Arise with Mathematics

Students' Book 3

Johns J. Mwale
Numeri Chimalizeni

CLAIM mabuku
A world of quality literature

CLAIM Limited

7 Glyn Jones Road

P.O. Box 503

Blantyre

Malawi

Editor : Blanzio Dayton

Proofreader : Thokozani Eston Mkwate

Designed by : Emmanuel Chikaonda
Joe Kima Phulusa

First Published 2014

ISBN : 978-99960-35-69-2

Contents

Dedication

This book is dedicated to Benjamin Chimalizeni and Clever Mwale who are the last born to Mr Chiamilzeni and Mr Mwale respectively. May you be inspired by this work as you journey in this life. God be with you and your brothers and sisters and not forgetting your mothers.

Acknowledgements

We would like to thank God for protecting and keeping us of good health and seeing us through the entire period of writing this book.

Thanks also go to Mrs Janet Mwale and Mrs Gladys Chimalizeni for their support and encouragement during the time the book was being written. We know, our time was dedicated to this work rather than you but you were patient with us. May God bless you all.

We would also like to thank Mr Chisamba, Mr Yohane, Mr Muphuwa , Mr Bob Tsapa and other members of CLAIM for their encouragement and support. We were about to give up but you came forward and encouraged us. May the Almighty God bless you.

We also extend our thanks to Mr Emmanuel Chikaonda, Mr Thokozani Mkwate for their wonderful work and making the book look like this.

Our gratitudes also go to the Head teachers and staff of Robert Blake, Madisi, and Mwansambo secondary schools for their moral support and encouragement. We are proud of you.

May we also thank all those we consulted and assisted in the development of this book and not forgetting Pastor Benjamin Makoto of Dowa for his guidance. May God bless you. We hope all those who use the book will find it useful.

Unit 1

QUADRATIC EQUATIONS

In your JCE Mathematics you learnt about quadratic expressions. You learnt how to factorise and expand such expressions. Recall that expanding and factorising of quadratic expressions are opposite operations. In this unit you will learn about quadratic equations and you will apply the skills you acquired in your JCE Mathematics. Specifically you will solve quadratic equations by making use of a number of methods such as factorisation, completing squares, and using a quadratic formula. You will finally apply quadratic equations to solving everyday problems.

Definition: Quadratic equation

Factorising quadratic expressions

A quadratic expression is any expression in mathematics which takes the form $ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$ .You will learn more about expressions of this form in this unit.

Activity 1:

Identifying quadratic expressions

Which of the following expressions are quadratic expressions?

a) $x^2 + 2x + 1$

b) $x^3 + 3x + 2$

c) $2x^2 + 1$

d) $x^2 - 1$

e) $x + xy + 2$

f) $x^4 + 3x + 2$

Understanding factorisation

Definition: Factorisation

In Mathematics, *factorisation* is the decomposing of a number or an expression into a product of other numbers or factors, which when multiplied together will you give the original number or expression.

For example, the number 20 can be factorised as 4 x 5 or 2 x10. And the expression x^2- 4 can be factorised as $(x + 2)(x – 2)$. In both of these examples, a product of simpler numbers or expressions is obtained.

Have you ever asked yourself why you need to learn factorisation? Or have you wondered why you should bother yourself factorising an expression at all?

The aim of factoring or factorisation is usually to reduce an expression to some simple expressions.

Before you learn how to factorise, it is important for you to know different types of quadratic expressions. Remember that in a quadratic expression, the highest power of the letters is a 2.

Types of quadratic expressions

Quadratic expressions are classified basing on how many terms are in the expression:

a) **Quadratic Monomial Expression**

A monomial is an expression in algebra which contains one term. A quadratic monomial expression has one term only. Examples of quadratic monomials include x^2, $2x^2$, ½ x^2, $5x^2$, and many others. Think about five more examples of quadratic monomials.

a) **Quadratic Binomial Expressions**

A binomial is an expression in algebra which contains two terms. **`*bi*` means `*2*`**. A quadratic monomial expression has two terms in it. Examples of quadratic monomials include

$x^2 + 2x$, $2x^2+3$, $5x^2 + 7$ and many others. Think about five more examples of quadratic binomial expressions.

b) **Quadratic Trinomial Expressions**

A trinomial is an algebraic expression with three terms. **`Tri` means `3`**. A quadratic trinomial has three terms in it. The following expressions are examples of trinomials:

i) $d^2+7d + 10$

ii) $x^2 + 11x + 10$

iii) $c^2 + 8c + 15$

iv) $x^2 + 5x + 6$

v) $f^2 + 7f + 6$

vi) $g^2 + 10g - 24$

vii) $x^2 + 2x\ 24$

Factorising quadratic expressions

Factoring trinomials of the type $x^2 + bx + c$

The standard format for a trinomial is $ax^2 + bx + c$. In this section, you will factorise trinomials where the coefficient of the x^2 term is 1, ($a = 1$).

When we factor trinomials of the form $x^2 + bx + c$ you are finding 2 binomials that will multiply out to give this initial polynomial. Use the following steps to factor out this type of trinomials:

1) Draw two sets of parentheses for the 2 binomials you are looking for: $x^2+6x+3+5= (\quad)(\quad)$

2) Put an x on the first terms of the two binomials: $x^2+2xy-15y^2 = (x-3y)\ (x+5y)$.

3) Find 2 integers whose product is c and whose sum is b. It helps to list all the possible factors of c and check to see which set of factors sum up to.

4) Put these integers on the last terms of the two binomials.

You can read the steps above, over and over again until you fully understand what we are saying about factorising quadratic trinomials.

Examples:

Factorising quadratic trinomials

Factorise the following quadratic trinomials:

a)	$x^2 + 6x + 5$	b)	$a^2 - 9a + 20$
c)	$m^2 + 4m - 21$	d)	$t^2 - 15 - 2t$
e)	$x^2 + 2xy - 15y^2$	f)	$2a^2 - 16a + 32$

Answers:

a) $x^2 + 6x + 5 = (\quad)(\quad)$

$= (x \quad)(x \quad)$

$=$

Factoring Trinomials in the form $ax^2 + bx + c$

In this section, you will factor out trinomials where the coefficient of the x^2 term is a number other than $1 (a \neq 1)$.

When you factor trinomials of the form $ax^2 + bx + c$ you are also finding 2 binomials that will multiply out to give the initial polynomial. There are 2 methods that you can use to factor this type of trinomials.

1. Factor with FOIL (Trial and Error)
2. Factor by Grouping – also called the Master Product Method or the "AC" Method

Definitions

Monomial is an algebraic expression containing one term. eg $2x^2$

Binomial is an algebraic expression containing two terms, for example $5x^2 + 2$.

Trinomial is an algebraic expression containing three terms, for example $x^2 + 5x + 6$.

Factorising quadratic expressions

Example 1:

Factorisation

Factorise the following;

(a) $x^2 + 7x + 12$

Solution;

Start by writing (x) (x)

Find numbers which multiply to give +12 and which add to give + 7

Which numbers are these?

$\therefore\ x^2 + 7x + 12 = (x + 3)(x + 4)$

The process of finding the numbers will not be very easy for a start but with more practice, you will find it not too hard.

(b) $x^2 + 4x - 12$

Solution;

Here the two numbers must multiply to give –12, this means one number is positive and the other negative. The same two numbers should add to give +4. The two numbers are +6 and –2.

$\therefore\ x^2 + 4x - 12 = (x + 6)(x - 2)$

(c) $x^2 - 11x - 12$

Solution:

Here the two numbers should multiply to give -12 and again one of the numbers is positive and the other negative. Since the two numbers add to -11, the smaller is positive and the larger is negative. The numbers are +1 and –12

$\therefore x^2 - 11x - 12 = (x + 1)(x - 12)$

(d) $h^2 - 8h + 12$

Solution

Here the two numbers should multiply to give +12 and add to give –8, this means the two numbers must both be negative the numbers are –2 and –6

$\therefore h^2 - 8h + 12 = (h - 2)(h - 6)$

Exercise 1a

Factorise the following quadratic expressions.

1. $d^2 + 7d + 10$
2. $x^2 + 11x + 10$
3. $c^2 + 8c + 15$
4. $x^2 + 5x + 6$
5. $f^2 + 7f + 6$
6. $g^2 + 10g - 24$
7. $x^2 + 2x - 24$
8. $b^2 + 5b - 24$
9. $x^2 + 23x - 24$
10. $m^2 - 23m - -24$
11. $v^2 + 12v + 36$
12. $x^2 - 2x - 24$
13. $y^2 - 10y - 24$
14. $m^2 - m - 30$
15. $g^2 - 6g - 40$

The expressions you have looked at so far have a **co-efficient of 1** on the square term. Sometimes the **co-efficient is more than 1**.

Activity 2:

Coefficients

a. What is the difference between the coefficient of x^2 in these two expressions

(i) $x^2 - 37x + 36$ and

(ii) $2x^2 - 17x - 10$

a. Factorise $6x^2 - 17x - 10$

Compare your work with friend and the examples given below.

Example 2:

Factorisation

Factorise the following:

(a) $2x^2 + 11x + 12$

Solution

To make $2x^2$ we require one bracket to contain 2x and the other x i.e.

$(2x \quad)(x \quad)$. We should look for two numbers which multiply to give +12 and when put in the two brackets, they should multiply with 2x and x and their sum simply to give 11x, by inspection the first number is 3 and the second is 4.

i.e. $2x^2 + 11x + 12 = (2x + 3)(x + 4)$

(b) $4x^2 + 12x + 9$

Trying $(4x \quad)(x \quad)$ does not work and we have to split the 4.

i.e. $(2x \quad)(2x \quad)$

And the numbers are +3 and +3

$\therefore\ 4x^2 + 12x + 9 = (2x + 3)(2x + 3)$

$= (2x + 3)^2$

Exercise 1b

Factorise the following expressions

1. $2b^2 + 3b + 1$
2. $2t^2 + 7t + 6$
3. $3q^2 + 8q + 4$
4. $3x^2 + 13x + 4$
5. $2d^2 + d - 6$
6. $3x^2 + 11x - 4$
7. $9y^2 - 6y + 1$
8. $5x^2 - 13x - 6$
9. $4w^2 + 17w - 15$
10. $6x^2 - 17x - 10$
11. $a^2b^2 + 7ab + 10$
12. $2u^2v^2 + uv - 6$
13. $3t^2 + 5st + 2s^2$
14. $m^2n^2 + 4mn - 21$
15. $35 - 2u - u^2$
16. $35 + 30d - 5d^2$
17. $x^2 - 2xy - 15y^2$
18. $x^2y^2 - xy - 30$
19. $10p^2 - 43p + 45$
20. $x^2 + 16xy - 36y^2$

Now look at these two expressions;

$9x^2 - 6x + 1$ and $9x^2 - 6x + 1 = 0$,

How do they differ?

Now you will go further solving such expressions.

Solving Quadratic Equations

Activity 3

a. Describe a quadratic equation.

b. Write an example of a quadratic equation.

c. Solve the equation you came up with.

Report your work to the class.

Definition;

Quadratic equations are equations of the form $ax^2 + bx + c = 0$.

There are different methods for solving quadratic equations. Can you mention them? You are going to look at some of them.

a. **The factor method.**

How do you factorise these quadratic expressions? Now look at these examples.

Example 3

Solve the following;

(a) $x^2 + x - 6 = 0$

Solution;

Factorise the left-hand side as before

i.e. $x^2 + x - 6 = (x + 3)(x - 2)$ …. These are the factors

$\therefore$ $(x + 3)(x - 2) = 0$

either $x + 3 = 0$ or $x - 2 = 0$

$\therefore x = -3$ or $x = 2$

(b) $3x^2 - 11x (-20) = 0$

Solution;

Multiply $3x^2 x(-20) = -60x^2$

Find two factors of $-60x^2$

Adding give $-11x$ and multiplying them give $-60x^2$

These are $-15x$ and $4x$

$3x^2 - 15x + 4x - 20 = 0$

$3x(x - 5) + 4x(x - 5) = 0$ ………………… factorise

$(3x + 4)(x - 5) = 0$ ……………………….. factor out $(x - 5)$

either $3x + 4 = 0$ or $x - 5 = 0$

$\therefore$ $x = -^4/_3$ or $x = 5$

Exercise 1c

Solve the following quadratic equations by factorisation

1. $f^2 + 3f + 2 = 0$
2. $s^2 + 11s + 18 = 0$
3. $x^2 + 7x + 6 = 0$
4. $r^2 + 16r + 15 = 0$
5. $x^2 - 8x + 12 = 0$
6. $t^2 + 3t - 10 = 0$

7. $p^2 - 2p - 15 = 0$

8. $x^2 - 3x - 54 = 0$

9. $t^2 + 12t + 27 = 0$

10. $m^2 - 3m - 10 = 0$

11. $x^2 + 4x - 32 = 0$

12. $2x^2 - 5x + 2 = 0$

13. $2n^2 - 10n + 12 = 0$

14. $3d^2 + 5d - 12 = 0$

15. $35 + 30d - 5d^2$

You will then turn to the other method.

b. Completing the square of a quadratic expression

Activity 4

1. What do you call an expression such as $(x + 1)^2$?

2. Make $x^2 + 8x + 16$ in the form as above and hence solve $(x+4)^2 = 0$

3. Brainstorm on other quadratic expressions which can be solved in the same way.

   Dicuss your finding to the class.

   The following are examples of the square terms: x^2, $(2x)^2$, $(x - 2)^2$, $(x + 3)^2$ etc.

   Recall also that $x^2 = 16$

   $$x = \pm\sqrt{16}$$
   $$x = \pm 4$$

   So $(x-3)^2 = 25 \therefore x - 3 = \pm\sqrt{25}$

   $$x = 3 \pm 5$$
   $$x = 8 \text{ or } x = -2$$

An expression $x^2 - 4x + 4$ can be factorised as $(x - 2)(x - 2) = (x - 2)^2$, a square.

Not all quadratic expressions can factorise into squares.

Expressions such as $(x - 2)^2$, $(x + 3)^2$ are called **Perfect squares**

You are going to learn a method of Completing the square of any quadratic expression.

Example 4:

Completing squares

Solve the following quadratic equations by completing the square.

a) $x^2 + 6x + 8 = 0$

Divide the co-efficient of x by 2, add to x and square the result.

i.e. $(x + 3)^2$, if we expand this we get $x^2 + 6x + 9$, we must take away 1

i.e. $x^2 + 6x + 8 = (x+3)^2 - 1 = 0$

$\therefore (x+3)^2 = 1$

$\therefore x + 3 = \pm\sqrt{1}$

$x + 3 = \pm 1$

$\therefore \quad x = -2 \text{ or } x = -4$

b) $2x^2 - 10x + 9 = 0$

The first step is to divide through by the co-efficient of x^2, in this case 2 i.e. $x^2 - 5x + 4\frac{1}{2} = 0$

then $x^2 - 5x = -4\frac{1}{2}$ (taking $-4\frac{1}{2}$ the other side)

$x^2 - 5x + (2.5)^2 = -3.5 + (2.5)^2$ (completing the square)

i.e $(x - 2.5)^2 = -4.5 + (2.5)^2$

$$= -4.5 + 6.25$$

$\therefore\ x - 2.5 = \pm\sqrt{1.75}$ (Removing the root)

$x = 2.5 + \sqrt{1.75}$ or

$x = 2.5 - \sqrt{1.75}$ (Simplifying the RHS)

To get the value of $\sqrt{1.75}$ quickly, you need to use a calculator.

Some of the questions in the exercise below require the use of a calculator.

Oral exercise

What must be added to the following expressions to make them into a perfect square? Factorise the results

a. $u^2 - u$

b. $d^2 - 6d$

c. $x^2 + 10xy$

d. $a^2 - 6ad$

e. $y^2 - 3y$

Exercise 1d

Solve the following quadratic equations by completing the square. In cases where answers cannot be given in exact form, give your answer correct to 2 d.p.

1. $b^2 + 4b + 3 = 0$
2. $c^2 - 2c + 1 = 0$
3. $t^2 - 14t + 48 = 0$
4. $e^2 - 6e - 16 = 0$
5. $x^2 - x - 6 = 0$
6. $n^2 - n - 13 = 0$

7. $r^2 - 4r - 11 = 0$

8. $v^2 + v - 18 = 0$

9. $x^2 + 2x - 7 = 0$

10. $2h^2 - 3h - 4 = 0$

11. $3x^2 - 4x - 2 = 0$

12. $4z^2 + 2z - 5 = 0$

13. $-2u^2 - 5u + 2 = 0$

14. $5t^2 - 8t - 1 = 0$

15. $-7x^2 - x + 15 = 0$

Having solved those above, can you also solve $x^2 - x - 5 = 0$, what do you notice? How can such equations be solved? Do the Activity 5 below.

The Quadratic formula

Activity 5:

A general quadratic equation is of form$ax^2 + bx + c = 0$, discuss with a friend and solve for x just like example 4b ie $2x^2 - 10x + 9 = 0$ above.

Compare with one given below.

Deriving quadratic formula

$ax^2 + bx + c = 0$ then $ax^2 + bx = -c$ (taking **c** the other side)

$x^2 + \frac{bx}{a} = -\frac{c}{a}$ (dividing by **a** on both sides)

(add both sides)

$$= \frac{-c}{a} + \frac{b^2}{4a^2}$$

$$\left(x+\frac{\mathrm{b}}{2a}\right)^2=\frac{\mathrm{b}^2}{4a^2}+\frac{-c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{\mathrm{b}^2-4ac}{4a^2}$$ (Simplifying the RHS)

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a}}$$ (Taking the square root on both sides)

$$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2}-4ac}{2a}$$

This is called the **quadratic formula** and it is used to solve quadratic equations.

Note, $\mathbf{b^2-4ac}$ under the root is called a **discriminant.** It indicates the number of solutions the quadratic expression has.

Example 5

Solve the following quadratic equation using the quadratic formula.

$x^2 + 2x - 15 = 0$

Here a = 1, b = 2 and c = –15

$\therefore$ Using the quadratic formula

$$x=-2\pm\frac{\sqrt{2^2-4\times1\times(-15)}}{2\times1}$$

$$x=-2\pm\frac{\sqrt{4+60}}{2}$$

$$x=-2\pm\frac{\sqrt{64}}{2}$$

$$x = \frac{-2 \pm 8}{2}$$

$x = 3$ or $x = -5$

Exercise 1e

Solve the following quadratic equations using quadratic formula. In some cases, you need to use a calculator or log tables. Give your answers to 2 d.p.

1. $a^2 + 7a + 12 = 0$
2. $x^2 - 2x - 24 = 0$
3. $p^2 + 6p - 40 = 0$
4. $y^2 - y - 6 = 0$
5. $x^2 + 2x - 8 = 0$
6. $e^2 + e - 9 = 0$
7. $x^2 - 3x - 3 = 0$
8. $t^2 - 2t - 2 = 0$
9. $x^2 - x - 5 = 0$
10. $x^2 + 2x - 7 = 0$
11. $5c^2 - 8c + 1 = 0$
12. $3z^2 - 4z - 2 = 0$
13. $x^2 - 6 = 0$
14. $x^2 - 8 = 0$

Formulating quadratic equations when roots are given

Activity 6a:

A car at times needs to reverse, so too in quadratic equations.

If a quadratic equation has roots $x = 3$ and $x = -5$, come up with its equation. Share your work with neighbour. Did you solve as those given below?

Example 6:

Formulating quadratic equations

Formulate the equation whose roots are:

a) $x = 2$ and $x = -3$

b) $x = 5$ and $x = ½$

a) If $x = 2$ then $x - 2$ was factor and

If $x = -3$ then $x + 3$ was factor

$\therefore$ the equation is $(x - 2)(x + 3) = 0$

i.e. $x^2 + x - 6 = 0$

b) $x - 5$ and $x - ½$ were the factor so $(x - 5)(x - ½) = 0$ is the equation

$x^2 - 5½x + {}^5/_2 = 0$ …multiply by 2 throughout to get rid of the fractions

$2x^2 - 11x + 5 = 0$

Exercise 1f

Find the quadratic equation whose roots are:

1. $x = 3$ and $x = 1$
2. $x = -7$ and $x = 2$

3. $f = -3$ and $f = -2$
4. $x = 0$ and $x = 4$
5. $x = -3$ and $x = 1$

Is it possible to have everyday life situation expressed as quadratic expression and solve it? Do the activity below.

Formulating quadratic equations from Practical problems

Activity 6b:

a. Get a piece of paper or card board which is rectangular.
b. Using a piece of paper or string equal to the width (y) of the rectangular paper.
c. Place the piece of paper or string along the length of the rectangle and mark where it ends.
d. Using a ruler, measure the remaining part of the length of rectangle and record its reading.
e. Derive an expression for finding the area of the rectangle.

In most cases, you have to formulate quadratic equations from everydaylife situations and solve them to find the values of the unknown like the one above.

Example 7:

Formulating quadratic equations

a. Benjamin is x years old and his sister Susan is 5 years younger. If the product of their ages is 36, form an equation in x and solve it to find Benjamin's and Susan's age.

   Benjamin is x years old Susan is $(x - 5)$ years old and

   $x(x - 5) = 36$ is the product of their ages

$x^2 - 5x = 36$

$\therefore \quad x^2 - 5x - 36 = 0$

Factorising we have $(x - 9)(x + 4) = 0$

$\therefore \quad x = 9$ or $x = -4$

-4 years does not make sense therefore $x = 9$

So Benjamin is 9 years old and Susan is 4 years old

b. When a number x is added to its square, the total is 12. Find two possible values of x.

Let the number be x, the other is x^2

The equation is $x + x^2 = 12$

$x^2 + x - 12 = 0$ collecting the terms to one side two factors of 12 when added give +1 and multiplied give –12, 4 and –3 $(x + 4)(x - 3)$ factorising

$\therefore x = -4$ or $x = 3$.

Exercise 1g

In the following questions, formulate equations from the information given and then solve to find the unknown.

1. If the area of the rectangle above is $28cm^2$, calculate the value of x and hence find the length of the rectangle.
2. A rectangle is 5cm longer than it is wide. If its width is x cm and its area is $66cm^2$ form an equation in x and solve it. Hence, find the dimensions of the rectangle.
3. A right - angled triangle has a length of x cm and a height of $(x - 1)$cm. If its area is $15cm^2$ calculate the base length and height.
4. The square of a number x is 16 more than six times the number. Form an equation in x and solve it.

5. When five times a number x is subtracted from the square of the same number, the answer is 14. Form an equation in x and solve it.

6. I think of a number x, if I square it and add to it the number I first thought of, the total is 56. Find the number I first thought of.

7. Grace has x^2 marbles. Duncan has x marbles. The sum of their marbles is 90. Find the number of marbles each one has.

8. Mercy is x years old and her sister is 5 years older. If the product of their ages is 104, form an equation in x and solve it to find Mercy's age.

9. A right-angled triangle ABC has $<B = 90°$, AB = x cm, BC = 2 cm longer than AB and AC is 4cm longer than AB.

a) Illustrate this information on a diagram.

b) Using this information show that $x^2 - 4x - 12 = 0$

c) Solve the above equation and find the length of each of the three sides.

Unit summary

- Quadratic equation is the expressions of the form $ax^2 + x + c$. These fall under monomial, binomial and trinomial. The unit factorisation of quadratic has looked at expressions, solving quadratic equations by factorisation, completing square as well using quadratic formula. The unit has also looked at formulating quadratic equations from real life problems. The next unit looks at circle geometry.

Unit review exercise

1. Factorise the following quadratic expressions:

   a. $n^2 - 12n + 36$

   b. $x^2 - 15x + 36$

c. $h^2 - 5h - 24$

d. $x^2 - 20x + 36$

e. $b^2 - 5b - 36$

2. Solve the following quadratic equations

a. $y^2 - 4 = 0$

b. $x^2 - 144 = 0$

c. $16r^2 - 25 = 0$

d. $4p^2 - 81 = 0$

e. $e^2 - 2e - 63 = 0$

1. In order to deal with problems of climate change, a certain village established a rectangular forest whose diagonal was 120m. If the width is 16m less than length, find the length of the forest.

2. During the 2014 tripartite elections for Member of Parliament Mr Chimuzu got 90 votes extra than Mr Chitsinde. The winner Mrs Masamba got 3600 votes which was the product of the votes of the first two. Find the total votes for Mr Chimuzu.

3. The number of HIV/AIDS patients in two consecutive years increased by 5. If the product of the number of patients in the two years was 50, find the number of HIV/AIDS patient in the first year.

4. The oversized television at the left has a 60-inch diagonal. The screen is 12 inches wider than it is high. Find the dimensions of the screen.

Glossary

Quadratic expression: is an algebraic expression of the form $ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$

Monomial: is an algebraic expression containing one term eg $2x^2$.

Binomial is an expression containing two terms, for example $5x^2 + 2x$.

Trinomial is an algebraic expression containing three terms, eg $2x^2 + 6x + 7$

Perfect square is an algebraic expression of the form $(a + b)^2$

References

Chanon J. B. & etl, *New General Mathematics 3*, *a Modern Course for Zimbabwe* London: Longman.

Hau S. and Saiti F. (2010). *Strides in Mathematics 3*. Blantyre: Longman Malawi.

Unit 2

IRRATIONAL NUMBERS

In your JCE Mathematics you learnt about numbers. You learnt about types of numbers such as natural numbers, whole numbers, and integers. In this unit you will learn about irrational numbers. You will learn how to recognise irrational numbers from a given set of numbers. You will also learn about a special type of irrational numbers called surds and how you can simplify them.

Irrational numbers are important as they assist in being exact with certain measurements. For example, irrational numbers are used in expressing angles in trigonometry, finding length, areas, and volumes of objects.

Recognising rational and irrational numbers

Numbers which can be expressed exactly as fractions are known as rational numbers. For example:

$\sqrt{4}$, $\frac{1}{2} = 0.5$, $\frac{3}{4} = 0.75$

However, there are certain numbers which cannot be expressed as exact fractions such as: $\sqrt{3} = 1.73205...$and $\sqrt{7} = 2.645751311....$ Pi or π.

Decimals, in these examples, extend forever and are non- recurring or repetitive. Most of these are expressed as roots. They are also known as **surds**.

From the statements above, you can see that some numbers, which contain a root sign, can be evaluated exactly while others cannot.

Now you are going to look at irrational numbers in details.

Activity1:

Recognising **rational** and **irrational numbers.**

1. Define rational and irrational numbers?
2. Give examples of irrational numbers.
3. Which of the following are irrational numbers?
   (a) 3.142857…
   (b) 0.75
   (c) $\sqrt{5}$
   (d) 5.252525…

Report your answers to the class.

Compare your work with the statements given below.

Rational and irrational numbers

- Rational numbers are numbers that that can be expressed as ratio of two integers. For example ¾ = 0.75, and 5.2525... are rational numbers.
- Irrational numbers are numbers that cannot be expressed as a ratio of two integers.

In the activity above (a) 3.142857 and (c) $\sqrt{5} = 2.23606..$are irrational numbers. These numbers extend forever without repetition or recurring.

Exercise 2a

1. Which of the following are rational and which are irrational?

   (a) 7, (b) $\sqrt{16}$, (c) 0.824, (d) 0.7, (e) $\sqrt{3}$, (f) $\sqrt{17}$ (g) $\sqrt{8}$, (h) $\sqrt{99}$, (i) $\sqrt{121}$

2. Draw aright triangle ABC with angle ABC = 90^0 and AB = BC =1cm . Find AC in surd form. Is the AC rational or irrational number?

Surds

Many roots are irrational, for example; $\sqrt{3} = 1.732050\ldots$ $\sqrt{21} = 4.5825\ldots$ Irrational numbers of this kind are called **surds.** Thus if the root of a number is irrational, then it is called a **surds**. In other words; a surd is a root of irrational number.

Rules for surds

Activity 2:

Deriving the rules for surd;

In groups do this activity

By putting N = 16 and M = 9, find which of the following pairs of expressions are equal:

a. $\sqrt{MN}$ and $\sqrt{M} \times \sqrt{N}$

b. $\sqrt{NM}$ and $\sqrt{N} \times \sqrt{M}$

c. $\sqrt{M} \times \sqrt{M}$ and $\sqrt{M^2}$

d. $\sqrt{\frac{M}{N}}$ and $\frac{\sqrt{M}}{\sqrt{N}}$

e. $M\sqrt{N}$ and $\sqrt{M^2N}$

From your results, what rules can you come up about surds?

Report your finding to class.

You should have noted that each of the pairs is equal. Therefore in general, here are **the rules of surds.**

a. $\sqrt{MN} = \sqrt{M} \times \sqrt{N}$

b. $\sqrt{\frac{M}{N}} = \frac{\sqrt{M}}{\sqrt{N}}$

c. $M\sqrt{N} = \sqrt{M^2N}$

d. $\sqrt{M} \times \sqrt{M} = \sqrt{M^2} = M$

c. $M\sqrt{N} + P\sqrt{N} = (M+P)\sqrt{N}$

d. $M\sqrt{N} - P\sqrt{N} = (M - P)\sqrt{N}$

Simplifying surds

In simplification of surds, numbers under the square root sign are reduced as much as possible. This is done by expressing the numbers under the square root sign as a product of two numbers in which one of them is a perfect square.

Activity 3:

Simplifying surds

In pairs, apply knowledge from activity above.

Simplify each of the following as far as possible by applying the rules above.

(a) $\sqrt{12}$

(b) $\sqrt{20}$

(c) $\sqrt{32}$

Check your answers against your friends' work.

Now look at the following examples and compare them with your work above.

Example 1:

Simplifying surds

Simplify each of the following as far as possible.

(a) $\sqrt{162}$

(b) $\frac{\sqrt{63}}{3}$

Solutions

In each case, you must look for two factors for a given number of which one should be a perfect square. I.e. $\sqrt{80} = \sqrt{16} \times \sqrt{5}$ and 16 is a perfect square.

(a) Solution;

$\sqrt{162} = \sqrt{81 \times 2}$ .... simplify $162 = 81 \times 2$

$= \sqrt{81} \times \sqrt{2}$ .... surd rule

$= 9 \times \sqrt{2}$ .... find root of 81

$= 9\sqrt{2}$

(b) Solution;

$\frac{\sqrt{63}}{3} = \frac{\sqrt{9 \times 7}}{3}$

$= \sqrt{9} \times \sqrt{7}$ .... Surd rule

$= \frac{3 \times \sqrt{7}}{3}$ .... Cancel out 3

$= \sqrt{7}$

Now do the following exercise.

Exercise 2b

Simplify each of the following:

(1) $\sqrt{45}$ **(2)** $\sqrt{147}$

(3) $\sqrt{112}$ **(4)** $\sqrt{50}$

(5) $\frac{\sqrt{98}}{7}$ **(6)** $\sqrt{72}$

(7) $2\sqrt{28}$ **(8)** $\sqrt{200}$

(9) $\frac{\sqrt{90}}{15}$ **(10)** $\sqrt{8}$

(11) $\sqrt{343}$ **(12)** $\sqrt{48}$

(13) $\sqrt{7500}$ **(14)** $\dfrac{\sqrt{18}}{9}$

Addition and subtraction of surds

You can subtract and add surds just like other ordinary numbers.

Activity 4:

Addition of surds

Find the solution of the following;

(a) $2b + b$

(b) $2\sqrt{m^2} + m$

(c) $3\sqrt{5} + \sqrt{5}$

Discuss your findings as a class and compare your answers with the examples given below:

The knowledge above also applies when adding surds. Check the following examples.

Example 2:

Simplifying surds

Simplify the following.

(a) $\sqrt{20} + \sqrt{5} = \sqrt{4 \times 5} + \sqrt{5}$

$= \sqrt{4} \times \sqrt{5} + \sqrt{5}$ ----------------- find factors of $20 = 4 \times 5$

$= 2\sqrt{5} + \sqrt{5}$ --------------------- this is the same as $2a + a$

$= 3\sqrt{5}$

(b) $5\sqrt{18} + 4\sqrt{50} = 5\times\sqrt{9 \times 2} + 4 \times \sqrt{25 \times 2}$

$= 5 \times \sqrt{9} \times \sqrt{2} - 4 \times \sqrt{25} \times \sqrt{2}$ since $\sqrt{MN} = \sqrt{M} \times \sqrt{N}$

$= 5 \times 3\sqrt{2} + 4 \times 5\sqrt{2}$ .... taking root of 9 and 25.

$= 15\sqrt{2} + 20\sqrt{2}$

$= 35\sqrt{2}$

(a) $3\sqrt{75} - \sqrt{12} = 3 \times \sqrt{25 \times 3} - \sqrt{4 \times 3}$ ....25 and 4 have roots

$= 3 \times \sqrt{25} \times \sqrt{3} - \sqrt{4} \times \sqrt{3}$ ....separating the roots

$= 15\sqrt{3} - 2\sqrt{3}$ ....subtract just like 15b-2b

$= 13\sqrt{3}$

Exercise 2c

Simplify each of the following

(1) $\sqrt{63} - 2\sqrt{28} + \sqrt{175}$

(2) $\sqrt{200} + \sqrt{18} - 2\sqrt{72}$

(3) $\sqrt{18} - \sqrt{32} + \sqrt{50}$

(4) $\sqrt{11} + \sqrt{55} - \sqrt{77}$

(5) $5\sqrt{18} + 4\sqrt{50}$

(6) $\sqrt{12} + 3\sqrt{75}$

(7) $\sqrt{18} + 3\sqrt{2} - \sqrt{2}$

(8) $5\sqrt{6} - \sqrt{24} + \sqrt{294}$

(9) $3\sqrt{27} + \sqrt{108} - \sqrt{48} - 2\sqrt{75}$

(10) $\frac{2}{\sqrt{2}} - \frac{1}{5\sqrt{2}}$

Multiplication of surds

When two or more surds multiply each surd is simplified as far as possible and then numbers in front of square roots signs multiply each other and then surds multiply with other surds.

Example 3

Simplify the following.

(a) $\sqrt{12} \times \sqrt{18}$ (b) $3\sqrt{5} \times \sqrt{75}$

Solutions

Where possible look for two factors for a given number under the root of which one should be a perfect square as you did above and find its root.

For example: $80 = \sqrt{16} \times \sqrt{5} = 4\sqrt{5}$ and multiply as you do with ab × ab.

(a) $\sqrt{12} \times \sqrt{18} = \sqrt{4 \times 3} \times \sqrt{9 \times 2}$ ....simplify

$= 2 \times \sqrt{3} \times 3 \times \sqrt{2}$ .... find root of 4 and 9

$= 3 \times 2\sqrt{3 \times 2}$ .... rule of surd

$= 6 \times \sqrt{3 \times 2}$

$= 6\sqrt{6}$

(b) $\sqrt{20} \times \sqrt{27} = \sqrt{4 \times 5} \times \sqrt{9 \times 3}$...............simplify

$= \sqrt{4} \times \sqrt{5} \times \sqrt{9} \times \sqrt{3}$ ....surd rule

$= 2 \times \sqrt{5} \times 3 \times \sqrt{3}$ ....find root of 4 and 9

$= 2 \times 3 \times \sqrt{5} \times \sqrt{3}$

$= 6 \times \sqrt{15}$

$= 6\sqrt{15}$

Proceed as in **a** and **b** above

(c) $3\sqrt{5} \times \sqrt{75} = 3\sqrt{5} \times \sqrt{25 \times 3}$

$= 3\sqrt{5} \times \sqrt{25 \times 3}$

$= 3\sqrt{5} \times 5 \times \sqrt{3}$

$= 3 \times 5 \times \sqrt{5} \times \sqrt{3}$

$= 15\sqrt{15}$

Exercise 2d

1. Simplify the following

   a) $\sqrt{5} \times \sqrt{10}$

   b) $\sqrt{2} \times \sqrt{6} \times \sqrt{3}$

   c) $\sqrt{30} \times \sqrt{5}$

   d) $\sqrt{12} \times \sqrt{3}$

   e) $\sqrt{32} \times \sqrt{12}$

   f) $\sqrt{10} \times 3\sqrt{2} \times \sqrt{20}$

   g) $\sqrt{5} \times \sqrt{24} \times \sqrt{30}$

   h) $(2\sqrt{3})^3$

   i) $(2\sqrt{7})^2$

   j) $\sqrt{6} \times \sqrt{8} \times \sqrt{10} \times \sqrt{12}$

2. Express each of the following as the square root of a single number, using rule number (c) for surds.

(1)	$2\sqrt{6}$	(2)	$3\sqrt{7}$	(3)	$2\sqrt{6}$
(4)	$12\sqrt{2}$	(5)	$2\sqrt{10}$	(6)	$7\sqrt{2}$
(7)	$3\sqrt{3}$	(8)	$2\sqrt{51}$	(9)	$6\sqrt{3}$
(10)	$2\sqrt{17}$	(11)	$5\sqrt{3}$	(12)	$2\sqrt{22}$
(13)	$2\sqrt{73}$	(14)	$4\sqrt{5}$	(15)	$3\sqrt{15}$

Division of surds

When surds divide, both the numerator and denominator should be simplified as far as possible and then denominator should be rationalised.

Example 4:

Simplifying surds

Simplify the following.

(a) $\dfrac{3}{\sqrt{27}}$

(b) $\dfrac{\sqrt{72}}{\sqrt{75}}$

Solutions

Find two factors of a number under root as before

(a) Solution; $\dfrac{3}{\sqrt{27}} = \dfrac{3}{\sqrt{9 \times 3}}$ .... simplify $27 = 9 \times 3$

$= \dfrac{3}{\sqrt{9} \times \sqrt{3}}$ .... rule of surds

$= \dfrac{3}{3 \times \sqrt{3}}$ .... find root of 9 which is 3

$= \dfrac{1}{\sqrt{3}}$ .... cancel out 3

$= \dfrac{1}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$ .... multiply numerator and denominator by $\sqrt{3}$

$= \dfrac{\sqrt{3}}{3}$

(b) Solution; $\dfrac{\sqrt{72}}{\sqrt{75}} = \dfrac{\sqrt{36 \times 2}}{\sqrt{25 \times 3}}$

$= \dfrac{\sqrt{36} \times \sqrt{2}}{\sqrt{25} \times \sqrt{3}}$ .... simplify by surd rule

$= \dfrac{6\sqrt{2}}{5\sqrt{3}}$ .... find root of 36 and 25

$$= \frac{6\sqrt{2}}{5\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$ .... Multiply numerator and denominator by √3

$$= \frac{6\sqrt{6}}{15}$$ ....divide 6 and 15 by 3

$$= \frac{2\sqrt{6}}{5}$$

Exercise 2e

Simplify the following;

1. $\frac{\sqrt{8}}{\sqrt{32}}$
2. $\frac{2\sqrt{18}}{3\sqrt{12}}$
3. $\frac{21}{\sqrt{6}}$
4. $\frac{\sqrt{27}}{\sqrt{8}}$
5. $\frac{\sqrt{12}}{\sqrt{50}}$
6. $\frac{3\sqrt{2}}{\sqrt{6}}$
7. $\frac{8}{\sqrt{18}}$
8. $\frac{\sqrt{12} x \sqrt{18} x \sqrt{20} x \sqrt{24}}{\sqrt{8} x \sqrt{30}}$
9. $\frac{30}{\sqrt{75}}$
10. $\frac{30}{\sqrt{72}}$
11. $\sqrt{\frac{4}{5}}$

12. $\sqrt{\frac{12}{50}}$

Conjugate surds

Activity 5:

Expanding conjugate surds

(a) Factorise $(a^2 - b^2)$

(b) What term is used to describe $a^2 - b^2$?

(c) Expand $(x + y)(x - y)$

Relate your answers to the explanation given below.

Conjugate surds are surds of the form $\left(\sqrt{8} - \sqrt{7}\right)$ and $\left(\sqrt{8} - \sqrt{7}\right)$, for example $\left(\sqrt{2} + \sqrt{5}\right) \textit{and} \left(\sqrt{2} - \sqrt{5}\right)$ when multiplying conjugate surds, they are expanded the same way as you do with factors of a difference of two squares.

Example 5:

Conjugates

Conjugate and expand the following surds

(a) $\sqrt{5} + \sqrt{3}$

(b) $\sqrt{5} + \sqrt{3}$

(c) $\sqrt{5} + \sqrt{3}$

Solutions

First come up with a conjugate of $\sqrt{5} + \sqrt{3}$ in (a) which is $\sqrt{5} + \sqrt{3}$

(a) $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$ expand as you do with (a + b) (a – b)

=√2 (√2 – √5) + √5 (√2 – √5) ………… as a(a – b)+ b(a – b)

$= \sqrt{2}\times\sqrt{2}-\sqrt{5}\times\sqrt{2}+\sqrt{5}\times\sqrt{2}-\sqrt{5}\times\sqrt{5}$

= 2 + 0 – 5

= –3

(b) $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$ do as above and this leads to

$= \left(\sqrt{5}\right)^2+\sqrt{10}-\sqrt{10}-\sqrt{4}$, since $+\sqrt{10}-\sqrt{10}$ gives zero

= 5 – 2

= 3

(c) $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$

$= \left(\sqrt{3}\right)^2-\sqrt{6}+\sqrt{6}-\sqrt{4}$, since $-\sqrt{6}+\sqrt{6}=0$

= 3 – 2

= 1

Exercise 2f

a. Find the conjugates and expand each of the following surds:

(1) $\left(a+\sqrt{b}\right)$ (2) $\left(a+\sqrt{b}\right)$

(3) $\left(\sqrt{8}-\sqrt{7}\right)$ (4) $\sqrt{5}+\sqrt{3}$

(5) $\left(a+\sqrt{b}\right)$ (6) $\left(\sqrt{11}-4\right)$

(7) $\left(\sqrt{13}+\sqrt{11}\right)$ (8) $\left(a+\sqrt{b}\right)$

(9) $3+\sqrt{8}$ (10) $\left(\sqrt{43}-\sqrt{23}\right)$

b. Simplify the following without using a calculator.

1. $(3\sqrt{2} - \sqrt{3})\ (3\sqrt{2} + \sqrt{3})$
2. $(2-\sqrt{7})\ 2(2 +\sqrt{7})$
3. $2/\sqrt{2} + 6/\sqrt{2}$

Rationalising surd denominators

When the denominator of a fraction is a surd, it is very necessary to remove the surd from the denominator. This is known as rationalizing the denominator i.e. the irrational denominator is changed to a rational denominator. The process involves multiplying the number by 1 ie , which takes different forms depending on the denominator in question.

Activity 6:

Rationalising denominator

Try to rationalise the denominator by multiplying numerator and denominator by the denominator in each of the following:

(a) $\frac{2}{\sqrt{5}}$ (b) $\frac{4\sqrt{5}}{3}$

Present your work to the whole class or show your friends and compare with the example given below.

Example 6

Rationalise the denominator of the following;

a. $\frac{2}{\sqrt{5}}$

In question (a), the 1 to be multiplied is $\frac{3y}{4}$

$$= \frac{2\sqrt{5}}{5} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$\frac{\sqrt{6}}{\sqrt{3}} \times 1$$

$$= \frac{2\sqrt{5}}{5}$$

b. $\frac{\sqrt{6}}{\sqrt{3}}$

In question (b), the 1 to be multiplied is $\frac{3y}{4}$

$$\frac{\sqrt{6}}{\sqrt{3}} \times 1 = \frac{\sqrt{6}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{7}{\sqrt{3} - \sqrt{3}}$$

$$= \frac{\sqrt{18}}{3}$$

This can be simplified further to give √2, work out!

Note: In general, if a fraction is in the form of $\frac{7}{\sqrt{5}}$, then multiply both the numerator and the denominator of the fraction by $\sqrt{8}$.

c. $\frac{20}{3\sqrt{5}}$

Solution

Do as in above ,multiply by $\frac{3y}{4}$

$$\frac{20}{3\sqrt{5}} \times 1 \qquad = \frac{20}{3\sqrt{5}} \times \frac{3y}{4}$$

$$= \frac{20 \times \sqrt{5}}{3 \times 5}$$

$$= \frac{20\sqrt{5}}{15}$$

$$= \frac{4\sqrt{5}}{3}$$

Sometimes, the fraction is in the form $\frac{1}{a + \sqrt{b}}$. In this case,

you have to multiply both the numerator and denominator by $\left(a + \sqrt{b}\right)$

Similarly, if the fraction is in the form $\left(\frac{x\sqrt{z}}{y^3}\right)$, then multiply

the numerator and denominator by $\left(a + \sqrt{b}\right)$.

d. $\frac{2}{\sqrt{5} + 1}$

Solution

$$\frac{1}{\left(2 + \sqrt{3}\right)} \times \frac{\left(2 - \sqrt{3}\right)}{\left(2 - \sqrt{3}\right)}$$ .... from statement above

$$= \frac{2 - \sqrt{3}}{4 - 2\sqrt{3} + 2\sqrt{3} - \sqrt{9}}$$ .... Expanding

$$= \frac{2 - \sqrt{3}}{4 - 3}$$

$$= \frac{2}{\sqrt{5} + 1}$$

$$= 2 - \sqrt{8}$$

Exercise 2g

Rationalise the denominator of the following

(1) $\frac{7}{\sqrt{5}}$ (2) $\frac{7}{\sqrt{5}}$ (3) $\frac{7}{\sqrt{5}}$

(4) $\frac{7}{\sqrt{5}}$ (5) $\frac{7}{\sqrt{5}}$ (6) $\frac{7}{\sqrt{5}}$

(7) $\frac{7}{\sqrt{5}}$ (8) $\frac{2}{\sqrt{5}+1}$ (9) $\frac{2}{\sqrt{5}+1}$

(10) $\frac{2}{\sqrt{5}+1}$ (11) $\frac{2}{\sqrt{5}+1}$ (12) $\frac{3y+}{4}$

(13) $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ (14) $\frac{\sqrt{13}-\sqrt{7}}{\sqrt{13}+\sqrt{7}}$ (15) $\frac{7}{\sqrt{3}-\sqrt{3}}$

Real life problems of irrational numbers

In most cases real life problems of irrational numbers involve finding lengths, areas, and volumes of given objects. Most of such measurements are actually irrational numbers. The use of which is irrational number is more common in finding volume and areas of certain objects.

Exercise 2h

1. The base length of an isosceles triangular truss of a house is 14m. The height is 2m. Find the approximate length of iron sheets that can be bought assuming that they are equal to the third side.
2. The two sides of a right angled triangle are $(5\sqrt{7}-3)$and (3 – 3) cm. Find the length of hypotenuse.
3. Find the perimeter of the a triangle whose sides are $2\sqrt{48}$ cm, $3\sqrt{75}$ and $4\sqrt{147}$ cm. The answer should be in simplified

surd form.

4. A right angled ABC with AB = 5, BC = 7cm. Find the length of hypotenuse side AC in surd form.

Unit summary

- In this unit you have learnt about rational numbers which are numbers that can be expressed as exact fractions. You have also learnt about irrational numbers which cannot be expressed as exact fractions. Most irrational numbers are under root and as such are called surds.You have learnt how you can simplify surds in addition, subtraction, multiplication and division. In the next unit you are going to learn about circle geometry.

Unit review exercise

1. Without using a calculator, simplify the following

   a $\frac{(24-\sqrt{6})^2}{\sqrt{6}}$

   b $\sqrt{125} + \sqrt{5} - \sqrt{45}$

   c $(3\sqrt{5} - \sqrt{2})/(2\sqrt{5} + 3\sqrt{2})$

   d $\frac{3\sqrt{5} - \sqrt{2}}{2\sqrt{5}+3\sqrt{2}}$

   e $\sqrt{99} - \sqrt{44} - \sqrt{11}$

   f $3\sqrt{28} - 5\sqrt{63} + 4\sqrt{252}$

   g $2\sqrt{150} - \sqrt{96} - 2\sqrt{24}$

   g $(8\sqrt{2})/(\sqrt{98} - 3\sqrt{2})$

   i $1/\sqrt{2} - \sqrt{2}/3$

   j $(\sqrt{2} + \sqrt{3})(\sqrt{8} - \sqrt{12})$

2. Given that $\sqrt{2} = 1.414$ and root $\sqrt{3} = 1.732$ evaluate the

following;

(a) $1/\sqrt{3}$

(b) $3/\sqrt{3}$

(c) $10/(\sqrt{2})$

(d) $6/(\sqrt{3})$

(e) $2/(\sqrt{2})$

3. Rationalise the denominator of the following

(a) $1/(5 + \sqrt{2})$

(b) $1/(4 - \sqrt{3})$

(c) $(5 + \sqrt{3})/(\sqrt{7} + \sqrt{5})$

(d) $\sqrt{(3 + 1)}/\sqrt{(3 - 1)}$

(e) $(3 + \sqrt{5})/(\sqrt{3} - \sqrt{2})$

Glossary

Rational numbers are numbers which can be expressed as exact fractions or ratios.

Irrational numbers are numbers which cannot be written as exact fractions. The decimals extend without end and without recurring.

Surds are square roots of irrational numbers.

Reference

Hau and Saiti F. (2010). *Strides in Mathematics 3*. Blantyre: Longman Malawi.

Elaine R., McAdams P, Hiddleson P ., (2013), *CHANCO Teach yourself series, Mathematics, Second Edition*, Chancellor College Publication, Zomba.

Hardwood Clarke and Norton F. G. J., (1984) Seventh Edition,

Oxford, Heinman Educational Book Ltd, London

Channon J. B. et al (1997). *New General Mathematics 3, A Modern Course for Zimbabwe*.Longman London.

Gunsaru et al, *Secondary Mathematics Book 3,* Dzuka Publising Company, Blantyre.

Unit 3

CIRCLE GEOMETRY (CHORD PROPERTIES)

In unit 2 you learnt about irrational numbers. In this unit, you will look at the chord properties of a circle. Specifically, you will describe chord properties of a circle and apply chord properties to solve problems.

Chord properties of a chord

A circle is an important shape in the field of geometry. A circle is a shape with all points the same distance from its centre. A circle is named by its centre. Before going in detail, do activity 1 below.

Activity 1:

Identifying parts of a circle

In pairs,

i. Define radius.

ii. Using a pair of campus, draw circle with centre O and label the following parts, diameter, radius, a chord, an arc, circumference and segment.

iii. Brainstorm on the labeled parts.

Compare your work with the drawing below.

Parts of a circle Now look at figure 3.1a below which is showing some parts of the circle.

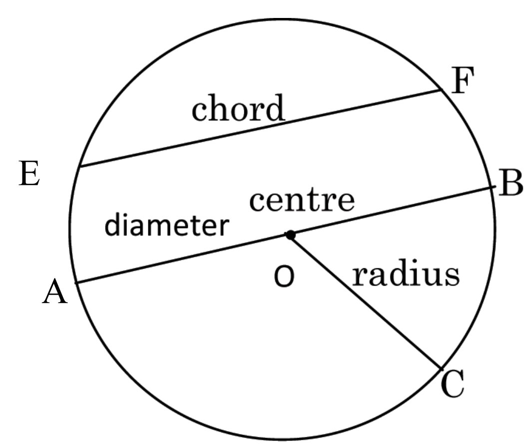


Figure 3.1a

The distance around a circle is called the ***circumference*** of a circle. Figure 3.1a is a circle with centre O.

A ***chord*** is a line segment that joins two points on a circle. Line *EF* is a chord.

A ***diameter*** is a chord that passes through the centre of the circle. Line *AB* is diameter.

A radius is a line segment that has the centre and a point on the circle as end points. In figure3.1a, *AO*, *OB* and *OC* are radii

Arcs of a circle A secant is a line containing a chord. If a secant intersects a circle in two distinct points, *A*, *B* for example in the diagram below, then the circle is divided into two sets of points.

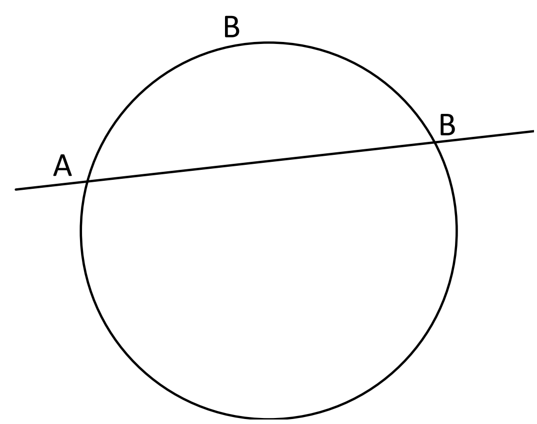


Figure 3.1b

APB is known as the minor arc while *AQB* is known as the major arc.

A sector of a circle is a region consisting of the union of an arc, the radii to the end points of the arc, and the interior points enclosed by the arc and the radii.

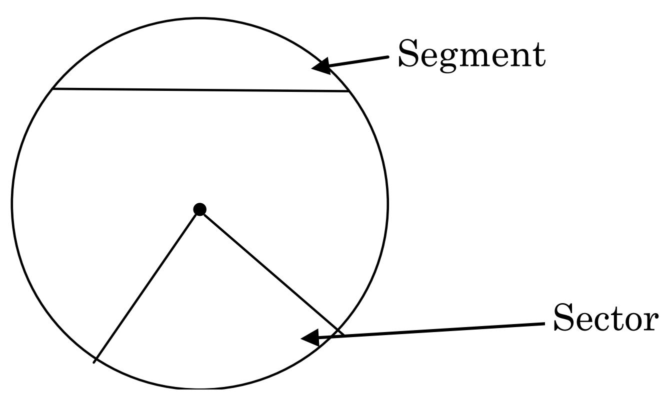


Figure 3.1c

An area bounded by chord and an arc is called a **segment** of a circle.

There might be major and minor segment. Identify these in the given circle.

The area bounded by chords and an arc is called a **sector.**

Chord property of circle

A chord is a straight line that join any two points of the on its circumference

A chord which passes through the centre of a circle is called a diameter.

(a) The straight line drawn from the centre of a circle to the mid – point of a chord is perpendicular to that chord. Conversely if a line is drawn from the centre of the circle perpendicular to a chord, it bisects the chord.

(b) If chords are equal in length then they are the same distance from the centre of a circle. Conversely if chords are the same distance from the centre of a circle then they are equal in length.

(c) If two chords of a circle are parallel, then the perpendicular bisector of one is also perpendicular bisector of the other.

Chord theorems: In your JCE Mathematics , you learnt about theorems. How did you define a theorem?

Now look at the following figure

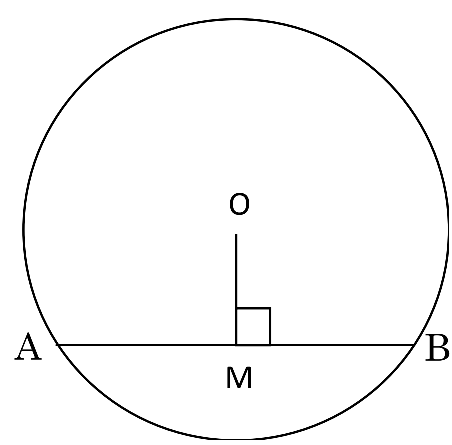


Figure 3.2

In figure 3.2, *AB* is a chord of the circle with Centre O. OM is perpendicular to a chord *AB*. Measure AM and BM. What do you notice?

Theorem 1: The line from the centre of the circle perpendicular to a chord bisects the chord.

You should show the effect of OM being perpendicular to AB.

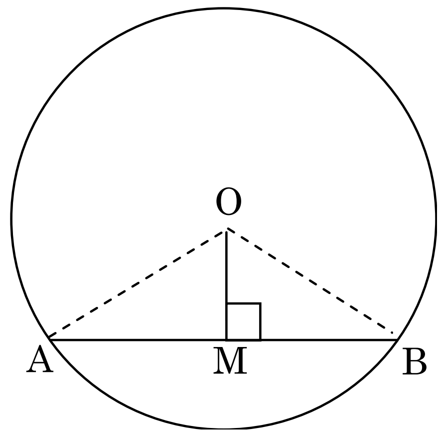


Figure 3.3

Given	:	Circle centre O, AB is a chord, $OMAB$.
To Prove	:	$AM = MB$.
Construction	:	Join OA, OB
Proof	:	In $\triangle\ OAM$ and $\triangle\ OBM$
		$OA = OB$ (radii)
		$\angle OMA = \angle OMB = 90^0$ (given)
		OM is common.
		$\therefore OAM \cong OBM$ (RHS)
		$\therefore\ AM = MB$

Conversely, the line from the centre of the circle bisecting the chord is perpendicular to the chord.

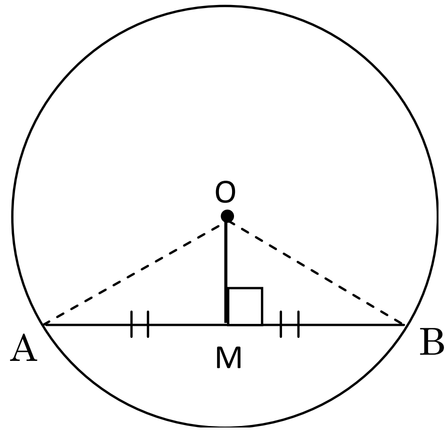


Figure 3.4

Given : Circle, centre O, AB a chord $AM = MB$

To Prove : $\angle OMA = \angle OMB = 90^0$

Construction : Join OA, OB

Proof : ΔOMA and ΔOMB

$OA = OB$ (radii)

$AM = BM$ (given)

OM is common

$\therefore\ OMA \cong \quad OMB$ (SSS)

$\therefore\ \angle OMA = \angle OMB$

But AMB is straight line

$\therefore \angle OMA = OMB = 90^0$ ($\angle$s on a straight line)

Equal chords: A circle can have several chords as shown in the following **figure 3.5** below.

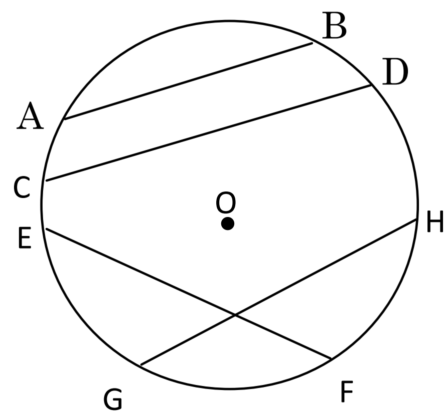


Figure 3.5

In this figure *CD*, *EF* and *GH* are all chords. In a diagram with equal chords, you want to find out what happens to their distances from the centre of the circle.

Activity 2:

Identifying distance of equal chords from the centre.

In pairs:

a. Draw a circle with centre O and radius of 5cm.

b. From the centre, draw a line of 3cm to any point E and another to any point F within the circle in opposite direction of the first line.

c. Draw a chord from a point A on one side of the circle passing through E to point B on the other side of the circle.

d. Produce another line from point C on the circle passing through F to point D on the other side of the circle.

e. Measure the length of the chord AB and CD.

f. What conclusion can you draw?

Report your findings to the whole class.

Now consider the following theorem.

Theorem 2: Equal chords are equidistant from the centre of the circle.

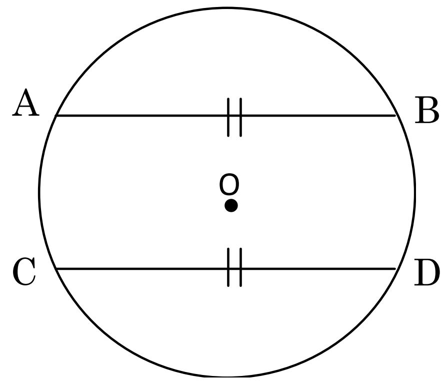


Figure 3.6

Figure 3.6 is a circle with centre *O*, and *AB* is a chord that is equal to chord *CD*. You want to show that the shortest distances (the perpendicular distances) from centre O to the chords, are equal.

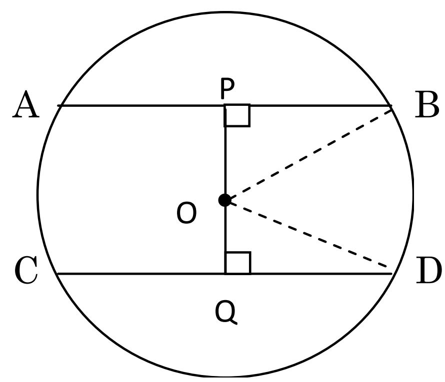


Figure 3.7

Given : Circle centre O, $AB = CD$

To Prove : $OP = OQ$

Construction : Join OB, OD

Proof : Since $OP \perp AB$

$\therefore AP = PB$ (since line from centre bisect the chord as proved already)

i.e. $PB = \frac{4}{3} AB$

And $OQ \perp CD$

$CQ = QD$ (as above)

i.e. $QD = \frac{4}{3} CD$

But $AB = CD$ (given)

$PB = QD$.

In OPB and OQD

$PB = QD$ (proved)

$\angle OPB = \angle OQD = 90^0$ (given)

$OB = OD$ (radii)

$\therefore OPB \cong OQD$ (RHS)

$OP = OQ$. (Corresponding sides)

Conversely, it can be proved that chords that are equidistant from the centre of the circle are equal.

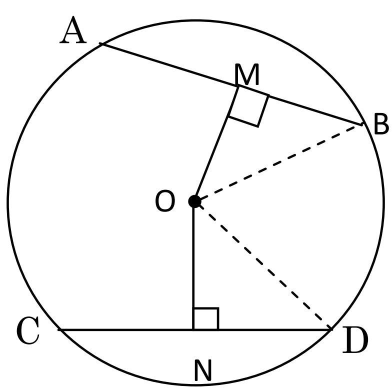


Figure 3.8

Given : Circle centre O, $OM = ON$

To Prove : $AB = CD$.

Construction : Join OB, OD

Proof : OMB and OND

$OM = ON$ (given)

$\angle OMB = \angle OND = 90^0$ (given)

$OB = OD$ (radii)

$\therefore \Delta OMB \cong \Delta OND$ (RHS)

$\therefore MB = ND$

But $OM \perp AB$

$MB = \frac{4}{3} AB$

Similarly, $ON \perp CD$

$\therefore ND = \frac{4}{3} CD$

But $MB = ND$ (proved)

$\therefore AB = CD$

Now do the following exercise

Exercise 3a

1. Given that a circle with centre O and any two points P and Q on the circumference of the circle such that PQ is not a diameter, E being the midpoint of PQ, prove that OE is perpendicular to PQ.

2. Prove that if two chords bisect each other, then the chords are diameters.

3. Prove that the perpendicular from the centre of a circle to a chord bisects the chord.

   For questions 4 and 5, refer to the diagram **Figure 3.9** below:

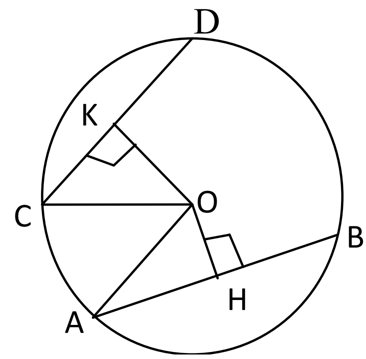


Figure 3.9

4. OH, OK are the perpendiculars from the centre O of a circle to two chords AB, CD, if AB = CD, prove that OH = OK.

(i) Explain why AH = CK

(ii) Prove triangle OHA ≡ triangle OKC

5. OH, OK are perpendiculars from the centre O of a circle to two chords AB, CD. If OH = OK, prove that AB = CD.

(i) Prove △ OHA △ OKC

(ii) Explain why AB = 2AH and complete the proof.

6. If the chord AB, length l cm, of a circle, radius r cm, is at a distance p cm from the centre, find l in terms of r and p

7. AB and CD are two equal chords of a circle; M, N are their mid – points. Prove that MN makes equal angles with AB and CD.

8. Two chords of a circle bisect each other at K. prove that K is the centre of the circle. Hint; if possible, join K to the centre.

Application of chord properties to solve problems

Using the knowledge from above, you can now find radius, length of the chord and distance from the centre to a chord.

Calculating the length of the chord given radius and distance from the centre

Activity 3

Calculating the length of the chord given radius and distance from the centre

In pairs, use the information given;

The figure 3.10 is circle with centre O and radius of 5cm and OE = 3cm. calculate the length of the chord AB.

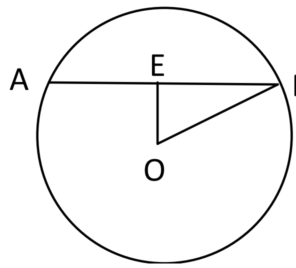


Figure 3.10

Present your work to class and compare your work with the given example.

Calculating the distance from the centre given the length of chord and radius

Example

Two parallel chords of a circle are of length 16cm and 12cm. If the radius of the circle of the circle is 10cm, what are the two possible perpendicular distances between the chords?

Solution

Draw the two chord in this circle; let AB = 16cm and CD = 12cm centre O, radius = 10cm

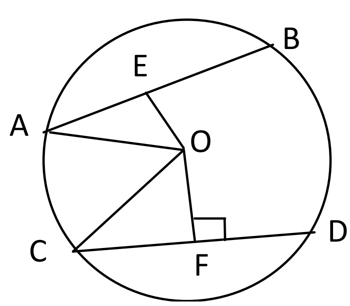


Figure 3.11

In $\triangle$ AOE, using Pythagoras theorem

$OE^2 = AO^2 - AE^2$

$= 10^2 - 8^2$ since AE = BE as OE bisect chord AB

$= 100 - 64$

$= 36$

$OE = \sqrt{36} = 6cm$

Similarly in $\triangle$ COF

$OF^2 = OC^2 - CF^2$

$= 10^2 - 6^2$

$= 100 - 36$

$= 64$

$OF = \sqrt{64} = 8cm$

The distances from the centre are OE = 6cm to chord AB and OF = 8cm to chord CD.

Calculating the radius given length of a chord and distance from the centre

Example

A chord 4.2cm long is 2.8cm from the centre of a circle. Calculate the radius of the circle.

Solution: draw diagram like the one below

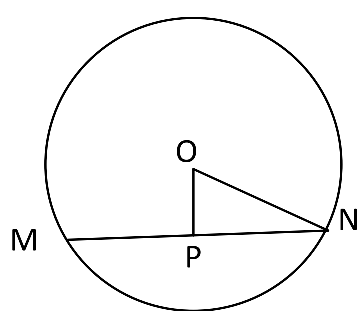


Figure 3. 12

In triangle OPN in figure 3.12

$ON^2 = OP^2 + PN^2$

$= 4.2^2 + 2.8^2$

$= 17.64 + 7.84$

$ON = \sqrt{25.28}$

$ON = 5.048\text{cm}$

Exercise 3b

1. Use figure3.13 to answer the questions given below.

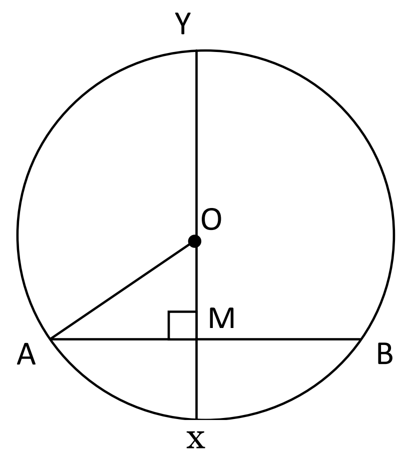


Figure 3.13

1. Using Pythagoras theorem in figure 3.13, calculate the following;
   a) OM if $AB = 10\text{cm}$ and $OA = 7\text{cm}$
   b) AM if $OM = 3\text{cm}$ and $OA = 8\text{cm}$
   c) OA if $AB = 12\text{cm}$ and $OM = 6\text{cm}$
2. Two parallel chords of lengths 6cm and 8cm are drawn in a 5cm radius circle. Calculate the two possible distances between them.
3. A circle has a radius of 10cm. Use Pythagoras Theorem to find the length of a chord of the circle that is 6 cm from the centre of the circle. How long is a chord of the same circle that is 8cm from the centre?

4. The figure below shows a circle ACB centre O. OM = 3y cm, MC = 2y cm and angle OMB = 90^0.

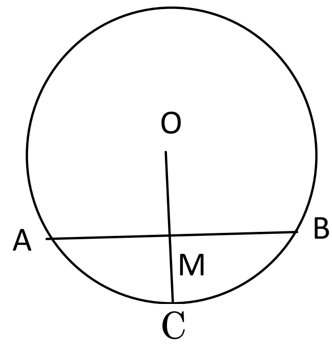


Figure 3.14

Find AB in terms of y

5. Chord is 6cm away from the centre of the circle. If the chord is 16cm long, calculate the radius of the circle.

6. A chord 20cm long is 20cm from the centre of a circle. Calculate the length of the chord which is 14cm from the centre.

7. In a circle of radius 2.5cm the lengths of two parallel chords are 1.4cm and 3cm. find the distance between the chords;
   (a) If they are on opposite sides of the centre
   (b) If they are the same side of the centre.

8. A chord of a circle of radius7cm is at a distance of 4cm from the centre. Calculate the length of the chord.

9. AB, CD are parallel chords of a circle, 3cm apart, on same side of the centre O; AB = 4cm, CD = 10cm, find OA. Hint; draw ONM perpendicular to AB to meet AB, CD in M, N; let ON = x cm.

10. A chord of length 10cm is at a distance of 12cm from the centre of the circle. Find the radius.

Unit summary

- In this unit you have learnt about chord properties of a circle and how to apply them to solving problems. You have also learnt on how to find radius, length of a chord and distance from the centre. In the next unit, you will learn about algebraic fractions with linear or quadratic denominators.

Unit review exercise

1. A chord 7cm long is drawn in a circle of radius 3.7cm. Calculate the distance of the chord from the centre of the circle.

2. A chord 6.6cm long is 5.6cm from the centre of the circle. Find the radius of the circle. Find also the length of a chord which is 6.3cm from the centre of the circle.

3. **Figure 3.15** shows a circle AYBX centre O. XY cuts AB at C such that angle ACY = 90^0.

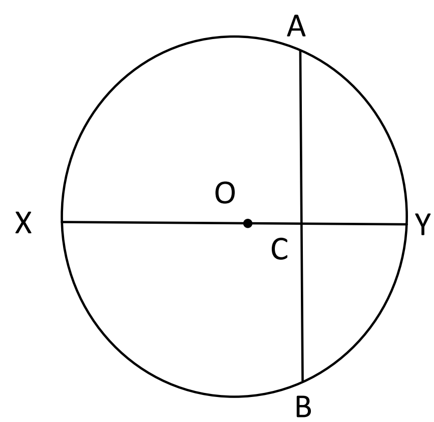


Figure 3.15

If OC = 8cm and CY = 9cm, calculate the length of AB.

4. Prove that if two chords of a circle are equal, then they are equidistant from the centre.

5. Two equal chords intersect inside a circle. Prove that the line joining their point of intersection to the centre of the circle bisects the angle between the chords.

6. XY is a diameter of a circle, and XZ is a chord. If O is the centre and OD is the perpendicular from O to XZ, prove that XY = 2OD.

Glossary

A chord is a line segment that joins two points on a circle.

A diameter is a chord that passes through the centre of the circle.

A secant is a line containing a chord

References

Hau and Saiti F. (2010). *Strides in Mathematics 3.* Blantyre: Longman Malawi.

Channon J. B. and et al (1997). N*ew General Mathematics 3, A Modern Course for Zimbabwe*. London,Longman.

Durell C. V . Volume 2. (1990) *Certificate Mathematics. 2nd Edition.* Bell and Hyman Ltd. London.

Gunsaru et al, *Secondary Mathematics Book 3.* Dzuka Publishing Company. Blantyre. Malawi.

Unit 4

ALGEBRAIC FRACTIONS WITH LINEAR OR QUADRATIC DENOMINATORS

You learnt about algebraic fractions in your JCE Mathematics with respect to LCM and HCF. In this unit, you are going to learn about the four operations: addition, subtraction, multiplication and division involving algebraic fractions . You are also going to express fractions in their simplest form.

Before all this, you must know how to simplify algebraic fractions.

Algebraic fraction help to stimulate critical thinking and reasoning ability as mostly it uses the brain since some fractions cannot be done by the calculator. Skills acquired assist in solving algebra and other problems in mathematics.

Expressing algebraic fractions to their lowest terms

An algebraic fraction is a fraction whose numerator and denominator are algebraic expressions. In other words it uses variables (unknowns) in the numerator and denominator

Can you write some examples of algebraic fractions? Here are examples of algebraic fraction as a reminder.

$\frac{a}{2}, \frac{a^2}{a+2}, \frac{x^2-4}{x+2}, \frac{x^2+5x+6}{x^2-9}$ Can you come up with more other example.

The rules for simplifying algebraic fractions are the same as those for numerical fractions. Now do the activity below.

Expressing algebraic fractions to their lowest terms

Activity 1:

Reducing algebraic fractions to their lowest terms.

In pairs, express the following algebraic fraction in their simplest form;

Simplify the following:

1. $\frac{3y^8}{12y^3}$ 2. $\frac{36p^5}{4p}$ 3. $\frac{1}{3}a + \frac{2}{3}a$

Present your work to class. Discuss the examples given below.

Example 1:

Expressing into lowest terms

Express the expression to its lowest term, $\frac{48x^4y^2}{18x^2y^6}$

Solution

Find factors that can go into the denominator and the numerator and cancel out.

$$\frac{\overset{8}{\cancel{48}}x^4y^2}{\underset{3}{\cancel{18}}x^2y^6} = \frac{8x^2}{3y^4}$$ and we cannot simplify any further.

If you have the same variables on denominator and numerator, subtract the powers. I.e. x^{4-2}.

Exercise 4a

Simplify the following

1. (a) $\frac{x^6}{x^4}$ (b) $\frac{y^{12}}{y^3}$

(c) $\frac{p^6}{p}$ (d) $\frac{m^5}{m^4}$

2. (a) $\frac{x^4y^6}{x^2y^2}$ (b) $\frac{a^2b^2c^4}{ab^2c}$

(c) $\frac{pq^2r^2}{p^2p^3r}$ (d) $\frac{4xy}{2yz}$

(e) $\dfrac{15pq^2}{5p}$ (f) $\dfrac{16m^2n}{24m^2n^2}$

Addition and Subtraction of Algebraic Fractions

When adding or subtracting algebraic fractions, follow the same procedure as in arithmetic fractions:

Activity 2:

Addition and subtraction

Simplify the following

(a) $\frac{3}{8} + \frac{1}{8}$

(b) $\dfrac{1}{2x} + \dfrac{2}{3x}$

(c) $\dfrac{3}{4} - \dfrac{1}{3}$

(d) $\dfrac{1}{4x} - \dfrac{x}{8}$

Compare your work with the other members of the class.

In simplifying these, did you follow the following steps?

(1) Find the LCM of the denominators.

(2) Express each fraction with the common denominator (divide the LCM by individual denominators and multiply by respective numerators).

(3) Add or subtract the fractions.

Now look at the following examples.

Example 2:

Simplifying fractions

Simplify

(a) $\dfrac{3y^8}{12y^3}$

Here, the denominators are x + 1 and x − 3 and their LCM is (x + 2) (x − 3)

$\therefore \quad \dfrac{(x(x-3) - \{(x+2)(x-2)\}}{(x+2)(x-3)}$ Divide LCM each denominator then multiply

$= \dfrac{(x^2 - 3x - \{x^2 - 4\})}{(x+2)\,(x-3)}$ expand by removing the denominator

$= \dfrac{(x^2 - 3x - x^2 + 4}{(x+2)\,(x-3)}$

$= \dfrac{4 - 3x}{(x+2)\,(x-3-)}$

(b) $\dfrac{3x-5}{3} + \dfrac{2x-3}{5}$

The LCM of 3 and 5 is 15

$\therefore \dfrac{3x-5}{3} + \dfrac{2x-3}{5}$

$= \dfrac{5(3x-5)+3(2x-3)}{15}$ ...... divide LCM by 3 and 5

$= \dfrac{15x-25+6x-9}{15}$ .... expand and add like terms together.

$= \dfrac{21x-34}{15}$

(c) $\dfrac{3}{x+1} + \dfrac{2}{x-2}$

The LCM of $x + 1$ and $x - 2$ is $(x + 1)\,(x - 2)$

$\therefore \dfrac{3}{x+1} + \dfrac{2}{x-2}$

$= \dfrac{3(x-2)+2(x+1)}{(x+1)\ (x-2)}$ ……. Divide the LCM with denominators

$= \dfrac{3x - 6 + 2x + 2}{(x+1)(x-2)}$ ……… expand and add like terms together

$= \dfrac{5x - 4}{(x+1)(x-2)}$

Sometimes, there is need for factorising in order to find the LCM of the denominators.

(c) $\dfrac{3}{x^2 - 4} - \dfrac{5}{x - 2}$

$x^2 - 4$ is not as simple as x-2, it factorises to $(x + 2)$ $(x - 2)$

$\therefore \quad \dfrac{3}{x^2 - 4} - \dfrac{5}{x - 2} = \dfrac{3}{(x+2)(x-2)} - \dfrac{5}{x-2}$

The LCM is $(x + 2)$ $(x - 2)$

$$\therefore \dfrac{3}{x^2-4} - \dfrac{5}{x-2} = \dfrac{3-5(x+2)}{(x+2)(x-2)}$$

$$= \dfrac{3-5x-10}{(x+2)(x-2)}$$

$$= \dfrac{-5x-7}{(x+2)(x-2)}$$

Now do the exercise given below.

Exerxise 4b

Simplify the following algebraic fractions.

1. (a) $\frac{a}{4}+\frac{a}{5}$ (b) $\frac{x}{3}-\frac{x}{4}$

(c) $\frac{x}{4}+\frac{x}{5}+\frac{x}{6}$ (d) $\frac{5p}{8}-\frac{7a}{12}$

(e) $\frac{3x}{4}-\frac{x}{6}$ (f) $\frac{x}{2}+\frac{y}{3}-\frac{z}{4}$

2. (a) $\frac{3}{5p}-\frac{2}{3p}$ (b) $\frac{3}{y}-\frac{5}{3y}+\frac{4}{5y}$

(c) $\frac{3}{x}+\frac{5}{2x^2}$ (d) $\frac{3r}{7s}+\frac{2r}{14s}-\frac{5r}{21s}$

(e) $\frac{3m}{15p}+\frac{4n}{5p}-\frac{11n}{30p}$

3. (a) $\frac{3x+2}{3}+\frac{2x+1}{4}$ (b) $\frac{3a+5b}{4}-\frac{a-3b}{2}$

(c) $\frac{x-2}{4}+\frac{2}{5}$ (d) $\frac{x-3}{3}-\frac{x-7}{6}$

(e) $\frac{3m-5n}{6}-\frac{3m+7n}{2}$

(f) $\frac{(2a+1)}{a}+\frac{(3b-2)}{b}-2$

4. (a) $\frac{2}{x+4}+\frac{3}{x+3}$ (b) $\frac{5}{p+3}-\frac{3}{p-5}$

(c) $\frac{3}{y+2}+\frac{2}{y+3}$ (d) $\frac{3}{m-1}-\frac{2}{m+3}$

(e) $\frac{5}{3x-2}-\frac{7}{5x+2}$

5. (a) $\dfrac{3}{x+3}+\dfrac{2}{2x-6}$ (b) $\dfrac{4}{x-1}-\dfrac{2}{x^2-1}$

(c) $\dfrac{3x}{x^2-4}+\dfrac{4x}{x-2}$ (d) $\dfrac{2}{y^2-9}+\dfrac{3}{y^2+x-12}$

(e) $\dfrac{3x+4}{x^2+9x+20}-\dfrac{2x-3}{x^2+2x-15}$

(f) $\dfrac{x+1}{x+3}-\dfrac{x+1}{x-2}$

Multiplication of algebraic fractions

The simplification involves multiplying two algebraic fractions. As with ordinary arithmetic fractions, numerators can be multiplied together as can denominators, in order to form a single fraction. Remember the following laws of indices

1. $a^m x\ a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $(a^m)^n = a^{mn}$

Activity 3:

Simplifying algebraic fractions

Individually, simplify (a) $\frac{2}{3}\times\frac{1}{6}$ (b) $\dfrac{a}{2}\times\dfrac{b}{3}$

Discuss your answer with a friend. Compare the way of solving with the examples below;

Example 3:

Simplifying fractions

Simplify the following; $\dfrac{3y^6}{4p^2}\times\dfrac{8p^3}{9y^2}$

Factors, which are common to both numerator and denominator, may be cancelled. It is important to realise that this cancelling means dividing the numerator and denominator by the same quantity.

.i.e. $$\frac{\cancel{3}y^6}{4p^2} \times \frac{\cancel{8}p^3}{\cancel{9}y^2} = \frac{2py^4}{3}$$

Exercise 4c

1. (a) $\frac{x}{4} \times \frac{y}{3}$ (b) $\frac{2}{p} \times \frac{q}{3}$

(c) $\frac{p}{q} \times \frac{q}{r}$ (d) $\frac{xy}{r} \times \frac{s}{xy}$

(e) $\frac{8}{x} \times \frac{x}{16}$

2. (a) $\frac{x^2y}{3py} \times \frac{4py}{x}$ (b) $\frac{3p^2}{2q} \times \frac{5q}{3p}$

(c) $\frac{6a}{b^2} \times \frac{b}{3a^2}$ (d) $\frac{6ab}{c} \times \frac{ad}{2b} \times \frac{8cd^2}{4bc}$

(e) $\frac{2z^2y}{3ac^2} \times \frac{6a^2}{5zy^2} \times \frac{10c^3}{3y^2}$

Division of algebraic fractions

Division and multiplication of algebraic fraction goes hand in with multiplication

Activity 4:

Dividing of algebraic fractions

In groups, simplify the following;

a. $\frac{1}{12} \div \frac{1}{4}$

b. $4x \div \frac{2x}{3}$

Present your answers to class. Discuss the example given below.

Example 4:

Expressing to lowest terms

Express to its lowest term $\frac{5x^2y^3}{8pq^3} \div \frac{10x^3y}{12p^2q}$

Solution:

Invert the fraction on the right hand side and put multiplication sign and then proceed in multiplicaion.

$$\frac{5x^2y^3}{8pq^3} \div \frac{10x^3y}{12p^2q} = \frac{5x^2y^3}{8pq^3} \times \frac{12p^2q}{10x^3y} = \frac{3py^2}{4q^2x}$$

Now do the exercise below.

Exercise 4d

Simplify the following;

(a) $\frac{ab^2}{bc^2} \div \frac{a^2}{bc^2}$ (b) $\frac{6ab}{5cd} \div \frac{4a^2}{7bd}$

(c) $\frac{3pq}{5rs} \div \frac{p^2}{15s^2}$ (d) $\frac{5x^2y}{8ab} \div \frac{10xy}{4a^2b}$

(e) $\frac{6pq}{5rs} \div \frac{4p^2}{7qs}$

Simplification of algebraic fractions involving factorisation

Some simplification involves factorisation as the case is in (d)

Example 5:

Simplifying algebraic fractions

(a) $\frac{x^2 + 2x}{3x}$

Factorising the numerator, see that x is common and can be cancelled out.

i.e. $\frac{x^2 + 2x}{3x} = \frac{x(x+2)}{3x} = \frac{x+2}{3}$

(b) $\frac{x+2}{x^2-4}$

Here you factorise the denominator i.e. the difference of two squares.

$$\frac{x+2}{x^2-4} = \frac{x+2}{(x+2)(x-2)} = \frac{\cancel{x+2}}{\cancel{(x+2)}(x-2)} = \frac{1}{x-2}$$

(c) $\frac{x^2-16}{x^2+x-12}$

Here factorise both the numerator and the denominator and cancel out common factor.

$$\frac{x^2-16}{x^2+x-12} = \frac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(x-3)} = \frac{x-4}{x-3}$$

Exercise 4e

Simplify the following;

1. (a) $\frac{3x + 6}{x + 2}$ (b) $\frac{x(x - 3)}{(x - 2)(x - 3)}$

(c) $\frac{x(x + 5)}{(x - 5)(x + 5)}$ (d) $\frac{y^2-1}{(y-1)(y+1)}$

(e) $\frac{3p(2p-3)}{(2p+3)(2p-3)}$

2. (a) $\frac{x^2-3x}{(x+3)(x-3)}$ (b) $\frac{x(x+4)}{x^2+5x+4}$

(c) $\frac{x^2+2x}{x^2+5x+6}$ (d) $\frac{x^2-x}{x^2-1}$

(e) $\frac{x^2+4x}{x^2+x-12}$ (f) $\frac{x^2-7x}{x^2-49}$

Now you will look at some equations

Equations Involving Algebraic Equations

In this section you will learn how to solve linear equations in our JCE mathematics and how to solve quadratic equations in unit one of this book. In this section, you will learn how to solve equations which involve algebraic fractions.

Activity 5:

Solving algebraic equation

Solve $\frac{x}{2} + \frac{x}{3} = \frac{10}{3}$

Report your findings to class. Now look at the examples given below.

Example 6:

Solving equations involving fractions

Solve the following equations

(a) $\frac{x}{2}+\frac{x}{3}=\frac{5}{6}$

The first step in solving such equations is to get rid of the denominators by multiplying each term by the LCM of the denominators. The resulting equation can then be solved by the methods learnt previously.

Now the LCM of the denominator of 2, 3 and 6 is 12

$$\therefore \qquad \frac{x}{2}+\frac{x}{3}=\frac{5}{6}=\frac{6x+4x=10}{12}$$

You can ignore the LCM after this step

i.e. $\quad 6x + 4x = 10$

$10x = 10$

$x = 1$

(b) $\frac{x+3}{2}=\frac{x-3}{3}$

The LCM of 2 and 3 is 6 and we have

$$\frac{x+3}{2}=\frac{x-3}{3}=\frac{3(x+3)=2(x-3)}{6}$$

$3x + 9 = 2x - 6$ (ignoring the denominator)

$3x - 2x = - 6 - 9$

$x = -15$

Exercise 4f

Solve the following equations

1. (a) $\frac{2x}{5} = \frac{x}{8} + \frac{1}{2}$

(b) $3x + \frac{3}{8} = 2 + \frac{3x}{3}$

(c) $\frac{3x}{4} + \frac{1}{3} = \frac{x}{2} + \frac{5}{8}$ (d) $\frac{5x}{7} - \frac{2}{3} = \frac{3}{7} - \frac{x}{3}$

(e) $\frac{1}{2x} + \frac{1}{3x} = \frac{3}{5}$

2. (a) $\frac{x+3}{2} = \frac{x-3}{3}$ (b) $\frac{3x+2}{2} - \frac{x-2}{2} = \frac{11}{4}$

(c) $\frac{x+2}{5} - \frac{3x-2}{4} = 2$ (d) $\frac{m-7}{5} = 3$

(e) $\frac{3x-5}{4} - \frac{9-2x}{3} = \frac{x-3}{2}$

Real life problem of algebraic fractions

Sometimes, the equations are given in words and you are required to write the equation and then solve it for the unknown.

The following procedure will help you to construct linear equations.

1) Represent the quantity to be found by a symbol (x is usually used).

2) Make up the equation, which conforms to the given information.

3) Make sure that both sides of the equation are in the same units.

Activity 6:

Solving algebraic fraction involving word problems

In groups, do the activity;

I think of a number, take $\frac{1}{3}$ of it and then add 4. The result is 7. Find the number I first thought of.

Let one member from your group go and see the work of other groups and see what they have done and report to the rest of the members in the group.

Did you it in this way?

Let the number be x

Then $\frac{1}{3}$ of $x + 4 = 7$

i.e. $\frac{x}{3} + 4 = 7$

This is the equation and you can solve it using methods learnt

i.e. $\frac{x}{3} = 3$

$x = 3 \times 3 = 9$

$x = 9$

Did you get this? Now look at the example below.

Example 7:

Word problems involving algebraic fractions

Mary uses $\frac{1}{12}$ of her salary to pay her bills, gives $\frac{1}{15}$ to her parents and she is left with K 6000.00. What is her salary?

Let her salary be x

$\therefore\ x - \dfrac{x}{12} - \dfrac{x}{15} = 6000$

Then proceed as follows;

$\dfrac{x}{1} - \dfrac{x}{12} - \dfrac{x}{15} = \dfrac{6000}{1}$

Note that the denominator of x and 6000 is 1

The LCM of 1, 12 and 15 is 60

i.e. $\dfrac{x}{1} - \dfrac{x}{12} - \dfrac{x}{15} = \dfrac{6000}{1}$

$\dfrac{60x - 5x - 4x = 360000}{60}$

$51x = 360000$

$x = 7058.82$

Can you now practice formulating algebraic equations from the exercise given below?

Exercise 4g

Construct equations from the following cases and solve them to find the value of the unknown.

(a) I think of a number, divide it by 4, add 5 and the result is 9. What is the number that I first thought of?

a) I think of a number, divide it by 3, subtract 4 and the result is 29. What is the number that I first thought of?

b) I think of a number. When $\dfrac{3}{7}$ is subtracted from $\dfrac{3}{4}$ of the

c) number the result is 1. Find the number.

d) When $\frac{1}{2}$ is subtracted from $\frac{2}{3}$ of a number the result is 4.

e) Find the number.

d) I think of a number. When $\frac{1}{15}$ of this number is subtracted from $\frac{2}{5}$ of the number the answer is $\frac{5}{9}$. What number did I

e) first think of?

f) Find the number which, when added to the numerator and denominator of the fraction $\frac{3}{5}$, gives a new fraction which is equal to $\frac{4}{5}$.

Unit summary

In this unit, you have looked at algebraic fractions, simplifying these by adding, subtracting, multiplying and dividing. The unit has extended by expressing algebraic fractions to their lowest term and simplifying algebraic fractions. In the next unit, you will look at sets.

Unit review exercise

Evaluate and simplify if possible the following:

1 a. $\frac{1}{x}+\frac{1}{x^2}$

b. $\frac{4}{t}-\frac{3}{2t^2}$

c. $\frac{2}{a}+\frac{3}{a\,b}$

d. $\frac{4}{y}-\frac{2}{3y}+\frac{5}{y}$

e. $\frac{4x}{3y} - \frac{5y}{7x}$

f. $\frac{1}{x} - \frac{1}{(x-1)}$

g. $\frac{1}{x^2-5x+6} + \frac{x}{x-3}$

h. $4x - \frac{1}{x}$

i. $\frac{x+1}{(x-1)} - \frac{x-1}{x+1}$

j. $\frac{2a+5b}{3} + \frac{3a+2b}{6}$

k. $\frac{2+d}{3} - \frac{5}{6}$

2. a. $\frac{y^2}{x} \times \frac{2x^2}{3y} \times \frac{4}{5yx}$

b. $\frac{x}{y} \times \frac{2y}{3x} \times \frac{4x}{5y}$

c. $\frac{pq}{r} \div \frac{3p}{2r}$

d. $\frac{5pq}{6t^2} \div \frac{3pq}{5t}$

e. $\frac{8xy+12xt}{12y^2-27t^2}$

Glossary

An algebraic fraction is a fraction whose numerator and denominator are algebraic expressions.

References

Chikwakwa, e t al (2002), *Senior Secondary Mathematics Student's Book 3*. Blantyre: Macmillan Malawi.

Hau S and Saiti F (2010). *Strides in Mathematics 3*. Blantyre: Longman Malawi.

G D Buckwell and B N Githua, *KCSE Golden Medal mathematics,* Macmillan, London.

Gunsaru, et al, *Secondary Mathematics Book 3,* Dzuka Publishing Company, Blantyre Malawi.

Unit 5

SETS

In form 1, you learnt that a set is a collection of objects. In this unit, you will learn more about sets . By the end of this unit you must be able to describe elements of a set and representsets in Venn diagrams. The idea of sets is used in many fields like manufacturing and other fields.

In everyday life we buy things in sets.

Describing elements of a set

A set is a collection of objects. The things or objects that make up a set are called **members** or **elements** of the set. It is possible to have a set which contains objects that do not have anything in common. What does mathematical instruments box contain? You may find objects such as a compass, divider, ruler, protractor and set square. Whether the objects are related in some way or not, as long as they form a collection, they comprise a set. Now do the activity below.

Activity 1:

Describing set builder notation

In pairs

1. Identify objects or things in a set in your class
2. Write set of objects in set language.
3. Write down the symbol for the following:

   Universal set

   Not a member of

   A union **B**

   A intersection **B**

Share your work with your friends and discuss your findings. Now look these in detail.

There are various symbols that are used in set language. Now look at the followings is a member of, $\notin$; is not a member of or $\{\ \}$; the empty set or null set;

$B \subset C$; B is a proper set of C

$A \subseteq B$; A is a subset of B

$A \supseteq B$; A contains B

$\not\subset$9; the negation of $\subset$, $\subseteq$ and $\supseteq$

$A \cup B$; union of A and B

$A \cap B$; intersection of A and B

n(A); number elements in set A

A'; complement of set A

Set builder notation is another way of representing sets. Inside the set, unknown or variable is used to represent the elements of a set in general. This is followed by a vertical line or a colon which means "such that". The vertical line or colon is followed by a description of the elements. Look at the examples below.

Example 1:

Set builder notation

Write the elements of the set given in set builder notation.

a. T = {t | t is all teachers at your school} or T = {t: t is all teachers at your school}.

   A = {Chimtengo, Chitedze, Chinthuzi, Chimwemwe, Getrude}

b. B = {y : y < 10, y is a whole number} or B = {y | y < 10, y is a whole number}

   B = {1, 2, 3, 4, 5, 6, 7, 8, 9}

c. X = {z | −5< z < 4, z is an integer} or X = {z : −5< z < 4, z is an integer}

   X = {−4, −3, −2, −1, 0, 2, 3}.

   Hence the elements in set X are integers greater than −5 and less than 4. Sets A, B, C are all **finite sets.** Now do the following exercise.

Now do the exercise below.

Exercise 5a

1. If A = {integers}, list the members of the following sets using… where appropriate.

   (a) $\{x : x\ 9\}$

   (b) $\{y \mid y\}$

   (c) $\{a : a \geq -3\}$

   (d) $\{x : x\ 0\}$

   (e) $\{y : y - 5 = 0\}$

   (f) $\{x : 2x + 4 = 16\}$

   (g) $\{y : -7 \leq y\ \ 5\}$

   (h) $\{x : -8\ x - 1\}$

   (i) {x : x 0 and has no remainder}

2. List the elements of the following sets:

   a. D = {d: d is public holiday days in Malawi}

   b. E = {e | e is even numbers less than 20}

   c. G = {g : g ≤ 18, g is an odd number}

   d. Y = {x : x is counting numbers less than 10}

   e. H = {d | d is district along the lake Malawi}

3. If D = {1; 2; 3; 4; 5; …20}, list the members of the following sets.

   a. {a : a is a square number, a ∈ D}

   b. $\{b \mid b + 2\ 15\ ,\ b\ D\}$

   c. {x : x is a factor of 36, x ∈ D}

4. Give that {a, c, e, h, I, l, m, s, t, w} is a universal set and

   Y = {a, c, e, h, i, m, s}, find n(Y’).

Definitions

Equal sets are sets that contain exactly the same members, regardless of the order in which the members are presented.

If A = {a, b, c, d, e} and B = {b, c, e, d, a} then A and B are equal sets.

Equivalent sets have the same number of elements.

If H = {1, 2, 3, 4, 5} and G = {a, d, e, f, g}, how many elements has G and H? These two sets are equivalent.

A subset contains wholly or part of the elements in a universal set.

If M = {a: a is whole number less than 10} and N = {1, 2, 3, 4, 5} then N is a subset of set M.

A set is said to be **proper subse**t if it has few element than those given in the universal set or if it is not exact subset of itself. From the above statement **N is a proper subset of M**

An improper subset is a subset which has all the elements in the original set. It is equal to the original set.

A set is said to be **finite** if the elements in there can be counted for example;

M = {a, c, b, g}, M has four elements while **infinite** set the elements in that particular set cannot be counted for example;

N = {1, 2, 3, 4, 5, 6…}; a set of natural numbers,

P = {1, 3, 5, 7, 11, 13, 17,.}; a set of prime numbers,

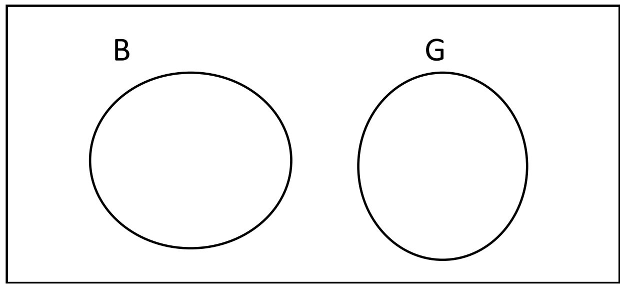
Z = {−4, −3, −2, −1, 0, 1, 2, 3…}; a set of integers.

The dots show that the elements continue.

The number of elements in a set is called the **cardinal number** of the set.

Empty or null set is a set that has no members. For example;

let B be all boys in class, G all girls in class and H neither a girl nor a boy. This can be also shown in this way;



H is an empty set since you can hardly find a person who has both sex. B and G are disjoint sets.

Oral exercise

Given the following sets; A = {2, 4, 5}, B = {1, 2}, C = {5, 2, 4},

D = {a, c,} and E = {x: x is a number greater than 1}

Identify sets that are equal, equivalent, finite and infinite.

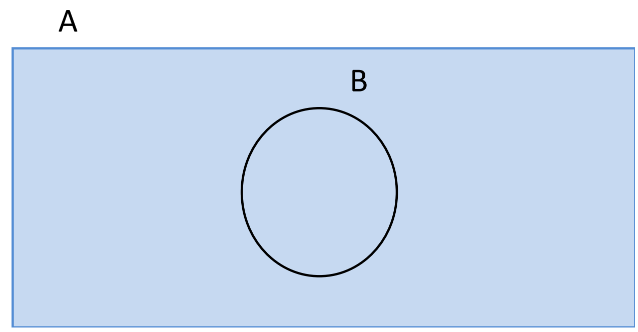
From exercise 5a above, you are able to come up with **subsets** from the main set. What do you call the main set? You are going to look at that now.

Universal set

The universal set is the set which contains all the possible elements. If

A = {y: y is a letters the alphabet} and B is a set of vowels of the alphabet then A = {a, e, i, o, u}, therefore A is the universal set and B is a subset of A.

A teacher, boys and girls in a class form a universal set of people present in a classroom. In this girls only or, boys only could be a subset of this set. Universal set is denoted ξ**.** Diagrammatically sets A and B could be represented as follows:



A= {a, b, c… z} and B = {a, e, i, o, u}

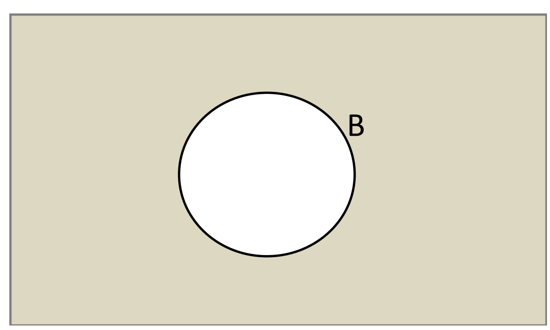
B is a proper subset of A or B ⊂ A.

Can you now come up with your own universal set?

Complement of a set

The complement of a set is the set containing all elements in a universal set but are not members of this given set.

In other words, the complement of B is the set which contains all those elements that are in a universal set, U but do not belong to set B. The complement of a set B is denoted B.



The shaded region represents the complements of set B denoted B´

Example 2:

Complement of a set

a. U= {1,2,3,4,5,6,7,8} and A = {2,4,6,8}. What set is represented by A?

Solution:

A consists of all elements in ∈ which are not in set A.

$\therefore A' = \{1, 3, 5, 7\}$

Show this diagrammatically as demonstrated above.

b. If ∈ = {all triangles} and Q = {equal sided triangles}. What set is represented by Q'?

Solution:

∈ = {Scalene, Equilateral, Isosceles △ s}

Q = {Equilateral △ s}

$\therefore Q' =$ {Scalene, Isosceles}

Note: A universal set, ∈ is the background set i.e. it contains all the possible elements. If a certain set Q, for example, contains some elements of the universal set, then Q is said to be **a subset** of ∈. But when it comes to Q, this means a set containing elements of the universal set which are not however contained in set Q.

Now do the exercise given below.

Exercise 5b

1. Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and A = {1, 2, 5, 7, 9}. What set is represented by A'?

2. Let U = {all quadrilaterals} and B = {quadrilaterals with all 4 sides equal}. What set is represented by B'? Give 2 examples of elements of B and give 3 examples of elements of B.

3. If U = {m, a, t, h, s} and C = {a, h, s}, what set is represented by C'?

4. If U = {1; 2; 3; 4; …10}, A = {1; 2; 5; 7}, B = {1; 3; 6; 7}, write down the sets A' and B'.

5. Given that the universal set U = {1, 2, 3, 4, 5, 6, 8, 9}, sets N and M of U such that N = {even numbers} and M = {perfect square}, find;

(a) N'

(b) M'

(c) n(N')

Union of sets

The union of two or three sets is everything which belongs to either, both or all the three sets and represented by the symbol $\cup$.

Activity 2:

Identifying elements of a union of two or three sets

In pairs,

(a) Get three mathematical boxes or sets of playing cards.

(b) Label them as set A, B and C.

(c) List the elements in each set.

(d) Write all elements in the three sets as a new set.

(e) Express the new set in terms of A, B and C.

(f) List A.

Present your work to the class. Look at the case below.

Consider A to be a set of all vowels of the alphabet. Let A = {a, e, i, o u} and B to be a set of whole numbers from 1 to 5. Then B = {1, 2, 3, 4, 5}.

Now write all the elements that are in A and B. You obtain the following set {a, e, i, o u, 1, 2, 3, 4, 5}. This is a set of all elements of A and B. Therefore, this is known as Union of Sets A and B.

In short $A \cup B$ = {a, e, i, o u, 1, 2, 3, 4, 5}.

Example 3:

Union of sets

a. Let A = {a, b, c, d}, B = {a, 1, d}. What is AB?

Solution:

$A \cup B = \{a, 1, b, c, d\}$

Note: *a* and *d* are not written twice since they are same elements.

b. Let C = {Malawi, Zambia, Zimbabwe}

D = {Zomba, Thomas, Hanna}. What is CD?

Solution:

C ∪ D = {Malawi, Zambia, Zimbabwe, Zomba, Thomas, Hanna}

Exercise 5c

1. Let $\in$ = {a, b, c, d, e, f}, X = {a, d, e}, Y = {a, b, e} and Z = {a, b, f}. List the elements of the following sets.

   (a) $X \cup Y$

   (b) $X \cup Y \cup Z$

   (c) $Y \cup Z$

   (d) $X \cup Z$

   (e) $X' \cup Z'$

2. If $\in$ = {a; b; c; d; e}, A = {a; c; e} and B = {a; e}, list the members of the following sets.

   (a) $A \cup B$

   (b) $A' \cup B$

   (c) $A' \cup B'$

   (d) $n(A \cup B)$

   (e) $(A \cup B)'$

3. If ∈ = {1, 2, 3, 4, 5}, N = {1, 3} and M = {3, 4}. Find U

(a) N'

(b) M'

(c) (N ∪ M)

4. Given that the universal set U = {1, 2, 3, 4, 5, 8, 9}, sets A and B of U such that A = {even numbers} and B = {perfect square}, find;

(a) A

(b) B

(c) n(A ∪ B')

(d) A' ∪ B'

(e) n(A' ∪B')

Having looked at union of sets you will now look at intersection of sets.

Intersection of sets

Intersecting sets are those sets which have some common elements in them. These sets can be two or more.

Activity 3:

Identifying elements of in an intersection of two or three sets

In pairs,

(a) Get three mathematical boxes or a set of cards of numbers.

(b) Label them as set A, B and C

(c) List the elements in each set

(d) Write elements common in the three sets, A, B and C.

(e) Express the new set in terms of A, B and C.

Present your work to the class. Look at the case below.

Consider set C = {first four letters of alphabet} and set D = {c, m, x, d}.

These sets are not equal.

The symbol $\in$ means "belongs to" or "a member of" e.g. Zomba $\in$ D, means Zomba belongs to D.

Set C = {a, b, c, d,}.

You see that a $\in$C, b $\in$C, c$\in$C and d$\in$C, also c$\in$D, m $\in$D, x$\in$D and d $\in$D.

Thus there are elements which are common to both sets. As already stated these are intersecting sets and symbol for intersection of sets is $\cap$.

Therefore C $\cap$ D = {c, d}. These are the only two elements, which are found in both sets C and D.

Example 4:

Intersection of sets

Let A = {a, b, c, d, 2, 3}

B = {1, 2, 3}

C = {p, q, r, t}

Find **(a)** A $\cap$ B

(b) A $\cap$ C

Solution:

(a) A $\cap$ B = {2, 3}

Since 3$\in$A, 3 $\in$B, 2 $\in$A and 2$\in$B.

(b) A $\cap$ C = Ø, (Ø is the symbol for an empty set. In other words, there is nothing in the given set).

Since A and C do not have any common element

Let P = {all positive whole numbers from 5 to 10}

Q = {even numbers between 2 and 16}

R = {natural odd numbers}

Find (a) $P \cap Q$

(b) $Q \cap R$

(c) $P \cap R$

Solution:

(a) P = {5, 6, 7, 8, 9, 10}

Q = {4, 6, 8, 10, 12, 14}

R = {1, 3, 5, 7, 9, 11, 13, 15,...}

$\therefore$ $P \cap Q$ = {6, 8, 10}

(b) $P \cap R$ = {5, 7, 9}

(c) $Q \cap R = \emptyset$ since there are no even numbers which are odd numbers and there are no odd numbers which are even.

Exercise 5d

1. Let ξ = {a, b, c, d, e, f}, X = {a, d, e}, Y = {a, b, e} and Z = {a, b, f}. List the elements of the following sets.

 (a) $X \cup (Y \cap Z)$

 (b) $X \cap Y \cap Z$

 (c) $(X \cup Z) \cap Y$

 (d) $X \cap Z'$

2. Let P = {2, 3, 5, 7, 11, 13, 17}. List the elements of the following sets:

 (a) $P \cup \emptyset$

 (b) $P \cap \emptyset$

3. If ξ = {a; b; c; d; e} and A = {a; c; e} and B = {b; e}, list the members of the following sets.

(a) $A' \cap B$

(b) $B' \cap B$

(c) $(A \cap B)'$

(d) $A' \cap B'$

(e) $(A \cup B)'$

(f) $A' \cup A$

4. If ξ = {days of the week}, S = { words which contain the letters s} and N = {words which contain six letters}.

(a) List the members of the sets S, N, S', N';

(b) List the members of the set,

(i) $(S \cup N)'$

(ii) $(S \cap N)'$

(c) Hence, without further working, list the members of the sets

(i) $S' \cap N'$

(ii) $S' \cup N'$

5. If ξ = {c; h; I; d, k; e; n}, P = {n; i; c; e}and Q = {h; e; n}, list the elements of the following;

(a) $P \cap Q$

(b) $P \cup Q$

(c) $(P \cup Q)'$

(d) $(P \cap Q)'$

(e) $P' \cap Q$

(f) $P \cup Q'$

Number of elements in a set

In pairs discuss

If set A = {1, 3, 4, 5, 6, 7, 9} and set B = {2, 3, 4, 6}. List the following;

(a) $A \cup B$

(b) $A \cap B$

(c) n (A)

(d) n (B)

(e) $n\ (A \cup B)$

(f) $n\ (A \cap B)$

Show that $n\ (A) + n\ (B) = n\ (A \cup B) + n\ (A \cap B)$.

From this activity, you might have seen the following;

$n\ (A) = 7$; number of elements in set A

$n\ (B) = 4$; number of elements in set B

$n\ (A \cup B) = 8$; total number of elements in A and B

$n\ (A \cap B) = 3$; number of elements common in both A and B.

Substituting; $n\ (A) + n\ (B) = n\ (A \cup B) + n\ (A \cup B)$

This gives $7 + 4 = 8 + 3 = 11$

Therefore for given sets A and B; $n(A) + n(B) = n(A \cup B) + n(A \cap B)$ usually written as $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Venn diagrams

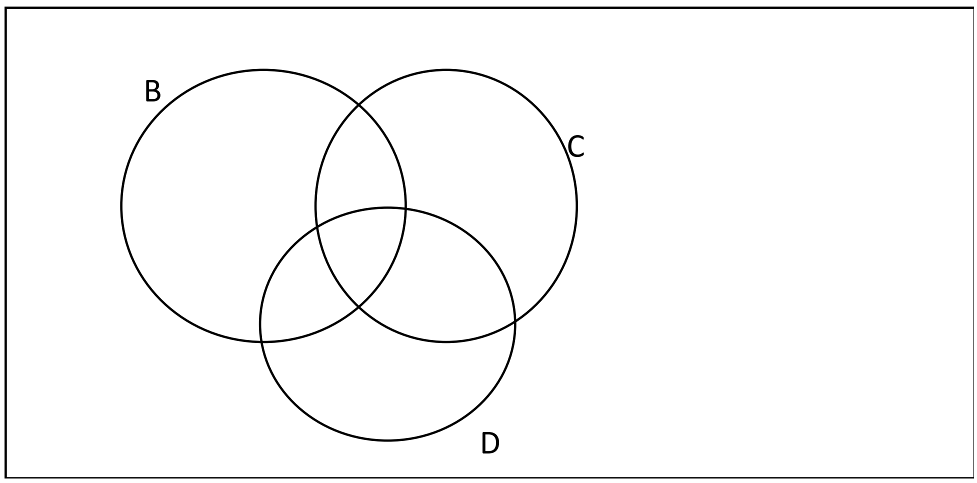
Venn diagrams are the principal way of showing sets diagrammatically. The method involves mainly entering the elements of a set into a circle or circles. You can use Venn diagrams for both union and intersection of sets.

Activity 4:

Illustrating elements of sets in a Venn diagram.

In pairs discuss; If B = {0, 2, 3, 5, 7, 10}, C = {2, 3, 4, 5, 6, 8, 9} and

D = {0, 1, 2, 8, 10, 13, 15}. Write the elements in the appropriate set in the Venn diagram below;



Write the set language for the listed elements above as follows;

(a) Elements found in B and C only

(b) Elements common in B, C and D

(c) Elements found B and D only

(d) Elements found in B only

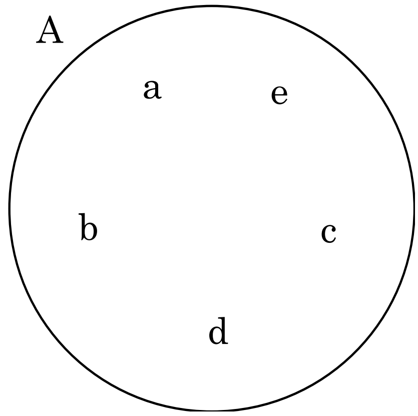
(e) Elements found in C only

(f) Elements found in D only

Present your work to the whole class. Now you will look at this in detail.

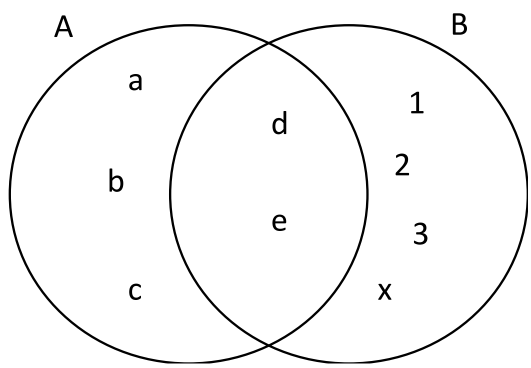
Intersection of Sets

Consider the sets, A = {a, b, c, d, e} and B = {1, 2, 3, d, e,x}.



All the elements of set A are inside the circle. Now look at the following sets:
A = {a, b, c, d, e} and B = {1, 2, 3, d, e, x}.

These sets can be represented as follows:

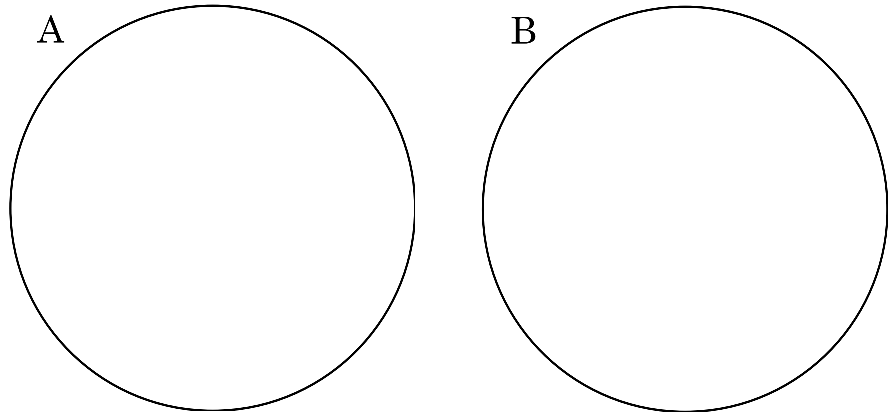


In this diagram, it can be seen that those elements which belong to both sets are placed in the region of the two circles. This overlapping shows that the two sets intersect i.e.

$A \cap B = \{d, e\}$.

What happens to two sets A and B whose intersection is empty i.e.

$A \cap B = Ø$? The Venn diagram looks like this:



The two circles do not intersect. This shows that the two circles have no element which is common to both of them.

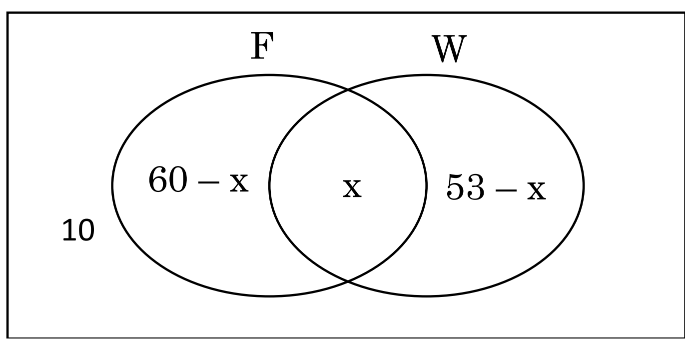
Example 5:

Venn diagrams

In a form 3 class of 108 students, 60 students like football, 53 like volleyball and 10 like neither. Calculate the number of students who like football but not volleyball.

Solution:

Let the Venn diagram be like this:



Let the number of students who like both volleyball and football be x.

$\therefore$ Students liking football only is 60 − x and those liking volleyball only is 53 − x.

See the diagram above

$\therefore\ 60 - x + x + 53 - x = 98$

$-x = 98 - 60 - 53$

$-x = 98 - 113$

$\therefore x = 15$ divide both sides by -1

$\therefore$ Students likes football but not volleyball = $53 - 15 = 38$

Exercise 5e

1. If A = {prime numbers less than 20} and B = {1, 2, 3, 4,..10} draw a Venn diagram illustrate the relation between A and B.

2. If M = {all integers from 2 to 15} and N = {Prime numbers less than 20}

   (a) Draw a Venn diagram to illustrate the information above

   (b) List the elements of the set $M \in N$.

3. P = {1, 4, 7, 11, 15, 17}

   Q = {5, 10, 15}

   R = {1, 4, 9}

   Represent this information on a Venn diagram.

4. Let A = {months of a year}, B = {months starting with M} and C = {months starting with N}

   (a) List elements of A, B and C

   (b) Draw a Venn diagram to illustrate

   (i) $A \cap B$

   (ii) $A \cap C$

   (iii) $B \cap C$

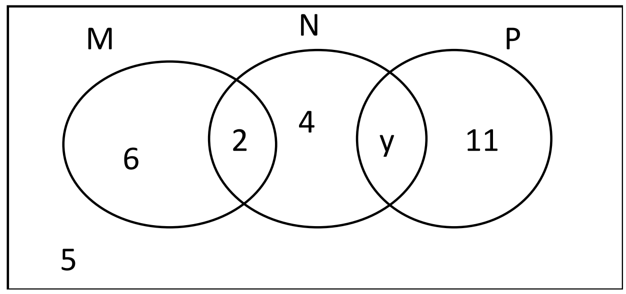
5. Given that the universal set ξ = {1, 2, 3, 4, 5, 8, 9}, sets A and B of ξ such that A = {even numbers} and B = {perfect square}, find;

   Find the following;

(a) n(A ∩ B)'

(b) Show that n(A ∪ B) = n(A) + n(B) − n(A ∩ B)

6. The number of elements in each region of the Venn diagram in the figure below are given as shown.



If n (ξ) = 36, find

(a) y

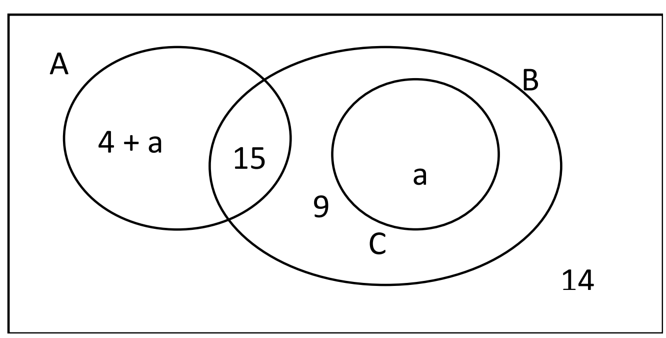
(b) n(N)

(c) n(M ∩ P)

(d) n(N' ∩ P')

7. Given that P and Q are sets such that n(P) = 9 and n(Q) = 13 and n(P ∩ Q) = y, find n(P' ∪ Q') in terms of y. If the total number of elements in P and Q is 15, find the value of y.

8. In the Venn diagram below, the numbers of elements are shown.



If the n(U) = 9a, find

(a) a (b) n(A) (c) n(A ∪ B)

(d) n(A ∩ B')

Union of Sets

Union of sets can also be presented in Venn diagram. Firstly do the activity below. Venn diagram is also used in Union of sets.

Activity 5:

Illustrating union of sets in the Venn diagram

In pair, discuss;

If $A = \{d, e\}$ and $B = \{a, b, c, d, e\}$, illustrate the given set in the Venn diagram;

Report your work to class.

Do more solving problems involving Venn diagram

Solving problems involving sets using Venn diagram

Venn diagrams in mathematics are also used to solve problems which might be more difficult to solve without the use of Venn diagrams. See the example given below.

Example 6:

Using Venn diagrams

In a group of 20 college students, 12 are taking Mathematics, 10 Physics and 14 Chemistry. 6 take Mathematics and Physics, 4 Chemistry and Physics and 8 Mathematics and Chemistry. Each student is taking at least one of these subjects. How many of the students are taking all the three subjects?

Solution

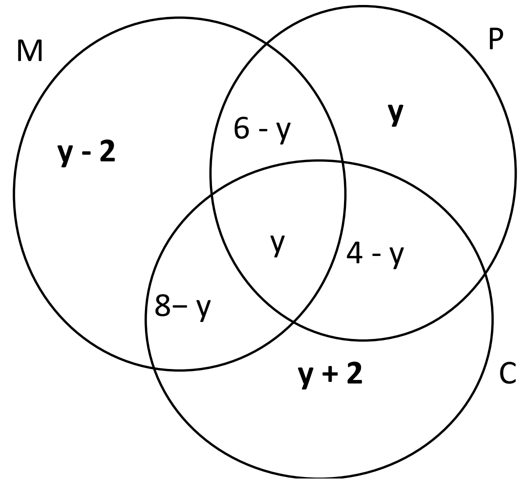
Let M = {students taking mathematics},

P = {students taking Physics}

C= {students taking Chemistry}

Illustrating as a Venn diagram;

The universal is $\xi = M \cup P \cup C$.



n (M ∩ P) = 6– y; students taking Mathematics and Physics only.

n (P ∩ C) = 4 − y; students taking Physics and Chemistry.

n(C ∩ M) = 8−y; students taking Mathematics and Chemistry.

n(M ∩ P ∩ C) = y; students taking all the three subjects.

n (M) = 12; students taking Mathematics, n (M ∩ P' ∩ C'); Students taking Mathematics only

$= 12 – (6 – y) – y – (8 – y)$

$= 12 – 6 + y – y – 8 + y$

$= y – 2$

n (P) = 10; students taking Physics

n(P) = P ∩ M' ∩ C' = Students taking Physics only

$= 10 – (6 – y) – y – (4 – y)$

$= 10 – 6 + y –y – 4 + y$

$= y$

n(C) = 14; students taking Chemistry

n(C ∩P'∩ M'); Students taking Chemistry only

$= 14 – (8 – y) – y – (4 – y)$

$= 14 – 8 + y – y – 4 + y$

$= y + 2$

But n(M ∪ P ∪ C) = 20; the total number of students

(y – 2) + (6 – y) + y + (8 – y) + y + (4 – y) + y + 2 = 20

9 +y = 20

y = 20 −18

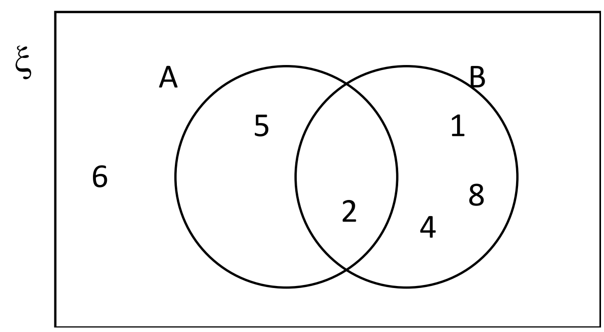
y = 2

n (M ∩ P ∩ C) = 2; students taking all the subjects.

Now do the following exercise.

Exercise 5f

1. Use the Venn diagram to answer the questions below.



List the elements of the following

(a) ξ

(b) A

(c) A ∩ B

(d) A ∪B

(e) (A ∪ B)'

(f) (A ∩ B)'

2. In a group of 100 students, 55 likes netball, 63 likes rugby and 15 likes neither, calculate the number of students who like only netball.

3. In a class of 30 students 20 takes History, 15 French and 2

takes neither. How many take;

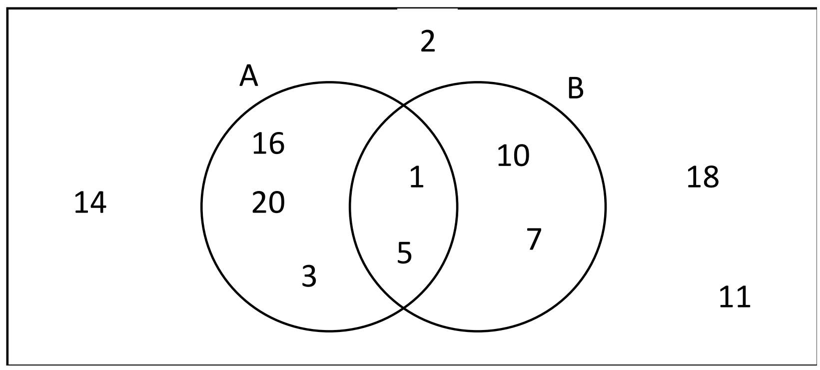
(a) both History and French?

(b) History only?

(c) French only?

4. Let B = {all boys in your class} and G = {all girls in your class}. Draw a Venn diagram to illustrate $B \cap G$.

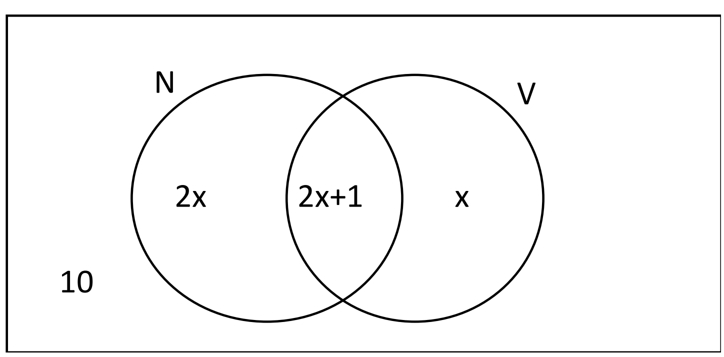
5. Consider the figure below:



List the elements of

(a)	ξ	(b)	A
(c)	B	(d)	$A \cap B$
(e)	$(A \cap B)'$	(f)	A'
(g)	B'	(h)	$A \cup B$

6. The figure below shows a Venn diagram.



In the Venn diagram, ξ = {girls in form three}

N = {girls that play netball}

V = {girls that play volleyball}

Given that there are 21 girls in the class, find how many girls play both netball and volleyball.

7. Given that universal set $\xi = \{11, 14, 15, 17, 18, 20, 23, 26\}$, and set $X = \{11, 14, 15, 17, 18, 20\}$ and set $Y = \{15, 17, 18, 20, 23, 26\}$, find $X' \cup Y'$

8. Given that the universal set $\xi = \{11, 12, 13, 14, 15, 16, 17, 18, 19\}$, Set B = {number greater than 16}, and set C = {multiples of 3}.

   Find the elements of:

   (a) Set B

   (b) Set C

   (c) Set $(B \cup C)'$

Unit summary

- A set is a collection of objects. The things or objects that make up a set are called members or elements of the set. It is possible to have a set which contains objects that do not have anything in common. Whether the objects are related in some way or not, as long as they form a collection, they comprise a set.
- Sets are denoted by a capital letters. There are various symbols that are used in set language. In the next unit, you will look at mapping and functions.

Unit review exercise

1. A class of 50 students wrote tests in mathematics, biology and physical science. The results of the tests were as follows; 12 passed mathematics and physical science, 19 passed mathematics and biology; 17; passed Biology and physical science; 2 passed physical science only, 5 passed mathematics only and 6passed Biology only. If 5 students failed all the three subjects and y passed all the subjects, use Venn diagram to calculate the value of y.

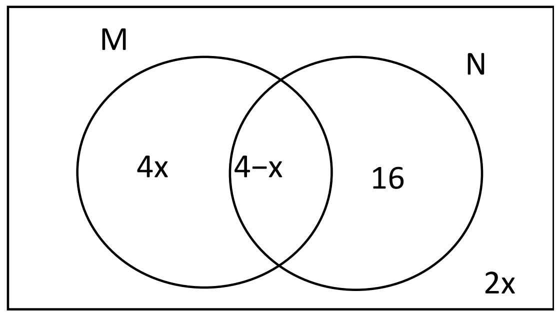
2. Given that A = {11, 12, 15,17}, B = {11, 15, 17, 19, 20} and C = {11, 17, 21}.

   (a) Show the three sets on a Venn diagram

   (b) Find n(A).

3. Given that X = {a, e}; Y = {b, c, d, e} and Z = {c, d, e, f}, find $\{X \cup Y \cup Z\}$

4. The figure below is a Venn diagram showing the number of elements in sets M, N and universal set ξ.



   If $n(M \cup N) = 29$, calculate the value of x.

5. At Chimwemwe secondary school, 72 students like watching football, 64 like watching basketball and 62 like watching netball, 18 like watching football and basketball, 24 like watching football and netball and 20 like watching basketball and netball, 8 students like watching all the three games and 56 don't like watching any game.

   (i) Draw a Venn diagram representing this information.

   (ii) From the Venn diagram, calculate the number of students at the school.

6. A church congregation has a youth group and a music group. There are 400 people in the congregation out of which 40 people belong to both the youth group and music group. There 60 members who belong to the youth group only, while 220 belong to neither the youth group nor the music group.

   (i) Draw a Venn diagram illustrating this information.

   (ii) Calculate the number of people who belong to the music group only.

7. In a family, six members eat meat; five members eat fish

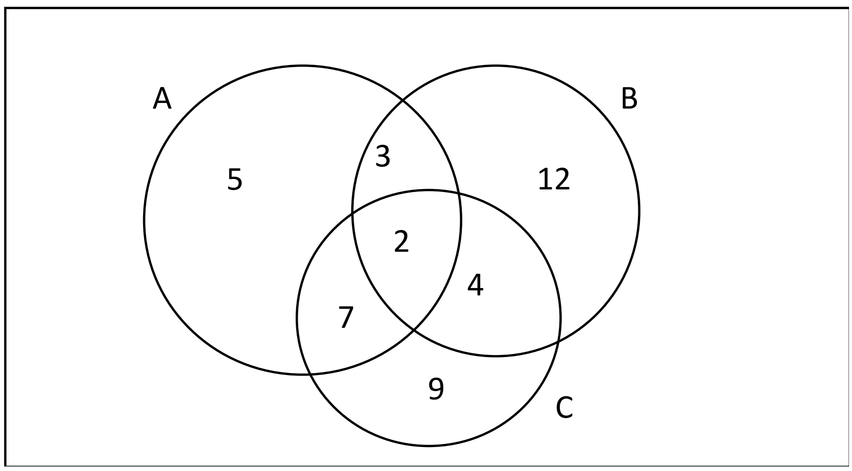
while two members eat both. Calculate the number of members in the family.

8. In form 3 class, students learn French, Latin and History. 20 students learn French, 55 learn Latin and 37 learn History. 7 students learn French and Latin only, 5 learn Latin and History only, 2 learn French and History only, 10 do not learn any of these subjects while x students learn all the three subjects. If there are 100 students in the class,

   (a) Draw a Venn diagram to represent this information.

   (b) Use your Venn diagram to calculate the number of students who learn Latin only.

9. The figure given below shows a Venn diagram of sets A, B and C.



   Find A’ (B C).

   (i) Draw a venn diagram and shade the region representing $A' \cap B' \cap C$.

   (ii) Find $n(A' \cap B' \cap C)$, if $n(A \cup B) = 8$ and $n(A \cup B \cup C) = 12$

10. In a class of 50 students, each of the students ate at least one of the following types of fruits: Banana, mango, and orange. It was found that: (x + 1) students ate all the three types of fruits, 9 students ate mangoes and oranges only, 8 ate bananas and mangoes only, 5 ate banana and oranges only, x students ate bananas only, (x − 1) students ate mangoes only and (x + 4) students ate oranges only.

   (i) Illustrate the information using a venn diagram.

(ii) Find the number of students who ate mangoes.

11. During National Examinations, 37 students sat for an examination in Mathematics, 48 Physical science and 45 sat for Biology. 15 students sat for Mathematics and Physical science. 13 sat for Physical science and Biology, 7 sat for Mathematics and Biology and 5 students sat for all three.

    Draw Venn diagram to represent this information and hence find the total number of candidates.

Glossary

A set is a collection of objects.

Empty or null set is set that has no elements.

Universal set is the main set that contains all the elements in it.

A Subset is a set that contains wholly or part of the elements in a universal set.

Equal sets are sets that contain exactly the same members, regardless of the order in which the members are presented.

A proper subset is a set that has few elements than those given in the universal set or if it is not exact subset of itself.

Finite set is a set with countable number of elements in it.

Infinite set is set whose elements are not countable

Union of set is the set of all elements that are of either.

Intersection of set is set that has elements that are common in all the given sets.

The complement of a set is the set containing all elements in a universal set but are not members of this given set.

References

Chikwakwa, e t al (2002), *Senior Secondary Mathematics Student's Book 3*. Blantyre: Macmillan Malawi.

Hau S and Saiti F (2010). *Strides in Mathematics 3.* Blantyre: Longman Malawi.

J B Channon et al, *New general Mathematics 3,* A modern Course for Zimbabwe, Longman group Ltd (1996), UK.

Elaine Ryder et al, *CHANCO Teach yourself series, Mathematics Questions and Model answers, second Edition (2013),* Chancellor College Publications, Zomba, Malawi

Unit 6

MAPPING AND FUNCTIONS

In the last unit you looked sets. In this unit you are going to look at functions. A function is a value which depends on and varies with another value. You are going to represent functions in different forms and also find the range and domain of functions.

Functions are the central object of investigation in most fields of modern mathematics.

The ideas of functions are used on daily basis knowingly or unknowingly. Functions will help to promote your thinking.

Defining mapping and functions

Before defining mapping and function, you will first of all do the activity.

Activity 1:

Identifying relations

a) Make two groups ,A and B of seven students each.

b) Each student in A should carry a card with a name of one student in group B.

c) Using a rope match a name in B with every person in A.

   (i) To how many persons is the rope being assigned?

   (ii) What do you call such linking?

From the above activity, you might have seen that every person of group A has a unique corresponding name in set B. Such a correspondence is called a relation.

Such kind of a correspondence also happens with numbers.

For example, if set A is all real numbers and set B is twice the number in A then this defines a relation between elements set of A and those of B. In this case;

1 corresponds to 2,

3 corresponds to 6,

4 corresponds to 8,

6 corresponds to 12

Notice that every number in set A corresponds to a unique (only one) number in set B in other words; each element in A maps onto one element in B.

Can you list the ordered pairs from the example above and come with own ordered pairs.

Thus a **relation** is a set of ordered pairs such as (1, 2), (3, 6), (4, 8) and (6, 12) from above.

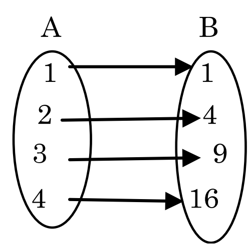
The set of ordered pair has input and output values. The set of inputs is called **domain** and the set of output is called the **range** also referred to as **image.**

Relation

There are of four types relations namely;

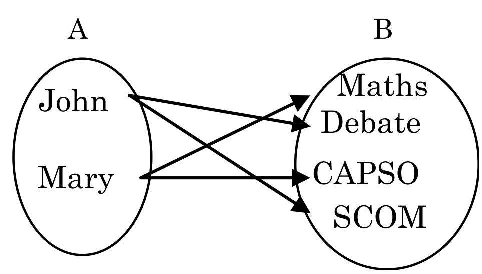
1. **One - to - one relation**

   One member in a domain relates or maps onto one member of the range. e.g. Square the input to get output.



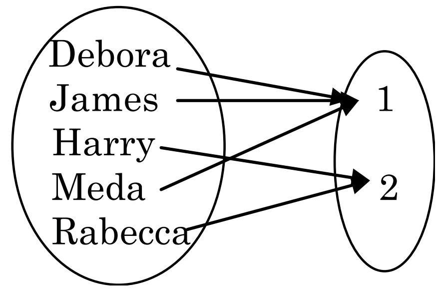
2. **One – to – many relation**

   A member of the domain A relates to more than one member of range B. Students who belong to more than one club at school.



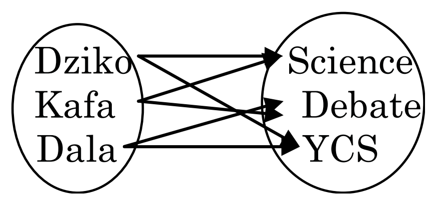
3. **Many – to – one relation**

   For one element in the range, there is more than one element of the domain that relate to it. This is also called multi-valued functions. For example, a number of students may obtain the same grade in a particular subject in an examination.



4. **Many – to – many relation**

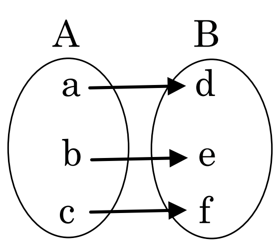
For one element of the domain, there is more than one member of the range that relates to it and one element in the range, there is more than one element of the domain relating to it. Students are members of more than one club society and a club society a club society has more than one student.

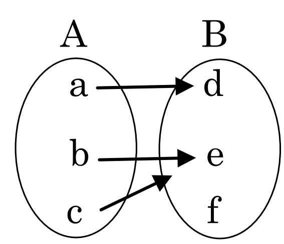


Example 1:

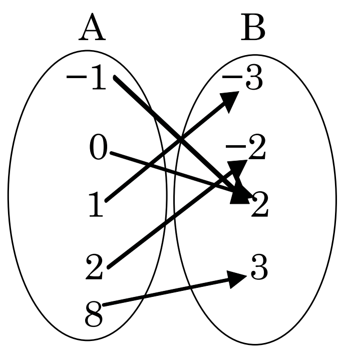
Identifying relations

Which of these arrow diagrams are relations?

(a) 

(b) 

(c)



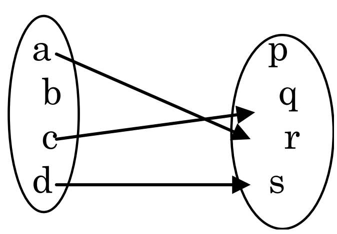
In diagrams (a), (c) represent functions, why? This is because every element in A has a unique element in B. Diagram (b) does not represent a relation. Why?

In these diagrams, the elements in set A are called the **inputs** and in set B are called the **outputs**.The set of inputs is called **domain** and the set of output is called the **range** also referred to as **image**

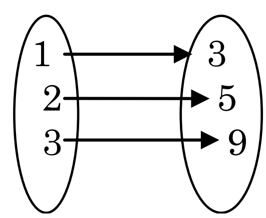
Exercise 6a

In the following diagrams, say whether the arrow diagrams represent relation and if not explain why not?

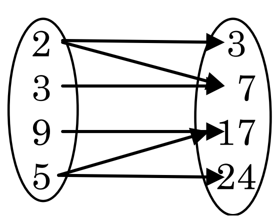
(a)



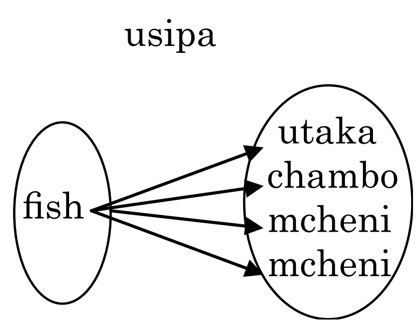
(b)

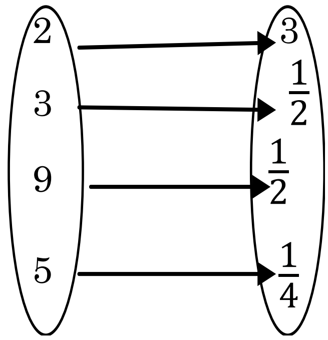


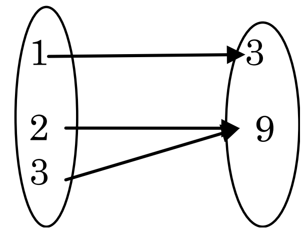
(c)



(d)



(e) 

(f) 

Mapping

A mapping is a special relation in which each element in one set is related to one element of the other set or pairing of input values with output values. Which of the four types of relation is a mapping? Certain statements are used in mapping such as,

If y is the operation "the square of", written as $A \xrightarrow{Y} B$. In this statement, elements in A are squared to get elements in set B.

Now look at the example below.

Example 2:

Elements of a relation

Let a be the operation of "add 3 to" A = {1, 3, 5, 7, 9}. What are the members of the new set?

Solution

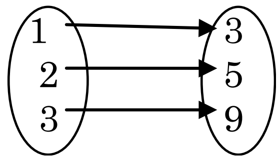
Let the image which is the new set be B

B = {4, 3, 8, 10, 12}; add 3 to input to get an output.

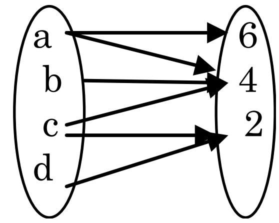
Exercise 6b

1. What type of relation is presented by the diagrams?

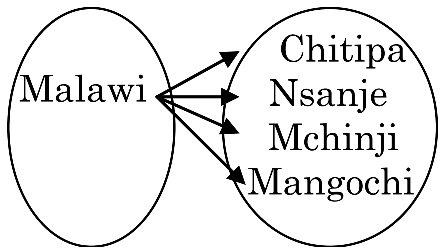
(a)



(b)



(c)



2. B is the operation “multiply by 2, add 1”, G = {0, 1, 2, 3, 5} and G $\xrightarrow{B}$ H. list the members of H.

3. X is the operation “square and add 2” A = {2, 3, 4, 5}. What are the members onto which A is mapped?

4. V is the operation “cubed” and B = {1, 2, 3, 4, 5}. What are the members of S, the set onto which the V is mapped?

5. Let S be the operation “the squaring of ” and T = {1, 2, 3, 4, 5, 6}. What is the image of T”?

Identifying functions

In mathematics **a function** is a relation between a set of permissible outputs with property that each input is related to exactly one output or

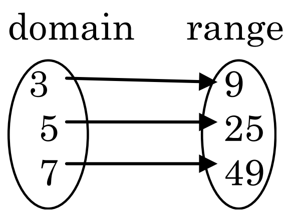
a rule which maps a single number to another single number. An example is the function that relates each real number x to its square x^2. Can you generate this function?

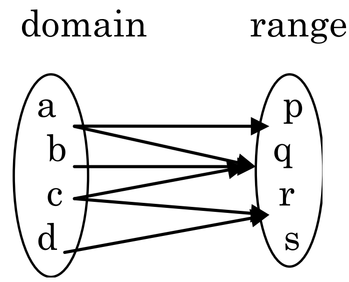
Which of the four types of relations are functions?

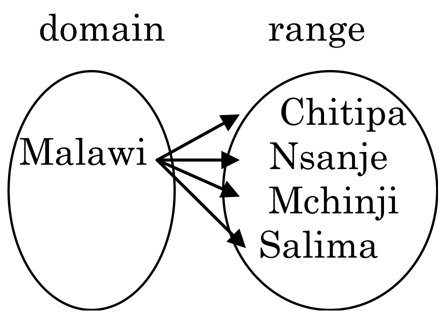
Only **one – to – one** mapping and **many – to – one** mapping represent function.

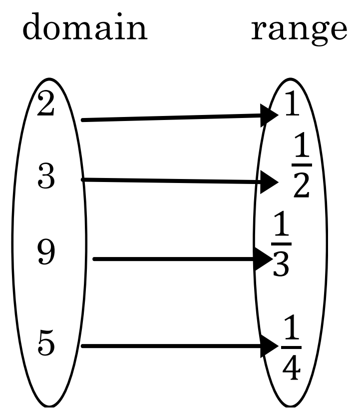
Oral exercise

Which of the following are functions? Give reason for your answer.

(a)


(b)


(c)


(d)


If a mapping is a function, it must satisfy these conditions;

(a) Each element of the domain has a unique image.

(b) The domain and the range have **one – to – one** correspondence or have **many – to – one** correspondence.

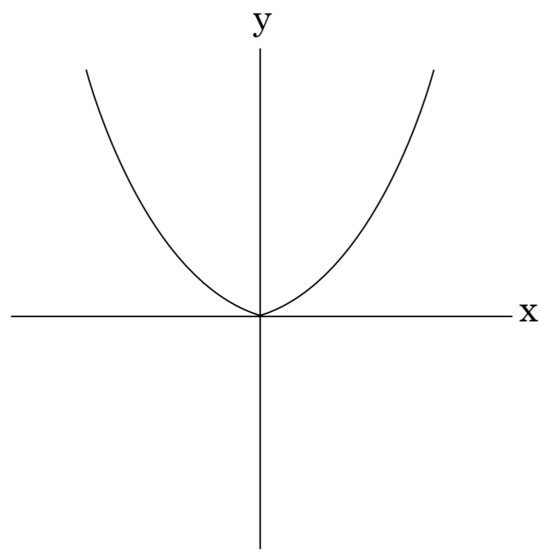
Function notation

There are several ways of describing a function. They are by means of;

(a) A mapping diagram like what you have done so far.

(b) an algebraic equation such as $y = 5x-4$

(c) A graph. $y = x^2$



(d) a set of ordered pairs like {(0,1), (1, 2), (2, 3)}

(e) as a table of values

x	0	1	2	3	5
y	0	3	6	9	15

The algebraic functions notation can take three different forms and these are;

(a) $y = 5x - 4$

(b) $f(x) = 5x - 4$ read as the function *f* of $x = 5x - 4$.

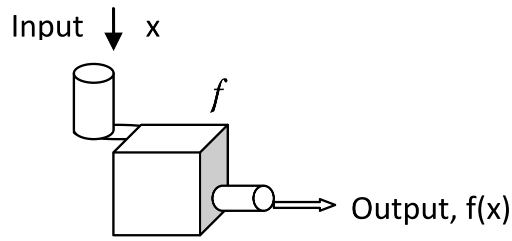
(c) $f : x \rightarrow 5x - 4$. which is read as the function *f* such that *x* maps onto 5*x* – 4.

All the three expressions mean that *y* is a function of $g(x), h(x)$ are also used to double functions.

A function *f* takes an input *x*, and returns an output *f*(*x*). *x* is the value taken by the function when you evaluate *f*(*x*) at a point. *f(x)* is commonly used.

One metaphor describes the function as a "machine" or as "black box" that for each input returns a corresponding output. Each element of the domain is said to be mapped onto the element of the second set (output) that corresponds to it.

You put the value of x into machine to get the output value. See the illustration.



Input and Output machine

Activity 2:

Finding the range given the domain

In pair, find the range given that f(x) = 5x – 4 if domain is {0, 1,2 ,3}

Present your work to class.

Now look at the example below.

Worked Examples 3:

Range and domain

a. Consider the function $f(x) = 3x + 5$ with domain {5, 8, 11, 14, 17}. The range is found by substituting each number of the domain into $3x + 5$.Thus when

$x = 0$ $\quad f(0) = 3(0) + 5 = 5$

$x = 1$ $\quad f(1) = 3(1) + 5 = 8$

$x = 2$ $\quad f(2) = 3(2) + 5 = 11$

$x = 3$ $\quad f(3) = 3(3) + 5 = 14$

$x = 4$ $\quad f(4) = 3(4) + 5 = 17$

Hence the range is the set 5, 8, 11, 14, 17

In this example, 0 is ***mapped onto*** 5, then 5 is said to be the **image** of 2, and 0 is a ***pre – image*** of 5

a) For the function $f: x \rightarrow x^2 - 3$

Evaluate:

(i) $f(1)$

(ii) $f(5)$

(iii) $f(-4)$

(iv) $f(0)$

Solution

Substitute x by the given x- value.

$f(1) = (1)^2 - 3 = -2$

$f(5) = (5)^2 - 3 = 22$

$f(-4) = (-4)^2 - 3 = 13$

$f(0) = 6^2 - 3 = -3$

Exercise 6c

1. If $f(1) = 2x + 3$, evaluate:

a) $f(1)$ b) $f(3)$

c) $f(7)$ d) $f(10)$

e) $f(40)$

2. If $f(x) = 4x - 3$, evaluate:

a) $f(0)$ b) $f(2)$

c) $f(5)$ d) $f(-1)$

e) $f(a)$

3. If $g(x) = x^2 + 6$, calculate:

a) $g(2)$ b) $g(4)$

c) $g(-3)$ d) $g(1)$

e) $g(-1)$

4. If $g(x) = 2x^2 - 5$ calculate:

a) $g(0)$ b) $g(1)$

c) $g(1/2)$ d) $g(-1/4)$

e) $g(-3/2)$

5. If $h(x) = \frac{3x-4}{2}$, calculate:

a) $h(0)$ b) $h(4)$

c) $h(6)$ d) $f(10)$

e) h(−2/3)

6. $f(x) = \frac{3x+2}{4}$, calculate:

a) $f(-6)$ b) $f(2.5)$

c) $f(-0.5)$ d) $f(0)$

e) $f(-1.6)$

7. If $h : x \to \frac{-6x+8}{4}$, calculate:

a) $h(i)$ b) $h(0)$

c) $h(4)$ d) $h(1.5)$

e) $h(-22)$

8. If $x \to \frac{(x+2)(x-4)}{-x}$, calculate

a) $g(1)$ b) $g(4)$

c) $g(0)$ d) $g(-2)$

e) $g(-8)$

9. For the function $f: x \to \frac{-x-2}{3}$. Evaluate

a) $f(4)$ b) $f(10)$

c) $f(-8)$ d) $f(0)$

e) $f(1)$

10. If $f(x) = 3x^2 - 5x - 8$, write down the value of

a) $f(-2)$ b) $f(1)$

c) $f(0)$ d) $f(2)$

e) $f(4)$

11. Copy and complete figure 6.1 if the relation is

a) $x \to 3x - 2$

b) $x \to x^2$

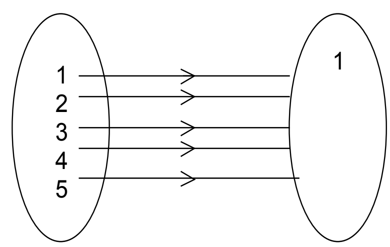


Figure 6.1

12. With {1, 3, 4, 5, 6} as domain draw mapping diagrams for $f: x \to 3x + 2$ and $f: x \to 3x^2 - 2$

Finding the domain given the range

Sometimes you know the range and you are asked to find the domain. To do this you solve the equation for *x*.

Activity 3:

Finding the element in the domain

In pairs do the following, If *f(x)* = 1 find the value of *x* in the function $f: x \to 3x^2 - 2$

Present your work to class.

Now look at the given example below, compare your way of working.

Example 3:

a) If $f(x) = 2x - 3$ find x if $f(x) = 5$ this means

$5 = 2x - 3$

$8 = 2x$

$4 = x$

b) Given g(x) = 3x +2 find the domain given the range is {2, 11, −1, −13} Solve equations, equating g(x) = 2, 11, −1, −13 in turns.

$2 = 3x + 2$

$0 = 3x$

$0 = x$

$11 = 3x + 2$

$3x = 9$

$\therefore x = 3$

$3x + 2 = -1$

$3x = -3$

$\therefore x = -1$

Therefore the range is $\{0, 3, -1, -5\}$

Exercise 6d

1. If $f(x) = 5x + 1$, find x if

a) $f(x) = 11$ b) $f(x) = 6$

c) $f(x) = 21$ d) $f(x) = -14$

e) $f(x) = -49$.

2. If $g(x) = 7x - 3$, find x if

a) $g(x) = 11$ b) $g(x) = -3$

c) $g(x) = 137$ d) $g(x) = -17$

e) $g(x) = -73$

3. If $h(x) = \frac{4x-2}{5}$, find x if

a) $h(x) = 2$

b) $h(x) = -2$

c) $h(x) = 7$

d) $h(x) = 10$

4 If $f(x) = x^2 + 2x - 3$, find x if

a) $f(x) = 21$

b) $f(x) = -3$

c) $f(x) = 0$

5 If $g: x \rightarrow x^2 - 1$, find x if

a) $g(x) = 3$

b) $g(x) = 8$

c) $g(x) = 1.25$

6. If $h(x) = x + 3$, find x if

a) $h(x) = 5$

b) $h(x) = 0$

c) $h(x) = 4.5$

Solving real life involving functions

In everyday life you unknowingly or knowing use relations in business and other areas.

Activity 4:

Identifying real life functions

In groups discuss; which of the following could be functions. State your reason.

(a) Petrol used by a car and distance travelled.

(b) Number of days and food used in the boarding school.

(c) Number of people and food used.

(d) Radius of a circle and area of the circle.

(e) Number of days spent in the boarding school and food left.

(f) Distance travelled and time taken in hours.

(g) Body mass and height of the person.

Report your answers to class in a plenary session and now look at the example below.

Example 4:

Functions in real life

If an orange costs K10, how many oranges will one buy with K30, K50, K80, and K100?

Solution

To get the number of oranges, divide the given amount by the cost.

Let the function be $f(x) = x/10$, the domain is {30, 50, 80, 100}

Then $f(30)$, $f(50)$, $f(80)$, $f(100)$

In table form, you have;

x	**30**	**50**	**80**	**100**
$f(x)$	**3**	**5**	**8**	**10**

Exercise 6e

1. In order to mitigate the impact of climate change, a village raised tree seedlings of which only half planted became fully grown. Let x represent the number of seedlings planted and y represent the number that become full grown trees.

(a) Write an equation that shows the relationship described above.

(b) Find four solutions of the equations. Write the solutions as ordered pair.

(c) Why does negative value not make sense?

2. The table below shows the tax rates provided by the MRA on salary earned.

Taxable income on monthly income(MK)	Rate of tax
First 20,000	0%
Next 5,000	15%
Excess of 25,000	30%

Three people gets the following; K15, 000, K22, 000 and K24, 000 as their monthly incomes.

Find how much will each pay to MRA. Give your answer as an ordered pair.

3. Grace is saving money to be used to buy school bag for K2200. She already has K1500 in her savings account. She plans to add K80 each day from the money she earns for selling freezes. The equation $f(x)$ =1500 + 80x describes total Grace's total savings $f(x)$ after x weeks.

After how many weeks will she have enough money to purchase the bag?

4. The temperature in degrees of the mine varies with depth. The temperature in degrees Fahrenheit, y is estimated by $y = 18x + 66.5$ where x is the depth in kilometres.

(a) Create a table of five ordered pairs of values that relate the depth of mine and the temperature of its walls.

(b) What would be the temperature of walls be if the mine is 3.6km deep.

5. The formula for finding the perimeter of a square with s units long is $P = 4s$. Find five ordered pairs of values that satisfy this condition.

Unit summary

- In this unit you have looked at functions. A function is rule of correspondence between two sets such that exactly one element in the second set corresponds to each element in the first set. The set of values in the domain is called the input and a range of values is called a output. The next unit looks at circle geometry where you will look at the angle properties.

Unit review exercise

1. Let $g(x) = \sqrt{x}$ and $h(x) = 7 + 2x$

a) $g(9)$

b) $h(9)$

c) $g(1)$

d) $h(1)$

2. If $f(x) = x^2 - 2x$, find

a) $f(1)$,

b) $f(2)$,

c) $f(4)$

3. If $f(x) = \frac{3}{\sqrt{24-5x}}$, find

a) $f(3)$

b) $f(-8)$

c) $f(-5)$

4. For the function f: x → $x^2 - 3$, find x if

a) $f(x) = -2$

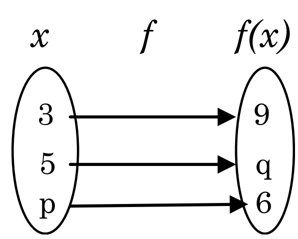
b) $f(x) = 22$

c) $f(x) = 13$

d) $f(x) = -3$

5. Given that $f(x) = 9^x$, calculate $f(\frac{3}{2})$

6. The figure below an arrow diagram for the function $f(x) = 4x - 2$.



Find the values of values of p and q.

7. Given that $f(x) = \frac{3x}{8x - b}$, find f (-b) in its simplest form.

8. Given that $g(x) = \frac{3x}{x+1}$, calculate the value of x when $g(x) = 2$.

9. The function $g(x) = \frac{2x - 1}{x}$ is defined on the domain $\{-1, \frac{1}{2}, 1\}$. Draw the arrow diagram to represent this function.

10. Given that $f(x) = ax - 6$ and $f(6) = 18$, find a.

11. Translate the sentence to equation and find four solutions as ordered pair. Some number is 4 more than the second

number.

12. Linda buys drinks for a morning meeting. She knows that the staff prefers fanta over coke. If she buys at least twice as many fanta as cocoa, write a relation to show different possibilities.

13. The table below shows the distance covered by a motorist from from Lilongwe to Blantyre.

t(h)	0	1	3	5	6
d (km)	0	50	150	250	300

What type of relation is this?

Draw distance time graph. Find the time travelled in 4hours time.

14. The table below shows approximate populations of certain local towns.

Town	A	B	C	D	E
Population	1600	3000	5000	10000	12000

If HIV prevalence rate in Malawi as of 2014 is at 10.5%. If $f(x) = 0.105x$, calculate the number of people who are infected in each town, assuming that the prevalence rate is the same in the entire town. Give your answer as an ordered pair.

15. As a thunderstorm approaches, you see lighting as it occurs, but you hear the accompanying sound of thunder a short time afterwards. The distance y in miles that sound travels in x seconds is given by the equation $y = 0.21x$.

(a) Create a table of five ordered pairs of values that relate the time it takes to hear thunder and the distance from the lighting.

(b) How far away is lighting when the thunder is heard 2.5s after the light is seen?

Summary

The unit has looked at functions. A function is rule of correspondence between two sets such that exactly one element in the second set corresponds to each element in the first set. The next unit looks at circle geometry where you will look at the angle properties.

Glossary

Function (or mapping) is rule of correspondence between two sets such that exactly one element in the second set corresponds to each element in the first set.

Domain is the first set of inputs of a function.

Range is the set of images of all the elements in the domain of a function.

References

Fawdry J. B. (1994), *Additional Mathematics a course of students*, Mathematics Association of Malawi.

Hau S. and Saiti F. (2010). *Strides in Mathematics* 3. Blantyre: Longman Malawi.

Elaine R. et al, *CHANCO Teach yourself series, Mathematics Questions and Model answers*, Second Edition (2013), Chancellor College Publications, Zomba, Malawi

Thomo F. et al (2011). *Excel and Succeed secondary Mathematics for* 3. Nairobi: Longhorn Publishers.

Unit 7

CIRCLE GEOMETRY ANGLE PROPERTIES

In unit 3, you learnt about circle geometry in relation to parts of the circle, and properties of a chord. In this unit, you will learn about angle property in which you will learn about the properties of a circle and prove theorems involving angles. You will describe the properties of cyclic quadrilateral and apply properties of a cyclic quadrilateral to solve problems.

The knowledge of circles is used in different fields such as production of wheels and tires, dinner plate and coins in industries.

Describing angle properties of a circle

You will begin the unit by first establishing the relationship between the angle at the centre and that at the circumference.

Activity 1:

Establishing that the angle at the centre is twice the angle at the circumference

In pairs Draw a circle of a reasonable radius with centre O as shown below.

(a) Draw angle AOC with A and C on the circumference.

(b) Draw another angle ABC with B at any point on the circumference.

(c) Measure angle AOB and angle ABC

(d) What conclusion can you draw on angle AOB and angle ABC?

Present your work to class.

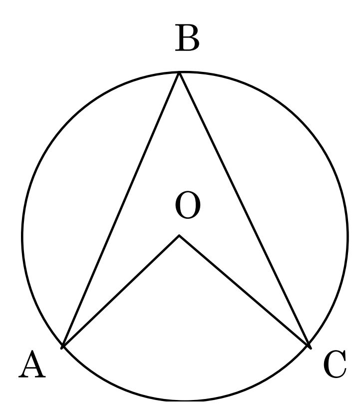


Figure 7.1

You might have noted that the angle at the centre is twice the angle at the circumference. I.e. angle AOB = 2angle ABC.

Can you try to show this by proving using the diagram above? Now compare your work the with the proof shown below.

Theorem:

Angle which an arc of a circle subtends at the centre is twice the angle which it subtends at any point on the remaining part of the circumference.

Using the drawings shown below

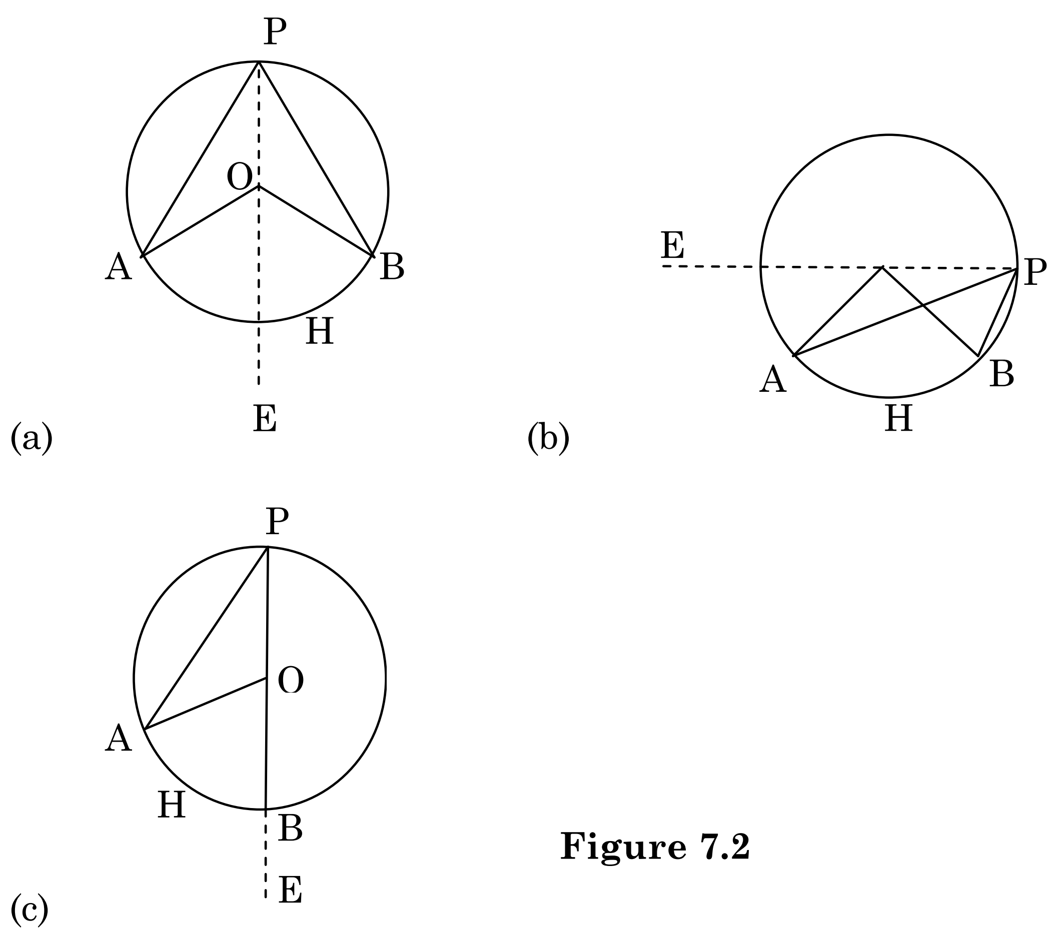


Figure 7.2

Given: a minor arc AHB of the circle, centre O, and a point P on the remaining part of the circumference

To proof: $\angle AOB = 2\angle APB$

Construction: join PO and produce it to any point E

Proof: $AO = OP$ (radii)

$\angle OAP = \angle OPA$ (base $\angle$s of isos.)

But $\angle EOA$ is an exterior angle of AOP,

$\angle EOA = \angle OAP + \angle OPA$ (2 opp. int. $\angle$s of equal to ext. $\angle$)

$\angle EOA = 2\angle OPA$

Similarly, $\angle EOB = 2\ \angle OPB$

But $\angle APB = \angle OPA + \angle OPA$

And also $\angle AOB = \angle EOA + \angle EOB$

$= 2\angle OPA + 2\angle OPA$

$\therefore\ \angle AOB = 2APB$

In pairs, using figure 7.3 below show that $\angle APB = \angle AOB$ consider the reflex $\angle AOB$.

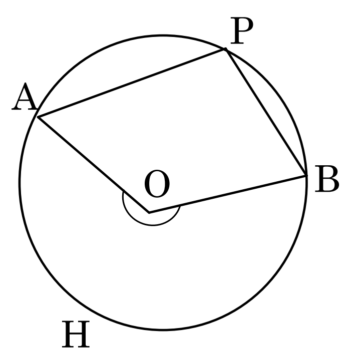


Figure 7.3

Now you will look at the angle in a semicircle.

Activity 2:

Establishing that the angle in a semicircle is a right angle

In pairs:Using a pair of compass and ruler, construct a circle of reasonable radius with centre O.

(a) Draw a diameter AB

(b) Mark point C at any point on the circumference on opposite side of the diameter

(c) Draw angle ACB and measure its size.

(d) What do you notice? Find out from your friends what they have come up with. Well, you might have found out that an angle in a semicircle is a right angle.

How can you prove that? Discuss with a friend if you can also show the proof.

Theorem: The angle in a semi - circle is a right angle.

Given: a circle with centre O and a diameter AB subtending ∠ACB at the circumference.

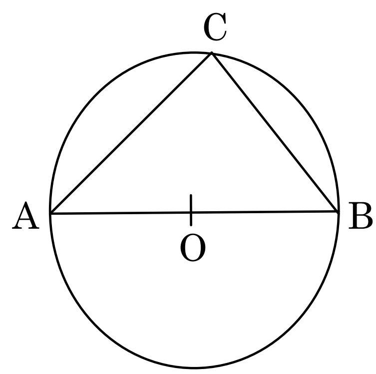


Figure 7.4

To prove: ∠ACB = 90

Proof: ∠AOB = 2∠ACB (∠ at the centre twice the ∠ at the circumference)

But ∠AOB = 180^0 (∠ on a straight line)

2 ∠ACB = 180^0

∠ACB = 90^0 (divide both sides of by 2)

Thus angle in a semicircle is a right angle.

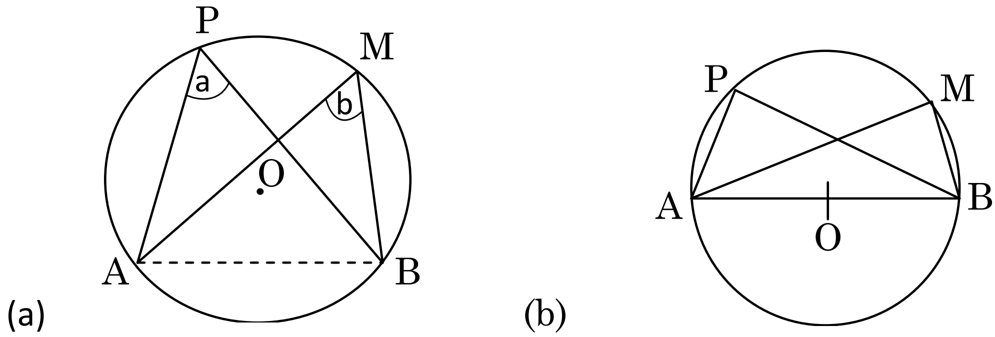
What is the relationship between the angles are in the same segment which are subtended by the same arc or chord.

Activity 3:

Proving that angles subtended by the same arc/chord are equal.

In groups discuss using the figure 7.5 below, show that a = b

Figure 7.5



Present your work to the class.

Now look at the theorem.

Theorem: angles subtended by the same arc / chord are equal or angles in the same segment of a circle are equal.

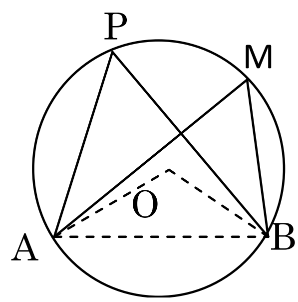


Figure 7.6

Given: a circle with centre O and ∠APB and ∠AMB on the circumference

To prove: ∠APB = ∠AMB

Construction: In (a) join AO and BO

Proof: ∠AOB = 2∠APB (∠ at centre twice ∠ at circ.)

Also ∠AOB = 2∠AMB (as above)

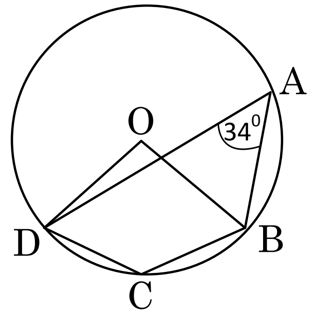
∴ ∠APB = ∠AMB

Similarly in (b) both angles are angles in a semicircle.

You will have to apply these theorems to solve some problems. Here are some examples.

Example 1

In figure below, O is the centre of circle ABCD.



If $\angle BAD = 34^0$, find $\angle BOD$ and $\angle$Bbncgn CD

Solution;

$\angle BOD = 2\angle BAD = 2 \times 34^0 = 68^0$ ($\angle$ at the centre = twice $\angle$ at the circum.)

Reflex $\angle BOD = 360^0 - 68^0 = 292^0$ ($\angle$at the point)

$\angle BCD = \frac{1}{2}\angle BOD = \frac{1}{2}(292^0) = 146^0$ ($\angle$ at the centre = twice $\angle$ at the circum.)

In figure 7.7, RT is a diameter of circle RSTV, centre O.

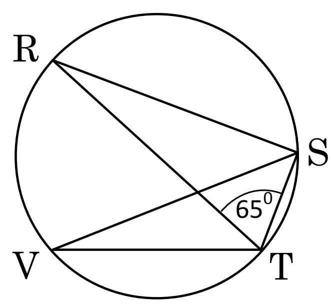


Figure 7.7

If $\angle RTV = 65^0$, find $\angle TVS$.

Solution;

In ΔTRS

$\angle RST = 90^0$ ($\angle$ in a semicircle)

$\angle TRS = 180^0 - (90^0 + 65^0)$ ($\angle$ sum of)

$= 25^0$

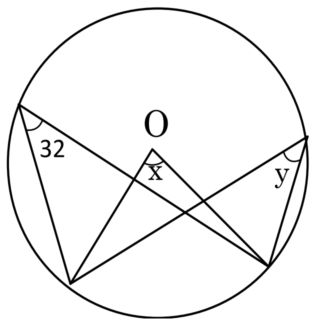
But $\angle TRS = \angle TVS$ ($\angle$s subtended by the same chord TS)

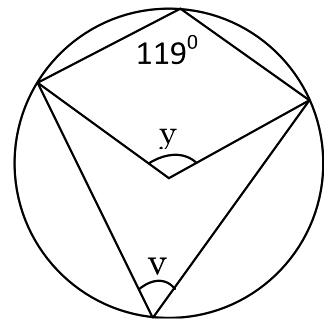
$\therefore\ \angle TVS = 25^0$

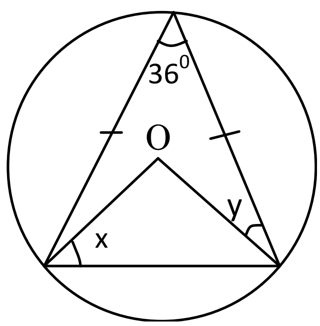
Now do the exercise below.

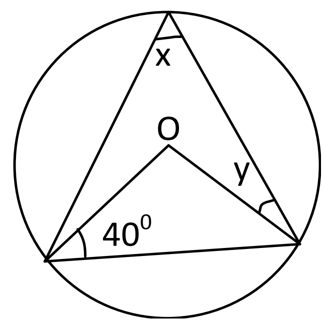
Exercise 7a

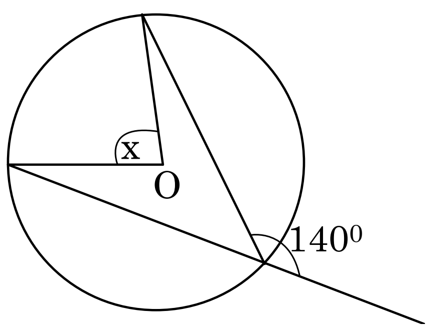
1. Find the lettered angles in the given circles.

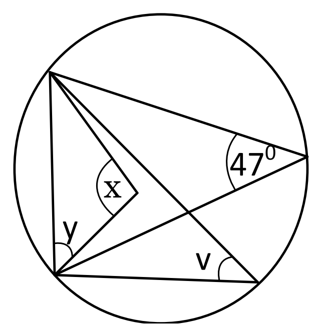
(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

Cyclic quadrilaterals

Properties of a cyclic quadrilateral

Now you will look at cyclic quadrilateral. Can you define a cyclic quadrilateral?

A cyclic quadrilateral is a quadrilateral in which all the four vertices lie on the circumference.

Activity 4:

Describing properties of a cyclic quadrilateral

In pairs;

(a) Draw a circle with centre O with reasonable radius as like the one below.

(b) In the circle draw a cyclic quadrilateral ABCD.

(c) Using a protractor, measure all the angles of the cyclic quadrilateral ABCD.

What can you say about the opposite angles of a cyclic quadrilateral?

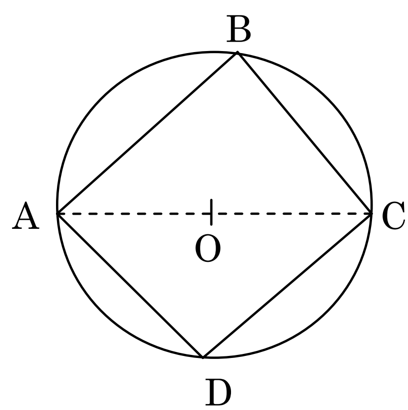


Figure 7.8

Now using the diagram above and your previous knowledge, show that $\angle A + \angle D = 180^0$.

Report your findings to the class.

Did you note that the opposite sides are supplementary? The sum of supplementary angle is 180^0. Now look at this theorem.

Theorem: the opposite angles of a cyclic quadrilateral are supplementary or angles in opposite segments are supplementary

Given: a cyclic quadrilateral ABCD

To prove: $\angle BAD + \angle BCD = 180^0$

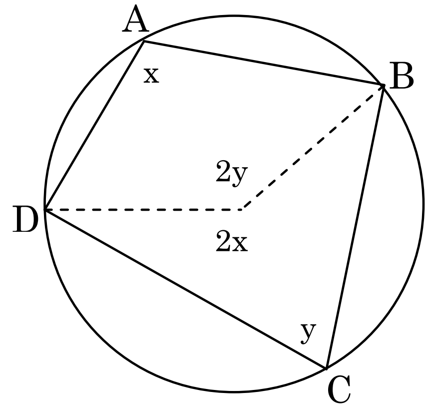


Figure 7.9

Join: Join OB and OD

Proof: with letters in the figure above

$\angle BOD = 2y$ ($\angle$ at the centre = twice $\angle$ at the circum.)

Reflex $\angle BOD = 2x$ (as above)

$2x + 2y = 360^0$ ($\angle$s at a point)

$x + y = 180^0$ divide by 2 both sides

Hence $\angle BAD + \angle BCD = 180^0$

Having looked at that, now you will look at the interior and exterior angle of a cyclic quadrilateral

Activity 5:

The interior and external angle of a cyclic quadrilateral

In groups

Study the figure below.

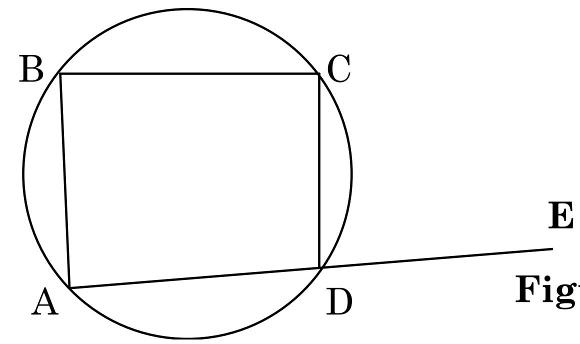


Figure 7.10

(a) Draw a similar figure and measure ∠ABC and ∠CDE.

(b) What do you notice?

(c) What is the relationship between these two angles?

(d) Identify the interior angle and exterior angle between ∠ABC and ∠CDE.

(e) Discuss and show theoretically that ∠ABC = ∠CDE.

(f) Present your work to the class.

You might have noted in the figure above that ∠ABC = ∠CDE.

Now look at the theorem below.

Theorem: The exterior angle of cyclic quadrilateral is equal to the interior opposite angle.

Given: a cyclic quadrilateral PQRS with PS extended to T

To prove: ∠PQR = ∠RST

Proof: with letters in the figure

$y + x = 180^0$ (opp. ∠s of cyclic quad,)

$y + v = 180^0$ (∠s on str. line)

$x = v \ (= 180^0 - y)$

∠PQR = ∠RST

Now you can apply the knowledge acquired and solve some problems. Look at the example below.

Example 2

In the figure A, B, C, D are points on a circle centre O. BA is produced to E. If ∠DAE = 76^0 and ∠ADO = 69^0, find ∠ABO.

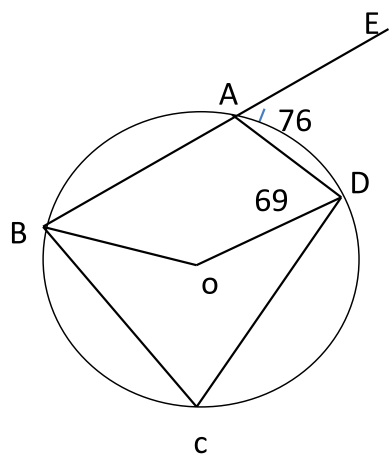


Figure 7.11

Solution;

∠BCD = 76^0 (= ext. ∠ of cycl. quad.)

∠BOD = 152^0

∠BAD = $180^0 - 76^0$ (∠s on str. line)

= 104^0

Now in quadrilateral. ABCD

∠ABO = $360^0 - 152^0 - 104^0 - 69^0$ (∠ sum of quad.)

= 35^0

∠ABO = 35^0

In the figure 7.12 below, CE is a diameter of a circle ABCDE. If ∠ABC = 126^0, find ∠ACE.

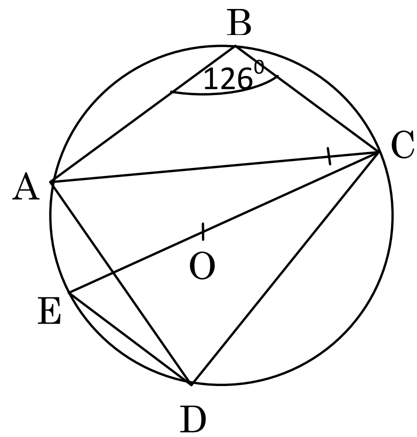


Figure 7.12

Solution;

$\angle ADC = 180 - 126^0$ (opp. $\angle$ of cycl. quad.)

$= 54^0$

$\angle EDC = 90^0$ ($\angle$ in semicircle)

$\angle ADE = 90^0 - 54^0 = 36^0$

But $\angle ADE = \angle ACE$ ($\angle$s in the same segment)

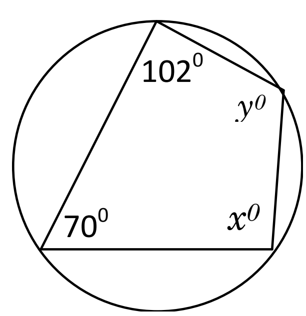
$\angle ACE = 36^0$

Now do the following exercise.

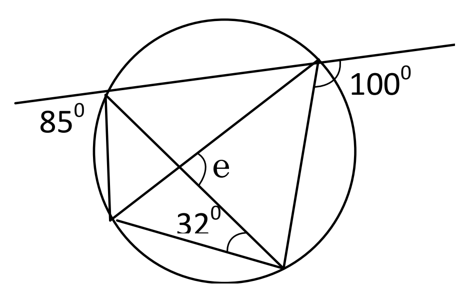
Exercise 7b

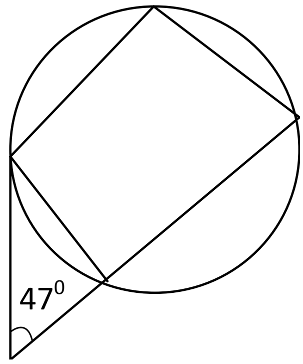
1. Find the value of the variables

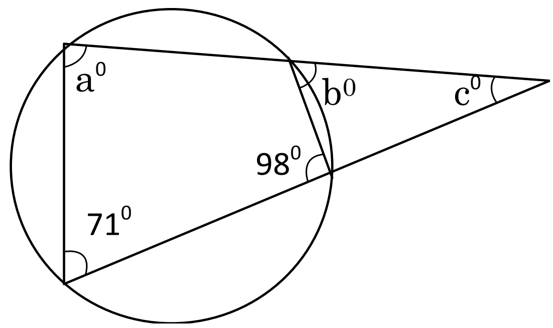
(a)

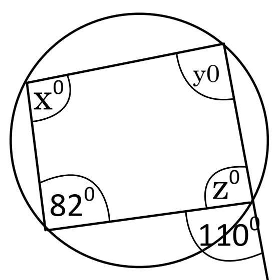


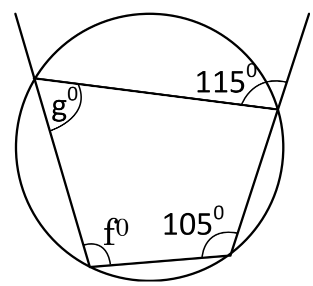
(b)



(c)


(d)


(e)


(f)


2.In the figure 7.13 below, if DC is a diameter and O is the centre of the circle, calculate angles BDC and DAB.

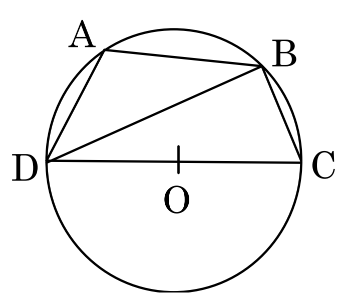


Figure 7.13

Now you will look at concyclic points.

Concyclic points and cyclic quadrilaterals

What do you understand by concyclic points? In last two theorems, you were looking at cyclic quadrilaterals. Do remember how you defined cyclic quadrilateral? Deduce the meaning of concyclic points from there.

Concyclic points are points which lie on the circumference of a circle.

Showing that points are concyclic

If the four points A, B, C, and D lie on a circle, then they are said to be concyclic

Activity 6:

Showing that points are concyclic

In groups;

(a) What are the properties of angles in cyclic quadrilaterals?

(b) Discuss three ways on how you can show that points are concyclic.

(c) Present your work to the class.

Now to show that points are concyclic points, you look at the **converse of the theorems** you looked earlier in this unit.

The following holds true for concyclic points

1. If the angles subtended by the same line are equal for example, angles APB, AQB, ARB are equal and subtended by AB.

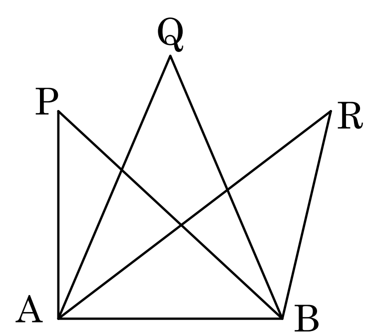
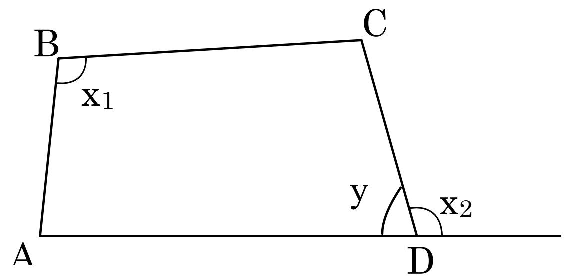


Figure 7.13

Which points are concyclic? Here $\angle APB = \angle AQB = \angle ARB$ and A, P, Q, R and B are concyclic points.

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.



2. If $x_1 + y = 180^0$, then this shows that ABCD is a cyclic quadrilateral and that A, B, C, D are concyclic points.

3. If the exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is a cyclic.

From the figure above if $x_1 = x_2$ then ABCD is a cyclic quadrilateral and A, B, C, D are concyclic points.

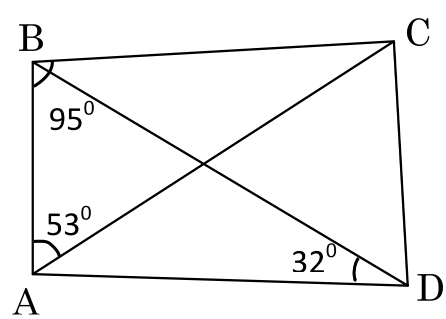
Now look at the example below.

Example 3

If ABCD is a quadrilateral in which $\angle ABC = 95^0$, $\angle BAC = 53^0$, $\angle ADB = 32^0$.Prove that ABCD is a quadrilateral.

Solution;

Sketch the quadrilateral



In Δ ABC

$\angle ACB = 180^0 - 95^0 - 53^0$ ($\angle$ sum in)

$= 32^0$

But $\angle ACB = \angle ADB = 32^0$ (as shown)

Also ∠ACB and ∠ADB are subtended the line AB

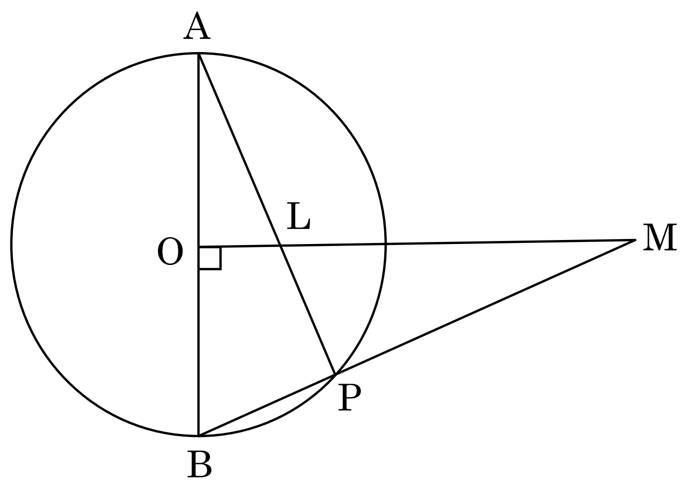
ABCD is a cyclic quadrilateral.

Exercise 7c

1. ABC is an equilateral triangle and ACD is an isosceles triangle drawn outside ∠ABC such that DA = DC and ∠DCB is a right angle. Prove that A, B, C, D are concyclic.

2. PQRS is a trapezium having PQ parallel to SR and ∠PSR = ∠QRS = 73^0. Prove that PQRS is a cyclic quadrilateral.

3.In the figure below, O is the centre of the circle ABP, MO is perpendicular to AB, and BPM is a straight line.

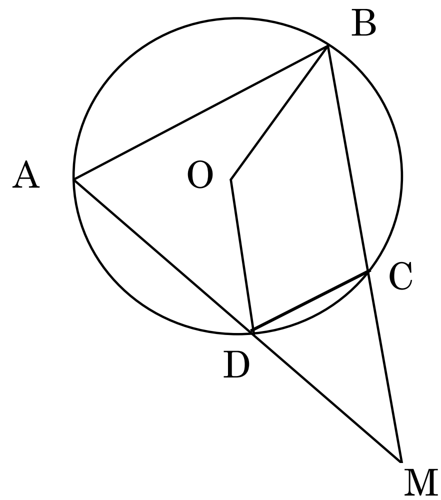


Prove that

(a) A, O, P, M are concyclic points.

(b) Angle OPA = angle OMB

4. In the figure below, ABCD is a cyclic quadrilateral and O is the centre of the circle. AB is parallel to DC and BC produced meets AD produced at M.



Prove that;

(a) MCD is isosceles triangle

(b) ODMB is a cyclic quadrilateral

5. MNYX is a circle centre O in which NM = YX. When MN and XY are produced, they meet at P. the mid – points of NM and YX are E and F respectively.

   Prove that;

   (a) Triangles OEP and OFP are congruent

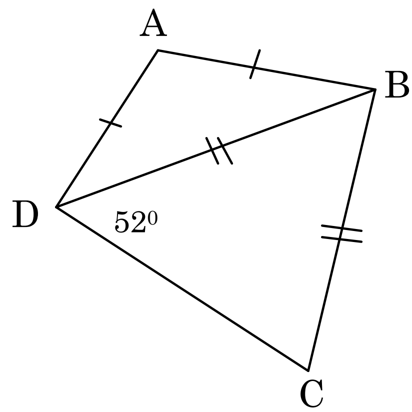
   (b) Points O, E, P, F are concyclic.

Unit summary

- The unit has so far covered work on properties of a cyclic quadrilateral where a number of theorems have been discussed and illustrated.
- The unit has finalized by looking at concyclic points. It has exercises for practice. The next unit looks at transformation.

Unit review exercise

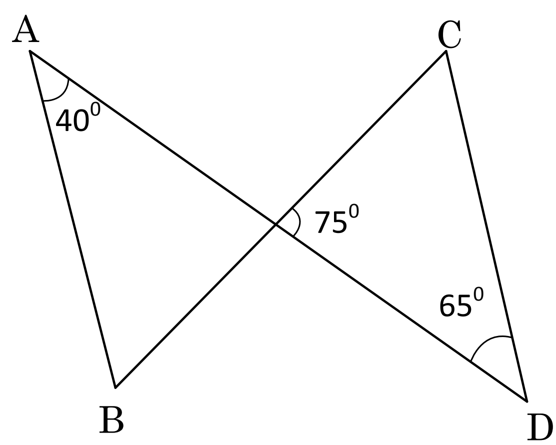
1. The figure is a cyclic quadrilateral ABCD where AD = AB. The diagonal BD = BC.



If $\angle BDC = 52^0$, find $\angle ABC$.

2. ABCD is a circle and the chords AC, BD cut at X. P and Q are points on XC, XD respectively such that PQ is parallel to CD. Prove that ABPQ is a cyclic quadrilateral.

3. Show that the figure ABCD below is a cyclic quadrilateral.

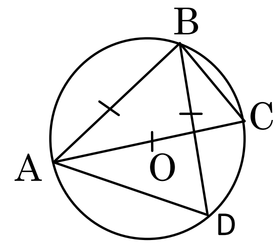


4. ABC is an equilateral triangle. A line from A meets BC at R, another line from B meets CA at S such that BR = CS. The two lines BS and AR, intersect at Q. Prove that;

(a) Triangles ABR and BCS are congruent.

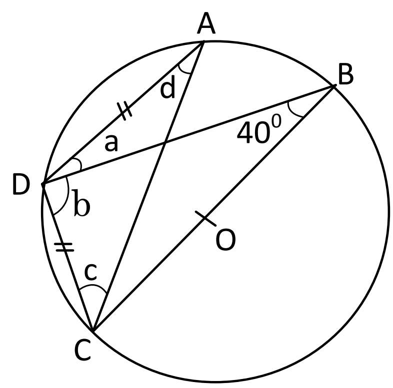
(b) Quadrilateral RCSQ is cyclic.

5. The figure ABCD below is a circle, centre O. AOC is a straight line and AB = BD.



If angle ABD = 70^0, calculate angle BAC.

6. A, B, C and D are points on a circle below with BOC a diameter, AD = CD and $\angle DBC = 40^0$.



Find the values of the angles marked a, b, and c.

Glossary

A cyclic quadrilateral is a quadrilateral in which all the four vertices lie on the circumference.

Concyclic points are points which lie on the circumference of a circle.

References

Geoff Buck well.(1997) *Mastering Mathematics*. Macmillan. London.

Channon J. B. et al. *New General Mathematics 3. A modern Course for Zimbabwe*. Longman. London.

Ric Pimentel and Terry Wall.(2011) *IGCSE Mathematics 2nd Edition*.John Murray. UK

Elaine Ryder et al.(2013) *CHANCO MSCE Mathematics Questions and Model Solutions. Chancellor College*. Zomba.

Gunsaru et al. *Secondary Mathematics Book 3*. Dzuka Publishing Company. Blantyre. Malawi.

Unit 8

TRANSFORMATION

Recall that in your JCE course, transformation was defined as the change in shape or position of an object. You studied reflections and rotations as examples of transformation. In this unit you will learn about more types of transformation: **translation** and **enlargement**. You will learn how to rotate a simple plane figure about a given point through a given angle clockwise or anticlockwise. You will also learn to translate a simple plane figure in a column vector. Finally, you will learn to enlarge a simple plane figure by a scale factor and a given a centre of enlargement.

The knowledge of transformations is used in artistic designs to make objects more appealing. People like, doctors and surveyors use enlargement in their day to day work.

Names and labels in transformation

Throughout this unit you will use the words **object** to mean the original figure before transformation and **image** to mean the figure after a transformation. All corresponding or matching points on the object and the image will have to be named using the same letter or same numbers. Where letters are used, those on the image should have a prime ´ on them. Where numbers are used, they should appear as subscripts.

Rotating a plane figure about a given point

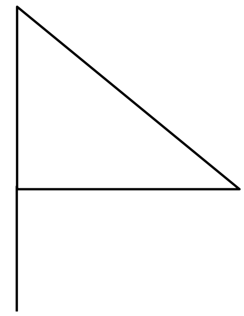
A rotation is a turn. In a rotation an object turns about a fixed point called **centre of rotation.** A rotation can be clockwise or anticlockwise. When the rotation is clockwise it is negative while if it is anticlockwise, it is positive. The rotation is given as a fraction of a turn or an angle in degrees.

Activity 1:

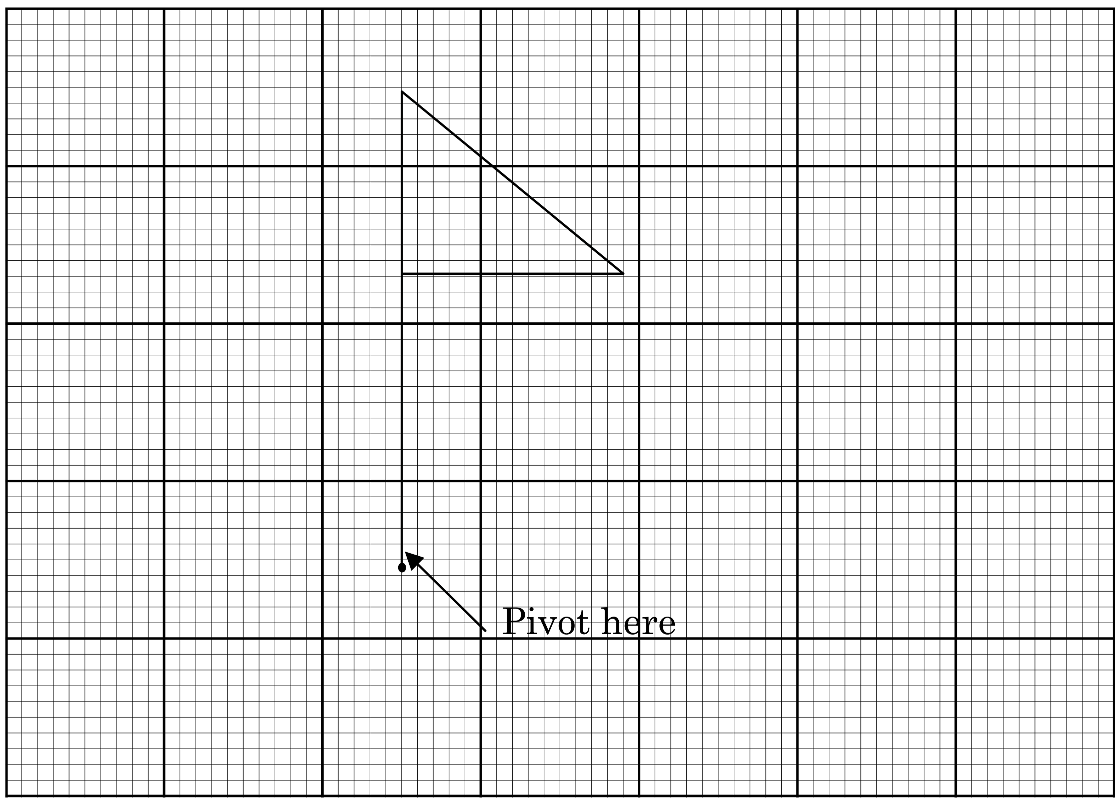
Drawing rotations

In groups

1. Draw a grid of square boxes (10 boxes by 10boxes) on a paper. You can also use a squared paper if you have it.
2. Using a cardboard paper or any other hard paper, cut out a right angled triangle and fix a stick to one corner as shown on following page:



3. Pivot one end of the stick at one point on the drawn square grid as shown below:



4. Now rotate the triangle through angles 30^0 and 45^0 clockwise, each time drawing the image of the triangle. Measure the angles that each vertex rotates through at the pivot centre.

5. Now draw square grids of your choice. Using different plane shapes e.g rectangles, trapeziums and your own choices of angles and directions, draw rotations of the plane shapes on the square grids. Again measure the angles that the vertices of your diagrams go through at the pivot centre.

6. Comment on your findings.

In a rotation, all points on the object move by the same measurement. To draw a rotation you must be given the centre of rotation, the angle of rotation and the direction of rotation. Here are two ways of drawing rotations:

Using a protractor and a ruler:

The assumption is that the initial position of the plane shape is given or the figure is drawn on the grid or squared paper.

- Join each vertex of the plane shape to the given centre of rotation.
- Choose any one vertex and using a protractor, draw the new position of this line by measuring the angle of rotation.
- Measure the distance of the vertex from the centre of rotation and use this distance to locate the new position of the vertex on the new position of the line. Do the same with other vertices.
- Join the new positions of the vertices on the new lines to obtain the image of the plane shape.

Using tracing transparent paper

- Join one point on the figure to the centre of rotation.
- Place the tracing paper on the squared paper on which the figure is drawn.
- Trace the shape, the line drawn in the first step above and the centre of rotation on the tracing paper.
- Measure the given angle from the line drawn in the first step.
- Place a sharp point say a pencil or pen or compasses on the traced centre and actual centre.
- Turn the traced figure through the given angle until the line drawn in the first step coincides with the line or mark you made when you measured the angle in step 4.
- Finally, trace the figure to obtain the new position of the object.

Exercise 8a

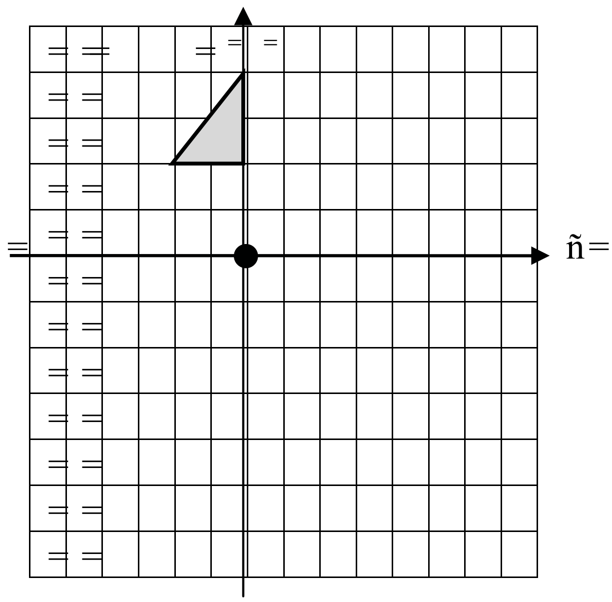
1. Use the above information and the grid below to draw the image of drawn triangle

a) after a rotation of 90^0 clockwise about (0,0)

b) after a rotation of 45^0 anticlockwise about the origin.

c) after a rotation of -180^0 anticlockwise about (0,0)

d) after a rotation of -270^0 clockwise about (0,0)



2. On a grid draw rectangle with vertices at (-2,1),(-6,1) (-6,3) and (-2,3). Draw the image of the rectangle after a rotation of 45^0 about (0, 0).

3. Draw and number x- and y- axes from -8 to +8. Show the position of a point Q (4,3) after each of the following rotations. In each case state the coordinates of the new point.

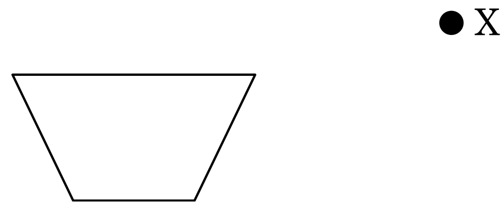
a. 180^0 about (0,0)

b. +90 about (1,4)

c. $+90^0$ about (-6,5)

d. -90^0 about (-3,0)

4. Trace the shape below. Find the image of the shape after a rotation $+120^0$ about X.



Activity 2:

Describing a rotation

To describe a rotation you must give the angle of rotation, the centre of rotation and the angle of rotation whether clockwise or anticlockwise.

Your teacher will provide you with a figure and its image on a squared grid. The centre of rotation will also be given. In groups,

1. Join one vertex on the object to the centre of rotation.
2. Join the corresponding vertex on the image to the centre of rotation.
3. Using a protractor, measure the angle between the lines joining the vertices to the centre of rotation.
4. Describe the rotation.

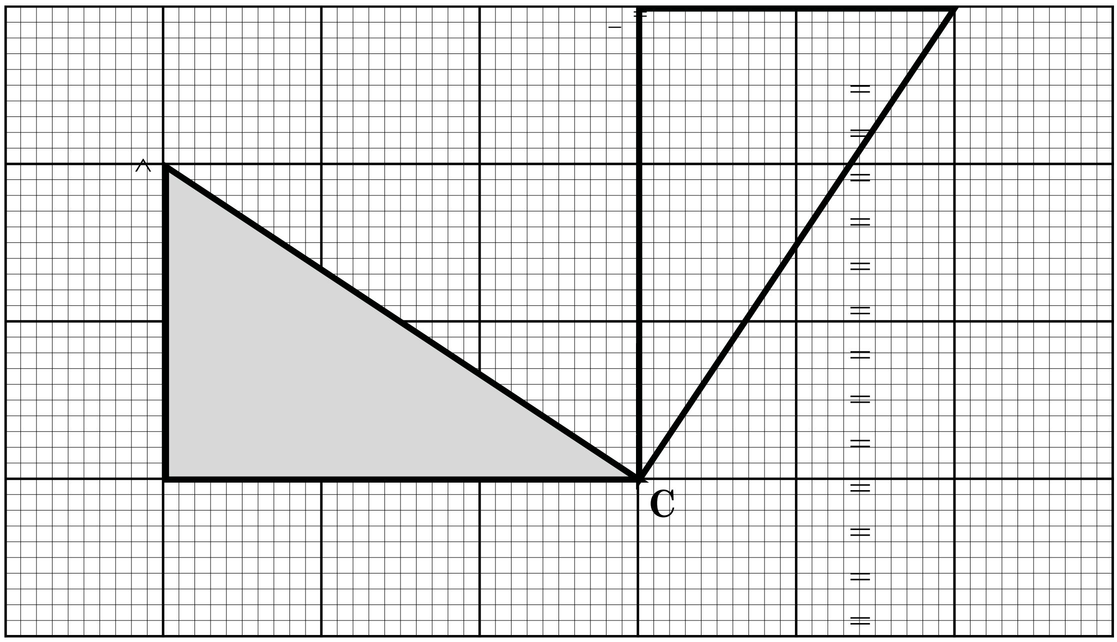
Sometimes you may not be given the centre of rotation. When this happens, you will have to proceed in the following way:

1. Join two pairs of matching points on the object and the image.
2. Using a pair of compasses, draw the perpendicular bisectors of the two lines joining the points.
3. Extend the lines until they intersect at a point. This point gives the centre of rotation.
4. To find the angle of rotation, join any one pair of matching points on the object and the image to the centre of rotation. Measure this angle using a protractor.

Example 1:

Describing rotation

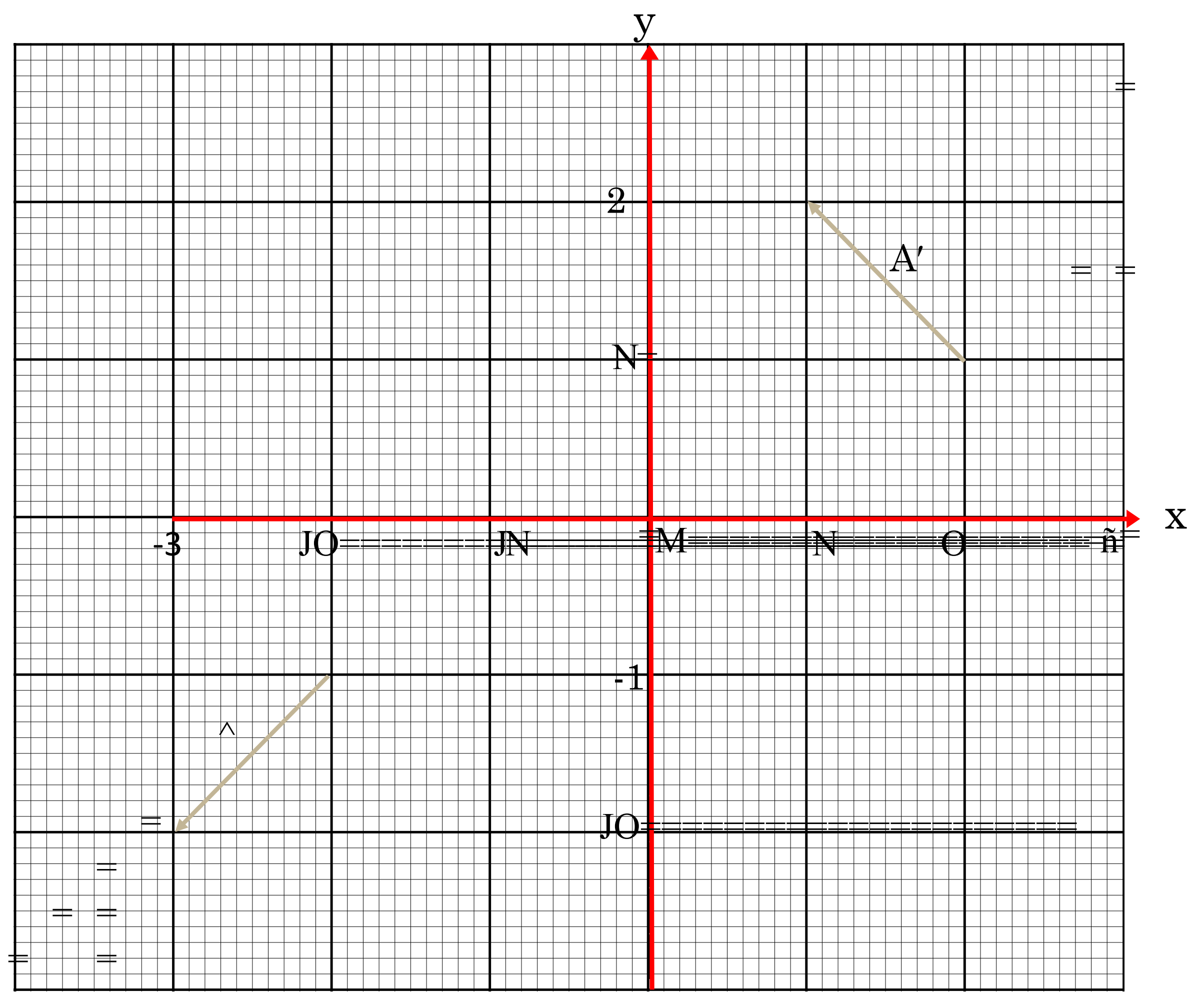
(a) Describe the rotation mapping triangle A onto B in each of the following diagram. C is the centre of rotation and the shaded triangle is the object.



Solution

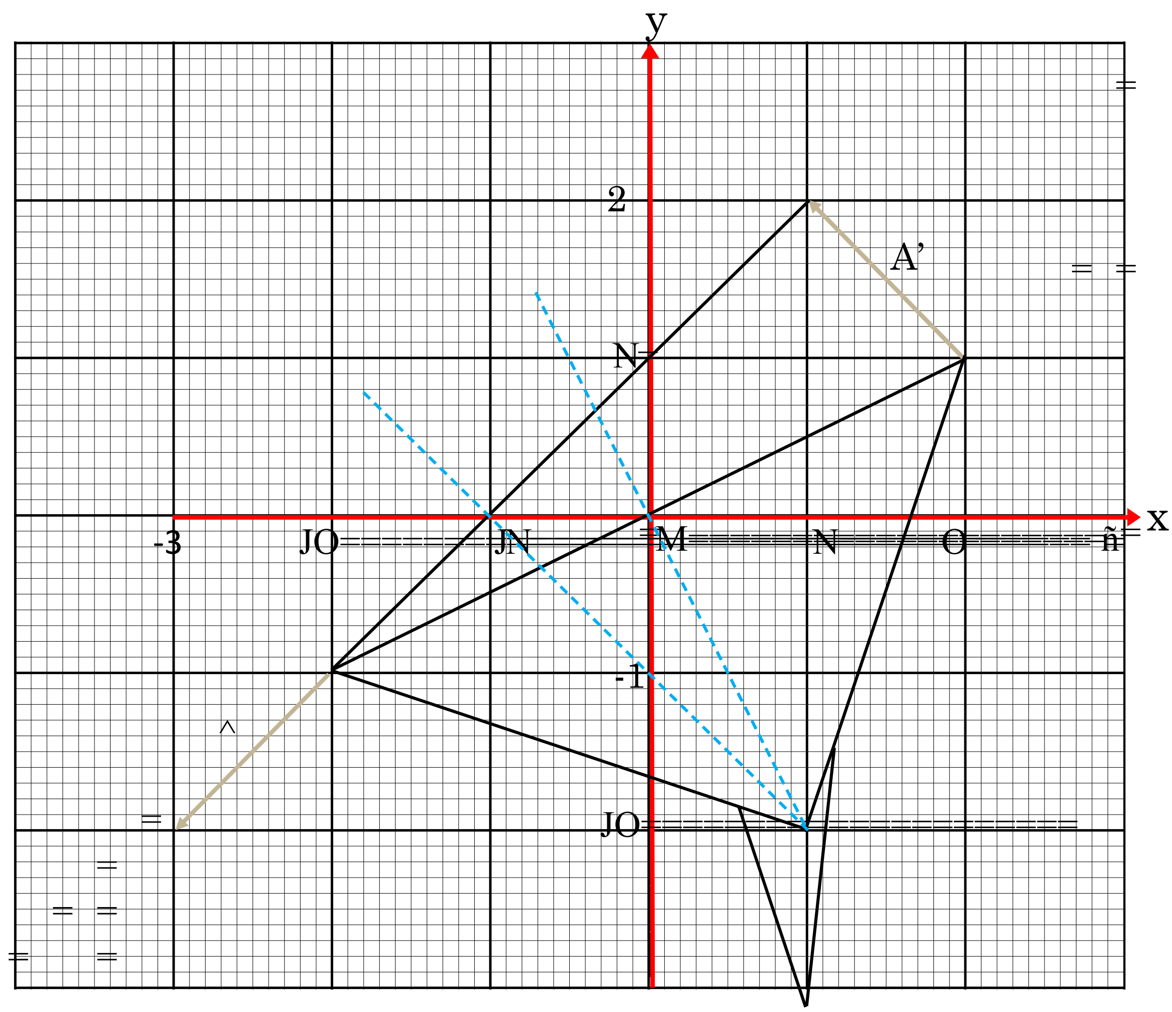
The rotation is 90^0 clockwise about centre C or you may say that the rotation is 270^0 anticlockwise about centre C. If you want to use – or + you can write -90^0about C or $+270^0$ about C.

(b) Describe the rotation that maps arrow A onto A´



Solution

Join the matching points on the two arrows and construct the perpendicular bisectors of the lines.



Lines joining matching points to the centre of rotation. The angle between them is the angle of rotation.

(Dashed lines are perpendicular bisectors of lines joining matching points on the object and the image)

From the diagram, the rotation is 90^0 clockwise about (1,-2).

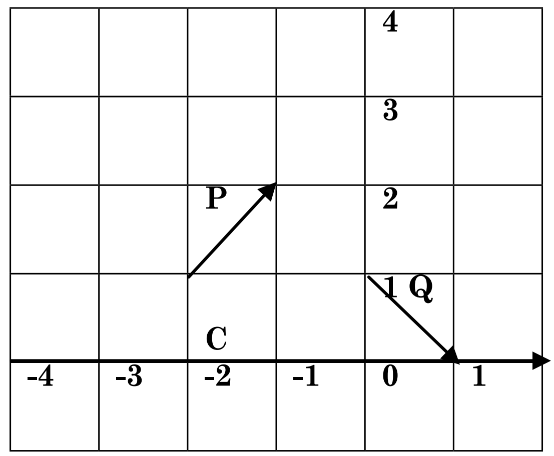
Exercise 8b:

1. The vertices of a triangle ABC are (3, 3), (1,-1) and (3,-2). The vertices of the image of triangle ABC are (-3, 3), (1, 1) and (2, 3) respectively. Graph the two triangles on a squared paper and describe the rotation mapping triangle ABC onto its image.

2. The vertices of a triangle ABC are (-1,1), (-2,1) and (-2.5,2) .

The vertices of the image of triangle ABC are (5,1), (6,1) and (-5.5,0) respectively. Graph the two triangles on a squared paper and describe the rotation mapping triangle ABC onto its image.

3.



In the above diagram, P is rotated anticlockwise about C to obtain imge Q. Describe the rotation.

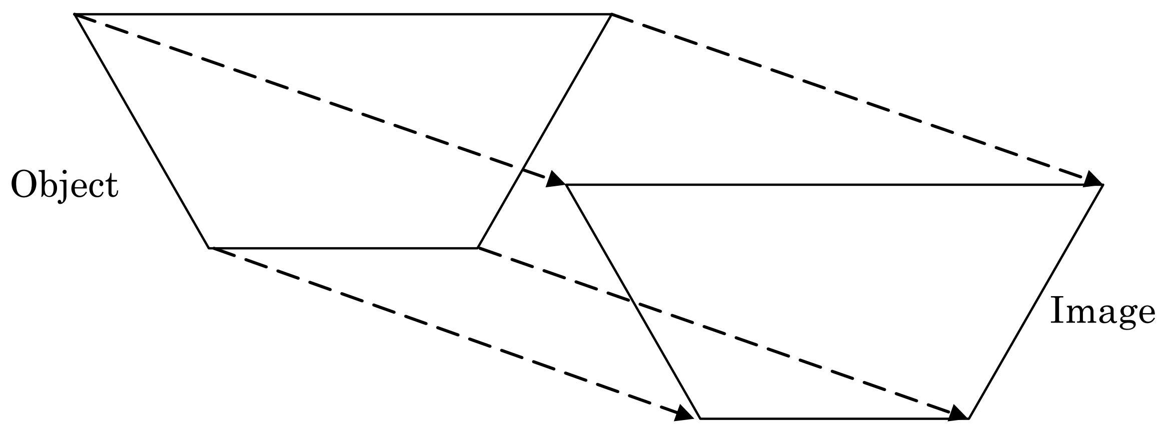
4. A quadrilateral WXYZ has coordinates (-4,0),(-3,1),(-2,0) and ((-2,0). The Image of the quadrilateral WXYZ are (0,-2),(-1,-3),(-2,-3) and (-2,-2). Draw the two quadrilaterals on a squared paper or grid and find the centre and angle of rotation.

5. Triangle ABC and PQR are congruent in that order. Triangle ABC has Vertices (-4, 3), (-3.8, 0) and (-2.5, 1). Triangle PQR has vertices (0.1,-2.5), (1.9, 0) and (0.2, 0.1). By drawing grids or by using a squared paper, describe the rotation that maps triangle ABC onto triangle PQR.

Translation

A translation sometimes called a **slide** or a **shift** moves the shape in a straight line. The shape of the object does not change and every point on the object moves by the same amount and in the same direction.

An example of a translation is shown below:



Activity 3:

Describing translation

Your teacher will provide you with a graph paper. In your groups,

1. Using a scale of 2cm to represent 1 unit on both axes, draw a triangle PQR such that the vertices are (-3, 3), (-2, 5) and (-5, 0) respectively.

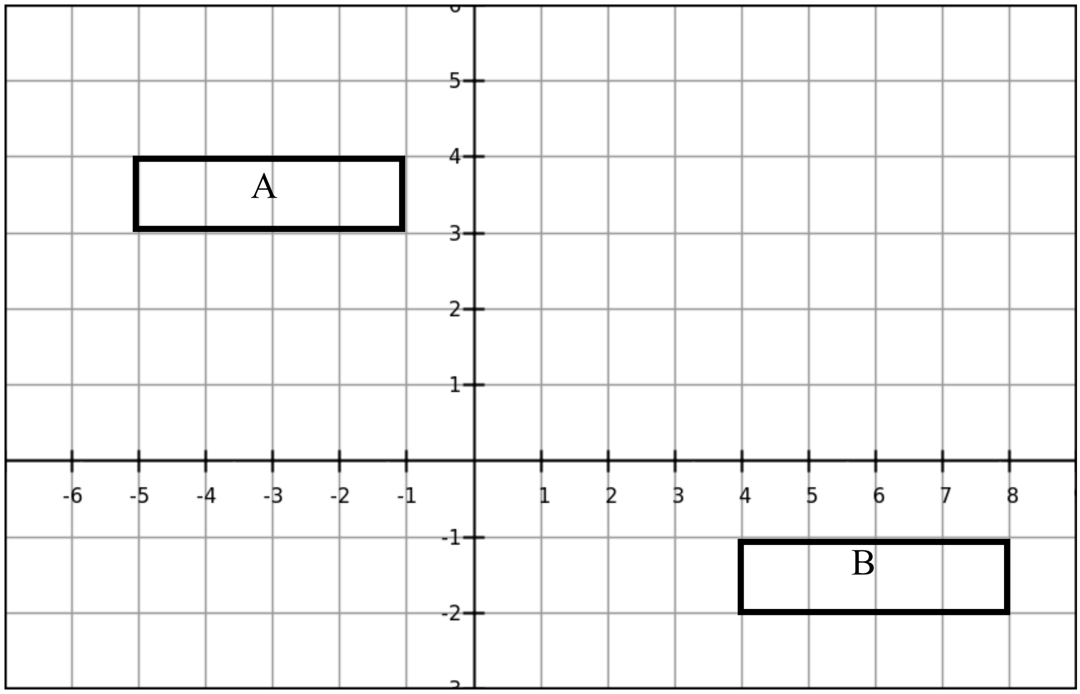
2. On the same axes draw the image of triangle PQR such that the vertices are (0,-1), (1, 1) and (-2,-4) respectively.

3. Use your drawings to describe the translation mapping triangle PQR onto P'Q'R'.

To describe a translation you must give the distance and the direction an object has moved from its old position to its new position. The movement is given in two parts: horizontal and vertical. You start by giving the movement in the horizontal direction followed by the movement in the vertical direction.

Example 2:

Translation

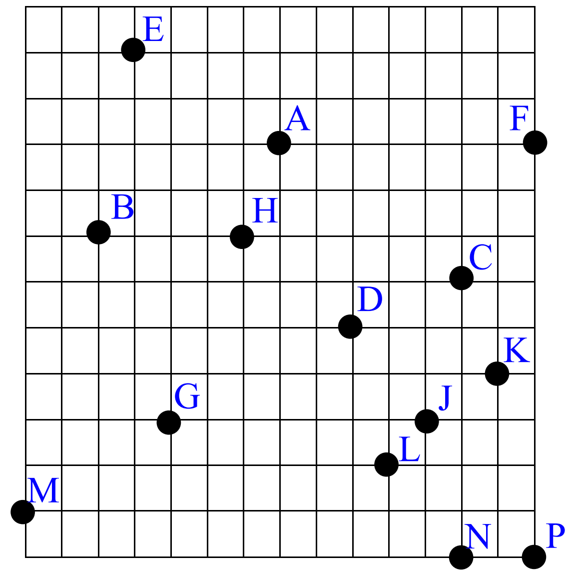


Describe the translation that maps rectangle A onto rectangle B in the grid above.

Solution:

As all points move by the same amount in a translation, you choose any one point on the object and trace the units the point moves to the corresponding point on the image. For example if you choose the top left corner, you will see that the rectangle moved 10 units horizontally to the right and then 8 units vertically downwards.

Exercise 8c



1. Using the above grid, describe the translation that moves the following points: Each box is 1 unit by 1 unit long. For parts k to 0, give a single translation.

   a. A to B

   b. C to F

   c. D to Q

   d. N to K

   e. H to G

   f. J to E

   g. L to M

   h. A to Q

   i. F to K

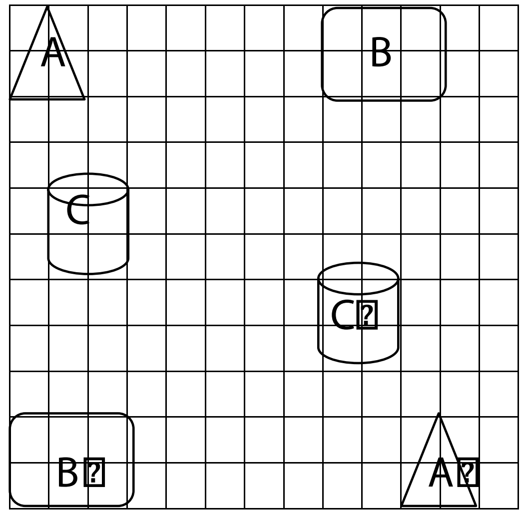
   j. E to M

   k. C to N.

   l. B to C to H

   m. Q to J to D

2.



In the above grid each grid is 1 unit by 1 unit .Describe the translation that maps each figure onto its image.

Activity 4:

Drawing a translation

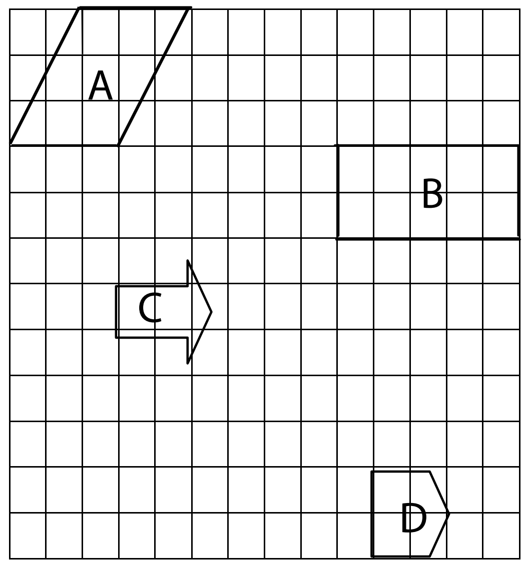
To draw a translation, you must be given the amount of movement of the object both horizontally and vertically. You must also be given the position of the figure to be translated.

Your teacher will show you a chart on which different plane shapes are drawn. In pairs, draw the grid and copy the figures. Translate the figures and draw the image of each figure as follows:

1. Translate A, 3units right 4units down to $A_{1.}$
2. Translate B, 5units left 2units up to B_1
3. Translate C, 4 units down to C_1.

Exchange your work with your partners and mark each others' work. Let your teacher check your work.

Exercise 8d



Draw the image of each of the above plane shapes after the following translations**:**

1. Shape A:

   (a) 4 units to the right and 2 units downwards.

   (b) 10 units to the right.

2. Shape B:

   5 units down.

3. Shape C:

   (a) 6 units to the right.

   (b) 3 units downwards.

4. Shape D:

   7units to the left and 9 units upwards.

Translation and column vectors

In Form 2, you learnt that vectors represent movements. In pairs discuss the meanings, in terms of movement, of the following: $\begin{pmatrix}2\\4\end{pmatrix}$, $\begin{pmatrix}-3\\-4\end{pmatrix}$, $\begin{pmatrix}1\\-3\end{pmatrix}$, $\begin{pmatrix}-2\\2\end{pmatrix}$. These are known as translation vectors. The translation vector shows how much the image has been moved in relation to the object. It is written in the form $\begin{pmatrix}x\\y\end{pmatrix}$, where *x* and y are real numbers. *x* represents the units in x axis, and *y* in y axis. When x is negative, movement is to the left while if it is positive, movement is to the right. Similarly, when y is negative, movement is downwards while if it is positive, movement is upwards.

Activity 5:

Writing down the coordinates of a translation in column vectors

In pairs, discuss how you can write the following coordinates of translation in column vectors basing on the paragraph above:

a. a movement of 2 units to the left followed by a movement of 1 unit upwards.

b. a movement of 3 units downwards.

c. a movement of 4 units to the right followed by a movement of 5 units downwards.

Present your work to class.

Example 3:

Writing coordinates

Write the following coordinates of translation in column vectors:

(a) A movement of 1 unit to the left followed by a movement of 2 units downwards.

(b) A movement of 5units upwards.

Solutions

(a) $\begin{pmatrix}-1\\-2\end{pmatrix}$

(b) $\begin{pmatrix}5\\0\end{pmatrix}$

Exercise 8e

Write the following coordinates of translation in column vectors:

A movement of

1. 2 units to the left followed 5 units down
2. 5 units to the right followed by 2 units up.
3. 5 units to the left.
4. 6 units to the right followed by 1 unit down.
5. 1 unit up.
6. 8 units down.
7. 4 units to the right.
8. 2 units to the right followed by 8 units down.
9. 5 units to the right followed by 4 units down.
10. 7 units to the left followed by 7 units up.

Activity 6:

Translating shapes using column vectors

In pairs,

1. On a squared paper or on a grid, draw x- and y- axes and number them from -5 to $+5$.
2. Draw triangle ABC such that A, B , C are points (-5,5), (-2, -4) and (3,3) respectively.
3. Draw the image of triangle of triangle ABC after a translation $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$.
4. Write down the coordinates of the vertices of image of the triangle.
5. What is the relationship between the coordinates of the object, the coordinates of the image and the translation vector?

6. Compare your work with other groups.

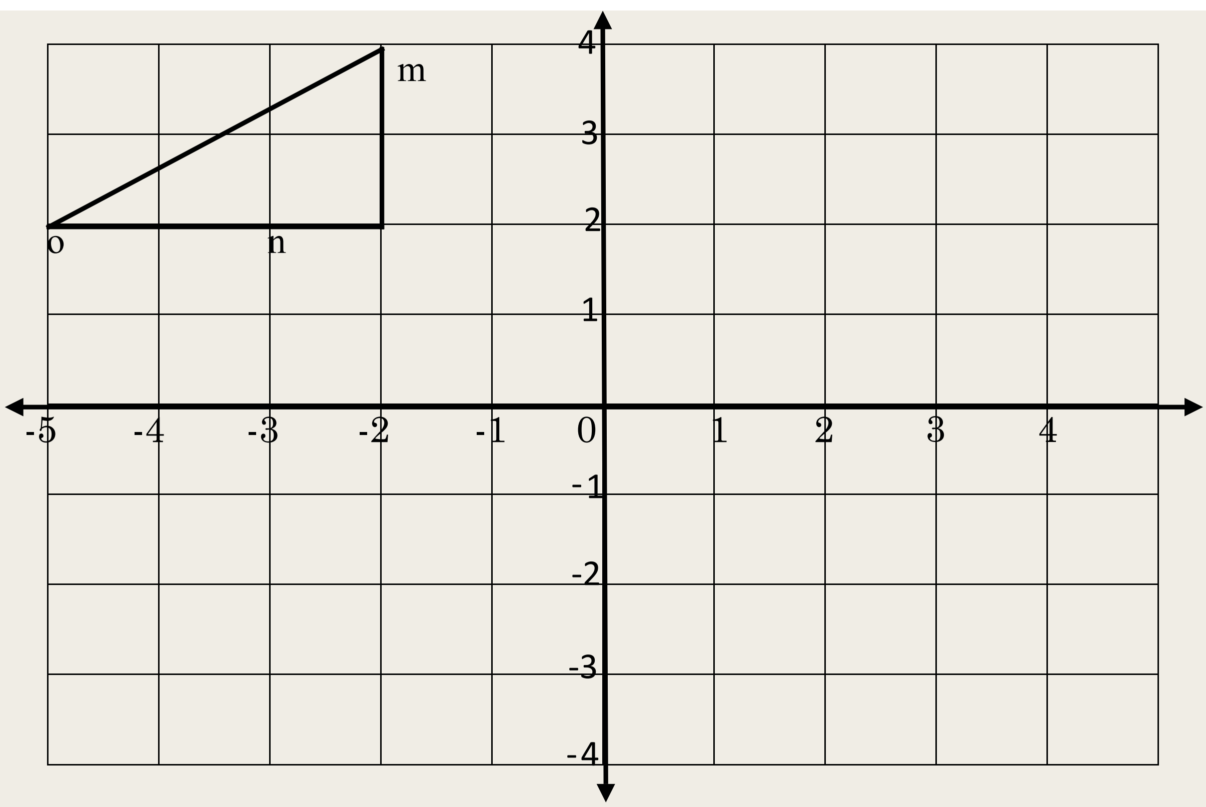
To translate a plane shape using a column vector, you need to first describe the vector itself. To describe the vector means to say what the vector means in terms of movement both horizontally and vertically. Once the vector has been described, the object can be moved by moving each vertex according to the translation vector. When all the vertices have been moved, the image can then be completed by joining the moved vertices.

You should also have seen from activity 6 that if the coordinates of the vertices are written as vectors, the coordinates of the object, the coordinates of the image and the translation vector are related as follows:

Vectors from the coordinates of object vertices + translation vector = Vectors of the coordinates of the image vertices.

Example 4:

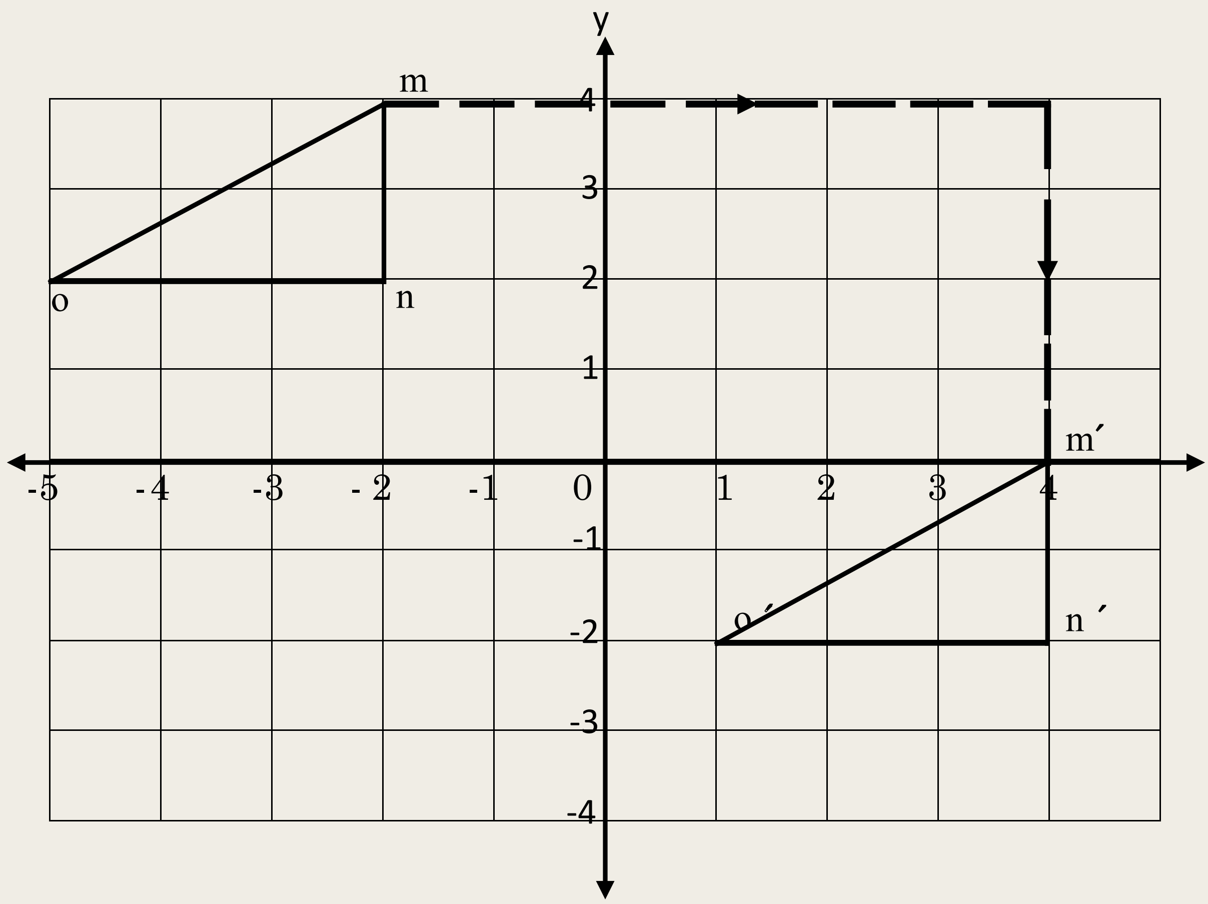
Drawing images after translation



Draw the image of triangle PQR after a translation of $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$

Solution

The vector $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$ means the object is moved 6 units to the right and then 4 units downwards. The image is shown below. Dotted lines have been included to show the movement of the object.



Example 5:

Finding coordinates

A point R (3,-7) is translated to R´ by a translation vector T $\begin{pmatrix} -5 \\ -4 \end{pmatrix}$ Find the coordinates of R´.

Solution

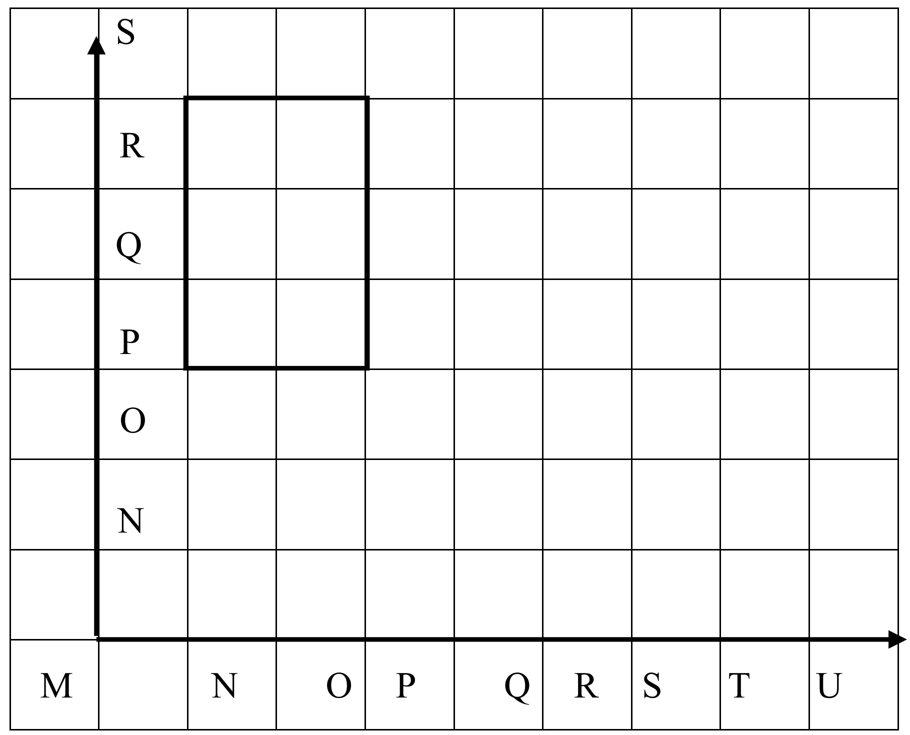
Vectors from the object vertices + translation vector → Vectors of the image vertices.

$$\begin{pmatrix} 3 \\ -7 \end{pmatrix} + \begin{pmatrix} -5 \\ -4 \end{pmatrix} \rightarrow \begin{pmatrix} -2 \\ -11 \end{pmatrix}$$

The coordinates of R´ are (-2,-11)

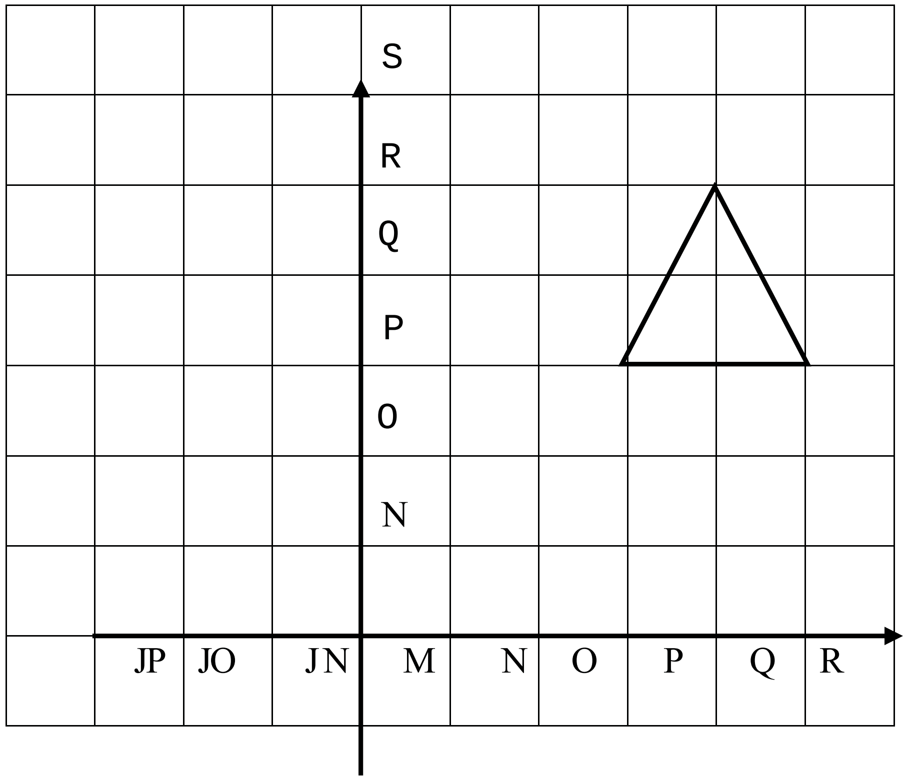
Exercise 8f

1. In questions 1 to 3, copy the diagram and draw the image of the object under the translation given by the column vector.



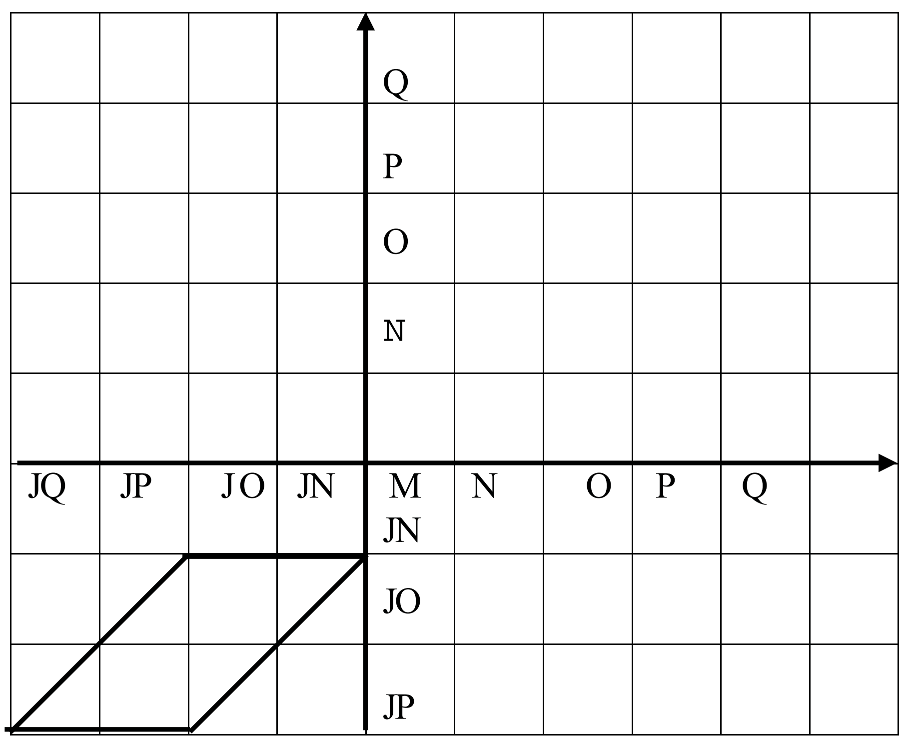
The vector is $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

2.



The vector is $\begin{pmatrix} -6 \\ 2 \end{pmatrix}$

3.



The vector is $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

4. A vector T$\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ translates a quadrilateral ABCD onto quadrilateral A´B´C´D´.

(a) If A and B are points (1,3) and (3,4) , find the coordinates of

A´ and B´.

(b) Given that the coordinates of C´ and D´ are (9,-3) and (7, -4) find the coordinates of C and D.

5. A translation vector T translates point A onto point A´. The coordinates of A and A´ are (4, -2) and (7, 5). Find T.

6. A =$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and B = $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ are translation vectors. Find the coordinates of a point P (3,2) after the Following translations:

(a) A + B

(b) 2A – B

Enlargement

All the transformations you have studied so far (reflections, rotations and translations) have moved the object or turned it over to produce the image, but its shape and size have not changed. In each case, the image and the object are congruent. In this section you shall learn about a transformation that keeps the shape of the object but alters its size – enlargement. Enlargement covers both making the image larger than the object and making the object smaller than the object.

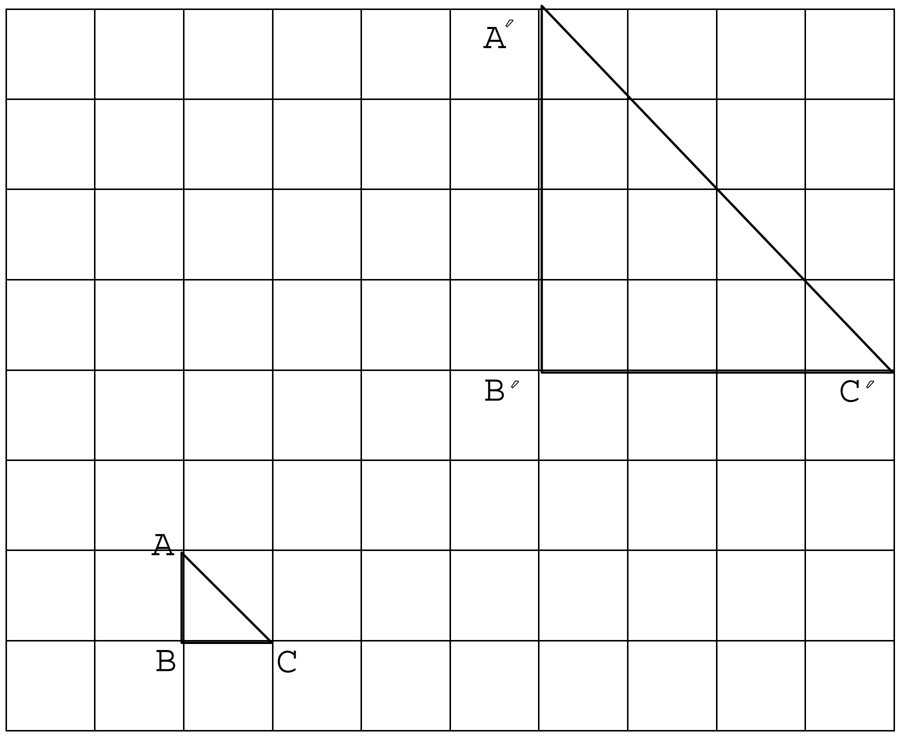
Finding the scale factor of an enlargement

In **unit 12** of this book you dealt with scale factor. In that unit, scale factor was defined as “ a number showing how many times an object has been enlarged”. You also learnt that scale factor is found by the formula

$$\text{Scale factor} = \frac{\textit{Length of the image}}{\textit{Corresponding length of the image}}$$

Example 6:

Finding a scale factor



Find the scale factor used in enlarging triangle ABC onto triangle A′B′C′ in the above grid.

Solution

By counting the grids, AB = BC = 1 unit long and A′B′ = B′C′ = 4 units long. The scale factor can be found by dividing A′B′ by AB or B′C′ by BC. You can also count the squares diagonally to find AC = 1 diagonal and A′C′ = 4 diagonals and then divide A′C′ by AC.

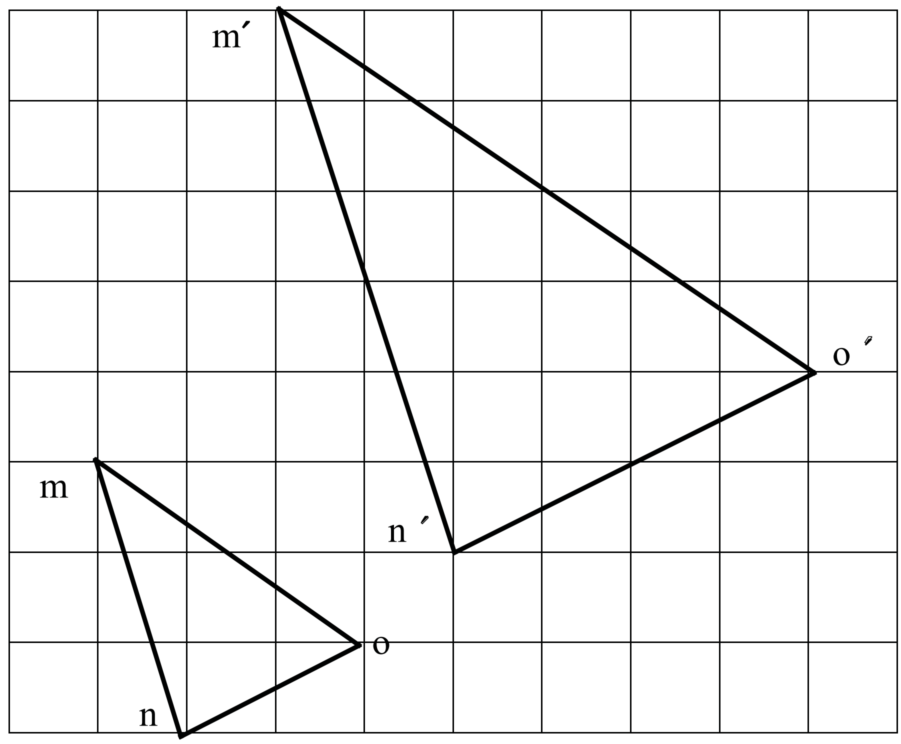
Hence Scale factor = $\frac{4}{1}$

$= 4$

Sometimes you may have to form right angled triangles and use them to find the scale factor as in example 7 below:

Example 7:

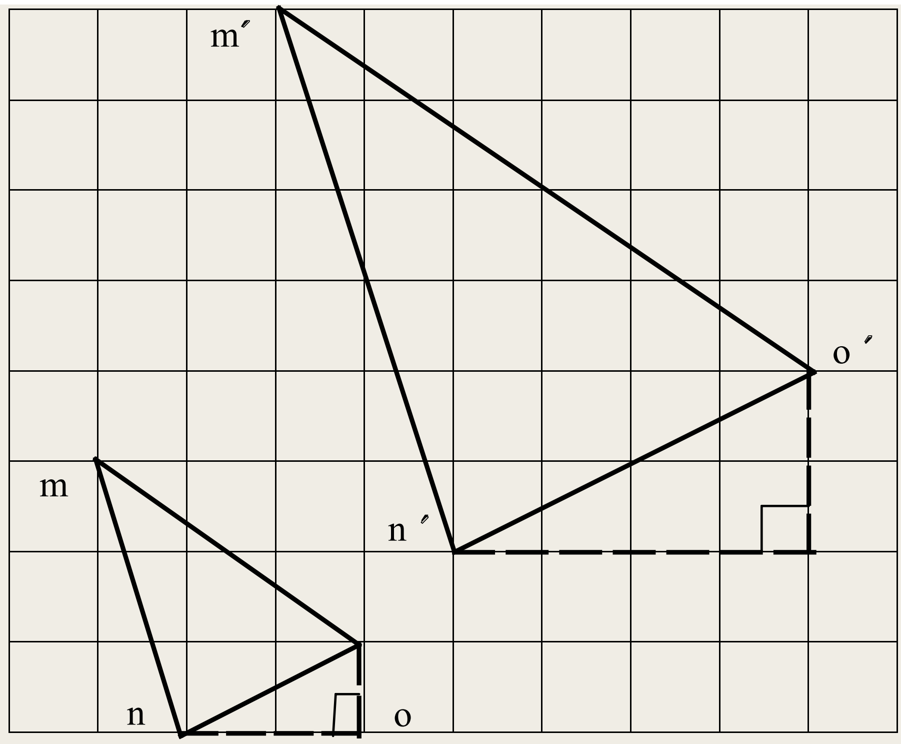
Finding scale factor



Find the scale factor used to enlarge triangle MNO to produce triangle M´N´O´.

Solution

Choose any two corresponding vertices and draw lines horizontally and vertically to produce right angled triangles as follows (dashed lines have been used so that you can clearly see how the triangles have been produced):



You can then divide the corresponding bases or heights of the formed triangles to find the scale factor.

Hence using bases, scale factor = $\frac{4}{2} = 2$

Example 8:

Finding a scale factor

The coordinates of A and B are (2, 4) and (-3, 6) respectively. The coordinates of A′ and B′ are (2.5, 5) and (-5, 8). Find the scale factor used to enlarge AB to A′ B′.

Solution

You can use distance formula as follows:

Scale factor $= \frac{\sqrt{(-5-2.5)^2+(8-5)^2}}{\sqrt{(-3-2)^2+(6-4)^2}}$

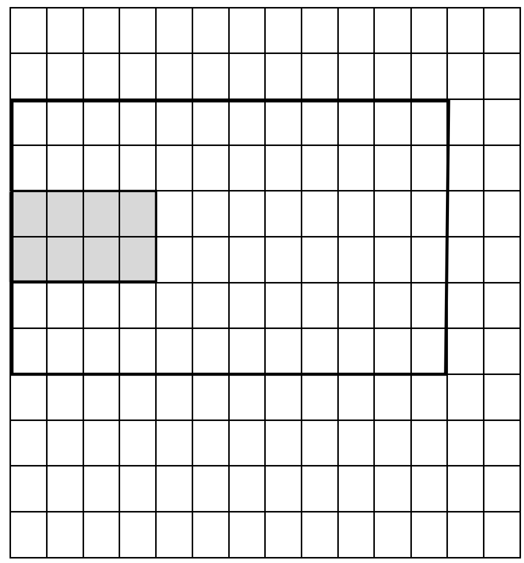
$= \frac{\sqrt{65.25}}{\sqrt{29}}$

= 1.5 (from the calculator)

Exercise 8g

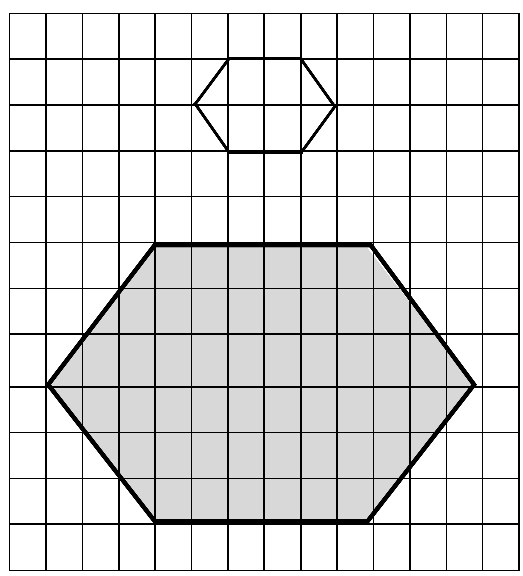
Find the scale factor of enlargement in each of the diagrams below(Questions 1 to 3).The shaded shape is the object.

1.

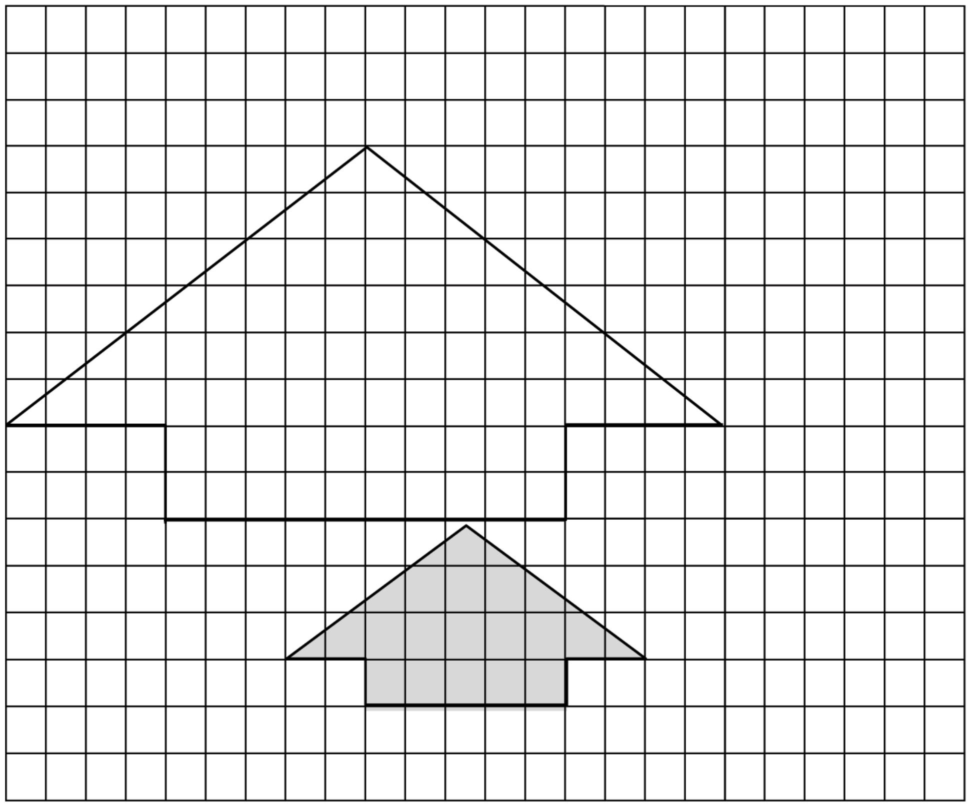


2.

$\frac{4}{2}$



3.



4. The coordinates of the vertices of a triangle XYZ are (2, 3), (4, 7) and (-1, -4).

   The coordinates of the vertices of the image X´Y´Z´ of triangle XYZ are (0.5, 4.5), (5.5, 14.5) and (-7,-13). Find the enlargement scale factor.

5. A scale factor of $\frac{1}{2}$ is used to enlarge a line segment with end points P(-1, -5) and Q(4,4). If the coordinates of Q´ are (2, 2) and of P´ are (x, -2.5), find the negative value of x.

6. A rectangle 20 cm long and 15cm wide is enlarged by a scale factor of 2. Find the length of the new rectangle.

7. A´B´C´ is the image of triangle ABC after an enlargement. AB = 7cm, AC = 8cm, A´B´ = 14cm and B´C´ = 18cm.

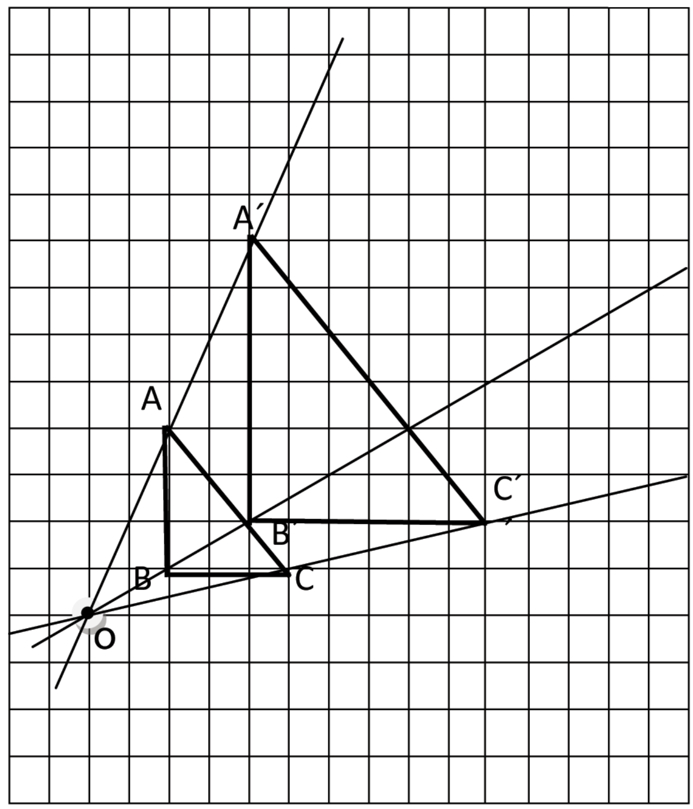
(a) Find the scale factor of the enlargement.

(b) Find BC and A´C´

8. Two places are 4.25km apart and are presented on the map by a distance of 18.5cm. Find the scale factor used.

Finding centre of enlargement

The centre of enlargement is found by drawing straight lines through matching points on the object and image. These lines are then extended until they meet. The point at which they meet is the centre of enlargement. Always draw three lines through matching points. Two points give the point you want but the third one acts as a check. See the diagram below.

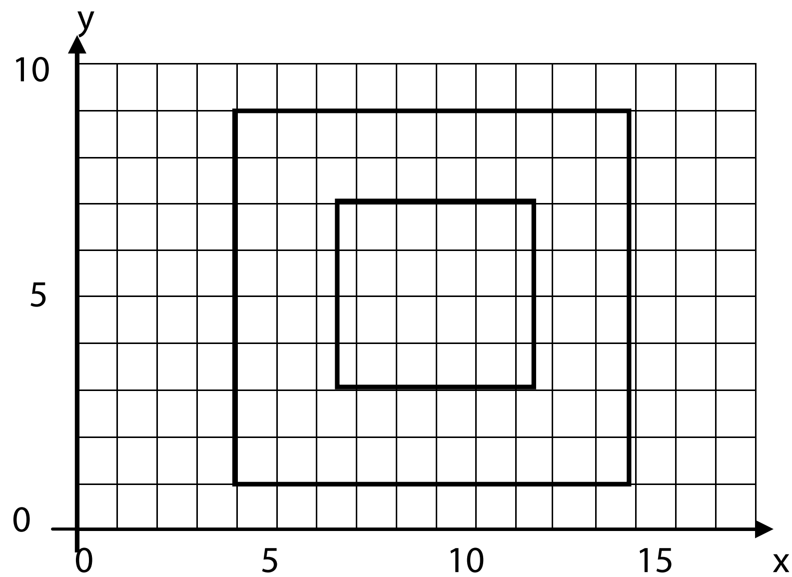


O is the centre of enlargement

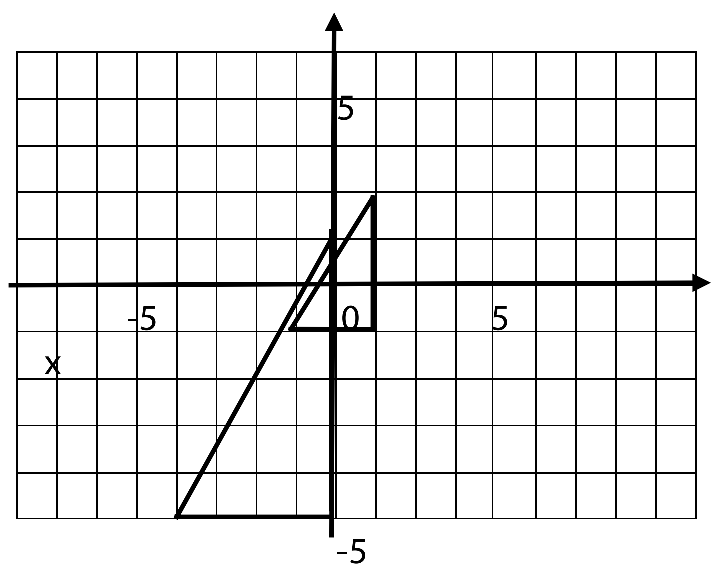
Exercise 8h

Find the centre of enlargement in each of the following diagrams (questions 1- 4).

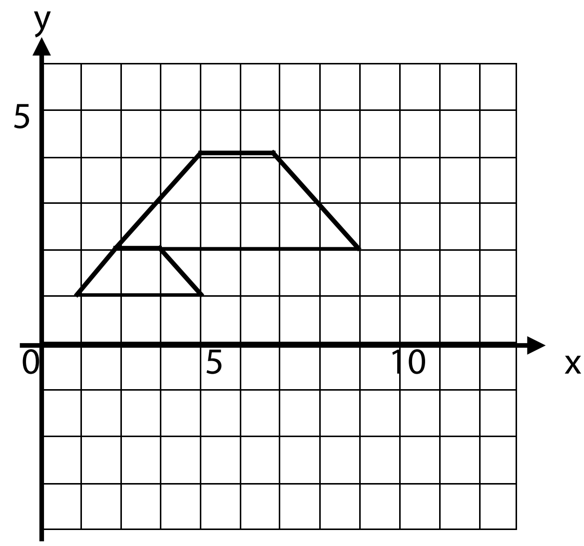
1.



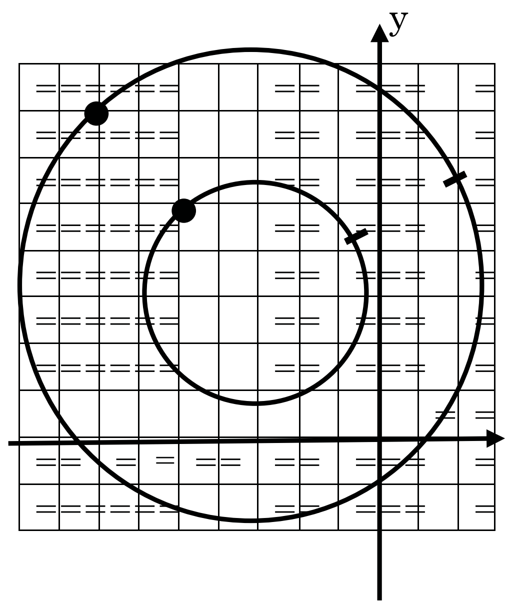
2.



3.



4.



(Each box is 1 unit long)

Drawing enlargement

In this section, you will learn to draw enlargement when given a positive whole or negative scale factor. You will do this in two ways:

(a) Drawing enlargement on a squared paper or grid when the object and scale factor are given.

(b) Drawing enlargement on a squared paper or grid when the object scale factor and centre of enlargement are given.

Activity 7:

Drawing enlargement on a squared paper or grid when the scale factor is given

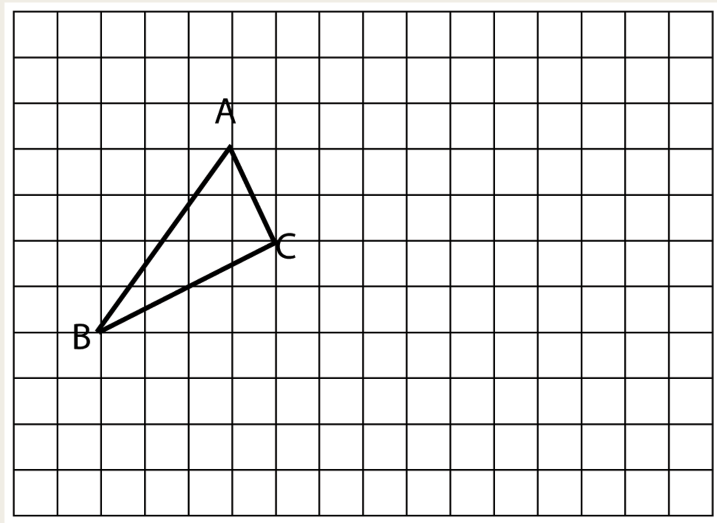
In your groups and using grids of 1 unit long,

1. Draw a rectangle ABCD in which the length = 4 units and the width = 2units.(Draw the rectangle parallel to the axes)
2. If you enlarge the rectangle by a scale factor of 2, how many units would be the lengths and the width?
3. If you enlarge the rectangle by a scale factor of $\frac{1}{2}$, how many units would be the lengths and the width?
4. Draw the images in 2 and 3 above on the same grid as the rectangle ABCD above.
5. How do the images compare with the rectangle?
6. If the rectangle were drawn at an angle to the axes, discuss how you would enlarge the rectangle.

When the lines forming the object run along the lines of the grids, the lengths of image are found by just multiplying the lengths of the object by the scale factor. If the lines are at an angle to the lines of the grids, you need to find the number of grids “across” and “up or down” each line forming the object. Multiply the number of the grids by the scale factor to find the lengths of the lines in the image.

In activity 7 you should also have seen that a positive whole number scale factor makes the object larger than the original object while a fractional scale factor reduces the size of the object.

Example 9



Enlarge triangle ABC above by the following scale factors

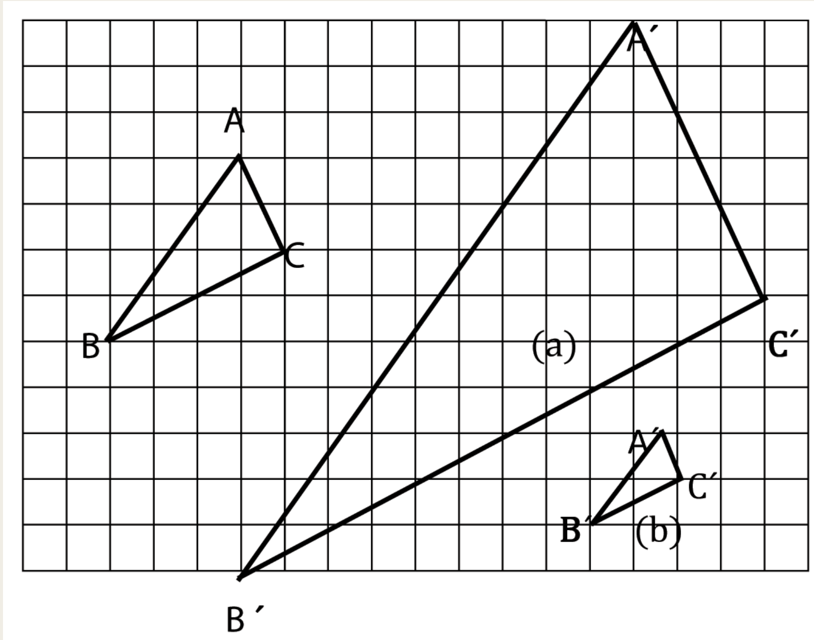
(a) 3

(b) $\frac{1}{2}$

Solution

(a) For line AB there are 3 grids across and 4 grids up. A′B′ will therefore be (3 × 3) = 9 grids across and (4 × 3) = 12 grids up. Similarly, line B′C′ will be (4 × 3) = 12 grids across and (2 × 3) = 6 grids up and line A′C′ will be (1 × 3) = 3 grids across and (2 × 3) = 6 grids down.

(b) Multiplying the number of grids across and up each line by $\frac{1}{2}$ as in (a) above, A′B′ will have 1.5 grids across and 2 grids up, B′C′ will have 2 grids across and I grid up while A′C′ will have 0.5 grid across and 1 grid down.



Activity 8:

Drawing enlargement on a squared paper or grid when the scale factor and centre of enlargement are given

In groups you are to enlarge a triangle ABC by a scale factor of 3.

1. On a square paper or grid draw triangle ABC with vertices in the corners of the grids. The triangle shouldn't be too big.
2. Choose one point O and from that point, draw and extend a straight line through each vertex of the triangle.
3. Using a pair of compasses or a ruler, measure the distance from O to vertex A.
4. Multiply the distance in 3 by the given scale factor and from O, mark off the new distance along the line OA and call the marked point A´.
5. Repeat 2,3,4, for all the vertices and join A´,B´ and C´.
6. Measure the lengths of new triangle and compare them to the matching lengths on the object. How do they compare?
7. Report your findings.

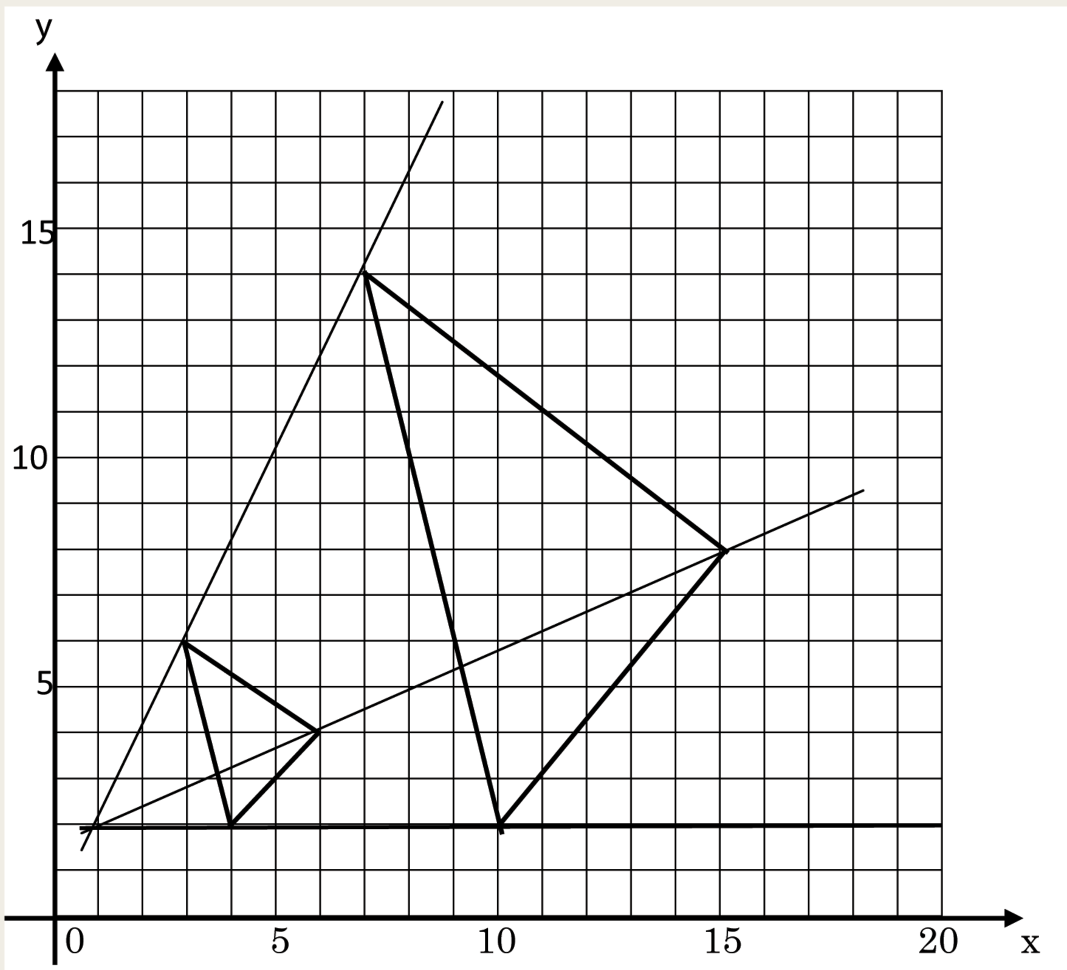
To draw an enlargement, you follow the following steps:

1. From the centre of enlargement, draw and extend straight line through each vertex of the object.
2. Using a pair of compass or a ruler, measure the distance from the centre of enlargement to each vertex of the object.
3. Multiply the distance in 2 by the given scale factor and from the centre of enlargement, mark off the new distance along each line.
4. Join the last marks to obtain the image.

Example 10

The vertices of a triangle are A(4,2) , (6,4) and (3,6). Draw the triangle on a grid of at least 17 by 17 boxes. On the same grid draw the image of triangle using scale factor 3 and centre of enlargement O (1,2). State the coordinates of the image.

Solution:

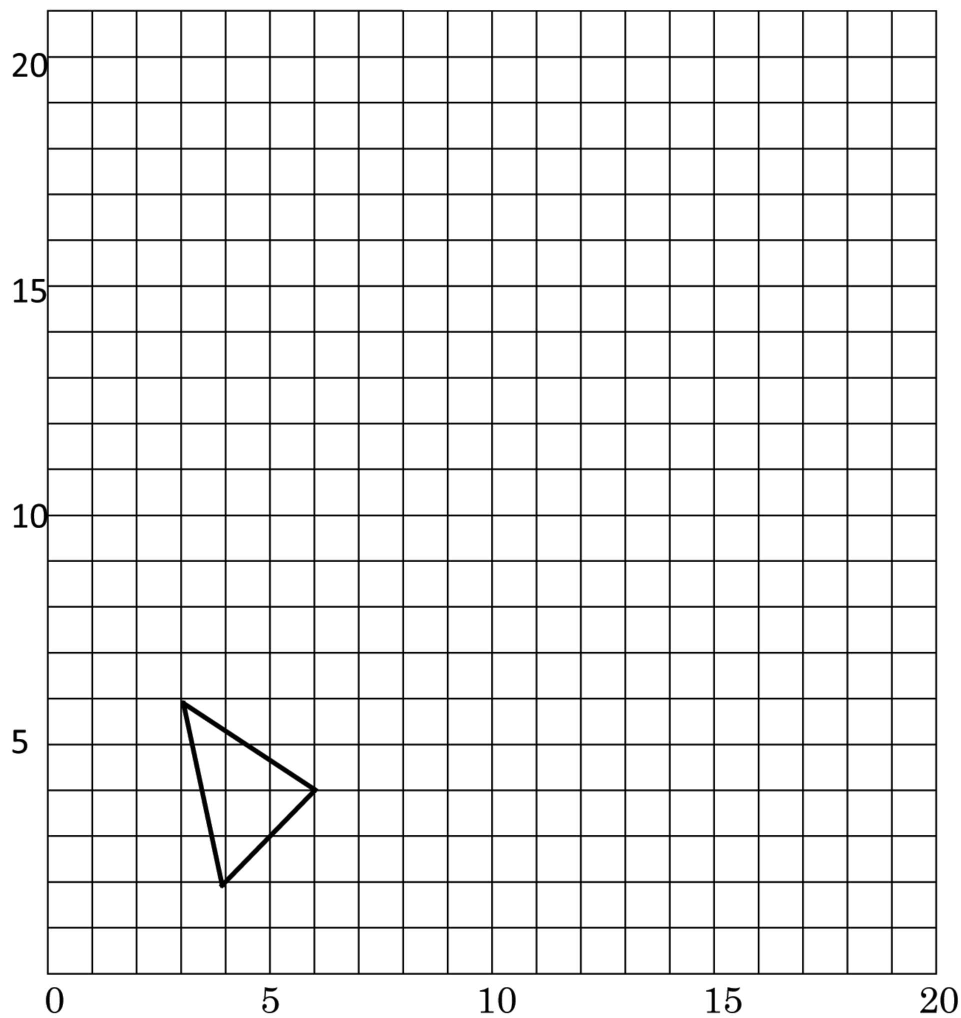


The coordinates of the image are (10,2) , (16,8) and (7,14)

Exercise 8i

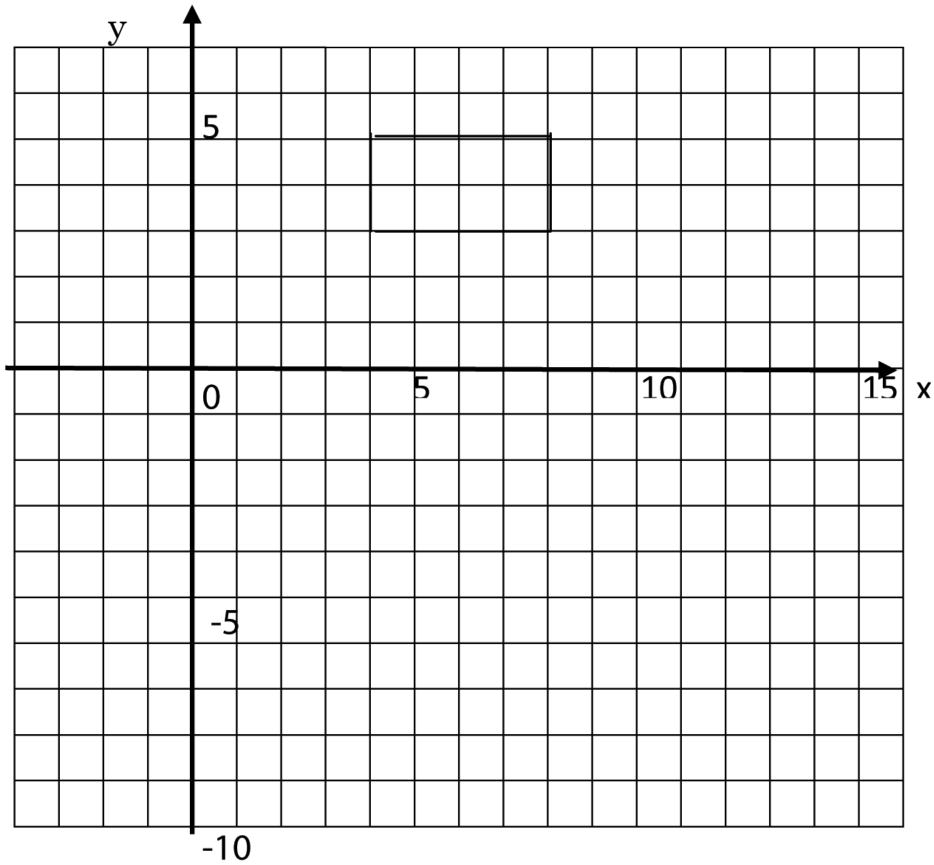
Copy the following diagrams and enlarge the objects by the scale factor given and from the centre of enlargement shown. Grids larger than those may be needed.

1.



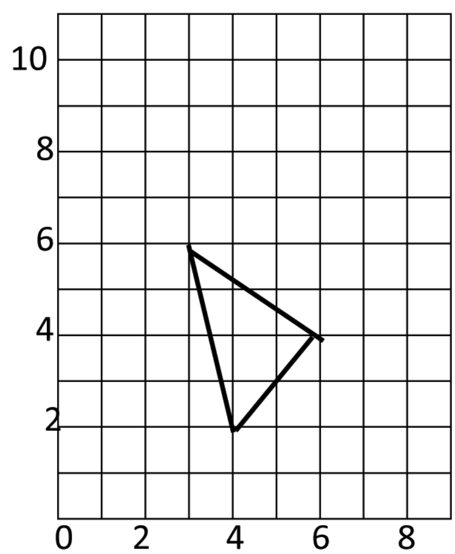
Enlarge triangle ABC above using scale factor 3 and (0,0) as centre of enlargement. What are the coordinates of the image?

2.



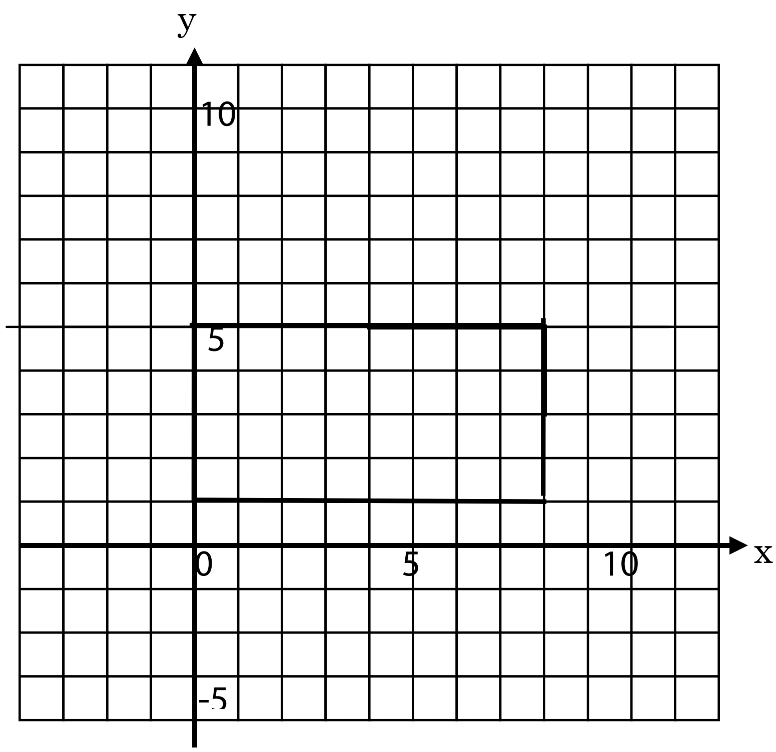
Draw the enlargement of the quadrilateral above using centre (6,7) and scale factor of 4. State the coordinates of the image.

3.



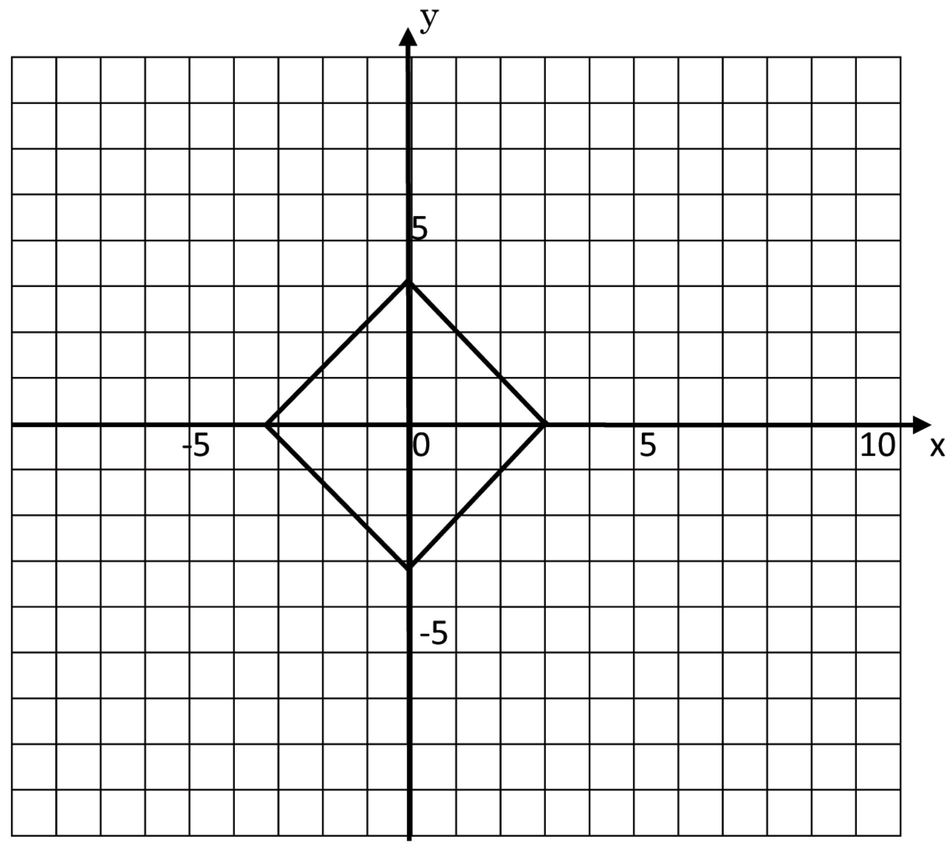
Draw the enlargement of triangle PQR using (4,2) as centre of enlargement and scale factor 2. Sate the coordinates of the image.

4.



Enlarge the quadrilateral above by the scale factor $\frac{1}{2}$ using (8,5) as centre of enlargement. State the coordinates of the image.

5.



Draw the enlargement of the quadrilateral above using (0,0) as centre of enlargement and scale factor 2. What are the coordinates of the image?

Unit summary

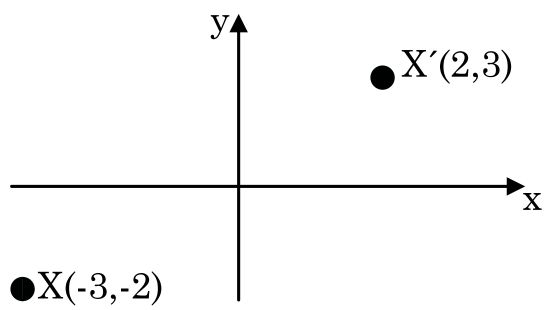
- In this topic you have learnt drawing rotations on a squared paper, describing rotations using directions and angles, describing and drawing translations, writing down coordinates of a translation in column vectors, translating shapes using column vectors, defining enlargement, finding the scale factor and centre of enlargement and enlarging shapes by a positive whole number scale factor and fractional scale factor.

Unit review exercise

1. Draw, on a squared paper, a rectangle ABCD with vertices at (-3,2),(-1,2), (-1,1) and (-3,1). Draw a rotation of the triangle 90^0 clockwise about O (0,0). What are the coordinates of the image of the image of the triangle?

2. T´(0,2) is the image of T(-1,-3) after a rotation about a fixed point C. Describe the rotation.

3.

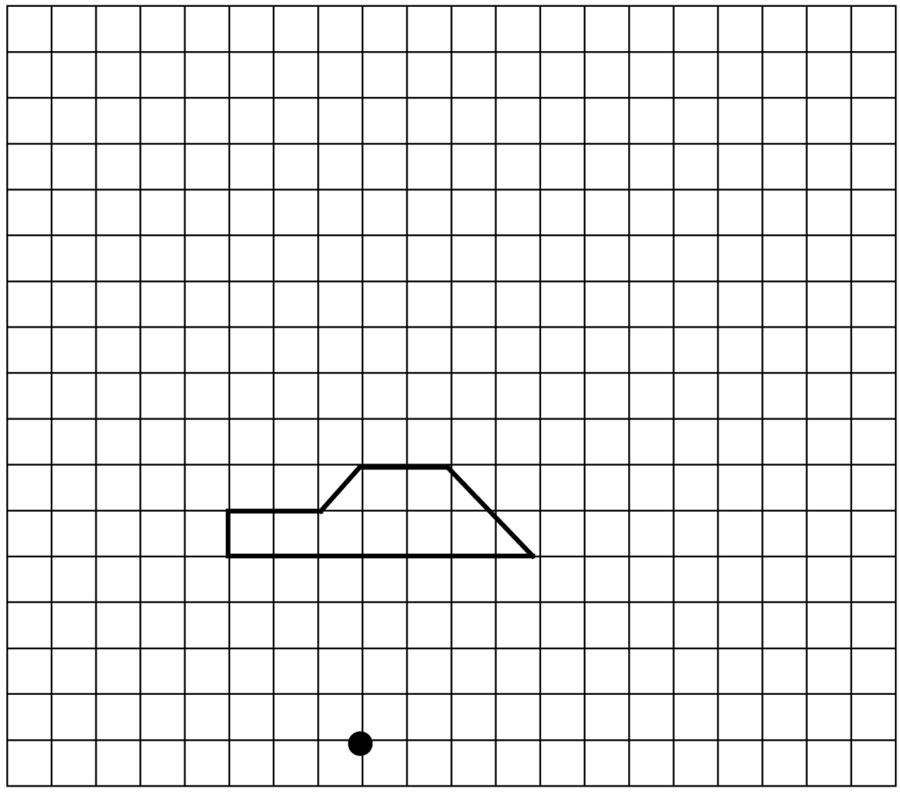


Describe the translation mapping point X onto X´ in the above diagram.

4. The coordinates of the vertices of a triangle ABC are (3,1), (1,2) and (1,5). Draw the image of the triangle after a translation of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. What are the coordinates of the vertices of the image?

5. A trapezium is moved 5 units to the left and 6 units down from a point T. Write this translation as a column vector.

6. On a squared paper, draw triangle ABC with vertices (3,5),(3,3) and (6,3) and the image of triangle ABC with vertices (1,7), (1,1) and (10,1). Use the two drawings to find the centre and scale factor of enlargement.



Draw the enlargement of the body of a toy car shown above using the black dot as the centre of enlargement and scale factor 2.

Glossary

Rotation: Transformation in which a figure changes the way it is put.

Centre of rotation: A point about which a figure rotates .

Translation: Transformation in which a figure changes its position.

Enlargement.: A transformation in which the figure changes in size.

Centre of enlargement: A point about from which a figure is enlarged.

Scale factor: The ratio of corresponding lengths of the image and the object.

References:

Ralge Chikwakwa et al, *Senior Secondary Mathematics Book3 (2002*), Mamillan, Malawi.

Duncan and Christine Graham, *Mainstream Mathematics for GSCE(1996),* Macmillan, London.

Suzanne et al, *Middle Grades Math Course 3(1999)*, Prentice Hall, United States of America.

Unit 9

CHANGE OF SUBJECT OF A FORMULA

In your JCE Mathematics , you learnt about linear equations. You learnt about some basic problems involving change of subject of formula. In this unit, you are going to learn more about changing the formula of literal equations and those involving powers. Most often, formulae can be used to solve any other unknown in the formula provided enough information is given.

The knowledge of change of formula will help you to solve a particular problem. For example you can find velocity of a car or find distance travelled given some variables.

Changing the subject of literal equations

In most cases banks and other money lending institution charge interest on money lent to people and as they such use formulae. Write down the formula for calculating interest. The formula for finding interest is; $I=\frac{PTR}{100}$ Can you come up with other formulae? Now do the activity below.

Activity 1:

Identifying the subject of the formula.

In pairs discuss the following:

1. Define a formula.
2 Write the formula for finding the following;

a. Perimeter of a rectangle

b. Volume of cylinder.

c. In these two formulae above, which one is the subject.

d. How can you find the length of the rectangle

Present your answers to the class.

A formula is an equation that shows a relationship between two or more variables. Changing or transposing the subject of a formula is the same as solving for an unknown or expressing the given letter in terms of other letters. This involves simple formula involving literal equations and also others that contain powers or roots.

Example 1:

Change of subject of formula

Make x the subject of the formula

(a) $3y = 4\boldsymbol{x} - 5$

Solution

Add 5 to both sides

$3y + 5 = 4\boldsymbol{x} - 5 + 5$ … like terms together

$3y + 5 = 4\boldsymbol{x}$

Divide both sides by 4

$$\left(a^3 \times^3\right)$$

$\therefore\ \boldsymbol{x} = \dfrac{3y+5}{4}$

(b) Make y subject of the formula $x = 2(\mathbf{y} + z)$

Solution;

Divide both sides by 2

$\dfrac{d}{t^2} = \mathbf{y} + z$

Subtract z from both sides

$\dfrac{d}{t^2} - z = y + z - z$

$\therefore\ \dfrac{d}{t^2} - z = y$

$\therefore\ y = \dfrac{d}{t^2} - z$

Example 2:

Change of subject of formula

Make ***R*** the subject of the formula

$$I = \frac{PT\mathbf{R}}{100}$$

Multiply both sides by 100

$$I \times 100 = PT\mathbf{R}$$

Divide both sides by PT

$$\frac{I \times 100}{PT} = \frac{PT\mathbf{R}}{PT}$$

$$\therefore \mathbf{R} = \frac{100\,I}{PT}$$

Now do the exercise.

Exercise 9a

Make the letter written in bold a subject of the formula.

(a) $A = \frac{4}{3}bh$ **b**

(b) $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ **u**

(c) $Y = kx^a$ **k**

(d) $x = b + d$ **d**

(e) $a = bd + f$ **b**

(f) $A = \frac{4}{3}(a + b)h$ **h**

(g) $V = \pi r^2 h$ **h**

(h) $P - Q = \frac{Q}{r}$ **Q**

(i) $S = (2n - 4) \times 90°$ **n**

All the examples above involve literal equations. These are easy to express in terms of any given letter. However others involve powers and roots.

Formula involving powers

Some formula will involve powers. From exercise 9a you can identify formulae that have a power in it.

Activity 2.

Identifying the subject of the formula.

Discuss with a friend the following;

(a) What is the formula for finding area of a circle? Identify the subject of the formula.

(b) Write the formula for finding the radius of the circle.

Now look at the following examples.

Example 3:

Change of subject of formula

Make **A** the subject of the formula $Y = \mathbf{A}^2 + BD$

$Y = \mathbf{A}^2 + BD$

Subtract BD from both sides

$Y - BD = \mathbf{A}^2$

$\therefore \mathbf{A}^2 = Y - BD$

Find the square root of both sides

$\sqrt{\mathbf{A}^2} = \sqrt{Y - BD}$

$\therefore \mathbf{A} = \pm\sqrt{Y - BD}$

Note: At $Y - BD = \mathbf{A}^2$, we cannot divide both sides by **A** since both sides will have an **A**. When we make a certain letter the subject of a formula, then that letter should only be found on one side of the formula, and not both sides.

Working with exponential and logarithmic equation

You will look at more example of that involve powers.

Example 4:

Given that $A = h\sqrt{r^3 + 1}$

(a) Make ***r*** the subject of the formula

(b) Find the value of ***r*** when $A = 10$ and $h = 4$

Solution

(a) $A = h\sqrt{\mathbf{r}^3 + 1}$

Divide both sides by *h*

$$\left(\frac{A}{h}\right)^2 = \mathbf{r}^3 + 1$$

Square both sides

$$\left(\frac{A}{h}\right)^2 = \mathbf{r}^3 + 1$$

Subtract 1 from both sides

$$\left(\frac{A}{h}\right)^2 - 1 = \mathbf{r}^3$$

Find the cube root of both sides

$$\mathbf{r} = \sqrt[3]{\frac{A^2}{h^2} - 1}$$

For part b substitute the given values in the formula and find the value of **r**.

Subject of formula that involve exponential and logarithmic functions

Some equation cannot be easily solved and such may require the use of logarithm. See the example below.

Example 5

a. Make n the subject of the formula $y = kx^n$

Working out:

Introduce log on both sides

logy = logk + logx^n

Take logk to other side

logy – logk = logx^n

changing n from an exponent

nlog$x$ = logy – logk

divide throughout by log$x$

therefore n = $\frac{logy - log\ k}{log\ x}$

Exercise 9b

In questions 1 − 18, a formula is given. A letter is printed in bold after it. Make that letter the subject of the formula.

1. $P = b + mN^2$ **N**
2. $b = 2a^c$ **c**
3. $V^2 = u^2 - 2as$ **s**
4. $m = \pi\sqrt{\frac{x}{y}}$ **y**
5. $A = \pi r\sqrt{h^2 - r^3}$ **h**
6. $x = \frac{M - m}{Mn + mp}$ **m**
7. $v = \sqrt{\frac{p + qr}{d}}$ **d**
8. $S = \frac{n}{2}[a + (n - 1)d]$ **a**

9. $x = \sqrt{\frac{s-a}{s-b}}$ **s**

10. $A = \frac{b^2 + c^2 - a^2}{2b}$ **a**

11. $P = \sqrt[5]{\frac{m}{m-n}}$ **m**

12. $V = \pi r\left(h + \frac{2}{3}r\right)$ **h**

13. $q^2 = p^2\left(a - q^2\right)$ **q**

14. $K = \frac{b^2 + t}{b - r}$ **r**

15. $A = 4\pi r^2$ **r**

16. $V = \frac{4}{3}\pi r^3$ **r**

17. $Y = kx^a$ **a**

18. $T = 2\pi\sqrt{\frac{l}{g}}$ **l**

Real life application of formula

Some of the problems solved in this unit involve real life situation. Can you identify some of them from those given in exercise above? Examples include;

Velocity, $V = u + at$

Volume of cylinder, $V = \pi r^2 h$

Area of a triangle; $A = \frac{1}{2}bh$

There many more formulae that are used in everyday life.

Unit summary

You have learnt how to identify the subject of formula. You also changed subject of formula of literal equations and those involving powers.

In the next unit, you will learn about exponential and logarithm equations.

Unit review exercise

1. Make $\boldsymbol{y}$ the subject of the formula in $x = 2\,(\mathbf{y} + z)$
2. Make p the subject of formula in the equation, $\log y = \log x^p + \log k$
3. Make x the subject of the formula $a^x = b$
4. Given that the area A of a triangle with base b and of height h is equal to $A = \frac{1}{2}bh$.
   Make h the subject of the formula.
5. Given that $A = P\left(1 + \frac{R}{100}\right)^T$, make T the subject of the formula.
6. Given that $R = a + bv^2$ make v the subject of the formula
7. The formula for finding velocity is $v = u + at$, make time (t) the subject of the formula.
8. In the formula, $W = \sqrt{\frac{Tv^3}{gx}}$ make x the subject of the formula.
9. In the formula $y = x - z^2$, make z the subject of the formula.
10. In the formula $\frac{a}{b} = \sqrt{\frac{c-1}{x+1}}$ make c the subject of the formula.

Glossary

A formula is an equation that shows a relationship between two or more variables

Literal equations are equations with several variables or unknowns that represent a value, for example V = u+ at.

Transpose refers to "change or rearrange" in this case, the subject of the formula.

References

Chikwakwa, et al (2002), *Senior Secondary Mathematics Student's Book 3.* Blantyre: Macmillan Malawi.

Hau S and Saiti F (2010). *Strides in Mathematics 3.* Blantyre: Longman Malawi.

John Fawdry (1994). *Additional Mathematics a Course for students.* Mathematics Association of Malawi

Mathteacher, subject of formula, *www.mathteacher.com.au/.../trans.html* retrieved on 24/06/2014

Unit 10

EXPONENTIAL AND LOGARITHMIC EQUATIONS

In unit 6you learnt about functions. In this unti you are going to solve exponential and logarithmic equations.

Exponential and logarithm equations are used in many fields like electrical engineering, metrology, communication and many fields of production.

Exponential equations

You have learnt about equations before. Can you define the word equation and give some examples of equations.

The expression such as $f(x) = 2^x$ is an equation. This can also be written as

$\mathbf{y = 2^x}$. What do you call such equation? This is known **exponential equation**.

Definition

The word ‘exponential’ is an adjective from the word exponent, which means **power** or **index**. An exponential equation is one which takes the form $\mathbf{y = x^n}$, where **x** is the base and **n** is **power** (**an exponent**).

In general:

Exponential equations are equations in which the exponent is the variable.

Activity1:

Expressing numbers as powers of given base by modelling.

Find the value of each box by completing table (b) below given on the right hand side.

Base	Index					
	0	1	2	3	4	5
2	2^0	2^1	2^2	2^3	2^4	2^5
3	3^0	3^1	3^2	3^3	3^4	3^5
4	4^0	4^1	4^2	4^3	4^4	4^5
5	5^0	5^1	5^2	5^3	5^4	5^5

Base	Index					
	0	1	2	3	4	5
2	1	2	4			32
3					81	
4			16			
5						

To evaluate $\mathbf{5^3}$**,** using calculator; follow these steps.

- Enter 5 on the calculator which is the base.
- Then punch the button with the symbol "$\mathbf{x^y}$"**.**
- Then punch 3 which is the power.
- You should be able to get 125.

Note Calculators differ as such you are supposed to follow its manual. The one above is for CASIO *fx*- 83ES

You will now learn the reverse process.

Expressing numbers as powers of given base

You can now do the reverse process of changing a given number to given base. First do the activity below.

Activity 2:

Expressing numbers as powers of a given base.

Express the given numbers to a given base

a. 32 to base 2

b. 27 to base 3

Share your work with your friends.

In form 1 you learned how to change a number from one base to another. Do you remember how you were doing that?

To change a given number to particular base you repeatedly divide the number by the base. Here is an example.

Example1:

Change of base

Change to 32 to base 2

Divide 32 by 2 continuously and find how many times you do so to get 1.

2	32
2	16
2	8
2	4
2	2
2	2
	1

You have divided 32 by 2 five times to get 1.

Hence 32 =2^5

Now do the following exercise.

Exercise 10a

Express the following to the given base

a. 81 to base 3

b. 243 to base 3

c. 125 to base 5

d. 16 to base 2

e. 625 to base 5

f. 1296 to base 6

g. 343 to base 7

h. 1024 to base 4

i. $\frac{1}{128}$ to base 2

Now you look at exponential equations.

Solving exponential equation

You learned about equations in the past. Can you write down some equations? Recall that an equation is an algebraic expression with the left hand side equal to the right hand side. On the other hand an exponential equation is an equation in which the exponent (index) is the unknown. Hence solving exponential equations implies finding the value of the exponent.

Do the following activity.

Activity 3:

Solving exponential equation

In pairs, find the value of x in the following:

(a) $2^x = 8$

(b) $2^x = 32$

Discuss your findings as a class. Now see how these can be done by going through these examples below.

Example 2:

Solving exponential equations

Solve the following exponential equations.

(a) $3^{2x} = 27$

Solutions:

To solve exponential equations, firstly you need to express the left and right hand sides of the equations to the same base and then equate the indices.

(a) $27 = 3^3$ .... change 27 to base 3.

$\therefore 3^{2x} = 3^3$ .... Equate the indices

$\therefore 2x = 3 \qquad \therefore x = \frac{3}{2}$

(b) $2^x = 0.0625$

$0.0625 = \frac{625}{10000}$ .... Expressing as a proper fraction

$= \frac{1}{2^4} = 2^{-4}$ ..... Simplify the fraction and change to base 2

$\therefore 2^x = 2^{-4} \qquad x = -4$ ... express to 2 and equate the bases

(c) $4^{x+2} = 64^{2x-1}$

Solving;

In this case, both 4 and 64 must be expressed to base 2

$2^{2(x+2)} = 2^{6(2x-1)}$ … expressing 4 and 64 to base 2

Then $2(x+2) = 6(2x - 1)$ ….equate the powers

$2x+4 = 12x - 6$ …..Remove the brackets

$12x – 2x = 6 +4$ …. Arranging like terms together and swapping

$10x =10$ …… subtracting and adding like terms

$\therefore\ x = 1$ ……. divide both sides by 10

Now do the following exercise.

Exerise 10b:

Solving exponential equation.

1. Solve the following exponential equations

(a) $2^x = 64$

(b) $9^x = 81$

(c) $10^x = 1000\,000$

(d) $5^x = 125$

(e) $2^x = 128$

(f) $3^x = \frac{1}{81}$

(g) $5^x = \frac{1}{25}$

(h) $4^x = 2$

(i) $64^x = 4$

(J) $10^x = 0.001$

(k) $8^x = 64$

(l) $(-2) = -8$

2. Solve the following;

(a) $5^{2n} - 6 \times 5^n + 5 = 0$

(b) $32^{t-1} = 192$

(c) $9^{p+1} = 27^p$

(d) $7^{2y-5} = 343$

(e) $2^{2x} - 5(2x) + 4 = 0$

(f) $12 + 2^b = 2^{2b}$

(g) $8^{2y-2)} = 4^{2y+1}$

(h) $27^{r-4} = 81^{2r+2}$

(i) $2^{2x} - 6 \times 2^x + 9 = 0$

In exercise 10b above, the equations are exponential equations. Now you will solve logarithmic equations.

Logarithm equations

Let a and x be the positive numbers, $a \neq 1$. The logarithm of **x** with **base a** is denoted by $\log_a x$ and is defined as follows; **$\log_a x = y$ if and only if $a^y = x$.**

The expression $\log_a x$ is read as "**log base a of x".** Thus logarithm is another word for "power".

We know that $10^2 = 100$.

In this case "the logarithm to base 10, of 100 is 2"

How can you write this as a logarithm equation?

Activity 4:

Expressing exponential equation as logarithm equations

In pairs, express the following as logarithm equation

a. $10^2 = 100$

b. $2^3 = 8$

Share your work with colleague in the classroom.

Look at the following example.

Example 3:

Expressing logarithm equations

Express as logarithm equation;

a. $10^4 = 10000$

Solution:

$\log_{10} 10000 = 4$

This is read as; 'the logarithm of 10000 to base 10 is 4.

b. $64 = 4^3$

Solution:

$\log_4 64 = 3$

In general;

If $a^m = P$ then **$\log_a P = m$.**

Now do the following exercises;

Exercise 10c

1. Rewrite as a logarithm equations.

(a) $10^3 = 1000$ (b) $10^6 = 1\,000\,000$

(c) $2^5 = 32$ (d) $5^2 = 25$

(e) $4^3 = 64$ (f) $3^4 = 81$

(g) $4^4 = 256$ (h) $7^1 = 7$

(i) $8^0 = 1$ (j) $3^{-2} = \frac{4}{3}$

2. Express the following from logarithm to exponential.

(a) $\text{Log}_{10}\ 100\,000 = 5$

(b) $\text{Log}_2\ 16 = 4$

(c) $\text{Log}_4\ 16 = 2$

(d) $\text{Log}_9\ 3 = \frac{4}{3}$

(e) $\text{Log}_5\ 0.2 = -1$

(f) $\text{Log}_{10}\ 10 = 1$

(g) $\text{Log}_3 1 = 0$

(h) $\text{Log}_{10}\ 0.1 = -1$

(i) $\text{Log}_2 6 \approx 2.585$

Activity 5:

Finding the value of a given logarithm

Discuss in pairs;

Find the value of $\log_2 8$.

Share your work with your friends in the class room.

Now look at the examples below if it is in line with what you have just done.

Example 4:

Finding logarithms

Find the value of

(a) $\log_2 64$

(b) $\log_{10} 100000$

(c) $\text{Log}_4 0.25$

(d) $\text{Log}_a (a^3)$

Solution:

Here find the number of times the base can multiply itself to get the number you want find its log.

Since $2^6 = 64 \therefore \log_2 64 = 6$. i. e multiplying 2 six times gives 64.

(b) $\log_{10} 100000$

Solution

Let $\log_{10} 100000 = a$

$10\,0000 = 10^a$ changing to exponential

But $100000 = 10^5$

Then $10^a = 10^5$

$\therefore \log_{10} 100000 = 5$

We know that $0.25 = \frac{4}{3} = 4^{-1} \therefore \log_4 0.25 = -1$

In general, if $\mathbf{a^n = a^m}$ then $\mathbf{n = m}$

Do the following exercise.

Exercise 10d

Find the value of

(a) $\log_3 81$

(b) $\text{Log}_3 243$

(c) $\log_7 343$

(d) $\log_5 25$

(e) $\log_{12} 12$

(f) $\log_2 0.25$

(g) $\log_{10}\sqrt{10}$

(h) $\log_a (a^8)$

(i) $\log_{\frac{1}{2}} 16$

(j) $\log_{\frac{4}{3}}\left(\frac{1}{8}\right)$

(k) $\log_5 1$

Rules of logarithm

When a logarithmic equation involves more than one term containing the unknown, you need to learn how to simplify such logarithmic expressions. To this end, you will look at the laws of logarithms which are similar to the laws of indices.

Multiplication rule

The first law states that 'the logarithm of a product of two numbers is equal to the sum of the logarithms of each of the numbers.'

Prove that $\log_a pq = \log_a p + \log_a q$

Let $m = \log_a p$. then $p = a^m$..... Changing to exponential equation

Let $n = \log_a q$ then $q = a^n$. ......changing to exponential equation

$pq = a^m \times a^n$......... substitute a^m for p and a^n for q

$= a^{m+n}$ .... Law of indices, same base, powers add

Then $\log_a pq = m + n = \log_a p + \log_a q$

$\therefore$ **$\log_a pq = \log_a p + \log_a q$**

You should use the multiplication rule to simplify logarithm. Do the activity 6 below.

Activity 6:

Simplifying logarithm of numbers to a given base

Express the following as single logarithms using multiplication rule

$\text{Log}_{10} 4 + \text{Log}_{10} 25$

Discuss your findings as a class.

Now look at the following examples;

Example 5:

Express the following as single logarithms

(a) $\log_3 6 + \log_3 7$

Solution

$\log_3 (6 \times 7)$ Using the multiplication law

$= \log_3 42.$

(b) $\log_6 6 + \log_6 2 + \log_6 3$

Solution;

$\log_6 (6 \times 2 \times 3) = \log_6 36$

Using multiplication law.

2. Division rule

The second law states that 'the logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor'

Prove that $\mathbf{\log_a \frac{4}{3} = \log_a p - \log_a q.}$

Let m = $\log_a p$. Then p = a^m ..... Changing to exponential equation

Let n = $\log_a q$ then q = a^n ...... changing to exponential equation

$\frac{4}{3} = \frac{a^m}{a^n}$ .... Substitute a^m for p and a^n for q

$= a^{m-n}$ ... division rule of indices

Then $\log_a \frac{4}{3} = m - n = \log_a p - \log_a q$. .... substitution

$\therefore$ **$\log_a \frac{4}{3} = \log_a p - \log_a q$.**

Look at the given examples below.

Example 6:

Expressing as single Logarithms

Express the following as single logarithms

(a) $\log_2 15 - \log_2 5$

Solution:

$\log_2 (15 \div 5)$ Using the division law

$= \log_2 3$

(b) $\log_{10} 175 - \log_{10} 25$

Solution:

$\log_{10} (175 \div 25)$

$= \log_{10} 7$

Powers

Prove that $\log_a p^n = n \log_a P$.

Let $m = \log_a p$. Then $p = a^m$ ..... Changing to exponential equation

$P^n = (a^m)^n$ .... Since p is raised to n

$= a^{mn}$ .... law of indices

Then $\log_a P^n = mn = n \log_a P$. ...... Substitute

$\therefore$ **$lo_a p^n = n \log_a P$.**

Example 7

Express the following as single logarithms

(a) $2 \log_5 3 + 3 \log_5 2$

Solution

$2 \log_{10} 3 = \log_{10} {}^2 = \log_5 9$

$3 \log_5 2 = \log_5 2^3 = \log_5 8$

$\therefore 2 \log_5 3 + 3 \log_5 \quad = \log_5 9 + \log_5 8$

$= \log_5 (9 \times 8)$ Using the multiplication law

$= \log_5 72$

(b) $\text{Log}_{10} 3 - 2 \log_{10}(\frac{1}{4})$

Solution

$\text{Log}_{10} 3 - 2 \log_{10}(\frac{1}{4})^2$ ....log rule

$\frac{4}{3} = \log_{10} 3 - \log_{10}\frac{1}{16}$

$= \log_{10} (3 \div \frac{1}{16})$ ......Using the division law.

$= \log_{10} 48.$

The base and number the same

Prove that $\log_a a = 1$

Let $\log_a a = n$ then $a = a^n$ to exponential equation

$\therefore a^1 = a^n$

$n = 1$ since the bases are the same

$\therefore$ **$\log_a a = 1$**

Example 8:

Example 8

Express as single logarithms

$\log_{10}150 - \log_{10}15$

Solution

$\log_{10}(15 \div 15)$

$\log_{10}10 = 1$

Note. The logarithm of numbers of the same base is always 1.

The log of 1

Prove that $\log_a 1 = 0$

Let $\log_a 1 = m$

$1 = a^m$ But $a^0 = 1.$

$\therefore\ a^m = a^0$

$\therefore\ m = o$

Hence $\mathbf{\log_a 1 = 0.}$

Note The logarithm of 1 is always 0.

Exercise 10e

. Write as a single logarithm

(a) $\log_3 6 + \log_3 4$

(b) $\log_2 48 - \log_2 6$

(c) $3\log_5 2 + \log_5 10$

(d) $2\log_6 8 - 4\log_6 3$

(e) $\text{Log}_{10} 5 + \log_{10} 6 - \log_{10}\left(\frac{1}{4}\right)$

(f) $\log_2 0 - \log_2 \frac{1}{8}$

(g) $\log_2 14 - \log_2 7$

(h) $2\log_a 3 + 3\log_a 2 - \log 4$

(i) $\log_3 8.1 + \log_3 10$

(i) $\dfrac{\log 25}{\log 5}$

Sometimes, you may require to perform the reverse process.

Example 9:

Expressing logarithms

1. Write the following in terms of $\log_a x$, $\log_a y$, and $\log_a z$

(a) $\log_a x^3 y^2 z^4$

(b) $\log_a\left(\frac{x^3\sqrt{z}}{y^2}\right)$

Solution

(a) $\log_a x^3 y^2 z^4 = \log_a x^3 + \log_a y^2 + \log_a z^4$ using the multiplication law.

$= 3\log_a x + 2\log_a y + 4\log_a z$ using the power law.

(b) $\log_a\left(\frac{x^3\sqrt{z}}{y^2}\right) = \log_a x^3 + \log_a \sqrt{8} - \log_a y^2$ using the multiplication and Division laws

$= 3\log_a x + \frac{4}{3}\log_a z - 2\log_a y$ since $\sqrt{z} = z^{1/2}$ using the power law.

2. Given that $\log_{10} 2 = 0.431$ and $\log_{10} 3 = 0.683$, find the value log 18

Solution:

$\text{Log}_{10} 18 = \log_{10} 2^3 + \log_{10} 3$

$= 3\log_{10} 2 + \log_{10} 3$

$= 3(0.431) + 0.683$

$=1.976$

Exercise 10f

1. Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$

(a) $\log_a (xy^3z^5)$

(b) $\log_a \left(\frac{x}{x-4}\right)$

(c) $\text{Log}\left(\frac{x\sqrt{z}}{y^3}\right)$

(d) $\log_a (a^3 \times^3)$

(e) $\log_a \left(\frac{x\sqrt{y}}{z}\right)$

(f) $\log_a \left(\frac{x}{x-4}\right)$

(g) $\log_a \left(\frac{x}{a^5}\right)$

2. Given that $\log_5 2 = 0.431$ and $\log_5 3 = 0.683$, find the value of

(a) $\log_5 6$ (b) $\log_5 1.5$

(c) $\log_5 8$ (d) $\log_5 12$

(e) $\log_5 \frac{1}{18}$

Having studied the laws of logarithms, you can now solve logarithmic equations with the unknown in more than one term.

Activity 7:

Solving log equation

(a) In pairs, solve the following

(i) $x = \log_2 8$

(ii)$\log_2 x = -3$

(iii)$\log_x 25 = 2$

Present your work to others members in the classroom.

In this activity, you change from logarithm equation to exponential equation and then solve.

Example 10:

Solving logarithmic equations

Solve the equations below

(a) $\log_5 x = 1 + \log (x - 4)$

(b) $2\log_3 x = \log_3 (x + 6)$

(c) $\log_2 x = -3$

Solution

(a) Since there is one term on each side, we can take the antilogarithm on both sides

$\therefore$ antilog $(\log_3 x)$ = antilog $(\log_3 (2x - 6)$.
$\therefore\ x = 2x - 6$. Arrange like terms together
$\therefore\ x = 6$

(b) Collect the logarithm terms to one side

i.e. $\log_5 x - \log_5 (x - 4) = 1$

$$\therefore\ \log_5\left(\frac{x}{x-4}\right) = 1$$

$$\therefore \log_5\left(\frac{x}{x-4}\right) = \log_5 5 \qquad \text{(since } \log_5 5 = 1\text{)}$$

then take antilogarithm on both sides

$$\therefore \frac{-b}{2a} = 5$$

$\therefore x = 5 (x - 4)$ … multiply (x- 4) both sides

$\therefore\ x = 5x - 20$

$\therefore\ 4x = 20$ …… like terms together and divide both sides by
$\therefore\ x = 5$

(c) $\log_3 x^2 = \log_3 (x + 6)$

$\therefore x^2 = x + 6$ ....take antilog both sides

$\therefore x^2 - x - 6 = 0$.... arrange like terms together

$\therefore (x - 3)(x + 2) = 0$.... Factorise LHS and solve for x

$\therefore x = 3 \; x = -2$

$\therefore$ x = 3 as it is not possible to take log (−2)

Exercise 10g

Solve the following logarithmic equations.

(a) $x = \log_3 27$ **(b)** $x = \log_5 625$

(c) $\log_x 125 = 3$ **(d)** $\log_x 81 = 2$

(e) $x = \log_2 \left(\frac{1}{8}\right)$ **(f)** $x = \log_{169} 13$

(g) $\log_3 1 = x$ **(h)** $\log_{2x} 36 = 2$

(i) $\log_4 (x - 2) = 3$

(j) $\log_{x-1} 8 = 3.$

(k) $\log_2 256 = x$

Real life problems of exponential and logarithm equations

As earlier pointed out, exponential equations are used for even more contexts, including population and bacterial growth, radioactive decay, compound interest, cooling of objects, and growth of phenomena such as virus infections, Internet usage and many others.

Example 11

Populations

a. A tree frog population doubles every three weeks. Suppose that currently, there are 10 tree frogs in your back yard. How many tree frogs will there be in 6 months, assuming that there are four weeks each month?

b. How long will it take this population to be 10,240?

Solutions

a. First figure out how many times this population will double in 4 months. Each month 4 weeks, then six months 6x4= 24 weeks.

Since the population doubles every three weeks, then

24/3 =8 times in 24 weeks. Look at the table below:

Number of weeks	Doubling period	population
O	0	10
3	1	20 =10 x 2or 10 x 2^1
6	2	40 = 10 x 2 x 2 or 10 x 2^2
9	3	80 = 10 x 2 x 2 x2 or 10 x 2^3

After 24 weeks, the population will be 10 x 2^8 = 2560 tree frogs!

If you look at the table above, you will notice that you could let n be the number of weeks. How would you get the number of doubling periods from n?

Solution:

After n weeks, the population would be P = 10 x $2^{(n/3)}$.

So you need to solve the equation

10,240 = 10 x $2^{(n/3)}$.

1024 = $2^{(n/3)}$… divide both sides by 10,

But 1024 = 2^{10}….expressing to base
$2^{10} = 2^{(n/3)}$. … equate the bases.

10 = n/3,
$\therefore$ n = 30.

That means that after 30 weeks, the population will be 10,240.

Unit summary

- In this unit so far, you have looked at exponential and logarithm equations. You have learned how to express a number as a power of a given base. You also modelled exponential equations and solved them. Furthermore, you looked at rules of logarithm and also solved logarithm equations. In the next unit you will learn about triangles in trigonometry.

Unit review exercise

1. Solve the following equations.

(a) $2 \log_7 x = \log_7 (x + 2)$

(b) $\log_5 x = 1 - \log_5 (x - 4)$

(c) $\log_3 (x + 3) = 2 \log_3 (x + 1)$

(d) $\log_7 2x = \log_7 (x + 2)$

(e) $\log_4 2 - \log_4 x = \log_4 2/3$.

(f) $\log_2 (x + 1) = 1$

(g) $\log_3 6 = \log_3 3 + \log_3 x$

(h) $8^{x-1} = 16$

(i) $3^{2y} - 4(3^y) + 3 = 0$

(j) $2^{2x} - 4(2^x) + 3 = 0$

(k) $25^{2b} \div 5^b = 5^6$

2. Currently, 80,000 bacteria are present in a culture. When an antibiotic is added to the culture, the number of bacteria is reduced by half every 3 hours.

   (a) How many bacteria are left after a day?

   (b) When will fewer than 1000 bacteria be present?

3 write as a single number and simplify if possible;

   (a) $\dfrac{\log 16 + \log 4}{\log 16 - \log 4}$

   (b) $\dfrac{\log 3 + \log 27}{\log 9 - \log 3}$

Glossary

Base is a number that is being raised to a power

Exponent is a number that has been raised as a power known also as power or indice

Logarithm equation is an expression of the form $\log_a x = y$ which is read as "log base **a** of **x** equals **y**"

Exponential equations is an equation in which the exponent is the variable.

References

Hau S and Saiti F (2010). *Strides in Mathematics 3*. Blantyre: Longman Malawi.

Elaine Ryder, Paul McAdams, Pat Huddleston, (2013), *CHANCO Teach yourself series, Mathematics, Second Edition,* Chancellor College Publication, Zomba.

Hardwood Clarke and Norton F G J, (1984) *Seventh Edition,* Heinemann Educational Book Ltd, Oxford, London

Unit 11

TRIGONOMETRY

In your JCE course you learnt about calculating sides of a right angled triangle using Pythagoras theorem. In this unit you will learn how to calculate sides and angles of a right angled triangle using trigonometry. You will also learn how to derive trigonometric ratios of 30^0, 45^0, 60^0, $90^{0\,Z}$and solve practical problems involving trigonometry.

The knowledge gained in learning this will be used in solving real life situations such as finding the height of a mountain.

Activity 1:

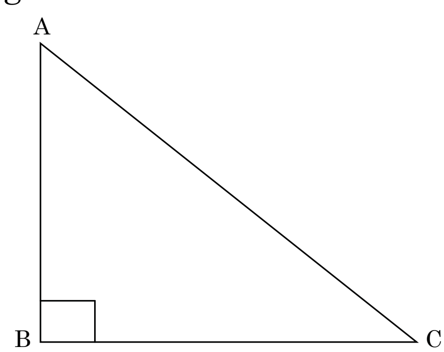
Identifying the sides of a right angled triangle

You are familiar with the side of the triangle called the hypotenuse. You will now be introduced to the names of two other sides in relation to the angles of the triangle.

In groups, study the definitions below which relate to a right angled triangle:

- An adjacent side to an angle is the side which together with the hypotenuse forms that angle.
- An opposite side to an angle is a side which together with the adjacent side forms a right angle of the triangle.

Basing on these two definitions, identify the adjacent side and the opposite side to angle A in the drawing below:



Which side is the hypotenuse in the above triangle?

Come up with your own right angled triangles labeling them with letters of your choices and identify the hypotenuse, the opposite sides, and the adjacent sides in relation to the angles of the triangles. Let your teacher check your answers.

Trigonometric ratios

In activity 1 you learnt to identify the sides of a triangle in relation to the given angles. Any two of these sides can be divided and the result is called the trigonometric ratio. You will now learn to define three of these ratios, sine ratio, cosine ratio and tangent ratio.

Activity 2:

Deriving sine ratio

In your groups,

1. Draw the following right angled triangles: 4cm by 3 cm by 5cm; 8cm by 6cm by 10cm; 12cm by 9cm by 15cm.
2. Label the angle opposite the shortest side in each triangle as θ (theta).
3. Now in relation to angle θ, find the ratio of the opposite side to the hypotenuse by dividing the side opposite angle θ by the hypotenuse in each of the three triangles simplifying the fractions to their lowest terms (or give the answer as decimal fractions to 4 decimal places). What do you find?
4. Report your findings.
5. Draw your own right angled triangles and practice getting sine ratio and in your group draw one triangle on a chart paper and hang the chart on the wall of your classroom.

The ratio you found above is called the **sine ratio** and as you have seen, it doesn't depend on the size of the triangle. From this, can you try to define what the sine of an angle is?

Because you found this ratio using angle θ you specifically call it sine of angle θ or in short S**in** θ.

Activity 3:

Deriving cosine ratio

1. Go back to the triangles you drew in activity 2. Repeat

step 3 but now divide the side adjacent angle θ by the hypotenuse. Again simplify the fraction to their lowest terms.

2. Report your findings.

The ratio you found is called the Cosine ratio and because you found this ratio using angle θ you specifically call it Cosine of angle θ or in short Cos θ. Again, it doesn't depend on the size of the triangle. As you did in activity 2, can you try to define the cosine of an angle?

Activity 4:

Deriving tangent ratio

1. Again go back to the triangles you drew in activity 2. Repeat step 3 but now divide the side opposite angle θ by side adjacent angle θ simplifying the fraction to its lowest term.
2. Report your findings.

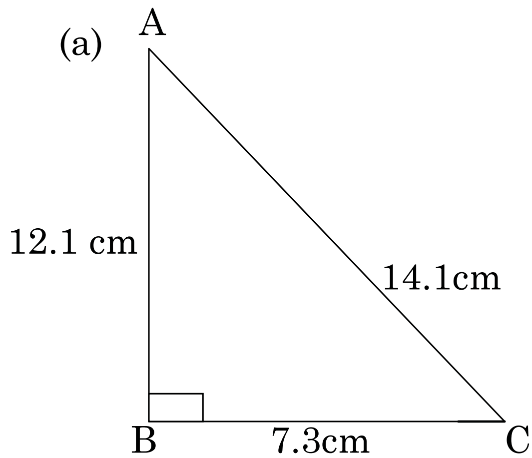
The ratio you found is called the tangent ratio. As in sine and cosine ratios, it doesn't depend on the size of the triangle.

The three trigonometric ratios above can easily be remembered by the word SOHCAHTOA. Discuss with a friend to see how this word has been formed. Ask your teacher to help you if you have problems.

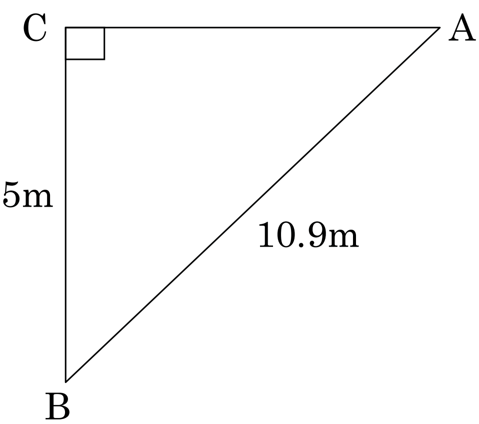
Exercise 11a

1. In each of the following triangles find the sine of angle A (Sin A): Give your answers correct to 4 decimal places:

(a)

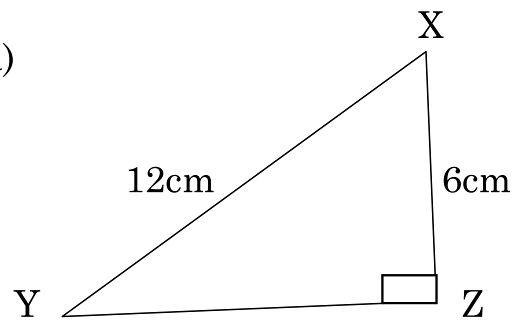


(b)

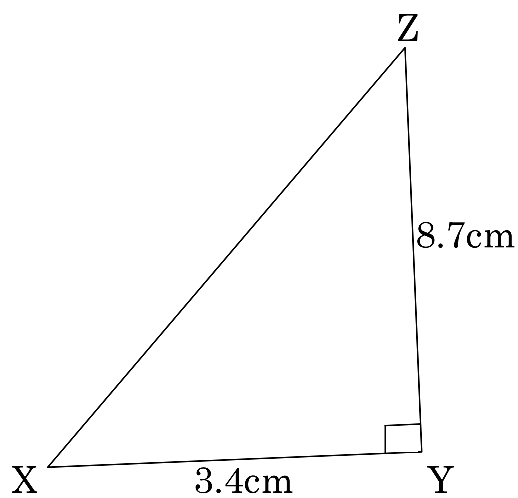


2. In each of the following triangles find the cosine of angle (Cos X): Give your answers correct to 4 decimal places.

(a)

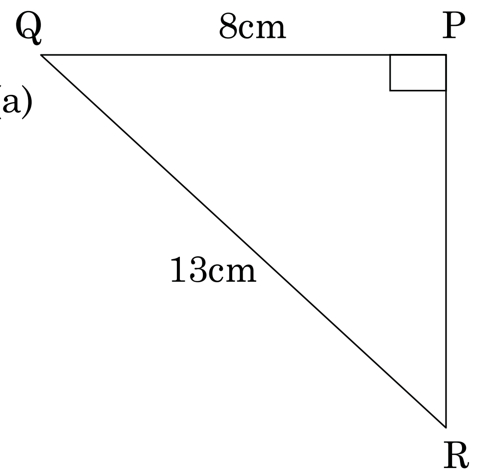


(b)

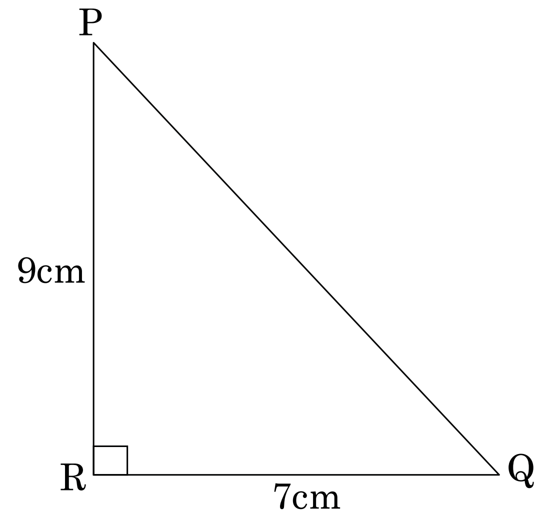


3. Find the tangent of angle PQR in each of the following diagrams. Give your answers correct to 4 decimal places.

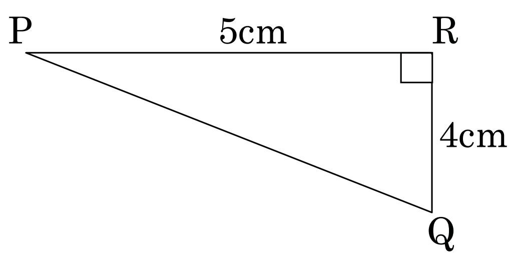
(a)



(b)



(c)



4. Given that in a right angled triangle ABC, the cosine of angle A = 0.8, angle B = 90^0 find

   (a) Sine of angle A

   (b) Cosine of angle C

   (c) Tangent of angle C

5. Draw your own triangles and write down sine, cosine and tangent of angles of your choices. Get a friend or your teacher to check your work.

Using a scientific calculator to work out problems involving trigonometry

You will quite often use a calculator to work out problems involving trigonometry in the sections that follow. You will use it especially in finding the tangent, sine and cosine of angles and finding angles whose ratios are given. The aim of this section therefore is to help you acquire the skills in using the calculator in these areas.

Activity 5:

Using a scientific calculator to find tangent, sine and cosine of angles

Before starting to work with your calculator, ensure that the calculator is in degree mode. Ask your teacher to help you set your calculator in this mode.

In pairs,

1. Study the keys on your scientific calculator. How are sine, cosine and tangent represented on the calculator?
2. Now press the “on” key and press “Cos” key followed by 53. What does your calculator display?
3. Now press the “=” key and write down the display. You can correct this display to four decimal places. This result is the cosine of 53^0.
4. Now using your calculator, find the following: tan 60^0, tan 40^0, sin 30^0, sin 90^0.
5. Choose your own angles and find their sines, cosines and tangents using the calculator.

Exercise 11b

Use a scientific calculator to find, correct to four decimal places

1. Tan 30^0
2. Cos 47^0
3. Sin 35^0
4. Tan 80^0
5. Sin 75^0
6. Cos 76^0
7. Sin 33^0

8. Tan 24^0

9. Cos 23^0

10. Sin 48^0

Activity 6:

Using a scientific calculator to find angles whose trigonometric ratios are given

Again in pairs, identify the key on the scientific calculator labeled "2^{nd}" or standing against "shift". You are to use this key together with other keys to find angles whose trigonometric ratios are given.

1. Press the key identified above and then press Cos. Write down what your calculator displays.

2. Repeat this step but now use Tan and Sin instead of Cos. Again write down what the calculator displays.

3. Discuss what the displays mean. Ask your teacher to help you if you cannot come up with the meaning.

4. In activity 5, you found Cos 53^0 as 0.6018 to four decimal places. Now press the 2^{nd} function key on your calculator followed by Cos followed by 0.6018 followed by " = " key. Write down what the calculator displays to the nearest degree.

5. Discuss the order of pressing the keys if you were to find the angle whose tangent or sine is given.

6. Repeat step 4 but now using the ratios you found in step 4 of activity 4. Did you manage to get the angles you started with?

7. In your own words write down the statement for finding angles whose trigonometric ratios are given.

Exercise 11c

Use a scientific calculator to find to the nearest degree the angle whose

1. Cosine is 0.5411
2. Cosine is 0.0089
3. Tangent is 2.3412
4. Sine is 0.7698
5. Tangent is 1.2300
6. Sine is 0.1543
7. Tangent is 0.4567
8. cosine is 0.7899
9. Sine is 0.9765
10. Tangent is 0.8354

Calculating sides and angles of a right angled triangle using trigonometry

In Form 2, you learnt how to calculate sides of a right angled triangle using Pythagoras Theorem. Remember that to use Pythagoras theorem you must be given two sides. However you may sometimes be given an angle or angles and a side and you may still be required to calculate the sides of a right angled triangle. This is when you can use trigonometry. In the activities that follow you will learn how to calculate sides and angles of a right angled triangle using trigonometry.

Activity 7:

Calculating sides of right angled triangles using trigonometric ratios

You learnt how to draw triangles in your JCE course using a ruler, a protractor and a pair of compasses. In groups,

1. Construct triangle PQR in which QR = 4cm, angle R = 90^0 and angle P = 32^0.Each one must draw his or her own

triangle.

2. Measure and record the lengths of PQ and AR.

3. Compare your work.

4. Report your findings.

You could also use scientific calculators to work out the lengths of PQ and AR. Again in your pairs,

1. Draw triangle PQR again. Just draw the triangle without using a compass or a protractor.

2. Using the triangle write down the relationship between 32^0 and the two sides you are to find.

3. Using the words hypotenuse, adjacent and opposite , complete the following:

   Tan 32^0 =

   Cos 32^0 =

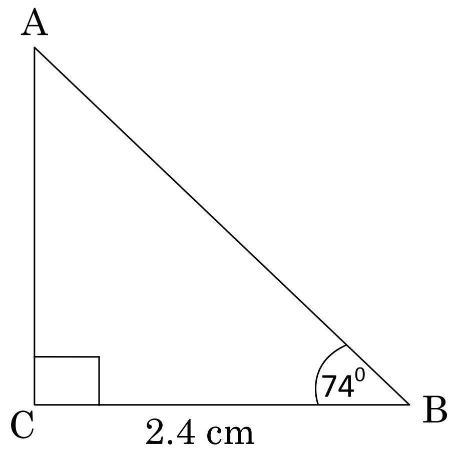
4. Discuss how you could solve the two equations to get the unknown sides using a calculator.

5. Using the calculator, find the unknown lengths. Compare your answers to the answers you obtained in step 2 in the triangle you constructed. Comment on your results.

To find the lengths of a triangle using trigonometry you need to first see the relationship between the given angle(s) and the given side and the side you are to find. This helps you to come up with the more direct trigonometric ratio to use. For example if the relationship is **adjacent** and **hypotenuse**, then you use Cosine ratio because it is the one which uses these sides. You then proceed as in the s that follows:

Example 1:

Calculating sides of a triangle



In the above triangle, find the length of AB.

Solution:

The relationship between 74^0 and 2.4cm and AB is adjacent and hypotenuse so you use Cosine ratio as follows:

Cos 74^0 = $\frac{2.4\text{cm}}{\text{AB}}$ --------- from the definition of Cosine of an angle.

AB Cos 74^0 = 2.4 --------- multiply both sides by AB

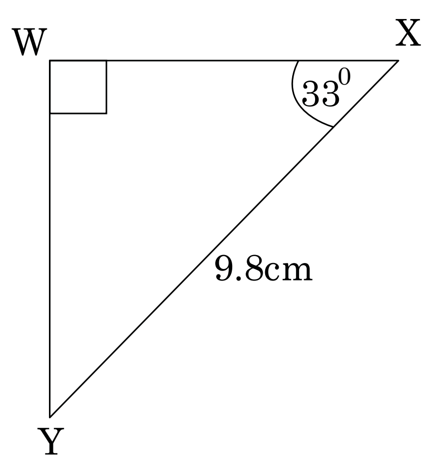
AB = $\frac{2.4\text{cm}}{\cos 74^0}$ ----------------------- divide by Cos 74^{0}both sides

AB = 8.71 cm approximately

Challenge

Discuss other ways in which the above question could have been solved.

Example 2



Calculate the length of WX in the above figure correct your answer to one decimal place.

Solution:

The relationship between 33^0 and 9.8cm and WY is **opposite** and **hypotenuse** so you use sine ratio as follows:

$\text{Sin } 33^0 = \frac{WY}{9.8cm}$ ------------From the definition of sine of an angle

$WY = 9.8 \text{ Sin } 33^0$ cm------Multiply by 9.8cm both sides and rearrange.

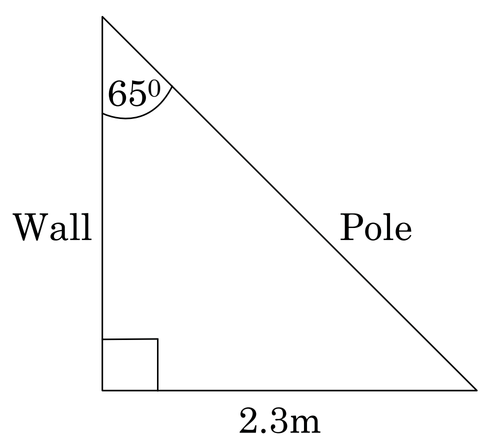
WY = 5.3 cm.

Example 3

A straight pole is rested against a wall with its one end 2.3m from the foot of the pole. The other end rests on top of the wall and the pole makes an angle of 65^0 with the wall. Calculate the length of the pole to the nearest metre.

Solution

You first need to make a sketch of the information. Straight lines are used to represent the wall and the ground on which the wall is standing. The assumption is that the wall and the ground are perpendicular.



$\text{Sin } 65^0 = \frac{2.3m}{\text{Lenght of the pole}}$ ----------------------- From the definition of sine of an angle

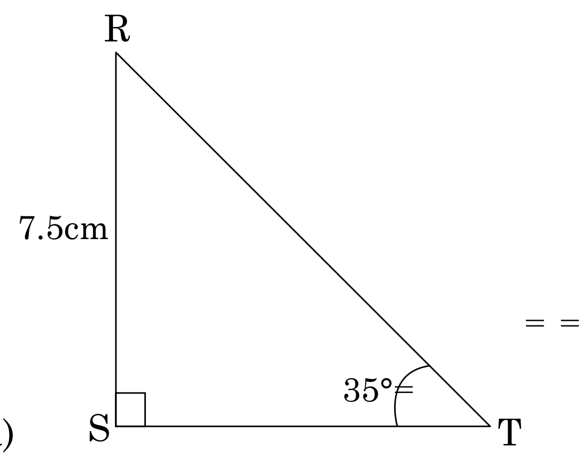
Length of the pole(Sin 65^0) = 2.3 m--------- Multiply by length of the pole both sides

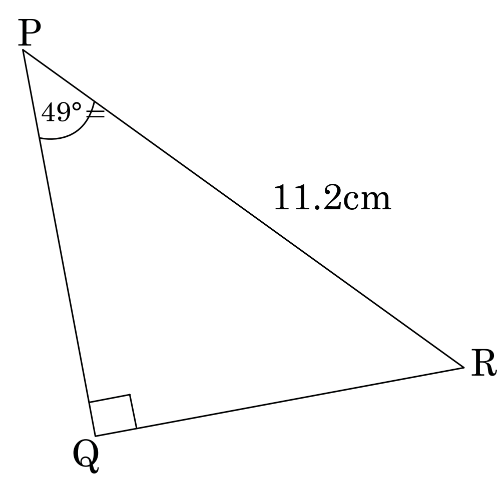
Length of the pole = $\frac{2.3m}{Sin\ 65^0}$ --------------------------- Divide by Sin 65^0 both sides

Length of the pole = 3m (to the nearest metre) ----- From the calculator.

Exercise 11d

1. Calculate, correcting your answers to 1 decimal place, the unknown sides in each of the triangles below:

(a) 

(b) 

2 A straight pole 10.5m long is leaning against a perpendicular wall. If the pole makes an angle of 65^0 with the top of the wall ,calculate the height of the wall.

3 A rectangle is 10m wide. One of the diagonals of the

rectangle makes an angle of 36^0 with the shorter side of the rectangle. Calculate the length of the rectangle.

4. Two wires are tied to the top of a pole which is standing perpendicular to the ground. They are then straight fixed to the ground along the same line so that the shorter wire makes an angle of $65^0$with the ground and the angle between the two wires is 10^0. Given that the shorter wire is 8.5m long, calculate the distance between the two wires on the ground.

5. A rectangular room is 10.2m long. The diagonals of the floor of the room each make an angle of 22^0 with the length of the room.

   Calculate

   (a) the length of the width of the room.

   (b) if the diagonal from the top of the corner to the bottom opposite corner of the room makes an angle of 60^0 with the height of the room, calculate the height of the room.

Activity 8:

Calculating angles of a right angled triangle using trigonometry

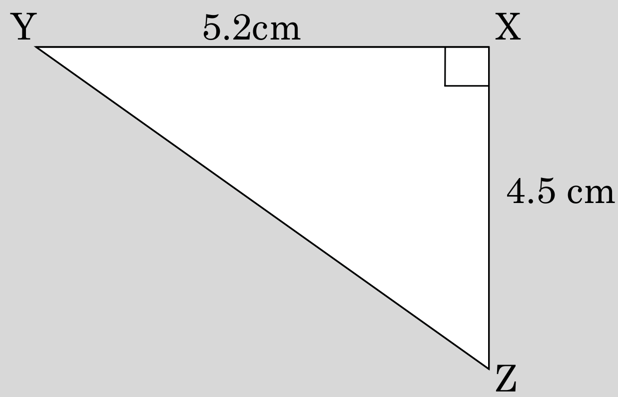
In the previous activities, you found sides of a right angled triangle by measurement and by using a calculator.

In this activity you will learn how to find angles of a right angled triangle using a calculator.

1. Draw triangle XYZ in which XY = 3cm, YZ = 4.8 cm, XZ = 5.6 cm and angle Y = 90^0
2. Write down the ratios for sin X and tan Z.
3. Discuss how you could use a calculator to find the values of angle X and angle Z. If you find problems go back to activity 5 of this topic.

Example 4:

Calculating angles



Find the value of angle XYZ in the triangle.

Solution:

There are more than one ways of solving this problem. Here is one of them:

Tan angle XYZ = $\frac{4.5\text{cm}}{5.2\text{cm}}$

Tan XYZ = 0.8653 (correct to 4 decimal places)

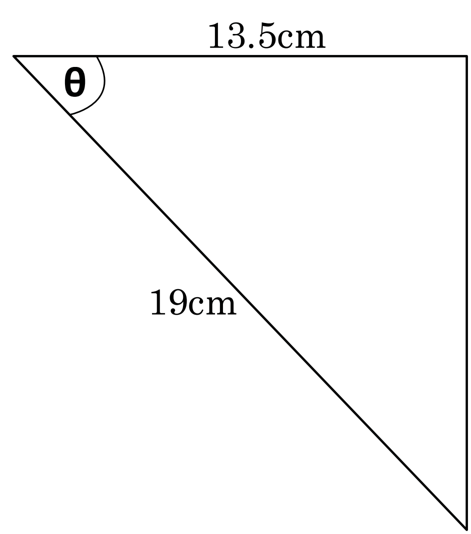
Angle XYZ= $\tan^{-1}$ 0.8653

Angle XYZ = 30^0 approximately (from a calculator)

Challenge: Find other methods of solving this problem and let your teacher check the methods.

Exercise 11e

1.



Calculate to the nearest degree, the value of angle marked θ (theta) in the above triangle.

2. In a right angled triangle PQR, PQ = 6 cm, PR = 15 cm and angle PQR = 90^0. Without using Pythagoras theorem, calculate the other two angles of the triangle giving your answers correct to the nearest degree.

3. A pole is resting in the corner of the room 4m by 5m. The top of the pole is 6.8m above the floor and the bottom is 2.5m from each wall. Calculate the angle that the pole makes with the floor.

4. A wheelchair ramp is to be built over steps up to a college entrance. Each step has a vertical rise of 12cm and a horizontal tread of 45cm. Calculate , to 1 decimal place, the angle that the ramp makes with the horizontal.

5. Madalitso tries to row straight across a river which is 46m wide. The current carries her downstream at an angle of 72^0 to the bank. How far downstream from the point she was trying to reach does she actually land?

Trigonometric ratios of special angles

Special angles are angles whose trigonometric ratios can be expressed as surds or as simple fractions. These angles are 30^0,45^0,60^0 and 90^0. The trigonometric ratios of these angles help us to solve problems involving trigonometry without having to use a calculator. In the activities that follow, you will learn how to derive the trigonometric ratios of these angles.

Activity 9:

Deriving fractional trigonometric ratios of 30^0 and 60^0

In groups,

1. Draw and label any one equilateral triangle.
2. From any one vertex in each triangle draw a perpendicular bisector of the side opposite that vertex.
3. What is the value of each angle in each triangle?

4. Work out the following ratios: sin 30^0,cos 30^0, sin60^0 and cos60^0. Leave your answers as simplified surds or as simplified fractions.

5. Summarise the information in the table below by filling the trigonometric ratios you found in step 4 above:

Sin 30^0	Cos 30^0	Tan 30^0	Sin 60^0	Cos 60^0	Tan 60^0

You must have discovered that despite the groups drawing their own equilateral triangles, all the trigonometric ratios reduce to the same results. However the amount of working varies from one triangle to another. If you draw an equilateral triangle whose length is an odd number you would work more than it was if the length was an even number. Furthermore, if you want to reduce the amount of working even further, it is advisable to use the smallest equilateral triangle of even numbered length of 2 units.

Activity 10:

Deriving trigonometric ratios for 45^0

In groups, draw any three isosceles right angled triangles.

1. Use Pythagoras theorem to calculate the lengths of the hypotenuse in each of the triangles, leaving them in simplified surd form.

2. Work out the following ratios for all the three triangles: tan 45^0, cos 45^0 and sin 45^0. Simplify your answers to their simplest forms.

3. Summarise the information in the table below:

Tan 45^0	Cos 45^0	Sin 45^0

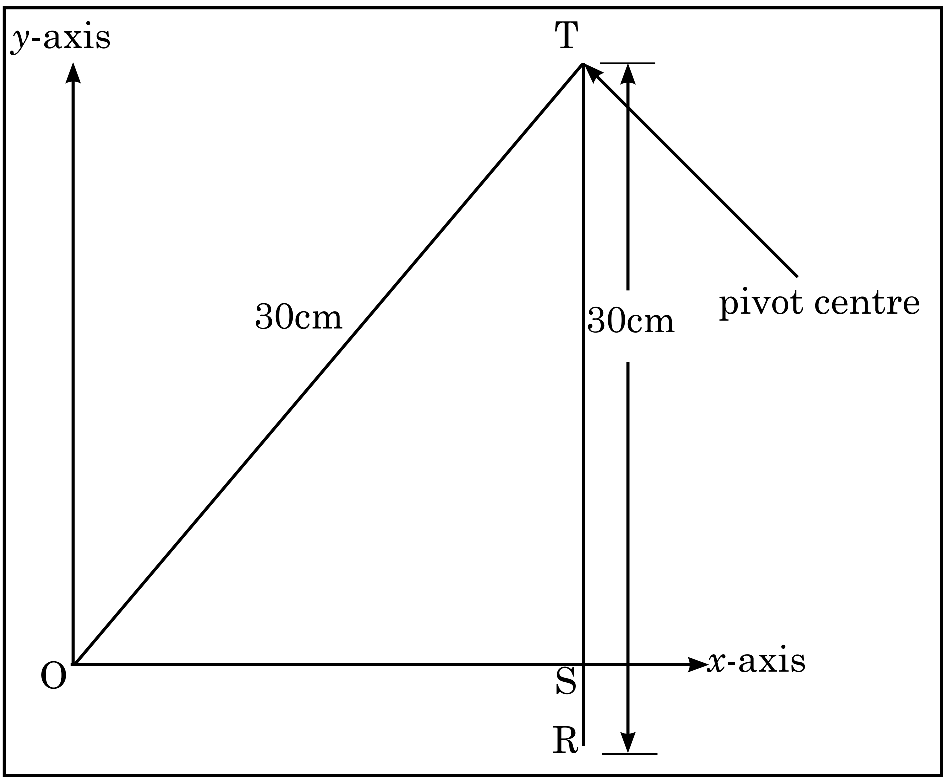
As in activity 9, you must have discovered that for any right angled isosceles triangle the fractional trigonometric ratios are the same.

Activity 11:

Deriving trigonometric ratios for 90^0

1. Working in groups, draw the y and x axes on a square sheet of Cardboard of not less than 40cm by 40cm.
2. Get two straight sticks each 30cm long, and pivot the two ends of the sticks.

Ensure that the two sticks can move freely on the pivot. Bring the two sticks onto the cardboard and pivot one end of the two sticks at the origin as shown below:



Square cardboard

3. Now pin the cardboard on to a wall so that TR is perpendicular to the x axis. Note that OS is the perpendicular distance of TR from the y axis.
4. Now write down the following ratios of acute angle O in terms of OT and OS: Cos angle TOS, Sin angle TOS and Tan angle TOS.
5. Slowly, rotate OT anticlockwise about O. What is happening to the size of acute angle O? What about the distance OS?
6. Continue rotating OT anticlockwise until R , S and O coincide. What will be the size of the acute angle O at this moment? What will be the length of OS?
7. Using your findings in step 6, make appropriate

substitutions into the ratios you found in step 4, simplify the results and give a summary in a table form.

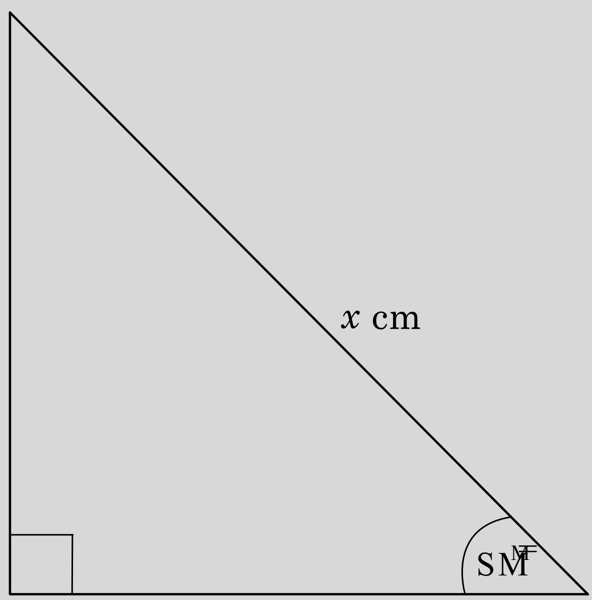
Challenge

In deriving trigonometric ratios for 90^0, you used sticks of length 30cm. Do you think you would obtain different results if sticks of other lengths say 10cm and 10cm, 20cm and 20cm e.t.c were used?

The trigonometric ratios of special angles are useful in solving some right angled triangles without having to use a calculator.

Example 5:

Solving for triangles



Find the value of x without using a calculator or tables.

Solution

$$\text{Cos } 60^0 = \frac{3\text{cm}}{\text{x}}$$

$$x \cos 60^0 = 3\text{cm} \text{ ---------- multiply both sides by x}$$

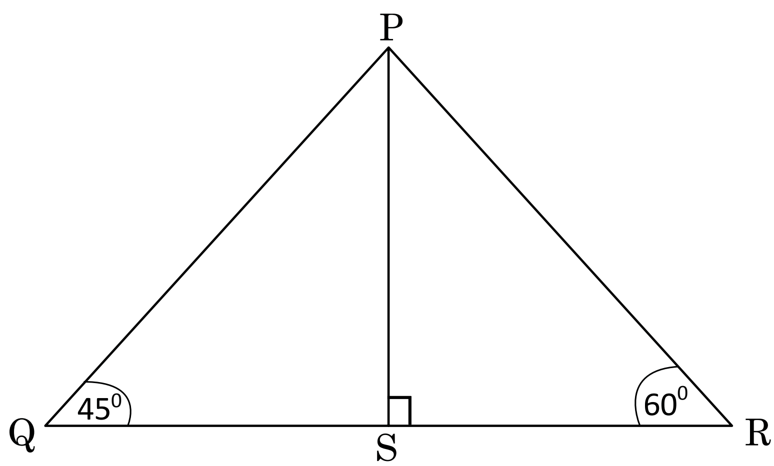
$$\text{x} = \frac{3\text{cm}}{\text{Cos } 60^0} \text{ ------ divide both sides by cos } 60^0$$

$x = \frac{3cm}{\frac{1}{2}}$ -------- substitute cos 60^0 for (activity 9)

$x = 6cm$

Exercise 11f

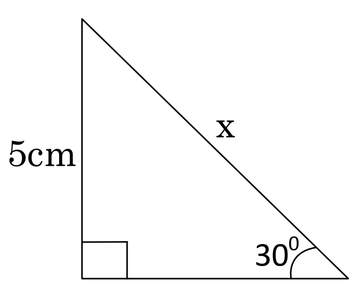
1.



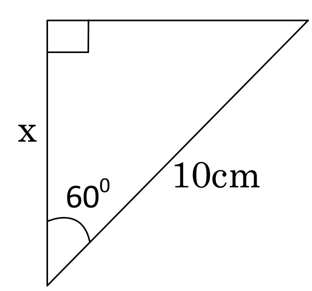
In the triangle above, PS is the height of the triangle PQR and QS = 40 cm.Without using a calculator, calculate, leaving your answer in surd form, the length of PR.

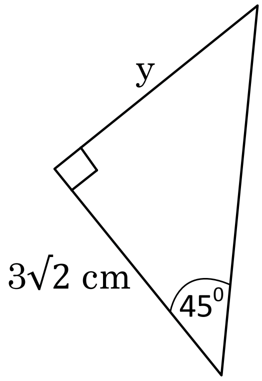
2. Calculate the lengths of the labeled sides in the triangles below leaving your answers in surd form:

(a)



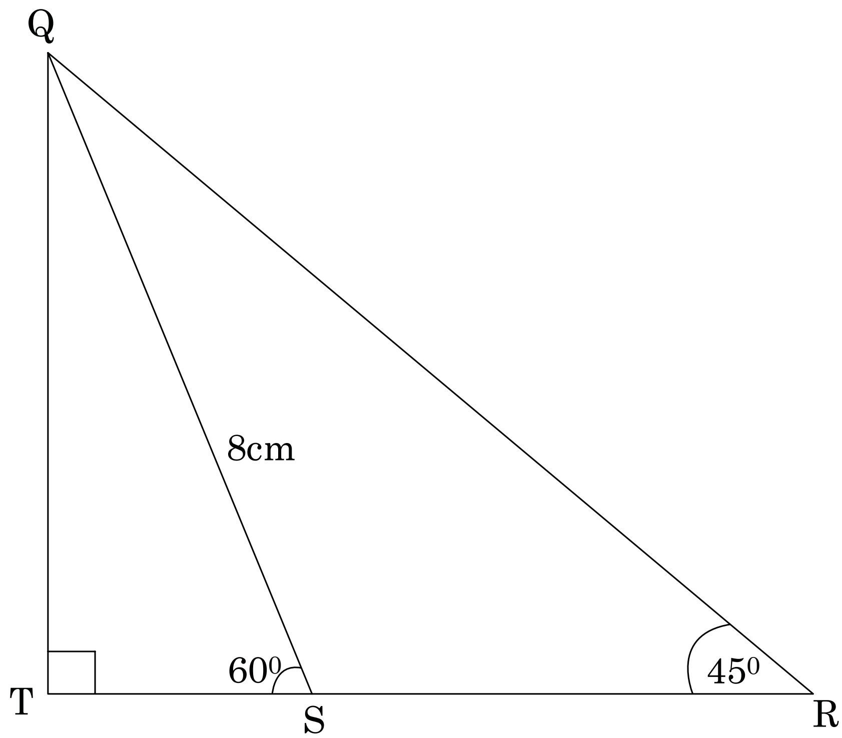
(b)





(c)

3.



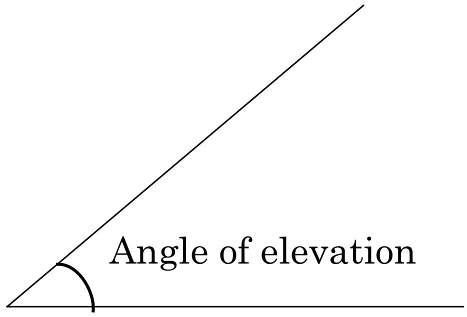
In the above figure, calculate, leaving your answers in surd forms, the lengths of TQ, RQ and RS.

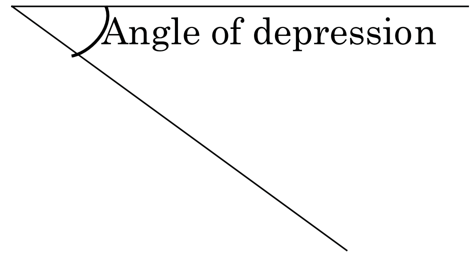
Angles of elevation and depression

An angle of elevation is the angle between a horizontal line and a straight line drawn above the horizontal line.

An angle of depression is the angle between a horizontal line and a straight line drawn below the horizontal line.

The two angles are shown below:





Calculating angles of elevation and depression

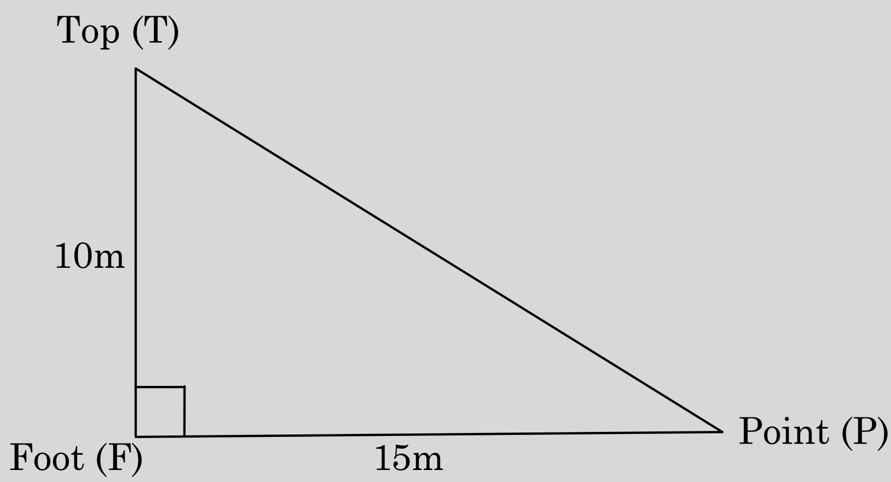
Basically what happens in calculating angles of elevation and depression is what you did in activity 8 of this topic.

Example 6

Calculate the angle of elevation to the top of a building 10 m high from a point 15m away from the foot of the building.

Solution

You need a sketch of the information. The building is assumed to be perpendicular to the ground.



You are required to calculate angle TPF.

Tan angle TPF = $\frac{10m}{15m}$

Tan angle TPF = 0.6667

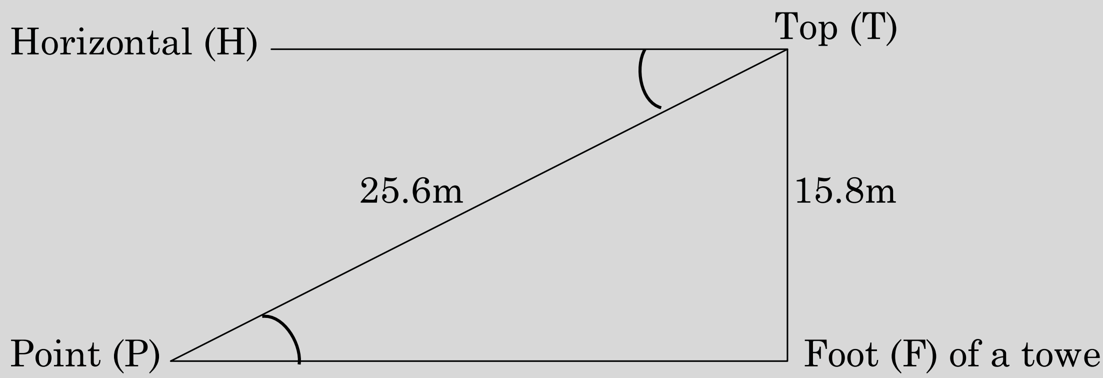
Angle TPF = 34^0 (to the nearest degree)

Example 7:

Angle of depression

Calculate the angle of depression of a point 25.6 m away from the top of a tower 15.8m high.

Solution



You are required to find angle HTP.

Angle HTP = angle TPF

Sine angle TPF = $\frac{15.8m}{25.6m}$

Sine angle TPF =0.6172

Angle TPF = 38^0

The angle of depression is therefore 38^0

Exercise 11g

Give your answers correct to the nearest degree in all the questions

1. Calculate the angle of elevation of the top of a mountain 300m high from a point 1.5Km from the foot of the mountain.
2. Find the angle of elevation of the tip of an aerial 1.5m high standing on a building 5m high from the point X 300m away from the foot of the building.
3. A point is 150m away from the foot of a tower. It is also

measured that this point is 350m from the top of the tower. Calculate the angle of depression of this point from the top of the tower.

4. A boy in a fruit tree 5.8m high throws a fruit straight to his friend who is standing 3m from the foot of the tree. If the boy on the ground catches the fruit at a height of 0.5m from the ground, at what angle to the tree did the other boy throw the fruit?

5. Two wires are tied to the top of a pole 10.5m high and then pegged straight on the ground. The two wires reach distances of 1.5m and 2m on the ground respectively. Calculate the angle between the two wires.

6. In a penalty shootout, a player shoots a ball straight onto a goalpost whose crossbar is 1.5 m above the ground. If the penalty spot is 11m away from the goal line, what is the maximum angle from which the player can score?

7. A plane flying at an attitude of 1500m is to land at an airport 10km away. At what minimum angle must the pilot lower his plane?

8. In a shoot a target competition, a shooter stands 35m away from the foot of a pole on which a target is placed. If a gun is held by the shooter at a height of 0.75m above the ground at an angle of 20^0 and if the pole is 12m high, show that the shooter will miss the target.

9. A rubber bullet is fired at an angle straight on to a perpendicular wall and hits a point 5.6m high. If the bullet was fired from the foot level of the wall and rebounds at an angle of $80^0$to its path onto the wall and at 30^0 to the wall, at what angle and distance from the wall was the bullet elevated?

10. The tip of the roof of a room stands 1.5 m midway above the last line of the room 6m wide. Calculate the angle of elevation of the roof.

Further problems involving angles of elevation and depression

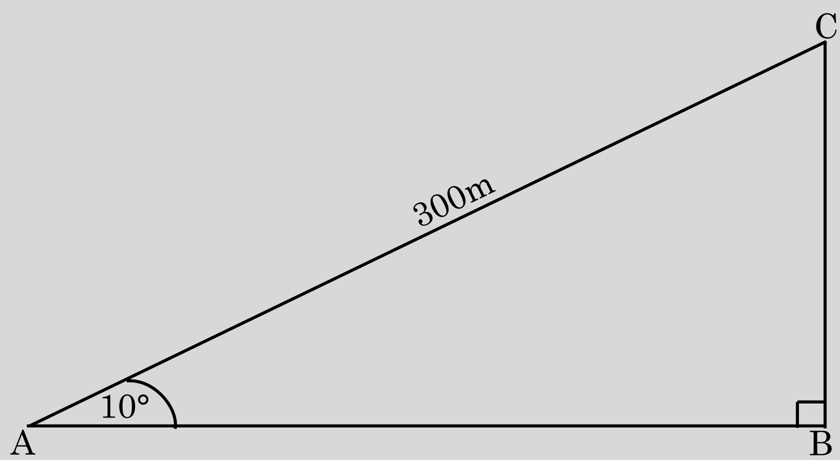
Example 8:

Problems involving angles of elevation and depression

A motorist travels a distance of 300m up a hill inclined at an angle of 10^0 to the horizontal. If the motorist was originally at sea level, find the height she is above sea level at the end of 300m.

Solution

It is important to make a sketch of the information. For purposes of calculations we use straight lines to represent objects.



$\sin 10^0 = \frac{BC}{300}$

$300\sin 10^0 = BC$ -------------------------------- (multiply by 300 both sides).

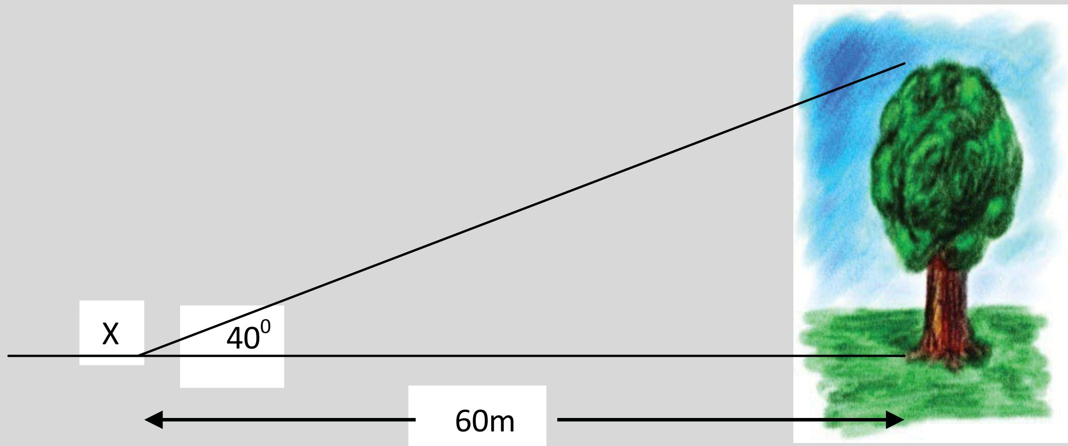
$BC = 52.1$m approximately.

Challenge

Discuss some other ways of solving this problem.

Example 9

The base of a tree is 60m away from point x on the ground. If the angle of elevation of the top of the tree from x is 40^0. Calculate the height of the tree. Give your answer to the nearest metre.



Let the height of the tree be h

$$\therefore \frac{h}{60cm} = \tan 40^0$$

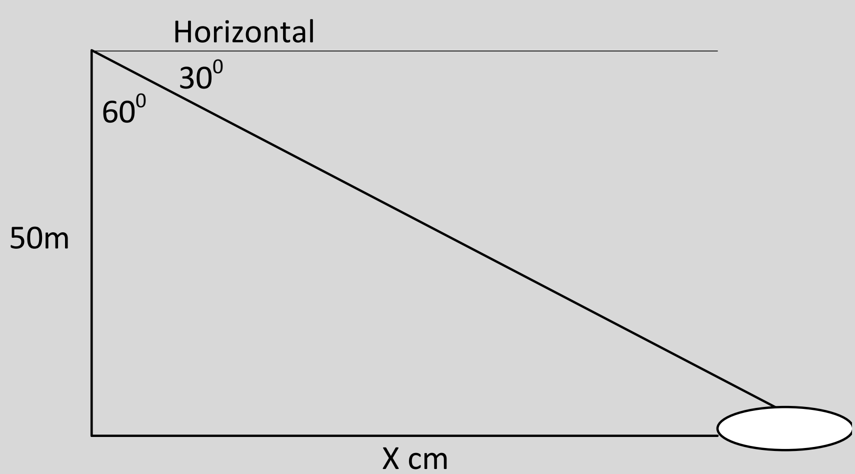
$h = 60m \times \tan 40^0$

$= 50.3m$ (to 3 significant figures)

Example 10:

Problems involving angles of elevation and deviation

A person on top of a cliff 50m high, observes the angle of depression of a boy to be 30^0. If he is in line with the boy, calculate the distance between the boy and the foot of the cliff (which may be assumed to be vertical).



Let the distance between the boy and the foot of the cliff be x.

The angle between horizontal line and vertical cliff is 90^0

$\therefore$ to find the angle for calculating x in the triangle, we subtract 30^0 from 90^0

i.e. $90^0 - 30^0 = 60^0$

$\therefore \frac{x}{50} = \tan 60^0$

$x = 50 \times \tan 60^0$

$x = 86.6$ m (to 3 significant figures)

Exercise 11h

1. From a point, the angle of elevation of a tower is 30^0. If the tower is 25m distant from the point, what is the height of the tower?

2. A woman 1.6m tall observes the angle of elevation of a tree to be 26^0. If she is standing 20m from the tree, find the height of the tree.

3. A boy 1.2m tall is 10m away from a tree 20m high. What is the angle of elevation of the top from his eyes?

4. A and B are two villages. If the horizontal distance between them is 12km and the vertical distance between them is 2km.

   Calculate

   (i) the shortest distance between the two villages

   (ii) The angle of elevation of B from A.

5. A surveyor stands 100m from the base of a tower on which an aerial stands. He measures the angles of elevation to the top and bottom of the aerial as 58^0 and 56^0. Find the height of the aerial.

6. A man 1.2m tall standing on top of the mountain 1200m

high observes the angle of depression of a steeple is 43^0. How far is the steeple from the mountain?

7. X and Y are two towns. If the vertical distance between them is 10km and the angle of depression of Y from X is 7^0,

   Calculate:

   (i) the shortest distance between the two towns

   (ii) the horizontal between the two towns.

8. An air plane receives a signal from a point X on the ground. If the angle of depression of point X from the airplane is 30^0, calculate the height at which the plane is flying given that the plane is 6km from X.

9. A girl sitting on a hill at A, overlooking a lake can see a small boat at a point B on the lake. If the girl is at height of 50m above B at a horizontal distance of 120m away from B, calculate:

   (i) The angle of depression of the boat from the girl

   (ii) The shortest distance between the girl and the boat.

10. A plane is flying at an altitude of 8km directly over the line AB. It spots two boats A and B, on the sea. If the angles of depression are 60^0 and 30^0 respectively, calculate the horizontal distance between A and B (two possible answers).

11. Mr Phiri uses 10m planks to offload some items from his lorry. If the lorry is 1.5m high, calculate the minimum angle of inclination of the planks.

Bearing

Activity 12:

Using a compass to name directions

In form 1geography, you learnt about how to use a compass.

1. Name the four cardinal points on a compass.

2. What is the initial reference line on the compass?
3. Discuss how you name points midway between the cardinal points. Are these the only points you can show on a compass?

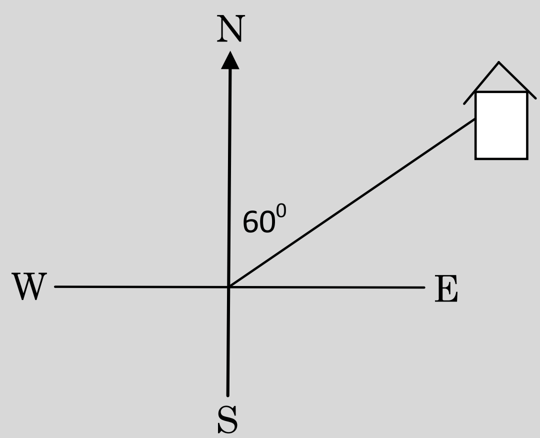
This method of naming points is used to show direction. Sometimes the cardinal points will be used together with angles. Bearing quoted in this way are always measured from N and S and never from E and W.

Example11:

Naming directions

State the directions of the huts in each of the following diagrams

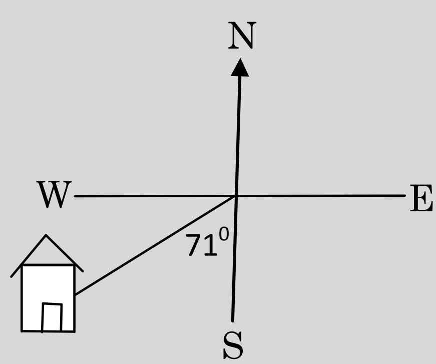
(a)



Solution:

N60^0E

(b)



Solution

S71^0W

Bearings are also measured from north in a clockwise direction, the north being taken 0^0. Three figures are always stated. For example 008^0 is written instead of 8^0. East will be 090^0, South 180^0 and West 270^0.

Activity 13:

Calculating the bearing of a point relative to a given point

1. On a piece of paper draw a north-south line.
2. Draw any object on the same piece of paper and join the object to the north - south line by a straight line. Let the two lines join at a point say O.
3. Use your compass to find the bearing of the object from O.
4. Your teacher will provide you with diagrams showing locations of point a point B in relation to another point A.
5. Use your compass to find the bearing of B from A.

Challenge

In the drawings provided by your teacher, how would you find the bearing of A from B?

Exercise 11i

1. Write each of the following as three-figure bearings

a) $N50^0E$

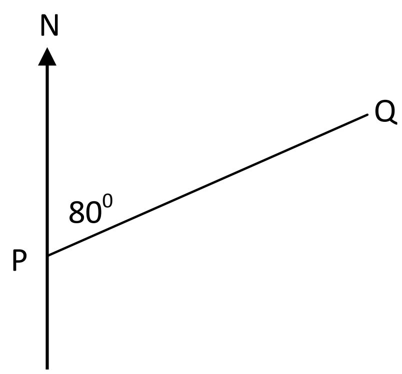
b) $N50^0W$

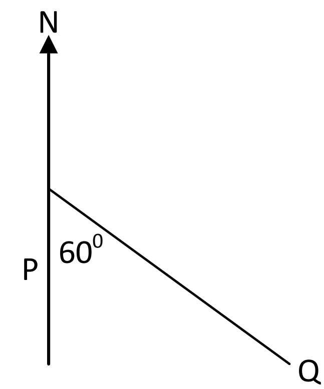
c) $S50^0W$

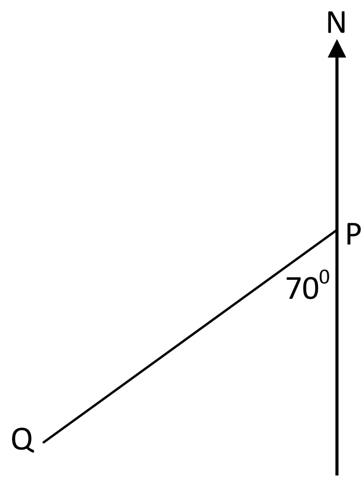
d) $S50^0E$

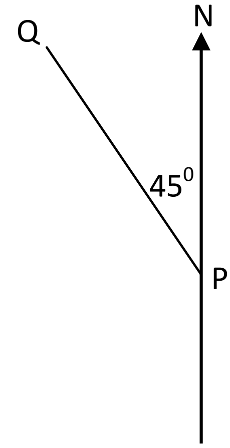
e) $S80^0E$

2. Find the three-figure bearing of Q from P in the sketches below.

a) 

b) 

c) 

d) 

3. Draw sketches representing the following:-

A is on a bearing of

a) 020^0 from B

b) 125^0 from B

c) 220^0 from B

d) 270^0 from B

e) 310^0 from B

Calculating the bearing of a point relative to a given point using trigonometry

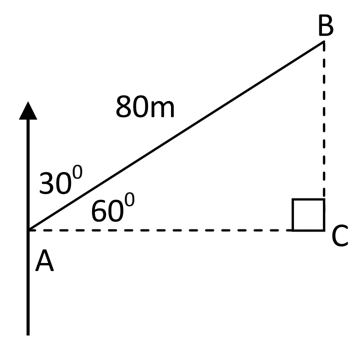
Problems involving bearing may be solved by making a scale drawing or by using trigonometry. When using the scale drawing method, you first choose a suitable scale. Since we have just studied trigonometry, we will solve the problems below using the trigonometry method.

Example 12

A boat starts from point A and sails to point B on a bearing of 030^0, given that the distance of A to B is 80m, find how far B is to the east of A.

Solution

You need to sketch the information. Form a right-angled triangle so that you can use SOHCAHTOA.



We are asked to find AC

$$\frac{AC}{80m} = \text{Cos}60^0$$

$$AC = 80\text{m} \times \text{Cos}60^0$$

$$AC = 40\text{m}$$

Example 13

A ship sets out from a point A and sails due north to a point B, a distance of 120km. It then sails due east to a point C. If the bearing of C from A is 037^0,

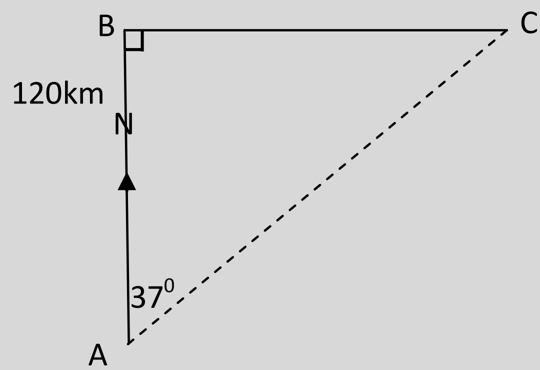
Find

a) the distance BC

b) The distance AC.

Solution

Again, you need to sketch what is going on. Join A to C to form the right-angled triangle ABC.



a) $\frac{BC}{120} = \tan 37^0$

bc = 120km $\times$ $\tan 37^0$

= 90.4km. (to 3 significant figures)

b) To find AC you can use the Pythagoras theorem or trigonometry. Using trigonometry,

$$\frac{120km}{AC} = \cos 37^0$$

$\therefore$ AC = $\frac{120km}{\cos 37^0}$ (making AC the subject of the formula)

= 150km (to 3 significant figures)

Exercise 11j

1. A ship is on a bearing 060^0 from a lighthouse. What is the bearing of the lighthouse from the ship?

2. A ship is on a bearing 200^0 from a lighthouse. What is the bearing of the lighthouse from the ship?

3. P is the point due west of a harbour H and Q is a point

which is 5km due south of H. If the distance PH is 7km, find the bearing of Q from P.

4. A boat leaves a harbour H on a bearing of 120^0 and it sails 100km on this bearing until it reaches a point B. How far is B east of A? What distance south of A is B?

5. X is a port due west of a point P. Y is a point due south of P. If the distances PX and PY are 10km and 15km respectively, find the bearing of X from Y.

6. A ship sails 35km on a bearing of 040^0.

a) How far north has it travelled?

b) How far east has it travelled?

7. A ship sails 200km on a bearing of 240^0.

a) How far south has it travelled?

b) How far west has it travelled?

8. An aircraft flies 400km from point O on a bearing of 025^0 and then 200Km on a bearing of 080^0 to arrive at B.

   a) how far north of O is B?

   b) how far east of O is B?

   c) Find the distance and bearing of b from o.

9. An aircraft flies 500 km on a bearing of 100^0 and then 600km on a bearing of 160^0. Find the distance and bearing of the finishing point from the Starting point.

Unit summary

- In this unit you have learnt calculating angles and sides of right angled triangles using trigonometric ratios. You have also learnt how to derive fractional trigonometric ratios of 30^0, 45^0, 60^0 and 90^0. You also worked out problems involving bearing and solved practical problems involving trigonometry.

Glossary

Tangent ratio: the ratio of the side opposite to a given angle in a right angled triangle to the side adjacent that angle.

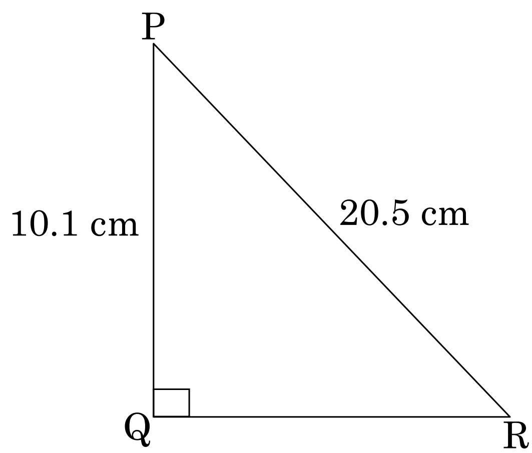
Sine ratio: the ratio of the side opposite to a given angle in a right angled triangle to the hypotenuse.

Cosine ratio: the ratio of the side adjacent to a given angle in a right angled triangle to the hypotenuse.

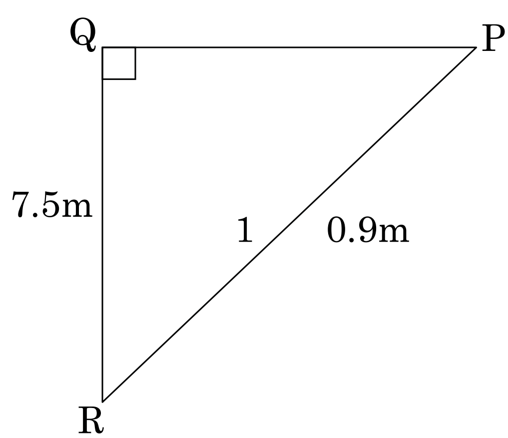
Special angles: angles whose ratios that can be expressed as surds or simple fractions.

Unit review exercise

1. In each of the following triangles find the sine of angle P : Give your answers correct to 4 decimal places:

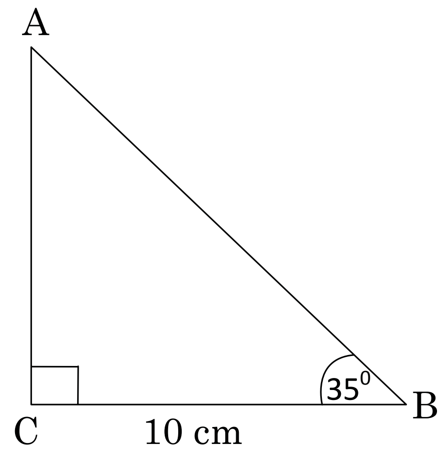


(a)



(b)

2.



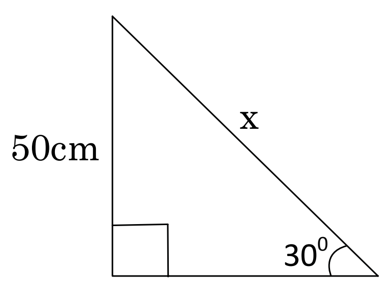
In the above triangle, find the length of AB.

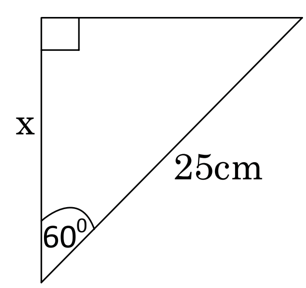
3. A surveyor stands at the end of the bridge across a river. The bridge is 20m long. The surveyor looks down at an angle of 68^0 to see the bottom of the river on the opposite side. Calculate the depth of the river to the nearest metre.

4. A 200m tower is to be built for relaying cellular phone signals. The tower is to be anchored by cables from the top of the tower that will each form a 65^0 angle with the ground. Find how far from the base of the tower each cable will be anchored.

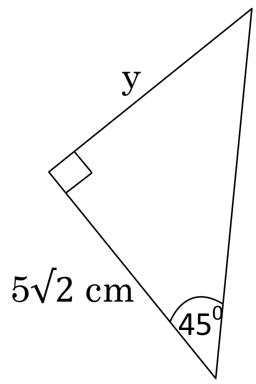
5. The longest ladder that a City Council fire department has is 415cm mounted onto the roof a truck. For the safety of those on the ladder, the fire department does not want to extend the ladder to an angle greater than 75^0 with the roof of the truck. If the roof of the truck is 300cm off the ground, find the highest point the ladder can reach.

6. Calculate the lengths of the labeled sides in the triangles below leaving your answers in surd form where possible:

(a) 

(b) 

(c)



7. A plane flying at an attitude of 10500m is to land at an airport 25Km away. At what minimum angle must the pilot lower his plane?

8. A ship left port X and travelled 70Km on a bearing $S40^0E$ to port Y. It then travelled 100Km on a bearing $S28^0W$ to port Z. Calculate

a. the shortest distance from X to Z correct to three significant figures.

b. the bearing of Z from Y to the nearest degee.

9. The bearing of ship H from town X is 145^0 36′ and from town Y is $055^0$34′. Given that XY =10Km and that X is due north of Y, find the distance from H to Y correct to one decimal place. = 5.7km

10. An aircraft flies 1300 km on a bearing of 210^0 and then 700km on a bearing of 060^0. Find the distance and bearing of the finishing point from the starting point.

References

R. Chikwakwa et al, *Senior Secondary Mathematics (2002*), Macmillan, Malawi.

L. Bostock et al, *National Curriculum Mathematics(1999),* Stanley Thornes, Cheltenham

Elain Ryder et al, *CHANCO Teach yourself Mathematics(2013*), Chancellor College Publications, Malawi.

Unit 12

SIMILARITY

In unit 9 of book 2, you studied similar figures. Recall that similar shapes are an enlargement of each other. This means the length of the larger shape are found by multiplying the scale factor by the lengths of the smaller shape.

In this unit, you will learn to apply the ratio of areas and volumes of similar figures to calculate areas, sides of similar figures and volumes of similar solids.

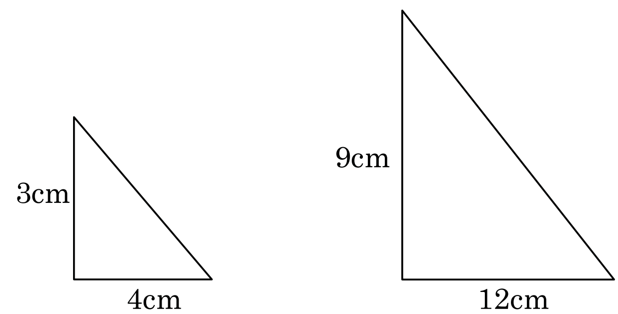
Similarity is used in many situations. It is used in solving real life problems such as finding lengths, volumes, and areas of similar objects such as triangles.

Ratio of areas of similar figures

Activity 1:

Identifying a scale factor of similar figures

1. In pairs discuss the meaning of "scale factor."
2. Identify the scale factor in the diagrams below (A is the object and B is the image)



3. Write the formula for finding scale factor.

Scale factor shows how many times an object has been enlarged. It is found by the formula scale Factor = $\frac{\textit{leng ht on the image}}{\textit{corresponding} \quad \textit{leng ht on the object}}$

When the scale faclvtor is greater than 1, the image is larger than the object, when it is less than 1, the image is smaller than the object and when it is 1 the image and the object are equal in size. A negative scale factor upsides down the image.

Example 1:

Scale factor

The height of a triangle ABC is 10cm and the height of a similar triangle A'B'C' is 14cm. Calc ulate the scale factor.

Solution

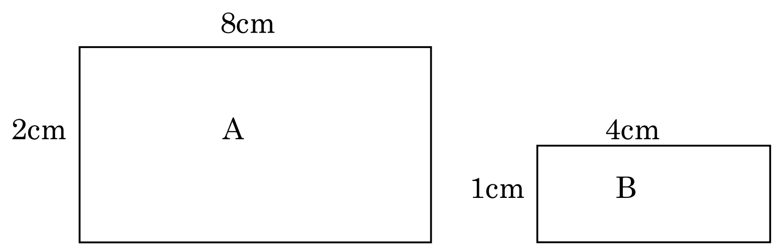
$$\text{Scale factor} = \frac{14}{10}$$

$$= 1.4$$

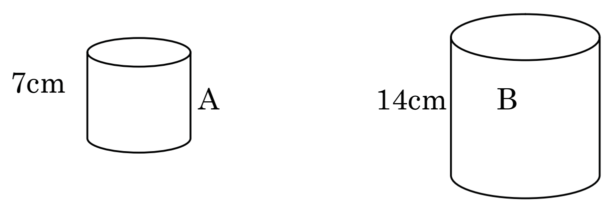
Exercise 12a:

Calculate the scale factor in each of the following pairs of figures. A is the object and B is the image:

1.



2.



3. The actual length of a line segment is 2.5 m but it is represented on a scale drawing by a line segment 2.5 cm. Calculate the scale factor.

4. A distance of 1km is represented on a map by a line 2cm long. find the Scale factor used.

5. The radius of a circle is 14cm. If the circle is enlarged by the scale factor of ½, find the radius of the corresponding circle.

Activity 2:

Finding area factor of similar figures

Working in pairs, discuss the meaning of area factor. Suggest how area factor can be found.

The area factor of similar figures is the ratio of areas of similar figures. It is found by the formula

$$\text{Area factor} = \frac{\text{Area of the image figure}}{\text{Area of the object}}$$

Example 2

The area of a triangle is $24cm^2$. The area of another triangle an enlargement of the first triangle is $72\ cm^2$. Find the area factor.

Solution

$$\text{Area factor} = \frac{72c\,m^2}{24c\,m^2}$$
$$= 3$$

Exercise 12b:

In each of the following, A is the area of the object and B is the area of the image . Find the area factor for each one of them:

a. $A = 32\ cm^2$ and $B = 48cm^2$

b. $A = 100\ cm^2$ and $B = 50\ cm^2$

c. $A = 54\ cm^2$ and $B = 90\ cm^2$

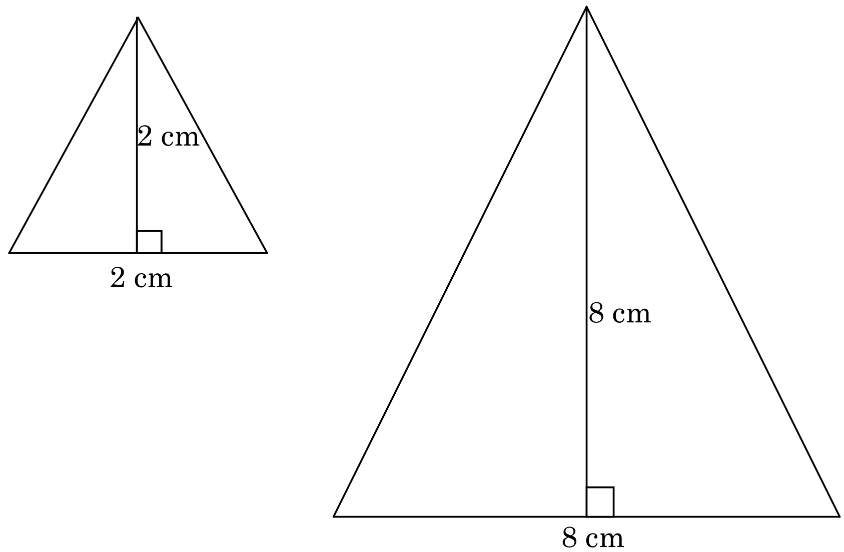
d. $A = 1.5m^2$ and $B = 1m^2$

e. $A = 100km^2$ and $B = 75km^2$

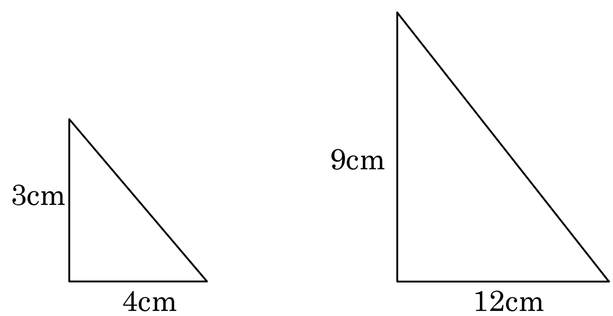
f. $A = 7.5\ m^2$ and $B = 2.5m^2$

1. Calculate the ratio of areas of the following pairs of solids.

a.



b.



2. A rectangular garden 120m by 100m. Another rectangular garden is150m by 120m. Find the ratio of the area of the first garden to that of the second garden.

3. The two parallel sides of a trapezium are 10cm and 20cm. the perpendicular distance between them is 8cm. Another trapezium has Parallel sides measuring 15cm and 30cm with a perpendicular distance of 12cm between them. Find the ratio of area of the second trapezium to the area of the first trapezium.

4. Calculate the ratio of area of two circles with radii 7cm and 21cm respectively

Activity 3:

Calculating areas of similar shapes

In the previous three activities you have learnt how to identify scale factor, area factor and how to calculate ratios of areas of similar solids. In this section you will learn how to calculate areas of similar shapes. But before you calculate areas of these similar shapes, you need to know the relationship between scale factor and area factor.

In groupsDraw any three pairs of similar figures of your choice. Show the dimensions of he figures.

1. For each pair find the scale factor and the area factor and complete the table like the one below:

	Scale factor	Area factor
First pair		
Second pair		
Third pair		

2. What is the relationship between the ratio of the lengths of three pairs of similar shapes above and the ratio of the areas?

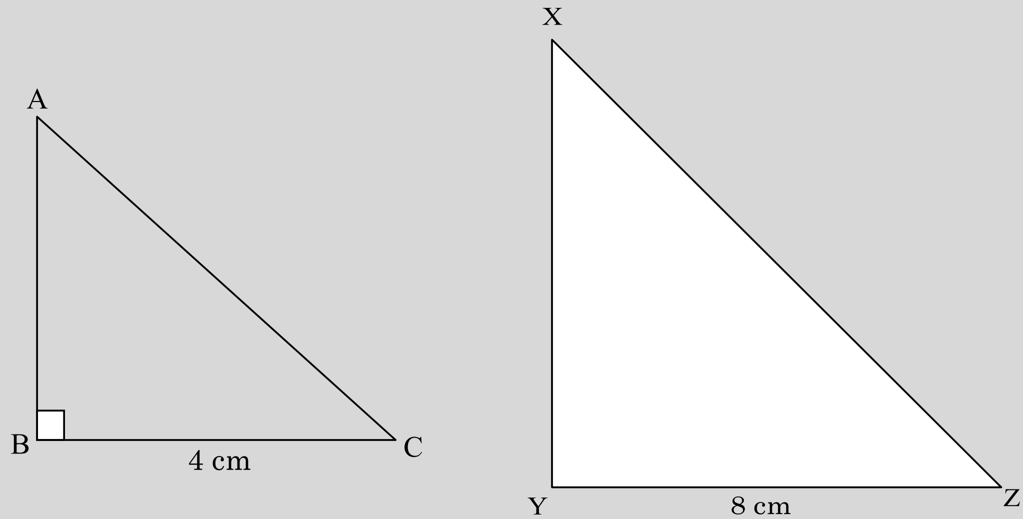
3. Report your findings to class.

You must have found out that the area factors are the squares of the scale factors or the scale factors are the square roots of the area factors.

Example 3:

Areas of triangle

The two triangles, ABC and XYZ are similar. Given that area of ABC is 5 cm^2, find the area of ΔXYZ.



Solution

Scale factor $= \frac{8}{4} = \frac{2}{1}$

$= \left(\frac{2}{1}\right)^2$ .... Relationship between scale factor and area factor

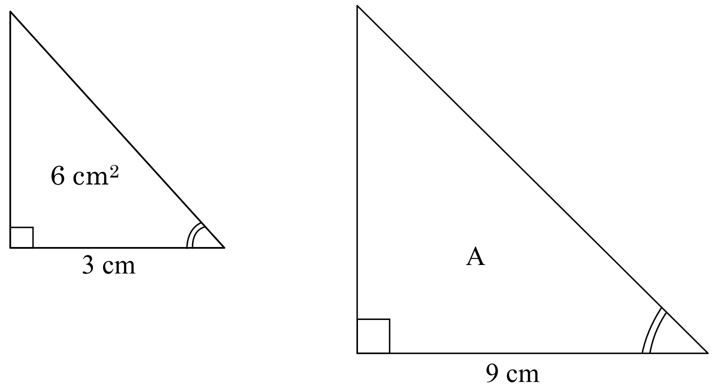
$= \frac{\text{Area of triangle XYZ}}{5\text{cm}^2} = \frac{4}{1}$ ------------- Substitute area of triangle ABC for 5cm^2

Area of triangle XYZ = 20cm^2 ---- After cross multiplication

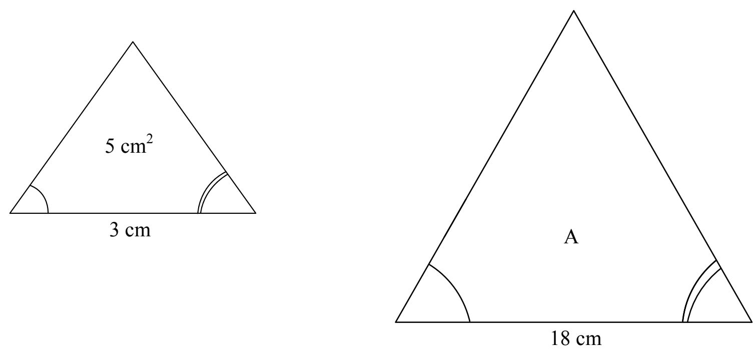
Exercise 12c

In this exercise, a number written inside a figure represents the area of the shape in cm^2. The number on the outside gives linear dimensions in cm. In question 1 to 10, find the unknown area A. In each case, the shapes are similar.

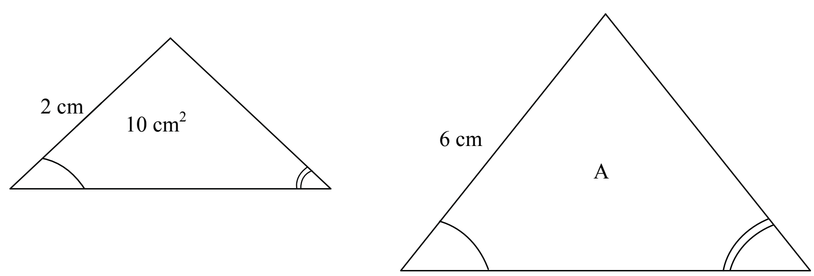
1.


6 cm²
3 cm
A
9 cm

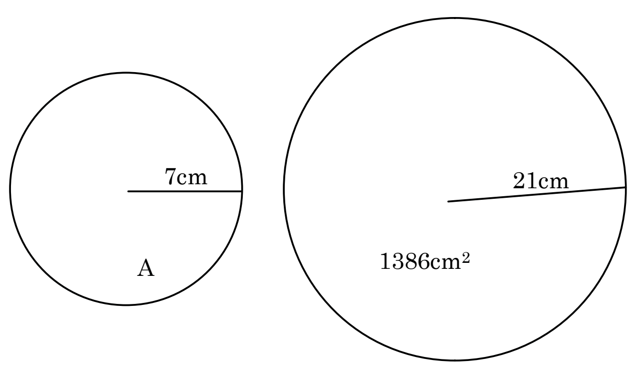
2.


5 cm²
3 cm
A
18 cm

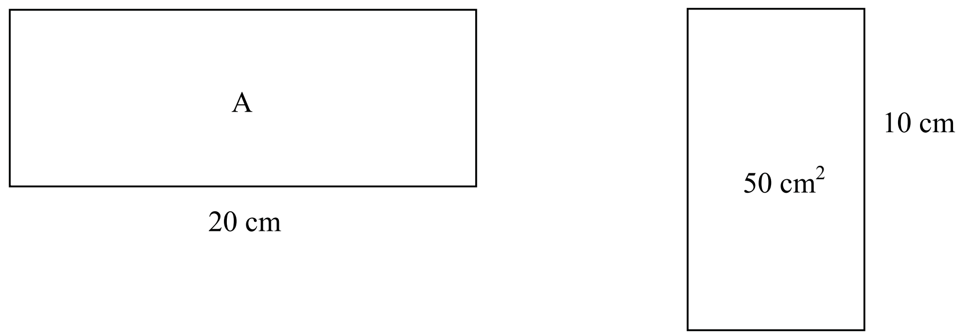
3.


2 cm
10 cm²
6 cm
A

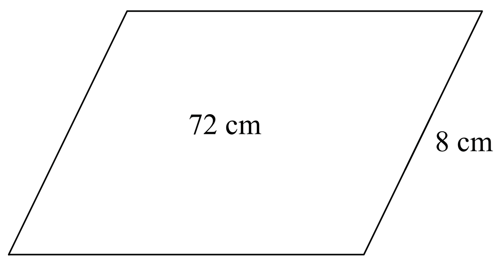
4.


7cm
A
21cm
1386cm²

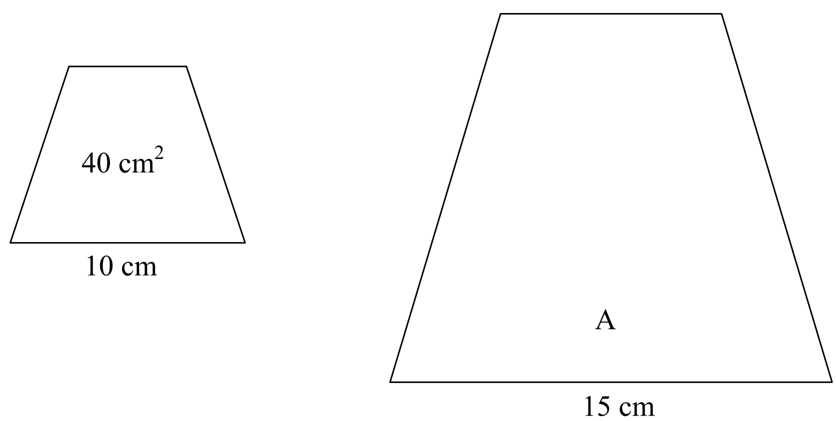
5


A
20 cm
50 cm^2
10 cm

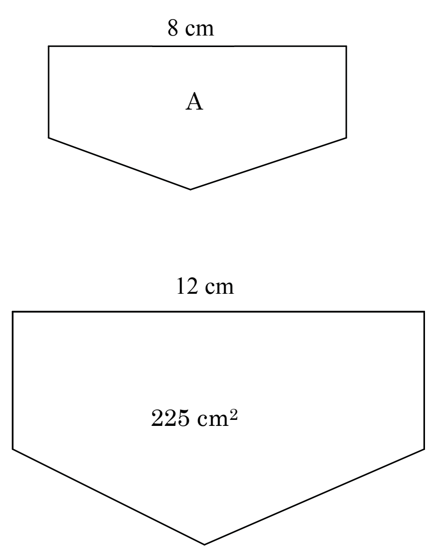
6


72 cm
8 cm
A
2 cm

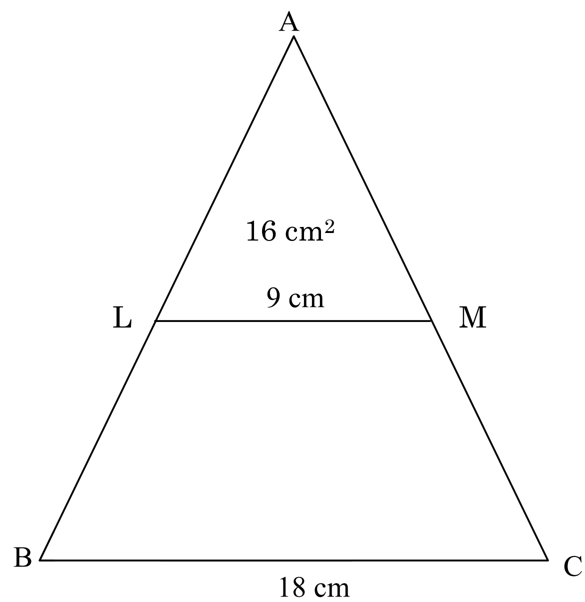
7


40 cm^2
10 cm
A
15 cm

8


8 cm
A
12 cm
225 cm^2

9



Find the area of LMCB.

Sometimes, you have the ratio of two similar shapes and you are required to find the ratio of the lengths. To do this, you must take the square root.

Example 4

Two similar shapes have area $10cm^2$ and $40cm^2$ respectively. If the length of the smaller shape is 6 cm, find the corresponding length in the larger shape.

Solution

Area factor $= \frac{40cm^2}{10cm^2} = \frac{4}{1}$

Scale factor

Hence $= \frac{4}{1} = \left(\frac{x}{6}\right)^2$ ------- Area factor, scale factor relationship

$= \frac{4}{1} = \frac{x^2}{36}$ --------- Remove the brackets

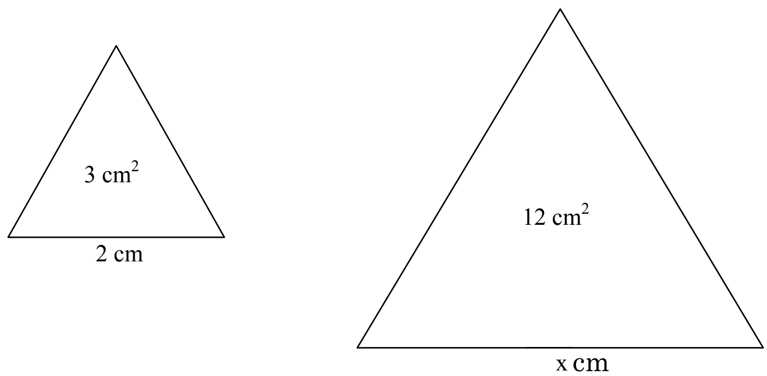
$x^2 = 144$ --------- Cross multiply

$x = 12cm$ ------- Take square root of both sides

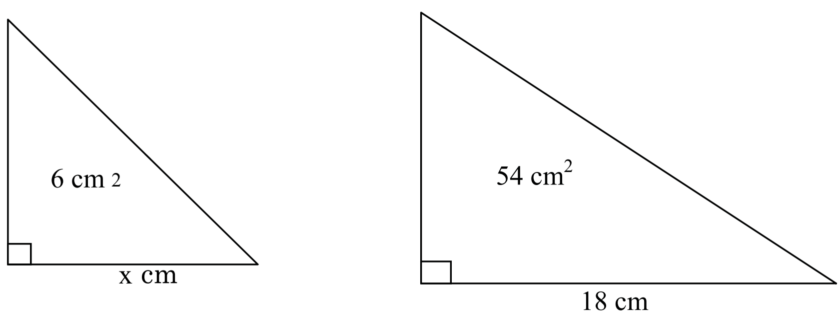
Exercise 12d

In questions 1– 6 find the lengths marked for each pair of similar shapes.

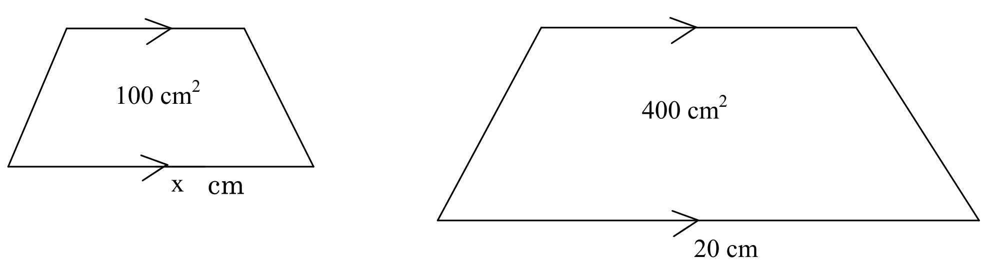
1.



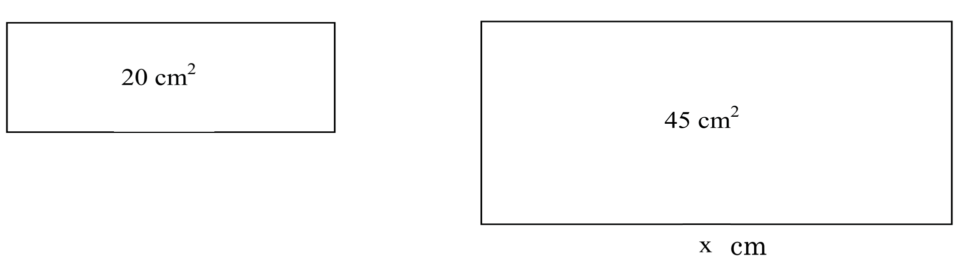
2.



3.



4.



5. P and Q are regular pentagons. Q is an enlargement of P

by a scale factor 3. If the area of pentagon Q is 180 cm^2, calculate the area of P.

6. The rectangular floor plan of a house measures 8 cm by 6 cm. If the scale of the plan is 1:50,

   Calculate:

   a) the dimension of the actual floor

   b) the area of the actual floor in m^2.

7. A garden has an area of 3025 m^2, and is represented on a plan by an area of 144 m^2. Find the actual length of a wall, which is represented on the plan by a line 8.4 m long.

2. **Ratio of volumes of similar solids**

   Having studied area factor and scale factor in the previous sections, you will now study the volume scale factor of similar figures and its application.

Activity 4:

Finding the volume scale factor of similar solids

1. Working in pairs, discuss the meaning of " volume scale factor".
2. Suggest how volume scale factor can be found.
3. Find the volume scale factor of the models and the drawn figures.

The volume scale factor of similar figures is the ratio of volumes of similar figures. It is found by the formula

$$\text{Volume factor} = \frac{\text{Volume of the image figure}}{\text{Volume of the object}}$$

You will later on learn in the coming sections the relationship between scale factor and volume scale factor.

Example 5

The volume of a container A is 300cm^3 and the volume of container B is 150cm^3. Given that container B is an enlargement of container A, find the volume scale factor of the two containers.

Solution

$$\text{Volume scale factor} = \frac{150cm^3}{300cm^3}$$

$$= \frac{1}{2}$$

Exercise 12e

1. Calculate the volume scale factor for each of the following pairs of Volumes. A is the volume of the object and B is the volume of the.

   a. A = 250cm^3; B = 100cm^3

   b. A = 240cm^3; B = 320cm^3

   c. A = 64m^3; B = 8m^3

   d. A = 576m^3; B = 1728m^3

   e. A = 244cm^3; B = 61cm^3

   f. A = 17.4m^3; B = 11.6m^3

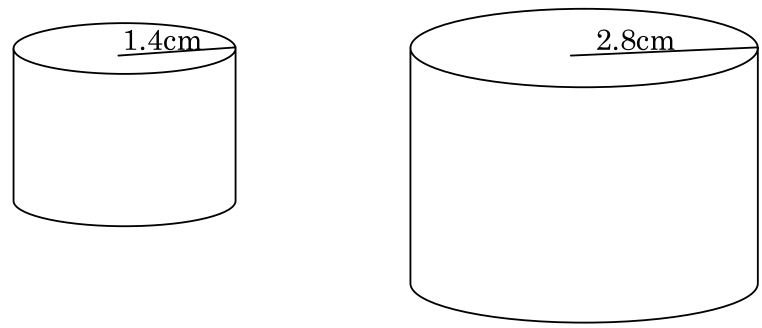
   g. A = 350m^3; B = 1050m^3

   h. A = 112cm^3; B = 28cm^3

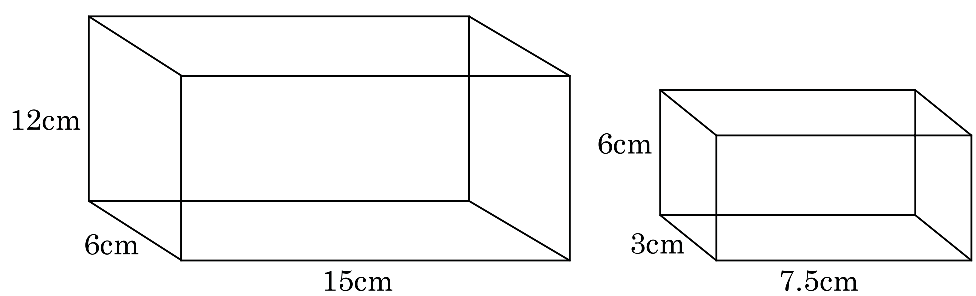
2. **Calculate the ratio of volumes of the following pairs of similar solids.**

   The figure to the right is the image

   a.



b.



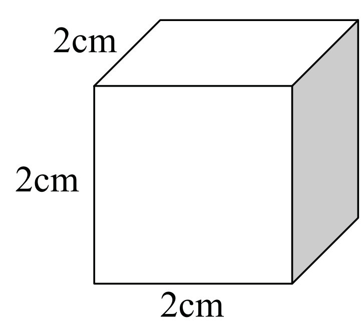
Activity 5:

Calculating of volumes of similar figures

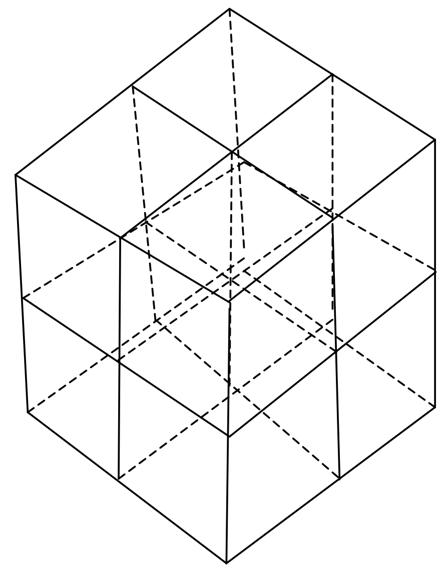
In the previous section you have learnt how to find and calculate the ratios of volumes of similar solids. You will now learn to calculate volumes of similar solids and to find lengths of sides of similar solids using the relationship between scale factor and volume factor. You will first find the relationship between volume scale factor and scale factor by doing the following activity:

Working in groups, perform the following activity

1. Find the volume of the cube below:



2. Using other similar cubes, build the solid below:



3. What is the length, width and height of the above solid?
4. Deduce the volume of the solid you have built above.
5. Build two more solids by adding one cube along the length, the width and the height each time to the preceding solid. Find the volumes of the resulting solids.
6. Now complete the table below:

	Length of a side	**Volume of the solid**
Solid 1	2 cm	8 cm^3
Solid 2	4 cm	6 4cm^3
Solid 3		
Solid 4		

Table 1

7. Now try dividing the lengths and volumes of any two cubes as follows:

   Solid 1 and solid 2 : Ratio of sides: $\frac{2cm}{4cm} = \frac{1}{2}$

   Ratio of volumes: = $\frac{(8\ cm^3)}{(64\ cm^3)} = \frac{1}{8}$

   Do the same with Solids 2 and 3 and Solids 3 and 4.

8. Present your work by filling the table below:

	Scale factor	Volumes factor
Solids 1 and 2		
Solids 2 and 3		
Solids 3 and 4		

Table 2

9. What is the relationship between the ratio of volume of similar figures and their sides?

   You might have discovered that the volume factor of similar solids is the cube of the scale factor or you may also say that the scale factor of similar figures is the cube root of the volumes factor s.

Example 6

Calculate the ratio of volumes of similar cylinders of diameter 6 cm and 9cm.

Solution

Let the volume of the smaller cylinder = x cm^3 and that of a larger

Cylinder = ycm^3

So x cm^3: ycm^3 = $6^3 : 9^3$

i.e $x : y = 216 : 729$

$= 8 : 27$

Example 7

Two spheres have volumes in the ratio 64:125. What is the ratio of their surface areas?

Solution:

The ratio of corresponding sides of the spheres = $\sqrt[3]{\frac{64}{125}}$

$= \frac{\sqrt[3]{64}}{\sqrt[3]{125}}$

The ratio of their surface areas $= \frac{4^2}{5^2}$

$= 16:25$

Example 8

Two similar cylinders have their radii in the ratio 1 : 2. If the smaller cylinder has a volume of $21.56m^3$, calculate the volume of the larger cylinder.

Solution

Since the volume factor of similar solids is the cube of scale factor of the similar solids, or since the scale factor of similar solids is the cube root of the ratio of the volume factor,

$$\frac{1}{2} = \sqrt[3]{\frac{21.56}{x}}$$

$$\frac{1^3}{2^3} = \frac{21.56}{x}$$ -------Cube both sides of the equation.

$$\frac{1}{8} = \frac{21.56}{x}$$

$x = 172.8m^3$ (After cross multiplication)

Exercise 12f

1. Calculate the ratio of volumes of similar rectangular tanks with lengths 12cm and 16cm.
2. Two similar solids have their surface areas as $160cm^2$ and $360cm^2$. Calculate the ratio of their volumes.
3. The volume of a tank 5m high is $343m^3$. Calculate the volume of a similar tank 10m high.
4. Two similar solids have surface areas in the ratio 9 : 25. If the volume of the smaller solid is $81cm^3$, what is the volume of the larger solid?
5. A cylindrical tin 1.5m high is filled with $50m^3$ of liquid and

the liquid rises to a level of 1m. Calculate how much more liquid the tank can hold so that it is completely full.

Unit summary

- In this unit you have learnt to calculate the ratios of areas and volumes of similar figures. You have also learnt to apply these ratios to calculate areas and sides of similar figures.

Unit review exercise

1. Two heaps of sand are both in a conical shape. The height of a smaller heap is 70cm and the height of a bigger heap is 56cm. If the smaller heap contain 128g of sand, calculate the mass of the bigger heap.
2. The ratio of the radii of two similar containers is 3:2. The larger container has a capacity of 4.05litres. Calculate the capacity of the smaller container.
3. The height of a triangle is 6cm and its area is $12cm^2$. Calculate base of a similar triangle whose area is $48cm^2$.
4. The areas of two similar parallelograms are in the ratio of 4:3. The height of the larger parallelogram is 8cm. Find the height of the smaller parallelogram.
5. In a scale drawing of a building, a wall 1.5m high is shown to be 10cm high. Calculate the actual width of the building which is shown by a width of 5cm on the drawing.
6. The volume of a balloon is Vcm^3 and its radius is 6cm. When the balloon is inflated further its volume increases by 40%. Find the new radius of he balloon.
7. A container is partially filled with $200cm^3$ of water and the water rises to a height of 50mm. calculate the volume of water that must be added to increase the depth by 20mm.
8. The fuel tank of a truck 1m long is in a rectangular form. When completely full the tank can hold 405litres of fuel. Calculate the length of similar tank for a small lorry that has a capacity of   of the truck.

Glossary

Scale factor: The ratio of the length of a side of an image to the corresponding side of a similar object.

Area factor: The ratio of the area of the image to the area of the object.

Volume factor: Ratio of volumes of similar figures.

References

S. Hau and F. Saiti (2002), *Strides in Mathematics Book 3*, Longman , Malawi

Larson etal (1998), *Heath Algebra an Integrated Approach*, Heath and company, Canada.

G. D. Buckwell and B.N Githua, *Gold Medal Mathematics,* Macmillan, London.

Unit 13

COORDINATE GEOMETRY

You are familiar with locating a point in a plane and using coordinates to describe the position of a point in a plane. These coordinates are called the Cartesian coordinates of the point. (The name comes from the French mathematician Rene Descartes (1596 –1650). The system of using a pair of coordinates to describe the position of a point in a plane is called Coordinate Geometry or Cartesian Geometry.

The coordinates measure the displacement (+ or –) of the point from two perpendicular axes, the y-axis (Oy) and the x – axis (Ox) where O is the origin.

In this unit, you are going to learn how to solve problems involving straight lines in an x – y plane. You will learn how to calculate the distance between two points on a straight line, how to find the equation of a straight line, and finally you will learn to describe the condition for two lines to be parallel and how to find the midpoint of a line segment.

Coordinate Geometry is applied in many situations. For example by recognising the relationship between two variables, you can learn more about real life situations and make reasonable predictions about future trends such as those concerning populations and business.

Calculating the distance between two points on a straight line

We can represent the position of a point with respect to the y axis and x axis by two numbers called **coordinates.** It is for this reason that the xy-plane is also called the **coordinate plane.** These numbers are enclosed in brackets and the first number in the brackets represents the position of a point with respect to the *x*-axis and the second number in the brackets represents the position of a point with respect to the *y*- axis.

If we want to find the distance between two points on the coordinate plane, then a simple right angled triangle can be constructed with sides parallel to the axes.

Activity 1:

Calculating the distance between two points by using Pythagoras theorem

Work in groups**:**

1. Using a scale of 2cm to represent 1 unit on both axes draw the straight line whose equation is $y = x + 2$ for the values of x =2 , 4 , 6
2. Using a ruler, measure the length of the line segment from x=2 to x = 6.
3. Record your findings.

   Now on the same graph paper draw straight lines y = 4 and x = 6. Let the two straight lines intersect at P.
4. What type of triangle is formed by the line $y = x + 2$ and the two lines you have just drawn?
5. Measure the lengths of the two legs of the triangle. Can you find a way of finding these lengths by using the coordinates on the two vertices of the triangle which are not right angles?
6. Using Pythagoras Theorem, find the length of the hypotenuse of the triangle and compare your result with the result you found in step 3.
7. Now generalize your results to finding the length of a straight line from A(x_1 , y_1) to B(x_2,y_2) where (x_1, y_1) and (x_2, y_2) are two points through which the straight line passes.

Example 1:

Distance between two points

Find the length of the line joining A (1, 2) and B (3, 4)

AB $= \sqrt{(3-1)^2 + (4-2)^2}$

$$= \sqrt{2^2 + 2^2}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2}$$

= 2.83 (to 3 significant figures)

Exercise 13a

Find, correcting your answers to 2 decimal places where necessary, the length of the line joining

1. A (1, 2) and B (4, 6)
2. A (4, 2) and B (2, 5)
3. A (3, 4) and B (0, 0)
4. A (−1, −3) and B (2, 1)
5. A (−4, −5) and B (1, 7)
6. A (0, −3) and B (4, 0)
7. A (−1, −3) and B (−2, −5)
8. A (−2, 1) and B (4, 2)
9. A (−5, −2) and B (0, −3)
10. A (−5, 0) and B (−7, −4)

The equation of a straight line

The equation of a straight line is a relationship between two variables x and y. You drew equations of straight lines in Form 2 when you were solving simultaneous equations graphically. In this section you shall continue to look at these lines in the xy-plane.

Activity 2:

Writing the equation of a straight line in the form $y = mx + c$

Generally, there are two forms of the equation of a straight line. These are $ax + bx = c$ and $y = mx + c$. In the first form, a, b and c are numbers. In the second form m is the gradient of the straight line and c is the y intercept.

In groups,

1. Write your own examples of equations on each of the above forms.
2. Present your answers on the chalkboard.
3. Together with your teacher, group the answers you have presented into the two equation forms.
4. Group the following equations into the two forms in 1 above:
   a. $3y = 2x + 1$
   b. $2y - 3x + 8 = 0$
   c. $y = 2x$
   d. $4x + 3y = 6$
   e. $2x - 1 = y$

All the equations that have y on one side of the equation is said to be in slope intercept form. Note that in this form the coefficient of y must be 1.You should also have seen that the basic idea in writing the equation of a straight line in slope intercept form is to make y the subject of the equation.

Example 2

Write the equation $y + 2x = 3$ in the form $y = mx + c$

Solution

$y = -2x + 3$ ------------- moving 2x from the left side to the right side of the equation.

Example 3:

Equation of a line

Write the equation $3y - 5x + 9 = 0$ in slope intercept form.

Solution:

Move -5x and +9 to the right side of the equation:

$$3y = 5x - 9$$

Divide by 3 throughout:

$$y = \quad x - 3$$

Exercise 13b

Write the following equations of straight lines in slope intercept from:

(1) $2y + 2x = 5$

(2) $3y = 6x - 13$

(3) $5y + 3x - 22 = 0$

(4) $2x - y = 9$

(5) $x + 2y + 6 = 0$

(6) $x + y = 6$

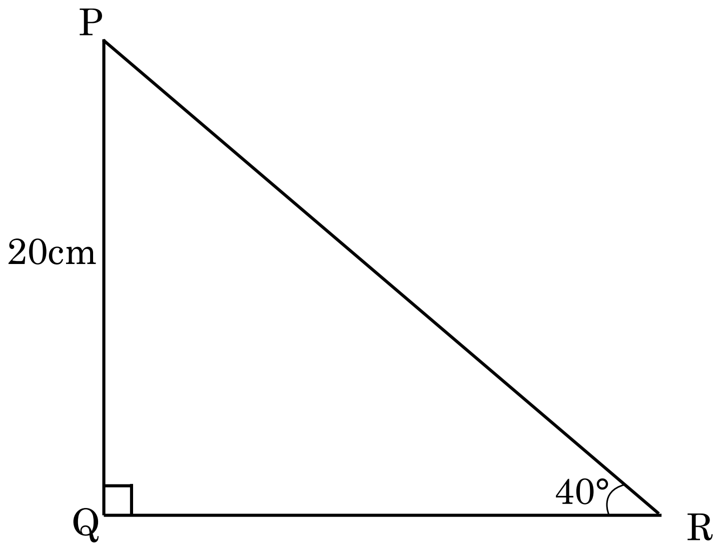
(7) $8x + 3y = 48$

(8) $2y - x + 1 = 0$

Activity 3:

Relating gradient to the tangent of an angle

Consider the triangle below:



1. Express tan 40^0 as a ratio of the two legs of triangle PQR.
2. Find the length of QR; give the answer correct to one decimal place.
3. Find the ratio $\frac{PQ}{QR}$ to 3 decimal places.
4. Now find tan 40^0 using a calculator.
5. Compare the results in 3 and 4. What conclusion do you draw from the results?

You might have noted that the gradient of a straight line PR is equal to the tangent of 40^0 or the tangent of its complement i.e. 50^0.

Example 4

Equation of a straight line

A straight line makes an angle θ with the x - axis. The line cuts the x axis at x = 8. If tan θ = ¼, find the equation of the straight line in slope intercept form.

Solution

Substitute m for ¼ into y = mx + c

$$y = \frac{1}{4}x + c$$

When y = 0, x = 8 so $\quad 0 = \frac{1}{4}(8) + c$

$$0 = 2 + c$$

$$c = -2$$

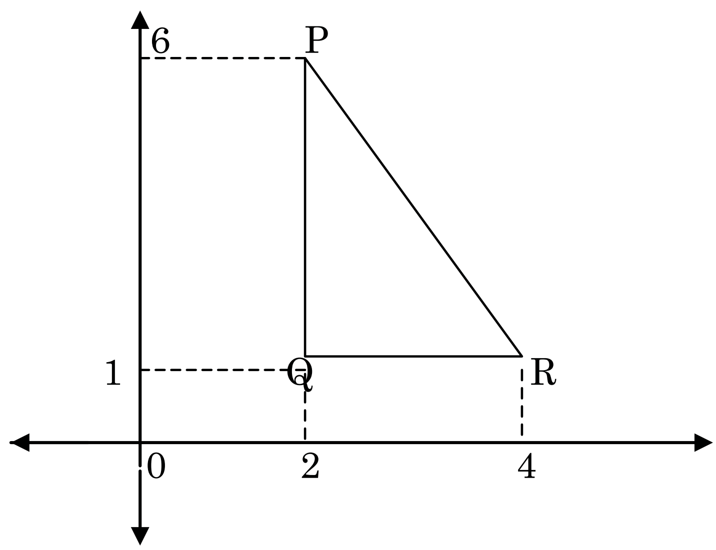
The equation is $y = \frac{1}{4}x - 2$

Activity 4:

Using the relationship between gradient of a line and tangent of an angle to find gradient of a straight line

In **Activity 3,** we have seen that the gradient of a line and the tangent of an angle (other than 90^0) in a right angled triangle are numerically equal. As long as the lengths of the legs of the triangle are known, we can find the gradient of the straight line.

Consider the triangle below:



1. Discuss how you can find the lengths of PQ and QR without having to use distance formula.

2. Use your findings to find the lengths of PQ and QR.

3. Hence find the gradient of line PR by dividing PQ by QR. leave the gradient as a fraction with a numerator and denominator only.

4. Write the coordinates of P and R and discuss how you can use the coordinates to find the result in 3 above.

5. Generalise the result in 4 above with $P(x_1,y_1)$ and $R(x_2,y_2)$.

The gradient of a straight line is found by dividing the difference between the two y-values by the difference between the two x-values. If the gradient is 0 the line is parallel to the x axis. The straight line parallel to the y-axis has an undefined gradient.

Example 5:

Gradient of a straight line

Find the gradient of a straight line passing through (3,6) and (8,3).

Solution

Gradient $= \dfrac{3-6}{8-3}$

$= -\dfrac{3}{5}$

Exercise 13c

Find the gradient of a straight line passing through each of the following pairs of points:

1. (2,3) and (6,9)
2. (4, 11) and (2, 5)
3. (2.5,3.5) and (8.5 , 9.5)
4. (1, 2) and (6,5)
5. (4, 3) and (2, 8)
6. (4.5 , 5) and (2 , 10)
7. (p , 2p) and (3p, 5p)
8. (2x,3y) and (x , -2y)

Activity 5:

Formulating the equation of a straight line with a given gradient and through a given point

We looked at this method when we solved example 3 on page 272. Carefully study this example again and rephrase the question to contain the word "tangent" and the phrase "passing through".

Compare your rephrased question with those from other groups.

You can use the slope intercept form or the point intercept form of a line to formulate the equation of a straight line with a given gradient and through a given point. This is illustrated in example 6 below:

Example 6:

Equation of a straight line

Find the equation of a straight line passing through (3, 5) and with gradient .

Solution

Using slope intercept form,

Substitute y for 5, x for 3 and m for $^2/_3$ into y = mx + c:

$5 = (3) + c$

$\therefore\ 5 = 2 + c$

$\therefore\ 5 - 2 = c$

$\therefore\ c = 3$

$\therefore$ The equation is $y = {}^2/_3 x + 3$

OR

Using point intercept form,

$y - y_1 = m(x-x_1)$ where (x_1, y_1) is the given point and m is the given gradient

$x_1=3$ and $y_1=5$

$$y - 5 = {}^{2}/_{3}(x - 3)$$

$$\therefore\ y - 5 = {}^{2}/_{3}x - 2$$

$$\therefore\ y = {}^{2}/_{3}x - 2 + 5$$

$$\therefore\ y = {}^{2}/_{3}x + 3$$

Exercise 13d

Find the equation, in slope intercept form, of a straight line passing through

1. (1,3) with gradient -3.
2. (-2,-5) with gradient
3. (-3, 1) with gradient ½.
4. (7, 5) with gradient -2.
5. (1,-2) with gradient
6. (0,0) with gradient 1
7. (2,0) with gradient 3
8. (a, 2a) with gradient 2

Activity 6:

Finding the equation of a straight line passing through two given points

1. On a graph paper, draw a straight line through A(-2 , 3) and B (4, -1)

2. Using your knowledge for finding gradients of straight lines, find the gradient of line segment AB.

3. Now choose any point P on the straight line .Call this point (x, y). Using this point and the coordinates of point A, find the gradient of line segment AP.

4. Do you think the gradients in steps 2 and 3 are different? Come up with a relationship for x and y using these results.

5. Try to use the point P(x, y) and point B (4, -1) to find the relationship between x and y. Compare your result to the result in step4 and comment on the result.

The relationship you found above is called the **equation of the line through A and B**. As you have seen, to find the equation you only need to find the gradient of the line (see Activity 4) and any one of the two points through which the line passes. You can then use either the slope intercept form or the point intercept form to find the equation of the line.

Example 7:

Equation of a straight line

Find the equation of a straight line passing through (1, 3) and (4, 9)

Solution

First find the gradient of the line $m = \frac{9-3}{4-1} = \frac{6}{3} = 2$

$\therefore$ The equation is of the form y = 2x + c.

To find the value of c substitute, the values of x and y from any one of the two points.

Using the first point we have $3 = 2 \times 1 + c$

$\therefore$ c = 1

$\therefore$ The equation is y = 2x + 1

Exercise 13e

Find the equation of the line passing through the points A and B in the form y = mx + c when

a. A = (2, 4) and B = (3, 8)

b. A = (0, 2) and B = (3, 5)

c. A = (−2, 0) and B = (2, 8)

d. A = (3, −1) and B = (7, 3)

e. A = (−4, −1) and B = (−3, −9)

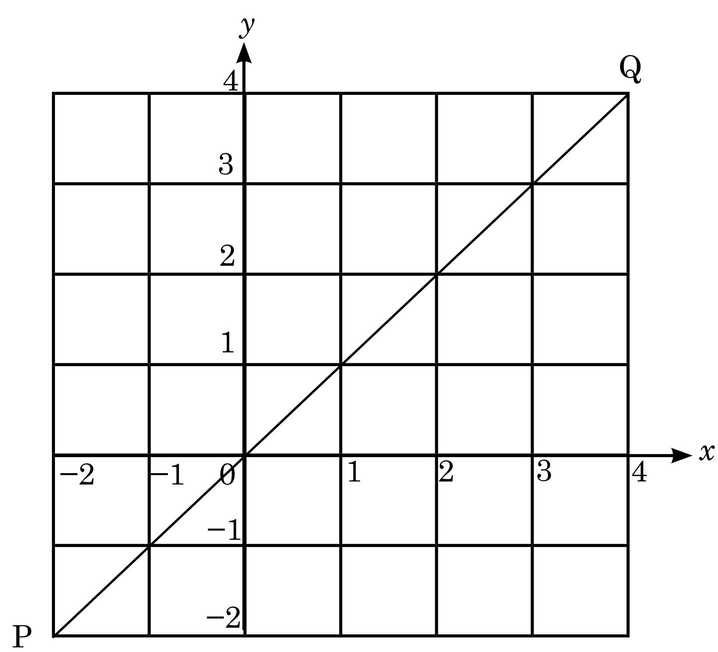
f. A = (0,0) and B = (2,3)

g. A = (3,5) and B = (1,1)

Activity 7:

Finding the equation of a straight line from the graph

Consider the graph below:

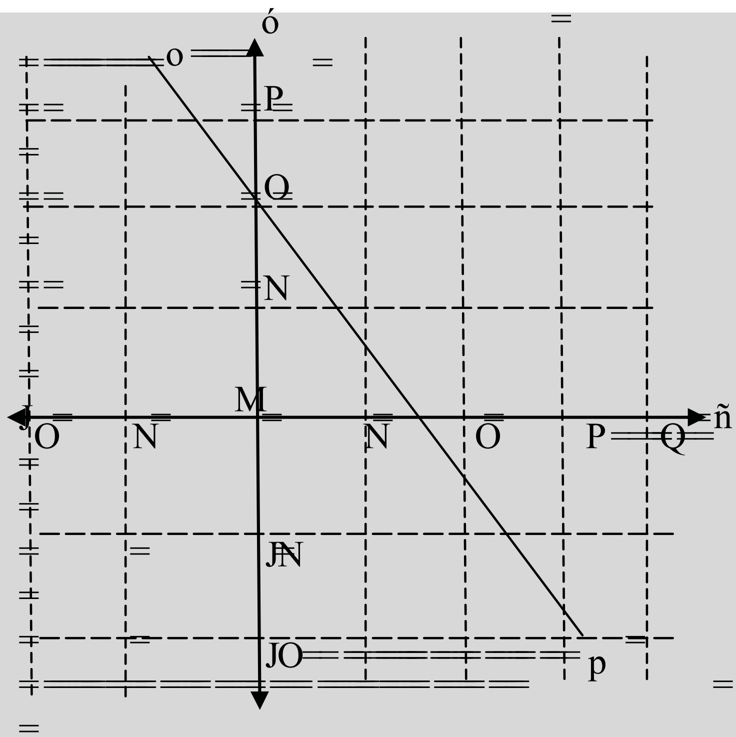


Discuss how you can find the equation of line PQ in the graph above.

To find the equation of a straight line from the graph, you need to obtain any two "**smart points**" on the line and use them to find the gradient of the line. Then the slope intercept or the point intercept form can be used to find the equation of the straight line.

Example 8:

Finding equation of a line from a graph



Find the equation of the straight line RS above.

Solution

Two smart points are (0, 2) and (3,-2)

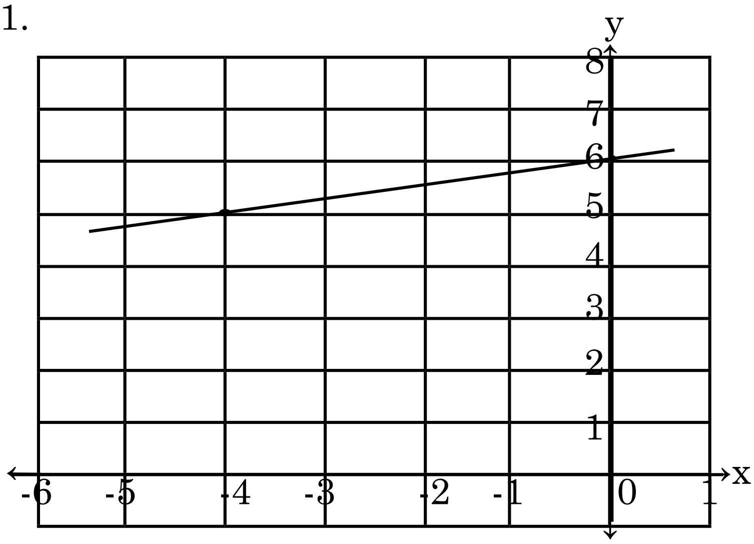
Gradient $= \dfrac{2 - -2}{0 - 3}$

$= -\frac{4}{3}$

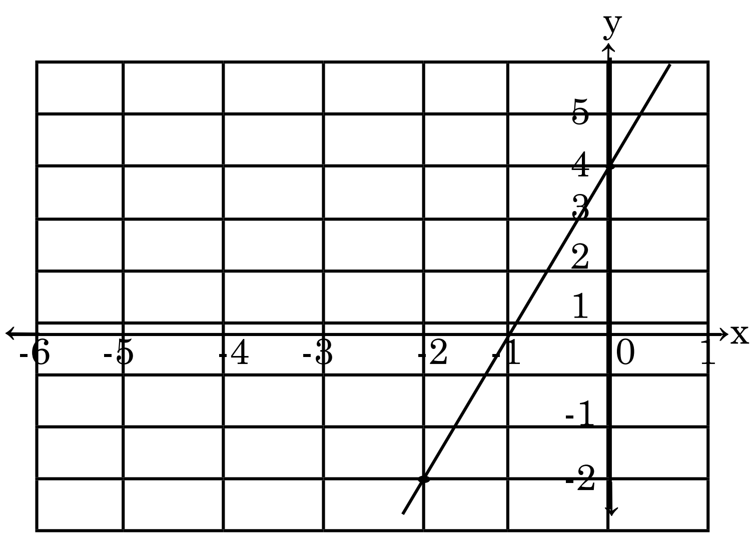
The equation is y - 2 = $-\frac{4}{3}$ (x – 0)

y = $-\frac{4}{3}$ x + 2

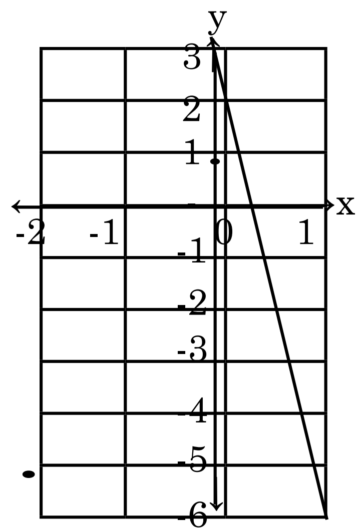
1.


y
x
8 7 6 5 4 3 2 1
-6 -5 -4 -3 -2 -1 0 1

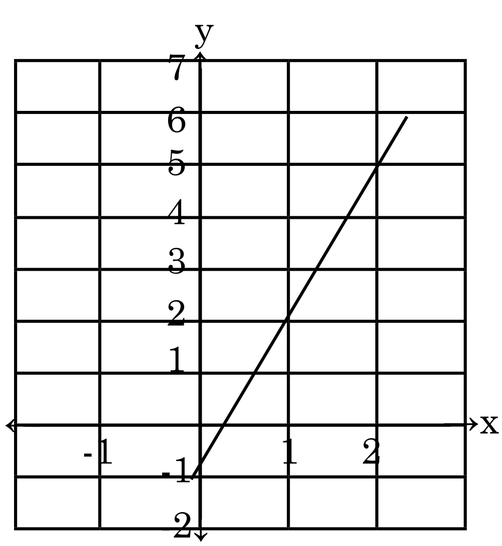
2.


y
x
5 4 3 2 1 -1 -2
-6 -5 -4 -3 -2 -1 0 1

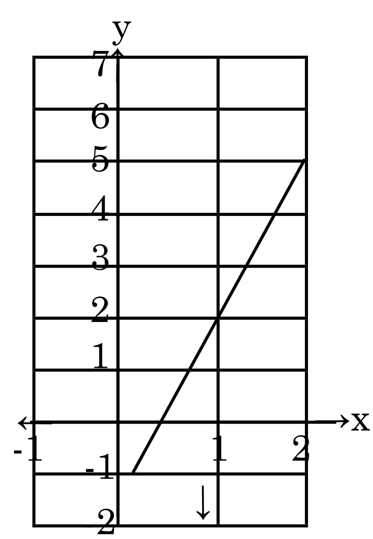
3.


y
x
3 2 1 -1 -2 -3 -4 -5 -6
-2 -1 0 1

4.


y
x
7 6 5 4 3 2 1 -1 2
-1 1 2

5.


y
x
7 6 5 4 3 2 1 -1 2
-1 1 2

Exercise 13f

Find the equations of each of the following straight lines:

Parallel lines

In this section we shall look at parallel lines. You have used the idea of parallel lines in many situations e.g. when you were learning about properties of parallelograms in your JCE Mathematics or when you are using alternate angle or corresponding angle properties. But what makes lines parallel? Let us investigate this question by looking at Activity 8.

Activity 8:

Gradients of parallel lines

In your JCE you learnt about drawing parallel lines.

1. On a graph paper, draw two straight parallel lines.
2. Choose two smart points on each line and find the gradient of each line.
3. Compare the gradients of the two lines. What do you notice?

In this activity you must have seen that parallel lines have the same gradient.

Example 9:

Showing that lines are parallel

Show that the line passing through (1,3) and (2, 7) is parallel to the line passing through (4,3) and (3,-1).

Solution

Gradient of the first line $= \frac{7-3}{2-1}$

$= 4$

Gradient of the second line $= \frac{3-(-1)}{3-4}$

$= \frac{4}{1}$

$= 4$

As the gradients of the two lines are the same, the two lines are parallel.

Example 10:

Equation of line parallel to another line

Find the equation of a line through the point (1, 2) which is parallel to the line $2x - 3y = 4$.

Solution

First, put the given equation of the line into standard form i.e.

$$y = \frac{2}{3}x + \frac{4}{3}$$

$\therefore$ The line has gradient $\frac{4}{3}$

This is also the gradient of the line whose equation we must find because the lines are parallel.

$\therefore$ Using slope intercept form, $y = \frac{4}{3}x + C$

Substitute $x = 1$, $y = 2$ into this form,

$\therefore 2 = \frac{4}{3} \times 1 + C \quad \therefore C = \frac{4}{3}$

$\therefore$ The equation is $y = \frac{2}{3}x + \frac{4}{3}$

Exercise 13g

1. Find the equation of the line which is

   a) parallel to $x - y = 1$ and passes through (2, 3)

   b) parallel to $2x + y = 3$ and passes through (3, 0)

   c) parallel to $y + 5x = 2$ and passes through (1, 3)

   d) parallel to $2x - y = 4$ and passes through (0, 3)

   e) parallel to $x - 3y = 1$ and passes through (−2, −1)

2 A straight line passes through A(5,7) and B(0,-1). Another straight line passes through Q(-3,-5) and R(2,3). Show that AB is parallel to QR.

3 Show that lines with equations $4x - 3y = 7$ and $9y - 12x - 1 = 0$ are parallel.

4 Which of the following lines are parallel?

   a. From (-1,3) to (4,5)

   b. From (2, -1) to (7,1)

   c. From (0,-1) to (2,2)

Midpoint of a line segment

The midpoint of a line segment is a point which lies halfway between the end points of a line segment. It is possible to find the midpoint of a line if we know the coordinates of the end points of the line. We first derive the formula for finding the midpoint of a line segment.

Activity 9:

Deriving the formula for the mid-point of a line segment

1. On a graph paper and using 1 cm to represent 1 unit on the horizontal axis, plot the points A (2, 6) and B (6, 3).

2. Use the distance formula to find the length of line AB and then get half of the length. Note that your answer is in cm.
3. From point A and along line AB measure the number of cm you found in step 2 above and write down the coordinates of the point you find.
4. What is the connection between the coordinates in 3 and the coordinates in 1?
5. Come up with a generalization for finding the midpoint of a line segment using (x_1, y_1) and (x_2, y_2) as two points on the straight line.

You have seen from this activity that the midpoint of a line segment is found by finding half of each of the sums of the x coordinates and the y coordinates i.e.

Midpoint $= \frac{1}{2}(x_1 + x_{2,}\ y_1 + y_2)$

Example 11:

Mid-point of a line

Find the midpoint of a line from (2, 6) to (4, 8)

Solution

The midpoint is $(\frac{2+8}{2}, \frac{6+8}{2}) = (5, 7)$

Example 12

The midpoint of line PQ is (4, 9). The coordinates of P are (a, 5) and of Q are (1, 7). Find the value of a.

Solution

$(\frac{4+a}{2}, \frac{9+5}{2}) = (1, 7)$

So $\frac{4+a}{2} = 1$

$4 + a = 2$ ------------ multiplying both sides by 2

$= -2$

Exercise 13h

1. Find the midpoints of the lines from:

   a. (3, 4) to (1, -5)

   b. (-1, -4) to (-8, -10)

   c. (0, 3) to (2, -5)

   d. (p, 2p) to (3p, -4p)

2. The midpoint of BC is (2, 4). If the coordinates of B are (1,8) find the Coordinates of C.

3. The coordinates of the point of intersection of the diagonals of a parallelogram ABCD are (3.5, 2.5). A is the point (2, 5) and B is a point (8, 8). Find the coordinates of C and D.

4. Find the distance between the midpoint of a line from (2, 2) to (4, 5) and a point (7, 9)

5. U(2, 4) , V(5, 4) , W (1, 4) and X(0, 6) are four points on a Cartesian plane. Find the gradient of the line joining the mid points of lines UV and WX.

6. The equation of a straight line is $2x + 3y = 7$. Find the equation of a line passing through the midpoint of A(8, 6) and B(2, 4) parallel to the line $2x + 3y = 7$.

Solving real life problems using coordinate geometry

The knowledge of coordinate geometry is useful many areas. The following few examples show how useful coordinate geometry can be in solving real life problems:

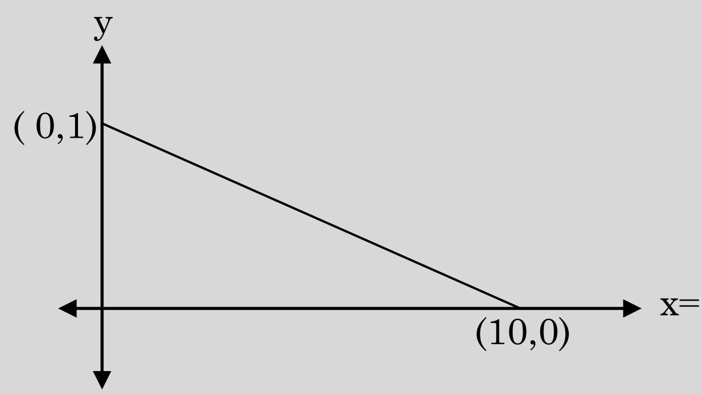
Example 13:

Real life problem using coordinate geometry

A contractor is to build an office for a company. He however has to build the office in such a way as to suit the physically challenged. Given that the recommended maximum slope for a wheelchair ramp is and the contractor is to build a ramp that rose to a height of 2m, calculate the minimum horizontal length of the ramp.

Solution

The fraction $^{1}/_{10}$ means y increases by 1m for every 10m increase in x. You can model an equation as follows:



The equation of the straight line is $y - 1 = (x - 0)$

$y - 1 = x$

$y = x + 1$

To find the minimum length of the ramp, put $y = 2$

$2 = x + 1$

$2 - 1 = x$

$1 = x$

$x = 10$

This means the minimum length of the ramp should be 10m.

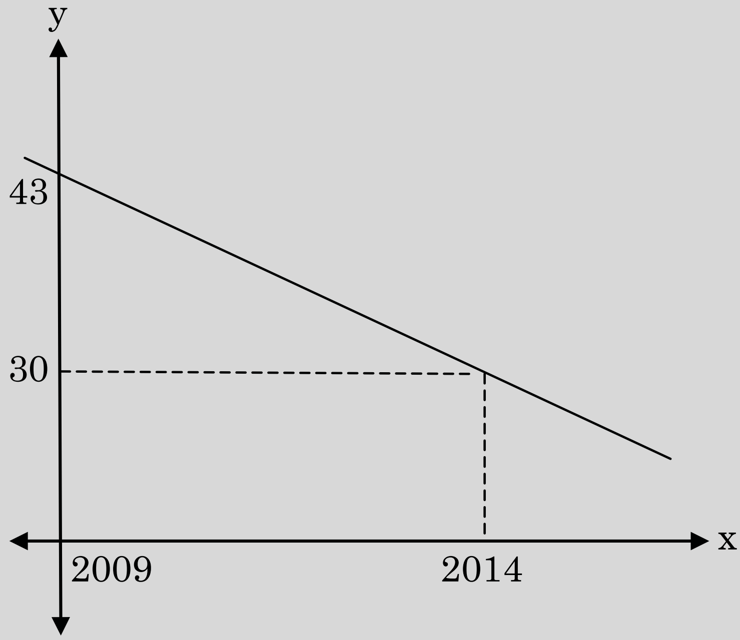
Example 14:

Real life problems using coordinate geometry

During the 2009 general elections in Malawi, 43 women were elected into parliament. In 2014, the number of elected women dropped to 30.Write a linear model for the number y, of elected women who were parliamentarians between 2009 and 2014.

Solution

Let x = 0 represent 2009. You can draw the following xy-plane



Gradient of the straight line $= \frac{43-30}{2009-2014}$

$= \frac{13}{-5}$

Y intercept is 43,so the equation is $y = -\frac{13}{5}x + 43$

Unit summary

- In this chapter you have learnt to find the distance between two points on a straight line, how to find the equation and gradient of a straight line and how to find the midpoint of a line segment. You have also learnt to write the equation of a straight line in slope intercept form and to find the gradients of parallel lines.

Glossary

Smart point: A point on the xy-plane where the straight line graph passes through a vertex of a grid box.

Unit review exercise

1. Find the length of a line joining the following points:

(a) A(3,8) and (3,3)

(b) P(-3,4) and Q (0, 6)

2. Write the following equations of straight lines in slope intercept form:

(a) $3x - 2y = 13$

(b) $4x + 2y - 7 = 0$

3. Find the equation of a straight line

(a) passing through (4, 5) and (1, 2).

(b) passing through (-2,4) with gradient

4. The vertices of a triangle are A(-1, 8) , B(-2 , 0) and C(0, 0). Show that triangle ABC is isosceles.

6. The line $y = mx + 4$ passes through the point (1 , 2). Calculate

a. the value of m

b. the angle that this line makes with the y – axis.

c. the coordinates of the point where this line cuts the x – axis.

Unit 14

VARIATIONS

In your JCE Mathematics you learned about direct and inverse proportion. You learnt that direct proportion is the relationship between related quantities such that they both increase or decrease in the same ratio. You also learnt that inverse proportion is the relationship between quantities such that as one quantity increases (decreases) the other decreases (increases) in the same but opposite ratio. In this book, these proportions will be referred to as direct variation and inverse variations. You will learn to solve problems involving direct and inverse variations. You will also solve problems involving joint and partial variations.

The knowledge of variation is used in many real life situations. For example, among others, it can be used to find the number of people required to do certain work, it can be used to find the speed at which a car must travel so as to cover a certain distance in a given time and it can be used in physics to find the force required to lift an object and in many other situations.

Direct variation

Two related quantities are said to be in direct variation if they have the same rate regardless of the variables. For example if the ratio of y: x = k then you can write y = kx or if you multiply both sides of this equation by x then you have y = kx. The number k is called the constant of variation.

Activity 1:

Modeling direct variation

In pairs, discuss how you can model direct variation from the following statement:

The speed, s (in km per hour), of a car varies directly with time, t (in hours). After the car has travelled for 2 hours its speed is 40km/h. write a model that gives speed of the car in terms of t.

Compare your work with the other pairs.

Let your teacher check your work.

Example 1:

Modeling direct variation

The exchange rate of Malawian Kwacha (K), to the United States Dollar ($) is 1: 475.

If the exchange rate is in direct variation, write a model that gives K in terms of ($). **Solution**

K = c ($), c is constant of variation

1 = 475c .... Substitute K for 1 and ($) for 475

$c = \frac{1}{475}$ .... Divide both sides by 475

$K = \frac{1}{475} \cdot (\$)$

Exercise 14a

1. If y varies directly as x, write a model that expresses y in terms of x when y = 2 and x = 4.
2. The pressure (p) of an object submerged under water is directly proportional to the depth (h). When p = 460, h = 5. Model an expression that gives p in terms of h.
3. The number of Calories, c, a person can burn and the time, t(in minutes), the person spends on doing activity vary directly. 150kg people can burn 75 Calories by sitting in a class for 50 minutes. Write a linear model that relates c and t.
4. The perimeter, p, of a square is proportional to the length, l. When l = 4, p =16. Create a linear model that give p in terms of l.
5. The number of bags (n) is directly proportional to the cost (C in Kwacha) per bag. If 200 bags cost K40000, model a linear expression which expresses n in terms of C.

Activity 2:

Deriving the general equation involving direct variation

In section 1.1 you modeled expressions representing direct variation. You used the general form $y = kx$. In this section you shall learn how this form can be derived. In groups go through the following activity:

1. Suppose a 50kg bag of maize costs K5000. Construct a table of number of bags (n) against total cost(c) for up to 5bags.
2. What happens to the total cost as the number of bags increases?
3. Pick any two numbers of bags and find their ratio in simplest form and then pick their corresponding costs and find their ratio in simplest form. How do the two pairs of ratios compare?
4. Try other pairs in your table in a similar manner and comment on the results.
5. Now try dividing corresponding number of Kwachas by the number of bags for all the entries giving the ratios in simplest form. Comment on your findings.

You have seen that as the number of bags increases, the total cost increases in the same ratio. You can then say that the number of bags (n) is directly proportional to the total cost(c). The symbol for variation is $\propto$. So you can write $c \propto n$ to mean "c is directly proportional to n" or "c varies directly as n" or "c varies as n."

Additionally, you have seen that the ratio $^c/_n$ is the same for all the corresponding entries or $^c/_n$ = constant. Hence if $n \propto c$ then $^c/_n = k$ where k is a Constant called the constant of variation.

Challenge:

Find your own example of a direct variation where the quantities decrease in the same ratio.

Example 2:

Direct variation

Given that y varies directly as x and that y =3 when x = 6 find the value of y when x = 10.

Solution

Since y varies directly as x then = k or y = xk where k is a constant

$\therefore$ 3 = 6k------------------substitute y for 3 and x for 6

$\therefore$ k = ½ -------------------divide by 6 both sides.

$\therefore$ the law of variation is y = ½ x

$\therefore$ When x = 10, y = ½ (10) i.e. y = 5

Example 3:

Derect variation

The mass (m) of each piece of log of wood that can be cut from the same log of wood is directly proportional to the length (l) of the piece. A piece 30cm long has a mass of 2kg. What will be the mass of a 45cm long log of wood?

Solution

If m $\propto$ l then m = kl

$\therefore$ 2 = 30l

$\therefore l = \frac{1}{45}$

The law of variation is $m = \frac{1}{15}l$

when l = 45, $m = \frac{1}{15}(45)$

m = 3kg

Exercise 14b

1. If d $\propto$ t and d = 80 when t = 5, find d when t = 3.
2. p $\propto$ q and p = 4.5 when q = 12. Find p when q = 16.
3. The time swing, t seconds, of a pendulum clock varies as the square root of its length, l, cm. If t = 1 when l = 25 calculate l when t = 1.5 seconds.

4. The cost (c) per metre of a cloth is proportional to the number of metres (n) bought. If 2metres cost K750.00,

(a) Find the relationship between c and n

(b) Hence find n when c = K3750.00

5. The cost, c, of painting a wall is directly proportional to the area, A, to Be painted. An area of $300m^2$ costs K3500. Find the cost of painting $210m^2$

6. In a given period, the cost, (C), of paying for accommodation at a hotel varies as the number (n) of participants present. The cost of paying for 60 participants is K18000. Find the cost for paying for the accommodation of 100 participants.

Activity 3:

Presenting direct variation graphically

The models you modeled in section 1.1 can be presented graphically. You first develop a table of values, plot them on a graph paper and then draw the graph of the model. In groups, perform the following activity:

1. Model a linear expression for a quantity y which varies directly as x given that x = 20 when y = 10.
2. Using the model, find the corresponding values of y when x = 0, 2, 4, 6, 8, 10.
3. Present your values in a table form.
4. Using the values in the table, draw the graph for the variation.
5. What type of graph is produced?

Graphs of direct variations are straight line graphs. The slope is represented by the constant of variation.

Example 4:

Direct variation

P α Q and P = 20 when Q = 100, find the relationship between P and Q and sketch the graph of the relation $P \propto Q$.

Solution

Since PQ, P = kQ, where k is a constant.

So, 20 = 100k ---- Substitute P for 20 and Q for 100

$$k = \frac{20}{100}$$

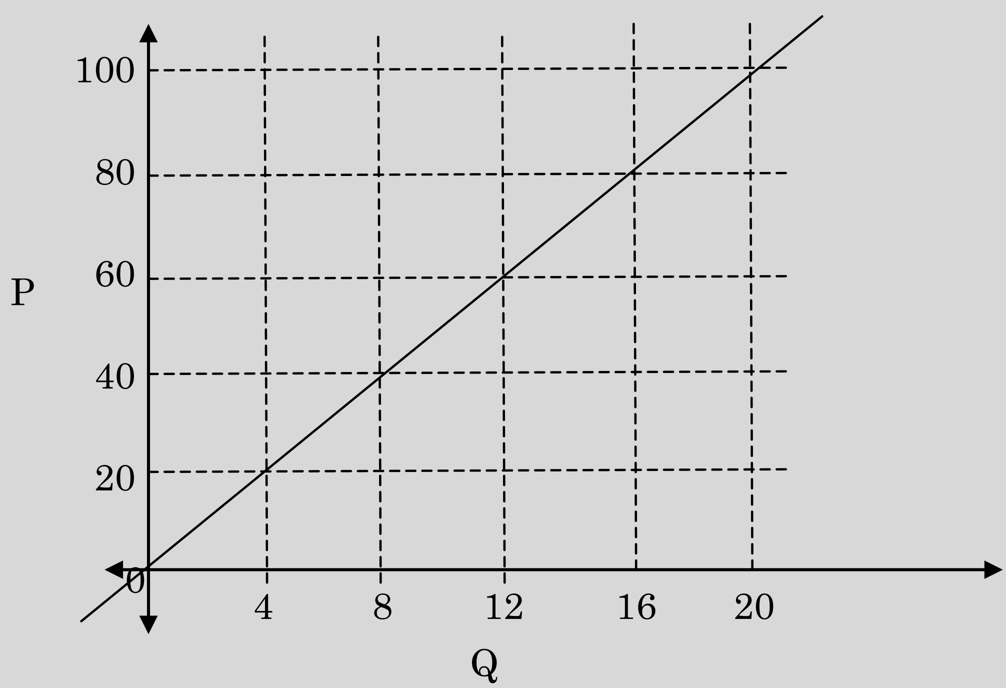
$$k = \frac{1}{5}$$

The relationship is $P = \frac{1}{5}Q$

To develop the table for the relation use multiples of 20 (P-value in the given question) for P. One of the ways is shown below:

P	0	20	40	60	80	100
Q	0	4	8	12	16	20

Then draw the graph as follows:



Exercise 14c

For each of the following, find the relationship between the variables and draw the graphs for the relationships:

1. $y \propto x$ and $y = 40$ when $x = 20$
2. $y \propto x$ and $y = 8$ when $x = 32$
3. $P \propto Q$ and $P = 20$ when $Q = -10$
4. $R \propto T$ and $R = 10$ when $T = -10$
5. $L \propto T$ and $L = 15$ when $T = 5$
6. $M \propto N$ and $M = -30$ when $N = 10$
7. $V \propto W$ and $V = 40$ when $W = 200$

Inverse variation

So far you have studied direct variation. You will now study one more type of variation called inverse variation. Variables x and y are said to be in inverse variation if the product of the two variables is constant i.e. $xy = k$, where k is the constant of variation.

Activity 4:

Modelling inverse variation

In pairs, discuss how you can model inverse variation from the statement below:

The number of days, (d), it takes to build a wall varies inversely as the number, (n), of boys available. If there are 20 boys the wall takes 10 days to build. Model an expression that gives d in terms of n.

Compare your work with the other pairs.

Example 5:

Inverse variation

The time (t), taken by a car to cover a given distance varies inversely as speed, (S in km/h). At 60km/h, the car takes 3hours. Model an expression that expresses t in terms of S.

Solution

t S = k, where k is a constant --------- Model for inverse variation

3 (60) = k ----- Substitute t for 3 and S for 60

180 = k ------- Simplify

So tS =180 ------- Substitute k for 180.

Exercise 14d

1. Model an inverse variation equation for each of the following for the given values of the variables:

   (a) P varies inversely as Q. (P = 0.025, Q = 0.04).

   (b) I varies inversely as R^2. (I = 10, R = 1).

   (c) T varies inversely as the square root of d. (T = 3, d = 64).

   (d) P varies inversely as q (s = 0.5, q = 8).

2. The time (t in seconds) taken to type a 2000 word document varies with the rate (r) at which you can type. When the rate is 40 words per minute, you can finish typing the document in 50 minutes. Write a model that relates t and r.

3. The distance (d metres) required to balance a seesaw is inversely proportional to ones weight (w in kg). A 120kg can balance the seesaw at 60cm away from the centre of the seesaw. Model an inverse variation expression that relates d and w.

4. Prepare your own questions on modeling inverse variation. Let your teacher check them.

Activity 5:

Deriving the general equation involving inverse variation

Having looked at modelling inverse variation, you will now learn how the general equation for this type of variation can be derived. In pairs, go through the following activity:

1. Suppose a motorist is to cover 20km. Construct a table of speed (s) against time (t) showing the time the motorist will take to cover the distance if she travels at 5km/h, 10km/h and 20km/h respectively.
2. What happens to the time taken as the speed increases?
3. Pick any two numbers of kilometres and find their ratio in simplest form and then pick their corresponding numbers of speeds and find their ratio in simplest form. How do the two pairs of ratios compare?
4. Try other pairs of numbers in your table in a similar manner and comment on the results.
5. Now try multiplying the number of kilometres by the corresponding number of hours for all the entries. Comment on your findings.

You have seen that as speed increases, the time it takes to cover the distance of 20km decreases in the same but opposite ratio. You can then say that the "time (t) is inversely proportional to speed(s)" or "time (t) varies inversely as speed (s)". You write $t \propto {}^{1}/_{s}$ to mean "time (t) is inversely proportional to speed (s)" or "time (t) varies inversely as speed (s)".

Additionally, you have seen that the product ts is the same for all the corresponding entries or ts = k where k is a constant.

Hence if $t \propto {}^{1}/_{s}$ then ts = k where k is a constant called the constant of variation.

Challenge

Can you find your own examples of inverse variation where one quantity decreases and the other increases in the same but opposite ratio.

Example 6:

Inverse variation

Given that $y \propto {}^{1}/_{x}$ and that when y = 60, x = 12. Find the value of x when y = 25.

Solution

If $y \propto {}^{1}/_{x}$ then xy = k, k constant

$\therefore$ $12 \times 60 = k$

$\therefore$ k = 720

$\therefore$ The law of variation is xy = 720

$\therefore$ When y = 25, then 25x = 720

$\therefore$ x = 28.8

Example 7:

Inverse variation

The number of days (d) it takes to complete to paint a wall is inversely proportional to the number (n) of men available to do the work. 12men take 15 days to paint the wall. How long will 18men take to paint the wall?

Solution

If $d \propto$ then dn = k, k constant

$\therefore$ 12x15 = k

$\therefore$ k = 180

$\therefore$ The law of variation is dn = 180

$\therefore$ When n = 18, then 18d = 180

$\therefore$ d = 10

$\therefore$ 18 men will take 10 days to paint the wall.

Exercise 14e

1. If $y \propto {}^{1}/_{x}$, and that when y = 100, x = 25. Find the relationship between x and y hence find x when y = 20.
2. Given that f is inversely proportional to d and that when f = 35, d = 7. Find d when f = 25.
3. q varies inversely as t. When q = 2, t = 8. Find the value of t when q is increased by 80%.
4. The time (t in minutes) it takes a typist to finish typing a document is inversely proportional to the number of words (w) she types per minute. if she types 250 words per minute she takes 30minutes. How long will it take her to finish typing the document if she types 100words per minute?
5. The effort (e in Newtons) applied to lift a load placed 100cm from the fulcrum is inversely proportional to the distance (d in cm) away from the fulcrum on the other side. An effort of 200N is required to lift the load when the distance is 60cm. What effort will be needed to lift the load if the distance is 40cm?

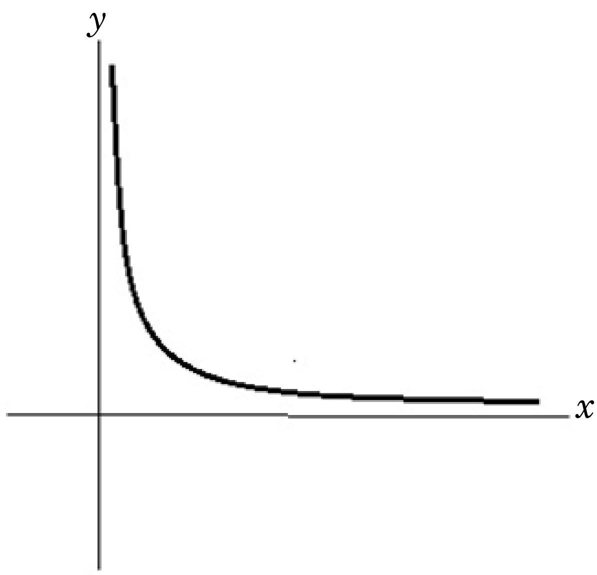
Activity 6:

Presenting inverse variation graphically

In this section you will learn how you can present the models of inverse variation graphically. In your groups

1. Draw the table of values for the model yx = 100 using the values x = 0, 5, 10, 15, 20, 25.
2. Using the values and a scale of 1cm to represent 5 units on the x – axis draw the graph of the equation xy = 100.
3. Present your work to the other groups. let also your teacher check your work.

   You have seen that the graph of inverse variation generally looks like this:



Exercise 14f

Rewrite xy = k starting with y and draw the same graph with negative values of x.

3. **Joint variation**

Joint variation is a variation whereby one variable varies with the product of two or more other variables. For example, if z = kxy, where k is a constant, then z varies jointly with the product of x and y. You will now learn how to derive the general equation involving joint variation.

Activity 7:

Deriving the general equation involving joint variation.

In form 2 you learnt the formula for volume of a cylinder. In pairs,

1. Write down the formula.
2. Does the formula depend on the size of a cylinder?
3. Write the formula as a direct variation.

4. What is the value of the constant of variation?

You have seen that the formula for the volume of the cylinder applies to all the cylinders. The volume depends on the square of the radius and the height i.e. V r^2h i.e $V = kr^2h$ where $k = \frac{22}{7}$. Hence the general equation of a joint variation can be derived from a direct variation.

Example 8:

Joint variation

p ∝ qr. When q = 2 and r = 5, p = 50

(a) Find the equation connecting p,q and r

(b) Find p when q = 5 and r = 4.

Solution (a) $9 = qr$

$\therefore p = kqr$ where k is a constant

Substitute $p = 50, q = 2$ and $\overline{9}$ in the equation above

$\therefore 50 = k \times 2 \times 5$

$\therefore 50 = 10k$

$\therefore \overline{9}$

$\therefore p = 5qr$

This is the equation connecting p, q and d

(b) $P = 5qr$ Substitute $\overline{9}$ and 9

$\therefore p = 5 \times 5 \times 4$

$162 =$

$\therefore k = 729$

Example 9:

Joint variation

m varies directly as d and inversely as the square of t. if $k = 729$ when $\overline{\quad 9 \quad}$ and $\overline{\quad 9 \quad}$, find:

(a) m when $\overline{\quad 9 \quad}$ and $\overline{\quad 9 \quad}$

(b) d when $m = 72$ and $\overline{\quad 9 \quad}$

Solution

(a) $\text{m} \propto \text{d}$ and $\text{m} \propto \dfrac{1}{t^2}$

$\therefore \text{m} \propto \dfrac{d}{t^2}$

$\therefore \text{m} = \dfrac{kd}{t^2}$ where k is a constant.

$\therefore 162 = \dfrac{k \times 2}{3^2}$

$\therefore\ 162 = \dfrac{4k}{6}$

$\therefore\ 9 \times 162 = 2k$

$\therefore\ 9 = \dfrac{729 \times 3}{4}$

$\therefore\ k = 729$

$\therefore\ m = \dfrac{729d}{t^2}$

$= \dfrac{729 \times 3}{4}$

(b) $M = \dfrac{729d}{t^2}$

$\therefore 72 = \dfrac{729 \times d}{3^2}$

$\therefore\ 72 = \dfrac{729d}{9}$

$\therefore 72 \times 9 = 729d$

$\therefore\ d = \dfrac{72 \times 9}{729}$

$= \dfrac{72}{81}$

$= \dfrac{8}{9}$

$= 0.9$ (to 2 decimal places)

$$= 9 =$$

$$= 546.75$$

Example 10:

Joint variation

m varies directly with the square of n and inversely with p. when *n=2* and *p=6, m=9*. Find m when *n=4* and *p=8*.

Solution

$$m \propto \frac{4k}{6}$$

$$\therefore \ m = \frac{kn^2}{p}$$

$$\therefore \ 9 = \frac{k \times 2^2}{6}$$

$$\therefore \ 9 = \frac{4k}{6}$$

$$\therefore \ 54 = 4k$$

$$\therefore \ k = \frac{54}{4}$$

$$= \frac{27}{2} \text{ or } 13.5$$

$\therefore$ The law of variation is $m = \frac{27n^2}{2p}$

$\therefore$ When n = 4 and p = 8, m= $\frac{27(4^2)}{2(8)}$

$\therefore$ m= 27

Exercise 14g

1. y varies directly with x and z. when x = 3 and z =10, y =15

Find the relationship between x, z and y.

2. $a \propto b^2c$. when a = 100, b = 2 and c = 5

(a) find a when b = 6 and c = 5

(b) find b when a=72 and c = 6.4

3. $p \propto \frac{Q}{\sqrt{R}}$ when Q = 75, R = 25 and p = 125

(a) find the equation connecting p, Q and R

(b) Find p when Q = 3 and R =100.

4. x, y and z are related quantities such that x varies directly as y and inversely as the square of z. When x =24 and y = 3, z = 4

(a) Find the value of x when y = 4 and z = 8

(b) Find the value of z when x =128 and y =$\frac{1}{4}$.

5. r varies directly with t and inversely with s. r =27 when t = 9 and s = 2. Find r when t = 7 and s = 6.

6. If $x \propto y$ and $y \propto \sqrt{z}$. How does x vary with z?

7. $P \propto qr$ and q varies inversely with the square of r. How does p vary with r?

8. x varies directly with y and z. When y = 6 and z = 3, x = $7\frac{1}{2}$.

Find (a) x when y =12 and z = 5

(b) Z when x =2.5 and y = 4

9. The mass m of a cylindrical tin varies jointly with its height (h) and the square of its radius (r). If the tin 50 cm high and of radius 2cm has a mass of 540g; Find the mass of a cylindrical tin 10cm high and of radius 4cm.

4. Partial **variation**

This is the type of variation which consists of two or more parts added together. The relationship in these parts varies from question to question. Since there are two or more parts added together, there are at least two constants which are used in solving these questions. Hence the knowledge of simultaneous linear equations is essential at this stage since there is need to solve for the unknowns which are used for constants in the equations.

Partial variation usually applies in situations where the value of one major quantity is the sum of two or more quantities varying in different ways. For example the cost of running a boarding school depends on two separate factors and these are overheads, such as electricity, water, telephone bills, and wages of employees and on the other hand the cost of food used in the feeding of students in the school. In most cases, overheads are constants i.e. the bills and wages are still paid without considering the fact that there are students in the school or not. The cost of food is the only quantity, which is proportional to the number of students being fed at the school. Therefore, we can say that the total cost is partly constant and partly varies as the number of students.

Activity 8:

Formulating the general equation of a partial variation

You have so far learnt to formulate the general equation of direct, inverse and joint variations. You will now learn to formulate the general equation of a partial variation.

Look at the equation $y = 3 + 2x$. In pairs, come up with your own examples of such equations. Discuss how similar the equations you have given are.

You might have noted that the right hands of all the equations are made up of two components: one is a direct variation and the other is a constant. So if $y = 3 + 2x$, you can say y is partly constant and partly varies as x. You should also note that in this equation and in the equations you gave in activity 8, there are two constants. You can therefore write y = a + kx as a general equation of partial variation where a and k are constants of variation.

Note: The knowledge of simultaneous linear equations is essential at this stage since there is need to solve for the unknowns which are used for constants in the equations.

Example 11:

Partial variations

A is partly constant and partly varies with B. When B = 2, A = 10 and when B = 4, A = 16

(a) Find the relationship between A and B

(b) Find A when B = 5.

Solution:

(a) From the first sentence A = a + b B where a and b are constants.

From the second sentence 10 = a + 2b (i) and 16 = a + 4b (ii)

Subtract (i) from (ii)

$$\begin{array}{r} 16 = a + 4b \\ -\ 10 = a + 2b \\ \hline 6 = 0 + 2b \end{array}$$

6 = 2b

$\therefore$ b = 3

Substitute b = 2 in equation (i) to find b

10 = a + (2 x 3)

10 = a + 6

$\therefore$ a = 10 – 6

= 4

Hence A = 4 + 3B is the required formula. (from A = a + bB)

(b) When B = 5

$\therefore$ A = 4 + 3 × 5

= 4 + 15

= 19

Example 12:

Partial variation

A varies partly as C and partly as the square root of C. When C = 4, A = 22 and when C = 9, A = 42

(a) 4

Solution

(a) From the first sentence $A = aC + b\sqrt{C}$ where a and b are constant

From the second sentence

$22 = 4a + b\sqrt{4}$ (i)

And $42 = 9a + b\sqrt{9}$ (ii)

i.e. 22 = 4a + 2b (i)

And 42 = 9a + 3b (ii)

Equation (i) x3 66 = 12a + 6b (iii)

Equation (ii) x2 84 = 18a + 6b (iv)

Subtract (iii) from (iv)

84 = 18a + 6b

<u>− 66 = 12a + 6b</u>

18 = 6a + 0

18 = 6a

$\therefore$ a = 3

Substitute a = 3 in equation (i) to find b

$22 = 4 \times 3 + 2b$

$22 = 12 + 2b$

$10 = 2b$

$\therefore$ b = 5

Hence $A = 3C + 5\sqrt{c}$ is the required equation.

(b) when C = 25

$A = 3 \times 25 + 5\sqrt{25}$

$= 75 + 5 \times 5$

$= 75 + 25$

$= 100$

Exercise 14h

1. P is partly constant and partly varies as Q. When Q = 3, P = 22 and when Q =2, P = 18.

(a) Find the equation connecting P and Q.

(b) Find P when Q =10.

2. x is partly constant and partly varies as the square of y. When y = 2, x =20 and when y = 3, x = 25; find the value of x when y = 7.

3. The cost of making a sofa set is partly constant and partly varies with the time it takes to make. If it takes 3 days to make, it costs K27, 000. If it takes 5 days to make, it costs K31, 000; find the cost if it takes 2 days to make?

4. The cost of running a boarding school is partly constant and partly varies as the number of students in the school. If there are 100 students, the total cost is K45,000 per month and if there are 240 students the total cost comes to K87, 000 per month. Find the total cost if there are 360 students in the school.

5. M varies partly as N and partly as the square root of N. When N = 4, M = 6 and when N = 16, M = 16. Find M when N = 25.

6. E is partly constant and partly varies with S. When S = 40, E = 50 and when S = 54, E = 92. Find S when E = 32.

7. The cost of producing mathematics books is partly constant and partly varies as the number of books produced. The total cost of producing 4 books is K2500 and that of producing 10 books is K5800. Find the total cost of producing 1000 books.

8. The cost of transporting people in a minibus from Blantyre to Lilongwe is partly constant and partly varies as the number of people transported. Transporting 15 people costs K8750 and transporting 20 people costs K10, 000. Find the cost of transporting 10 people within the same distance.

9. P is a quantity, which varies partly as V and partly as V^2. When V = 4, p =52.8 and when v =5, p = 81. Find p when v = 3.

10. The resistance to motion of a bicycle is partly constant and partly varies as the square of the speed. At 8 km/hr, the resistance is 80N and at 12km/hr, the resistance is 100N. Find the resistance if the speed of the bicycle is 20km/hr.

Activity 9:

Presenting partial variation graphically

You are familiar with equations of the form y = mx + c.

In pairs,

1. Compare this form with the general equation involving partial variation.

2. From your comparison, what type of graph is a partial variation?

You must have seen that partial variation graphs are straight lines in slope intercept form.

Example 13:

Partial variations

y is partly constant and partly varies as x. When x = 10, y = 2 and when x = 4, y = 14. Express y in terms of x and draw the graph of the variation.

Solution

$y = a + kx$ ---------- General partial variation equation.

$2 = a + 10k$ (i) ---------- Substitute y for 2 and x for 10

$\underline{14 = a + 4k}$ (ii) ---------- Substitute y for 16 and x for 4
$-12 = 6k$ ----------------------- Subtract (ii) from (i)

$k = -2$ ------------------------ Divide by 6 both sides

$2 = a - 2\ (10)$ ---------------- Substitute k for -2 into equation(i)

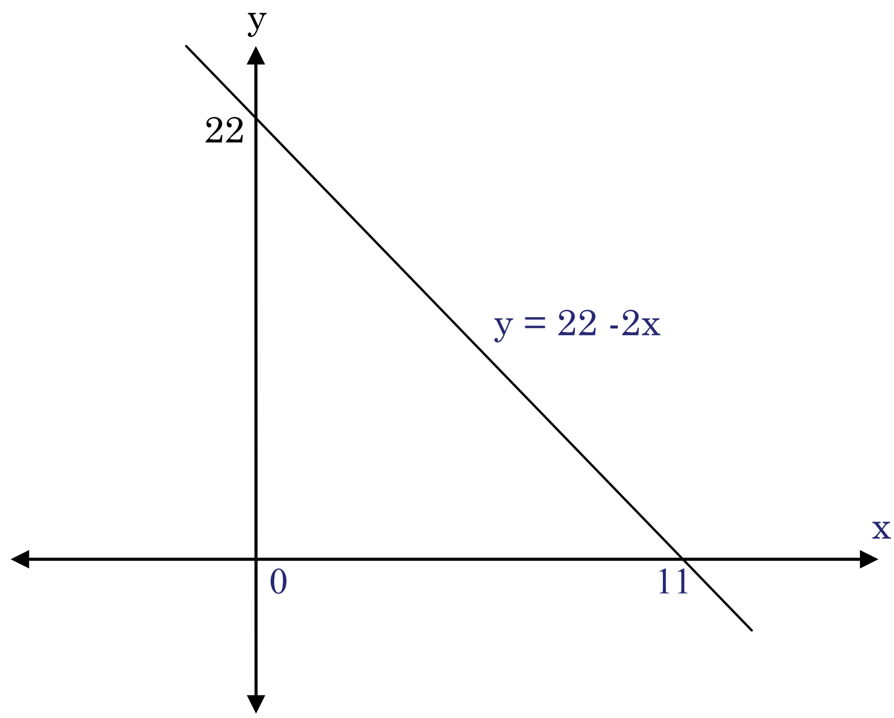
$2 = a - 20$

$a = 22$ ------------- Rearrange

The expression is $y = 22 - 2x$

The graph is a straight line whose y – intercept is 22 . The graph also cuts the x axis at −11 (i.e when y = 0)

The graph is drawn below:



Exercise 14i

Draw partial variation graphs for the following questions:

1. A quantity y is partly constant and partly varies as x. When $y = 5$, $x = 2.5$ and when $y = 3$, $x = 1.5$.
2. p is partly constant and partly varies as q. When $p =3$, $q = 1$ and when $p = 6$, $q = 2.5$
3. A is partly constant and partly varies as B. When $A = 11$, $B = 7$ and when $A = 15$, $B = 9$.
4. The cost (C in Malawi Kwacha) of making an item is partly constant and partly varies with time(t in minutes) it takes to make the item. When $t = 100$, C = K498 and when $t = 30$, C = K148.
5. T is partly constant and partly varies as C. When $C =5$, $T = 37$ and when $C = 15$, $T = 57$.

Unit Summary

- In this unit you have learnt direct, inverse, joint and partial variations. You have learnt to model direct and inverse variations and to derive general equations involving the four variations. You also learnt how you can present variations graphically and solve variation problems.

Unit review exercise

1. If y varies directly as x, write a model that expresses y in terms of x when $y = 9$ and $x = 3$.
2. Q is directly proportional R . When $Q = 100$, $R = 5$. Model an expression that gives Q in terms of R.
3. If $y \propto x$ and $y = 80$ when $x = 5$, find y when $x = 3$.
4. $V \propto W$ and $V = 4.5$ when $W = 12$. Find V when $W = 16$.
5. Given that $y \propto$ and that when $y = 20$, $x = 10$. Find the value of x when $y = 25$.
6. Given that $y \propto$ and that when $y = 2$, $x = 3$. Find the value of x when $y = 1$.
7. If T varies directly as the cube of S and that $T = 6$ when $S = 2$, find T when $S = 8$.
8. The mass (m in grams) of a solid varies directly as the volume(V in cm^3) of the solid. The mass is 12g when the volume is 8 cm^3. Find the volume for which the mass is 30g.
9. The resistance , R , of a wire varies inversely with the cross sectional area of the wire, A. A 500cm wire length copper wire with a radius of 0.1cm has a resistance of 0.1ohms. find the resistance of the same length of copper wire with a radius of 0.2cm.
10. The intensity, I in watts per m^2, of jet engine noise at Kamuzu International Airport varies inversely as the square of the distance, r in m^2. At a distance of 1 metre the intensity is 10 watts per square metre. An airport cargo worker is 15metres from the jet engine. Calculate the intensity of the jet noise at that distance.
11. The royalties, in Kwacha, that an author receive on the sale of a book are partly constant and partly varies as

the number of books sold. On sales of 10000 she receives K80000 and on the sales of 6000 she receives 50000. Find the amount she will receive on the sales of 15000copies.

12. The cost of producing a component is partly proportional to the area of the component and inversely proportional to the number of components produced each day. On a day 120 components of area 10cm^2 were produced and the cost was K100 per component. On the other day 180 components of area 20cm^2 were produced and the cost per component was K80. Find the cost per component of producing 50 components per day of area 50cm^2.

Glossary: None

References

S. Hau and F. Saiti (2002), *Strides in Mathematics Book 3*, Longman , Blantyre

Larson etal (1998),*Heath Algebra An Integrated Approach*, Heath and company, Ottawa.

G. D. Buckwell and B.N Githua, *Gold Medal Mathematics,* Macmillan, London.

Unit 15

GRAPHS OF QUADRATIC FUNCTIONS

Recall from unit 1 that a quadratic function is a function of the form $ax^2 + bx + c$ where a, b and c are constants and $a \neq o$.

Youlearnt how to factorize quadratic expressions and how to solve quadratic equations by factorisation, completing the square and by the quadratic formula.

In this unit, you are going to learn about the graphs of quadratic functions and the properties of such graphs. You shall learn how to draw and interpret graphs of quadratic functions. You shall also learn how to solve quadratic equations and linear and quadratic equations graphically and how to formulate quadratic equations given quadratic graphs which cut the x – axis.

The knowledge of quadratic graphs is used in many situations such as in the study of falling objects and in studying quantities related to time among others.

Drawing graphs of quadratic functions

Activity 1:

You are familiar with completing or drawing a table of values of a given equation from your JCE Course.

In groups,

1. Discuss how you construct a table of values for any given equation.
2. Now construct a table of values for the equation $y = x^2$ for $-4 \leq x \leq 4$.
3. Using a scale of 2cm to represent 1 unit on both axes, plot the points on the graph. What shape have the points formed?
4. Using a free hand draw the graph of $y = x^2$ through all the points.
5. Repeat steps 1, 2 and 3 but now use the equation $y = -x^2$.
6. Comment on the similarities and differences between the two graphs.

You should have noted that if the graph of a quadratic function is plotted and drawn, a smooth curve is produced. The curve is called a parabola. The parabola is either cup shaped when the coefficient of x^2 is positive *or* inverted(cap shaped) when the coefficient of x^2 is negative.

Exercise 15 a

For each of the following quadratic functions, construct a table of values and then draw the graph.

1. $y = \frac{1}{2}x^2$ for $-4 \le x \le 4$
2. $y = -\frac{1}{2}x^2$ for $-4 \le x \le 4$
3. $y = 2x^2$ for $-4 \le x \le 4$
4. $y = -2x^2$ for $-4 \le x \le 4$
5. $y = x^2 + 1$ for $-4 \le x \le 4$
6. $y = x^2 - 1$ for $-4 \le x \le 4$
7. $y = -x^2 - 1$ for $-4 \le x \le 4$
8. $y = x^2 - x - 2$ for $-4 \le x \le 4$
9. $y = -x^2 + 2x + 3$ for $-3 \le x \le 5$
10. $y = 3x^2 - 3x - 6$ for $-2 \le x \le 3$

Interpreting graphs of quadratic functions

In this section, you shall learn to interpret graphs of quadratic functions. You shall learn to describe the effect of a, b and c on the nature of graph of $y = ax^2 + bx + c$, find maximum and minimum values of a quadratic function and to find the equation of the line of symmetry.

Activity 2:

The effect of changing the value of a in $y = ax^2$

In Activity 1 you drew the graph of $y = x^2$. You will now see more closely what happens to this graph as the value of a change.

In groups,

1. Draw the graphs of $y = x^2$, $y = 2x^2$, and $y = 3x^2$ on the same axes and using the same scale.
2. How do the three graphs compare to the graph of $y = x^2$?
3. What is the major difference amongst the graphs?
4. Write down the coefficients of the three graphs in order of their steepness starting with the less steep.
5. What can you say about the effect of the change in the value of a on the graph of $y = x^2$?

You have seen that as the value of a decrease, the graphs become more and more open or they become less and less steep. In all the three graphs, the line $x = 0$ divides each graph into equal halves. In other words, the line $x = 0$ is the **line of symmetry.**

Activity 3:

The effect of adding or subtracting a constant from the quadratic squaring function i.e. the equation $y = x^2$

In your groups,

1. Draw on the same axes and the same scale, the graphs of $y = x^2$, $y = x^2 - 1$and $y = x^2 + 1$ for values of x from -4 to $+4$.
2. What are the similarities and the differences between the graphs?
3. How do the shapes of the two graphs compare to that of $y = x^2$?
4. What can you say is the effect of adding or subtracting a constant to x^2?

You have seen from this activity that all the three graphs have the same shape and that in all the three graphs, the line $x = 0$ is the line of symmetry. The three graphs only differ in the points where they cut the y axis. This enables you to come up with the following interpretation about the graphs:

(a) Adding a constant to the quadratic squaring function i.e to x^2 shifts the graph vertically upwards along the line of symmetry. The graph shifts by the value of the constant.

(b) Subtracting a constant from the quadratic squaring function i.e. to x^2 shifts the graph vertically downwards along the line of symmetry. Again, the graph shifts by the value of the constant.

(c) The constant also gives the minimum value of the quadratic function for cup shaped parabola or maximum value for cap shaped parabola.

Example 1:

Sketching graphs

Draw sketches of the following graphs on the same system of axes using the same scale on both axes:

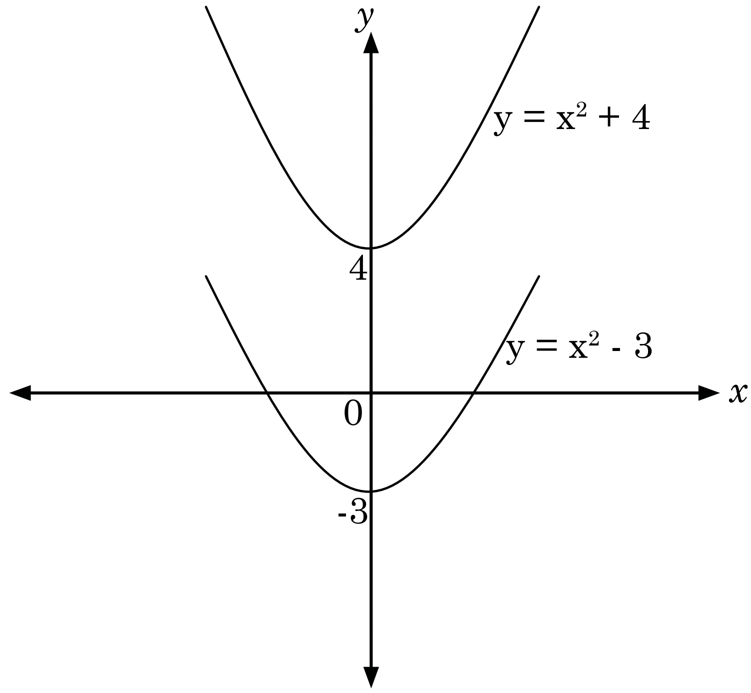
a. $y = x^2 + 4$ b. $y = x^2 - 3$

Solution

From activity 3,

- The graph of $y = x^2 + 4$ has the shape of $y = x^2$ but shifted vertically upwards by 4 units and that $y = 4$ is the minimum value of the function.
- The graph of $y = x^2 - 3$ has the shape of $y = x^2$ but shifted vertically downwards by 3 units and that $y =-3$ is the minimum value of the function.
- $x = 0$ is the line of symmetry

The graphs are drawn below:



Activity 4:

The effect of adding or subtracting a constant from x and then squaring the result

Again in your groups,

1. Draw, on the same axes and scale, the following graphs: $y = x^2$, $y = (x - 1)^2$ and $y = (x + 2)^2$ for values of x from −4 to +4.
2. What are the similarities and the differences between the graphs?
3. What is the equation of the line of symmetry for the graphs?
4. What can you say is the effect of adding or subtracting a constant from x and then squaring the result?

You have seen that the graph of $y = (x - 1)^2$ has the same shape as the graph of $y = x^2$ but has been shifted to the right by 1 unit. The graph of $y = (x + 2)^2$ also has the same shape as the graph of $y = x^2$ but has been shifted to the left by 2 units. The lines of symmetry are the lines $x = 1$ and $x = 2$ respectively. The minimum value of each function is 0. The square of the constant gives the y-intercept.

You can therefore make the following interpretations about adding or subtracting constants to x before squaring:

(a) Adding a constant to x before squaring shifts the graph of $y = x^2$ horizontally to the left by the value – *constant (minus the given constant)*.

(b) Subtracting a constant from x before squaring shifts the graph of $y = x^2$ horizontally to the right by the value *positive constant (plus the given constant)*.

(c) The minus constant and the plus constant are also the new lines of symmetry.

(d) Y-intercept is the square of the constant.

Example 2:

Sketching graphs

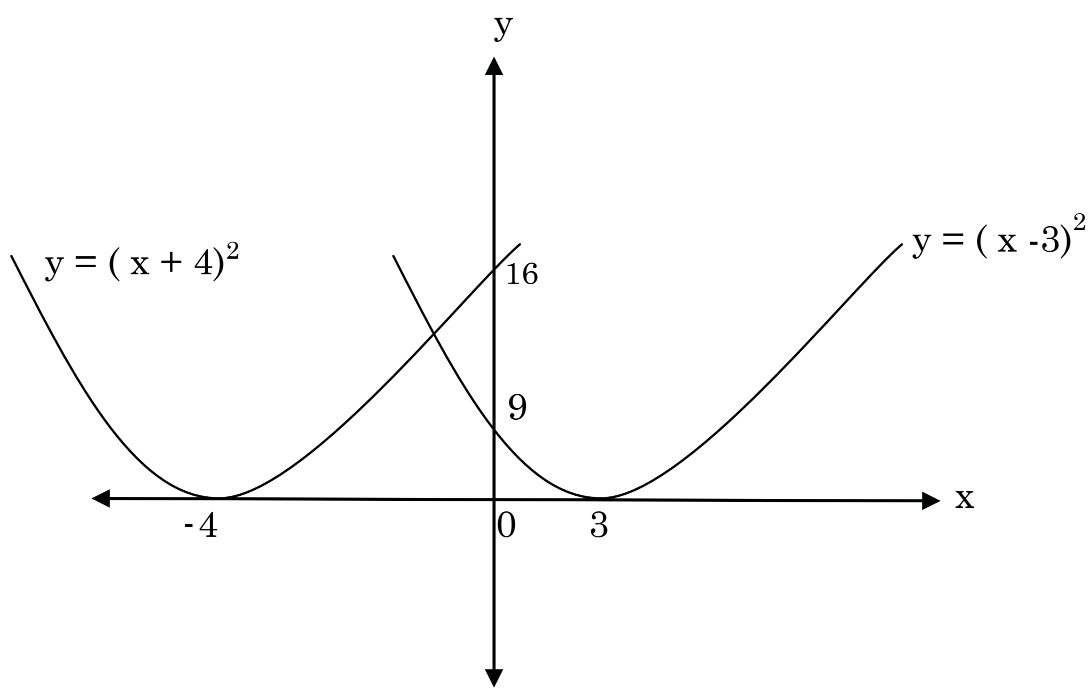
On the same axes draw sketches of the graphs of $y = (x - 3)^2$ and $y = (x + 4)^2$

Solution

From activity 4,

- The graphs of $y = (x - 3)^2$ and $y = (x + 4)^2$ are the same as the graph of $y = x^2$ but shifted horizontally to the right by 3 units and to the left by 4 units respectively.
- 0 is the minimum value of each function.
- $x = 4$ and $x = 3$ are the lines of symmetry respectively.

The graphs are drawn below:



=

Example 3:

Drawing graphs

Draw a sketch of the graph of $y = x^2 - 2x + 3$

Solution

You need to relate this function to the forms $y = ax^2$ or $y = (x - a)^2$ first. These are the functions that you have looked at in activity 3 and 4. You do this by writing the given function into a perfect square plus a number by adding and subtracting (coefficient of $x)^2$ as follows:

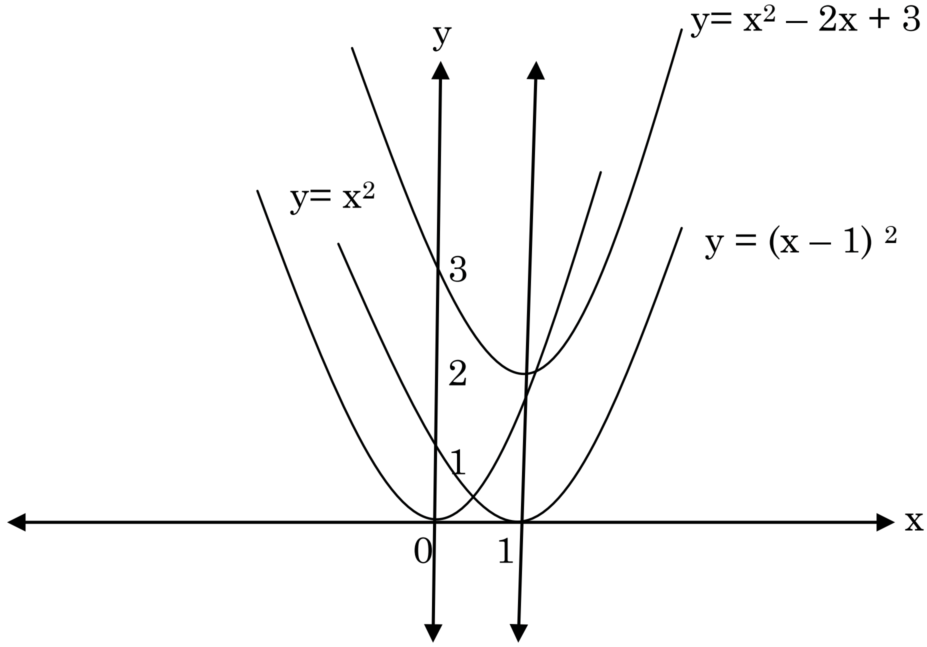
$y = x^2 - 2x + \{½(-2)\}^2 - \{½(-2)\}^2 + 3$

$y = x^2 - 2x + 1 - 1 + 3$

$y = x^2 - 2x + 1 + 2$

i.e. $y = (x - 1)^2 + 2$

You can see that expressed in this form, $y = x^2 - 2x + 3$ is the graph of $y = x^2$ shifted in two directions: 1 unit to the right and then 2 units upwards or in short, it is the graph of $y = (x - 1)^2$ shifted upwards by 2 units. The equation of the axis of symmetry (which is also the x –value at the turning point of the graph) is x = 1 and the minimum value of the function is y = 2.the graph is drawn below together with the graphs $y = x^2$ and $y = (x - 1)^2$ for you to see their relationship:



You will now investigate the effect of increasing or decreasing the values of a , b and c in the equation $y = ax^2 + bx + c$.

Activity 5 :

Finding the effect of increasing or decreasing the values of a , b and c in the equation $y = ax^2 + bx + c$.

Copy the following table in your exercise books before you go into your groups. Fill in this table as you go through the activity:

	Increasing the value a	Decreasing the value of a	Increasing the value of b	Decreasing the value of b	Increasing the value of c	Decreasing the value of c
Effect on turning point						
Effect on line of symmetry						

1. In your groups, draw the graph of $y = x^2 - 2x - 8$, clearly showing the line of symmetry and the turning point of the graph. Note that in this equation a = 1, b = -2 and c = -8.

2. Now, keeping the values of b and c constant, increase the value of a and draw the graph of the resulting equation on the same axes.
3. Show the equation of the line of symmetry and the turning point of this graph.
4. How do the two compare with their positions in the first graph?
5. Increase the value of **a** even further and repeat the process.
6. Fill the first column of your table.
7. Repeat the above steps and fill column 2 by decreasing the value of a.
8. Investigate the effects of b and c in a similar manner and complete the table. Note that as you investigate the effect of b, a and c must be kept constant and when you investigate the effect of c, a and b must be kept constant.

Investigate the effect of a on one graph the effect of b on another graph and similarly the effect of c.

Challenge

Use the equation of a cap shaped parabola to investigate the effects of a , b, c on the nature of the parabola. Come up with a table similar to the one you used on the cup shaped parabola in activity 5.

Exercise 15b

Draw sketches of the following graphs. In each case, state the equation of the axis of symmetry and the coordinates of the turning point of the function:

1. $y = x^2 - 2$
2. $y = -x^2 + 1$
3. $y = -(x - 1)^2$
4. $y = 3x^2 + 6x$
5. $y = 2x^2 + 5x + 2$

Activity 6:

Finding the maximum and minimum values of a quadratic function

A minimum or maximum value of a quadratic function is the value of y at the turning point of the graph. Earlier on in activity 1 of this unit, you saw that a quadratic graph is either cup shaped or cap shaped. A cup shaped parabola has a minimum point i.e. lowest point while the cap shaped parabola has a maximum point i.e. highest point. In short, a cup shaped parabola has a minimum value while cap shaped parabola has a maximum point. Each of the two points lies on the line of symmetry. You will now learn how to find the maximum and minimum values of a quadratic function by doing two group activities.

In groups,

1. Draw up a table of values for the quadratic functions $y = x^2 + 2x - 3$ and $4 + 3x - x^2$ for -4 x 4.
2. Using the tables, draw the graph of the quadratic function $y = x^2 + 2x - 3$ and $4 + 3x - x^2$
3. What is the value of x at the minimum or maximum point from the graphs?
4. Now, find the roots of the two quadratic functions and complete the table below:

Equation of parabola	Roots of the parabola	x value at the turning point of the parabola
$x^2 + 2x - 3$		
$4 + 3x - x^2$		

5. Now find the relationship between the roots and the value of x at the turning Points of each graph.

You must have seen that the value of x at the turning point of the graph is half the sum of the two roots.

Example 4:

Turning points

Find the x value at the turning point of the graph
$y = 2x^2 - 7x + 3$

Solution

First find roots of the equation $2x^2 - 7x + 3 = 0$.

$(x - 3)(2x - 1) = 0$ ------------ By factor method

$x = 3$ or $x = \frac{1}{2}$

Hence the x vale at the turning point is $\frac{1}{2}(3 + \frac{1}{2}) = 1.75$

Exercise 15c

Find the value of x at the turning points of the following graphs:

1. $y = x^2 - 5x + 6$
2. $y = x^2 - 4x - 5$
3. $y = x^2 + 8x + 15$
4. $y = 4x^2 - 5x - 6$
5. $y = 4x^2 - 3x - 10$

Having seen the relationship between the roots and the value of x at the turning point of a graph, you will now see the relationship between the x value and the constants a and b of a parabola $ax^2 + bx + c$.

Activity 7

In groups, study the table below which shows the roots of parabolas, the turning points, the x value at the turning points and the values of a and b in the parabola. (One fraction in column 3 has deliberately not been simplified)

Equation of parabola	Roots of the parabola	x value at the turning point of the parabola	Coefficients of x^2 and x
$y = x^2 - 8x + 15$	3 , 5	$^8/_2$	a = 1 and b = –8
$y = 6x^2 - 7x + 2$	$^1/_2$, $^2/_3$	$^7/_{12}$	a = 6 and b = –7
$y = 3x^2 - 13x + 12$	$^4/_3$, 3	$^{13}/_6$	a =3 and b = -13
$y = 6x^2 + 7x - 5$	$^1/_2$, $^{-5}/_2$,	$^{-7}/_{12}$	a = 6 and b = 7

Discuss the relationship that you see between the x value at the turning point and the constants a and b of a parabola in each row. Write down this relationship. Use your own equations and verify that the relationship is true.

You must have seen that in column 3 the numerator *is the value of b with opposite sign* and that the *denominator is twice the value of a.* The results can be generalised as follows: the x – value at the turning point (maximum or minimum) of the parabola $y = ax^2 + bx + c$, where a, b and c are constants is

$x = \frac{-b}{2a}$ and this also gives the equation of the line of symmetry.

To find the corresponding value of y, substitute $x = \frac{-b}{2a}$ into the quadratic equation. This gives the maximum or minimum value of the quadratic function.

Example 5:

Maximum and minimum points

For the function $y = x^2 + x - 6$, what is

(i) The value of x at the minimum point?

(ii) The equation of the line of symmetry?

(iii) The minimum value of y.

(i) at the minimum $x = \frac{-b}{2a}$, and for this quadratic function $a = 1$, $b = 1$, $c = -6$

$$\therefore x = -\frac{1}{2 \times 1} = -\frac{1}{2}.$$

(ii) The equation of the line of symmetry is $x = -\frac{1}{2}$

(iii) $y = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 6 = -6\frac{1}{4}.$

Exercise 15d

Find the maximum or minimum values of the following functions and the equation of the line of symmetry.

1.	$y = x^2 + 2x - 6$	2.	$y = 2x^2 + 3x - 5$
3.	$y = x^2 - 4x + 1$	4.	$y = x^2 + 2x - 3$
5.	$y = 1 - 2x - x^2$	6.	$y = 4 - 8x - 2x^2$
7.	$y = -3x^2 - 12x + 1$	8.	$y = 5 - 2x - 4x^2$
9.	$y = (x - 1)^2$	10.	$y = (2x + 1)^2$

4. Solving quadratic graphs graphically

   You learnt to solve quadratic equations in unit 1 of this book. In that unit you used factorisation and quadratic formula. You will now learn to solve quadratic equations graphically.

Activity 8:

Finding roots of a quadratic equation $ax^2 + bx + c = 0$ graphically

In your groups,

1. Draw the table of values for the function $y = x^2 + x - 6$ for the values -4 x 4.

2. Using a scale of 2cm to represent 1 unit on both axes draw the graph of $y = x^2 + x - 6$ on the graph paper.

3. Read off the values of x at the two points where the graph cuts the x axis i.e where the graph cuts the line $y = 0$.(note that since $y = 0$, you have actually drawn the graph for the equation $0 = x^2 + x - 6$ or $x^2 + x - 6=0$)

4. Now solve the equation $x^2 + x - 6 = 0$ by factors and compare your result with the values you found in 3 above. What do you find?

It can be seen from the results that to solve the equation $ax^2 + bx + c = 0$, the quadratic graph is drawn and the roots of the equation are the x values at which the graph cuts the x-axis.

Exercise 15e

Solve each of the quadratic equations below by plotting a graph for $-4 \leq x \leq +4$. On the horizontal axis, use a scale of 1cm to represent 1 unit. On the vertical axis, use the scales indicated in the brackets against each question:

1. $x^2 - x - 6 = 0$ (2cm to represent 5units)
2. $-x^2 + 1 = 0$ (2cm to represent 5units)
3. $x^2 - 6x + 9 = 0$ (2cm to represent 10units)
4. $-x^2 - x + 12 = 0$ (2cm to represent 5units)
5. $x^2 - 4x + 4 = 0$ (2cm to represent 10units)
6. $2x^2 - 7x + 3 = 0$ (2cm to represent 20units)
7. $x^2 + 3x - 10 =0$ (2cm to represent 10units)
8. $3x^2 - 5x - 2 = 0$ (2cm to represent 20units)
9. $2x^2 - 7x = 0$ (2cm to represent 20units)
10. $x^2 - 4x + 3 = 0$ (2cm to represent 10units)

Activity 8:

Solutions to simultaneous linear and quadratic equations

In activity 7, you solved quadratic graphs by finding the points of intersection of a quadratic graph and any horizontal line. You will now learn how to solve graphically equations of the form $ax^2 + bx + c = d$, where d is less than 0 or is greater than 0. You will also learn to solve quadratic equations by finding the points of intersection of the quadratic graph and other linear equations of the form $y = mx + c$.

Suppose you want to solve graphically the equation $x^2 + 3x - 6 = 4$ for -6 x 3

1. Draw up the table of values for the equation $y = x^2 + 3x - 6$ for -6 x 2.
2. Using a scale of 2 cm to represent 1 unit on both axes draw the graph of $y = x^2 + 3x - 6$
3. Now draw the line $y = 4$ on the same axes as $y = x^2 + 3x - 6$.
4. Read off the values of x at the points where the two graphs intersect.
5. Now solve the quadratic equation $x^2 + 3x - 6 = 4$ by factors and compare the results with the results they found in 3. Comment on your findings.

You have learnt from the above activity that provided that solutions exist, all quadratic expressions equated to a constant can be solved by drawing the graphs of y = the quadratic expression and y = the constant and their point(s) of intersection give the solution to the quadratic equation.

Some equations that can be solved in a similar way are those of the form $ax^2 + bx + c = ax + c$ where a, b and c. Go through activity 9 below:

Activity 9:

Graphical solutions by points of intersection of the quadratic graph and a linear expression of the form $y = mx = c$

Suppose you want to solve graphically the equation $x^2 + 2x - 1 = x + 1$ for $-6\ x\ 3$.

In groups,

1. Using the values of x in the given range, draw up the tables of values for the two equations.
2. Using a scale of 2cm to represent 1 unit on both axes draw the graph of $y = x^2 + 2x - 1$ and $y = x + 1$
3. Read off the values of x at the points where the two graphs intersect.
4. Now solve the quadratic equation $x^2 + 2x - 1 = x + 1$ by factors by grouping and compare the results with the results you found in 3. What do you find?

You have seen that the results in 3 and 4 are the same. This means that provided that solutions exist, all quadratic expressions equated to a linear expression can be solved by drawing the graphs of y = the quadratic expression and y = the linear expression and their point(s) of intersection give the solution to the quadratic equation.

Example 6:

Graphical solutions

Solve graphically the equation $x^2 + x - 6 = 2x + 3$

Solution

In this case you draw the graphs of $y = x^2 + x - 6$ and $y = 2x + 3$ on the same pair of axes.

$y = x^2 + x - 6$

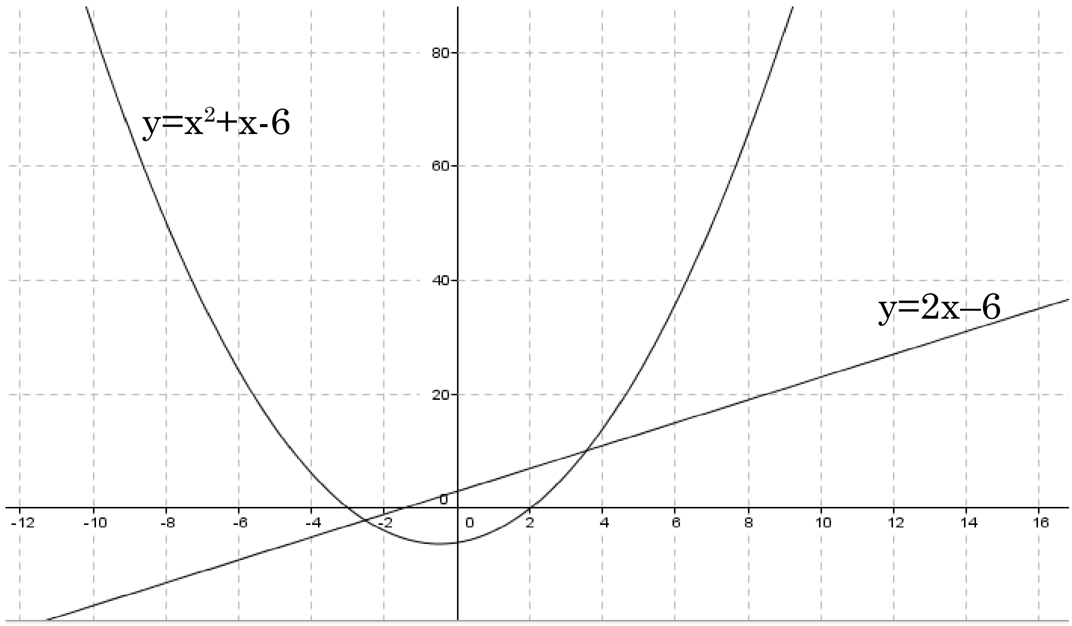
x	−4	−3	−2	−1	0	1	2	3	4
y	6	0	−4	−6	−6	−4	0	6	14

For graphs of straight lines, you only need three pairs of points.

$y = 2x + 3$

x	−3	0	3
y	−3	3	9

Now you can draw the two graphs



The two graphs intersect when x = −2.5 and when x = 3.5. Hence the roots of the equation x2 + x – 6 = 2x + 3 are x = −2.5 and x = 3.5.

Sometimes you are asked to draw a graph and use it to solve other quadratic equations.

Example 7:

Grapical solutions

(a) Draw the graph of $y = 3x^2 - x - 2$ for $-3 \le x \le 3$.

(b) Use the graph to solve the following equations:

(i) $3x^2 - x = 12$

(ii) $3x^2 - 7 = 0$

(iii) $3x^2 = 2x + 5$

(iv) $2 + x - 3x^2 = 0$

Solution notes:

Draw the graph of $y = 3x^2 - x - 2$ as in the previous activities. Then proceed as follows to answer part (b) of the question:

i. Rearrange $3x^2 - x = 12$ to get $3x^2 - x - 12 = 0$. Then draw the graph of $y = 3x^2 - x - 2 - (3x^2 - x - 12)$ i.e. $y = 10$. The roots of the equation $3x^2 - x = 12$ are the x values at the points of intersection of the two graphs.

ii. As before, draw the graph of $y = 3x^2 - x - 2 - (3x^2 - 7)$ i.e. $y = 5 - x$. The roots of the equation $3x^2 - 7 = 0$ are the x values at the points of intersection of the two graphs.

Now get graph papers and solve the two remaining equations. Let your teacher check your work.

Note that to solve one graph using the other, the former graph must first be rearranged so that it equals to 0. The quadratic expression of this arranged graph is then subtracted *from* the quadratic expression of the drawn graph not vice versa. Usually, the result of subtraction is a linear graph which is then drawn using any three x values in the given range.

Exercise 15f

1. Using the scale of 2 cm to represent 2 units on the x axis and 2cm to represent 1 unit on the y axis, draw the graph of $y = x^2 + 5x - 6$ for -6 x 1.

   Use the graph to solve the equations

   (a) $x^2 + 5x - 6 = 0$

   (b) $x^2 + 5x - 6 = -10$

   (c) $x^2 + 2x - 7 = 0$

2. Solve graphically the simultaneous equations $y = x^2$ and $y = x + 3$ by drawing the two graphs on the same axes. Use the scale of 2 cm to represent 2 units on both axes.

3. On the same axes and scale, draw the graphs of $y = x^2 – 3x + 10$. Use this graph to solve the equation $x^2 – 4x + 3 =0$ for integral values of x from -1 to 5

4. On the same axes and scale draw the graph of $y = 2x^2 - 3x - 7$ for the following values of x : -2 , -1 , 0, 1 , 2 , 3, 4, 5. Use this graph to solve the following equation $2x^2 -5x - 6 = 0$

5. By drawing the graphs of $y = 2x^2 + x – 3$ and $y = 2x – 1$ on the same axes, solve the equation $2x^2 + x -3 = 2x – 1$

6. Draw the graph of $y = 4 + 3x – 2x^2$ for values of x from -3 to 5. Use your graph to solve equation $5 + x – 2x^2 = 0$

Activity 5:

Formulating quadratic equation from roots

Study the solution to the quadratic equation $x^2 – x – 2 =0$ below:

$x^2 – x – 2 = 0$

$(x^2 – 2x) (+ x – 2) = 0$

$x(x – 2) + 1(x – 2) =0$

$(x + 1)(x – 2) = 0$

$x = –1$ or $x = 2$

Now suppose you are given the roots (-1 and 2) of the quadratic equation, discuss in groups how you can find the quadratic equation $x^2 – x – 2 = 0$. Report your findings.

To find the quadratic equation $x^2 – x – 2 = 0$, you will have to work backwards from the roots.

Example 4:

Equation from roots

Formulate a quadratic equation whose roots are $\frac{2}{3}$ and $-\frac{1}{5}$.

Solution:

Let $x = \frac{2}{3}$ or $x = -\frac{1}{5}$

i.e. $3x = 2$ or $5x = -1$

i.e. $3x - 2 = 0$ or $5x + 1 = 0$

i.e. $(3x - 2)(5x + 1) = 0$

$15x^2 - 7x - 2 = 0$

Exercise 15g

Formulate a quadratic equation whose roots are:

a. 2 and 4

b. -1 and -5

c. -4 and $\frac{1}{2}$

d. $-\frac{3}{4}$ and $-\frac{2}{3}$

e. $\frac{1}{3}$ and -2

Activity 10:

Formulating the quadratic equation given a quadratic graph which cuts the x-axis

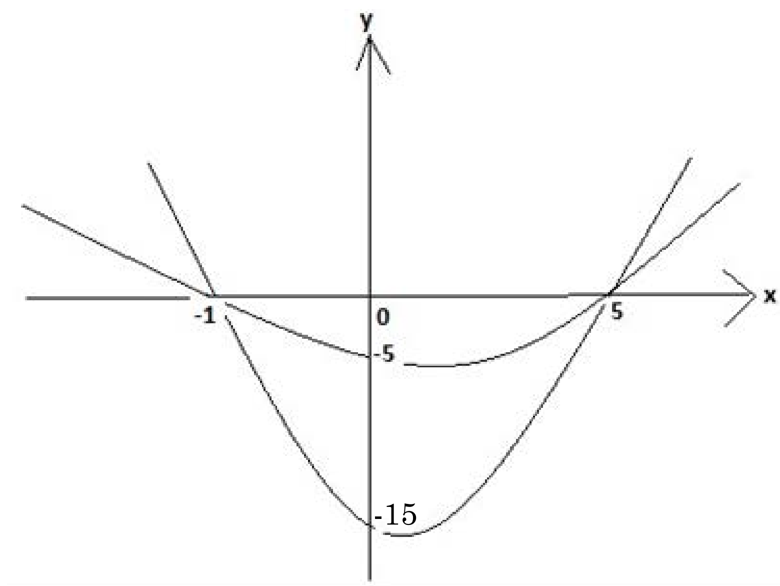
Working in groups,

1. Draw the graphs of $y = x^2 - x - 12$ and $y = 2x^2 - 2x - 24$ on the same axes.
2. From your graphs what are the roots of each of the two equations?
3. Now use the method of formulating the quadratic equation given roots in activity 4.
4. Are you able to get both equations?

You may have discovered that there are more than one equation to the given roots. In fact, there are more than two equations whose roots are 4 and -3. You should also discovered that the method of working backwards works only where the product of the roots equals the y-intercept of the graph. Where the product of the roots is not the same as y intercept of the graph we need another method.

Example 8

Find the equations of the graphs in the diagram below:



Solution

Since $-1 \times 5 = -5$, the equation of the graph which cut the y axis at -5 is

$(x + 1)(x - 5) = 0$ i.e. $x^2 - 4x - 5 = 0$.

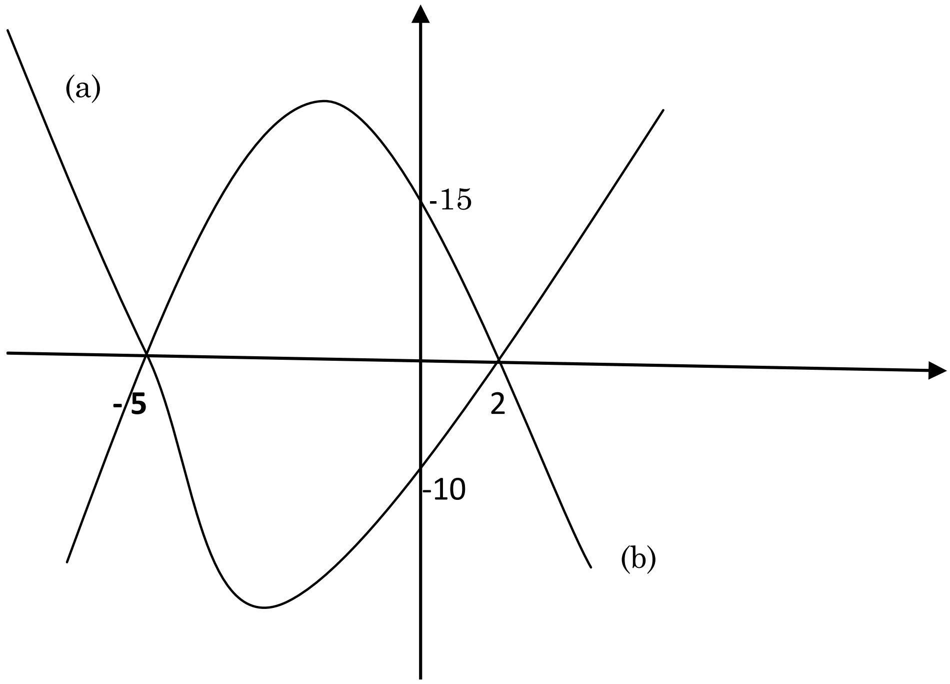
Now -15 is $3x - 5$, hence the equation of the other graph is $3(x^2 - 4x - 5 = 0)$ i.e.

$3x^2 - 12x - 15 = 0$.

Example 9:

Finding equations of graphs

Find the equations of the graphs in the diagram below:



Note that $x = -5$ and $x = 2$ are the x-intercepts.

Since $-5 \times 2 = -10$, the equation of graph (a) is
$y = (x + 5)(x - 2) = x^2 + 3x - 10$.

For graph (b), the y intercept is $\frac{15}{-10}$ or $\frac{3}{-2}$ or times the y intercept of graph (a) so the equation of graph (b) is $y = \frac{3}{-2}(x^2 + 3x + 10)$ i.e.
$y = -\frac{3}{2}x^2 - \frac{9}{2}x - 15$

Example 10:

Finding equations of graphs

Find the equation of the parabola which has roots $x = -2$ and $x = -6$ and cuts the y axis at $y = 4$

Solution

$-2x - 6 = 12$

Now $4 = \frac{1}{3}$ times 12

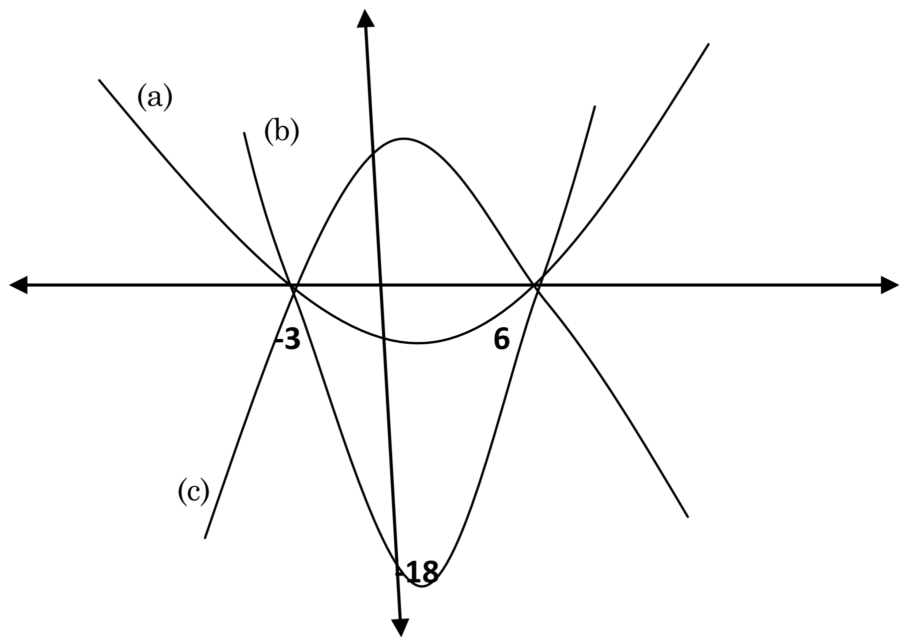
Hence the equation of the parabola is $y = \left\{\frac{1}{3}(x + 2)(x + 6)\right\}$

$$y = \frac{1}{3}\left\{ x^2 + 8x + 12 \right\}$$

$$y = \frac{1}{3}x^{2} + \frac{8}{3}x + 4$$

Exercise 15h:

Find the equation that defines each of the following graphs.



1. Find the equation of the parabola which has

a. Roots $x = 3$ and $x = -4$ and its y-intercept is -12. Find the equation of the parabola.

b. x-intercepts 3 and 4 and its y-intercept is -6. Find the equation of the parabola.

c. has x-intercepts 4 and -3 and its y-intercept is 24. Find the equation of the parabola.

2. One of root of a quadratic equation is $x = 0.5$. The graph has a turning point at (2, 3.5). Find the equation of the parabola.

3. Two quadratic graphs cut the x axis at $x = -2$ and at $x = 5$. The y-intercept of the first graph is 5 and the y-intercept of the second graph is -10. Find the equation of the second graph. ($y = -1/2x^2 + 1.5x + 5$

Prepare your own questions on finding the equations of parabolas which cut the x-axis.

Unit summary

In this chapter, you have learnt to draw and interpret graphs of quadratic equations, how to solve graphically quadratic equations and linear equations and how to formulate quadratic equations given roots and given graphs which cut the x axis. You also learnt to find minimum and maximum value of a quadratic function.

Unit review exercise

1. Draw the graph of $y = x^2 – 3x + 2$, taking the values of x between o and 4. Use a scale of 2cm to represent to represent 1 unit on the horizontal axis and 1cm to represent 1 unit on the vertical axis. From the graph, what is the minimum value of the quadratic function?

2. By drawing the quadratic graph, find the maximum value of the quadratic function $y = 1 – 2x – 3x^2$. Draw the graph in the range of x values from -3 to 3 and using a scale of 2cm to 1 unit in the x axis and 1cm to 1 unit in the y axis.

3. Solve graphically the equation $x^2 – 4x + 7 = x + 1$. Use 2cm to represent 1 unit in the x axis and 2cm to represent 5 units in the y axis.

4. Draw, using the same scales and axes, the graphs of $y = x + 3$ and $y = x^2 – x + 1$ for values of x from –3 to +4. Use the graph to solve the following equations:

   (a) $x^2–2x – 2 = 0$

   (b) $x^2 – x – 2 = 0$

5. By drawing the graphs of $y = –x^2 + 4$ and $y = x + 2$ solve the equation
   $−x^2 + 4 = x + 2$.

6. The equation $y = – 0.035x^2 + 1.4x + 1$ where x and y are in measured in cm is a model of the path taken by a bullet

fired from the ground level. What is the maximum height reached by the bullet?

7. Members of a science club at a secondary school launch a model rocket from ground level and the velocity of the rocket is given by the formula $h = -16t^2 + 96t - 128$ where h is the height in metres and t is the time in seconds. After how many seconds will the rocket reach 128m above the ground?

8. A Blantyre City Council firefighter aims a hose at a window 25m above the ground . The equation $y = 5 + 2x - 0.05x^2$ describes the path of the water.

   How far from the building is the firefighter?

Glossary

Line of symmetry: A line that divides the quadratic graph into two congruent halves.

References:

1. Rheta N. Rubenstein et al (1995), *Intergrated Mathematics* McDoughtal Little, New York

2. Larson, Kanold, Stiff (1998), *Heath Algebra 2*, Heath and Company, New York

Unit 16

INEQUALITIES

In your JCE mathematics, you learnt how to show inequalities on a number line. You also learnt how to formulate and solve inequalities.

In this unit, you will learn to graphically present inequalities, how to graphically illustrate simultaneous linear inequalities, and how to graphically illustrate the solution to simultaneous inequalities in two variables. You will also learn how to present inequalities in two variables graphically.

Inequalities are used in modeling real life situations such as those that involve area, finding dimensions of rectangles, triangles and other geometrical shapes. They are also used in finding profits in a business.

Presenting inequalities graphically

Activity 1:

Sketching linear inequalities in one variable graphically

In pairs, discuss the following questions:

1. What do you think a linear inequality in one variable is?
2. Write at least five examples of linear inequalities in one variable and let your teacher check your answers.

A linear inequality in one variable is an expression that contains one of the four inequality symbols and has its *one* variable raised to the power of 1. The examples of linear inequalities in one variable are $x > 5$, $x \leq 2.5$, $x > 0$, $y \leq 5$, $y > 3$, $x \leq 1.5$. To sketch a linear inequality in one variable you use either a dashed line or a solid line. A dashed line is used when the inequality symbol is $>$ or $<$. The dashed line shows that the number through which the line passes is not included in the solutions. When the symbol is $\leq$ or $\geq$ use the solid line to show that the number through which the line passes is included in the solution set.

Example 1:

Present the graph of the following inequalities on xy-plane.

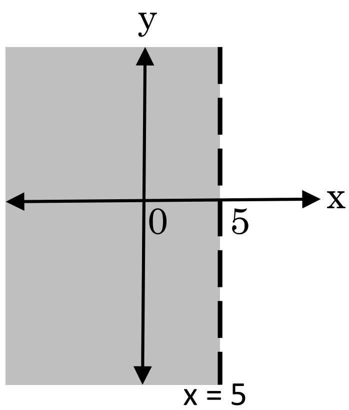
(a) $x > 5$

(b) $y \leq 3$

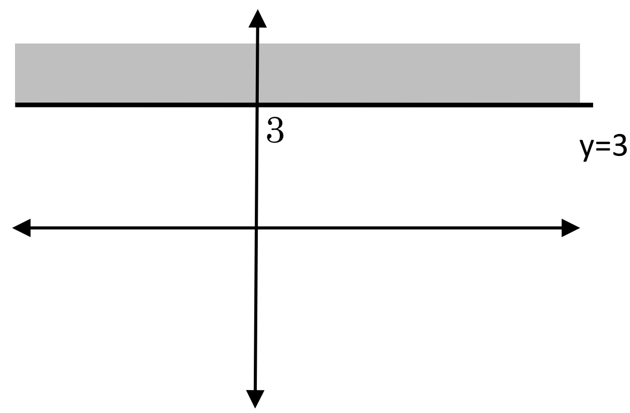
Solutions

To present the above graphs, first replace the inequality symbol by " = " and draw the line of the resulting equation. Note that the line will depend on the inequality symbol in the given question. Then identify the side of this line containing the solutions and shade the unwanted side of the line. The graphs are shown below:

(a)



(b)



Example 2:

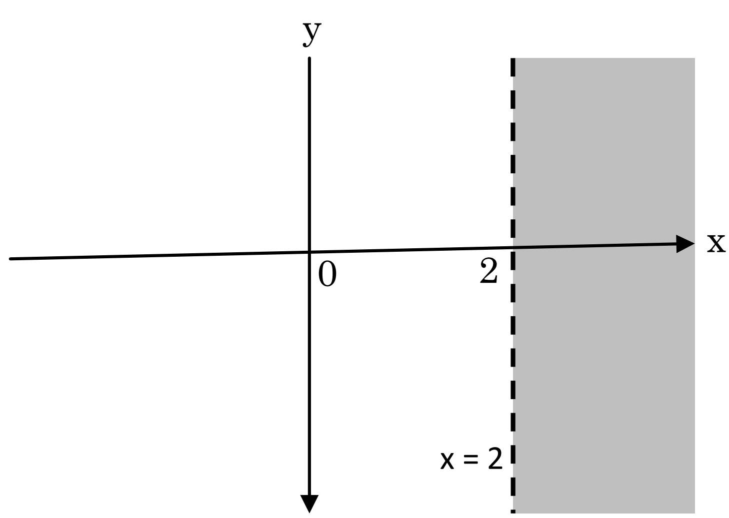
Graph the solution of x –2 > 0, where x is a real number in the xy -plane.

Solution

$x - 2 < 0$

$\therefore x < 2 + 0$

$\therefore x < 2$



Exercise 16a

Sketch the graphs to show the region represented by the following inequalities in the xy-plane.

1. $x < 4$
2. $y < -2$
3. $y \leq -1$
4. $x + 2 > 5$
5. $3 < 7 - x$
6. $x - 2 > 1$
7. $y \leq 4$
8. $y \leq 5$

Illustrating simultaneous linear inequalities graphically

Simultaneous linear inequalities are inequalities composed of two inequality statements in one sentence. They may be in one, two or even in three variables. In this book you shall only look at the first two.

Activity 2:

Illustrating simultaneous linear inequalities in one variable graphically

1. In your groups discuss the meaning of simultaneous linear inequality in one variable.
2. Give examples of simultaneous linear inequality in one variable.

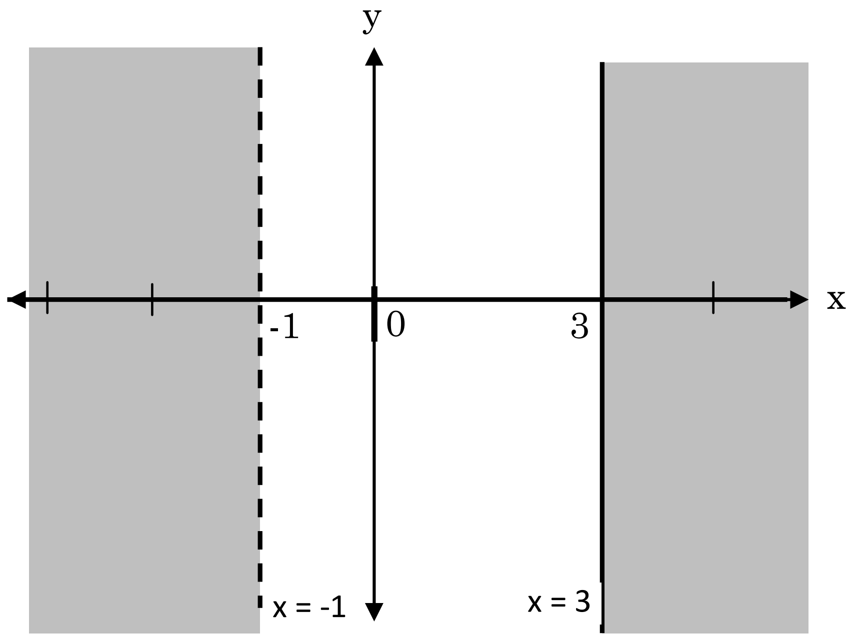
Example 3:

Sketching graphs of inequalities

Sketch the graph to show the region represented by the inequality $-1 < x \leq 3$, in the xy-plane.

Solution:

Note that this inequality is composed of two inequalities $-1 < x$ or $x > -1$ and $x \leq 3$. Inequalities like these are called simultaneous linear inequalities in one variable.



Exercise 16b

Sketch the graph to show the region represented by the following inequalities in the xy-plane:

1. $2< x \leq 4$
2. $4\leq x \leq 5$
3. $1< y < 4$
4. $3\leq y< 3$
5. $1 \leq y \leq 3$
6. $2< x < 3.5$
7. $1 < x < 2$
8. $2.5 <y < 3.5$

Activity 3:

Illustrating linear inequalities in two variables graphically

1. Discuss in groups what linear inequalities in two variables are. Give examples of these inequalities.

   You are given four set of points in the xy – plane above and the line whose equation is $y = -x + 4$ drawn from an inequality $y \leq -x + 4$. Note that this line divides the plane into regions called half planes. By making substitutions into the inequality, find out which of the four sets of solutions satisfy the inequality. Complete the table below:

Inequality	Set of points	Satisfies/doesn't satisfy the inequality(write true/not true)
$y \leq -x + 4$	(-4,6)	True
$y < -x + 4$	(-4,-6)	
$y > -x + 4$	(4, 6)	
$y \geq -x + 4$	(8, 2)	Not true

2. Make a general statement about shading the unwanted side basing on the results you found in 2.

3. Try other points and verify that the statement you have made in 2 is correct.

4. Now replace the symbol $\leq$ in the inequality by one other symbol and make substitutions as in 1, each time recording which set of solutions satisfies the new inequality.

You should have seen that for the inequities $y \leq -x + 4$ and $y < -x + 4$,the set of points are true only for the points below the line $y = -x + 4$. This means all the points below this line will satisfy the inequalities. Hence the region above this line is an unwanted region.

For the inequalities $y > -x + 4$ and $y \geq -x + 4$, the set of points are not true only for the points below the line $y = -x + 4$. This means all the points below this line will not satisfy the inequalities. Hence this region is an unwanted region. In other words, for the inequities $y > -x + 4$ and $y \geq -x + 4$, the wanted region is above the line $y = -x + 4$.

You can now generalise the results as follows:

As long as y (or any variable representing the vertical axis) is on the left side of the inequality statement and is positive,

1. Shade above the line if the inequality symbol is $\leq$ or $<$.
2. Shade below the line if the inequality symbol is $\geq$ or $>$.

If y is on the right side of the inequality symbol, you may need to rearrange the inequality for you to apply the results above.

Example 4:

Illustrate graphically the region represented by the following inequalities on the xy plane:

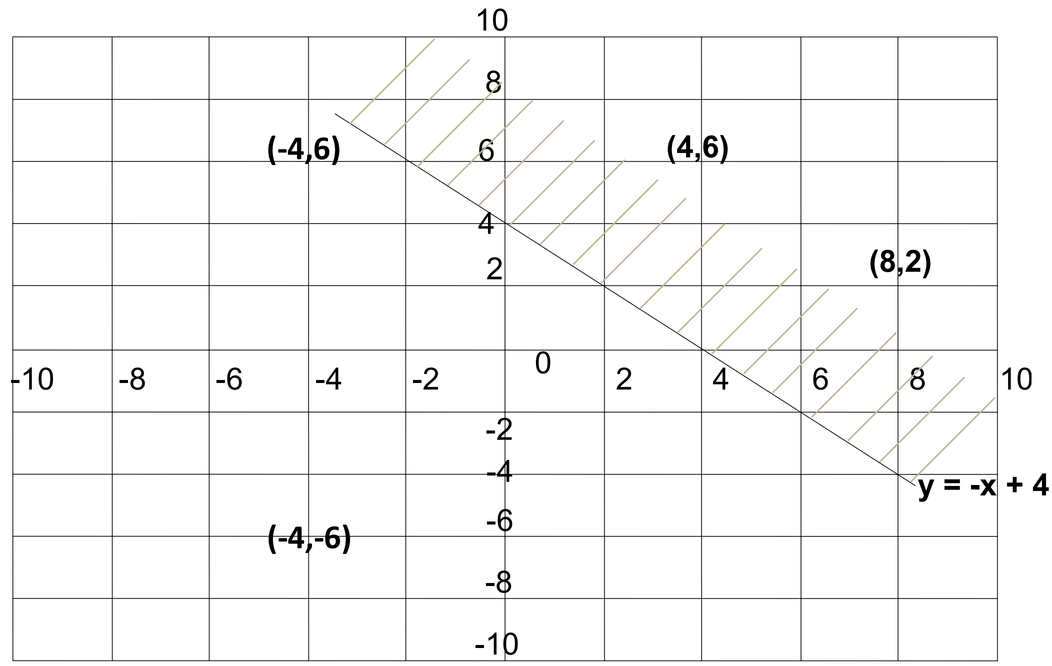
(a) $x + y < 4$

(b) $x + y \leq 4$

Solutions:

(a)
- First know where the line $x + y = 4$ will cross the x − axis by putting $y = 0$ in the equation i.e. $x + 0 = 4$, $\therefore\ x = 4$
- Also know where the line $x + y = 4$ will cross the y − axis by Putting $x = 0$ in the equation i.e. $0 + y = 4$, $\therefore\ y = 4$.
- You draw the graph of $x + y = 4$ as a dashed line because the Inequality symbol is $<$.
- As y is on the left side of the inequality and is positive, you shade above the drawn line (Rule 1 activity 3)

Graphically the line will look like this

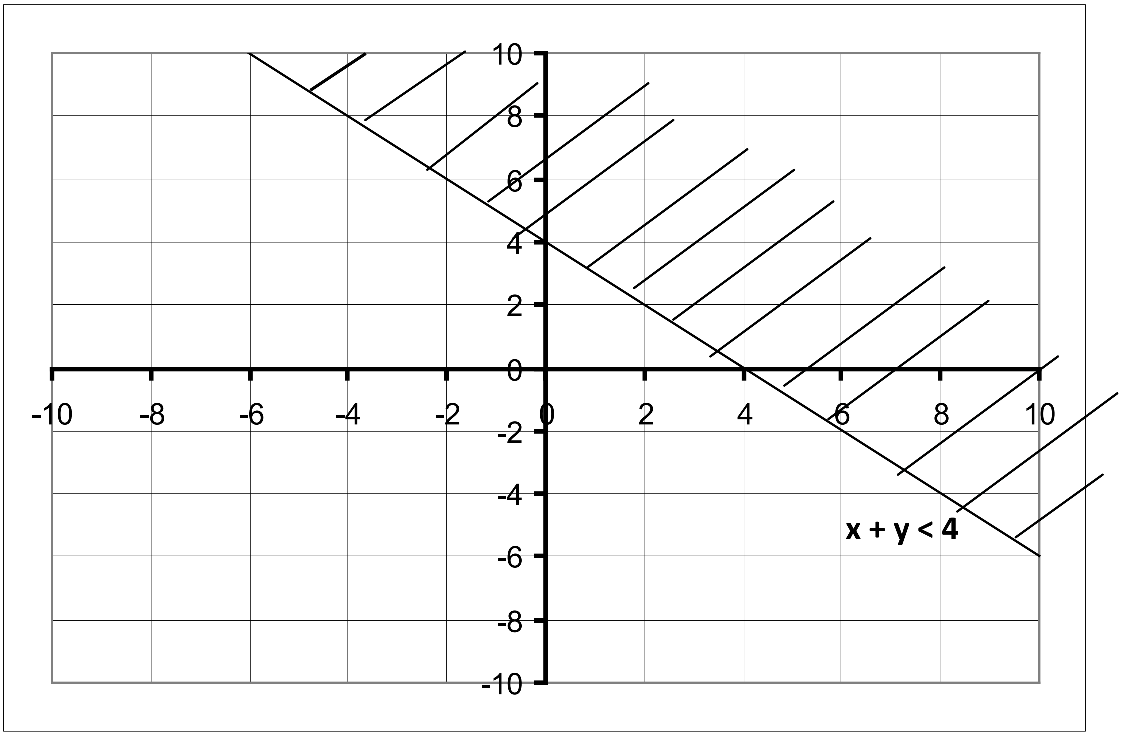


In figure above, all points in the unshaded area satisfy the inequality

$x + y < 4$.

(b) Similarly, the line to be used for the inequality is $x + y \leq 4$. The only Difference is that the line will not be dashed but a solid line because the boundary is included in the

The graph will look like this.



Example 5:

Graph the inequality 2x + y > 6

Solution:

(a) Lets know where the line 2x + y = 6 crosses the x and y axes by putting y and x = 0 respectively into the equation
$2x + y = 6$

$2x + 0 = 6$

$2x = 6$

$x = 3$

$\therefore$ It will cross the x axis at x = 3

Similarly putting x = 0

$2 \times 0 + y = 6$

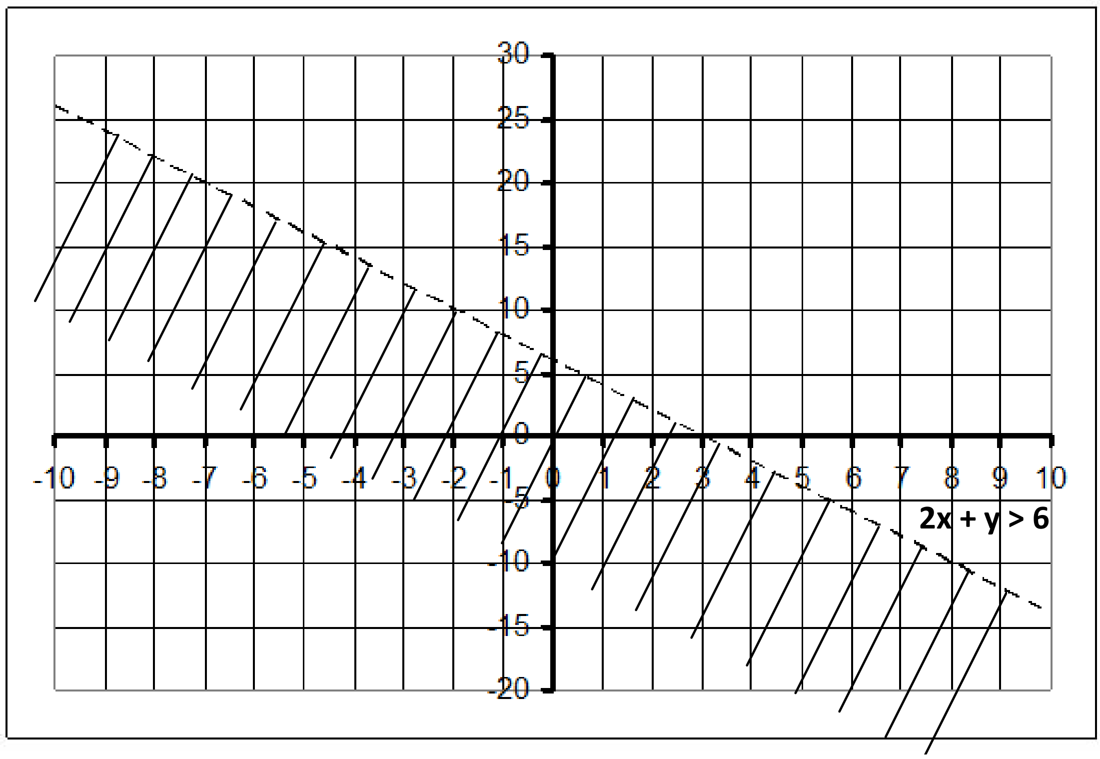
$0 + y = 6$

$y = 6$

$\therefore$ It will cross the y-axis at y = 6

From result 2 of activity 3, you shade below the line 2x + y = 6.

The graph will look like this:



All points in the unshaded area satisfy the inequality 2x + y > 6.

Example 6:

Graph the inequality x – y ≤ 3

Solution:

For line x – y = 3

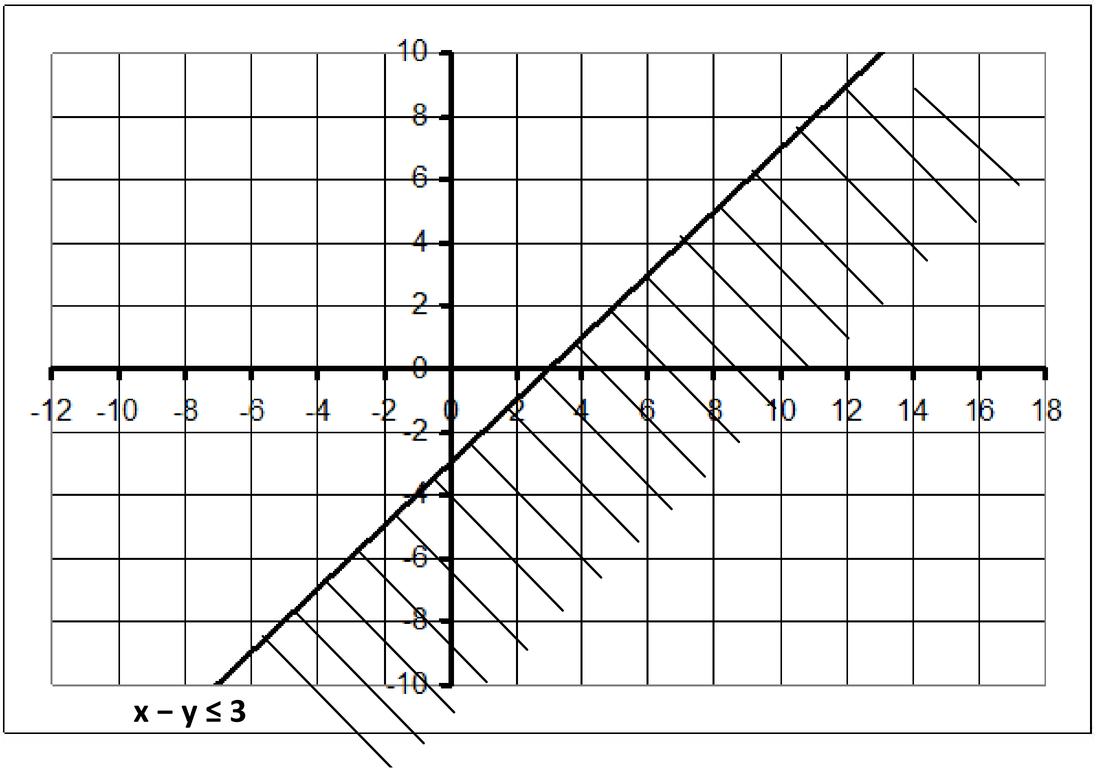
If x = 0, then y = −3

If y = 0, then x = 3

∴ The line will cross points (0, −3) and (3, 0)

Also note that y is on the left side of the inequality and is negative. Rearranging, you shall have y ≥ 3 – x and so you shade below the line.

Graphically, it will look like this:



Exercise 16c

Graph the following inequalities in the xy-plane:

1. $x + 3y < 6$
2. $2x + y > 4$
3. $x + y > 5$
4. $2x - y \geq 4$
5. $x - y \geq 0$
6. $-3x - 2y < -6$
7. $x - 2y < 0$
8. $-2x - 2y > 0$
9. $3x - 5y \leq 15$
10. $x + y \geq 2$

Illustrating simultaneous linear inequalities in two variables graphically

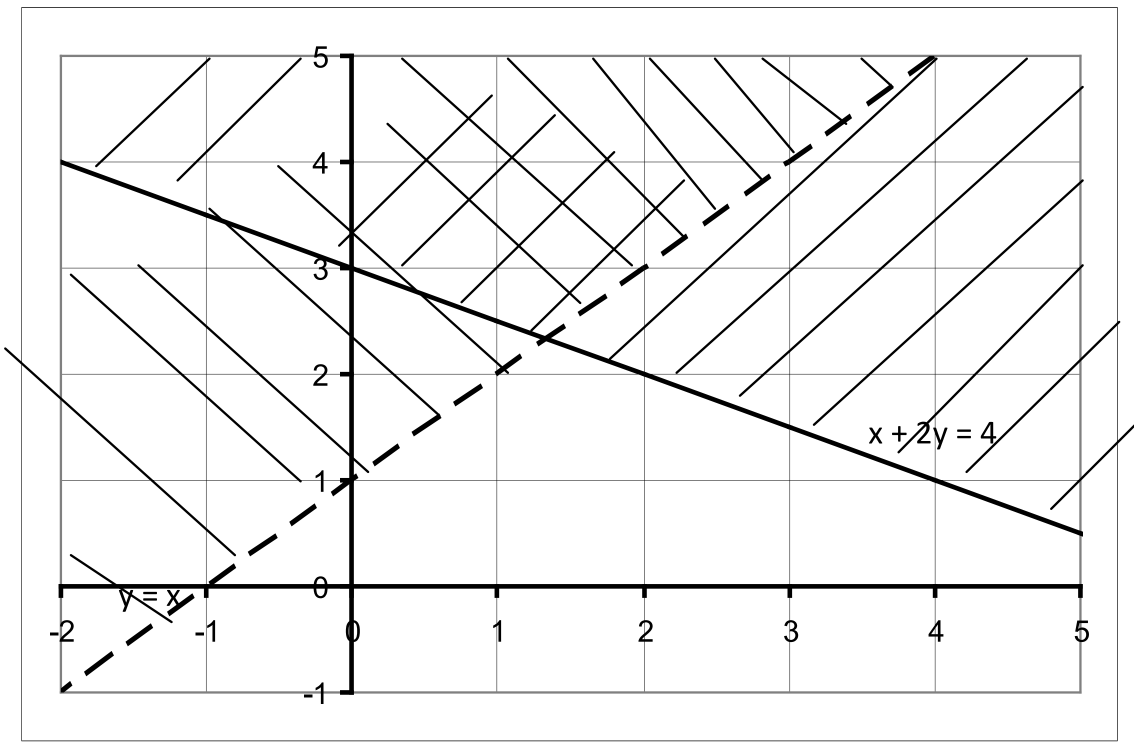
Two or more inequalities may be graphed in the same coordinate plane. The intersection of these graphs is the solution set for the system of inequalities. In activity 3, you learnt how to graph linear inequalities in two variables graphically. This is the same method used to illustrate simultaneous linear inequalities in two variables.

Example 7:

Show graphically the solution set of the system y < x and x + 2y ≤ 4.

Solution

Using ideas of activity 3 you come up with the following graph:



The solution set for $y < x$ and $x + 2y \leq 4$ is the unshaded area.

Exercise 16d

Graph each of the following simultaneous linear inequalities:

1. $x + y \geq 2$ and $x - y < 2$
2. $x + y \leq 3$ and $x - y < 3$
3. $x \geq y$ and $x + y \leq 6$

4. $-2x - 3y > -6$ and $3x - 2y > 6$

5. $y \geq 2x$ and $y \geq -x$

6. $2x - 5y \leq 10$ and $x + y \leq 5$

7. $y - x \leq 2$ and $x + y < 1$

Writing down inequalities that describe a given region

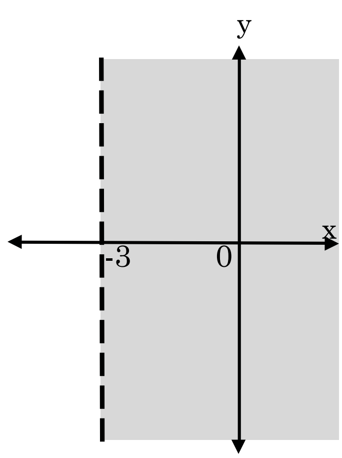
To write down an inequality describing a given region, you need to consider a number of things;

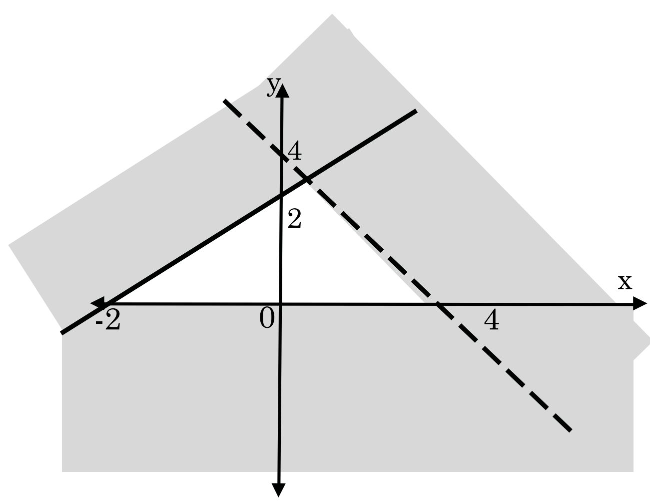
- whether the line dividing the plane into half planes is dashed or solid.
- which side of the line is the wanted side or contains the solutions of the inequalities.
- the equation of the line dividing the plane into half planes.

Example 8:

Inequalities for a given region

Write down the inequalities that describe the unshaded regions below;





Solutions

(a) The line passes through $x = -3$ so its equation is $x = -3$. The line is dashed so the inequality symbol must be $<$ or $>$. The wanted side is to the left of the line where numbers are less than -3. Hence the inequality is $x < -3$.

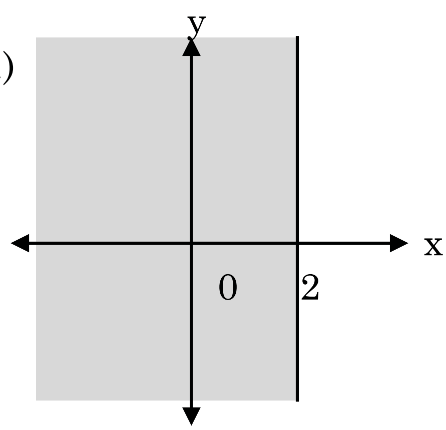
(b) You must note that in this example there are three lines bordering the unshaded region. The gradient of the line passing through (–2, 0) and (0, 2) is $\frac{2-0}{0- -2} = 1$. Therefore its equation is $y - 0 = 1(x - -2)$ i.e. $y = x + 2$. The line is solid so the inequality symbol must be $\leq$ or $\geq$. The line is shaded above and in the equation, y is on the left and is positive. Therefore the inequality is $y \leq x + 2$ (activity 3).

By a similar method, the equation of the line passing through (4, 0) and (0, 4) is $y = -x + 4$. The line is dashed so the inequality symbol must be < or >. The line is shaded above and y is on the left of the equation and is positive therefore the inequality is $y \leq -x + 4$.The equation of the third line is $y = 0$ and is shaded below. Therefore the inequality is $y \geq 0$.

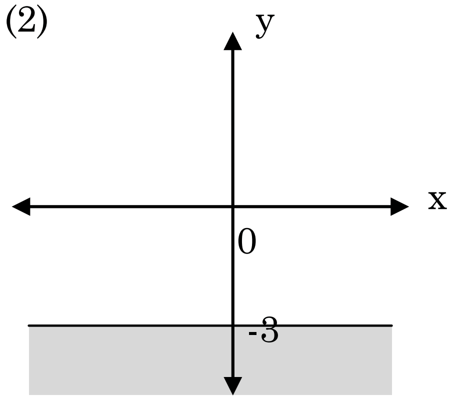
Exercise 16e

Write down the inequalities that describe the unshaded region in each of the following:

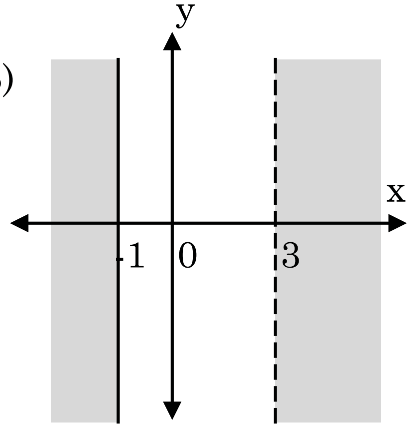
(1)



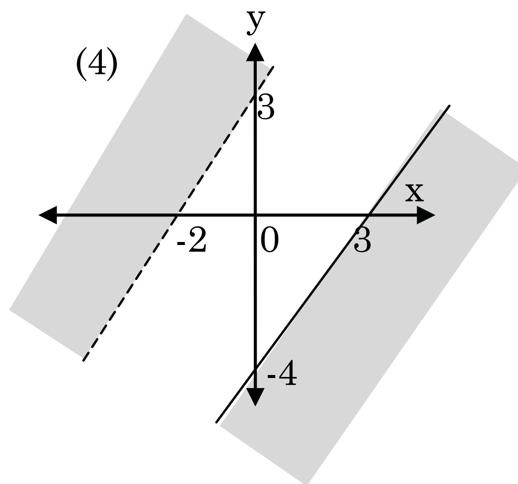
(2)



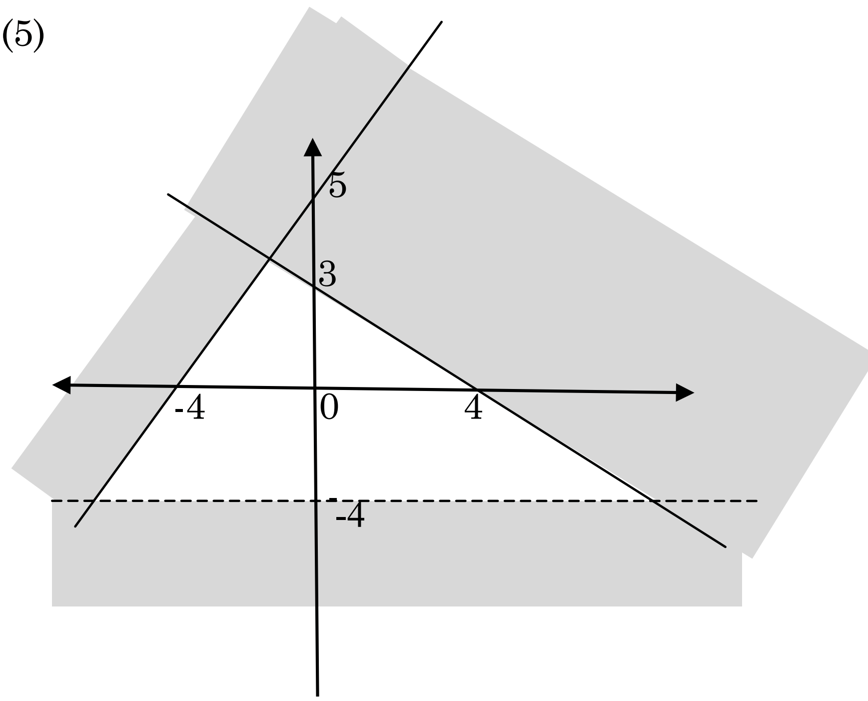
(3)



(4)



(5)



Unit summary

In this unit, you have learnt to present inequalities in one and two variables graphically. You have also learnt to illustrate graphically the solutions to simultaneous linear inequalities in one and two variables and writing down inequalities that describe a given region.

Glossary:

Wanted region: The region that contains solutions to an inequality or a set of inequalities.

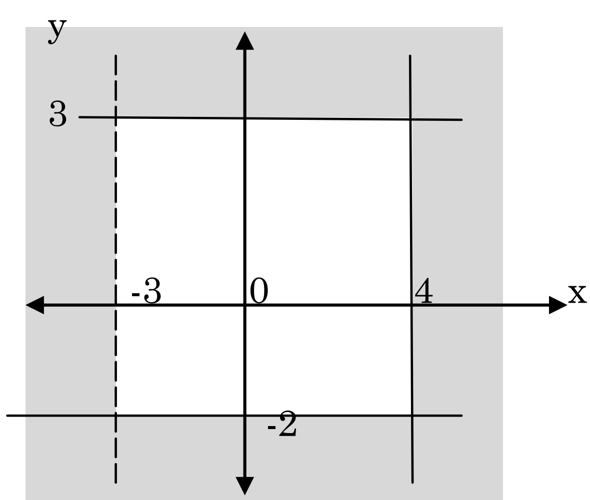
Linear inequalities: Inequalities whose equations produce a straight line graph.

Unit review exercise

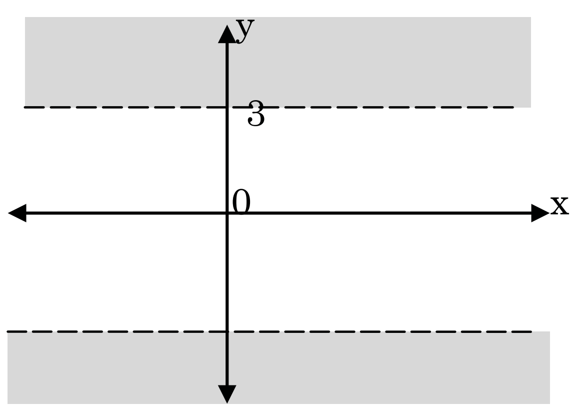
1. Draw diagrams to represent the following inequalities:

    1. $x \geq 2$
    2. $y \geq -3$
    3. $x < 5$
    4. $y < 2$
    5. $-2 < x < 5$
    6. $2 < x + y \leq 5$

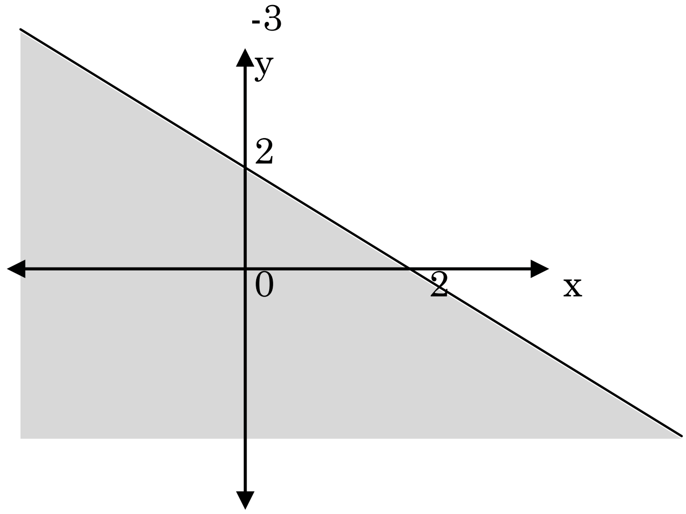
2. Write down the inequalities that describe the unshaded regions in each of the following:



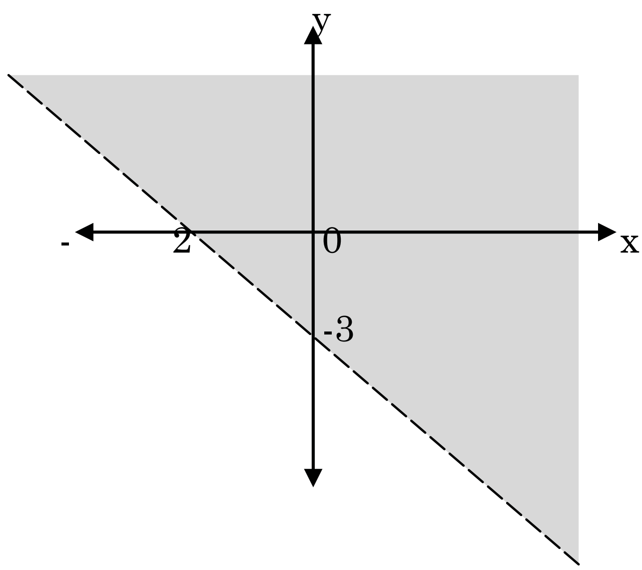
(a)



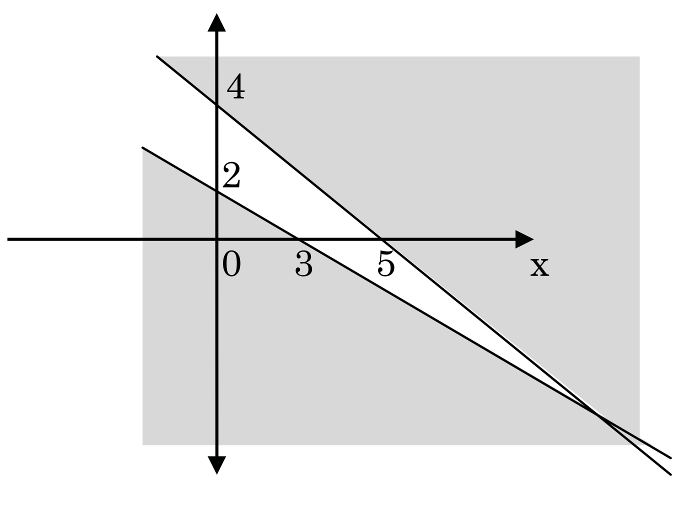
(b)



(c)



(d)



(e)

Unit 17

STATISTICS

In your JCE Mathematics you learnt about some statistics. You learnt about collecting and classifying and presenting data in form of graphs.,In this unit, you will learn about organisation of data, presenting data in form of charts and tables, calculating measures of central tendency and spread and interpreting data.

Organisation of data

When you first look at some data, all you can see is a jumble. Such data is called **raw data** or **unprocessed data. Raw data** is hard to analyse. Therefore data must be organized in such a way that it can easily be understood. In the activities that follow, you will learn how to organize data so that it can easily be understood.

Activity 1:

Classifying grouped data

There are two types of data: Qualitative and **Quantitative** data.

Qualitative data are non-numerical data e.g. the texture and colour of a fabric.

Quantitative data are numerical data e.g. the numbers of people in a room or the height of a person.

Quantitative data is further classified into two groups: **Discrete and Continuous data.**

Discrete data are countable for example: The number of rooms in a house or a person's shoe size. Continuous data are measured data, for example: length, weight, temperature and time are all measured on a continuous scale.

Now in pairs, discuss whether the following data are discrete or continuous: Explain your answers.

1. Your weight from birth to age 14.
2. The time you get up each morning for one month.
3. The number of mangoes you sell at the market each week.
4. The number of peas in a pod.
5. The age of pupils at your school.

Example 1:

Classifying data

Classify the following groups of data as discrete or continuous:

a. a group of 16-20 years of age.

b. a group of students who attended lessons for 15-30 periods in a week.

Solutions

a. It is continuous data because between any two weights you measure any value is possible.

b. It is discrete data because number of periods is distinct. You cannot have one and half periods or two and three-quarter periods e.t.c.

Exercise 17a

Classify the following grouped data as discrete or continuous. The data properties or what the data stand for is shown in the brackets against each group**:**

1. 50 – 80 (marks in %)
2. 20 – 30 (temperature in degrees Celsius)
3. 40 – 60 (speed in km/h)
4. 61 – 70 (heights in cm)

5. 100 – 150 (weight in kg)
6. 10 – 15 (Number of rooms in houses)
7. 4 – 6 (Shoe sizes)
8. 2 – 10 (number of people in cars)
9. 0 – 9 (number of absentees)
10. 19 – 20 (age group)
11. 0 – 99 (monthly salary in Kwacha)

Class intervals

In many cases you may have a set of many items e.g. numbers. It is hard to make an analysis of the numbers or it is hard to make any interpretation from the set. To get a clearer picture of the data you group the data within **class intervals**. Each class interval has the beginning and end. The beginning is called the **lower limit** and the end is called the **upper limit**. The number of items from the lower limit to the upper limit is called **class width.**

The following things need to be considered when grouping data within class intervals:

1. The starting point of the first class interval must be the smallest number in the given set or a number just below that.
2. Each class interval must be of the same class width.
3. The class intervals must not overlap.
4. The last class interval must contain the highest number in the given data.

Activity 2:

Forming class intervals

You are given a set of different numbers representing marks scored by students in a mathematics test marked out of 100. The lowest mark is 1 and the highest mark is 100. In pairs form class intervals given that

(a) The first class interval is 1 – 10.

(b) The first class interval is 1 – 20.

Compare your work with that from other pairs.

Suppose the first class interval was 1 – 5, why would the intervals 1 – 5, 5 – 9, 9 – 13 e.t.c. be wrong?

Example 2:

Class intervals

The table below gives masses, in kg, of 30 students:

43	45	50	47	51	58	52	47	42	54
61	50	45	55	57	41	46	49	51	50
59	44	53	57	49	40	48	52	51	48

Table 1

Form class intervals with 40 – 45 as the first class interval.

Solution

First note that this is continuous data and as such the class intervals must cover all possible masses of data. Hence you can have the following intervals: 40 – 45, 45 – 50, 50 – 55, 55 – 60 and 60 – 65.

Example 3:

Class intervals

The number of people passing through a check point was recorded after every 30 minutes for 24hours. The smallest number recorded was 3 and the highest number recorded was 40. Form class intervals for the data with 1 – 5 as the first interval.

Solution

Unlike in example 1, this is discrete data and so there are no half values between the class intervals. Hence the intervals are 1 – 5,

6 – 10, 11 – 15, 16 – 20, 21 – 25, 26 – 30, 31 – 35, and 36 – 40.

Exercise 17b

In questions 1 – 5, the first class interval and the class of data are given. Form the next 5 class intervals for each question.

1. 1 – 9 ;(discrete data)
2. 1 – 15; (discrete data)
3. 40 – 45; (continuous data)
4. 20 – 30; (continuous data)
5. 40 – 59; (continuous data)

In questions 6 – 10 the lower limit of the first class interval, the class width and the class of data are given. Form 5 class intervals for each question.

6. Lower limit = 1, class width = 5, discrete data.
7. Lower limit = 10, class width =10, discrete data.
8. Lower limit = 0, class width = 20, discrete data.
9. Lower limit = 50, class width = 5, continuous data.
10. Lower limit = 8, class width = 6, discrete data
11. 50 cars were tested to see how far they had travelled on 10

Litres of certain petrol and the distances travelled in km were recorded as shown in table 2 below:

Table 2

100	110	130	120	140	121	127	142	126	143
130	134	107	131	145	111	146	112	147	144
131	130	132	129	108	132	131	125	149	140
128	106	133	128	128	148	123	133	141	122
135	139	134	122	143	101	128	138	121	149

The first two of the five class intervals for the data are 100 – 109 and 110 – 119. Write down the other intervals.

Activity 3:

Determining class boundaries

A class boundary separates one class from the other i.e. the ending point of one group is the starting point of the other. In groups, discuss how you can determine the class boundaries for the following class intervals: The data is continuous.

1 – 9, 10 – 19, 20 – 29, ---

Present your work to class.

Each class boundary lies halfway between the upper limit stated in one class interval and the lower limit stated in the next. This means the class boundaries are the averages of the two limits in the intervals given. The number of values from the lower limit to the upper of each class gives the class width.

Example 4:

Class boundaries

The following are intervals of temperature measured in degrees :

1 – 5, 6 – 10, 11 – 15 … Determine the class boundaries for the groups.

Solution

Get the averages of the upper and lower limits for adjacent class intervals as follows:

$\frac{0+1}{2}$, $\frac{5+6}{2}$, $\frac{10+11}{2}$ = 0.5, 5.5, 10.5, ---

So the class boundaries are 0.5, 5.5, 10.5 and so on.

Example 5:

Class intervals

The following are class intervals of ages of a group of students.

0 – 9, 10 – 19, 20 – 29.

Determine the class intervals for the groups.

Solution

Ages are usually given in completed years. For example you say someone is 9 years old from his or her 9^{th} birthday until just under their 10^{th} birthday. This means someone who is 9 years 364days i.e. 10 years belongs to a group

0 – 9. So for ages, 0 – 9 means from 0 to just less than 10 years. So the class boundaries for the groups are 0, 10, 20, 30.

Exercise 17c

1. The intervals below are weights (in grams) of a number of tomatoes for grading purposes:

   60 – 65, 65 – 70, 70 – 75, 75 – 80, 80 – 85.

   Determine the class boundaries for the intervals.

2. The following are the intervals of lengths (in mm) of 30 leaves measured by students in a statistics lesson:

   30 – 40, 40 – 50, …, 90 – 100.

   What are the class boundaries for the groups?

3. The ages of people attending a lesson on environment are grouped as follows: 0 – 9, 10 – 19, 20 – 29, 30 – 39, 40 – 59,

60 – 69 , 70 – 79.

Determine the class boundaries of the intervals.

4. The temperature, measured to the nearest degree Celsius, recorded at a certain place over a period of time is grouped as follows:

   0 – 14, 15 – 29, 30 – 34, 35 – 49, 50 – 64, 65 – 79.

   What are the class boundaries for the data?

5. Several books were collected to see how many pages each book had. The pages of the books were put in the following categories:

   1 – 5, 6 – 10, 11 – 15, 16 – 20, 21 – 25 , 26 – 30.

   Determine the class boundaries of the groups.

Activity 4:

Finding mid-points of class intervals

In pairs, discuss what "mid-point of a class interval" is.

Suppose 40 – 59 is a class interval, what will be the mid-point of this Interval? Explain how you got it.

The mid-point of a class interval is the value midway between the lower limit and the upper limit of each class interval. It may be calculated by finding the arithmetic mean of the two numbers.

Example 6

Find the mid-point of the interval 86 – 100.

Solution

Mid-point $= \frac{86+100}{2}$

$= 93$

Exercise 17d

Find the mid-points of the following class intervals:

1. 20 – 24
2. 72.8 – 72.9
3. 12 – 30
4. 1 – 10
5. 80 – 100
6. 30 – 39
7. 140 – 144
8. 0 – 99
9. 100 – 119
10. 40 – 50

Presenting data in forms of charts and tables

Diagrams and tables are often used to display data. They are simpler to understand than just crumbled data. Diagrams look more attractive and interesting and help you to spot any patterns and compare things easily. In the activities that follow, you will learn to present data in form of frequency tables, histogram, pie charts and frequency polygons.

Activity 5:

Presenting data in the form of a frequency table

Frequency is how many times a value in the given data occurs. Frequency of things can be shown in a table called **frequency table.**

In your JCE you learnt how to tally the given data and draw frequencies from them. In pairs, discuss how you can come up with a frequency table for the data below. Draw the table. The data shows marks obtained by 25 students in a test.

14	19	9	17	15	20	17	10	15
12	17	11						
17	15	16	17	19	17	12	8	17
10	12	15						
18	15	10	7	10				

From your table

(a) What is the total frequency?

(b) What are the frequencies of the following scores? 12,17 and 20 Present your findings to class.

Exercise 17e

1. Consider the data below:

1	1	2	0	0	3	1	1	2
2	2	1	1	0	4	1	0	2
3	1	2	1	3	0	1	2	1
4	0	1	2	2	0	1	2	2
0	1	2	2	3	3	0	1	4

(a) Draw a frequency table for the distribution.

(b) What are the frequencies of the following: 0 , 4, 1?

Construct a frequency table for the distribution below:

1	1	3	3	3	0	2	0	1	2
0	3	0	4	2	3	0	3	1	3
3	4	0	0	6	3	5	0	1	0
3	1	0	7	4	3	3	0	2	3
1	3	3	2	5	0	3	0	6	3
1	0	3	5	0	1	4	2	3	4

Which value ha the highest frequency?

3. Consider the scores below:

26	12	24	42	16	18	30	24	36	8
34	24	20	16	24	16	18	26	10	32
24	12	26	12	12	20	24	18	14	22
24	14	34	24	20	12	20	24	20	26
32	12	16	42	26	22	24	20	16	18
18	30	24	36	41	35	30	41	10	36

(a) In groups, form class intervals for the data using intervals 5 – 9, 10 – 14, ---

(b) Draw the frequency table for the data.

(c) Which group has the highest frequency?

4. The following marks were obtained by 80 candidates in a Mathematics test which was marked out of 65**.**

54	52	31	47	24	36	27	15	44	26
8	20	46	32	27	31	33	57	39	32
43	32	23	33	31	21	38	28	40	19
52	37	38	39	9	30	47	29	8	13
33	35	48	18	36	39	23	58	34	35
16	21	32	38	34	13	27	32	37	23
37	49	25	38	24	27	48	36	45	18
41	34	43	12	47	24	61	29	37	33

Using intervals 0 – 9, 10 – 19, ---, draw the frequency table for the data.

Activity 6:

Presenting data in the form of the histogram

In groups,

1. Find the class boundaries of the following class intervals: 1 – 5, 6 – 10, 11 – 15, 16 – 20.

2. Given that the frequencies of the above intervals are 2,3,5,7 draw a bar graph on a chart paper with the class boundaries on the horizontal axis and frequencies on the vertical axis.

3. Find the area and the width of each bar.

4. Now for each bar, divide the area by the width. What do you notice?

5. Report your findings to class.

A histogram often looks like a bar chart that you dealt with in your JCE. However the essential characteristic of a histogram is that it represents frequency by area and that there is a numerical relationship between quantities whose frequencies it represents as you saw in instruction 5 in the above activity. In addition there are no gaps between adjacent bars in the histogram.

Example 7:

Drawing histogram

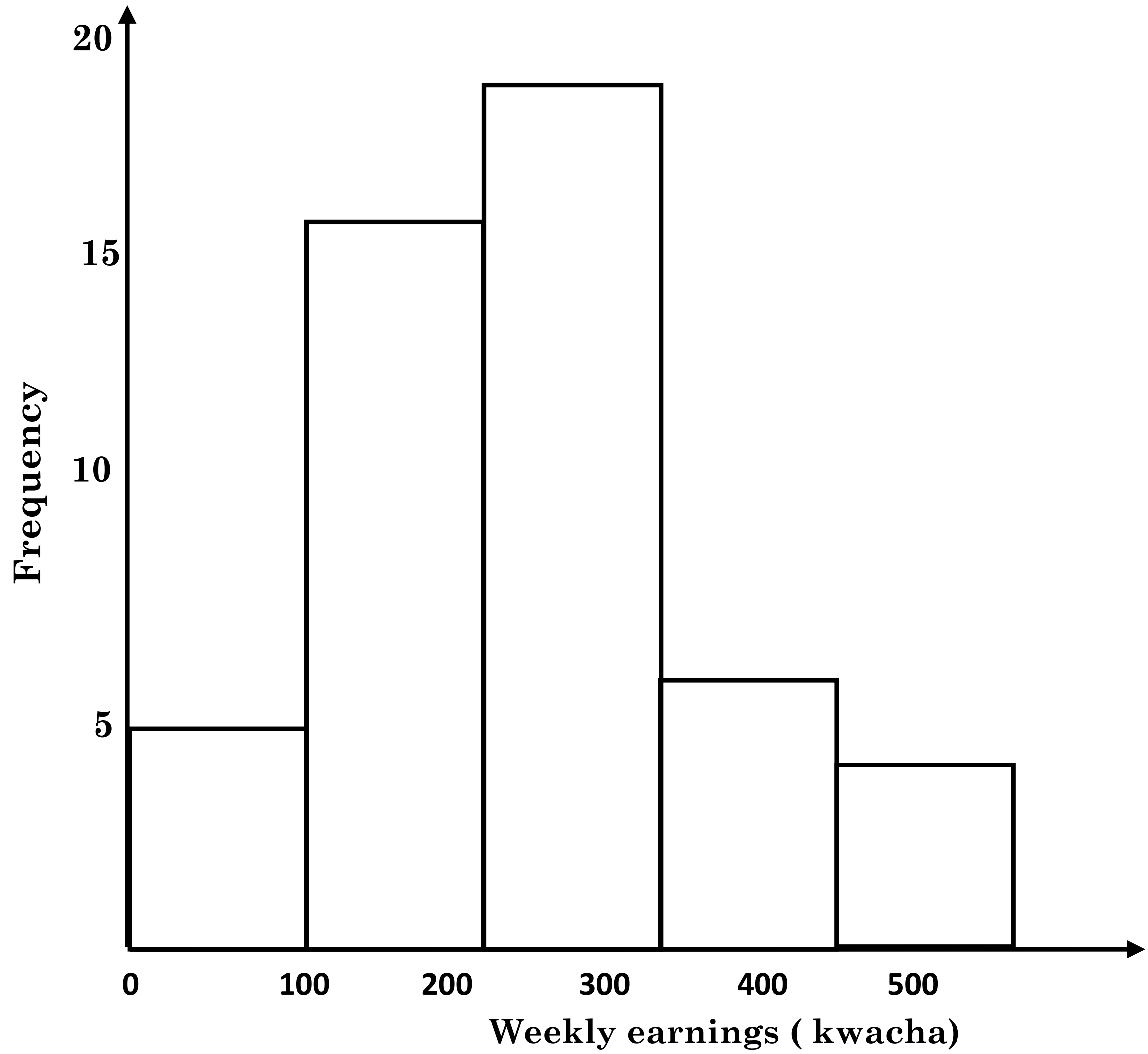
The frequency distribution below shows weekly earnings of 50 people.

Weekly earnings (Kwacha)	0– 99	100 – 199	200– 299	300– 399	400– 500
Number of people	5	16	19	6	4

Draw a histogram for the frequency distribution.

Solution:

As the first interval starts from 0,using the method of finding class boundaries shown in example 3 on page 359 would lead into the first class having a class width different from the others so use the lower limits of the intervals as boundaries. The histogram is shown below:



Example 8:

Drawing a histogram

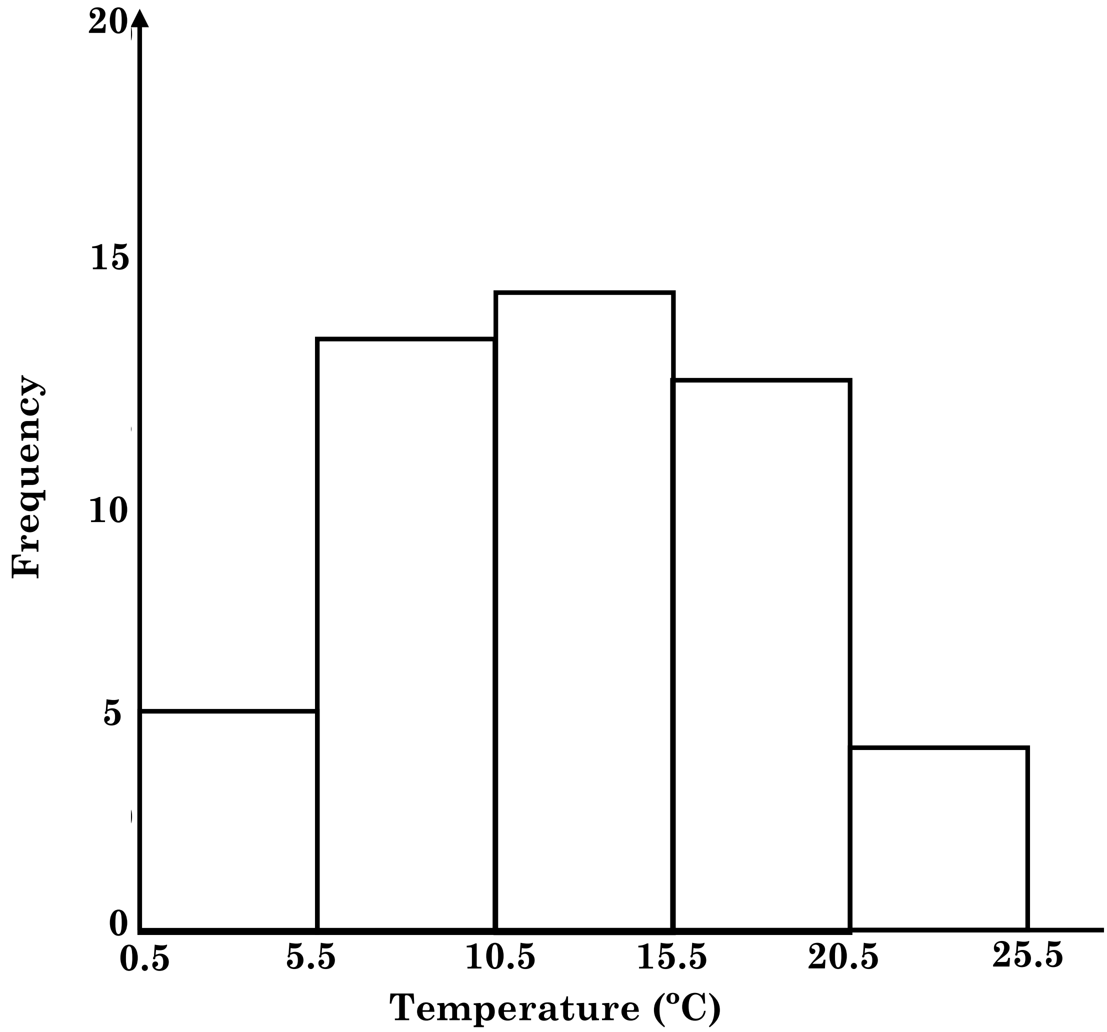
The data below shows a frequency distribution of temperature in degrees Celsius recorded at a weather station over a period of time.

Class interval (t minutes)	Frequency
1 – 5	5
6 – 10	13
11 – 15	14
16 – 20	12
21 – 25	6

Draw a histogram for the distribution.

Solution

The class boundaries are 0.5, 5.5, 10.5, 15.5 and 25.5



Activity 7:

Presenting data in the form of a frequency polygon

If you draw straight lines through the mid-points at the top of each bar of a histogram, the resulting graph is a frequency polygon. Basing on this statement, in groups discuss how you would construct a frequency polygon using the data provided in example 8. Draw the polygon.

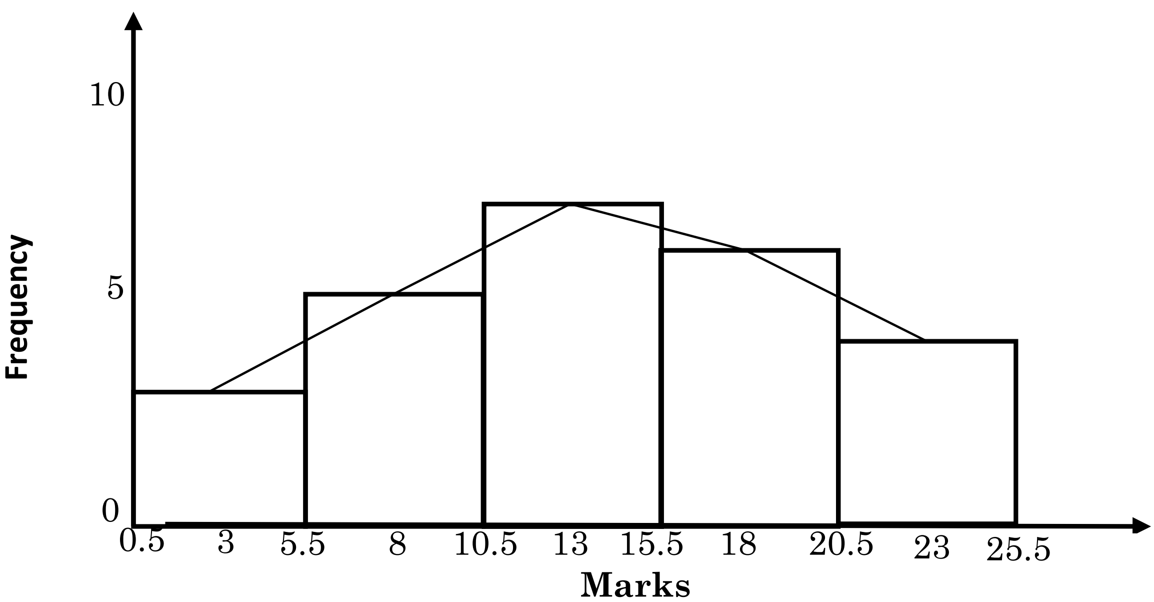
Example 9

Use the frequency table below to draw a frequency polygon.

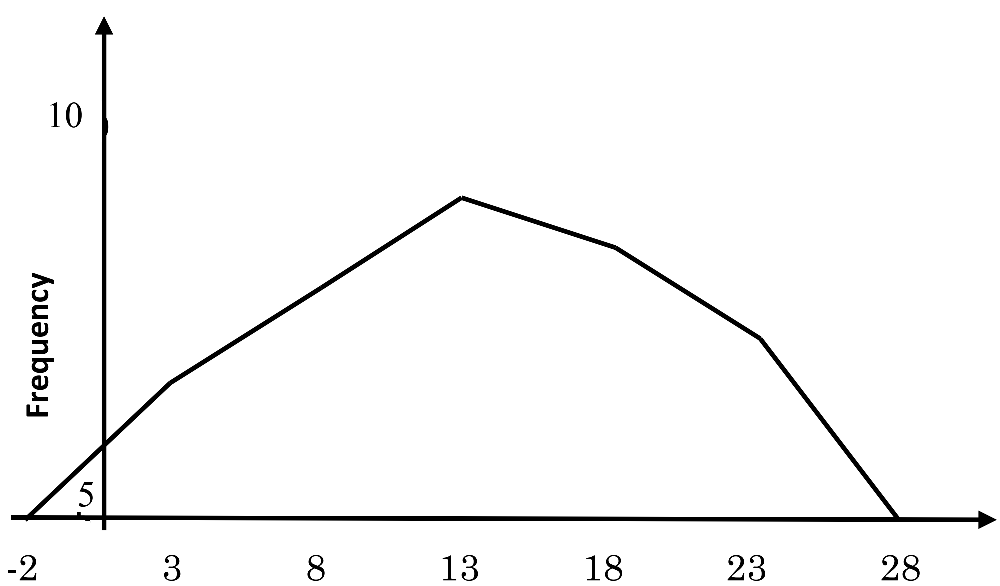
Class interval (Marks)	1 – 5	6 – 10	11 – 15	16 – 20	21 – 25
Frequency	3	5	7	6	4

Solution

The class boundaries are 0.5, 5.5, 10.5, 15.5, 20.5 and 25.5 .The mid points of the class intervals are 3, 8, 13, 18 and 23. You then draw a histogram for the data and then a frequency polygon as below:



If you only use the midpoints, you get the following frequency polygon. Note that extra intervals have been added at the beginning and at the end to remove "hangings" of the polygon.



Exercise 17f

1. Draw a histogram hence a frequency polygon for the table below:

Height, x cm	Frequency
$130 \leq x < 135$	8
$135 \leq x < 140$	14
$140 \leq x < 145$	17
$145 \leq x < 150$	12
$150 \leq x < 155$	4
	Total: 55

2. The temperatures recorded one summer were as follows

Temperature ^{0}F	68	69	70	71	72	73	74	75	76	77	78
Number of days	3	5	12	14	13	8	5	9	6	3	2

Draw a histogram hence a frequency polygon.

3. The lengths of pupils' forearms were recorded as follows:

Length of forearm (cm)	24	25	26	27	28	29	30	31	32	33
Number of pupils	8	9	21	24	20	27	16	22	12	9

Draw a histogram hence a frequency polygon.

4. The year of manufacturing 100 cars in a car factory is:

Year	1994	1995	1996	1997	1998	1999	2000
Number of cars	42	48	57	79	74	85	52

Draw a histogram for the data.

5. The table gives the time taken for students to travel to school.

Time (minutes	$0 \leq t < 10$	$10 \leq t < 20$	$20 \leq t < 30$	$30 \leq t < 40$
Frequency	12	36	8	5

Draw a histogram for the data.

6. The actual time between departure of minibuses at rush hour was recorded to the nearest minute.

Time (minutes)	7	8	9	10	11	12
Number of minibuses	13	18	35	12	19	3

Draw a histogram for the data.

Activity 8:

Presenting data in a form of the pie chart

In your groups do a research on the following:

1. Definition of a pie chart.
2. The steps in drawing a pie chart.

   Report your findings to class.

A pie chart is a circular diagram used to display data. The whole pie stands for the whole amount of data being dealt with and each slice stands for a named part of the data. To draw a pie chart you follow the following steps:

1. Find the total frequency of the given data.
2. Divide 360^0 by the total frequency.
3. Multiply each frequency by the result in 3. This gives the size of each slice.
4. Measure the angles in 3 above anticlockwise at the centre of the circle.

Example 10:

Drawing pie charts

Draw a pie chart to show the following daily life of Takondwa:

Activity	Number of hours
Lessons	5
Meals	1
Homework	3
TV	2
Travel	1

Sleep	8
Other	4

Solution

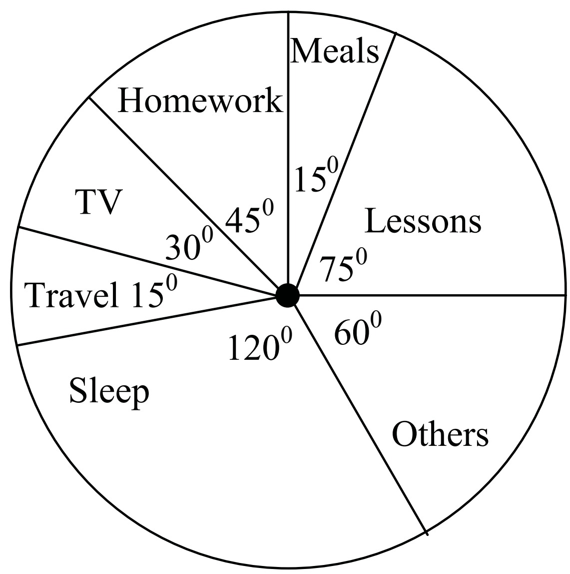
Total frequency = 5 + 1 + 3 + 2 + 1 + 8 + 4 = 24

Angle for 1 of each activity $= \frac{360^0}{24^0}$

$= 15^0$

Angles for each activity are $5x15^0$, 1×15^0, $3x15^0$, $2x15^0$, $1 x15^0$, $8x15^0$ and $4x15^0$ i.e. 75^0, 15^0, 45^0, 30^0, 15^0, 120^0 and 60^0

The pie chart is shown below:



Exercise 17g

1. The frequency distribution below shows how a manufacturing firm used its income in 2004:

Area of spending	Wages	Building	Raw materials	Profit
Amount spent (%)	5	15	75	5

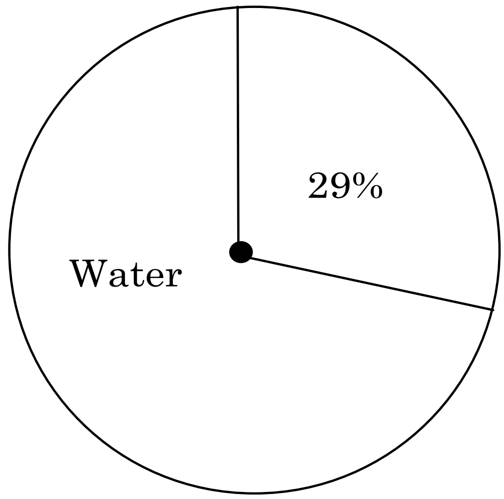
Draw a pie chart to show the distribution.

2. A school allocates its weekly timetable to pupils in Form 3 as follows:

Subject	English	Mathematics	Physical science
Number of periods	12	15	18

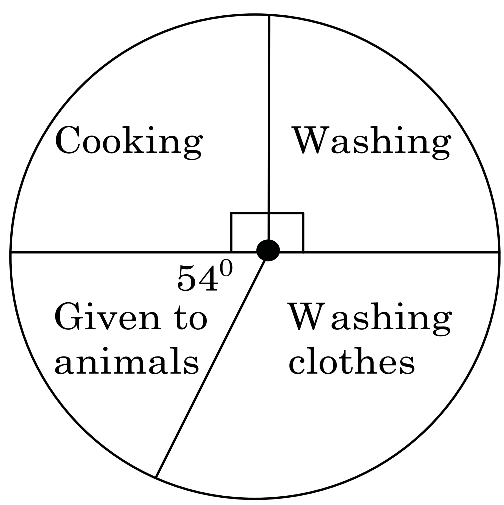
Represent the above information in a pie chart.

3. The total surface area of the earth is approximately 510 million km^2. The area is composed of water and land as shown in the pie chart below:



What angle of the pie chart represent water?

4. In a village 324 litres of water are used each day. The pie chart below shows how the water is used.



How much water is used in washing clothes?

Measures of central tendency

In your JCE you learnt about three measures of central tendency, mean , mode and median. You learnt how to calculate them from an ungrouped data. You will now learn how to calculate the measures from grouped data.

Activity 9:

Calculating median of grouped data

In groups, discuss how you can find the middle value (median) from the frequency distribution below:

Age (years)	0-4	5-9	10-14	15-19	20- 24	25-29
Number of people	2	3	6	9	8	5

You cannot find median directly from a frequency distribution table. The table helps you to find the class that contains the median (median class).

You add the frequencies cumulatively to find the position of the middle value(s). If the sum (n) of frequencies is even, the median is $\frac{n}{2}$ th and $(\frac{n}{2}+1)^{th}$ values while if the sum of frequencies is odd the median is $\frac{n+1}{2}$ th value.

Example 11

Find the median of the frequency distribution below.

Class interval	0-9	10-19	20-29	30-39	40-49	50-59
Frequency	1	2	11	9	14	3

Solution

Sum of frequencies = 1 + 2 + 11 + 9 + 14 + 3 = 40

Hence the middle values are 20^{th} and 21^{st} values. Add the frequencies cumulatively as follows:

1 + 2 = 3 values

3 + 11= 14 values

14 + 9 = 23 values

Hence the 20th and 21st values are in the class in the class interval 30-39. Hence 30 - 39 is the median class.

But if we need to estimate a single **Median value** we can use this formula:

$$\text{Estimated Median} = L + \frac{(n/2) - cf_b}{f_m} \times w$$

where:

- **L** is the lower class boundary of the group containing the median
- **n** is the total number of data
- $\mathbf{cf_b}$ is the cumulative frequency of the groups before the median group
- $\mathbf{f_m}$ is the frequency of the median group
- **w** is the group width

So the estimated median of the above data $= 30 + \frac{\frac{40}{2} - 14}{9} \times 10$

$= 30 + 6.7$

$= 36.7$

Exercise 17h

1. The frequency distribution below shows the heights in cm of 50 students.

Height (cm)	**Frequency**
120 -129	8
130-139	3
140-149	9
150-159	22
160-169	7
170-179	1

a. What is the median class of the data?

b. Estimate the median of the data.

2. The following are the weights 30 pupils in kg.

45	62	35	54	48	35
48	59	52	40	54	46
59	51	32	37	49	42
53	38	37	35	53	46
48	44	33	52	54	44

Construct a frequency distribution using intervals 30 – 39, 40 – 49,--- and find the median class and the median of the data.

3. The frequency distribution below shows number of students who scored marks within 10-mark class intervals in a test.

Marks	**Frequency**
1-10	2
11-20	7
21-30	9
31-40	11
41-50	13

a. Find the median class of the distribution.

b. Calculate the median of the data.

4. Find the median of the frequency distribution which shows diameters in mm of 24tins.

Diameter(mm)	60-69	70-79	80-89	90-99	100-109
Frequency	1	3	7	8	5

5. Find the median class and median of the frequency distribution below.

Class interval	31 - 50	51-70	71-90	91-110
Frequency	1	11	4	3

Activity 10 :

Calculating mean of grouped data

The table below is given in example 1 of this topic.

43	45	50	47	51	58	52	47	42
54	61	50	45	55	57	41	46	49
51	50	59	44	53	57	49	40	48
52	51	48	1					

In groups,

1. Calculate the mean of the above data using the method you learnt in form1 to the nearest whole number.
2. Group the data in the following intervals:
   40 – 44, 45 – 49,---.

   Construct a frequency table using intervals for the data using the intervals.
3. Find the midpoint of each class interval.
4. Multiply each midpoint by the frequency of that group.
5. Sum up the frequencies and the products.
6. Divide the sum of all the products by the total frequency correcting their answer to the nearest whole number.
7. Comment on your findings.

The method of finding mean by grouping data only gives an approximation but is a quicker and less tiring way of finding the mean where the data is so large. The method can be summarised into the following steps for finding mean of grouped data:

1. Find the frequency and midpoint of each class interval.
2. Multiply each midpoint by the frequency of that group.
3. Sum up the frequencies and the products.
4. Divide the sum of all the products by the total frequency.

Example 12

Calculate the mean of the frequency distribution below.

Number of cars	10-14	15-19	20-24	25-29	30-34	35-39	40-44	45-49
Number of days	1	1	2	8	19	14	4	1

Solution

Class interval	Midpoint of class interval	Frequency	Midpoint x Frequency
10-14	12	1	12
15-19	17	1	17
20-24	22	2	44
25-29	27	8	216
30-34	32	19	608
35-39	37	14	518
40-44	42	4	168
45-49	47	1	47
		Sum=50	Sum= 1630

Mean = $\frac{1630}{50}$ 32.6 cars/day.

Sometimes you can use what is known as a **working mean**. A working mean is the central value of the class with the highest frequency in this case 30 -34. The central value is 32. You then proceed as follows:

Class interval	Class central Value(x)	Frequency (f)	Central value-Working mean. This is a called deviation.	Frequency x Deviations
10-14	12	1	−20	−20
15-19	17	1	−15	−15
20-24	22	2	−10	−20
25-29	27	8	−5	−40
30-34	32	19	0	0
35-39	37	14	5	+70
40-44	42	4	10	+40
45-49	47	1	15	+15
Total deviations				30

Mean deviation = 30 ÷ 50 = 0.6

Then add this mean deviation to the working mean.

Mean of the data = 32 + 0.6 = 32.6 cars/day.

Exercise 17i

Find the mean of each of the following frequency distributions:

1.

Class interval	1- 5	6-10	11-15	16-20	21-25
Frequency	3	5	10	6	4

2.

Class interval	21-30	31-40	41-50	51-60	61-70
Frequency	2	5	7	9	11

3.

Class interval	100-109	110-119	120-129	130-139	140-149
Frequency	5	15	25	35	20

Find the working mean and hence the mean of the following frequency distributions:

4.

Class interval	0-4	5-9	10-14	15-19	20-24
Frequency	5	20	5	10	10

5.

Class interval	0-49	50-99	100-149	150-199	200-249
Frequency	6	8	11	9	4

Activity 11:

Calculating mode of grouped data

You cannot give the exact value of mode from grouped data. However, you can make a reasonable estimate of it by using the formula below.

$$\textbf{Estimated Mode} = L + \frac{f_m - f_{m-1}}{(f_m - f_{m-1}) + (f_m - f_{m+1})} \times w$$

Where:

- L is the lower class boundary of the modal group(the group with the highest frequency)
- f_{m-1} is the frequency of the group before the modal group
- f_m is the frequency of the modal group
- f_{m+1} is the frequency of the group after the modal group
- w is the group width

Go back to activity 10 above and find the modal class and the mode of the data.

1. Using the method of mode of ungrouped data find the mode of the data in activity 9 and compare it to the mode you have just found.
2. Comment on your findings.

Example 13

The following frequency table was drawn up for the marks in a mathematics test.

Class	Frequency
50-53	7
54-57	8
58-61	9
62-65	6
66-69	4
70-73	1

a. What is the modal class?

b. Estimate the mode of the data

Solution

a. Modal class = 58-61

b. Mode $= 58 + \frac{9-8}{(9-8)+(9+6)} \times 3$

$= 58 + 0.2$

$= 58.2$

Exercise 17j

For each of the following frequency distributions find the modal class and hence estimate the mode.

1.

Class interval	frequency
0 – 9	3
10 – 19	5
20 – 29	11
30 – 39	9
40 – 49	8

2.

Class interval Height (cm)	frequency
40 – 44	7
45 – 49	9
50 – 54	10
55 – 59	6
60 - 64	2

3.

Mass of nails (g)	0-4	5-9	10-14	15-19	20-14
Number of nails	15	12	10	5	3

4.

Length of life (hours)	201-300	301-400	401-500	501-600	601-700
Number of bulbs	10	16	32	54	88

Activity 12:

Calculating the range of grouped data

In groups, discuss how you can get the range from the frequency distribution in example 13.

In a grouped frequency distribution, you can also make a reasonable estimate of the range. It is found by subtracting the lower limit of the first class interval from the upper limit of the last class interval.

Example 14

Estimate the range of the data in example 13.

Solution

Range = 73 – 20 = 23.

Exercise 17k

Estimate the range of the data in **question 1** to **question 4** of exercise 17i.

Unit summary

- In this unit you have learnt about organisation of data, presenting data in form of charts and tables, calculating measures of central tendency and spread.
- In this unit you have also seen that you cannot give exact values of mean, mode and median. You can only give estimates of them.

Unit review exercise

1. The first class interval of discrete data is 0- 4. Form the next 5 class intervals.

2. The lower limit of the first class interval of continuous data is10, the class width is 8. Form the first three class intervals for the data.

3. Two first class intervals of data are 15-19 and 20-24. Find (a) the class boundary and(b) the midpoint of the class intervals.

4. The weights (in kg correct to the nearest tenth) of a group of thirty dogs were recorded as follows:

15.5	14.8	15.8	14.3	14.6	15.0
16.2	13.9	15.2	15.1	16.0	15.2
14.4	15.4	15.7	16.2	14.9	14.7
15.5	13.7	15.5	14.3	14.7	15.1

   Represent the data in the form of a frequency distribution using intervals 13.5 -13.9,14.0-14.4,---

5. 45 children working in groups, were asked to time each other's estimates of the length of a minute. Their estimates correct to the nearest second , are given below.

53	47	77	63	59	54	62	65	71
77	42	68	67	51	72	57	73	48
61	46	51	50	63	68	54	50	65
53	78	69	44	56	77	58	55	79
66	58	67	52	48	70	49	71	73

   Using intervals 40 – 44, 45 – 49,---, draw a histogram for the data.

6.

Number of games	**Frequency**
1 - 5	2
6 - 10	7
11 - 15	8
16 - 20	3

Using the frequency distribution above find

a. the mode

b. the median

c. the modal class

d. the median class

e. the range.

f. the mean.

Glossary

Qualitative data: Data of properties that are numbers.

Quantitative data: Data of properties that are not numbers.

Discrete data: Countable data.

Continuous data: Measured data

Lower limit: The beginning of one class interval.

Upper limit: End of class interval.

Class width: Number of values from the lower to the upper limit.

Class boundary: A point where one class ends and the other begins.