \frac{\sum fx}{\sum f} = \frac{843}{50}$$

$$= 16.86$$

(b) (i) $$\text{Variance} = \frac{\sum fx^2}{\sum f} - \bar{x}^2$$

$$= \frac{15\,075}{50} - (16.86)^2$$

$$= 301.5 - 284.26$$

$$= 17.24$$

(ii) $$\text{Standard deviation} = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

$$= \sqrt{17.24}$$

$$= 4.152$$

Note that using the formula $s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$ involves less work, and hence, it is quicker. This formula is known as the **computational formula** for the standard deviation.

An alternative method of calculating standard deviation

Suppose we add or subtract a constant to/from each of the values of a distribution. What is the effect of this on the standard deviation? The following example will enable us to answer this question.

Example 3.6

Calculate the standard deviation (d) in Table 3.16, where $d = x - 9$. Compare the value obtained by that of the standard deviation of x obtained in Example 3.2.

Solution

The working is as shown in Table 3.19.

x	f	$d = x - 9$	fd	fd^2
5	2	−4	−8	32
7	4	−2	−8	16
9	8	0	0	0
11	6	2	12	24
13	4	4	16	64
	$\sum f = 24$		$\sum fd = 12$	$\sum fd^2 = 136$

Table 3.19

Mean, $\bar{d} = \frac{\sum fd}{\sum f} = \frac{12}{24} = 0.5.$

Standard deviation (d) $= \sqrt{\frac{\sum fd^2}{\sum f} - \bar{d}^2}$

$= \sqrt{\frac{136}{24} - 0.5^2}$

$= \sqrt{5.667 - 0.25} = \sqrt{5.417}$

$= 2.327$ (4 s.f).

This value is the same as the standard deviation of x in Example 3.2. Note that the constant 9 was arbitrary and any other constant could have been used. This constant is often referred to as a work mean.

Example 3.7

Table 3.20 shows the heights of 50 pupils in a certain primary school.

Height (cm)	105-109	110-114	115-119	120-124	125-129	130-134	135-139
Frequency	3	5	8	12	14	5	3

Table 3.20

Find the mean and standard deviation (d) of the distribution where $d = x - 122$.

Solution

Table 3.21 shows the required working.

Height	Mid. intervals value, x	Frequency, f	$d = x - A$ ($A = 122$)	fd	fd^2
105-109	107	3	−15	−45	675
110-114	112	5	−10	−50	500
115-119	117	8	−5	−40	200
120-124	122	12	0	0	0
125-129	127	14	5	70	350
130-134	132	5	10	50	250
135-139	137	3	15	45	675
		$\sum f = 50$		$\sum fd = 30$	$\sum fd^2 = 2650$

Table 3.21

Mean, $\bar{d} = \frac{\sum fd}{\sum f} = \frac{30}{50} = 0.6$

s.d.(d) $= \sqrt{\frac{\sum fd^2}{\sum f} - \bar{d}^2}$

$= \sqrt{\frac{2\,650}{50} - 0.6^2}$

$= \sqrt{53 - 0.36}$

$= \sqrt{52.64}$

$= 7.26.$

From Examples 3.6 and 3.7, we see that subtracting a constant from the values in a distribution does not alter the value of the variance or standard deviation. Hence, the following formula may also be used to find the standard deviation.

$$s = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2},$$

where $d = x - A$ where A is a working/assumed mean.

Exercise 3.3

1. Find the standard deviation of the distribution in Table 3.22 using the computational formula.

x	9	10	11	12	13	14	15
f	1	2	4	4	5	3	1

Table 3.22

2. Using a suitable working mean, find the standard deviation of the distribution in Table 3.23. Check your result using the computational formula.

Class	Frequency
5	2
15	4
25	4
35	8
45	6
55	3
65	2

Table 3.23

3. In a study of the characteristics of cockroaches, a student of zoology measured lengths of antennae of a number of cockroaches and recorded them as in Table 3.24. Find the mean and standard deviation of the antenna lengths, using a working mean of 2.75.

Antenna length (cm)	Number of cockroaches
1.25	5
1.75	11
2.25	25
2.75	36
3.25	30
3.75	20

Table 3.24

4. Using the assumed mean method, find the mean and standard deviation of the distribution of marks scored in a certain test by a number of students (Table 3.25).

Marks	Frequency
72	4
72	8
82	11
87	15
92	9
92	3

Table 3.25

5. Some AIDS sufferers were weighed and their masses were recorded as in Table 3.26. Find the mean and standard deviation of the masses.

Mass (kg)	No. of patients
34.5	4
44.5	26
54.5	40
64.5	26
74.5	2
84.5	2

Table 3.26

! CAUTION: AIDS has no cure. However, there is a sure way of avoiding it: TOTAL ABSTINENCE before marriage, and once married, STICK TO YOUR PARTNER!

6. Using an assumed mean of 32, find the mean and standard deviation of the distribution in Table 3.27.

Marks	Frequency
17	8
22	10
27	16
32	26
37	22
42	12
47	6

Table 3.27

7. Table 3.28 shows the masses of eighty students in a certain college. Calculate the mean and standard deviation of the masses.

Mass (kg)	Frequency
52	12
57	14
62	24
67	15
72	8
77	7

Table 3.28

8. The frequency distribution in Table 3.29 shows the masses of some biological specimens.

Mass (g)	2	9	20	35	60
Frequency, f	14	41	59	70	15

Table 3.29

Calculate:

(a) the mean mass,

(b) the standard deviation of the masses.

9. A movie was rated "unsuitable for under 16". Table 3.30 shows the age distribution of those who attended one sitting. What is the mean age and standard deviation of the ages of the attendants?

Age	Frequency
19.5	6
28.5	21
38.5	45
48.5	66
63.5	51

Table 3.30

4 SIMULTANEOUS LINEAR AND QUADRATIC EQUATIONS

Success criteria

By the end of this topic, the student must be able to:
- *Solve simultaneous linear and quadratic equations.*
- *Solve practical problems involving simultaneous linear and quadratic equations.*

Introduction

In Form 2, we solved linear simultaneous equations using three different methods namely;

(i) elimination

(ii) substitution

(iii) graphical methods

In Form 3, we solved simultaneous equations, one linear and one quadratic by graphical method. In this chapter we will learn how to solve simultaneous equations, one linear and one quadratic using the substitution method.

We will begin the topic with a brief revision of how to solve simultaneous equations by substitution method; and how to solve quadratic equations using various methods.

Example 4.1

Solve the following simultaneous equations.

$x - 3y = 6$

$x + y = 10$

Solution

Use substitution method to find the value of x in terms of y.

$x - 3y = 6$ ...... (i)

$x + y = 10$ ...... (ii)

$x = 6 + 3y$ (*from equation (i)*)

Substituting the value of x in equation (ii).

$(6 + 3y) + y = 10$

$6 + 4y = 10$

$4y = 10 - 6$

$4y = 4$

$y = 1$

Substitute the value of y in equation (i) to get the value of x.

$x - 3(1) = 6$

$x - 3 = 6$

$x = 6 + 3$

$x = 9$

$\therefore$ *the solution is* $x = 9, y = 1$.

Example 4.2

Use factor method to solve the equation
$8x^2 - 2x - 3 = 0$

Solution

Rearrange the equation $8x^2 - 2x - 3 = 0$ *to facilitate factorisation.*

$8x^2 - 2x - 3 = 0$

$8x^2 - 6x + 4x - 3 = 0$

$2x(4x - 3) + 1(4x - 3) = 0$ (*factorise LHS by grouping*)

$(4x - 3)(2x + 1) = 0$

$4x - 3 = 0$ or $2x + 1 = 0$

$4x = 3$ $\quad$ $2x = -1$

$\therefore \quad x = \frac{3}{4}$ $\quad$ $x = \frac{-1}{2}$

$\therefore$ *The solutions of* $8x^2 - 2x - 3 = 0$ *are* $x = \frac{3}{4}$, *or* $\frac{-1}{2}$.

Example 4.3

Solve the equation $2x^2 - x - 8 = 0$ using the quadratic formula.

Solution

$2x^2 - x - 8$ *is of the form* $ax^2 + bx + c = 0$.

So, $a = 2$, $b = -1$ *and* $c = -8$.

Using the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - (4 \times 2 \times -8)}}{2 \times 2}$$

$$= \frac{1 \pm \sqrt{1 + 64}}{4}$$

$$= \frac{1 \pm \sqrt{65}}{4}$$

$$x = \frac{1 + \sqrt{65}}{4} \text{ or } \frac{1 - \sqrt{65}}{4}$$

$$= 2.27 \text{ or } -1.77 \text{ (2 d.p.)}$$

Exercise 4.1

1. Use substitution method to solve the simultaneous equations

   (a) $x - 2y = 10$, $3x + 2y = 6$ (b) $3x - 2y = 10$, $4x + 3y = 2$

2. Use the factor method to solve the following equations.

   (a) $4x^2 - 9 = 0$ (b) $x^2 + 6x + 9 = 0$

3. Use the quadratic formula to solve the following equations.

   (a) $2x^2 - 7x - 21 = 0$ (b) $4x^2 - 12x + 9 = 0$

Solving simultaneous linear and quadratic equations by substitution method

Simultaneous equations involving a linear and a quadratic equation are solved either graphically or by substitution methods. At the beginning of this chapter, both methods were revised and some revision exercise given.

In this section, we will concentrate on the substitution method where one equation is linear and the other is quadratic.

The approach of this method is to eliminate one of the two variables so that we deal with a quadratic equations in one unknown.

Now, consider the equations

(i) $y = ax^2 + bx + c$

(ii) $y = mx + n$ where a, b, c, m and n are constants.

In equation (ii), y is already expressed in terms of x.

If we substitute $mx + n$ in equation (i), we shall have an equation in x only.

Thus, $y = ax^2 + bx + c$ becomes

$mx + n = ax^2 + bx + c$

$ax^2 + bx - mx + c - n = 0$ (rearranging the equation so that it is equal to zero)

$ax^2 + (b - m)x + (c - n) = 0$

$\therefore ax^2 + (b - m)x + (c - n) = 0$ $(b - m)$ can be represented by a single constant like $\boldsymbol{k}$ and $(c - n)$ by another constant say, $\boldsymbol{t}$.

Thus we have $ax^2 + \boldsymbol{k}x + \boldsymbol{t} = 0$.

The resulting equation is a simple quadratic equation which may be solved by **factor method** if possible or by the **quadratic formula** or by **completing the square method**.

Usually, we use the simplest form of the linear equation in order to minimize the amount of computation involved in the process.

For example, in our case, if we had decided to express x in terms of y, we would have ended up with a fractional expression which is not suitable.

i.e. $y = mx + n$

$mx = y - n$

$$x = \frac{y - n}{m}$$

It is easier to substitute $y = mx + n$ than to substitute $x = \dfrac{y - n}{m}$

The examples below illustrate use of substitution method to solve simultaneous equations where one is linear and the other is quadratic. This method is combined with factor method, completing the square method or quadratic formula whichever may be appropriate.

Example 4.4

Solve the simultaneous equations
$y = x^2 - 4x + 5$ *and* $y = 8 - 2x$.

Solution

$y = x^2 - 4x + 5$
$y = 8 - 2x$

Substitute $y = 8 - 2x$ *into* $y = x^2 - 4x + 5$ *in order to obtain a quadratic equation in one unknown.*

Thus, $y = x^2 - 4x + 5$ *becomes*

$8 - 2x = x^2 - 4x + 5$
$x^2 - 4x + 5 + 2x - 8 = 0$
$x^2 - 2x - 3 = 0$

The resulting equation is a simple quadratic equation, which can be solved by factor method.

$x^2 - 2x - 3 = 0$
$x^2 - 3x + x - 3 = 0$
$x(x - 3) + 1(x - 3) = 0$
$(x + 1)(x - 3) = 0$
$\therefore\ x = -1 \text{ or } 3$

Substitute -1 *and* 3 *in the linear equation to find the corresponding values of* y.

Using $y = 8 - 2x$,
When $x = {}^-1,\ y = 8 - 2(-1)$
$= 8 + 2 = 10$

When $x = 3,\ y = 8 - 2(3)$
$= 8 - 6 = 2$

Solutions are: when $x = {}^-1,\ y = 10$
$x = 3,\ y = 2$

Example 4.5

Solve the simultaneous equations
$y = 2x^2 - 13x + 15$ *and* $y = {}^-x + 2$

Solution

Substitute $y = {}^-x + 2$ *into* $y = 2x^2 - 13x + 15$

$y = 2x^2 - 13x + 15$ *becomes*

${}^-x + 2 = 2x^2 - 13x + 15$
$0 = 2x^2 - 13x + x + 15 - 2$
$0 = 2x^2 - 12x + 13$

The resulting quadratic equation has no factors. So, we use the quadratic formula.

$$x = \frac{{}^-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In $2x^2 - 12x + 13 = 0,\ a = 2,\ b = -12,\ c = 13$

$$\therefore\ x = \frac{-({}^-12) \pm \sqrt{({}^-12)^2 - 4(2)(13)}}{2 \times 2}$$

$$x = \frac{12 \pm \sqrt{144 - 104}}{4}$$

$$= \frac{12 \pm \sqrt{40}}{4}$$

$$= \frac{12 \pm 6.325}{4}$$

$$= \frac{12 + 6.325}{4} \text{ or } \frac{12 - 6.325}{4}$$

$x = 4.581 \text{ or } 1.419$

Now, substitute the values of x *in the linear equation.*

When $x = 4.581,\ y = {}^-x + 2$
$= {}^-4.581 + 2$
$= {}^-2.581$

when $x = 1.419,\ y = -x + 2$
$= -1.419 + 2$
$= 0.581$

Solutions are:
When $x = 4.581,\ y = -2.581$
When $x = 1.419,\ y = 0.581$

Example 4.6

Solve the simultaneous equations
$y = 2x^2 - 4x + 5$ *and* $y = {}^-2x + 6$.

Solution

Substitute $y = -2x + 6$ *in* $y = 2x^2 - 4x + 5$

$y = 2x^2 - 4x + 5$ *becomes*

$$2x^2 - 4x + 5 = -2x + 6$$

$$2x^2 - 4x + 2x + 5 - 6 = 0$$

$$2x^2 - 2x - 1 = 0$$

This equation has no factors.

This time, we will use completing the square method.

$$2x^2 - 2x - 1 = 0$$

$$2x^2 - 2x = 1$$

$$x^2 - x = \frac{1}{2}$$

$$x^2 - x + (^-\tfrac{1}{2})^2 = \frac{1}{2} + (^-\tfrac{1}{2})^2$$

$$= \frac{1}{2} + \frac{1}{4}$$

$$(x - \tfrac{1}{2})^2 = \frac{3}{4}$$

$$x - \frac{1}{2} = \pm\sqrt{\tfrac{3}{4}}$$

$$x - \frac{1}{2} = \pm\frac{\sqrt{3}}{2}$$

$$x = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \text{ or } \frac{1}{2} - \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2} + \frac{1.732}{2} \text{ or } \frac{1}{2} - \frac{1.732}{2}$$

$$x = 0.5 + 0.866 \text{ or } 0.5 - 0.866$$

$$= 1.366 \text{ or } ^-0.366$$

Using the linear equation, we will find the corresponding values of y.

When $x = 1.366$, $y = -2(1.366) + 6$

$$= -2.732 + 6$$

$$= 3.268$$

When $x = {}^-0.366$

$$y = 2(^-0.366) + 6$$

$$= {}^-0.732 + 6$$

$$= 5.268$$

$\therefore$ *The solutions are*

when $x = 1.366$, $y = 3.268$

$x = {}^-0.366$, $y = 5.268$

Exercise 4.2

In this exercise, use the substitution method only to solve the simultaneous equations.

1. $3y = 16x + 24$
   $y = 4x^2 - 12x + 9$

2. $2y = x + 6$
   $y = x(4 - x)$

3. $y = {}^-2x^2 + 3x + 9$
   $y = 2x + 2$

4. $x + y = 0$
   $x^2 + y^2 - xy = 12$

5. $xy = 4$
   $x + y = 5$

6. $y = 2x^2 + x - 2$
   $y = -x + 1$

7. $y = (1 - 2x)(4 + x)$
   $y = 2 - 3x$

8. $y = 2x^2 - x + 2$
   $y = 4x + 2$

9. $y = 4x^2 - x + 3$
   $y = 5x + 2$

10. $y = 2x^2 - 3x - 7$
    $y = 2x - 1$

11. Solve the simultaneous equations.
    $x^2 + y^2 = 10$
    $x - y = 2$

12. Solve the simultaneous equations
    $y = -x^2 + 5x - 3$ and $y = 3x - 15$

13. Solve the simultaneous equations
    $2x - y = 3$ and $x^2 - xy = -4$

Problems leading to simultaneous linear and quadratic equations

We have already seen how to solve problems leading to simple equations. The important points to remember are:

(i) Choose a letter to represent the required number.

(ii) Translate each statement given in the question into a statement containing the letter.

(iii) By linking up the parts of the question, form an equation.

(iv) Solve the equation.

When dealing with simultaneous equations, we use the same steps *except* that we use two letters and hence form two equations in the two unknowns and solve them simultaneously. The following worked examples illustrate this method.

Example 4.7

The sum of ages of two children, a brother and a sister is 6. The sum of the square of their ages is 20. Find their ages if the brother is the eldest.

Solution

Let the brother's age be x

Let the sister's age be y

Forming simultaneous equations:

$x + y = 6$ ....(i)

$x^2 + y^2 = 20$ ......(ii)

From equation (i)

$x = 6 - y$

Substituting for x in equation (ii)

$(6 - y)^2 + y^2 = 20$

$36 - 12y + y^2 + y^2 = 20$

$2y^2 - 12y + 16 = 0$ (simplifying)

$y^2 - 6y + 8 = 0$

$y^2 - 4y - 2y + 8 = 0$

$y(y - 4) - 2(y - 4) = 0$

$(y - 2)(y - 4) = 0$

$y = 2$ or 4

if $y = 2, x = 6 - 2 = 4$

if $y = 4, x = 6 - 4 = 2$

Since the brother is the eldest, the ages are brother = 4 years and sister = 2 years.

Example 4.8

A rectangle has a perimeter of 42 cm and an area of 68 cm². Find the dimensions of the rectangle.

Solution

Let the length of the rectangle be x cm and the width be y cm.

Perimeter: $2x + 2y = 42$......... (i)

Area: $xy = 68$ .... (ii)

Equation (i) can simplify to $x + y = 21$ .... (iii)

Solve simultaneously

$x + y = 21$

$xy = 68$

Using equation (iii)

$x + y = 21$

$y = 21 - x$

Substituting in equation (ii)

$xy = 68$

$x(21 - x) = 68$

$21x - x^2 = 68$

$x^2 - 21x + 68 = 0$

$x^2 - 17x - 4x + 68 = 0$

$x(x - 17) - 4(x - 17) = 0$

$(x - 17)(x - 4) = 0$

$x = 17$ or 4

if $x = 17, y = 21 - 17 = 4$

$x = 4, y = 21 - 4 = 17$

The rectangle has a length of 17 cm and a width of 4 cm.

Exercise 4.3

1. A boy covers a distance of 4 km partly by walking and partly by cycling. If he cycles at

12 km/h and walks at 4 km/h and takes a total of 35 minutes, find the distance he walked.

2. A two digit number is increased by 36 when the digits are reversed. The product of their digits is 21. Find the number.

3. The sum of the squares of the ages of a father and a son is 964 years. Four years ago, the difference of their ages was 22 years. Find their present ages.

4. Peter covers a distance of 10 km partly by walking and partly by cycling. If he walks at 4 km/h and cycles at 12 km/h and takes a total of 1 hr and 10 minutes to complete the journey, find the distance he walks.

5. The sum of two numbers is 14 and the sum of their squares is 100. Find the numbers.

6. The perimeter of a rectangle is 26 cm and one of its sides is x cm long. If the area of the rectangle is 36 cm^2, calculate the lengths of the sides.

5 PROGRESSIONS

Success criteria

By the end of this topic, the student must be able to:

- *Find the terms of an arithmetic progression.*
- *Find the sum of term of arithmetic progression.*
- *Apply AP in real life situations.*
- *Find the terms of a geometric progression.*
- *Find the sum of the terms of a geometric progression.*
- *Apply GP in real life situations.*

Introduction

You have already learnt that a sequence is an ordered list of numbers. Remember also that, any term in a sequence can be represented in short as t_n or u_n meaning the n^{th} term. In this chapter, we are going to explore further and distinguish between geometric and arithmetic progressions and use them to solve problems.

Arithmetic sequences

Examine the following sequences:

(a) 1, 3, 5, 7, 9, … (b) 2, 4, 8, 16, 32, …

(c) 1, 4, 9, 16, 25, …

In the sequence 1, 3, 5, 7, 9, …, any two consecutive terms differ by 2, i.e.

$$3 - 1 = 5 - 3 = 7 - 5 = 9 - 7 = 2, \text{ etc.}$$

Note that, the difference in consecutive terms of sequences (b) and (c) is not constant.

A sequence in which any two consecutive terms differ by the same number, i.e. a constant, is called an **arithmetic sequence**. The constant number by which the consecutive terms differ is called the **common difference**.

An arithmetic sequence is generally expressed in the form $a, a + d, a + 2d, a + 3d, \ldots$, where $\boldsymbol{a}$ is the first term and $\boldsymbol{d}$ is the common difference.

Note that 1st term = a, 2^{nd} term = $a + d$, 3^{rd} term = $a + 2d$ and 4^{th} term = $a + 3d$. What is the 5^{th} term, 10^{th} term, 15^{th} term, n^{th} term?

In general,

The n^{th} term (denoted as U_n or T_n) of an arithmetic sequence is $\boldsymbol{a + (n - 1)d}$, i.e.

$$T_n = a + (n - 1)d \text{ and } d = T_n - T_{n-1}.$$

Given a number of terms in an arithmetic sequence, we can find the n^{th} term of the sequence

Example 5.1

In the sequence $^-2, 2, 6, 10, \ldots$ what is the 16^{th} term?

Solution

First term $a = {^-2}$, common difference $d = 2 - (^-2) = 4$, n^{th} term $= a + (n - 1)d$

$\therefore$ *16^{th} term $= {^-2} + (16 - 1) \times 4 = 58$.*

Example 5.2

The seventh term of an arithmetic sequence is 80. If the eleventh term is 68, what is the 4^{th} term?

Solution

In this example there are two unknowns, the first term and the common difference.

Let a = first term, d = common difference

The n^{th} term $= a + (n-1)d$

7^{th} term $= a + 6d = 80$ ……………(i)

11^{th} term $= a + 10d = 68$ ………… (ii)

Equation (ii) – equation (i) gives $4d = {}^-12$

$\therefore\ d = {}^-3$

Substituting in (i) $a + 6d = 80$

$a = 80 - 6d = 80 - 6({}^-3)$

$= 98$

Thus, 4^{th} term is $a + 3d = 98 + 3(-3)$

$= 98 - 9$

$= 89.$

Exercise 5.1

1. For each of the following sequences, find the indicated terms.
   (a) $1, 3\frac{1}{2}, 6, 8\frac{1}{2}, \ldots$; 20^{th} and 30^{th} terms.
   (b) $4, 3\frac{1}{2}, 3, 2\frac{1}{2}, \ldots$; 15^{th} and 25^{th} terms.
   (c) $1, 2\frac{1}{2}, 4, 5\frac{1}{2}, \ldots$; 10^{th} and 20^{th} terms.
   (d) $1, 0.8, 0.6, \ldots$; 10^{th} and 15^{th} terms.

2. How many terms are in each of the following sequences?
   (a) $1, 4, 7, \ldots, 61$
   (b) $4, 4\frac{1}{4}, \ldots, 7$
   (c) $20, 17\frac{1}{2}, \ldots, {}^-15$
   (d) $5, 5.9, 6.8, \ldots, 23$

3. The 10^{th} term of an arithmetic sequence is ${}^-15$, and the 20^{th} term is -35. What is:
   (a) the first term,
   (b) the common difference?

4. The 9^{th} term of an arithmetic sequence is 37 and the 16^{th} is 65. What is the 20^{th} term?

5. An arithmetic sequence consists of all integers between 1 and 100 which are divisible by 3. How many terms are there?

6. Which term of the arithmetic sequence 4, 13, 22, … is 139?

7. What are the next three terms of the sequence 1, 5, 13, 25, 31, …?
   [Hint: Find the difference between consecutive terms of the sequence. What type of sequence do you get?]

Arithmetic series/progression (A.P.)

A **series** is the sum of the terms of a sequence. Thus, from the sequences (a) 1, 3, 5, 7, … (b) 2, 4, 8, 16, … (c) 1, 4, 9, 16, … we obtain the series

(a) $1 + 3 + 5 + 7 + \ldots$

(b) $2 + 4 + 8 + 16 + \ldots$

(c) $1 + 4 + 9 + 16 + \ldots$ respectively.

Note that the series $1 + 4 + 9 + 16 + 25 + \ldots + 100$, has a finite number of terms. A series with a finite number of terms is called a **finite series.**

A series such as $1 + 3 + 5 + 7 + \ldots$ has many terms that cannot be counted exhaustively. Such a series is called an **infinite series**.

When the terms of an arithmetic sequence are added, the series obtained is called an **arithmetic progression** (**A.P.**) or an arithmetic series.

Thus, the arithmetic sequence a, $a + d$, $a + 2d$, $a + 3d$, … becomes the Arithmetic progression $a + (a + d) + (a + 2d) + (a + 3d) + \ldots$

Just as for an arithmetic sequence, for an A.P, the first term is a, the common difference is d and the **nth term** is $\boldsymbol{a + (n-1)d}$.

We use the symbol S_n to denote the sum of the first n terms in a series. Thus for the sum of the arithmetic series $1 + 3 + 5 + 7 + \ldots$, S_1 is equivalent to the first term, S_2 is the sum of the first 2 terms, S_3 is the sum of the first 3 terms, etc. Thus:

$S_1 = 1$, $S_2 = 1 + 3 = 4$, $S_3 = 1 + 3 + 5 = 9$

$S_4 = 1 + 3 + 5 + 7 = 16,$

$S_5 = 1 + 3 + 5 + 7 + 9 = 25$, and so on.

$\therefore S_{10} = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100$

We can express the sum S_{10} in two ways:

$S_{10} = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$ and

$S_{10} = 19 + 17 + 15 + 13 + 11 + 9 + 7 + 5 + 3 + 1$

Adding the two vertically gives:

$2S_{10} = 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 = 200.$

$\therefore S_{10} = \frac{200}{2} = 100.$

We use the same method to find the sum (S_n) of n terms of an A.P.:

$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + [a + (n - 1)d].$

For simplicity, we write the last term $[a + (n - 1)d]$ as l.

So we have $S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$ .... (i)

Writing the same equation for S_n but starting with the last term, we have

$S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a$ .... (ii)

Adding the terms on the left of equations (i) and (ii) together and also adding the terms on the right hand side of the two equations, gives:

$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots\ldots + (l - 3d) + (l - 2d) + (l - d) + l$ (i)

$S_n = l + (l - d) + (l - 2d) + (l - 3d) + \ldots\ldots\ldots + (a + 3d) + (a + 2d) + (a + d) + a$ (ii)

$2S_n = (a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l)$

i.e. $2S_n = n(a + l)$

$\therefore S_n = \frac{1}{2}n(a + l)$ where a is the first term and l is the last term of the series.

Substituting $a + (n - 1)d$ for l gives

$S_n = \frac{n}{2}[a + a + (n - 1)d]$: Thus,

$$\boxed{S_n = \frac{n}{2}[2a + (n - 1)d]}$$

Example 5.3

In the arithmetic series $5 + 9 + 13 \ldots$, find

(a) the 100^{th} term

(b) the sum of the first 90 terms

Solution

(a) n^{th} term $l = [a + (n - 1)d]$, $a = 5$, $d = 4$, $n = 100$

$\therefore 100^{th}$ term $= [5 + (100 - 1) \times 4]$

$= 5 + 99 \times 4$

$= 401$

(b) sum of the first 90 terms $= S_{90}$

$S_n = \frac{n}{2}[2a + (n - 1)d]$ where $n = 90$, $a = 5$ and $d = 4$.

Thus, $S_{90} = \frac{90}{2}[2 \times 5 + (90 - 1) \times 4]$

$= 90(5 + 89 \times 2)$

$= 16\,470.$

Example 5.4

The third and the fifth terms of an arithmetic progression are 10 and –10 respectively.

(a) Determine the first term and the common difference.

(b) Find the sum of the first 15 terms.

Solution

(a) *If the first term is a and the common difference is d, third term is*

$a + 2d = 10$ *and fifth term is*

$a + 4d = -10$

Solving the equations simultaneously,

$$\begin{array}{r} a + 2d = 10 \\ a + 4d = -10 \\ \hline -2d = 20 \\ \hline \end{array}$$

$\therefore d = -10$

$a + 2d = 10$

$a + 2(-10) = 10$

$$a = 10 + 20 = 30$$
$$a = 30$$
(b) $S_{15} = \frac{n}{2}(2a + (n-1)d)$

$= \frac{15}{2}[2 \times 30 + (15-1) \times (-10)]$

$= \frac{15}{2}(60 - 140) = \frac{15}{2} \times (-80)$

$S_{15} = -600$

Exercise 5.2

1. State which of the following are A.P.s, and state their common difference.
   (a) $1 + 2 + 4 + 8 + \ldots$
   (b) $^-9 - 7 - 5 - 3 + \ldots$
   (c) $1^2 + 2^2 + 3^2 + 5^2 + \ldots$
   (d) $19 + 20 + 21 + \ldots$
   (e) $2 + 5 + 8 + \ldots$
   (f) $1^4 + 2^4 + 3^4 + \ldots$
   (g) $8 + 14 + 20 + \ldots$
   (h) $7 + 14 + 28 + \ldots$
2. Given the A.P. $7 + 13 + 19 + \ldots$
   (a) write down the 15th term
   (b) find the sum of 15 terms and n terms
3. Given the A.P. $9 + 13 + 17 + \ldots + 85$, find:
   (a) the number of terms in the series,
   (b) the sum of the series.
4. The first term of an A.P. is 10 and the sum of the first 15 terms is 465. Find an expression for the sum of the first n terms.
5. The second term of an A.P. is 18 and the 10^{th} term is 106. What is the sum of the first 10 terms?
6. The sum of the A.P. $10 + 7 + 4 + \ldots$ is -4. How many terms does the A.P. have?
7. How many terms of the A.P. $19 + 16 + 13 + \ldots$ must be taken, at most, before the sum becomes less than zero?
8. Find the sum of the even numbers between 100 and 300 that can be divided by 7?
9. Which term of the A.P. $15 + 12 + 10 + \ldots$ is the first to be less than zero?

Geometric sequence

Examine the sequences:

(i) 2, 4, 8, 16, 32, … and

(ii) 4, 9, 16, 25, …

Neither of them is an arithmetic sequence. In sequence (i), it can be seen that each term is obtained by multiplying the preceding one by 2 and so, if we divide each term by the preceding one, we get a constant value. Thus, we have $4 \div 2 = 8 \div 4 = 16 \div 8 = 32 \div 16 = 2$.

A sequence in which the ratio between any two consecutive terms is a constant value is called a **geometric sequence**. This constant value is called the **constant ratio** or **common ratio**.

Note that sequence (ii) above is neither an arithmetic nor a geometric sequence.

In general, the first term is represented by ***a*** and the common ratio is represented by ***r***. Since in a geometric sequence each term is obtained by multiplying the preceding one by the common ratio, then we have:

1^{st} term $= a$, 2^{nd} term $= ar$,

3^{rd} term $= ar \times r = ar^2$, 4^{th} term $= ar^2 \times r = ar^3$.

What is the 5^{th} term, 12^{th} term, 15^{th} term, n^{th} term?

We note that:

The n^{th} term (U_n or T_n) of a geometric sequence is ar^{n-1}, i.e. $T_n = ar^{n-1}$.

In general, a geometric sequence is in the form

$$a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}$$

Example 5.5

What are the 8^{th}, 10^{th} and n^{th} terms in the sequence $^-8, 24, ^-72, 216, \ldots$?

Solution

This is a geometric sequence with the first term $a = -8$ and common ratio $r = \frac{24}{-8} = -3$.

$\therefore$ *8^{th} term* $= -8 \times (-3)^{8-1} = -8(-3)^7$

$= 17\,496$ (by calculator)

$= 17\,500$ (3sf)

10^{th} term $= 8 \times (-3)^{10-1} = -8(-3)^9$

$= 157\,464$ (by calculator)

$= 157\,400$ (4sf)

n^{th} term $= {^-8} \times (-3)^{n-1}$

Example 5.6

The 3^{rd} term of a geometric sequence is 3 and the 5^{th} term is $\frac{3}{4}$. Write down the first 4 terms of the sequence.

Solution

3^{rd} term $= ar^{3-1} = ar^2 = 3$ .... (i)

5^{th} term $= ar^{5-1} = ar^4 = \frac{3}{4}$ ....(ii)

(ii) ÷ (i) gives $r^2 = \frac{1}{4} \Rightarrow r = \frac{1}{2}$ or $^-\frac{1}{2}$

$\therefore ar^2 = 3 \Rightarrow a \times (\frac{1}{2})^2 = 3$

$\therefore a = 12.$

There are two sequences: 12, 6, 3, $\frac{3}{2}$, ... and 12, $^-6$, 3, $^-\frac{3}{2}$, ...

Exercise 5.3

1. For each of the following sequences, find the common ratio and the indicated terms.
   (a) 1, 3, 9, ...; 5^{th} and 9^{th} terms
   (b) 1, $\frac{2}{3}$, $\frac{4}{9}$, ...; 7^{th} and 10^{th} terms
   (c) 1, 0.8, 0.64, ...; 4^{th} and 6^{th} terms
   (d) 2, $^-4$, 8, ...; 8^{th} and 12^{th} terms
   (e) $\frac{2\,401}{16}$, $\frac{343}{8}$, $\frac{49}{4}$, ...; 8^{th} term
   (f) 6, –0.6, 0.06, ...; 6^{th} term

2. Find how many terms each of the following sequences has.
   (a) 2, $\frac{1}{2}$, $\frac{1}{8}$, ..., $\frac{1}{512}$
   (b) 20, 4, 0.8, ..., 0.006 4
   (c) 60, 20, $6\frac{2}{3}$, ..., $\frac{20}{729}$
   (d) 80, 32, 12.8, ..., 0.819 2

3. 12, b, 75 are consecutive terms of a geometric sequence. What is the common ratio for the sequence?

4. The second term in a geometric sequence is 64. The fifth term is $\frac{1}{8}$. Find the first 4 terms of the sequence.

5. The first term of a geometric sequence is 16 and the fifth term is 9. What is the value of the 9^{th} term?

6. The first term of a geometric sequence is 1.1 and the fourth term is 1.464 1. What is the common ratio for the sequence?

7. The second term of a geometric sequence is $^-1$ and the fifth term is $\frac{1}{8}$. What is the tenth term?

8. Ounga saves K 100 on his son's first birthday. He saves K 200 on the second birthday and K 400 on the third birthday and so on doubling the amount on every birthday. How much will he have saved on the boy's 10th birthday?

9. Which is the first term of the geometrical sequence 2 187, 729, 243, ... which is less than 1?

Geometric series/Progression (G.P.)

When the terms of a geometric sequence are added, the series obtained is called a **Geometric Progression** (G.P.) or **Geometric series**.

Thus from the geometric sequence 2, 4, 8, 16, ..., we obtain the G.P. 2 + 4 + 8 + 16 + ...

In general, a geometric sequence $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$ gives the geometric progression
$\boldsymbol{a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}}$ where $\boldsymbol{a}$ is the first term and $\boldsymbol{r}$ is the **common ratio**.

Taking S_n as the sum of the first n terms of a G.P. we have

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

Thus, for the series $2 + 4 + 8 + 16 + \ldots$
$S_1 = 2,\ S_2 = 2 + 4 = 6,\ S_3 = 2 + 4 + 8 = 14,$
$S_4 = 2 + 4 + 8 + 16 = 30$, etc.

Consider the sum

$$S_{10} = 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1\,024 = 2\,046.$$

We can also obtain S_{10} in the following way.

$$S_{10} = 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1\,024 + \ldots\ldots\ldots \text{(i)}$$

Multiplying each term of equation (i) by 2, the common ratio, we have

$$2S_{10} = 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1\,024 + 2\,048 \ldots\ldots\ldots \text{(ii)}$$

Rewriting the two sums and subtracting as shown, we have

$$\begin{array}{rl} 2S_{10} = & 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1\,024 + 2\,048 - \\ S_{10} = & 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1\,024 + \\ \hline 2S_{10} - S_{10} = & -2 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 2\,048 \\ \hline \end{array}$$

Thus, $2S_{10} - S_{10} = 2\,048 - 2$

$$(2 - 1)S_{10} = 2\,046$$

$$\therefore\ S_{10} = 2\,046.$$

We use the same method to find the sum S_n of n terms of a G.P.

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} \ldots \text{(i)}$$

Multiplying (i) by r, the common ratio, we have

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \ldots\ldots\ldots \text{(ii)}$$

Thus, (ii) – (i) gives $rS_n - S_n = ar^n - a$

$$S_n(r - 1) = a(r^n - 1)$$

$$\therefore\ S_n = \frac{a(r^n - 1)}{r - 1}$$

Alternatively, (i) – (ii) gives

$$S_n - rS_n = a - ar^n$$

$$S_n(1 - r) = a(1 - r^n)$$

$$\therefore\ S_n = \frac{a(1 - r^n)}{1 - r}$$

Note:

When the common ratio $|r| > 1$, we use the formula $sn = \dfrac{a(rn - 1)}{r - 1}$

and when the common ratio $|r| < 1$, we use the formula $sn = \dfrac{a(1 - rn).}{1 - r}$

This helps avoid having to divide by a negative denominator.

$|r|$ means the absolute value of r.

Example 5.7

Find the sum of the first 10 terms in the following G.P.s.

(a) $3 + 6 + 12 + 24 + \ldots$

(b) $12 + 4 + \quad + \ldots$

Solution

(a) $3 + 6 + 12 + 24 + \ldots$

First term $a = 3$, common ratio $r = 2$.

Since $r = 2$ i.e is greater than 1, we use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$\therefore\ S_{10} = \frac{3(2^{10} - 1)}{2 - 1} = 3 \times 1\,023 = 3\,069$$

(b) $12 + 4 + \frac{4}{3} + \ldots$

$a = 12,\ r = \frac{1}{3}$

Since $r = \frac{1}{3}$ i.e is less than 1, we use the formula $S_n = \dfrac{a(1 - r^n)}{1 - r}$

$$\therefore\ S_{10} = \frac{12[1 - (\frac{1}{3})^{10}]}{1 - \frac{1}{3}} = \frac{12[1 - (\frac{1}{3})^{10}]}{\frac{2}{3}}$$

$$= 18[1 - (\tfrac{1}{3})^{10}]$$

$$= 18(1 - 0.000\,02) \approx 18 \times 1$$

Thus, $S_{10} \approx 18.$

Example 5.8

The sum of the first three terms of a geometric series is 42. If the common ratio is 4, find the sum of the first seven terms.

Solution

$S_3 = 42, r = 4, n = 3$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$42 = \frac{a(4^3 - 1)}{4 - 1}$$

$$42 = \frac{a(64 - 1)}{3}$$

$$42 = a\left(\frac{63}{3}\right) \therefore a = \frac{42 \times 3}{63} = 2$$

$$S_7 = \frac{2(4^7 - 1)}{4 - 1}$$

$$= \frac{2(1\,6384 - 1)}{3} = \frac{2 \times 1\,6383}{3}$$

$$= 10\,922$$

Exercise 5.4

1. Find the last term and the sum of the following G.P.s.
   (a) $\frac{2}{3} + 2 + 6 + \ldots$; (6 terms)
   (b) $9 + 3 + 1 + \ldots$; (8 terms)
   (c) $\frac{1}{2} - 1 + 2 - \ldots$; (10 terms)
   (d) $1 + \frac{2}{3} + \frac{4}{9} + \ldots$; (8 terms)
   (e) $1 + 0.9 + 0.81 + \ldots$; (5 terms)
   (f) $9 - 6 + 4 - \ldots$; (9 terms)
   (g) $\frac{40}{3} + 10 + \frac{15}{2} + \ldots$; (10 terms)
2. Find the sums of the following G.P.s.
   (a) $1 - 2 + 4 - \ldots + 1\,024$
   (b) $100 - 10 + 1 - \ldots + 0.000\,001$
   (c) $50 + 25 + 12.5 + \ldots + 0.781\,25$
   (d) $20 - 4 + \ldots - 0.000\,256$
   (e) $0.5 - 2.5 + 12.5 - \ldots - 1\,562.5$
   (f) $9 + 3 + \ldots + \frac{1}{243}$
3. The second term of a G.P. is 8 and the 4th term is 128. Write down the first 4 terms of the GP.
4. The 3rd term of a GP is $\frac{-1}{3}$ and the 6th term is $\frac{1}{81}$. What is the first term of the G.P.?
5. The 3rd term of a G.P. is $\frac{1}{2}$ and the 5th term is $\frac{1}{32}$. Find the sum of the first six terms.
6. The first three terms of a G.P. are the first, fourth and tenth terms of an A.P. Given that the first term is six, and that all the terms of the G.P. are different, find the common ratio.
7. What is the smallest number of terms of the G.P. $1 + 4 + 16 + 64 + \ldots$, for which the sum is more than 25 000?

Application of A.P. and G.P. to real life situations

There are many situations that require solving a problem using your knowledge of geometric and arithmetic series. However, we must identify the type of series that is useful.

Examples 5.9 and 5.10 should be useful in helping you to distinguish between the two situations.

Example 5.9

An entertainment hall has 25 seats in the front row. Each row behind has two more seats than the row before it. Find the total number of seats if the hall has 18 rows.

Solution

Since each row increases by two seats, this is an arithmetic series.

The series is 25 + 27 + 29 + ... (18 rows)

$a = 25, d = 27 - 25 = 2, n = 18$

$S_n = S_{18}$ *(total number of seats)*

$$\therefore S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$= \frac{18}{2}[(2 \times 25 + (18 - 1)2]$$

$= 9[50 + 17 \times 2]$

$= 9(50 + 17 \times 2]$

$= 9(50 + 34)$

$= 9 \times 84$

$= 756$

Thus, there are 756 seats in the hall.

Example 5.10

It is said that the population of some of the developed countries are declining. Suppose that the population of one such country was 75 million eight years ago and has been declining at a constant rate of 3% p.a. Use a G.P. to find the current population of that country.

Solution

Each year the population declines by 3% to 97% of its size at the end of the previous year.

Thus, the common ratio, r, is 0.03.

Now, the first term, a, is 75 million and the number of the terms, n, is 8 (equal to the number of years).

∴ The current population is

$S_n = ar^{n-1}$

$= 75\ 000\ 000 \times 0.03^{8-1}$

$= 75\ 000\ 000 \times 0.03^7$

$= 67{,}511{,}056$

Exercise 5.5

1. Mr. Mlenga was employed in January 1994 at a basic salary of K 79 200 per annum. If he was given an annual increment of K 6 600 each year, in which year would his salary be double his starting salary?
2. Chipo starts a savings account by depositing K 250 in the first month. Each subsequent month, his deposit will be K 75 more that the preceding one. After how many months will his savings be K 1 825 a month?
3. Ms Sigele joined a savings and co-operative society and deposited K 3 000 in the first month. She intended to save a total of K 27 000 in 16 months. How much did she have to contribute per month?
4. The value of a machine depreciated each year by 10% of its value at the beginning of that year. If its value when new was K 15 000, what was its value after 8 years?
5. A machine depreciated from K 36 000 to K 12 000 in 12 years. What was the yearly rate of depreciation, assuming it to be constant?
6. A man opens a savings account and deposits K 2 000 each year at 5% interest p.a. Find an expression for the amount in his account at the end of
   (a) the first year,
   (b) the second year, and
   (c) the third year.
   Hence, find the amount that he will have at the end of the tenth year.
7. A researcher finds that the population of bacteria in a culture that he is studying doubles itself every hour. At one point, he estimated that the population was 80 million. Estimate the number of bacteria in the culture four hours earlier.
8. For some unexplained reason, the population of the country in Example 5.10 has started growing at a steady rate of 1.5% p.a. Assuming that it will continue to grow at that rate, after how many years (to the nearest whole number) would you expect it to hit the 75 million mark again?

1–5 REVISION EXERCISES 1

Revision exercise 1.1

1. Evaluate $\begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} + 3\begin{pmatrix} 4 & 8 \\ 3 & 1 \end{pmatrix}$?

2. Find the matrix **A** such that

   $2\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -3 & 5 \end{pmatrix} + 2\begin{pmatrix} 1 & 4 \\ 4 & 0 \end{pmatrix}$.

3. In Fig. R.1.1, points O and P are centres of intersecting circles ABD and BCD respectively. Lines ABE is a tangent to circle BCD at B. Angle BCD = 42°

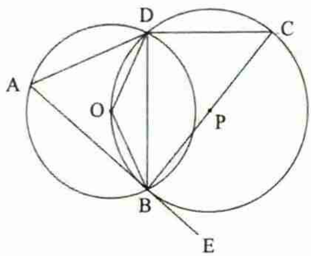


Fig. R.1.1

   (a) Stating reasons, determine the size of:
      (i) ∠CBD
      (ii) reflex ∠BOD
   (b) Show that ABD is isosceles.

4. Two circles of radii 8 cm and r cm touch externally. The common tangent XY to the two circles at X and Y respectively is 12 cm long. Find *r*.

5. Caculate the mean and standard deviation of the values:

   87, 89, 90, 93, 95, 96, 98, 101, 102

6. In an agricultural centre, the lengths of a sample of 50 maize cobs were measured and recorded as shown in the Table R.1.1.

Length (cm)	Number of cobs
9	4
12	7
15	11
18	15
21	8
24	5

Table R.1.1

   Calculate:
   (a) the mean,
   (b) (i) the variance,
   (ii) the standard deviation.

7. An arithmetic progression has a sum of 120. Given that its third term is 7 and its seventh term is 15, how many terms does the series have?

8. The third and the sixth terms of a geometric sequence are 24 and 192 respectively. Find the sum of the first 10 terms.

9. Table R.1.2 shows masses of 100 young birds. The records were stated to the nearest gram.

Mass(g)	frequency
79.5	3
99.5	7
119.5	34
139.5	43
159.5	10
179.5	2
199.5	1

Table R.1.2

Use the table to find;

(a) the mean,

(b) the standard deviation.

10. (a) Find the values of x and y if

$$\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} y & 5 \\ 2 & x \end{pmatrix} = \begin{pmatrix} 30 & 23 \\ 10 & 7 \end{pmatrix}$$

(b) Given that $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, find the column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$.

11. Use the method of completing the square to solve:

(a) $2x^2 - 14x + 9 = 0$

(b) $3x^2 - 4x = 5$

12. A right angled triangle has its longest side as 25 cm and the two shorter sides as x cm and y cm. If one of the shorter sides exceeds the other by 5 cm, form two equations in x and y and hence find the lengths of the shorter sides of the triangle.

Revison exercise 1.2

1. Given that $A = \begin{pmatrix} 2 & 10 \\ 3 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$ show that $A - B = -(B - A)$.

2. The matrices A, B and C are such that

$A = \begin{pmatrix} 2 & 4 \\ 4 & -8 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$ and $CA = B$.

Determine C.

3. ΔABC is inscribed in a circle. AB = 6 cm, $\angle BAC = 40°$ and $\angle ABC = 60°$. A tangent to the circle at C meets AB produced at D. Calculate CD.

4. In Fig. R.1.2, A and B are the centres of the circles. PQ = 12 cm is an internal tangent, AB = 15 cm and the ratio of the radii is 2:3. Calculate:

(a) the radii of the circles,

(b) AT and TQ.

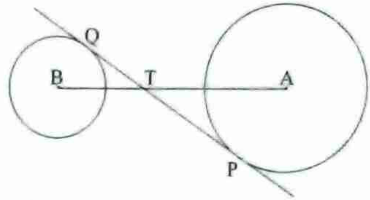


Fig. R.1.2

5. Calculate the mean and standard deviation of:

(a) 10, 13, 14, 16, 17, 20.

(b) 100, 130, 140, 160, 170, 200.

(c) 120, 150, 160, 180, 190, 220.

6. The following is a set of marks scored by a group of eleven pupils: 13, 7, 5, 16, 3, 9, 2, 20, 6, 13, 5. Use the data to find;

(a) the range.

(b) the mean deviation.

(c) the standard deviation.

7. Find the sum of the following sequences.

(a) 8, 5, 2, ... to 28th term

(b) $1, \frac{1}{2}, \frac{1}{4}, \ldots$ to 10th term

8. A motorist passes a street clock on his way to work every morning at 7.00 am. On 2nd February, he noticed that the clock had lost 2 minutes compared to the time shown on 1st. On the 3rd, it had lost 4 minutes. On 4th, it had lost 8 minutes and on 5th it had lost 16 minutes. If the clock continued losing time that way, what time was the clock showing when the motorist passed by on the 8th February.

9. Using a suitable working mean, calculate the mean of the values:
166, 171, 163, 169, 174, 172, 175, 168, 171

10. Ekari and Tadala went shopping. Ekari bought 2 kg of sugar and 5 kg of beef. Tadala bought 3 kg of sugar and 2 kg of beef.

(a) Write this information in 2×2 matrix.

(b) If the prices were K 150 per kg of sugar and K 500 per kg of beef, use matrix multiplication to calculate how much each person spent.

11. Use the quadratic formula to solve

(a) $2x^2 + 7x - 2 = 0$

(b) $x^2 - 6x + 9 = 0$

(c) $x^2 + 6x + 13 = 0$

12. What must be added to each of the following to make it a perfect square? Of what expression is each a square?

(a) $y^2 + 5y$ (b) $x^2 - 13x$

Revision exercise 1.3

1. Three matrices A, B and C are such that BA = AB + C. Determine matrix C, given that $A = \begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$

2. A and B are two matrices. If $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$, find B given that $A^2 = A + B$.

3. (a) Two circles of radii R and r touch externally. Find an expression for the length of the direct common tangent to the circles.

(b) Two circles of radii R and r have their centres d cm apart. A direct common tangent meets the line of centres, produced, at A. If $R > r$ what is the distance from the centre of the larger circle to A?

4. The two circles in Fig. R.1.3 have radii 6 cm and 4 cm and centres A and B respectively. They touch at Q and RS is a chord to the larger circle as well as a tangent to the smaller one. If TB = 5 cm, determine:

(a) the perpendicular distance from A to the chord RS.

(b) the angle subtended by chord RS at A.

(c) the area of the segment RPS.

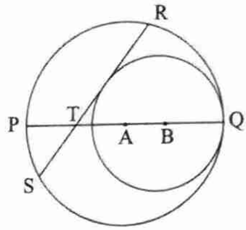


Fig. R.1.3

5. Calculate the mean and mean deviation of the data in Table R.1.3.

x	70	71	72	73	74	75	76	77	78	79	80
f	2	5	10	11	15	17	14	12	8	5	1

Table R.1.3

6. The marks scored by ten students are, 18, 17, 15, 16, 12, 19, 14, 16, 15.

(a) Find the standard deviation of the set of marks.

(b) If each mark is reduced by three marks, find:

(i) the mean of the new set of marks.

(ii) the new standard deviation.

7. On his birthday celebration, Mapiko blew out 1 candle. On his second birth day, he blew out 2 candles. On his third birth day celebration, he blew out 3 candles, and so it went on each year. His father would, each time, wish him to live long enough to have blown 1 001 candles. If each year a new set of candles were used, by which birthday will he have blown 1 001 candles in total?

8. The second term of an arithmetic sequence is 1. Given that the seventh term of the same sequence is 11, find the:

(a) first term and the common difference,

(b) sum of the first seven terms of the sequence.

9. Table R.1.4 shows the heights of some seedlings measured and recorded to the nearest millimetre. Use it to calculate:

(a) the mean.

(b) the median of the given data.

Height (mm)	28	33	38	43	48	53	58	63
f	4	5	23	58	61	30	3	3

Table R.1.4

10. Given that $A = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 8 & 1 \\ 2 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 5 \\ 7 & 1 \end{pmatrix}$ Show that:

(a) $A(BC) = (AB)C$

(b) $A(B + C) = AB + AC$

11. Use the method of factorisation to solve:

(a) $2x^2 + 5x + 2 = 0$

(b) $2x^2 - 7x + 6 = 0$

(c) $2x^2 - 4x - 6 = 0$

(d) $(x - 2)(2x + 5) = 5$

12. Use the method of factorisation to solve:

(a) $y^2 - 8 = xy$
$x + y = 0$

(b) $x^2 - y^2 = 25$
$x + y = 7$

6 TRAVEL GRAPHS

Success criteria

By the end of this topic, the student must be able to:

- *Draw velocity-time graphs.*
- *Interpret velocity-time graphs.*
- *Calculate speed, time, acceleration/deceleration and distance using velocity-time graph.*

Introduction

In Form 2, we learnt about travel graphs where we dealt with distance-time graphs. This involved drawing, interpreting and use of distance-time graphs in solving problems on distance, time and speed.

In this unit, we will learn how to draw and interpret velocity-time graphs. Let us first understand the terms used to describe the motion of a body represented in a velocity time graph. These terms are distance, speed, displacement, velocity and acceleration.

Distance

Distance is the length of the path between two points.

Speed

Speed is the **rate of change of distance** (i.e the distance covered) **per unit time**. Since in most cases the speed of a moving object keeps on varying, we usually calculate the average speed.

$$\text{Average speed} = \frac{\text{Distance covered}}{\text{Time taken}}$$

$$\text{Distance} = \text{Average speed} \times \text{Time}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Average speed}}$$

Since distance covered is a scalar quantity i.e it is not in a specific direction, speed is also a scalar quantity.

If the distance is in kilometres and the time is in hours, then the speed is given in **kilometres per hour** (**km/h** or **kmh**$^{-1}$ or **kph**).

If the distance is in metres and the time in seconds, then speed is given in **metres per second** (**m/s or ms**$^{-1}$).

Speed is therefore the rate of change of distance per unit time. Average speed is then the speed an object would move at if the motion was constant.

Example 6.1

A man walked 10.8 km in 2 h. Find his average speed in: (a) km/h (b) m/s.

Solution

(a) $\text{Average speed} = \dfrac{\text{Distance covered}}{\text{Time taken}}$

$= \dfrac{10.8 \text{ km}}{2 \text{ h}}$

$= 5.4 \text{ km/h}$

(b) Average speed $= 5.4$ km/h

$= \dfrac{5.4 \times 1\,000 \text{ m}}{1 \times 60 \times 60 \text{ s}}$ $(5.4 \times 1\,000$ m covered in 60×60 s$)$

$= 1.5$ m/s

Example 6.2

Mwanza ran 100 m in 12.5 s. What was his average speed in: (a) m/s (b) km/h?

Solution

(a) $\text{Average speed} = \dfrac{\text{Distance covered}}{\text{Time taken}}$

$$= \frac{100\ m}{12.5\ s}$$

$$= 8\ m/s$$

(b) Average speed $= 8\ m/s$

$$= \frac{8 \times 60 \times 60}{1\ 000}$$ *(Multiplying by 60 × 60 to get distance covered in 1 hour, divide by 1 000 to change distance into km)*

$$= 28.8\ km/h$$

Exercise 6.1

1. Change the following speeds to m/s.
   (a) 60 km/h (b) 5.4 km/h
   (c) 108 km/h
2. Convert the following speeds to km/h.
   (a) $3\frac{3}{4}$ m/s (b) 2.75 m/s (c) 10 m/s
3. A mini bus takes 5 seconds to cross a bridge $\frac{1}{10}$ km long. Calculate its speed in km/h.
4. A train 182 m long takes 14 seconds to pass a signal post. Calculate its speed in km/h.
5. A motorist drove 72 km on tarmac road for 10 minutes and a further 90 km on earth road for 2 h. Find the average speed for the whole journey.
6. A mini bus travelling at 60 km/h took 2 h 25 min between two towns. Find the distance between the towns.
7. Mr. Banda took a total of $2\frac{1}{4}$ h to walk from his home to his work place and back. If his average speeds for the to-and-fro journeys were 2 km/h and 3 km/h respectively, find the distance between his home and the work place.

Displacement

Displacement is the distance covered by an object moving in a particular direction. It is measured from some initial position in a particular direction.

Velocity

Velocity is the **rate of change of displacement** i.e displacement covered per unit time.

Thus,

$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Time taken}} = \frac{\text{Distance moved in a particular direction}}{\text{Time taken}}$$

Like speed, velocity is measured in **km/h** or **m/s**.

Note that velocity has both **magnitude** and **direction** while speed has only magnitude but **no** direction.

Quantities, such as velocity, which have both magnitude and direction are known as **vector quantities**.

Quantities, such as speed, which have only magnitude are known as **scalar quantities**.

The velocity of an object changes when either the magnitude or the direction changes. When the magnitude and direction remain the same, the object is said to have constant or uniform velocity. A vehicle moving round a bend (or a corner) may have constant speed but not constant velocity since the **direction** is changing.

Acceleration

When the velocity of an object changes, the object is said to have an **acceleration or deceleration**. Acceleration is defined as the **rate of change of velocity** and is given by:

$$\text{Average acceleration} = \frac{\text{Change in velocity}}{\text{Time taken}} = \frac{\text{Final velocity} - \text{Initial velocity}}{\text{Time taken}}$$

Acceleration is measured in **m/s²** when velocity is in m/s or **km/h²** when velocity is in km/h.

Example 6.3

A motor bike starts from rest and reaches a velocity of 20 m/s in 4 s. Find its acceleration.

Solution

Initial velocity $= 0$ *m/s (starts from rest)*

Final velocity $= 20$ *m/s*

Time taken $= 4$ *s*

$$\text{Acceleration} = \frac{\text{Final velocity} - \text{Initial velocity}}{\text{Time taken}}$$

$$= \frac{(20 - 0)\ m/s}{4\ s} = 5\ m/s/s = 5\ m/s^2$$

Example 6.4

A vehicle slows down from 17 m/s to 11 m/s in 10 s. What is its acceleration?

Solution

$$\text{Acceleration} = \frac{\text{Final velocity} - \text{Initial velocity}}{\text{Time taken}}$$

$$= \frac{(11 - 17)\ m/s}{10\ s} = -0.6\ m/s^2$$

Note that when acceleration has a negative value, it is called **deceleration** or **retardation**. It means that the body is slowing down.

Example 6.5

An object moving at 30 m/s accelerated at 8 m/s² for 6 s. What is its final velocity?

Solution

$$\text{Acceleration} = \frac{\text{Final velocity} - \text{Initial velocity}}{\text{Time taken}}$$

Let the final velocity $= v$ *m/s*

$$\text{Then, } 8\ m/s^2 = \frac{(v - 30)\ m/s}{6\ s}$$

$$8\ m/s^2 \times 6\ s = (v - 30)\ m/s$$

$$v = 48 + 30$$

$$= 78$$

$\therefore$ *the final velocity* $= 78$ *m/s*

Exercise 6.2

1. An object travelled 1 800 m in 90 s. What was its average velocity in:
   (a) m/s (b) km/h?
2. An aeroplane travelled 180 km eastwards in 50 minutes. What was its average velocity in:
   (a) km/h (b) m/s?
3. A supersonic jet travels at 300 m/s. What is its average velocity in km/h?
4. A ball travelled 45 m towards the goal area in 2.5 s. What was its average velocity in km/h?
5. A motor cyclist travelled for 30 s at an average velocity of 120 km/h. What distance did he cover?
6. A ball moves at a velocity of 60 km/h. How much time will it need to cover 30 m?
7. A vehicle starts from rest and reaches a velocity of 20 m/s in 30 s. What is its acceleration?
8. A particle increased its velocity from 30 km/h to 75 km/h in 10 s. What was its acceleration?
9. A vehicle accelerated at 12 m/s² to reach a velocity of 115 m/s in 6 s. What was its initial velocity?
10. A motorist reduced the velocity of a vehicle from 125 km/h to 75 km/h at a rate of 4 m/s². How long did this take?

Velocity–time graph

Velocity- time graphs are usually drawn with the velocity on the vertical axis and time on the horizontal axis

Describing motion in velocity-time graphs

Fig 6.1 shows an example of a velocity-time graph.

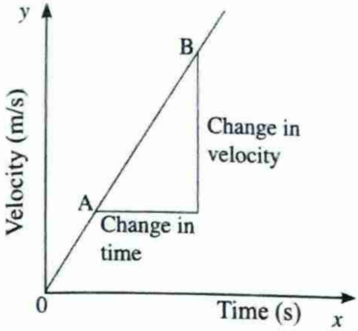


Fig. 6.1

The gradient (slope) of the graph is given by

$$\text{Gradient} = \frac{\text{change in velocity}}{\text{change in time}}$$

$\therefore$ Gradient = Acceleration

Thus, at any instance or interval, the gradient of the velocity – time graph gives the acceleration of the object. By looking at the trend in the gradient (slope) of the graph, we can tell whether the object is at rest, accelerating/decelerating uniformly or non-uniformly or is at a constant velocity.

Let us now look at the sketches of velocity-time graphs for objects moving in particular states of motions.

(a) Object at rest

$V = 0$ m/s

The graph of an object at rest is as shown in Fig. 6.2.

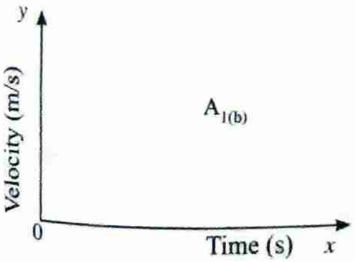


Fig. 6.2

(b) Motion at constant velocity

The graph has a gradient = 0, hence, a horizontal line (Fig 6.3).

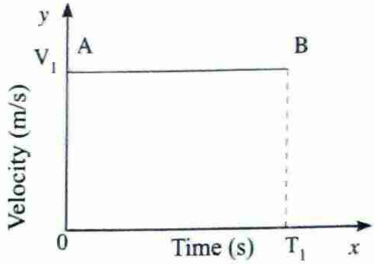


Fig. 6.3

The object is moving at a constant velocity V_1 from point A to B in time T_1.

Thus, the distance covered by the object in moving from A to B is given by:

Distance = constant velocity × time taken

$= V_1 \times t_1$

= Area of rectangle OT, BA

Thus the area under a velocity time graph (i.e. area of the region between the graph line and the time axis) gives the distance covered by the object.

(c) Uniform acceleration

The graph has a constant positive gradient (Fig 6.4).

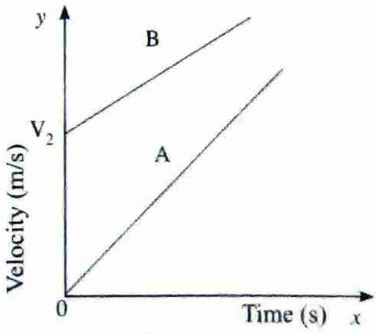


Fig 6.4

Object A started at rest ($V = 0$ m/s) and accelerated uniformly.

Object B started moving with an initial velocity V_2 and accelerated uniformly.

Note that, the gradient of A is greater than that of B. Hence, object A has a greater acceleration than object B.

(d) Uniform deceleration

The graph has a constant negative gradient (Fig. 6.5).

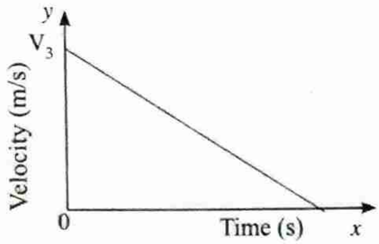


Fig. 6.5

The object started at an initial velocity V_3 and decelerated uniformly to rest (0 m/s).

(e) Consider the graph in Fig. 6.6

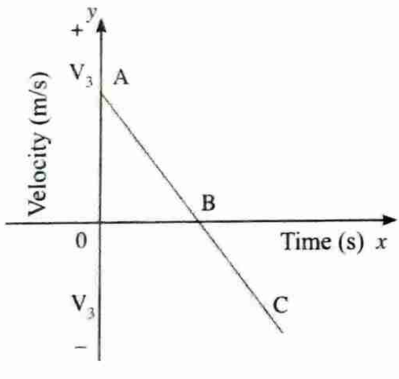


Fig. 6.6

At A: The objects start moving at an initial velocity V_1 in the forward direction.

AB: The object moves forward while decelerating uniformly to rest (0 m/s) at B.

BC: The object starts at rest and starts accelerating uniformly while moving in the opposite (reverse) direction.

(f) Consider the motion of the object represented in the velocity-time graph in Fig 6.7.

The motion can be described with reference to velocity at various times.

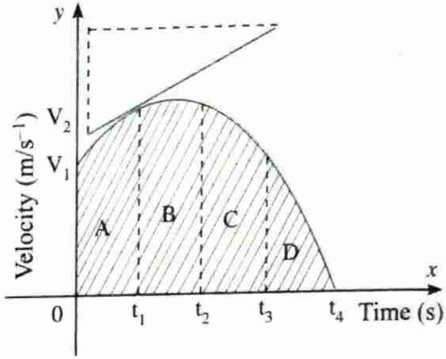


Fig. 6.7

At time t = 0, the velocity is V_1, known as the initial velocity.

At t = t_1, velocity is v_2, and it is at maximum. At this point, acceleration is momentarily equal to zero.

At t = t_4, velocity is at 0. It is the **final velocity**.

Therefore in this graph, the velocity changes with time. It is therefore not possible to find a uniform acceleration. We can only calculate the velocity at an instant using gradient of a tangent at various points. For example, to find the acceleration at t = t_1, we draw a tangent to the curve at t = t_1. The gradient of the tangent to the curve at t = t_1 gives the acceleration of the object at t = t_1.

To find the distance travelled, we calculate the area between the curve and the time axis. We divide the area under the curve into strips and approximate them to trapezia and then estimate the area.

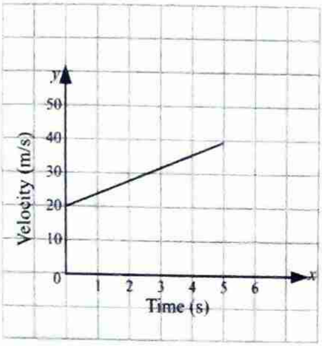
Total area = Area A + area B + area C + area D

= distance travelled.

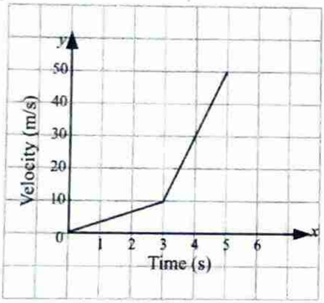
Exercise 6.3

1. Describe the motions in the graphs shown in Fig. 6.8.

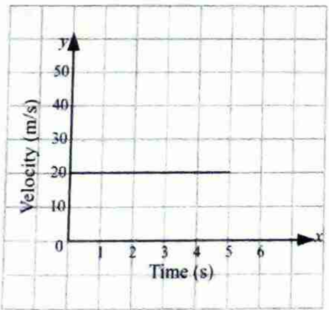
(a)



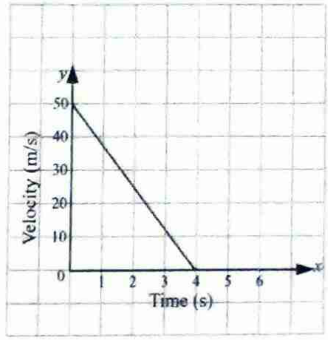
(b)



(c)



(d)



(e)

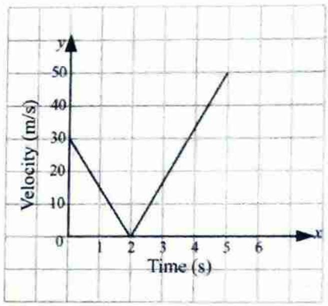
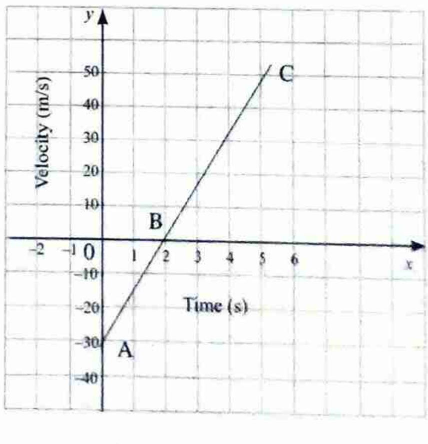


Fig. 6.8

2. Describe the motions in the velocity-time graphs in Fig. 6.9.

(a)



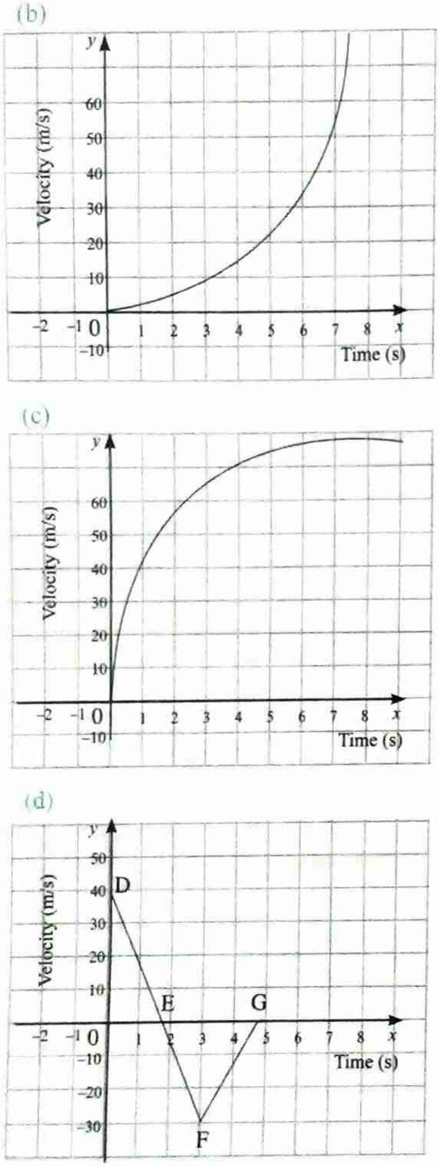


Fig. 6.9

Drawing and interpreting velocity-time graph

Example 6.6

Fig. 6.10 is the velocity-time graph for a car. Describe the motion of the car as shown in the following parts of the graph:

(a) (i) AB (ii) BC (iii) CD

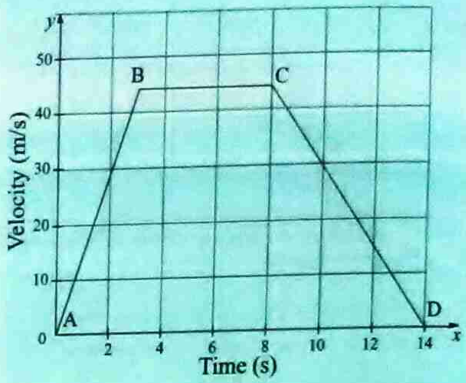


Fig. 6.10

(b) Find the total distance covered by the car.

(c) Calculate the area of trapezium ABCD and compare it with the value you got in (b). What do you notice?

Solution

(a) The velocity recorded when time is 0 seconds is called the **initial velocity**. This is 0 m/s at A.

(i) AB indicates a constant increase in velocity from 0 m/s at A to 45 m/s at B. The time taken for the velocity to increase from 0 m/s to 45 m/s is 3 seconds.

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$
$$= \frac{(45 - 0)\ \text{m/s}}{(3 - 0)\ \text{s}}$$
$$= \frac{45}{3}\ \text{m/s}^2$$
$$= 15\ \text{m/s}^2$$

This is the gradient of the line AB.

(ii) Over BC, the car maintained a constant velocity of 45 m/s for 6 seconds.

(iii) Over CD, the velocity of the car decreased at a constant rate from 45 m/s to 0 m/s in 6 seconds.

$$\text{Acceleration over } CD = \frac{(0-45)\ m/s}{(14-8)\ s}$$

$$= \frac{^-45\ m/s^2}{6}$$

$$= -7.5\ m/s^2$$

The negative acceleration indicates that the car's velocity is decreasing at 7.5 m/s.

(b) *Since average speed* $= \dfrac{\text{Distance covered}}{\text{Time taken}}$,

then, Distance = average speed × time

$\therefore$ *the distance covered for the various sections of the graph is worked out as follows:*

Section AB: (We find the area of the triangle formed from AB to the time axis).

$$\text{Distance} = \left(\frac{0+45}{2}\right) m/s \times 3\ s = 67.5\ m.$$

Section BC: (We find the area of the rectangle formed by section BC to the time axis).

$$\text{Distance} = 45 + 0\ m/s \times (8-3)\ s = 225\ m$$

(velocity over this section is constant).

Section CD: (We find the area of the triangle formed by CD and the time axis).

$$\text{Distance} = \left(\frac{45+0}{2}\right)\ m/s \times (14-8)\ s$$

$$= 135\ m.$$

The total distance covered

$$= (67.5 + 225 + 135)\ m$$

$$= 427.5\ m$$

Note: *The average velocity is calculated as 'final velocity* **minus** *initial velocity' only if the acceleration is constant.*

(c) *The area of trapezium ABCD is given by*

$$A = \tfrac{1}{2}(14+5) \times 45.$$

$$= 427\ m$$

We notice that the area under the graph OABCD (trapezium) is equal to the distance covered by the car.

Example 6.7

An electric train was initially moving at a uniform velocity of 18 ms^{-1}. It maintained this velocity for 8.0 s. It then decelerated at 4 ms^{-2} for 2.0 s, after which it travelled at the velocity attained for 5.0 s. It then accelerated at 3.0 ms^{-2} for 6.0 s. The brakes were finally applied and brought it to rest in 9.0 s.

(a) *Sketch a velocity-time graph representing the motion of the train.*

(b) *Use the graph to determine the*

(i) *total distance travelled.*

(ii) *average velocity for the journey. (Assume that the motion is in a straight line).*

Solution

(a) *To sketch the velocity-time graph, we first obtain the velocity after decelerating at a rate of 4 m/s^2 for 2 s as follows:*

$$a = \frac{v-u}{t} \Rightarrow {}^-4 = \frac{v-18}{2}$$

$$\Rightarrow -8 = v - 18, \Rightarrow v = 18 - 8 = 10\ m/s$$

Similarly, we obtain the velocity after accelerating at 3 m/s^2 for 6 s.

$$a = \frac{v-u}{t} \Rightarrow 3 = \frac{v-10}{6}$$

$$\Rightarrow 18 = v - 10,\ \text{hence},\ v = 18 + 10 = 28\ m/s$$

Fig. 6.11 is a sketch of the velocity – time graph for the train.

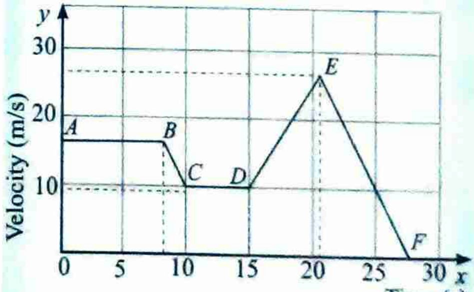


Fig 6.11: Velocity - time graph for the train

(b) (i) Total distance travelled

The total distance travelled is given by the total area under the graph and the x-axis (sum of the areas of portions shown with dotted lines on the graph)

$$= \left[8 \times 18\right] + \left[\frac{1}{2}(18 + 10) \times 2\right] + \left[5 \times 10\right]$$

$$+ \left[\frac{1}{2}(10 + 28) \times 6\right] + \left[\frac{1}{2} \times 9 \times 28\right]$$

$$= 144 + 28 + 50 + 114 + 126$$

$$= 462 \text{ m.}$$

(ii) Average velocity for the journey

The train did not change the direction hence its displacement is equal to the total distance covered.

$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time}}$$

$$= \frac{462 \text{ m}}{30 \text{ s}} = 15.4 \text{ m/s}$$

Example 6.8

Table 6.1 gives the speed, v (m/s) of a particle observed at 1 second intervals.

T(s)	0	1	2	3	4	5	6	7	8
Vm/s	32	35	36	35	32	27	20	11	0

Table 6.1

(a) Draw a velocity – time graph to represent the information.

(b) Find (i) t when v = 30 m/s

(ii) v when t = 3.5 s

(c) Find the acceleration at (i) 2.0 s

(ii) t = 4

(d) Find the total distance covered by the particle.

(e) Find the average speed of the particle.

(f) Find the initial velocity.

(g) Find the final velocity.

Solution

(a) Fig. 6.12 shows the required distance-time graph.

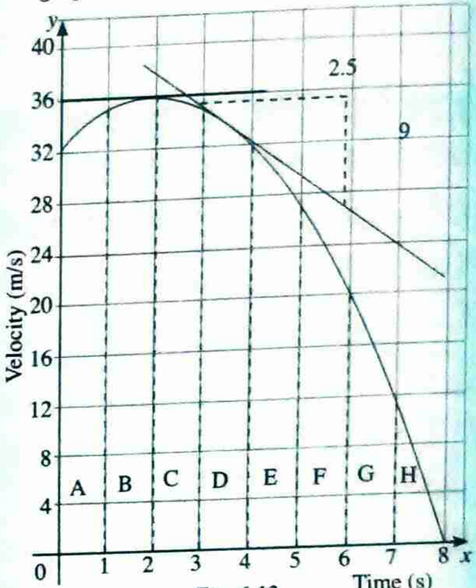


Fig. 6.12

(b) (i) When $v = 30$ m/s. $t = 4.5$ s

(ii) When $t = 3.5$. $v = 34$ m/s

(c) (i) at 2.0; $a = 0$

(ii) draw the tangent at $t = 4$

$$\text{Gradient} = \frac{34 - 25}{6 - 3.5} \ \frac{\text{m/s}}{\text{s}}$$

$$= \frac{9}{2.5} = \frac{9 \times 2}{5} \ \frac{\text{m/s}}{\text{s}}$$

$$= 3.6 \text{ m/s}$$

Acceleration at $t = 4$ is 3.6 m/s^2

(d) Total distance covered is the area between the graph, the x-axis and the y axis.

$$= \frac{1}{2} \times 1(32 + 35) + \frac{1}{2} \times 1(35 + 36) + \frac{1}{2} \times 1(36 + 35)$$

$$+ \frac{1}{2} \times 1(35 + 32) + \frac{1}{2} \times 1(32 + 27) + \frac{1}{2} \times 1(27 + 20)$$

$$+ \frac{1}{2} \times 1(20 + 11) + \frac{1}{2} \times 1(11 + 0)$$

$$= \frac{67}{2} + \frac{71}{2} + \frac{71}{2} + \frac{67}{2} + \frac{59}{2} + \frac{47}{2} + \frac{31}{2} + \frac{11}{2}$$

$$= 211$$

Distance covered = 211 m

(e) *Average speed* $= \dfrac{\text{Total distance}}{\text{Total time}}$

$= \dfrac{211 m}{8 s}$

$= 26.375$ *m/s*

$= 26.4$ *m/s*

(f) *Initial velocity* $= 32$ *m/s*

(g) *Final velocity* $= 0$ *m/s*

Exercise 6.4

1. Fig 6.13 shows a velocity-time graph of a car.

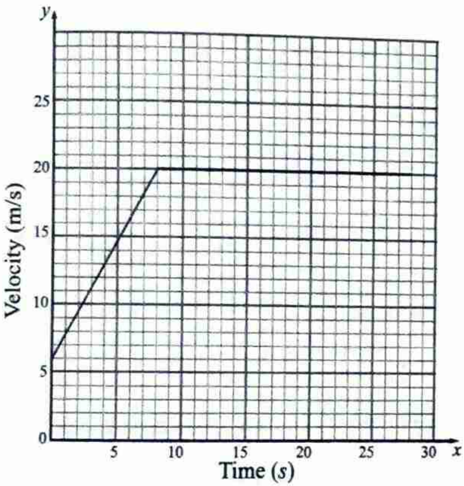


Fig. 6.13

   (a) Describe the motion of the car.

   (b) How long was the car in motion?

   (c) Find the distance travelled and the total time that the car was in motion.

2. A body accelerates uniformly from 4 m/s to 10 m/s in 8 s. Calculate its acceleration and draw its velocity-time graph.

3. From Fig. 6.14, calculate:

   (a) the acceleration

      (i) at P,

      (ii) between 23[rd] and 45[th] seconds,

      (iii) during the last 15 seconds.

   (b) the distance covered between

      (i) the 10[th] and 23[rd] seconds,

      (ii) the 23[rd] and 45[th] seconds.

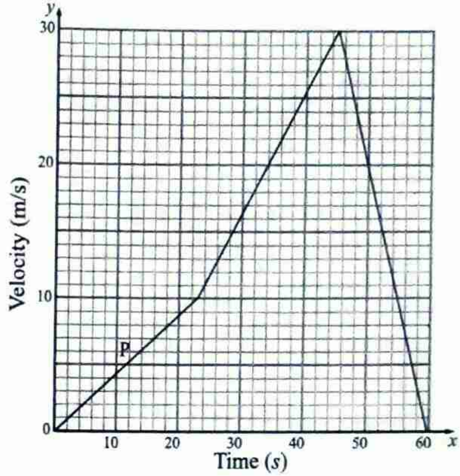


Fig. 6.14

4. A car starts from rest and attains a maximum velocity of 30 m/s in 5 seconds. It is maintained at that velocity for the next 10 seconds before it decelerates to 0 m/s in the next 10 seconds.

   (a) Represent the given motion in a velocity-time graph using a scale of 1 cm to 5 seconds on the horizontal and 1 cm to 5 m on the vertical.

   (b) Use your graph to find:

      (i) the acceleration of the car

      (ii) the deceleration of the car.

   (c) The distance travelled over the whole period of time.

5. A car accelerates from rest to a speed of 10 m/s in 10 seconds. It travelled at this speed for 20 s and then came to a stop in 5 s. Find:

   (a) the initial acceleration,

   (b) the distance travelled,

   (c) the average speed.

6. A lift accelerated from rest for 3 s and reached a speed of 4 m/s. It then immediately

decelerated taking 5 s to come to rest. Calculate:

(a) the acceleration.

(b) the retardation.

7. The speed (v) of a particle in metres per second at various times (t) is shown in Table 6.2

t	0	2	3	4	5	6	7	8
v	0	2	6	12	20	30	42	56

Table 6.2

(a) Draw a graph of *v* against *t* and use the graph to find acceleration after 2 seconds.

(b) Find v when t = 1.

(c) Estimate the distance travelled in the eight seconds.

(d) Find average speed for the eight seconds.

8. A vehicle starts from rest and attains a velocity of 16m/s in 8 seconds, with a uniform acceleration. The vehicle continues at that speed for another 15 seconds, and then it slows down with a uniform retardation until it finally stops in 10 seconds.

(a) Draw the velocity-time graph.

(b) Find the acceleration for the vehicle.

(c) Find the acceleration for the last 10 seconds of the travel.

(d) Find the distance travelled hence find the average speed.

! Overspeeding leads to fatal accidents. Reduce road carnage by driving at low and controllable speeds.

7 TRIGONOMETRY II

Success criteria

By the end of this topic, the student must be able to:

- *Calculate the area and angles of a triangle using area rule.*
- *Calculate value of a side and an angle of a triangle using sine and cosine rules.*
- *Solve problems using sine and cosine rules.*

Introduction

In Form 3, we used right angled triangles to define and state trigonometric ratios of acute angles. We used their ratios to find angles and lengths of right angled triangles.

In this chapter, we are going to extend the use of these ratios to find angles and sides of obtuse angled triangles.

Using Fig. 7.1, can you recall the three ratios as already learned?

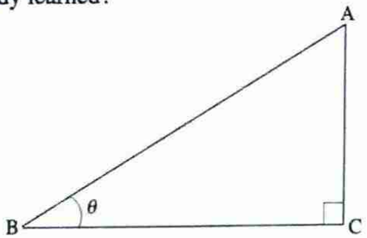


Fig. 7.1

Using Fig. 7.1 we learnt that;

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{AC}{AB},$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{BC}{AB},$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{AC}{BC}.$$

The area rule

In Form 1, we learnt that the area of a triangle is given by the formula

$$\text{Area} = \frac{1}{2}\,bh,$$

where b is the length of the base and h is the height or altitude (perpendicular to the base) – see Example 7.1. BC is the length while AC is the height.

Example 7.1

Calculate the area of the triangle shown in Fig. 7.2.

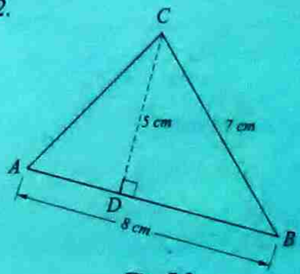


Fig. 7.2

Solution

The height is 5 cm and the corresponding base is 8 cm. For purposes of this question, we do not need the 7 cm side.

Area of the triangle $= \frac{1}{2}bh$

$= \frac{1}{2} \times 8 \times 5\ cm^2$

$= 20\ cm^2$

Suppose that, in Example 7.1, we are not given the height CD but we are given $\angle B = 45.58°$ (Fig. 7.3). How would we find the area?

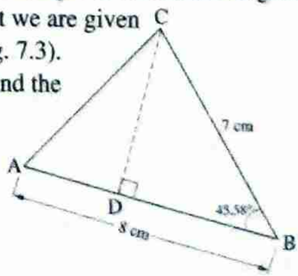


Fig. 7.3

We would proceed as follows:

In ΔBCD, $\sin B = \frac{CD}{7}$

i.e. $\sin 45.58° = \frac{CD}{7}$

$\Rightarrow CD = 7 \sin 45.58°$

Hence, area of the triangle is $\frac{1}{2} \times AB \times CD$

$$= \underbrace{\frac{1}{2} \times 8}_{\frac{1}{2}\text{ base}} \times \underbrace{7 \sin 45.58°}_{\text{height}}$$

$= 20 \text{ cm}^2$ (19.998 395 410 9 by calculator)

Now consider any triangle ABC. In Fig. 7.4, ΔABC has sides a, b, c and height h.

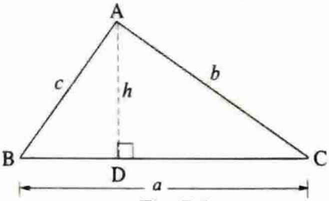


Fig. 7.4

Using ΔACD, $\frac{h}{b} = \sin C$.

$\therefore \quad h = b \sin C$.

Thus,

> Area of ABC $= \frac{1}{2} ah$
>
> $= \frac{1}{2} ab \sin C$
>
> Similarly, it can be shown that:
>
> Area of $\Delta ABC = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B$

Now consider the obtuse-angled triangle ABC in Fig. 7.5.

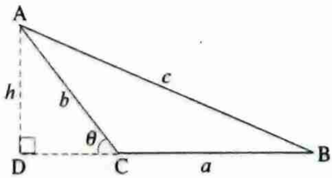


Fig. 7.5

Using ΔACD, $h = b \sin \theta$

Area of $\Delta ABC = \frac{1}{2} ah$

$= \frac{1}{2} ab \sin \theta$

But $\theta = 180° - C$, where $C = \angle ACB$.

> Thus area of ΔABC
>
> $= \frac{1}{2} ab \sin (180° - C)$, where C is the obtuse $\angle ACB$.

Note that these formula apply when two sides and the included angle of a triangle are given.

Example 7.2

Find the area of ΔABC to the nearest cm^2, given that $AB = 6$ cm, $BC = 9$ cm and $\angle ABC = 37°$.

Solution

Let the height of ΔABC be h cm (Fig. 7.6).

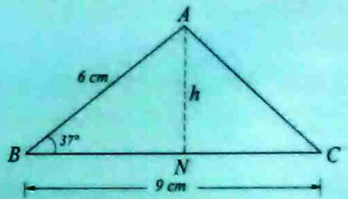


Fig. 7.6

In ΔABN, $\frac{h}{6} = \sin 37°$

$\therefore \quad h = 6 \sin 37°$

Area of $\Delta ABC = \frac{1}{2} \times BC \times AN$

$= \frac{1}{2} \times 9 \times 6 \sin 37°$

$= \frac{1}{2} \times 9 \times 6 \times 0.601\,8$

$= 16.\,248\,6 \text{ cm}^2$

$= 16 \text{ cm}^2$ *(to the nearest cm^2)*

Example 7.3

Find $\angle ACB$ given that $AC = 6$ cm, $BC = 7$ cm and area of $\Delta ABC = 16.1 \text{ cm}^2$, and that $\angle C$ is an obtuse angle.

Solution

Fig. 7.7 is a sketch of ΔABC.

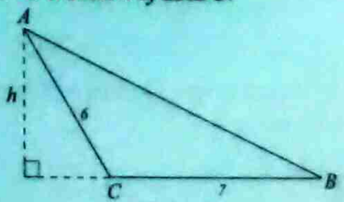


Fig. 7.7

$$\begin{aligned}
\text{Area of } \Delta ABC &= \tfrac{1}{2} \times 7 \times 6 \sin C \\
\tfrac{1}{2} \times 7 \times 6 \sin \theta &= 16.1 \\
\Rightarrow \sin \theta &= \frac{32.2}{42} \\
&= 0.766\,7 \\
\therefore \angle ACB &= 50.1^\circ
\end{aligned}$$

But $\angle ACB$ is obtuse

$$\begin{aligned}
\therefore \text{ obtuse } \angle ACB &= 180^\circ - 50.1^\circ \\
&= 129.9^\circ
\end{aligned}$$

Exercise 7.1

1. Find the area of ΔABC given that:
   (a) $AB = 10$ cm, $BC = 13$ cm, $\angle ABC = 48^\circ$
   (b) $AB = 6$ cm, $BC = 8.5$ cm, $\angle ABC = 37^\circ$
   (c) $AC = 7$ cm, $AB = 6$ cm, $\angle BAC = 57^\circ$
   (d) $AC = 4$ cm, $BC = 6$ cm, $\angle ACB = 130^\circ$
   (e) $BC = 9$ cm, $BA = 7$ cm, $\angle ABC = 145^\circ$
   (f) $AC = 10$ cm, $BA = 10$ cm, $\angle CAB = 165^\circ$

2. Find $\angle B$ given that:
   (a) $AB = 3$ cm, $BC = 8$ cm, area of $\Delta ABC = 7.05$ cm^2
   (b) $AB = 10$ cm, $BC = 11$cm, area of $\Delta ABC = 51$cm^2
   (c) $AB = 12$ cm, $BC = 35$ cm, area of $\Delta ABC = 210$ cm^2
   (d) $AB = 7$ cm, $BC = 24$ cm, area of $\Delta ABC = 84$ cm^2
   (e) $AB = 6$ cm, $BC = 6$ cm, area of $\Delta ABC = 10.3$cm^2
   (f) $AB = 5$ cm, $BC = 9$ cm, area of $\Delta ABC = 18.4$cm^2 and $\angle B$ is an obtuse angle.
   (g) $AB = 8$ cm, $BC = 10$ cm, area of $\Delta ABC = 25.7$ cm^2 and $\angle B$ is an obtuse angle.

3. Find the areas of the triangles in Fig. 7.8. All the measurements are in centimetres.

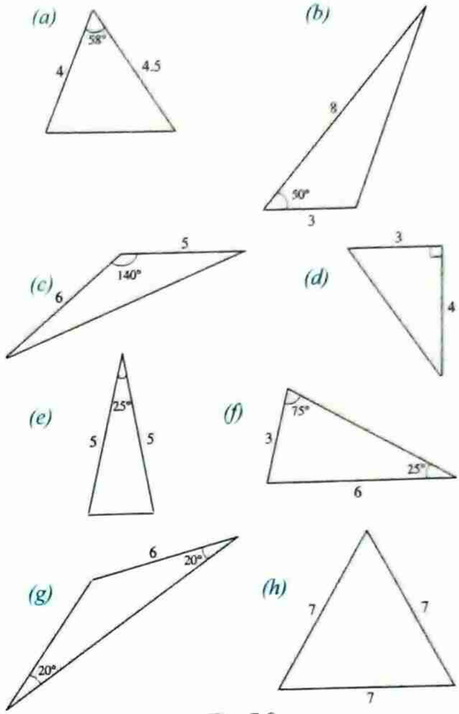


Fig. 7.8

The sine rule

This is a rule that can be used to solve any triangle given the necessary information.

In Fig. 7.9, ΔABC has sides of lengths a, b, c. AD and CE are the altitudes from A to BC and C to AB respectively.

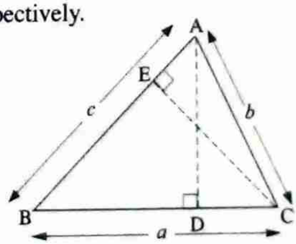


Fig. 7.9

Using ΔADC: $\frac{AD}{AC} = \sin C \Rightarrow AD = AC \sin C$

i.e. $AD = b \sin C$.

$$\text{Area of } \Delta ABC = \tfrac{1}{2}\, BC \cdot AD$$
$$= \tfrac{1}{2}\, ab \sin C \text{ ....(i)}$$

Using ΔCBE: $\frac{CE}{BC} = \sin B \Rightarrow CE = BC \sin B$

i.e. $CE = a \sin B$

$$\therefore \text{ Area of } \Delta ABC = \tfrac{1}{2}\, AB \cdot CE$$
$$= \tfrac{1}{2}\, ac \sin B \text{ ....(ii)}$$

It can similarly be shown that, area of ΔABC

$$= \tfrac{1}{2}\, bc \sin A \text{ ....(iii)}$$

Combining (i) and (ii), we obtain

$$\text{Area of } \Delta ABC = \tfrac{1}{2}\, ab \sin C = \tfrac{1}{2}\, ac \sin B$$

Hence $\frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B$

$\Leftrightarrow b \sin C = c \sin B$ .... dividing both sides by $\sin B \sin C$

$$\text{Thus, } \frac{b}{\sin B} = \frac{c}{\sin C} \text{ ....(iv)}$$

Combining (ii) and (iii), we obtain

$$\text{Area of } \Delta ABC = \tfrac{1}{2}\, ac \sin B = \tfrac{1}{2}\, bc \sin A$$

Hence $\frac{1}{2} ac \sin B = \frac{1}{2} bc \sin A$

$\Leftrightarrow a \sin B = b \sin A$ .... dividing both sides by $\sin A \sin B$

$$\text{Thus, } \frac{a}{\sin A} = \frac{b}{\sin B} \text{ ....(v)}$$

Combining (iv) and (v), we obtain

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ which is called the **sine rule**.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

is known as the sine rule.

Note: If the given angle θ is obtuse, then $\sin \theta = \sin(180 - \theta)$.

Example 7.4

Solve ΔPQR in which $QR = 5.12$ cm, $\angle Q = 43°$ and $\angle R = 74°$ (Fig. 7.10) and find its area.

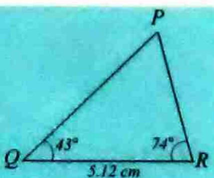


Fig. 7.10

Solution

*To **solve a triangle** means to find the sides and angles of the triangle, which are not known, and the area of triangle.*

Since $\angle Q = 43°$ and $\angle R = 74°$, then

$$\angle P = 180° - (43° + 74°) \text{ (angle sum of } \Delta)$$

Using sine rule: $\frac{PQ}{\sin \angle R} = \frac{QR}{\sin \angle P}$

$$\Rightarrow \frac{PQ}{\sin 74°} = \frac{QR}{\sin 63°}$$

$$\Rightarrow PQ = 5.12 \frac{\sin 74°}{\sin 63°} = 5.524 \text{ cm}$$

$$\frac{PR}{\sin \angle Q} = \frac{QR}{\sin \angle P} \Rightarrow \frac{PR}{\sin 43°} = \frac{5.12}{\sin 63°}$$

$$\Rightarrow PR = 5.12 \frac{\sin 43°}{\sin 63°} = 3.919 \text{ cm}$$

$$\text{Area of } \Delta PQR = \tfrac{1}{2}\, PQ \times QR \sin 43°$$
$$= \tfrac{1}{2} \times 5.524 \times 5.12 \sin 43°$$
$$= 9.644 \text{ cm}^2.$$

Example 7.5

Solve ΔABC Fig. 7.11 in which $a = 5.7$ cm, $b = 4$ cm and $\angle B = 19°$, and find two possible areas of ΔABC.

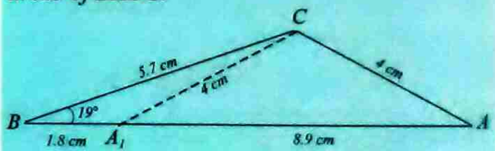


Fig. 7.11

Solution

Using ΔABC and the Sine Rule: $\frac{5.7}{\sin A} = \frac{4}{\sin 19°}$

$$\Rightarrow \sin A = 5.7 \frac{\sin 19°}{4} = 0.464\,0$$

$$\therefore A = 27.65° \text{ or } 152.35° \approx 152.4°$$

(i) *When $A = 27.65°$, then*

$$C = 180° - (19° + 27.65°)$$
$$= 133.35° \approx 133.4°.$$

Using ΔABC

$$\text{Then, } \frac{c}{\sin 133.4°} = \frac{4}{\sin 19°}$$

$$\Rightarrow c = 4\frac{\sin 133.4^\circ}{\sin 19^\circ} = 8.927 \text{ cm}$$
$$= 8.9 \text{ cm (1 d.p.)}$$

(ii) When $A = 152.4^\circ$, then

$$C = 180 - (19^\circ + 152.4^\circ) = 8.6^\circ$$

Then $\frac{c}{\sin 8.6^\circ} = \frac{4}{\sin 19^\circ}$

$$\Rightarrow c = 4\frac{\sin 8.6^\circ}{\sin 19^\circ} = 1.837 \text{ cm}$$
$$\approx 1.8 \text{ cm (1 d.p.)}$$

The two possible areas are 8.282 cm² and 1.704 cm².

Note:

We use the sine rule when given:

1. two sides and an angle opposite one of the given sides, or
2. two sides and any two angles, or
3. two angles and a side.

Exercise 7.2

In this exercise, state your answers correct to 3 s.f.

1. Solve ΔABC where:
   (a) $a = 2.2$ m, $c = 4$ m, $C = 36^\circ$
   (b) $b = 5$ cm, $A = 45^\circ$, $C = 25^\circ$
   (c) $a = 11$ cm, $A = 120^\circ$, $B = 24^\circ$
   (d) $b = 4$ cm, $c = 7$ cm, $B = 28^\circ$
   (e) $a = 6$ cm, $b = 6$ cm, $A = 50^\circ$
   (f) $a = 5$ cm, $A = 80^\circ$, $B = 30^\circ$
2. In ΔPQR, $p = 10$ cm, $q = 5$ cm, $P = 30^\circ$. Find Q and r.
3. In ΔXYZ, $X = 50^\circ$, $Y = 60^\circ$, $z = 20$ cm. Find x and y.
4. Solve ΔPQR, where
   (a) $p = 5$ cm, $P = 70^\circ$, $Q = 30^\circ$.
   (b) $P = 105^\circ$, $Q = 36^\circ$, $q = 4.17$ cm.
5. In ΔPQR, $p = 10$ cm, $Q = 15^\circ$, $R = 45^\circ$. Find r.
6. In ΔKLM, $k = 15$ cm, $L = 25^\circ$, $M = 120^\circ$. Find m.
7. In ΔXYZ, $y = 10$ cm, $Z = 84^\circ$, $Y = 20^\circ$. Calculate x.
8. At one instant, the distance from the earth to the sun is 149 million kilometres. The distance from the sun to Venus is 107 million kilometres. The line from earth to Venus and the line from the earth to the sun make an angle of 37°. How far is Venus from the earth at that instant?
9. In Fig. 7.12, find the length of SR.

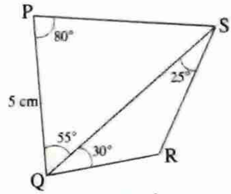


Fig. 7.12

10. ΔPQR has sides $PR = 30$ cm, $\angle QPR = 52^\circ\ 30'$, $\angle PQR = 66^\circ\ 10'$. Find PQ and QR.

The cosine rule

Consider ΔPQR (Fig. 7.13) in which $PQ = 10$ cm, $QR = 12$ cm and $\angle Q = 58^\circ$. Can you solve it?

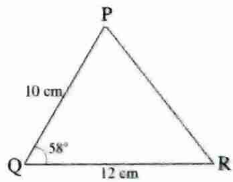


Fig. 7.13

Note that ΔPQR is not right-angled and the sine rule does not help one to solve it. So another relationship between the sides and angles is needed.

Consider Fig. 7.14 in which $\angle C$ is an acute angle and $AD \perp BC$.

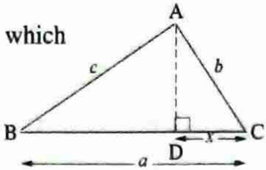


Fig. 7.14

In ΔABD,

$$AD^2 = AB^2 - BD^2$$
$$= c^2 - (a - x)^2 \quad \text{.......................(i)}$$

(Pythagoras theorem).

In ΔACD, $AD^2 = AC^2 - DC^2$

$$= b^2 - x^2 \quad \text{.................(ii)}$$

(Pythagoras theorem).

$\therefore$ From (i) and (ii) we obtain

$$c^2 - (a - x)^2 = b^2 - x^2$$
$$\Rightarrow c^2 - (a^2 - 2ax + x^2) = b^2 - x^2$$
$$\Rightarrow c^2 - a^2 + 2ax - x^2 = b^2 - x^2$$
$$\Rightarrow c^2 = a^2 + b^2 - 2ax$$

In ΔACD: $x = b \cos C$

$$\therefore \quad c^2 = a^2 + b^2 - 2ab \cos C.$$

Consider Fig. 7.15 in which $\angle ACB$ is an obtuse angle. $AD \perp BC$ produced.

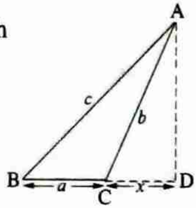


Fig. 7.15

Using Pythagoras theorem in ΔABD,

$$\Rightarrow AD^2 = AB^2 - BD^2$$
$$= AB^2 - (BC + CD)^2$$
$$= c^2 - (a + x)^2$$
$$= c^2 - (a^2 + 2ax + x^2)$$
$$= c^2 - a^2 - 2ax - x^2 \quad \text{.................(i)}$$

Using Pythagoras theorem in ΔACD,

$$AD^2 = AC^2 - CD^2 = b^2 - x^2 \quad \text{...........(ii)}$$

From (i) and (ii) we obtain

$$c^2 - a^2 - 2ax - x^2 = b^2 - x^2$$
$$\Rightarrow c^2 = a^2 + b^2 + 2ax$$

But in ΔACD, $x = b \cos \angle ACD$

$$= b \cos (180° - C)$$
$$= {}^-b \cos C.$$
$$\therefore \quad c^2 = a^2 + b^2 - 2ab \cos C.$$

So in either case $c^2 = a^2 + b^2 - 2ab \cos C$.

It can similarly be shown that

$$a^2 = b^2 + c^2 - 2bc \cos A$$

and $b^2 = a^2 + c^2 - 2ac \cos B$.

The expression $a^2 = b^2 + c^2 - 2bc \cos A$
or $b^2 = a^2 + c^2 - 2ac \cos B$
or $c^2 = a^2 + b^2 - 2ab \cos C$

is known as the **cosine rule**.

If θ is obtuse, then $\cos\theta = -\cos(180 - \theta)$.

To find the angles of a triangle given the lengths of the three sides, we need to rearrange the cosine rule. Thus,

$a^2 = b^2 + c^2 - 2bc \cos A$ becomes

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Similarly,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}, \text{ and}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

We only require to evaluate the right hand side of each of the formulas and then find the angle whose cosine has been calculated.

Example 7.6

Find the angles of a triangle whose sides measure 3 cm, 5 cm and 7 cm.

Solution

Fig. 7.16 is a sketch of the given triangle.

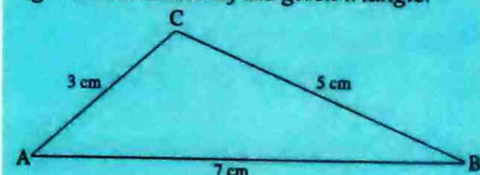


Fig. 7.16

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{7^2 + 3^2 - 5^2}{2 \times 7 \times 3} = \frac{33}{42} = 0.785\,7 \ (4 \text{ s.f.})$$

$$\Rightarrow A = \cos^{-1} 0.7857 = 38.21° \ (4 \text{ s.f.})$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$\cos B = \frac{5^2 + 7^2 - 3^2}{2 \times 5 \times 7} = \frac{65}{70} = 0.928\,6 \ (4 \text{ s.f.})$$

$$\Rightarrow B = \cos^{-1} 0.9286 = 21.79^\circ \ (4 \text{ s.f.})$$

C must be an obtuse angle

$$\therefore \quad C = 180^\circ - (38.21^\circ + 21.79^\circ) = 120^\circ$$

Check: $A + B + C = 38.21^\circ + 120^\circ + 21.79^\circ$

$$= 180^\circ.$$

Example 7.7

Solve ΔPQR in which $PQ = 10$ cm, $QR = 12$ cm and $\angle Q = 58^\circ$.

Solution

The triangle is as shown in Fig. 7.17.

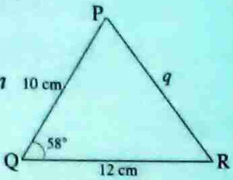


Fig. 7.17

$$q^2 = p^2 + r^2 - 2pr \cos Q$$
$$q^2 = 12^2 + 10^2 - 2 \times 12 \times 10 \times \cos 58^\circ$$
$$= 144 + 100 - 240 \times 0.530$$
$$= 116.8$$
$$q = 10.81 \text{ cm.}$$

Using the cosine rule,

$$r^2 = p^2 + q^2 - 2pq \cos R$$
$$10^2 = 12^2 + 10.81^2 - 2(12)(10.81) \cos R$$
$$100 = 144 + 116.856\,1 - 259.44 \cos R$$
$$100 = 260.856\,1 - 259.44 \cos R$$
$$^-160.856\,1 = {}^-259.44 \cos R$$
$$\Rightarrow \cos R = \frac{160.856\,1}{259.44}$$
$$\cos R = 0.620$$
$$\angle R = 51.68^\circ \text{ and } \angle P = 70.32^\circ$$

Note

We use the **cosine rule** when given:

(i) two sides and the included angle, or

(ii) all the three sides.

Exercise 7.3

In this exercise, give answers correct to 3 s.f.

1. Solve ΔABC where:

   (a) $a = 5$ cm, $b = 8$ cm, $c = 7$ cm,

   (b) $b = 6$ cm, $c = 14.5$ cm, $\angle A = 95^\circ$,

   (c) $a = 17$ cm, $c = 12$ cm, $\angle B = 80^\circ$,

   (d) $a = 4.1$ cm, $b = 8.5$ cm, $c = 5.9$ cm,

   (e) $a = 6$ cm, $b = 6$ cm, $\angle c = 50^\circ$,

   (f) $a = 3.49$ cm, $b = 4.62$ cm, $c = 6.93$ cm.

2. In a triangle, two sides are 2.8 cm and 12 cm long, and the angle between them is 60°. Find the length of the third side.

3. Find the value of x in Fig. 7.18.

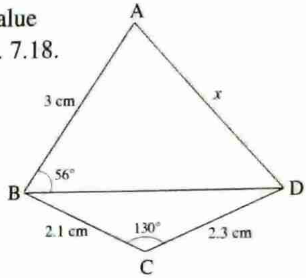


Fig. 7.18

4. Find $\angle$ABC in Fig. 7.19.

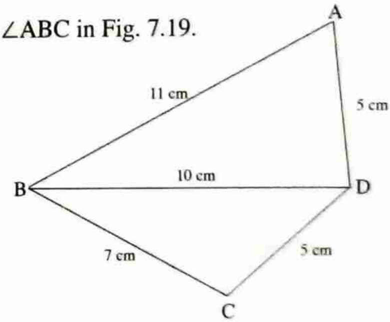


Fig. 7.19

5. Solve ΔABC where $a = 5$ cm, $b = 3$ cm, and $\angle B = 30^\circ$.

6. The sides of a triangle are 7 cm, 9 cm and 14 cm. What is the length of the shortest median?

7. The sides of a triangle are 11 cm, 8 cm and 9 cm. What are the lengths of the two shorter medians?

8. In Fig. 7.20, AB = 4 cm, BC = 5 cm, AC = 8 cm and CD = 5 cm. Find AD.

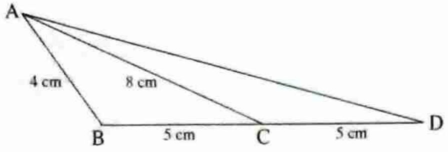


Fig. 7.20

9. Find the lengths of the diagonals of a parallelogram whose sides are 3.6 cm and 4.8 cm and one angle which is 54.2°.

Bearings using sine and cosine rule

We can use sine and cosine rules to solve problems involving bearing.

Example 7.8

Four towns A, B, C and D are such that B is 750 km from A on a bearing of 050°. C is 500 km from B on a bearing of 340° and D is 1 500 km from C on a bearing of 260°. Calculate:

(a) the distance from B to D.

(b) the bearing of D from B.

(c) the bearing and distance of A from D.

Solution

We need a sketch to show the relative positions of A, B, C and D. Fig. 7.21 is the required sketch.

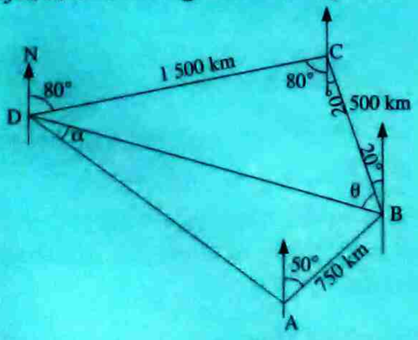


Fig. 7.21

(a) Using ΔBCD, and the cosine rule,

$BD^2 = BC^2 + CD^2 - 2BC.CD \cos C$

$= 500^2 + 1\,500^2 - 2 \times 500 \times 1\,500 \times \cos 100°$

$= 250\,000 + 2\,250\,000 + 260\,400$

$= 27\,640\,400$

$BD = \sqrt{27\,640\,400}$

$= 1\,661$ (4 s.f.)

The distance from B to D is 1 661 km

(b) Using ΔBCD, $\dfrac{BD}{\sin 100} = \dfrac{DC}{\sin \theta}$

$$\frac{1\,661}{\sin 100} = \frac{1\,500}{\sin \theta}$$

$1\,661 \sin \theta = 1\,500 \sin 100$

$1\,661 \sin \theta = 1\,500 \times 0.9848$

$\sin \theta = \dfrac{1\,500 \times 0.9848}{1\,661}$

$\sin \theta = 0.8894$

$\theta = 62.79°$

bearing of D from B $= 360 - (20 + 62.79)$

$= 360 - 82.79$

$= 277.21°$

(c) In ΔABD, $\angle ABD = 180 - (50 + 20 + 62.79)$

$= 180 - 132.79$

$= 47.21°$

$AD^2 = BD^2 + AB^2 - 2.BD.AB \cos B$

$= 1\,661^2 + 750^2 - 2 \times 1\,661 \times 750 \times \cos 47.21°$

$= 2\,758\,921 + 562\,500 - 1\,692\,476$

$AD = \sqrt{1\,628\,945}$

$= 1276.3$ km

Using ΔABD, $\dfrac{750}{\sin \alpha} = \dfrac{1\,276.3}{\sin 47.21}$

$\sin \alpha = \dfrac{750 \sin 47.21}{1\,276.3}$

$= \dfrac{750 \times 0.7338}{1\,276.3}$

$= \dfrac{550.39}{1\,276.3}$

$= 0.4312$

$\alpha = 25.55°$

$\angle NDA = 80° + 180° - (100° + 62.79°) + 25.55°$

$= 80° + 17.21° + 25.55°$

$= 122.76°$

So, bearing of A from D is 122.76°

distance of A from D is 1 276.3 km

Exercise 7.4

1. Two boats A and B leave a port at 0700 h. Boat A travels at 25 km/h on a bearing of 037°. Boat B travels at 15 km/h on a bearing of 140°. After 3 hours, how far is A from B?

2. At one instant, the distance from the Earth to the Sun is 149 million kilometres and the distance from Mars to the Sun is 225 million kilometres. The line from the earth to Mars and the line from the Earth to the Sun form an angle of 132.5°. How far is Mars from the Earth?

3. An aeroplane flies 120 km in the direction 113°, then turns and flies 160 km in the direction 156°. Find its distance from the starting point.

4. The distance from the Earth to the Sun is 149 million kilometres. The distance from the Earth to Venus is 160 million kilometres and the distance from the Sun to Venus is 107 million kilometres. What is the angle between the line joining the sun to Venus and the line joining the Earth to Venus?

5. Two aeroplanes start from an airport at the same time. One plane flies West at 400 km per hour while the other flies at 500 km per hour on a bearing of 040°. What is the distance between the two planes after 15 minutes?

6. In a soccer game, players A and B are 15 m apart. Player C has the ball and wants to pass it either to A or B, whoever is nearest to him. If the angle CAB = 45.6° and the angle ABC = 37.9°, find by calculation, between A and B, is nearer to C?

7. Mary was driving along a straight level road in the direction 053°. She saw a billboard on a bearing 037°. After covering a distance of 800 m, the billboard was on a bearing of 296°. How far is the billboard from the road?

8. A ranger was walking towards a tower in a village and noticed that the angle of elevation of the top of the tower was 10°. After walking a distance of 20 m, she noticed that the angle of elevation was 15°. What is the height of the tower?

9. Funsani walked from a point A along a straight path that meets a straight road at point B. At point B he turned right and walked 300 m along the road to point C. He was then 400 m from A. If at point B he had turned left and walked 300 m, he would have been 700 m from A.
   (a) What is the distance from point A to point B.
   (b) What is the size of $\angle ABC$?

8 POLYNOMIALS

Success criteria

By the end of this topic, the student must be able to:
- *Divide a polynomial of a higher degree by a polynomial of lower degree.*
- *Find the reminder by using reminder theorem.*
- *Factorise polynomials.*
- *Find the roots of polynomial equations of third degree.*
- *Find coefficients in identical polynomials.*

Definition of a polynomial

Many algebraic expressions consist of a group of terms all of the form ax^n where $\boldsymbol{a}$ and $\boldsymbol{n}$ are constants, $\boldsymbol{a}$ being the coefficient of x while $\boldsymbol{n}$ is a positive integer. Usually, these expressions are arranged in descending order of size of powers of x. Occasionally, they can be written in the reverse order.

Expressions such as $4x^4 + 3x^3 - 6x^2 - x + 2$, $x^2 + 5x - 6$, $6x - 3$ are called **polynomials**.

The highest power of x in a polynomial defines the **degree** of the polynomial. Thus, in the examples above, the polynomials are of degree 4, 2 and 1 respectively.

The rules that govern basic operation on numbers also apply in polynomials.

In this chapter, we are going to concentrate on division of polynomials, but addition, subtraction and multiplication cannot be avoided.

Addition and subtraction of polynomials

Terms in a polynomial of the same degree are collected together and combined as may be appropriate.

Example 8.1

Given that P is a polynomial $2x^3 - 3x^2 + 4x - 2$ and Q is $x^2 + 3x + 4$, evaluate

(a) $P + Q$ *(b)* $P - Q$

Solution

(a) $P + Q = (2x^3 - 3x^2 + 4x - 2) + (x^2 + 3x + 4)$

$= 2x^3 \underbrace{- 3x^2 + x^2} + \underbrace{4x + 3x} \underbrace{- 2 + 4}$

(collect like terms together)

$= 2x^3 - 2x^2 + 7x + 2$

(b) $P - Q = (2x^3 - 3x^2 + 4x - 2) - (x^2 + 3x + 4)$

$= 2x^3 - 3x^2 + 4x - 2 - x^2 - 3x - 4$

(open the brackets)

$= 2x^3 - \underbrace{3x^2 - x^2} + \underbrace{4x - 3x} - 2 - 4$

(group like terms together)

$= 2x^3 - 4x^2 + x - 6$

Multiplication of polynomials

Multiplication often involves removal of brackets as illustrated in the example below.

Example 8.2

Multiply $x^2 + x - 3$ by $x^2 - 3x + 2$

Solution

$(x^2 + x - 3)(x^2 - 3x + 2) = x^2(x^2 - 3x + 2) + x(x^2 - 3x + 2) - 3(x^2 - 3x + 2)$

$= x^4 - 3x^3 + 2x^2 + x^3 - 3x^2 + 2x - 3x^2 + 9x - 6$

$= x^4 - 3x^3 + x^3 + 2x^2 - 3x^2 - 3x^2 + 2x + 9x - 6$

$= x^4 - 2x^3 - 4x^2 + 11x - 6$

This multiplication can be performed using a long multiplication format as in arithmetic.

Division of polynomials

Multiplication and division are inverse processes and so we can relate division of one polynomial by another to the long division process in arithmetic.

In division, order of the terms is important.

> (i) Both the dividend and the divisor must be written in descending powers of the variable.
>
> (ii) If a term is missing, a zero term must be inserted in its place.

For the division of one polynomial by another to work, the degree of the dividend must be higher than that of the divisor.

Remember:

Just as in division of numbers, not all polynomials divide exactly, some will have remainders.

Suppose the polynomial to be divided is denoted by $f(x)$, and the divisor by the function $g(x)$, we can denote the result of division as,

$$\frac{f(x)}{g(x)} = Q + \frac{R}{g(x)}$$

$$f(x) = Q.g(x) + R$$

Where Q is the quotient and R is the remainder.

The division process terminates as soon as the degree of R is less than the degree of $g(x)$.

> If a polynomial f(x) is divided by a polynomial g(x) of a lower degree then, f(x), the remainder is another polynomial of a lower degree than g(x).

Examples 8.3

Given that $f(x) = x^2 + 7x + 12$ and $g(x) = x + 4$, divide $f(x)$ by $g(x)$.

Solution

Using skills of long division of numbers we write $(x^2 + 7x + 12) \div (x + 4)$ in the form

$$\begin{array}{r|l}
 & \quad\; x + 3 \\
\hline
x + 4 & x^2 + 7x + 12 \\
 & -\underline{x^2 + 4x} \\
 & \qquad 3x + 12 \\
 & \quad -\underline{3x + 12} \\
 & \qquad\qquad 0
\end{array}$$

(i) Divide x^2 by x to obtain x.

(ii) Multiply $x(x + 4)$ and subtract then bring down the next term.

(iii) Divide $3x$ by x to obtain and multiply by $x + 4$.

(iv) Subtract.

Thus, $(x^2 + 7x + 12) \div (x + 4) = x + 3$

The division is exact, the quotient is $x + 3$ and there is no remainder.

Note: Each term in the **quotient** is obtained by making the first term in the **divisor** divide exactly each time.

Example 8.4

Divide $-20 + 6x^3 - 4x^2$ by $2x - 4$ and state the quotient and the remainder.

Solution

(i) Rearrange the terms in the dividend and write them in descending order.

(ii) Insert $0x$ for the missing.

$-20 + 6x^3 - 4x^2 \div 2x - 4$ becomes

$(6x^3 - 4x^2 + 0x - 20) \div (2x - 4)$

$$\begin{array}{r|l}
 & \quad 3x^2 + 4x + 8 \\
\hline
2x - 4 & 6x^3 - 4x^2 + 0x - 20 \\
 & -\underline{6x^3 - 12x^2} \\
 & \qquad\quad 8x^2 + 0x \\
 & \qquad -\underline{8x^2 - 16x} \\
 & \qquad\qquad\quad 16x - 20 \\
 & \qquad\qquad -\underline{16x - 32} \\
 & \qquad\qquad\qquad\quad 12
\end{array}$$

(i) $6x^3 \div 2x = 3x^2$

(ii) $3x^2(2x - 4) = 6x^3 - 12x^2$

(iii) $8x^2 \div 2x = 4x$

(iv) $4x(2x - 4)$

(v) $16x - 20$

(vi) $8(2x - 4)$

(vii) Subtract

Thus,

$$(6x^3 - 4x^2 - 20) \div (2x - 4) = 3x^2 + 4x + 8 + \frac{12}{2x-4}$$

This division is not exact. The quotient is $3x^2 + 4x + 8$ and the remainder is 12.

Example 8.5

Divide $2x^4 + x^3 - 3x^2$ by $x^2 + 2$ and state the quotient and the remainder.

Solution

In the dividend, the constant and the x term are missing, so we replace them with 0x, and 0 respectively.

In the divisor, the term in x is missing, so we replace it with 0x.

Dividend: $2x^4 + x^3 - 3x^2$ becomes $2x^4 + x^3 - 3x^2 + 0x + 0$

Divisor: $x^2 + 2$ becomes $x^2 + 0x + 2$

$$\begin{array}{r|l} & 2x^2 + x - 7 \\ \hline x^2 + 0x + 2 & 2x^4 + x^3 - 3x^2 + 0x + 0 \\ & -\underline{2x^4 + 0 + 4x^2} \\ & \quad x^3 - 7x^2 + 0x \\ & -\underline{\quad x^3 + 0 + 2x} \\ & \qquad -7x^2 - 2x + 0 \\ & -\underline{\qquad 7x^2 + 0x - 14} \\ & \qquad\qquad -2x + 14 \end{array}$$

The quotient is $2x^2 + x - 7$, remainder is $^-2x + 14$.

Exercise 8.1

1. Arrange each of the polynomials in descending powers of the variable.
   (a) $^-12x^3 + 3x^2 - x^4 + 7x + 10$
   (b) $4x^3 - 3x^5 + 6x^2$
   (c) $4a + a^2 - 12 + 12a^3$
   (d) $4a^5 - 5a + 3a^3 + a^4 - 7a^2$

2. Arrange the following polynomials in descending order, inserting zero for any missing term.
   (a) $8 + 4x^3 - 2x$
   (b) $^-3x^3 + 4x^5 - 2x^2 + 7x$
   (c) $8x^5 - 3x^2 + 4x^3 - 7$
   (d) $7a + 3a^4 - 8a^2$

3. Simplify the following.
   (a) $(m^2 + 5m) - (m^2 + m)$
   (b) $(4x^2 + 6x) - (4x^2 - 10x)$
   (c) $(^-6x^2 - 3x) - (-6x^2 - 8x)$
   (d) $(11x^2 + 0x) - (11x^2 - 10x)$

4. Verify that the statements given in this question are all true.
   (a) $a^2 - 7a - 18 = (a - 9)(a + 2)$
   (b) $b^3 - 8 = (b^2 + 2b + 4)(b - 2)$
   (c) $3x^3 - 2x^2 + 16x - 8 = (3x^2 + 4x + 11)(x - 2) + 18$
   (d) $3p^2 - 3p + p^3 - 8 = (p^2 + 5p + 7)(p - 2) + 6$

5. Divide the given polynomials and in each case state the quotient and the remainder.
   (a) $(13x + 14 + 3x^2) \div (x + 2)$
   (b) $(t + 20t^2 - 4) \div (4 + 5t)$
   (c) $(17a + 14a^2 + 7) \div (2 - 7a)$
   (d) $(9 - 16r^2) \div (4r - 3)$

6. Find the quotient and the remainder in each of the following.
   (a) $(x + 2 - 3x^2 - 2x^3) \div (1 + 2x)$
   (b) $(a + 2a^4 - 14a^2 + 5) \div (a^2 + 5)$
   (c) $(3y^2 + 4 - 10y) \div (3y - 2)$
   (d) $(3a^3 - 1) \div (a - 1)$
   (e) $(2h^3 + 10 - 13h^2 + 16h) \div (2h - 5)$

The remainder theorem

We have already seen that in the division of polynomials, four quantities are involved namely dividend, divisor, quotient and the remainder.

Let the dividend $= f(x)$

Divisor $= g(x)$

Quotient = Q which may also be function of x

Remainder = R

So, $f(x) \div g(x) = Q + \frac{R}{g(x)}$

$$\frac{f(x)}{g(x)} \equiv Q + \frac{R}{g(x)}$$

Multiplying each term by $g(x)$ gives

$f(x) = Q(g(x)) + R$

$= Q(x - a) + R$ where $g(x) = (x - a)$

This expression represents an identity which is true for all values of x.

If we replace $\boldsymbol{x}$ by $\boldsymbol{a}$,

$f(x) = Q(x - a) + R$ becomes

$f(a) = Q(a - a) + R$

$= Q(0) + R$

$= R$

This shows the remainder $R = f(a)$ where $\boldsymbol{a}$ is the value of x that makes the divisor equal to zero. This result is called the **remainder theorem**.

If a polynomial $f(x)$ is divided by a factor $(x - a)$, the remainder is equal to $f(a)$ i.e. $f(a) = R$.

$\therefore f(x) = (x - a)\,g(x) + f(a)$

Examples 8.6

Use the remainder theorem to find the remainder when $x^3 - 3x^2 + 4x - 2$ is divided by:

(i) $x - 1$ *(ii)* $x + 2$

Solution

$f(x) = x^3 - 3x^2 + 4x - 2$

(i) If $x - 1$ is the divisor, the value of x that makes the divisor zero is 1 i.e. $x = 1$ substituting 1 for x in $f(x)$,

$f(1) = (1)^3 - 3(1)^2 + 4(1) - 2$

$= 1 - 3 + 4 - 2$

$= 0$

$\therefore$ *The remainder is 0.*

(ii) If $x + 2$ is the divisor, the value of x that makes $x + 2$ zero is -2 i.e. $x = {}^-2$ substituting ${}^-2$ in $f(x)$.

$R = f(-2) = (-2)^3 - 3(-2)^2 + 4(-2) - 2$

$= -8 - 12 - 8 - 2$

$= -30$

The factor theorem

When we divide a polynomial $f(x)$ by $(x - a)$, the remainder $f(a) = 0$ implies that $x - a$ is a factor of $f(x)$.

The **factor theorem** is based on the **remainder theorem**.

If $x - a$ is a factor of $f(x)$, R will be zero when $f(x) \div (x - a)$ i.e. $R = f(a) = 0$

Therefore for a given polynomial $f(x)$, $f(a) = 0$ implies $(x - a)$ is a factor of $f(x)$.

The factor theorem therefore can be used to factorise polynomials which have factors. This method can be used in conjunction with long division.

Suppose $(x - a)$ is a factor of the polynomial $x^3 - 3x^2 - 10 + 24$, then a must be a factor of 24. So, to divide $x^3 - 3x^2 - 10x + 24$ by $(x - a)$, we try the smallest factors such 1, -1, ± 2, ± 3 first.

Let $f(x) = x^3 - 3x^2 - 10x + 24$

So,

$f(1) = 1 - 3 - 10 + 24 \neq 0 \Rightarrow (x - 1)$ is not a factor.

$f(-1) = -1 - 3 + 10 + 24 \neq 0 \Rightarrow (x + 1)$ is not a factor.

$f(2) = 8 - 12 - 20 + 24 = 0 \Rightarrow (x - 2)$ is a factor.

Trying all the possible factors of 24 is quite a tedious process.

So, it is wise to carry out a division by $(x - 2)$ so as to get a simpler polynomial.

$$
\begin{array}{r}
x^2 - x - 12 \\
x - 2\,\overline{\big)\, x^3 - 3x^2 - 10x + 24} \\
\underline{x^3 - 2x^2} \qquad\qquad\quad \\
-x^2 - 10x \qquad\;\; \\
\underline{-x^2 + 2x} \qquad\;\; \\
-12x + 24 \\
\underline{-12x + 24} \\
0 \quad
\end{array}
$$

We can see by inspection that the quotient $x^2 - x - 12$ is factorisable.

i.e. $x^2 - x - 12 = (x - 4)(x + 3)$

$\therefore x^3 - 3x^2 - 10x + 24 = (x - 2)(x - 4)(x + 3)$

So we can combine factor theorem and long division to factorise polynomials.

Example 8.7

Which of the following are factors of the polynomial $2x^3 - 3x^2 - 11x + 6$
$x + 2, x - 3, 2x - 1$

Solution

Test $x + 2, x - 3, 2x - 1$ for factors of $2x^3 - 3x^2 - 11x + 6$, using one factor at a time.

(i) Using $x + 2$, $x = -2$ makes the divisor zero

$f(-2) = 2(-2)^3 - 3(-2)^2 - 11(-2) + 6$
$= -8 \times 2 - 12 + 22 + 6$
$= -16 - 12 + 22 + 6 = 0$

$\therefore x + 2$ *is a factor of* $2x^3 - 3x^2 - 11x + 6$

(ii) Using $x - 3$, substitute $x = 3$

$f(3) = 2(3)^3 - 3(3)^2 - 11(3) + 6$
$= 54 - 27 - 33 + 6$
$= 54 + 6 - (27 + 33) = 0$

$\therefore x - 3$ *is a factor of* $2x^3 - 3x^2 - 11x + 6$

(iii) Using $2x - 1$, $2x - 1 = 0$

$\Rightarrow 2x = 1$

$\therefore x = \frac{1}{2}$

$f(\frac{1}{2}) = 2(\frac{1}{2})^3 - 3(\frac{1}{2})^2 - 11(\frac{1}{2}) + 6$
$= 2 \times \frac{1}{8} - 3 \times \frac{1}{4} - \frac{11}{2} + 6$
$= \frac{1}{4} - \frac{3}{4} - 5\frac{1}{2} + 6 = {}^{-}\frac{1}{2} - 5\frac{1}{2} + 6$
$= {}^{-}6 + 6 = 0$

$x + 2, x - 3, 2x - 1$ are factors of $2x^3 - 3x^2 - 11x + 6$

So,

$2x^3 - 3x^2 - 11x + 6 = (x + 2)(x - 3)(2x - 1)$

Notice that, the constant term usually gives a limit as to the factors we should try.

Example 8.8

Use the factor theorem to factorise
$x^3 - 6x^2 + 11x - 6$

Solution

Possible factors of ${}^{-}6$ that we can try are ${}^{-}2, 3$; $2, {}^{-}3$; $1, {}^{-}6$; ${}^{-}1, 6$

We try these factors in turn

$f({}^{-}2) = ({}^{-}2)^3 - 6({}^{-}2)^2 + 11({}^{-}2) - 6$
$= {}^{-}8 - 24 - 22 - 6 \neq 0 \Rightarrow x + 2$ *is not a factor*

$f(3) = 3^3 - 6(3)^2 + 11(3) - 6$
$= 27 - 54 + 33 - 6 = 0 \Rightarrow (x - 3)$ *is a factor*

$f(2) = 2^3 - 6(2)^2 + 11(2) - 6$
$= 8 - 24 + 22 - 6 = 0 \Rightarrow x - 2$ *is a factor*

$f(1) = 1^3 - 6(1)^2 + 11(1) - 6$
$= 1 - 6 + 11 = 0 \Rightarrow x - 1$ *is a factor*

Since a cubic function can have a maximum of three factors, one needs not test the other factors of six, since one has ***three*** *factors.*

$\therefore$ *the factors are* $x - 1, x - 2, x - 3$

So, $x^3 - 6x^2 + 11x - 6 \equiv (x - 1)(x - 2)(x - 3)$

Exercise 8.2

1. (a) Given that $f(x) = 6x^2 - 7x - 3$, find the values of: (i) $f(0)$ (ii) $f(2)$ (iii) $f(-1)$
   (b) If $f(x) = 2x^3 - 7x^2 + 4x - 3$, evaluate $f(1)$ and $f(7)$.

2. Given that $(x + 1)$ and $(x - 2)$ are factors of $x^4 - 3x^3 + ax^2 + bx + 4$ find the values of *a* and *b*.

3. Given that $f(x) = x^3 + 2x^2 - 7x + 5$, find the remainders when $f(x)$ is divided by $(x - 2)$, $x + 3$ and $(x - 4)$ respectively.

4. Use the remainder theorem to find the remainder in the following.

(a) $(x^2 - 5x + 4) \div (x - 1)$

(b) $(x^3 - 2x^2 + 3x + 6) \div (x - 2)$

(c) $(2x^3 - x^2 - 3x + 1) \div (2x + 1)$

(d) $(x^3 - 7x^2 + 4x - 2) \div (x + 1)$

5. Show that:

(a) $x - 1$ is a factor of $3x^3 - x^2 - 2x + 1$.

Write down the other factors.

(b) $x + 2$ is a factor of $x^3 - x^2 - 10x + -8$

What are the other factors?

(c) Given that $x^3 - x^2 - 9x + 9$ is exactly divisible by $x - 3$, find the other factors.

(d) Find the other factors of $x^3 - 2x^2 - 5x + 6$ given that $x - 3$ is one of the factors.

6. Factorise the following.

(a) $5x - 28x^2 - 15x^3 + 2$

(b) $x^3 - 4x^2 + x + 6$

(c) $x^3 - 8x^2 + 19x - 12$

(d) $x^3 - 2x^2 - 5x + 6$

(e) $x^3 + 1$

(f) $x^3 - 1$

7. Find the remainder when

(a) $3x^3 - 11x^2 + 2x + 5$ is divided by $3x + 1$

(b) $8x^3 + 1$ is divided $(2x + 1)$

8. Use the factor theorem to factorise

(a) $3x^3 - x^2 - 6x + 4$

(b) $2x^3 - 4x^2 - 9x + 9$

(c) $4x^3 - 5x^2 - 18x - 9$

(d) $x^3 + 4x^2 - 4x - 16$

Solving cubic equations

An equation of the form $ax^3 + bx^2 + cx + d = 0$ where $a \neq 0$ is called a cubic equation. Cubic equations can be solved either by factor method or by graphical method.

In this section we are going to use the factor method. We shall only deal with equations in which $f(x)$ is factorisable.

Example 8.9

Factorise the polynomial $x^3 - 9x^2 + 23x - 15$ and hence solve the equation

$x^3 - 9x^2 + 23x - 15 = 0$

Solution

Factors of $^-15$ that we can try are $3, ^-5$; $^-3, 5$; $^-1, 15$; $1, ^-15$.

$f(3) = (3)^3 - 9(3)^2 + 23(3) - 15 = 0 \Rightarrow x - 3$ *is a factor*

$f(5) = (5)^3 - 9(5)^2 + 23(5) + 23(5) - 15$

$= 125 - 225 + 115 - 15 = 0 \Rightarrow x - 5$ *is a factor*

$f(-1) = (-1)^3 - 9(-1)^2 + 25(-1) - 15$

$= -1 - 9 - 25 - 15 \neq 0 \Rightarrow x + 1$ *is not a factor*

$f = (1)^3 - 9(1)^2 - 23(1) - 15$

$= 1 - 9 + 23 - 15 = 0 \Rightarrow x - 1$ *is a factor*

$\therefore x^3 - 9x^2 + 23x - 15 = (x - 3)(x - 5)(x - 1)$

$\therefore f(x) = 0 \Rightarrow x^3 - 9x^2 + 23x - 15$

$= (x - 3)(x - 5)(x - 1) = 0$

$\therefore x - 3 = 0 \Rightarrow x = 3$

$x - 5 = 0 \Rightarrow x = 5$

$x - 1 = 0 \Rightarrow x = 1$

The solution set is $(1, 3, 5)$ i.e. $x = 1, 3$ or 5.

Example 8.10

Factorise the polynomial $x^3 + 2x^2 - 13x + 10$ and hence solve the equation

$x^3 + 2x^2 - 13x + 10 = 0$

Solution

$x^3 + 2x^2 - 13x + 10$

Factors of 10 are $1, ^-1, 2, ^-2, 5, ^-5, 10$ and $^-10$.

$f(1) = (1)^3 + 2(1)^2 - 13(1) + 10 = 0 \Rightarrow (x - 1)$ *is a factor.*

$f(-1) = (^-1)^3 + 2(^-1)^2 - 13(^-1) + 10 \neq 0 \Rightarrow (x + 1)$ is not a factor.

$f(2) = (2)^3 + 2(2)^2 - 13(2) + 10 = 0 \Rightarrow (x - 2)$ is a factor.

$f(-2) = (^-2)^3 + 2(^-2)^2 - 13(^-2) + 10 \neq 0 \Rightarrow (x + 2)$ is not a factor.

$f(5) = (5)^3 + 2(5)^2 - 13(5) + 10 \neq 0 \Rightarrow (x - 5)$ is not a factor.

$f(-5) = (^-5)^3 + 2(^-5)^2 - 13(^-5) + 10 = 0 \Rightarrow (x + 5)$ is a factor.

$\therefore x^3 + 2x^2 - 13x + 10 = (x - 1)(x - 2)\ (x + 5)$

$\therefore f(x) = 0 \Rightarrow x^3 + 2x^2 - 13x + 10$

$= (x - 1)(x - 2)(x + 5) = 0$

$\therefore x - 1 = 0 \Rightarrow x = 1$

$x - 2 = 0 \Rightarrow x = 2$

$x + 5 = 0 \Rightarrow x = -5$

The solution set is $(1, 2, -5) \Rightarrow x = 1, 2$, 0r -5

Note

If in the course of factorisation one of the factors is quadratic in the form $ax^2 + bx + c$ which has no factor, then we use the quadratic formula or completing the square method to solve that part of the equation.

Exercise 8.3

1. Solve the following cubic equations
   (a) $(x - 1)(x + 2)(x - 3) = 0$
   (b) $(x + 1)(x - 3)(8 - x) = 0$
   (c) $(x + 4)(x - 1)(1 + 5x) = 0$

Factorise and solve the following polynomials

2. $x^3 - 7x - 6 = 0$
3. $2x^3 - 5x^2 - 4x + 3 = 0$
4. $2x^3 + 13x^2 - 48x - 27 = 0$
5. $6x^3 - x^2 - 31x - 10 = 0$
6. $x^3 + 2x^2 - 5x - 6 = 0$
7. $x^3 - 3x^2 - 10x + 24 = 0$
8. $2x^3 + {}^-x^2 + {}^-8x - 4 = 0$
9. $3x^3 - 2x^2 - 3x + 2 = 0$
10. $x^3 - 4x^2 + 5x - 2 = 0$

Identities

An identity is an equation whose solution set is the set of all real numbers.

An identity is denoted by the symbol $\equiv$. For example, $2(x - 3) + 14 \equiv 2(x + 4)$ is an identity and it is true for all values of x.

Suppose $f(x)$ and $g(x)$ are polynomials;

$f(x) \equiv g(x)$ only if:

(i) they are of the same degree,

(ii) they have the same number of terms,

(iii) the coefficients of the corresponding terms are equal.

If $f(x) = g(x)$, then $f(a) = g(a)$ for all $\boldsymbol{a}$, and the coefficient of x^n in $f(x)$ = coefficient of x^n in $g(x)$ for all $\boldsymbol{n}$.

Example 8.11

Given that $a(x + 3)^2 + b(x - 2) + 1 \equiv 3x^2 + 20x + 24$, find the values of a and b.

Solution

Since the identity is true for all values of x, we substitute (i) $x = -3$ and (ii) $x = 2$, one at a time.

When $x = {}^-3$,

$a(x + 3)^2 + b(x - 2) + 1 \equiv 3x^2 + 20x + 24$

Becomes

$$1 + a(^-3 + 3)^2 + b(^-3 - 2) \equiv 3(^-3)^2 + 20(^-3) + 24$$
$$^-5b + 1 \equiv +27 - 60 + 24$$
$$\equiv 51 - 60$$
$$^-5b \equiv -9 - 1 = -10$$
$$\therefore\ b = 2$$

when $x = 2$,

$$a(2 + 3)^2 + 2(2 - 2) + 1 \equiv 3(2)^2 + 20(2) + 24$$
$$25a + 1 \equiv 12 + 40 + 24$$
$$a = \frac{75}{25}$$
$$= 3$$

Alternatively we can find the values of a and b by first expanding the left hand side of the identity, and then comparing coefficients of appropriate terms.

$a(x + 3)^2 + b(x - 2) + 1 \equiv 3x^2 + 20x + 24$

LHS: $a(x^2 + 6x + 9) + b(x - 2) + 1$

$ax^2 + 6ax + 9a + bx - 2b + 1$

If $ax^2 + (6a + b)x + 9a - 2b + 1 \equiv 3x^2 + 20x + 24$,

We can now equate the coefficient of like terms.

Thus: $a = 3$ …… *the coefficients of x^2 in the two expressions*

$6a+b = 20$

and $9a - 2b + 1 = 24$

The first two equations of the coefficient give $a = 3$ and $b = 20 - 18 = 2$.

To confirm that our values of a and b are correct, we substitute in the third equation

LHS $9a - 2b + 1 = 9 \times 3 - 2 \times 2 + 1$

$= 27 - 4 + 1$

$= 24$

$\therefore a = 3$ and $b = 2$

Example 8.12

*Given that $x + 1$ and $x - 2$ are factors of the polynomial $x^4 - 3x^3 + ax^2 + bx + 4$, use the factor theorem to find the values of **a** and **b**.*

Solution

If $x + 1$ is a factor, $f(-1) = 0$

$f(-1) = (-1)^4 - 3(-1)^3 + a(-1)^2 + b(-1) + 4 = 0$

$= 1 + 3 + a - b + 4 = 0$

$a - b + 8 = 0$ … (i)

If $x - 2$ is a factor, $f(2) = 0$

$f(+2) = (2)^4 - 3(2)^3 + a(2)^2 + b(2) + 4 = 0$

$16 - 24 + 4a + 2b + 4 = 0$

$20 - 24 + 4a + 2b = 0$ …(ii)

$2a + b = 2$

(i) and (ii) are simultaneous equations

Rearrange to obtain $a - b = -8$

$$\begin{array}{r} a - b = -8 \\ +\ 2a + b = 2 \\ \hline 3a \quad = -6 \end{array}$$

$a = -2$

Substituting $a = -2$ in (i), $-2 - b = 8$

$b = 6$

Application of polynomials to real life

In earlier mathematics topics, we used linear, quadratic and trigonometric functions to model real life problems. There are many other types of functions that we can use to model real life relations. Polynomials are among them.

Example 8.13

A toy manufacturer has created a new card game. Each game is packed in an open-top cardboard box, which is then wrapped with clear plastic paper. The box for the game is made from a 20 cm × 30 cm piece of cardboard. Four equal squares are cut from the corners, one from each corner of the cardboard piece. The sides are then folded and the edges that touch each other are glued. What must be the dimensions of each square so that the resulting box has maximum volume?

Solution

To develop a model of this problem, we will begin by defining variables.

Let x represent the side length of each square cut from the cardboard. Let v represent the volume of the box. Then draw and label diagrams (Fig. 8.1).

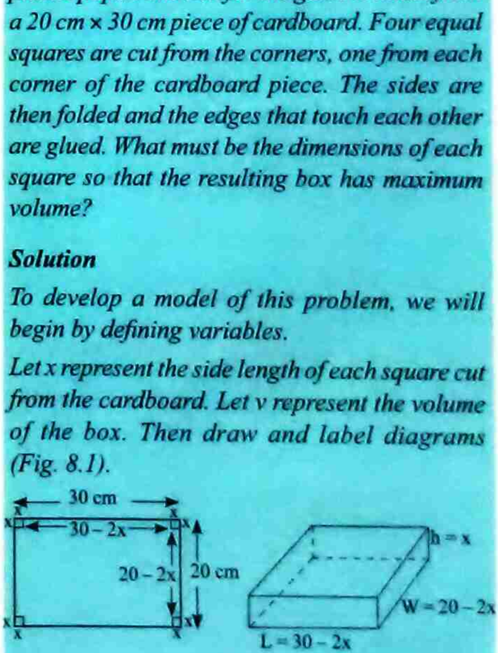


Fig. 8.1

Fig. 8.1 is a two dimensional and a 3 dimensional sketches of the cardboard box.

Volume = length × width × height

$V(x) = (30 - 2x)(20 - 2x)(x)$ *(expand and multiply binomials)*

$= (600 - 60x - 40x + 4x^2)(x)$ *(expand and multiply by x)*

$= 600x - 60x^2 - 40x^2 + 4x^3$ *(collect like terms)*

$= 600x - 100x^2 + 4x^3$ *(Rearrange powers of x in descending order)*

$4x^3 - 100x^2 + 600x$

The resulting algebraic model

$V(x) = 4x^3 - 100x^2 + 600x$

is a polynomial

To find the dimensions of the squares, we find the value of x in $4x^3 - 100x^2 + 600x = 0$ *(By letting* $V(x) = 0$*)*

$4x(x^2 - 25x + 150) = 0$ *(factor out common factor 4x)*

$4x = 0$ *or* $x^2 - 25x + 150 = 0$

$x = 0$

$x^2 - 25x + 150 = 0$ *(find factors of +150 that add up to −25)*

$x^2 - 10x - 15x + 150) = 0$

$x(x - 10) - 15(x - 10) = 0$

either $x - 10 = 0$ *or* $x - 15 = 0$

$x = 10$ *or* $x = 15$

$x = 0$ *or* $x = 10$ *or* $x = 15$

Now to ensure maximum volume the dimensions of the squares should be 15 cm by 15 cm.

Exercise 8.4

1. $x + 2$ is a factor of the polynomial $x^4 - 2x^2 + k$. Find the value of k.

2. Factorise $x^3 + 3x^2 - 4x - 12$. Hence state the resulting identity.

3. Given that $f(x) = ax^3 + ax^2 + bx + 12$ and that $f(^-2) = f(3) = 0$, find the values of $\boldsymbol{a}$ and $\boldsymbol{b}$.

4. Find the value of k if $x^3 + 4x^2 + kx + 6$ divided by $x + 5$ leaves a reminder of -4.

5. If $f(2) = f(^-3) = 0$, use the identity $f(x) \equiv x^3 + 2x^2 + ax + b$ to find the values of $\boldsymbol{a}$ and $\boldsymbol{b}$. Hence, the remainder when $x^3 + 2x^2 + ax + b$ is divided by $x - 4$.

6. Use the identity $5x^3 + ax^2 + bx - 12 \equiv (5x^2 + 2)(x + 2)$ to find the value of $\boldsymbol{a}$ and $\boldsymbol{b}$.

7. Use the identity $x^2 + 7x + 12 \square (x + a)(x + a)$ to find the value of **a** and **b.**

8. Find the value of a, b and c in the identity $2x^2 - x + 1 \square a(x - 1)^2 + b(x - 1) + c$

9. When the expression $3x^3 + ax^2 - 4x + 6$, is divided by $(x - 4)$, the reminder is 22. Find the value of **a**.

10. If $x^3 + ax^2 + bx + c \square (x + d)^3$ where a, b and d are constants, express **ab** in terms of **c**.

9 PROBABILITY II

Success criteria

By the end of this topic, the student must be able to:

- *Predict the occurrence of events.*
- *Calculate the probability of two or more events.*

Introduction

In Form 2, we were introduced to both experimental and theoretical probability involving single events and simple cases. In this chapter, we will extend our knowledge to experimental and theoretical probability involving two events. We will also construct and use tree diagrams in solving problems.

The possibility or sample space

When a coin is tossed, the only possible outcomes are head (H) or tail (T). When a coin is tossed twice or when two coins are tossed at the same time, the only possible outcomes are HH, HT, TH, TT.

The list of all possible outcomes of an experiment is called the **possibility space** or the **sample space**. Each outcome is called a **sample** or a **sample point**.

Thus, we have the following as examples:

	Experiment	Possibility space
(a)	Tossing a coin once	H,T
(b)	Tossing a coin twice (or two coins at the same time)	HH, HT, TH, TT
(c)	Tossing a die once	1, 2, 3, 4, 5, 6

What would be the possibility space when a coin is tossed three times?

Table 9.1 shows another way of displaying the possibility space when a coin is tossed twice.

Second toss	First toss		
		H	T
	H	HH	HT
	T	TH	TT

Table 9.1

Example 9.1

Two dice are tossed at the same time. What is the possibility space?

Solution

Note that each die has the faces marked 1, 2, 3, 4, 5, 6. Taking the first number to refer to the outcome on the first die and the second number for the second die, the possibility space could be written as:

(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), ...,(6,4), (6,5), (6,6).

The same possibility space could be presented as shown in Table 9.2. Copy and complete the table.

	2nd die					
1st die	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Table 9.2

A possibility space may contain a finite number or sample point (countable). It is said to be finite.

For example tossing a coin can only give a head or a tail hence finite. Otherwise, possibility space may be infinite (uncountable). For example, choosing an odd integer from the set of integers is uncountable.

Exercise 9.1

Define a suitable possibility space for each of the following experiments.

How many outcomes are there in each case? Discuss:

1. A coin and a die are tossed at the same time and the faces showing recorded.
2. Two dice are tossed at the same time. The sum of the faces showing up is recorded.
3. Three coins are tossed and the faces showing recorded.
4. A natural number is chosen at random.
5. Two coins are tossed and the number of heads showing recorded.
6. Two coins are tossed onto the floor and the distance between the coins recorded.
7. Three coins are tossed and the number of heads observed is recorded.
8. The height of each pupil in your class is recorded.
9. Four coins are tossed and the number of heads observed is recorded.
10. The life time of an electric bulb is measured.

Events

An **event** is a set of outcomes which are a part of the possibility space of an experiment.

For example, when two coins are tossed, the possibility space is HH, HT, TH, TT. We may describe event A as getting one head. Then the outcomes of event A are HT, TH.

Let us take event B as getting at least one head. Then the outcomes of event B are HH, HT, TH.

Equally likely outcomes

We have seen that because of the symmetry of a coin, " heads" and "tails" have equal chances of occurring when the coin is tossed. We say that the two outcomes are **equally likely**. Hence, $P(H) = P(T) = \frac{1}{2}$.

In general,

> If an experiment has N equally likely outcomes $a_1, a_2, \ldots, a_n$, then the probability of each outcome is
>
> $$P(a_1) = P(a_2) \ldots = P(a_n) = \frac{1}{N}.$$
>
> If an event E can occur in a number of ways (outcomes), then the probability of event E is:
>
> $$P(E) = \frac{\text{No. of outcomes in favour of E}}{\text{Total No. of outcomes in the possibility space}}$$

Example 9.2

A die is tossed and the number showing on top is observed. What is the probability that the number is greater than 4?

Solution

The possibility space is 1, 2, 3, 4, 5, 6.

Let E be the event that the outcome is greater than 4. Then, E has outcomes 5 and 6.

$$P(E) = \frac{\text{No. of outcomes in favour of E}}{\text{Total number of outcomes}} = \frac{2}{6} = \frac{1}{3}.$$

Example 9.3

A card is selected from an ordinary pack of playing cards. Let A be the event that "the card is a Spade" and B be the event "the card is a Jack (J), Queen (Q) or King (K)".

Find: (a) P (A) (b) P (B).

Solution

Total number of outcomes

= total number of cards = 52 cards.

(a) No. of spades = 13

$$\therefore \quad P(A) = \frac{\text{No. of spades}}{\text{Total no. of cards}} = \frac{13}{52} = \frac{1}{4}$$

(b) Number of (Jack + Queen + King) = 12

$$P(B) = \frac{\text{No. of } (J + Q + K)}{\text{Total No. of cards}} = \frac{12}{52} = \frac{3}{13}$$

Note:

1. The probability that an event occurs may also be expressed as

   "The odds that an event occurs are p to q".

   This means that the probability that the event occurs is $\frac{p}{p+q}$.

   For example, in Example 9.3 the odds that the event A occurs are 1 to 3 and the odds that the event B occurs are 3 to 10.

2. There are certain terms that are commonly used in probability.

They include:

	Term	Meaning
(a)	Selection at random	Each item has the same chance of being selected.
(b)	At least	Starting with the given value onwards.
(c)	At most	Up to and including the given value.
(d)	Not more than	Same as "at most".
(e)	Not less than	Same as "at least".

Exercise 9.2

1. A die is tossed and the number showing up on top is observed. E_1 is the event "the score is 5", E_2 is the event "the score is less than 5". Find:

   (a) $P(E_1)$ (b) $P(E_2)$.

2. Two dice are rolled and the sum of the numbers showing on top is recorded. You are given the following events:

   E_1: the score is a prime number;
   E_2: the score is a multiple of 4;
   E_3: the score is greater than 8;
   E_4: the score is a factor of 9.

   Find:

   (a) $P(E_1)$, (b) $P(E_2)$,
   (c) $P(E_3)$, (d) $P(E_4)$.

3. Two dice are rolled and the numbers showing on top observed. You are given the following events.

   E_1: the sum is 5;
   E_2: both dice show the same score;
   E_3: the total score is greater than 7.

   Find:

   (a) $P(E_1)$ (b) $P(E_2)$ (c) $P(E_3)$.

4. In a room, there are 4 men and 6 women. One person is picked at random. What is the probability that the person is a woman?

5. Mleng bought shirts from a wholesaler. He found that 10 were good, 4 had minor defects and 2 had major defects. One shirt was chosen at random. Find the probability that:

   (a) it had no defects
   (b) it had no major defects
   (c) it was either good or had major defects.

6. An integer is chosen at random from the numbers 1, 2, 3, ..., 50. What is the probability that the chosen number is divisible by 6 or 8?

7. John had 3 black biro pens and 5 blue ones. He took one at random and gave it to Evans. What is the probability that Evans got a blue biro pen?

8. Two coins are tossed. What is the probability of at least one tail appearing?

9. Two sets of cards X and Y are numbered from 1 to 5. A card is drawn at random from each set. The two cards are placed side by side to form a two-digit number. What is the probability that the number formed:

   (a) is divisible by 5?
   (b) contains at least one 4?
   (c) is prime?

10. In a school of 700 students, 150 are in Form 4. There are 25 prefects in Form 4. What is the

probability that a student chosen at random is in Form 4 but is not a prefect?

11. What is the probability that your teacher was born on a Monday?

12. A boy made savings by keeping one-kwacha and five-kwacha coins in a bag. A coin is picked from the bag at random. The probability that the coin is a five-kwacha coin is $\frac{3}{8}$.
    (a) What is the probability that the coin is a one-kwacha coin?
    (b) If the boy's savings were K 500, how much of it was in five-kwacha coins?

13. A class consists of 5 Malawians, 4 Ugandans, 8 Tanzanians and 3 Zambians. A student is chosen at random to represent the class. What is the probability that the student is:
    (a) Ugandan? (b) Zambian?
    (c) Tanzanian or Zambian?

14. Find the probability of an event if the odds that it will occur are:
    (a) 3 to 1 (b) 6 to 7.

Discrete probability space

Earlier on, we said that the list of all possible outcomes of an experiment is called a possibility space or a sample space.

Let the possibility space of an experiment be a_1, a_2, ..., a_n.

Let the corresponding probabilities be $P(a_1)$, $P(a_2)$, ..., $P(a_n)$ respectively.

Then, the list $P(a_1)$, $P(a_2)$, ..., $P(a_n)$ is called the **probability space** of the experiment. Since the sample points are countable, we call such a sample space a **discrete sample space**.

Note:

1. For each $P(a_i)$, $0 \leq P(a_i) \leq 1$.
2. $P(a_1) + P(a_2) + \ldots + P(a_n) = 1$.

Example 9.4

A die is tossed. What is the probability space?

Solution

The possibility space is 1, 2, 3, 4, 5, 6.
All the outcomes are equally likely.
So the probability of each is $\frac{1}{6}$.
The probability space is $\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$.

Example 9.5

Two coins are tossed and the number of heads observed. What is the probability space?

Solution

When two coins are tossed, all the possible outcomes are HH, HT, TH, TT. Since all the outcomes are equally likely, each has a probability of $\frac{1}{4}$.
Thus, $P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$.
Note that we are interested in the number of heads appearing. Thus, the possibility space is 0, 1, 2, since TT means 0 heads, HT or TH means 1 head and HH means 2 heads. We get 1 head by getting HT or TH.
Thus $P(0) = P(TT) = \frac{1}{4}$, $P(1) = P(HT \text{ or } TH) = \frac{1}{2}$, $P(2) = P(HH) = \frac{1}{4}$.
$\therefore$ *Probability space is* $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.

Exercise 9.3

1. Suppose the possibility space of an experiment is a_1, a_2, a_3, a_4. Which of the following will define a probability space? Explain why others do not define probability spaces.
    (a) $P(a_1) = \frac{1}{4}$, $P(a_2) = -\frac{1}{4}$, $P(a_3) = \frac{1}{2}$, $P(a_4) = \frac{1}{2}$
    (b) $P(a_1) = \frac{1}{2}$, $P(a_2) = \frac{1}{2}$, $P(a_3) = \frac{1}{4}$, $P(a_4) = \frac{1}{15}$
    (c) $P(a_1) = \frac{1}{4}$, $P(a_2) = \frac{1}{8}$, $P(a_3) = \frac{1}{5}$, $P(a_4) = \frac{1}{10}$
    (d) $P(a_1) = 0$, $P(a_2) = \frac{1}{4}$, $P(a_3) = \frac{1}{2}$, $P(a_4) = \frac{1}{4}$

2. A coin is made in such a way that heads is three times as likely to appear as tails. Find:
    (a) P(H), (b) P(T)

3. Let the possibility space of an experiment be x_1, x_2, x_3, x_4.
   (a) Find $P(x_4)$ if $P(x_1) = \frac{1}{6}$, $P(x_2) = \frac{1}{8}$, $P(x_3) = \frac{1}{9}$.
   (b) If $P(x_1) = 3\,P(x_2)$, $P(x_3) = \frac{1}{4}$ and $P(x_4) = \frac{1}{5}$, find $P(x_1)$ and $P(x_2)$.

4. Two boys and three girls are in a table tennis tournament. Those of the same sex have equal chances of winning, but each boy is three times as likely to win as any girl. Find the probability of a girl winning the tournament.

Continuous possibility space

At the end of Exercise 9.1, you were asked how many outcomes there were in each possibility space in Questions 1–10. What were the results?

You should have noticed that in some possibility spaces:

(a) the outcomes are specific numbers. Such a possibility space is said to be a **finite possibility space**.

(b) the outcomes are countable but infinite. Such a possibility space is said to be a **countably infinite possibility space**.

(c) the outcomes cannot be counted. For example, height, area, volume, etc. would not, under normal circumstances, be specific values. A height, an area or a volume could be a value within a certain range. Such outcomes make up a **continuous possibility space** or **uncountable possibility space**.

Note: A finite or countably infinite space is said to be **discrete** and a continuous (uncountable) space is said to be **non-discrete**.

If S is a continuous possibility space and A is a given range within S, then the probability of the event A (i.e. the probability that a point selected at random belongs to A) is given by

$$P(A) = \frac{\text{Length of A}}{\text{Length of S}} \text{ or } P(A) = \frac{\text{Area of A}}{\text{Area of S}} \text{ or}$$

$$P(A) = \frac{\text{Volume of A}}{\text{Volume of S}}.$$

Example 9.6

What is the probability that, at a given moment, the minute hand of a clock is between 2 and 3?

Solution

Fig 9.1 shows the face of a clock. The area shaded is the region the minute hand would be in.

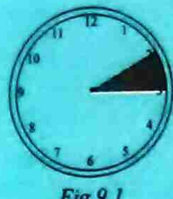


Fig 9.1

Let A be the event "minute hand is between 2 and 3"

$$P(A) = \frac{\text{Area of A}}{\text{Area of the face of the clock}}.$$

Since area of A is $\frac{1}{12}$ of the area of the face of the clock, then $P(A) = \frac{1}{12}$.

Exercise 9.4

1. A point is selected at random inside a circle of radius 4 cm. What is the probability that the point is nearer the centre of the circle than the circumference?

2. Two concentric circles have radii of 4 cm and 6 cm respectively. A point is selected at random inside the bigger circle. What is the probability that the point is outside the smaller circle?

3. A point is selected at random inside an equilateral triangle ABC whose side is 5 cm. What is the probability that the point is more than 2 cm away from any vertex?

4. Fig 9.2 shows a hollow cylinder of radius 1 m and length 6 m, having lids on both ends. The circular openings A and B, which are the

only openings into the hollow of the cylinder, has a radius of 0.3 m each. A butterfly enters through A and gets out through B. Given that a butterfly does not keep on a straight path as it flies, find the probability that at any given time the butterfly is in the shaded space A and B?

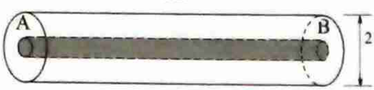


Fig 9.2

5. Fig 9.3 shows an inscribed circle radius 1.2 cm, of a ΔPQR in which PQ = 3.6 cm, QR = 4.6 cm and PR = 5.8 cm.

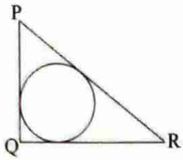


Fig 9.3

A point is selected at random in the triangle. What is the probability that the point lies outside the circle?

6. Fig 9.4 shows a circle, centre O, radius 10 cm and $\angle AOB = 70°$. An insect is observed to be moving around within the circle. What is the probability that the insect is in the shaded region?

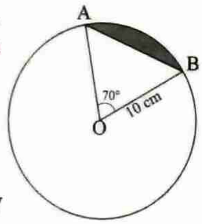


Fig. 9.4

7. The school clock is faulty. It works and stops at random. What is the probability that it stops when the minute hand is between 10.43 a.m and 11.02 a.m?

8. Fig 9.5 shows a darts board of radius 24 cm. The inner ring A has inner radius of 7 cm and is 1 cm thick. The next ring B has an inner radius of 15 cm and is 1cm thick.

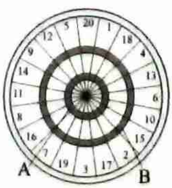


Fig 9.5

Angela throws darts which land on the board in a random manner. What is the probability that the dart lands:

(a) in the sector labelled 20?

(b) either in the sector labelled 20 or in the one labelled 12?

(c) on ring A?

(d) on ring B?

(e) on the part of ring A which is in sector 1?

(f) on ring A or in sector 11?

Combined events

We have so far been dealing with cases involving single events. Let us consider cases in which two or more events are involved.

Mutually exclusive events

We have seen that, when a coin is tossed, the outcomes are H,T. It is **not possible** to have a head and a tail appear at the same time.

When a die is tossed, the outcomes are 1, 2, 3, 4, 5, 6. If the outcome is 1, then none of the others can occur at the same time. Similarly, when any other number appears, none of the others can occur.

Thus, when a coin is tossed, the events "getting heads" and "getting tails" are mutually exclusive. Similarly, when a die is tossed, the events "getting a one", "getting a two", "getting a three", "getting a four", "getting a five" or "getting a six" are mutually exclusive events.

Two or more events in which the occurrence of one eliminates the possibility of the other one occuring are called **mutually exclusive events**.

In set language, we say that events A and A' are mutually exclusive for any event A which means that $P(A \cup A) = P(A) + P(A)$ since one of the event A or A' must occur

$P(A \cup A') = 1 = P(A + P(A')$

Example 9.7

A box contains red and green apples. There are 8 red and 28 green apples. One apple is picked at random from the box. Find the probability that the apple picked is (a) red (b) green

Solution

(a) $P(red) = \dfrac{\text{Number of red apples}}{\text{Total number of apples}}$

$= \frac{8}{36} = \frac{2}{9}$.

(b) $P(green) = \dfrac{\text{Number of green apples}}{\text{Total number of apples}}$

$= \frac{28}{36} = \frac{7}{9}$.

Example 9.8

A bag contains 8 oranges, 4 mangoes and 6 lemons. A fruit is taken from the bag at random. What is the probability that it is:

(a) an orange, (b) a mango,

(c) a lemon, (d) an orange or a mango,

(e) an orange or a lemon,

(f) a mango or a lemon?

What relationship is there between the answers you got in (a)–(c) and those you got in (d)–(f)?

Solution

(a) $P(\text{an orange}) = \dfrac{\text{Number of oranges}}{\text{Total number of fruits}}$

$= \frac{8}{18} = \frac{4}{9}$.

(b) $P(\text{a mango}) = \dfrac{\text{Number of mangoes}}{\text{Total number of fruits}}$

$= \frac{4}{18} = \frac{2}{9}$.

Similarly,

(c) $P(\text{a lemon}) = \frac{6}{18} = \frac{1}{3}$.

(d) $P(\text{an orange or a mango})$

$= \dfrac{\text{Number of oranges and mangoes}}{\text{Total number of fruits}}$

$= \frac{12}{18} = \frac{2}{3}$.

(e) $P(\text{an orange or a lemon})$

$= \dfrac{\text{Number of oranges and lemons}}{\text{Total number of fruits}}$

$= \frac{14}{18} = \frac{7}{9}$.

Similarly,

(f) $P(\text{a mango or a lemon}) = \frac{10}{18} = \frac{5}{9}$.

The relationship between the answers is as follows:

$\frac{4}{9} + \frac{2}{9} = \frac{6}{9}$ i.e. $P(\text{an orange}) + P(\text{a mango})$

$= P(\text{an orange or a mango})$

$\frac{4}{9} + \frac{1}{3} = \frac{7}{9}$ i.e. $P(\text{an orange}) + P(\text{a lemon})$

$= P(\text{an orange or a lemon})$

$\frac{2}{9} + \frac{1}{3} = \frac{5}{9}$ i.e. $P(\text{a mango}) + P(\text{lemon})$

$= P(\text{a mango or a lemon})$

We note that:

If A and B are **mutually exclusive** outcomes of a random experiment, the probability that A **or** B will occur is the **sum** of their probabilities. We write

$P(A \text{ or } B) = P(A) + P(B)$

or $P(A + B) = P(A) + P(B)$

This property is also called the **additon** rule for probabilities of mutually exclusive events. It follows that if events A, B, C, ... are mutually exclusive, P(A or B or C or ...)

$= P(A) + P(B) + P(C) + \ldots$

Exercise 9.5

1. A group of tourists arrived at Malawi International Airport. 5 were English, 4 were French, 8 were American and 3 were German. One tourist was chosen at random to be their leader. What is the probability that the one chosen was:

   (a) English (b) American

   (c) German (d) French or German

   (e) English or French

   (f) not English?

2. In a bag, there are some blue pens, some red pens and some of other colours. The probability of taking a blue pen at random is $\frac{1}{7}$. If the probability of taking a blue pen or a red pen at random is $\frac{8}{21}$. What is the probability of taking:

   (a) a red pen?

   (b) a pen which is neither red nor blue?

3. In a factory, machines A, B and C produce identical balls. The probability that a ball was produced by machine A or B is $\frac{11}{15}$. The probability that a ball was produced by machine B or C is $\frac{2}{3}$. If the probability that a ball was produced by machine A is $\frac{1}{3}$. What is the probability that a ball was produced by machine C?

4. A card is chosen at random from an ordinary pack of playing cards. What is the probability that it is:

   (a) either hearts or spades?

   (b) either a club or a jack of spades?

5. When playing netball, the probability that only Ann scores is $\frac{1}{4}$, the probability that only Betty scores is $\frac{1}{8}$ and the probability that only Carol scores is $\frac{1}{12}$. What is the probability that none of them scores?

6. Two dice are tossed. Find the probability that:

   (a) an odd number shows on the second die,

   (b) a two or a five shows on the first die,

   (c) a two or a five shows on the first die and an odd number on the second die.

   What connection is there between the answers to parts (a), (b) and (c)?

7. In a certain school of 1 000 students, 20 are colour blind and one hundred are overweight. A student is chosen at random. What is the probability that the students is:

   (a) colour blind? (b) overweight?

8. In a certain race, the odds that Edwin wins are 2 to 3 and the odds that Richard wins are 1 to 4. What is the probability that Edwin or Richard wins the race?

Independent events

Examine Question 6 of Exercise 9.5 again.

The question shows that the events A "a two or five on the first die" and B "an odd number on the second die" are not mutually exclusive.

Note that in this question P(A) × P(B) equals the probability of the events occuring together [answer to part (c)].

Two events A and B are said to be **independent** if the probability of them occuring together is the product of their individual probabilities.
We write:

$P(A \text{ and } B) = P(A) \times P(B) = P(A)\,P(B)$ or

$P(A.B) = P(A) \times P(B) = P(A)\,P(B)$.

This property is also called the **product law** for probabilities of independent events.

Given a number of events, we confirm for their independence by:

1. checking that any two of them are independent, and
2. checking that the probability of all of them occuring together is equal to the product of their individual probabilities

Example 9.9

A box contains 10 bolts. It is found that 4 of them are substandard. If two bolts are taken from the box at random, what is the chance that both are substandard?

Solution

Probability that the first bolt taken is substandard is $\frac{4}{10}$.
Since 3 substandard bolts are remaining in the box, the probability that the second bolt taken is substandard is $\frac{3}{9}$.

∴ *Probability that both bolts are substandard is* $\frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$.

Example 9.10

During registration of candidates for a national examination, it is observed that a form four class comprising of 20 candidates has five under age learners. If three learners are chosen at random, what is the probability that all the three learners are underage?

Solution

Probability that the first chosen learner is underage $= \frac{5}{20}$.

Four underage learners remains.

The probability that the second chosen learner is under age $= \frac{4}{19}$.

The probability that the third chosen learner is underage $= \frac{3}{18}$.

The probability that all the three learners are under age

$= \frac{5}{20} \times \frac{4}{19} \times \frac{3}{18} = \frac{1}{114}$

Exercise 9.6

1. A bag contains 7 black and 3 white balls. If two balls are drawn from the bag, what is the probability that:
   (a) one is black and one is white?
   (b) they are of the same colour?
2. Three students were asked to solve a problem. Their chances of solving the problem independently were $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.
   (a) What is the chance that all of them solved the problem independently?
   (b) What is the probability that only two solved the problem independently?
3. Two dice are tossed giving the events: A: the first die shows a six, B: the second die shows a three, C: the sum of the numbers on the two dice is 7. Check these events for independence.
4. A class has 18 boys and 12 girls. Three students are chosen at random from the class. What is the probability that:
   (a) they are all boys?
   (b) one is a boy and the others are girls?
5. In an office, there are 3 men and 7 women. Three people are chosen at random. What is the probability that two are women and one is a man?
6. Events A and B are such that $P(A) = \frac{1}{5}$ and $P(A \text{ and } B) = \frac{2}{15}$. What is P(B) if A and B are independent?
7. A bag contains 7 lemons and 3 oranges. If they are drawn one at a time from the bag, what is the probability of drawing a lemon then an orange, and so on, alternately until only lemons remain?
8. The probability that machine A will be working at the end of the year is $\frac{3}{5}$. The probability that machine B will be working at the end of the year is $\frac{2}{3}$. Find the probability that:
   (a) both will be working,
   (b) only machine B will be working at the end of the year.
9. A school has two old typewriters. One is under repair 10% of the time and the other for 15% of the time. What is the probability of both being out of action at the same time?

Tree diagrams

Consider the case where we are given three bags as follows:

Bag A contains 10 watches of which 4 are defective.

Bag B contains 12 watches of which 2 are defective.

Bag C contains 15 watches of which 3 are defective.

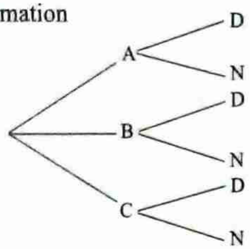
If a watch is chosen at random from a bag, what is the probability that the watch is defective?

Note that, a bag must be selected first before a watch is chosen. Hence, there are two experiments:

(i) Selecting a bag.

(ii) Selecting a watch from the bag. It is either defective (D) or not defective (N).

We represent this information on a **tree diagram** as shown in Fig. 9.6



Since the bags have equal chances of being selected, the probability of selecting a bag is $P(A) = P(B) = P(C) = \frac{1}{3}$.

Fig. 9.6

Let P(D) be the probability of a watch being defective and P(N) be the probability of a watch not being defective.

From bag A, $P(D) = \frac{4}{10}$ and $P(N) = \frac{6}{10}$

From bag B, $P(D) = \frac{2}{12}$ and $P(N) = \frac{10}{12}$

From bag C, $P(D) = \frac{3}{15}$ and $P(N) = \frac{12}{15}$

Fig. 9.7 shows the same tree diagram as in Fig. 9.6, but with probabilities shown on it.

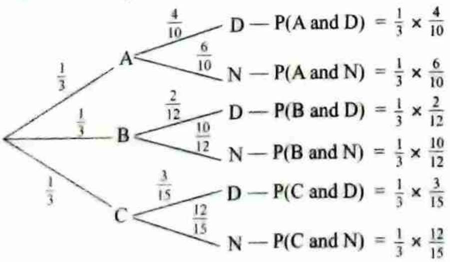


Fig. 9.7

Since choosing a bag and selecting a watch from the bag are independent events, we multiply the probabilities along the branches.

Since the branches are mutually exclusive alternatives, we add the products on the branches.

Thus, the probability of having a defective watch is,

P(D) = P(A) × P(defective watch in A) + P(B) × P(defective watch in B) + P(C) × P(defective watch in C).

i.e. $P(D) = \frac{1}{3} \times \frac{4}{10} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{5}$

$= \frac{2}{15} + \frac{1}{18} + \frac{1}{15}$

$= \frac{23}{90}$.

A tree diagram is very useful when working out problems involving successive experiments.

Example 9.11

A bag contains 3 black, 5 red and 4 white marbles. Two marbles are drawn from the bag without replacement. Find the probability that:

(a) they are both the same colour.

(b) they are of different colours.

Solution

The tree diagram in Fig. 9.8 represents the situation.

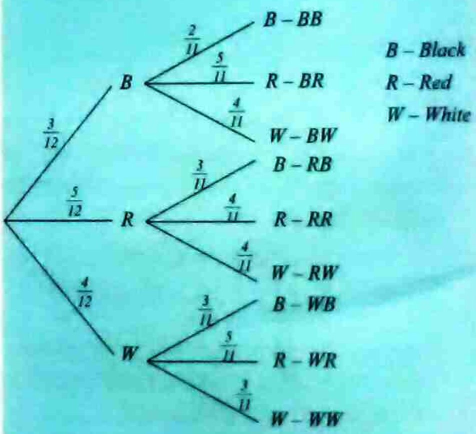


Fig. 9.8

(a) P(both same colour)

$= P(BB) + P(RR) + P(WW)$

$= \frac{3}{12} \times \frac{2}{11} + \frac{5}{12} \times \frac{4}{11} + \frac{4}{12} \times \frac{3}{11} = \frac{19}{66}$

(b) P(different colours)

$= P(BR) + P(BW) + P(RB) + P(RW) + P(WB) + P(WR)$

$= \frac{3}{12} \times \frac{5}{11} + \frac{3}{12} \times \frac{4}{11} + \frac{5}{12} \times \frac{3}{11} + \frac{5}{12} \times \frac{4}{11} + \frac{4}{12}$
$\frac{3}{11} \times \frac{4}{12} + \frac{5}{11}$

$= \frac{15}{132} + \frac{12}{132} + \frac{15}{132} + \frac{20}{132} + \frac{12}{132} + \frac{20}{132} = \frac{94}{132} = \frac{47}{66}$

or

P(different colours) $= 1 -$ *P(≈same colour)*

$= 1 -$ *P(same colour)*

$= 1 - \frac{19}{66} = \frac{47}{66}$

Example 9.12

A science club is made up of 5 boys and 7 girls. The club has three officials. Using a tree diagram, find the probability that

(a) *the club officials are all boys.*

(b) *two of the officials are girls.*

Solution

Fig. 9.9 is the required tree diagram.

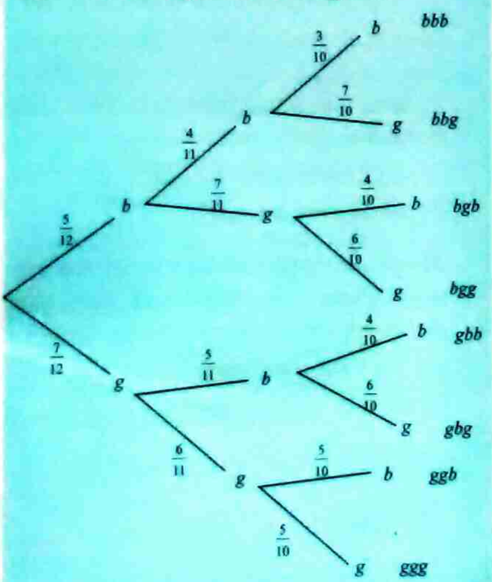


Fig. 9.9

(a) *P(all boys)* $= \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{1}{22}$

Probability that all officials are boys is $\frac{1}{22}$.

(b) *P(two girls and one boy)*

$= P(bgg) + P(gbg) + P(ggb)$

$= \frac{5}{12} \times \frac{7}{11} \times \frac{6}{10} + \frac{7}{12} \times \frac{5}{11} \times \frac{6}{10} + \frac{7}{12} \times \frac{6}{11} \times \frac{5}{10}$

$= \frac{7}{44} + \frac{7}{44} + \frac{7}{44} = \frac{21}{44}$

Probability that two of the officials are girls is $\frac{21}{44}$.

Example 9.13

The probability that a student goes to school on foot is $\frac{1}{6}$. *If he goes on foot, the probability that he will be late is* $\frac{1}{2}$. *If he does not go on foot, the probability that he will be late is* $\frac{1}{3}$. *Find the probability that:*

(a) *The student will go on foot and be late*

(b) *The student will not go on foot and be late*

(c) *The student will be late.*

Solution

Let F stand for going on foot and F' represents not going on foot. L stands for being late and L' for not being late (Fig 9.10)

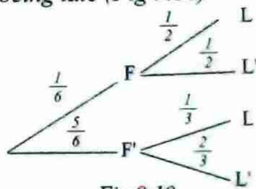


Fig 9.10

(a) *The probability that the student will go on foot and be late*

$= P(F \text{ and } L)$

$= P(F) \times P(L)$

$= \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

(b) *The probability that the student will not go on foot and be late*

$= P(F' \text{ and } L)$

$= P(F') \times P(L)$

$= \frac{5}{6} \times \frac{1}{3} = \frac{5}{18}$

(c) *The probability that the student will be late*

$= P$ *(F and L or P (F' and L)*

$= P(F \text{ and } L) + P(F' \text{ and } L)$

$= (P(F) \times (L)) + P(F') \times (L)$

$= \frac{1}{12} + \frac{5}{18} = \frac{13}{36}$

Exercise 9.7

1. A class has 15 boys and 10 girls. If three pupils are selected at random from the class, what is the probability that:
   (a) they are all boys?
   (b) two are boys and one is a girl?
   (c) they are all of the same sex?

2. A container has 7 red balls and 3 white balls. Three balls are taken from the container one after the other. What is the probability that the first two are red and the third is white?

3. Box A contains 8 oranges 3 of which are unripe. Box B contains 5 oranges 2 of which are unripe. An orange is taken from each box. What is the probability that:
   (a) both oranges are unripe?
   (b) one orange is unripe and the other is ripe?

4. The probabilities that three soccer players score penalties are $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{3}$ respectively. If each player shoots once, find the probability that only one of them scores.

5. The probability that the school team wins a match is 0.6. The probability that the team loses is 0.3 and the probability that the team ties is 0.1. The team plays 3 games. What is the probability that the team:
   (a) wins two matches?
   (b) either wins all the matches or loses all the matches?
   (c) wins one match, loses one and ties in one?

6. On average, Langa Estate misses water once a fortnight and experiences power failure once a week. What is the probability that there is either lack of water or a power failure on a given day if the events are independent?

7. The probability that it will rain tomorrow is $\frac{2}{3}$. The probability that Domasi football team wins is $\frac{3}{7}$. What is the probability that:
   (a) it rains and the team loses?
   (b) it does not rain and the team wins?

8. There are 7 men and 3 women waiting for an interview. They are called for the interview one by one at random. What is the probability that:
   (a) the first one is a woman?
   (b) the second one is woman?
   (c) of the first two, only one is a woman?

9. Some cards numbered 1 to 9 are put in a bag. Three cards are taken from the bag one after the other without replacement. What is the probability that:
   (a) only the first two are odd?
   (b) the last two are odd?

10. The chance that the school's volleyball team wins a game is $\frac{3}{5}$. If the team plays three games, what is the chance that the team:
    (a) wins only two games?
    (b) loses at least one game?

10 VECTORS

Success criteria

By the end of this topic, the student must be able to:

- *Describe vectors.*
- *Subtract vectors.*
- *Calculate the magnitude of a vector.*
- *Show that points are collinear using the vector method.*
- *Solve problems involving application of a parallelogram law.*
- *Add vectors.*
- *Multiply a vector by scalar.*
- *Find the mid-point of vector.*

Vector and scalar quantities

A group of rangers is preparing to go on a treasure hunt trip from their camping base. Before they leave the camp, they **must** know:

1. the specific distance to the treasure, and
2. the direction in which they have to move.

If they do not know **both** the **distance** and the **direction**, they are not likely to locate the exact position of the treasure.

Quantities which have both magnitude (size) and direction are called **vector quantities** or simply **vectors**. Velocity and acceleration are examples of vectors.

Quantities which have magnitude but no direction, e.g. a packet of biscuits, the cost of a pen, etc., are called **scalar quantities** or simply **scalars**.

The direction of a vector, in a diagram, is shown by means of an arrow.

Example 10.1

Represent the vector 30 km/h due west using a diagram.

Solution

Fig. 10.1 shows the required representation.

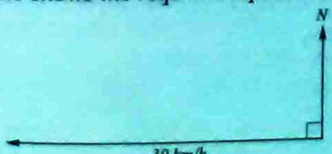


Fig. 10.1

Speed in a specific direction is known as **velocity**.

Exercise 10.1

1. State whether the following are vector or scalar quantities.

(a) A speed of 70 km/h due west.

(b) A distance of 50 km due east.

(c) 30 cows.

(d) 15 km.

(e) A speed of 400 km/h.

(f) A distance of 10 km south east of the city centre.

(g) 86 litres of milk.

(h) A distance of 70 km.

(i) A speed of 370 km/h on a bearing of 050°.

2. Represent the vectors in Question 1 using well labelled diagrams.

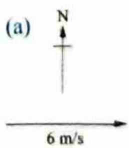
3. Name two examples, other than those in question 1, of:

(a) vector quantities,

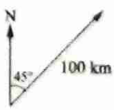
(b) scalar quantities.

4. Write down the magnitude and direction of each of the vectors in Fig. 10.2.

(a)



(b)



(c)

(d)

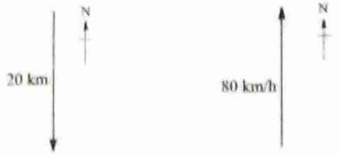


Fig. 10.2

Displacement vector and notation

We have seen that a **directed distance** is a **vector**. To travel from town A to town B, along the shortest distance, we must travel in a specific direction and for a definite distance (Fig. 10.3).

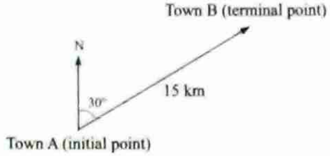


Fig. 10.3

The distance from A to B in the given direction is called the **displacement vector AB**, denoted as **AB**. Sometimes, vectors are denoted by specified small letters e.g. vector $\underset{\sim}{a}$ or **a**.

In print, vector AB is shown in bold **AB**.

In our handwriting we use an **arrow** or a **wavy line** notation since we cannot write in bold. For example, vector **AB** is written as $\overrightarrow{AB}$ or $\underset{\sim}{AB}$; vector **a** is written as $\underset{\sim}{a}$.

A vector whose initial and terminal points coincide is a **null vector** denoted as **O**. Its magnitude is 0 (zero).

Equivalent vectors

If two vectors have the same magnitude and the same direction, then they are **equivalent vectors**.

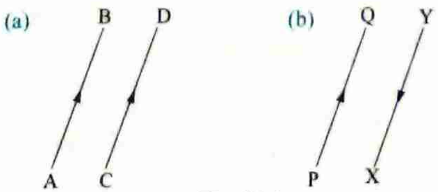


Fig. 10.4

(i) $\overrightarrow{AB}$ and $\overrightarrow{CD}$ (Fig. 10.4(a)) are parallel. They have the same direction, same magnitude or length, denoted as $|\overrightarrow{AB}|$ and $|\overrightarrow{CD}|$ respectively. $\overrightarrow{AB}$ is therefore equivalent to $\overrightarrow{CD}$. We write, $\overrightarrow{AB} = \overrightarrow{CD}$ (sometimes written as **AB ≡ CD**).

(ii) $\overrightarrow{PQ}$ and $\overrightarrow{XY}$ (Fig. 10.4(b)) are anti parallel. They also have the same magnitude, but are in opposite directions. Therefore, $\overrightarrow{PQ}$ and $\overrightarrow{XY}$ are **not equivalent**. We write $\overrightarrow{PQ} \neq \overrightarrow{XY}$. We have to change the direction of either $\overrightarrow{PQ}$ or $\overrightarrow{XY}$ so that their sense of direction is the same. A negative sign is used to reverse the direction. Thus, $\overrightarrow{PQ} = {}^{-}\overrightarrow{XY}$.

In general, two vectors are equivalent if they have the **same magnitude** and the **same sense of direction**.

Addition of vectors

Triangle ABC (Fig. 10.5) represents routes joining three towns A, B and C.

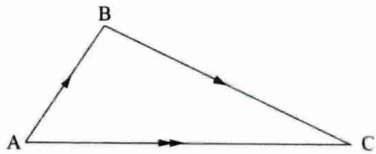


Fig. 10.5

If you go from A to B, then from B to C, the effect is the same as going from A to C directly.

The required effect is to reach town C from A.

Since the effect is the same, then

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

Vector $\overrightarrow{AC}$ is called the **resultant vector** of $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Such a vector is usually represented by a line segment with a double arrowhead, as in Fig. 10.5.

Subtraction of vectors

We have already seen that equivalent vectors must have equal magnitude and same sense of direction. For example, vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in Fig 10.6 are equal in magnitude but directions are different.

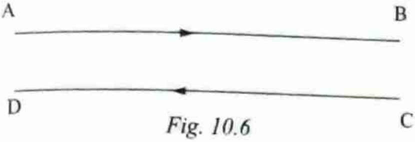


Fig. 10.6

But we can reverse the direction of $\overrightarrow{CD}$ so that $\overrightarrow{DC} = -\overrightarrow{CD}$. This means that the arrow for $\overrightarrow{CD}$ is facing the opposite direction, which is the direction of $\overrightarrow{AB}$.

$\therefore \overrightarrow{AB} \neq -\overrightarrow{CD}$

Now, consider Fig 10.7

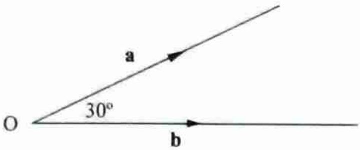


Fig. 10.7

Given the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ as shown in the figure, we want to show the relative positions of the vectors **a** – **b**.

We can write **a** – **b** as **a**+ -**b** = -**b**+ **a**

This means, we reverse the direction of vector -**b** and proceed as follows (Fig. 10.8).

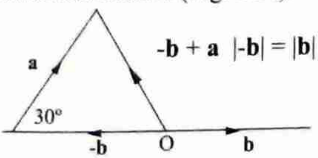


Fig. 10.8

We can also represent –**b** + **a** as **a** + -**b** as in Fig. 10.9

$\therefore$ **a** + (-**b**) = -**b**+ **a**= **a** – **b**

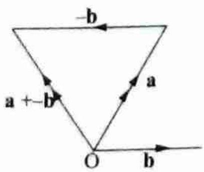


Fig 10.9

Example 10.2

Using Fig. 10.10, write down the single vector equivalent to:

(a)	***AB + BC***	*(b)*	***AE + ED***
(c)	***BC + CD + DE***	*(d)*	***ED + DC + CB***
(e)	***AB + BA***	*(f)*	***CD + DC***
(g)	***AE + EB + BC***	*(h)*	***CD + DE + EB***

(i) ***AB + BC + CD + DE***

(j) ***DE + EA + AB + BC + CD***

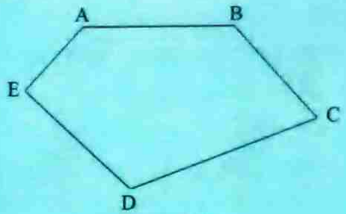


Fig. 10.10

Solution

(a) ***AB + BC = AC*** *(Moving from A to B, then from B to C is equivalent to moving from A to C directly.)*

(b) ***AE + ED*** = *A to E then to D.*
= *A to D*
= ***AD***

(c) ***BC + CD + DE = BD + DE = BE***

(d) ***ED + DC + CB = EC + CB = EB***

(e) ***AB + BA = AB – AB = 0***

(f) ***CD + DC = 0*** *(from A to B then back to A)*

(g) ***AE + EB + BC = AB + BC = AC***

(h) ***CD + DE + EB = CE + EB = CB***

(i) ***AB + BC + CD + DE = AC + CD + DE***
= AD + DE = AE

(j) ***DE + EA + AB + BC + CD***
= DA + AB + BC + CD
= DB + BC + CD
= DC + CD
= 0 *(Start from D and back to D).*

Exercise 10.2

1. Which of the vectors in Fig. 10.11 are equivalent?

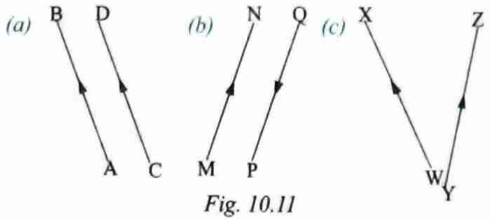


Fig. 10.11

2. STUR is a quadrilateral (Fig. 10.12). Use it to write down the single vector equivalent to:

(a) $\overrightarrow{ST} + \overrightarrow{TU}$ (b) $\overrightarrow{TS} + \overrightarrow{SR}$
(c) $\overrightarrow{RS} + \overrightarrow{ST}$ (d) $\overrightarrow{UR} + \overrightarrow{RS}$
(e) $\overrightarrow{UT} + \overrightarrow{TR}$ (f) $\overrightarrow{UR} + \overrightarrow{RT}$
(g) $\overrightarrow{TS} + \overrightarrow{ST}$ (h) $\overrightarrow{UR} + \overrightarrow{RU}$
(i) $\overrightarrow{RS} + \overrightarrow{ST} + \overrightarrow{TU}$
(j) $\overrightarrow{UT} + \overrightarrow{TS} + \overrightarrow{SR}$
(k) $\overrightarrow{ST} + \overrightarrow{TU} + \overrightarrow{UR} + \overrightarrow{RS}$
(l) $\overrightarrow{UT} + \overrightarrow{TS} + \overrightarrow{SR} + \overrightarrow{RU}$

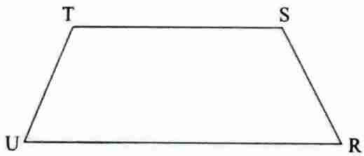


Fig. 10.12

3. Draw a triangle STR and put arrows on its sides to show that **TS** + **SR** = **TR.**

4. Draw a quadrilateral ABCD and on it show $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$. State a single vector equivalent to $\overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$.

5. A man walks 10 km in the NE direction, and then 4 km due north. Using an appropriate scale, draw a vector diagram showing the man's displacement from his starting point. When he stops walking, how far from the starting point will he have walked?

6. Vectors **a**, **b** and **c** are such that **a** = **b** and **b** = **c**. What can you say about **a** and **c**?

7. Mrs. Mandondo's family planned a sight seeing trip which was to take them from Blantyre to Lilongwe, then to Zomba and back to Blantyre. Draw a vector triangle to show their trip. What vector does **BL** + **LZ** + **ZB** represent if B stands for Blantyre, L for Lilongwe and Z for Zomba?

8. PQRS (Fig. 10.13) represents a parallelogram.

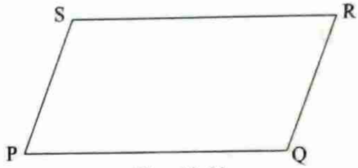


Fig. 10.13

(a) Copy the figure. Mark with arrows and name two pairs of equal vectors.

(b) Simplify:

(i) **PQ** + **QR** (ii) **PS** + **SR**

9. For each of the following equations, use Fig. 10.14 to find a directed line segment which can replace **PQ**.

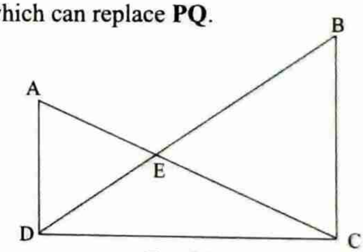


Fig. 10.14

(a) **AE** + **PQ** = **AB** (b) **DE** + **PQ** = **DB**
(c) **DB** + **PQ** = **O** (d) **EB** + **PQ** = **EC**
(e) **EB** + **PQ** = **ED** (f) **PQ** + **DA** = **CA**
(g) **AE** + **ED** + **PQ** = **AD**
(h) **AD** + **PQ** + **EC** = **AC**
(i) **DC** + **PQ** + **ED** = **O**

10. Use Fig. 10.15 to simplify the following.

(a) **a** − **w** (b) **u** + **a** (c) −**w** + **u**
(d) **u** + **v** (e) **u** − **b**

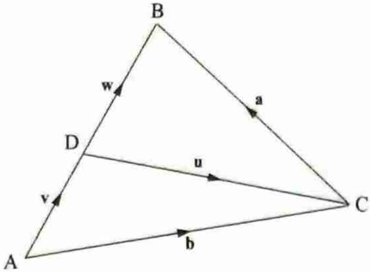


Fig. 10.15

Multiplication of vectors by scalars

Consider Fig. 10.16 below.

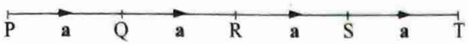


Fig. 10.16

$$\begin{aligned}\overrightarrow{PT} &= \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST}\\ &= \mathbf{a} + \mathbf{a} + \mathbf{a} + \mathbf{a}\\ &= 4 \times \mathbf{a}\\ &= 4\mathbf{a}\end{aligned}$$

This means that the length (magnitude) of **PT** is four times that of **a**.

Note that 4**a** and **a** have the same direction.

Similarly, $\overrightarrow{PR} = 2\mathbf{a}$ and

$\overrightarrow{PS} = 3\mathbf{a}$.

If X is a point halfway between P and Q, **PX** would be half of **a**.

i.e., $\overrightarrow{PX} = \frac{1}{2}\mathbf{a}$.

In vectors $\overrightarrow{PT}$, $\overrightarrow{PS}$, $\overrightarrow{PR}$ and $\overrightarrow{PX}$, the values 4, 3, 2 and $\frac{1}{2}$ are scalars.

A scalar multiplier can take any value, positive or negative, whole number or fraction, or even zero (Fig. 10.17).

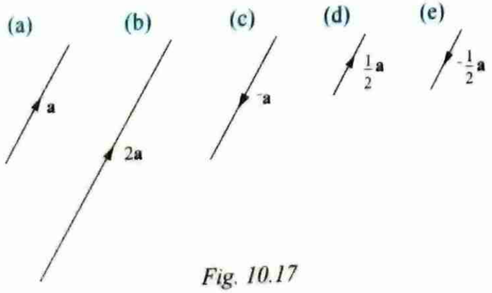


Fig. 10.17

In Fig. 10.17 parts (b) to (e), vector **a** has been multiplied by 2, $^-1$, $\frac{1}{2}$ and $^-\frac{1}{2}$ respectively.

If m is a scalar, and **a** is a vector, then $m\mathbf{a}$ is a vector parallel to **a**, and m times its length.

If $m > 0$, then $m\mathbf{a}$ has the same sense of direction as **a**, and its magnitude is m times that of **a**.

If $m = 0$, then $m\mathbf{a}$ is a null vector whose magnitude is zero.

If $m < 0$, then $m\mathbf{a}$ is a vector whose direction is opposite that of **a** and whose length is $|m|$ times that of **a**, where $|m|$, read as the **modulus of *m***, means the value of m irrespective of the sign.

Multiplying a vector by a negative number reverses the direction of the vector.

Example 10.3

Towns A, B and C are along a straight road which runs due north from A. From A to B is 6 km and from B to C is 12 km.

Express the following vectors in terms of $\overrightarrow{AB}$.

(a) $\overrightarrow{BC}$ *(b)* $\overrightarrow{AC}$ *(c)* $\overrightarrow{CB}$

Solution

Fig. 10.18

(a) $\overrightarrow{AB}$ *is 6 km due north.*

$\overrightarrow{BC}$ *is 12 km due north.*

$\overrightarrow{BC}$ = *twice* $\overrightarrow{AB}$

= $2\overrightarrow{AB}$ *(see Fig. 10.18)*

(b) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ *due north*

= *6 + 12 km due north*

= *18 km due north*

$\therefore \overrightarrow{AC} = 3\overrightarrow{AB}$

(c) $\overrightarrow{CB}$ *is 12 km due south. CB has same magnitude as* $\overrightarrow{BC}$ *but is in the opposite direction.*

$\therefore \overrightarrow{CB} = {^-}\overrightarrow{BC}$

$= {^-}2\overrightarrow{AB}$

Exercise 10.3

1. Represent each of the following vectors using a diagram.
   (a) A velocity of 400 km/h due west.
   (b) A speed of 60 km/h on a straight road due east increased in the ratio 5 : 4.
   (c) Half the reverse of a speed of 80 km/h due north on a straight road.

2. Four railway stations L, M, E and R are on a straight railway line (Fig. 10.19).

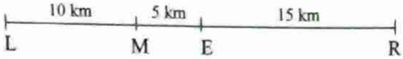


Fig. 10.19

(a) Express the following in terms of **ME.**

(i) **LM**	(ii) **LE**	(iii) **ER**
(iv) **2ER**	(v) **MR**	(vi) **3LM**

(b) Express the following in terms of **LM.**

(i) **MR**	(ii) **LE**	(iii) **ML**
(iv) **ME**	(v) **ME**	(vi) **LR**
(vii) **RM**	(viii) **ER**	

(c) Express the following in terms of **RE.**

(i) **EM**	(ii) **ME**	(iii) **LE**
(iv) **RL**	(v) **ML**	(vi) **LM**
(vii) **EL**	(viii) **LR**	

(d) Write the following in terms of **ME.**

(i) **LM + EM**	(ii) **LM + RE**
(iii) **ER + ML**	(iv) **ME + RE**
(v) **ME + ML**	(vi) **ER + $^-$3ME**
(vii) **LM – ER**	
(viii) **LM + ME + ER**	

Vectors in the Cartesian plane

In Fig. 10.20, $\overrightarrow{AB}$ represents a vector, known as a translation of the points on the plane, which moves A to B, H to K, P to Q, etc. This translation transfers the points (A, H, P) on the plane, 1 unit to the right and 3 units up. We can indicate this vector by an ordered pair of numbers which we write in column form as $\begin{pmatrix}1\\3\end{pmatrix}$.

This is called a **column vector** for the said translation. 1 and 3 are called **components** of the vector.

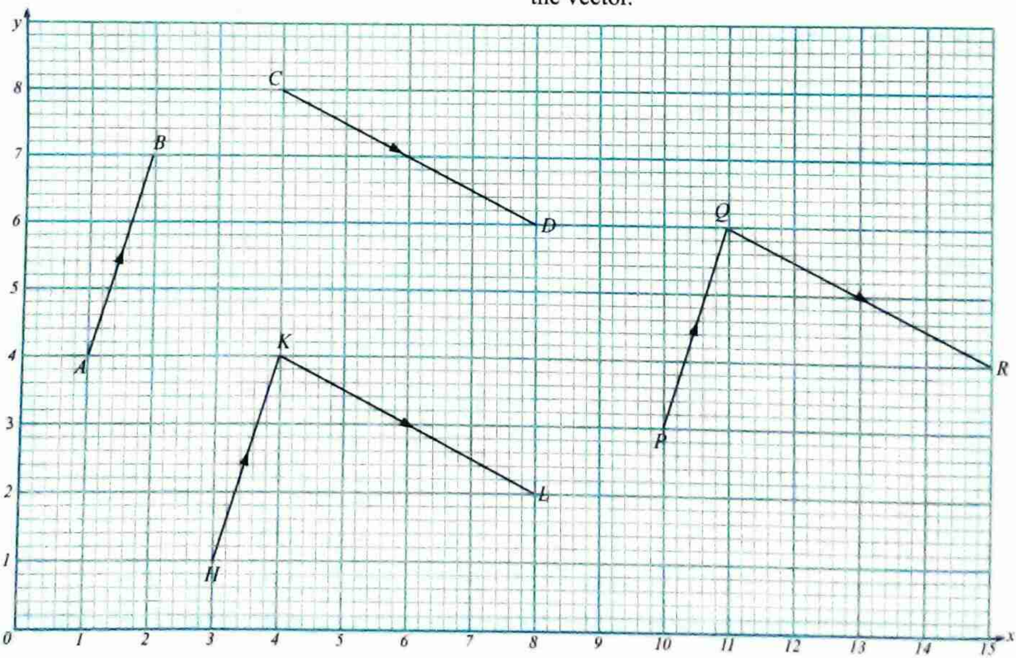


Fig. 10.20

Note: The vectors shown in Fig 10.20 are examples of free vectors. They are called free vectors because they can be drawn anywhere on the Cartesian plane, as long as the direction and the magnitude are right.

Example 10.4

Using Fig. 10.20 in the previous page, find the column vectors which represent

(a) (i) $\overrightarrow{AB}$ (ii) $\overrightarrow{CD}$ (iii) $\overrightarrow{BA}$

(iv) $\overrightarrow{DC}$ (v) $\overrightarrow{HL}$ (vi) $\overrightarrow{PR}$

(vii) $\overrightarrow{LH}$

(b) *How can we obtain the column vectors for HL and PR from those of*

(i) $\overrightarrow{HK}$ *and* $\overrightarrow{KL}$

(ii) $\overrightarrow{PQ}$ *and* $\overrightarrow{QR}$ *respectively?*

Solution

(a) (i) *Point A moves 1 unit to the right and 3 units up to map onto B.*

$$\therefore \quad \overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

(ii) *Point C moves 4 units to the right and 2 units down to map onto D.*

$$\therefore \quad \overrightarrow{CD} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

(iii) *Point B moves 1 unit to the left and 3 units down to map onto A.*

$$\therefore \quad \overrightarrow{BA} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

(iv) *D moves 4 units to the left and 2 units up to C.*

$$\therefore \overrightarrow{DC} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

(v) *H moves 5 units to the right and 1 unit up to L.*

$$\therefore \quad \overrightarrow{HL} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

(vi) *P moves 5 units to the right and 1 unit up to R.*

$$\therefore \overrightarrow{PR} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

(vii) *L moves 5 units to the left and 1 unit down to H.*

$$\therefore \overrightarrow{LH} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$

(b) (i) *Since* $\overrightarrow{HL} = \overrightarrow{HK} + \overrightarrow{KL}$,

then $\overrightarrow{HL} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

(We add corresponding components in the column vectors $\overrightarrow{HK}$ *and* $\overrightarrow{KL}$ *)*

(ii) $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Note that, given two vectors **u** and **v** such that $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} c \\ d \end{pmatrix}$ where a, b, c and d are scalars, $\mathbf{u} + \mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$.

Also, $\mathbf{u} - \mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a-c \\ b-d \end{pmatrix}$.

Example 10.5

If $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ *and* $\mathbf{c} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, *evaluate:*

(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{c} - \mathbf{b}$ (c) $2\mathbf{c} + \mathbf{a}$

(d) $\frac{1}{3}\mathbf{b} - \mathbf{c}$ (e) $\mathbf{a} - 2\mathbf{c}$ (f) $\frac{1}{2}\mathbf{c} + \frac{1}{3}\mathbf{b}$

Solution

(a) $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2+6 \\ -1+3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$

(b) $\mathbf{c} - \mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -2-6 \\ 1-3 \end{pmatrix} = \begin{pmatrix} -8 \\ -2 \end{pmatrix}$

(c) $2\mathbf{c} + \mathbf{a} = 2\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} -4+2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

(d) $\frac{1}{3}\mathbf{b} - \mathbf{c} = \frac{1}{3}\begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 - {}^{-}2 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

(e) $\mathbf{a} - 2\mathbf{c} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} - 2\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 + {}^{-}4 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

(f) $\frac{1}{2}\mathbf{c}+\frac{1}{3}\mathbf{b}=\frac{1}{2}\begin{pmatrix}-2\\1\end{pmatrix}+\frac{1}{3}\begin{pmatrix}6\\3\end{pmatrix}=\begin{pmatrix}-1\\\frac{1}{2}\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}-1+2\\\frac{1}{2}+1\end{pmatrix}=\begin{pmatrix}-1\\1\frac{1}{2}\end{pmatrix}=\begin{pmatrix}-1\\\frac{3}{2}\end{pmatrix}$

Exercise 10.4

Use Fig. 10.21 to answer Questions 1 to 3.

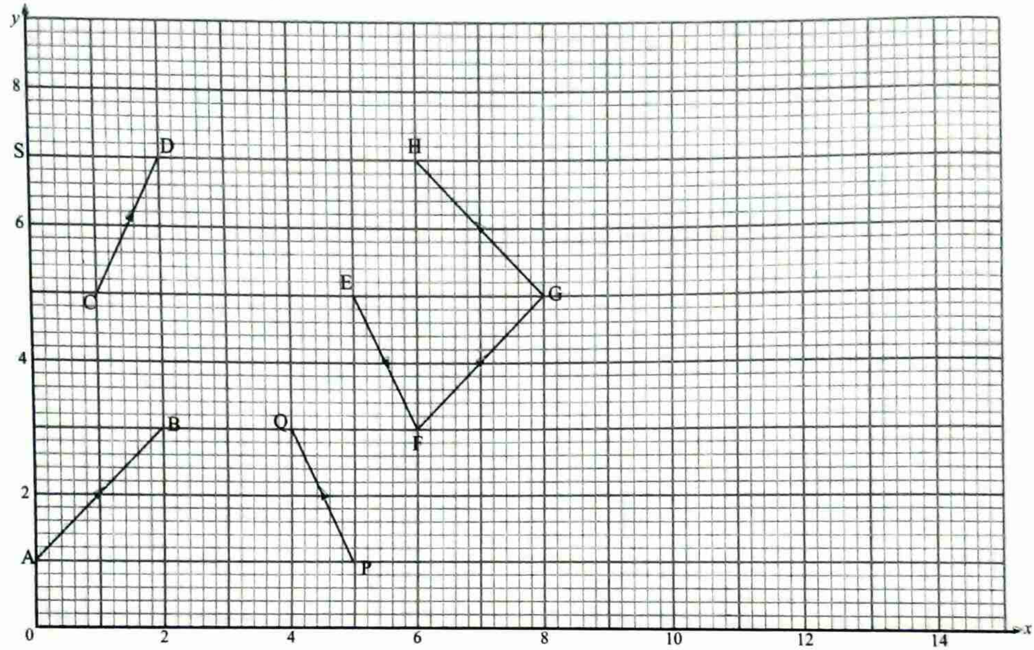


Fig. 10.21

1. (a) Name the vector that is equal to **AB** and state its column vector.
   (b) Simplify **EF** + **FG** + **GH** and give your answer as a column vector.

2. Write all the vectors in Fig. 10.21 as column vectors.

3. Simplify **FG** + **GH** giving your answer in column vector form. What is the meaning of the first component in your answer?

4. Draw diagrams on a squared paper to show:
   (a) $\begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}3\\3\end{pmatrix}=\begin{pmatrix}6\\4\end{pmatrix}$ (b) $\begin{pmatrix}0\\2\end{pmatrix}+\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$
   (c) $\begin{pmatrix}2\\2\end{pmatrix}+\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}0\\3\end{pmatrix}$

5. Simplify:
   (a) $\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}4\\3\end{pmatrix}$ (b) $\begin{pmatrix}0\\4\end{pmatrix}+\begin{pmatrix}6\\1\end{pmatrix}$
   (c) $\begin{pmatrix}-2\\4\end{pmatrix}+\begin{pmatrix}-3\\-8\end{pmatrix}$ (d) $\begin{pmatrix}-10\\-4\end{pmatrix}+\begin{pmatrix}-6\\8\end{pmatrix}$

6. Given that 2**u** means **u** followed by **u**, $2\begin{pmatrix}1\\2\end{pmatrix}$ means $\begin{pmatrix}1\\2\end{pmatrix}$ followed by $\begin{pmatrix}1\\2\end{pmatrix}$ which means a total of 2 units to the right and 4 units up i.e. $2\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}2\times1\\2\times2\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}$.

   Simplify:
   (a) $3\begin{pmatrix}5\\2\end{pmatrix}$ (b) $2\begin{pmatrix}5\\6\end{pmatrix}$ (c) $3\begin{pmatrix}-1\\5\end{pmatrix}$
   (d) $\frac{1}{3}\begin{pmatrix}-9\\0\end{pmatrix}$ (e) $-\frac{1}{2}\begin{pmatrix}-6\\4\end{pmatrix}$ (f) $-3\begin{pmatrix}2\\-1\end{pmatrix}$
   (g) $4\begin{pmatrix}-2\\3\end{pmatrix}$ (h) $4\begin{pmatrix}3\\1\end{pmatrix}$

7. If $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} ^{-}3 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} ^{-}1 \\ 2 \end{pmatrix}$, evaluate:

(a) $\mathbf{a} + \mathbf{b}$ (b) $2\mathbf{a}$ (c) $3\mathbf{b}$

(d) $\mathbf{b} + \mathbf{c}$ (e) $4\mathbf{a} + 3\mathbf{b}$ (f) $^{-}\mathbf{a} + 2\mathbf{c}$

(g) $3(\mathbf{a} + \mathbf{b})$ (h) $\mathbf{a} - \mathbf{b}$ (i) $\mathbf{a} + \mathbf{c}$

(j) $\mathbf{a} + \mathbf{b} + \mathbf{c}$ (k) $2\mathbf{b} + 5\mathbf{c}$ (l) $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

(m) $^{-}2(\mathbf{a} + \mathbf{c})$ (n) $\mathbf{a} - \mathbf{c}$

Position vector

On a Cartesian plane, the position of a point is given with reference to the origin, O, the intersection of the x and y- axes. Thus, we can use vectors to describe the position of a point (Fig. 10.22).

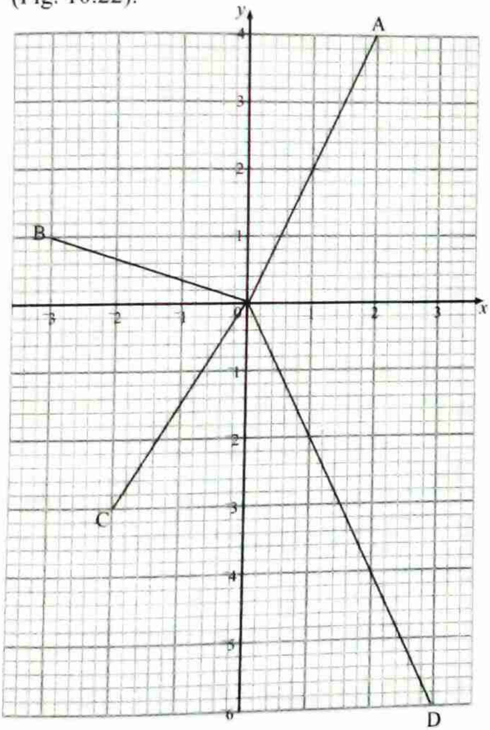


Fig. 10.22

From the origin, A is $^{+}2$ units in the x direction and $^{+}4$ units in the y direction.
Thus, A has coordinates (2, 4) and **OA** has column vector $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

Similarly, B is -3 units in the x direction and $^{+}1$ units in the y direction.
Thus, B has coordinates $(-3, 1)$ and **OB** has column vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

C is -2 units in the x direction and -3 units in the y direction.
Thus, C is $(-2, -3)$ and $\mathbf{OC} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

D is $+3$ units in the x direction and -6 units in the y direction.
Thus, D is $(3, -6)$ and $\mathbf{OD} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$.

OA, OB, OC and **OD** are known as **position vectors** of A, B, C, and D respectively.
All position vectors have O as their initial point.

Example 10.6

P has coordinates (2, 3) and Q (7, 5).

(a) Find the position vector of: (i) P, (ii) Q.

*(b) State the column vector for **PO**.*

*(c) Find the column vector for **PQ**.*

Solution

(a) (i) P is $(2, 3)$ (ii) Q is $(7, 5)$

$\therefore \mathbf{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\therefore \mathbf{OQ} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

(b) $\mathbf{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\mathbf{PO} = -\mathbf{OP}$

$= -\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

(c) $\mathbf{PQ} = \mathbf{PO} + \mathbf{OQ}$

$= \begin{pmatrix} -2 \\ -3 \end{pmatrix} + \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} -2+7 \\ -3+5 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

Note that:

$\mathbf{PO} + \mathbf{OQ} = \mathbf{OQ} + \mathbf{PO} = \mathbf{PQ}$

$\therefore \mathbf{PQ} = \mathbf{OQ} - \mathbf{OP}$ *(since* $\mathbf{PO} = {}^{-}\mathbf{OP}$*)*

$= \textit{position vector of } Q - \textit{position vector of } P$

$= \begin{pmatrix} 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

Example 10.7

If $\mathbf{OA} = \begin{pmatrix} 2 \\ ^-5 \end{pmatrix}$ *and* $\mathbf{OB} = \begin{pmatrix} ^-4 \\ ^-3 \end{pmatrix}$, *find:*

(a) *the coordinates of A,*

(b) *the coordinates of B,*

(c) *the column vector for* **AB**.

Solution

(a) $\mathbf{OA} = \begin{pmatrix} 2 \\ ^-5 \end{pmatrix}$

$\therefore$ *A is* $(2, {^-5})$

(b) $\mathbf{OB} = \begin{pmatrix} ^-4 \\ ^-3 \end{pmatrix}$

$\therefore$ *B is* $({^-4}, {^-3})$

(c) $\mathbf{AB} = \mathbf{AO} + \mathbf{OB}$

$= {^-\mathbf{OA}} + \mathbf{OB}$ (*since* $\mathbf{AO} = {^-\mathbf{OA}}$)

$= \mathbf{OB} - \mathbf{OA}$

$= \begin{pmatrix} ^-4 \\ ^-3 \end{pmatrix} - \begin{pmatrix} 2 \\ ^-5 \end{pmatrix} = \begin{pmatrix} ^-4-2 \\ ^-3-{^-5} \end{pmatrix} = \begin{pmatrix} ^-6 \\ 2 \end{pmatrix}$.

In general, if P is (a, b) then $\mathbf{OP} = \begin{pmatrix} a \\ b \end{pmatrix}$.

Similarly, if Q is (c, d) then $\mathbf{OQ} = \begin{pmatrix} c \\ d \end{pmatrix}$.

$\mathbf{PQ}$ = position vector of Q − position vector of P

$= \mathbf{OQ} - \mathbf{OP}$

$= \begin{pmatrix} c \\ d \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c-a \\ d-b \end{pmatrix}$.

Exercise 10.5

1. Use Fig. 10.23 to write down the position vectors of the marked points.

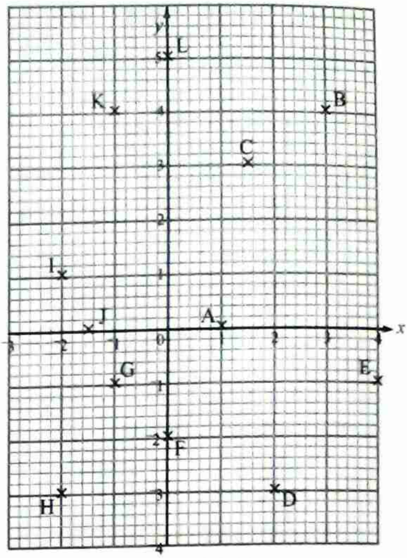


Fig. 10.23

2. On squared paper, mark the points whose position vectors are given below.

(a) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(d) $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (e) $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$ (f) $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

(g) $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ (h) $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

3. (a) If $\mathbf{OP} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{OQ} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, find the column vector for **PQ**.

(b) If $\mathbf{OP} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\mathbf{OQ} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$, find the column vector for: (i) **PQ** (ii) **QP**.

(c) If $\mathbf{OF} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$ and $\mathbf{OG} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, find the column vectors for **FG** and **GF**.

(d) If $\mathbf{OM} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\mathbf{ON} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\mathbf{OP} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, find the column vector:

(i) **MN** (ii) **MP** (iii) **NM**

(iv) **NP** (v) **PN** (vi) **PM**.

4. If $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, evaluate the following and illustrate them on a Cartesian

plane given that **a** and **b** are position vectors.

(a) $\mathbf{a}+\mathbf{b}$ (b) $\mathbf{a}-\mathbf{b}$ (c) $2\mathbf{a}+3\mathbf{b}$

(d) $2\mathbf{b}-\mathbf{a}$ (e) $^{-}2\mathbf{a}+3\mathbf{b}$ (f) $^{-}4\mathbf{a}-\mathbf{b}$

5. Given that $\mathbf{AB}=\begin{pmatrix}6\\4\end{pmatrix}$ and $\mathbf{OA}=\begin{pmatrix}1\\8\end{pmatrix}$, find the column vector of **OB**.

6. Given that $\mathbf{a}=\begin{pmatrix}2\\3\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}2\\4\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}-3\\3\end{pmatrix}$, find:

(a) $\mathbf{a}+2\mathbf{c}$ (b) $\mathbf{b}-\mathbf{c}$ (c) $3\mathbf{a}+\frac{1}{2}\mathbf{c}$

Unit vectors

Consider Fig. 10.24 below.

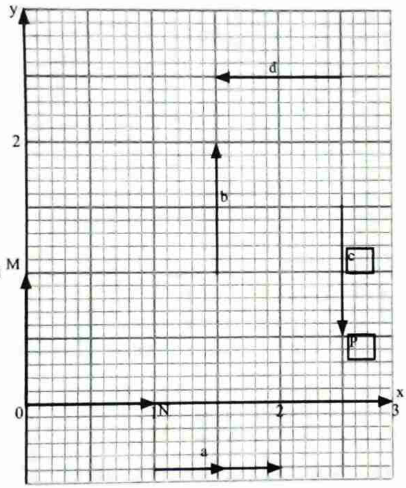


Fig. 10.24

$\mathbf{a}=\begin{pmatrix}1\\0\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}0\\1\end{pmatrix}$, $\mathbf{c}=\begin{pmatrix}0\\-1\end{pmatrix}$ and $\mathbf{d}=\begin{pmatrix}-1\\0\end{pmatrix}$.

Each of the vectors a, b, c and d has a magnitude of 1 unit. Hence they are unit vectors.

In the same Fig. 10.24, $\mathbf{ON}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{OM}=\begin{pmatrix}0\\1\end{pmatrix}$ are special unit vectors since;

(i) each has a length of 1 unit,

(ii) they start from the origin,

(iii) they lie on the x and y-axes respectively.

The point N(1, 0) is normally denoted as I(1, 0) and the vector **ON** is denoted as $\mathbf{i}=\begin{pmatrix}1\\0\end{pmatrix}$, similarly, the point M is denoted as J(0, 1) and the vector **OM** denoted as $\mathbf{OJ}=\mathbf{j}=\begin{pmatrix}0\\1\end{pmatrix}$.

Note: the vectors **a**, **b**, **c**, and **d** in Fig. 10.24 can be expressed in terms of **i** or **j**.

i.e $\mathbf{a}=\begin{pmatrix}1\\0\end{pmatrix}=\mathbf{i}$ $\quad \mathbf{c}=\begin{pmatrix}0\\-1\end{pmatrix}=-\begin{pmatrix}0\\1\end{pmatrix}=-\mathbf{j}$

$\mathbf{b}=\begin{pmatrix}0\\1\end{pmatrix}=\mathbf{j}$ $\quad \mathbf{d}=\begin{pmatrix}-1\\0\end{pmatrix}=-\begin{pmatrix}1\\0\end{pmatrix}=-\mathbf{i}$

Magnitude of a vector

We have already learnt that, a vector can be represented by a directed line segment using a horizontal and a vertical component.

For example, $\mathbf{AB}=\begin{pmatrix}6\\8\end{pmatrix}$ means that, starting from A, the horizontal distance covered is 6 units while the corresponding vertical distance is 8 units (Fig. 10.25).

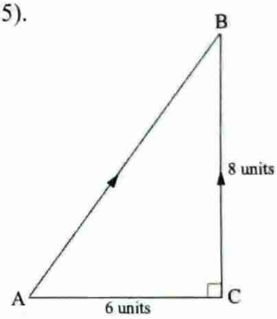


Fig. 10.25

AB represents the hypotenuse of a right-angled triangle whose two shorter sides are 6 and 8 units long.

Therefore, the **magnitude** of **AB**, written as $|\mathbf{AB}|$, is found by using Pythagoras' theorem.

$$\begin{aligned}\text{Thus, } |\mathbf{AB}|^2 &= 6^2+8^2\\ &= 36+64\\ &= 100\\ |\mathbf{AB}| &= \sqrt{100}\\ &= 10 \text{ units}\end{aligned}$$

In general, if P is (x, y) and Q is (a, b), then

$\mathbf{PQ}=\begin{pmatrix}a-x\\b-y\end{pmatrix}$ and

$|\mathbf{PQ}|=\sqrt{(a-x)^2+(b-y)^2}$

Example 10.8

Given that P is the point $(7, {}^{-}9)$ and Q is the point $(10, {}^{-}5)$, find:

(a) $|\vec{OP}|$ (b) $|\vec{OQ}|$

(c) $|\vec{PQ}|$ (d) $|\vec{OP}| + |\vec{OQ}|$

(e) $|\vec{OP} + \vec{OQ}|$

Solution

(a) $\vec{OP} = \begin{pmatrix} 7 \\ {}^{-}9 \end{pmatrix}$

$\therefore \quad |\vec{OP}| = \sqrt{7^2 + ({}^{-}9)^2}$

$= \sqrt{49 + 81} = \sqrt{130}$

$= 11.4$ units

(b) $\vec{OQ} = \begin{pmatrix} 10 \\ {}^{-}5 \end{pmatrix}$

$\therefore \quad |\vec{OQ}| = \sqrt{100 + 25}$

$= \sqrt{125} = 11.18$ units

(c) $\vec{PQ} = \vec{OQ} - \vec{OP}$

$= \begin{pmatrix} 10 \\ {}^{-}5 \end{pmatrix} - \begin{pmatrix} 7 \\ {}^{-}9 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\therefore \quad |\vec{PQ}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ units

(d) $|\vec{OP}| + |\vec{OQ}| = 11.4 + 11.18 = 22.58$ units

(e) $|\vec{OP} + \vec{OQ}| = \begin{pmatrix} 7 \\ {}^{-}9 \end{pmatrix} + \begin{pmatrix} 10 \\ {}^{-}5 \end{pmatrix} = \begin{pmatrix} 17 \\ {}^{-}14 \end{pmatrix}$

$\therefore \quad |\vec{OP} + \vec{OQ}| = \sqrt{17^2 + ({}^{-}14)^2}$

$= \sqrt{289 + 196}$

$= \sqrt{485} = 22.02$ units

You should have noticed that:

$|\vec{OP}| + |\vec{OQ}| \neq |\vec{OP} + \vec{OQ}|$.

In general, $|\mathbf{a} + \mathbf{b}| \neq |\mathbf{a}| + |\mathbf{b}|$.

Note: A vector which has no magnitude is called **a zero or null vector**. Usually we do not deal with zero vectors, but an operation on vectors might give rise to a zero vector. For example, Fig. 10.27 represents a quadrilateral ABCD.

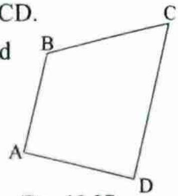


Fig. 10.27

Use Fig. 10.26 to simplify and state the single vector equivalent to:

(a) **AB** + **BC** + **CD** + **DA**

(b) **BC** + **CD** + **DA** + **AB**

(c) **AB** + **BC** + **CA**

All these vector sums simplify to O. This means that, each vector expression starts and ends at the same point.

Exercise 10.6

1. Calculate the length of each of the following vectors.

(a) $\begin{pmatrix} {}^{-}3 \\ {}^{-}4 \end{pmatrix}$ (b) $\begin{pmatrix} 6 \\ {}^{-}8 \end{pmatrix}$

(c) $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ {}^{-}8 \end{pmatrix}$

2. Calculate the distances between the following pairs of points.

(a) A (5, 0), B (10, 4)

(b) C (7, 4), D (1, 12)

(c) E $({}^{-}1, {}^{-}1)$, F$({}^{-}5, {}^{-}6)$

(d) P $(4, {}^{-}1)$, Q$({}^{-}3, {}^{-}4)$

(e) H $(b, 4b)$, K$({}^{-}2b, 8b)$

(f) M $({}^{-}2m, 5m)$, N $({}^{-}4m, {}^{-}2m)$

3. State which of the following expressions represent the distance between the points A (a, b) and B (c, d).

(a) $\sqrt{(b-d)^2 + (a-c)^2}$

(b) $\sqrt{(a-b)^2 + (c-d)^2}$

(c) $\sqrt{(c-a)^2 + (d-b)^2}$

(d) $\sqrt{(a-d)^2 + (b-c)^2}$

4. Which of the following statements are true and which ones are false?

(a) Given A (3, 1), B (6, 5), P $({}^{-}1, {}^{-}2)$ and Q $(3, {}^{-}5)$, $|\vec{AB}| = |\vec{PQ}|$

(b) For the points given in (a) above, $\overrightarrow{AB} = \overrightarrow{PQ}$.

(c) If A is (3, 1), B is $(^-2, 4)$, C is (0, 8) and D is $(^-5, 11)$, then AB = CD.

(d) If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent u and v respectively, then $\overrightarrow{OR}$ represents $(u - v)$, where R is the midpoint of $\overrightarrow{PQ}$.

5. Calculate the length of the vectors whose column vectors are:

(a) $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (c) $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

6. Find the vector **p** such that:

(a) $\begin{pmatrix} 5 \\ -6 \end{pmatrix} + \mathbf{p} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, hence, find $|\mathbf{p}|$.

(b) $\begin{pmatrix} 5 \\ -6 \end{pmatrix} + \mathbf{p} = \mathbf{i}$. Hence, find $|\mathbf{p}|$.

(c) $\begin{pmatrix} 7 \\ -8 \end{pmatrix} + \mathbf{p} = \mathbf{j}$. Hence, find $|\mathbf{p}|$.

7. Find (i) vector a and hence find (ii) $|\mathbf{a}|$ in each case:

(a) $\begin{pmatrix} -3 \\ 1 \end{pmatrix} - \mathbf{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} -3 \\ 2 \end{pmatrix} - \mathbf{a} = \mathbf{i}$

(c) $\begin{pmatrix} -4 \\ 2 \end{pmatrix} - \mathbf{a} = \mathbf{j}$ (d) $\begin{pmatrix} -3 \\ 1 \end{pmatrix} - \mathbf{a} = \begin{pmatrix} -9 \\ 4 \end{pmatrix}$

8. Given that $\mathbf{a} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}$ find:

(a) $|\mathbf{a} - \mathbf{b}|$ (b) $|\mathbf{a} + \mathbf{b}|$ (c) $|\mathbf{b} - \mathbf{a}|$

Midpoints

In Fig. 10.27, O is the origin and M is the midpoint of AB.

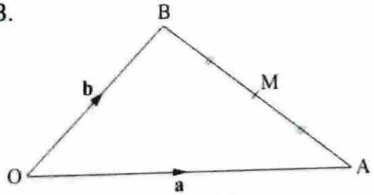


Fig. 10.27

Since O is the origin, **OA**, **OB** and **OM** are position vectors.

$$\begin{aligned} \mathbf{AB} &= \mathbf{AO} + \mathbf{OB} \\ &= {}^-\mathbf{OA} + \mathbf{OB} \quad (\text{since } \mathbf{AO} = {}^-\mathbf{OA}) \\ &= \mathbf{OB} - \mathbf{OA} \\ &= \mathbf{b} - \mathbf{a} \end{aligned}$$

Since M is the midpoint of AB, then

$$\begin{aligned} \mathbf{AM} = \mathbf{MB} &= \tfrac{1}{2}\mathbf{AB} \\ &= \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) \end{aligned}$$

$$\begin{aligned} \therefore \quad \mathbf{OM} &= \mathbf{OA} + \mathbf{AM} \\ &= \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) \\ &= \mathbf{a} + \tfrac{1}{2}\mathbf{b} - \tfrac{1}{2}\mathbf{a} \\ &= \mathbf{a} + \tfrac{1}{2}\mathbf{b} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) \end{aligned}$$

If A is (7 , −3) and B is (−5 , 5),

then $\mathbf{OA} = \mathbf{a} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$ and $\mathbf{OB} = \mathbf{b} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$.

$$\begin{aligned} \text{Thus, } \mathbf{OM} &= \tfrac{1}{2}\left(\begin{pmatrix} 7 \\ -3 \end{pmatrix} + \begin{pmatrix} -5 \\ 5 \end{pmatrix}\right) \\ &= \tfrac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Since **OM** is a position vector, M is (1, 1).

In general, if A is (a, b) and B is (c, d), the midpoint M of **AB** has coordinates
$$M\left(\frac{a+c}{2}, \frac{b+d}{2}\right).$$

Example 10.9

The points A, B, C and D have coordinates (-2, 3), (6, 2), (-4, -5) and (x, y) respectively. Given that M is the midpoint of both AB and CD.

Find:

(a) The co-ordinates of M

(b) The coordinates of D.

Solution

(a) Since A(-2, 3) and B(6, 2)

The coordinates of M are $\left(\frac{-2+6}{2}, \frac{3+2}{2}\right)$

$= \begin{pmatrix} \frac{4}{2}, \frac{5}{2} \end{pmatrix}$

$= (2, 2\frac{1}{2})$

(b) C(-4, -5) D(x, y) and midpoint M(2, $2\frac{1}{2}$)

$$\left(\frac{-4+x}{2}, \frac{5+y}{2}\right) = (2, 2\tfrac{1}{2})$$

$$\therefore \left(\frac{-4+x}{2}\right) = 2$$

$$-4 + x = 4$$

$$\therefore x = 8$$

$$\frac{-5+y}{2} = 2\tfrac{1}{2}$$

$$-5 + y = 5$$

$$y = 10$$

$$\therefore (x, y) = (8, 10)$$

∴ the co-ordinates of D are (8, 10).

Exercise 10.7

1. Calculate the coordinates of the midpoints of the line segment joining the following pairs of points.
   (a) A (2, 1), B (5, 3)
   (b) A (0, 3), B (2, 7)
   (c) A (4, –1), B (4, 3)
   (d) A (–2, 3), B (2, 1)

2. In each of the following cases,
   (i) Find the column vector of **PQ**.
   (ii) Hence or otherwise, find the coordinates of the midpoint.
   (a) P (3, 0), Q (4, 3)
   (b) P (–3, 1), Q (5, 1)
   (c) P (–2, –1), Q (–12, –8)
   (d) P (–9, 1) Q (12, 0)
   (e) P (–8, 7), Q (–7, 8)
   (f) P (–3, 2), Q (3, –2)

3. P is (1, 0), Q is (4, 2) and R is (5, 4). Use the vector method to find the coordinates of S if PQRS is a parallelogram. Find the coordinates of the midpoints of the sides of the parallelogram.

4. In triangle OAB, P is the midpoint of OA and Q is the midpoint of **OB**. Given that **OA** = a and **OB** = b, find in terms of a and b:
   (a) (i) **OP** (ii) **OQ** (iii) **PQ** (iv) **AB**
   (b) Describe **PQ** in relation to **AB**.

5. ABCD is a quadrilateral P,Q,R,S are the midpoints of AB, BC, CD and AD respectively. Find the position vectors of P,Q,R and S in terms of **a**, **b**, **c**, **d**. Hence, describe the quadrilateral PQRS.

6. OACB is a trapezium in which BC is parallel to **OA** and **BC** = **3OA**.
   (a) Find **c** in terms of **a** and **b**
   (b) Given that E and F are the midpoints of AB and OC respectively, find their position vectors in terms of **a** and **b**.
   Describe the quadrilateral OEFA, giving reasons for your answer.

Parallelogram law of addition

Consider two vectors **a** and **b** as shown in Fig. 10.28.

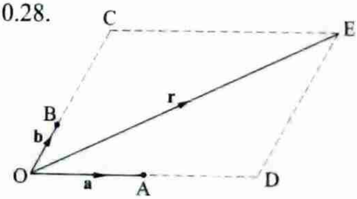


Fig. 10.28

In the figure, a parallelogram OCED has been drawn with the sides being parallel to the vectors **a** and **b**, and with a diagonal vector r = **OE**.

Given that **AD** = **OA** and **BC** = 2**OB**, then

OD = 2 **OA** = 2**a**, and

OC = 3 **OB** = 3**b**

Since **DE** = **OC**, then **OE** = **OD** + **DE** becomes

OE = **OD** + **OC**

i.e. **r** = 2**a** + 3**b**

The vector $\mathbf{r} = \mathbf{OC} + \mathbf{OD} = 2\mathbf{a} + 3\mathbf{b}$ is called the resultant vector. It represents the diagonal OE of the parallelogram whose sides have magnitude $|\mathbf{OD}|$ and $|\mathbf{OA}|$.

Thus:

If $\mathbf{r} = m\mathbf{a} + n\mathbf{b}$, where m and n are scalars, then $\mathbf{r}$ is the resultant vector of a and b whose magnitude is the diagonal of a parallelogram.

Example 10.10

Given that $\mathbf{p} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 5 \\ 16 \end{pmatrix}$, use the parallelogram law of addition to show that $\mathbf{r}$ is a resultant vector of $\mathbf{p}$ and $\mathbf{q}$.

Solution

Let m and n be scalar multipliers such that

$$m\mathbf{p} + n\mathbf{q} = \mathbf{r}$$

$$m\begin{pmatrix} 5 \\ 3 \end{pmatrix} + n\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 16 \end{pmatrix}$$

$5m + n = 5$ ... *(i)*

$3m - 2n = 16$ ... *(ii)*

$10m + 2n = 10$ ... *multiply (i) by 2*

$$\begin{aligned} 3m - 2n &= 16 \\ \hline 13m \quad &= 26 \end{aligned}$$

$$\therefore m = \tfrac{26}{13} = 2$$

$$5m + n = 5$$

$$5(2) + n = 5$$

$$n = 5 - 10 = -5$$

$\therefore \mathbf{r} = 2\mathbf{p} - 5\mathbf{q}$

$\therefore$ *$\mathbf{r}$ is the resultant vector of $\mathbf{p}$ and $\mathbf{q}$.*

Parallel vectors and collinear points

If **two vectors** are such that one is a scalar multiple of the other, then they are parallel. If they are not multiples of each other, they are not parallel (Fig. 10.29).

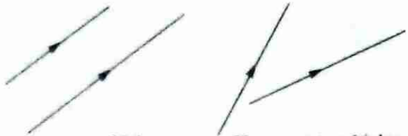


Fig 10.29

This can be summarised as follows:

If **a** and **b** are two vectors and $\mathbf{a} = k\mathbf{b}$, then **a** is k times as long as **b**, and **a** and **b** are parallel.

The idea of parallel vectors may be used to test if any three given points are collinear (i.e. if they lie on the same straight line).

How?

1. Determine any two vectors using the three points.
2. Show that one vector is a scalar multiple of the other. This indicates that the vectors are parallel.
3. Since the vectors share a common point, and are parallel, then conclude that the vectors lie in the same straight line (Fig. 10.31).

Fig. 10.31

Say, $|\mathbf{AB}| = 1$ unit long and $|\mathbf{BC}| = 2$ units long. Then, $\mathbf{BC} = 2\mathbf{AB}$ and B is a common point.

Example 10.11

Show that the points A(⁻2 , ⁻2), B(2 , 1) and C(10 , 7) are collinear.

Solution

$$\mathbf{AB} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\mathbf{BC} = \begin{pmatrix} 10 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

i.e ***BC*** $=$ *2AB*

$\therefore$ ***AB*** *is parallel to* ***BC***

Thus, ***AB*** *and* ***BC*** *are in the same direction and since B is a common point, then A, B and C are collinear, i.e. they are on the same line.*

Exercise 10.8

1. Given $\mathbf{a}=\begin{pmatrix}5\\3\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}2\\-2\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}3\\4\end{pmatrix}$, express:
   (a) **a** as a linear combination of **b** and **c**.
   (b) **b** as a linear combination of **a** and **c**.

2. Express **c** as a resultant vector of **a** and **b** in each of the following cases.
   (a) $\mathbf{a}=\begin{pmatrix}3\\2\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\1\end{pmatrix}$, $\mathbf{c}=\begin{pmatrix}1\\2\end{pmatrix}$
   (b) $\mathbf{a}=\begin{pmatrix}3\\4\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}2\\2\end{pmatrix}$, $\mathbf{c}=\begin{pmatrix}0\\1\end{pmatrix}$

3. (a) Given that $3\mathbf{p}+2\mathbf{q}-3\mathbf{r}=\mathbf{0}$, express each of the following as a linear combination of the other two:
   (i) **p** (ii) **q** (iii) **r**
   (b) If $k\mathbf{p}+m\mathbf{q}+n\mathbf{r}=\mathbf{0}$, express **q** as a linear combination of the other two vectors.

4. Given that $\mathbf{p}=\begin{pmatrix}4\\5\end{pmatrix}$ and $\mathbf{q}=\begin{pmatrix}1\\2\end{pmatrix}$, find values of m and n such that $m\mathbf{p}+n\mathbf{q}=\begin{pmatrix}15\\24\end{pmatrix}$.

5. Given that $\mathbf{p}=8\mathbf{a}+6\mathbf{b}$, $\mathbf{q}=10\mathbf{a}-2\mathbf{b}$ and $r=2m\mathbf{a}+2(m+n)\mathbf{b}$, where m and n are scalars, find values of m and n such that $\mathbf{r}=6\mathbf{p}-8\mathbf{q}$.

6. If $\mathbf{OA}=6\mathbf{p}-4\mathbf{q}$, $\mathbf{OB}=2\mathbf{p}+14\mathbf{q}$ and $\mathbf{AB}=4m\mathbf{p}+(2m-n)\mathbf{q}$, find the scalars m and n.

7. Show that the following points are collinear.
   (a) A(3 , 1), B(6 , –1) and C(–3 , 5).
   (b) A(7 , 20), B(1 , 2) and C(3 , 8).

8. Determine if the following points are collinear.
   A (5, 3), B (–3, 2), C (9, 5).

9. In Fig. 10.31, $\mathbf{DE}=2\,\mathbf{EB}$, $\mathbf{AB}=\mathbf{p}$, $\mathbf{DC}=2\mathbf{p}$ and $\mathbf{DA}=\mathbf{q}$.
   (a) Find in terms of **p** and **q**
   (i) **DB**
   (ii) **EB**
   (iii) **CB**
   (iv) **AE**
   (v) **EC**

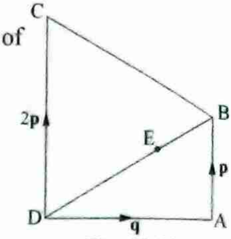


Fig. 10.31

   (b) What do your answer to (iv) and (v) tell you about the points A, E and C?

10. Given that A, B, C and D are points such that A = (3, 8) B = (11, 14) C = (11, 4) and D = (5, 2). Consider the lengths of the sides of quadrilateral ABCD to show that it is a kite.

11. The position vector for A, B and C are $\begin{pmatrix}2\\1\end{pmatrix}$, $\begin{pmatrix}6\\-2\end{pmatrix}$ and $\begin{pmatrix}3\frac{1}{3}\\0\end{pmatrix}$ respectively. Use vectors to show that:
   (a) AB and AC are parallel,
   (b) Points A, B and C are collinear.

Parallelogram and triangle laws

From trigonometry, we know that for every triangle ABC, the lengths AB + BC ≠ AC. In this chapter we have shown earlier on that for any three points on the plane, **AB** + **BC** = **AC.**

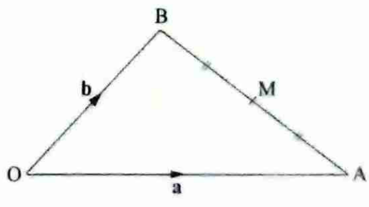


Fig. 10.32

In this case, we are only interested in the initial position and the destination moving from A to B and then from B to C which has the same effects as moving from A to C. This is called the triangle law of vectors.

For any triangle ABC, **AB** + **BC** = **AC**

Considering a parallelogram, we can also define any parallelogram in terms of vectors. Consider parallelogram ABCD in Fig. 10.33.

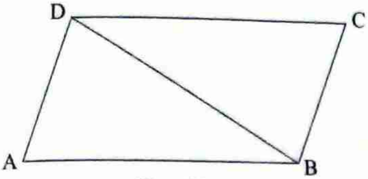


Fig. 10.33

We know that, ABCD is a parallelogram. Using vectors, we can prove that **AB** = **DC** and **AD** = **BC**.

ABCD has a pair of sides say AB and DC for which

AB = **DC** …. Opposite sides of a parallelogram are equal in length.

BD a is diagonal such that

AB + **BD** = **AD** .... triangle law of vectors

BD + **DC** = **BC** .... triangle law of vectors

∴ **BD** + **AB** = **BC** .... **AB** = **DC**

AB + **BD** = **BC**

∴ **AD** = **BC**

Similarly **AB** = **DC.**

For any parallelogram ABCD,
AB = **DC** and **AD** = **BC**.

Application of parallelogram law

Vector methods can be used to establish some well known geometric results as illustrated in the following example.

Example 10.12

*Draw a square ABCD. Let **AB** = **b** and **AD** = **d**. If P and Q are the midpoints of $\overrightarrow{BC}$ and $\overrightarrow{AD}$ respectively, show that APCQ is a parallelogram.*

Solution

Fig. 10.34 shows the required square.

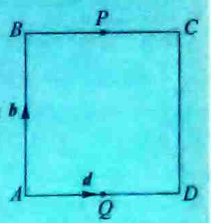


Fig. 10.34

$\boldsymbol{PC} = \frac{1}{2}\boldsymbol{BC} = \frac{1}{2}\boldsymbol{AD} = \frac{1}{2}\boldsymbol{d}$

$\boldsymbol{AQ} = \frac{1}{2}\boldsymbol{AD} = \frac{1}{2}\boldsymbol{d}$

*Thus, vectors **PC** and **AQ** are equal and in the same direction, therefore parallel.*

$$\boldsymbol{AP} = \boldsymbol{AB} + \boldsymbol{BP} = \boldsymbol{AB} + \tfrac{1}{2}\boldsymbol{BC}$$
$$= \boldsymbol{b} + \tfrac{1}{2}\boldsymbol{d}$$

$$\boldsymbol{QC} = \boldsymbol{QD} + \boldsymbol{DC} = \tfrac{1}{2}\boldsymbol{AD} + \boldsymbol{DC}$$
$$= \tfrac{1}{2}\boldsymbol{d} + \boldsymbol{b}$$
$$= \boldsymbol{b} + \tfrac{1}{2}\boldsymbol{d}$$

*Thus, vectors **AP** and **QC** are equal and in the same direction. Since AP and QC are opposite sides of the quadrilateral APCQ, then the quadrilateral is a parallelogram.*

Exercise 10.9

In this exercise, use vector methods only.

1. Show that the diagonals of a rhombus bisect each other.

2. In Fig. 10.35, **OA** = **a**, **OB** = **b**, **OA** = $\frac{1}{3}$ **OC** and **OB** = $\frac{1}{3}$ **OD**.

   Show that **AB** is parallel to **CD**.

   If A and B are the midpoints of $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively, show that even in this case, AB//CD.

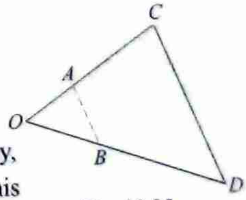


Fig. 10.35

3. The midpoints of the sides AB, BC, CD and DA of a quadrilaterial are P, Q, R and S, respectively. Taking **AB** = **b**, **AC** = **c** and **AD** = **d**, show that PQRS is a parallelogram.

4. A(3, 8), B(11, 14), C(11, 4) and D(5, 2) are the vertices of a quadrilateral. If P,Q, R and S are the midpoints of $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$ respectively show that $\overrightarrow{PQ} // \overrightarrow{AC}$ and $\overrightarrow{QR} // \overrightarrow{BD}$.

5. Fig. 10.36 shows the edges of a box. **AB** = **x**, **AD** = **y** and **AE** = **z**.

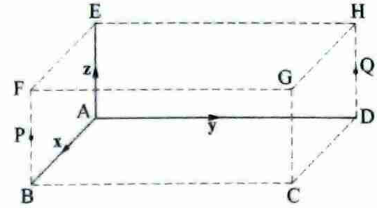


Fig. 10.36

If P and Q are the midpoints of BF and DH respectively, show that EPCQ is a parallelogram.

6. Use vectors to prove that diagonals of a parallelogram bisect each other.

6-10 REVISION EXERCISES 2

Revision exercise 2.1

1. Find the area of ΔPQR in which $PQ = 6$ cm, $QR = 7$ cm, and $\angle PQR = 34°$.

2. In ΔABC, $AC = 5$ cm, $BC = 6$ cm and $\angle ACB = 118°$. Find the area of the triangle.

3. Vectors **a**, **b** and **c** are expressed in terms of vectors **p** and **q** as follows;
$\mathbf{a} = 3\mathbf{p} + 2\mathbf{q}$, $\mathbf{b} = 5\mathbf{p} - \mathbf{q}$ and
$\mathbf{c} = h\mathbf{p} + (h - k)\,\mathbf{q}$,
where h and k are constants. Given that $\mathbf{c} = 2\mathbf{a} - 3\mathbf{b}$, find the values h and k.

4. Given that A and B are the points whose position vectors, are $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\4\end{pmatrix}$ respectively, determine AB and |AB|.

5. ABCD is a parallelogram such that $\mathbf{OA} = \begin{pmatrix}4\\2\end{pmatrix}$, $\mathbf{AB} = \begin{pmatrix}1\\-12\end{pmatrix}$ and C is the point (–3 , 0). P, Q, R, S and T are the midpoints of **AB**, **BC**, **CD**, **DA** and **BD**, respectively.
   (a) Find the coordinates of D.
   (b) Find the coordinates of P, Q, R, S and T.
   (c) Evaluate | **AT** | and | **CT** |.
   (d) Show that PQRS is a parallelogram.

6. Simplify
$(3x^4 + 4x^3 - 5x^2 + 18) - (7x^5 + 3x^3 - 12x^3 - 3x^2 + 7x - 9)$

7. Evaluate $\frac{x^3 - 12x^2 - 42}{x - 3}$.

8. Evaluate: $(x^2 + 6x - 4)(x^3 - 6x)$ and state the degree of the polynominal formed.

9. The probability that a husband and a wife will be alive 25 years from now are 0.7 and 0.9 respectively. Find the probability that in 25 years time,
   (a) both will be alive.
   (b) neither will be alive.
   (c) one will be alive.
   (d) at least one will be alive

10. Three soccer teams A, B, and C participated in qualifying matches. The probability of A, B and C qualifying are $\frac{1}{2}$, $\frac{3}{10}$ and $\frac{3}{20}$ respectively. Find the probability that:
   (a) all the three teams qualify,
   (b) only one of them qualifies,
   (c) at most one team fails to qualify.

11. Given that $OB = \begin{pmatrix}3\\2\end{pmatrix}$, $CB = \begin{pmatrix}2\\-1\end{pmatrix}$ and $CD = \begin{pmatrix}1\\-4\end{pmatrix}$ and O is the origin, find the coordinates of D. Show that OCBD is a parallelogram.

12. The speed of an object at t seconds after the commencement of motion is given by the relation v = (t + 1) m/s. Represent speed against time graphically for the first 5 s. Use your graph to calculate:
   (a) the initial speed,
   (b) the acceleration of the object,
   (c) distance travelled by the object in the 5 s.

Revision exercise 2.2

1. Find the area of the quadrilateral in Fig. R.2.1.

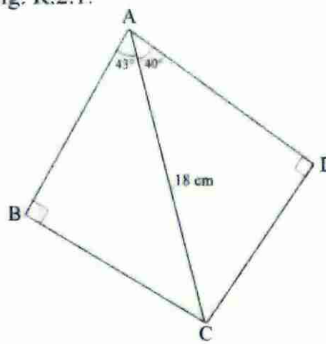


Fig. R.2.1

2. Find the angles of ABC whose sides are 4 cm, 5 cm and 6 cm in that order.

3. In ΔABC, E lies on BC such that $\frac{BE}{EC} = \frac{2}{3}$ F lies on CA such that $\frac{CF}{FA} = \frac{3}{4}$ and G lies on AB produced such that $\frac{GB}{GA} = \frac{1}{2}$.
The position vectors of A, B and C, relative to the origin O, are **a**, **b** and **c** respectively.
   (a) Determine the position vectors of E, F and G in terms of **a, b** and **c**.
   (b) Hence, deduce that E, F and G lie on a straight line.

4. Given that $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, find the value of m such that

   $\mathbf{a} + m\mathbf{b} = \mathbf{c}$

5. Given that $\mathbf{a} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -9 \\ -1 \end{pmatrix}$, and M is

   a point on AB such that AM:MB = 2 : 2, find:
   (a) the position vector of M.
   (b) the magnitude of OM.

6. Divide $(x^3 + 2x - x - 2)$ by $(x - 2)$

7. Evaluate:
   $(4x^5 + x^4 - 12x^3 + x - 6)(3x^{14} + 8x^3 + 6x^2 - x)$.

8. Given that $f(x) = x^3 - x^2 - 5x + 5)$ and $g(x) = x - 2$, find $\frac{f(x)}{g(x)}$ and state the quotient and the remainder.

9. Two fair dice are thrown simultaneously and the sum of their outcomes recorded. Find the probability of a score:
   (a) greater than 3.
   (b) of 7 or 11.
   (c) that is not divisible by 4.

10. The probability that a certain student passes her examination is $\frac{4}{5}$. If she passes, the probability that she does not get a job is $\frac{3}{8}$. If she does not pass, the probability that she gets a job is $\frac{1}{4}$. Use a tree diagram to find the probability that she:
   (a) passes and gets a job.
   (b) gets a job.
   (c) does not get a job.

11. A coin is biased so that the probability of "Heads" is $\frac{3}{4}$. Find the probability that when the coin is tossed three times, it shows:
   (a) 3 tails (b) 2 heads and 1 tail
   (c) no tails

12. A particle starts from rest and moves with an acceleration, a m/s^2, given by $a = t - 6$, where t represents time in seconds. Given that its initial speed is 2 m/s^2, find expressions for its speed, v, and displacement, s, in terms of t.

Revision exercise 2.3

1. Determine, in ΔABC
   (a) $\angle$A given that $\angle$B = 90°, AC = 9 cm and BC = 5 cm.

(b) $\angle C$ given that $\angle B = 90°$, AC = 10 cm and BC = 4 cm.

2. A ship sails 15 km from a port A on a bearing of 045° and a further 10 km on a bearing of 125°. Find the distance and the bearing of the port from the ship after the sailing.

3. In Fig. R.2.2, **OV** = $\frac{1}{3}$ **OP** and **PX** = $\frac{3}{4}$ **PQ**.

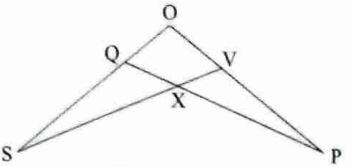


Fig. R.2.2

(a) Given that **OP** = 12**p** and **OQ** = 4**q**, express the following in terms of **p** and **q** as simply as possible:

(i) **PQ** (ii) **PX**

(iii) **OX** (iv) **VX**

(b) Also, given that **VS** = a**VX** and **PS** = h**p** + k**q**, express h and k in terms of a.

4. Points A, B, C, and D have coordinates (–1 , 2), (–3, –2), (1, 6) and (2,1) respectively.

(a) Show that points A, B and C are collinear.

(b) Find the coordinates of point E such that 2**AC** + **BE** = **AD**.

(c) Find the values of the scalars m and n given that m**AB** + n**CD** = **BD**.

5. Given that points P and Q are ($^-2$, $^-4$) and (6, 0) respectively, determine:

(a) PQ (b) |2PQ|.

6. Identify the degree of each of the following polymonials:

(a) $(x^6 - x^4 - 2x^3 - 6)$

(b) $x^2 + 2x^3 - 6x^2)$

7. Evaluate:

$(2x^3 + 6x^3 - 20x) \div (2x - 4)$

8. Use the remainder theorem to find the remainder when $3x^2 + x^3 + 3x + 2$ is divided by $(x + 1)$.

9. A poultry farmer had a total of 1 260 birds. The owner vaccinated 540 birds against fowl pox. Shortly after, an outbreak of the disease affected 5% of the vaccinated birds and 80% of the unvaccinated birds. Find the probability that a bird chosen at random is healthy.

10. Two bags A and B each contain a mixture of identical red and green balls. A contains 26 red balls and 3 green balls, while B contains 18 red balls and 15 Green balls. A bag is picked at random and then a ball picked from it at random. Determine the probability that the ball picked is red.

11. Use the graph in Fig. R.2.3 to answer the following questions.

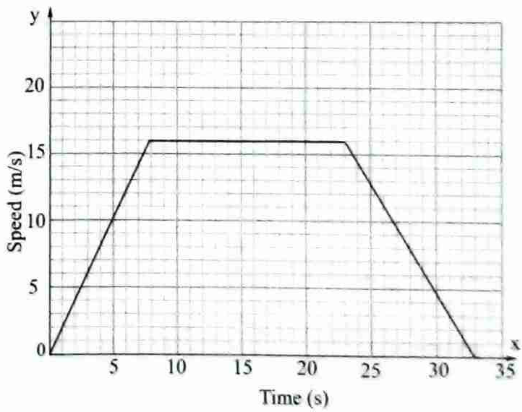


Fig. R.2.3

(a) Describe the motion represented by the graph.

(b) Determine the acceleration of the object:

  (i) for the first 8 s,

  (ii) when the speed was at its maximum,

  (iii) for the last 10 s.

(c) Use your graph to find the distance travelled by the object from the spinning until it stops.

12. Pempho is learning how to play darts. The probability that he hits the mark when he throws a dart is $\frac{1}{10}$. If he tries four times, find the probability that he:

  (a) hits the mark four times,

  (b) does not hit the mark at all,

  (c) hits the mark at least once.

11 LINEAR PROGRAMMING

Success criteria

By the end of this topic, the student must be able to:

- *Formulate inequalities from given situations.*
- *Solve linear programming problems.*

Introduction

In Form 3, we drew graphs of linear inequalities in two variables. We also formulated linear inequalities that satisfy given regions. In this chapter, we are going to form inequalities and use them to solve given problems. We will begin by reviewing graphical representation of given linear inequalities in two variables. We will also formulate inequalities from given regions.

Review of graphical solution of linear inequalities

It is important to recall that, in order to represent an inequality graphically, we must first identify the **boundary line**. The line is drawn solid if the boundary is included in the required region, or broken if the boundary is not included.

In order to identify the required region, we pick any point, not on the line, and substitute its coordinates in the given inequality to test whether it satisfies the inequality or not. We then shade the unwanted region. If the required region is an intersection of two or more regions, the individual regions are illustrated, one at a time, but on the same graph.

Example 11.1

Represent, on the same graph, the solution of the three simultaneous inequalities $x < 7$, $y < 5$ and $8x + 6y \geq 48$.

Solution

We are looking for the intersection of the three regions: $x < 7$, $y < 5$ and $8x + 6y \geq 48$.

The boundary lines are $x = 7$, $y = 5$ and $8x + 6y = 48$ respectively.

Thus, we draw the line

(i) $x = 7$ (broken) and shade the region $x > 7$.

(ii) $y = 5$ (broken) and shade the region $y > 5$.

(iii) $8x + 6y = 48$ (solid) and shade the region $8x + 6y < 48$ (Fig. 11.1).

Substituting $(0, 0)$ in $8x + 6y \geq 48$ gives $0 \geq 48$, which is false. Therefore, the point $(0, 0)$ does not belong to the region we want so we shade the region that contains $(0, 0)$. For $y < 5$, it is obvious that any number below 5 satisfies the inequality, therefore we shade the region above the line $y = 5$. Similarly, for $x < 7$, we shade the region to the right of $x = 7$. This is shown in Fig. 11.1.

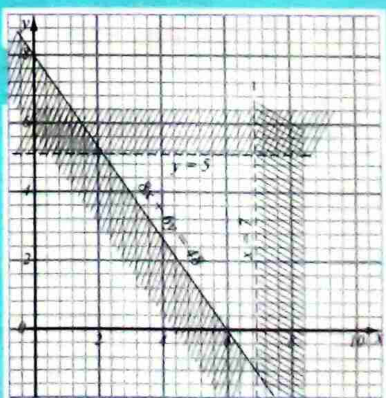


Fig. 11.1

Example 11.2

Use graphical method to solve, simultaneously, the inequalities

$x \geq 0,\ y \geq 0,\ x + y \leq 4,\ x + 2y < 6.$

Solution

The required boundary lines are $x = 0,\ y = 0,$ $x + y = 4,\ x + 2y = 6$

On the same axes, we draw the boundary lines and shade the unwanted regions, one at a time (Fig. 11.2).

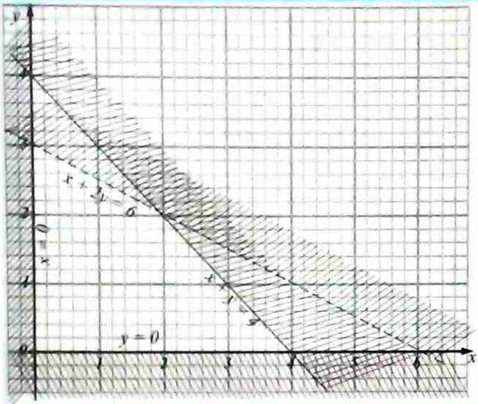


Fig. 11.2

The unshaded region represents the solution set.

Exercise 11.1

1. Draw separate diagrams to show the regions representing the following inequalities.

   (a) $5x + 4y < 60$ (b) $3x - y > -6$

   (c) $8x + 3y \geq 24$

2. A region R is given by the inequalities $x \leq 6,\ y \leq 6,\ x + y < 9$ and $6x + 5y \geq 30$. Represent this region graphically and list all the points in the region which have integral coordinates.

3. Show the regions and list the points whose coordinates are integers and which satisfy the following simultaneous inequalities.

   (a) $x < 4, 3y - x \leq 6, 3y + 2x > 6$

   (b) $0 \leq x \leq 4,\ 0 \leq 3y - x \leq 6$

   (c) $x \geq 0,\ 5x + 6y \leq 60,\ 9x + 6y \leq 72,$ $x + y \geq 8$

4. Find the points with integral coordinates which satisfy the simultaneous inequalities $x \leq 4,\ 3y \leq x + 6,\ 2x + 3y > 6.$

5. R represents the region in a Cartesian plane whose points satisfy the inequalities $0 \leq x < 5,\ 0 \leq 3y + x \leq 9$. Solve the inequalities graphically.

6. On the same graph, show the region R that is satisfied by the inequalities $x \geq 0,\ y \geq 0,$ $x + y \leq 12,\ x + 2y \leq 16$ and $y \geq -\frac{4}{5}x + 4.$

Variables: Formulating inequalities

Given a narrative and we are required to form inequalities from it, our first task is to define the variables that we intend to use. Since our method is graphical, we can only deal with problems that can be fully defined in terms of two variables. Once we explain the meaning of the variables we introduce, then we write down as many inequalities as we can from the given information.

Example 11.3

Sunshine Academy is a mixed school with a maximum capacity of 450 students. There are at least 150 girls, and more than 160 boys. Write down as many inequalities as you can to represent the given information.

Solution

Let x be the number of girls and y be the number of boys in the school.

At least 150 girls, means there could be more than 150 girls.

$\therefore x \geq 150$

Boys must be more than 160, i.e.

$\therefore y > 160$

The capacity of the school is 450.

This means no more than 450 students.

$\therefore x + y \leq 450$

All the inequalities are

$$x \geq 150$$
$$y > 160$$
$$x + y \leq 450$$

Example 11.4

A games master wishes to buy new sports shoes for his students. He has K 24 000 to spend. In his town, only two shops, A and B, stock the kind of shoes he wants. At shop A, they cost K 1 000 a pair and in shop B they cost K 1 200 a pair. In shop A, only six pairs are remaining.
Write down as many inequalities as possible from the given information.

Solution

Though it is reasonable to argue that the games master should buy the six pairs from shop A and the rest from shop B, he does not have to.
Let x be the number of pairs of shoes bought from shop A and y be the number of pairs of shoes bought from shop B.
The cost of the shoes is $1\,000x + 1\,200y$

$\therefore\ 1\,000x + 1\,200y \leq 24\,000$ *(maximum he can spend is 24 000)*

$x \leq 6$ *(since there are only 6 pairs of shoes in shop A)*

$x \geq 0$ *and* $y \geq 0$ *(since x and y cannot be negative)*

$\therefore$ *the inequalities are*
$x \geq 0,\ y \geq 0,\ x \leq 6,\ 5x + 6y \leq 120.$

Exercise 11.2

In each of Questions 1 to 5, explain the meaning of the variables you introduce, and then write down as many inequalities as you can from the given information.

1. Jane went shopping, with K 540, to buy some Christmas cards for her friends. She found two types of cards, one costing K 60.00 each and the other costing K 90.00 each. She decided to buy some of each type, but not less than four cards altogether.

2. A small aircraft company is required to transport 600 people and 45 tonnes of baggage. Two types of aircraft are available. Type A carries 60 people and 6 tonnes of baggage. Type B carries 70 people and 3 tonnes of baggage. Only eight aircrafts of type A and seven of type B are available for use.

3. A milk transporter distributes 900 crates of milk per day, using trucks and vans. A truck carries 150 crates while a van carries 60 crates. The cost of each trip by a truck is K 500 and that of a van is K 400. He has K 4 400 available to use for transport. It is advisable that he uses both modes of transport.

4. Amin went shopping, with K 720 only to buy some fireworks. He bought two different types, A and B, at K 60 and K 90 each respectively. For every one of type B, he bought at least two of type A.

5. Joan wishes to buy x kg of beans and y kg of maize. The cost of beans is K 80 per kg while that of maize is K 25 per kg. She wishes to buy more maize than beans, and she has only K 400 to spend.

6. A poultry farmer plans to keep some layers and broilers. Layers cost K 65 per day old chick while broilers cost K 45 per day old chick. He cannot afford to spend more than K 40 000. He finds it uneconomical to keep less than 400 birds. He wishes to keep more broilers than layers and not less than 300 broilers.
   Write down as many inequalities as possible to describe the given situation.

7. The sum of three consecutive integers is less than 999. If the smallest of these integers is n, what are the other two integers in terms of n? Write down an inequality involving n.

8. The sum of the lengths of any two sides of a triangle is greater than the length of the third side. The lengths of the sides of such a

triangle are 5 cm, 9 cm and x cm. Write down as many inequalities in x as possible.

Maximising and minimising a function

Sometimes, when we are solving problems graphically, we may be required to find the maximum or minimum value of a function in the region obtained as we will see later in this chapter.

Example 11.5

A region, A, given by the inequalities $x \geq 0$, $y \geq 0$, $x + y \leq 10$ and $x + 2y \leq 16$ represents the solution set of a certain problem. Find the maximum value of the function $3x + 4y$ in this region.

Solution

Fig. 11.3 illustrates the region that satisfies all the given inequalities. This is our feasible region.

We could find the maximum value by listing the coordinates of all the points in the region and substituting them in the function $3x + 4y$ (which is our search line). But this is not practical since the number of points in the region is infinite. Therefore, we use the method described below.

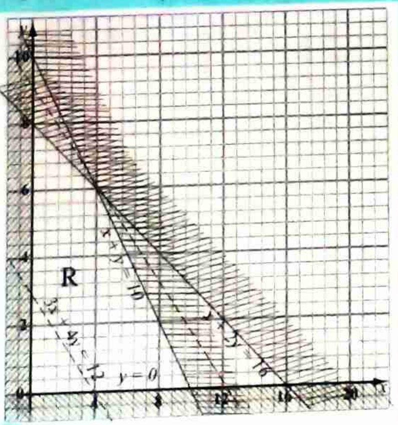


Fig. 11.3

R is our feasible region.

If we draw line $3x + 4y = k$, where k is a constant, say $k = 12$ or 24, etc., we notice that the value of k gets larger as the line moves further away from the origin. Thus, to find the maximum value of $3x + 4y$, we draw a line $3x + 4y = k$ on our graph and using a ruler and a set square, we translate the line as far away from the origin as the conditions will allow. The furthest point in the region which falls on the line gives the values of x and y which make $3x + 4y$ a maximum.

In our example, the furthest point is (4, 6), at the intersection of the lines $x + y = 10$ and $x + 2y = 16$.

$\therefore$ the maximum value of $3x + 4y$ is

$$3 \times 4 + 4 \times 6$$
$$= 12 + 24$$
$$= 36$$

Note: The verticles (corners) of our feasible region also form part of the solution set to our inequalities. The verticles are: (0,0), (0,8), (10,0) and (4,6).

Example 11.6

Draw the region that satisfies the inequalities $y \leq 4$, $x \leq 7$, $x + y \leq 9$, $4x + 5y \geq 30$. Hence, find:

(a) the maximum value of the function $5x + 8y$.

(b) the minimum value of the function $5x + 8y$.

Solution

On the same axes, draw the lines $y = 4$, $x = 7$, $x + y = 9$ and $4x + 5y = 30$ (Fig. 11.4).

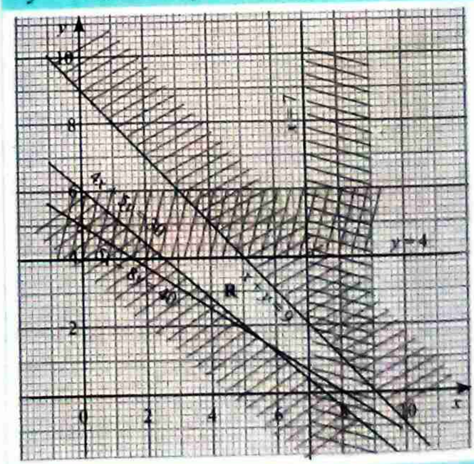


Fig. 11.4

(a) *In order to maximise 5x + 8y, we write it in the form ax + by = c where C is a constant i.e. 5x + 8y = 40 (choosing C = 40 for easy plotting)*

On the same graph, we draw the line 5x + 8y = 40, and translate it to the furthest point in the region. The furthest point is (5, 4) as shown by the broken line.

∴ maximum value of the function

5x + 8y is 5(5) + 8(4) = 25 + 32 = 57

(b) *Using the search line 5x + 8y = 40, the point in the region nearest to the origin is (7 , 0.4)*

∴ minimum value $= 5 \times 7 + 8 \times 0.4$

$= 35 + 3.2$

$= 38.2$

Note that, if we needed the minimum value, we would translate the line towards the origin. We would then pick the point within the region that is nearest to the origin.

A line such as 3x + 4y = k in Example 11.5 is known as the search line.

Exercise 11.3

1. A region is defined by the inequalities $x \geq 0, y \geq 0,\ 3x + 2y \leq 12$ and $7x + 3y \leq 21$. Show the region on a graph and find, within the region,
   (a) the maximum value of $x + y$,
   (b) the maximum value of $5x + y$.

2. Solve the simultaneous inequalities $x \geq 0$, $y \geq 0,\ 2x + 2y \geq 11,\ 2x + y \geq 8$ and $2x + 5y \geq 18$ and find, within the solution set, the minimum value of:
   (a) $3x + 2y$, (b) $2x + 3y$, (c) $x + 3y$.

3. Find the minimum value of:
   (a) $y - x$ (b) $6x + 5y$ (c) $4x + 6y$
   in the region defined by $y \leq 2x,\ x \leq 6$, $y \geq 2$ and $2x + 3y \leq 30$.

4. 500 men and 42 tonnes of equipment are to be transported to a new camp. There are two types of trucks that can be used. Truck type A carries 50 men and 5 tonnes of equipment, while truck type B carries 40 men and 3 tonnes of equipment. Given that x trucks of type A and y trucks of type B were used, write down, from this information, as many inequalities as you can and represent them on a graph.
   Find the minimum value, in the region, of $1050x + 900y$.

5. A famine relief food agent has to transport 900 bags of maize and beans from the city to one of the distributing centres. He intends to use trucks which can carry 150 bags at a time and vans which can carry 60 bags at a time.
   (a) If the number of lorries used is l, and the number of vans v, write down an inequality in terms of l and v to represent the given situation.
   (b) The cost of running a truck for the journey is K 5 000, and that of running a van is K 4 000. If the agent has a maximum of K 44 000 to spend, write down another inequality in terms of l and v.
   (c) Represent the inequalities in (a) and (b) graphically, and clearly label the region which must be satisfied by the inequalities in l and v.
   (d) (i) Use your graph to list all the possible combinations of vehicles that could be used.
   (ii) Identify the combination that would keep the cost at a minimum, and hence state the minimum cost.
   (iii) Find the combination that would be most expensive.

Objective function

We are now in a position to look at some problems which may arise from real life situations and which can be solved by means of graphs of linear inequalities. The whole process of finding the possible solutions to a given problem is called **linear programming**.

By drawing appropriate graphs, we first find a region containing the points which represent possible solutions ie. we identify the feasible region. When the solution set has been found, the next task is to decide which element of the solution set best meets the requirements of the problem. This process is called **optimisation**. Usually, the best solution is the one that will make, say, the profit as large as possible, the cost as little as possible, the time taken for a process as short as possible, and so on.

In order to minimise or maximise a value, we use a method similar to the one we used in the previous section. This involves drawing the graph of the function we wish to maximise or minimise. This function is known as the **objective function** and is usually of the form $C = ax + by$, where a, b and C are constants.

We must identify what quantity we wish to maximise or minimise and use the appropriate information available in the problem being solved. After obtaining the function we draw the line in the same graph and continue as illustrated in Example 11.5.

The following example illustrates this method.

Example 11.7

A transport company has two types of lorries, 9 of type A and 5 of type B. There are 11 drivers available. The company has been contracted to transport at least 3 600 bags of coffee from a certain co-operative store to the Coffee Association of Malawi stores in Lilongwe. Type A lorries can each make 4 trips and carry 90 bags per trip. Type B lorries make 3 trips per day and carry 150 bags each per trip. It costs K 5 000 per day to run a type A lorry and K8000 per day to run a type B lorry. How should the contractor organise the use of his lorries so as to

(a) *run the lorries at a minimum cost?*

(b) *carry the maximum number of bags each day?*

(c) *use the minimum number of drivers?*

Solution

Let x be the number of type A lorries used per day and y be the number of type B lorries used per day. Due to the limitation on the number of lorries and the number of drivers available,

$x \leq 9,\ y \leq 5$ *and* $x + y \leq 11$

are some of the required inequalities.

In one day, a type A lorry can carry 90×4 *bags, while a type B can carry* 150×3 *bags.*

A minimum of 3 600 bags are to be transported, using x type A lorries and y type B lorries.

$\therefore$ $360x + 450y \geq 3\,600$ *which can be simplified as* $4x + 5y \geq 40$

$\therefore$ *all the possible inequalities from the given information are*

$x \leq 9,\ y \leq 5,\ x + y \leq 11$ *and* $4x + 5y \geq 40$

Now, we graph the inequalities to find the solution set of the problem (Fig. 11.4).

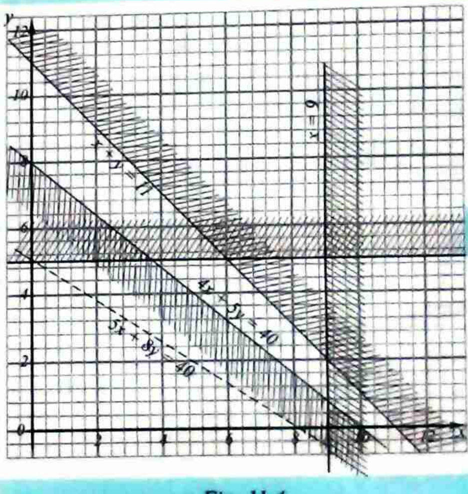


Fig. 11.4

From our graph, it is clear that the points

(4, 5), (5, 4), (5, 5), (6, 4), (6, 5), (7, 3), (7, 4), (8, 2), (8, 3), (9, 1) and (9, 2) represent possible solutions.

Using these possible solutions, we must now find the solution that suits the transporter's requirements best.

(a) The expression that gives the total cost is $5\,000x + 8\,000y$. In order to minimise the cost, we use the objective function (Fig. 11.5)

$C = 5\,000x + 8\,000y$,
where C represents the total cost.
By choosing an appropriate value of C, we get the search line. For example, if $C = 40\,000$, we get the search line $5000x + 8000y = 40\,000$, i.e. $5x + 8y = 40$.
By translating this line towards the solution region, we find that the solution point which gives the minimum cost as (9, 1).

$\therefore$ The contractor should use 9 lorries of type A and 1 of type B so as to incur a minimum cost of K $(5\,000 \times 9 + 1 \times 8\,000)$ = K 53 000.

(b) In order to transport the maximum number of bags, our objective function is $4x + 5y = k$ where k is a constant.
This function gives the best solution at (6 , 5), i.e. use 6 type A lorries and 5 type B lorries to carry a maximum of 1 290 bags per day.

(c) In order to use the minimum number of drivers, the objective function is $x + y = d$ where d is a constant.
The best solution is 9 drivers which is given by the point (5 , 4).

Example 11.8

A draper is required to supply two types of shirts, A and B. The total number of shirts must not exceed 400. He has to supply more of type A than of type B. The number of type A must not be more than 300 and the type B shirts must not be less than 80. Let x be the number of type A shirts and y be the number of type B shirts.

(a) Write down, in terms of x and y, the linear inequalities representing the information above.

(b) On the grid draw the inequalities and shade the unwanted regions.

Type A cost: K 600 per shirt
Type B cost: K 400 per shirt

(i) Use the graph to determine the number of shirts of each type that should be made to maximise the earnings.

(ii) Calculate the maximum possible earnings.

Solution

(a) $x + y \leq 400$, $x > y$, $x \leq 300$, $y \geq 80$

(b) Fig. 11.6 shows the correctly shaded graph.

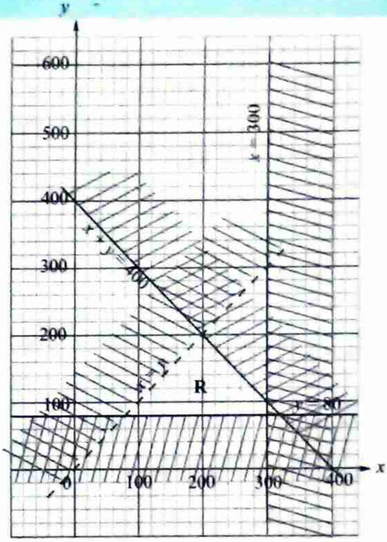


Fig. 11.6

(i) The combination of the two types of shirts will be found within the unshaded region. As shown earlier, the point that gives maximum profit will be found at a corner within the region, furthest from the origin. The point that gives the suitable combination is given by the coordinates (300, 100).

Thus, maximum combination is 300 type A shirts; 100 type B shirts.

(ii) Each type A shirt sells at K 600 and that of type B shirt at K 400

Thus earning on type A shirts = K 300×600
earning on type B shirts = K 100×400
$\therefore$ maximum possible earning
$= 300 \times 600 + 100 \times 400 =$ K 220 000

Example 11.9

A theatre has a sitting capacity of 250 people. The charges are K 100 for an ordinary seat and K 160 for a special seat. It costs K 16 000 to stage a show and the theatre must make a profit. There are never more than 200 ordinary seats and for a show to take place, at least 50 ordinary seats must be occupied. The number of special seats is always less than twice the number of ordinary seats.

(a) Taking x to be the number of ordinary seats, and y the number of special seats, write down all the inequalities representing the above information.

(b) On the grid provided, draw a graph to show the inequalities above.

(c) Determine the number of seats of each type that should be booked in order to maximise the profit.

Solution

(a) $x + y \leq 250,\ x \geq 50,\ x \leq 200, y < 2x$

$100x + 160y > 16\,000,\ y \geq 0.$

(b) Fig. 11.7 shows the correct graph.

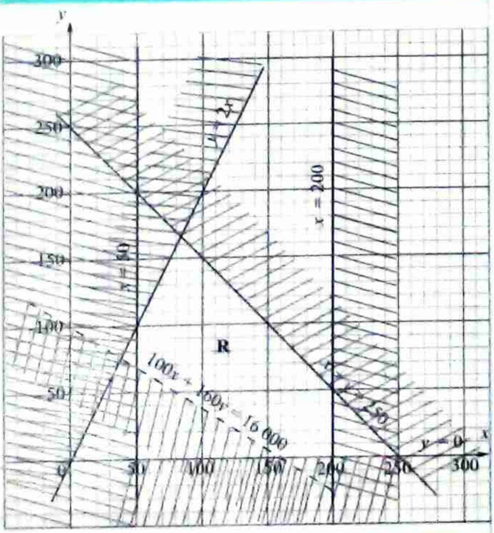


Fig. 11.7

The line 100x + 160y = 16000 is the objective function in this case.

(c) Number of ordinary seats is 85

Number of special seats is 165

It is important to note that the maximum or minimum usually occurs at one of the corners of the feasible region. Thus, instead of using an objective function, one could get the optimum solution by testing the coordinates of the corner points.

Re-do Example 11.8 using this technique.

Exercise 11.4

In this exercise, the solution set will consist of only positive numbers or zero.

1. At a charity show, 600 tickets are available at two different prices. Type A is offered at K 300 each and type B at K 500 each. The organiser wishes to raise a minimum of K 150 000. He also wishes to sell at least twice as many of the dearer tickets as the cheaper ones. Write down as many inequalities as you can from the given information.
   (a) Represent all the inequalities graphically.
   (b) Find the maximum possible collection.
   (c) If the organiser has to cover his expenses of K 12 000, find the minimum possible amount that goes to charity.

2. Mr. Onani has an 18 ha piece of land. He wishes to plant part of the land with beans and the other part with potatoes. The total cost per ha for beans is K 1 000, and the cost per ha for potatoes is K 800. He has to hire 2 men per ha for beans, and 1 man per ha for potatoes. He cannot hire more than 18 men. For the whole project he has a maximum of K 12 000 available. If he plants x ha with beans and y ha with potatoes, write down all the inequalities that must be satisfied by x and

y. Hence, find his maximum profit if profit per ha of beans is K 2 400 and K 1 600 per ha of potatoes.

3. A factory is in the process of installing two new types of machines. For machine type A, the floor space available is 500 m^2, labour needed per machine is 9 men, and the output per week is 300 units. For type B, floor space available is 600 m^2, labour per machine is 6 men, and the output per week is 200 units. The factory has 4 500 m^2 floor space available and only 54 skilled workers who can work on the new machines.
   (a) If the manager buys x machines of type A and y machines of type B, write down the inequalities satisfied by x and y.
   (b) Represent the inequalities graphically, and identify clearly the possible solutions open to the manager.
   (c) How many machines of each type should he buy in order to:
      (i) achieve maximum output?
      (ii) utilise all his skilled workers?

4. A factory makes tables and chairs. One table takes 1 hour of machine time and 3 hours of craftman's time. A chair takes 2 hours of machine time and 1 hour of craftman's time. In a day, the factory has a maximum of 28 machine hours and 24 hours of craftman's time. The factory makes x tables and y chairs on a particular day.
   (a) Write down all the inequalities satisfied by x and y.
   (b) Represent the inequalities graphically.
   (c) How many tables and chairs must be made in a day if the factory is to work at full capacity?
   (d) If a chair gives a profit of K 500 while a table gives a pofit of K 800, find the maximum profit to the company in a day if it produces:
      (i) chairs only,
      (ii) tables only,
      (iii) an equal number of tables and chairs,
      (iv) at full capacity.

5. Lily is a fresh fruit juice supplier. She is particular about flavour and colour density. She uses x mangoes and y oranges. For a strong good flavour, she ensures that $5x + 2y$ is at least 80. For an attractive colour, $3x$ must be less than $2y$.
   (a) Write down all the possible inequalities which satisfy the given conditions, assuming that both types of fruits must be used.
   (b) Illustrate your inequalities graphically.
   (c) Given that one mango costs K 20 while an orange costs K 5, state the objective function, and hence find the cheapest combination of fruits for making the juice.

6. In a musical concert, a local musician was to sing x classicals and y raps. Each classical takes 3 minutes while each rap takes 4 minutes. Allowing for applause and change over, the musician is expected to perform for a maximum of 36 minutes. His manager advises him to sing more classicals than raps. His fans demand that he sings more than 3 classicals and at least 2 raps.
   (a) Write down all the inequalities that satisfy the given conditions in x and y.
   (b) Represent the inequalities in part (a) graphically.
   (c) Use your graph to write down all the possible combinations of songs.

7. A science laboratory is to be improved by adding some extra science kits. There are two types available. Each type A kit occupies 3 m^2 of floor space while type B occupies 0.5 m^2 of floor space. The available floor area does not exceed 40 m^2. Each type A kit costs K 12 000 and each of type B costs K 30 000. The purchasing

officer can only approve a maximum of K 270 000 for this purpose.

(a) If the school plans to buy x kits of type A and y of type B, form as many inequalities in x and y as possible to represent this information.

(b) Use graphical method to find the greatest number of kits of each type that the school may be able to buy.

8. A transport company uses two types of trucks, P and Q. Type P carries 200 bags of maize while type Q carries 300 bags of maize per trip. There are more than 12 000 bags to be moved, and the trucks are to make no more than 60 trips. Type Q trucks are to make at most twice the number of trips made by type P trucks.

(a) If x represents the number of trips made by type P trucks and y the number of trips made by type Q trucks, write down all the inequalities representing this information.

(b) The transporter makes a profit of K 1 000 per trip on type P truck and K 2 000 per trip on type Q truck. Write down an objective function for the profit. State the number of trips he should make in order to maximise his profit.

12 MENSURATION I: SURFACE AREA AND VOLUME OF SOLIDS

Success criteria

By the end of this topic, the student must be able to:

- *Calculate the surface area of cubes, cuboids ,pyramids, cones, spheres and prisms.*
- *Find volumes of 3-D shapes.*
- *Solve word problems involving surface areas and volumes.*

Introduction

In Forms 2, we constructed models of two common solids i.e. a cylinder and a pyramid. We used the nets of these solids to calculate their surface areas. We also calculated the volume of a cylinder. Solids such as these are called three dimensional figures.

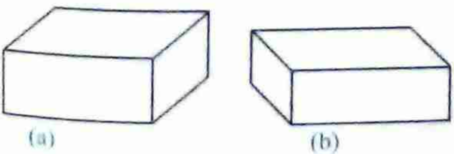
In this chapter, we are going to find surface areas and volumes of some more three dimensional figures. Sketching three dimensional figures is an essential part of this chapter.

Sketching solids

Many times, we find it more useful to work with a drawing of a solid than with a mental image of it. Professionals like carpenters, architects, engineers, etc., who are involved in making solid objects often start by making drawings of them. It is thus important to be able to make clear drawings of solids. Given below are the procedures of drawing sketches of some solids.

Parallels and verticals

Fig. 12.1 shows different ways of drawing a sketch of rectangular block in which the bottom face is horizontal. The block is drawn from five different viewpoints. This block represents a cuboid.



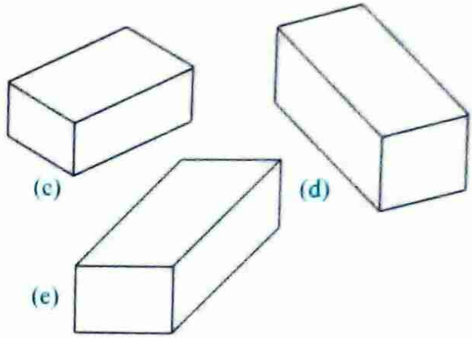


Fig. 12.1

There are certain details that are common with all the drawings in Fig. 12.1. Discover them by answering the following questions.

(i) What do you notice about the lines which represent vertical edges?

(ii) What do you notice about the lines which represent parallel edges?

(iii) All the faces of the block are rectangular. What shape do we draw them?

(iv) Certain faces of the block are parallel. How do we draw them?

You should have discovered the following.

1. Edges of a solid which are vertical are drawn straight up the page i.e. vertical.
2. Edges of a solid which are parallel are drawn parallel.
3. Faces of a solid which are parallel are drawn parallel.
4. Certain rectangular faces are drawn as parallelograms.

These are the rules most commonly used when drawing solids.

Drawing a cube of edge 2 cm

Activity 12.1

1. Draw a square with side 2 cm to represent the front face of the cube (Fig. 12.2)

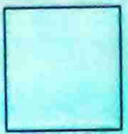


Fig. 12.2

2. Draw the four edges of the cube which are perpendicular to this, as shown in Fig. 12.3. These should be drawn parallel.

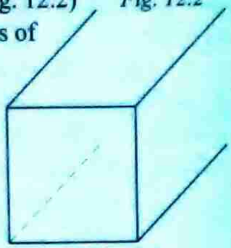


Fig. 12.3

In Fig. 12.3 these edges are long while in Fig. 12.4, they are 1 cm long. Why do we choose Fig. 12.4 rather than Fig. 12.3 when we complete the drawing?

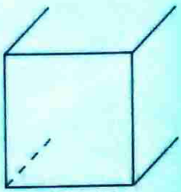


Fig. 12.4

Edges which are not visible may be drawn as broken lines.

3. Complete the figure by drawing the square which represents the rear face of the cube as in Fig. 12.5.

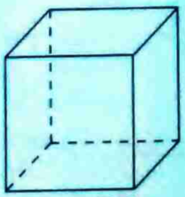


Fig. 12.5

What shape represents the base of the cube?

Which other faces are represented by the same shape?

How are parallel faces represented?

Drawing a square-based pyramid

Activity 12.2

The drawing is made easier by using squared paper.

1. Draw the base ABCD as a parallelogram. (Fig. 12.6(a)).
2. Draw the diagonals of the base,to find its centre. (Fig. 12.6(b))
3. Draw a vertical line (straight up the page) through this centre. (Fig. 12.6(c))

   Note: The diagonals of the base and the vertical line are only construction lines and should therefore be drawn lighter than the other lines of the drawing. They may also be drawn as broken lines.
4. Locate the vertex V as a point on the vertical line. Complete the drawing by joining V to A, B, C and D. Fig. 12.6(d) is the required pyramid.

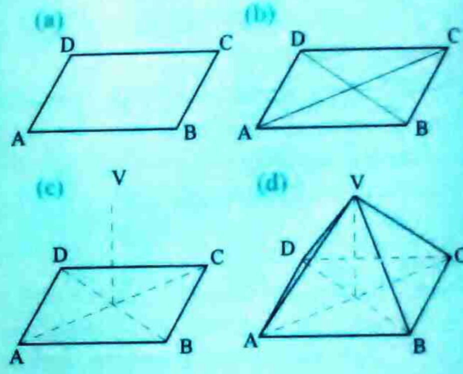


Fig. 12.6

Considering the work done on solids so far, have you observed that some of the faces of a solid must be drawn distorted? e.g.

(a) Some rectangles are drawn as parallelograms, some circles are drawn as ovals, etc. to give the figure the impression of a solid.

(b) Invisible edges are usually drawn using broken lines to indicate that they are hidden.

Exercise 12.1

1. Make drawings of the following.

   (a) A prism with a triangular cross-section.

   (b) A regular tetrahedron.

(c) A rectangular four-legged table.

(d) A right pyramid on a hexagonal base.

2. Fig. 12.7 shows a cuboid.

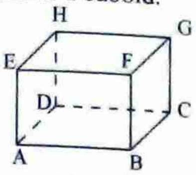


Fig. 12.7

(a) What shape is face EFGH in the drawing? What shape is it in the real solid?

(b) Name the faces which are drawn with their true shapes.

(c) Name the edges parallel to FG in the drawing. Are they parallel in the real solid?

(d) Line segment AC is a diagonal of the base. Name two other line segments in the drawing which appear to be of different length but which, in fact, have the same length as AC.

3. (a) Make a drawing of the wedge in Fig. 12.8 as though you were facing the triangular end PQR.

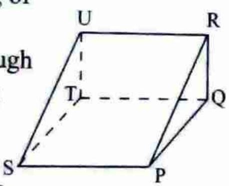


Fig. 12.8

(b) What is the name of this solid?

(c) What can you say about the lengths PU and SR?

(d) Name two line segments which appear shorter in the drawing than they actually are?

(e) Do any line segments appear longer than they actually are? If so, name them.

4. Make drawings of each of the solids in Fig. 12.9 from two different points of view.

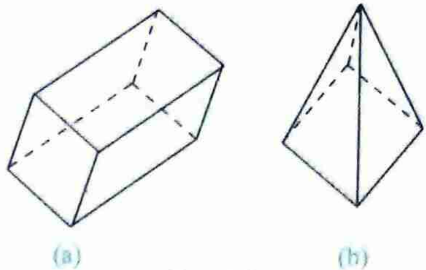


Fig. 12.9

Prisms

Now, consider the model of the cylinder you made in Form 2. It has a uniform thickness and there are two opposite faces that are identical. A solid of such description is called a prism, and each of the identical opposite faces is called a cross-section.

Fig. 12.10 shows some examples of prism, where the cross-section is shown by shading.

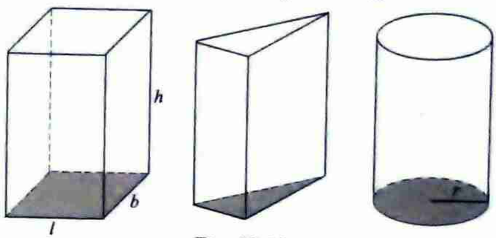


Fig. 12.10

Surface areas of prisms

We begin by identifying all the faces that compose the solid, calculate the individual areas and then find the total.

> The surface area of a prism is found as follows:
>
> 1. Find the area of the cross-section and multiply it by 2.
> 2. Find the area of each rectangular side face and add up these areas, or find the area of the curved surface in the case of a cylinder.
> 3. Add up the results to get the surface area of the prism.

The surface area of a solid is equal to the sum of the areas of all its faces.

Surface area of a cuboid

A cuboid of length l, width w and height h (Fig. 12.11), the surface area is

$SA = 2lw + 2lh + 2wh = 2(lw + lh + wh)$

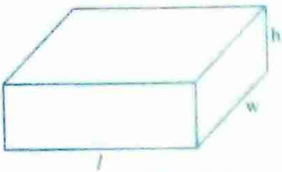


Fig. 12.11

Example 12.1 illustrate how to calculate surface areas of a cuboid from its net.

Example 12.1

The net of a cuboid consists of a series of rectangles. How many rectangles are there? What is the surface area of the cuboid if it measures 6 cm by 3 cm by 2 cm?

Solution

Fig 12.12 (a) shows a possible net of a cuboid which measures 6 cm by 3 cm by 2 cm.

It is composed of three pairs of rectangles i.e. 6 rectangles.

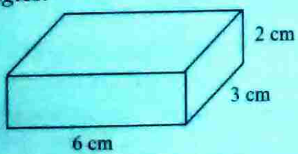


Fig. 12.12(b)

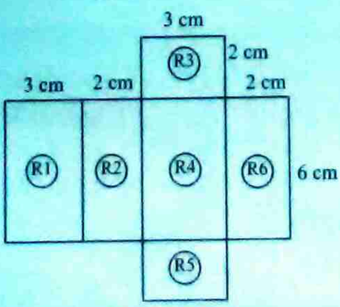


Fig 12.12 (a)

The surface area of the cuboid = sum of the areas of all the rectangles that comprise the net.

Area of rectangle R1 $= 6\ cm \times 3\ cm = 18\ cm^2$

Area of rectangle R2 $= 6\ cm \times 2\ cm = 12\ cm^2$

Area of rectangle R3 $= 3\ cm \times 2\ cm = 6\ cm^2$

Area of rectangle R4 $= 6\ cm \times 3\ cm = 18\ cm^2$

Area of rectangle R5 $= 3\ cm \times 2\ cm = 6\ cm^2$

Area of rectangle R6 $= 6\ cm \times 2\ cm = 12\ cm^2$

Surface area of the cuboid $= 72\ cm^2$

Fig. 12.12 (b) shows the sketch of cuboid described in this example.

Surface area of a cube

A cube has 6 identical faces. If the length of each face is l unit.

Area $= l \times l \times 6$

$= 6l^2$ sq. units

Example 12.2

Find the surface area of a cube of sides 5 cm.

Solution

Fig. 12.13 is the sketch of the cube.

The cube has 6 identical faces each of area $5 \times 5 = 25\ cm^2$

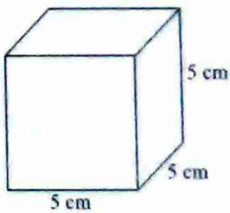


Fig. 12.13

$SA = 6l^2$

$= 6\ cm \times 5\ cm \times 5\ cm$

$= 150\ cm^2$

Surface area of triangular-based prism

Example 12.3

Find the surface area of the prism in Fig. 12.14.

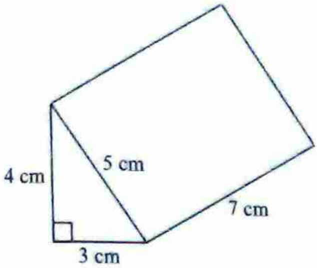


Fig. 12.14

Solution

The area of this prism is made up of 3 rectangles and 2 triangles. Fig. 12.15 is a sketch of the net of the prism.

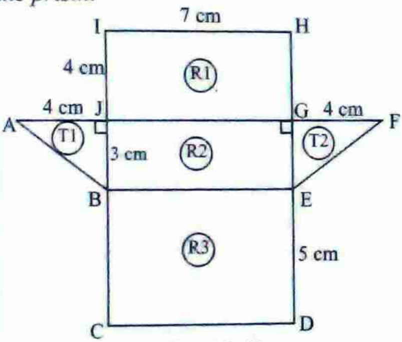


Fig. 12.15

Area of rectangle R1 $= 7\ cm \times 4\ cm = 28\ cm^2$

Area of rectangle R2 $= 7\ cm \times 3\ cm = 21\ cm^2$

Area of rectangle R3 $= 7\ cm \times 5\ cm = 35\ cm^2$

Area of triangle T1 $= \frac{1}{2} \times 3\ cm \times 4\ cm = 6\ cm^2$

Area of triangle T2 $= \frac{1}{2} \times 3\ cm \times 4\ cm = 6\ cm^2$

Surface area of the prism $= 96\ cm^2$

Note: A prism may have varied cross-sections e.g triangular, hexagonal, trapezium and so on.

Surface area of a cylinder

Example 12.4

A very thin sheet of metal is used to make a cylinder of radius 5 cm and height 14 cm. Using $\pi = 3.142$, find the total area of the sheet that is needed to make:

(a) a closed cylinder

(b) a cylinder that is open on one end.

Solution

(a) Radius of the circular face of the cylinder

$= 5\ cm$

$\therefore$ *area of a circular face*

$= \pi r^2 = 3.142 \times 5^2\ cm^2$

$\therefore$ *area of the two circular end faces*

$= 2 \times \pi r^2 = 2 \times 3.142 \times 5 \times 5\ cm^2$

Recall that when a cylinder is opened up to form its net, the curved surface becomes a rectangle of length $2\pi r$ (i.e. the circumference of the cylinder) and width h (the height of the cylinder)

Thus, area of curved surface $= 2\pi r \times h$

$= 2 \times 3.142 \times 5 \times 14\ cm^2$

Now, total area of the metal sheet

$= 2 \times 3.142 \times 5 \times 5 + 2 \times 3.142 \times 5 \times 14\ cm^2$

$= 2 \times 3.142 \times 5(5 + 14)\ cm^2$

$= 596.98\ cm^2$

(b) Surface area of open cylinder $= \pi r^2 + 2\pi rh$

$= 3.142 \times 5 \times 5 + 2 \times 3.142 \times 5 \times 14$

$= 3.142 \times 5(5 + 2 \times 14)$

$= 3.142 \times 5 \times 33$

$= 518.43\ cm^2$

Note: From the working in the example above, we are reminded that:

Total surface area of closed cylinder
$= 2\pi r^2 + 2\pi rh = 2\pi r(r + h)$

Example 12.5

A closed cylinder whose height is 18 cm has a radius 3.5 cm. Draw a net of the cylinder and use it to find the total surface area of the cylinder.

Solution

Fig. 12.16 (a) shows the diagram of the cylinder and Fig. 12.16 (b) shows its net.

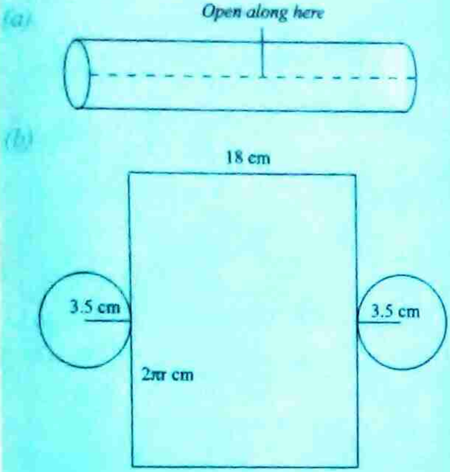


Fig. 12.16

The net consists of two end circles, and a rectangle measuring 22 cm by 18 cm.

Circumference of the end face (circle)

$= 2\pi r$

$= 2 \times \frac{22}{7} \times \frac{7}{2}$

$= 22$ *cm*

Total surface area $= 2 \times \pi r^2 + (2\pi r)h$

$= 2 \times \pi \times 3.5^2 + 22 \times 18$

$= 76.98 + 396$

$= 472.98\ cm^2$

Note that it is not always necessary to draw the net of the solid whose area is required. In our examples, we did the nets for the purposes of demonstration.

Exercise 12.2

1. Calculate the total surface areas of the solids in Fig. 12.17. (All measurements are in cm.)

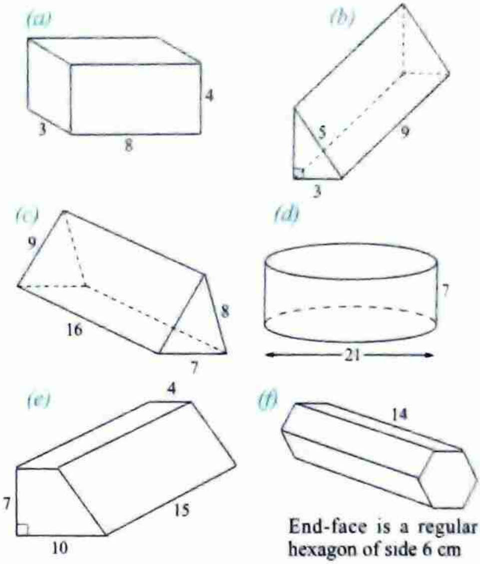


Fig. 12.17

2. Calculate the total surface area of a solid cylinder whose radius and height are 9 cm and 12 cm, respectively, leaving π in your answer.
3. A paper label just covers the curved surface of a cylindrical can of diameter 14 cm and height 10.5 cm. Calculate the area of the paper label.
4. The surface area of a cuboid is 586 cm^2. Given that its length and height are 12 cm and 7 cm respectively, find its breadth.
5. A closed cylinder whose height is 18 cm has a radius 3.5 cm. Calculate the total surface area of the cylinder.
6. The solid in Fig. 12.18 has a total surface area of 257 cm^2. Find its length, *l*.

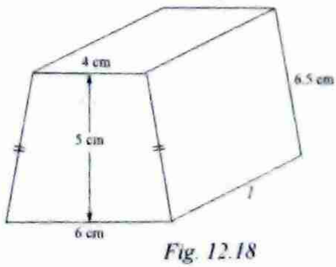


Fig. 12.18

Surface area of a pyramid

As you learned in Form 2, the surface area of a pyramid is obtained as the sum of the areas of the slant faces and the base.

Example 12.6

Find the surface area of the right pyramid shown in Fig. 12.19

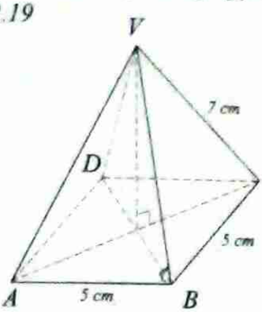


Fig. 12.19

Solution

Area of the base $= 5 \times 5 = 25\ cm^2$

Each slanting face is an isosceles triangle (Fig. 12.20).

Its height is $\sqrt{7^2 - (2.5)^2}$ cm

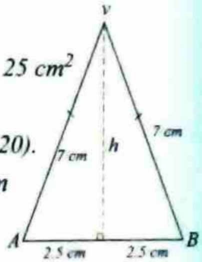


Fig. 12.20

Since $h^2 = 7^2 - 2.5^2$

$$h = \sqrt{7^2 - (2.5)^2}$$

Area of each slanting face

$$= \frac{1}{2} \times 5 \times \sqrt{7^2 - 2.5^2}$$
$$= \frac{1}{2} \times 5 \times \sqrt{49 - 6.25}$$
$$= \frac{1}{2} \times 5 \times \sqrt{42.75}$$
$$= \frac{1}{2} \times 5 \times 6.538$$
$$= 16.345\ cm^2$$

$\therefore$ *Total area of the slanting faces*

$$= 4 \times 16.345$$
$$= 65.38\ cm^2$$

Total surface area of the pyramid

$$= 25 + 65.38$$
$$= 90.4\ cm^2.\ (3\ s.f)$$

Note: The area of a slant face could also be found using Hero's formula.

Example 12.7

The base of a right pyramid is a square of sides 4 cm. The slant edges are all 6 cm long.

(a) *Draw and label a sketch of the solid.*

(b) *Draw a net of the pyramid.*

(c) *Use your net to find:*

(i) *the slant height of the pyramid.*

(ii) *the total surface area of the pyramid.*

Solution

(a) *Let the base of the pyramid be ABCD, and its vertex be V. Fig. 12.21 shows the sketch of the pyramid.*

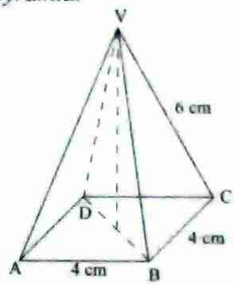


Fig. 12.21

(b) *Fig. 12.22 shows the net of the pyramid. (Drawn to a scale of 1 : 2).*

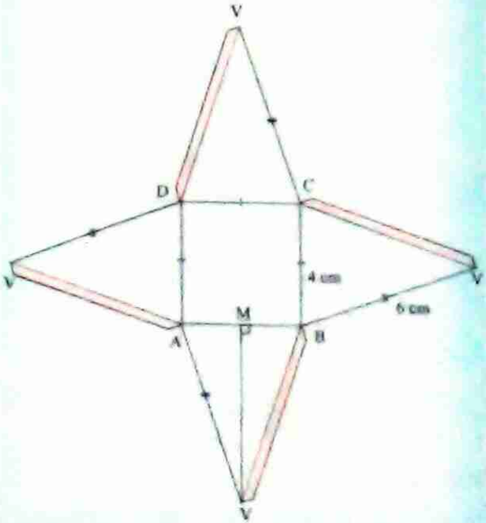


Fig. 12.22

(c) (i) Line segment VM on the net in part (b) represents the slant height of the pyramid.

$$VM = \sqrt{6^2 - 2^2}$$
$$= \sqrt{32} = 5.7 \text{ cm.}$$

(ii) Total surface area of the pyramid

$$= (\tfrac{1}{2} \times 4 \times 5.7) \times 4 + 4 \times 4$$
$$= 45.6 + 16$$
$$= 61.6 \text{ cm}^2.$$

Exercise 12.3

In questions 1 to 9 calculate the total surface area of the given right pyramids.

1. Height is 4 cm, square base of sides 6 cm.
2. Height 6 cm, rectangular base of sides 4 cm by 5 cm.
3. Height 6 cm; square base, side 9 cm.
4. Height 5 cm; rectangular base, 6 cm by 4 cm.
5. Height 16 cm; triangular base, sides 6 cm, 8 cm, 10 cm.
6. Slant edge 12 cm; rectangular base 6 cm by 8 cm.
7. Height 10 cm; equilateral triangular base, side 6 cm.
8. Slant edge 4cm; square base, side 4 cm.
9. Slant height 8 cm; square base, side 5.3 cm.
10. The base of a right pyramid is a square of sides 4 cm. The slant edges are all 6 cm long.
    (a) Draw and label a sketch of the pyramid.
    (b) Calculate the total surface area of the pyramid.

Surface area of a cone

Activity 12.3

1. Draw a circle, radius l (say $l = 10$ cm). At the centre O of the circle, measure an angle AOB (say of 150°) and form a sector as shown in Fig. 12.23 (a) (shaded part). Cut out the sector.
2. Fold the sector so that OA and OB coincide. This forms the curved surface of a cone as shown in Fig. 12.23(b).

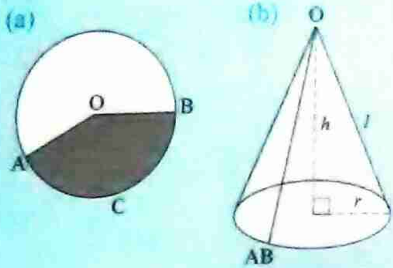


Fig. 12.23

Such a cone is said to be a **right circular cone** since the base is a circle and the vertex is vertically above the centre of the base. The word 'right' here means 'upright'.

3. What fraction of the circumference is arc ACB in Fig. 12.23(a)? Calculate the length of the arc.
4. What relationship is there between the length of arc ACB and the circumference of the base of the cone in Fig. 12.23(b)?
5. Using the relationship in 4, calculate the length of the radius of the base of the cone in Fig. 12.23(b).
6. Using your result in 5, calculate the ratio $\frac{r}{l}$.
7. Find, in terms of l the circumference of the circle in Fig. 12.23(a).

   Also, find in terms of, r the circumference of the base of the cone in Fig. 12.23(b). Hence, write down, in terms of l and r, the ratio

   $$\frac{\text{circumference of base of cone}}{\text{circumference of circle}}$$
8. What fraction of the area of the circle is the sector shaded in Fig. 12.17(a)? What is this fraction in terms of l and r?
9. Hence, what is the area of the curved surface of the cone? Give your answer in terms of l and r.

You should have found out that:

(a) Area of curved surface of the cone = area of sector used to model the cone.

(b) $\dfrac{\text{circumference of base of cone}}{\text{circumference of circle}}$

$= \dfrac{\text{area of sector}}{\text{area of circle}} = \dfrac{r}{l}$

Hence, area of sector $= \frac{r}{l}$ of the area of the circle $= \frac{r}{l}$ of $\pi l^2 = \pi r l$.

$\therefore$ Area of curved surface of a cone $= \pi r l$.
Hence, total surface area of a closed cone $= \pi r^2 + \pi r l$.

Note that by pythagoras' theorem, the values h, r and l (Fig 12.23) are connected by the relation $l^2 = h^2 + r^2$

Example 12.8

Find the surface area of a cone whose height and slant height are 4 cm and 5 cm respectively. (use $\pi = 3.142$).

Solution

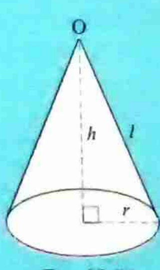


Fig. 12.24

Using Fig. 12.24

$l = 5$ cm, $h = 4$ cm

Since $l^2 = h^2 + r^2$, then

$r^2 = l^2 - h^2$

i.e. $r^2 = 5^2 - 4^2$

$\therefore r = 3$ cm

Area of curved surface $= \pi r l$.
$= 3.142 \times 3 \times 5$ cm^2
$= 47.13$ cm^2

Area of circular base $= \pi r^2$
$= 3.142 \times 3^2$ cm^2
$= 28.278$ cm^2

Hence, total surface area $= \pi r^2 + \pi r l$
$= 28.278 + 47.13$
$= 75.408$ cm^2
$= 75.41$ cm^2 (2 d.p)

Exercise 12.4

In this exercise, take $\pi = 3.142$.

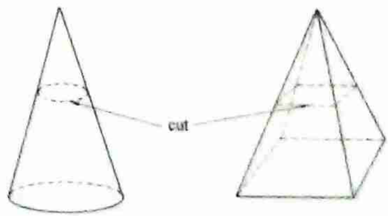
In Questions 1 to 7, find the surface area of the given cone.

1. Slant height 8 cm; base radius 6 cm.
2. Slant height 13 cm; height 5 cm.
3. Height 8 cm; base diameter 12 cm.
4. Height 8 cm; base radius 3 cm.
5. Slant height 8.5 cm; height 6.5 cm.
6. Slant height 9 cm; perimeter of base 12 cm.
7. Height 4 cm; area of base 15 cm^2.
8. A circle has a radius of 5 cm. The length of the arc of a sector of the circle is 6 cm. Find the:
   (a) area of the sector
   (b) surface area of the closed cone made using this sector.
9. The height of a conical tent is 3 m and the diameter of the base is 5 m. Find the area of the canvas used for making the tent.

Frustrums

Note: If a cone or pyramid is cut through a plane parallel to its base, the top part will be a smaller cone or pyramid. The bottom part is called a **frustrum of the cone** or **pyramid** (see Fig. 12.25).

(a)



(b)

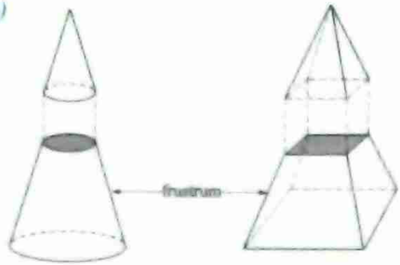


Fig. 12.25

How can we find the surface area of a frustrum?

1. Extend the slant height of a frustrum of a cone or the slant edges of the frustrum of a pyramid to obtain the solid from which the frustrum was cut.
2. Find the curved (side) surface area of the complete solid.
3. Find the curved surface area of the small cone that was cut off or the total area of the side faces of the small pyramid that was cut off.
4. Subtract the area obtained in (3) from that obtained in (2).
5. Find the area of the top and the bottom faces of the frustrum and add them to the result in (4).

Example 12.9

Fig. 12.26 shows a lampshade in the form of a frustrum whose top and bottom diameters are 18 cm and 27 cm. Find the area of the material used in making it, if the vertical height is 12 cm.

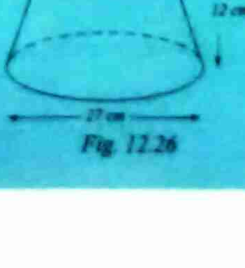


Fig. 12.26

Solution

Fig. 12.27 shows the complete cone of which the lampshade is a frustrum, which is open at both ends.

$BC = 9$ cm, $DE = 13.5$ cm, $BD = 12$ cm.

Let $AB = x$ cm

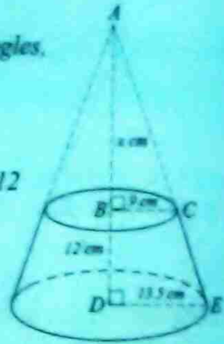


Fig. 12.27

By use of similar triangles,

$$\frac{AB}{BC} = \frac{AD}{DE}$$

$$\Rightarrow \quad \frac{x}{9} = \frac{x+12}{13.5}$$

$$\Rightarrow 13.5x = 9x + 9 \times 12$$

$$\Rightarrow \frac{27x}{2} = 9x + 108$$

$$\Rightarrow 27x = 18x + 216$$

$$9x = 216$$

$$x = 24$$

i.e. $AB = 24$ cm

By Pythagoras' theorem $AC^2 = AB^2 + BC^2$

$$AC = \sqrt{24^2 + 9^2}$$

$$\therefore AC = 25.63 \text{ cm}$$

Similarly, $AE^2 = 36^2 + 13.5^2$

$$\therefore AE = 38.45 \text{ cm}$$

Surface area of the lampshade (frustrum)

$$= (\pi \times 13.5 \times 38.45 - \pi \times 9 \times 25.63) \text{ cm}^2$$

$$= \pi(519.1 - 230.7) \text{cm}^2$$

$$= 3.142 \times 288.4 \text{ cm}^2$$

$$= 906.2 \text{ cm}^2$$

Example 12.10

Fig. 12.28 is a solid frustrum of a rectangular based pyramid. $AB = 12$ cm, $BC = 10$ cm, $EF = 6$ cm, $FG = 5$ cm, $FB = 8$ cm and vertical height is 6 cm. Calculate the total surface area of the frustrum.

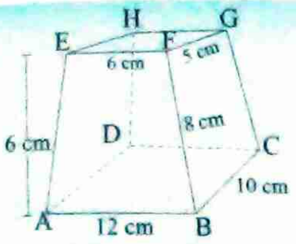


Fig. 12.28

Solution

Fig. 12.29 is the complete pyramid.

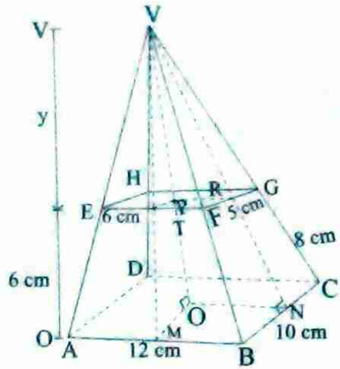


Fig. 12.29

Using similarity to find VF,

$\frac{AB}{EF} = \frac{VB}{VF} \quad \Rightarrow \frac{12}{6} = \frac{x+8}{x}$ *where* $VF = x$

$12x = 6x + 48$

$6x = 48$

$x = VF = 8 \text{ cm}$

Similarly, to find VO

$\frac{AB}{EF} = \frac{OV}{PV} \quad \Rightarrow \frac{12}{6} = \frac{6+y}{y}$

$12y = 36 + 6y$

$6y = 36 \Rightarrow y = 6$

$\therefore VO = 6 + 6 = 12 \text{ cm.}$

Surface area of the pyramid is given by the sum of areas of the Δs + base area.

VM is the height of Δs VAB and VDC

Using ΔVOM to find VM,

$VM = \sqrt{VO^2 + OM^2}$ *(By Pythagoras' theorem)*

$= \sqrt{12^2 + 5^2} = \sqrt{169} = 13 \text{ cm}$

VT = *height of Δs VEF and VHG*

$= \frac{12}{6} = \frac{13}{VT}$ *(Using similarity)*

$= 6.5 \text{ cm}$

VN is the height of Δs VBC and VAD

Using ΔVON to find VN,

$VN = \sqrt{VO^2 + ON^2}$ *(By Pythagoras' theorem)*

$= \sqrt{12^2 + 6^2} = \sqrt{144 + 36}$

$= \sqrt{180} = 13.416 \text{ cm}$

VR, height of Δs VFG and VEH is given by,

$\frac{VN}{VR} = \frac{VO}{VP} \Rightarrow \frac{13.416}{VR} = \frac{12}{6} \quad = 6.708 \text{ cm}$

S.A of ABFE and DCGH

$= \left[\left(\frac{1}{2} \times 12 \times 13\right) - \left(\frac{1}{2} \times 6 \times 6.5\right)\right] \times 2$

$= 58.5 \text{ cm}^2 \times 2 = 117 \text{ cm}^2$

S.A of BCGF and ADHE

$= \left[\left(\frac{1}{2} \times 10 \times 13.416\right) - \left(\frac{1}{2} \times 5 \times 6.708\right)\right] \times 2$

$= 50.31 \text{ cm}^2 \times 2 = 100.62 \text{ cm}^2$

S.A of top and bottom base

$= (6 \times 5) + (12 \times 10) = 150 \text{ cm}^2$

S.A of the frustrum

$= (117 + 100.62 + 150) \text{ cm}^2 = 367.62 \text{ cm}^2$

Exercise 12.5

1. A frustrum of a solid pyramid has a square base of side 8 cm and a square top of side 6 cm. The height between the two ends is 2 cm. Calculate the surface area of the frustrum.

2. A bucket is in the shape of a frustrum of a cone. Its diameters at the bottom and top are 30 cm and 36 cm respectively. Its depth is 20 cm, find the area of the sheet of material used in making the bucket.

3. Fig. 12.30 shows a frustrum of a solid cone. Find the surface area of the frustrum.

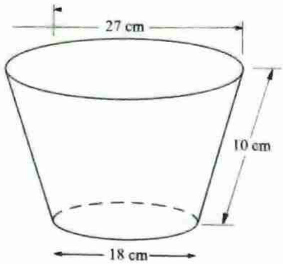


Fig. 12.30

4. A dustbin is in the shape of a frustrum of a right pyramid, with a square top of side 32 cm and square bottom of side 20 cm. If the dustbin is 24 cm deep, find the area of the sheet of material used to make it.

5. Fig. 12.31 shows a frustrum of a square based pyramid. The base ABCD is a square of sides 10 cm. The top A′B′C′D′ is a square of sides 4 cm and each of the slant edges of the frustrum is 5 cm long.

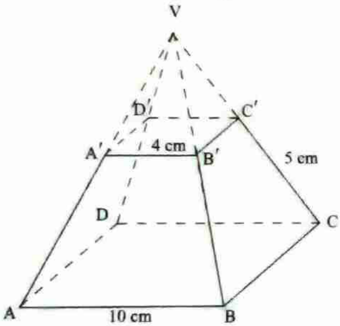


Fig. 12.31

Determine:

(a) the height of the frustrum.

(b) the total surface area of the frustrum.

6. PQRSV is a right pyramid on a rectangular base PQRS such that PQ = 16 cm and QR = 12 cm. Points M and O are the midpoints of QR and PR respectively, and VP = VQ = VR = VS = 18 cm.

(a) Sketch and label the pyramid.

(b) A frustrum is formed by cutting the top of the pyramid, 6 cm from V.

(i) What is the height of the frustrum?

(ii) Find the volume of the frustrum.

7. Fig. 12.32 shows the frustrum of a cone with dimensions as indicated in the figure. A and B are centres of the circular end faces. Find the total surface area of the frustrum.

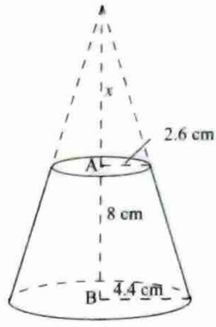


Fig. 12.32

Surface area of a sphere

Fig. 12.33 represents a solid sphere of radius r units.

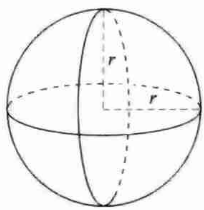


Fig. 12.33

The surface area of the sphere is given by:

Surface area $= 4\pi r^2$ square units.

Note that the derivation or proof of this formula is beyond the scope of this course, and so we shall just adopt it as it is.

Recall that half of a sphere is known as a **hemisphere**. Its surface area is given by:

Surface area of hemisphere

$=$ half the area of the sphere + area of flat surface

$= \frac{1}{2} \times 4\pi r^2 + \pi r^2 = 3\pi r^2$

Example 12.11

A solid hemisphere has a radius of 5.8 cm. Find its surface area.

Solution

Surface area of hemisphere $= 3\pi r^2$

$= 3 \times 3.142 \times 5.8 \times 5.8 \text{ cm}^2$

$= 317.1 \text{ cm}^2$ *(4 s.f.)*

Exercise 12.6

In this exercise, use $\pi = 3.142$ or $\frac{22}{7}$ depending on the measurements given.

1. Calculate the surface area of a sphere whose radius is:

   (a) 3.2 cm (b) 1.2 cm (c) 4.2 cm

2. Find the surface area of Fig. 12.34 shown.

   (a) (b)

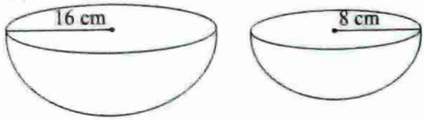


Fig. 12.34

3. Find the radius of a sphere whose surface area is:

   (a) 78.5 cm^2 (b) 181 cm^2

4. Find the total surface area of a solid hemisphere of diameter 10 cm.

5. A hollow sphere has an internal diameter of 18 cm and a thickness of 0.5 cm. Find the external surface area of the sphere.

Surface area of composite solids

A composite solid is a solid formed by two or more solids. For example, a cone and a hemisphere joined together will form a composite solid.

To get the total surface area of a composite solid, we get the surface area of all the faces exposed. Where the two solids are joined together is not an exposed surface and therefore we do not get the area of such a face.

Example 12.12

Fig. 12.35 shows a composite solid composed of a cuboid and a pyramid. Find its total surface if all the slanting heights of the pyramid are equal.

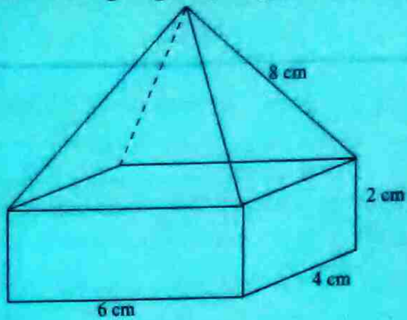


Fig. 12.35

Solution

To find the total surface area of the composite solid, we will find the surface area of the four triangular faces of the pyramid and the five exposed rectangular faces of the base cuboid.

Surface area of the pyramid part: Area of the two similar triangles + area of the two similar triangles.

Area of triangle $= \frac{1}{2} b \times h$ *(Fig. 12.36)*

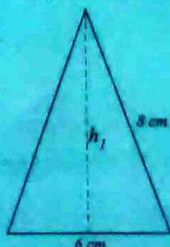


Fig. 12.36

$h_1^2 = (8^2 - 3^2)^2$ *(using pythagora's theorem to get* h_1.

$h_1^2 = \sqrt{8^2 - 3^2} \quad = \sqrt{64 - 9}$

$\therefore \quad = 7.4$ *cm*

Area $= (\frac{1}{2} \times 6 \times 7.4) \times 2$

$= 44.5 \text{ cm}^2$

Area of other two similar triangles $= \frac{1}{2} \times b \times h$ *(Fig. 12.37).*

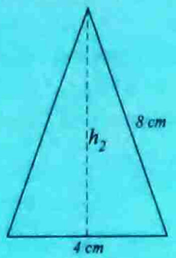


Fig. 12.37

We will use pythagora's theorem to get h_2.

$h_2 = \sqrt{8^2 - 2^2} \quad = \sqrt{64 - 4}$

$= 7.75$ *cm*

Area $= (\frac{1}{2} \times 4 \times 7.75) \times 2$

$= 30.98$ *cm*^{2}

Area of pyramid part $= 44.5$ *cm*^{2} $+ 30.98$ *cm*^{2}

$= 75.48$ *cm*^{2}

Area of the cuboid (base)

$= (2 \times 6)\ 2 + (2 \times 4)\ 2 + (6 \times 4)$

$= 24 + 16 + 24$

$= 64$ *cm*^{2}

Total surface area of the composite solid

$= 75.48^2 + 64$ *cm*^{2}

$= 139.48$ *cm*^{2}

Exercise 12.7

1. Fig. 12.38 shows a composite solid made of a cylinder and a hemisphere. Find its total surface area.

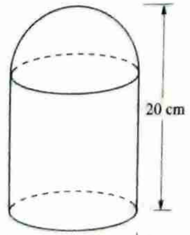


Fig. 12.38

2. Fig. 12.39 shows a conical flask whose external base diameter is 8 cm. The external diameter of the mouth is 2 cm. Assuming that the neck is cylindrical and ignoring the brim, find the external surface area of the flask.

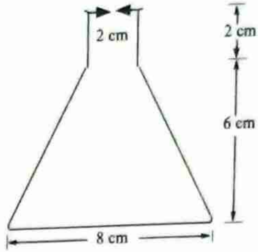


Fig. 12.39

3. A certain solid is a combination of a conical frustrum and a hemispherical bowl, as shown in Fig. 12.40. The dimensions are as indicated.

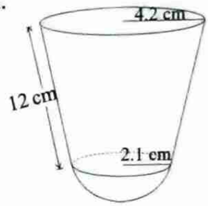


Fig. 12.40

   (a) Find the area of:
      (i) the circular top,
      (ii) the curved surface of the frustrum,
      (iii) the hemispherical surface.
   (b) Find the cost of painting the whole solid if 1 cm² costs K 16.

4. A solid is made up of a hemisphere mounted on a right cone both of radius 5 cm (Fig. 12.44). Calculate the surface area of the solid.

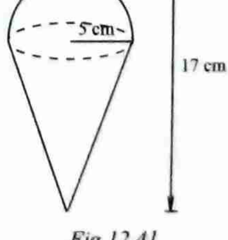


Fig 12.41

5. Fig. 12.42 shows solids which comprise of cubes surmounted with pyramids. Calculate the surface area of each solid.

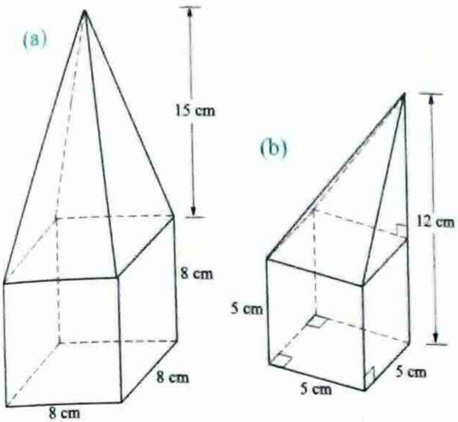


Fig. 12.42

Volume of a prism

In Form 2, we learnt how to find the volume of a cube, a cuboid and a cylinder. Volume of other prisms are calculated in a similar way.

For example,

Volume of a cube
$= l^3$, where l is the length of a side.

Volume of a cuboid
$= lbh$, where l is the length, b is the breadth and h is the height of the cuboid.

Volume of a cylinder
$= \pi r^2 h$, where r is the radius and h is the height of the cylinder.

We also learnt that a cuboid has a uniform cross-section (shaded in Fig. 12.43) of area bh.

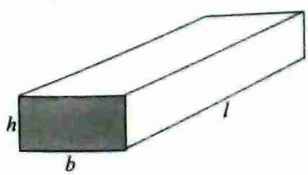


Fig. 12.43

Since the volume of a cuboid = lbh, then
volume of cuboid = $l \times$ area of cross-section.

We also learnt that the volumes of **solids which have uniform cross-sections** which are not rectangular are calculated in the same way as that of a cuboid, i.e.

Volume = cross-section area × length.

Do you recall that solids which have a uniform cross-section are known as **prisms**? The following are some examples. (Fig. 12.44).

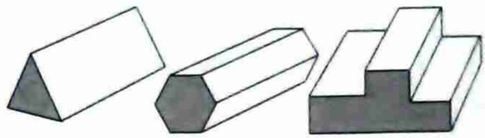


Fig. 12.44

In general, for any prism:

Volume
= area of uniform cross-section × length (or height) of the prism.

Example 12.13

A beam is shaped as in Fig. 12.45, with the measurements being in centimetres. If the length of the beam is 5 m and all angles are right angles, find its volume.

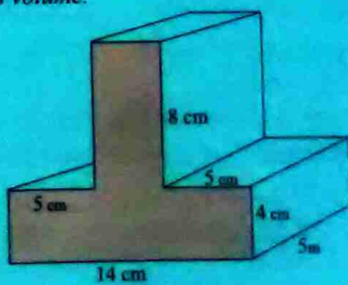


Fig. 12.45

Solution

The end face forms our uniform cross-section. The end-face may be taken as comprising of two rectangles.

$\therefore$ *area of end-face* $= (14 \times 4 + 8 \times 4)$

$= 88\ cm^2$

length of beam $= 5\ m = 500\ cm.$

$\therefore$ *volume of beam* $=$ *Base area* $\times$ *length*

$= 88 \times 500$

$= 44\ 000\ cm^3$

Exercise 12.8

1. Calculate the volume of the solids in Fig. 12.46.

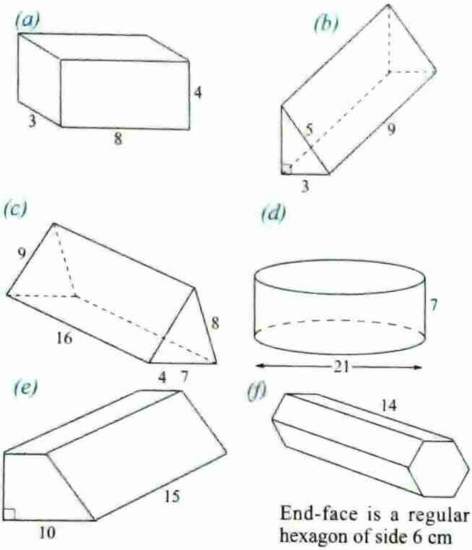


Fig. 12.46

2. A wooden beam has a rectangular cross-section measuring 21 cm by 16 cm and is 4 m long. Calculate the volume of the beam, giving your answer in cm^3 and in m^3.

3. A block of concrete is in the shape of a wedge whose triangular end-face is such that two of its sides are 16 cm and 19 cm long and the angle between them is 50°. If the block is 1 m long, find its volume.

4. Fig. 12.47 shows the shapes of steel beams often used in construction of buildings. Calculate the volume of a 6 m length of each beam, given that all dimensions are in centimetres and that all angles are right angles.

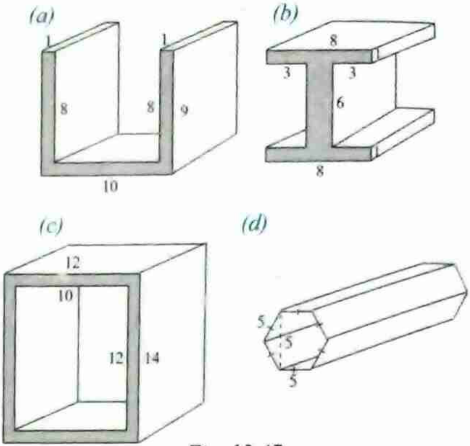


Fig. 12.47

5. A cylindrical container has a diameter of 14 cm and a height of 20 cm. Using $\pi = \frac{22}{7}$, find how many litres of liquid it holds when full.

6. 8 800 litres of diesel are poured into a cylindrical tank whose diameter is 4 m. Using $\pi = \frac{22}{7}$, find the depth of diesel in the tank.

7. The volume of the prism in Fig. 12.48 is 1 170 cm^3. Find its length.

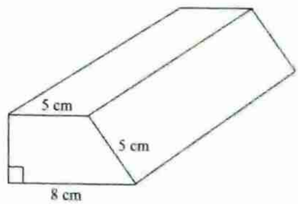


Fig. 12.48

8. The volume of a prism with a regular pentagonal base is 4 755 cm^3. If the prism is 1 m long, find, in centimetres, the distance from the centre of the base to any of its vertices.

Volume of a pyramid

By going through the following activity, let us find out how to determine the volume of a pyramid.

Activity 12.4

Work in groups of three. Every member in the group should construct a net of a square based pyramid as shown in Fig. 12.49.

All measurements are in centimetres.

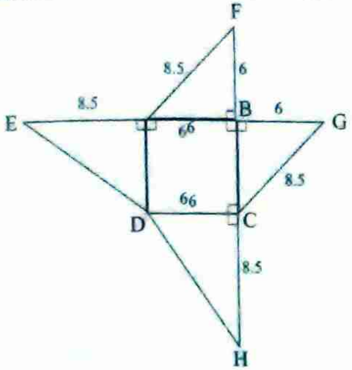


Fig. 12.49

Cut the net out and fold the triangles up to form the pyramid as shown in Fig. 12.50.

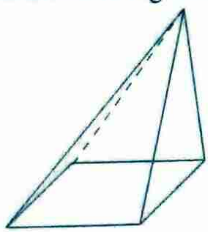


Fig. 12.50

Rotate and arrange the three pyramids to make a cube as shown in Fig. 12.51.

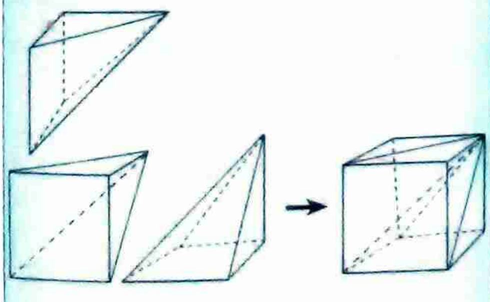


Fig. 12.51

What is the volume of the cube?

What is the volume of each pyramid?

You should have found in Activity 12.4 that:

volume of the pyramid

$= \frac{1}{3} \times$ volume of the cube

(since the three pyramids are identical)

Do you think this result is true for a cube of any size?

Since the volume of a cube

= base area × height (Ah),

then the volume (V) of a pyramid is given by

$V = \frac{1}{3} Ah$

where A = Area of the base, and

h = the vertical height of the pyramid

Example 12.14

Fig. 12.52 shows a pyramid on a rectangular base. Find its volume. The dimensions are in centimetres.

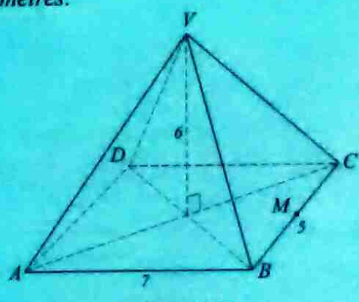


Fig. 12.52

Solution

Volume of a pyramid** = $\frac{1}{3}$ **(base area) × height

$\therefore \; V = \frac{1}{3}(7 \times 5) \times 6 \text{ cm}^3$

$= 70 \text{ cm}^3$

Note: If, in Fig. 12.52, we are given the slant edge, say VB, we need to use Pythagoras' theorem to find half the diagonal of the base and hence the vertical height. Similarly, given the slant height, say VM, we use Pythagoras' theorem to find the vertical height.

Exercise 12.9

In Questions 1 to 9, find the volume of the given right pyramid.

1. Height 4 cm; square base, side 6 cm.
2. Height 6 cm; square base of side 9 cm
3. Height 5 cm; rectangular base, 6 cm by 4 cm.
4. Height 6 cm; rectangular base, 4 cm by 5 cm.
5. Height 16 cm; triangular base, sides 6 cm, 8 cm and 10 cm.
6. Slant edge 12 cm; rectangular base, 6 cm by 8 cm.
7. Height 10 cm; equilateral triangle base, side 6 cm.
8. Slant edge 4 cm; square base, side 4 cm.
9. Slant height 8 cm; square base, side 5.3 cm.
10. A pyramid whose height is 8 cm has a volume of 48 cm^3. What is the area of its base?
11. A pyramid has a square base of side 5 cm. What is its height if its volume is 100 cm^3?
12. A square based pyramid has a height of 6 cm and a slant edge of 8 cm. What is its volume?

Volume of a cone

Since a cone may be regarded as a right pyramid with a circular base, its volume is given by:

Volume of cone
$= \frac{1}{3} \times \text{base area} \times \text{height} = \frac{1}{3}\pi r^2 h$

Example 12.15

Find the volume of a cone whose height and slant height are 4 cm and 5 cm, respectively. (Take $\pi = 3.142$).

Solution

Fig 12.53 is a sketch of a cone

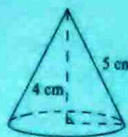


Fig. 12.53

$l = 5$ cm and $h = 4$ cm

Hence, $r^2 = l^2 - h^2 = 5^2 - 4^2$

$\therefore\ r = \sqrt{9} = 3$ cm

Volume of cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times 3.142 \times 3 \times 3 \times 4$

$= 37.7\ cm^3$

Exercise 12.10

In Questions 1 to 7, find the volume of the given cone. Take $\pi = 3.142$.

1. Height 4 cm; area of base 15 cm^2.
2. Slant height 8 cm; base radius 6 cm.
3. Slant height 13 cm; height 5 cm.
4. Height 8 cm; base diameter 12 cm.
5. Height 8 cm; base radius 3 cm.
6. Slant height 8.5 cm; height 6.5 cm.
7. Slant height 9 cm; perimeter of base 12 cm.
8. Find the height of a cone whose base radius is 3.72 cm and whose volume is 143 cm^3.
9. The area of a sector of a circle of radius 4 cm is 20 cm^2. What is the length of the arc of the sector? Find the radius and the volume of the cone made using this sector.
10. The height of a conical tent is 3 m and the diameter of the base is 5 m. Find the volume of the tent.
11. The capacity of a conical tank is 66 m^3. The area of the circular base on which it stands is 18 m^2. Find the area of the surface of the tank.

Volume of a frustrum

Earlier in the chapter, we learnt that a frustrum is the solid obtained when the top of a cone or pyramid is cut off along a plane parallel to the base.

How can we find the volume of a frustrum?

1. Extend the slant height of the frustrum of a cone or the slant edges of the frustrum of a pyramid to obtain the solid from which the frustrum was obtained.
2. Find the volume of the complete solid.
3. Find the volume of the small cone or pyramid that was cut off.
4. Find the difference between the two volumes to get the volume of the frustrum.

Example 12.16

A bucket that is in the shape of a frustrum of a cone has a top radius of 12 cm and a bottom radius of 8 cm. If it is 20 cm deep, find its capacity in litres.

Solution

Fig. 12.54 shows a sketch of the bucket with the cone from which the frustrum was obtained shown in dotted lines.

Recall that in similar triangles, the ratios of corresponding sides are equal.

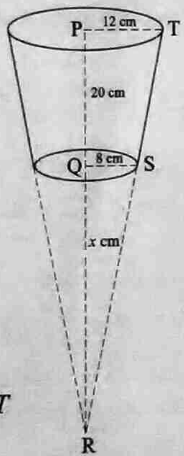


Fig. 12.54

Now, in Fig. 12.54, Δs PRT and QRS are similar.

$$\therefore \quad \frac{QR}{PR} = \frac{QS}{PT}$$

$$\text{i.e.} \quad \frac{x}{x+20} = \frac{8}{12}$$

$$\Rightarrow \quad 12x = 8x + 160$$

$$4x = 160$$

Hence, $\quad x = 40$

Volume of complete cone

$$= \tfrac{1}{3} \times \pi \times 12^2 \times 60 \text{ cm}^3$$

Volume of smaller cone

$$= \tfrac{1}{3} \times \pi \times 8^2 \times 40 \text{ cm}^3$$

$\therefore$ *Volume of bucket (the frustrum)*

$$= \left(\tfrac{1}{3} \times \pi \times 12^2 \times 60 - \tfrac{1}{3} \times \pi \times 8^2 \times 40\right) \text{cm}^3$$

$$= \tfrac{1}{3} \times \pi\,(8\,640 - 2\,560) \text{ cm}^3$$

$$= 6\,368 \text{ cm}^3 \text{ (4 s.f.)}$$

Hence, capacity of bucket = 6.368 litres

Example 12.17

Fig. 12.55 represents a frustrum of a right rectangular-based pyramid of side 12 cm by 8 cm. The top rectangular face measures 6 cm by 4 cm. The vertical height of the frustrum is 7 cm. Calculate the volume of the frustrum in cm³.

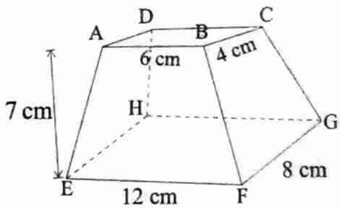


Fig. 12.55

Solution

We need to sketch the pyramid from which the frustrum was obtained. Fig. 12.56 is the required sketch.

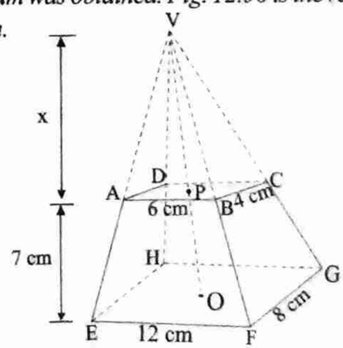


Fig. 12.56

Pyramid VABCD is similar to pyramid VEFGH.

$$\therefore \frac{EF}{AB} = \frac{OV}{PV} \Rightarrow \frac{12}{6} = \frac{x+7}{x}$$

$$12x = 6x + 42$$
$$\therefore x = 7\ cm$$
$$\therefore OV = h = x + 7 = 7 + 7 = 14\ cm$$

Volume of frustrum = volume of Pyramid VEFGH − volume of Pyramid VABCD

$$\therefore\ V = \left(\tfrac{1}{3} \times 12 \times 8 \times 14\right) - \left(\tfrac{1}{3} \times 6 \times 4 \times 7\right)$$
$$= 448\ cm^3 - 56\ cm^3 = 392\ cm^3$$

Exercise 12.11

1. A frustrum of a solid pyramid has a rectangular base of sides 8 cm and 6 cm and a rectangular top of side 4 cm and 3 cm. Given that the vertical height of the frustrum is 5 cm, calculate the volume of the frustrum.

2. A bucket, that is in the shape of a frustrum, has a diameter of 21 cm at the bottom and 28 cm at the top and its height is 40 cm. Find the capacity of the bucket.

3. Find the capacity of the conical flask in Fig. 12.57 shown.

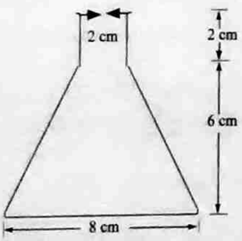


Fig. 12.57

4. Fig. 12.58 shows a conical salt-shaker which contains some salt. Calculate the volume of the salt required to fill the shaker.

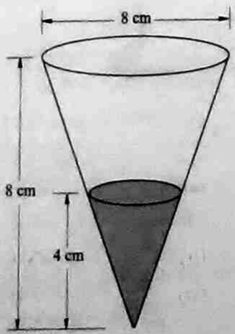


Fig. 12.58

5. Find the volume of the frustrum of a solid cone shown in Fig. 12.59.

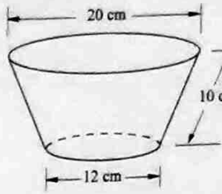


Fig. 12.59

Volume of a sphere

Let A represent a small square area on the surface of a sphere of radius r, centre O (Fig. 12.60).

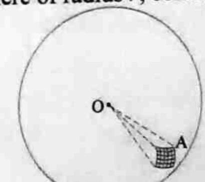


Fig. 12.60

If A is a very small area, we can look at it as almost flat.

The solid formed by joining the vertices of A to the centre O is a small 'pyramid'.

Volume of the small 'pyramid' = $\frac{1}{3}Ar$

Let there be such small 'pyramids' with base areas A_1, A_2, A_3, ...

Their volumes are $\frac{1}{3}A_1r$, $\frac{1}{3}A_2r$, $\frac{1}{3}A_3r$, ...

Total volume $= \frac{1}{3}r(A_1 + A_2 + A_3 + \ldots)$.

For the whole surface of the sphere, the sum of all the base areas is $4\pi r^2$, (by definition)

i.e., $A_1 + A_2 + A_3 + \ldots = 4\pi r^2$.

$\therefore\ \frac{1}{3}r(A_1 + A_2 \ldots) = \frac{1}{3}r \cdot 4\pi r^2 = \frac{4}{3}\pi r^3$

Hence, total volume V of a sphere $= \frac{1}{3}r \times 4\pi r^2$

Volume of a sphere $= \frac{4}{3}\pi r^3$

Example 12.18

Determine the radius of a sphere of volume 12 m^3. Use $\pi = 3.142$.

Solution

Volume $= \frac{4}{3}\pi r^3 \Rightarrow \frac{4}{3} \times 3.142 \times r^3 = 12$

$$r^3 = \frac{12 \times 3}{4 \times 3.142}$$

$$= \frac{36}{12.568} = 2.864\ 4$$

$$r = \sqrt[3]{2.864} = 1.42\ m$$

Example 12.19

A solid hemisphere of radius 5.8 cm has density 10.5 g/cm³.

Calculate the (a) volume,

(b) mass, in kg, of the solid

Solution

(a) *Volume of hemisphere*

$$= \frac{1}{2} \times \text{volume of sphere}$$

$$= \frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$$

$$= \frac{2}{3} \times 3.142 \times 5.8^3\ cm^3$$

$$= 408.7\ cm^3\ (4\ s.f.)$$

(b) $\text{Density} = \frac{\text{Mass}}{\text{Volume}}$

$\therefore$ $\text{Mass} = \text{Density} \times \text{Volume}$

$$= 10.5 g/cm^3 \times 408.7\ cm^3$$

$$= 4\ 291.35\ g$$

$$= 4.291\ kg\ \ (4\ s.f.)$$

Example 12.20

Find the surface area of a sphere whose volume is given as 1 000 cm³.

Solution

Let V = volume, r = radius, S = surface area

$$V = \frac{4}{3}\pi r^3 \Rightarrow 1\ 000 = \frac{4}{3}\pi r^3$$

$$r^3 = \frac{3\ 000}{4\pi} = 238.7$$

$$r = \sqrt[3]{23.87} = 6.2\ cm$$

$$S.A = 4\pi r^2$$

$$\Rightarrow S.A = 4\pi \times 2.88^2 = 483.11\ cm^2$$

Exercise 12.12

In this exercise, use $\pi = 3.142$ or $\frac{22}{7}$, depending on the measurements given.

1. Calculate the volume of a sphere whose radius is:

   (a) 3.2 cm (b) 1.2 cm (c) 4.2 cm

2. Find the radius of a sphere whose volume is

   (a) 73.58 cm³ (b) 463 cm³

3. Find the volume of a sphere whose surface area is:

   (a) 21.2 cm² (b) 972 cm²

4. Find the volume of a solid hemisphere of diameter 10 cm.

5. What is the mass of a solid gold hemisphere of diameter 4 cm if the density of gold is 19.3 g/cm³?

6. A hollow sphere has an internal diameter of 18 cm and a thickness of 0.5 cm. Find the volume of the material used in making the sphere.

7. Ten plasticine balls of diameter 2.8 cm are rolled together to form one large ball. What is the radius of the large ball?

8. Ten marbles, each of radius 1.5 cm, are placed in an empty beaker of capacity 150 ml. Water is then added to fill up the beaker. What volume of water is added?

> ⚠ CAUTION: Conserve water by ensuring that all taps are closed after use.

9. A solid cylinder has a radius of 18 cm and height 15 cm. A conical hole of radius r is drilled in the cylinder on one of the end faces. The conical hole is 12 cm deep. If the material removed from the hole is 9% of the volume of the cylinder, find:

   (a) the surface area of the hole,

   (b) the radius of a spherical ball made out of the material.

Volume of composite solids

It is a solid made of two or more distinct recognisable solids

Example 12.21

Fig. 12.61 is a model representing a storage container. The model has a total height of 15 cm. It is made up of a conical top, a hemispherical bottom and the middle part is cylindrical. The radius of the base of the cone and that of the hemisphere are 3 cm.

The cylindrical part is 8 cm high. Calculate the volume of the container.

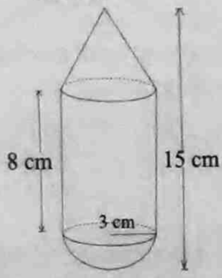


Fig. 12.61

Solution

Volume of container = Volume of hemisphere + Volume of cylinder + Volume of cone

$$\text{Volume of hemisphere} = \frac{2}{3}\pi r^3$$
$$= \frac{2}{3} \times 3.142 \times 3 \times 3 \times 3$$
$$= 56.556 \text{ cm}^3$$
$$\text{Volume of cylinder} = \pi r^2 h = 3.142 \times 3 \times 3 \times 8$$
$$= 226.224 \text{ cm}^3$$
$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times 3.142 \times 3 \times 3 \times 4$$
$$= 37.704 \text{ cm}^3$$
$$\text{Total volume} = (56.556 + 226.224 + 37.704) \text{ cm}^3$$
$$= 320.484 \text{ cm}^3$$

Exercise 12.13

1. A container in the shape of cylinder has a radius of 1.5 m. It contains water to a depth of 3.5 m. A solid plastic sphere of radius 0.8 m is placed inside the container and the new level of water rises to x cm. Calculate the value of x to the nearest unit.

2. A solid metal cylinder has a radius of 7 cm and height 4 cm. A conical cavity of base radius 7 cm is hollowed out such that the apex of the cone is 1 cm from the bottom of the cylinder as shown in Fig. 12.62. Calculate the volume and mass of the remaining solid if 1 cm^3 of the metal weighs 7.8 g.

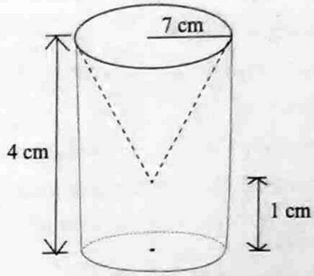


Fig. 12.62

3. A spherical glass vessel has a cylindrical neck 8 cm long, 2 cm diameter. The diameter of the spherical part is 8.5 cm. Neglecting any overlaps at the neck, determine the volume of water that the vessel can hold correct to 2 significant figures.

4. Fig. 12.63 below shows a model of a solid structure in the shape of a frustrum of a cone with a hemispherical cap. The diameter of the hemispherical caps is 75 cm which is equal to the diameter of the top part of the frustrum. The base radius of the frustrum is 25 cm and its slant height is 60 cm.

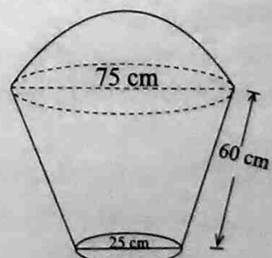


Fig. 12.63

Calculate:

(a) the surface area of the solid,

(b) the volume of the solid.

13 MENSURATION II: THREE DIMENSIONAL GEOMETRY

Success criteria

By the end of this topic, the student must be able to:

- *Calculate angles between lines, planes and angle between planes and lines in 3-D shapes.*
- *Calculate lengths of sides in 3-D shapes.*

Points, lines (edges) and planes

In Unit 12, we looked at some common solids. Since solids have length, area and volume (or since measurements on them can be taken in three directions), they are said to be **three dimensional**.

A vertex, on a solid, is a point where three or more edges meet while an edge is a line along which two faces meet.

In many solids, some faces, lines, and faces are parallel while others are not.

Fig. 13.1 shows a cuboid.

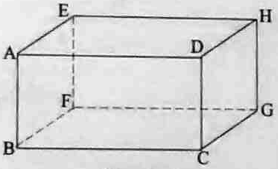


Fig. 13.1

(a) Name lines which are parallel to:
(i) BC (ii) EF (iii) BE (iv) ED

(b) Name a face which is parallel to face:
(i) ABCD (ii) BCGF (iii) CDHG

(c) Name lines which are parallel to the faces in (b).

(d) Name lines which are not parallel to and do not intersect with AD however much they are extended.

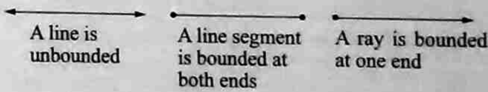
(e) Name all the lines which are perpendicular to face CDHG.

(f) Name the point of intersection between faces ADHE, ABCD and DCGH.

(g) Name the line where faces EFGH and BCGF intersect.

In geometry, a **point** is said to mark a particular position, and it therefore has no size. Since it has no length, breadth or thickness, it is said to be **dimensionless**.

A **line** is made of a set of points: It is straight and extends indefinitely in two directions as in Fig. 13.2(a). A **line segment** is a part of a line with two definite ends [Fig. 13.2(b)] and a **half-line** (or **ray**) is a part of a line with one definite end and extending indefinitely in one direction [Fig. 13.2(c)].

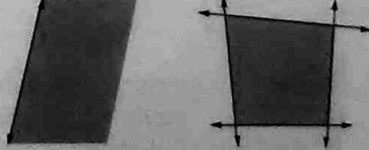


(a) *(b)* *(c)*

Fig. 13.2

A line has length but no breadth or thickness. It is therefore, said to be **one-dimensional.**

A **plane** is a set of points in a flat surface and extends indefinitely in all directions. When bounded by one straight line, it is called a **half-plane**. [Fig. 13.3(a)]. When bounded by any number of lines or curves, it is said to be a **region** [Fig. 13.3(b)]. However, a region does not necessarily have to be bounded all the way round.



A half-plane A region

(a) *(b)*

Fig. 13.3

A plane has length and breadth but no thickness, and it is therefore said to be **two dimensional**.

A **solid** occupies space. It has length, area and volume, and has a definite (i.e. fixed) shape. It is therefore said to be **three-dimensional**.

Note: The word 'line' is often loosely used when referring to a half-line or a line-segment. Likewise, the word 'plane' is also loosely used when referring to a half-plane or a region.

Skew lines

Two distinct parallel lines have no point of intersection. Is it always true that two lines which do not intersect are parallel?

In point (d) of the previous section, you found lines such as BF and CG which do not intersect with line AD however much they are produced. Yet, these lines are not parallel to AD. Such pairs of lines as AD and BF, AD and CG are called **skew lines**.

> **Skew lines** are lines in space which are not parallel and do not intersect however much they are produced.

Identification of a plane

Three points which are in a straight line are said to be **collinear**. But if any three points are non-collinear, they determine a plane.

However, four points are not necessarily coplanar (i.e. in the same plane), i.e. points A, B, C and D are not coplanar while points A, B and C are coplanar (Fig. 13.4). This is the reason why many tables, desks and chairs are not stable while a tripod is very stable.

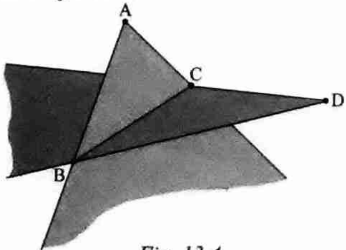


Fig. 13.4

Two parallel lines determine a plane [Fig. 13.5(a)]. Also, two intersecting lines determine a plane [Fig. 13.5(b)].

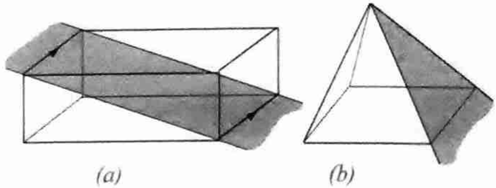


Fig. 13.5

Note:

1. A plane is determined by either
   (a) three non-collinear points, or
   (b) two parallel or intersecting lines.
2. Three collinear points in space determine an infinite number of planes.
3. A pair of skew lines does not determine a plane.
4. Two lines in space are either parallel, intersecting or skew.

In summary, given the cuboid in Fig. 13.6 below, we can identify faces, edges, vertices, planes and skew lines.

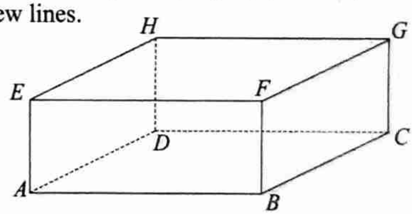


Fig. 13.6

- **Vertex** – A point where two or more lines or edges meet. E, F, G, H, A, B, C and D are the vertices of the cuboid in Fig. 13.6.
- **Edge** – A line along which two faces of a solid meet. AB, BC and FG are examples of edges in Fig. 13.6.
- **Face** – A flat surface of a solid. Faces form the outer surfaces of a solid. ABCD, BCGF and ABFE are examples of faces in Fig. 13.6.
- **Planes** – A flat surface enclosed by three or more lines. Faces of a solid are examples of planes. A plane can be an outer surface of a

solid or lie inside the solid. EFGH, ABCD, AFGD, BEHC and AGC are examples of planes of the cuboid in Fig. 13.6.

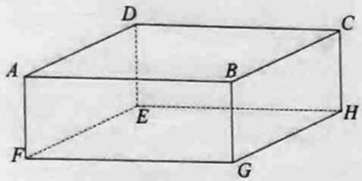
ABGH, BEHC, AFGD are diagonal planes. ABFE and BFGC are adjacent planes. AEHD and BFGC are parallel side planes.

- **Skew lines** – Are lines that are not parallel but do not meet because they lie on different planes. BF and HG, BC and DH, AG and BC, DC and BH are examples of skew lines in Fig. 13.6.

Example 13.1

Fig. 13.7(a) and (b) below shows a cuboid and a pyramid.

(a)



(b)

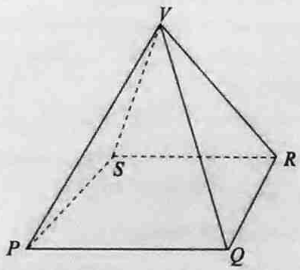


Fig. 13.7

In each case, in Fig. 13.7, name:

(i) the vertices *(ii) faces*

(iii) the edges *(iv) three planes*

(v) three pairs of skew lines.

Solution

(a) (i) Vertices are A, B, C, D, E, F, G and H.

(ii) Faces are ABCD, FGHE, ABGF, DCHE, BCHG and ADEF.

(iii) Edges are AB, BC, CD, DA, AF, FE, ED, EH, HC, HG, BG and FG.

(iv) Three planes are FEBC, ADHG and FDCG.

(v) Three pairs of skew lines are: BC and DE, FG and CH, FC and BG.

(b) (i) Vertices are P, Q, R, S and V.

(ii) Faces are PQRS, QVR, SRV, PSV and PQV.

(iii) Edges are PQ, PS, SR, RQ, VQ, VR, VS and VP

(iv) Three planes are PVQ, PQRS and VRS.

(v) Three pairs of skew lines are VS and RQ, VQ and SR, PS and VR.

Exercise 13.1

In this exercise, restrict your answers to the points, lines and planes named (not necessarily drawn) in Fig. 13.8. Fig. 13.8(a) is a model of a cuboid while Fig. 13.8(b) is a model of a right square-based pyramid. You may find framework models of these useful.

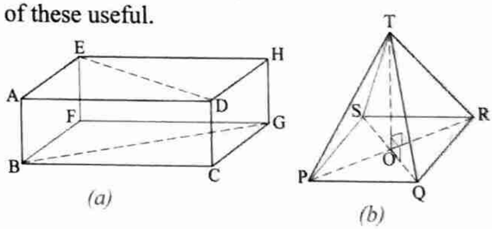


Fig. 13.8

1. Name all lines that are skew with:
   (a) line BG (b) line SQ

2. Name the lines which are parallel to plane ACGE.

3. Which of the following pairs of lines determine a plane?
   (a) AD and FG (b) BD and EF
   (c) BD and HF (d) BG and HG
   (e) FC and HG (f) PR and OT
   (g) ST and PR (h) PT and SP

4. Which of the following sets of points determine a plane?
   (a) A, B, F, E (b) B, C, H, F
   (c) A, E, G, C (d) A, F, D, H
   (e) B, D, F (f) C, E, H

(g) T, S, R, Q (h) P, O, T
(i) P, S, O, T (j) O, Q, T, R
(k) O, P, R, T (l) S, O, R

5. If a plane cut is made through AD to come out at F, does it come out through any other vertex? If so, which one? What solids result from this cutting?

6. If the top of the square-based pyramid [Fig. 12.43 (b)] is sawn off, with the cut being parallel to the base, what is the shape of the exposed surface? What name is given to the remaining lower part of the pyramid?

Projections and angles

Projections

Fig. 13.9 shows a pole standing on horizontal ground. It is kept vertical by three taut wires attached to the pole at S and to the ground at P, Q and R.

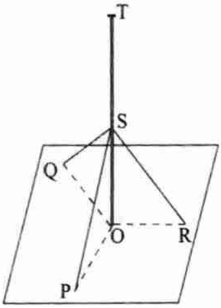


Fig. 13.9

The pole TO is perpendicular to the ground. It is said to meet the ground **normally** (i.e. perpendicularly).

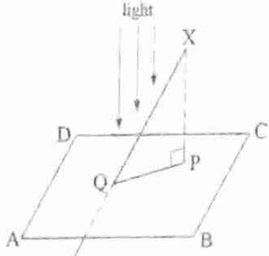
The wires SP, SQ and SR meet the ground **obliquely** (i.e. **not** perpendicularly).

Suppose that the sun is vertically overhead. What is the shadow of

(i) point T (ii) wire SP (iii) wire SQ
(iv) wire SR on the ground?

These are the respective **projections** of the point T and lines SP, SQ and SR on the horizontal ground.

> The **projection of a point onto a plane (or a line)** is the foot of the perpendicular from the point to the plane (or line). In Fig. 13.10, P is the projection of point X onto plane ABCD or onto line PQ, drawn on the plane.

> If plane ABCD is thought of as being lit from a perpendicular direction, the shadow that line QX forms on the plane (i.e. QP) is the **projection of the line onto the plane**.
>
> 
>
>
> *Fig. 13.10*

For example look at Fig. 13.11

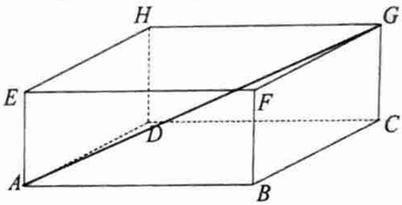


Fig. 13.11

The projection of:

(a) AG onto plane ABCD is AC.
(b) AG onto plane BCGF is BG.
(c) AG onto plane EFGH is EG.
(d) AG onto plane DCGH is DG.

Consider also Fig. 13.12. M,N,X and Y are midpoints to lines HG, DC, EF and AB respectively.

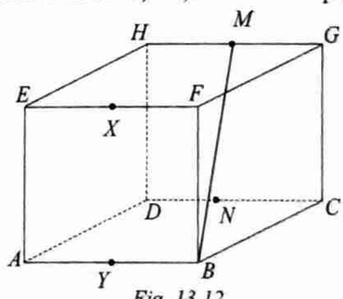


Fig. 13.12

Now, the projection of:

(a) BM onto plane BCGF is BG.
(b) BM onto plane ABCD is BN.

(c) BM onto plane ABFE is BX.

(d) BM onto plane EFGH is XM.

Angle between two lines - Identification

The lines m and n in Fig. 13.13 are co-planar and θ is the angle between them, i.e.

> The angle between two intersecting lines is defined as the acute angle formed at their point of intersection.

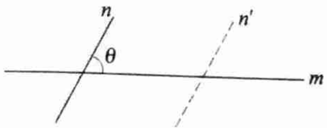


Fig. 13.13

Line n' is the image of line n under a translation and is also co-planar with line m. What can you say about θ and the acute angle between m and n'? This idea, that angles are unchanged by a translation, is used to define the angle between two skew lines.

> If the line a, in Fig. 13.14, is translated so that its image a' intersects b, then the angle between a and b is defined to be the angle between a' and b.

Fig. 13.14

For example, the angle between TV and PS, in Fig. 13.15, is ∠SPR. It is obtained by translating TV to PR. Alternatively, the angle between the two lines could be obtained by translating PS to TW or to UV. State the size of the angle between UV and PQ and between TU and SR.

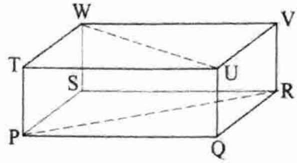


Fig. 13.15

Exercise 13.2

1. Referring to the framework ABCDV, in Fig. 13.16, state the projection of:
   (a) $\overline{AV}$ onto ABCD
   (b) $\overline{AV}$ onto BDV
   (c) $\overline{BV}$ onto ACV
   (d) $\overline{CV}$ onto AC

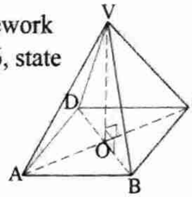


Fig. 13.16

2. Referring to the framework of the cube ABCDEFGH in Fig. 13.17, name the projection of:

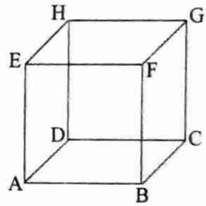


Fig. 13.17

   (a) $\overline{AG}$ onto ABCD
   (b) $\overline{AG}$ onto ADHE
   (c) $\overline{FD}$ onto EFGH
   (d) $\overline{BH}$ onto DCGH
   (e) $\overline{FH}$ onto ABCD
   (f) $\overline{ED}$ onto BCGF
   (g) $\overline{EG}$ onto HG
   (h) $\overline{EF}$ onto AB

3. Referring to Fig. 13.16, the angle between $\overline{VG}$ and $\overline{AB}$ is ∠VDC. Name the angle between:
   (a) $\overline{VC}$ and $\overline{BE}$
   (b) $\overline{VA}$ and $\overline{DC}$
   (c) $\overline{VA}$ and $\overline{BC}$
   (d) $\overline{VC}$ and $\overline{AD}$

4. Referring to Fig. 13.17, state the size of the angle between:
   (a) $\overline{AB}$ and $\overline{BE}$
   (b) $\overline{GH}$ and $\overline{BE}$
   (c) $\overline{AB}$ and $\overline{DH}$
   (d) $\overline{HD}$ and $\overline{FC}$
   (e) $\overline{EH}$ and $\overline{BC}$
   (f) $\overline{ED}$ and $\overline{BG}$
   (g) $\overline{AH}$ and $\overline{BG}$
   (h) $\overline{AH}$ and $\overline{BE}$

5. Name the lines, in Fig. 13.17, which are skew to HG and make an angle of 90° with HG.

6. Name the lines, in Fig. 13.17, which are skew to EG and make an angle of 45° with EG.

7. Fig. 13.18 shows a triangular prism.

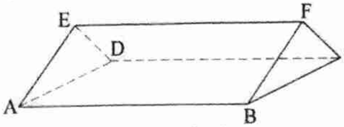


Fig. 13.18

(a) Name the lines which are skew to $\overline{AE}$.

(b) Name two angles which are equal to the angle between:

(i) $\overline{BF}$ and $\overline{AD}$ (ii) $\overline{BC}$ and $\overline{ED}$

(iii) $\overline{EF}$ and $\overline{AC}$

Angle between a line and a plane - Identification

Activity 13.1

Consider Fig. 13.19(a). It shows a piece of paper on which a point O is marked and five half-lines drawn from the point. The paper is placed on a flat surface and a piece of straight stiff wire stood vertically at O.

What angle does the wire make with each of the half-lines?

Answer this by taking your piece of paper and drawing half-lines and measuring the angles using a set square.

Now consider Fig. 13.19(b). The wire is now placed in a sloping position. What angle does the wire now make with each of the half-lines?

Again, answer this by placing your own wire in a similar position and measuring the angles.

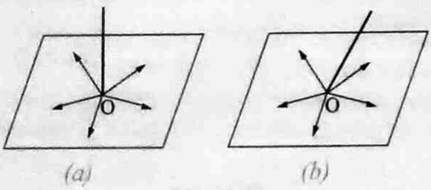


Fig. 13.19

You should observe that:

(a) For any half-line in the plane of the paper in Fig. 13.19(a), the angle is 90°. We say that the wire is perpendicular to the plane.

(b) In Fig. 13.19(b), the wire makes different angles with the different half-lines.

In the second case, we need to define which one is the angle between the wire and the plane.

Thinking of the plane as being lit from a perpendicular direction, the projection of the wire onto the plane is OP (Fig. 13.20).

We define the angle θ, between the wire and its projection onto the paper, as the angle between the wire and the plane.

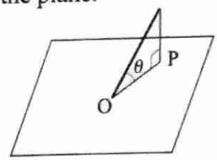


Fig. 13.20

Thus:

> The angle between a line and a plane is defined as the angle between the line and its projection onto the plane.

Exercise 13.3

1. Fig 13.21 represents a square-based pyramid. Name the angle between

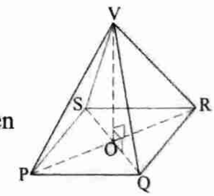


Fig. 13.21

(a) $\overline{VQ}$ and PQRS

(b) $\overline{VR}$ and VQS

(c) $\overline{VS}$ and VPR

2. Fig. 13.22 shows the framework of a cube ABCDEFGH. The angle between EC and plane CDHG is ECH. Name the angle between

(a) $\overline{EC}$ and ABCD

(b) $\overline{FD}$ and BCGF

(c) $\overline{CF}$ and ABCD

(d) $\overline{AH}$ and CDHG

(e) $\overline{EB}$ and BDHF

(f) $\overline{GB}$ and ACGE

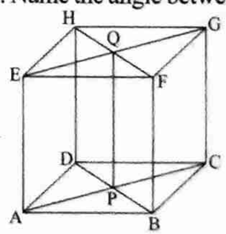


Fig. 13.22

3. Referring to Fig. 13.22, state the size of the angle between

(a) $\overline{GF}$ and CDHG (b) $\overline{EF}$ and ABCD

(c) $\overline{CH}$ and EFGH (d) $\overline{BF}$ and ACGE

Calculating angles

In three dimensional geometry, unknown lengths and angles can, in most cases, be determined by solving right-angled triangles. The following examples illustrate this. Note that it is quite helpful to sketch the triangles separately from the solids. The following examples illustrate how to calculate angle between two lines and between a line and a plane

Example 13.2

Fig. 13.23 shows a right pyramid ABCDV on a rectangular base. Each of the slant edges VA, VB, VC, VD is 13 cm. Given that AB = 12 cm and BC = 9 cm

(a) *State the projection of VA onto the base (plane ABCD).*

(b) *Calculate the length of the projection of the line VA to the base ABCD.*

(c) *Calculate the angle between the line VA and the plane ABCD.*

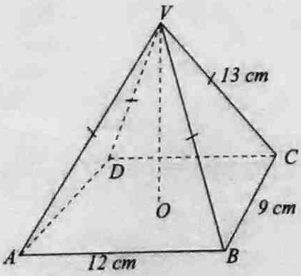


Fig. 13.23

Solution

Note: right pyramids are such that the vertex V is vertically above the centre of the base O.

(a) *OA is the projection of VA onto the base (plane ABC).*

(b) $AC = \sqrt{AB^2 + BC^2} = \sqrt{12^2 + 9^2}$

$= \sqrt{225} = 15 \text{ cm}$

$AO = \frac{AC}{2} = \frac{15}{2} = 7.5 \text{ cm}$

(c) *Angle between VA and plane ABCD is* $\angle VAO$

Let $\angle VAO = \square$ *(Fig. 13.24)*

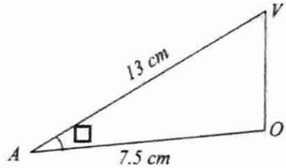


Fig. 13.24

$\cos \square = \frac{AO}{OV} = \frac{7.5}{13} = 0.576\,9$

$= 54.77^\circ$

Example 13.3

ABCDEFGH is a cuboid with dimensions as shown in Fig. 13.25.

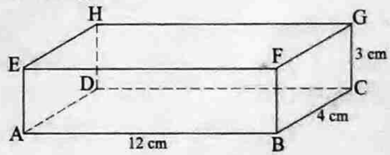


Fig. 13.25

Calculate:

(a) *the length of AG,*

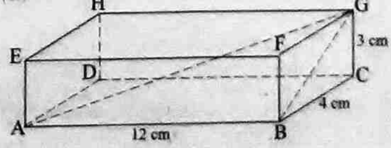
(b) *the angle that AG makes with plane BCGF,*

(c) *the shortest distance between line BF and plane ACG.*

Solution

(a) *In Fig. 13.26(a), the diagonal AG and its projection BG onto the plane BCGF are drawn in. Fig. 13.26(b) and (c) show the triangles used to calculate the length of AG.*

(a)



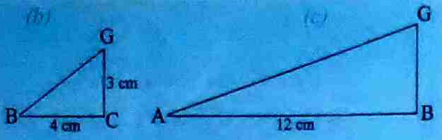


Fig. 13.26

In ΔBCG, $\angle BCG = 90°$.

$\therefore$ By Pythagoras' theorem,

$$BG^2 = BC^2 + CG^2 = 4^2 + 3^2 = 25$$

$$\therefore BG = \sqrt{25} = 5 \text{ cm.}$$

In ΔABG, $\angle ABG = 90°$

$\therefore$ By Pythagoras' theorem,

$$AG^2 = AB^2 + BG^2$$
$$= 12^2 + 5^2 = 169$$
$$\therefore AG = \sqrt{169} = 13 \text{ cm.}$$

We could also use the projection of AG on ABCD.

> Note that $AG^2 = AB^2 + BC^2 + CG^2$
> $= 12^2 + 4^2 + 3^2$
> $= 144 + 16 + 9$
> $= 169$
> $AG = \sqrt{169} = 13$ cm long

(b) The angle that AG makes with plane BCGF is $\angle AGB$ since BG is the projection of AG onto plane BCGF. ΔAGB is right angled at B (Fig. 13.27).

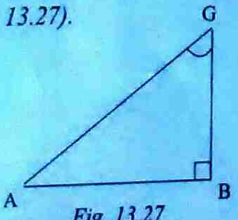


Fig. 13.27

$$\text{In } \Delta ABG,\ \tan A\square B = \frac{AB}{GB}$$

$$\text{Tan } A\square B = \frac{12}{5} = 2.4$$

$$\therefore \angle AGB = \tan^{-1} 2.4 = 67.38° \text{ (2 d.p.)}$$

(c)

> *The shortest distance between a line and a plane is the distance between a point on the line and its projection onto the plane.*

In Fig. 13.28(i), BP is the shortest distance between line BF and plane ACG.

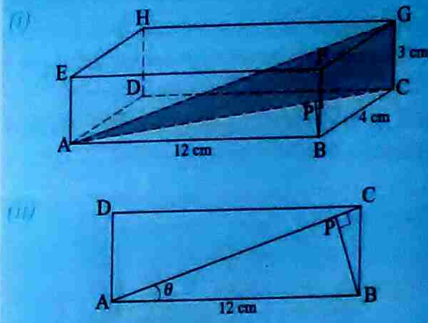


Fig. 13.28

In Fig. 13.28(ii)

$$AC^2 = AB^2 + BC^2$$
$$= 12^2 + 4^2 = 160$$
$$\therefore AC = \sqrt{160} \text{ cm}$$

$$\text{In } \Delta ABP,\ \sin\theta = \frac{BP}{AB} = \frac{BP}{12}$$

$$\text{In } \Delta ABC,\ \sin\theta = \frac{BC}{AC} = \frac{4}{\sqrt{160}}$$

$$\text{Hence, } \frac{BP}{12} = \frac{4}{\sqrt{160}}$$

$$\therefore BP = \frac{4 \times 12}{\sqrt{160}} = \frac{48}{\sqrt{16 \times 10}}$$

$$= \frac{48}{4\sqrt{10}} = \frac{12}{\sqrt{10}}$$

$$= \frac{12\sqrt{10}}{10} \text{ (rationalising the denominator)}$$

$$= \frac{12 \times 3.162}{10}$$

$$= \frac{37.944}{10}$$

$$= 3.79 \text{ cm (2 d.p).}$$

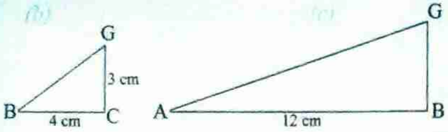


Fig. 13.26

In ΔBCG, $\angle BCG = 90°$.

$\therefore$ By Pythagoras' theorem,

$$BG^2 = BC^2 + CG^2 = 4^2 + 3^2 = 25$$

$$\therefore\ BG = \sqrt{25} = 5 \text{ cm}.$$

In ΔABG, $\angle ABG = 90°$

$\therefore$ By Pythagoras' theorem,

$$AG^2 = AB^2 + BG^2$$
$$= 12^2 + 5^2 = 169$$
$$\therefore\ AG = \sqrt{169} = 13 \text{ cm}.$$

We could also use the projection of AG on ABCD.

> Note that $AG^2 = AB^2 + BC^2 + CG^2$
> $= 12^2 + 4^2 + 3^2$
> $= 144 + 16 + 9$
> $= 169$
> $AG = \sqrt{169} = 13$ cm long

(b) The angle that AG makes with plane BCGF is $\angle AGB$ since BG is the projection of AG onto plane BCGF. ΔAGB is right angled at B (Fig. 13.27).

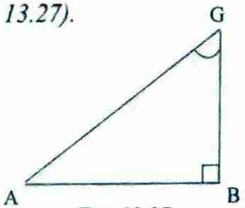


Fig. 13.27

$$\text{In } \Delta ABG,\ \tan A\Box B = \frac{AB}{GB}$$

$$\text{Tan } A\Box B = \frac{12}{5} = 2.4$$

$$\therefore\ \angle AGB = \tan^{-1} 2.4 = 67.38° \text{ (2 d.p.)}$$

> (c) The shortest distance between a line and a plane is the distance between a point on the line and its projection onto the plane.

In Fig. 13.28(i), BP is the shortest distance between line BF and plane ACG.

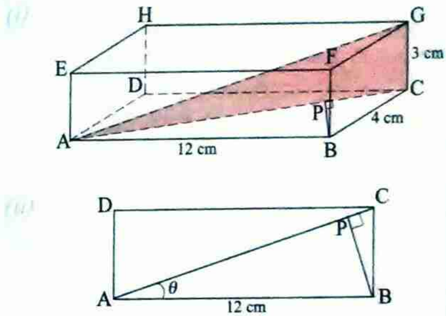


Fig. 13.28

In Fig. 13.28(ii)

$$AC^2 = AB^2 + BC^2$$
$$= 12^2 + 4^2 = 160$$
$$\therefore\ AC = \sqrt{160} \text{ cm}$$

$$\text{In } \Delta ABP,\ \sin\theta = \frac{BP}{AB} = \frac{BP}{12}$$

$$\text{In } \Delta ABC,\ \sin\theta = \frac{BC}{AC} = \frac{4}{\sqrt{160}}$$

$$\text{Hence, } \frac{BP}{12} = \frac{4}{\sqrt{160}}$$

$$\therefore\ BP = \frac{4 \times 12}{\sqrt{160}} = \frac{48}{\sqrt{16 \times 10}}$$

$$= \frac{48}{4\sqrt{10}} = \frac{12}{\sqrt{10}}$$

$$= \frac{12\sqrt{10}}{10} \text{ (rationalising the denominator)}$$

$$= \frac{12 \times 3.162}{10}$$

$$= \frac{37.944}{10}$$

$$= 3.79 \text{ cm (2 d.p).}$$

Exercise 13.4

1. Fig. 13.29 shows a prism whose cross-section is a right-angled triangle. Find the angle between EB and plane ABCD.

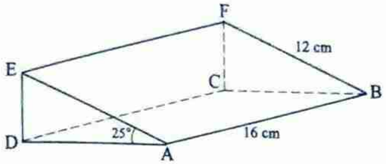


Fig. 13.29

2. Fig. 13.30 shows a cuboid. Given that M is the midpoint of EH, find the inclination of BM to ADHE.

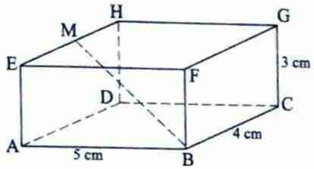


Fig. 13.30

3. Fig. 13.31 shows a wedge-shaped cutting blade whose cross-section is a right-angled triangle.

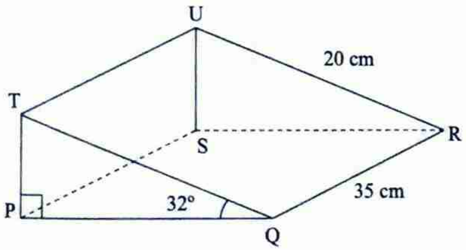


Fig. 13.31

Calculate:

(a) the angle between UQ and the plane PSUT.

(b) the angle between UQ and the plane PQRS.

4. Fig. 13.32 shows a pyramid VABCD with a rectangular base ABCD. The vertex V is vertically above point A. VA = 14 cm, AB = 10 cm, BC = 6 cm. M is the midpoint of BC.

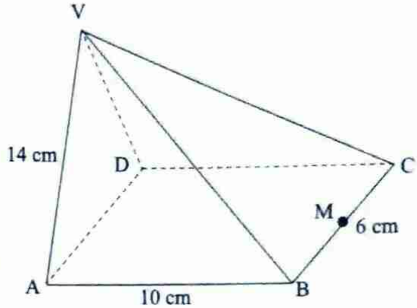


Fig. 13.32

Calculate:

(a) the length of VM,

(b) the angle between VM and the base ABCD,

(c) the angle between VC and the base ABCD and

(d) the volume of the pyramid.

5. Fig. 13.33 shows a cuboid with AB = 12 cm, BC = 9 cm and CH = 6 cm

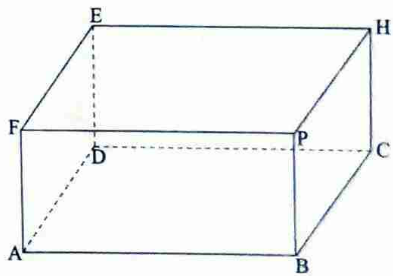


Fig. 13.33

Calculate:

(a) the length BE,

(b) the angle between BE and the base ABCD,

(c) the angle between BE and the plane DEHC and

(d) the angle between BE and the plane EFAD.

Angle between two planes - Identification

Activity 13.2

Open a book and stand it on a flat surface, as shown in Fig. 13.34. Using a light pencil, mark the indicated points and draw the lines shown on the two facing pages. (You will need to rub them out when you are through with this activity).

The angle between planes PABQ and PCDQ is the angle through which you would turn PABQ to fit onto PCDQ.

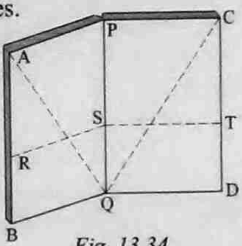


Fig. 13.34

(a) Is ∠BQD equal to
  (i) ∠APC, (ii) ∠AQC?

(b) What angle do BQ and DQ make with PQ?

(c) How does ∠BSD compare in size with ∠BQD?

(d) RS and TS are both perpendicular to PQ. How does ∠RST compare in size with ∠APC?

(e) Which of the angles APC, RST and BQD is equal to the angle between planes PABQ and PCDQ?

We define the angle between two planes as follows:

> The angle between two planes is the angle between any two lines, one in each plane, which meet on and at right angles to the line of intersection of the planes.

In Fig. 13.35, lines OX and OY, drawn at point O, are perpendicular to the line of intersection, PQ, of the planes ABCD and EFGH. Hence, the angle between the planes ABCD and EFGH is ∠XOY = θ.

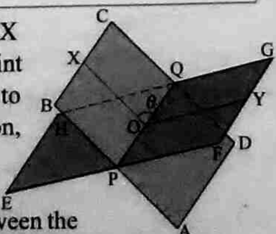


Fig. 13.35

Exercise 13.5

1. In Fig. 13.36, name the line of intersection between the planes:

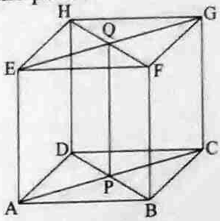


Fig. 13.36

(a) ABCD and ADHE
(b) ABFE and BCGF
(c) CDHG and ACGE
(d) ACGE and BDHF

2. In Fig. 13.37, name the line of intersection and state the size of the angle between planes

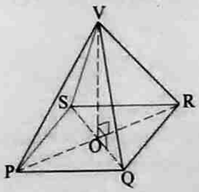


Fig. 13.37

(a) PQRS and QSV
(b) PRV and QSV

3. State the size of the angle between each of the pairs of planes in Question 1.

4. Referring to Fig. 13.37, name the angle between PQRS and QRV.

Calculating angle between two planes

The following examples shows how to find and calculate angles between two planes.

Example 13.4

A rectangular-based pyramid with vertex V is such that each of the edges VA, VB, VC, VD is 26 cm long. The dimensions of the base are AB = CD = 16 cm and AD = BC = 12 cm.

Calculate:

(a) *the height VO of the pyramid,*

(b) *the angle between the edges AD and VC,*

(c) *the angle between the base and a slant edge,*

(d) *the angle between the base and face VBC.*

Solution

Fig. 13.38 is an illustration of the pyramid.

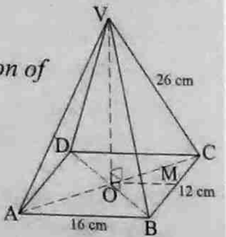


Fig. 13.38

(a) Fig. 13.39 shows the triangles used to calculate VO.

In ΔABC,

$$AC^2 = AB^2 + BC^2 \quad \text{(Pythagoras' theorem)}$$
$$= 16^2 + 12^2 = 400$$
$$\therefore\ AC = \sqrt{400} = 20 \text{ cm}$$
$$OC = \tfrac{1}{2}AC = 10 \text{ cm.}$$

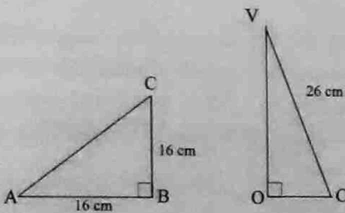


Fig. 13.39

In ΔVOC,

$$VO^2 = VC^2 - OC^2 \quad \text{(Pythagoras' theorem)}$$
$$= 26^2 - 10^2 = 576$$
$$\therefore\ VO = \sqrt{576} = 24 \text{ cm.}$$

(b) AD and VC are skew lines. We therefore translate AD to BC to form the required angle VCB.

Fig. 13.40 shows the triangle used to calculate $\angle VCB$.

In ΔVMC, where M is the midpoint of BC,

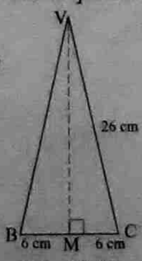


Fig. 13.40

$$\cos \angle VCB = \frac{6}{26} = 0.2308$$
$$\therefore\ \angle VCB = \cos^{-1} 0.2308 = 76.66°.$$

(c) Since VO is perpendicular to the base, VCO is one of the angles between the base and a slant edge.

In ΔVCO (Fig. 13.42),

$$\tan \angle VCO = \frac{VO}{CO} = \frac{24}{10} = 2.4$$
$$\therefore\ \angle VCO = \tan^{-1} 2.4 = 67.38°.$$

(d) BC is the line of intersection between the two planes, and M is the mid-point of BC. VM and OM are lines, in the planes, which are both perpendicular to BC. Thus, $\angle VMO$ is the angle between the base and face VBC.

Fig. 13.41 shows the triangle used to find $\angle VMO$.

In ΔVMO,

$$\tan \angle VMO = \frac{24}{8} = 3$$
$$\therefore\ \angle VMO = 71.57°.$$

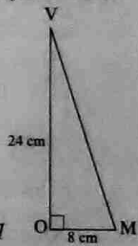


Fig. 13.41

Example 13.5

Calculate the angle between the faces VAB and VBC of the pyramid in Example 13.4.

Solution

VB is the line of intersection of the two planes.

AP and CP are lines, on the planes, that are both perpendicular to VB (Fig. 13.42). Thus, $\angle APC$ is the angle between faces VAB and VBC.

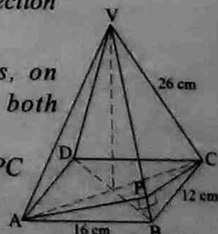


Fig. 13.42

Fig. 13.43 shows the triangles used to calculate $\angle APC$.

In ΔVAB, N is the mid-point of AB.

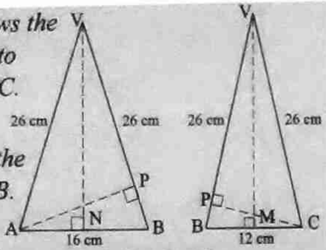


Fig. 13.43

Thus,

$VN^2 = VB^2 - NB^2$

$= 26^2 - 8^2 = 612$

$\therefore\ VN = \sqrt{612}$

$\therefore$ *Area of* $\Delta VAB = 8\sqrt{612} = \frac{1}{2} \times 26 \times AP$

$$\therefore\ AP = \frac{2 \times 8\sqrt{612}}{26}$$

$= 15.22$ *cm.*

In ΔVBC, M is the mid-point of BC.

$VM^2 = VC^2 - MC^2$

$= 26^2 - 6^2 = 640$

$\therefore\ VM = \sqrt{640}$

$\therefore$ *Area of* $\Delta VBC = 6\sqrt{640} = \frac{1}{2} \times 26 \times CP$

$$\therefore\ CP = \frac{2 \times 6 \times \sqrt{640}}{26}$$

$= 11.68$ *cm.*

Fig. 13.44 shows the triangle used in finding $\angle APC$. Since the triangle is not right-angled, we find $\angle APC$ using the cosine rule.

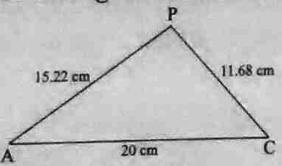


Fig. 13.44

Thus,

$$\cos \angle APC = \frac{PC^2 + PA^2 - AC^2}{2 \times PC \times PA}$$

$$= \frac{11.68^2 + 15.22^2 - 20^2}{2 \times 11.68 \times 15.22}$$

$$= \frac{136.4 + 231.6 - 400}{355.5}$$

$$= \frac{-32}{355.5} = {}^{-}0.090\,0$$

$\therefore\ \angle APC = \cos^{-1} {}^{-}0.900 = 1\,95.16°$

Thus, the angle between faces VAB and VBC is 195.16°.

Exercise 13.6

1. ABCDEFGH is a cuboid. The base ABCD is such that AB = DC = 8 cm and AD = BC = 6 cm. The height of the cuboid is 4 cm. Calculate the angle between:
   (a) AG and plane ABCD,
   (b) EC and plane ADHE,
   (c) ED and BG,
   (d) AG and EF,
   (e) planes EFGH and EBCH.

2. The slant edges VA, VB, VC, VD of a square-based pyramid are each 20 cm long. The base is of side 16 cm. Calculate:
   (a) the height VN of the pyramid,
   (b) the angle between a slant edge and the base,
   (c) the angle between a sloping face and the base.

3. Fig. 13.45 shows a rectangle ABCD on horizontal ground with AB = 4 m and BC = 3 m. AP is a vertical pole to which three taut wires PB, PC and PD are attached. Calculate:
   (a) the angle that PC makes with the ground,
   (b) the angle between planes PBD and ABCD.

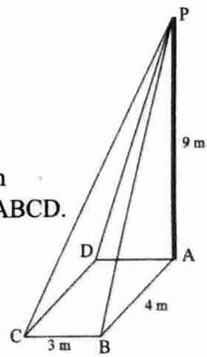


Fig. 13.45

4. The edges of a regular tetrahedron are all equal in length. Find:
   (a) the angle between an edge and a face,
   (b) the angle between two faces of the tetrahedron.
   (Hint: Let the length of an edge be $2l$).

5. VABCD is a square-based pyramid. The base is of side 12 cm and each slant edge is 18 cm long. Calculate the angle between the faces VAB and VBC.

6. Fig. 13.46, shows a cuboid, calculate:

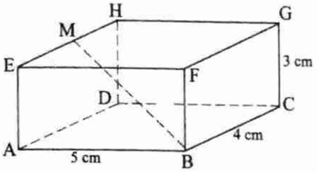


Fig. 13.46

(a) the length of the diagonal of the cuboid,

(b) the inclination of plane CDEF to plane CDHG,

(c) the shortest distance between AE and plane BDHF.

7. A right pyramid ABCDV on a square base of sides 8 cm has a vertical height of 12 cm. Draw a sketch of the pyramid.

Calculate:

(a) the slant height of the pyramid

(b) the angle the slant face makes with the base

(c) the total surface area of the pyramid.

8. A rectangular room has a length of 6 m, a breath 2 m and a height 2 m. One of the end walls of the room is labelled PQRS in that order, and a diagonal is drawn from vertex P to the opposite corner on the floor and labelled T:

(a) Draw a sketch of the room and label the five vertices.

(b) Calculate:

(i) the length of the diagonal

(ii) the angle between the diagonal and the plane PQRS.

9. A model of a conical tent stands on a circle radius 80 cm and a slant height of 1.6 m long. Its vertex (V) is vertically above the centre of the base. Diameters AC and BD of the base meet at right angles at the center O.

(a) Sketch the cone.

Calculate:

(b) the angle between VA and the horizontal,

(c) angle BVD,

(d) angle between VA and VB.

10. Fig. 13.47 is a sketch of a roof with a rectangular base. The edges AE, DE, BF, CF are equal. If BC = 6 m, EF = 10 m the perpendicular height is 4 m and AB = 25 m,

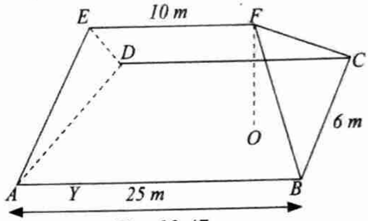


Fig. 13.47

(a) Calculate:

(i) the length BF,

(ii) the angle that line BF makes with plane ABCD.

(b) Find the angle that plane ABFE makes with the base ABCD.

11. Fig. 13.48 shows a right pyramid in which VA = VB = VC = VD = 18 cm. Its base is a rectangle measuring AB = 10 cm and BC = 6 cm. Points M, N, P and Q are the midpoints of AB, DC, VD and VC respectively. Calculate the following:

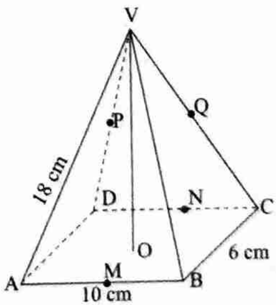


Fig. 13.48

(a) height of the pyramid to 2 significant figures.

(b) volume of the pyramid,

(c) angle between VM and VO,

(d) angle between planes VAC and the base,

(e) angles between planes VOC and VOB.

12. VABCD is a right pyramid with V as the vertex. The base of the pyramid is a rectangle ABCD with AB = 4 cm, BC = 3 cm, and the height of the pyramid is 6 cm.

    (a) Calculate:

        (i) the length of the projection of VA on the base.

        (ii) the angle between the face VAB and the base.

    (b) P is the midpoint of VC and Q is the midpoint of VD.

    Find the angle between the planes VAB and the plane ABPQ.

13. Fig. 13.49 represents a right pyramid with vertex V and a rectangular base PQRS. VP = VQ = VR = VS = 18 cm, PQ = 16 cm and QR = 12 cm. M and O are the midpoints of QR and PR respectively.

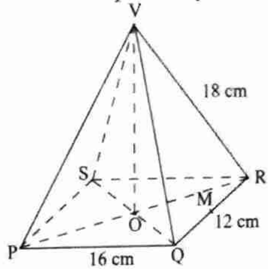


Fig. 13.49

Find:

(a) the length of the projection of line VP on the plane PQRS.

(b) the size of the angle between VP and the plane PQRS.

(c) the size of the angle between the planes VQR and PQRS.

14 GRAPHS OF CUBIC FUNCTIONS

Success criteria

By the end of this topic, the student must be able to:

- *Draw graphs of cubic functions.*
- *Solve cubic equation graphically.*

Introduction

In Book 1, we learnt that a relation of the form $ax + b$ where a and b are constants is called a **linear expression**. Thus, an equation of the form $y = ax + b$ is called a **linear equation** and the graph of y against x is a straight line.

In chapter 14 of Book 3, we dealt with quadratic expressions, quadratic equations and graphs of quadratic relations.

In this chapter, we shall further our knowledge of graphs to drawing of graphs of cubic functions.

But first we do a brief review of solving linear and quadratic simultaneous equations graphically.

Example 14.1

On the same axes, draw the graphs of $y = 9 + 3x - 2x^2$ and $y = 2x + 2$ for values of x from $^-2$ to $^+3$.

Use your graphs to solve the equations

(a) $^-2x^2 + 3x + 9 = 0$,

(b) $^-2x^2 + 3x + 9 = 2x + 2$.

Solution

In order to draw the graph of $y = {^-2x^2} + 3x + 9$, we make a table of values of x and y for $^-2 < x < 3$ (Table 14.1).

x	$^-2$	$^-1$	0	1	2	3
$^-2x^2$	$^-8$	$^-2$	0	$^-2$	$^-8$	$^-18$
$3x$	$^-6$	$^-3$	0	3	6	9
9	9	9	9	9	9	9
y	$^-5$	4	9	10	7	0

Table 14.1

To draw the graph of $y = 2x + 2$, either:

(i) use the gradient and y-intercept form method, or

(ii) make a table of values using at least three points (Table 14.2).

x	$^-1$	0	1
y	0	2	4

Table 14.2

Both graphs are shown in Fig. 14.1.

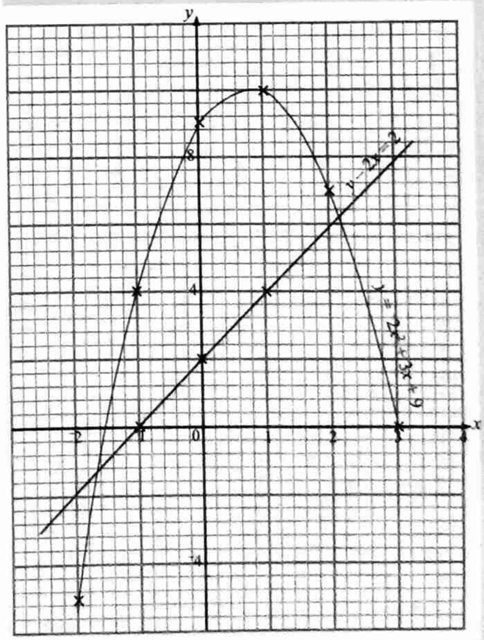


Fig. 14.1

(a) *Solutions of $^-2x^2 + 3x + 9 = 0$ are found at the points where the graph of $y = {}^-2x^2 + 3x + 9$ meets the x–axis, i.e. they are values of x when $y = 0$.*

Thus, $x = {}^-1.5$ or $x = 3$.

(b) *Solutions of $^-2x^2 + 3x + 9 = 2x + 2$ are found at the intersection of the two graphs.*

Thus, $x = 2.1$ or $^-1.6$ (1 d.p.).

Exercise 14.1

1. (a) Copy and complete Table 14.3 for the relation $y = 5 - x$.

x	$^-3$	$^-2$	0	1	2	3	5
y			5			2	0

Table 14.3

Use your table to draw the graph of $y = 5 - x$.

(b) Make a table of values for the relation $y = x^2 - 1$ for $^-3 \leq x \leq 3$. Hence draw the graph of $y = x^2 - 1$.

2. Two variables x and y are related by the equation (i) $y = x^2$ (ii) $y = 25 + \frac{36}{x}$.

(a) Make a table of values for each relation for $1 \leq x \leq 6$.

(b) On the same axes draw their graphs.

(c) Read the values of x and y at the point where the graphs intersect.

(d) State the equation whose solution the value of x represents.

3. Water is poured into a container at a steady rate. The height h cm of water after t seconds is as shown in Table 14.4.

t	0	0.5	1.0	1.5	2.0	2.5	3.0
h	0	0.1	0.3	0.6	1.0	1.7	3.0

3.5	4.0	4.5	5.0	5.5	6.0
4.3	5.0	5.4	5.7	5.9	6.0

Table 14.4

Draw the graph of h against t, taking 1 cm for 1 unit on each axis. From your graph state the height of water after:

(a) 1.7 seconds (b) 4.3 seconds.

Graphs of cubic relations

This section aims at using graphs of cubic functions solutions to solve cubic equations.

A **cubic expression** is one in which the highest power of x is 3. Thus, x^3, $2x^3 + 2$, $2x^3 + 5x^2 - 3x + 4$ are examples of cubic expressions. The equation $y = 2x^3 + 5x^2 - 3x + 4$ is a **cubic relation**.

The graph of $y = x^3$

The method of drawing a curve such as $y = x^3$ is exactly the same as that used to draw the graph of a quadratic relation.

The relation $y = x^3$ is the simplest cubic relation and its graph is equally simple to draw.

We start by making a table of corresponding values of x and y within a given range.

Look at the following example:

Example 14.2

Draw the graph of $y = x^3$ for values of x from $^-3$ to $^+3$.

Solution

1. *First, take integral values of x in the given range and tabulate your results for x and y (Table 14.5).*

x	$^-3$	$^-2$	$^-1$	0	1	2	3	$^-2\frac{1}{2}$	$^-1\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{2}$
$y=x^3$	$^-27$	$^-8$	$^-1$	0	1	8	27	$^-15.63$	$^-3.38$	3.38	15.63

Table 14.5

2. *To make the drawing of the graph easier and more accurate, find the values of y corresponding to $x = {}^-2\frac{1}{2}, {}^-1\frac{1}{2}, 1\frac{1}{2}$ and $2\frac{1}{2}$. These extra values of y will reduce the large gaps between some consecutive values of y.*

3. *Choose a suitable scale, large enough to accommodate all the values on both x- and y- axes, and plot the points.*
4. *Join the points using a smooth continuous curve (Fig. 14.2).*

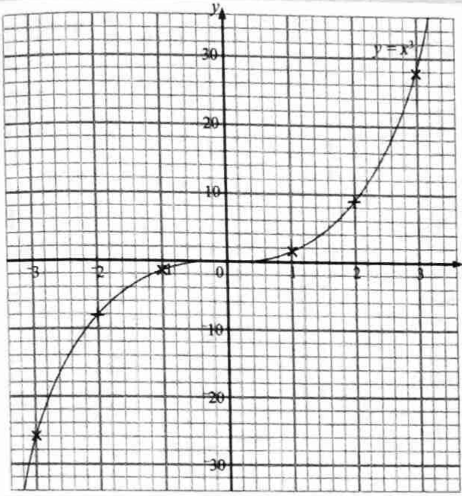


Fig. 14.2

As the values of x increase, the corresponding values of y increase very rapidly. Similarly, as the values of x become larger in magnitude, though negative, the values of y also become larger in magnitude and negative.

Use the graph in Fig. 14.2 to answer the following questions.

1. Find the value of y when $x = 2.8$.
2. Find the value of x when $y = 25$.
3. Find the number whose cube root is 2.3.
4. Find the number whose cube root is 24.

The graph of $y = ax^3 + bx^2 + cx + d$.

The relation $y = ax^3 + bx^2 + cx + d$ is the general cubic relation. We now draw the graph of such a relation and observe its properties.

Example 14.3

Draw the graph of the relation.

$y = x^3 - 3x^2 - 9x + 2$

Solution

Using the relation $y = x^3 - 3x^2 - 9x + 2$, we work in the same way as in Example 14.2.

1. *Make a table of values for $^-3 \leq x \leq +5$ (Table 14.6).*

x	$^-3$	$^-2$	$^-1$	0	1	2	3	4	5
x^3	$^-27$	$^-8$	$^-1$	0	1	8	27	64	125
$^-3x^2$	$^-27$	$^-12$	$^-3$	0	$^-3$	$^-12$	$^-27$	$^-48$	$^-75$
^-9x	27	18	9	0	$^-9$	$^-18$	$^-27$	$^-36$	$^-45$
$^+2$	2	2	2	2	2	2	2	2	2
y	$^-25$	0	7	2	$^-9$	$^-20$	$^-25$	$^-18$	7

Table 14.6

To avoid unnecessary errors, the values of y are calculated systematically, one term at a time and finally combined.

2. *If the gap between any two consecutive y values is too large, it is advisable to use extra values of x within the range, for a better curve.*
3. *Now, plot the corresponding values of x and y, and join the points with a smooth curve (Fig 14.3).*

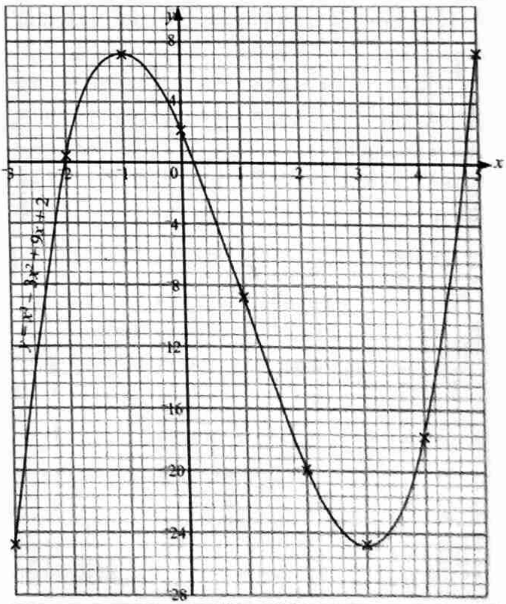


Fig. 14.3

Note that the **peak** and the **trough** in the graph in Fig. 14.3 is a common property of cubic curves. The curve has **rotational symmetry** of order 2, about the point (1 , ⁻9), which is **halfway point** between the peak and the trough along the curve.

The graph of the general cubic relation is of the form shown in Fig. 14.4(a) if ***a* is positive** and is of the form shown in Fig 14.4(b) if ***a* is negative**.

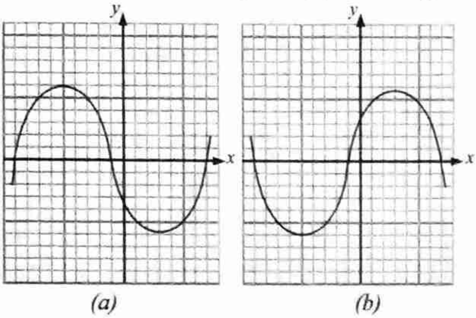


Fig. 14.4

Note: Some cubic curves do not exhibit the peak and trough property. A good example of such a curve is the curve of $y = x^3$. In general, any cubic relation which does not have a quadratic term (i.e. $b = 0$) results in a curve without a peak and a trough.

Exercise 14.2

For each cubic relation given in this exercise, make a table of values for the given range, and hence draw the graphs accurately. Save your graphs for later use.

1. $y = x^3$ for $0 \le x \le 6$
2. $y = x^3 - x$ for $^-3 \le x \le 3$
3. $y = x^3 + 3x$ for $^-4 \le x \le 4$
4. $y = x(x-1)(x-2)$ for $^-2 \le x \le 3$
5. $y = x^2(4-x)$ for $^-1 \le x \le 4$
6. $y = x^3 - 3x + 1$ for $^-3 \le x \le 3$
7. $y = x^3 - 2x^2 + 3x - 4$ for $^-2 \le x \le 3$
8. $y = {^-2}x^3 + x^2 - 5x + 2$ for $^-2 \le x \le 2$
9. $y = x^3 - 4x^2 + 7x - 2$ for $^-1 \le x$

Solving cubic equations

The graph of $y = x^3$ may be used to solve an equation such as $x^3 = 10 - 5x$. This involves drawing an appropriate line on the same axes as $y = x^3$.

Using the same scale and on the same axes, draw the graphs of $y = 10 - 5x$ and $y = x^3$ as in Fig. 14.5.

To draw the graph of $y = x^3$, we can use the table of values used in Example 14.2.

For the line whose equation is y = 10 − 5x, we only need to find 3 points on the line i.e. (0, 10), (1, 5), (2, 0).

At the point of intersection of the curve and the line, the value of y on the curve is equal to the value of y on the line,

i.e. $y = x^3 = 10 - 5x$.

At the point of intersection, $x = 1.4$. Thus, 1.4 is an approximate root of the equation $x^3 = 10 - 5x$.

Note: If such an equation has more than one real root in the given range, then the line would meet the curve at more than one point.

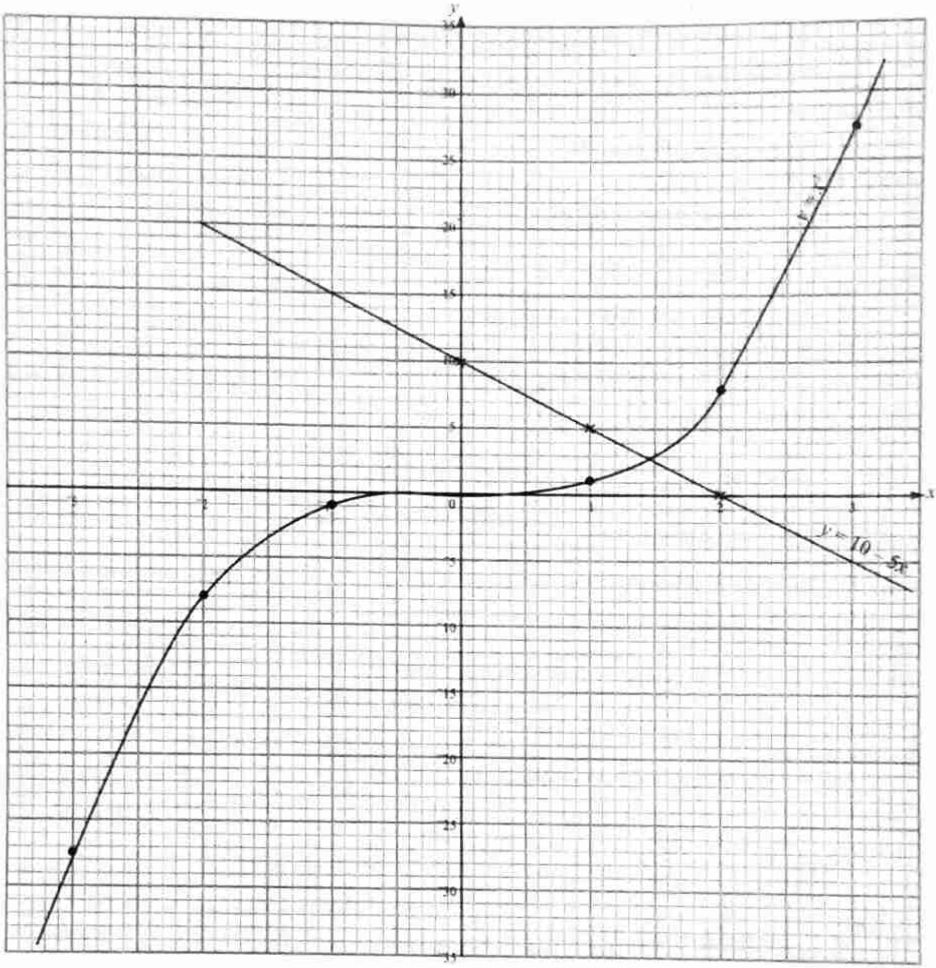


Fig. 14.5

Example 14.4

(a) Draw the graph of $y = x^3 + 5x - 10$.

(b) Use your graph to solve the equation $x^3 + 5x = 15$.

Solution

(a) To draw the graph of $y = x^3 + 5x - 10$,

(i) Make a table of values for $^-2 \leq x \leq {}^+3$.

x	$^-2$	$^-1$	0	1	2	3	$^-\frac{1}{2}$	$\frac{1}{2}$	$1\frac{1}{2}$	$^-1\frac{1}{2}$
x^3	$^-8$	$^-1$	0	1	8	27	$^-\frac{1}{8}$	$\frac{1}{8}$	$\frac{27}{8}$	$\frac{27}{8}$
$5x$	$^-10$	$^-5$	0	5	10	15	$2\frac{1}{2}$	$2\frac{1}{2}$	$\frac{15}{2}$	$\frac{15}{2}$
$^-10$	$^-10$	$^-10$	$^-10$	$^-10$	$^-10$	$^-10$	$^-10$	$^-10$	$^-10$	$^-10$
y	$^-28$	$^-16$	$^-10$	$^-4$	8	32	$^-12.6$	$^-7.4$	0.88	$^-20.9$

Table 14.7

If the gap between any two consecutive y values is too large, it is advisable to use extra values, of x within the given range, for a better curve.

(ii) *Plot the corresponding values of x and y and join the points with a smooth curve (Fig 13.6).*

(b) *The equation $x^3 + 5x = 15$ can be written as $x^3 + 5x - 15 = 0$.*

Adding 5 to both sides:

$$x^3 + 5x - 15 + 5 = 0 + 5$$

$$\Rightarrow x^3 + 5x - 10 = 5.$$

But, we have already drawn the graph of

$$y = x^3 + 5x - 10.$$

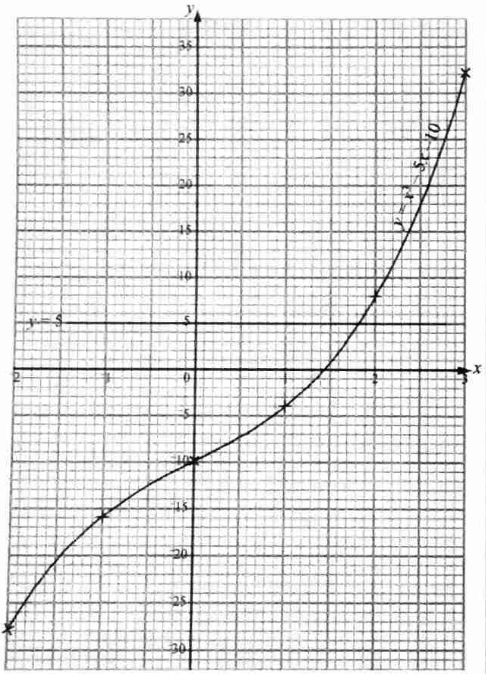


Fig. 14.6

So, on the same axes, draw the graph of $y = 5$ (see Fig. 14.6).

From your graph, find the value of x where the line meets the curve. The value of x at this point is 1.8.

$\therefore$ The root of the equation $x^3 + 5x - 15 = 0$ is 1.8 (approximately).

***Note:** If an equation has more than one root, the line meets the curve as many times as the number of roots.*

Example 14.5

Draw the graph of $y = 4x^2 - x^3$ from $x = -1$ to $x = 4$, plotting points at half unit intervals.

(a) *State the* (i) *maximum value of y.*

(ii) *minimum value of y.*

(b) *Find the roots of $x^3 - 4x^2 + 4 = 0$*

(c) *Draw the tangent to the curve at (3 , 9) and use it to find the gradient of the curve at that point.*

(d) *On the same axis, draw the graph of $2y = x + 6$ and state the values of x at which the two graphs intersect. What equation has these values as its roots?*

(e) *What linear graph would enable you to solve the equation*

$2x^3 - 8x^2 - x + 8 = 0$ using the existing graph.

Solution

Make a table of values (Table 14.8) for the graph.

x	-1	-0.5	0	0.5	1	1.5	2	2.5	3	3.5	4
$4x^2$	4	1	0	1	4	9	16	25	36	49	64
$-x^3$	1	0.125	0	$-.125$	-1	-3.375	-8	-15.625	-27	-42.875	-64
y	5	1.125	0	0.875	3	5.625	8	9.375	9	6.125	0

Table 14.8

To be able to draw the graph, round off the y values to 1 d.p (Table 14.9). Then plot the points and join them with a smooth curve (Fig. 14.7).

x	–1	–0.5	0	0.5	1	1.5	2	2.5	3	3.5	4
y	5	1.1	0	0.9	3	5.6	8	9.4	9	6.1	0

Table 14.9

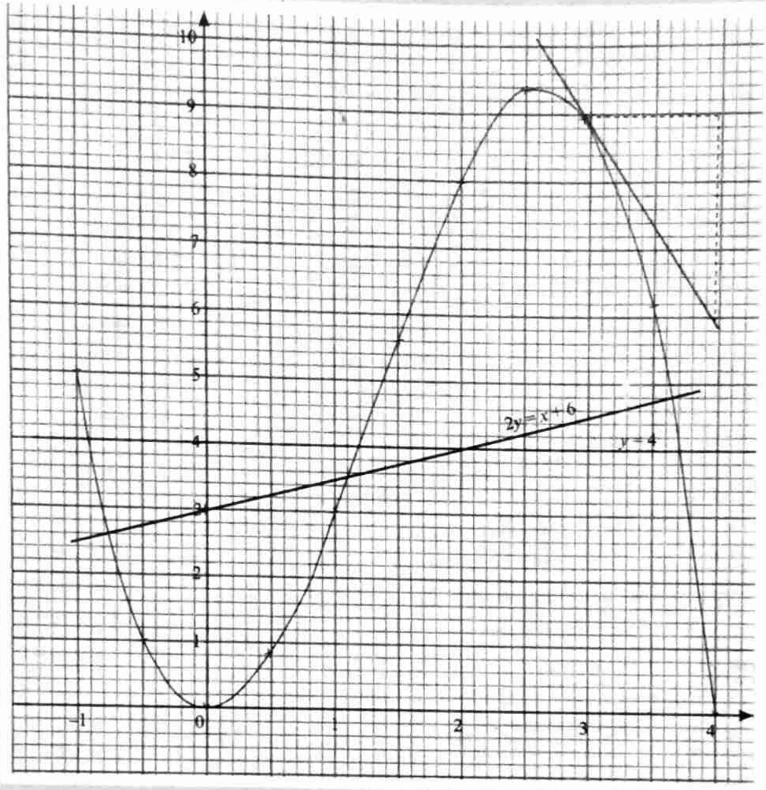


Fig. 14.7

(a) (i) Maximum value of y is 9.4.

(ii) Minimum value of y is 0.

(b) The roots of $x^3 - 4x^2 + 4 = 0$ are found by using an appropriate line graph.

Thus, $y = -x^3 + 4x^2$
$$\underline{0 = x^3 - 4x^2 + 4} \quad +$$
$y = 4$ is the suitable line.

The intersection of the line $y = 4$ and the curve $y = -x^3 + 4x^2$ are where $x = -0.9$, $x = 1.2$ and 3.8 (1d.p.)

$\therefore$ the roots of $x^3 - 4x^2 + 4 = 0$ are (0.9, 1.2, 3.8) 1dp

(c) Points (4 , 6) and (3 , 9) lie on the tangent to the curve at the point (3 , 9)

The gradient of the tangent is $= \dfrac{9-6}{3-4}$

$= \dfrac{3}{-1} = -3$

Note:

Any other two points on the line could be used to find the gradient.

(d) $2y = x + 6$ meets the curve $y = -x^3 + 4x^2$ at $x = -0.8, 1.1, 3.7$ (1 d.p)

$2y = x + 6 \Rightarrow y = \frac{1}{2}x + \frac{6}{2}$

At the intersection point,

$$-x^3 + 4x^2 = \frac{1}{2}x + 3$$

$$-x^3 + 4x^2 - \frac{1}{2}x - 3 = 0$$

$$-2x^3 + 8x^2 - x - 6 = 0$$

$$2x^3 - 8x^2 + x + 6 = 0$$

The equation is $2x^3 - 8x^2 + x + 6 = 0$

(e) *The existing graph is* $y = -x^3 + 4x^2$

$\Rightarrow 2y = -2x^3 + 8x^2$

Equation to be solved is $0 = 2x^3 - 8x^2 - x + 8$

$$\begin{array}{r} 2y = -2x^3 + 8x^2 \\ 0 = 2x^3 - 8x^2 - x + 8 \\ \hline 2y = -x + 8 \end{array} \quad +$$

The graph is $2y = 8 - x$

Exercise 14.3

1. Use the graph of $y = x^3 - x$ to solve the equation
   (a) $x^3 - x = \frac{1}{4}$ (b) $x^3 - x = 1$
2. Using the graph of $y = x^3 + 3x$, solve the equation
   (a) $x^3 + 3x = 4$ (b) $x^3 + 3x = {}^{-}4$
3. Using the graph of $y = x(x - 1)(x - 2)$, solve the equation $x^3 - 3x^2 + 2x - 4 = 0$
4. Use the graph of $y = \frac{1}{2}x^2(4 - x)$ to find the
   (a) maximum value of $\frac{1}{2}x^2(4 - x)$ in the given range.
   (b) values of x for which $x^2(4 - x)$ is equal to 2.
   (c) roots of the equation $x^2(4 - x) = 1$.
5. (a) Use the graph of $y = x^3 - 3x + 1$ to solve the equation $x^3 - 3x + 1 = 0$.
   (b) Solve the equation in (a) using a different cubic curve and a line.

Simultaneous equations: one linear, one cubic

We have seen how to solve simultaneous equations involving linear and quadratic equations by graphical method.

Example 14.4 demonstrates how to solve simultaneous equations involving cubic and linear equations.

Example 14.6

Solve the simultaneous equations

$y = x^3 - 9x$

$y = 2x - 3$

Solution

To solve the equations, we must draw the two graphs on the same axes.

Make a table of values for the cubic function.

x	-4	-3	-2	-1	0	1	2	3	4
x^3	-64	-27	-8	-1	0	1	8	27	64
$-9x$	36	27	18	9	0	-9	-18	-27	-36
$y=x^3-9x$	-28	0	10	8	0	-8	-10	0	28

Table 14.10

Fig 14.8 shows the graphs of $y = 2x - 3$ *and* $y = x^3 - 9x$

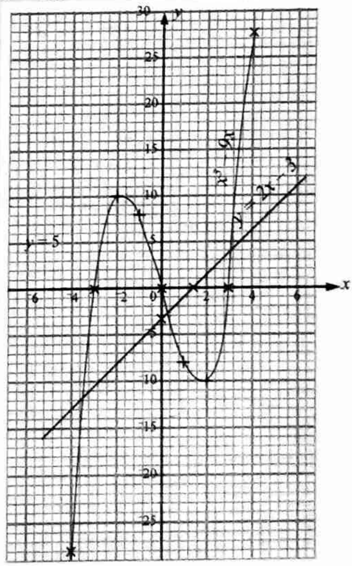


Fig. 14.8

The two graphs intersect at three points whose coordinates are $({}^{-}3.4, {}^{-}11)$, $(0.2, {}^{-}3)$ *and* $(3.2, 4)$. *From these points we pick the solutions of the equations.*

$x = {}^{-}3.4$ when $y = {}^{-}11$

$x = 0.2$ when $y = {}^{-}3$

$x = 3.2$ when $y = 4$

Example 14.7

Solve the simultaneous equations

$y = x^3 - 3x^2 - 13x + 15$

$y = \frac{-5}{2}x + \frac{25}{2}$

Solution

To solve the two simultaneous equations, we will draw the two graphs on the same axes.

A table of values for the cubic function will be as shown in Table 14.11.

x	-3	-2	-1	0	1	2	3	4	5
x^3	-27	-8	-1	0	1	8	27	64	125
$-3x^2$	-27	-12	-3	0	-3	-12	-27	-48	-75
$-13x$	39	26	13	0	-13	-26	-39	-52	-65
15	15	15	15	15	15	15	15	15	15
y	0	21	24	15	0	-15	-24	-21	0

Table 14.11

For $y = \frac{-5}{2}x + \frac{25}{2}$

x	0	5
y	$\frac{25}{2}$	0

Table 14.12

Fig 14.9 shows the graphs of $y = \frac{-5}{2}x + \frac{25}{2}$ *and* $y = x^3 - 3x^2 - 13x + 15$

The two graphs intersect at three points: (-2.2, 18), (0.2, 12) and (5, 0). The solution of the equations x - -2.2 when y = 18, x = 0.2 when y = 12, x = 5 when y = 0

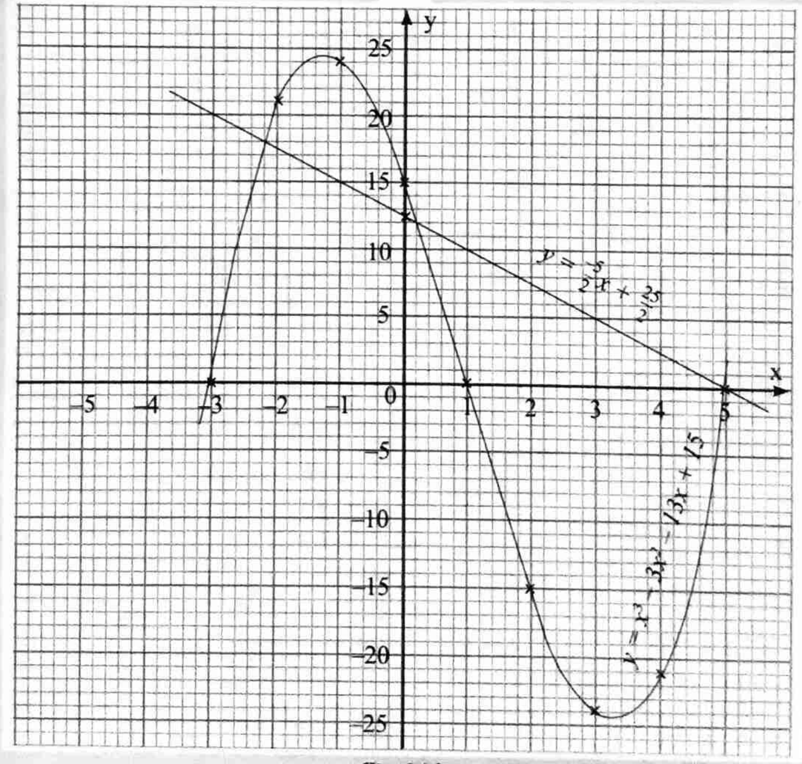


Fig. 14.9

Exercise 14.4

1. (a) Given that $y = \frac{1}{4}x^3$, use graphical method to find approximate values of $\sqrt[3]{100}$ and $\sqrt[3]{200}$.

   (b) On the same axes as the graph of $y = \frac{1}{4}x^3$, draw the graph of $y = 30 - 5x$, and find the value of x at the intersection of the two graphs. State the equation which has this value as one of its roots.

2. (a) Draw the graph of $y = x^2(4 - x)$ for $^-1 \leq x \leq 4$, plotting points at half unit intervals. Use your graph to state the roots of $x^3 - 4x^2 + 4 = 0$.

   (b) On the same axes draw the graph of $2y = x + 6$ and state the values of x at its intersections with the graph of $y = x^2(4 - x)$. What equation has these values as its roots?

   (c) What linear graph would enable you to find the roots of $2x^3 - 8x^2 - x + 8 = 0$ using the graph in (a)?

3. Use graphical method to solve the simultaneous equations: $y = {^-x^3} + 4x^2$
   $y = \frac{1}{2}x + 3$

4. Solve the simultaneous equations:
   $y = x^3 - 4x$
   $y = {^-x} - \frac{1}{2}$

5. Solve the simultaneous equations:
   $y = {^-x^3} + 9x + 10$
   $y = x + 8$

11-14 REVISION EXERCISES 3

Revision exercise 3.1

1. Draw the graph of $y = x(x-1)(x-2)$ for values of x from $^-2$ to $^+4$. From your graph solve the equations
   (a) $x^3 - 3x^2 + 2x - 4 = 0$
   (b) $x^3 - 3x^2 + 2x = 0$

2. Using the same scale on both axes, draw the graph of $y = 0.4x + 0.1$ and $y = 0.1x^3$, for values of x from $x = -2$ to $x = 3$.
   (a) Use your graph to state the values of x for which the functions $y = 0.4x + 0.1$ and $y = 0.1x^3$ are equal.
   (b) Using the two functions write a single equation whose solutions are the values of x in part (a).

3. A cone is made from the sector of a circle of radius 4 cm. The angle of the sector is 270°. Find
   (a) the radius of the base of the cone.
   (b) the curved surface area of the cone.

4. Fig. R. 3.1 shows a door opened at an angle of 60°. Calculate
   (a) DE
   (b) ∠DCE
   (c) ∠DBE

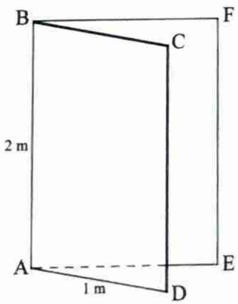


Fig. R. 3.1

5. With reference to Fig. R. 3.2, state
   (a) all the lines that are skew with line BC,
   (b) all the lines that are parallel to plane ADHE,
   (c) which of the following sets of lines or points determine a plane:
   (i) A, H, G, B (ii) AD and FG
   (iii) FB and HD (iv) A, H, G, C
   (d) the projection of line
   (i) EG onto plane ABCD,
   (ii) FG onto line BC.

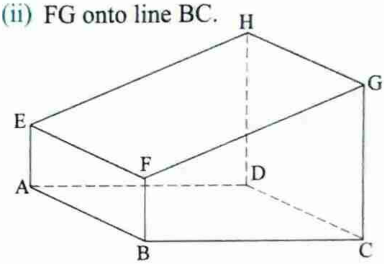


Fig. R. 3.2

6. ABCD is a rectangular base of a pyramid with vertex V. The edges VA, VB, VC, and VD are each 25 cm long. AB = CD = 14 cm and AD = BC = 10 cm. Calculate
   (a) the height of the pyramid,
   (b) the angle between the base and an edge,
   (c) the angle between the base and plane VBC,
   (d) the angle between the base and plane VCD.

7. The unshaded region in Fig. R. 3.3 represents the solution set of three simultaneous inequalities. Find the inequalities.

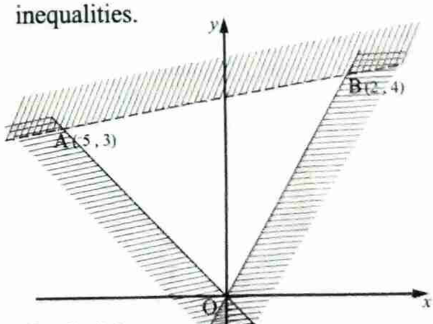


Fig. R. 3.3

8. Solve the following inequalities and illustrate each solution set on a number line.
   (a) $3(x-2)+4 > -2-(3+2x)$
   (b) $\frac{1}{2}(x+2)+\frac{1}{4}(x+3) < \frac{1}{4}(x+4)$
   (c) $\frac{1}{3x} \leq \frac{1}{2x}+\frac{1}{4}$

9. Draw a graph to illustrate the region satisfying the inequalities $y \geq 0$, $y+2x \geq 8$, $4y+3x < 24$ and $3y \leq 2x$.
   (a) List the coordinates of all the points that give integral solutions.
   (b) Find the minimum value of $y - \frac{2}{3}x$ in this region.
   (c) Find the maximum value of $2y+3x$ in this region.

10. A farmer plans to grow two types of crops, X and Y. He has identified up to 70 ha of land for this purpose. He has 240 man-days of labour available and he can spend up to a maximum of K 36 000. The requirements for the crops are as follows.

	X	Y
Minimum number of hectares to be sown	10	20
Man-days per ha	2	4
Cost per ha in K	600	400

If x and y represent the number of hectares to be used for crops X and Y respectively, write down, in their simplest form, the inequalities which x and y must satisfy.

On a graph paper, using a suitable scale, show the region within which the point (x, y) must lie if the inequalities are to be satisfied simultaneously.

11. Table R. 3.1 gives corresponding values of two variables x and y.

x	1.1	1.2	1.3	1.5	1.6
y	0.3	0.5	1.4	3.8	5.2

Table R. 3.1

The variables are known to satisfy an equation of the form $y = ax^3 + c$, where a and c are constants. By drawing a graph of y against x^3, find the values of a and c. Hence, write down the relationship connecting x and y.

12. Make a table of values for the graph of $y = -2x+3$ from $x = -4$ to $x = 4$.
   (a) Using a scale of 1 cm represents 1 unit on both axes, draw a graph.
   (b) From the graph, find the value of
      (i) y when $x = 3\frac{1}{2}$
      (ii) x when $y = -2$
   (c) Find the gradient of your graph.

Revision exercise 3.2

1. (a) Complete the table below for the equation

$$y = x^3 - 2x^2 - 4x + 7$$

x	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3	4
x^3	$^{-}27$	$^{-}8$		0		8	27	64
$^{-}2x^2$	$^{-}18$	$^{-}8$	$^{-}2$	0	$^{-}2$	$^{-}8$	$^{-}18$	$^{-}32$
$^{-}4x$	12			0				$^{-}16$
7	7	7	7	7	7	7	7	7
y	$^{-}26$	$^{-}1$		7		$^{-}1$		23

Table R. 3.2

   (b) Using the scale 1 cm to represent 1 unit on the x-axis and 1 unit to represent 5 units on the y-axis, draw the graph of $y = x^3 - 2x^2 - 4x + 7$
   (c) Use your graph to estimate the roots of the equation $x^3 - 2x^2 - 4x + 7 = 0$
   (d) By drawing appropriate straight lines, use your graph to solve the equations
      (i) $x^3 - 2x^2 - 4x + 2 = 0$
      (ii) $x^3 - 2x^2 - 3x + 3 = 0$

2. A sphere of radius 3 cm is dropped into a cylindrical container of radius 5 cm. The sphere is completely submerged in water. Find the rise in the level of the water.

3. Fig. R. 3.4 shows a right pyramid with a triangular base. The base ABC is an equilateral triangle of side 8 cm. The edges VA, VB and VC are 12 cm each. Find
   (a) the angle between VA and the base,
   (b) the angle between face VBC and the base,
   (c) the height of the pyramid.

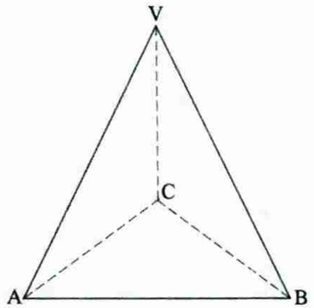


Fig. R. 3.4

4. Fig. R. 3.5 shows a cage in which base ABCD and roof PQRS are both rectangular. AP, BQ, CR and DS are perpendicular to the base. Calculate
   (a) QR (b) ∠QRC
   (c) the angle between planes ABCD and PQRS,
   (d) the inclination of PR to the horizontal.

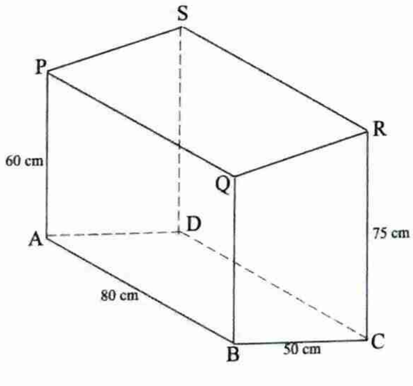


Fig. R. 3.5

5. Fig. R. 3.6 represents a wedge in which the base and back are rectangular. Find the angle indicated θ.

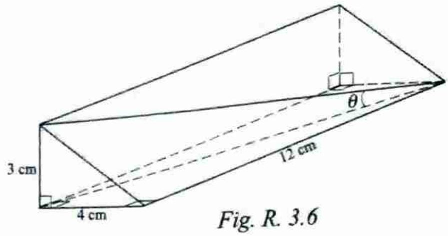


Fig. R. 3.6

6. Form linear inequalities to describe each of the following conditions:
   (a) The perimeter of a rectangular plot of land must be at least 80 m and the length must not exceed twice the breadth.
   (b) The sum of the ages of two sisters is not more than 20 years and one of them is at least 3 years older than the other.

7. By shading the unwanted region, indicate the region satisfied by the inequalities $y > 0$, $0 \leq x \leq 3$ and $x + y < 4$.

8. Illustrate the region that satisfies the inequalities, $y < 9 - \frac{1}{3}x$, $2x + y < 24$, and $x + y > 12$. Find, within the region,
   (a) the maximum value of
      (i) $x + 2y$ (ii) $3x + y$
   (b) the minimum value of $x + 2y$.

9. Find the inequalities that satisfy the region represented in the figure R. 3.7.

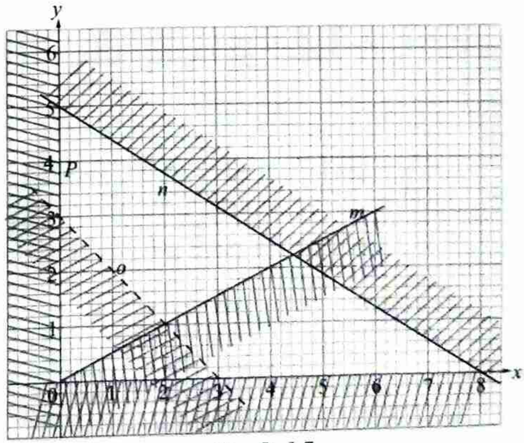


Fig. R. 3.7

10. (a) Complete Table R. 3.3 of values for the equation

$y = 2x^3 + 5x^2 - x - 6.$

x	-4	-3.5	-3	-1.75	-1.5	-1	0	1	1.5	2
$2x^3$	-128		-54				0	2		16
$5x^2$	80		45	20		5	0	5		20
$-x$	4		3			1	0			
-6	-6		-6	-6		-6	-6	-6		-6
y	-50						-6			

Table R. 3.3

(b) Draw the graph $y = 2x^3 + 5x^2 - x - 6$ for $^-4 \leq x \leq 2$. Use 2 cm to represent 1 unit on the x-axis and 1 cm to represent 5 units on the y-axis

(c) By drawing a suitable line, use the graph in (b) to solve the equation

$$2x^3 + 5x^2 + x - 4 = 0$$

11. Solve the following pairs of simultaneous equations graphically.

(a) $y = \frac{2}{3}x + 2$
$y = -\frac{1}{2}x + 2$

(b) $y + 2x = 8$
$3y + 2x = 12$

12. Draw the graph of $y = 4x^2 - x^3$ from $x = {}^-1$ to $x = 4$, plotting points at half unit intervals.

(a) State the (i) maximum value of y,
(ii) minimum value of y.

(b) Find the roots of $x^3 - 4x^2 + 4 = 0$

Revision exercise 3.3

1. Two variables, x and y, are related by the equation $y = 4x(4 - x)(3 - x)$.

(a) Copy and complete Table R. 3.4 for this relation.

x	0	0.25	0.5	1.0	1.5	2	2.5	3
y		10.3	17.5				7.5	

Table R. 3.4

(b) Draw the graph of y against x.

(c) Use your graph to find the
(i) values of x when $y = 15$.
(ii) range of x for which $y \leq 17.5$.
(iii) gradient of the curve at $x = 2$.

2. The length of the edge of a cube is 10 cm. Calculate the angle between its main diagonal and the base.

3. The curved surface of a closed cylinder is formed from a rectangle of sides 12 cm by 9 cm

(a) Draw a net of the cylinder.

(b) Find
(i) the radius of the base
(ii) the total surface area of the cylinder.

4. Fig. R. 3.8 shows a right pyramid on a rectangular base ABCD, where AB = 14 cm and BC = 10 cm. Its slant edge is 24 cm long. Calculate

(a) the height of the pyramid,
(b) the angle between faces ADV and DCV,
(c) the angle between VA and AD.

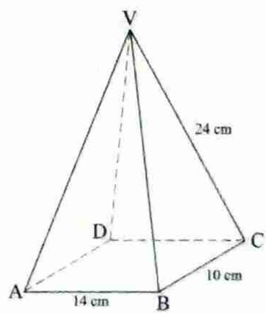


Fig. R. 3.8

5. ABCDEFGH, in Fig. R. 3.9 is a frustum of a pyramid whose base is a square of side 12 cm. M and N are the centres of ABCD and EFGH respectively. Given that MN = 5 cm, calculate,

(a) NT, where T is the midpoint of FG.

(b) the inclination of the face BCGF to the horizontal.

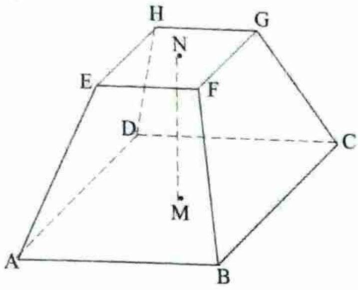


Fig. R. 3.9

6. Show, on the xy-plane, the region representing the inequalities $4y + 3x \geq 18$, $y < 2x + 2$, $5y + 2x < 32$, $y \geq 0$. Maximize $x + y$ in this region.

7. A town council plans to build a car park for x cars and y buses. Cars are allowed 10 m² of ground space, and buses 20 m². There is only 500 m² available. The park can hold a maximum of 40 vehicles at a time, and only a maximum of 15 buses at a time.

(a) Write down three inequalities other than $x \geq 0$ and $y \geq 0$.

(b) On a graph paper, show the region that satisfies the given conditions.

(c) If the parking charges are K 50 for cars and K 200 for buses per day, find the number of vehicles of each type that should be parked to obtain maximum income.

(d) Calculate the maximum income.

8. Given that p, q and r are numbers such that $2 \leq p \leq 3$, $4 \leq q \leq 5$ and $6 \leq r \leq 7$, find

(a) the maximum value of $\dfrac{r-p}{q}$,

(b) the minimum value of $q + r - p$.

9. Use the information given in the graph to find the inequalities that are satisfied by the unshaded region, R.

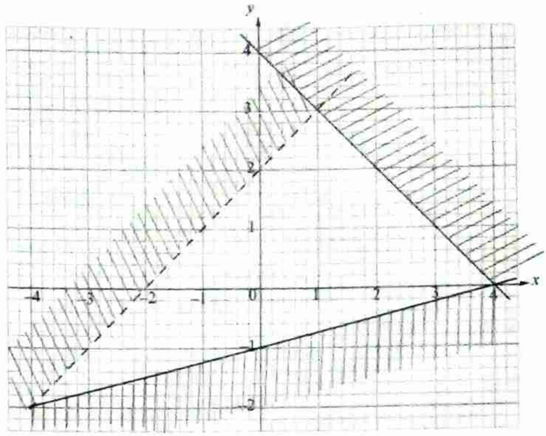


Fig. R. 3.10

10. Draw the graph $y = 2x^3 + \frac{1}{2}x^2 - 4x + 1$. Find the gradient of the curve at the point $(1, -\frac{1}{2})$.

11. The table R. 3.5 shows the distance covered and the corresponding time taken for a cyclist moving downhill.

Time(s)	0	0.5	1	1.5	2	2.5	3	3.5	4
Distance	0	2	8	17	28	41	60	81	104

Table R. 3.5

Draw a distance-time graph and use it to find

(a) the distance travelled in the first 3.8 seconds.

(b) the total distance travelled in the four seconds.

(c) the time when the cyclist was 68 m from the starting point.

12. Table R. 3.6 gives values of two variables x and y.

x	0.05	0.075	0.150	0.250	0.500	0.750
y	1.00	0.660	0.330	0.200	0.100	0.067

Table R. 3.6

(a) Draw a graph to show how $\frac{1}{y}$ varies with x.

(b) Explain the kind of relationship existing between x and y.

(c) Given that $xy = c$, determine the value of c.

MODEL PAPERS
SET I PAPER I

Answer all the questions.

1. Find the quadratic equation whose roots are

   (a) $^-1, 2$ (b) $\left(^-\frac{3}{4}, \frac{2}{3}\right)$.

   Give your answer in the form
   $ax^2 + bx + c = 0$ *(4 marks)*

2. In Fig. MP1.1, PS = 3 cm, PQ = 4 cm and QS is a diameter of the circle. Find $\angle$PRQ. *(3 marks)*

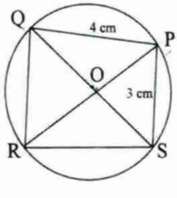


Fig. MP1.1

3. From the top A of a tower AN, 75 m high, the angle of depression of a point B is 49°. Find the distance of B from N, giving your answer correct to 1 d.p. *(3 marks)*

4. Find the surface area of the tetrahedron in Fig. MP.1.2, given that all measurements are in cm. *(6 marks)*

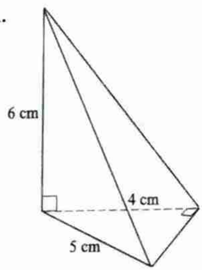


Fig. MP.1.2

5. Nyondo drove South from town P at a speed of 90 km/h for 30 minutes. He then changed direction and drove for another 52 km on a bearing of 330°. How far was he from town P? *(4 marks)*

6. X(2 , 2), Y(4 , 2) and Z(3 , 4) are the vertices of a triangle. The triangle is enlarged by a scale factor 2 and centre the origin. What are the coordinates of the vertices of the image $\Delta X'Y'Z'$? *(3 marks)*

7. Two cuboids are similar, with linear scale factor 2. One has a surface area of 88 cm^2 and a volume of 48 cm^3. What are the two possible areas and volumes of the other? *(5 marks)*

8. Given that the position vectors of A, C and D are $\begin{pmatrix} 3 \\ ^-2 \end{pmatrix}$, $\begin{pmatrix} ^-9 \\ ^-3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ respectively,

   (a) find, by calculation, the coordinates of the midpoint of CD. *(3 marks)*

   (b) find the position vector of the midpoint of AC. *(3 marks)*

9. Table MP 1.1 shows the number of vehicles that were observed passing a certain point, and the number of passengers in the vehicles.

Number of passengers	Number of vehicles
0 – 4	50
5 – 9	25
10 – 18	18
19 – 25	14
26 – 50	12
51 – 70	10

Table MP 1.1

   (a) Draw a histogram for this data. *(4 marks)*

   (b) Draw a frequency polygon. *(3 marks)*

10. (a) Given that y partly varies inversely as x and partly constant, determine, the relationship between x and y if $x = 4$ when $y = 11$ and $x = 7$ when $y = 7$. *(3 marks)*

(b) A variable y is jointly proportional to the square of x and the cube of z. When $y = 45$, $x = 3$ and $z = 2$. Find the value of y when $x = 2$ and $z = 2$. *(3 marks)*

11. (a) Solve for x in the following:

(i) $1\,024 \times 2^x = 1$ *(2 marks)*

(ii) $4 \log x + \log 81 = 2 \log 6x$ *(2 marks)*

(b) Express n in terms of x and y given that $\log y = \log(10x^n)$ *(3 marks)*

12. A triangle ABC is right angled at B and has sides AB = $(\sqrt{5} + 2)$ cm and AC = $(3\sqrt{2})$ cm. Find, in surd form, the length of side BC, and hence the area of the triangle. *(5 marks)*

13. Table MP 1.2 shows the distribution of masses of a sample of pupils in a certain academy.

Mass (kg)	22	27	32	37	42
No. of pupils	1	5	9	11	20

Mass (kg)	47	52	57	62	67
No. of pupils	20	19	8	4	3

Table MP 1.2

Calculate

(a) the mean mass. *(3 marks)*

(b) the standard deviation of the masses. *(3 marks)*

14. ABCD is a quadrilateral. The midpoints of AB, BC, CD and DA are P, Q, R and S respectively. Using vector method, prove that PQRS is a parallelogram. *(3 marks)*

15. The speed of an object at time t seconds after commencement of motion is given by the relation v = $(t + 1)$ m/s. Draw the graph of speed against time for the first five seconds. Use your graph to find; *(3 marks)*

(a) the initial speed, *(1 mark)*

(b) the acceleration of the object, *(2 marks)*

(c) the distance travelled by the object in 5 seconds. *(2 marks)*

16. The roof of a warehouse is in the shape of a triangular prism, as shown in Fig. MP 1.3.

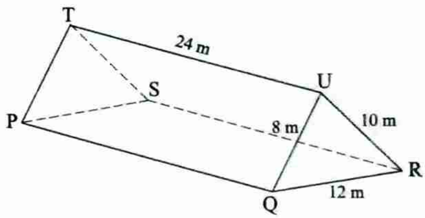


Fig. MP 1.3

Calculate

(a) the angle between faces RSTU and PQRS, *(3 marks)*

(b) the space occupied by the roof, *(3 marks)*

(c) the angle between plane QTR and PQRS. *(4 marks)*

17. A triangle ABC is such that $\angle ACB = 30°$ and BC = 5 cm. It has an area of 10 cm^2. Find AC. *(3 marks)*

18. Solve the following simultaneous equations.

$$x + y = 0$$
$$x^2 + y^2 - xy = 24$$ *(4 marks)*

19. In Fig. MP 1.4, TA and TB are tangents to the circle at A and B. AD is parallel to BC and $\angle CAD = 51$. Calculate the value of x.

(3 marks)

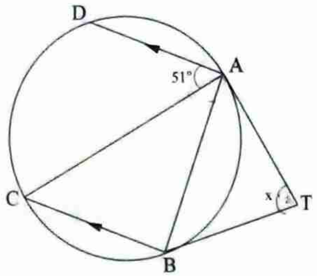


Fig. MP. 1.4

20. (a) State the conjugate of

(i) $\sqrt{3}$ (ii) $2 - \sqrt{5}$

(iii) $\sqrt{3} + 3$ (iv) $2\sqrt{2} + 3\sqrt{3}$ *(4 marks)*

(b) By first rationalising the denominator, evaluate $\frac{2}{1 + \sqrt{2}}$ to 4 d.p. *(3 marks)*

MODEL PAPERS
SET I PAPER II

SECTION A (55 marks)

Answer all the questions.

1. Factorise the following expression
   $15 - 13x + 2x^2$ *(3 marks)*

2. Given that $\log_x 12\frac{1}{4} = 2$, Solve for x.
   (2 marks)

3. Solve the equation
   $\frac{x}{6} - \frac{2x-3}{5} = \frac{x-5}{3}$. *(3 marks)*

4. Table MP 1.3 shows the number of goals scored in 40 soccer matches during a certain season.

No. of goals	0	1	2	3	4	5	6	7
No. of matches	3	9	6	8	5	5	2	2

Table MP 1.3

   Calculate the mean number of goals scored per match. *(3 marks)*

5. A right pyramid is cut along a plane parallel to the base such that the height of the resulting frustrum is $\frac{1}{3}$ that of the original pyramid. Express the volume of the resulting smaller pyramid as a fraction of the volume of the initial pyramid.
   (3 marks)

6. Solve the equation
   $\log 5 + \log(2x + 10) - 2 = \log(x - 4)$.
   (3 marks)

7. Use logarithms to evaluate
   $\frac{68.53}{(13.8 \times 0.07421)^{\frac{1}{2}}}$. *(4 marks)*

8. Solve the equation $8^x + 2^{3x} + 3 = 35$.
   (3 marks)

9. Find the length of AC in Fig. MP 1.5.
   (3 marks)

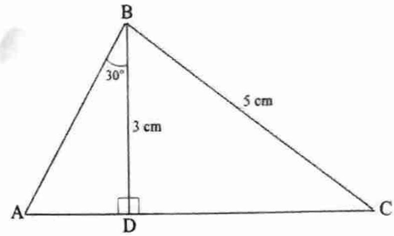


Fig. MP 1.5

10. Three points A(2 , 3), B(2k , 5) and C($^-$3 , 6) lie on a straight line. Find the value of k. *(3 marks)*

11. The unshaded region in Fig. MP 1.6 is bounded by the lines $y = 4$, $x = {}^-2$ and $y = x + 3$.

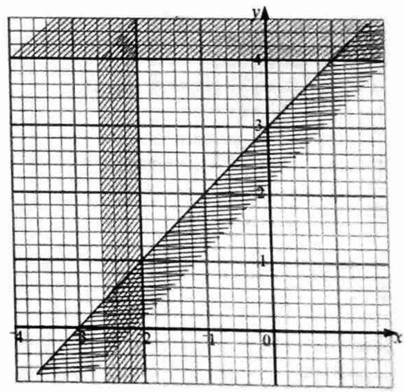


Fig. MP 1.6

State the inequalities that satisfy the region.
(2 marks)

12. A spherical ball is deflated so that its volume decreases in the ratio 8:27. Find the ratio in which the radius decreases.
    (2 marks)

13. Joan has some money in two denominations only: twenty-kwacha notes and one hundred-kwacha notes. She has four times as many twenty-kwacha notes as one hundred-kwacha notes. Altogether she has K 2 160. How many one hundred-kwacha notes does she have? *(3 marks)*

14. Solve the simultaneous equations

$$xy = 4$$
$$x + y = 5$$

(4 marks)

15. Simplify the following by rationalizing the denominator

$$\frac{2}{\sqrt{5}+\sqrt{3}}$$

(3 marks)

SECTION B (45 marks)

Answer any 3 questions.

16. The length and breadth of a rectangle are given as $(6x - 1)$ and $(x - 2)$ cm respectively. If the length and breadth are each increased by 4 cm, the new area is three times that of the original rectangle.

(a) Form an equation in x and solve it. *(7 marks)*

(b) Find the dimensions of the original rectangle. *(5 marks)*

(c) Express the increase in area as a percentage of the original area. *(3 marks)*

17. Jean is a college student and she stays alone. She usually sets the alarm clock to wake her up in the morning. The probability that she remembers to set the alarm before going to sleep is $\frac{1}{3}$. If she does not set the alarm, she never wakes up before 7.30 a.m. If she sets the alarm for 7.00 am, the probability that it wakes her up is only 0.8. Whenever she wakes up at 7.00 am, she is never late for class, but if she wakes up at 7.30 am, the probability that she will be late for class is 0.8.

Calculate the probability that;

(a) Jean wakes up at 7.00 am, *(4 marks)*

(b) she forgets to set the alarm the night before but manages to reach college on time, *(3 marks)*

(c) she sets the alarm, but it fails to wake her up and yet she reaches college punctually, *(4 marks)*

(d) she is late for college. *(4 marks)*

18. A rhombus has vertices at A($^-1$, 1), B(0 , 8), C(5 , 3) and D(x , y). T is the intersection of the diagonals of the rhombus.

(a) Find the coordinates of D and T. *(3 marks)*

(b) Given that $\angle CBT = a$, express $\angle BAD$ in terms of a. *(3 marks)*

(c) Calculate the lengths of diagonals AC and BD. *(4 marks)*

(d) Calculate the area of the rhombus. *(5 marks)*

19. (a) In Fig. MP 1.7, O is the centre of a circle whose radius is 6 cm and PQ is 9 cm.

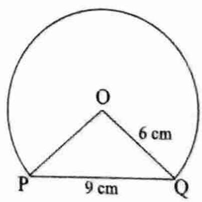


Fig. MP 1.7

Calculate the area of the major segment. *(7 marks)*

(b) Find the area of a triangle ABC with sides 7 cm, 9 cm and 11 cm long. *(8 marks)*

20. A bus company runs a fleet of two types of buses operating between Lilongwe and Blantyre. Type A bus has capacity to take 52 passengers and 200 kg of luggage. Type B carries 32 passengers and 300 kg of luggage. On a certain day, there were 500 passengers with 3 500 kg of luggage to be transported. The company could only use a maximum of 15 buses altogether.

(a) If the company uses x buses of type A and y buses of type B, write down all the inequalities satisfied by the given conditions. *(5 marks)*

(b) Represent the inequalities graphically and use your graph to determine the smallest number of buses that could be used. *(5 marks)*

(c) If the cost of running one bus of type A is K 7 200 and that of running one bus of type B is K 6 000, find the minimum cost of running the buses. *(5 marks)*

21. In a residential estate, 100 heads of families were interviewed regarding patronage of the social clubs A, B, C in their neighborhood. The results were as follows:
10 people said they patronise all the three,
17 patronise clubs A and B,
19 patronise B and C,
18 patronise A and C,
15 patronise A only,
18 patronise B only,
17 patronise C only,
Using a Venn diagram, find *(7 marks)*

(a) how many people patronise club A.

(b) how many people patronise B and C but never A.

(c) how many people do not patronise any of the three clubs. *(8 marks)*

22. A flag post stands on top of a church tower. From a point on level ground, the angles of elevation of the top and bottom of the flag post are 40° and 30° respectively. Given that the flag post is 6 m long,

(a) draw a sketch of the above *(2 marks)*

(b) calculate the height of the church tower *(5 marks)*

(c) calculate the distance of the observer from the church tower. *(3 marks)*

(d) Use the sine rule to calculate the distance from the observation point to the top of the flag post. *(5 marks)*

MODEL PAPERS
SET II PAPER 1

Answer all the questions.

1. (a) Given that $\tan \theta = \frac{15}{8}$ and that θ is acute, find sin θ and cos θ without using tables. *(3 marks)*

   (b) If 5 sin θ = 3 cos θ, find tan θ without evaluating the angle. *(2 marks)*

2. Consider two similar containers having capacities of 64 litres and 27 litres respectively. If the smaller container has a height of 18 cm, what is the corresponding height for the larger one? *(3 marks)*

3. Δ ABC has coordinates A (–3, 1) B (–3, 4), C (–1, 4). Δ A′B′C′ has vertices A′ (–1,–3), B′ (–1, 3) and C′ (3, 3). Draw the two triangles on the same axis. Find the centre and enlargement factor of an enlargement that maps Δ ABC on to Δ A′B′C′. *(4 marks)*

4. Find the gradient of a line that passes through points A (–2, 1) and B (4, –4). Hence find the equation of the line. *(3 marks)*

5. A line passes through point (5, 3) and is parallel to the line $4x + 2y = 7$. Find its equation. *(3 marks)*

6. Fig. MP 2.1 is a graph of motion of a particle over a period of 20 seconds. Calculate the distance travelled by the particle and its average speed. *(5 marks)*

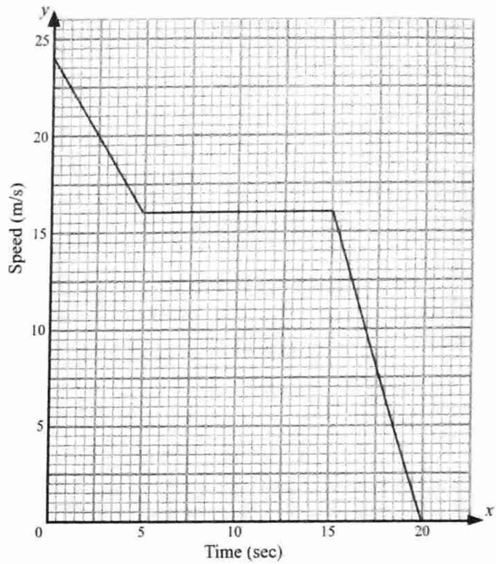


Fig. MP 2.1

7. Draw the graph of the function $x \rightarrow x^2$ for $x = \{-3, -2\frac{1}{2}, -2 \ldots\ldots 2, 2\ \ , 3\}$. Join the points with a smooth curve and hence estimate the value of $\sqrt{5}$. *(4 marks)*

8. A chord, 15 cm long, subtends an angle of 60° at the centre of a circle. Using π = 3.142, calculate:

   (a) the radius of the circle, *(3 marks)*

   (b) the length of the minor arc. *(2 marks)*

9. Find the inequalities which are satisfied by the unshaded region in the Fig. MP 2.2. *(6 marks)*

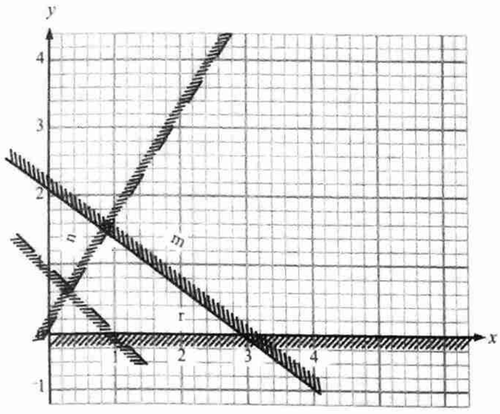


Fig. MP 2.2

10. The probability that a husband and his wife will be alive 25 years from now is 0.7 and 0.9 respectively. Using a tree diagram, find the probability that in 25 years time,

(a) both will be alive, *(2 marks)*

(b) neither will be alive, *(2 marks)*

(c) one of them will be alive, *(3 marks)*

(d) at least one will be alive. *(3 marks)*

11. (a) **A** and **B** are matrices such that

$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. Given that $\mathbf{A}^2 = \mathbf{AB}$,

find **B.** *(2 marks)*

(b) Given that $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 5 & ^-1 \end{pmatrix}$ and

$\mathbf{C} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ and that $\mathbf{AB} = \mathbf{BC}$,

determine the value of p and q. *(2 marks)*

12. Solve for x in the equation $\log(x - 1) = \log 2 - \log(x - 2)$ *(4 marks)*

13. Use the factor theorem to solve the cubic equation $6x^3 - 25x^2 + 3x + 4 = 0$ *(4 marks)*

14. Use the substitution method to solve the simultaneous equations. $x^2 + y^2 = 25$ and $x + y = 7$ *(4 marks)*

15. Use the quadratic formula to solve the equation $2x^2 + 7x - 2 = 0$ *(4 marks)*

16. (a) Make a table of values for the cubic function $y = x^3 - 3x^2 - 4x + 12 = 0$ for values of x such that $^-3 \leq x \leq 4$.

(b) Draw the graph of the function and use it to solve the equation $x^3 - 3x^2 - 4x + 12 = 0$ *(4 marks)*

17. Find the sum of the series $5 + 9 + 13 + \ldots + 12^{\text{th}}$ term. *(3 marks)*

18. Find the sum of the series

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ \ldots\ldots + \frac{1}{2^{10}}$ *(3 marks)*

19. (a) In ΔABC, BC = 6 cm, AC = 10 cm and $\angle C = 48°$. Calculate the area of the triangle. *(3 marks)*

(b) A cuboid has a rectangular base of sides 3 cm by 8 cm and a height of 5 cm. Find the total surface area of the cuboid. *(3 marks)*

(c) An arc of a circle of radius 6 cm subtends an angle of 64° at the centre of the circle. Find the area of the minor sector. *(2 marks)*

20. With reference to Fig. MP 2.3, state

(a) all the lines that are perpendicular to BC, *(2 marks)*

(b) all the lines that are parallel to plane ADHE, *(2 marks)*

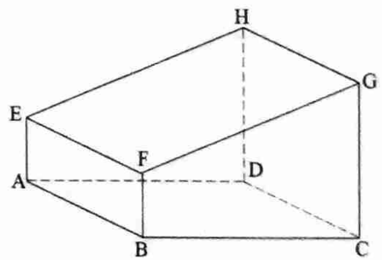


Fig. MP 2.3

(c) which of the following sets of lines or points determine a plane: *(1 mark)*

(i) A, H, G, B (ii) AD and FG

(iii) FB and HD (iv) A, H, G, C

(d) the projection of line

(i) EG onto plane ABCD,

(ii) FG onto line BC. *(2 marks)*

MODEL PAPERS
SET II PAPER II

SECTION A (55 marks)

Answer all the questions.

1. Make n the subject of the formula

$$P = {}^{-}3\left(\sqrt{\frac{m-n}{1+mn}}\right). \qquad \textit{(3 marks)}$$

2. Find the standard deviation of the following values, giving your answer correct to 3 s.f.

   4, 7, 6, 4, 8, 6, 9, 11, 10, 11.

   (3 marks)

3. (a) If $f(x) = -2 + x$, find $f(0)$ *(2 mark)*

   (b) If $f(x) = x^2 - 3x + 1$, find $f(-4)$ *(3 marks)*

4. If $n(P) = 13$, $n(Q) =$ and $n(PQ) = 16$, draw a venn diagram to show sets P and Q and find $n(PQ)$. *(5 marks)*

5. A variable V varies jointly as the variables A and h. When A = 63, and $h = 4$, V = 84. Find

   (a) the value of V when A= 9 and $h = 7$. *(4 marks)*

   (b) the value of A when V = 4.5 and $h =$ 0.5. *(2 marks)*

6. AB and XY are two intersecting chords of a circle. They meet at R such that AR = 4 cm, XR = 5 cm and RY = 3 cm. Calculate the length of AB. *(4 marks)*

7. A science club is made up of 5 boys and 7 girls. The club has 3 officials. Using a tree diagram, find the probability that

   (a) the club officials are all boys

   (b) two of the officials are girls

8. Find without using tables or a calculator the value of:

   (a) $\dfrac{\cos 45}{\sin 30}$ *(2 marks)*

   (b) $\dfrac{\sin 60^\circ}{\cos 30^\circ + \sin 30^\circ}$ *(3 marks)*

   Leave your answers in their surd form where possible.

9. Find the value of y in the equation:

$$\frac{243 \times 3^{2y}}{729 \times 3^{y} \div 3^{(2y-1)}} = 81 \qquad \textit{(3 marks)}$$

10. (a) I think of number x, square it and subtract three times the original number. My answer is ${}^{-}2$. Find the number x. *(3 marks)*

    (b) Two consecutive odd numbers have a product of 195. Find the numbers. *(3 marks)*

11. A car travels 280 km in $3\frac{1}{2}$ hours. It is also given that a train travels 60 km in 48 minutes. Find

    (a) Average speed of the car in km/h. *(2 marks)*

    (b) Average speed of the train in km/h. *(2 marks)*

    (c) The ratio of the average speed of the car to that of the train. *(1 marks)*

12. The first term of a geometric progression is 3.4. Given that the fifth term is 54.4, find the sum of the first ten terms of the series. *(5 marks)*

GLOSSARY

Acceleration: It is the rate of change of velocity.

An event: It is a set of outcomes that are part of the possibility space of an experiment.

An identity: It is an equation whose solution set is the set of all real numbers.

Arithmetic progression (A.P.): It is the series obtained when the terms of an arithmetic sequence are added.

Arithmetic sequence: It is a sequence in which any two consecutive terms differ by the same number, i.e. a constant.

Collinear: Three or more points that are in a straight line.

Column matrix: It is a matrix of order $m \times 1$.

Common difference: It is the constant number by which the consecutive terms in an arithmetic sequence differ.

Common ratio: It is the constant value between any two consecutive terms in a geometric sequence.

Composite solid: It is a solid formed by two or more solids e.g. a cone and a hemisphere.

Continuous possibility: It is an uncountable possibility space i.e. the outcome cannot be counted.

Cubic equation: It is an equation of the form $ax^3 + bx^2 + cx + d = 0$, where a is not equal to zero.

Discrete sample space: It is a sample space where the sample points are countable.

Displacement: It is the distance covered by an object moving in a particular direction.

Edge: It is a line along which two faces of a solid meet.

Equivalent vectors: These are two or more vectors that have the same magnitude and the same sense of direction.

Face: It is a flat surface of a solid.

Finite possibility space: It is a possibility space with specific numbers of outcomes.

Frustum: It is the bottom part that remains when a cone or pyramid is cut through a plane parallel to its base.

Geometric Progression (G.P.): It the series obtained when the terms of a geometric sequence are added.

Geometric sequence: It is a sequence in which the ratio between any two consecutive terms is a constant value.

Identity matrix: It is a matrix which has each elements in the leading diagonal as ones and all the other elements as zeros.

Independent events: These are two or more events whose probability of them occurring together is the product of their individual probabilities.

Linear programming: It is the process of finding the possible solutions to a given problem (an inequality).

Matrix: It is a rectangular array of numbers whose value and position in the arrangement is significant.

Mutually exclusive events: These are two or more events in which the occurrence of one eliminates the possibility of the other one occurring.

Objective function: It is a function we wish to maximise or minimise when drawing a graph in linear programming.

Planes: It is a flat surface enclosed by three or more lines.

Range: It is the difference between the largest and smallest values in a distribution.

Row matrix: It is a matrix of order $1 \times n$.

Sample space: The list of all possible outcomes of an experiment (possibility space).

Scalar quantity: It is a quantity that has no direction.

Secant: It is a line, which cuts a circle at two distinct points.

Series: It is the sum of the terms of a sequence.

Skew lines: These are lines in space that are not parallel and do not intersect however much they are produced.

Speed: It is the rate of change of distance (i.e. the distance covered) per unit time.

Square matrix: It is a matrix of order $n \times n$.

Standard deviation: The square root of the variance (root mean squared deviation).

Tangent: A line that has one, and only one point in contact with a circle.

The order of a matrix: It denotes the number of rows and columns in the matrix.

Three-dimensional solids: These are solids that have length, area and volume.

Variance: The mean of the squares of the deviations from the mean (the mean squared deviation).

Vector quantity: It is a quantity that has both magnitude (size) and direction.

Velocity: It is the rate of change of displacement.

Vertex: It a point where, two or more lines or edges meet.

Volume: It is the area of uniform cross-section multiplied by length (or height) of the prism.

Zero or null matrix: It is a matrix whose elements are zero.

Zero or null vector: It is a vector that has no magnitude.