Unit 1 NUMBER SYSTEMS

Introduction

One of the oldest ways of writing **numbers** was using **tally marks** (also called **tally scores**). Ancient cave paintings, like the one shown in Fig. 1.1, show a series of strokes after the picture. The eight strokes probably represent 8 antelopes killed in a hunt.

Fig. 1.1

Writing larger numbers such as 37 using tally scores, soon proved difficult and people had to invent other ways of writing them.

Numbers are mathematical symbols used to denote **quantity** or **value** as in counting things. 0, 1, 2, 3…… are some of the symbols (or digits or numerals) in common usage.

These symbols are known as **Hindu-Arabic** numerals because they were developed by Hindus (from India) and letter refined by the Arabs (mainly from Egypt).

In a more strict sense, “numerals” and “digits” have slightly different meanings.

Numerals refer to any symbol that represents a number, while **digit** usually refers to any of the ten symbols in **Hindu-Arabic system**.

Numbers and numerals

In base ten number system, numbers are grouped in tens (probably because people have ten figures). The digits in base ten numeration are: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

Zero (0) has no value and is simply used to **hold place** in a number.

In the Hindu-Arabic numeral system, the value given to any digit, depends on its position, as in the example illustrated in Table 1.1 below.

Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
9	0	4	7	6	3	0	3	3

Table 1.1

Digit 3 is written three times, yet each 3 has a different value. Starting from the right, there are three ones (units), three tens and three thousands respectively. The zero in the third (hundreds) column preserves the place value. Without it, the number 3 033 would look like 333, thus losing the original value.

Similarly, the zero (0) in the ten millions column holds that place value in order to preserve the value of the number 904 763 033.

The Abacus

An abacus is a basic calculating device consisting of beads or balls strung on wires or rods set on a frame. Fig. 1.2, shows a typical abacus on which the place value concept can be developed very effectively in a practical approach.

On each wire, there are ten beads. Let us consider the beads at the bottom of each wire.

Beginning from the right, 10 beads on wire 2.

Similarly, 10 beads on wire 2 can be represented by 1 bead on wire 3 and so on.

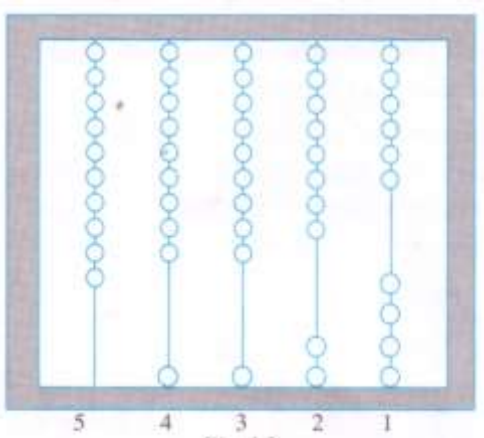


Fig. 1.2

This means:

1 beads in wire 1 represents a single bead.

1 bead in wire 2 represents 10 beads.

1 bead in wire 3 represents (10 × 10) beads.

1 bead in wire 4 represents (10 × 10) × 10 beads.

So the number shown in Fig. 1.2 is 1 124.

If we had x beads in each wire such that x is less than 10, it would mean that;

In wire 2 we had 10.x beads

In wire 3 we had 10^2.x beads

In wire 4 we had 10^3.x beads and so on.

The place values from right to left are:

$$\begin{tikzcd}
10^0 \arrow[d] & 10^1 \arrow[d] & 10^2 \arrow[d] & 10^3 \arrow[d] & 10^4 \arrow[d] \\
\text{Ones} & 10\text{s} & 100\text{s} & 1\ 000\text{s} & 10\ 000\text{s etc.}
\end{tikzcd}$$

Any number can be written in an **expanded form**, in terms of place values, as illustrated in Example 1.1 and 1.2.

Example 1.1

Write the number 95 704 in expanded form, in terms of place values.

Solution

The number 95 704 can be illustrated in place values as shown in Table 1.2

Ten thousands (10 000)	Thousand (1 000)	Hundred (100)	Tens (10)	Ones (1)
9	5	7	0	4

Table 1.2

From table 1.2, the expanded form of the number is

95 704 = 9 × 10 000 + 5 × 1 000 + 7 × 100 + 0 × 10 + 4 × 1

Or 95 704 = 9 × 10 000 + 5 × 1 000 + 7 × 100 + 4 × 1

Example 1.2

Write the number whose expanded form is 1 × 10 000 + 0 × 1 000 + 6 × 100 + 1 × 10 + 1× 1

Solution

1 × 10 000 + 0 × 1 000 + 6 × 100 + 1 × 10 + 1× 1

= 10 000 + 0 + 600 + 10 +1

= 10 611

Note: The value of a digit in a number is 'that digit multiplied by the place value', e.g. the value of 5 in the number 95 704 is 5 × 1 000 = 5 000.

Numbers in words

It is important to know how to read and write numbers in words. Example 1.3 illustrates this.

Example 1.3

Write the following in words.

(a) 80 743
(b) 537 409 260

Solution

(a) We begin by identifying the place value for each of the digits in the number 80 743 as in Table 1.3

Ten thousands (10 000)	Thousand (1 000)	Hundred (100)	Tens (10)	Ones (1)
8	0	7	4	3

Table 1.3

80 743 can be written as follows: eight ten thousand seven hundred four tens and three ones

i.e. 80 743 is ‘Eighty thousand seven hundred forty three.’

(b) We can write the number 537 409 260 showing place value of each digit as in table 1.4

Hundred million (100 000 000)	Ten million (10 000 000)	Million (1 000 000)	Hundred thousand (100 000)	Ten thousands (10 000)	Thousands (1 000)	Hundreds (100)	Tens (10)	Ones (1)
5	3	7	4	0	9	2	6	0

Table 1.4

537 409 260 can be written as follows: five hundred million three ten millions seven millions four hundreds nine thousands two hundreds and six tens

i.e. 537 409 260 is ‘Five hundred thirty seven million four hundred nine thousand two hundred and sixty.’

Exercise 1.1

1. What is the place value of 5 in:
   a) 453
   b) 570
   c) 705
   d) 5 160
   e) 75 326
   f) 753 689 004
2. Write down the following numbers in figures and state the number of digits in each case.
   a) Two hundred five thousand and four
   b) Nine hundred thirty one thousand seven hundred and thirty one.
   c) Thirty one million thirty one thousand and thirty one.
   d) Thirty one million five hundred three thousand seven hundred and three.
3. Write down the following numbers in words.
   a) 10 001 001
   b) 972 500
   c) 1 079 900
   d) 205 830 652
4. Write down the place value of the digit in bold.
   a) 248 964
   b) 2 046 924

c) 4 305 600　　d) 670 084

5. Write down in figures the number that is nine more than one hundred thousand.
6. Write the following numbers in expanded form.
   - a) 71 409　　c) 4 837 463
   - b) 643 012　　d) 289 364 918

Natural numbers and whole numbers

Natural numbers

When counting different objects people normally start with 1, followed by 2, then 3, and so on. These numbers 1, 2, 3, 4, 5… are called **counting numbers** or **natural numbers**. The three dots after 5 imply that these numbers continue indefinitely and in the same pattern. These numbers can be represented on a number as shown in fig. 1.3

A number line is a pictorial representation of numbers on which the numbers are marked at unit intervals (equal single steps) increasing from 0 to the right.

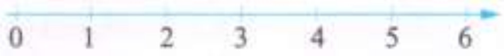


Fig. 1.3

Taking two numbers on the number line, the one to the **right** is the **greater** one. For example, 6 is to the right of 5 and so 6 is greater than 5.

The arrow means that these numbers continue in the direction shown, and in the same pattern indefinitely.

Whole numbers

If the group of natural numbers is extended to both left and right of 0, -1, -2, -3, -4, -5…., 0, 1, 2, 3, 4, 5, …, the resulting new group of numbers is called whole numbers. Whole numbers can also be represented on a number line as shown in Figure 1.4.

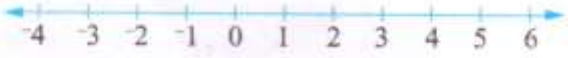


Fig. 1.4

The group of whole numbers can be classified into three smaller categories. Each of the three groups have properties as follows.

Even numbers

An even number is a number that is exactly divisible by 2 i.e. the remainder is zero.

The firs ten positive even numbers are:

2, 4, 6, 8, 10, 12, 14, 16, 18 20.

All even numbers end with digits 0, 2, 4, 6, or 8.

Now list the next ten even numbers.

Odd numbers

An odd number is a number that is not exactly divisible by 2 i.e. the remainder is not zero.

The first ten positive odd numbers are:

1, 3, 5, 7, 9, 11, 15, 17, 19.

All odd numbers end with digits 1, 3, 5, 7 or 9.

List the next ten odd numbers.

Note: The difference between any two consecutive even numbers is 2. The same applies to any two consecutive odd numbers e.g.

4 – 2 = 2, 8 – 6 = 2,

3 -1 = 2, 19 – 17 = 2.

Prime numbers

A prime number is a number other than 1 that can be divided by only 1 and itself.

The first ten prime numbers are:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29.

Now list the next ten prime numbers.

Exercise 1.2

1. Write down all the natural numbers
   a) Between 6 and 11
   b) Between 5 and 6
2. List all the even numbers between 7 and 17
3. List down all the odd numbers between 16 and 28.
4. The following is a list of numbers:
   12, 30, 27, 35, 56, 48, 19, 49, 81, 51, 33, 144, 111, 243, 72, 125, 107, 127, 41.
   From the list, pick out all the
      a) Even numbers
      b) Odd numbers
      c) Prime numbers
5. Is it true that the sum of two odd numbers is also odd? Support your answer with an example.
6. Between the groups of natural numbers and whole numbers, which one is bigger?

Number bases

This far, we have discussing a system of numbers which counts in groups of ten. We say we are using base ten numeration. Now we are going to discuss and use some other **bases**. Why do you think we count in groups of ten?

If we had 6 fingers most probably we would count using group of 6, if 8 fingers, groups of 8 and so on. In the system that we use, we have seen that every ten items make one basic group which is represented in the next place value column to the left as shown in Fig. 1.5 (a).

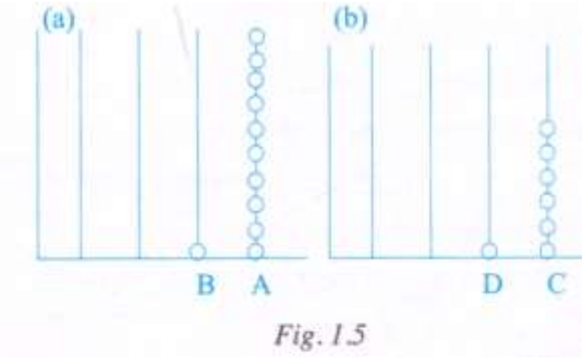


Fig. 1.5

(a) The 1 bend in wire B represents the 10 beads in wire A i.e. it represents a group of 10 beads.
(b) The 1 bead in wire D can represent 6 beads in wire C Fig. 1.5 (b), thus making a group of 6 beads.

Counting in different groups of numbers such as 10, 6, 5, 8 etc. means using different number systems. We call them base ten, base six, base five, base eight etc.

Now consider Fig. 1.6

Fig. 1.6

Counting in base six, what numbers do the beads on each wire represent?

i. There are 4 beads in wire A. This represents 4 ones.
ii. There are 5 beads in wire B. This means 5 groups of 6 beads each.

   i.e. 5 × 6 = 30 beads written as 50_{six}

iii. There are 3 beads in wire C. This means 3 groups of six sixes i.e. 3 × 6 × 6 = 108 beads, written as 300_{six}.
iv. There are 2 groups of 6 × 6 × 6 = 216 × 2 written as $2\ 000_{six}$

The whole number represented in Fig. 1.6 is $4_{six} + 50_{six} + 300_{six} + 2\ 000_{six} = 2\ 354_{six}$

The answer $2\ 354_{six}$ is read as; two three five four base six.

Example 1.4

Given that the number represented in Fig. 1.7 is in base five, find the number in base 10.

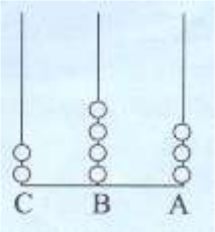


Fig. 1.7

Solution

Column A represents 3 ones.

Column b represents 4 fives.

Column C represents 2 five fives.

The number = $(3 \times 1) + (5 \times 5) + (2 \times 5^2)$

$= 3 + 25 + 50$

$= 78_{ten}$

$253_{five} = 78_{ten}$

Note that 253_{five} and 78_{ten} are two different symbols for the same number.

The binary system (base two)

A binary system is a number that uses only two digits 0 and 1. Numbers are expressed as powers of 2 instead of powers of 10 as in the decimal system, the digits corresponding to two switching position, on and off, in the individual electronic devices in the logic circuits.

Compare the first ten numbers in base ten with those of base two.

Remember; in any base there is no numeral equal to the base. Such a digit or numeral always takes the form of 10.

Base ten:	0	1	2	3	4	5	6	7	8	9
Base two:	0	1	10	11	100	101	110	111	1000	1001

Table 1.5

Example 1.5

Calculate in binary

(a) 101 + 10101
(b) 1001 – 101

Solution

(a) 101
+ 10101
11010

(b) 1001
- 101
100

Exercise 1.3

1. Evaluate the following giving your answers in base two:
   (a) $15_{ten} + 23_{ten}$
   (b) $35_{ten} - 12_{ten}$
2. Evaluate:
   (a) $1011_{two} + 1101_{two}$
   (b) $10001_{two} + 110011_{two}$
   (c) $11101_{two} + 11_{two} + 10101_{two}$
   (d) $1_{two} + 11_{two} + 1011_{two} + 110011_{two}$

Other number bases

Table 1.6 shows various bases and the numerals appropriate to each.

Base	**Numerals**
Ten	0 1 2 3 4 5 6 7 8 9
Nine	0 1 2 3 4 5 6 7 8
Eight	0 1 2 3 4 5 6 7
Seven	0 1 2 3 4 5 6
Six	0 1 2 3 4 5
Five	0 1 2 3 4

Four	0 1 2 3
Three	0 1 2
Two	0 1
And so on.	

Table 1.6

In any base, the numeral equal to the base is represented by 10.

i.e. $5_5 = 10$ $6_6 = 10$ 10_{10} 10 8 = 10 etc.

When a base is greater than 10, say 12, we need to create and define a symbol to represent 10 and 11.

Change of base

Numbers in base ten have place values denoted by powers of 10 i.e.

10^0	10^1	10^2	10^3	10^4....
1	10	100	1 000	10 000

Similarly, in base six, place values are denoted by powers of 6 i.e. 6^0, 6^1, 6^2, 6^3, 6^4...

A number such as $135_{six} = (5 \times 6^0) + (3 \times 6^1) + (6^2 \times 1)$

Also, a number as 65_{ten} can be expressed as a number in base six as

65 ÷ 6 = 10 Rem 5 → 5 ones

10 ÷ 6 = 1 Rem 4 → 4 sixes

1 ÷ 6 = 0 Rem 1 → 1 six sixes

$65_{ten} = 145_{six}$

Example 1.6

Express

(a) 415_{six} as a number in base ten

(b) 85_{ten} to base six

Solution

(a) We use place values to change from base six to base 10.

$415 = (5 \times 1) + (1 \times 6) + (4 \times 6^2)$
$\quad = 5 + 6 + 144$
$\quad = 155$
$415_{\text{six}} = 155_{\text{ten}}$

(b) To change from base ten to base six or any other we do successive division by the required base noting the remainder at every step.

$85 \div 6 = 14$ Rem 1 → 14 groups of six and 1 one.
$14 \div 6 = 2$ Rem 2 → 2 groups of six sixes and 2 groups of six
$2 \div 6 = 0$ Rem 2 → 2 groups of six sixes

$85_{\text{ten}} = 221_{\text{six}}$

Example 1.7

Convert 526_{eight} to base five

Solution

To change a number from base eight to base five, we convert first to base ten then from base ten to base five.

Thus:

$526_{\text{eight}} = 6 \times 1 + 2 \times 8 + 5 \times 8 \times 8$

$\quad = 6 + 16 + 320$

$\quad = 342_{\text{ten}}$

Now we can convert 342_{ten} to base five by the method of successive division by 5.

Thus:

$342 \div 5 = 68$ rem 2

$68 \div 5 = 13$ rem 3

$13 \div 5 = 2$ rem 3

2 ÷ 5 = 0 rem 2

$526_{eight} = 2\ 332_{five}$

To convert from base ten.

1. Do successive division by the required base noting the remainders at every step.
2. Write down the remainders beginning with the last on the left.
3. These remainders make up the number required.

To convert from any base x to base 10

1. Multiply, every digit in the number by its place value i.e. 1, x, x^2, x^3 etc.
2. Add the results
3. To change from base x to base y, change from base x to base ten, then from base ten to base y.

Exercise 1.4

1. Write the first twenty numerals of
   a) Base six
   b) Base seven
   c) Base eight
2. What does 8 mean in
   a) 108_{ten}
   b) 180_{ten}
   c) 801_{ten}
   d) $88\ 801_{ten}$

3. Write down in words;
   a) 203_{six}
   b) 302_{four}
   c) 15_{six}
   d) $3\ 215_{eight}$
4. Convert the number 703_{eight} to
   a) Base 6
   b) Base 10
   c) Base 9

5. Convert the following into decimal system:
   a) 411_{five}
   b) 321_{six}
   c) 207_{eight}
   d) 750_{nine}
6. Express 63_{seven} to base 5
7. Convert the following to the binary system
   a) 18_{ten}
   b) 135_{six}
   c) 65_{seven}
   d) 35_{eight}
8. Write in words the meaning of
   a) 12_{three}
   b) 142_{five}
   c) 21_{four}
   d) 180_{nine}
9. Use abacus to show place values for the numerals in
   a) 211_{five}
   b) 615_{seven}
   c) 173_{eight}
   d) $1\,254_{\text{ten}}$
10. Convert 118_{nine} to base 5.
11. What number in base ten does the digit 4 represent in each of the following numeral?
   a) 864_{nine}
   b) 846_{nine}
   c) 486_{nine}
12. Express $2\,222_{\text{four}}$ in binary
13. Given that $23_{\text{seven}} = 32_{a}$, $34_{\text{nine}} = 43_{b}$, $45_{\text{ten}} = 54_{c}$, find the value of a, b and c.

Operation using bases

Addition

In performing addition, whatever the base, the digits to be added must be in the same place value. For example in $65_{ten,} + 18_{ten,}$ 5 and 8 have the same place while 6 and 1 have another place value.

Example 1.8

Evaluate $332_{six} + 25_{six}$

Solution

It is best to set work vertically so that the place values correspond.

$332_{six} + 25_{six} = 332_{six}$

$\underline{+ 25_{six}}$

(1) Illustrate the two numbers on different abaci (Fig. 1.8).

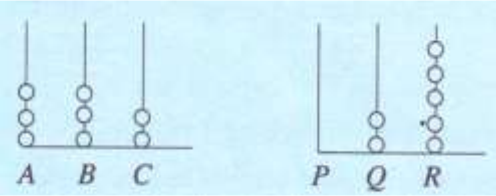


Fig. 1.8

(2) Remove all the 5 beads from R and place them in C. one bead remains at C another goes to B to represent another group of six (Fig. 1.9).

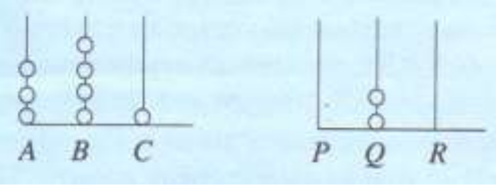


Fig. 1.9

(3) Remove the two beads from Q and place them on B to make 6 beads. Bo bead remains at B, but one bead goes to A to represent another group of six sixes.

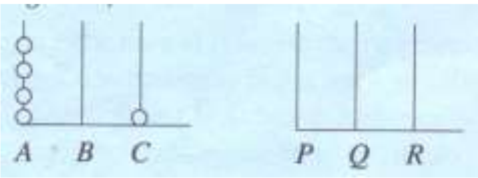


Fig. 1.10

(4) The result of addition is 401_{six}

Alternatively.

$332 \longrightarrow 330 + 2$

$+$

$25 \longrightarrow 20 + 5$

$350 + 11$

$= 350$

$+ 11$

$4\ 0\ 1_{\text{six}}$

0 + 1

5 + 1 = 0 carry 1

3 + 1 (carried above) = 4

Subtraction

Example 1.9

Use abacus to evaluate

$52_{\text{eight}} - 23_{\text{eight}}$

Solution

Fig. 1.11 shows the two numbers on different abaci

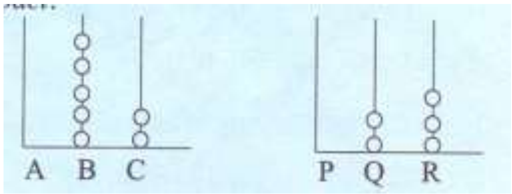


Fig. 1.11

Since we cannot subtract beads in R from beads in C.

(1) Remove one bead from B and place it on wire C so that there is a total of 10 in C (Fig. 1.12)

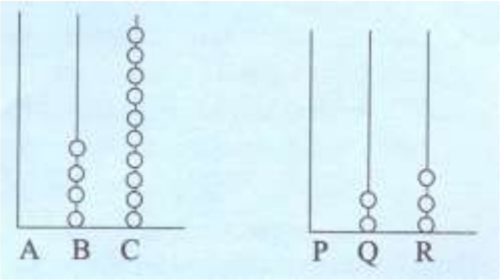


Fig. 1.12

(2) Remove 3 beads from C and R (Fig. 1.13).

(3) Remove 2 beads from B and Q so that the result is as represented in Fig. 1.13.

$52_{eight} - 23_{eight} = 27_{eight}$

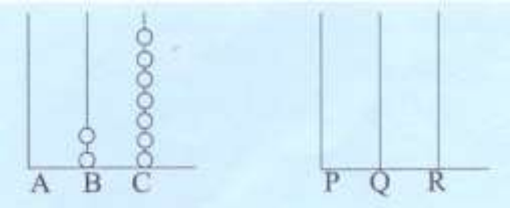


Fig. 1.13

Alternatively

52 → 50 + 2 40 + 10

-

-23 → 20 + 3 20 + 3

20 + 7

Exercise 1.5

1. Work out the following in base eight
   a) 17 + 211
   b) 106 + 12
   c) 257 + 462
2. Evaluate the following in base six
   a) 31 – 25
   b) 145 – 51

c) 55 – 43 d) 403 – 54

3. Evaluate the following in base nine
   a) 122 + 85
   c) 17 – 8
   b) 103 – 86
   d) 66 + 35
4. The following calculations are correct. State the base used in each case.

a) 36	b) 53	c) 3	d) 172	e) 82	d) 65
+26	+ 36	+23	- 69	-53	-36
64	111	31	3	26	26

5. The following calculations were done using a certain base. Two of them are correct while two are wrong.
   Identify (i) the base (ii) the incorrect i=ones and explain why?

b) 22	b) 68	c) 100	d) 172
-16	+ 15	-64	+69
6	84	25	207

Directed numbers or Integers

When we discussed number lines and whole numbers. We dealt with positive numbers only. We saw that as we move along the number line from right to left, numbers decrease in values as we approach zero. We can locate another group of numbers. Since these new numbers cannot be positive, they can only be the opposite of counting numbers i.e. negative. Subtraction of natural numbers led to negative numbers and zero. All these numbers, negative, positive and zero are called integers and can be located (marked) on the number line as in Fig. 1.14. Fig. 1.14 shows a number line of integers. The numbers increase in value from left to right. For example, +2 has greater value than -5, i.e. +2 is greater than -5 or -5 has lesser value than +2, i.e. -5 is less than +2.

Fig. 1.14

Integers which are next to each other and differ by 1 are called **consecutive** integers. For example, -5 and -4, 1 and 0, 2 and 3, etc. are pairs of consecutive integer

Numbers to the left of zero on a number line are called negative numbers. **Negative numbers** are written with minus (-) signs before them (e.g. -2).

Numbers to the **right of zero** are called **positive numbers** and may be written with or without plus (+) signs before them, (e.g. +5 is the same as 5).

Negative and positive whole numbers together with zero are called **integers.**

Negative and positive numbers are called directed numbers because they carry signs which show their directions from zero.

The **difference** between two numbers means the number of steps, on the number line, between the two numbers. For example, the difference between 3 and 4 or 4 and 3 is 1. The difference between two numbers is always positive. Fig. 1.15 shows a number line of integers. It is helpful to think of it as sloping upwards from left to the right as the numbers increase in value. For example, +2 has greater value than -5, i.e. +2 is greater than -5, or -5 has less value than +2, i.e. -5 is less than +2.

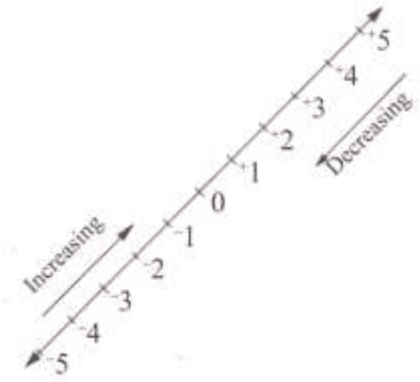


Fig. 1.15: A number line sloping upwards to show increase in value

Exercise 1.6

1. Arrange the following numbers in increasing order.
   a) -3, -2, +6, -2, -3
   b) -1, -8, -5, +1, -7

2. Write true or false for each of the following.
    a) -1 is greater than -2
    b) -2 is greater than -1
    c) +2 is less than -5
    d) -5 is greater than -2
    e) 0 is greater than -6
    f) -6 is less than 0
3. In each of the following pairs of numbers, insert the correct phrase, “is greater than” or “is less than”, to make the statement true.
    a) -5…. +3
    b) +1 … -1
    c) +6 … -7
    d) -100... +1
    e) -3 … 0
    f) -4 … 40
4. Find the difference between:
    a) 5 and 2
    b) 1 and 7
    c) 8 and 0
    d) -8 and 3
    e) -2 and -9
    f) -5 and -4

Operations on integers

Addition involving negative integers

To add a **positive integer** using a number line, start from the first number and **move right** a number of steps equal to the integer being added.

To add a **negative integer**, start from the first number and move left a number of steps equal to the integer being added.

Example 1.10

Perform the addition:

a) -4 + 6
b) 5 + -7
c) -2 + -3

Solution

a) -4 + 6

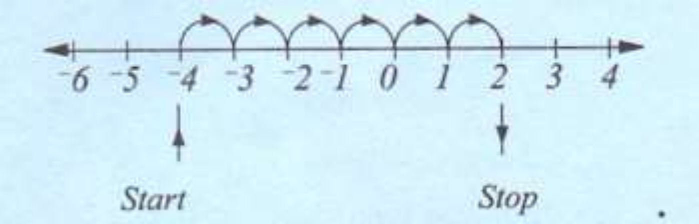


Fig. 1.16

So, -4 + 6 = 2

Since 6 – 4 = 2, then -4 + 6 = 6 + -4

= 6 – 4

(Order does not matter in addition)

b) 5 + -7

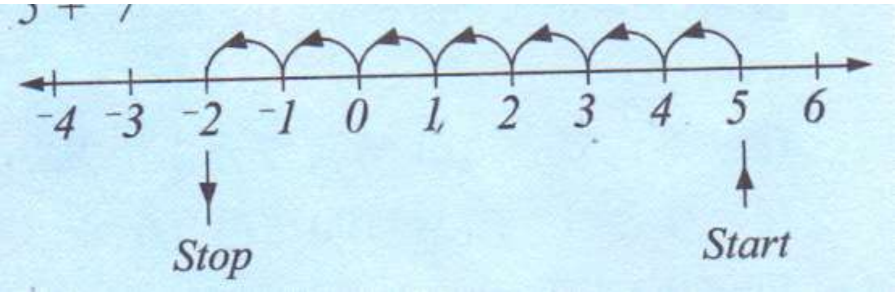


Fig. 1.17

So, 5 + -7 = -2

Since 5 -7 = -2

Then 5 + -7 = 5 – 7

= -(7 – 5)

= -2

(Subtract the smaller number from the larger one and change the sign of the answer).

c) -2 + -3

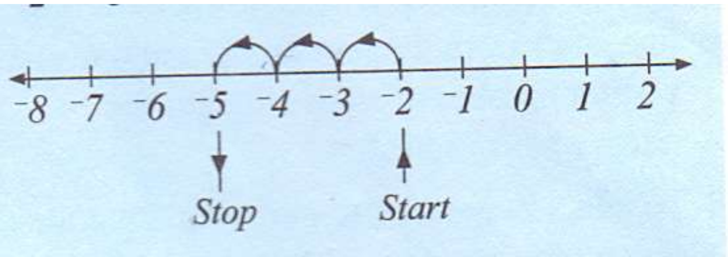


Fig. 1.18

So -2 + -3 = -5 (Two negative numbers must add up to a negative).

Note:

a) If the sum of two numbers is zero (0), the numbers are said to be **addition inverse** of each other, e.g. 5 is the addition inverse of -5 and vice versa.

b) If we add zero (0) to any number, the number remains the same. Zero (0) is said to be **addition identity.**

Exercise 1.7

1. Copy and complete the following number pattern.

   6 + (+4) = 10
   6 + (+2) = 8
   6 + 0 = 6
   6 + (-2) =
   6 + (-4) =
   6 + (-6) =
   6 + (-8) =

2. Using a number line, work out each of the following.

   a) 4 + -3
   b) 9 + -8
   c) 9 + -15
   d) -3 + 6
   e) -4 + 8
   f) -2 + -7
   g) -6 + -5
   h) -3 + +3

3. What must be added to:

   a) 3 to make 7
   b) -1 to make 3
   c) 15 to make 5
   d) 2 to make -7
   e) -30 to make -22
   f) 24 to make -2?

4. Copy and complete the addition Table 1.7.

Second number

+	-6	-4	-2	0	2	4	6
-6		10					
-4			-6				
-2							
0						4	
2		-2			4		
4				4			
6							12

Table 1.7

5. Copy and fill in the boxes in the following.
    a) 2 + ☐ = 5
    b) ☐ + 4 = 7
    c) 6 – 2 + 4 = ☐
    d) 3 + ☐ + 2 = 9
    e) 5 + 3 - ☐ = 3
    f) 6 + ☐ = 6

Subtraction involving negative numbers

Consider 5 – 2. This is the same as asking, ‘what must be added to 2 to make 5?’

Since 2 +3 = 5, then 5 – 2 = 3.

Similarly, 6 - -3 is the same as asking, ‘what must be added to ‘-3 to make 6?’. Since -3 + 9 = 6 (Fig. 1.19), then the answer is 9, i.e. 6 - -3 = 9.

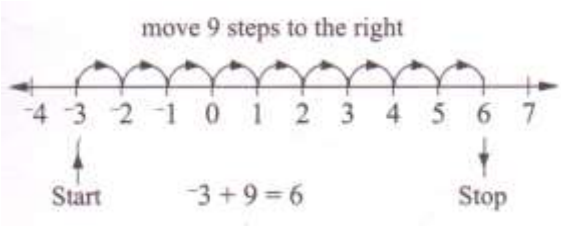


Fig. 1.19

This is the same as adding 3 to 6, i.e.

6 - -3 = 6 + 3 = 9

Now consider -7 = -4 + ☐

We have to find the number that must be added to -4 to make -7. This is -3 (since if we start at -4, we move 3 step to the left to stop at -7 (Fig. 1.20)

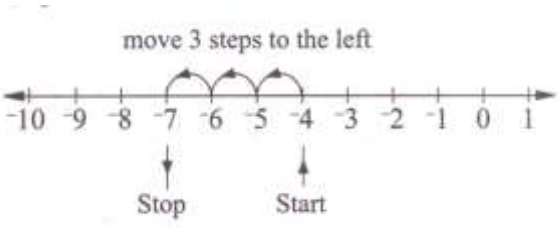


Fig. 1.20

Thus, -7 - -4 = -3, which is the same as adding 4 to -7, i.e.

-7 - -4 = -7 + 4 = -3.

Note that:

Subtracting a negative number from any number is the same as adding the equivalent positive number.

An alternative approach, when subtracting a number from another using the number line is:

1. Locate the two numbers on the number line.
2. Find the difference between the two numbers.

Example 1.11

Use the number line to perform the following subtractions.

a) 5 - -2
b) -5 - +2
c) -8 - -3
d) Temperature in a freezer rose from -10˚ to -2˚. What is the rise temperature?

Solution

a) 5 - -2

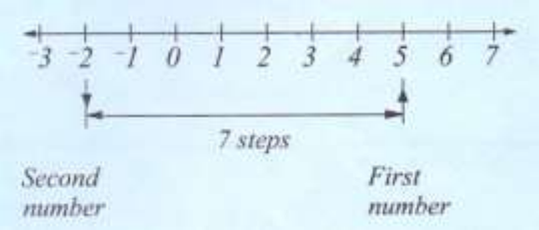


Fig. 1.21

So, 5 - -2 = 7.

Since 5 + 2 = 7, then 5 - -2 = 5 + 2.

b) -5 - +2

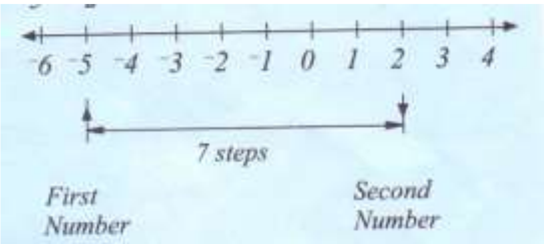


Fig. 1.22

So -5 - +2 = 7 (negative answer since +2 > -5)

Since -5 + -2 = -7, then -5 - +2 = -5 + -2.

c) -8 - -3

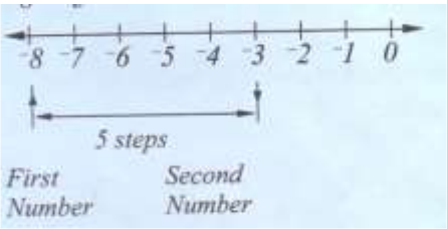


Fig. 1.23

So, -8 - -3 = -5 (since -3 > -8)

Since -8 + 3 = -5, then -8 - -3 = -8 + 3

d) Temperature in a freezer rose from -10˚ to -2˚. What is the rise in temperature?

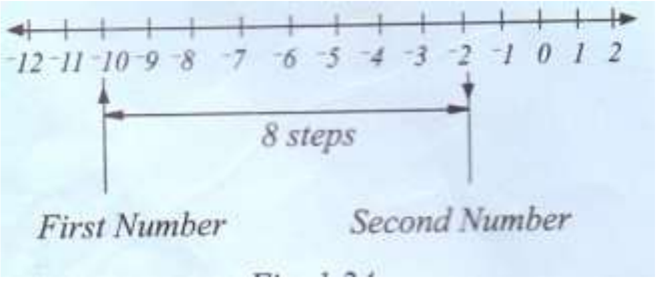


Fig. 1.24

So, -10 -2 = -8 (since -2 > -8)

Since -10 + 2 = -8, then -10 - -2 = -10 +2

Exercise 1.8

1. Copy and complete the following number pattern.
   -4 - +4 = -8
   -4 - +2 = -6
   -4 – 0 = -4
   -4 - -2 =
   -4 - -4 =
   -4 - -6 =
   -4 - -8 =
2. Using a number line, work out the following.
   - a) 5 – 2
   - b) 3 – 7
   - c) -3 – 6
   - d) -6 – 2
   - e) -3 - -5
   - f) -8 - -1
3. Copy and complete the subtraction Table 1.8

Second number

+	-6	-4	-2	0	2	4	6
-6							
-4		0	-4				
-2							
0			2				
2							
4					2		
6							

Table 1.8

4. Work out the following.
   - a) 4 – 2
   - b) 8 – 6
   - c) 9 – 10
   - d) 7 – 12
   - e) 5 – -3
   - f) 6 – -7
   - g) 8 – -10
   - h) -2 – -5
   - i) -6 – -12
   - j) -10 – -3
   - k) -24 – -17
   - l) -3 – 4
   - m) -13 – 7
   - n) 0 – 4
   - o) 0 – -3
5. Evaluate the following by first changing the subtraction into addition
   - a) +3 – 8
   - b) -5 – (-6)
   - c) -5 – +5
   - d) 9 – 6
   - e) -5 – (-3)
   - f) -8 – -6
   - g) 9 – -4
   - h) 6 – (-9)
6. Write down the answers to the following.
   - a) 2 – -2
   - b) -2 – (-2)
   - c) -2 – 2
   - d) 0 – 5
   - e) (-3) – 0
   - f) 0 – -5
   - g) -3 – -5
   - h) 5 – 6
7. Work out the following.
   - a) 2 + -3 + 6
   - b) 8 + 9 + -10
   - c) 4 + -15 + 8
   - d) -5 + -4 + 12
   - e) -9 + 15 + -3
   - f) 4 + -15 + 8
   - g) -6 – -6 - -10
   - h) 0 – -4 – 3
8. Evaluate the following
   - a) 15 – 16 – 10
   - b) 8 – 5 – (-2)
   - c) 39 – 12 – (-29)
   - d) 12 – (-15) – (-1)
   - e) -12 – (-13) – -6
   - f) 19 – 4 – 20
   - g) 3 – (-17) – -8
   - h) 60 – -30 + 6
9. Work out the following by first changing the addition into subtraction.
   - a) 4 + -3
   - b) 9 + -8
   - c) 9 + -15
   - d) -3 + 6
   - e) 6 + -6 - +3
   - f) 17 = (-7)
   - g) 64 – (+6)
   - h) -35 + -35
10. Simplify the following.
   - a) -3 – 4 + 10
   - b) -8 + 12 – (-2)
   - c) +3 – (+7) + (-2)
   - d) -3 – -4 – -10
   - e) -9 + -4 + -3
   - f) -12 – -7 – -5

g) -11 – (-8) + 3

h) 56 – (-40) – -8

Multiplication of integers

Multiplication of numbers can be treated as repeated addition or grouping.

For example, 3 × 5 means 3 groups of 5 i.e. 3 × 5 = 5 + 5 + 5 = 15.

Also, 3 × 5 means 5 groups of 3 i.e. 3 × 5 = 3 + 3 + 3 + 3 + 3 = 15.

Thus, 3 × 5 = 5 × 3 i.e. order does not matter in multiplication. We say that multiplication of numbers is commutative. See Fig. 1.25

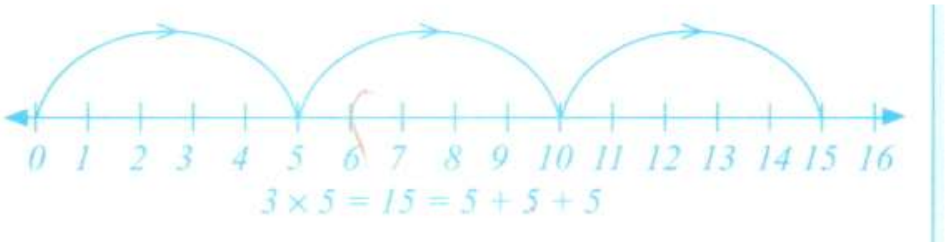


Fig. 1.25

Thus:

The product of two numbers with **like signs** is **positive** while that of two numbers with **unlike signs** is **negative.**

Example 1.12

Work out 3 × -5

Solution

3 × -5 = 3 groups of -5

= -5 + -5 + -5 = -15

Also, -5 × 3 = 15 (since order does not matter)

Study the number patterns in Table 1.9 below:

Pattern (a)	Pattern (b)
+3 × +2 = +6	+3 × -2 = -6
+2 × +2 = +4	+2 × -2 = -4

+1 × +2 = +2	+2 × -2 = -2
0 × +2 = 0	0 × -2 = 0
-1 × +2 = -2	-1 × -2 = 2
-2 × +2 = -4	-2 × -2 = 4
-3 × +2 = -6	-3 × -2 = 6
-4 × +2 = -8	-4 × -2 = 8

Table 1.9

In both patterns (a) and (b),

1. When both numbers are positive, the product is positive
2. When one number is negative, the product is negative
3. When both numbers are negative, the product is positive.

Exercise 1.9

1. Copy and complete the following number patterns.
    - a) 5 × 4 = 20
      4 × 4 = 16
      3 × 4 = 12
      2 × 4 =
      1 × 4 =
      -1 × 4 =
      -2 × 4 =
      -3 × 4 =
      -4 × 4 =
      0 × 4 =
    - b) 5 × -2 = -10
      4 × -2 = -8
      3 × -2 = -6
      2 × -2 =
      1 × -2 =
      0 × -2 =
      -1 × -2 =
      -2 × -2 =
      -3 × -2 =
      -4 × -2 =
      -5 × -2 =
    - c) -5 × 5 = -25
      -5 × 4 = -20
      -5 × 3 =
      -5 × 2 =
      -5 × 1 =
      -5 × 0 =
      -5 × -1 =
      -5 × -2 =
      -5 × -3 =
      -5 × -4 =
2. Work out the following:
    - a) (-3) × (+7)
    - b) +4 × -6
    - c) 13 × 9
    - d) -15 × 3
    - e) -6 × 8
    - f) -8 × 5

g) 7 × -5
h) -9 × +9

3. Evaluate the following:
   a) (-3) × (-2)
   c) -11 × -5
   b) -4 × -6
   d) -8 × -5
4. Copy and complete the multiplication Table 1.10

×	-3	-2	-1	0	1	2	3
-3						-6	
-2			2				
-1	0				-1		
0							
1							
2		4	0				
3							9

Table 1.10

5. Work out that:
   a) -3 × -4 × 2
   c) (+2) × (-7) × 3
   b) -5 × 4 × -3
   d) -6 × -2 × -3

Division of integers

We know that:

(=3) × (+4) = +12
(-3) × (-4) = +12
(+3) × (-4) = -12
(-3) × (+4) = -12

Division simply means the **reverse of multiplication** e.g. 6 ÷ 2 is the same as asking 'What number multiplied by 2 gives 6?'

Thus,

(+12) ÷ (+3) = +4 since (+3) × (+4) = (+12)

(-12) ÷ (-3) = +4 since (-3) × (+4) = (-12)

(+12) ÷ (-3) = -4 since (-3) × (-4) = (+12)

(-12) ÷ (+3) = -4 since (+3) × (-4) = (-12)

Note that:

The quotient (i.e. result of division) of two numbers with **like signs** is **positive** while that of two numbers with **unlike signs** is **negative.**

Example 1.13

Divide: (a) -24 by 3 (b) -18 by -6

Solution

(a) -24 ÷ 3 = -8 since -8 × 3 = -24
(b) -18 ÷ -6 = 3 since -6 × 3 = -18

Exercise 1.10

1. Divide:
    a) (+10) by (+2)
    b) 42 by 6
    c) 126 by 9
    d) -1 by -1
    e) -140 by -20
    f) (-24) by (-8)
2. Evaluate:
    a) -16 ÷ 2
    b) -90 ÷ 10
    c) (-96) ÷ (+4)
    d) (+96) ÷ (-4)
3. Work out:
    a) 50 ÷ -25
    b) 36 ÷ -9
    c) 24 ÷ -8
    d) (+96) ÷ (-4)
4. Simplify the following:
    a) (+12) ÷ -6
    b) (-10) ÷ (-4)
    c) (-28) ÷ (+7)
5. Copy and fill in the boxes in the following:
    a) 12 ÷ ☐ = -2
    b) -27÷ ☐ = 3
    c) ☐ ÷ (+4) = -6
    d) ☐ ÷ -5 = +7

Order of operations and brackets

Addition and subtraction

Consider 23 – 7 – 2

Do we:

(i) Subtract 7 from 23 and then subtract 2 from the result, or
(ii) Subtract 2 from 7 and then subtract the result 5 from 23?

We need to know the order in which the operations are to be carried out. A way of doing this is to use brackets, and work out what is in brackets first.

If the operations are to be carried out as in (i), we would write (23 – 7) – 2.

If the operations are to be carried out as in (ii), we would write 23 – (7 – 2).

Normally, however, we do the operations in the order in which we read them, so that 23 – 7 -2 means ‘start with 23, take away 7, then take away 2’ which gives the value 14.

Thus, 23 – 7 – 2 means (23 – 7) -2 and not 23 – (7 – 2). In the same way.

22 – 5 + 6 = (22 – 5) + 6 = 23

19 – 6 + 2 = (19 – 6) + 2 = 15

4 – 9 + 11 = (4 – 9) + 11 = 6

Work out the following pairs:

(i)	23 + (7 + 11),	23 + 7 + 11
(ii)	23 – (9 + 15),	23 – 9 + 15
(iii)	23 + (7 – 11),	23 + 7 – 11
(iv)	23 – (9 – 15),	23 – 9 – 15

In which pair(s) are the results the same?

Is 16 + (11 – 5) the same as 16 + 11 – 5?

Conclusion: The omission of brackets makes a difference only when we are subtracting what is in brackets.

Multiplication and division

Work out the following pairs:

(i)	24 ÷ (2 × -3),	(24 ÷ 2) × -3
(ii)	24 ÷ (4 ÷ 2),	(24 ÷ 4) ÷ 2
(iii)	4 × (3 × -2),	(24 × 3) × -2
(iv)	24 ÷ (4 ÷ -2),	(24 × 4) ÷ -2

In which pair(s) are the result the same?

Is 12 × (4 ÷ 2) the same as (12 × 4) ÷ 2?

Is 12 ÷ (4 × 2) the same as (12 × 4) × 2?

Is 12 ÷ (4 ÷ 2) the same as (12 × 4) ÷ 2?

Again, we see that different positions of brackets can give different answers. Thus,

We normally do the operations of multiplication and division in the order in which we read them, so that 12 × 4 ÷ 2 means ‘start with 12, multiply by 4 and then divided by 2’ i.e.

12 × 4 ÷ 2 = (12 × 4) ÷ 2.

24 ÷ 4 × 2 means ‘start with 24, divide by 4 and then multiply by 2’ i.e. 24 ÷ 4 × 2 = (24 ÷ 4) × 2.

Note: If multiplication comes before division, we can do the operations in that order, or do division first and then multiply. However, if division comes before multiplication, the operations should be done strictly in the order we read them.

Note: If division is followed by division, we do the divisions in the order we read them, e.g. 32 ÷ 8 ÷ 2 = (32 ÷ 8) ÷ 2 = 2 and **not** 32 ÷ (8 ÷ 2) = 8.

Combining addition or subtraction with multiplication or addition

5000 – 300 × 2 is understood to mean 5000 – (300 × 2) and not (5000 – 300) × 2, i.e. we do the multiplication before the subtraction.

If there are no brackets, we do multiplication (and division) before addition (and subtraction). If there is doubt, it is better to put the brackets in, so that

7 + 2 × 3 means 7 + (2 × 3)

3 × 4 + 2 means (3 × 4) + 2

72 ÷ 2 + 6 means (72 ÷ 2) + 6

4 × 3 – 8 ÷ 2 means (4 × 3) – (8 ÷ 2)

Exercise 1.11

1. Which of the following are correct?
   a) 27 + 7 – 3 = (27 + 7) – 3
   b) 27 – 7 – 3 = 27 – (7 – 3)
   c) 27 + 7 × 3 = (27 + 7) × 3
   d) 27 × 7 + 3 = (27 × 7) + 3
   e) 27 ÷ 3 × -2 = (27 ÷ 3) × -2
   f) 27 × -2 ÷ 3 = 27 × (-2 ÷ 3)
2. Copy and complete the following by putting in signs to make them correct.
   a) 14 + 10 – 2 = 14 (10   2)
   b) 14 – 12 – 3 = 14 (12   3)
   c) 12 + 8 + -9 = 12 (8   -9)
   d) 12 – 6 + 7 = 12 (6   7)
3. Write the following expressions without brackets.
   a) 16 + (1 + 5)
   b) 25 + (8 – 3)
   c) 9 – (6 + 12)
   d) 3 × 2 – (7 – 4)
4. Find the value of:
   a) -2 × 3 + 5
   b) -2 – 3 × 5
   c) -2 × -2 + 3
   d) -2 × 5 ÷ 2 + -2
   e) -2 × -2 + 3 × 3 – 5 ÷ 5
5. Simplify:
   a) -5 × (-2 + -8)
   b) -6 × (-3 × -5)
   c) 12 × (-7 + 7)
   d) (-2 × -3) + (-5 × -5)
   e) (4 × -4) + (-6 × +7)

Removing brackets

Brackets are used to indicate which part of an expression must be worked out first.

If a positive sign occurs before brackets containing only + or – signs, then the omission of brackets does not alter the result.

i.e. 7 + (3 + 2) = 7 + 3 + 2;

7 + (3 – 2) = 7 + 3 – 2

Now work out the following pairs:

(i) 7 – (3 + 2), 7 – 3 – 2
(ii) 7 – (3 – 1), 7 – 3 + 1
(iii) 7 – (3 + 2), 7 – 3 – 2
(iv) 7 – (3 – 1), 7 – 3 + 1

You must have discovered that:

If a negative sign occurs before brackets, then the brackets may be removed and the operation inside altered as follows:

18 – (5 + 3) = 18 – 5 – 3;

18 – (5 – 3) = 18 – 5 + 3.

Example 1.14

Work out:

(a) 101 – (100 + 1)
(b) 101 – (100 -1)
(c) 56 – (10 + 4)
(d) 576 – (10 – 18)

Solution

(a) 101 – (100 + 1) = 101 – 100 – 1 = 0
(b) 101 – (100 – 1) = 101 – 100 + 1 = 2
(c) 56 – (10 + 4) = 56 – 10 – 4 = 42
(d) 576 – (10 – 18) = 576 – 10 + 18 = 584

Mixed operations

When the operations of addition, subtraction, multiplication and division occur in the same expression without brackets, we do division first, followed by multiplication, then addition and finally subtraction. This can easily be remembered using the abbreviation **DMAS.**

Example 1.15

Simplify 39 – 5 + 6 × 8 ÷ 2

Solution

First divide 8 by 2: we get 39 – 5 + 6 × 4

Multiply 6 by 4: we get 39 – 5 + 24

Addition before subtraction: we get 63 – 5

Finally subtraction gives us 58

Or

First divide 8 by 2. We get 39 – 5 + 6 × 4

Multiply 6 by 4. We get 39 – 5 + 24

Working in the order in which subtraction and addition occur we get 34 + 24 = 58.

Exercise 1.12

1. Copy and complete the multiplication Table 1.11

×	1	2	3	4	5	6	7	8	9	10
1										
2					10					
3								24		
4										
5							35			
6				24						
7									63	
8						48				
9										90
10		20								

Table 1.11

2. Use the completed Table 1.11 to show that:
   a) 7 × 9 = 7(10 – 1) = 7 × 10 – 7 × 1
   b) 6 × 7 = (5 + 1)7 = 5 × 7 + 1 × 7
3. Work out the following products by first introducing brackets.
   a) 98 × 73
   b) 46 × 999
   c) 102 × 56
   d) 33 × 1 003
4. State the products of the following (Hint: Identify the pair that gives a product which easily multiplies with the third number):
   a) 2 × 9 × 5
   b) 4 × 78 × 25
5. Use the completed Table 1.11 to find:
   a) 7 × 16
   b) 9 × 17
   c) 13 × 15
   d) 18 × 16

   (You will have to split the (16)
6. Work out by first removing the brackets, then check your answers by evaluating what is within the brackets first.
   a) -8(10 – 1)
   b) 30 + -2(-1 – 4)
   c) 5(-10 + 2)
7. Evaluate the following by removing the brackets first.
   a) 8 + (4 + 6)
   b) 8 – (4 + 6)
   c) -8 – (4 – 6)
   d) -8 – (-4 + -6)
   e) 8 – 2(-4 + 6)
8. Evaluate:
   a) 18 ÷ -2 × 3
   b) -24 × 36 ÷ -9
   c) -12 ÷ -2 ÷ -3
9. Simplify:
   a) -2 × +15 ÷ -6
   b) -6 × -8 ÷ -6

c) +9 × +12 ÷ -3

10. What number gives -8, when divided by 3?

11. What number divides 40 to give -8?

12. Calculate:

a) 56 – 7 × 8 ÷ 4

b) 56 – 7(8 ÷ 4)

c) 56 – (7 × 8) ÷ 4

d) 56 – [(7 × 8) ÷ 4]

e) (56 – 7) × 8 ÷ 4

f) (56 – 7)(8 ÷ 4)

13. Simplify the following by first removing the brackets.

a) 30 – (14 – 8)

b) 25 – (-8 + 2)

c) 16 – (-8 – -4)

14. Copy and insert the missing signs or numbers to make the following statements true.

a) 10 – 8 - _ = 10 – (8 _ 1)

b) 12 – (_ - -4) = 12 – 6 _ 4

c) (9 – 1) – (6 _ 2) = 4

d) (18 - _) – (10 _ 3) = 10

15. Evaluate:

a) 3 × 4 + 18 ÷ -3

b) 12 ÷ 6 + 8 × 3 + -4 + 2

c) -2 × 3 + -8

d) -3 × -8 ÷ (4 + -2)

e) 8 × 4 – 2 × 4 + 21 ÷ 3

f) 90 ÷ -6 + (8 × 15) – 14 × -15

16. Work out:

a) 3 × -2 – 3

b) -2 × 4 ÷ 4 – 3

c) -2 × 3 × 4 – 3

d) 2 × 4 – 3 × -2 + 3

e) 2 × 12 × -2 × 3 – 4

Operations on larger integers

Addition and subtraction

It is easy to add or subtract small numbers using a number line. However, for larger integers using a number line is not practical. We therefore need to know how to add or subtract numbers by arranging them column wise by place value.

Example 1.15

Work out

(a) 348 + 75
(b) 5 083 – 567

Solution

(a) 3 4 8	5 0 8 3
+ 7 5	- 5 6 7
4 2 3	4 5 1 6

Exercise 1.13

1. Work out the following:

a) 38 + 82	b) 2 4 8	c) 6 3 3
	+ 5 9	+6 1 5
	7 2 6	7 4 3

2. Calculate the following:
   a) 87 – 32
   b) 1 083 – 767
   c) 1 000 – 4 283
   d) 20 827 – 9 308
3. Chafulumisa had 170 cows and 139 goats. He teamed up with Chubale who had 165 cows and 44 goats. They sold 56 cows and 13 goats because the grazing field was small. How many animals of each type did they remain with?
4. For subtraction from 100, 1 000, 10 000, etc., simply proceed as follows.
   1 000 becomes 999 + 1
   - 873 ,,, - 873
   126 + 1 = 127

Use this method to calculate the following.

a) 100 – 87
b) 1 000 – 647
c) 10 000 – 5 382
d) 6 000 – 1 264

5. Write down the answers to the following. (Hint: first add the pair that will give a sum which adds easily with the third number).
   a) 9 + 7 + 1
   b) 36 + 78 + 64
   c) 102 + 327 + -2
   d) 382 + 796 + 618
   e) 1 725 + -568 + 568

6. Work out:
   a) 62 + -45
   b) -29 – 69 – (-40)
   c) 71 + -58 + -23
   d) 379 – 480
   e) -273 + (+324)
   f) -726 – (-96)

Copy and fill in the boxes in Question 7 and 8

7. (a) -3 201 + ☐ = -943
   (b) +296 + ☐ + -69 = 78

8. (a) ☐ - -546 = 36
   (c) ☐ - _491 – 702 = -111

Multiplication and division

Multiplication and division of integers are done in the same way as the multiplication and division of natural numbers. However, when negative integers are involved, disregard the sign(s) during the working and insert the appropriate sign only after getting the result, recalling that two numbers with like signs give a positive result while two unlike signs give a negative result.

Example 1.17

Evaluate:

a) -372 × 645
b) -2 784 ÷ 13

Solution

a)
$$\begin{array}{r} -3\ 7\ 2 \\ \underline{\times\ 6\ 4\ 5} \\ 1860 \end{array}$$

$$\begin{array}{r} 1488 \\ 2232 \\ \hline -239940 \\ \hline \end{array}$$

b) Divide as positive numbers:

$$\begin{array}{r|l} & \;\;214 \\ \hline 13 & 2784 \\ & \underline{26}\downarrow\;\downarrow \\ & \;\;18 \\ & \;\;\underline{13}\;\downarrow \\ & \;\;\;\;54 \\ & \;\;\;\;\underline{52} \\ & \;\;\;\;\;\;2 \end{array}$$

Since unlike signs always give a negative quotient and a negative remainder, then -2 784 ÷ 13 = -214 rem -2

Exercise 1.14

1. (a) Work out the following:
   - (i) 1 000 × 28
   - (ii) -723 × 5 016
   - (iii) -2 740 × -59

   (c) Work out the following stating the remainder if it exists.
   - (i) -27 ÷ 8
   - (ii) 1 620 ÷ 36
   - (iii) 4 004 ÷ 44
   - (iv) -70 599 ÷ 485
   - (v) -18 792 ÷ 87
2. Work out the following by doing multiplication first. Work them out again by taking the order of precedence of operations into consideration.
   - (i) 5 × 8 ÷ 2
   - (ii) 36 × 42 ÷ 7
   - (iii) 18 × 425 ÷ 17

a) Are the two answers the same?
b) Which operation performed first, will make the work easier?

3. Evaluate:
   a) 100 ÷ 10 × 5
   b) 72 ÷ 12 × 3
   c) 216 ÷ 24 × 9
4. Find the least number that must be
   a) Subtracted from
   b) Added to 4 651 to get a number divisible by 62.
5. When 1 597 is divided by a certain number, the quotient is 25 and the remainder is 47. Find the number.
6. Thenga miscopied 98 as 89. He multiplied 89 by a certain number and got 4 005. Find that number and the correct product.
7. Calculate how much could be received from 12 sacks of beans each 80 kg if packed in 1 kg packets each costing K 200.
8. 2 438 books are parceled in groups of 10. The parcels are then arranged in piles of 8. How many parcels are not in piles and how many books at=re not parceled?
9. A farmer has 63 goats and twenty times as many sheep. If the number of sheep he has is thirty times the number of his chicken, find the total number of legs of the animals (including those of chicken).
10. Write true or false for each of the following statements and where false, give an example to support your answer.
    a) The product of any negative number by zero is always zero.
    b) The product of two negative numbers is always positive.
    c) The product of zero ad any positive number is always positive.
    d) The product of a negative number and a positive number is always negative.
    e) If the product of two numbers is negative, then one of them is negative and the other is positive.
    f) If the product of two numbers is positive, then both numbers must be positive.
    g) If the product of two numbers is positive then both numbers must be negative.

h) If the product of two numbers is positive, either the two numbers are both positive or they are both negative.

Application of integers

The knowledge of integers finds a lot of application in real life situations. In such applications, measurements below zero are written as negative numbers while those above zero are written as positive numbers. The words used, such as left/right, above /below, east/west, etc., suggest a reference point as well as what is positive or negative. For example, a temperature of 5˚C below freezing is said to be -5˚C.

Positive and negative integers are constantly being used in real life situations. For example, when measuring temperatures, we find that temperature is sometimes above 0˚C meaning that it is positive e.g. 10˚C and sometimes it is below 0C meaning that it is negative e.g. -9˚C. In this case, 0 is the reference point.

Similarly, when we refer to east or west, north or south, right or left of a point, we are actually applying the knowledge of integers. For example, in drawing graphs, we use two perpendicular axes which meet at a point referred to as the origin or point O. Point to the left of the origin are considered to be negative while points to the right of the origin are considered to be positive. Points above the origin are positive while those below the origin are negative.

Similarly, the distance below sea level can be considered as negative while above sea level is positive. There are many such situations that make use of positive and negative numbers.

Example 1.18

On a certain day, the temperature at the top of Mulanje Mountain was -10˚C and down at Mkhoto was +35˚C. What was the difference between the two temperatures?

Solution

Difference = 35˚C – -10˚C = 45˚C

Exercise 1.15

1. A man owes a bank K 10 000. Another has K 2 400 in the bank. What is the difference in the deposits of the two men?
2. A lake is 1 340 m deep at its deepest point. The surface of the lake is 1 000 above sea level. What is the height of the deepest point above sea level?
3. A school in a semi-arid region sunk a bore hole 40 m deep. The water was to be pumped into an overhead tank whose top is 20 m above the ground. The level of water in the pipe when pumping started was 30 m below the ground and it rose by 5 m every second (See Fig. 1.26)
   a) What is the vertical length of the pipe?

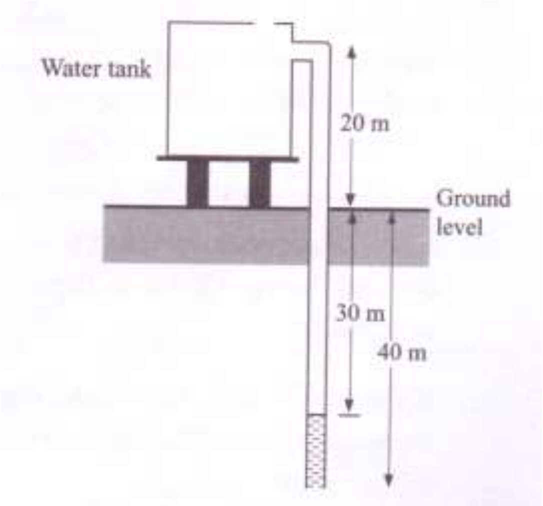


Fig. 1.26

   b) How long will it take for the water to start entering the tank? (Ignore the horizontal distance travelled by the water).
   c) State the height of the water levels at intervals of 2 seconds after the pumping started.
   d) How long will it take for the water level to rise from -30 m to 10 m above the ground?
4. Table 1.12 shows the Malawian education system that was completed by various learners in December 1996.

Name	Level
Mphatso	Primary 2
Mabvuto	Primary 4

Kausiwa	Primary 5
Mesi	Primary 7
Kwayera	Form 2
Mwai	Form 3
Mleng	Form 4

Table 1.12

By taking completion of primary education as the reference point (zero),

a) Illustrate the information in the table on a number line and use the number line to work out the number of years.
   (i) Mleng is ahead of Mesi
   (ii) Mphatso is behind Kausiwa

b) Name the opposite pairs (i.e. one beyond Primary 7 just as the other one is behind).

5. (a) if the temperature of a liquid changes from -15˚C to -8˚C, find the rise in temperature.
   (c) Town A is 250 m above sea level and point B is 25 m below sea level. How far, vertically, is A above B?
   (d) In a financial institution, if a pay-in is denoted by + an a withdrawal by -, find the state of the account after the following transactions:
   +K 8 300, -K61.40, +K280, -K4 040

6. In a maths quiz, every correct answer scores 2 marks, -1 mark for every wrong answer, and no mark for no answer. The test has 30 questions.
   a) Find
      (i) The maximum possible score,
      (ii) The minimum possible score.
   b) If June has 20 correct answers, 8 wrong answers and 2 no answers, what mark does she get?
   c) Mary scores 23 marks, having got 5 answers wrong. How many answers does she get right?

7. At a meteorological station, temperatures are recorded (in degrees Centigrade/Celsius) every three hours as in Table 1.13.

Time	6 a.m.	9 a.m.	12 noon	3 p.m.	6 p.m.	9 p.m.	Mid-night
Temp (˚C)	-8						

Table 1.13

a) Copy the table and fill in the missing temperatures if the temperature rose by
   (i) 6˚C between 6 a.m. and 9 a.m.
   (ii) 12˚C between 9 a.m. and noon.
   (iii) 4˚C between noon and 3 p.m.
   (iv) 16˚C between 3 p.m. and 6 p.m.
   (v) 6˚C between 6 p.m. and 9 p.m.
   (vi) -5˚C between 9 p.m. and midnight.

b) In which three hour period was
   (i) The rise in temperature greatest?
   (ii) The fall in temperature greatest?

8. An account is overdrawn by K 160 000. How much must be deposited in order to have a credit of K 300 000?
9. An aircraft flying at a height of 1 5000m above sea level lands on an air field 600 m above sea level. What height does it loose on landing?
10. What is the longitude difference between two places 70˚ W and 30˚ E?

Rational numbers

Any integer can be expressed in the form $\frac{a}{b}$ where by ≠0. Any number that can be expressed in such a form is called a rational number. Thus:

A rational number is any number that can be written as a **ratio** $\frac{a}{b}$, where a and b are integers and b ≠ 0.

All integers are rational numbers as each can be written in the form $\frac{a}{b}$. For example 3 =$\frac{3}{1}$, -8 =$\frac{-8}{1}$, etc. other examples of rational numbers are

$\frac{-8}{5}, \frac{22}{7}, \frac{11}{3}, \frac{-8}{101}$

Activity 1.1

Give more examples of rational numbers. Is zero (0) a rational number?

Example 1.19

Show that the following are rational numbers.

(a) 0.134 (b) 3.45 (c) 3.55

Solution

Writing each decimal as an exact fraction we get

(a) 0.134 = $\frac{134}{1\,000}$ = $\frac{67}{500}$

Thus, 0.134 is a rational number.

(b) 3.45 = $\frac{345}{100}$ = $\frac{69}{20}$

Hence, 3.45 is a rational number.

(c) 3.55 is converted to a fraction as follows:

Let f = 3.55

100f = 355.55

100f - f = 355.55 – 3.55

99f = 352

f = $\frac{352}{99}$

3.55 = $\frac{352}{99}$

Hence, 3.55 is a rational number.

Some decimals neither terminate nor recur. An example of this type of decimals is 0.818 118 118 …. This decimal cannot be written as an exact fraction and is referred to as irrational. An **irrational number** is any number that cannot be expressed as an exact fraction since it neither terminates nor recurs.

Representing rational and irrational numbers on a number line

To represent any positive rational number $\frac{a}{b}$ on a number line, divide the unit interval into 'b' equal parts and then take 'a' of these parts along the number line to reach the point corresponding to $\frac{a}{b}$, to the right of zero.

The rational number $\frac{-a}{b}$ is similarly represented, but to the left of zero.

Example 1.20

Represent the following rational numbers on number lines.

(a) $\frac{3}{5}$ (b) $\frac{7}{4}$ (c) $-\frac{5}{4}$

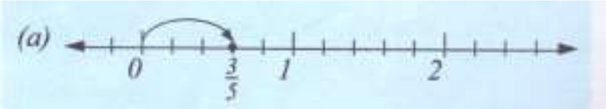


Fig. 1.27

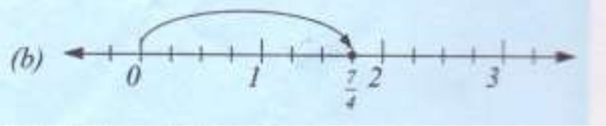


Fig. 1.28

Fig. 1.29

Since irrational numbers contain decimals that neither terminate nor repeat. It is not possible to exactly locate them on a number line. They can be represented only by approximation.

Example 1.21

Locate π on a number line correct to 2 decimal places.

Solution

π = 3.141 592 6…. = 3.14(2 d.p)

Fig. 1.30

Exercise 1.16

1. Find the rational number $\frac{a}{b}$ represented by each of the following numbers, where $\frac{a}{b}$ is in its simplest form.
   a) 0.625
   b) 3.25
   c) -0.04
   d) 5
   e) $4\frac{3}{8}$
   f) $-3\frac{5}{6}$
2. What recurring decimal is represented by each of the following rational numbers?
   a) $\frac{2}{9}$
   b) $\frac{5}{6}$
   c) $\frac{3}{11}$
   d) $1\frac{5}{27}$
3. Indicate whether each of the following numbers is rational or irrational.
   a) 2.18
   b) $\frac{3}{77}$
   c) 5.151 151 115
   d) $-2\frac{4}{13}$
   e) 7.217 217 217
   f) 2.718 281 828 26
4. Locate each of the following rational numbers on a number line.
   a) 3.4
   b) $\frac{-5}{4}$
   c) $1\frac{2}{3}$
   d) 1.27
5. Indicate the position of each of the following numbers on the number line correct to 2 decimal places.
   a) 0.353 553 555
   b) 2.929 229 222
6. Given that Q represents the set of all rational numbers, Z all integers, T all terminating decimals, and P all recurring decimals, which of the following statements are true?
   a) Z is wholly contained in Q
   b) Z, P and T put together make Q
   c) There are no numbers that are in T as well as in P.

Unit 2 SETS 1

Definition and description of sets

Definition of a set

The term “set’ refers to a collection of things or items that have a common characteristics.

In everyday use of the term set, one can come up with sets such as sofa set, dinner set, etc. However, in Mathematics, sets are treated in an algebraic manner, mostly using letters and numbers. For example, the collection of natural numbers, a group of boys in a class both represent sets.

Notation

A set is denoted by curly brackets as {3, 5}. They are named using capital letters such as A, B, C etc. For example,

A = {all tomatoes in a garden} is read as “A is the set of all tomatoes in a garden.”

In every set, there are well defined items, which are referred to as “**elements”** or “**members”** of the set.

For example, if a, b, c and d are all elements of a set B, then B = {a, b, c, d}.

The symbol ϵ is used to denote that a given element is a member of a given set B. for example, a ϵ B means and is read as “ a is a member of set B”.

C ∉ B means that “C is not a member of set B”.

Describing a set

There are various methods of describing a set:

1. Describing a set using words; e.g. B = {a, b, c, d} can be described as “the set of the first four letters of the English alphabet.
2. Describing a set by listing all the elements in it; e.g. B = {a, b, c, d}.
3. Describing a set by using a formula; e.g. D = {x: x^2 = 9}. This means that D is “the set of all x values such that x is a solution to the equation x^2= 9

Example 2.1

List all the elements of set A given that

$A = \{x: -3 < x < 8, x \text{ is an integer}\}$.

Solution

The elements of set A are -2, -1, 0, 1, 2, 3, 4, 5, 6 and 7.

Example 2.2

Describe set B using words, given that

$B = \{1, 3, 5, 7….\}$.

Solution

B is the set of all positive odd numbers.

Example 2.3

Describe set T in words, given that

$T = \{x: x = 2n – 1, \text{ where } n = 1, 2, 3…\}$

Solution

See Table 2.1

n	1	2	3	4	5	6	7
x	1	3	5	7	9	11	13

Table 2.1

Thus, T is the set of all positive odd numbers.

Example 2.4

List down the set G of vowels in the word “algebra”.

Solution

$G = \{a, e\}$

Note:

Elements in a set must be written without repetition e.g. there are two ‘a’s in the word ‘algebra’ that belong to set G, but we only listed one “a”.

Number of members in a set

If A is a set, n (A) means number of members in A. for example, in set W, where W = {days of the week}, n (W) = 7 (since there are 7 days in a week). If a set B is such that n (B) = 0, then B is an empty set.

For example, B = {four legged men}.

There are no known men with four legs.

Therefore, the set has no members, meaning n(B) = 0

Exercise 2.1

Give a precise description in words for each of the following sets.

1. {Hastings Banda, Bakili Muluzi, Bingu wa Mutharika, Joyce Banda}
2. {Daily Times, Nation, Nyasa Times}
3. {Africa, Europe, Asia, Australia, North America, South America}
4. {Machinga, Nsanje, Balaka , Zomba}
5. {Lake Malawi, Lake Chirwa, Lake Malombe}

! Clean water is key to good health. Let us avoid all acts that pollute our water reservoirs

6. {Zambia, Tanzania, Mozambique}
7. {Vwaza Marsh, Nkhotakota, Majete, Mwabvi}
8. {Escom United, Silver Strikers, Moyale Barracks}

Give a precise description of each of the following set using listing method.

9. {The set of natural numbers}
10. {The set of squares of natural numbers}
11. {x: x is the set of even numbers less than 15}
12. {x: $x^2 – 1 = 0$}
13. {x: x – 1 = 1
14. X: $\frac{1}{2}x – 3 = 4$}

15. Copy the following and insert the correct symbol in each case.
   a) {4} …………….. {even numbers}
   b) -5 ………………. {counting numbers}
   c) 3 ………………. {even numbers}
   d) {5, 6} …………. {odd numbers}
   e) 7 ………………. {odd numbers}
   f) {6, 7, 8} ……… {counting numbers}

Describe each of the following sets using a rule.

16. {2, 4, 6, 8}
17. {0, 3, 8, 15}
18. {1, 3, 5, 7 …}
19. {2, 5, 10, 17}

Types of sets

Sets may be grouped into two broad categories. These are **finite** and **infinite** sets.

Finite sets

A set such as {days of the week}, {first six letters of the alphabet}, {months of the year}, {girls in a class} and so on, have a finite number of elements: these are called **finite** sets.

Infinite sets

These are those sets with unknown number of elements. The elements in these sets cannot be exhaustively counted.

In the number patterns below, the elements are shown to continue indefinitely. Thus,

N = {1, 2, 3, 4…} i.e. natural numbers,

E = {2, 4, 6 …..} i.e. even numbers,

P = {1, 3, 5, 7, 11, 13 ….} i.e. prime numbers are examples of **infinite** sets.

Other sets

Empty set or Null set

A set which does not contain any element is called an empty set or null set. Examples of empty sets are:

{A set of natural numbers between -10 and 0}.

{A set of girls in a boys only school}, etc.

An empty set is denoted {} or Φ.

Activity 2.1

Describe ten empty sets

Disjoint sets

Suppose A = {1, 2, 3, 4, 5} and

B = {-1, -2, -3, -4, -5}

The members of set A and those of set B are very different. No members of A belongs to set B and vice versa. Set A and set B are said to be disjoint and can be illustrated in a diagram as in fig. 2.1

Fig. 2.1

Sets which have no members in common are called disjoin sets. For example, the set of alphabets and the set of natural numbers are disjoint sets.

Exercise 2.2

State whether the following sets are empty, finite or infinite.

1. {Malawi, Mozambique, Zambia}
2. {All integers between 1 and 2}
3. {x: x<-2}
4. {All trees in your class}
5. {All real numbers}.
6. {Prime numbers between 1 and 10}.

Define a universal set that connects the given sets below.

7. {Boys in a class} and {Girls in a class}.
8. {Set of male teachers in a school} and {Set of female teachers in a school}
9. {a, b, c,} and {a, d, e}
10. {Consonants in the word "class"} and {vowels in the word "class}

Comparison of sets

Equivalent sets

If two sets P and Q are such that n (P) = n (Q) but the elements are different, the two sets are called **equivalent** sets. For example,

P = {1, 2, 3, 4, 5}

Q = {a, e, i, o, u}

The elements in P are different from the elements in Q.

But n(P) = n(Q)

Therefore P and Q are equivalent.

Two or more sets are said to be equivalent if they have the same number of elements. For instance set A = {1, 6, 3} is equivalent to set B = {a, b, c} as they both have three elements each.

Example 2.5

Show that G = {7, 3, 2, 1} is equivalent to M = {a, z, w, n}

Solution

The number of elements in set G is 4 and the number of elements inset M is 4, so the two sets G and M are equivalent.

Equal sets

Two sets which contain exactly the same elements are called **equal sets.**

For example if,

N = {natural numbers less than or equal to 6},

P = {1, 2, 3, 4, 5, 6}

N and P contain the same elements

Therefore N = P ➡ n (P) = n(N)

Two or more sets are equal if they are equivalent and their elements are exactly the same. For instance set N = {7, 8, 9} is equal to set M = {7, 8, 9}

Number of elements

The number of elements in a set is called cardinal number. If A is a set, then the cardinal number of set A is donated as n(A), for example if A = {a, b, c, d}, then the cardinal number of set A is 4. This is written as n(A) = 4.

Subsets

Consider two sets M and N such that

M = {m, a, t, h, e, i, c, s}

N = {a, e, i}

You will notice that, all the elements of set N are contained in set M. in this case, N is said to be a subset of M, i.e. a set within a set. Fig. 2.2 shows two such sets.

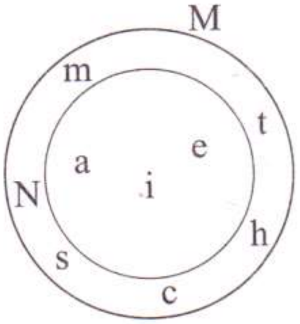


Fig. 2.2

For any two sets N and M, N is a subset of M if all elements of N are contained in M. A subset is donated by the symbol " ⊂ ".

Thus, N ⊂ M means, and is read as "set N is a subset of set M."

Note:

1. An empty set is a subset of all sets.
2. If all elements in N are in M and N≠M, N is said to be a **proper subset of M** i.e. N⊂ M.
3. If all elements in N are in M and N = M then either of the two sets is said to be an **improper subset** of the other; denoted as N⊆M or M⊆N

Example 2.6

List all subsets of the set {1} and state their number

Solution

The subsets of {1} are {} and {1}. So there are 2 subsets.

Example 2.7

List all the subsets of {1, 2} and state their numbers.

Solution

The subsets are {}, {1}, {2} and {1, 2}. So there are 4 subsets.

Example 2.8

List all the subsets of {1, 2, 3} and state their number.

Solution

The subsets of {1, 2, 3} are { }, {1}, {2}, {3} {1, 2}, {1, 3}, {3, 2}, {1, 2, 3}

There are 8 subsets.

The number of subsets in a set

The number of subsets of a given set depends on the number of elements. See Table 2.2

Set	Number of subset
{ }	$1 = 2^0$
{1}	$2 = 2^1$
{1, 2}	$4 = 2^2$
{1, 2, 3}	$8 = 2^3$

Table 2.2

If the number of subsets of a given set is represented by N_s then the number of subsets is given by;

$N_s = 2^n$

Where n equals number of elements in the set.

Example 2.9

Set C has elements. How much subsets does it have?

Solution

Since the number of elements is five (n = 5) then from $N_s = 2^n$, the number of subsets is

$N_s = 2^5$

$= 32$

i.e. the set has 32 subsets.

Example 2.10

A certain set has 64 subsets. How many elements are there in the set?

Solution

Number of subsets (N_s) = 64

$N_s = 2^n$

$64 = 2^n$

$2^6 = 2^n$

N = 6

i.e. there are 6 elements

Exercise 2.3

State whether the pairs of sets in questions 1 – 6 below are equivalent, equal or neither.

1. {3, 2, 1} and {1, 2, 6}
2. {a, b, z} and {z, b, a}
3. {r, s, t} and {r, s, m, t}
4. {Leah, Carol} and {Carol, Leah}
5. {Mzuzu} and {Lilongwe, Blantyre}
6. {Set of positive integers} and {Set of natural numbers}.

List the subsets of each of the sets in questions 7 and 8.

7. {w, z, y}
8. {a, 3, 1, 2}
9. If there are 128 subsets in a set, how many elements does the set contain?
10. Suppose a set has $4^{(n-3)}$ subsets. How many elements are in this set? (n is the number of elements in a set.)

Union and intersection of sets

Consider sets A and B

Set A = {a, b, c, d, e, f, n} and

B = {a, d, e, f, i, k, l, n}

Set A and B have common elements i.e. a, d, e, f, n. we can illustrate each set as in Fig. 2.3

Fig. 2.3

Union

Sets a and B can be combined in a single diagram without repeating the common elements as in Fig. 2.4

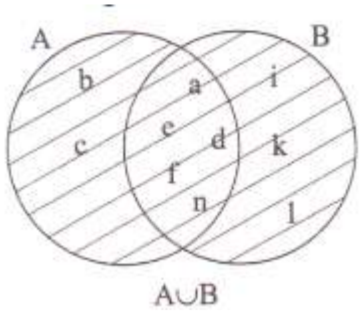


Fig. 2.4

Fig. 2.4 has three regions, a region containing elements belonging to both sets, a region containing elements belonging to a set A only and the region containing elements belonging to set B only. The three regions are shaded and make a union of set A and B denoted as A∪B.

Thus, A∪B = {a, e, d, f, n, b, c, i, k, l}

The union of two sets A and B denoted as A∪B is the set of elements which are in A or B, including those that are in both A and B. these elements of A∪B are listed without repetition.

Example 2.11

If A = {1, 3, 4, 8} and B = {3, 4, 7} determine A∪B. list the elements of the union.

Solution

A∪B = {1, 3, 4, 7, 8}

Therefore, the elements of A∪B are 1, 3, 4, 7, 8.

Intersection

Two sets A and B can be combined in a single diagram if they have at least one element in common.

Suppose A = {2, 4, 6, 8, 10} and

B = {1, 2, 3, 4, 5, 6}

Set A and B have three common members 2, 4, 6.

We say that the intersection of A and B contains the elements 2, 4, 6. Fig. 2.5 shows the intersection of the sets denoted as A∩B.

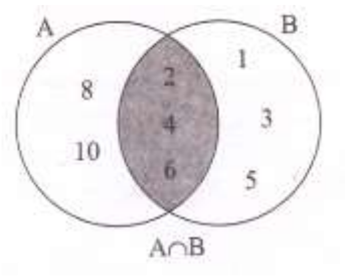


Fig. 2.5

The shaded region represents the intersection of the two sets A and B.

The intersection of two sets A and B, denoted as A∩B is the set of elements that are common to both sets A and B. if the two sets do not have elements in common. i.e. A∩B = { }, the two sets are said to be disjoint.

Example 2.12

Given that A = {1, 3, 5, 9} and B = {3, 5, 7}, determine A∩B. list the elements of the intersection.

Solution

Since A∩B is the set of elements that are in both A and B, then A∩B = {3, 5}.

Therefore, the elements of A∩B are 3, 5.

Exercise 2.4

Given each of the sets A and B in questions 1 – 4, express each pair in the form $A \cap B$.

1. A = {3, 1} and B = {3}
2. A = {3, 1} and B = {3, 4, 1}.
3. A = {x : x even, 1 < x <21} and
   B = {x : x even, 3 < x <11}
4. A = {factors of 60} and
   B = {x : x^2 = 1 < x < 5

Given the sets in questions 5 – 8, form the set $A \cup B$

5. A = {4, 2} and B = {4}.
6. A = {4, 2} and B = {4, 5, 2}.
7. A = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20} and
   B = {4, 6, 8, 10, 11}.
8. A = {factors of 30} and
   B = {x : x^2, 1 < x < 4}.
9. If A = {4, 6, 8} and B = {2, 4, 9}.

   List the elements of the following sets.

   a) $A \cup B$ b) $A \cap B$
10. If n (A) = 11, n (B) = 7 and n ($A \cap B$) = 5
    Find n ($A \cup B$).
11. Given that A and B are sets such that $A \cap B$ = B and $A \cup B$ = A what can you say about A and B.

Venn diagrams

The relationship between sets may be shown using diagrams known as Venn diagrams named so after the English Mathematics John Venn (1834 – 1923) who initiated their use.

Different set expressions can be showing using this method.

Example 2.13

Represent the following on Venn diagrams

(i) $A \cup B$ (ii) $A \cap B$

Solution

Fig. 2.6 to 2.7 show the respective sets. The convention is to shade the region.

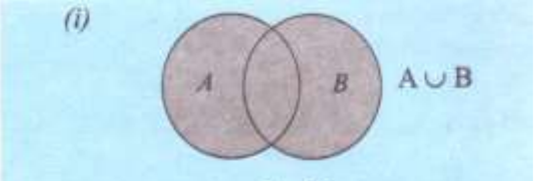


Fig. 2.6

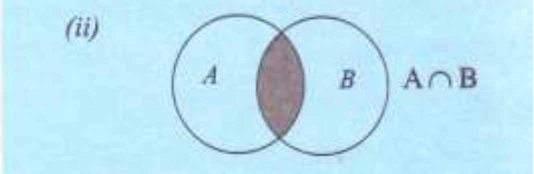


Fig. 2.7

Fig. 2.8 shows a combination of two sets in two different diagrams

A = {2, 4, 6, 8, 10, 12, 14}

B = {1, 2, 3, 4, 5, 6, 7}

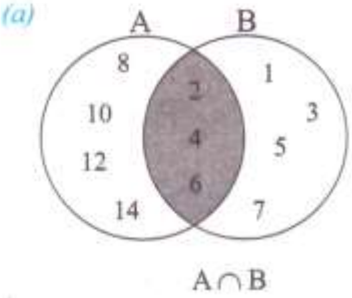


Fig. 2.8 (a)

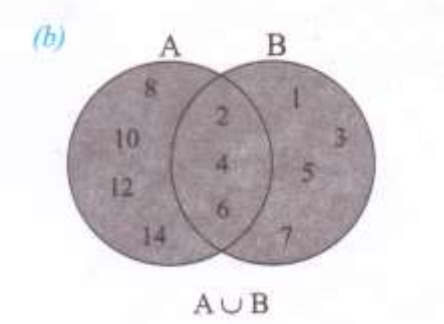


Fig. 2.8 (b)

Fig. 2.8(a) shows the relationship between sets A and B, they have common members 2, 4, 6,. The shaded region shows the intersection of the two sets, written as AᴖB = {2, 4, 6}

Fig. 2.8(b) shows a combination of set A and B into one set, written as.

AᴗB {1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14}

Note that in a union between any two or more sets, the numbers that are common to the sets are listed only once.

Example 2.14

In a school of 150 students, 90 of them registered for chemistry, 110 for biology and some registered for both. Use a Venn diagram to determine how many students registered for both subjects.

Solution

Let B = {students who registered for chemistry}

C = {those who registered for biology

X = {those who registered for both}

So, n (C) = 90

N (B) = 110

CᴖB = {those who registered for both}

N (CᴖB) = x

We can draw a Venn diagram to illustrate this information. See Fig. 2.9.

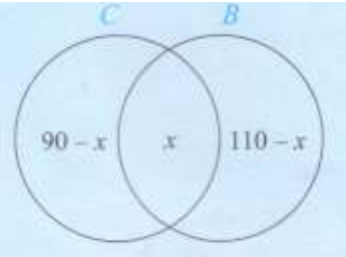


Fig. 2.9

Since CᴗB = {all the pupils in the school} then n(CᴗB) = 150

Therefore 90 – x + x + 110 – x = 150

200 – x = 150

-x = -50

X = 50

Therefore 50 students registered for both subjects

Exercise 2.5

1. Draw Venn diagrams to illustrate the following sets.
   a) A = {a, b, d, f, n, m}
      B = {j, k, m, l, n, d}
   b) C = {the first ten natural numbers}
      D = {the first ten even numbers}
   c) G = {0, 2, 6, 8, 10, 12, 14}
      H = {1, 4, 9, 16, 25}
   d) P = {1, 2, 4, 5, 9}
      Q = {2, 4, 6, 8}
2. Copy and complete the following.
   a) {p, q, _} ∩ {p, r, s} = {s, }
   b) {9, _, 8} ∩ {4, 7,_} = {9, 4}
   c) [a, b, _, d, _} ∩ {_, q, _, r, s} = {c, f}
   d) {6, _, _} ∪ {8, _, 3} = {8, 6, 3, 9}
3. What can you say about each of the following pairs of sets if,
   a) A ∩ B = A
   b) A ∪B = B
   c) (P ∩ Q) = { }

   In each case illustrate the sets in a Venn diagram where possible.

4. Fig. 2.10 illustrates two set in a Venn diagram. If the entries given represent the number of members in each region, use the information to find: n(A), n(B), n(A∩B), n(A∪B)

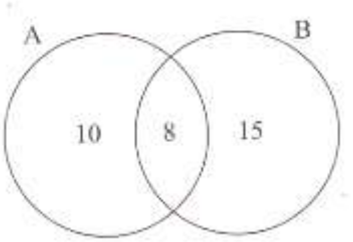


Fig. 2.10

5. Fig. 2.11 shows two sets A and B. Given that n(A ᴗ B) = 30

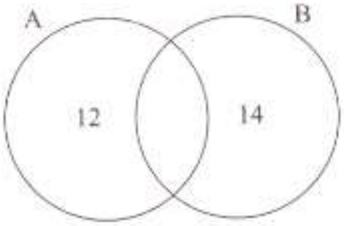


Fig. 2.11

Find,

(a) n(A⁀B)
(b) n(A)
(c) n(B)

6. 120 candidates took an examination in Physics and Chemistry. 75 candidates passed in Physics and 55 passed in Chemistry.
   Draw a Venn diagram and show the given information. Show the candidates who passed in both subjects.
7. Given that P and Q are intersecting sets and that n(A) = 20, n(Q) = 30 and n(PᴗQ) = 45, find n(P ⁀ Q).
8. In a class of 40 students, 33 play volleyball, 27 play soccer. How many students play?
   (a) Volleyball only?
   (b) Soccer only?
   (c) Both games?
9. (a) A = {a, b, c}
   B = {b, c, d, e}
   C = {A ⁀ B}
   D = {A ᴗ B}

Find n(A), n(B), n(C), n(D)

(a) For each of the sets in (a) above, write true or false in the following.

- (i) $C \subset A$
- (ii) $A \subset D$
- (iii) $D \subset A$
- (iv) $C \subset D$
- (v) $A \in C$
- (vi) $5 \in C$
- (vii) $\{5\} \subset C$
- (viii) $B \subset D$

10. Pupils of a given class in certain school belong to one or more school clubs. If 21 pupils belong to the music club, 23 to the mathematics club and 18 belong to both clubs, how many pupils are there in the question.

11. On a certain day, there were 36 visitors in a certain hotel. Given that 24 of them had visited America, 26 had visited Britain and some had visited both countries. Use Venn diagram to determine how many had visited only one country.

12. P = {All people who like pork}

C = {All people who like chicken}

If G = {All people who like pork but not chicken}

And N = {All people who like chicken but not pork}

Represent each of the following on separate Venn diagrams by shading the appropriate region.

(a) Set G

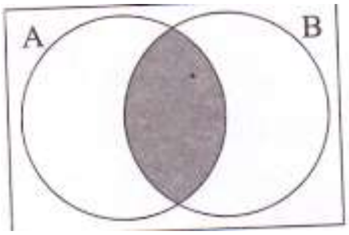
(b) Set N

(c) Those who like either chicken or pork

(d) Those who like both chicken and pork

13. Set A and B are such that n(A) = 3 and n(B) = 8. Describe and illustrate in separate Venn diagrams three possible relations between A and B.

14. Using the set notation. Describe each of the shaded sets Fig. 2.12 below).



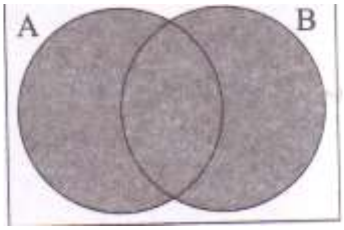


Fig. 2.12

Unit 3 ALGEBRAIC EXPRESSIONS

Symbolic representation

Algebra is an extension of arithmetic. In addition of numbers, letters are used to represent items or numbers. The letters are treated in the same way numbers are treated in arithmetic.

An expression in which letters are used to represent numbers is called an **algebraic expression**. The following are some examples in Table 3.1.

Statement	**Algebraic expression**
The number of mangoes in a bag	m
The number of bananas in a bunch	b
The number of fruits in 2 bags of mangoes and 3 bunches of bananas	2m + 3b
The number of fruits in $\frac{1}{2}$ a bag of mangoes subtract the number of fruits in 5 bunches of bananas	$\frac{1}{2}$m – 5b

Table 3.1

The choice of the letter should, as far as possible, remind one of what it represents, as in this case, m for number of mangoes and b for number of bananas.

Note:

1. 2m means 2 × m = m + m;
   3b means 3 × b = b + b + b
2. a × b = b × a = ab = ba
   (Usually we write the letters of a product in alphabetical order and without multiplication symbols).
3. a × a × a = a^3 (power form).

Example 3.1

Mr. Kafele bought 5 bananas and 3 oranges. By choice of appropriate letters for the price per frit. Find an expression for his expenditure.

Solution

Let K x be the price of a banana.

Let K y be the price of an orange.

Cost of 5 banana = K 5x

Cost of 3 oranges = K 3y

Total expenditure = K (5x + 3y).

In this case, letter o for oranges could be confused with the number 0 (zero), hence the choice of the letters x and y.

We have used letters to represent quantities of different values. When letters are used to represent nay values, they are referred to as **variables**. In examples 3.1, x and y represent the cost of a banana and an orange respectively so x and y are variables.

In algebra, we use letters and symbols to represent numbers in various ways;

1. In formulae e.g. πr^2 = Area of a circle
2. As constants e.g. π in πr^2 is of constant value.
3. As unknown numbers as in Exercise 3.1
4. As numbers in general terms e.g. a, b, c, d … representing numbers.

We have applied some of these on a smaller scale but we shall apply more of this later.

Exercise 3.1

1. Write down the following products in short form.
   a) $2 \times b$
   b) $P \times 5$
   c) $m \times n$
   d) $x \times x$
2. find the total mass of 2 boxes weighing
   a) 4 kg and 20 kg
   b) t kg and 20 kg
   c) x kg and y kg
3. Out of K 50 pocket money, a student spent K x. express his balance algebraically.
4. Represent the following statements algebraically.
   a) The number of days in x weeks.

   b) The total number of exercise books issued to a class of 40 students, if each gets x books.
   c) The total cost of x kg of sugar at K 104 per kg and 3 packets of maize meal at K y per packet.
5. Write down the number which is 12 times as big as m.
6. Write down the number that is 8 more than m.
7. Write down the number that is 12 more than 3 times x.
8. Think of a number. Write down an expression for 5 more than 3 times the number.
9. A certain number is multiplied by 2. By how much does 5 exceed the result?
10. A boy takes 10 minutes longer going to the shop than when returning home. Write down an expression for the total time taken.
11. The length of a classroom is x m and the width is 2 m shorter. Find the total distance round the classroom.
12. A woman had y sweets. She wanted to share the sweets equally amongst her four children. If she had to remain with 6 sweets, write down an expression of the number of sweets each child got.

Terms, coefficients, variables and constants.

In any algebraic expression, parts connected by + or – signs are called terms. Consider the algebraic expression.

$7x - 3y + 2$

It has 3 terms

7 is the **coefficient** of x and -3 is the **coefficient** of y. x and y can represent any values and are referred to as **variables**. The last term, 2 is fixed (Constant) while the terms 7x and -3y will vary depending on the values of x and y. The **constant** term in the expression is thus 2.

If the coefficient is 1 or -1, just write the letter e.g. in 4 + x – y the constant term is 4 while the coefficients of x and y are 1 and 1 respectively.

Exercise 3.2

What is the coefficient of x in each of the expressions in Questions 1 and 2?

1. (a) x

(b) 3x

(c) 18x

(d) -2x

(e) –x

2. (a) $\frac{1}{4}$ x

   (b) $\frac{-2}{3}$ x

   (c) $\frac{2}{5}$ x

   (d) $\frac{6}{5}$ x

   (e) $\frac{-10}{13}$ x

3. State the number of terms in the following expressions.

   (a) 2

   (b) m

   (c) 4m + 2n – 3

   (d) $\frac{2}{3}$ x z

   (e) 5 × y + x × 3 – z ÷ 2

4. Write down the constant term, if any =, in the following expressions.

   (a) a + 3b – 1

   (b) 2x - $\frac{2}{3}$ + 7w

   (c) 3m

5. How many variables are there in each of the following expressions?

   (a) 2x – y

   (b) 26w + 4 – y

   (c) 6m + 3n – 1 + 2k

Simplification of algebraic expressions

Like and unlike terms

Terms are said to be **like** if they have exactly the same variable(s) to the same power, otherwise they are **unlike.**

For example:

2a and 3a are like,

2ab and 4ab are like,

2a and 5b are unlike

a and a^2 are unlike

When simplifying algebraic expressions, first collect the like terms together. Simplification is usually easier if the positive like terms are separated from the negative ones.

Examples 3.2

Simplify:

(a) 2a – 8a + 5a + 9a – 3a
(b) 4m + 8n – m – 3n + 2n – 5 – n + 9

Solution

(a) 2a – 8a + 5a + 9a – 3a
= 2a + 5a + 9a – 8a – 3a (Grouping positive and negative like terms).
= 16a – 11a
= 5a

(b) 4m + 8n – m – 3n + 2n – 5 – n + 9
= 4m – m + 8n – 3n + 2n – n -5 + 9 (Collecting like terms together)
= 4m – m + 8n + 2n – 3n – n + 9 – 5 (Grouping positive and negative like terms)
= 3m + 10n – 4n + 4
= 3m + 6n + 4

Exercise 3.3

1. In each of the following, pick out the term which is unlike the others.
   (a) 2x, 4x, 6x, $3x^2$, 8x
   (b) M^3, $5m^2$, $6m^3$, $3m^3$, $10m^3$
   (c) X^2y, xy, yx^2, $3x^2y$, $4yx^2$
   (d) 3mn, nm, -mn, m^2n, $\frac{2}{3}$ mn

2. Simplify the following expressions.
   (a) y + y + y

(b) $n + n - n + n + n - n - n$
(c) $f - f + f - f + f$
(d) $d + d + d - d - d + d$

3. simplify:
(a) $3a + 3a$
(b) $4b - b$
(c) $6z - 2z$
(d) $k - k$
(e) $q + 2q$
(f) $5p + 7p$
(g) $9r - 8r$
(h) $w - 5w$

4. simplify:
(a) $5a + a + 3a$
(b) $C + 2c + 4c$
(c) $3b + 3b + 3b$
(d) $5y - 4y + y$
(e) $12w - 6w + 6w$
(f) $9n - 3n + 2n$
(g) $2m + 8m - 4m + m - 2m$
(h) $8t - 2t - 3t + 4t - 7t$

5. Simplify the following by first collecting like terms together and grouping those with the same signs.
(a) $x + y + y + x$
(b) $3w + 8w + 9z - 4z$
(c) $11n + 11 + n - 10$
(d) $4s - 2t + 5t - 3s$
(e) $2p - 7 - 4 + 5p$
(f) $14b - 9c - 6b$
(g) $4m - m + 5n - 4n$
(h) $10 - 5d + 2d - 15 + 4d$

6. Simplify where possible
(a) $2e + 2f - 2$
(b) $d - 7e + 3f + 8e + 2d$
(c) $48n - 24$
(d) $6m + 3n - 2 - 6m + 5$
(e) $7h - 2g + 4 - 3h + 7g + 2$
(f) $12 - 4e + 6e - 10h + 7k - 3e - 14$

Multiplication and division with algebraic terms

Note the following points regarding multiplication.

(i) We have seen that $3 \times a = 3a$

So, $2 \times 3a = 2 \times 3 \times a$

$= 6a$

Also, $3a \times 2 = 3 \times a \times 2 = 3 \times 2 \times a$

$= 6a$

Thus, $2 \times 3a = 3a \times 2$

(ii) Just as $5 \times 5 = 5^2$,
So also, $a \times a = a^2$
Similarly, $2a \times 2a = (2a)^2$
And $2a \times 2a = 2 \times a \times 2 \times a$
$= 2^2 \times a^2$

$= 4a^2$

Therefore, $(2a)^2 = 4a^2$

Example 3.3

Simplify:

(a) $5 \times 2x$
(b) $3a \times 6b$
(c) $8p \times 6pq$

Solution

(a) $5 \times 2x = 5 \times 2 \times x$
$= 10 \times x = 10x$
(b) $3a \times 6b = 3 \times a \times 6 \times b$
$= 18 \times ab$
$= 18ab$
(d) $8p \times 6pq = 8 \times p \times 6 \times p \times q$
$= 48p^2 \times q$
$= 48\ p^2q$

Just as we can divide a number by itself, we can also divide a letter by itself, i.e. just as $5 \div 5 = 1$, so also $x \div x = 1$. This fact is useful in division with algebraic terms as shown in Example 3.4

Example 3.4

Simplify:

(a) 10x ÷ 5
(b) 6xy ÷ 2y
(c) $\frac{3}{4}$ of 8x

Solution

(a) 10x ÷ 5 = $\frac{10x}{5}$
= 2 × x (Dividing 10 by 5)
= 2x

(b) 6xy ÷ 2y = $\frac{6xy}{2y} = \frac{6 \times x \times y}{2 \times y}$
= 3x (Dividing 6 by 2 and y by y)

(c) $\frac{3}{4}$ of 8x = $\frac{3}{4}$ × 8 × x ‘of means multiply’
= 3 × 2 × x (Dividing 8 by 4)
= 6x

Exercise 3.4

Simplify the following

1. (a) 3 × 4a (b) 4m × 5
   (c) 6x × 9 (d) 11 × 3q
2. 3x × 2y
3. 2a × 7b
4. 7y × 4yx
5. 13st × 11t
6. 8q × 5p
7. Y × 8x
8. a × 3ab
9. 2a × 7ab
10. 5 × $2m^2$
11. 15pq × p^2

12. $(3q)^2 \times pq$
13. $3p^2 \times 2q^2$
14. $24m \div 8$
15. $18x \div 6$
16. $27p^2 \div 9$
17. $143t \div 13$
18. $16xy \div x$
19. $P^2 \div p$
20. $27w^2m \div 9w$
21. $54p^2q \div 9pq$
22. $\frac{2}{5}$ of 15y
23. $\frac{1}{3}$ of 36m
24. $\frac{3}{4}$ of 36 pq
25. $\frac{2}{11}$ of 132qt

Subtraction involves replacing variables, in an algebraic expression, with specific values. The expression may then be evaluated.

Example 3.5

If x = 3, y = -2 and z = 5, find the value of

(a) $xy + z^2$ (b) $(x + y)(3x - 4z)$

Solution

(a) $xy + z^2 = 3 \times -2 + 5^2$
$= -6 + 25$
$= 19$

(b) $(x + y)(3x - 4z) = (3 + -2)(3 \times 3 - 4 \times 5)$
$= (1)(9 - 20)$
$= 1 \times -11$
$= -11$

Exercise 3.5

1. If m = 4 and n = 3, evaluate:
    - (a) $3m + 3$
    - (b) $4m – 5n$
    - (c) $\frac{1}{2}m + n$
    - (d) $\frac{1}{4}m - \frac{1}{3}n$
    - (e) $5m - 5$
    - (f) $6m + 2n$
    - (g) $3m – 4n$
    - (h) $\frac{2}{3}m – n$
    - (i) $3n^2$
    - (j) $2mn^2$
    - (k) $mn - n$
    - (l) $m(n – m)$
    - (m) $2m^3$
    - (n) $\frac{1}{3}mn$
    - (o) $m^2 - n^2$
2. If a = 5, b = 9 and c = 1, evaluate:
    - (a) $a \div (b + c) + 6$
    - (b) $(b – 2c) \div (4a – 2b)$
    - (c) $\frac{a^2 – c^2}{b + 3}$
    - (d) $(\frac{3a+c}{3a+c})^2$
3. If $E = \frac{1}{2}mv^2$, find E when m = 27 and $v = \frac{1}{3}$.
4. If xy = 5 and y = 2, find
    - (a) x
    - (b) $2(x + y)$
5. If $F = 32 + \frac{9}{5}C$, find the value of F given that:
    - (a) C = 20
    - (b) C = 42
    - (c) $C = 75\frac{1}{4}$

Expanding algebraic expressions

The terms inside brackets are intended to be taken as one term, e.g. 9 + (7 + 3) means that 7 and 3 are to be added together, and their sum added to 9, so that,

$9 + (7 + 3) = 9 + 10 = 19$

Also, $9 + 7 + 3 = 19$

Hence, the bracket may be removed without changing the result,

i.e. $9 + (7 + 3) = 9 + 7 + 3$

In general,

$\boxed{x + (y + z) = x + y + z}$

9 + (7 – 3) = 9 + 4 (first subtracting 3 from 7)

= 13

Also, 9 + 7 – 3 = 16 – 3 = 13

Hence, the bracket may again be removed without changing the result.

$\boxed{x + (y - z) = x + y - z}$

9 – (7 + 3) = 9 – 10 (First adding 3 to 7) = -1

Also, 9 – 7 – 3 = 2 – 3 = -1

Hence, in this case, when the bracket is removed, the sign of each term inside the bracket is changed.

i.e. 9 – (7 + 3) = 9 – 7 – 3

In general

$\boxed{x - (y + z) = x - y - z}$

Caution!

9 – 7 + 3 = 2 + 3 = 5

Note that this is **not** the same as 9 – (7 + 3).

This is common mistake, which must be avoided. In general.

$\boxed{x - (y + z) \neq x - y + z}$

9 – (7 – 3) = 9 -4 (First subtracting 3 from 7)

= 5

Also, 9 – 7 + 3 = 2 + 3 = 5

Hence, again, when the bracket is removed, the sign of each term inside the bracket is changed,

i.e. 9 – (7 – 3) = 9 – 7 + 3.

In general,

x – (y – z) = x – y + z

Caution!

9 – (7 – 3) ≠ 9 – 7 – 3!

Therefore, in general,

x – (y – z) = x – y – z

The rules, therefore, are:

1. If there is a + (plus) sign just before a bracket, the sign of each term inside is unchanged when the bracket is removed (i.e. when the expression is expanded).
2. If there is a – (minus) sign just before a bracket, the sign of each term inside must be changed when the bracket is removed. Removing the bracket is like multiplying each term by -1

Example 3.6

Remove the bracket and simplify:

(a) 7g + (3g – 4h) – (2g – 9h)
(b) (6x – y + 3z) – (2x + 5y – 4z)

Solution

(a) 7g + (3g – 4h) – (2g – 9h)
= 7g + 3g -4h – 2g + 9h
= 7g + 3g – 2g + 9h – 4h
= 8g + 5h
(b) (6x – y + 3z) – (2x + 5y – 4z)
= 6x – y + 3z – 2x – 5y + 4z
= 6x – 2x – y – 5y + 3z + 4z
= 4x – 6y + 7z

Note: in (b) there is no sign before the first bracket so a + sign is assumed.

In the expression $9 \times (7 + 3)$, the bracket means that 7 and 3 are to be added together, and the result multiplied by 9.

Thus, $9 \times (7 + 3) = 9 \times 10 = 90$.

But $9 \times 7 + 9 \times 3 = 63 + 27 = 90$.

Therefore, $9 \times (7 + 3) = 9 \times 7 + 9 \times 3$, which means that 7 and 3 may each be multiplied by 9 and the products added together.

The multiplication sign is usually omitted, so that $9(7 + 3)$ means exactly the same as $9 \times (7 + 3)$, just as $a \times b = ab$.

In general, when expression in a bracket is multiplied by a number in order to remove the brackets, every term inside the bracket must be multiplied by that number.

Thus,

$a(x + y) = a \times x + a \times y = ax + ay$ and

and $a(x - y) = a \times x - a \times y = ax - ay$

Example 3.7

Remove brackets and simplify:

$2(3x - y) + 4(x + 2y) - 3(2x - 3y)$

Solution

$2(3x - y) + 4(x + 2y) - 3(2x - 3y)$

$= 2 \times 3x - 2 \times y + 4 \times x + 4 \times 2y - 3 \times 2x + 3 \times 3y$

$= 6x - 2y + 4x + 8y - 6x + 9y$

$= 6x + 4x - 6x + 8y + 9y - 2y$

$= 4x + 15y$

Notice how the signs obey the rules obtained earlier.

When simplifying expressions containing brackets enclosed in other brackets, **remove the innermost bracket first** and collect like terms (if any) before removing the next outer bracket.

Example 3.8

Simplify {3y – (x – 2y)} – {5x – (y + 3x)}

Solution

{3y – (x – 2y)} – {5x – (y + 3x)}

= {3y – x + 2y)} – {5x – y _ 3x}

= {5y – x} – {2x – y}

= 5y – x – 2x + y

= 6y – 3x

Note: different shapes of brackets are usually used to make the meaning of the hyphen expression more easily understood.

Exercise 3.6

1. Remove the brackets and simplify:
   (a) 592x + 3)
   (b) 4(3m – 2n)
   (c) 7(2b – 3c + 1)
   (d) 3w(4x – 1)
   (e) 6(3x – 5y – 1)
   (f) 2(4r – 3) + 3(s – 1)
   (g) 3a(2b + c) – 2a(2x + y)
   (h) 5(c + 4) -2(3c – 8)
   (i) –(x + y) + x
   (j) (7a + 5b) – (3a – 10b)
   (k) (x – 3y)9y + 2(y^2 – 3xy)
   (l) xy(x – xy) –x(xy – x^2)
2. Write algebraic expressions for the following. Do not remove brackets.
   (a) Add a to 2y and multiply the result by 4.
   (b) Divide 12e + 30d – 18 by 6.

   (c) The product of 3 consecutive even numbers, the largest of which is p.
   (d) The number by which a + b exceeds a – b
3. Copy and complete the following:
   (a) $m + n - 1 = m + (\_)$
   (b) $a - b - c = a - (\_)$
   (c) $x - y + z = x - (\_)$
   (d) $p - q - r + s = p - (\_)$
   (e) $u - v + w + x = u - (\_)$
   (f) $x - y + v - w = u - (\_)$
   (g) $a + 2b - 2c = a + 2(\_)$
   (h) $a - 3b - 6c = a - 3(\_)$
   (i) $2a - 8c - 3x - 9z = 2(\_) -3(\_)$
   (j) $K^2 + 2kl - 3m^2 + 4mn = k(\_) - m(\_)$
4. Saliza is 10 years older than her brother, Pambuka. Find an expression, in terms of Saliza's age, for
   (a) The sum of their ages
   (b) The sum of their ages in 5 years' time.
   (c) The product of their ages 3 years ago.
5. A tin, which weighs w kg when empty, holds 5 kg of jam when full. What is the mass of 12 such full tins? The 12 full tins are packed in a box weighing m kg. What is the total mass?
6. At a hotel, a man is charged K 7 500 a day for the first 4 days and K 5 000 a day afterwards. Find the total charge if he stays
   (a) A week
   (b) N days, where $n < 4$
   (c) N days, where $n>4$.

Algebraic fractions

An algebraic fraction is one in which either the numerator or denominator (or both) is an algebraic expression. For example,

$\frac{x}{4},\frac{3y}{4},\frac{m-5}{3},\frac{3(p-1)}{q},\frac{5}{x+y}$, etc.

Are algebraic fractions.

Since the letters in these fractions stand for numbers, we deal with algebraic fractions in the same way as with fractions in arithmetic. Thus:

1. Dividing or multiplying the numerator and denominator of an algebraic fraction by the same number does not change the value of the fraction.
2. When combining fractions by addition or subtraction, the fractions are first expressed over a common denominator.

In each of the following cases, each fraction is reduced to a simpler form by dividing the numerator and denominator by a common divisor.

- i. $\frac{8}{20} = \frac{2}{5}$ (dividing numerator and denominator by 4)
- ii. $\frac{3x}{20x} = \frac{3}{20}$ (dividing numerator and denominator x)
- iii. $\frac{9xy}{12xz} = \frac{3y}{4z}$ (dividing numerator and denominator by 3x)

In the following cases, equivalent fractions are obtained by multiplying the numerator and denominator by the same number.

- i. $\frac{2}{5} = \frac{8}{20}$ (multiplying numerator and denominator by 4)
- ii. $\frac{2a}{5d} = \frac{6a}{15d}$ (multiplying numerator and denominator by 3)
- iii. $\frac{2a}{5d} = \frac{4ab}{10bd}$ (multiplying numerator and denominator by 2b)

Addition and subtraction of fractions

To add or to subtract fractions, the denominator must be the same. Therefore, we make use of equivalent fractions as follows:

Consider $\frac{a}{4} + \frac{a}{5}$. The denominator of these fractions are 4 and 5.

Find the equivalent fractions of $\frac{a}{4}$ and $\frac{a}{5}$ so that their denominators are the same.

$\frac{a}{4} = \frac{2a}{8} = \frac{3a}{12} = \frac{4a}{16} = \frac{5a}{20} = \frac{6a}{24}$ …. And

$\frac{a}{5} = \frac{2a}{10} = \frac{3a}{15} = \frac{4a}{20}$

$\frac{5a}{20}$ and $\frac{4a}{20}$ are the required equivalent fractions.

$\frac{a \quad 5}{4 \quad 5} + \frac{a \times 4}{5 \times 4} = \frac{5a}{20} + \frac{4a}{20}$

Since the denominators are the same, we can now add the numerators.

Thus, $\frac{5a}{20} + \frac{4a}{20} = \frac{5a \quad 4a}{20}$

Similarly, $\frac{4x}{3y} - \frac{2x}{5y}$ can be simplified by expressing them using a common denominator

So $\frac{4x}{3y} - \frac{2x}{5y} = \frac{4x \times 5}{3y \times 5} - \frac{2x \times 3}{5y \times 3}$

$$= \frac{20x}{15y} - \frac{6x}{15y}$$

$$= \frac{14x}{15y}$$

Note: we can find many common denominators in any case but we use the smallest possible i.e. 15y.

Remember, that an expression such as

$5 + \frac{x-y}{2}$ can be written as $\frac{5}{1} + \frac{x-y}{2}$ so that the denominators are 1 and 2.

Example 3.9

Simplify:

(a) $\frac{5}{6} + \frac{3}{8}$

(b) $\frac{7}{10} + \frac{3}{15}$

(c) $\frac{5a}{6} + \frac{2b}{9} - \frac{7b}{12}$

(d) $\frac{4x}{2a} + \frac{1}{5a} - \frac{3}{a}$

(e) $\frac{d}{ab} + \frac{e}{bc}$

Solution

(a) $\frac{5}{6} + \frac{3}{8} = \frac{20}{24} + \frac{9}{24} = \frac{20+9}{24} = \frac{29}{24}$

(b) $\frac{7}{10} + \frac{3}{15} = \frac{21}{30} - \frac{6}{30} = \frac{21-6}{30} = \frac{15}{30} = \frac{1}{2}$

(c) $\frac{5a}{6} + \frac{2b}{9} - \frac{7b}{12} = \frac{30a}{36} - \frac{8b}{36} = \frac{21b}{36}$

$$= \frac{30a+8b-21b}{36} = \frac{30a- 13b}{36}$$

$$(d)\ \frac{4x}{2a} + \frac{1}{5a} - \frac{3}{a} = \frac{20a}{10a} - \frac{2}{10a} - \frac{30a}{10a}$$

(LCM of 2a, 5a and a is 10a)

$$\frac{20a+2-30}{10a}$$

$$= \frac{8}{10a} = \frac{4}{5a}$$

$$(e)\ \frac{d}{ab} + \frac{e}{bc} = \frac{cd}{abc} + \frac{ae}{abc}$$

(LCM of ab, bc is abc)

$$\frac{cd+ae}{abc}$$

Example 3.10

Simplify $\frac{3a}{3} - \frac{a-2}{2}$

Solution

$$\frac{5a}{3} - \frac{a-2}{2} = \frac{5a}{3} - \frac{a-2}{2}$$

$$= \frac{5a \times 2}{3 \times 2} - \frac{a-2 \times 3}{2 \times 3}$$

$$= \frac{10a}{6} - \frac{3(a-2)}{6}$$

$$= \frac{10a-3a+6}{6}$$

$$= \frac{7a + 6}{6}$$

Example 3.11

Simplify the following fractions to the lowest terms.

(a) $\frac{bx}{b^2}$

(b) $\underline{6a^2 d}$

$15ad^2$

Solution

(a) $\frac{bx}{b^2}$ (Divide the numerator and the denominator by b

We get, $\frac{bx}{b^2} = \frac{b \times x}{b \times b}$

$= \frac{x}{b}$

(b) $\frac{6a^2\,d}{15ad^2}$ (Divide the numerator and the denominator by 3ad)

$\frac{6a^2\,d}{15ad^2} = \frac{6\times a\times a\times d}{15 \times a \times d \times d}$

$= \frac{2a}{5d}$

Example 3.12

Simplify the following algebraic fractions.

(a) $\frac{2y}{x} + \frac{3}{2x} + \frac{4y}{3x}$

(b) $\frac{3x-5y}{6z} - \frac{2x-7y}{2z}$

Solution

(a) $\frac{2y}{x} + \frac{3}{2x} + \frac{4y}{3x}$

LCM of x, 2x and 3x is 6x

$= \frac{12y+9+8y}{6x}$ (Add like terms)

$= \frac{20y+9}{6x}$

(b) $\frac{3x-5y}{6z} - \frac{2x-7y}{2z}$

LCM of 6z and 2z is 6z

$= \frac{(3x-5y)-3(2x-7y)}{6z}$ (Remove brackets)

$=\frac{3x-5y-6x+21y}{6z}$ (Subtract like terms)

$=\frac{-3x+16y}{6z}=\frac{16y-3x}{6z}$

Exercise 3.7

1. Reduce the following fractions to their lowest terms.

   (a) $\frac{3a}{8a}$

   (b) $\frac{7x}{35x}$

   (c) $\frac{14m}{21m}$

   (d) $\frac{16b}{24b}$

   (e) $\frac{2mx}{12my}$

   (f) $\frac{16ab}{24bc}$

   (g) $\frac{6dk}{18dh}$

   (h) $\frac{25m}{30mp}$

   (i) $\frac{27abx}{36adx}$

2. Simplify the following fractions:

   (a) $\frac{99y^2}{132y}$

   (b) $\frac{42p^2}{63p}$

   (c) $\frac{108xy^2}{132x^2y}$

3. Copy and fill in the blanks in each of the following.

   a) $\frac{7}{4}=\frac{7}{\quad}$

   b) $\frac{3x}{5}=\frac{\quad}{25}$

   c) $\frac{3x}{5}=\frac{\quad}{25}$

   d) $\frac{5a}{8}=\frac{15a}{\quad}$

   $\frac{3}{4a}=\frac{\quad}{12a}$

e) $\frac{3}{9b} = \frac{15}{\quad}$

f) $\frac{4}{7} = \frac{4x}{\quad}$

g) $\frac{a}{b} = \frac{\quad}{by}$

h) $\frac{bn}{\quad} = \frac{b}{yz}$

4. Simplify:

a) $\frac{5a}{13} + \frac{4a}{13}$

b) $\frac{2e}{7} + \frac{5e}{7}$

c) $\frac{7b}{10} - \frac{4b}{10}$

d) $\frac{3x}{5} - \frac{2x}{7}$

e) $\frac{5y}{8} + \frac{3y}{8}$

f) $\frac{2p}{3} - \frac{3p}{5}$

g) $\frac{a}{m} + \frac{3c}{m}$

h) $\frac{8}{mn} + \frac{5}{mn}$

i) $\frac{4}{x} - \frac{3}{y}$

j) $\frac{m}{2p} - \frac{n}{3p}$

k) $\frac{5}{ab} + \frac{2}{bc}$

l) $\frac{2}{pq} - \frac{5}{mq}$

m) $\frac{5}{fg} + \frac{3}{fh} + \frac{2}{kh}$

n) $\frac{x}{ab} - \frac{y}{ac} + \frac{z}{bc}$

5. Simplify:

a) $\frac{p+3}{3} + \frac{2x}{3}$

b) $\frac{1}{x} + \frac{1}{x+1}$

c) $\frac{y+x}{x} + \frac{y}{2x}$

d) $\frac{x+3}{6} + \frac{x+5}{4}$

e) $\frac{a+b}{c} - \frac{a-c}{b}$

g) $\frac{x-1}{x} - \frac{2x+1}{3x}$

f) $\frac{2x-4y}{5z} - \frac{2x+6y}{z}$

h) $\frac{mn}{4} - \frac{1}{3}$ my

6. (a) $\frac{2ab}{3} + \frac{3bc}{4}$

(c) $4 + \frac{a-b}{c}$

(d) $5 + \frac{2a}{a-b}$

(e) $\frac{3}{2} + \frac{x}{x+2y}$

7. (a) $\frac{3m-n}{5mn} - \frac{7}{6n}$

(c) $\frac{2b+3}{6b} - \frac{3-2a}{15a}$

(d) $2 - \frac{a}{x+zy}$

(e) $\frac{x-2y}{6} - \frac{2y-x}{4}$

In questions 8 and 9, express each part with a single denominator in one simplest form.

8. (a) $\frac{3}{2x} + \frac{2}{3x}$

(c) $\frac{a}{3} + \frac{2a-1}{4}$

(d) $5c - \frac{m-2}{3}$

9. (a) $\frac{2b-3}{4} - \frac{1}{3c}$

(c) $\frac{4}{p} - \frac{5p}{2} + \frac{3}{4p}$

(d) $\frac{6x-6m}{6} - \frac{x-3}{2}$

(e) $\frac{2x+2y}{3} - \frac{-2b}{4}$

(f) $\frac{m}{d} - \frac{3m}{2d} + \frac{5mn}{5d}$

Multiplication of Algebraic fractions

Fractions, in arithmetic, may be simplified using the cancellation process. For example,

$\frac{9}{16} \times \frac{20}{21}$ may be simplified as follows:

$\frac{9}{16} \times \frac{20}{21}$

$\frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$ (9 and 21 are both divided by 3 while 16 and 20 are both divided by 4)

We do likewise in algebra. For example,

$\frac{4ab}{9} \times \frac{6m}{acm}$ may be simplified as follows:

$\frac{4ab}{9} \times \frac{6m}{acm} = \frac{8b}{3c}$ (The a's in the numerator are both divided by a, each giving 1; 6 and 9 are both divided by 3; the two m's are both divided by m).

Example 3.13

Multiply and simplify the algebraic expression

$36a^2bc \times \frac{3}{4}ab^2$

Solution

$36a^2bc$ means $36 \times a^2 \times b \times c$ and

$\frac{3}{4}$ ab^2 means $\frac{3}{4} \times a \times b^2$

$= 36 \times a^2 \times b \times c \times \frac{3}{4} \times a \times b^2$

$= 9\ 36 \times a^2 \times b \times c \times \frac{3}{4} \times a \times b^2$

$= 9 \times 3 \times a^2 \times a \times b \times b^2 \times c$

$= 27 \times a^3 \times b^3 \times c$

$= 27a^3 \times b^3 \times c$

$= 27a^3b^3c$

Dividing Algebraic fractions

Division of fractions in algebra follows the same process as in arithmetic. For example,

$\frac{9}{4} \div \frac{3}{7} = \frac{9}{4} \times \frac{7}{3} = \frac{21}{4}$

Likewise,

$\frac{16ab}{21cd} \div \frac{24a}{35d} = \frac{16ab}{21cd} \times \frac{35d}{24a} = \frac{10b}{8c}$

In order to perform division of fractions, we must be able to find reciprocals of numbers. If the product of two rational numbers is 1, one number is the reciprocal of the other and vice versa.

For example, if $a \times \frac{1}{a} = 1$, then $\frac{1}{a}$ is the **reciprocal** of **a.**

If $\frac{a}{b} \times \frac{b}{a} = 1$

Then $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$

Thus, $\frac{1}{\frac{a}{b}}$ is the reciprocal of? $\frac{a}{b}$

Therefore, $\frac{1}{\frac{a}{b}} = \frac{1 \times b}{4} = \frac{b}{a}$

Now consider 6x ÷ 2, we can write this as,

$6x \div 2 = \frac{6x}{2}$

$= 6x \times \frac{1}{2}$

$= 3x$

To divide by a rational number or a fraction is the same as multiplying by the reciprocal of the divisor.

Example 3.14

Simplify the expression

$\frac{6x^2}{20by} \div \frac{9xy}{10by^2}$

Solution

The divisor is $\frac{9xy}{10by^2}$

Therefore, the reciprocal of $\frac{9xy}{10by^2}$ is $\frac{10by^2}{9xy}$

Therefore; $\frac{6x^2}{20by} \div \frac{9xy}{10by^2} = \frac{6x^2}{20by} \times \frac{\frac{1}{9xy}}{10by^2}$

$= \frac{6x^2}{20by} \times \frac{10by^2}{9xy} = \left(\frac{\frac{1}{9xy}}{10by^2} = \frac{10by^2}{9xy} \right)$

$= \frac{6x^2}{2y} \times \frac{y^2}{9xy}$

$= \frac{3x^2}{9x}$

$= \frac{x}{3}$

Example 3.15

Simplify the expression

$$\frac{27x^2z}{9x^2} \div \frac{3x^3}{8x^3z^2}$$

Solution

$$\frac{27x^2z}{9x^2} \div \frac{3x^3}{8x^3z^2}$$

$$\frac{27x^2z}{9x^2} \times \frac{8x^3z^2}{3x^3}$$ (Multiplying by the multiplication inverse of the divisor)

$$\frac{16x^2z}{27}$$

Exercise 3.8

1. Simplify

(a) $\frac{4a}{9} \times \frac{3}{2}$

(b) $\frac{12}{7x} \times \frac{3}{2}$

(c) $\frac{5x}{6} \times \frac{12}{x}$

(d) $\frac{6n}{4b} \times \frac{8}{n}$

(e) $\frac{8p}{21q} \times \frac{28q}{16r}$

(f) $\frac{3}{20}\ of\ \frac{16}{6}$

2. Simplify:

(a) $\frac{4p}{5} \div \frac{12p}{25}$

(b) $\frac{35}{8x} \div \frac{5}{12x}$

(c) $\frac{3pq}{14} \div \frac{12q}{7}$

(d) $\frac{8m}{5pq} \div \frac{32n}{15pq}$

(e) $\frac{12ab}{11mb} \div \frac{4a}{33}$

(f) $\frac{14rs}{33} \div \frac{35rst}{22}$

(g)

3. Simplify:

(a) $3\frac{1}{2}p \text{ of } \frac{15q}{8p^2}$

(b) $3\frac{3}{5}xy^2z \times \frac{5}{6yt}$

(c) $\frac{9pqr}{10p} \times \frac{16m}{21q^2}$

(d) $\frac{m}{5}$ of $\frac{15n}{16m^2}$

(e) $\frac{16fg^2h}{27} \times \frac{8g}{45}$

(f) $\frac{8abc}{15p^2q} \times \frac{24c}{25p}$

(g) $\frac{2x-2}{x^2} \div \frac{x-1}{x}$

(h) $\frac{2x^2}{3} \div \frac{4}{9xy}$

4. Simplify:

(a) $\frac{20xy}{16yz} \times \frac{24de}{20zd}$

(b) $\frac{a+b}{a} \times \frac{21}{3(a+b)}$

(c) $\frac{8a^2c^2}{15bc^2} \times \frac{8b^2d}{20da^3}$

(d) $\frac{ab}{3(a-2b)} \times \frac{4(a-2b)}{a^2b}$

(e) $\frac{8abc}{8ab} \times \frac{24ab}{5(a-b)}$

5. Simplify:

a) $\frac{6ab}{5bc} \div \frac{24de}{20cd}$

b) $\frac{12ax}{15cx^2} \div \frac{9cx}{10c^2x^2}$

c) $\frac{18a^2b}{16b^3c^2} \div \frac{24a}{15b^3c}$

d) $\frac{1}{16mn} \div \frac{1}{24m^2n^3}$

e) $\frac{10abc}{6b^2c} \div \frac{10acd}{12bc^2d}$

Unit 4 LINES AND ANGELS

Angles

Fig. 4.1 shows a wheel as it turns, starting from position (i) to position (vi).

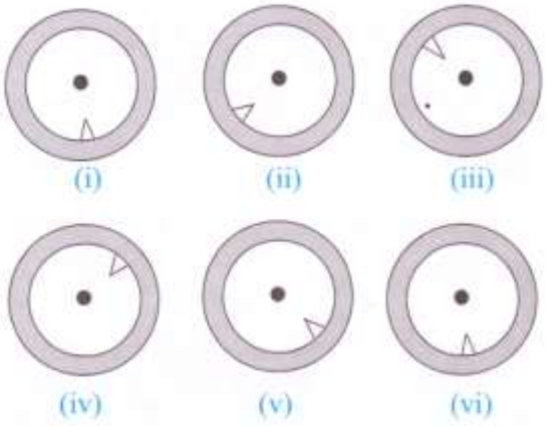


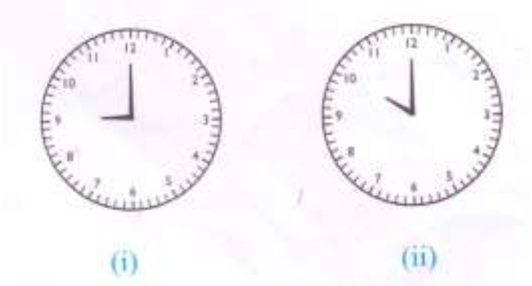
Fig. 4.1

For the amount the wheel turns from one position of the wheel to another we use the word ‘angle’ or ‘angle of rotation’.

When the wheel turns from the position in (i) to the position in (vi), it has made ‘one complete turn’ or ‘a whole turn’ or ‘one revolution’.

The angle turned = 1 revolution.

Look at the clock faces below. From 9 o’clock (Fig. 4.2(i) to 10 o’clock (Fig. 4.2 (ii).



The minute hand of the clock makes one revolution as shown in Fig. 4.2 (iii).

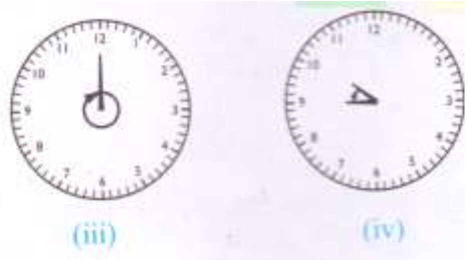

(iii) (iv)

Fig. 4.2

Over the same duration, the hour hand turns through $\frac{1}{12}$ of a revolution as shown in Fig. 4.2 (iv)

Exercise 4.1

Use ‘revolutions’ or ‘whole turns’ and ‘fractions of turns’ to measure angles in this exercise.

1. Approximately, through what angle does the wheel in Fig. 4.1 turn:
   (a) From position (i) to (iii),
   (b) From position (ii) to (iv),
   (c) From position (iii) to (v)?
2. Approximately, through what angle do you rotate;
   (a) Your head, when you turn it from facing the front of the class to face the person sitting next to you on your right (or left)?
   (b) A key to open a door (discussion)?
   (c) The school gate for a car to pass through it (discussion)?
3. Through what angle does a soldier turn when he is given the order?
   (a) ‘left turn’
   (b) ‘Right turn’
   (c) ‘About turn’?
4. Through what angle does the minute hand of a clock turn in:
   (a) 3 hours
   (b) 30 min
   (c) 20 min
   (d) 12 min?
5. Through what angle does the hour hand of a clock turn:

(a) From 9 o’clock to 12 o’clock
(b) From noon to midnight
(c) From noon to 4 p.m.

6. A bicycle travelled 3 m during one complete turn of each wheel. Through what angle does each wheel turn when the bicycle travels:
   (a) 10 m
   (b) 16 m
   (c) 18 m
   (d) 24 m
   (e) 1 m
   (f) $\frac{1}{2}$ m
   (g) 2 m?
7. In one day, i.e. 24 hours, the earth makes one complete turn about its axis. Through what angle does it turn about its axis in;
   (a) 18 hours
   (b) 12 hours
   (c) 3 hours
8. What angle is between the hour hand and the minute hand of a clock at the following times?
   a) 6 o’clock
   b) 2 o’clock
   c) 4 o’clock
   d) 9 o’clock
   e) $\frac{1}{2}$ past 1
   f) $\frac{1}{2}$ past 5
   g) $20\ minutes\ past\ 3$

Degree measure of an angle

Many people, for example surveyors and engineers, need a smaller unit of measuring angles than ‘revolution or turn’. The unit used is called a **‘degree’.** There are 360 degrees in a revolution.

This unit possibly came from the Babylonians who thought that a year consisted of 360 days.

The symbol (˚) is used for degrees. 360 degrees is written in short as 360˚.

Example 4.1

What is the angle between the hour hand and the minute hand of a clock at 10 o'clock?

Solution

The angle between the two hands is the amount the minute hand, for example, would be moved to reach where the hour-hand is (Fig. 4.3). That is, $\frac{1}{6}$ of a revolution.

Angle between the hands = $\frac{1}{6}$ of a revolution

$= \frac{1}{6}$ of 360˚

$= 60^\circ$

Fig. 4.3

Note that Example 4.1, if the minute hand was allowed to move freely as shown by the arrow in Fig. 4.4, the angle would be $\frac{5}{6}$ of 360˚ = 360˚

Fig. 4.4

Exercise 4.2

1. The following are angles measured in revolutions (revs). What are they in degrees?
   - (a) 1 rev
   - (b) $2\frac{1}{2}$ revs
   - (c) 5 revs
   - (d) 20 revs
   - (e) $\frac{1}{2}$ rev
   - (f) $\frac{1}{4}$ rev
   - (g) $\frac{1}{3}$ rev
   - (h) $\frac{2}{3}$ rev
   - (i) $\frac{1}{10}$ rev
   - (j) $\frac{1}{60}$ rev
   - (k) $\frac{3}{4}$ revs

2. What is the angle, in degrees, between the hour hand and the minute hand of a clock at the following times?
   - (a) 9 o’clock
   - (b) 7 o’clock
   - (c) 6 o’clock
   - (d) 1 o’clock
   - (e) 4 o’clock
   - (f) 12 o’clock
3. What is the angle, in degrees, between the hour hand and the minute hand of a clock at the following times?
   - (a) Half past twelve
   - (b) Half past five
   - (c) Half past two
   - (d) Half past seven
   - (e) Half past nine
   - (f) Half past ten
4. Find the angle, in degrees, between the hour hand and the minute hand of a clock at the following times.
   - (a) Quarter past one
   - (b) Quarter past four
   - (c) Quarter of three
   - (d) Quarter of six
   - (e) 20 minutes past eleven
   - (f) 25 minutes to twelve
   - (g) 10 minutes to five
   - (h) 10 minutes past seven
   - (i) 25 minutes past two
   - (j) 20 minutes to ten
5. The scale of a weighing machine is marked in kilograms from 0 to 180. In turning from the mark to the 180 kg mark, the pointer turns through an

angle of 270. Through what angle does the pointer turn in weighing a person whose mass is:

(a) 95 kg
(b) 56 kg
(c) 40 kg
(d) 130 kg

6. Using the information in Questions 5, find the mass of a person if the pointer turns through an angle of:

(a) 90
(b) 57
(c) 120
(d) 210

7. The speedometer needle of a car turns clockwise through 180 when the speed of the car increases from 0 and 90 km/h. through which angle and in which direction will the needle turn when the car changes speed from:

(a) 0 to 15 km/h
(b) 15 to 40 km/h
(c) 60 to 3o km/h
(d) 40 to 30 km/h
(e) 25 to 30 km/h
(f) 48 to 46 km/h

Types of angles

Fig. 4.5 shows two line segments AB and BC. They form an angle at point B. the symbol or is used to denote an angle.

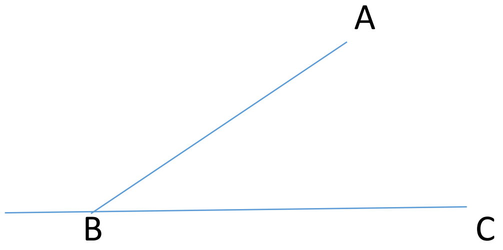


fig. 4.5

Point B is called the 'vertex' of the angle between the lines AB and BC. The angle is denoted as either ABC or B or AB̂C or B̂.

In Fig. 4.5, AB and BC are arms we get angles of different sizes. Some of these angles have special names.

In Fig. 4.5 ABC is less than 90˚ ($\frac{1}{4}$ revolution). Such an angle is known as an **acute angle.**

In Fig. 4.6 ABC = 90˚ it is called a **right angle**.

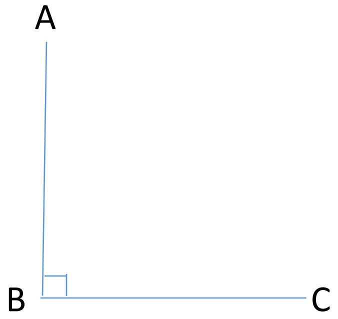


Fig. 4.6

A right angle is usually shown on a diagram by drawing a small square at the vertex of the angle.

In Fig. 4.7, ABC is greater than 90˚ but less than 180˚ (half turn). Such an angle is called an **obtuse angle**.

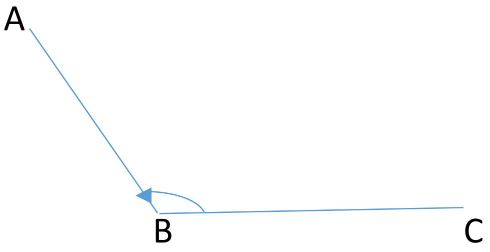


fig. 4.7

In Fig. 4.8, ABC = 180˚ = 2 right angles. This is called a **straight angle.**

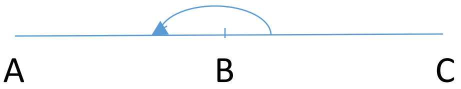
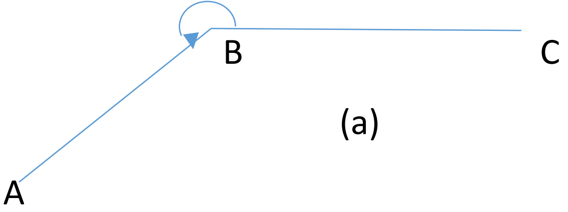


Fig. 4.8

In both Fig. 4.9 (a) and (b), ABC is greater than 180˚ but less than 360˚. Such an angle is called a **reflex angle.**



(a)

(b)

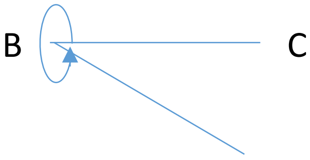


Fig. 4.9

In Fig. 4.10, ABC will refer to the acute angle in (a) or the obtuse angle in (b)

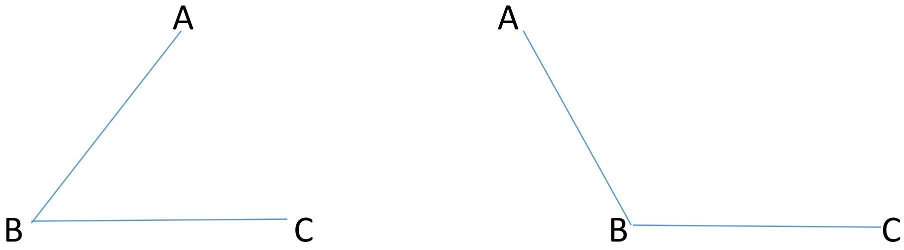


fig. 4.10

To refer to the reflex angle, we must specify ‘**reflex** BC’ or ‘**reflex** B’

Exercise 4.3

1. In Fig. 4.11, name all the
   (a) Acute angles (b) obtuse angles

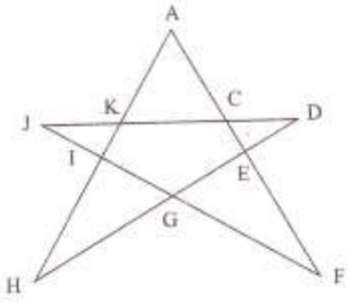


Fig. 4.11

2. In Fig. 4.12, name all the
   (a) Acute angles
   (b) Obtuse angles
   (c) Right angles

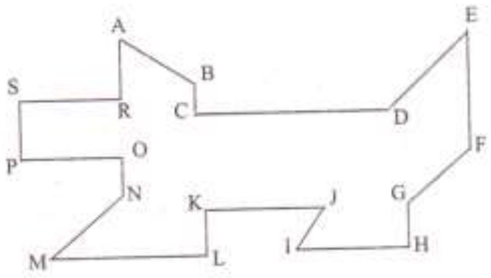


Fig. 4.12

3. In Fig. 4.13, state the type of angle for each of a, b, c, d, e,

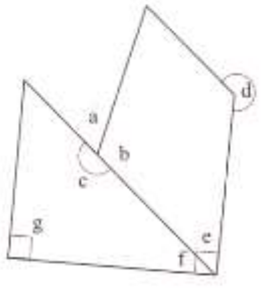


Fig. 4.13

Using a protractor to measure and draw angles

Fig. 4.14 shows a semi-circular **protractor** used to measure angles accurately.

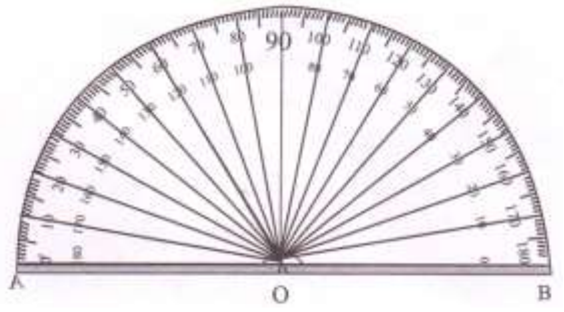


Fig. 4.14

This kind of protractor measures angles from 0˚ to 180˚ either from left to right (clockwise), or from right to the left (anti-clockwise).

The line AOB is **the zero-line.**

The measurements are made from the direction OA on the outer scale, or from OB on the inner scale. The part of the protractor shaded in the fig. 4.14 helps to protect the zero-line. We do not use the shaded part when measuring.

There are some protractors which are made in complete circles. These will measure angles from 0˚ to 360˚.

Example 4.2

(a) Measure the size of the acute PQR in Fig. 4.15
(b) What is the size of the reflex PQR?

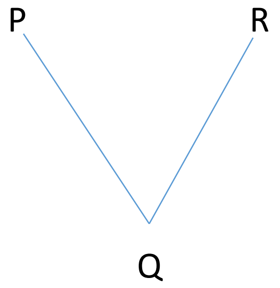


Fig. 4.15

Solution

i. Place the protractor over the angle so that its centre O is exactly over point Q, and the zero-line is exactly along PQ (or QR) as shown in Fig. 4.16.
ii. Count the degrees from the zero-line to the arm QR (or PQ) of the angle.

In Fig. 4.16 (a), count the degrees clockwise using the outer scale of the protractor. In Fig. 4.16 (b) count the degrees anticlockwise using the inner scale of the protractor.

PQR = 60

Reflex PQR = 360˚ - 60˚ = 300˚

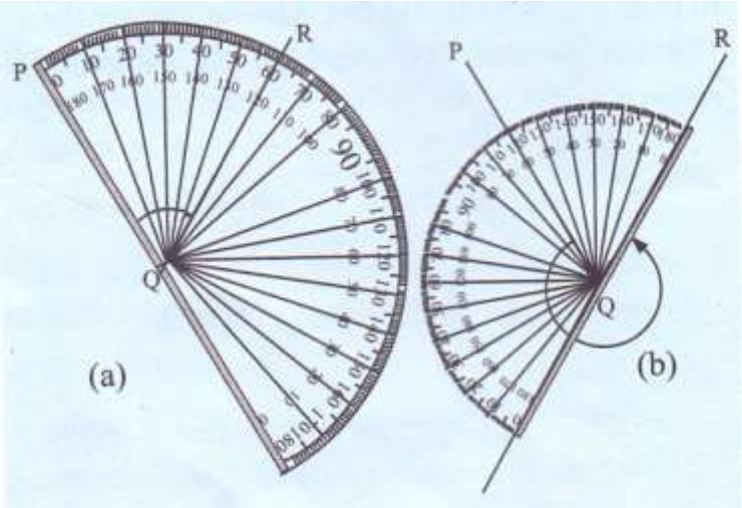


Fig. 4.16

Example 4.3

Draw an angle BC = 84˚

Solution

Draw a line and mark on it points B and C. as shown in Fig. 4.17

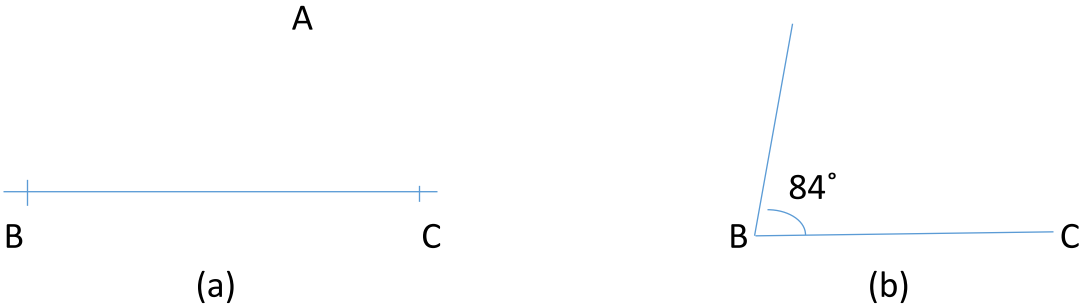
Fig. 4.17

Place the protractor so that the zero-line is exactly over the line BC and the centre of the protractor is over B.

Count 84˚ round the edge of the protractor from the zero-line and put a dot A this point (Fig. 4.18(a)).

Remove the protractor and join AB (Fig. 4.18 (b)).

ABC is the required angle.

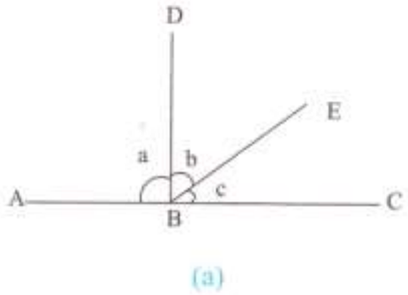


(a) (b)

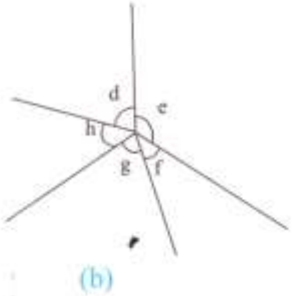
Fig. 4.18

Exercise 4.4

1. Use a protractor to draw an angle of:
   - (a) 35˚
   - (b) 47˚
   - (c) 53˚
   - (d) 77˚
   - (e) 23˚
   - (f) 15˚
   - (g) 97˚
   - (h) 108˚
   - (i) 133˚
   - (j) 145˚
   - (k) 165˚
   - (l) 170˚
2. Use a protractor to draw an angle of:
   - (a) 210˚
   - (b) 260˚
   - (c) 290˚
   - (d) 31˚
   - (e) 330˚
   - (f) 285˚
3. Make a tracing of a set square in your book. Using a protractor, measure all the angles.
4. Measures the angles indicated in Fig. 4.19 (a) and (b). In each case, find the sum of the angles.



(a)



(b)

Fig. 4.19

5. Measure the angles indicated in Fig. 4.20. what do you notice about angles (a) I and k, 9b) j and l?

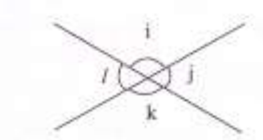


Fig. 4.20

Angles on a straight line

It was pointed out earlier that for a straight line AOB, AOB is a **straight angle** (Fig. 4.21 (a)) and AOB = 180˚.

If follows that AOC, COD, DOE and EOB are angles on the straight line AOB (Fig. 4.21 (b)).

Their sum is 180˚ i.e.

AOC + COD + DOE + EOB = 180˚

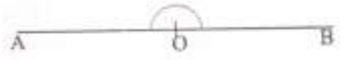


Fig. 4.21

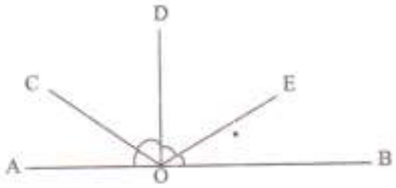


Fig. 4.21

Did you discover the same when you did Question 4(a) of Exercise 4.4?

Two angles having a common line and a common vertex are called **adjacent angles**. In Fig. 4.21 (b), AOC is adjacent to COD, COD is adjacent to DOE etc.

Can you name another pair of adjacent angles?

Angles at a point

When two lines intersect, they form four angles having a common vertex. The two angles which are on opposite sides of the vertex are called **vertically opposite angles.**

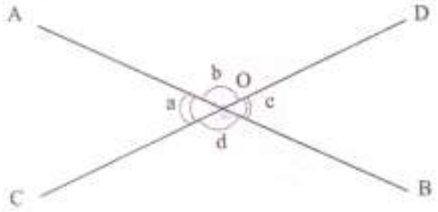


Fig 4.22

In Fig. 4.22, AOC is vertically opposite BOD.
AOD is vertically opposite COB.
Let AOC = a, AOD = b, BOD = c and BOC = d. AOD and AOC are adjacent angles on a line, it follows that:
a + b = 180˚ (angles on a straight line)
Also b + c = 180˚ (angles on a straight line).
Then a + b = b + c

Therefore, a = c
Similarly, we can show that b =d.
We conclude that **vertically opposite angles are equal**. Did you discover the same result when you did Question 5 of Exercise 4.4?
Also, since a + b = 180˚ (angle on a straight line)
And c + d = 180˚ (angle on a straight line)

Then a + b + c + d = 180˚ + 180˚

= 360˚

This shows that **angles at a point add up to 360˚**. Thus, in Fig. 4.23:

AOB + BOC + COD + DOA = 360˚

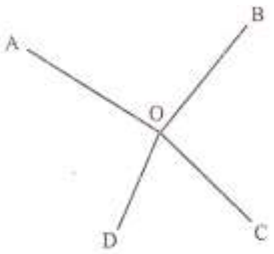
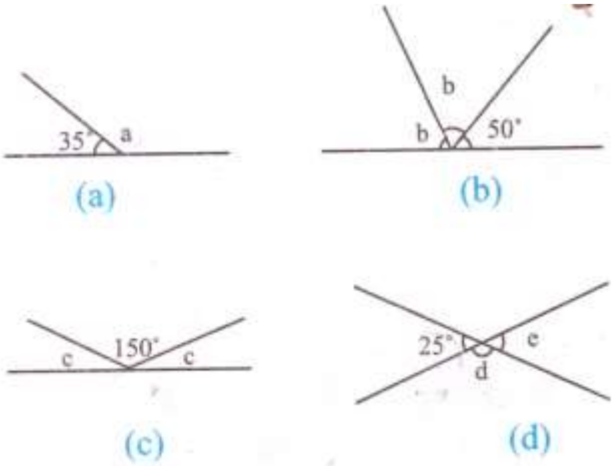


Fig. 4.23

Exercise 4.5

1. Find the sizes of the angles marked by letters in Fig. 4.24 (not drawn accurately).



(a) (b) (c) (d)

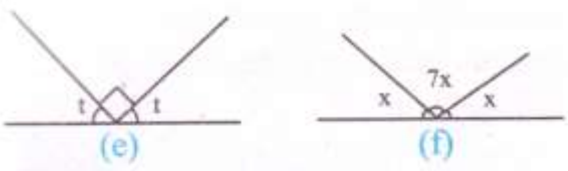


Fig.4.24

2. Using Fig. 4.25, find z.

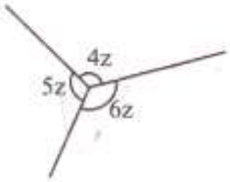


Fig. 4.25

3. Using Fig. 4.26, XYZ is a straight line. Find a if it exceeds b by $\frac{1}{2}$ of a right angle.

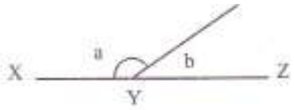


Fig. 4.26

4. Fig. 4.27 represents three lines intersecting at a point. Find p and q.

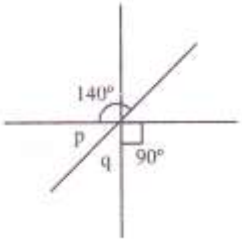


Fig. 4.27

5. Fig. 4.28 is not drawn accurately.
   (a) Find PAT
   (b) An angle bisector is a line that divides the angle into two equal parts. Find the angle between the bisectors of RAQ and RAS.s

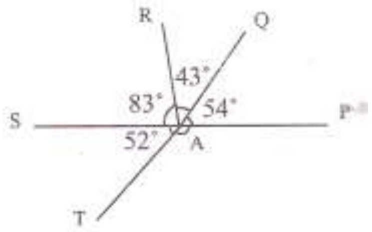


Fig. 4.28

Complementary and supplementary angles

If A = 37 and B = 53, then A + B = 90.

Any two angles A and B are said to be **complementary angles** if A + B = 90˚. A and B are said to be **complements** of each other.

If A = 37 and B = 143, then A + B = 180.

If A + B = 180˚, then the angles are said to be **supplementary angles**. A and B are **supplements** of each other.

Example 4.4

Given that x and y are complimentary angles and y is twice as big as x, find the value of x and y

Solution

Complementary angles add up to 90˚.

We also know that y is twice as big as x.

y = 2x

So, x + y = 90˚

x + 2x = 90

3x = 90˚

x = $\frac{90˚}{3}$

= 30˚

Since $y = 2x$

$$y = 2x$$

$$y = 2(30^\circ)$$

$$= 60^\circ$$

The two angels x and y are 30˚ and 60˚ respectively.

Example 4.5

Find two angles A and B such that $A = 2y^\circ$, $B = 3y^\circ$ given that A and B are supplementary.

Solution

Supplementary angles add up to 180˚

$$A + B = 180$$

But $A = 2y^\circ$ and

$$B = 3y^\circ$$

$$A + B = 2y^\circ + 3y^\circ = 180^\circ$$

$$5y^\circ = 180^\circ$$

$$y^\circ = \frac{180}{5}$$

$$= 36^\circ$$

Since $A = 2y^\circ$

$$= 2 \times 36^\circ$$

$$= 72^\circ$$

Since $B = 3y^\circ$

$$= 3 \times 36^\circ$$

$$= 108^\circ$$

The two angles A and B are 72˚ and 108˚ respectively.

Exercise 4.6

1. State the complements of the following angles.

i.	44°	iv.	82°
ii.	36°	v.	22°
iii.	57°	vi.	17°

State the supplements of the following angles.

i.	44°	v.	124°
ii.	32°	vi.	176°
iii.	132°	vii.	180°
iv.	85°		

2. Given that ABC and BCA are complementary and that BCA = 33°, find the value of ABC.
3. Given that B = y° and D = 3y° and that angles B and D are supplementary, find the size of angles B and D.
4. Identify which of the following pairs of angles are:
   - i. Complimentary
   - ii. Supplementary

a.	36° and 54°	f.	150° and 30°
b.	110° and 70°	g.	80° and 20°
c.	28° and 152°	h.	90° and 0°
d.	52° and 32°	i.	180° and 0°
e.	70° and 35°	j.	140° and 40°

5. Find the compliments of the following angles.

(a)	10°	(e)	13°
(b)	65°	(f)	78°
(c)	88°	(g)	25°
(d)	59°	(h)	9°

6. Find the supplements of the following angles.

(a)	21°	(e)	170°
(b)	77°	(f)	110°
(c)	120°	(g)	130°
(d)	108°	(h)	123°

Angles on parallel lines

Fig. 4.29 shows AB and CD which meet at P when produced.

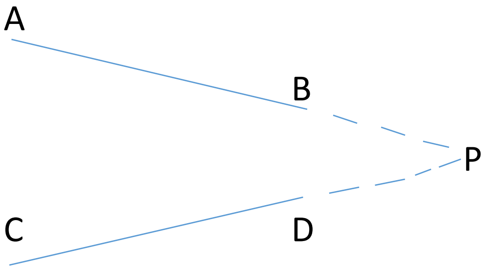


Fig. 4.29

In contrast, Fig. 4.30 shows lines KL and MN which do not meet however far they are produced. Such lines are said to be parallel. For example the horizontal lines in your exercise book are parallel to each other. The opposite edges of your desk are parallel to each other.

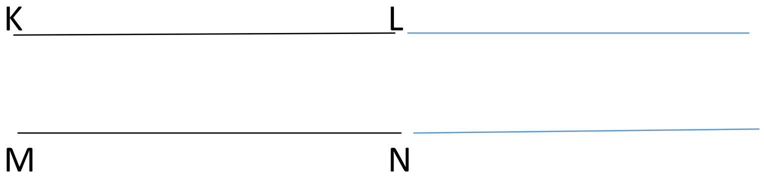


Fig. 4.30

We show that lines are parallel by drawing arrow heads on them as shown in Fig. 4.31.

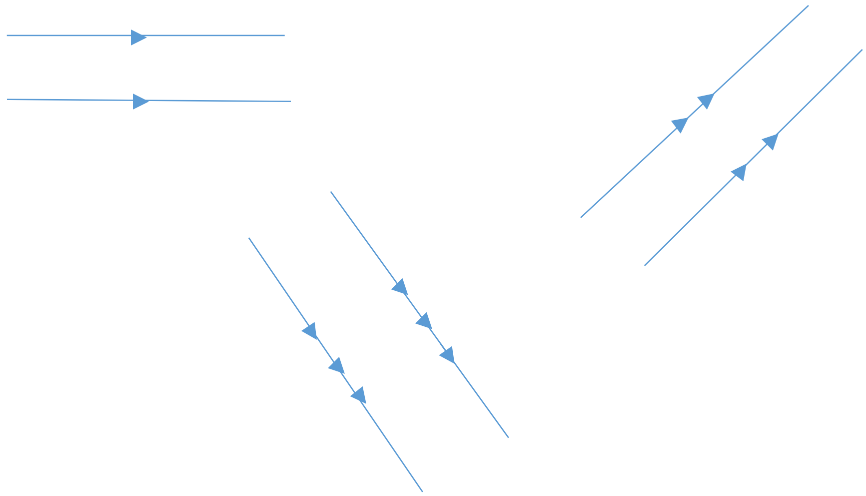


Fig. 4.31

‘Line KL is parallel to line MN’ written in short as KL// MN (sometimes as KL∥ MN).

A line which cuts through a pair of parallel lines is called a **transversal** (See Fig. 4.32).

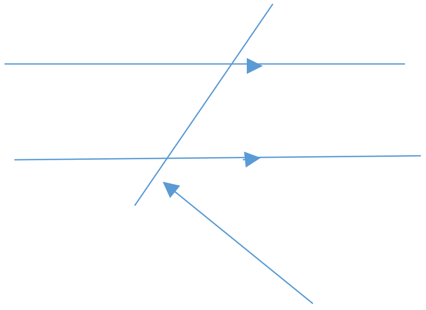


Fig. 4.32

Fig. 4.33 shows parallel lines AB and CD and transversal PQ. The marked angles are called **corresponding angles (or F – angles)**

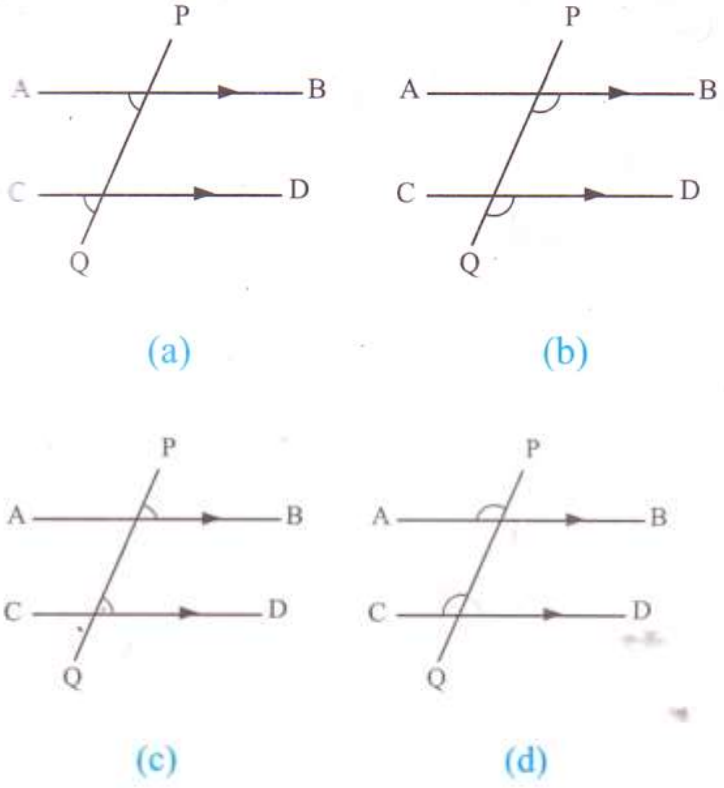
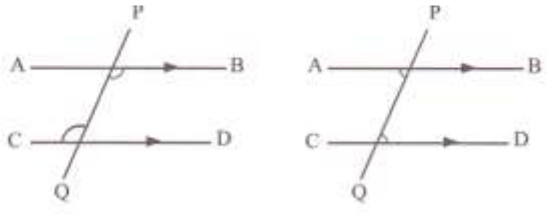


Fig. 4.33

Measure the corresponding angles in each case. What do you notice?

Fig. 4.34 shows parallel lines AB and CD, and a transversal PQ. The marked angles are called alternate angles (or Z – angles).

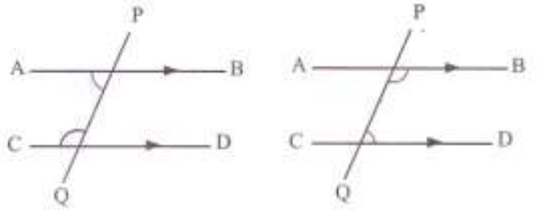


(a) (b)

Fig. 4.34

Measure the alternate angles in each case. What do you notice?

In Fig. 4.35, the marked angles are called **co-interior or allied angles**.



(a) (b)

Fig. 4.35

Measure the co-interior angles in each case. What can you say about their sum?

Did you notice the following?

1. Corresponding angles are equal.
2. Alternate angles are equal.
3. Co-interior angles add up to 180˚

Note that is possible to have corresponding, co-interior and alternate angles with lines which are not parallel. In Fig. 4.36, angle s corresponds to angle t, angle v and w are co-interior and angle u is alternate to angle v. since the lines are not parallel the angles are **not equal** i.e. $s \neq t$, $u \neq v$ and $v + w \neq 180^\circ$

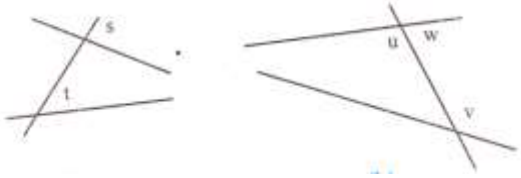


Fig. 4.36

Example 4.5

Fig. 4.37 shows a parallelogram ABCD. Calculate the size of angles marked a, b, c and d.

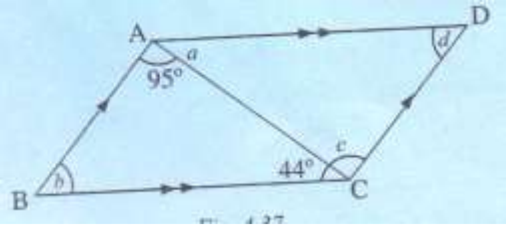


Fig. 4.37

Solution

Using ABC,   A = 95˚

C = 44˚

B = b

95 + 44 + b = 180˚…. Angles in a triangle add up to 180˚.

B = 180˚ – (95 + 44)

= 180˚ – 139˚

= 41˚

Since Ad is parallel to BC alternate S

CÂD and ACB are equal

CÂD = ACB

a = 44˚

In parallelogram ABCD

ABC = ADC…. Opposite angles of a parallelogram are equal

b = d

d = 41˚

Since AB is parallel to CD and AC is a transversal BAC and ACD are alternate

BAC = ACD

95 = c

C = 95˚

Exercise 4.7

1. State 5 pairs of examples of parallel lines in your classroom.
2. 2. Fig. 4.38 shows a section of honeycomb.

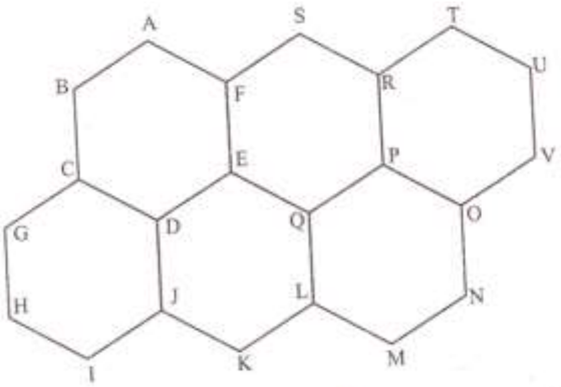


Fig. 4.38

   (a) Name all the lines parallel to AB.
   (b) Name all the lines parallel to ON.
   (c) Which lines are parallel to SR?
3. Using Fig. 4.39,
   (a) Write down all the pairs of
      i. Corresponding angles
      ii. Alternate angles
      iii. Co-interior angles.
   (b) Write down the sizes of the angles marked with letters.

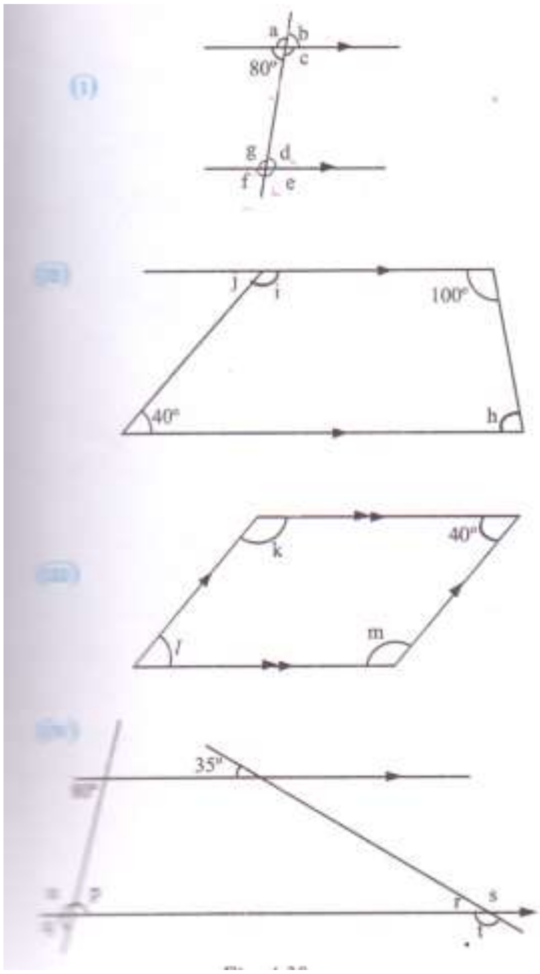


Fig. 4.39

4. In Fig. 4.40, find pairs of parallel lines.

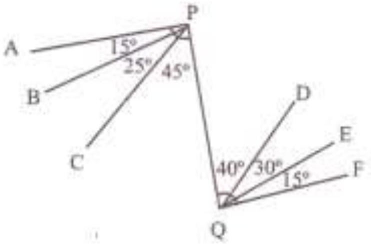


Fig. 4.40

5. Find the angles marked with letters in Fig. 4.41. [Hint: Copy the given figures and insert other parallel lines at M and C respectively].

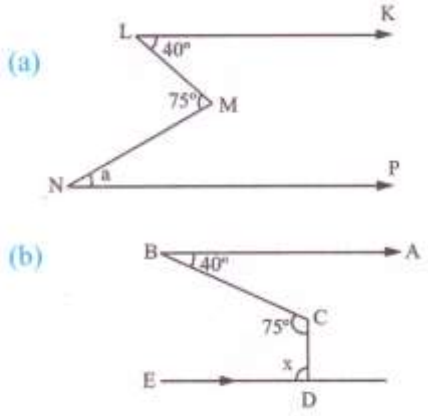


Fig. 4.41

Unit 5 APPROXIMATION, ESTIMATION AND ACCURACY

Rounding off numbers

In real life situations, there are times when obtaining or using exact answers is not practical. For example, the accuracy of a measurement depends on its purpose or the method employed in obtaining it. Usually, we indicate the degree of accuracy by means of:

(a) Rounding off numbers to the required number of;
  i. Decimal places,
  ii. Significant figures.
(b) Stating the smallest unit of measurement used.

Rounding off to a given place value

Consider the number 7 352

This number lies between 7 350 and 7 360. It is closer to 7 350 than it is to 7 360. So the number is approximately equal to 7 350 since we have considered the tens only. This is written symbolically as

7 350 = 7 350 (to the nearest ten).

The same number 7 352 lies between 7 300 and 7 400. It is closer to 7 400 than to 7 300. So we can write.

7 352 = 7 400 (to the nearest hundred).

When we write 7 352 as 7 350 or as 7 400, we say that we have rounded off the number to the nearest ten or to the nearest hundred, respectively.

To rounding off to a given place value, locate the required place value and:

1. If the first digit after the required place value is from 5 to 9, add 1 to the last digit in the required place value (i.e. round up the number) and replace the digit (s) after it with zero (s).
2. If the first digit after the required place value is 4 or less, leave the digit in the required place value unaltered (i.e. round down the number) and replace the digit (s) after it with zero (s).

Example 5.1

Round off 764 293 to the nearest:

(a) Hundred (b) ten thousand

Solution

Required place value (hundreds)

(a) 764 293

First digit after the required place value is more than 5.

Therefore, add 1 to 2.

Therefore, 764 293 = 764 300 (to the nearest hundred)

(b) Required place value (ten thousands)

764 293

First digit after the required place value is 4.

Therefore, 6 is to be unaltered.

Therefore, 764 293 = 760 000 (to the nearest ten thousand)

Note: The digits after the required place value must be replaced with zeros in order to keep the place values in the number correct.

Decimal places

When a fraction is written with a power of 10 as its denominator, it is called a **decimal fraction** or simply a **decimal**. For example:

$\frac{1}{10} = 0.1$; $\frac{2}{100} = 0.02$; $1\frac{3}{1000} = 1.003$; $\frac{4}{10\,000} = 0.000\,4$; $23\frac{5}{100\,000} = 23.000\,05$; etc.

The dot is called the decimal point and it is used to separate whole numbers from the fractional parts. A digit after the decimal point is said to stand in a certain decimal place.

Rounding off to a number of decimal places

Consider the following case:

The length of a line segment is stated as 15.695 cm.

It is highly unlikely that a measurement can be taken to the nearest thousandth of a centimeter. For all practical purposes, this length can be stated as 15.7 cm (to the nearest tenth or to 1 decimal place (1 d.p). we say that the length has been **rounded off** to the nearest tenth or correct to 1 decimal place.

Example 5.2

Write the number 7.852 63 correct to:

(a) 4 d.p
(b) 3 d.p
(c) 2 d.p
(d) 1 d.p
(e) The nearest whole number.

Solution

(a) 7.852 63 = 7.852 6 (to 4 d.p)
(b) 7.852 63 = 7.853 (to 3 d.p)
(c) 7.852 63 = 7.85 (to 2 d.p)
(d) 7.852 63 = 7.9 (to 1 d.p)
(e) 7.852 63 = 8 (to the nearest whole number)

In writing a number correct to a given number of decimal places, count from the first digit after the decimal point.

(a) If the first digit after the required decimal place is a digit from 5 to 9, add 1 to the digit in the required decimal place (i.e. round up the number).
(b) If the first digit after the required decimal place is 4 or less, then the digit in the required decimal place remains unaltered (i.e. round down the number).

Exercise 5.1

1. Round off the numbers in Table 5.1 to the indicated nearest place value.

Number	Nearest ten	Nearest hundred	Nearest thousand	Nearest million
(a) 687 182				
(b) 3 628 435				
(c) 28 199 846				

(d) 418 500 037				

Table 5.1

2. To what nearest place value has each of the following numbers been rounded?
    a) 876 = 1000
    b) 580 694 = 580 700
    c) 564 444 = 564 440
    d) 64 293 = 60 000
    e) 3 456 023 = 3 000 000
    f) 954 999 = 950 000
3. Express each of the following numbers to 1, 2, 3 and 4 decimal places.
    a) 2.456 2
    b) 94. 126 7
    c) 192.090
    d) 8.000 38
    e) 18.982 09
    f) 6.809 91
    g) 4.590 84
    h) 205.498
    i) 7.999 98
4. Write the following correct to 1, 2 and 3 decimal places.
    a) 19.045
    b) 43.588
    c) 31.52
    d) 0.034 56
    e) 0.054 6

Significant figures

Consider the following statements:

1. A new car costs K 1 994 751.90
2. The distance between Cairo and New York is 8 891.74 km.
3. My height is 168.782 cm.

All these measurements are given with what we might call ‘unreasonable accuracy’.

The following would probably be more reasonable statements:

1. A new car costs K 1 990 000.
2. The distance between Cairo and New York is 8 900 km.
3. My height is 169 cm.

In each case, it is only the first few figures which are **important** (or **significant**) for each measurement to be accurate enough.

K 1 990 000 is the price of the car to 3 significant figures (3. S.f)

The distance between Cairo and New York is 8 900 km to 2 s.f

169 cm is my height correct to 3 s.f.

Rounding off to a number of significant figures

Consider the number 652.73

1. 652.73 = 700 since is closer to 700 than 600.
2. 652.73 = 650 since 652.73 is closer to 650 than to 660
3. 652.73 = 653 since 652.73 is closer to 653 than 652.
4. 652.73 = 652.7 since 652.73 is closer to 652.7 than 652.8

In 1, **importance (i.e. significance)** is given to only the hundreds place value, so that 652.73 = 700 to 1 significant figure.

In 2 significance is given to both the hundreds and tens place values, so that 652.73 = 650 to 2 significant figures.

Likewise, 652.73 = 653 to 3 significant figures, and 652.73 = 653.7 to 4 significant figures.

Thus, the phrase '**significant figures** (s.f)' refers to the number of place value, starting from the left.

Note:

1. If the next digit after the last significant digit is 4 or less, the number is rounded down. If it is 5 or more, the number is rounded up.
2. Zeros that lie between non-zero digits are significant.
   e.g.

$$0.\underbrace{00}_{\text{Non-significant}}8\ 7\underbrace{00}_{\text{significant}}\ 2$$

Example 5.3

Approximate the following to

i. 1 s.f
ii. 2 s.f
iii. 3 s.f
iv. 4 s.f

(a) 40 256
(b) 69.048 2
(c) 0.008 421 09

Solution

The approximate values are as shown in Table 5.2

Number	*1 s.f*	*2 s.f*	*3 s.f*	*4 s.f*
(a) 40 256	*40 000*	*40 000*	*400 300*	*40 260*
(b) 69.048 2	*70*	*69*	*69.0*	*69.05*
(c) 0.008 421 09	*0.008*	*0.008 4*	*0.008 42*	*0.008 421*

Table 5.2

Example 5.4

Write the following correct to the number of significant figures given in brackets.

a) 546.52 (3)
b) 546.52 (4)
c) 8.029 6 (1)
d) 8.029 6 (2)
e) 0.009 25 (1)
f) 997 375 (3)

Solution

We start counting the significant figures from the first non-zero digit at the left of the number.

(a) 546.52 = 547 to 3 s.f
(b) 546.52 = 546.5 to 4 s.f
(c) 8.029 6 = 8 to 1 s.f
(d) 8.029 6 = 8.0 to 2 s.f. (In this case, the zero must be given after the decimal point; it is significant.)

(e) 0.009 25 = 0.009 to 1 s.f. (9 is the first non-zero digit. The two zeros after the decimal point are not significant figures. However, they must be written down to keep the place value correct.)

(f) 997 375 = 997 000 to 3 s.f (The last 3 zeros must be written down to keep the place value correct.)

Example 5.5

Write 0.081 043 correct to 5, 4, 3, 2, 1

(a) Decimal places

(b) Significant figures

Solution

Table 5.3 shows the approximations.

Approximations	*d.p.*	*Approximations*	*s.f.*
0.081 04	*5*	*0.081 043*	*5*
0.081 0	*4*	*0.081 04*	*4*
0.081	*3*	*0.081 0*	*3*
0.08	*2*	*0.081*	*2*
0.1	*1*	*0.08*	*1*

Table 5.3

Exercise 5.2

1. Express each of the following correct to
   i. One decimal place
   ii. One significant figure.
      a. 7.82
      b. 12.19
      c. 50.701
      d. 38.09
      e. 4.98
      f. 8.056
2. Write each of the following correct to the number of significant figures indicated in brackets.
   a) 3.141 6 (2)
   b) 0.589 63 (3)

c) 19.189 8 (4)
d) 0.006 193 (2)
e) 10.046 (3)
f) 4.078 6 (4)

3. Express each of the following numbers to 1, 2, 3 and 4 significant figures.
   a) 60 539
   b) 78 949
   c) 20 909
   d) 80 099
   e) 79 990
   f) 49 801
   g) 90 199
   h) 45 999
   i) 209 009
4. Express each of the following numbers to 1, 2 and 3 significant figures.
   a) 0.145 2
   b) 0.567 05
   c) 0.036 06
   d) 0.009 680 9
   e) 0.060 59
   f) 0.999 9
5. Copy and complete Table 5.4

Number	Number of d.p.			Number of s.f.			Nearest whole number
	3	2	1	3	2	1	
0.043 5	0.044	0.04	0.0	0.043 5	0.44	0.04	0
(a) 0.023 45							
(b) 0.005 130 7							
(c) 0.082 056							
(d) 6.893 4							
(e) 4.624 7							

Table 5.4

6. Express the following correct to one decimal place.
   a) 3.43
   b) 5.28
   c) 6.48
   d) 5.46
   e) 8.37
7. State the number of (i) decimal places (ii) significant figures in:

a) 0.23
b) 0.1
c) 18.000 6
d) 0.565
e) 0.006 5
f) 33. 076

8. Evaluate the following giving your answers correct to
   i. 3 decimal places
   ii. 3 significant figures.
   a. 15.043 ÷ 0.8
   b. 11.78 ÷ 0.6
   c. 12.47 ÷ 0.03
   d. 105.7 ÷ 2.45
   e. 613.2 ÷ 3.75

Estimation

A friendly match between the Tigers FC and the Blue Eagles FC at the Kamuzu Stadium was attended by 55 298 people. A television commentator told his viewers that about 55 000 people attended the match. Do you think he lied to the viewers?

The commentator did not lie. His interest was to inform the viewers of the number of spectators correct to the nearest 1 000. He was stating the number correct to 2 significant figures (2 s.f). The commentator would still have not lied had he given the number as 60 000 or 55 300.

Thus, 55 298 = 60 000 correct to the nearest 10 000 (or 1 s.f.)

= 55 000 correct to the nearest 1 000 (2 s.f)

= 55 300 correct to the nearest 100 (or 3 s.f. or 4 s.f.)

We use significant figures when estimating numbers or amounts and when we are doing rough calculations or estimates.

We use estimates to help us make decisions. Estimates are only meant to give rough guidelines.

In order to estimate the answer to a calculator we round off each number to its highest place value and then compute.

Example 5.6

Find a rough estimate of the sum of 1 497 + 2 565.

Solution

Number	Estimate
1 497	1 000
2 565	3 000
4 062	4 000

In this case, the sum 4 000 is estimate of (1 497 + 2 565), i.e. 4 062.

Example 5.7

Find a rough estimate of $\frac{8.76 \times 0.495}{17.8}$

Solution

Round of each number to 1 s.f. then proceed.

$$\frac{8.76 \times 0.495}{17.8} = \frac{9 \times 0.5}{20}$$

$$= 0.225$$

Approximation

Consider the following:

a) Calculate the following exactly:
   i. 7.2 × 1.2
   ii. 0.64 × 0.914
   iii. 0.75 ÷ 1.25
   iv. Area of a square of sides 11.2 cm

b) Measure the width of your Mathematics textbooks using a ruler and compare your answer with those of other members of your class.

In (a), your answers will agree with those of other members of your class, i.e.

i. 8.64
ii. 0.584 96
iii. 0.6

iv. 125.44 cm^2

In (b), some answers may agree with yours, while others may be quite different. So, whose answer is exactly correct? None of the answers are exact, but all may be correct depending on the degree of accuracy stated.

All measurements, no matter how sophisticated the measuring instruments are, are always approximated. When we say, for example, that the length of a line segment AB = 7.4 cm (1 d.p.), we mean that the length of AB can be anywhere between 7.35 cm and 7.45 cm.

So, $7.35 \leq 7.4 < 7.45$, which means that the true length must lie between 7.35 inclusive and 7.45 exclusive i.e. $7.35 \leqslant 7.4 < 7.45$.

Activity 5.1

1. Use your ruler to measure the length of your Mathematics textbook correct to (i) 1 d.p. (ii) the nearest cm (iii) 2 s.f.
   Compare your answers with those of other members of your class.
   Determine which of the answers give the best approximation and why.

2. Do you think it is more accurate to use a pair of dividers and a ruler to measure lengths than to use a ruler only? Why? Do you think there are any limitations to the use of dividers? Discuss.

Example 5.8

A lorry driver is contracted to transport 20 000 building blocks to a site. The lorry has a capacity to hold 5 120 blocks. Approximate the number of lorry loads necessary to transport the blocks.

Solution

In this case, the most significant figure in 5 120 is 5.

So, 5 120 may be estimated as 5 000 (1 s.f.)

Total number of blocks = 20 000

Therefore, Number of lorry loads is $\frac{20\,000}{5\,000} = 4$

We see that approximately 4 lorry loads are required to transport 20 000 blocks.

This is an example of an estimate necessary to help make a decision and plan.

Exercise 5.3

1. Express 67 348.07 m to the nearest
    a. 1 m
    b. 10 m
    c. 100 m
    d. 1 000 m
2. A factory worker making precision instruments is told that the diameter of a piston he is making is to be 6.7 cm with a 0.01 cm tolerance. State
    a) The least possible diameter.
    b) The largest possible diameter
3. Work out the rough estimates of each of the following.
    a) 275 × 992
    b) 8 806 × 0.845
    c) 67.4 × 0.001 2
    d) 25.04 ÷ 0.45
    e) 375 × 569 ÷ 42.3
    f) 0.006 43 ÷ 5.81
4. Estimate how much it would cost to buy a copy of this textbook for every student in your school given that a copy costs K825.
5. You wish to paint two surfaces measuring 6.4 m by 4.6 m and 10.8 m by 6.7 m. each tin of paint is expected to cover 12 m^2 and costs K 2 500. Estimate the number of tins of paint you should buy. Roughly, how much do you expect the paint to cost?
6. Estimate the total area of paper, in square metres, used to produce this textbook. Explain your answer.
7. A metal rod is measured as 3 m long (1 s.f.) and another as 4 m long (1 s.f.). Do you think it is true to say that the total length of the rods is 7 m (1 s.f.)?

Squares and square roots

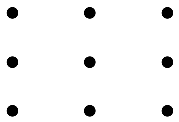
Squares

Number that can be represented by a square pattern of dots are called perfect squares or simply **square numbers**.

(a) •   •
      •   •

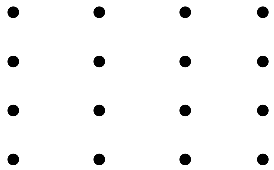
This pattern is a square with two dots on each side. It has 2 × 2 dots = 4 dots

(b)



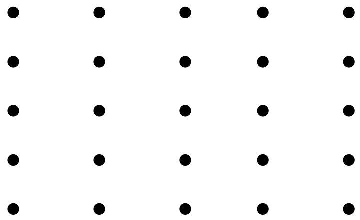
This pattern is a square with 3 dots on each side. It has 3 × 3 dots = 9 dots.

(c)



This pattern is a square with 4 dots on each side. It has 4 × 4 dots = 16 dots.

(d)



This pattern is a square with 5 dots on each side. It has 5 × 5 dots = 25 dots.

This pattern of dots can be continued indefinitely.

Finding squares by estimation

Numbers such as 1, 4, 9, 16, 25… are known as square numbers or perfect squares.

The square of a number is found by multiplying the number by itself.

The squares of the first 10 natural numbers are:

$1 \times 1 = 1^2 = 1$

$2 \times 2 = 2^2 = 4$

$9 \times 9 = 9^2 = 9$

$4 \times 4 = 4^2 = 16$

$5 \times 5 = 5^2 = 25$

$6 \times 6 = 6^2 = 36$

$7 \times 7 = 7^2 = 49$

$8 \times 8 = 8^2 = 64$

$9 \times 9 = 9^2 = 81$

$10 \times 10 = 10^2 = 100$

Example 5.9

Estimate the squares of:

(a) 42
(b) 48

Solution

(a) 42 lies between 40^2 and 50^2.

Therefore, 42^2 lies between 402 and 50^2

$40^2 = 1\ 600$ and $50^2 = 2\ 500$

Thus, 42^2 lies between 1 600 and 2 500.

Since 42 = 40 (1 s.f)

Then $42^2 = 40^2 = 1\ 600$

(b) 48 lies between 40 and 50.

Therefore, 48^2 lies between 1 600 and 2 500

Since 48 = 50 (1 s.f)

Then $48^2 = 40^2 = 2\ 500$

Exercise 5.4

1. Find the squares by estimation:
    - (a) 3
    - (b) 4.5
    - (c) 25.8
    - (d) 55
    - (e) 720
    - (f) 3 150
    - (g) 0.031 7
    - (h) 318
    - (i) 70.8
2. Use approximation to find the squares of the numbers in questions a to x.
    - (a) 8
    - (b) 39
    - (c) 839
    - (d) 8.39
    - (e) 5.34
    - (f) 669
    - (g) 7.432
    - (h) 446.9
    - (i) 0.851 6
    - (j) 5.276 4
    - (k) 0.504 3
    - (l) 0.092 7
    - (m) 3
      8.426
    - (n) 8341
    - (o) 59 648
    - (p) 0.0064 72
    - (q) 100.7
    - (r) 0.899 98
    - (s) 99.99
    - (t) 530 829
    - (u) 0.07 264
    - (v) 9.090 11
    - (w) 0
      .020 67
    - (x) 227.84

3. Se estimation to find the value of $a^2 - b^2$ if:
    a) a = 43.2 and b = 21.9
    b) a = 0.724 and b = 0.516
    c) a = 8.72 and b = 5.41
4. Find the square of 228 by estimation.

Square roots

If x and y are numbers such that $y^2 = x$, we say that y is the square root of x, written as y =$x = \sqrt{x}$.

For example:

$2^2 = 4$ $\quad$ $2 = \sqrt{4}$

$3^2 = 9$ $\quad$ $3 = \sqrt{9}$

$4^2 = 16$ $\quad$ $4 = \sqrt{16}$

This pattern continues indefinitely.

We know that $\sqrt{36} = 6 \text{ and } 36 = 4 \times 9$.

Therefore, $\sqrt{36} = \sqrt{4 \times 9} = 6$

Also $\sqrt{4} \times \sqrt{9} = 2 \times 3 = 6$.

Therefore $\sqrt{4 \times 9} = \sqrt{4} \times \sqrt{9}$

In general,

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

Square roots of numbers by estimation

Example 5.10

Find the square root of 30 by estimation

Solution

30 lies between 25 and 36

Therefore, $\sqrt{30}$ lies between 5 and 6

Taking a rough estimate of $\sqrt{30}$ *as 5.5, we get* 5.5^2*= 30.25.*

Since 30 is less than 30.25,

Then $\sqrt{30}$ *lies between 5 and 5.5 and* $\sqrt{30}$ *is closer to 5.5 than 5.*

To get a better estimate of $\sqrt{30}$, *we take a rough estimate as* 5.475

5.475^2 *= 29.920 which is closer to 30.*

Hence $\sqrt{30}$ *= 5.475*

Exercise 5.5

1. Use estimation to find the square roots of each of the following.
   - a) 69
   - b) 220
   - c) 580
   - d) 1 300
   - e) 780
   - f) 3 970
2. Find the square roots of the following.
   - a) 0.09
   - b) 0.64
   - c) 0.81
   - d) 0.004 9
   - e) 1.21
   - f) 2.56
   - g) 1.69
   - h) 2.25
3. State the two immediate perfect squares between which each of the following numbers lie.
   - a) 2.54
   - b) 44.6
   - c) 4.977
   - d) 51.32
   - e) 15.663
   - f) 67.45
4. Use a calculator to evaluate the square roots of each of the following.
   - a) 25.7
   - b) 8.84
   - c) 41.84
   - d) 96.62
   - e) 32.32
   - f) 59.7
   - g) 94.36
   - h) 50.7
   - i) 4.007 4
5. Using a calculator, evaluate the square roots of the following.
   - a) 562.4
   - b) 8 161
   - c) 3 274
   - d) 842
   - e) 9 216
   - f) 4 040
6. Use a calculator to solve the following;
   - a) If $x^2 = 130$, find x.
   - b) If c = $x = \sqrt{a^2 + b^2}$, evaluate c given that a = 7 and b = 12.
   - c) If p = $\sqrt{2ab}$ and a = 9 and b = 11, calculate the value of p correct to 1 decimal place.

Cube and cube root

Cubes

Activity 5.2

Study the number pattern 1, 8, 27, 64. State the next five terms. Compare your answers with those of other members of your class and discuss how you arrived at your answers.

From your discussion, you should have noticed that each of the numbers can be written as a product of three identical numbers; that is, multiplying a number by itself thrice.

i.e. $1 = 1 \times 1 \times 1 = 1^3$

$8 = 2 \times 2 \times 2 = 2^3$

$27 =3 \times 3 \times 3 = 3^3$

$64 = 4 \times 4 \times 4 = 4^3$

Numbers such as 1, 8, 27, and 64 …… are known as **cubes**.

Finding cubes by estimation

The cubes of the first 10 natural numbers are:

$1 \times 1 \times 1 = 1^3 = 1$

$2 \times 2 \times 2 = 2^3 = 8$

$3 \times 3 \times 3 = 3^3 = 27$

$4 \times 4 \times 4 = 4^3 = 64$

$5 \times 5 \times 5 = 2^3 = 125$

$6 \times 6 \times 6 = 6^3 = 216$

$7 \times 7 \times 7 =7^3 = 343$

$8 \times 8 \times 8 = 8^3 = 512$

$9 \times 9 \times 9 = 9^3 = 729$

$10 \times 10 \times 10 = 1\,000$

Example 5.11

Find the cube of 13 by estimation

Solution

13 lies between 10 and 20

$10^3 = 1\ 000$ and $20^3 = 8\ 000$

Thus, 13^3 lies between 1 000 and 8 000

Since 13 = 10 (1 s.f)

Then, $13^3 = 1\ 000$

Exercise 5.6

1. Find the cube of each of the following numbers by estimation.
   a) 2.3
   b) 5.4
   c) 52.0
   d) 79
2. Use estimation to find the cubes of the following numbers
   a) 7
   b) 29
   c) 398
   d) 1 238
   e) 3.891
   f) 0.817
   g) 0.002 5
   h) 0.206 388
   i) 6.831 9
   j) 3.999 9
   k) 80.901
   l) 288.48
3. Use estimation to evaluate $x^3 + y^3$ given that
   a) x = 63.2 and y = 41.9,
   b) x = 0.842 1 and y = 0.615 8,
   c) x = 9.872 and y = 4.518 7.

Cube roots

If a and b are two non-zero numbers such that $a^3 = b$, we say that a is the cube root of b,

Written as a $\sqrt[3]{b}$ For example,

$2 \times 2 \times 2 = 2^3 = 8$ ➡ $2 = \sqrt[3]{8}$ and

$-3 \times -3 \times -3 = -3^3 = -27$ ➡ $-3 = \sqrt[-3]{27}$.

Cube roots of numbers by estimation

Example 5.12

Find the cube root of 25 by estimation

Solution

Since $2^3 = 8$ *and* $3^3 = 27$

$\sqrt[3]{25}$ *lies between 2 and 3*

25 is nearer 27 than 8 thus, we take a rough estimate nearer 3 than 2 as 2.8

$2.8^3 = 21.952$, *thus* $\sqrt[3]{25}$ *lies between 2.8 and 3.*

We now take an estimate of 2.9

$2.9^3 = 24.389$

Thus, $\sqrt[3]{25} = 2.9$

Exercise 5.7

1. Use estimation to find the cube root of:
   - a) 225
   - b) 350
   - c) 648
   - d) 576
2. Each of the numbers given in this question lies between two consecutive perfect cubes. In each case, state the two cubes.
   - a) 5.831
   - b) 85.76
   - c) 985.3
   - d) 503.77
3. Use estimation to evaluate:
   - a) $\sqrt[3]{98.36}$
   - b) $\sqrt[3]{628.5}$
   - c) $\sqrt[3]{1\,977}$
   - d) $\sqrt[3]{4.247\,8 \times 10\hat{}4}$
4. Use estimation to evaluate:
   - a. $\sqrt[3]{0.062\,16}$
   - b. $\sqrt[3]{0.004\,32}$
   - c. $\sqrt[3]{0.618\,79}$
   - d. $\sqrt[3]{16.8 \times 10\ \hat{}6}$

5. Given that $x^3 = 158$, estimate the value of x.
If a = 8 and b = 14, estimate:

(i) $\sqrt[3]{a^2 + b^2}$

(ii) $\sqrt[3]{3a^2b^2}$

(iii) $\sqrt[3]{b^2 - a^2}$

Unit 6 ALGEBRAIC PROCESSES 1

Factors

If a number can be divided by a second number without leaving a remainder, the second number is said to be a **factor** of the first. For example

6 ÷ 3 = 2

Thus, 3 is a factor of 6.

Similarly, 2 divides into 6 without leaving a remainder. So, 2 is also a factor of 6.

All the factors of 6 are: 1, 2, 3 and 6.

In the same way, we can list the factors of algebraic terms such as 6x.

Factors of 6x are: 1, 2, 3, 6, x, 2x, 3x and 6x since any of them can divide 6x without leaving a remainder.

Example 6.1

List down all the factors of 16ab.

Solution

Factors of 16ab: 1, 2, 4, 8, 16, a, 2a, 4a, 8a, 16a, b, 2b, 4b, 8b, 16b, ab, 2ab, 4ab, 8ab and 16ab.

Exercise 6.1

List down all the factors of:

1). 24 2. 42 3. 210 4. 3 x 5).15 xy 6).18 pq

7). 5 mn 8). 13 st 9). 21 pqr 10). 85 cde 11). 192 efj

12). 40 apq

Prime factors

The prime factors of a number are those factors which themselves have no except 1 and themselves.

For example, 2 is a prime factor since it has only two factors 1 and 2. Similarly a term can be written as a product of its prime factors. Thus, 4x = 2 × 2 × x

When finding the prime factors of a term, it is best to try the prime numbers in turn, starting with the least, i.e. 2, and if possible, repeating each until it is no longer a factor.

Example 6.2

Express 60xy as a product of prime factors, hence list down all its factors.

Solution

$60xy = 2 \times 30xy = 2 \times 2 \times 15zy$

$= 2 \times 2 \times 3 \times 5xy = 2 \times 2 \times 3 \times 5 \times xy$

$= 2 \times 2 \times 3 \times 5 \times x \times y$

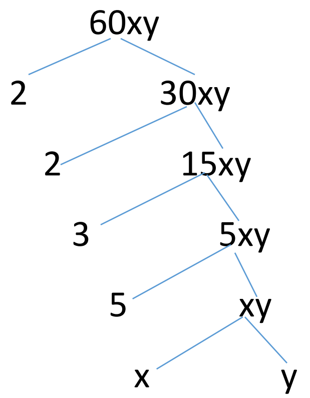
Prime factors of 60xy are 2, 3, 5, x, y

The working can be conveniently set out as a successive (or continued) division as follows

2	60xy
2	30xy
3	15xy
5	5xy
x	xy
y	y
	1

$60xy = 2 \times 2 \times 3 \times 5 \times x \times y$

Prime factors of 60xy are 2, 3, 4, 5, x and y. alternatively, the working can be set out as a factor **tree diagram** as follows.



Thus, $60xy = 2 \times 2 \times 3 \times 5 \times x \times y$

Prime factors of 60xy are 2, 3, 5, x, y

Exercise 6.2

1. Express each of the following as a product of its prime factors.
   a) 72k
   b) 350zy
   c) 344hg
   d) 243p
   e) 180qr
   f) 45rw
   g) 165mn
   h) 78xyz

2. List down all the prime factors of:
   a) 42xyz
   b) $12pq^2$
   c) 2 520rs
   d) 6 006qt
   e) ab^2c
   f) j^2kl^2

Highest common factor (HCF)

if two or more terms can be divided by a smaller factor without leaving a remainder, the factor (i.e. the smaller number) is called a **common factor** or a **common divisor** of the terms. For example, 3x is a common divisor of 12x, 15xy, 21x and 27xy.

A group of terms may have more than one common factor. For example, 2, 3, and x are common factor of 18x and 24x. Also, since 2, 3 and x are common factor and are prime factors, 6x must also be a common factor of 18x and 24x.

The highest (or greatest) of the common factors of a group of terms is called the **highest common factor (HCF)** or the **greatest common divisor (GCD**) of the terms. Thus, 6x is the HCF of 18x and 24x.

The HCF of two or more terms can be found using any of the following methods:

1. Listing all the factors of each term and identifying the highest factor common to all.
2. Expressing each term as a product of its prime factors and identifying all prime factors common in all.
3. Successive division by common factors.

The three methods are illustrated in Examples 6.3 and 6.4

Example 6.3

Find the HCF of 4p, 6pq and 14p.

Solution

Factors of 4p are 1, 2, 4, p, 2p, 4p.

Factors of 6pq are 1, 2, 3, 6, p, 2p, 3p, 6p, q, 2q, 3q, 6q, pq, 2pq, 3pq, 6pq.

Factors of 14p are 1, 2, 7, 14, p, 2p, 7p, 14p.

Common factors are 1, 2, p, 2p.

The HCF is 2p.

Example 6.4

Find the GCD of 432x, 288xy and 1 080x^2

Solution

$432x = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times x$

$288xy = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times x \times y$

$1\,080x^2 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times x \times y$

Therefore, HCF $= 2 \times 2 \times 2 \times 3 \times 3 \times x$

$= 72x$

Using successive division:

2	432x	288xy	1 080x^2	← (Dividing each by least common prime factor 2 until it
2	216x	144xy	540 x^2	it cannot divide further)
2	108x	72xy	270 x^2	
3	54x	36xy	135 x^2	← (Next least common prime factor is 3)
3	18x	12xy	45 x^2	
x	6x	4xy	15 x^2	← (Next least common prime factor is x)
	6	4y	15x	← (No common prime factor)

HCF = 2 × 2 × 2 × 3 × 3 × x = 72x

This method is also known as the method of **continuous division** by common prime factors. HCF = 2 × 2 × 2 × 3 × 3 × x = 72x

This method is also known as the method of **continuous division** by common prime factors.

Exercise 6.3

Find the HCF of each of the following.

1. Xy, 4xy
2. 8p, 6pq
3. 2x, 4x, 6x
4. 9y, 18xyz
5. P, 15pq, 27pq^2
6. 16k, 18kl, 12kn
7. 6nab, 4bm, 10cb, 20mb
8. 18xy, 24xz, 36xz
9. 40ef, 70e, 56ef
10. 14st, 21ps, 35stu

Least common multiple (LCM)

If a number is a factor of another, then the second number is said to be **multiple** of the first number.

For example, 5 is a factor of 15 and so 15 is a multiple of 5.

Similarly, x and y are factors of xy, so xy is a multiple of x and y.

Example 6.5

List the multiple of 6p, 8p and 12p and identify the common multiples.

Solution

6p: multiples are: 6p, 12p, 18p, 24p, 30p, 36p, 42p, **48p**…

8p: multiples are: 8p, 16p, **24p**, 32p, 40p, **48p**…

12p: multiples are: 12p, **24p**, 36p, **48p**, 60p.

Common multiples are: 24p, 48p…

Unlike common factors, we cannot list all the common multiples of two or more numbers, they are unlimited. The most useful of them is the smallest, usually known as the **least** or **lowest common multiple (LCM).**

In example 6.5, 24p is the LCM of 6p, 8p and 12p.

The LCM of two or more numbers can be found using any of the following methods.

1. Listing multiples of each of the numbers and identifying the smallest multiple that is common to all (Example 6.5)
2. Prime factorization
3. Successive division by prime factors.

Examples 6.6 and 6.7 illustrate the methods of two or three respectively.

Example 6.6

Find the LCM of 40n, 180nm and 210n^2m by expressing each term as a product of its prime factors.

Solution

After expressing the terms as products of their prime factors, we pick the highest number of each factor. (Note that all appearing factors must be represented.)

$40n = 2 \times 2 \times 2 \times 5 \times n$

$180nm = 2 \times 2 \times 3 \times 3 \times 5 \times n \times m$

$210n^2m = 2 \times 2 \times 5 \times 7 \times n \times n \times m$

$LCM \text{ is } 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times n \times n \times m$

$= 2\,520\ n^2m$

Example 6.7

Find the L.C.M of 15pq and 27pqr by the method of successive division.

Solution

3	15pq	27pqr
3	5pq	9pqr
3	5pq	3pqr
5	5pq	pqr
p	pq	pqr
q	q	qr
r	1	r
	1	1

(Start with the smallest prime number that divides at least one of the numbers)

(Bring 5pq down as it is not divisible by 3)

(3 cannot divide again, so pick the next prime number)

LCM = $3 \times 3 \times 3 \times 5 \times p \times q \times r$

= 135 pqr

Exercise 6.4

Find the LCM of the following.

1. 4a, 6a
2. 3a, 6b
3. 3p, 6pq
4. 6xy, 2y
5. 9m, 10mn
6. 36pq, 24qt
7. 27pq, 132q
8. 3x, 5y, 9xy
9. 7ab, 8pq, 12rs
10. 3d, 4de, def
11. abc, abd, 3acd
12. 3mn, m^2n, mny

Factorization of algebraic expressions

In Unit 3, we defined the terms algebra and algebraic expressions and performed some computations involving algebraic expressions. In this unit, we are going to factorize different types of algebraic expressions.

Consider the expressions:

(a) 3ab
(b) 2a + 4ab
(c) A + 2ab + 3abc

Can you identify any difference between the three expressions?

They are all algebraic expressions. Part (a) is a one term expression. An algebraic expression containing only one term is called a monomial expression.

The main difference between the expressions 3ab and 2a + 4ab is that the first expression has only one term while the second is a sum of two terms. An algebraic expression that is a sum of a difference of two terms is called a **binomial expression.**

Now the expression a + 2ab + 3abc is a sum of three distinct terms. Such an expression is called a **trinomial expression.**

A trinomial expression has three distinct terms which may be positive or negative.

To factorize an algebraic expression means to find the factors of that expression and write it as a product of prime factors.

For example, in an expression such as 3ab, there are three factors: **3, a, b.**

Thus, 3ab has been factorized.

To factorize the binomial expression 2a + 4ab, we identify the common factor in the two terms.

The factors of 2a are 2 and a.

The factors of 4ab are 2, a, b and 4.

The common factors of 2a + 4ab are 2 and a. i.e. 2a. To complete the factorization, we introduce brackets and divide the whole expression by 2a.

Thus, (2a + 4ab) ÷ 2a = 1 + 2b (dividing each term by the common factor).

Thus, 2a and (1 + 2b) are the factors of 2a + 4ab

Therefore, 2a + 4ab = 2a(1 + 2b)

To factorize a trinomial expression such as a + 2ab + 3abc, we first identify the factors of each term.

So, 1st term **a** has one factor a

2nd term **2ab** has prime factors **2, a, b**

3rd term **3abc** has prime factors **3, a, b, c**

In all the three terms, there is only one common factor **a.**

Dividing a + 2ab + 3abc by a means dividing each term by a.

Thus, (a + 2ab + 3abc) ÷ a = 1 + 2b + 3abc.

Note that in the quotient, there is no common factor in the three terms.

The factors of a + 2ab + 3abc are a and dividing each term by a, we obtain 1 + 2b + 3bc.

So a + 2ab + 3abc = a(1 + 2b + 3bc).

Example 6.8

Factorize the expressions

(a) 4xyz *(b) -6P^2QR*

Solution

(a) 4xyz is a monomial expression the prime factors of 4xyz are 2, x, y, z

Therefore, 4xyz = 2 × 2 × x × y × z

(b) -6P^2QR is also a monomial expression. Since the expression is negative, one of the factors must be negative

So, the prime factors of -6P^2QR are

-2, 3, P^2, Q, R or

2, -3, P^2, Q, R or

-2, 3, $-P^2$, Q, R etc.

$-6P^2QR = -2p(3PQR)$ or

$= 3p(2PQR)$ or

$= -3Q(2P^2R)$

$= -3PR(2PQ)$

Note that any of the factors can be negative to give a correct expression.

Example 6.9

Factorize: (a) $3m + 3$ (b) $3m + 6n$

(c) $a^2 – ab$ (d) $4xy – 6y^2$

Solution

(a) $3m + 3 = 3(m +1)$ (common factor is 3)
(b) $3m + 6n = 3(m + 2n)$ (common factor is 3)
(c) $a^2 – ab = a(a – b)$ (common factor is a)
(d) $4xy – 6y^2 = 2y(2x – 3y)$ (common factors are 2 and y)

Example 6.10

Factorize: (a) $2xy – 6yz + 2y$

(b)$2ab^2+ 4abc – 6ab$

Solution

(a) $2xy – 6yz + 2y$ is a trinomial expression

There are two factors 2 and y in the three terms

$(2xy – 6yz + 2y) \div 2y = x – 3z + 1$

Therefore, $2xy – 6yz + 2y = 2y(x – 3z + 1)$

(b) $2ab^2+ 4abc – 6ab$

2ab is the common factor in the three terms

$(2ab^2 + 4abc – 6ab) \div 2ab = (b + 2c – 3)$

(dividing each term by 2ab)

Therefore $2ab^2 + 4abc - 6ab = 2ab(b + 2c - 3)$

Exercise 6.5

Factorize each of the following:

1. (a) $3a + 6b$ (b) $12p + 4$
2. (a) $10x - 15y$ (b) $2p + 6q$
3. (a) $20p - 15q$ (b) $4p + 12pq$
4. (a) $q - p$ (b) $12p^2 - 8pq$
5. (a) $8y^2 - 6x$ (b) $6pq^2 - 9p^2q$
6. (a) $15n - 20mn$ (b) $6xy - 4xz$
7. $3a + 6ab$
8. $4x - 8xy$
9. $5p^2 + 10pq$
10. $-3m - 3n$
11. $2a + 2b$
12. $6m + 15n$
13. $4ab - 6bc$
14. $-9p + 6q$
15. $33p + 9q$
16. $-6x + 4y - 16$
17. $6x^2 + 9y - 12z$
18. $4ab^2 + a^2b + 12bc$
19. $2x^2 + 6xy - 12xz$
20. $9x - 3xy - 6xyz$
21. $-3x + 12y + 9$
22. $a^2 + 2a^3 - 3a^4$
23. $3ad + 12bd - 18dq$
24. $4p + 4r + 4q$

Unit 7 SOCIAL AND COMMERCIAL ARITHMETIC

Taxation

Taxation is the means by which a government of any country raises funds to finance its spending such as defence, health, and education and so on. Taxation is levied annually on all individuals and companies who earn an income either by employment or business or nay other means that generates an income. In Malawi, a government institution called Malawi revenue Authority (MRA) is mandated to collect taxes and review the tax rules whenever it is necessary. There are two major categories of tax namely **direct** and **indirect** tax. Two examples of direct taxes are **income Tax** and **Pay-as-you-earn (PAYE).**

Direct taxes

Income tax

This is a tax payable by individuals, companies, partnerships and sole proprietors assessed on the income generated. The total amount which an individual or a company earns is called **gross income**. This includes salaries, bonuses, commissions, overtime, gratuities, pensions, cash allowances and so on. Before tax is calculated, deductions are made from the gross income and tax is then calculated on the remaining amount. Some examples of such deductions include professional subscriptions as well as individual donations to approved charitable organizations. Such donations must not be less than K250. Some payments by the government are also exempted from tax, the exemptions include:

i. Salary and payments of the president and vice president received from the government in respect to their offices.
ii. Allowances made to the members of the national assembly
iii. Up to K50 000 of any amount paid to an employee on retrenchment or commutation of leave.

The amount on which the tax is calculated is called the **taxable income.**

Taxable income = gross income – allowances deductions, plus any other employment benefit

Tax rates

Individual tax payers are assessed using graduated scale rates. Table 7.1 below shows the income tax rates applicable in Malawi.

Income tax levied on companies, is charged at a constant rate of 30%.

TAXABLE INCOME IN KWACHA	**RATE OF TAX**
ANNUAL INCOME	%
First K240 000	0%
Next K60 000	15%
Excess of K300 000	30%
MONTHLY INCOME	
First K20 000	0%
Next K 5 000	15%
Excess of K25 000	30%
FORTNIGHTLY INCOME	
First K9 230.7 692	0%
Next K2 307.6 923	15%
Excess of K11 538.461	30%
WEEKLY INCOME	
First k4 615.3 846	0%
Next k1 153.8 461	15%
Excess of K5 769.2 307	30%

Table 7.1

Notes

1. Earnings in Malawi Kwacha are either yearly, monthly, fortnightly or weekly, hence PAYE can be paid accordingly.
2. Any other allowance (income) i.e. house allowance, is grossed up (added) with the basic pay then taxed using the above rates.
3. Income from non-local sources is not taxed.

PAYE

This tax is deducted by the employers from their employees' salaries every month fortnightly or weekly. This money is then remitted to the Malawi Revenue Authority. It therefore forms another category of direct tax. What the employee receives after PAYE is deducted is called **net salary.**

Example 7.1

Mrs. Kamageni earns a salary of K26 000 per month. How much tax does she pay?

Solution

From PAYE Tax Table

Tax on first K 20 000 at 0% = 0

Tax on next K5 000 at 15% = K 750

Tax on remaining K (2 6000 – 2 5000) =1 000 × 30% =K 300

Therefore, Tax K 1 050

Example 7.2

A company secretary earns a monthly basic salary of K 300, 000, K 20 000 for a gardener allowance, K80 000 entertainment allowance K20 000 for security. He also receives K100 000 disturbance allowances for working away from home. Given that the disturbance allowance is exempted from tax. Calculate the secretary's tax for the month

Solution

Basic salary	*K300 000*
Gardener allowance	*K 20 000*
Entertainment allowance	*K80 000*
Disturbance allowance	*K100 000 +*
Security allowance	*K 20 000*
Gross income	*K 520 000*

Since disturbance allowance is not taxable.

Taxable = gross income – allowance deductions

= K520 000 – K100 000

= 420 000

Taxable income = K420 000

First K20 000 = Free of tax

Next K5 000 = 5 000 × 15%

= K750

In excess of K25 000 = 420 000 – 25 000

= K 395 000

Tax on K395 000 = $\frac{395\ 000 \times 30}{100}$

= K 118 500

Total tax paid = K 118 500 + K 750

= K119 250

The secretary's tax paid K 119 250

Exercise 7.1

1. Find the tax due by the following:
    a) A man with an income of K25 000 per month.
    b) A person who earns K36 500 per month
    c) A worker who earns K 3 500 per week
    d) A person on contract earning K 50 000 fortnightly.

2. A business lady earns k36 000 per month. She pays her tax monthly and repays a loan at the rate of K 15 000 per month. How much is she left with?

3. Mrs. Malavi has a monthly salary of K30 400 and gets a house allowance of K 1 700 and bonus of K2 400 monthly. How much tax does he pay?

4. Mr. Jones works with an international organization and is paid in US dollars. Annually, he is paid US$ 100 000 as salary. US$ 20 000 bonus, US$ 10 000 living allowance, US$ 12 000 house allowance, US$ 5 000 interest income from non-local sources. How much tax does he pay annually?
5. Masamba pays K 22 950 as tax from his monthly salary. How much is he paid as salary?

6. Find the tax due by:
    a) A man who earns K50 000 per month
    b) A lady whose income is K73 000 per month
7. At the beginning of year 2009, the consumer price index was calculated to be 250.5. find the consumer price index at the beginning of year 2010 given that the rate of inflation for that year was 45%

Indirect taxes

Indirect taxes are levied on goods and services. Examples of such taxes include: value added tax (VAT), excise tax, stamp duty, withholding tax etc. Other sources of taxable income include loyalties, dividends, interest and professional fees.

The details concerning rates of these taxes are outlined in the country's tax guidelines.

Customs duty: This is a tax levied on goods and services. This tax is collected by the customs authority and comprise of import duty, import excise duty, VAT other assessed taxes etc.

Custom duty is payable on imports of foreign goods. There are guidelines on the criteria of assessing tax. Just like any other tax, this tax is used to raise revenue for the government expenditure and also as a means to restrict/reduce imports to protect local industries.

Custom duty is charged depending on the value of the goods, weight, dimensions and source of those goods. In case of cars and machinery, tax is charged depending on the size of the engine. This tax is a certain percentage of cost of goods.

Excise duty: It is a tax charged on specific goods produced or consumed within the country. Petrol and intoxicating liquor and tobacco, perfumes etc. are some examples of goods subjected to excise duty. **Assessed taxes** are duties not strictly in the category of excise duty but are classified under that heading because they are levies on particular services within a country e.g. motor vehicles licenses, trade licenses etc. The rates and the amounts to be paid are determined by the tax authority.

Example 7.3

A car worth K 900 000 was subjected to the import duty (Custom duty) at a rate of 40% followed by a VAT at 16.5%. Calculate the total amount of tax charged.

Solution

Custom duty = $\frac{40 \times 900\ 000}{100}$

= K 360 000

New value of the car is

= 900 000 + 360 000

= K 1 260 000

VAT = $\frac{1\ 260\ 000 \times 16.5}{100}$

= 207,900

Total cost of the car = 1 260 000 + 207 900

= K 1 467, 900

Total tax = 1 467 900 – 900 000

= 567 900

Total tax can also be obtained by adding the two taxes i.e. Customs duty + VAT

= 360, 000 + 207 900

= K 567 900

Withholding tax

This is a type of tax deducted from certain specified payments. The deduction is done by the person doing the payment and then remitted to the commissioner of income tax. Currently, this tax is charged on dividends, interests, royalties, technical service fee among others. Tax on dividends is charged at 10%; tax on interest and loyalties at 20% as an advance tax. The tax on royalties, technical service fees paid to non-residents are charged a withholding calculating tax of 15%. More rates of withholding tax are specified according to specific commodities and goods, Table 7.2.

WITHHOLDING TAX RATES	
Nature of payment	**Rate in percentage**
Royalties	20
Rent	15
Payment of any supplies to traders and institutions e.g. food	3
Commission	10
Payment of carriage and haulage	20
Payment to contractors/ subcontractors	4
Payment for public entertainment	20
Payment in excess of K15 000 casual labour	20
Payment of services	20
Payment of tobacco sales	3
Bank interest in excess K 10 000	20
Fees	10

Table 7.2

VAT (Value Added Tax)

This is an indirect tax on goods calculated by adding a percentage to the value added to a product at each stage of production. The whole cost of the tax is eventually passed on to the consumer of the finished products. Currently the standard rate of VAT in Malawi is 16.5%. VAT is payable by those who supply taxable goods and services and those who make a turnover of taxable goods of K 10 million or more per annum. VAT is also charged on some imported goods and services as may be guided by the tax authority. The importer is mandated to change VAT and then remit it to the tax department.

Example 7.4

A manufacturer produces a product at a cost of K20 000 and sold it at K 28 000 to a wholesaler. Calculate the value of VAT that he paid to the tax authority.

Solution

Cost of manufacturing and materials

= K 20, 000

Selling price of the product

= K 28 000

Value added = K 28 000 – 20 000

= K 8 000

Therefore VAT is charged on K 8 000 at the rate of 16.5%

Therefore, VAT = $\frac{8\,0\cancel{00} \times 16.5}{1\cancel{00}}$

= 80 × 16.5

= K 1 320

Example 7.5

An author earned a royalty of K980 250 before an advance tax was deducted. If the tax was charged at a rate of 20%,

Calculate:

- *(a) How much tax he was charged*
- *(b) His earning after the tax*

Solution

(a) Advance tax means withholding tax.

Therefore the rate of 20% means for every K100, the tax is K20

Tax on K 100 = 20

Tax on K 980 250 = $\frac{980\,250 \times 20}{100}$

= K 196 050

(b) *Earning before tax = K 980 250*

Tax = K 196 050

Earning after tax = 980 250 – 196 050

= K 784 200

Exercise 7.2

1. A machine is sold at K8 00 plus VAT. If VAT is charged at 16.5%, how much would one pay for the machine?
2. The price of a three seat set inclusive of VAT is K58 250. What is the price of those seats exclusive of VAT?
3. A washing machine costs K48 000 exclusive of VAT. Given that VAT is charged at 16.5%, how much will the machine cost including VAT?
4. If the value added tax is charged at 16.5%, find the tax payable if a commodity is priced at K 5 000 inclusive of VAT.
5. A businessman imported a car valued at K 1 000 000. An import duty was levied at 50% and VAT on the value after duty at 16.5%. He then sold the car making a profit of 30%. How much money did he receive?
6. A machine costs K 699 000 with a VAT at 16.5% inclusive. Calculate the cost of the machine before VAT was added.
7. A town council imposes different taxes on different fixed assets as follows:
   Commercial property 15% per year
   Residential property 5% per year
   Industrial property 10% per year
   An investor owns a residential building on a plot all valued at K 8 000 000 an industrial plot worth K 7 500 000 and a commercial premises worth K12 500 000. How much tax does the investor pay annually?
8. A company declares final dividend of K0.32 per share. Given that a shareholder holds 18 000 shares, calculate:
   a) His gross dividend
   b) The withholding tax charged on the dividend at 10%.
   c) The net amount due.
9. A shareholder holds a total of 30 000 shares. In a certain year, he received a net dividend of K13 500. Given that dividends are taxed at 10%, Calculate:
   a) The shareholder’s net dividend
   b) The dividend declared by the company per share

c) The amount of money he paid in withholding tax.

10. An author paid a withholding tax of K108 000 on his royalty. Calculate
    i. Her gross royalty
    ii. Her net royalty
11. An author earned a royalty of K 1 176 300 after tax. Calculate:
    i. The gross royalty
    ii. Tax charged on the royalty

Insurance

A home is probably one of the most expensive purchases a person can make. To enable a person to protect himself/herself against risks of heavy loses, the owner makes annual payments to an insurance company so that the company can compensate the owner should the risk occur. The owner is paid a sum of money at which the damage or loss is assessed. The sum to cover the risk is paid annually and is called the **premium**. The contract signed between the company and the insured is called the **insurance policy**. The premium is calculated based on the value insured and the likelihood of the risk occurring.

Some forms of insurance are compulsory. For example, the owner of a car must insure against **third party risks** so that anyone who is injured, or his or her car damaged as result of his negligent driving may receive compensation.

Another principal form of insurance is life insurance. This may be classified as:

i. A **whole life policy**, which in return for an annual premium secures a lump sum payment at death. This form of insurance benefits the dependents of the policy holder.
ii. An **endowment policy**, which in return for an annual premium for a fixed period of time secures a lump sum payment at a definite age e.g. at retirement. In this case, the policy holder is the beneficiary.

However, if the insured person dies before the policy matures, the whole or a fraction of the amount paid in premiums is returned to the insured person's estate.

Benefits of insurance

In return for the premiums that the insured person pays during the period of insurance, there are certain benefits that he enjoys. For example, when a car is, insured against third party risks or even comprehensively, there is possible compensation in case of an accident. Similarly, in life assurance, the sum assured accrues a bonus annually which is paid together with the assured amount when the policy matures.

In asset management, the invested funds accrue interest which is payable annually. The investor has a choice to receive his interest at the end of the year or to re-invest it to earn more.

There are many other schemes that offer specific benefits depending on the individual companies and the products that they offer.

Points to note

1. It is illegal to knowingly insure a property for more than its value.
2. It is illegal to insure against a risk in which the insurer has no financial interest.

Other forms of insurance include: medical, burglary and loss of household property and so on.

Below are some examples to illustrate some of these forms of insurance.

Example 7.6

A businessman insures goods valued at K42 625 000 at the rate of K13 640 for every K341 000 worth of goods. What annual premium does he pay?

Solution

Premium for K341 000 is K13 640

Premium for K42 625 000

$$= \frac{42\,625\,000}{341\,000} \times 13\,640$$

$$= 1\,705\,000$$

He pays an annual premium of K1 705 000

Example 7.7

a man took a life insurance policy for K272 800 on his 25th birthday. He paid a premium of K 3 574 yearly. He died at the age of 65 years. The company paid into his estate the whole amount of his insurance policy. How much more than his premium contribution did the company pay into the man's estate?

Solution

From age 25 years to age 65; 40 premiums were paid. Amount paid in premiums:

574 × 40 = 142 960

Policy value was K272 800

Difference: 272 800 – 142 960 = 129 840

The company paid K129 840 above his contribution premiums.

Example 7.8

The annual insurance on a car is K18 000. The owner is allowed a 20% non-claim bonus because he had not made any claims during the previous year. How much premium did he pay for the year.

Solution

No-claim bonus = $\frac{20}{100}$ *×18 000*

The owner will pay in premium less the no-claim bonus.

Insurance premium = 18 000 – 3 600

= K14 400

Exercise 7.3

1. Goods in transit worth K95 480 were insured against damage at K682 in every K27 200 worth of goods. Find the premium for insuring the goods.
2. John takes out a 20- year K836 000 endowment policy and pays K7 160 in K167 200 of the coverage. Find the total premiums.
3. Anne's policy is K527 per year for each K167 200 of goods coverage. If the goods are estimated to have a replacement value of K11 787 600, find the premium.

4. A property sells for K81 092 000. The new owner has taken a policy coverage of K41 800 000. What would be your advice to him regarding the coverage?
5. A business premises was insured for K2 450 000. The insurance company promised to pay 50% of the sum insured in case the premises got destroyed. How much money will the company pay if the premises is destroyed?
6. A businesswoman insures her business at 14%. If the business was worth K1 250 000. What annual premium does she pay?
7. A man takes out a whole life insurance policy for K48 000 000 on his 25th birthday by paying a premium of K628 800 a year. If he dies at the age of 64 years and 8 months, how much more does the company pay his estate than he has paid in premiums to the company?
8. A car owner pays an annual premium of K9 000. He is allowed a no-claim bonus of $25\frac{1}{2}\%$. How much does he pay in premiums that year?
9. A home valued at K30 000 000 is to be insured. The insurance company quotes a premium of K300 per K100 000. How much will the owner pay in premiums per year?

Simple budgets

Consider a family with a monthly total income of K78 370 it may spend as follows:

Food	K21 600
Clothing	K3 200
Rent	K10 800
Medical expenses	K 4 300
School fees	K7 600
Taxes	K17 250
Utility bills	K4 340
(Water and electricity)	
Leisure	K3 220

Savings	K6 000

Note:

The amount spent on an item may vary from month to month.

A budget is a list of the total income and the planned expenses. Household budgeting involves identifying the sources of income and planning for the expenditure. It ensures appropriate spending and saving.

Some families do not make a budget and are likely to overspend on unimportant items or underspend on important items. It is therefore, necessary to budget on a weekly or monthly basis so as to ensure the money is reasonably spent.

Schools, organizations, institutions and the government have also to prepare annual budgets so as to use the available resources as effectively as possible.

Example 7.9

Lorna has k 4 800 to spend on provisions. She makes the following list.

Rice	*K1 500*
Maize meal	*K1 200*
Bananas	*K 120*
Soap	*K 125*
Cooking oil	*K 590*
Tomatoes	*K 150*
Paraffin	*K 1 030*
Salt	*K 85*

What percentage is spent on (i) rice (ii) maize meal (iii) other non-food items.

Solution

i. *Fraction of money spent on rice:*
$= \frac{1\ 500}{4\ 800} = \frac{5 \times 100}{16} = 31.25\%$

ii. *Fraction spent on maize meal:*

$\frac{1\,200}{4\,800} = \frac{1 \times 100}{4} = 25\%$

iii. *Non-food items are soap and paraffin*

Therefore, Fraction spent $= \frac{1\,155}{4\,800} = 24.06\%$

Exercise 7.4

1. June buys 3 packets of milk @ K103 each, 2 loaves of bread @ K131 each, 5 eggs at K33 each and a packet of tea leaves at K140. How much change would she get if she had 1000 kwacha note?
2. Find the total cost of the following:
   3 trays of eggs at K990 per tray, 2 kg of meat at K608 per kg, 4 litres of kerosene at K120 per litre.
3. Nahimba bought 12 oranges at K40 each, 6 bunches of bananas at K120 per bunch, 4 pineapples at K400 each.
   a) How much did she pay?
   b) What percentage of her total was spent on bananas?
4. Solomon deposited 37% of his savings in a cooperative society, 43% in his business and the rest he deposited in a bank. How much did he invest in each of the three ways if his total savings was K240 00?
5. Nyondo's one month's salary was K43 200. Of this, he spent 50% on food, 8% on rent, 15% on his car repairs, 12% on fuel and the remainder on miscellaneous items. How much money did he spend on food, rent and car repairs?
6. Suzzy earns K36 000. She receives a house allowance of K12 000. Her January expenses were as follows:

Rent	K10 000
Hair styling	K 2 400
Fee for her brother	K15 000
Travel	K 2 000
Clothes	K 2 960
Food	K 6 960
Repayment of loan	K 4 000
Entertainment	K 2 000

   a) If she saved the remainder, how much did she save?
   b) What percentage did she spend on her brother?

7. Msowoya’s monthly salary is K43 200. The food bill was 55% of his salary, he spent 14% on school fees, car consumed 9% of his salary and car fuel cost 7%, while the rent took K3 440. How much did he spend on:
   a) Food b) Fuel c) Scholl fees?
8. Lucy earns K40 000 per month and saves 10% of this. How long will it take her to save K86 400?
9. Noel earns K800 000 annually. He saved K50 000 in a year. What percentage does he need to save each month?
10. Judy saves K4 000 per month. 20% of her salary is deducted for tax. Of the remainder, 10% is spent on rent and 80% on other expenses. How much does she earn in a month?

Bills

Water bills

In most towns and centres, water is piped to people’s homes, industries, institutions and commercial premises. Water connection and distribution is managed by water boards. It is sold in cubic metres (m^3) at different rates depending on the user, such as residential, institutional, commercial and industrial and consumption.

Table 7.3 (a) and (b) shows water tariffs for Lilongwe and Blantyre.

Lilongwe Central region Water Board (Effective January 1, 2011)	
Category	Tariff
Residential	
First 4 cubic metres (minimum charge)	K 360.24
5 to 30 cubic metres (per cubic metre)	K 85.12
Thereafter (per cubic metre)	K 92.72
Institutional	
First 4 cubic metres (minimum charge)	K 1266.16
5 to 30 cubic metres (per cubic metre)	K 148.96
Thereafter (per cubic metre)	K 177.84
Commercial and industrial	

First 4 cubic metres (minimum charge)	K 1267.16
5 to 30 cubic metres (per cubic metre)	K 148.96
Thereafter (per cubic metre)	K 177.84

Table 7.3(a)

<table>
<tr><td colspan="2">Blantyre
Blantyre Water Board
(Prices Effective From January 1, 2011)</td></tr>
<tr><td>Category</td><td>Tariff</td></tr>
<tr><td colspan="2">Residential</td></tr>
<tr><td>First 5 cubic metres or part thereof per month</td><td>K 456</td></tr>
<tr><td>Exceeding 5 cubic metres upto 10 cubic metres, rate applicable from zero upto 10 cubic meters (per cubic metre)</td><td>K 101.84</td></tr>
<tr><td>Exceeding 10 cubic metres upto 40 cubic metres(per cubic metre)</td><td>K 121.6</td></tr>
<tr><td>Exceeding 40 cubic metres (per cubic metre)</td><td>K 130.12</td></tr>
<tr><td colspan="2">Institutional</td></tr>
<tr><td>First 10 cubic metres or part thereof per month</td><td>K 1368</td></tr>
<tr><td>Exceeding 10 cubic metres (per cubic metre)</td><td>K 142.88</td></tr>
<tr><td colspan="2">Commercial and industrial</td></tr>
<tr><td>First 10 cubic metres or part thereof per month</td><td>K 1368</td></tr>
<tr><td>Exceeding 10 cubic metres (per cubic metre)</td><td>K 142.88</td></tr>
</table>

Table 7.3(b)

Example 7.10

Useni's family used 25 m^3 of water in January 2011. How much were they supposed to pay in Kwacha if they were living in Lilongwe? (Use table 7.3)

Solution

First 4 m^3 cost 360 = 360.24

Next 26 m^3 cost 26 × 85.12 = 2213.12

Next 5 m^3 × cost 5 × 92.72 = 463.60

Total = 3036.96

Electricity bills

Electricity distribution is managed by Electricity Supply Corporation of Malawi (ESCOM). The company sells electricity in units called kilowatt-hour (Kwh). Consumer usage differs depending on the appliances they use.

Table 7.4 shows charges applied to a domestic consumer.

The bills are calculated by charging a fixed charge plus the consumption charge per Kwh.

Domestic Tariff Consumer	**(k)**
Fixed charge per month	124.71
Charge for each unit consumed of the first 30 units per month	2.67
Charge for each unit consumed in excess of 30 units and less than 750 units per month	3.91
Charge for each unit consumed in excess of 750 units per month	5.55

Table 7.4

Example 7.11

A family consumed 55 Kwh of electricity in a month. How much were they supposed to pay?

Solution

Fixed monthly charge = K 124.71

First 30 Kwh @ K2.6708 = K 80.12

Remaining 25 Kwh @ K3.9146 = K 97.87

= K 302.70

= K 303

Exercise 7.5

1. James lives in a residential house in Lilongwe and consumes an average of 30 $m^3$of water and 40 Kwh of electricity. How much does he pay for each in Kwacha?
2. Susan works in an institution in Blantyre. Water consumption in the institution in February 2011 was 105 m^3. How much did the institution pay?
3. Dr. Usi who lives in Blantyre consumed 38 $m^3$of water in his residence in March 2011. How much did he pay for it?
4. Sanudi has a commercial property in Lilongwe. In February 2011, 154 $m^3$of water was used. How much was paid for it in Kwacha?
5. School L in Lilongwe consumed 25 m^3 of water while School B in Blantyre consumed 30 m^3in the same period of time. Which of the two schools paid more and by how much?
6. Find the monthly electricity bills for each of the following families, given that they used the units indicated.
    a) Sangala – 450 Kwh
    b) Mponda – 725 Kwh
    c) Chiukepo – 827 Kwh
    d) Ng’ambi – 992 Kwh
    e) Kondowe – 324 Kwh
    f) Sani – 1 115 Kwh

Inflation

Suppose in 2008, the price of a bar of soap was K156. In 2009, the price was K 171 and in 2010 it was K 188.

We notice that the price of the soap increases over time. This change can be represented as a percentage by finding the percentage increase in each year for the price of the bar soap.

Thus, in 2009 % increase

= $\frac{(171-156)}{156}$ × 100% = 9.6%

In 2010 % increase = $\frac{(188-171)}{171}$ × 1005 = 9.9%

We can hence describe the price of the bar soap as increasing by 9.6% in 2009 and 9.9% in 2010.

The continued general increase in prices of goods and services over time is referred to as **inflation**. It is measured as an annual percentage increase. If the general increase of prices is high, then the rate of inflation will be high.

Similar rise in prices of goods and services over a period of time is called **inflation**. This means each unit of currency buys fewer goods and services. One general method of calculating the inflation rate is by finding the percentage rate change of consumer price index over a specific period of time. The consumer price index is estimated using a sample of goods and services from different stores. For example, we can use a sample of 3 000 items from 1 500 stores and 5 000 rental units to find the total cost of the items and then find the average cost. The result is known as the **consumer price index.**

Example 7.12

In January 2010, the consumer price index in a country was 305.816. In January the following year, the consumer price index in the same country was 476.081. Calculate the rate of inflation of the year 2010.

Solution

Consumer price index for 2010 = 305.816

Consumer price index for 2011 = 476.081

The change in CPI

= 476.081 – 305.816

$$= 170.265$$

Thus, the percentage rate of inflation

$$= \frac{The\ change\ in\ CPI}{CPI\ in\ 2010} \times 100\%$$

$$= \frac{(170.265}{305.816} \times 100$$

$$= 55.67563\%$$

$$= 55.68\%$$

The inflation rate for the CPI in this one year is 55.68%.

Note:

This means that generally the level of consumer prices in this country rose by approximately 56% within a period of one year. This means that the consumers have to pay more for fewer goods and services.

Devaluation

When importing or exporting commodities, transactions are made in US dollars. This means that the Kwacha will be exchanged for the dollar.

Suppose that at a time when the US dollar exchanges at US$ = K152, the demand for the dollar goes up and there are not enough dollars in the country to meet the demand. The Reserve Bank of Malawi may decide to devalue the Kwacha by 10% relative to the US dollar.

Hence, US$ = K167.20

The situation whereby the government decides to decrease or reduce the value when it is exchanged for the currency of another country is known as **devaluation**. Devaluation is done by the government through the Reserve Bank of Malawi.

Now consider this;

On May, 7th 2012, The Reserve Bank of Malawi devalued the Malawian Kwacha exchange rate from K168 to K250 per dollar. By what percentage did the Malawian kwacha drop?

Before: $ 1 = K168

Now: $ 1 = K250

The drop in value of Kwacha

= K250 – 168

= K82

Therefore, percentage drop = $\frac{82}{168} \times 100$

= 48.80952

= 48.81%

This means the buying power of the Kwacha has dropped by almost 50%

Example 7.13

On a certain day in April 2012 a visiting American exchanged 300 US dollar for Malawian Kwacha. If the exchange rate was K168 to a US, find:

a) *How much in Kwacha he received.*
b) *How much in Kwacha the same US dollar 300 would buy today given that the exchange rate is US$ 1.00 = K480. Find the percentage change.*

Solution

a) *In April the exchange rate was K168 to a dollar*

Thus, for every $1, he received K168

Therefore for $300 buy 168 × 300

= K 50 400

b) *Today $ 1 = K480 (the new exchange rate)*

Therefore, $300 = 480 × 300 = K 144 000

The change = K 144 000 – 50 400

= K 93 600

Percentage change $= \frac{93\ 600}{50\ 400} \times 100$

$= 185.71\%$

Appreciation and depreciation

When the value of an asset increases with time, we say it appreciates in value. Land is an example of an asset that **appreciates in value**. If an asset decreases in value with time, we say that its value **depreciates**. A car is a good example of an asset whose value depreciates with time.

Given the rate of appreciation or depreciation, the value of such assets may be calculated.

Example 7.14

A car K1 200 000when new. If its value depreciated at the rate of 10% p.a, calculate its value after 3 years.

Solution

When new, the value of the car was K1 200 000. At the end of the first year, the value of the car would be $\frac{90}{100}$ *of K1 200 000.*

= K 1 080 000

At the end of the 3rd year, the value of the car was $972000 \times \frac{90}{100} = 874\ 800$

= K874 000

After 3 years, the value of the car was K874 800.

Example 7.15

A residential plot in town appreciates in value at the rate of 15% pa. Its current value is K 3 500 000. Calculate its value after 2 years (answer to the nearest K1 000)

Solution

Current value = K 3 500 000

Value at end of first year

$$= K\ 3\ 500\ 000 + \frac{15}{100}\ of\ 3\ 500\ 000$$

$$= K\ 3\ 500\ 00 + \frac{15}{100} \times 3\ 500\ 000$$

$$= K\ 3\ 500\ 000 + 525\ 000$$

$$= K\ 4\ 025\ 000$$

Value at end of 2nd year

$$= K\ 4\ 025\ 000 + \frac{15}{100}\ \%\ of\ 4\ 025\ 000$$

$$= K\ 4\ 025\ 000 + \frac{15}{100} \times 4\ 025\ 000$$

$$= K\ 4\ 025\ 000 + 603\ 750$$

$$= K\ 4\ 628\ 750$$

$$= K\ 4\ 629\ 000$$

Exercise 7.6

1. A painting increases in value by $12\frac{1}{2}$% each year. If its value now is K 30 000, find its value after 2 years. (To the nearest whole number).
2. The value of a car depreciates by 8% each year. A man pays K 900 000 for his second hand car. Find its value after 2 years.
3. A diamond ring is priced at K 4 000. It appreciates by 10% each year. What will its value be 3 years after purchase?
4. The initial cost of a sewing machine was K90 000. It's depreciated by 8%. Find the value of the machine after the first year.
5. A town residential plot was sold at K 5 000 000. Thereafter, its value appreciated at the rate of 2% pa. Find the value of the property at the end of the year.
6. An amount of K1 500 000 is invested at a rate of 4% p.a. How much will it be at the end of the year?

7. The present population of a country is 24.65 million. If the population increases at a rate of 3% p.a, what will be the end of the year?
8. At the beginning of the year 2009, the consumer price index was calculated to be 250.5. Find the consumer price index at the beginning of the year 2010 given that the rate of inflation for that year was 45%.

Investment options

To invest means to use money to obtain an income, or capital gain for the future. We can deposit money in a bank, in a building society or a cooperative society to earn an income in form of interest. Government stocks and bonds provide some limited incomes, while shares and unit trusts provide an opportunity for the larger capital gains and income with an element of speculation. Investment also includes the purchase of assets whose value is expected to appreciate with tome for future gains. Such purchases include buying and selling land.

In this section, we are going to discuss a few investment options.

Interest

If you put money in a bank or in any other financial institution, you are lending money to the institution and they will pay you a percentage of that money for being allowed to make use of your money. This payment is called **interest**. Similarly, if you will be required to pay a fee for being allowed to use the money. Any money lent or borrowed is called the **principal** and the charge made for its use is called **interest**, which depends on the amount borrowed and the length of the period of borrowing. Interest paid at fixed intervals, usually annually or half yearly or even quarterly on the principal at a given rate (R) only is called **simple interest**. At the end of the interval (T), the principal (P) together with the interest is called the **amount** (A).

Thus, $A = P + I$, $I = P \times T \times \frac{R}{100}$

Example 7.16

a) A man invested K30 000 at 15% simple interest per annum. How long will it take for the investment to reach K57 00?

b) What principal must be invested to earn an interest of K1 920 if the money is invested for 5 years at 6% per annum?

Solution

a) *Principal = K 30 000*
Amount = K 57 000

Therefore, Interest = K57 000 – K30 000

= K27 000

Thus, I = K27 000, P = 30, 000, R = 15%

And I = $\frac{P\times T \times R}{100}$

27 000 = 30 000 × T × $\frac{15}{100}$

$\frac{27\ 000}{30\ 000}$ *= T ×* $\frac{15}{100}$

$\frac{27}{30}\times\frac{100}{15}$ *= T ×* $\frac{15}{100}\times\frac{100}{15}$

$\frac{90}{15}$ *= T*

6 = T

It takes 6 years.

b) *We are given that I = 1920 R = 6% T = 5 years*

We know that I = $\frac{PTR}{100}$

1 920 = $\frac{P\times 5\times 6}{100}$

1 920 = $\frac{30\times P}{100}$

$\frac{1\ 920\times 100}{30}$ *= P … multiply both sides by* $\frac{100}{3}$

Therefore, P = K6 400

Hence, the principal required is K6 400.

Exercise 7.7

For questions 1 to 2, find the time of investment

1. K20 000 is the interest on K100 000 at a simple interest of 5% p.a.
2. K40 000 earns an interest of K9 600 when invested at simple interest of 8% p.a

For question 3 and 4 =, find the percentage rate per annum simple interest.

3. K8 400 is the interest on K24 000 invested for 5 years.
4. K24 000 is the interest on K60 000 invested for 4 years.
5. K10, 000 was invested for 7 years at 10% p.a. simple interest. How much will the investment be worth after this time?
6. If you invested K40 000 at 12% p.a. simple interest to get an investment worth K59 200, how long would you have invested the money?
7. Calculate the compound interest earned on K6 000 invested for 2 years at 5% p.a.

Stocks and shares

Stocks represent fixed-interest loans made to the national government, local authorities and companies etc.

Shares represent equal amount of capital subscribed to a company in return for membership rights. Ordinary shareholders are the last to receive their dividends which fluctuate according to the level of profits. They usually have voting rights in the company. Shares in public companies can be bought and sold in a stock exchange market.

Stocks

If a Government does not raise enough money through taxes, it can borrow from the public by issuing **stock**, bearing a fixed rate of interest for a fixed period of time. Other institutions may borrow using the same method. The tender of say, K200 is said to hold K200 stock. A stock holder is entitled to:

I. Receive income at the stipulated rate of interest
II. Repayment of principal at the date of maturity

It is possible for a stock holder to **sell his stock** to another person. The cash value of the stock varies from time to time.

If the market value of say K100 stock is K100, the stock is said to be at **par**. If the market value is below K100, e.g. the stock is at a discount of K8, and if is above K100 e.g. K105 it is at a premium of K5

The income from stock is based on the **nominal value** of the stock. Note also stock of 5% at 95 means:

I. K100 stock costs K95 (market value).
II. K100 (nominal value) gives a yearly income of K5. The buyer invests K95 to buy K100 stock.

Example 7.17

Find the cost K15 600 stock of 9% at K95 and the yearly income from the investment.

Solution

K 100 stock costs K95

Number of shares at nominal value = $\frac{15\,600}{100}$

$= 156$

Cost of 156 shares at market value = 156 × 95

= k14 820

Income is calculated as 9% of the nominal value.

Therefore, Income $= \frac{9}{100}$ *of nominal value*

$= \frac{9}{100} \times 15\,600$

= K 1 404

Example 7.18

How much stock of 8% at K 105 can be bought for K2 995? Find also the yearly income derived from it.

Solution

K105 buys K100 stock

K29925 buys $\frac{29\,925}{105}$ *× 100 = K28 500 stock*

K100 stock gives an income of K8%.

K28 500 gives an income of $\frac{28\,500}{100}$ *× 8*

= K2 280

K105 gives an income of K8

K29 925 gives an income of $\frac{29\,925}{105}$ *× 8*

= K2 280

Shares

Just as we can invest money in banks and other financial institutions, to earn interest, we can also invest in shares. A company can raise its capital by issuing shares of definite amount or value such as K5, K10, K20, and K100 and so on.

This means a company's capital of K200 000 may consist of K40 000 Kwacha shares, or 20 000 Kwacha shares or 100 000 Kwacha shares etc.

If a company is said to have 5% of K10 shares standing at K12, means the **market value** of K10 shares is K12. K10 is called the **nominal value** and K12 the market value or cash value, and the dividend is 5% of the **nominal value** of the share.

K10 share standing at K12 is said to be at K2 **premium.** K10 shares standing at K9 are said to be at K1 **discount.**

K10 shares standing at K10 are said to be at par.

The yield of an investment = $\frac{Dividend}{Investment}$ × 100

Example 7.19

If K100 shares in a company are sold at K128, find:

a) *How many shares can be bought for K19 200?*
b) *The dividend at 8% on them.*
c) *The yield on the investment.*

Solution

a) *The cash price of 1 share = K128*

Amount of money available to invest is K19 200

Therefore, number of shares = $\frac{19\,200}{128}$

= 150 shares

Therefore, K19 200 will buy 150 shares.

b) *Number of shares = 150*

Nominal value of each share = K100

Therefore, nominal value of 150 shares = 150 × 100

= K15 000

Dividend at 8% is calculated on the nominal value

Therefore, dividend = $\frac{8}{100}$ *× 15 000*

= K 1 200

c) *Amount invested = K19 200*

Amount earned = K1 200

The yield on K100 = $\frac{1\,200}{19\,200}$ *× 100*

= 6.35%

The yield of the investment is $6\frac{1}{4}\%$

Note: Expressing the dividend as a percentage of the minimal investment, we obtain the yield.

Treasury Bills (T-Bills)

Treasury Bills are short term investments which mature within a year from the date of issue. The purpose of T-bills is to raise money from the public to finance government projects and services to the citizens.

There are two methods of buying T-Bills, either through competitive bidding process or through non-competitive bidding. Majority of individuals use non-competitive bidding methods where they accept whatever interest rate is decided at the auction. The calculation of the interest of a T-Bill is based on its price and time to maturity.

Treasury Bonds

A bond is a fixed income security which pays the holder a regular fixed interest for a fixed period of time. Treasury bonds are a long term investment whose maturity is between 20 and 30 years. Treasury bond investors lend the government money for a fixed period of time at a fixed interest rate. A bond investment earns interest on the **principal**, also known as **face value** at agreed intervals, e.g. twice a year. At maturity, the investor receives back his principal.

Capital Assets and Capital Gain

A Capital Asset is a fixed asset such as land, building, expensive equipment, furniture and fixtures etc. which have longer useful life than 1 year and are not meant for sale during the normal course of business. Capital assets could belong to individuals or to firms or companies.

A capital gain is the profit that results from a disposition of a capital asset where the amount realized exceeds the purchase price. The gain is not realized until the asset is sold as it is the difference between a higher selling price and a lower purchase price. A capital loss can also be incurred when there is a decrease in the capital asset value compared to the assets' purchase price. Capital gain is a taxable income.

Exercise 7.8

1. Find the cash value of K15 100 stock at K95.
2. Find the amount of stock which can be bought for K56 000 stock at K98.
3. How much cash will be earned by selling K91 800 stock of 9% at K109?
4. Janet invests K11 440 in 9% stock at K104 and K10 868 in $13\frac{1}{2}$% stock at K143, which was the better investment and by how much?
5. A man invests K4 950 in 9% stock at K99. Find his income.
   He sells out his stock at K102 and buys 10% stock at par. Find his income.
6. Find the yield obtained by investing in 11.25% stock at 90.
7. John invested K1365 in 10% stock at 91 and sells out K1000 of the stock at $92\frac{1}{2}$% and the remainder at 88. How much does he lose or gain in this transaction?
8. Find the cost and the income derived from 88; (K 10) shares paying 16% at a premium of K14.

9. Find the number of shares bought, the income and the yielding obtained from investing:
    a) K10 800 in (K10) shares at 13.50 dividend at 8%
    b) K320 400 in (K) shares at K427.20 dividend at $8\frac{1}{2}$%
10. A man invested K3 300 in $7\frac{1}{2}$% (K20) shares at K2 per premium.
    a) How many shares did he buy and what income does he receive from resale?
    b) At what price must he sell all his shares to gain K375 on the transaction?

11. Grace sells out her K152 000 stock of 6% at K96 and invests the proceeds in shares which pays an annual dividend of K760 per share which increases her income by 25%. What was the price of each share?
12. A shareholder invests K456 000 in K152 shares costing K190 each and giving a dividend of 12% p.a. He also buys stock with a normal value K304 000 at K98. The stock paid 10% per annum dividend. Calculate:
    I. The number of shares bought
    II. His investment in stock

Calculate his yield expressed as a percentage from each investment and identify the better investment.

Unit 8 TRIANGLES AND POLYGONS

Types of triangles

A triangle is a closed plane figure with three straight sides. It has three angles.

There are different kinds of triangles. A **scalene** triangle is one in which there are no equal sides and no equal angles as shown in Fig. 8.1

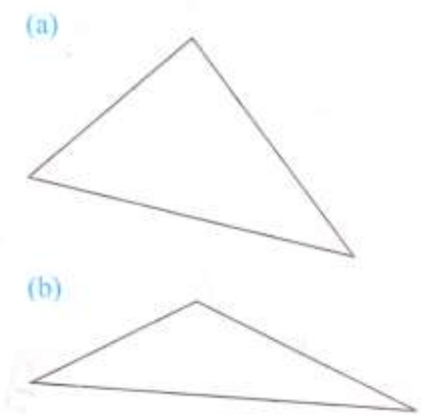


Fig. 8.1

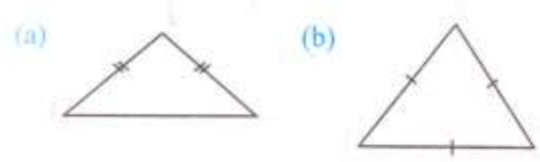
An **acute-triangle** (or **acute-angled triangle**) is one whose interior angles are all acute

(Fig. 8.1 (a)).

An **obtuse-triangle** (or **obtuse-angles triangle**) is one that contains an obtuse interior angle (Fig. 8.1 (b)).

An **isosceles triangle** is one with two equal sides (Fig. 8.2 (a)). The third side is called the **base** and the angle opposite to it the **vertex**. The base angles are equal.

An **equilateral triangle** is one with all the sides equal. (Fig. 8.2 (b)). All its angles are also equal.



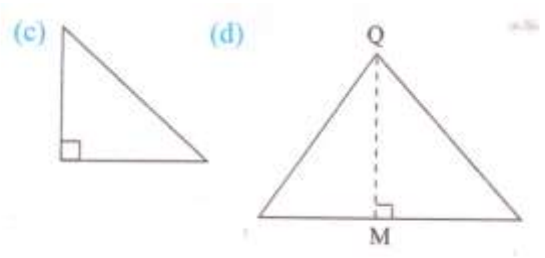


Fig. 8.2

A right triangle or **right-angled triangle** is a triangle with one right angle (Fig. 8.2 (c)). The side opposite the right angle is called the **hypotenuse**.

A triangle which contains no right angles is called an **oblique triangle**. Thus, scalene and acute triangles are both oblique.

A line from the vertex of a triangle perpendicular to the opposite side such as QM in fig. 8.2(d) is known as **altitude** of the triangle. An altitude may be drawn from any vertex to the opposite side.

The symbol Δ (Greek letter 'delta') is used to denote a triangle, e.g. ΔABC means 'triangle ABC'.

Exercise 8.1

1. (a) Draw an acute-angled ΔABC. Measure the angles in the triangle. Find the sum of the angles.
   (b)Draw an obtuse angled ΔPQR. Measure the angles in the triangle. Find the sum of the angles.
   (c) What do you notice about the results in (a) and (b)?

   (d) Draw a triangle ABC (Fig. 8.3). Cut it out along its sides.

(e)Cut off the corners along the indicated lines (Fig. 8.3(a)). Arrange them against a straight edge as shown in Fig. 8.3(b)

(f) What do you notice? What can you say about the sum of the angles of the triangle?

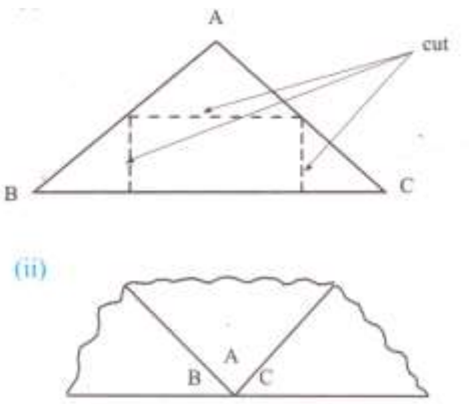


Fig. 8.3

2. Draw any triangle PQR (Fig. 8.4). Cut it out along its sides.
(c) Fold the triangle through angle P such that Q lies on the side QR. Mark the point where the fold meets QR as M.

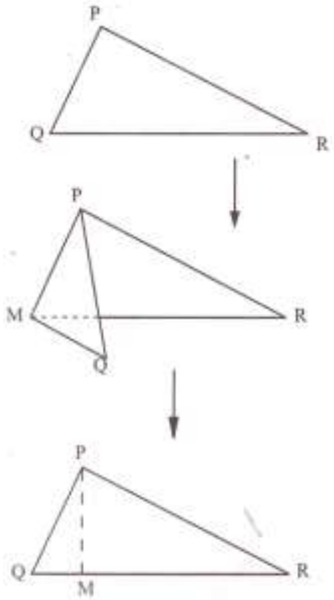


Fig. 8.4

(c) Open up the triangle and measure <PMR.
(d) What type of triangle is △PMR?

Draw a triangle ABC. Produce AB to P, BC to Q and CA to R (See Fig. 8.5).

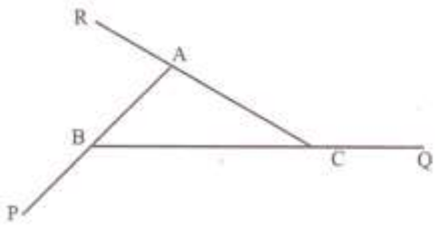


Fig. 8.5

What is <PBC + <QCA + <RAB

3. Copy triangle ABC (Fig.8.6) on a sheet of paper. Carefully cut it out. By folding the triangle so that A lies on C, obtain the line BM. Similarly, by folding the triangle so that B lie on C, obtain line AN.

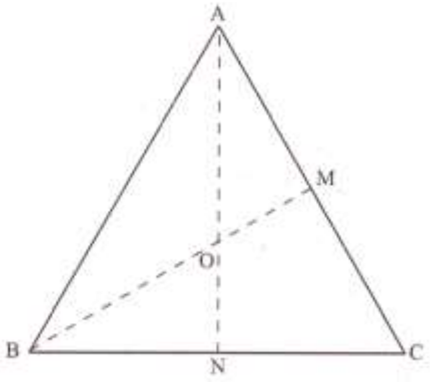


Fig 8.6

a) Measure angles <AMB and <ANC.
b) What is the size of <AOM?
c) Measure: (i) <BAN and <CAN
(ii) <ABM and <CBM
What do you notice?

From exercise 8.1 you should have noticed that following

1. The sum of the interior angles of a triangle is 180˚.
2. If the three side of a triangle are produced in the same sense (i.e. clockwise or anti-clockwise) as shown in Fig. 8.7, the sum of the exterior angles so formed is 360˚.

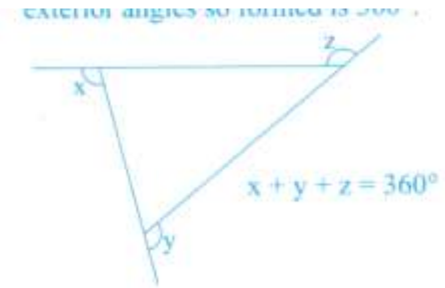


Fig. 8.7

3. In an equilateral triangle,
    i. Any two altitudes intersect at 60˚
    ii. An altitude bisects the angle from which it is drawn.

Angle property of triangles

Activity 8.1

Draw a triangle PQR. Produce PQ to X, QR to Y and RP to Z (See Fig. 8.8)

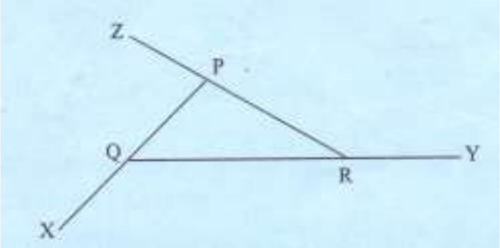


Fig. 8.8

From Activity 8.1 we note the following:

If a side of a triangle is produced, the exterior angle so formed is equal to the sum of the interior opposite angles.

Exercise 8.2

Calculate the angles marked with letters a to n in Fig. 8.9.

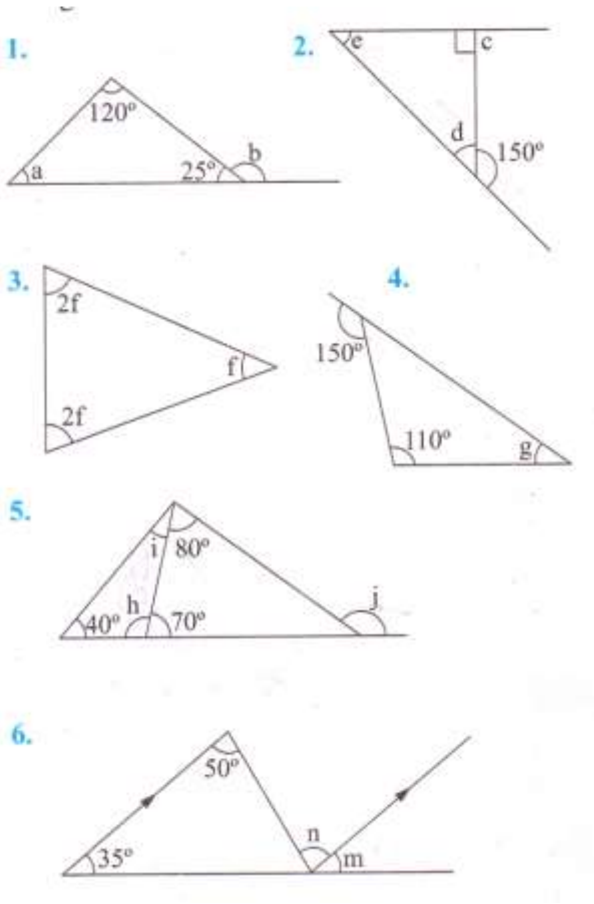
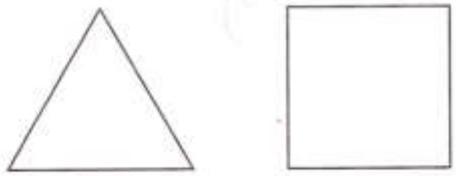


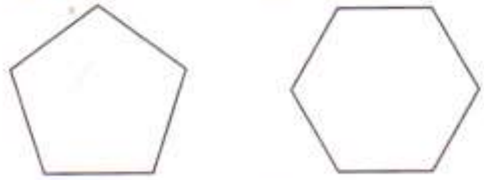
Fig. 8.9

Polygons

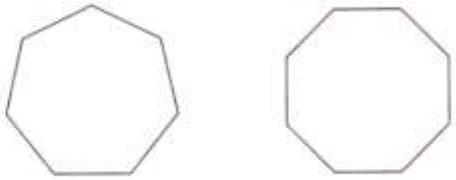
A polygon is a geometric figure that has straight sides. The number of sides determines the name of the polygon. (Fig. 8.10 (a) to (f)).



(a) Triangle (b) Quadrilateral



(c) Pentagon (d) Hexagon



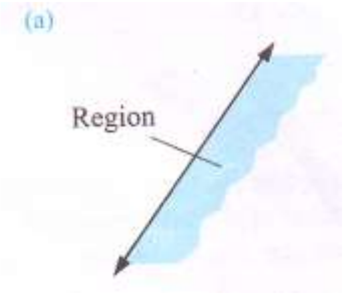
(e) Heptagon (f) Octagon

Fig. 8.10

Note that the number of vertices and the number of sides are equal in each case. Note also that a polygon must be closed and all the sides must be straight line segments.

A region is the space enclosed within the sides of a polygon. However, a region may also be described as the space between two or more intersecting lines, or simply the space on one side of a line or curve. If a region is defined by the polygon enclosing it, then it is said to be a bounded region.

A region may be fully bounded (as in case of polygons) or partially bounded (as in Fig 8.11).



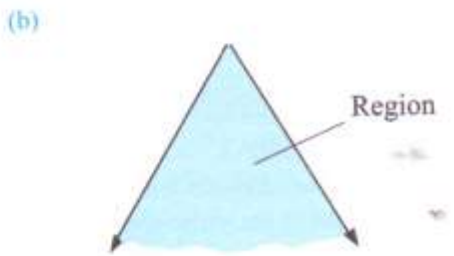


Fig. 8.11

Note that a **polygon** and a **fully bounded** region are often loosely referred to using the same name. For example, use 'triangle ABC' or 'triangular region ABC when describing the shape in Fig. 8.12

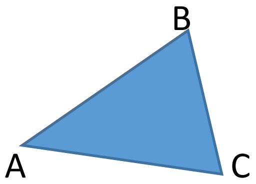


Fig. 8.12

In most cases, there is no ambiguity as to what is to be referred.

Types of polygons

There are two types of polygons; **convex** and **re-entrant** polygons.

A **convex polygon** has no interior angle greater than 180° (See Fig. 8.13).

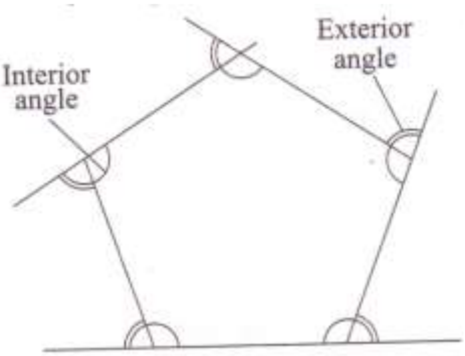


Fig. 8.13

A **regular polygon** has all its sides and angles equal. (Fig. 8.13).

A **re-entrant polygon** has at least one interior angle greater than 180° (Fig. 8.14).

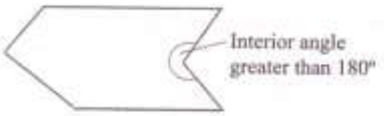


Fig. 8.14

Note that the shapes shown in Fig. 8.15 are not polygons

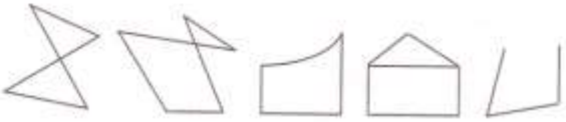


Fig 8.15

Names of polygons

Polygons are named according to the number of sides they have. (Table 8.1).

Number of sides	**Name of polygon**
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon

Table 8.1

Constructing regular polygons

(a) Constructing a regular polygon from the centre of a circle

A regular polygon has equal sides. If the vertices are joined to the centre of the polygon, the number of triangles formed are equal to the number of sides of the polygon. The number of angles at the centre will also be equal to the number of sides of the polygon and are of equal size. Each angle at the centre is $\frac{360^\circ}{n}$.

To draw a regular pentagon using a circle of radius 2 cm.

Procedure:

a) *With centre O, draw a circle radius 2 cm. draw a radius OA (Fig. 8.16).*
b) *From centre O, mark an angle of 72˚ as shown i.e. <BOA = 72˚, where A and B are on the circumference of the circle.*
(Angle at the centre = $\frac{360^\circ}{5}$ *= 72˚)*
c) *With centre B and radius AB. Draw an arc cutting the circle at point C.*
d) *With centre C and the same radius, draw an arc cutting the circle at D.*
e) *With centre D and the same radius, draw an arc cutting the circle at E.*
f) *Join AB, BC, CD, DE, EA (Fig. 8.16).*
ABCDE is the required regular pentagon.

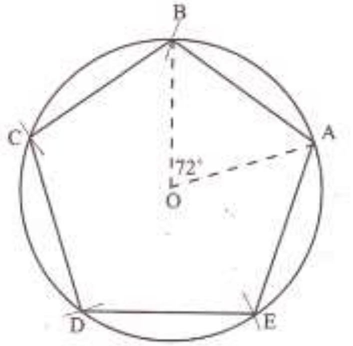


Fig. 8.16

(b) Construction of a regular polygon on a given line

A regular polygon has all its interior angles equal. Remember that the sum of interior angles in an n-sided polygon is (2n – 4) right angles.

Therefore, One interior angle = $\frac{(2n-4)right\ angles}{n}$

Construct a regular pentagon of sides 3 cm.

Procedure:

a) *Draw a line PQ = 3 cm long*
b) *At Q, construct an angle of 108˚.*
One interior angle = $\frac{(10-4)90^\circ}{5}$
= 108˚
c) *With centre Q and radius 3 cm, make an arc on the line drawn in (b) to cut it at point R.*

d) *At R make an angle of 108˚. With the same radius drawn in (c) make an arc to cut the line drawn at S.*
e) *Repeat steps (b) and (c) at S to get point T.*
f) *Join PT (See Fig. 8.17)*

PQRST is the required pentagon

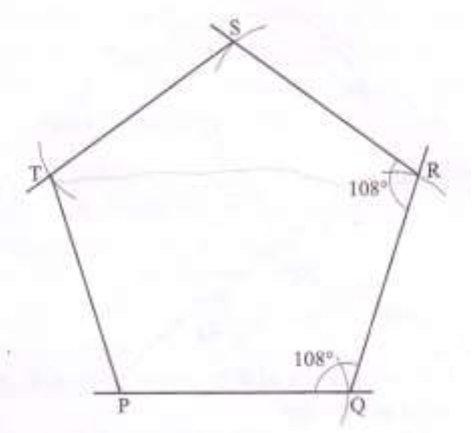


Fig. 8.17

Recall that:

A polygon with n sides has an exterior angles. If the polygon is regular, then the exterior angles are equal, each being$\frac{360^{\circ}}{n}$.

The, the following is an alternative procedure for constructing a regular polygon on a given line.

Procedure

a) *Draw a line of the specified length.*
b) *At one end of the line, make an exterior angle of* $\frac{360^{\circ}}{n}$ *(Fig. 8.18)*
c) *On the new line, mark off a length equal to the length of the side.*
d) *At the end of the line drawn in (c), draw an exterior angle, as in (b).*
e) *Continue on each new line till the required polygon is formed.*

Now repeat the construction of a regular pentagon using this procedure.

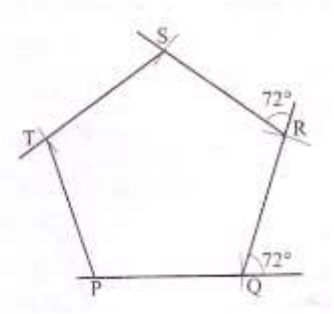


Fig. 8.18

Constructing an irregular polygon

In an irregular polygon, the sides are not equal. Thus, to construct one of n sides, the sizes of at least (n – 2) consecutive angles and lengths of (n – 1) sides must be known.

The procedure then is the same as that of constructing a regular polygon, only that the angle and length to be marked off at each vertex keep changing.

Exercise 8.3

1. By first drawing a circle, make an accurate drawing of a:
    a) Regular quadrilateral,
    b) Regular hexagon,
    c) Regular octagon.
2. Construct a regular octagon of side 3.5 cm. measure its interior angles.
3. Construct a regular decagon of side 3 cm. measure its interior angles.
4. Construct a regular polygon with 12 sides (duodecagon).
5. Using a ruler and a pair of compasses only, Construct a regular octagon starting with a circle of radius 4.5 cm. Measure one side of the octagon.
6. Construct a pentagon ABCDE such that
    AB = 3.4 cm, BC = 3.8 cm, <ABC = 110˚,
    CD = 5.5 cm, <BCD = 125˚, DE = 3.2 cm,
    <CDE = 80˚. Measure AE and <AED.

Sum of interior angles of a polygon

Example 8.1 illustrates two ways of calculating the sum of interior angles of a polygon.

Example 8.1

Calculate the sum of the interior angles of a hexagon.

Solution

A hexagon can be divided into triangles as shown in Fig. 8.19

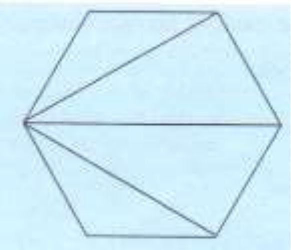


Fig. 8.19

Sum of angles in a triangle $= 180^\circ$

Therefore, Sum of angles in 4 triangles $= 4 \times 180^\circ$

$= 720^\circ$

$= 8 \times 90^\circ$

$= 720^\circ$

Therefore, Sum of interior angles of a hexagon is 8 right angles.

Alternative method

A hexagon can also be divided into 6 triangles as shown in Fig. 8.20. O is a point anywhere inside the hexagon.

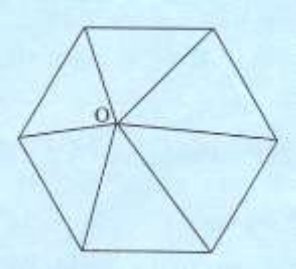


Fig. 8.20

Sum of angles in a triangle $= 180^\circ$

Therefore, Sum of angles in 6 triangles $= 6 \times 180^\circ$

= 1 080˚

Sum of angles at O = 360˚

Therefore, Sum of interior angles of the hexagon

= 1 080˚ - 360˚

= 720˚ = 8, right angles

Activity 8.2

Using the first method of Example 8.1, determine the sum of the interior angles of a

- *i. Quadrilateral*
- *ii. Pentagon*
- *iii. Heptagon*
- *iv. Octagon*
- *v. Nonagon*

Using the results above, complete Table 8.2.

Number of sides of polygon	Number of triangles to make the polygon	Sum of interior angles in degrees	Sum of interior angles in terms of right angles
3	1	180	2
4	2	-	-
5	-	-	-
6	4	720	8
7	-	-	-
8	-	-	-
9	-	-	-
10	-	-	-
:	:	:	:
.	.	.	.
n	-	-	-

Table. 8.2

Summary on sum of angles of polygons:

For a polygon with n sides, the sum of interior angles is **(2n – 4) right angles**.

Sum of exterior angles of a convex polygon

Consider a polygon ABCDE Fig. 8.21

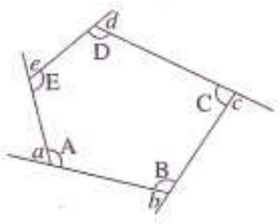


Fig. 8.21

In Fig. 8.21, let the interior angles be denoted by the letters A, B, C, D, E. The exterior angles are marked by the small letters a, b, c, d and e.

Now, A + a = 180˚ ……….. angles on a straight line.

Similarly, B + b = C + c = D + d = E + e = 180˚

For a polygon n sides, sum of interior <s + sum of exterior <s = 180n = 2n right angle.

Since the sum of interior angles = (2n – 4) right angles.

The sum of exterior angles

= (2n – (2n – 4) right angles.

= (2n – 2n + 4) right angles

= 4 right angles

= 4 × 90˚ = 360˚

This result is true regardless of the number of sides that a polygon has and whether it is a regular polygon or not.

Example 8.2

Each interior angle of a regular polygon is 140˚. How many sides does the polygon have?

Solution

Since the interior angle + exterior angle = 180˚ and the interior angle is 140˚

Exterior angle = 180˚ – 140˚

= 40˚

The sum of exterior angles is 360˚

Each exterior angle is equal to 40˚

If n is the number of sides

Then, n = $\frac{360}{40}$

= 9

The polygon has 9 sides.

Example 8.3

A regular polygon has 10 sides. Find the size of each interior angle.

Solution

If there are 10 sides, there must be 10 vertices and consequently 10 angles.

Sum of exterior angles = 360˚

Therefore, each exterior angle = $\frac{360^\circ}{140}$ = 36˚.

Sum of exterior angle + interior angle = 180˚

Thus, interior angle + 36 = 180˚

Therefore, Interior angle = 180˚ - 36˚

= 144˚

Each interior angle = 144˚

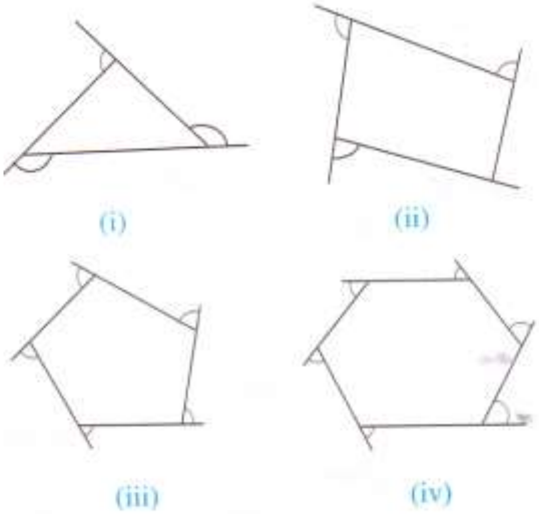
Exercise 8.4

1. Copy and complete table 8.3. The second column refers to the triangle formed when all the diagonals are drawn from one vertex.

Number of sides of polygon	Number of triangles to make the polygon	Sum of interior angles in degrees	Sum of interior angles in terms of right angles
3	1	180°	180°÷ 3 = 60°
4	2	4 × 180°=360°	360°÷ 4 = 90°
5	-	-	-
6	4	4 × 180°=720°	720°÷ 6 = 120°
7	-	-	-
8	-	-	-
9	-	-	-
10	-	-	-
12	-	-	-
20	-	-	-
n	-	-	-

Table. 8.3

2. Two exterior angles of a triangle are 110° and 140°. Calculate the size of the third one.
3. (a) Draw figures as shown in Figure 8.22. Carefully measure the indicated angles.
   What is the sum of the angles in each case?



(i) (ii) (iii) (iv)

Fig. 8.22

(c) Write down an expression for the size of an exterior angle of an n-sided regular polygon.

4. Which regular polygons have the following exterior angles?
   a) 120˚
   b) 72˚
   c) 45˚
   d) 36˚
   e) 24˚
5. Which regular polygons have the following interior angles?
   a) 135˚ b) 156˚ c) 160˚ d) 165˚
6. The angles of a pentagon are x, 3x, 3x, 4x and 4x in that order. Find the angles in degrees.
7. The interior angle of a regular polygon is $6\frac{1}{2}$ times the exterior angle. How many sides has the polygon?
8. A regular polygon with 3x sides has interior angle 40˚ greater than that of one with x sides. What is x?

Unit 9 EQUATIONS

Equations

A mathematical sentence with the symbol = is called an **equation**. Such a statement express the equality of things.

An equations may be **conditional** or an **identity**.

A **conditional equation** is one that is satisfied by at least one value of the unknown. For example there is one value of the unknown x which satisfies the equation x + 11 = 15.

An **identity** is an equation which is true regardless of what values are substituted for the unknown. For example;

I. $(x + 3)^2 = x^2 = 6x + 9$ and $4(x – 3) = 4x – 12$ are identities. Whatever value x takes, the statement is true.

II. $(x + 7)^2= 2x^2+ x + 1$ is a conditional equation since it is true only when x = -3 or x = 16.

A letter such as x used above is called an **unknown**.

An equation in which the unknown has power 1 (i.e. it is just x or y, etc.) is known as a **linear equation**. For example.

x + 3 = 17, 2**m** – 5 = 25, $\frac{n}{4}$ – 3 = 8, are linear equations.

$x^2 + 4 = 8$, $2y^3- y = 7$, are non-linear equations.

Exercise 9.1

1. Which of the following are equations?
   - a) 3 – 5 + 1
   - b) 5 + 2 – 6
   - c) 6 – 5 = 1
   - d) 4 × 6
   - e) 3 – 9
   - f) 3 × 8 = 24
   - g) 16 – 9 = 3 + 4
   - h) 4 + 17 – 3
   - i) 6 × 4 = 3 × 8
   - j) 17 = 13 + 4
2. State whether the following are true or false.
   - a) 3 + 6 = 7 + 2
   - b) 7 – 3 = 6 – 2

   c) 6 + 2 = 6
   d) 5 + 6 + 1 = 11
   e) 10 + 7 + 4 = 13 + 4
   f) 5 + 6 – 3 = 9
   g) 3 + 5 – 2 = 8

3. State whether the following are true, false or open.
   a) 2 – 5 = 3
   b) 8 + 13 = 23
   c) -2 + 9 = 7
   d) 7 – x = 0
   e) 2 = 2x – 3
   f) 13 – 17 = +7 + -3
   g) (-24) ÷ (+3) = (-8)
   h) $\frac{(-6)\times(-5)}{10} = 3$
4. Copy and complete the following to make them true.
   a) 4 + 7 =
   b) 5 + 1 =
   c) 4 – 1 =
   d) 6 – 6 =

Meaning of letters in Algebra

In algebra, we use letters and symbols to represent unknown quantities. When used, the letters may have different meanings depending on the type of problem to be solved. Generally, the first letters of alphabet a, b, c…. are used to represent constant values or numbers. The last letters of the alphabet x, y, z are used to represent unknown values to be solved. As was pointed out in chapter 3, some letters and symbols stand for;

I. Constants e.g. π as in Area of a circle = π^2.
II. General numbers e.g. a and b in a + b = b + a
III. Unknowns e.g. x in 5x + 9 = 19. Here x is a quantity whose value is to be found by solving an equation.
IV. Variables e.g. x and y in y = 4x – 7, when x varies (changes) y also varies. Although x and y are unknown, they are also referred to as **variables** because they can assigned a set of values.

Solving linear equations

Consider the equation x + 13 = 19.

This equation may be true or false depending on the value of the unknown. For example, it is true when x = 6, but false when x = 3.

The value of the unknown that makes an equation true is called the **solution** or **root** of the equation.

To **solve** an equation means to find the value of the unknown which makes the equation true.

Example 9.1

Solve the equations

a) $X + 6 = 10$
b) $X - 4 = 14$
c) $2x = 14$

Solution

a) We are required to find a number to which we add 6 to get 10.
The number is 4.

Therefore, the solution is $x = 4$.

b) We are required to find a number from which we subtract 4 to get 14.
The number is 18.

Therefore, the solution is $x = 18$

c) We are required to find a number which when multiplied by 2 gives 14.

The number is 7.

Therefore, the solution is $x = 7$.

The method used in Example 9.2 is known as the cover-up technique. Sometimes we use the **cover-up technique** more than once.

Example 9.2

Solve (a) $2x + 3 = 17$

(b) $-3 + 2x = 5x$

Solution

(a) $2x + 3 = 17$
□ $+ 3 = 17$
$14 + 3 = 17$

So, $2x = 14$ Check $2x + 3 = 2 \times 7 + 3$

$2 \times \square\ 14$

$2 \times 7 = 14$

$X = 7$

$= 14 + 3$

$= 17$

Therefore, 7 is the solution

(b) $-3 + 2x = 5x$
$\square + 2x = 5x$

So $5x – 2x = -3$

$3x = -3$

$x = \frac{-3}{3}$

$x = -1$

Exercise 9.2

1. Copy and complete the following to make them true.
   a) $18 \div 3 = \square$
   b) $5 + \square = 7$
   c) $9 - \square = 7$
   d) $3 \times \square = 6$
   e) $6 + \square = 9$
   f) $24 \div \square = 6$
   g) $4 \times \square = -12$
   h) $7 + 4 + \square = 13$
   i) $3 + 4 + \square = 10$
   j) $8 + 2 + \square = 6$
   k) $4 + 5 – \square = 2$
   l) $4 + 6 – \square = 2$
   m) $6 + \square + 4 = 3$
   n) $3 + 7 + 1 = 9 – \square$
   o) $8 + 3 + 2 = 10 + \square$
   p) $6 + 5 = 11 – \square$
   q) $6 + 6 – 3 = 12 – \square$
   r) $3 + 12 – 8 = 15 + \square$

2. State whether the following are true or false.
   a) $\frac{x}{4} = 5$ when $x = 24$
   b) $x - 2 = 9\ when\ x = 7$
   c) $20 + x = 28$ when $x = 8$
   d) $4x = 20$ when $x = -5$
   e) $12 – 3x = 0$ when $x = 4$

f) $8 = 9 - x$ when $x = 1$

3. Solve the following equations.
   a) $x + 6 = 20$
   b) $x + 9 = 23$
   c) $16 + x = 25$
   d) $17 + x = 25$
   e) $x - 3 = 12$
   f) $x - 5 = 1$
   g) $12 - x = 10$
   h) $14 - x = 8$
   i) $4x = 28$
   j) $7x = 42$
   k) $3x + 4 = 12$
   l) $5 + 5x = 5$
   m) $2x + 16 = 32$
   n) $7 + x = 63$
   o) $2x - 8 = 15$
   p) $18 - 3x = 15$
   q) $X - 12 = 60$
   r) $18 - x = 10$
   s) $30 + 3x = 9x$
   t) $12x - 60 = 0$

Solving equations by the balancing method

Since an equation states the equality of two things, it may be compared to a pair of scales.

If the content of the two scale-pans balance each other, they will still do so if:

i. Equal amounts are added to both sides.
ii. Equal amounts are taken away from both sides.
iii. The contents of both sides are doubled, or tripled, or halved, and so on.

Thus, if the two sides balance, they will still do so if **what is done on one side is also done on the other.**

Example 9.3

Solve the equation $8x - 6 = 5x + 9$

Solution

Imagine a pair of scales with $8x - 6$ on one side balanced by $5x + 9$ on the other (Fig. 9.1(a)).

$8x - 6 = 5x + 9$

Add 6 to both sides: $8x - 6 + 6 = 5x = 9 + 6$

$$8x = 5x = 15$$

Subtract 5x from both sides:

$$8x - 5x = 5x + 15 - 5x$$

$$3x = 15$$

(See Fig. 9.1 (b) and (c)

Divide both sides by 3:

$\frac{3x}{3} = \frac{15}{3}$

i.e. $x = 5$ (Fig. 9.1 (d))

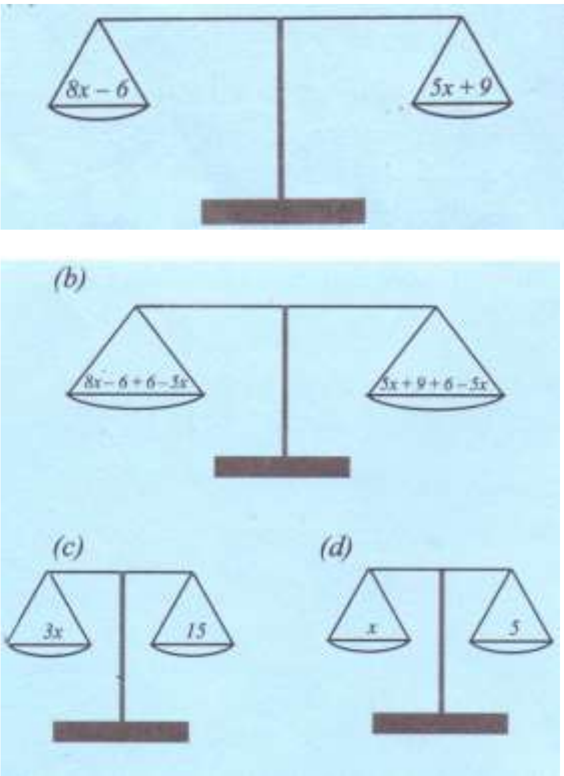


Fig. 9.1

Check: If $x = 5$, $8x - 6 = 8 \times 5 - 6$

$= 40 - 6$

$= 34$

and $5x + 9 = 5 \times 5 + 9 = 25 + 9$

$= 34$

Therefore, $8x - 6 = 5x + 9$ when $x = 5$

The reason for adding 6 to both sides and subtracting 5x from both sides is to get rid of the -6 on the left-hand side (LHS) and the 5x on the right-hand side (RHS). A simpler equation is obtained, in which the LHS contains only a term in x, while x does not appear on the RHS. The LHS of the equation 3x = 15 is divided by 3 to give x. the RHS also has to be divided by 3.

Example 9.4

Solve the equation

$3x + 34 - 8x = 11 - 9x - 13$

Solution

$3x + 34 - 8x = 11 - 9x - 13$

Simplifying both sides:

$3x - 8x + 34 = 11 - 13 - 9x$

i.e. $-5x + 34 = -2 - 9x$

Adding 9x to both sides:

$-5 + 34 + 9x = -2 - 9x + 9x$

$4x + 34 = -2$

Subtracting 34 from both sides:

$4x + 34 - 34 = -2 - 34$

$4x = -36$

Dividing both sides by: 4

$\frac{4x}{4} = \frac{-36}{4}$

Therefore, $x = -9$

Example 9.5

Solve $\frac{2\,x}{5} = 10$

Solution

$\frac{2x}{5}$ = 10

Multiplying both sides by 5:

$\frac{2x \times 5}{5}$ = 10 × 5

2x = 50

Dividing both sides by 2:

$\frac{2x}{2} = \frac{50}{2}$

Therefore, x = 25

Exercise 9.3

Solve the following equations using the balance method and stating the steps as in Examples 9.4 and 9.5.

1. (a) x + 113 = 153 (b) 24 = x + 13
   (c) x + 8 = -12 (d) x + -7 = -19
2. (a) x – 17 = 37 (b) 11 = x – 7
   (c) x – 2 = -4 (d) x - -2 = -7
3. (a) x – 72 = -24 (b) –x + 72 = -30
   (c) 4 = x – 5 (d) 15 + 2x = -3x
4. (a) 24x = 72 (b) +10 = -5x
   (c) -25x = 200 (d) -7 = -84x
5. (a) 6x – 5 = 25 (b) 9x + 8 = 35
   (c) -5x + 5 = -15 (d) -7x – 6 = -20

Short cut to the balance method

Consider the equation 8p – 4 = 3p + 11.

Adding 4 to both sides, the equation becomes

$8p - 4 + 4 = 3p + 11 + 4$

i.e. $8p = 3p + 11 + 4$ ……………. (1)

Subtracting 3p from both sides, the equation becomes;

$8p - 3p = 3p + 11 + 4 - 3p$

i.e. $8p - 3p = 11 + 4$ …………. (2)

Comparing (2) with the original equation, it looks as if.

i. The -4 has been transferred to the RHS and its sign changed to +, while
ii. The 3p has been transferred to the LHS and its sign changed to -.

In actual practice, this is usually what is done, i.e. a term may be moved from one side of an equation to the other, but its sign must be changed.

! Caution: The short cut method should only be used after mastering the balance method.

Exercise 9.4

1. (a) $7y = 3y + 20$ (b) $5q - 14 = 3$
   (c) $40 - 3e = 2e$ (d) $1 + 2f = 22 - 5f$

2. (a) $3 + 2t - 24 = 14 - 3t$

   (b) $2p + 19 - 5p = p - 5$

   (c) $12 - 3m - 3 = 9 - 5m$

   (d) $7 - 4n - 3 = 3n + 9 - 15$

3. (a) $17d - 30 + 18d + 14 = 18d + 19 + 10d$

   (b) $41 + 52u - 3 - 13u = 41u + 51 - 12u - 7$

Solving linear equations involving fractions

When solving the fractions using the LCM.

Example 9.6

Solve the following equations.

(a) $\frac{2x+7}{3} - \frac{5x+6}{4} = 0$

(b) $\frac{2x+7}{3} - \frac{5x+6}{4} = 0$

Solution

(a) $\frac{2x+7}{3} - \frac{5x+6}{4} = 0$

$\frac{4(2x+7)-3(5x+6)}{12} = 0$ (Use LCM to get same denominator)

$4(2x+7) - 3(5x+6) = 0$ (Multiply both sides by 12)

$8x + 28 - 15x - 18 = 0$ (Open brackets)

$-7x + 10 = 0$

$-7x = -10$

Therefore, x = $\frac{10}{7} = 1\frac{3}{7}$

(b) $\frac{2a + 36}{a} - \frac{4}{5} = 0$

$\frac{5(2a+36)-4a}{5a} = 0$

$5(2a + 36) - 4a = 0$

$10a + 180 - 4a = 0$

$6a + 180 = 0$

$6a = -180$

$a = -30$

Example 9.7

Solve $\frac{2}{5}\,y + \frac{3}{4} = 10 + \frac{y}{2}$

Solution

$\frac{2}{5}\,y + \frac{3}{4} = 10 + \frac{y}{2}$

LCM of 5, 4 and 2 = 20

Multiplying both sides by 20:

$20\,(\frac{2}{5}\,y + \frac{3}{4}) = 20(10 + \frac{y}{2})$

$8y + 15 = 200 + 10y$

Subtracting 10y from both sides:

8y + 15 – 10y = 200 + 10y – 10y

-2y + 15 = 200

Subtracting 15 from both sides:

-2y + 15 – 15 = 200 – 15

-2y = 185

Dividing both sides by -2

$$\frac{-2y}{-2} = \frac{185}{-2}$$

Therefore, y = -92$\frac{1}{2}$

Note: Should there be a term like $1\frac{1}{2}x$ in the equation, always write it in the improper fraction form, as$\frac{3}{2}x$, and then proceed as in Example 9.7

Exercise 9.5

Solve the following equations using the balance method and stating the steps as in example 9.6

1. (a) $\frac{x}{6}$ – 2 = 10 (b) $9 - \frac{x}{2} = 5$
   (c) $\frac{k}{5} = 0$ (d) $p - 2\frac{1}{2} = 6\frac{1}{2}$
2. (a) $-3\frac{3}{4} = x + 1\frac{2}{5}$ (b) $4\frac{1}{2} = 5q - -$
   (c) 2p – 8 = p – 3 (d) t + 7 = 17 – 4t
3. (a) $3\frac{1}{2} + 2 - f = 17\frac{1}{2} - 1 - f$
   (b) $1\frac{1}{2}\,x + \frac{1}{4} = 1\frac{1}{2}\,x + 3\frac{1}{4}$
   (c) $\frac{0.1}{x} + \frac{3.9}{x} = 12$
4. (a) $2\frac{1}{2}\,b - 4 = 10 - \ 3\frac{1}{4}\,b$

(b) $1\frac{1}{2}h - 4\frac{1}{2} = 1\frac{1}{2} + \frac{1}{2}h$

5. (a) $\frac{x}{3} + 3 = 3$ (b) $-3 - \frac{x}{2} = 4$

(c) $\frac{1}{x} = 2\frac{1}{2}$ (d) $\frac{2}{p-1} = \frac{5}{p}$

Equations involving brackets

Recall that:

i. a + (b + c) = a + b + c
ii. a + (b – c) = a + b – c
iii. a – (b + c) = a – b = c
iv. a – (b – c) = a – b + c

Use these rules to remove brackets before solving equations involving brackets.

Remember that the dividing line, as in $\frac{3m+2}{4}$, is both a division and a bracket.

Example 9.8

Solve the equation

9x – (4x – 3) = 11 + 2(2x – 1)

Solution

9x – (4x – 3) = 11 + 2(2x – 1)

Removing brackets:

9x – 4x + 3 = 11 + 4x – 2

Simplifying both sides:

5x + 3 + 4x + 9

Subtracting 4x and 3 from both sides:

5x – 4x = 9 – 3

i.e. x = 6

Example 9.9

Solve the equation $\frac{5x+2}{4} - \frac{3}{2} = \frac{7x+1}{3}$

Solution

$\frac{5x+2}{4} - \frac{3}{2} = \frac{7x+1}{3}$

Multiplying both sides by 12 (LCM of 4, 2 and 3)

$12\{(\frac{5x+2}{4}) - \frac{3}{2}\} = 12\ (\frac{7x+1}{3})$

i.e. $12\ \times \left\{\left(\frac{5x+2}{4}\right) - \frac{3}{2}\right\} = \ 12\ \times (\frac{7x+1}{3})$

i.e. 3(5x + 2) – 6 × 3 = 4(7x + 1)

Removing brackets:

15x + 6 – 18 = 28x – 4

Simplifying LHS:

15x – 12 = 28x – 4

Subtracting 28x from both sides and adding 12 to both sides:

15x – 28x = -4 + 12

i.e. -13x = 8

Dividing both sides by -13:

$\frac{-13}{-13} = \frac{8}{-13}$

i.e. $x = \frac{8}{-13}$

Exercise 9.6

1. *4d + (5 – d0 = 17*
2. *12m + (1 – 7m) = 23*
3. *23 = 7 – (3 -4t)*
4. *(8w – 7) (5w + 13) = 0*
5. *24 – (5 + 3x) = 8x + (4 – 5x)*

6. *3(5x – 1) = 4(3x + 2)*
7. *5(3c + 4) – 3(4c + 7) = 0*
8. *4(3k – 1) = 11k – 3(k – 4)*
9. *4(3x – 5) – 7(2x + 3) + 2(5x + 11) = 5*
10. 7(5x – 3) 10 = 2(3x – 5) 3(5 – 7x)
11. $\frac{x+1}{2} - \frac{x-2}{4} = \frac{1}{8}$
12. $\frac{3y}{2} - \frac{14y-3}{5} = \frac{y-4}{4}$
13. $\frac{5-c}{3} = \frac{c-4}{2} + 6$
14. $\frac{3x+1}{2} = \frac{4x-3}{3} + 3$
15. $3y - \frac{2}{5}(2y-5) = 12$
16. $\frac{p-3}{12} - \frac{3(p-1)}{8} = \frac{2}{3}$
17. $\frac{3(2x+3)}{5} = 2(3x-2) + 4$
18. $\frac{x-1}{7} + 1 = \frac{5x+1}{5}$
19. $\frac{2}{3}(4-2y) - \frac{3}{5}(2-4y) = 2$
20. $\frac{3}{4}(8-4z) - \frac{2}{3}(3-2z) = 1$
21. $\frac{x}{5} + 1\frac{1}{2}x - \frac{11}{20} = \frac{5x-1}{3}$
22. $\frac{7e-5}{5} - \frac{2e-1}{10} = \frac{4e-3}{15} = 0$

Forming and solving linear equations

To solve a word problem in which a number is to be found:

i. Introduce a letter to stand for the number to be found (the unknown).
ii. Form an equation involving this letter by expressing the given information in symbols instead of words.
iii. Solve the equation to get the required number.

Example 9.10

The sum of two numbers is 120 and their difference is 18. Find the two numbers

Solution

Let the smaller number be x.

Then, the larger number is x + 18

Sum of the two numbers = x + (x +18).

i.e. x + (x + 18) = 120

x + x + 18 = 120

2x + 18 = 120

2x = 102

Therefore, x = 51

Thus, the smaller number is 51 and the larger number is 51 + 18 = 69

Example 9.11

Tom and Mary share K450 so that Mary gets K54 less than Tom. Find their shares.

Solution

Let Tom's share be K x.

Then, Mary's share is K (x – 54).

Together, they have K450.

Therefore, x + x – 54 = 450

2x = 450 + 54

2x = 504

Therefore, x = 252

Thus, Tom's share is K252 and Mary's share is K54 less i.e. K198

Example 9.12

Jack has K116 and June has K64. How many Kwacha must June give Jack so that Jack shall have 4 times as much as June?

Solution

Suppose June gives Jack K x.

Then Jack has K (116 + x) and June has K (64 – x).

116 + x is 4 times as big 64 – x,

i.e. 116 + x = 4(64 – x)

$116 + x = 256 - 4x$

$x + 4x = 256 - 116$

$5x = 140$

$x = 28$

Thus, June must give jack K 28.

Example 9.13

When 55 is added to a certain number and the sum is divided by 3, the result is 4 times the original number. What is the original number?

Solution

Let the number be x.

Adding 55 and dividing the sum by 3 gives

$$\frac{x+55}{3} = 4x$$

Therefore, $x + 55 = 12x$

$55 = 12x - x$

$55 = 11x$

$X = 5$

Thus, the number is 5.

Exercise 9.7

Find an answer to each of the following problems by forming an equation and solving and solving it.

1. When I double a number and add 17, the result is 59. What is the number?
2. When a number is added to 4 times of itself, the result is 30. Find the number.

3. The difference of two numbers is 5 and their sum is 19. Find the two numbers.
4. Mr. Ali has 7 marbles less than Mohammed and they have 29 between them. How many does each boy have?
5. When a number is doubled and 4 added, the result is the same as when it is tripled and 9 subtracted. Find the number.
6. A rectangle is 3 times as long as it is wide. The total length round its boundary is 56 cm. find its length and width.
7. Mr. Ngunda is twice as old as Useni, Alile is 3 years younger than Ngunda. The sum of their ages is 32. Find their individual ages.
8. Mt Chirwa and Mr. Dziko share K 1 470 such that Dziko receives K190 less than Mr. Chirwa. Find their individual shares.
9. Find a number such that when it is divided by 3 and 2 added, the result is 17.
10. A number is such that when 3 is subtracted from three-quarters of it, the result is two thirds of the number. Find the number.
11. The result of adding one third of a number to itself is 28. What is the number?
12. From a certain number, subtract 3, multiply the result by 5, and then add 9. If the final result is 54, find the number.
13. Find two consecutive even numbers such that the sum of 3 times the smaller and 5 times the larger is 106.
14. Two tanks contain diesel. The first tank contains 5 times as much as the second. When 20 litres of diesel are allowed to flow from the first tank into the second, the first contains 3 times as much as the second. What were the original contents of the tanks?
15. A woman earns three times as much as her husband earns. After spending three fifths of their combined income, the couple have K20 000 left. How much does the husband earn?
16. A profit of K232 000 is shared amongst three business partners Ann, Betty and Charles. Charles got K12 000 more than Betty while Ann gets twice as much as Charles. Find how much each got.
17. A man is 30 years old while his daughter is 4. In how many years' time will the daughter be half the age of her father?

18. A member of a wildlife club is charged a third of the normal fee for entry into a game park. What is the normal entry fee if such a member pays K 144 less than normal?
19. Mr. Kassim has a money box containing 100 mixed K 5 and K 10 coins with a total value of K 600. How many of each type of coin does the box contain?
20. The fraction $\frac{8}{9}$ is obtained after the same number is added to the numerator and denominator of the fraction $\frac{3}{5}$. What number is added?

What is a formula?

You have already met relations such as A = lb, $A = \frac{1}{2}bh$, $A = \pi r^2$, P = 2(l + b), V = lbh and so on. Identity, what each of the letters in each case stands for.

Suppose a rectangle is l cm long and b cm wide. We know that is area is l × b cm^2. If we call the area A cm^2, then A = l × b, i.e. A =lb.

This relation is called a **formula**. We use it whenever we wish to find the area of a rectangle. The formula is said to express A in terms of l and b.

Most formulae are given or stated in a standard form. However, we can write our own formulae given specific situations and conditions.

When writing a formula which involves two or more different symbols, an equation is obtained. For example, in the formula C = 2πr, 2 and π are called constants while C and r are called variables.

Example 9.14

If a motorist covers a distance, D km in T hours at an average speed of S km/h, write down a relation between D, T and S.

Solution

We know that $\frac{Total\ distance\ travelled\ (km)}{Total\ time\ (h) taken}$

Gives average speed in km/h

Therefore, $S = \frac{D}{T}$

Example 9.15

At the end of the year, a business lady observed that she had K s in her bank account. For the next one year, she deposited k t every month and did not withdraw. At the end of the year her account had K A. write a formula connecting s, t and A.

Solution

Initial amount was K s

In 12 months she deposits K 12t

Therefore, at the end of the year amount in the account is s + 12t

Therefore, A = s + 12t

Exercise 9.8

1. The length and breadth of a rectangle is q cm and r cm respectively. If its perimeter is P cm and its area A cm^2, find the formula connecting:
    a) P, q and r
    b) A, q and r
2. A class has p boys and q girls. In a mathematics test, the average mark for boys was m marks and n marks for girls. Find
    a) The total marks for the whole class.
    b) The average for the whole class.

Subject of simple formula

When writing a formula, we have seen that a formula express one quantity in terms of others i.e. in V = lbh, V is expressed in terms of l, b and h.

In this case, V is called the subject of the formula.

In order to solve a particular problem, it may be necessary to express the given formula differently. Suppose we are given the length, volume and height of a rectangular tank and are required to find the breadth. We know that the formula connecting these variables is V = lbh. To find the breadth b, we express b in terms of V, I and h.

Ibh = V

Dividing each side by lh:

b=$\frac{v}{lh}$.

Example 9.16

Make x the subject of the formula a+ mx = b.

Solution

a + mx = b

Subtracting a from both sides, we get

mx = b – a

Dividing both sides by m, we get

x = $\frac{b-a}{m}$

Example 9.17

The formula for area of a circle is given by $A = \pi r^2$.

a) *Express* π *in terms of A and r.*
b) *Express r in terms of A and* π

Solution

a) $A = \pi r^2$

Dividing both sides by r^2, we get

$$\frac{A}{r^2} = \pi \Rightarrow \pi = \frac{A}{r^2}$$

b) $A = \pi r^2$.

Dividing both sides by π, we got

$\frac{A}{\pi} = r^2$

$finding\ the\ square\ roots\ of\ both\ sides, we\ get$

r = $\sqrt{}\frac{A}{\pi}$.

Substitution in simple formula

As seen earlier in chapter 4, substitution involves replacing variables with known values in algebraic expressions. The expressions can then be evaluated. The same method is employed when carrying out substitution in simple formulae.

Example 9.18

The area of a triangle ABC is given by

A = $\frac{1}{2}$bh.

a) Express b in terms of A and h.
b) Find the value of b if h = 4 units and A = 6 units.

Solution

a) A = $\frac{1}{2}$bh.
   2A = bh (multiplying both sides by 2)
   $\frac{2A}{h}$ = b (dividing both sides by h)
   $therefore, b = \frac{2A}{h}$
b) h = 4, a = 6
   by substitution
   b =$\frac{2 \times 6}{4}$.
   = 3 units

Exercise 9.9

1. solve for x in:
   a) x – p = q
   b) px + q = r
   c) $\frac{1}{2}x = s$
   d) $a(x-b) = cx$
   e) $\frac{x}{a} = 1$
   f) $\frac{x}{r} = t$
2. If a rectangle is x cm long and y cm wide,
   a) Find a formula for its perimeter p.
   b) Express y in terms of p and x.

3. A rectangular box has a square base of sides x cm and a height of y cm. the sum of the length of its edges is s cm, and the sum of the areas of its faces in A. write a formula for: (a) s (b) A
4. The surface area of a closed cylinder is given by the formula A = $2\pi r^2$+ 2 πrh. Express h in terms of r and A.
5. Fig. 9.2 represents the scale drawing of an athletics track. The dimensions are given in metres, and the enclosed area is made up of a rectangle and two semi-circles. The area is A m^2, and its perimeter is p m.

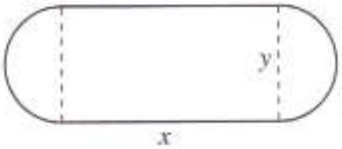


Fig. 9.2

Find:

a) A formula for P in terms of x and y.
b) A formula for A in terms of x and y.
c) (i) y in terms of x and p
(ii) x in terms of y and A.
d) The value of (i) p and (ii) A given that x = 100 m and y = 63.6 m.

6. Mrs. Umi has x dairy cows in her farm. Each cow produces p litres of milk per day.
   a) How many litres of milk do the cows produce in seven days?
   b) If in 23 days the cows produce 100 litres, express x in terms of p.
7. Mrs. Mponda planted m trees in his farm in the year 2009. In 2010 she planted q trees.
   a) If 14 trees dried in 2011, how many trees did the farmer remain with?
   b) If the total number of trees in Mrs. Mponda's farm in 2011 were 156, express m in terms of q.

Unit 10 COORDINATE GEOMETRY

Position of a point on a line

You are already familiar with the number line. The number line is a **graph** or a picture of all the negative and positive numbers.

If we mark points on the number line, we can say exactly where they are with reference to zero.

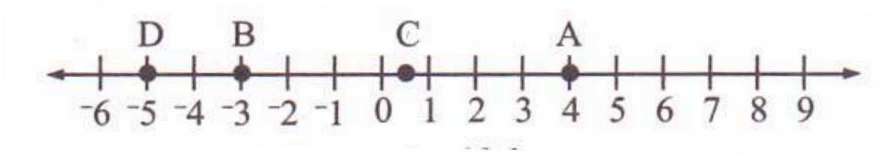


Fig. 10.1

In Fig. 10.1, A is 4 units to the right of zero and B is 3 units to the left of zero. This can be shortened to A (4) and B (-3). Similarly, C is the point C ($\frac{1}{2}$) and D is the point D (-5). A (4) and B9-3) give the **positions** of A and B on the number line with reference to zero (0).

Note that brackets here are used in a different way from the way they are used in algebra and arithmetic.

Position of a point on a plane surface

Fig. 10.2 shows a plan of the desks in a classroom. Each rectangle represents a desk. Describe the position of the shaded desk.

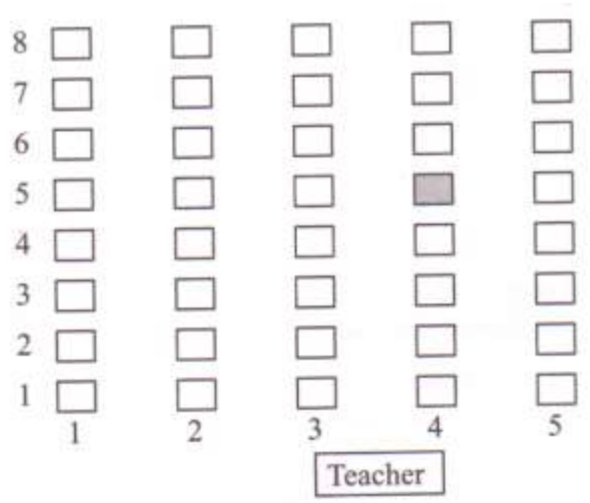


Fig. 10.2

You might describe the desk as being in the fourth column and fifth row. If we agreed to write this as (4, 5), it would mean 4 columns across from the left-hand side and then 5 rows up (i.e. from the front toward the back). Does (5, 4) mean the same position?

It is clear that the **order of the numbers inside brackets is important** and so we call pairs like these **ordered pairs.**

If all the desks in the classroom were removed and then you were asked to put your desk back in the room, exactly where it was before, how would you do it?

If you knew it was 5 metres from one wall and 6 metres from another, could you do it then? How many places on your classroom floor are 5 m from one wall and 6 m from another? We have to decide which walls we are going to use.

Fig. 10.3 shows one way in which you might fix the position of your desk.

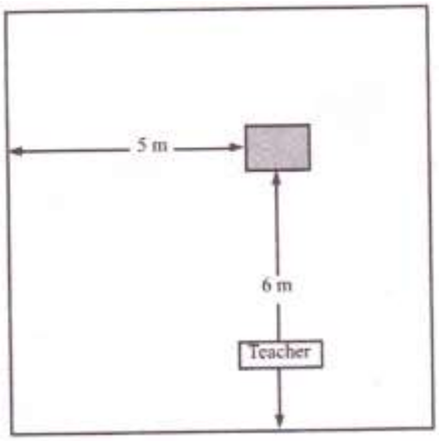


Fig. 10.3

We could say that the desk has an ordered pair (5, 6), to mean five metres from the left-hand wall and six metres from the front wall.

Drawing and labelling axes

In our classroom example, without the desks in place, it would be difficult to locate the exact position of any desk. To locate relative position of any point, we use a more accurate method where we have a reference point as well as a reference pair of axes. The axes are two perpendicular lines, one which is vertical and the other is horizontal, the two axes meet at a point generally referred to as the **origin** denoted by the capital letter O. for an accurate representation we use a standard grid system (1 cm squares) known as the graph paper.

Using a pair of axes on a graph paper and the information in Fig. 10.4, we can show the relative position of the shaded desk. We will use a suitable scale on both axes.

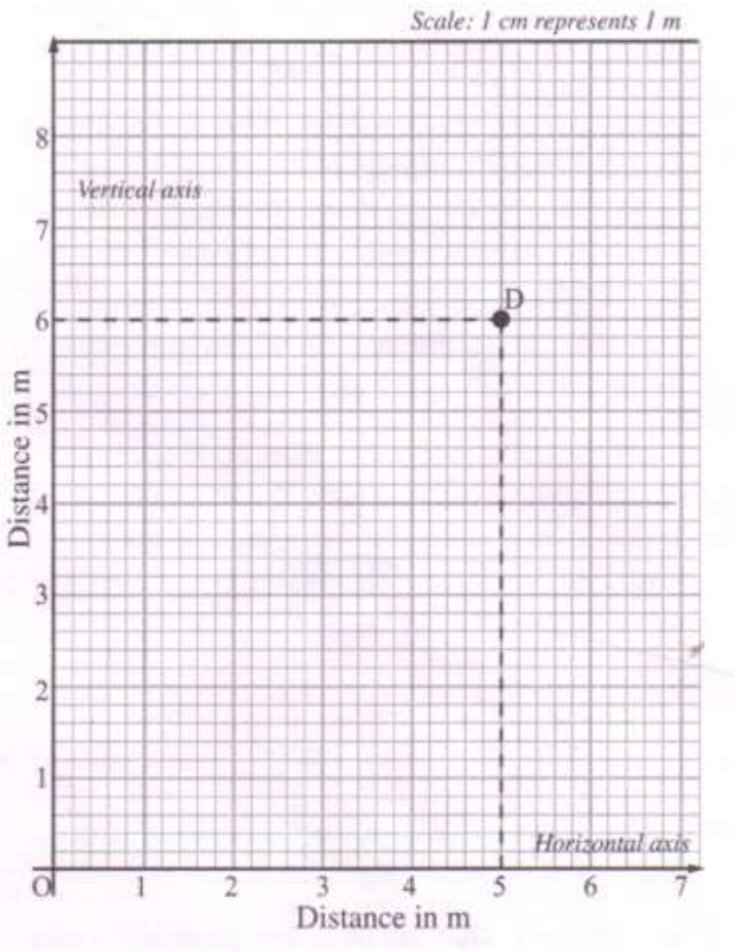


Fig. 10.4

The point marked D represents the relative position of the desk with reference to the two axes and the origin.

Drawing vertical and horizontal lines

The pair of axes discussed above refers only to positive direction of the number line.

Fig. 10.5 shows the axes extending to both positive and negative directions.

We have used a different scale to draw and label vertical and horizontal lines Fig. 10.5

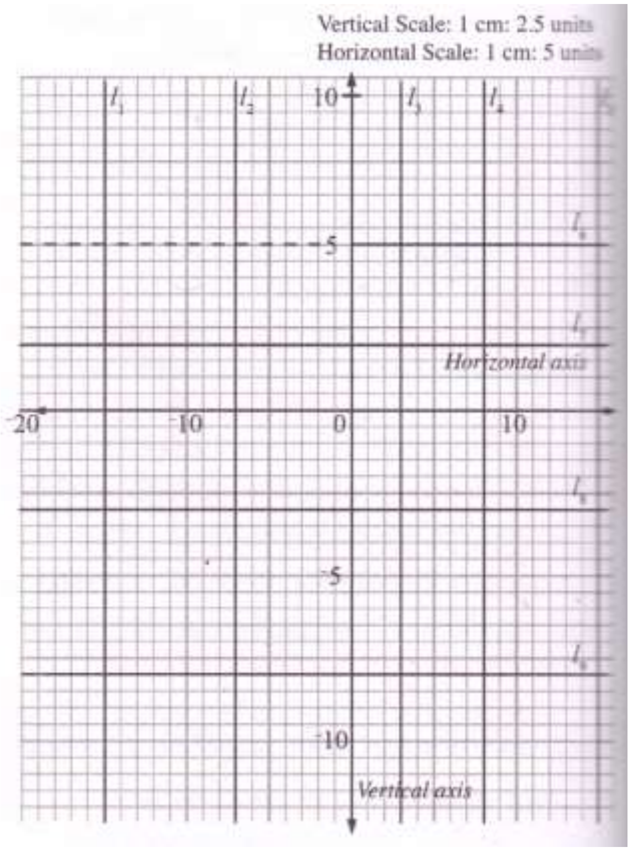


Fig. 10.5

Line l_1 to l_9 in Fig. 10.5 can be described with reference to the axes.

For example,

l_1 is a vertical line 15 units from the vertical axis and perpendicular to the horizontal axis.

Similarly,

l_6 is a horizontal line, parallel to the horizontal axis and perpendicular to the vertical axis.

You can now describe the remaining lines in a similar way as done above.

Exercise 10.1

1. In Fig. 10.6, A(2) gives the position of A and B(-4) gives the position of B. state the positions of C, D and E in the same way.

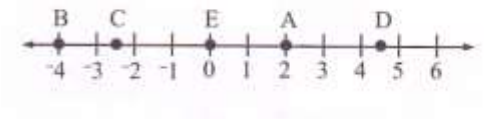


Fig. 10.6

2. Draw a number line from -5 to 5. On it, mark the points A (3), B (-1), C $(-4\frac{1}{2})$ and D$(3\frac{1}{2})$.
3. On a graph paper, draw a number line from -2 to 2. On it, mark the points A (1.0), B (0.4), C (-1, 6), D(-0.7) and E(1, 3).
4. Fig. 10.7 shows points on a centimeter square grid. Starting from the point marked O, and using ordered pairs, the position of R is (-2, 3), meaning that R is 2 cm to the **left** and 3 cm up. In a similar way, describe the positions of points P, Q, S and T.

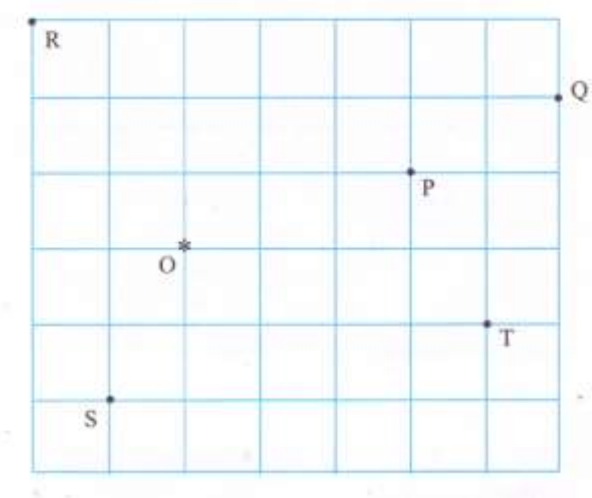


Fig. 10.7

The Cartesian plane

The positions of points on a line are found by using a number-line, and are written down using a single number e.g. P(-7).

The positions of points on a plane surface are found using **two** fixed number lines (i.e. two directions), usually at right angles, and are written down using two numbers called **ordered pairs.**

In Fig. 10.8, starting from the zero point, O, A is in position 2 units to the **right** and 3 units **up**, written as A (2, 3);

B is in position 4 units to the **right** and 0 units **up**, written as B (4, 0);

C is in position 0 units to the **right** and 2 units **down**, written as C (0, -2);

D is in position 2 units to the **left** and 2 units **down**, written as D (-2, -2).

In a similar way, find the positions of points E, F, G, H, I and J.

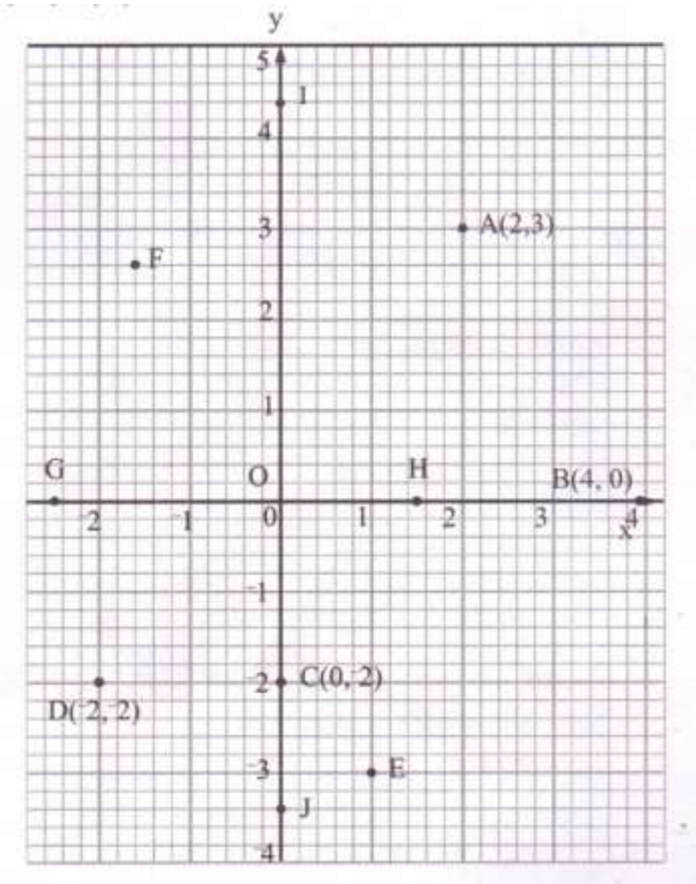


Fig. 10.8

Fig. 10.8 is a **graph**, or picture of the points A – J. in a graph like this, the number lines are called **axes**. They cross at the zero point of each axis. This point is called the **origin**. The axis going across from left to right is called the **x-axis**. It has a positive scale to the right of 0 and a negative scale to the left of 0.

The axis going up the page is called the y-axis. It has a positive scale upwards from 0 and a negative scale downwards from 0.

A plane surface with axes drawn on it, such as Fig. 10.8, is called **Cartesian plane.**

Point of interest

Cartesian coordinates is named after the great French philosopher and mathematician Rene' Descartes (1596-1650) (whose Latin name was Cartesians). Descartes' was the first person ever to use coordinates, and he is therefore considered to be the founder of **coordinate geometry**.

The ordered pairs that describe the positions of points on a Cartesian plane are called **Cartesian coordinates**. The first number is called the x-coordinate and it gives the distance of the point from the origin along the x-axis. The second number is called the y-coordinate and gives the distance of the point from the origin along the y-axis. For example, in (5, 7), 5 is the x-coordinate and 7 is the y-coordinate.

Exercise 10.2

1. Name the points in Fig. 10.9 which have the following coordinates.
   a) (0, 1.5)
   b) (0, -0.5)
   c) (-1, -2)
   d) (-2, 1)
   e) (0.5, 2.5)
   f) (1, -2.2)
   g) (-2, -1)
   h) (-2, 3)
   i) (2.5, 1)
   j) (4, -1)
   k) (-1.4, -0.6)
   l) (0, 0)
   m) (1, 0)
   n) (2.5, -1)
   o) (-3, 0)

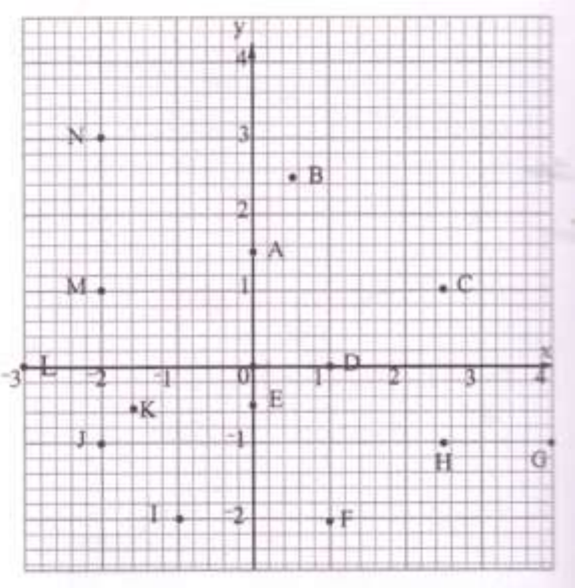


Fig. 10.9

2. What are the coordinates of the points A – K in Fig. 10.10?

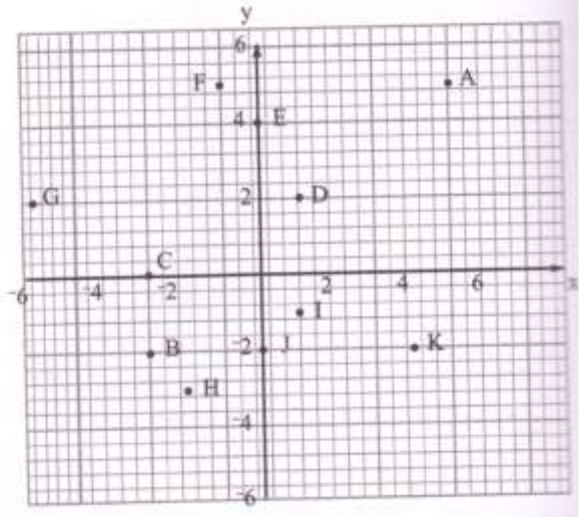


Fig. 10.10

3. In Fig. 10.11 write down the coordinates of the vertices of:
   a) Triangle ABC
   b) Parallelogram PQRS.

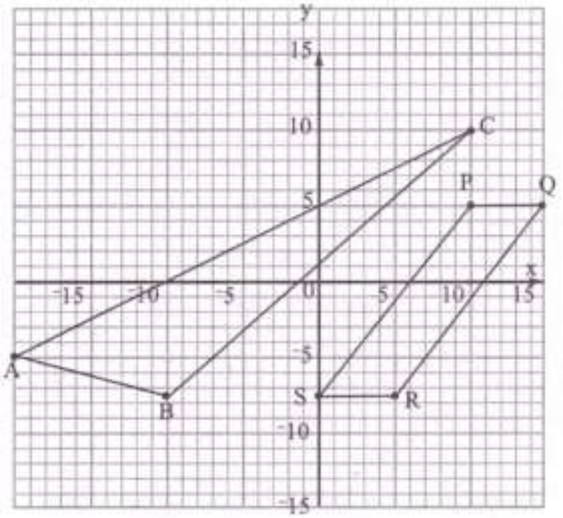


Fig. 10.11

4. Fig. 10.12 shows part of a map drawn on a Cartesian plane.

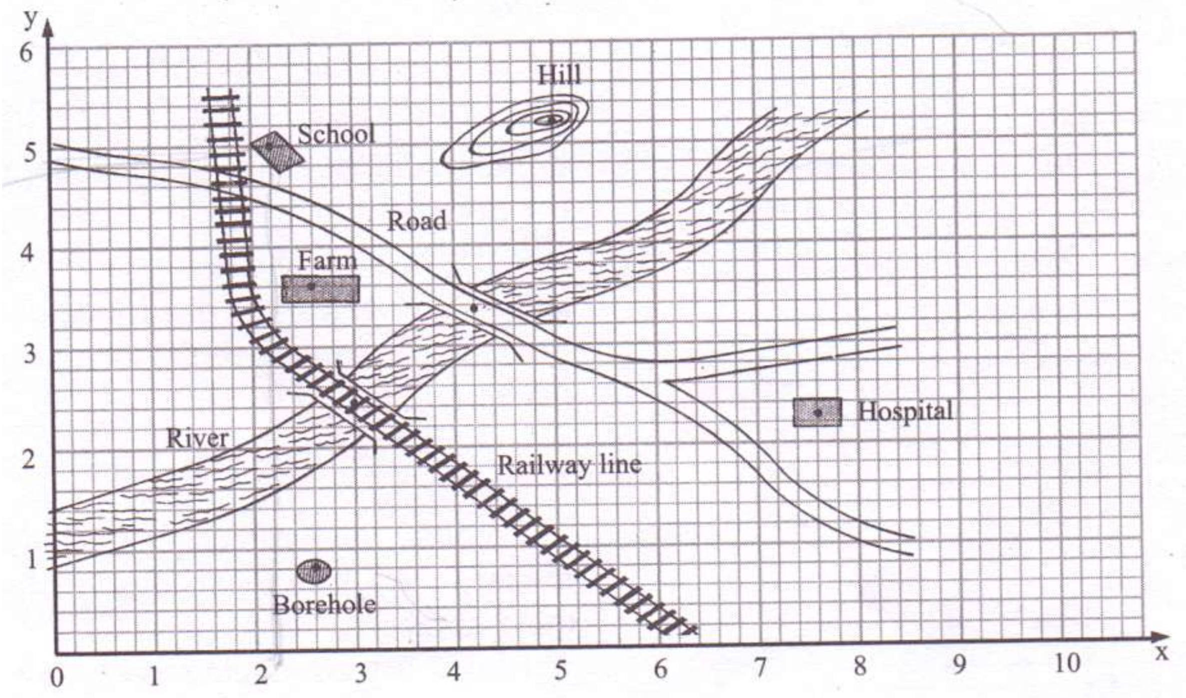


Fig. 10.12

Find the coordinates of the point in:

a) The school
b) The borehole
c) The farm
d) The hospital
e) The top of the hill
f) The intersection of the road and railway line.
g) The point where the railway line crosses the river.
h) The point where the road crosses the river.

5. State the coordinates of the vertices of the ‘arrow’ in Fig. 10.13.

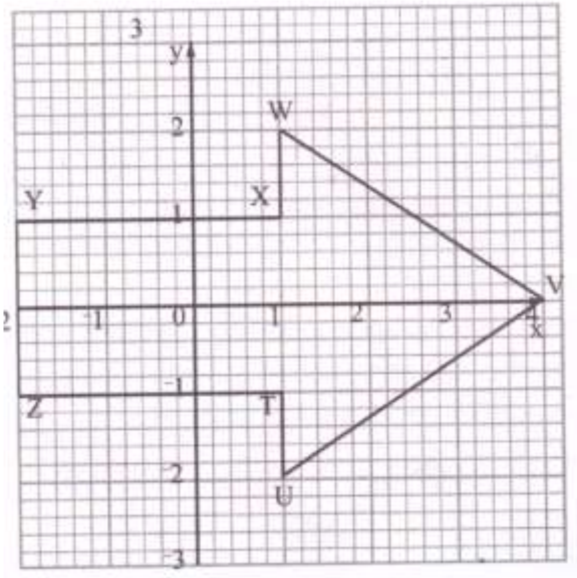


Fig. 10.13

Plotting points and choice of scale

To plot a point means to mark its position on a Cartesian plane.

Procedure:

1. Start at the origin
2. Move along the x-axis the number of steps, and in the direction given by the x-coordinate of the point.
3. Move up or down parallel to the y-axis the number of steps, and in the direction given by the y-coordinate of the point.
4. Mark the point with a dot (.) or a cross (x).

Example 10.1

Plot the points (2, -3) and (-2.4, 1.8) on a Cartesian plane.

Solution

The dotted lines in Fig. 10.14 show the method of plotting.

For (2, -3):

x-coordinates is 2, i.e. 2 steps in the positive direction of x-axis. Y-coordinate is -3, i.e. 3 steps in the negative direction of the y-axis.

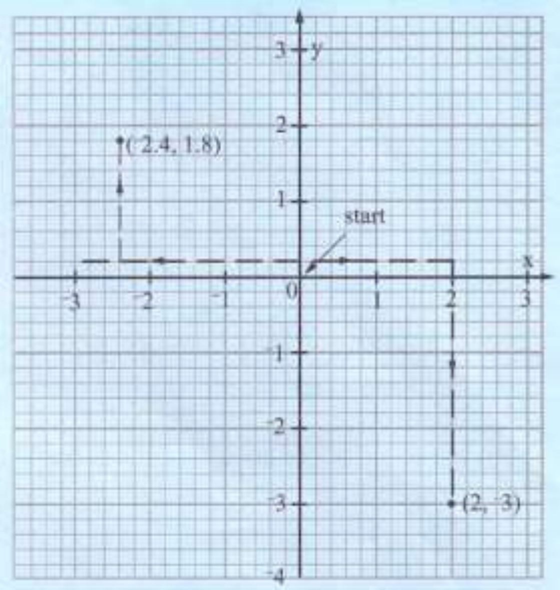


Fig. 10.14

For (-2.4, 1.8):

Move 2.4 steps to the left along the axis.

Move 1.8 steps up parallel to the y-axis.

Note that the dotted arrows in Fig. 10.14 are not normally put on the graph. They are used here only to show the method of plotting the points.

To choose the **scale** of a graph, check what the highest and lowest values of x and y are in the given points. Choose the scale such that the axes include all the numbers. The scale should be as large as possible to make it easy to plot and read coordinates accurately. We normally use 1 cm (square) to represent 1, 2, 5, 10, 20, 50, 100… Units. Do not use scales in multiples of 3 or 4 or such numbers as 7, 11, etc.! They are not easy to subdivide.

Example 10.2

The vertices of a quadrilateral are

A(-7.5, -5), B(0, -5), C(7.5, 7.5), D(0, 7.5).

a) Using a suitable scale, plot the plot the points A, B, C and D.
b) Join the vertices of quadrilateral ABCD and state what kind of quadrilateral it is.

c) Find the coordinates of the intersection of the diagonals of ABCD.

Solution

a) The highest x-coordinate is 7.5 and the lowest is -7.5. The x-axis must include these numbers.

The highest y-coordinate is 7.5 and the lowest is -5. With a scale of ‘1 cm represents 1 unit’ on each axis, it should be possible to include all the numbers but the figure would be unnecessarily large. A scale of ‘2 cm represent 5 units’ should be appropriate so that the figure is neither too large nor too small. The points are plotted as in Fig. 10. 15

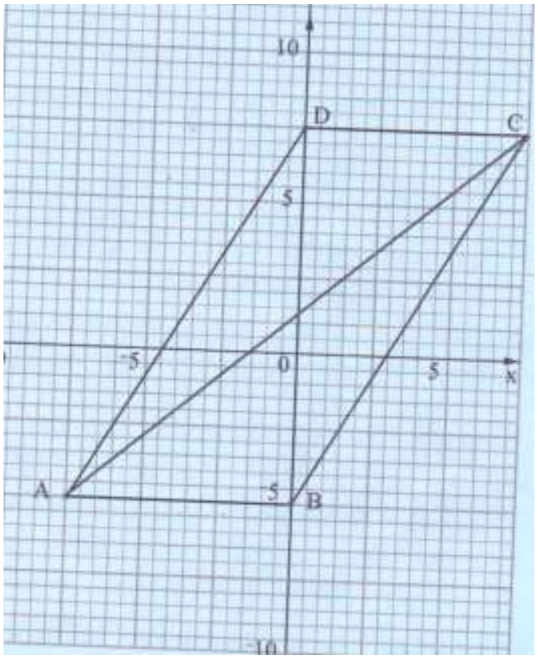


Fig. 10.15

b) ABCD is a parallelogram.
c) The point of intersection of the diagonals is (0, 1.25).

Exercise 10.3

1. On squared or graph paper, plot the following points, taking one square to represent a unit on both axes.
   A (-1.5, -0.5), B (0.5, -2.5) and C (4, 1.5). Given that ABCD is a parallelogram, find the possible coordinates of D.

2. Using a scale of ‘1 cm represents 2 units’ on both axes, plot the following points.

   A (8, 10), B (-8, -10), C (3, -5)
   D (-6, 9), E (-4, -7), F (1, 8), G (2, 0),
   H (0, -6), I (-2.4, 5.2), J (-4, 3.8),
   K (0, 6.6), L (0.8, -7.8).
3. The vertices of a triangle are X9-0.5, -0.8), Y (-0.5, -2.4) and Z (2, 0.8). Find the area of the triangle.
4. Plot the following points, joining each point to the next in the order they are given. A (1, 5), B (3, 5), C (5, 3) D (5, 1), E (3, 1), F (1, 3), G (3, 3). Draw in BE, GC and GA. What is the name of the solid formed?
5. Look carefully at the following sets of coordinates. Decide, without drawing, what shape they will make when they are joined together.
   a) (4, 2), (4, 4), (4, 5) (4, 6)
   b) (2, 3), (4, 3), (5, 3), (7, 3)
   c) (1, 1), (2, 2), (3, 3), (4, 3)

Now plot the given coordinates. Join each set of points. Were you right?

Linear graphs

We have learnt how to choose appropriate scale and use it to plot points whose coordinates are given. In this unit, we will expand our graphical work restricting ourselves to linear graphs only.

The ordered pair (x, y) represents coordinates of any point (i.e. a general point) on the Cartesian plane.

Consider the equation y = 3. For all values of x, y is always equal to 3. Thus (x, y) may lie anywhere on the line shown in Fig. 10.16

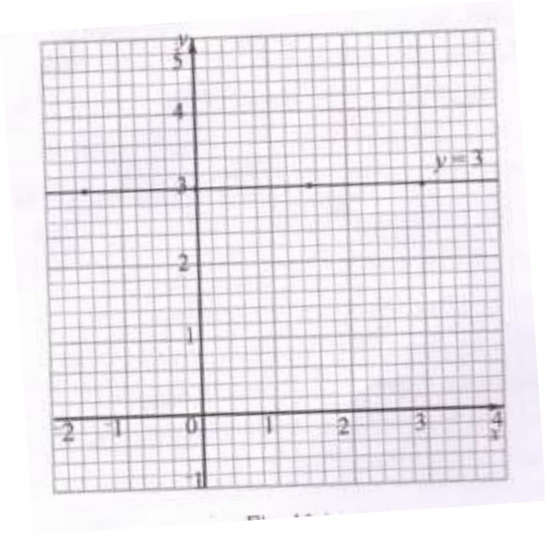


Fig. 10.16

Y = 3 is called the **equation of the line** and Fig. 10.16 is the graph of the line y = 3.

Fig. 10.17 shows lines which are parallel to the axes. The equation of each line is written beside it.

Note that the **equation of the x-axis is** y = 0, i.e. the line on which all points have the y-coordinate as 0.

Likewise x = **0 is the equation of the y-axis.**

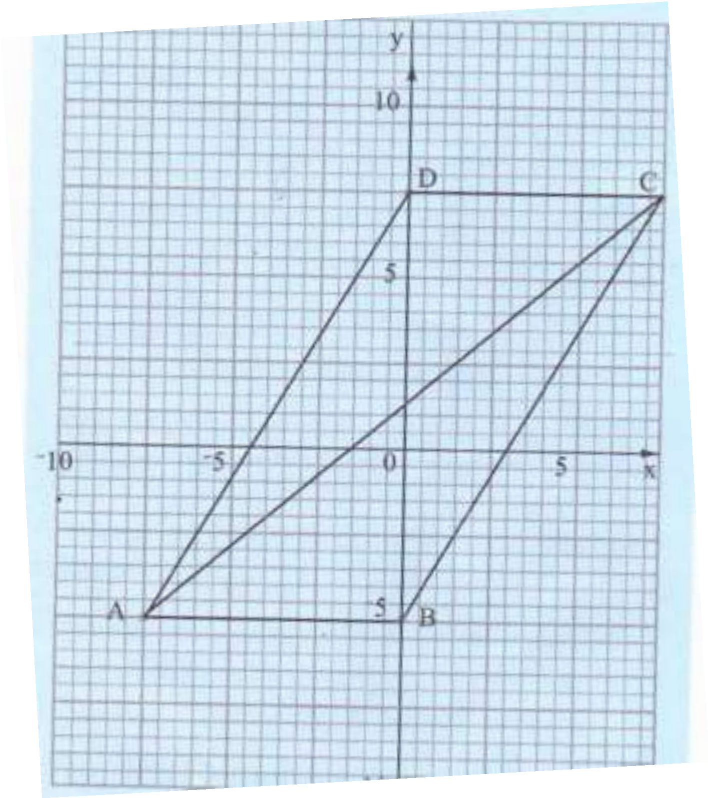


Fig. 10.17

Consider the equation y = x + 2 (a linear equation in x and y). For every value of x, there is a corresponding value of y, x and y are called **variables**. The equation gives the connection (or relation) between the variables.

For example, if x = 0, y = 2; if x = 1, y = 3; if x = 3, y = 5, etc. these values can be written as ordered pairs of corresponding x and y values as: (0, 2), (1, 3), (3, 5), etc.

If these pairs are plotted as point on the Cartesian plane, they give the **graph of the equation y = x + 2**, as shown in Fig. 10.18. Since the points lie on a straight line, they are joined using a straight edge or ruler.

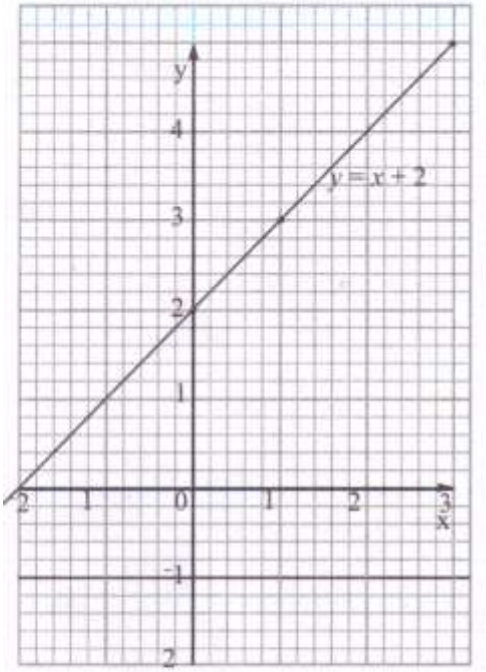


Fig. 10.18

Graphs of linear equations (Table of values)

When drawing a graph of a linear equation, it is sufficient to plot only two points. In practice, however, it is wise to plot three points. If the three points lie on the same line, the working is probably correct, if not you have a chance to check whether there could be an error in calculation.

Consider the equation y = 2x + 3.

If we assign x any value we please, we can easily calculate the corresponding value of y.

Y = 2x + 3,

When x = 0, $y = 2 \times 0 + 3 = 3$

When x = 1, $y = 2 \times 1 + 3 = 5$

When x = 2, $y = 2 \times 2 + 3 = 7$ and so on.

For convenience and ease while reading, the calculations are usually tabulated as shown in Table 10.1

x	0	1	2	3	4
2x	0	3	4	6	8
+3	3	3	3	3	3
Y = 2x + 3	3	5	7	9	11

Table. 10.1

From the table the coordinates (x, y) are (0, 3), (1, 5), (2, 7), (3, 9), (4, 11)…

A table such as Table 10.1 is called a **table of values** for y = 2x + 3.

From the calculation, the value of y depends on the value of x. y is therefore called the dependent variable and x the independent variable.

When drawing the graph, the dependent variable is marked on the vertical axis generally known as the **y – axis**. The independent variable is marked on the horizontal axis also known as the **x – axis.**

Fig. 10.19 shows the line whose equation is y = 2x + 3.

The graph of y = 2x + 3 is neither vertical nor horizontal. It meets the two axes at distinct point A and B.

The point A is called the **x-intercept** and has coordinates (-1.5, 0).

The point B is called the **y-intercept** and has coordinates (0, 3).

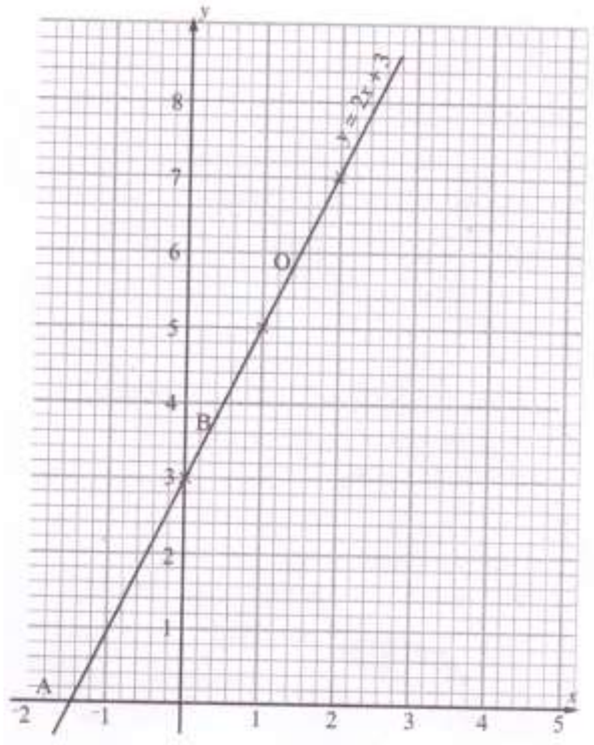


Fig. 10.19

Drawing and interpreting graphs

Graphs have a wide application in science and in many other fields. Graphs should be easy to draw, read and interpret if the following points are adopted.

i. Write a brief explanatory heading (title) above the graph.
ii. The quantity whose values are selected (independent variable) should be placed along the horizontal axis while the quantity whose values are observed or calculated (dependent variable) should be placed along the vertical axis.
iii. Choose as large a scale as the paper allows. This will make plotting and reading easy. Ensure that you accommodate all the data in the table.
iv. Graduate and clearly label the axes and write the units used.
v. If two graphs are drawn on the same axes, label each clearly.

Example 10.2

Table 10.2 shows the volume of a gas at various temperatures when heated from 0˚C

Temperature ˚C	20	40	60	80	100
Volume (litres)	1.82	1.95	2.07	2.20	2.35

Table 10.2

a) Using a suitable scale draw a graph of volume against temperature.
b) Use your graph to find:
   i. *The initial volume of the gas*
   ii. *The volume of the gas when the temperature is 48˚C and 70˚C*
   iii. *The temperature of the gas when the volume is 1.8 litres and when it is 2.1 litres.*

Solution

a) Fig. 12.7 shows the required graph. 1 cm represents 0.1 litre on the vertical axis while 1 cm represents 20˚ on the horizontal axis. The zero mark is not shown on the vertical axis. It is necessary to shown the zero mark on the horizontal axis because of question (b) (i).

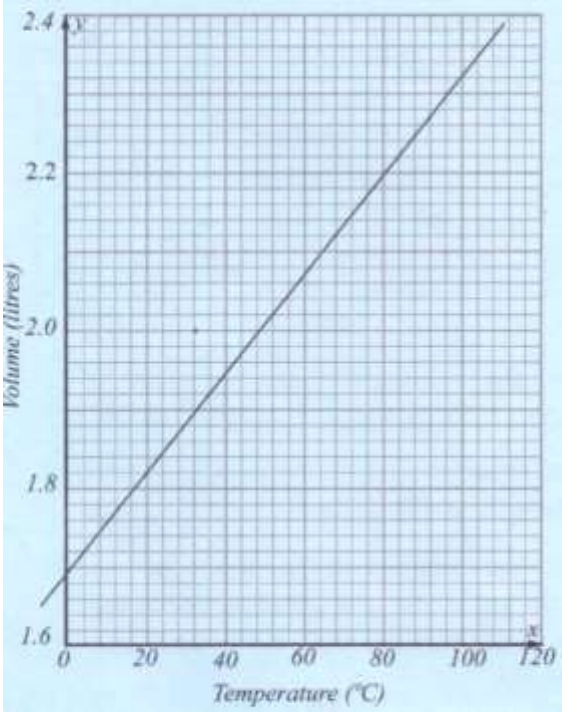


Fig. 10.20

b) (i) Initial volume of gas is 1.7 litres.
(ii) At 48˚C volume is 2.0 litres and at 70˚C volume is 2.13 litres.

(iii) When volume is 1.8 litres, temperature is 16˚C and when the volume is 2.1 litres, temperature is 64˚C

Exercise 10.4

1. Electricity cost consists of a standing charge and an additional amount which depends on the number of watts used. Table 10.3 shows the amounts payable for various number of watts used.

Number of watts (W)	Amount (K)
10	60
20	100
30	140
40	180
50	220
60	260

Table 10.3

   a) Choose a suitable scale and draw a graph of amount paid against the number of watts (W) used.
   b) Use your graph to find:
      i. The payment for consumptions of 47 watts, 25 watts and 43 watts.
      ii. The consumptions for payment of K 40, K 124 and K 248.
      iii. The standing charge

2. Table 10.4 shows the extension on a spring for different masses.

Mass(g)	3.2	4.5	5.5	6.3	7.1	7.7	8.4	8.9
Extension (mm)	6.3	8.9	10.9	12.6	14.2	15.4	16.8	17.8

Table 10.4

a) Choosing a suitable scale for each axis, draw a graph of extension against mass.
b) Using your graph, find the extensions for masses of 3 g, 5.2 g and 7.3 g.
c) Using the graph, find the mass that would give an extension of 4.0 mm, 9.2 mm and 16.2 mm.

3. Table 10.5 shows temperatures in degrees Celsius (˚C) and their equivalents in degrees Fahrenheit (˚F).

Degrees Celsius (˚C)	-50	-25	0	25	50	75	100
Degrees Fahrenheit (˚F)	-58	-13	32	77	122	167	212

Table 10.5

a) Use a suitable scale to draw a graph of ˚F (Fahrenheit) against ˚C (Celsius).

b) Use your graph to convert the following temperatures in degrees Fahrenheit to degree Celsius.

i. O ˚F

ii. -25 ˚F

iii. 90 ˚F

iv. 25 ˚F

c) Use your graph to convert the following temperatures in degrees Celsius to degrees Fahrenheit.

i. 20 ˚C

ii. -10 ˚C

iii. 39 ˚C

iv. -40 ˚C

4. The measured volumes of a gas at various temperatures are shown in Table 10.6.

Temperature (˚C)	25	50	75	100	125
Volume (litres)	1.82	1.945	2.075	2.20	2.33

Table 10.6

a) Using a suitable scale, draw the graph of volume against temperature.

b) Use your graph to find:

i. The initial volume of the gas 9i.e. volume when temperature is 0˚C.

ii. The volume of the gas when the temperature is 60˚C and 83˚C.

iii. The temperature of the gas when the volume is 2 litres and when it is 2.3 litres.

5. In Table 10.7, d is the number of days that food lasts a given number of students (n).

$\frac{1}{d}$	0.1	C 0.05	0.025	0.02	0.0125	0.01	0.008
n	1000	500	250	200	125	100	80

Table 10.7

Using a scale of 1 cm to 100 students on the vertical axis and 1 cm to 0.0125 units on the horizontal axis, draw a graph of n against $\frac{1}{d}$ and use it to find how long the food will last for 100, 350 and 870 students.

Gradient of a line

Consider the straight line shown in Fig. 10.21 below.

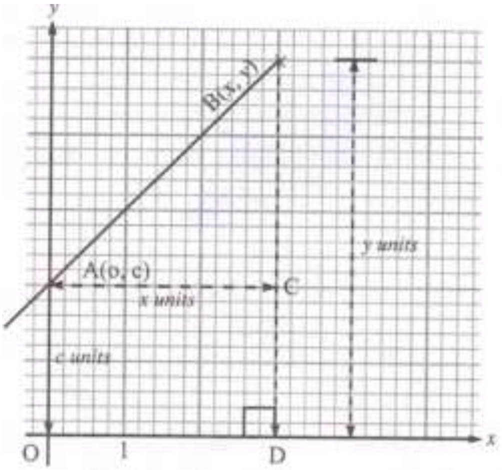


Fig. 10.21

The points A and B lie on a line and this line can be extended both sides indefinitely.

From point A to B, the horizontal distance AC = x units and vertical distance is BC = BD – OA = Y – C

The slope of this or any other line is given by the ratio of the vertical distance and the horizontal distance between any two points on the line.

In this case from A to B.

The vertical distance + BC

And the horizontal distance = AC

Therefore, The measure of the slope of the line also known as the gradient, $= \frac{BC}{AC} = \frac{y-CD}{x-0} = \frac{y-CD}{x}$

The gradient can be given by any other two points on the line. Fig. 10.22.

Suppose P(x, y) and Q(x_1, y_1)

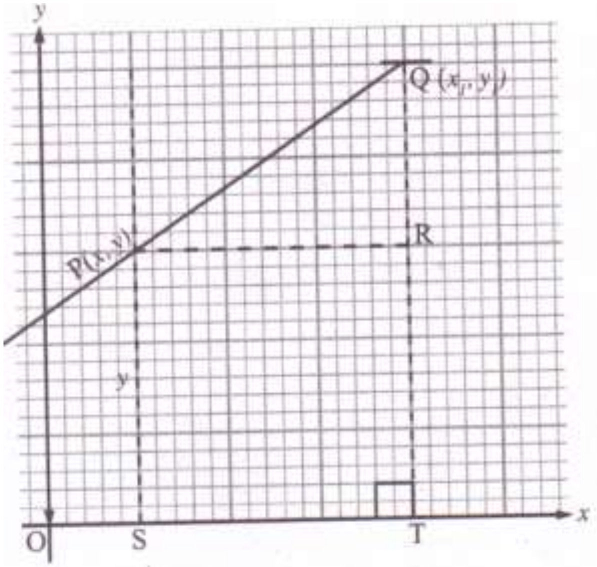


Fig. 10.22

From P and Q,
Vertical distance = QT – TR = $y_1 - y$
Horizontal distance = OT – OS = $x_1 - x$
The gradient of the line on which P and Q lie

$$= \frac{QR}{PR}$$

But QR = $y_1 - y$ and PR = $x_1 - x$

$$\text{The gradient} = \frac{QR}{PR} = \frac{y_1 - y}{x_1 - x}$$

Note: A line may have a positive or a negative gradient depending on the way it slopes.

Line *l* has **positive** gradient. See Fig. 10.23.

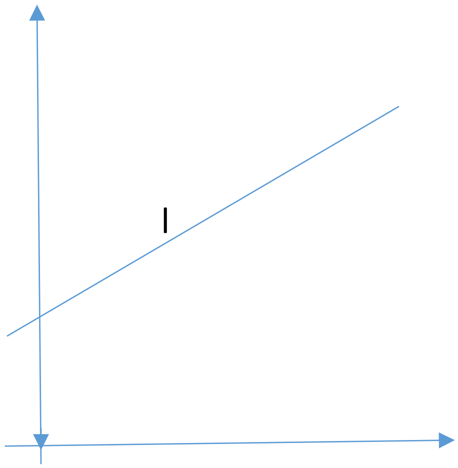


Fig. 10.23

Any line drawn from bottom left hand corner to top right hand corner has a positive gradient i.e. the line slopes upwards from left to right.

Line *l* has **negative** gradient. See Fig. 10.24.

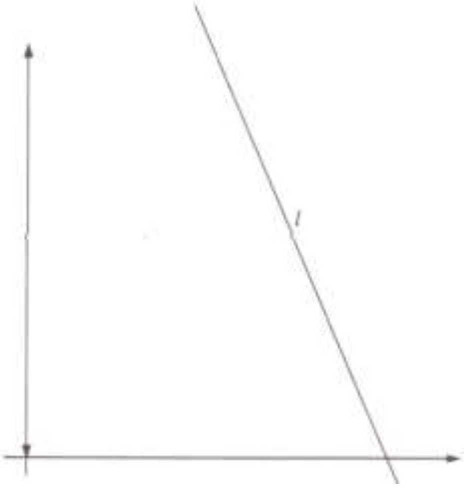


Fig. 10.24

Any line drawn from top left hand corner to bottom right hand corner has a negative gradient i.e. the line **slopes downwards from left** to right.

Example 10.3

Use any two points on the line to calculate the gradients of the following lines

a) Y = 2x + 2
b) Y = -3x + 1

Solution

a) Using equation y = 2x + 2 make a table of values (three points only) Table 10.8

x	1	2	3
2x	2	4	6
3	2	2	2
y	4	6	8

Let the three points on the line be
A (1, 4)
B (2, 6)
C (3, 8)

Fig. 10. 8

Using points A and C,

Gradient = $\frac{vertical\ distance}{horizontal\ distance}$

$= \frac{8-4}{3-1}$

$= \frac{4}{2}$ = 2

Therefore, the gradient of the line is 2.

b) Using equation y = -3x + 1 make a table of values for x and y. Table 10.9.

x	0	1	2	3
2x	0	-3	-3	-3
3	1	1	1	1
y	1	-2	-5	-8

Fig. 10.9

Let the four on the table be
A (0, 1)
B (1, -2)
C (2, -5)
D (3, -8)

Using points B and D

Gradient = $\frac{vertical\ distance}{horizontal\ distance}$

$= \frac{-8-(-2)}{3-1}$

$= \frac{-8+2}{3-1} = \frac{-6}{2}$ = -3

Therefore, the gradient of the line is -3.

How does this gradient compare with the coefficient of x?

Use any other two points to confirm that the gradient is correct.

Example 10.4

On the same axis, draw the graphs of

Y = -1.5, x = 2, y = 3x and 5x + 7y = 14, for values of x from -4 to +4.

For each of the lines drawn, calculate the gradient.

Solution

Choose values of x, within the given range, that enable easy calculation of the corresponding y values, and which result in points that are easy to plot. It is convenient to show the working and corresponding values in tables, as in Tables 10.4 to 10.7

y = -1.5

x	-2	0	3
y	-1.5	-1.5	-1.5

Table 10.4

x = 2

x	2	2	2
y	-2	0	2

Table 10.5

Y=3x

x	-4	0	3
y	-12	0	9

Table 10.6

To make it easier to find values of y,

5x + 7y = 14 may be arranged as:

7y = 14 – 5x (subtracting 5x from both sides)

$y = 2 - \frac{5}{7}x$

(dividing both sides by 7)

x	-3.5	0	3.5
2	2	2	2
$\frac{5}{7}x$	2.5	0	-2.5
y	4.5	2	-0.5

Table 10.7

The graphs are as shown in Fig. 10.24

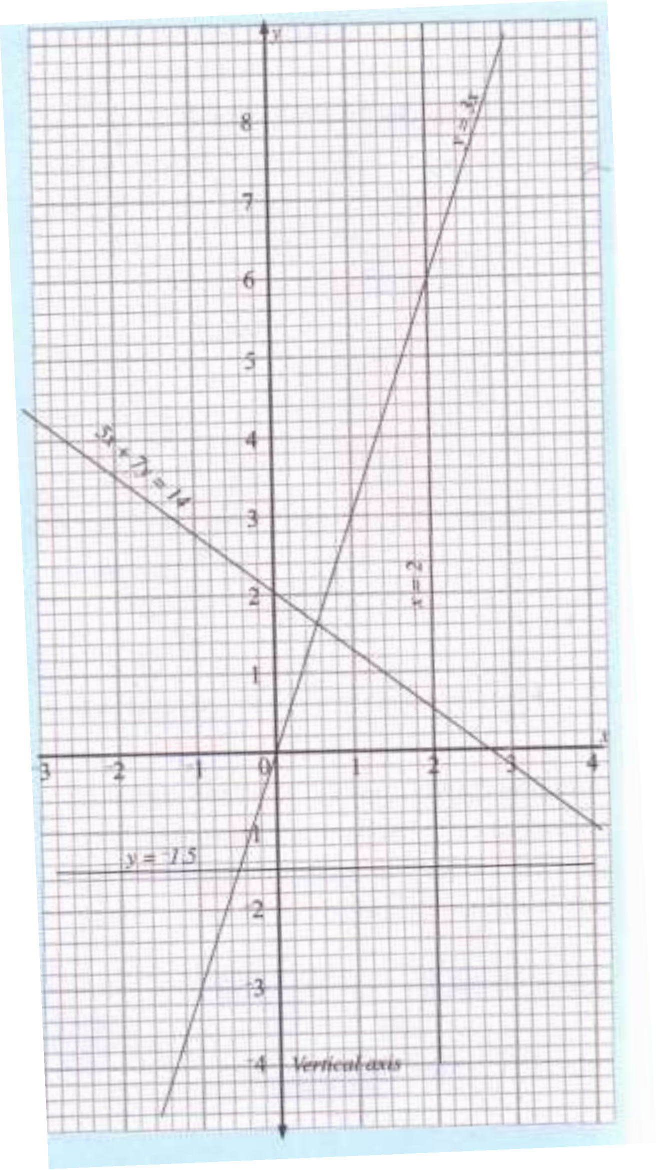


Fig. 10.24

Solution

For the line y = -1.5, using points (-2, -1.5) and (3, -1.5)

Vertical distance = -1.5 – (-1.5) = 0

Horizontal distance = 3 – (-2) = 5

Gradient = $\frac{0}{5} = 0$

Therefore, Gradient of the line y = -1.5 is 0

For the line x = 2, for any two points e.g. (2, 0) and (2, 6)

Vertical distance = 6 – 0 = 0

Horizontal distance = 2 – 2 = 0

But division by zero does not exist.

Therefore, we say the line has no gradient

For the line y = 3x, using points (0, 0) and (2, 6)

Gradient = $\frac{6-0}{2-0} = \frac{6}{2} = 3$

Note: that the gradient is equal to the coefficient of x.

For the line 5x + 7y = 14, using points (-3.5, 4.5) and (0, 2)

Gradient =$\frac{2-(4.5)}{0-(-3.5)}$

$= \frac{-2.5}{3.5} = \frac{-5}{2} \times \frac{2}{7}$

$= \frac{-5}{7}$

If we solve for y in terms of x in the equation

5x + 7y = 14

One gets 5x + 7y = 14

7y = -5x + 14

y = $\frac{-5}{7}x + \frac{14}{7}$

$y = \frac{-5}{7}x + 2$

The coefficient of x in the equation is $\frac{-5}{7}$ *which is equal to the gradient of the line.*

Note*: In general, to find the gradient of a line ax + by = c where a, b, and c are constants, we make y the subject of the formula then the coefficient of x represents the gradient of the given line.*

Point to note

1. A line whose equation is of the form y = c where c is a constant, implies that the line:
   - (i) is parallel to the x-axis
   - (ii) meets the y-axis at a point whose coordinates are (0, c).
   - (iii) has zero gradient.
2. A line whose equation is of the form x = k where k is a constant, implies that the line;
   - (i) Is parallel to the y-axis.
   - (ii) Meets the x-axis at a point whose co-ordinates are (k, 0).
   - (iii) Its gradient is not defined.
3. A line whose equation is of the form y = mx, implies that:
   - (i) the line passes through the origin, (0, 0)
   - (ii) The steepness of the line is determined by the value of m. i.e. gradient = m
4. A line whose equation is of the form y = mx + c, where m and c are constants, implies that the line meets the y-axis at a point whose coordinates are (0, c) and the gradient of the line is **m**.

In note (1) and (4), point (0, c) is called the y-intercept i.e. the point at which a line meets the y-axis.

Similarly, in note (2), a point such as (k, 0) is called the x-intercept, a point where a line meets the x-axis.

Activity 10.1

Using Fig. 10.24 identify the intercept for each line. Hence copy and complete the table below.

Equation	y-intercept	x-intercept	Parallel to x-axis	Parallel to y-axis
y = -1				
y = 0	(0, 0)	(0, 0)	Yes. Same line	No
y = 2				
x = -1				
x = 0				
x = 3				
y = x + 2				

Exercise 10.5

1. State the equations of the lines (a), (b), (c) and (d) shown in Fig. 10.25.

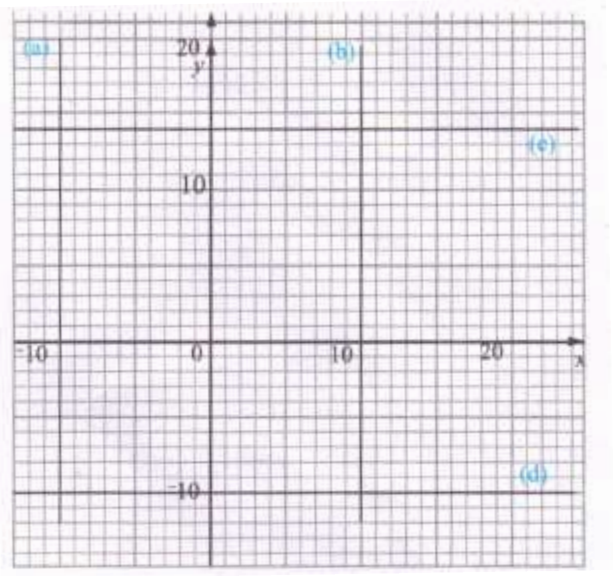


Fig. 10.25

2. State the equations of the x and y-axes.
3. On a Cartesian plane, draw and label the lines represented by the following equations.
   a) x = 5
   b) y = -2
   c) y = 3.2
   d) x = -3

4. Copy and complete Table 10. 8 (a) to (d) for each of the given linear equations.

a) Y = x + 1

x	0	1	2	3
y				

b) Y = 5 – 2x

x	0	2	4
y			

c) $y + \frac{1}{4}x = 2$

x	-4	0	4
y			

d) 2x – y + 3 = 0

x	0	1	2
y			

Table 10.8

Hence,

State the coordinates of points where lines;

y = x + 1 and y = 5 – 2x, $y + \frac{1}{4}x = 2$ and 2x – y + 3 = 0 intersect.

On squared paper, draw the graphs and in each case, find the gradient.

5. Table 10.9 gives corresponding values of x and y for the equation y = x + 3.

x	-2	0	2
y	1	3	5

Table 10.9

a) Use a scale of 2 cm to 1 unit on both axes and draw the graph of y = x + 3.
b) Extend the line to find the coordinates of the point where it cut the x-axis.

6. Copy and complete Table 10.10 for the equation 3y + 2x = 21.

x	-6	-3	0	3	6
y	11				3

Table 10.10

a) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of 3y + 2x = 21.

b) Use your graph to find the value of:
   i. y when x = -1.5
   ii. x when y = 4

c) Write down the coordinates of the points where the line cuts the axes.

d) Calculate the gradient of the line.

7. Using a scale of 1 cm to represent 1 unit on both axes, draw the graphs of the following equations, for values of x from -3 to 3.
   a) y = x + 2
   b) y = x
   c) y = 2x
   d) y = 2 – 3x
   e) Find the gradient of each line.
8. (a) Using a scale of 2 cm to represent 1 unit on both axes, draw on the same axes the lines y =$\frac{1}{3}$ x; y = $\frac{1}{3}$ $x + 2$ and y = $\frac{1}{3}$ $x - 2$ for values of x from -6 to 6.

What do you notice about the three lines?

(b) Find the gradient of the lines.

9. (a) For values of x from -2 to 2, draw the graph of $\frac{y-x}{4} - \frac{2x-y}{3} = 0.$

(b) Calculate the gradient of the line.

10. State whether the following points lie on the line y = 7x – 1.

a) (1, 6) b) (3, 2) c) (-1, 6)

if they do, calculate their gradient.

Unit 11 NUMBER PATTERN 1

Introduction

Number patterns have been used since ancient times. For example, the early Arabs, Hindus, Greeks and Babylonians studied certain number patterns which they believed had mystic powers.

In this unit, we shall look at some numbers which have some interesting observations and usefulness in real life.

Number patterns

The number 6 can be represented using patterns of dots as shown in Fig. 11.1

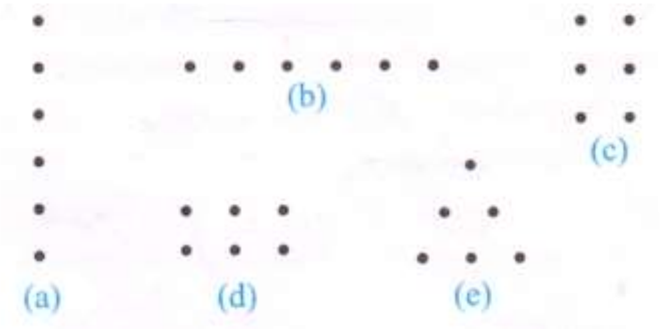


Fig. 11.1

Fig. 11.1 9a) and (b) are said to be **linear patterns**. Fig. 11.1 (c) and (d) are said to be **rectangular patterns**. Fig. 11.1 € is said to be a **triangular pattern**.

Numbers which can be represented by a rectangular pattern of dots are called **rectangle numbers**. Each rectangle number has two or more factors other than itself.

Numbers which can be represented by a square pattern of dots are called **square numbers**.

Fig. 11.2 shows some square numbers and their patterns of dots.

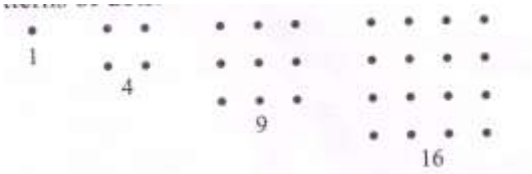


Fig. 11.2

Since $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$ and so on, we can write the square numbers 1, 4, 9, 16…. as 1^2, 2^2, 3^2, 4^2,…..

We can generate other number patterns from the number pattern in Fig. 11.2

Note that $4 = 2^2$ is obtained by adding 3 to 1^2, $9 = 3^2$ is obtained by adding 5 to 2^2, and so on.

So we have $2^2 = 1^2 + 3$ or $2^2 – 1^2 = 3$

$3^2 = 2^2 + 5$ or $3^2 – 2^2 = 5$

$4^2 = 3^2 + 7$ or $4^2 – 3^2 = 7$.

Now, copy and fill in the gaps in the following:

$5^2 = 4^2 + …$ or $5^2 – 4^2 =$

$6^2 = 5^2 + …$ or $6^2 – 5^2 =$

$7^2 = 6^2 + …$ or $7^2 – 6^2 =$

.

.

.

$50^2 = 49^2 + …$ or $50^2 – 49^2 =$

.

.

.

$99^2 = 98^2 + …$ or $99^2 – 98^2 = …$

Thus, from this pattern, we see that

$a^2 – b^2 = a + b$, provided that a and b are consecutive positive integers and $a > b$.

Using the pattern, evaluate:

(a) $111^2 – 110^2$
(b) $888^2 – 887^2$

Copy and complete the following pattern

1 = 1^2

1 + 3 = 2^2

1 + 3 + 5 =

1 + 3 + 5 + 7 =

1 + 3 + 5 + 7 + 9 =

1 + 3 + 5 + 7 + 9 + 11 =

Numbers like 6 which form a triangular pattern [as in Fig. 11.1 €] are called **triangle numbers**.

Copy and complete the following pattern

1 = 1

1 + 2 = 3

1 + 2 + 3 = 6

1 + 2 + 3 + 4 =

1 + 2 + 3 + 4 + 5 =

1 + 2 + 3 + 4 + 5 + 6 =

1 + 2 + 3 + 4 + 5 + 6 + 7 =

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 =

What type of numbers do you get?

Exercise 11.1

1. Examine the numbers 1 to 40. List down all:
    a) Rectangle numbers
    b) Square numbers
    c) Triangle numbers
2. Using your for Question 1 (c),
    a) List the differences, between pairs of consecutive triangle numbers.

   - b) List the sums, s, of consecutive triangle numbers.
   - c) What is the connection between difference and sum?
3. From Question 1 (c), how many prime numbers are also triangle numbers?
4. Copy and complete the following numbers pattern.

   $3^2 - 1^2 = 8$
   $4^2 - 2^2 = 12$
   $5^2 - 3^2 = 16$
   $6^2 - 4^2$
   $7^2 - 5^2$
   $8^2 - 6^2$

   What relationship can you deduce from the set of numbers 3, 1, 8; 2, 12; 5, 3, 16; ….
5. When 11 is squared, we get 121 which is said to be symmetrical. Similarly, $11^3 = 1\,331$ is symmetrical
   - a) Which of the following is symmetrical 11^4 , 11^5, 11^6 ?
   - b) Which integer between 1 and 11 has a cube which is a symmetrical number?

Sequences

General sequences

Examine the following lists of numbers

- a) 1, 2, 3, 4, 5….
- b) 2, 4, 6, 8, 10…
- c) 1, 3, 5, 7, 9….
- d) 1, 3, 9, 27, 81….
- e) 21, 19, 17, 15, 13….
- f) 15, 14, 12, 9, 5…..
- g) 1, 3, 10, 4, 19…..
- h) 0, 5, 1, 10, -47…..

Five numbers are given in each list. We refer to each number in a list as a **term**. Thus, in (a), 1 is the **first term**, 2 is the **second term**, etc.

What could be the next three terms in each list? How do you obtain them?

The terms could be obtained as shown below.

a) 1, 2, 3, 4, 5, 6, 7, 8:
(any term) + 1 gives the next term or
Since, 1st term = 1,
2nd term = 2,
3rd term = 3 and so on,
Then, 6th term = 6,
7th term = 7,
8th term = 8.

b) 2, 4, 6, 8, 10, 12, 14 , 16:
(any term) × 2 = next term; or
Since, 1st term = 1 × 2,
2nd term = 2 × 2,
3rd term = 3 × 2 and so on,
Then, 6th = 6 × 2,
7th = 7 × 2,
8th = 8 × 2.

c) 1, 3, 5, 7, 9, 11, 13, 15:
(any term) + 2 = next term; or
1st term = 1 × 2 – 1
2nd term = 2 × 2 – 1
3rd term = 3 × 2 – 1, so on
6th term = 6 × 2 – 1,
7th term = 7 × 2 – 1
8th term = 8 × 2 – 1.

d) 1, 3, 9, 27, 81, 243, 729, 2 187:
(any term) × 3 = next term
1st term = 1
2nd term = $3^1 = 3^{(2-1)}$
3rd term = $3^2 = 3^{(3-1)}$
4th term = $3^3 = 3^{(4-1),}$ so on
6th term = $3^5 = 3^{(6-1)}$
7th term = $3^6 = 3^{(7-1)}$

8th term = $3^7 = 3^{(8-1)}$

e) 21, 19, 17, 15, 13, 11, 9, 7:

(any term) -2 = next term or.

1st term = 21
2nd term = 21 – (2 × 1),
3rd term = 21 – (2 × 2),
4th term = 21 – (2 × 3),
5th term = 21 – (2 × 4),
6th term = 21 – (2 × 5),
7th term = 21 – (2 × 6),
8th term = 21 – (2 × 6).

f) 15, 14, 12, 9, 5, 0, -6, -13: 1st term -1 = 2nd term, 2nd term -2 = 3rd term, 3rd term – 3 = 4th term.

1st term = 15,
2nd term =1st term – 1,
3rd term = 2nd term – 2,
4th term = 3rd term – 3,
5th term = 4th term – 4,

g) And (h) have no pattern that pattern that would help us obtain the next term.

A list of numbers, arranged in order and having a rule which give the terms in the list [as in (a) – (f) is called a **sequence.**

Thus, the lists of numbers in (a) to (f) are **sequences** while 9g) and (h) are **not sequences.**

Since the terms in the sequences 1, 2, 3, 4, 5 are increasing in magnitude, the sequence is an **increasing sequences**. It follows that the sequence 21, 19, 17, 13… Is a **decreasing sequence.**

Note that the sequence 1, 3, 5, 7, 9, 11 has 6 terms only while the sequence 1, 3, 5, 7, 9, 11…. Has many terms that cannot be exhausted. The first sequence is called a **finite sequence** while the second one is an **infinite sequence.**

Examine the following sequences again.

a) 1, 2, 3, 4, 5…
b) 2, 4, 6, 8, 10….
c) 1, 3, 5, 7, 9….
d) 1, 3, 9, 27, 81….

How would you obtain the 10th, 20th, 100th and nth terms?

a) 1, 2, 3, 4, 5….

1st term = 1,
2nd term = 2,
3rd term = 3,
4th term = 4, etc.

It follows that;

10th term = 10,

20th term = 20,

100th term = 100

And nth term = n.

b) 2, 4, 6, 8, 10 ……

We saw that
1st term = 1 × 2,
2nd term = 2 × 2,
3rd term = 3 × 2,
4th term = 4 × 2, and so on.
Thus
10th term = 10 × 2,
20th term = 20 × 2,
100th term = 100 × 2
nth term = n × 2= 2n

c) 1, 3, 5, 7, 9…..

We saw that,
1st term = 1 × 2 – 1

2nd term = 2 × 2 – 1
3rd term = 3 × 2 – 1
4th term = 4 × 2 – 1 and so on.
Therefore, 10th term = 10 × 2 – 1
20th term = 20 × 2 – 1
100th term = 100 × 2 - 1

and hence,

nth term = n × 2 – 1 = 2n – 1

d) 1, 3, 9, 27, 81 ….
1st term $= 1 = 3^0$
2nd term $= 3 = 3^1$
3rd term $= 9 = 3^2$
4th term $= 27 = 3^3$
5th term $= 81 = 3^4$
It follows that
10th term $= 3^9$
20th term $= 3^{19}$
100th term $= 3^{99}$
nth term $= 3^{n-1}$

We see that the nth term in (a) to (d) gives n, 2n, 2n – 1 and 3^{n-1} respectively.

We say that these are the rules or laws which give the terms of the sequences or the **general terms of the sequences**.

Exercise 11.2

1. For each of the following sequences, find the next three terms and the rule which gives the terms.
   a) 0, 3, 6, 9, 12…..
   b) 0, 4, 8, 12, 16….
   c) 1, 4, 7, 10, 13…
   d) 90, 85, 80, 75, 70….
   e) 88, 84, 80, 76….
   f) 30, 20, 10, 0, -10….

2. For each of the following sequences, find the next three terms and the rule which gives the terms
    a) 1, 4, 16, 64, 256….
    b) 2, 4, 8, 16, 32…..
    c) 3, 6, 9, 12, 15….
    d) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$
    e) $1, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \frac{1}{16},$
    f) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$
3. For each of the following sequences, find the indicated terms.
    a) 5, 8, 11, 14, 17….; 8th and 12th terms
    b) 11, 9, 7, 5, 3….; 9th and 13th terms
    c) -1, 3, -5, 7, -9….; 10th and 12th terms
    d) 39, 33, 27, 21, 15….; 8th and 10th terms
    e) -2, -5, -8, -11…; 12th and 15th terms
    f) 2, -6, -14, -22, -30….; 10th and 12th terms.

4. For each of the following sequences, find the indicated terms and the general of the sequence.
    a) 1, 4, 9, 16, 25…..; 8th and 11th terms
    b) 2, -6, 18, -54, 162…; 7th and 10th terms.
    c) 2, 4, 8, 16, 32…; 7th and 11th terms.
    d) 1, 8, 27, 64, 125….; 10th and 15th terms.
    e) $4, 2, 1, \frac{1}{2}, \frac{1}{4}$….. ; 9th and 14th terms.
    f) $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}$……; 8th and 11th terms.

5. For each of the following sequences, find the next three terms.

    a) $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}$…….
    b) $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}$…..
    c) $1, 2\frac{1}{2}, 1\frac{3}{4}, 2\frac{1}{8}$…
    d) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$, …..

e) $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \frac{25}{6}, \ldots.$

f) $\frac{1}{6}, 1\frac{11}{6}, \frac{8}{3}, \frac{7}{2}, \ldots.$

6. Write down the first four of the sequence for which the general term is

a) $2n - 3$ d) $6n + \frac{1}{2}$ f) $3 \times 4n$

b) $n^2 + 1$ e) $2^n(n - 1)$ g) $2 \times 3^{(n-1)}$

c) $\frac{n}{n+1}$ h) $6 + \frac{1}{n}$

7. The sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{4}{4}$, is called a **Farey Sequence** of order 4. It consists of all fractions whose denominators are 4 or less, arranged in order of size. Write down the Farey Sequence of order 8.

8. The sequence 1, 1, 2, 3, 5 … is obtained by adding two preceding terms to get the next term. Thus, to get the next term, add 3 and 5 to get 8. Such a sequence is called the Sequence of Fibonacci numbers or the **Fibonacci sequence**. Write the next 4 terms in the following Fibonacci sequences

a) 1, 3… c) 2, 2 … d) 0, 3…

Real life Application of number patterns

There are many application of number patterns in real life. Some of them include arrangements of bricks by a builder. Saving regular amounts, depreciation or inflation by regular amounts, heights of buildings where each floor above the ground floor is the same height and removing regular equal amount of substances from a container.

The following building blocks have been arranged in an orderly way. See Fig. 11.3

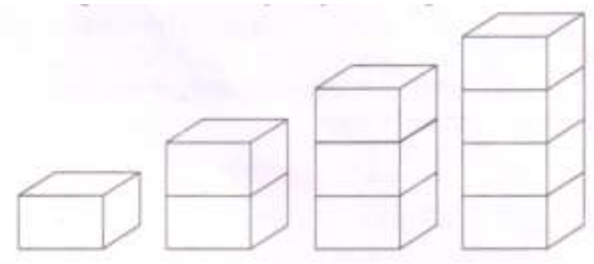


Fig. 113

This arrangement is in increasing order. The number of blocks in each column 1, 2, 3, 4.

Now, the following Fig. 11.4 is the arrangement of bricks on part of a wall.

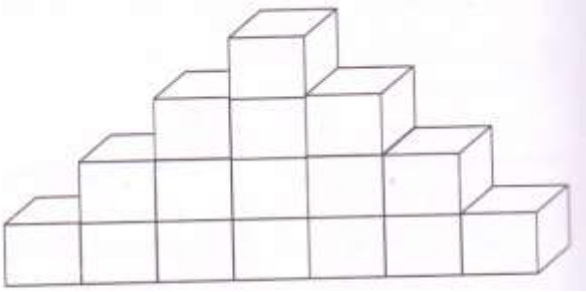


Fig. 11.4

The number sequences for this arrangement is 1, 3, and 5.

If a bricklayer has 400 bricks and wishes to build a wall of their pattern, how many layers high will the wall be if she plans to use all her blocks?

Exercise 11.3

1. A shopkeeper stacks some tins in triangular patterns as shown in Fig. 11.5

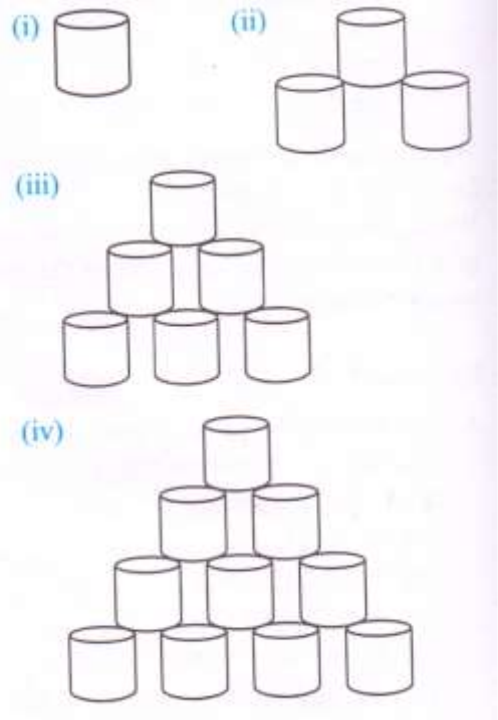


Fig. 11.5

Copy and complete Table 11.1

Number of tins in bottom row	1	2	3	4
Number of this altogether				

Table 11.1

(c) Draw triangular patterns for 5, 6, 7, 8, tins and draw a table similar to Table 11.1

2. Tiyamike's mother bought a piece of cloth with the following pattern in Fig. 11.6

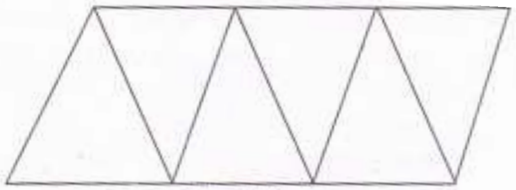


Fig. 11.6

A teacher demonstrated this arrangement using matchsticks. Starting with 3 sticks, she made a triangle and continued the pattern by adding 2 sticks each time. How many matchsticks did she use for this arrangement? Develop a number sequence from this arrangement.

Unit 12 GEOMETRICAL CONSTRUCTIONS 1

Definition of a geometric construction

A **geometric construction** is **an accurate** drawing of a geometric figure.

In unit 4, we learnt how to draw angles and measure given ones. We will use this knowledge to do some geometric constructions using instruments in the mathematical set.

Bisecting an angle

To bisect an angle means to divide an angle into two equal parts. For example, if given angle ABC = 60˚, when it is bisected, each part should measure 30˚. After bisecting an angle, we use the protractor to measure the two parts of that angle to ensure that the construction was accurate. By the end of activity 12.1, you will be able to bisect any given angle.

For geometric constructions, the following will be needed: a pencil, a ruler, a pair of compasses, a protractor and set a square.

Activity 12.1

Draw any angle ABC and bisect it using a ruler and pair of compasses only.

Procedure:

a) With centre B and any radius, draw arcs to cut AB at P and BC and Q. see Fig. 12.1

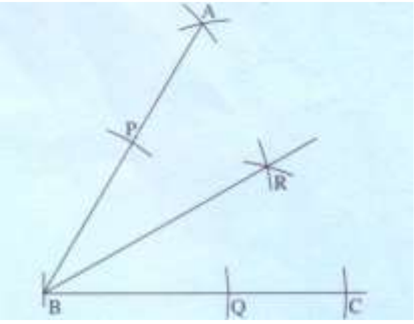


Fig. 12.1

b) With centre P and any suitable radius, draw an arc between the lines AB and BC.
c) With centre Q and same radius used in (b) above, draw an arc to cut the first arc at R
d) Join BR.

Measure <ABC and <RBC. You should find that <ABC = 2 <RBC = 2 <RBA.

BR is called the **angle bisector** of <ABC.

Exercise 12.1

1. Draw an angle ABC and bisect it. Use a protractor to check your working.
2. Draw an obtuse angle. Divide it into four equal angles.
3. Draw a circle of any radius and mark three points P, Q, R on the circumference.
    a) Construct the bisector of <PQR
    b) Construct the bisector of <PRQ.

In exercise 12.1 questions 3, you should have observed that:

Angle bisector of a triangle meet at a common point.

Constructing angles

Constructing an angle of 60˚ Using a pair of compasses and ruler only

Carry out Activity 12.2 to learn how to construct an angle of 60˚ using a pair of compasses and a ruler only.

Activity 12.2

Construct an angle of 60˚.

Procedure:

1. Draw a line segment AB
2. With centre A and any radius, draw an arc to cut AB at a point X and extend it upwards (Figure 12.2)

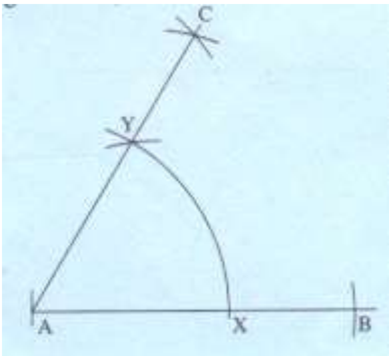


Fig. 12.2

3. With centre X and the same radius as used in step 2, draw an arc to cut the first one at a point Y.
4. Join A to Y and extend C and measure <YAB to confirm that it is actually 60˚ (Fig. 12.2).

Constructing an angle of 90˚

Activity 12.3 gives a step by step method of constructing an angle of 90˚. Carry out the activity as accurately as possible.

Activity 12.3

Construct a line through point O perpendicular to line segment AB (Fig. 12.3)

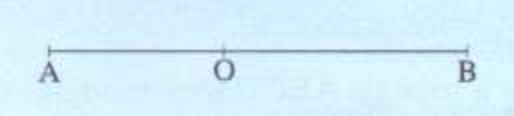


Fig. 12. 3

Procedure

1. Draw line segment AB and mark the point O, anywhere between A and B.
2. With centre O and a convenient radius, draw arc to cut AB at point P and Q (Fig. 12.4).
3. With centre P and radius greater than PQ, draw an arc on one side of AB. Using the same radius, and centre Q, draw an arc to cut the first arc at point R. join RO. Check that <AOR is 90˚ = <ROQ (Fig. 12.4).

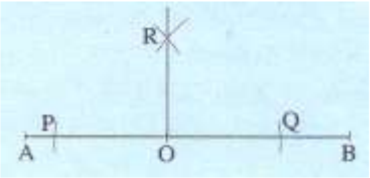


Fig. 12.4

Constructing angles of 45˚, 30˚, 150˚, 120˚, 135˚, by combining constructions

Now, to construct angles of 45˚, 30˚, 150˚, 120˚ and 135˚, combine constructions as recommended in Activity 12.4.

Activity 12.4

1. To construct an angle of 30˚, first construct an angle of 60˚ and then bisect it. To construct an angle of 15˚, bisect the constructed 30˚ angle.
2. To construct an angle of 45˚, first construct an angle of 90˚ and then bisect it.
3. To construct an angle of:
    a) 120˚, construct an angle of 90˚ and combine it with an angle of 30˚ (60˚ bisected) or construct an angle of 60˚ at the same point twice.
    b) 150˚, construct an angle of 120˚ and combine with an angle of 30˚ (60˚ bisected) or construct an angle of 90˚ and combine it with an angle of 60˚.
    c) 135˚, construct an angle of 90˚ and combine it with an angle of 45˚ (90˚ bisected).

Exercise 12.2

1. Construct the following angles:
    a) 60˚
    b) 30˚
    c) 45˚
    d) 75˚
    e) 120˚
    f) 135˚
    g) $22\frac{1}{2}^{\circ}$
    h) 15˚
2. Construct a right angle and construct two lines dividing the right angle into three equal angles.
3. Draw a line AC = 10 cm, <BAC = 30˚, <ABC = 90˚. Measure AB and BC?
4. Construct <ABC = 67.5˚, BC = 8 cm, <BCA = 75˚. Measure AC.
5. Draw lines AB = BC = 6.4 cm, <ABC = 105˚.Measure AC.

Copying an angle using a ruler and compasses only

Activity 12.5

Construct <ABC on line BC equal to <XYZ, Fig. 12.5

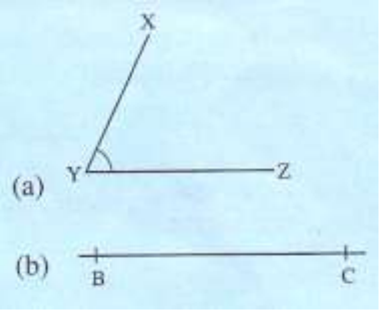


Fig. 12.5

Procedure

a) With Y as centre and any radius, draw an arc to cut XY and YZ at M and N respectively (Fig. 12.6 (a)).
b) With YN as radius and B as centre, draw an arc to cut BC at N' (Fig. 12.6 (b)).
c) With MN as radius and N' as centre draw an arc to cut the first arc in (b) above at M'.
d) Join B to M'. The required angle is ABC (Fig. 12.6 (b)).

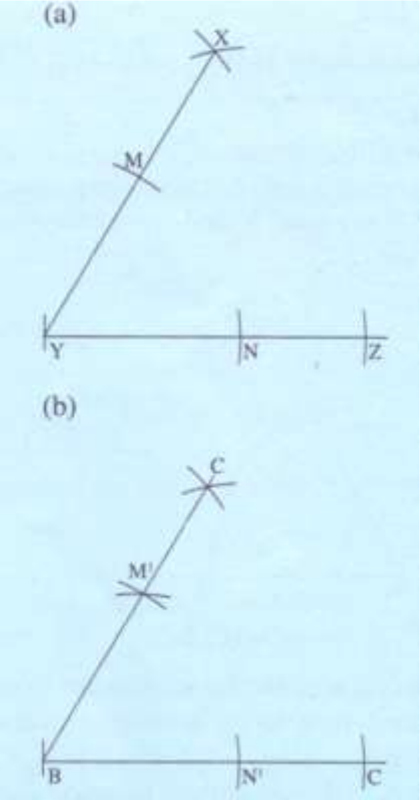


Fig. 12.6

Exercise 12.3

In this exercise, use a ruler and a pair of compasses only.

1. Draw an angle ABC and a straight line DE. Construct a line DF such that <EDF = <BAC.
2. Draw an angle PQR. Draw an isosceles triangle KLM such that <KLM = <PQR.
3. Draw an angle PQR. Draw a right angled ΔABC such that <ABC = <PQR.
4. Draw a triangle ABC. Draw triangle ACD such that <DAC = <BCA and <ACD = <BAC. What type of shape is figure ABCD?
5. Draw a triangle PQR and take a point X inside the triangle. Construct a point Y on QR such that <XYR = <PQR.

Bisecting a given line segment

Bisect a line segment AB whose length is 5.3 cm.

Activity 12.6

Procedure:

a) Draw a line and on it, mark points A and B, 5.3 cm apart (Fig. 12.7 (a)).

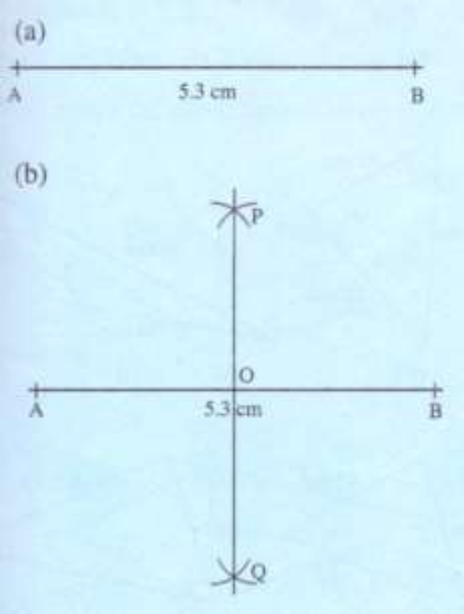


Fig. 12.7

b) With centre A and a radius greater than half the length of AB, mark arcs on either side of AB. With centre B, and the same radius, mark arcs which cut the first arcs at points P and Q.

c) Join PQ (fig. 12.7 (b).
Check that AO = OB and <POB = 90˚

Line PO (or PQ) is called **perpendicular bisector** of AB or the **mediator** of AB.

Any point on the mediator is **equidistant** (i.e. at equal distances) from points A and B. check this.

Exercise 12.4

1. Draw line AB = 7.5 cm. By construction, find its mid-point.
2. Draw any circle centre. Draw two chords PQ and RS which are not diameters. Construct perpendicular bisectors of the chords. Where do they meet?
3. On a clean page, mark points A, B and C such that they are not on a straight line. Find a point O such that AO = BO = CO.

(i) The perpendicular bisector of a chord passes through the centre of the circle.

(ii) The perpendicular bisectors of the sides of a triangle are concurrent.

Construction of triangles

A triangle can be constructed given measurements of:

i) All the three sides,
ii) Two sides and one angle, or
iii) One side and two angles.

Construction of a triangle, given three sides

Activity 12.7

Using a ruler and a pair of compasses only, construct triangle PQR such that PQ = 2.5 cm, QR = 3.5 cm and PR = 5 cm.

Procedure:

Using a ruler and a pair of compasses only, construct triangle PQR such that PQ = 2.5 cm, QR = 3.5 cm and PR = 5 cm.

Procedure:

a) Draw a rough sketch of the triangle to be constructed and on it, indicate all the given measurements (Fig. 12.8).

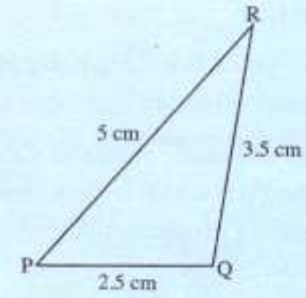


Fig. 12.8

b) Construct the triangle accurately using the following steps.
   i) Draw a line and mark a point P on it.
   ii) On the line, mark off a point Q, 2.5 cm from P, using a pair of compasses.

iii) With P as the centre and radius 5 cm, draw another arc to intersect the arc in (iii) at R.

iv) Join P to R and Q to R. PQR is the required triangle. (Fig. 12.9)

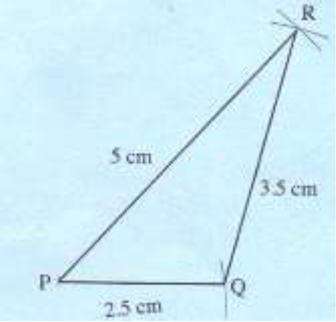
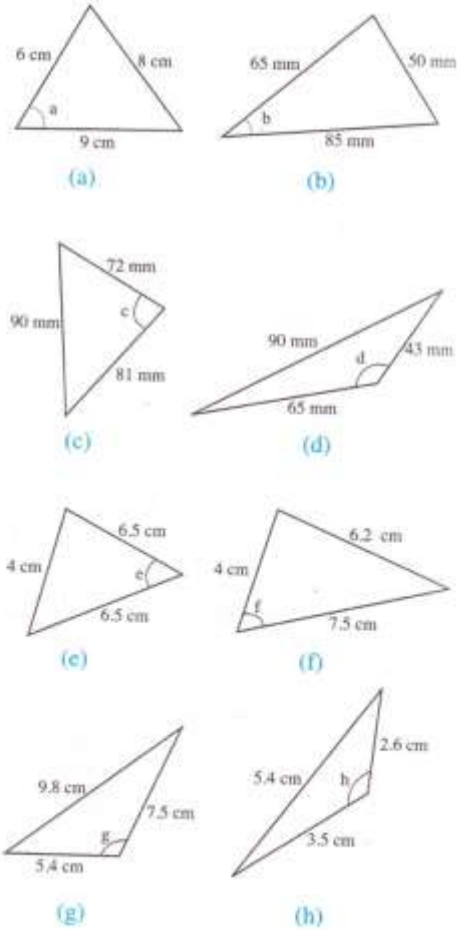


Fig. 12.9

Exercise 12.5

1. Construct the triangles sketched in Fig. 12.10 and measure the angles marked with letters.



(a) (b) (c) (d) (e) (f) (g) (h)

Fig. 12.10

2. Construct ABC such that,
   a) AB = 40 mm, BC = 50 mm,
      AC = 60 mm
   b) AB = 50 mm, BC = 85 mm,
      AC = 75 mm
   c) AB = 65 mm, BC = 45 mm,
      AC = 60 mm
   d) AB = 8.3 cm, BC = 8.3 cm, AC = 5 cm
   e) AB = 7 cm, BC = 6 cm, AC = 4 cm
   f) AB = 5.6 cm, BC = 5 cm, AC = 7.5 cm

In each case measure <ABC .

3. Make accurate constructions of the diagrams in Fig. 12.11 and
   i) Measure <BCD in each case.
   ii) Name the plane figures in (a) and (b).

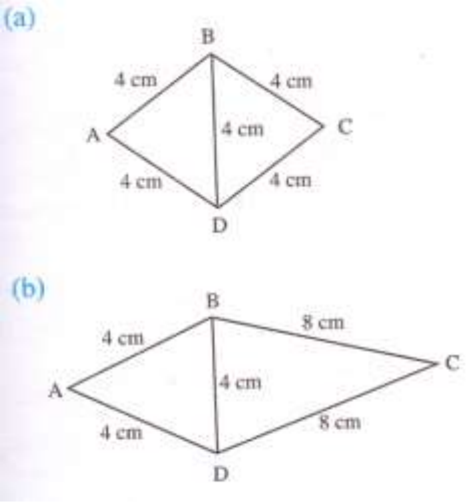


Fig. 12.11

4. Construct triangles using the following measurements.
   a) 3 cm, 4 cm, 8 cm
   b) 10 cm, 7 cm, 3 cm
   c) 5 cm, 5 cm, 12 cm

What do you notice in each case?

5. Make an accurate construction of the diagram in Fig. 12.12. Measure <BAC and <ACD. What can you say about lines AB and DC?

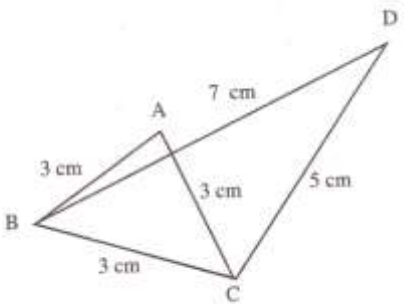


Fig. 12.12

Construction of a triangle, given two sides and an angle.

Activity 12.8

Construct Δ ABC such that <ABC = 50˚, AB = 4 cm, BC = 6 cm.

Procedure:

a) First make a rough sketch (Fig. 12.13).

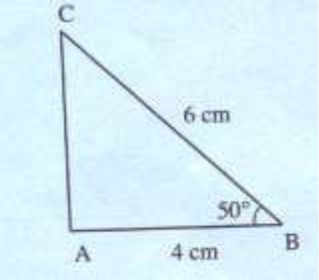


Fig. 12.13

b) Construct ΔABC as follows.
   - i) Draw a line and mark a point A on it.
   - ii) On the line, mark off a point B, 4 cm from A, using a pair of compasses.
   - iii) At point B, use your protractor to measure an angle of 50˚ and draw the line BX.
   - iv) Mark a point C on line BX such that BC = 6 cm.
   - v) Join AC to complete ΔABC

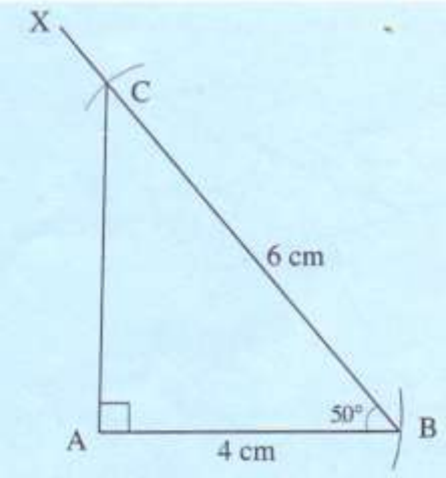


Fig. 12.14

Exercise 12.6

1. Construct the triangles sketched in Fig. 12.15 and measure the sides marked with letters.

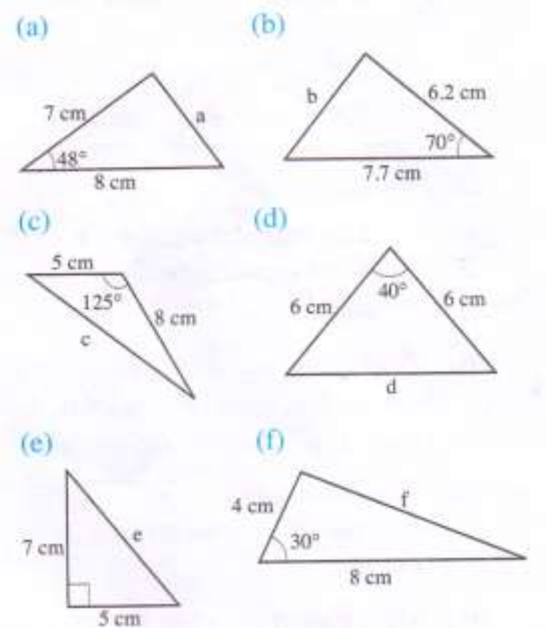


Fig. 12.15

2. Construct the following triangles. Two sides and the angles between them are given. Measure the third side.
   a) 7 cm, 48˚, 9 cm
   b) 8.8 cm, 75˚, 6.5 cm

   c) 5 cm, 110˚, 6 cm
   d) 7.8 cm, 70˚, 4.8 cm
   e) 9 cm, 40˚, 6 cm
   f) 8.5 cm, 45˚, 8.5 cm

3. Construct the given triangles.
   a) ΔABC; AB = 7 cm, BC = 4.5 cm, <B = 105˚. Measure AC.
   b) ΔPQR; PQ = 5.5 cm, QR = 4 cm, <R = 85˚. Measure PR.
   c) ΔXYZ; XY = 5 cm, YZ = 6 cm, <X = 70˚. Measure XZ.
   d) ΔJKL; JK = 5.5 cm, KL = 3.5 cm, <L = 130˚. Measure JL.
   e) ΔSTU; ST = 5 cm, TU = 6 cm, <T = 65˚. Measure SU.
   f) ΔXYZ; XY = 6.5 cm, ZY = 7.5 cm, <X = 35˚. Measure <Y.

4. (a) Construct ΔABC such that AB = 4.5 cm, BC = 3.5 cm and <A = 40˚. Measure <C. is there more than one possible triangle?

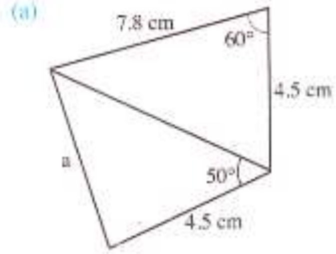
   (b) Construct Δ ABC such that AB = 4.5 cm, <A = 40˚and

   i) BC = 4.8 cm

   ii) BC = 2.5 cm

   Measure angle B in each case. What do you notice?

5. Construct the diagrams in Fig. 12.16 and measure the sides marked with letters.



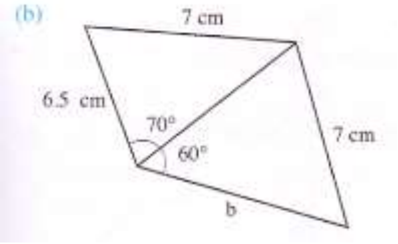


Fig. 12.16

From this exercise you should have noticed that:

When two sides and the angle between them (i.e. two sides and the included angle) are given, we are always able to construct a unique triangle (as in Question 2). But if the angle is not included, we do not necessarily get a unique triangle (Question 4).

Construction of a triangle, given one side and two angles

Activity 12.9

Construct Δ PQR such that PQ = 5 cm, <P = 35˚, <R = 65˚.

Procedure:

a) Make a rough sketch of the triangle (Fig. 12.17). Note that <Q = 80˚. Why?

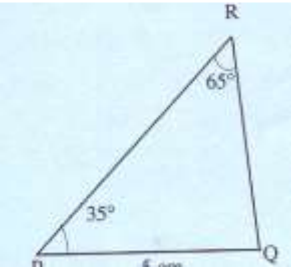


Fig. 12.17

b) Make an accurate construction as follows.
   - i) Draw a line and mark a point P on it.
   - ii) On the line, mark off a point Q, 5 cm from P, using a pair of compasses.
   - iii) At point P, draw a line at an angle of 35˚.
   - iv) At point Q, draw a line at an angle of 80˚.

Where the two lines meet is the point R. check that <R = 65˚. PQR is the required triangle. (See Fig. 12.18).

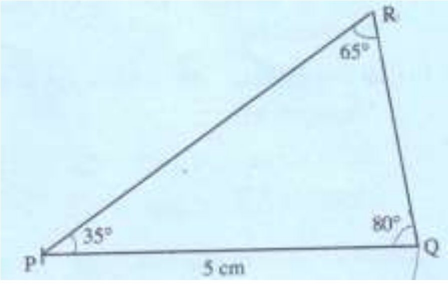


Fig. 12.18

Note that, given two angles and a side, we need to find the third angle so that the given side is between two angles. It is only then that we shall be able to construct the triangle.

Exercise 12.7

1. Construct the triangles sketched in Fig. 12.19 and measure the sides marked with letters.

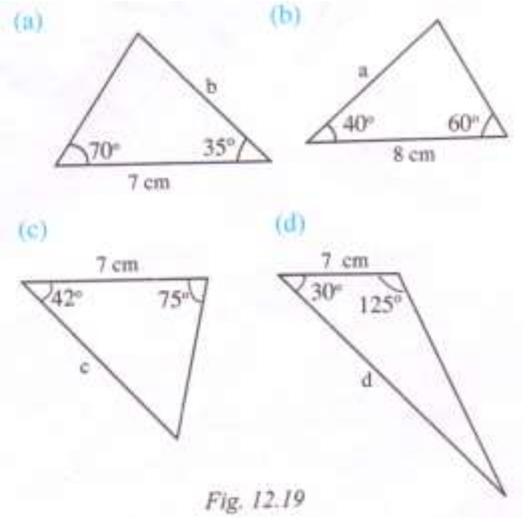


Fig. 12.19

2. Construct ΔABC such that:
   a) <A = 30˚, AB = 5.5 cm, <B = 70˚. Measure AC
   b) <A = 25˚, <B = 65˚, AC = 6.5 cm. Measure AB
   c) AC = 8 cm, <B = 55˚, <C = 65˚ Measure BC
   d) BC = 45˚, <C = 40˚, AB = 7.8 cm Measure AC
   e) BC = 6.8 cm, <B = 65˚, <A = 65˚

    Measure AB

  f) BC = 6 cm, <A = 35˚, <B = 115˚.
     Measure AB

3. Construct each of the shapes in Fig. 12.20 and find the lengths marked with letters.

   Fig. 12.20

4. Construct the shapes in Fig. 12.21 and measure the sides marked with letters.

   Fig. 12.21

5. On the same diagram, construct triangles PQS and PRS from the given measurements.
   a) PQ = 5 cm, PR = 6 cm, <Q = 40˚,
      QS = 7 cm, RS = 3 cm, Measure QR.

   b) PQ = 5 cm, QR = 9 cm, RS = 3 cm,
      SP = 3.8 cm, <QPS = 100˚. Measure angle Q.

   c) PS = 5 cm, <RPS = 35˚, <PSQ = 47˚,
      <PRS = 65˚, <PQS = 54˚. Measure QR.

What is the name of the plane figure PQRS in (a), (b) and (c)?

Constructing parallel lines

a) Using a ruler and a pair of compasses only

Activity 12.10

Given line AB, construct a line through P parallel to AB (See Fig. 12.22).

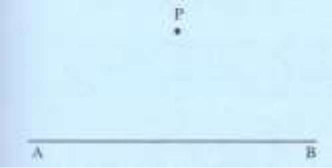


Fig. 12.22

Method 1

i) On line AB mark two points X and Y.

ii) With centre P and radius XY, draw an arc roughly to the left of P.

iii) With centre X and radius PY, draw an arc to intersect the first arc at Q (Fig 12.23).

iv) Join P to Q. PQ is the required line.

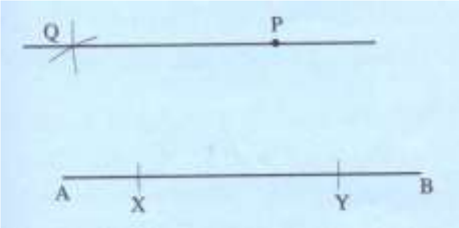


Fig. 12.23

Method 2

i) Take any point X on AB. Join PX.

ii) With centre X and any radius, draw an arc to cut AB at M and PX at N.

iii) With centre P and same radius as in (ii), draw an arc to cut PX at K and another one roughly to the left of P.

iv) With centre K and radius MN, draw an arc to cut the other arc to the left of P at L.

v) Join P to L. this is the required line (Fig. 12 24).

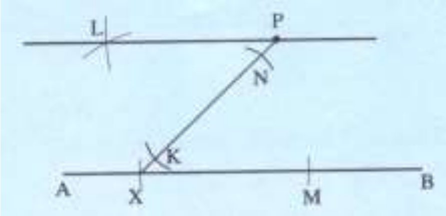


Fig. 12.24

b) Using a ruler and a set square

Activity 12.11

To draw a line parallel to line AB passing through a given point P,

i) Place a set square along AB.

ii) Place a ruler against the set square as shown in Fig. 12.25.

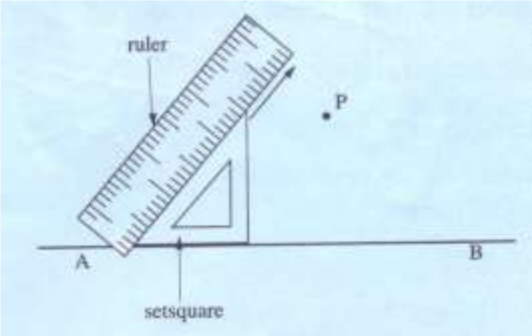


Fig. 12.25

iii) Hold the ruler firmly, then slide the set square in the direction shown by the arrow (Fig. 12.25) until it reaches the position shown in Fig. 12.26.

iv) Hold the set square in position and draw a line through P.

Fig. 12.26

This will be the required line (See Fig. 12.27)

Fig. 12.27

Proportional division of a line segment

A line segment can be divided into a given number of equal parts as shown below.

Line divided into equal parts

Activity 12.12

Divide line segment AB in Fig. 12.28 (a) into 5 equal parts.

Procedure:

a) Through A, draw any line AL.

b) Using a suitable radius on a pair of compasses, starting at A, mark off 5 equal lengths AA_1, A_1A_2, A_2A_3, A_3A_4, A_4A_5, along AL.

c) Join BA_5.

Fig. 12.28

B_1, B_2, B_3, B_4 are the points that divide AB into the required 5 equal parts.

Exercise 12.8

1. Draw a straight line PQ. Mark a point X above the line. Construct a line through X parallel to PQ using a ruler and a setsquare only.
2. Draw a line XY. On it, mark a point M. through point M, construct a perpendicular. On this perpendicular, mark off points A and B such that MA= AB = 3 cm. through points A and B, construct lines parallel to the line XY. (Use a ruler and a pair of compasses only).
3. Draw a line PQ, 6 cm long. Construct <SPQ = 60˚ with PS = 4 cm. through points S and Q, construct lines parallel to PQ and PS respectively, to meet at a point R. measure SR and QR, and all the remaining angles. Name the figure obtained.
4. Draw triangle ABC. Construct D such that AD = BC and CD = BA. What shape is the figure?

Construction of perpendicular lines

When two lines meet at right angles, they are said to be **perpendicular** to each other. For example, the adjacent edges of a text book are perpendicular to each other.

Constructing a perpendicular to a line from a given point on the line

Activity 12.13

Construct a line through point O perpendicular to line AB (Fig. 12.29).

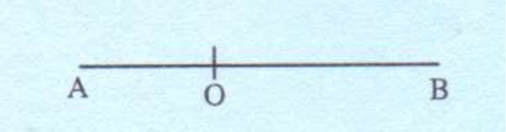


Fig. 12.29

Procedure:

a) Draw line AB and mark the point O, anywhere between A and B.
b) With centre O and a convenient radius draw arcs to cut AB at P and Q (Fig. 12.30)

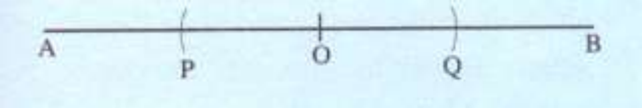


Fig. 12.30

c) With centre P and radius greater than PO, draw an arc on one side of AB. Using the same radius and centre Q draw an arc to cut the first arc at point R. join RO (Fig. 12.31). Check that <AOR and <BOR are 90˚ each.

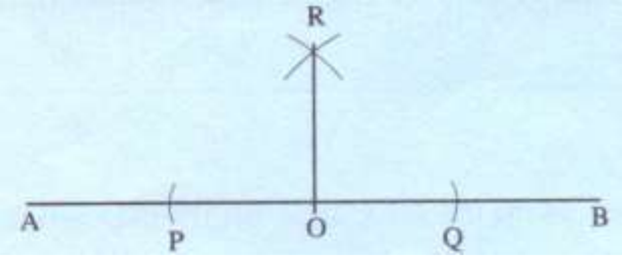


Fig. 12.31

Exercise 12.9

Use a ruler and a pair of compasses only in this exercise.

1. Given that AB = 8 cm, and that O is a point on AB such that AO = 3 cm, construct a line CO = 4 cm which is perpendicular to AB. Measure AC and BC.
2. Draw line PQ = 8 cm. draw PR = 5 cm such that PR is perpendicular to PQ. Measure RQ.
3. Draw line PQ = 7 cm.
   a) Draw QR = 3 cm such that QR is perpendicular to PQ.
   b) Draw PS = 8 cm such that PS is perpendicular to PQ.

Join and measure RS. What is the name of figure PQRS?

4. Given AD = 6 cm, construct an equilateral triangle ABC of sides 6 cm such that AD is perpendicular to AB. Measure CD.
5. Make an accurate drawing of the shape shown in Fig. 12.32 and measure ADE.

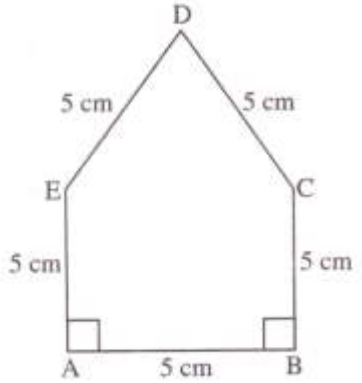


Fig. 12.32

6. Construct:
   a) A square of side 4 cm
   b) A rectangle of sides 4 cm by 7 cm.

Constructing a perpendicular to a line from an external point using a ruler and a pair of compasses

Activity 12.14

Construct a perpendicular from point M to line PQ (Fig. 12.33)

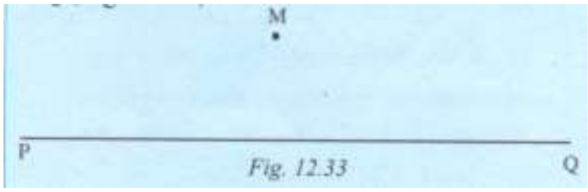


Fig. 12.33

Procedure:

a) With centre M and a suitable radius on a pair of compasses, draw arcs to cut line PQ at two points R and S (See Fig. 12.34).

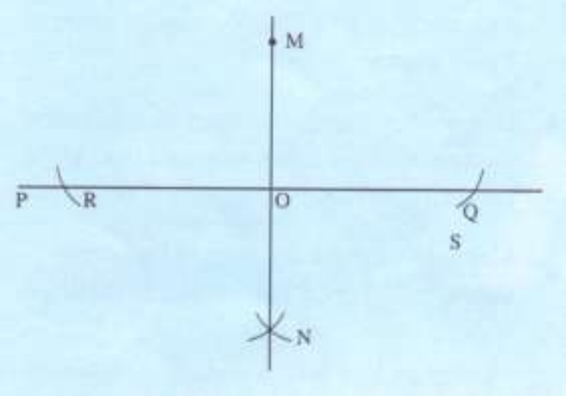


Fig. 12.34

b) With centre R and a convenient radius, draw an arc on the other side of PQ away from M.

c) With centre S and the same radius, draw an arc to cut the first arc at N. join MN (Fig. 12.34). Check that <MOP and <MOQ are 90˚ each.

Do you think the construction would have been possible if the arcs were on the same side of PQ as M (but not through M)?

Note that the length MO is called the **perpendicular distance** of point M from the line PQ. It is also the shortest distance of point M from line PQ.

Constructing a perpendicular to a line from an external point using a ruler and a set square

In Fig. 12.34, the perpendicular from M to line PQ could have been drawn using a ruler and a set square as follows.

Activity 12.15

a) Place a ruler along the given line PQ. (Fig. 12.35).

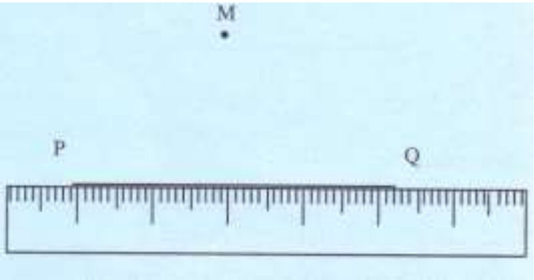


Fig. 12.35

b) Place a set square so that t rests against the ruler in any position S_1 (Fig. 12.36). hold the ruler firmly and slide the set square along the ruler until the edge reaches M (position S_2)

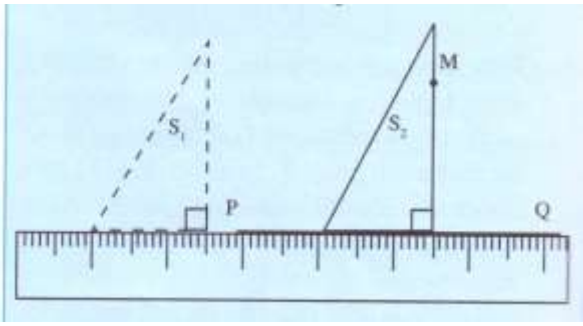


Fig. 12.36

c) Hold the set square firmly in position S_2, remove the ruler and draw a line through M to cut line PQ, at a point O.

This is the required perpendicular (See Fig. 12.36).

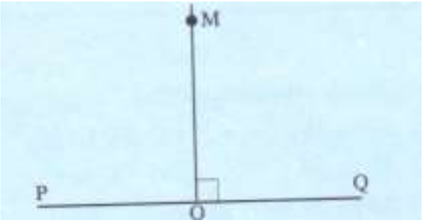


Fig. 12.37

Exercise 12.10

1. Draw any line AB. Mark a point P on one side of AB. Draw a perpendicular from P to AB using ruler and compasses. Confirm your working using a ruler and a set square.
2. (a) On a clean page, mark three points A, B and C. join A to B, B to C and C to A using straight line segments. Construct the perpendicular bisectors of the three lines using a pair of compasses and a ruler only. What is to be noticed about the bisectors?
   (b) Measure the distances from points A, B and C to the point where the perpendicular bisectors meet. What do you notice?

Unit 13 STATISTICS 1

What is statistics?

For a group of people. E.g. students in a class, a lot of information can be collected about each one of them. Such information could be their names. Names, where they live, what food they like most etc. They could also state their ages, heights, masses etc.

Information in which numbers are used is called **statistics**. Each such number is called a statistic (plural statistics). Any other number that may be derived from some computation using the original numbers is also a statistics.

Thus, **statistics** is the study of the information represented in numerical form.

For example, Table 13.1 shows marks scored by 40 students in a Biology test.

78	46	55	47	77	63	52	52	62	46
77	47	40	35	67	61	58	52	42	40
48	57	66	54	75	78	75	59	75	47
59	35	62	53	72	57	51	69	55	57

Table 13.1

a) What is the highest score?
b) What is the lowest score?
c) What is the difference between the highest and the lowest scores?
d) What mark was scored by most students?
e) How many students scored above 70?
f) If the pass mark was 45, how many students failed the test?
g) How many students scored
    i. 30 to 39,
    ii. 40 to 49,
    iii. 50 to 59,
    iv. 60 to 69,
    v. 70 to 79 marks?
h) How did the students perform in that test in general?

To answer questions (d) – (g), we have to count correctly, while question (h) requires some general conclusions.

Note

1. Information such as in Table 13.1 is called **raw data** or simply **data**.
2. Statistics, in general, is concerned with collecting data, organizing it, answering questions about the data and making sensible conclusions and decisions based on the analysis of the data.

Collecting and organizing data

In statistics, data must be collected and recorded accurately. The data in Table 13.1 could have been recorded first in a class list showing marks scored by each student e.g.

Hanka 78

Dziko 46

Fatsani 55

The list could then have been used to make Table 13.1

Information could also be collected as shown in Table 13.2. The table shows types of vehicles which passed at a given point over a given time interval.

Vehicle	**Tally marks**	**Totals (Frequency)**
Tractor	*II*	2
Motor bike	*III*	3
Car	*~~IIII~~ ~~IIII~~ II*	12
Minibus	*~~IIII~~*	5
Lorry	*II*	2
Bus	*~~IIII~~ II*	7

Table 13.2

When a vehicle passes, a stroke (*I*) is put in the ‘Tally’ column in the row corresponding to that type of vehicle. Every fifth stroke against a given type of vehicle is made to cross the other four, so that instead of having five strokes as *IIII* We have *~~IIII~~*. The next stroke will be next to the group of five strokes, i.e. *~~IIII~~*, *I* etc. these strokes are recorded in the ‘Tally’ column until the time for collecting the information is over. The total number of strokes against each type of vehicle is then recorded in the ‘Totals’ column.

Other methods of collecting data include interviews, prepared questionnaires and so on. The information is then tabulated and organized for the intended purpose.

Note

1. The tally method of recording makes counting easier and more accurate especially when big numbers are involved.
2. Bundling strokes into fives (~~IIII~~) makes it easier to count them.

Frequency distribution table

A table such as Table 13.2 is called a **frequency distribution table** or simply a **frequency table. Frequency** means the number of times an item or value occurs.

In Table 13.2, the last column shows the total number of each type of vehicle. This number is the frequency for each type of vehicle. Thus we usually write ‘Frequency’ in that column instead of ‘Totals.’

Example 13.1

Table 13.3 shows the grades scored by a class of 30 students in a Mathematics examination.

C	B	C	A	C	B	B	B	D	A
B	B	C	C	B	D	B	C	B	D
A	B	A	C	B	A	C	C	B	C

Table 13.3

Make a frequency table for the information in this table.

Solution

Table 13.4 is the required frequency table.

Grade	Tally	Frequency
A	*𝍸 I*	6
B	*𝍸 𝍸 I*	11
C	*𝍸 𝍸*	10
D	*III*	3
		30

Table 13.4

Table 13.4 is made by putting a stroke (I) in the 'Tally' column against the grade, for each grade in Table 13.3. This is systematically done so that in Table 13.4, there is a stroke for each grade. All the strokes against each grade are counted and the counts written in the Frequency column.

Exercise 13.1

1. Work in groups of 5 or 6. Collect the following data from each member of your group: height, mass, size of shoes worn, favourite subject, favourite sport. Each group leader should record information about his/her group as shown in Table 13.5. Each leader should then collect the data from other groups so that each group has data for the whole class.

Name	Height (cm)	Mass (Kg)	Size of shoes	Favourite Subject	Favourite Sport

   Table 13.5

   Make a frequency table for each statistic except the name.
2. Record the temperature outside the classroom, at the following times: 7.30 a.m., 10.30 a.m., 1.30 p.m. and 4.30 p.m. This should be done every day for a whole week (Different groups could be assigned different days).
3. Choose a place near your school where traffic passes. Count how many different types of vehicles pass there in the morning (7 – 8 a.m.), at lunch time (1 – 2 p.m.), and in the evening (4 – 5 p.m.). Keep different records for different directions. Different groups should be assigned different days of the week so that the data is recorded for a whole week. Each group should fill their data in a table such as Table 13.6(a). Table 13.6 (b) could be used to compile information from all groups.

`

Group no. ………….
Date/Day …………………….. Time …………….
Direction of traffic …………………………………………………….

Vehicle	Tally	Frequency
Bicycle Car Motor cycle Mini bus Lorry Bus Others		

Table 13.6(a)

Place …………………………….
Time …………………………….
Direction of traffic …………………….

Vehicle	Mon	Tue	Wed	Thur	Fri

Table 13.6(b)

4. Find out from the school library how many books there are in each of the following subjects categories.
   a) English
   b) Other languages (French, German, etc.)
   c) Mathematics
   d) Science
   e) Religion
   f) Others

5. On this age that you are reading, count all the words having one letter, two letters, three letters, four letters and five letters. Make a frequency table for this data.

6. Table 13.7 shows the amount of milk in litres produced by 36 cows in Kaphiri's farm in one day.

8	10	11	9	9	4	8	18	13
16	15	12	13	4	6	11	15	4
8	7	9	13	7	10	11	8	5
14	12	9	14	9	12	5	14	14

Table 13.7

Make a frequency distribution table for this data.

7. Table 13.8 shows the sizes of shoes worn by 40 students in a Form 2 class in St. Paul's High school.
   a) Make a frequency table for the data.

6	10	7	6	7	8	7	9	11	11
10	7	8	6	8	7	9	11	6	7
9	9	8	7	10	10	8	8	7	8
8	7	8	8	7	9	8	7	10	9

Table 13.8

   b) Whom do you think would be interested in this information?

8. Table 13.9 shows the grades scored by 40 pupils in a mathematics examination. Make a frequency table for the data.

E	A	B	D	C	C	B	C
B	C	C	B	D	C	B	D
A	C	B	C	D	A	B	B
C	B	D	A	C	B	B	C

Table 13.9

Presentation of data

Once data have been collected, they may be presented or displayed in various ways. Such displays make it easier to interpret and compare the data. The following are some of the ways.

Ranks order list

A **rank order list** is a list showing items that have been arranged in order from the highest to the lowest or from the lowest to the highest.

For example, five pupils had the following scores in a Mathematics test: 15, 12, 21, 13, and 18. The rank order is 21, 18, 15, 13, 12 or 12, 13, 15, 18, 18, 21.

The rank order list helps us find the:

a) Highest value.
b) Lowest value.
c) Most common value.
d) Value which is in the middle.
e) Number of those above or below a given value, etc.

Frequency distribution table

As we saw earlier, a frequency distribution table is a table which shows data items and the number of times (frequency) they occur. Such a table helps us to see the:

a) Highest value
b) Lowest value
c) Most common value, etc.

Pictogram (or pictograph)

A **pictogram** (pictograph) is a diagram that represents statistical data in a pictorial form. Each picture or drawing represents a certain number or value from the data.

The picture to be used is chosen to represent the data subject as closely as possible, e.g. a shoe to represent the number of pairs of shoes, a car to represent the number of cars etc.

Example 13.2

Represent the data in Table 13.10 in a pictogram.

Size	6	7	8	9	10	11
No. of shoes	4	11	11	6	5	3

Table 13.10

Solution

Let ___ represent 2 pairs of shoes. Since 4 pupils wore shoes of size 6, then 4 will be represented by _____, etc. So the data in Table 13.10 would be represented as in Fig. 13.1

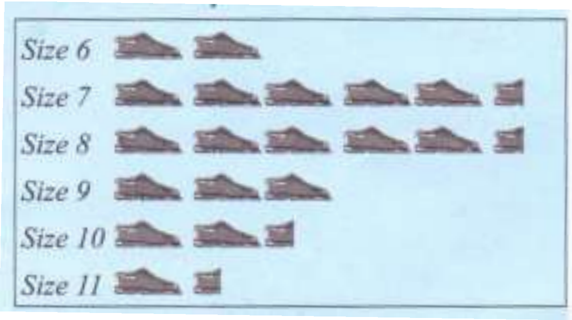


Fig. 13.1

Note that a fraction of 2 represented by a fraction of the drawing.

Although the pictogram is used to display information, it is not an accurate way of representing data. For instance, suppose each picture represented 10 pairs, how would you represent 2, 3, 7 or 8 pairs?

Pie-chart

A **pie-chart** is a graph or diagram in which different proportions of a given data distribution are represented by sectors of a circle.

Since the diagram is a circle, it is looked at as a circular 'pie', hence the name pie chart.

Example 13.3

Table 13.11 shows grades scored by 15 candidates who sat for a certain test.

Grade	A	B	C	D	E
No. of candidates	2	5	4	1	3

Table 13.11

Draw a pie chart for this data.

Solution

Work out the fractions of numbers of candidates who scored each grade. For example, for example, for grade A we have $\frac{2}{15}$.

Since the angle at the centre of a circle is 360°, we calculate the angle to represent grade A as $\frac{2}{15}$ of 360°

i.e. $\frac{2}{15} \times 360^\circ = 48^\circ$.

A is represented by an angle of 48°.

Table 13.12 shows all the angles.

Grade	No. of candidates	Fraction of total	Angle at the centre of the circle
A	2	$\frac{2}{15}$	$\frac{2}{15} \times 360^\circ = 48^\circ$.
B	5	$\frac{5}{15}$	$\frac{2}{15} \times 360^\circ = 120^\circ$.
C	4	$\frac{4}{15}$	$\frac{2}{15} \times 360^\circ = 96^\circ$.
D	1	$\frac{1}{15}$	$\frac{2}{15} \times 360^\circ = 24^\circ$.
E	3	$\frac{3}{15}$	$\frac{3}{15} \times 360^\circ = 72^\circ$.

Table 13.12

Fig. 13.2 shows the required pie chart

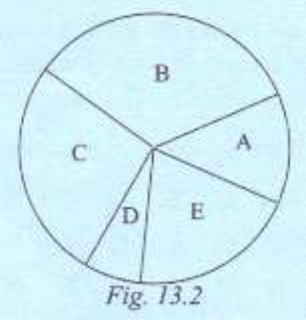


Fig. 13.2

Note

1. Usually there are no numbers on a pie chart.
2. The sizes of the sectors give a comparison between the quantities represented.
3. The order in which the sectors are presented does not matter.
4. Sectors may be shaded with different patterns (or colours) to give a better visual impression.

Bar chart

A **bar chart** (or bar graph) is a graph consisting of rectangular bars whose lengths are proportional to frequencies in a data distribution

Example 13.4

Table 13.13 shows the sizes of sweaters worn by 30 Form 2 students in a certain schools.

Represent the data on:

a) A horizontal bar chart
b) A vertical bar chart

Size	Small (S)	Medium (M)	Large (L)	Extra-large (XL)
No. of pupils	5	13	8	4

Table 13.13

Solution

a) Horizontal bar chart Fig. 13.3(a). In a horizontal bar chart, frequency is represented on the horizontal axis. Bars are drawn with spaces between them (as in Fig. 13.3 (a) and they may be shaded or not.
b) Vertical bar chart Fig. 13.3 (b). In a vertical bar graph, frequency is represented on the vertical axis.

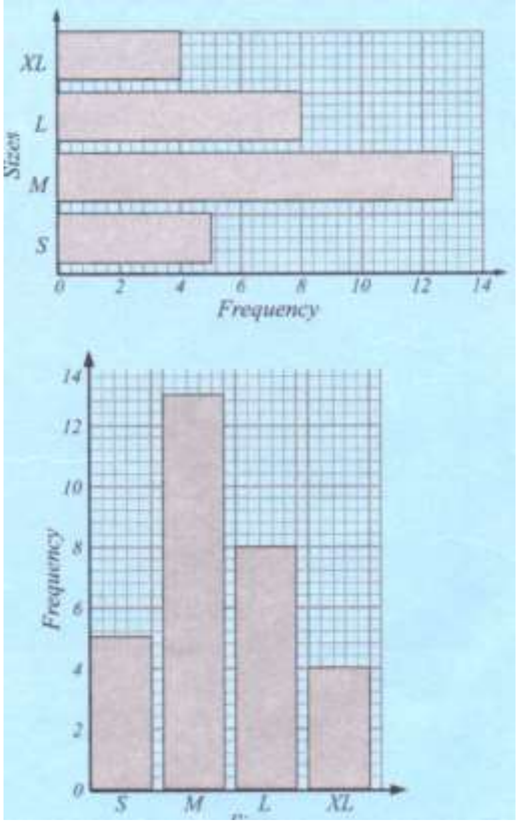


Fig. 13.3

Note: In a bar chart:

1. The widths of the bars are the same.
2. The height of a bar is proportional to the corresponding frequency.

Example 13.5

Fig. 13.4 shows marks scored in a test by a Physics class.

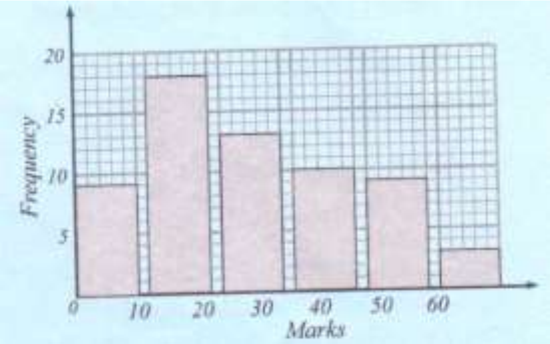


Fig. 13.4

a) How many students took the test?

b) How many students scored less than 20 marks?
c) How many students scored more than 29 but less than 50 marks?

Solution

a) To get number of students who took the test, add all the frequencies i.e. 9 + 18 + 12 + 10 + 8 + 3 = 60
b) No. of students who scored less than 20 marks = those who scored 0 – 9 and 10 – 19 = 9 + 18 = 27
c) No. of students who scored more than 29 but less than 50 marks = 10 + 8 + 3 = 21

Exercise 13.2

1. In a survey on soft=drinks, 180 people were asked to state the brand they preferred. 35 chose brand A, 30 chose brand B., 100 chose brand C and 15 chose brand D. draw a pie-chart to display this information.
2. At the semi-final stage of a football competition, 72 neutral observers were asked to predict which team they thought would win. Table 13.14 shows their predictions.

Team	No. of predictions
Team A	9
Team B	40
Team C	22
Team D	1

Table 13.14

Draw a pie-chart to display the predictions.

3. Mr. Onani has a monthly income of K12 000. The pie-chart in Fig. 13.5 shows how he spends the money.

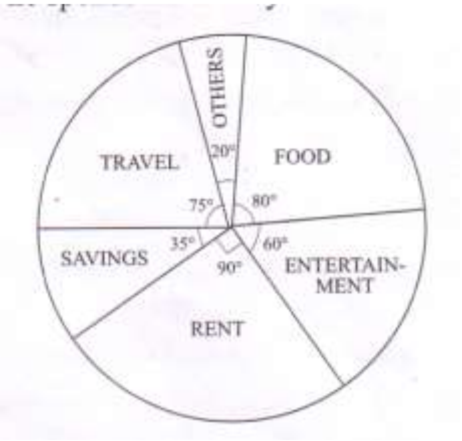


Fig. 13.5

How much does he spend on:

a) Food
b) Rent
c) Rent
d) Entertainment
e) Travel

4. Of the animals on Sigele's farm, 35% are cows, 20% are goats, 15% are sheep, 2% are donkeys and 28% are pigs. A pie chart is to be drawn to illustrate this information. Find the angle of the sector representing each type of animal.
5. In Exercise 13.1 Question 1, you collected information about your class, i.e.
    i. Size of shoes worn
    ii. Favourite subject
    iii. Favourite sport of each pupil.

    Display this information in a pie-chart.

6. Display the information you collected in Exercise 13.1 Question 4, in a pie chart.
7. Represent the information in Exercise 13.1
   Question 5 on: (a) a pie-chart
   (b) a bar chart.

8. Display the information in Table 13.9 of Question 8 (Exercise 13.1) on a bar chart.

9. Fig. 13.6 shows the heights of pupils in a certain class.

   a) How many pupils are over 150 cm tall?

b) How many pupils have a height between 125 cm and 145 cm?

c) How many pupils are in the class?

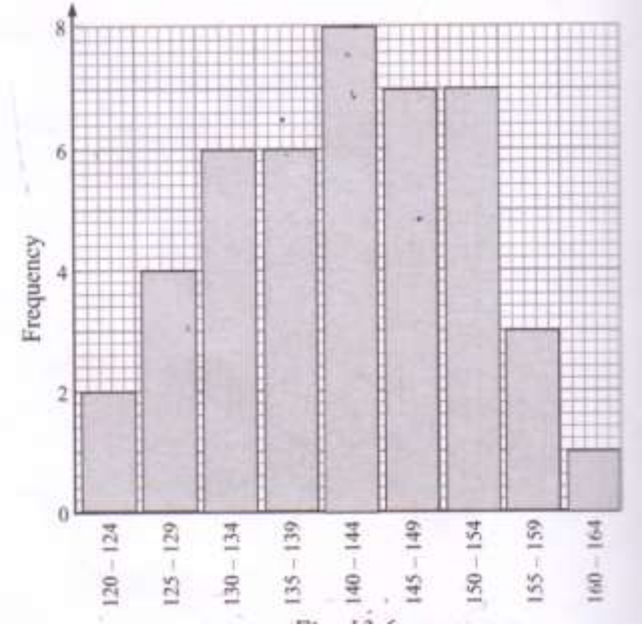


Fig. 13.6

9. Some children were picked at random and their handspans measured in centimeters.

This data was recorded as in Table 13.15

18.4	17.4	21.2	18.7	21.7	17.5	18.1
19.2	16.6	14.3	19.3	17.8	20.7	18.5
16.9	20.0	15.9	19.8	16.0	14.8	19.0

Table 13.15

a) Draw a bar-chart for this data using class intervals 14.0 – 15.9, 16.0 – 17.9…
b) How many children had a handspan greater than or equal to 18 cm?

Line graphs

Table 13.16 shows the temperatures, in degrees Celsius, observed at 2-hourly intervals, of a patient who was admitted at a hospital.

Time	6 a.m.	8 a.m.	10 a.m.	12 noon	2 p.m.	4 p.m.
Temp ˚C	38.2	38.6	38.9	38.8	38.8	38.5

Time	6 p.m.	8 p.m.	10 p.m.	Mid-night	2 a.m.	4 a.m.
Temp ˚C	38.2	37.8	37.1	37.0	36.8	36.8

Table 13.16

When a graph of time against temperature is plotted using the data in Table 13.16 and the points joined by dotted line segments, Fig. 13.7 is obtained.

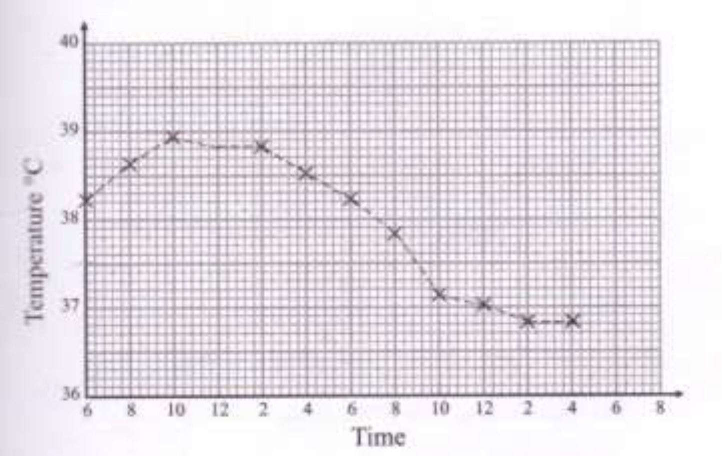


Fig. 13.7

A graph, such as Fig. 13.7, which is formed by broken line segments joining the points representing given data is known as a **line graph.**

a) Why are points joined by dotted lines? When could such points be joined by continuous line segments?

b) Estimate the patient's temperature at:
   i. 9 a.m.
   ii. 3 p.m.
   iii. 11 p.m.
   iv. 3 a.m.

c) What can you say about the patient's temperature;
   i. During the day?
   ii. During the night?

Note

A line graph helps us appreciate the pattern or trend of a given variable i.e. how the variable changes with time.

Exercise 13.3

1. The average masses of pupils of different ages in a certain school were obtained and recorded as in Table 13.17.

Age (years)	11	12	13	14	15	16	17
Average mass (kg)	36.7	38.6	42.4	46.8	51.9	60.4	65.6

Table 13.17

   a) Represent this data on a line graph
   b) What is the estimate for the average mass of pupils who are:
      i. $11\frac{1}{2}$ years,
      ii. $13\frac{1}{2}$ years,
      iii. $15\frac{1}{2}$ Years old?

2. Table 13.18 shows the population of a certain country, in thousands, between the years 1901 to 1966.

Year	1901	1911	1921	1931	1941	1951	1961	1966
Pop.	5 400	7 200	8 800	10 400	11 500	14 000	18 200	20 000

Table 13.18

   a) Draw a line graph for the data.
   b) Estimate the population in:
      i. 1926
      ii. 1971

3. Table 13.19 shows, the maximum and minimum temperatures in degrees, recorded for the first 12 days of September in a certain town.

Day	1	2	3	4	5	6	7	8	9	10	11	12
Max (˚C)	26	21	17	23	26	21	20	21	19	19	19	17
Min (˚C)	15	13	12	13	13	14	16	11	11	10	10	12

Table 13.19

   a) Using the same axes, draw line graphs to represent these temperatures.
   b) Which of the two sets of temperatures shows the greater variation?

c) If the temperatures for 5th and 11th September had been omitted, could you have estimated them?

d) Using the trends shown by the graph, make a forecast for 13th September.

4. Table 13.20 shows deaths, in thousands, from two diseases during a period of 10 years.

Year	1947	1948	1949	1950	1951	1952	1953	1954	1955	1956
TB deaths	23.9	22.9	17.5	14.1	12	9.3	7.9	7.1	5.8	4.9
Pneumonia deaths	26.7	20.7	23.6	20.3	25.6	21	23	20.4	23.8	24.8

Table 13.20

a) On the same axis, draw line graphs to represent this data.

b) What can you say about deaths from each disease?

5. Table 13.21 shows the time taken by a certain car test to accelerate from 0 to the given speeds.

Acceleration	Time taken(s)
0 to 50 km/h	4.8
0 to 65 km/h	6.4
0 to 80 km/h	9.0
0 to 95 km/h	12.3
0 to 110 km/h	16.1
0 to 125 km/h	21.4

Table 13.21

a) Represent the data on a line graph with speed on the horizontal axis.

b) How long would the car take to accelerate from:

   i. 0 to 70 km/h,

   ii. 0 to 120 km/h?

c) What speed would the car gain from rest in:

   i. 8 s,

   ii. 14 s?

Averages

The **average** of a group of numbers is a single number which can be used to represent the entire group. For example:

1. If the **average age** of pupils in a class is 6 years, it means that most pupils in that class would have their ages close to that value. A child of age 11 or 3 years would ordinarily not be expected to be in that class.
2. If the **average life span** of a certain type of light bulb is 600 hours, it is expected that a light bulb of that make should last that long when used, maybe a little more or little less.

The most common types of averages are the mean, the mode and the median.

Averages are also called measures of **central tendency** because they show how the values in a data distribution tend to cluster around a central value.

Mean

The **arithmetic mean** (or simply the **mean**) of a group of values is the most common average and it is given by:

Arithmetic mean

$$= \frac{\textit{the sum of all the values in the group}}{\textit{the total number of values in the group}}$$

In other words, if the group has n values x_1, x_2, x_3……. x_n then the mean of those values is given by,

$$\text{Mean} = \frac{x_1, x_2, x_3 \ldots\ldots. + x_n}{n}$$

Example 13.5

Calculate the mean of K2 500, K 4000, K5 500, K7 500 and K 3 000, the pocket money of some 5 students.

Solution

$$\text{Mean} = \frac{K\ 2\ 500 + 4\ 000 + 5\ 500 + 7\ 500 + 3\ 000}{5}$$

$$= K\ \frac{22\ 500}{5} = K\ 4\ 500$$

Example 13.6

Table 13.22 shows the masses, in grams, of some 40 mangoes sold in a shop. Calculate the mean mass of the mangoes.

70	100	90	120	110	80	110	80	100	90
120	110	120	80	110	100	110	90	110	100
80	70	100	80	90	110	110	90	100	90
100	100	90	100	90	90	80	100	100	80

Table 13.22

Solution

Method 1

$$Mean\ mass = \frac{Sum\ of\ masses\ of\ all\ mangoes}{Total\ number\ of\ mangoes}$$

Add all the values in Table 13.22. The total mass is 3 850 g.

Therefore, $mean\ mass = \frac{3\ 850}{40}\ g = 96.25\ g$

Method 2

Make a frequency table as shown in Table 13.23.

Mass x (g)	Tally	Frequency	fx
70	II	2	140
80	𝍸 II	7	560
90	𝍸 IIII	9	810
100	𝍸 𝍸 I	11	1 100
110	𝍸 III	8	880
120	III	3	360
		$\sum f = 40$	$\sum fx = 3850$

Table 13.23

The column ‘fx’ represents the total mass of mangoes. For example, if there are 2 mangoes each of mass 70 g, their total mass is

(70 × 2) g = 140 g.

Similarly, there are 7 mangoes, each of mass 80 g, having a total mass of (80 × 7) g = 560 g, etc.

The symbol $\sum$ (Greek letter 'sigma'), stands for 'sum of'.

Thus, $\sum f$ means 'sum of frequencies' and $\sum fx$ means 'sum of products of f and x'.

The mean, denoted by $\bar{x} = \dfrac{\sum fx}{\sum f}$

In the example, $\sum f = 40, \sum fx = 3\,850$

Therefore, x = = $\dfrac{\sum fx}{\sum f} = \dfrac{3\,850}{40}$ = 96.25 g

Exercise 13.4

1. Calculate the mean of the following.
   a) 1, 3, 5, 7
   b) 2, 4, 6, 8
   c) 1, 2, 3, 4, 5
   d) 6, 7, 8, 9, 10
   e) 2, 2, 3, 4, 4, 5
   f) 3, 4, 4, 7, 8, 9
   g) 3, 9, 4, 7, 2, 8, 7, 9
   h) 10, 4, 11, 13, 15, 19, 21, 5
   i) 8, 0, 3, 3, 1, 7, 4, 1, 4

2. Calculate the mean of the following.
   a) 2.1, 1.4, 3.5, 2.7
   b) 4.8, 4.5, 3.2, 1.8, 2.2
   c) 0.7, 0.9, 0.3, 0.8, 0.7, 0.9, 0.8, 0.6, 0.5, 0.2
   d) $3\frac{1}{2}, 4\frac{1}{4}, 2\frac{1}{8}, 3\frac{3}{4}$

3. Eight ladies had masses of 51 kg, 44 kg, 57 kg, 63 kg, 48 kg, 49 kg, 45 kg, and 53 kg. Find the mean of their masses.

4. The mean mark scored by 5 students in a mathematics test was 19. Four students had the following scores 15, 18, 17 and 16. What score did the 5th students have?

5. Table 13.24 shows the lengths, in centimetres, of a sample of 50 seedlings found in a certain seedbed.

2	7	4	3	2	6	5	5	4	6
5	7	3	5	4	7	3	4	3	2
4	6	4	7	5	3	5	5	6	3
6	5	7	6	6	4	6	6	5	7
5	4	2	7	4	3	6	5	5	7

Table 13.24

Construct a frequency table and use it to calculate the mean the mean length of the seedlings.

Trees are very important in our environment. Let us join hands to plant as many trees as possible.

6. Table 13.25 shows the number of goals scored in a series of football matches.

Number of goals	1	2	3
Number of matches	8	8	x

Table 13.25

If the mean number of goals is 2, what is x?

7. In this question, letter grades are assigned the values shown in Table 13.26.

A	A-	B+	B	B-	C+	C	C-	D+	D
12	11	10	9	8	7	6	5	4	3

Table 13.26

Use the values in Table 13.26 to determine the mean grade for each subject as obtained in an examination by students in St Peter's High school.

	A	A-	B+	B	B-	C+	C	C-	D+	D
Biology	20	15	26	12	16	7	4	1	-	-
English	14	9	16	11	9	3	-	-	-	-
German	1	2	6	7	10	1	3	3	-	-
French	1	1	6	9	5	5	1	7	3	5
Art	-	-	-	3	2	5	3	4	2	1
Maths	-	-	-	5	4	13	22	35	6	10

Table 13.27

Mode

In a given data distribution, the value or item that has the highest frequency is called the mode (from the French 'a la mode' meaning 'fashionable')

Example 13.7

What is the mode of: 71, 71, 72, 75, 73, 75, 76, 76, 75, 72, 78, 79, 75, 78, 79, 75, 71, 73, 75, and 76?

Solution

Table 13.28 is the frequency table for the data.

Number	Frequency
71	3
72	2
73	2
75	6
76	3
78	2
79	2
	20

Table 13.28

Table 13.28 shows that 75 is the value with the highest frequency (i.e. 6). Thus, 75 is the mode and 6 is the **modal frequency.**

Median

When data is arranged in order from the smallest to the largest or the largest to the smallest, the middle value is called the **median.**

Example 13.8

Find the median of the following numbers

a) 15, 12, 13, 13, 9, 10, 8, 11, 10, 8, 7, 9, 10, 10, 11
b) 12, 8, 21, 11, 4, 12, 13, 18, 20, 19, 21, 11

Solution

Arrange the values in order from the smallest to the largest.

a) 7, 8, 8, 9, 9, 10, 10, 10, 10, 11, 11, 12, 13, 13, 15

This is the middle value i.e. median = 10

b) 4, 8, 11, 11, 12, 12, 13, 18, 19, 20, 21, 21
In this case, there is no one value that is in the middle. In such a case the median is the mean of the two middle values.
Thus, median = $\frac{12+13}{2} = 12.5$
Note: When the number of values is **odd**, the median is the **middle value** but when the number is **even**, the median is the mean of the **two middle** values.

Example 13.9

Table 13.29 shows the masses of some tomatoes bought from a farmer.

Mass (in g)	58	59	60	61	62	63
Frequency	2	6	12	9	8	3

Table 13.29

Find: (a) the mean, (b) the median, (c) the mode of the masses of the tomatoes.

Solution

a) Mean mass = $\frac{(58\ 2) + (59\ 6) + (60\ 12) + (61\ 9) + (62\ 8) + (63\ 3)}{(2 + 6 + 12 + 9 + 8 + 3)}$

$= \frac{2\,424}{40} = 60.6\ g.$

b) Since there are 40 tomatoes, the median mass is the mass between those of the 20th and 21st tomatoes.
From Table 13.30, we see that there are 20 tomatoes with mass 60 g and less. The 21st tomato must, therefore, have a mass of 61 g.

Mass (g)	58	59	60	61	62	63
frequency	2	6	12	9	8	3
Cumulative frequency	2	8	20	29	37	40

Table 13.30

Cumulative frequency is a running total of frequencies showing what the total frequency is at the end of each class.

Mass of the 20th tomato = 60g, and mass of the 21st tomato = 61 g.

Therefore, median = $\frac{60 + 61}{2}$ $60.5\ g.$

c) The mode is 60 g.

Exercise 13.5

1. Find the mean, median and mode of each of the following groups of numbers:
   a) 4, 5, 8, 6, 4, 12, 3
   b) 9, 9, 7, 7, 4, 13, 3, 17, 2, 21, 7
   c) 7, 9, 4, 9, 3, 9, 10, 1, 1, 12
   d) 5, 4, 5, 5, 5, 4, 2, 3, 2, 1, 3, 1, 4, 0, 2, 2, 0, 1

2. Write down 5 numbers such that the mean is 6, the median is 5 and the mode is 4.
3. Seven pieces of luggage have masses of 48 kg, 45 kg, 49 kg, 63 kg, 57 kg, 44 kg and 51 kg.
   a) Find the mean mass of the seven pieces.

b) If the lightest and the heaviest piece are taken away, what is the mean mass of the remaining ones?

c) What is the median mass if the lightest piece is removed?

4. Table 13.31 shows lengths of some nails picked at random.

Length of nail (cm)	2	3	4	5	6	7
Frequency	3	6	7	11	8	5

Table 13.31

Find: (a) the mean

(b) the mode and

(c) the median length of the nails.

5. A student carried out a survey on the number of people in cars passing at a certain point. Table 13.32 shows the data he collected.

No. of occupants	1	2	3	4
No. of cars	7	11	7	x

Table 13.32

a) Find x if the mean number of occupants is$2\frac{1}{3}$.

b) What is the largest possible value of x if the mode is 2?

c) What is the largest possible value of x if the median is 2?

5. Table 13.33 shows data from students who participated in blood donation at Maera Secondary School in a particular year.

	Month 1	Month 2	Month 3	Month 4
Boys	4	8	9	10
Girls	6	5	11	12

Table 13.32

a) What type of graph would you draw using this data?

b) Draw the graph for the data.

c) Use the graph to determine

i. The least number of students who participated in the blood donation.
ii. The largest number of students who participated in the blood donation.
iii. The mean number of students who participated in the blood donation.

Unit 14 TRANSFORMATION 1

Transformation

With reference to geometrical shapes, to transform means to make change in form, outward appearance, size, position and so on. The operation applied to the given shape is called **transformation.**

The shape that is being transformed is called the **object** and the transformed figure, the **image**.

Referring to the object, the image and the transformation, we can say that a transformation maps the object onto the image. Some examples of common transformations are **translation, reflection, rotation**, and **enlargement**. In this chapter, we will deal with reflection and its properties.

What do the pictures in Fig. 14.1 have in common?

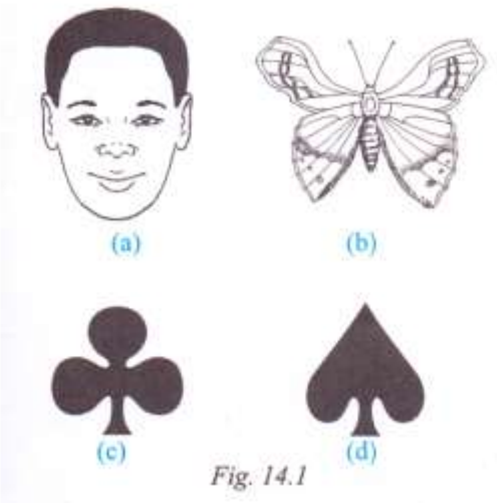


Fig. 14.1

The left hand side of each picture is exactly the same shape and size as the right hand side. This property is known as **symmetry** and the pictures are said to be **symmetrical**.

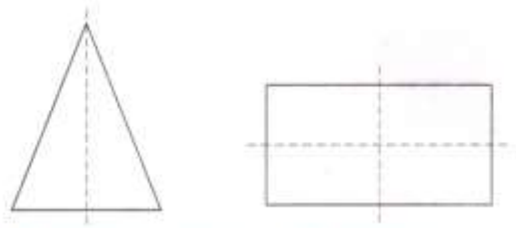
Choose a picture from Fig. 14.1 and make a tracing of it. Draw a line which divide the picture into two identical parts. This line is called a **line of symmetry** and the figure is said to have **line symmetry**.

Place the edge of a mirror along the line of symmetry. What do you notice?

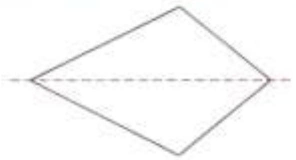
A line which divides a shape into two equal parts i.e. one part is a mirror image of the other, is called a **line of symmetry**. Each part is a **reflection** of the other.

A shape which has **one or more** lines of symmetry is said to be **symmetrical** about the line(s).

Fig. 14.2 shows some symmetrical geometric shapes with the lines of symmetry drawn as broken lines.



(a) an isosceles triangle (b) a rectangle



(c) a kite

Fig. 14.2

How many lines of symmetry has:

a) An equilateral triangle
b) A square
c) A rhombus
d) A circle?

Paper folding and cutting to make symmetrical shapes

Activity 14.1

Fold a rectangular piece of paper once so that the corners coincide. Cut off a corner as shown in Fig. 14.3

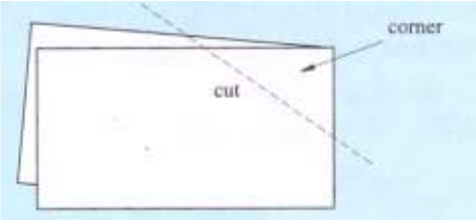


Fig. 14.3

Unfold the corner which you have cut off.

What shape do you get?

What can you say about:

i. The shape and angles of your figure,
ii. The two parts of the shape on each side of the fold.
iii. The distances of the corresponding points on opposite sides of the line?

What line does the fold line represent?

Activity 14.2

Fold a rectangular sheet of paper twice as shown in Fig. 14.4 ensuring that the corners coincide. Cut off the corner as shown.

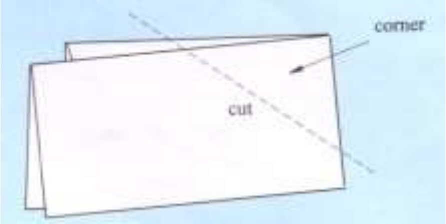


Fig. 14.4

Unfold the corner which you have cut off.

What shape do you get?

What can you say about the sides and angles of your figure?

Place the edge of a mirror on the fold lines in turn. What do you notice?

How many lines of symmetry does your figure have?

Since one half of a symmetrical shape is a mirror image, i.e. a reflection of the other, line symmetry is sometimes called mirror symmetry is sometimes called

mirror symmetry, **reflection symmetry** or **bilateral symmetry** (**bilateral** means two-sided).

Properties of symmetrical shapes

From Activities 14.1 and 14.2, you should have discovered that a line of symmetry divides a shape into two parts with the properties that:

i. The corresponding sides of the two parts are equal;
ii. The corresponding angles of the two parts are equal; and
iii. The corresponding points on the two parts are the same distance away from the line of symmetry.
iv. The two parts are mirror images.

For example, in Fig. 14.5, m is the line of symmetry of the given shape.

The left-hand is the mirror image of the right-hand side. Hence,

<A’ = <A (corresponding angles)

A’B’ = AB (corresponding sides)

$\overline{OB'} = \overline{OB}$ (corresponding distances).

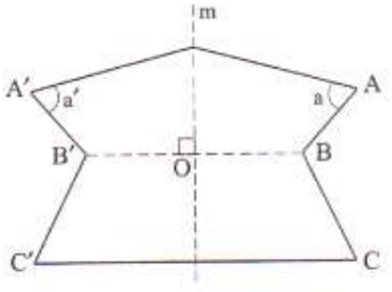


Fig. 14.5

Lines of symmetry in isosceles and equilateral triangles

An isosceles triangle

Activity 14.3

a) Cop triangle ABC (Fig. 14.6) on a sheet paper. Carefully cut it out.

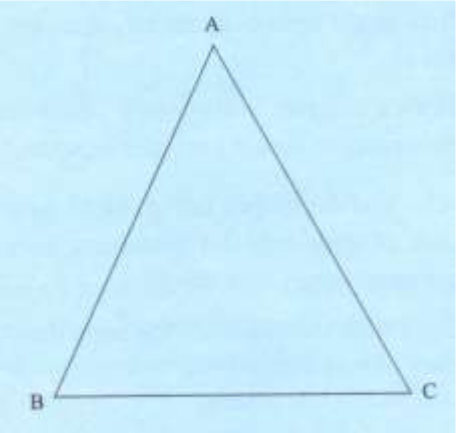


Fig. 14.6

b) Fold the triangle such that B lies on C. what do you notice?
c) Fold the triangle such that A lies on B. what do you notice?
d) Fold the triangle such that A lies on C. what do you notice?

You should have noticed that, by folding the triangle such that B lies on C, we obtain two identical triangles. The fold line formed in this case is a line of symmetry.

An equilateral triangle

Activity 14.4

a) Copy triangle ABC (fig. 14.7) on a sheet of paper. Carefully cut it out.

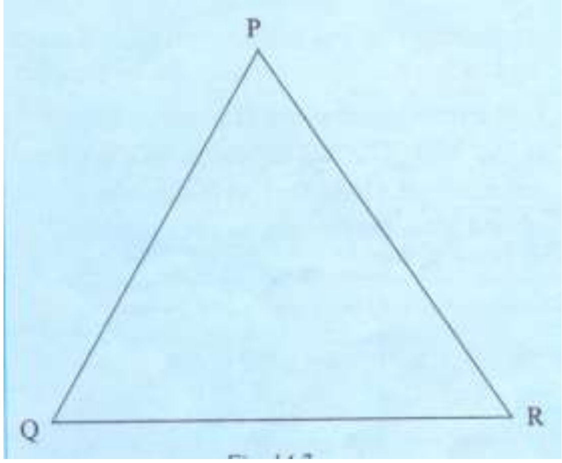


Fig. 14.7

b) Fold the triangle such that Q lies on R. what do you notice?
c) Fold the triangle such that P lies on Q. what do you notice?

d) Fold the triangle such that P lies on R. what do you notice?

You should have noticed that each fold line divide triangle PQR into two identical triangles. From activities 14.3 and 14.4, we conclude that;

1. An isosceles triangle has only one line of symmetry.
2. An equilateral triangle has three lines of symmetry.

Exercise 14.1

1. Fold a rectangular sheet of paper as shown I Fig. 14.8 and cut off the corner as shown. (Note that the corners do not coincide). Is the fold line a line of symmetry?

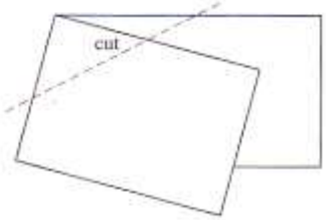


Fig. 14.8

2. Fold a rectangular piece of paper as shown in Fig. 14.9. Cut off the corner as shown and unfold it. (Note that on folding the second time, the corners do not coincide).

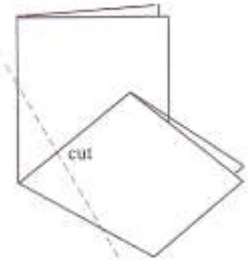


Fig. 14.9

a) How many lines of symmetry does your figure have?
b) Do all the folds represent lines of symmetry? If not, what are they?

3. Fold a piece of paper as in Fig. 14.10 ensuring that the corners coincide. Cut as shown and unfold the part indicated with an arrow.

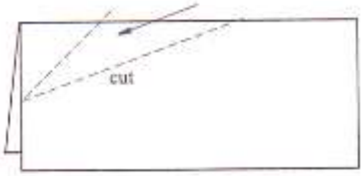


Fig. 14.10

a) What is the shape of the figure obtained?
b) How many lines of symmetry does it have?

4. Fold a sheet of paper twice as in Fig. 14.4 and then fold again as in Fig. 14.11. Cut as indicated and unfold the corner that you have cut off. (Note the equal angles).

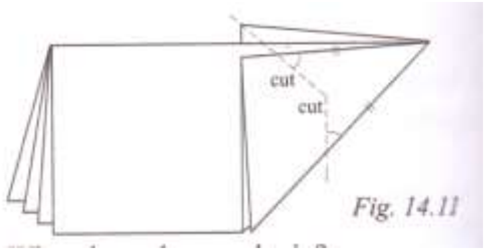


Fig. 14.11

a) What shape do you obtain?
b) Are the fold lines the only lines of symmetry of your figure? If not, where are the others?

How many lines of symmetry does the figure have?

Draw a diagram to show how you would cut the corner to obtain a regular octagon.

5. Which of the shapes in Fig. 14.12 have lines of symmetry and how many lines of symmetry does each have?
Copy each shape and on the copy draw the lines of symmetry.

Fig. 14.12

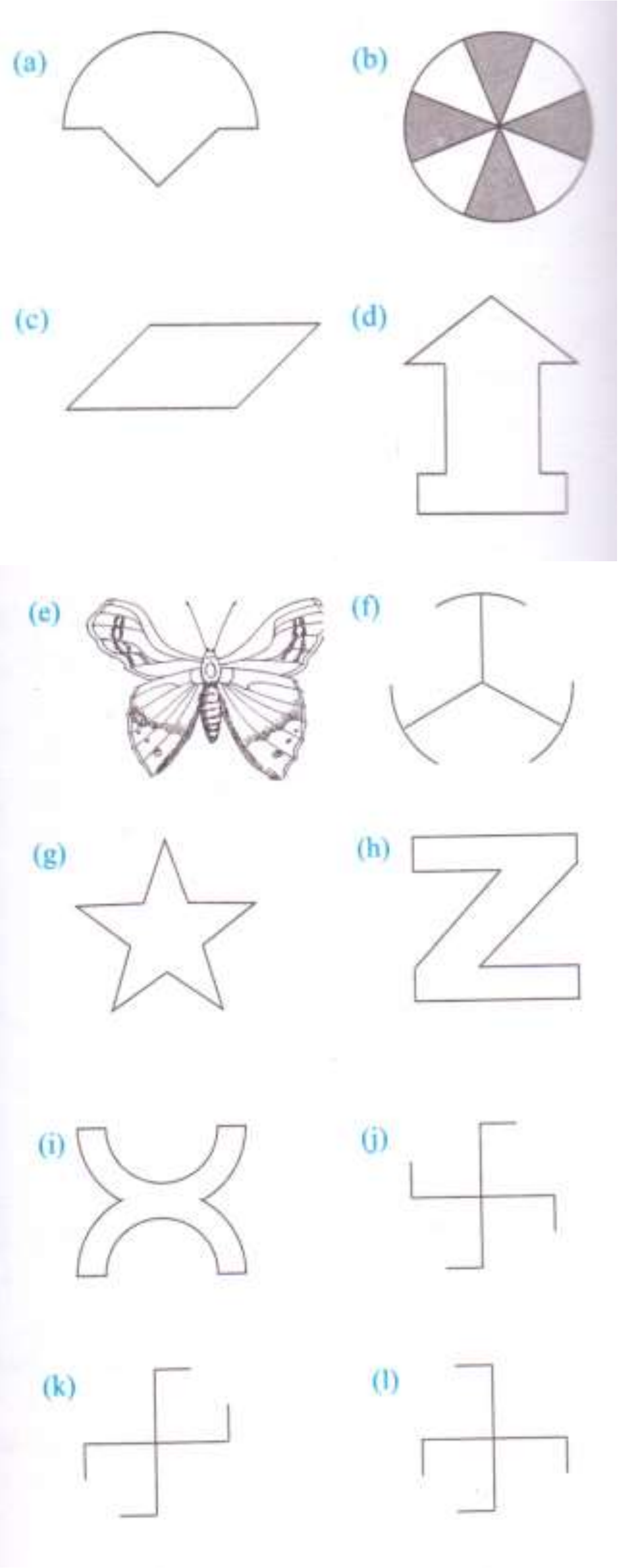


Fig. 14.12

6. Draw a line segment PQ on a piece of paper. Does PQ have a line of symmetry? Fold the paper so that the fold is a line of symmetry of PQ. What is the size of the angles between the fold and PQ? What can you say about the distances of P and Q from you say about the distances of P and Q from any point on the line of symmetry?

Reflection

We have already seen that the two parts of a shape on opposite sides of a line of symmetry, are mirror images of each other.

Now consider looking at yourself in a mirror. Dou you see yourself as others see you? In what ways does your reflection differ from yourself?

Now answer the following questions.

1. If you raise your right arm, which arm is raised in your reflection?
2. Which is taller, you or your reflection?
3. Imagine a line joining the tip of your nose to its reflection in the mirror. What angle does this line make with the mirror?
4. If you stand 3 m in front of the mirror, where does your reflection appear to be?
5. If you walk towards the mirror, what happens to your reflection?

Now look at the pictures in Fig. 14.13(a) and (b) each of which is a picture of a car and its reflection.

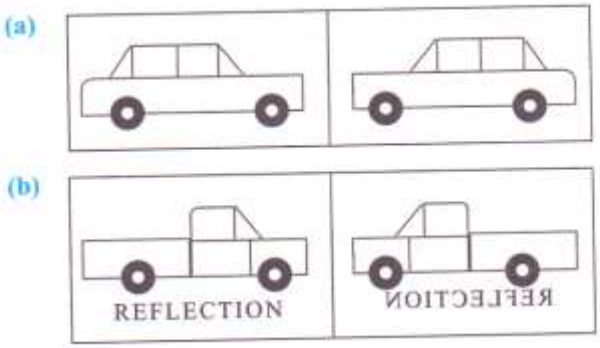


Fig. 14.13

In Fig. 14.13(a), it is not possible to tell which the reflection of the other is. Why? However, we can easily see which the reflection is in Fig. 14.13 (b). Why? Which letters in Fig. 14.13(b) remain the same when reflected? What can you say about their lines of symmetry?

Do your observation apply also to Fig 14.14?

The figure being reflected is called the **object** and its reflection the **image**.

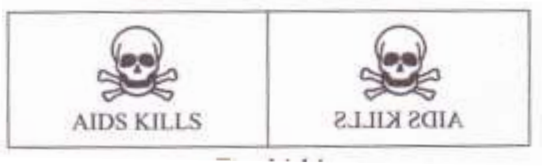


Fig. 14.14

The process or act of reflecting an object is a **transformation.**

In mathematics, a transformation is said to be a **change in the position or dimensions (or both) of a shape**. The figure being transformed is referred to as the **object** and the figure which results after transformation as the **image.**

You are going to encounter other transformations later.

Under reflection, an object and its image have reflection symmetry and the mirror acts as the line or plane of symmetry.

Note that if an object has a line of symmetry that is parallel to the mirror line, as with the letters T, I and O in Fig. 14.13 (b) it is unchanged when reflected. Note that it is the same case with the letters A, I and H and in the skull bones in Fig. 14.14

In mathematics, we usually study only the reflection of two-dimensional (plane) figures. It is easier to see and study the mathematics involved. The mirror line (m) is usually indicated by an arrow at each end.

Properties of reflection

From the foregoing discussion, you should have noticed that an object and its image are on opposite sides of the mirror line, and that:

1. An object and its image have **the same shape and size**.

2. A point on the object and a corresponding point on the image are **equidistant** from the mirror line.
3. The image is **laterally inverted**, i.e. what is the object's left-hand side becomes the image's right-hand side and vice versa.
4. The line joining a point and its image is **perpendicular** to the mirror line.
5. A point on the mirror line is an image of itself. Such a point is said to be **invariant** since its position has not changed.

Note: we think of a mirror as two-sided so that if B is on the same side as the image A', then its image B' is on the same side as the object A (Fig. 14.15).

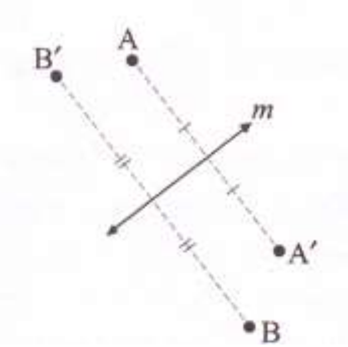


Fig. 14.15

Example 14.1

Draw the image of triangle PQR (Fig. 14.16) under reflection in the mirror line m.

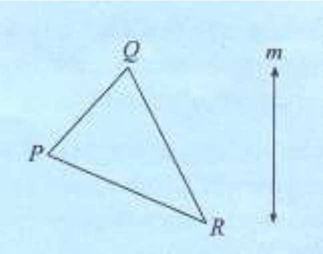


Fig. 14.16

Solution

i. To obtain the image of point P, draw a perpendicular from P to the mirror line and produce it (Fig. 14.17).
ii. Mark off P', the image of P, equidistant from the mirror line as P.

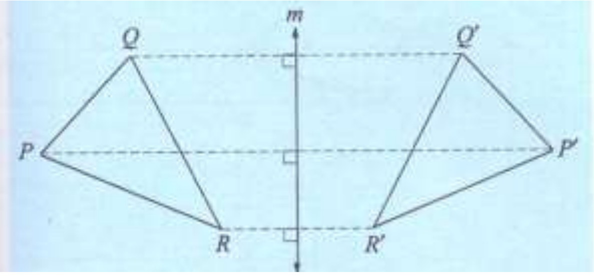


Fig. 14.17

iii. Similarly, obtain Q’ and R’, the images of Q and R respectively.
iv. Join P’, Q’ and R’ to obtain the image of ΔPQR.

Exercise 14.2

1. Make a tracing of ach of the drawing in Fig. 14.18 and construct their images under reflection in the indicated mirror line m.

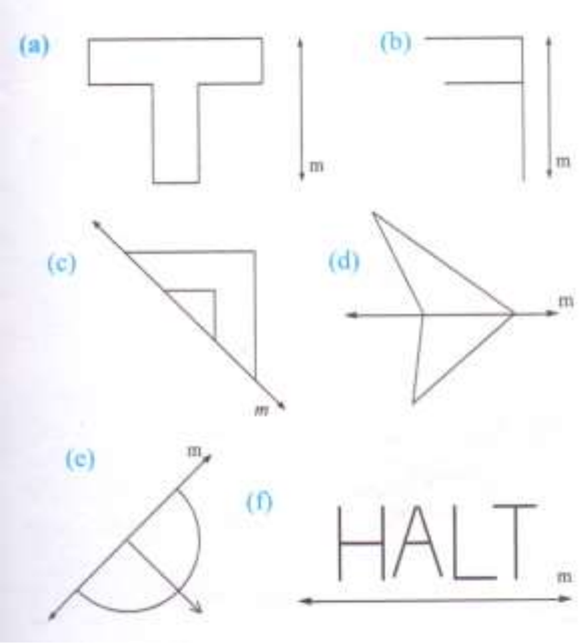


Fig. 14.18

2. Fig. 14.19 shows objects and their images under reflection. Trace each of the drawings and construct the mirror line in each case.

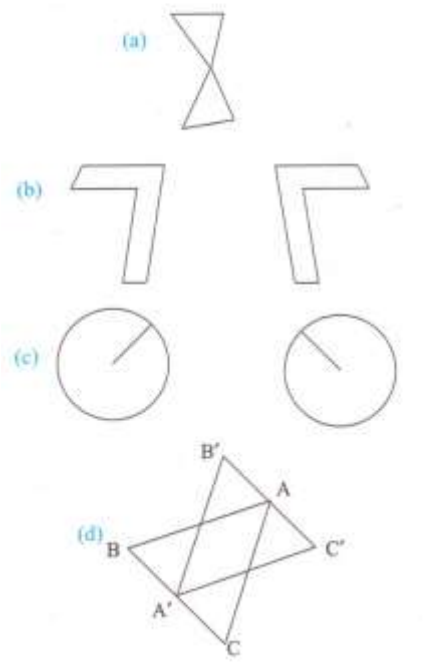


Fig. 14.19

Reflection on the Cartesian plane

Reflection in the mirror lines x-axis (y = 0) and y-axis (x = 0)

Example 14.2

A (2, 4), B (6, 4) and C (7, 2) are the vertices of a triangle. Find the image of the triangle under reflection in the line (i) x-axis, (ii) y-axis, labelling them respectively as A” B” C”.

Solution

Fig. 14.20 shows ΔABC and its images.

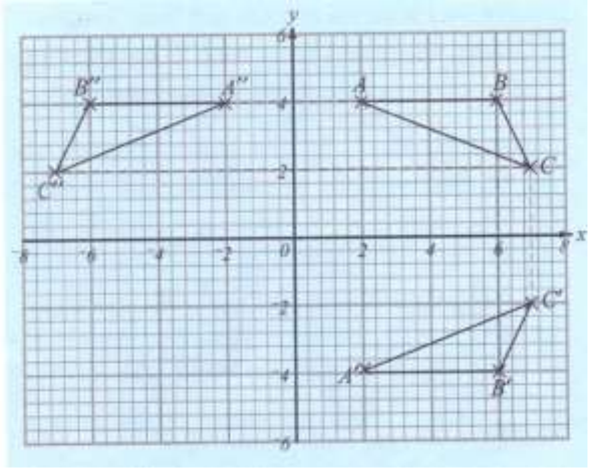


Fig. 14.20

How are the x-coordinates of an object point and its image related?

How are the y-coordinates related?

You should notice that:

Reflection in the mirror line

1. X-axis (y = 0) maps a point P (a, b) onto P’ (a, -b).
2. Y-axis (x = 0) maps a point P(a, b) onto P’(-a, b)

Reflection in the mirror lines x = k and y = k

Example 14.3

Find the images of ΔABC with vertices A(-1, -2), B(1, 5) and C(2, 3) under reflection in the mirror lines (i) x = -1 and (ii) y = 1, labelling them as ΔA’B’C’ respectively.

Solution

ΔABC and its image are shown in Fig. 14.21

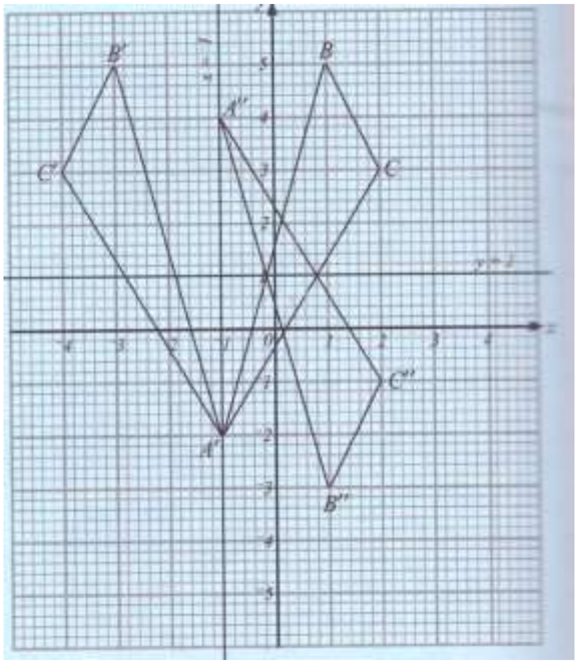


Fig. 14.21

State the relationship between the x-coordinates of an object point and its image. Do likewise for the y-coordinates.

You should notice that:

Reflection in the mirror line

1. X = k maps a point P (a, b) onto P' (2k – a, b).
2. Y = k maps a point P(a, b) onto P'(a, 2k – b).

Reflection in the mirror lines y = x and y = -x

Example 14.4

A(-1, 2), B(1, 5) and C(3, 4) are the vertices of a triangle. Find the images of the triangle when it is reflected in the mirror lines (i) y = x and (ii) y = -x, labelling them as A'B'C' and A"B"C" respectively.

Solution

Fig. 14.22 shows ΔABC and its images.

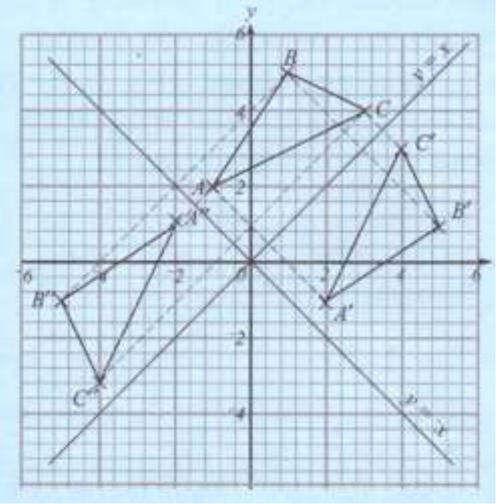


Fig. 14.22

How are the x-coordinates of an object point and its image related? How about the y-coordinates?

You should notice that:

Reflection in the mirror line

1. y = x maps a point P(a, b) onto P'(b, a).
2. y = -x maps a point P(a, b) onto P' (-b, -a).

Example 14.5

The vertices of a quadrilateral are A(2, 0.5), B(2, 2), C(4, 3.5) and D(3.5, 1). Find the image of the quadrilateral under reflection in line y = 0 then reflect the image in the line y = -x.

Solution

We first obtain the image under reflection in the line y = 0. Then we reflect this image in line y = -x.

This is shown in Fig. 14.23. In the figure, A'B'C'D' is the reflection of ABCD in line y = 0. A"B"C"D" is the reflection of A'B'C'D' in line y = -x.

Thus the required image vertices are:

A"(0.5, -2), B"(2, -2), C"(3.5, -4) and D"(1, 3.5).

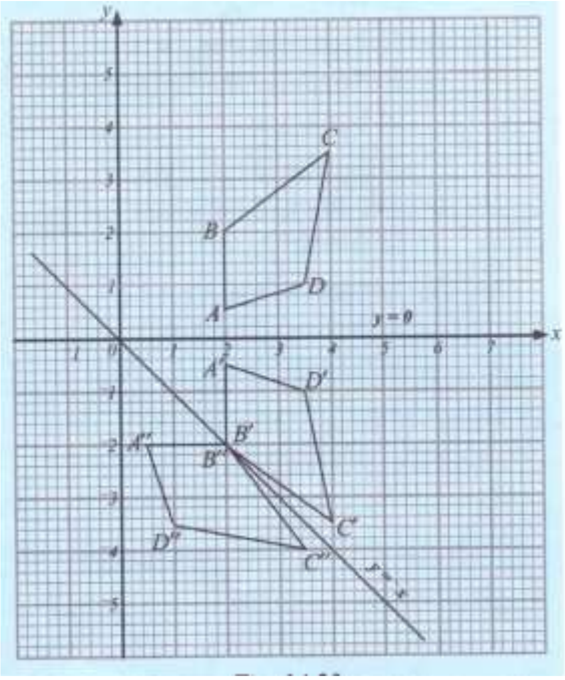


Fig. 14.23

Exercise 14.3

1. a quadrilateral has vertices P($, 2), Q(7, 3), R(6, 2) and S(4, 0). Draw, on the same axes, the quadrilateral and its images under reflection in
   a) the x-axis,
   b) the line y = x,
   c) the y=axis,
   d) the line y = -x,

   labelling the images as P’Q’R’S’, P”Q”R”S”, P”’Q”’R”’S”’ and $P^{iv}Q^{iv}R^{iv}S^{iv}$ respectively.

   State the coordinates of each image point.

2. A(-4, 1), B(-2, -1), C(1, 0) are the vertices of a triangle. Find the image of the triangle when it is reflected in the mirror line:
   a) Y = 1
   b) Y = -2
   c) X = -3
   d) X = 1.5
3. The vertices of a triangle are A(-4, 6), B(-3, 2) and C(-7, 1). Find the final image of the triangle under

a) (i) Reflection in line y = 0
   (ii) Reflection of the image in (i) in line y = x.

b) (i) Reflection in line y = -x
(ii) Reflection of the image in (i) in line x = 0

c) (i) Reflection in line y = x
(ii) Reflection of the image in (i) in line y = 1.

d) (i) Reflection in line x = 1.5
(ii) Reflection of the image in (i) in the same line.

4. Under reflection, which properties of an object are invariant?

Geometric deductions using reflection angle bisector

Draw two intersecting lines on tracing paper. Thinking of the lines as indefinitely long., fold to form lines of symmetry (mirror lines) for the pair lines. What do you notice about the angles?

Did you notice that the angles marked *a* in Fig. 14.24 are equal? What about the angles marked b?

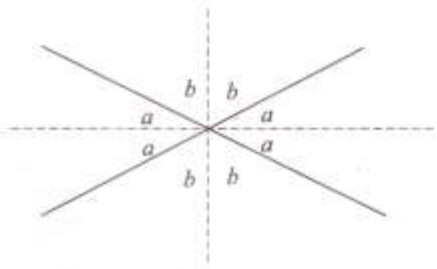


Fig. 14.24

a + a + b + b = 180˚ (angles on a straight line)

2a + 2b = 180˚

2(a + b) = 180˚

a + b = 90˚

Thus, **the two angle bisectors are perpendicular.**

If a figure has only two lines of symmetry, are they always at right angles to each other?

Vertically opposite angles

Two lines intersect as shown in Fig. 14.25. The broken line is a mirror line m. what can you say about the angles of marked c and d?

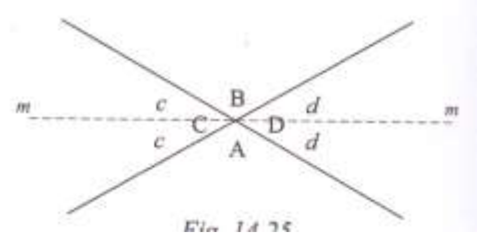


Fig. 14.25

c + c + B = 180˚

i.e. 2c + B = 180˚ (angles on a straight line)

Also, d + d + A = 180˚

i.e. 2d + A = 180˚ (angles on a straight line)

But c = d

Therefore, 2c + B = 180˚

And 2c + A = 180˚

i.e. 2c + A = 2c + B

Therefore, A = B

Also C = D since 2c = 2d or c = d

Thus, if two lines intersect, the **vertically opposite angles** formed **are equal.**

An isosceles triangle

Fig. 14.26 (a) shows an isosceles triangle ABC. Draw a similar figure on tracing paper. Fold to form a line of symmetry for $\overline{BC}$ as in Fig. 14.26

(b) What do you notice about angles B and C?

Label as D the point where the fold line cuts BC.

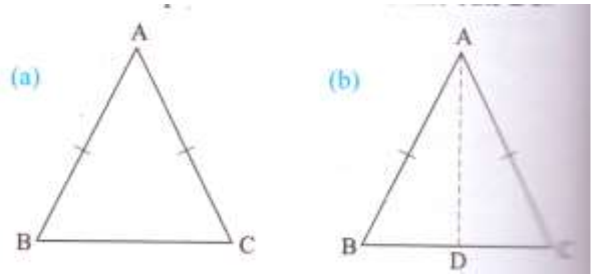


Fig. 14.26

What can you say about <ADC?

<A is called the vertical angle of the isosceles triangle. Does your fold line bisect this angle? You should notice that:

1. The base angles of an isosceles triangle are equal.
2. The perpendicular from the vertices of an isosceles triangle to the base bisects the vertical angle. It also bisects the base.

Mediators of the sides of a triangle.

Draw a triangle ABC on tracing paper. Fold to form lines of symmetry for each of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ in turn, (See Fig. 14.27). The lines of symmetry divide the respective sides of the triangle equally. These lines are called **mediators**. What do you notice about the fold lines?

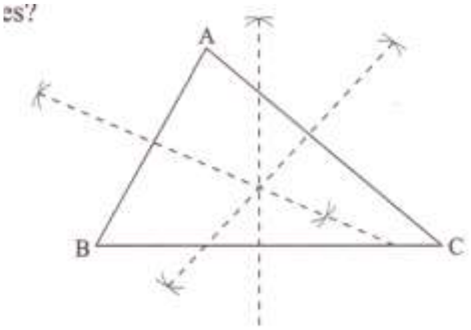


Fig. 14.27

You should notice that:

1. The three mediators of the sides of a triangle meet at a common point. The common point is called the circumcentreof the triangle.
2. If the triangle was equilateral, each mediator would pass through the vertex of the triangle opposite the side it mediates.

Chord of a circle

Draw a circle centre O on a tracing paper. Draw a chord AB on the circle (See Fig. 14.28).

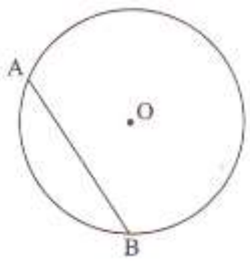


Fig. 14.28

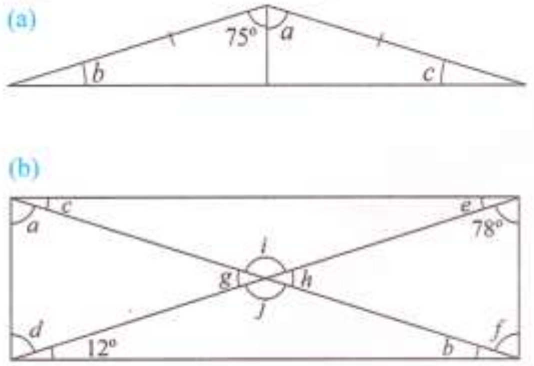
Fold the circle so that A coincides with B. does your fold line pass through O? What angle does the fold line make with chord AB?

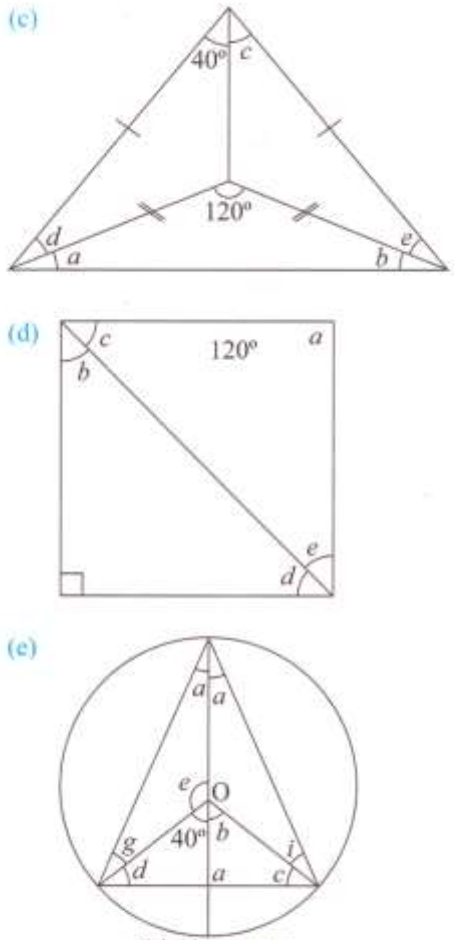
We see that:

1. The fold is a line of symmetry and meets AB at angle of 90˚.
2. The fold passes through the centre of the circle.

Exercise 14.4

1. Draw a circle on tracing paper by drawing round a circular object. Find the centre by folding. Check with a pair of compasses.
2. Use reflection properties to find the angles marked by small letters in each part of Fig. 14.29.





O is the centre

Fig. 14.29

3. In Fig. 14.30, PQRS is a circle, centre O. OP = 5 cm., TR = 4 cm, and ST = 2 cm

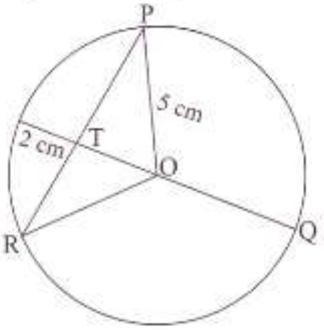


Fig. 14.30

a) State the size of <PTO.

b) State the lengths of the following:

i.	OR	iv.	OQ
ii.	OS	v.	QT
iii.	OT		

4. Draw a triangle ABC. With BC as the mirror line, construct the reflection image of ABC. What special figure is formed by the object triangle ABC combined with its image? What is the sum of the interior angles of this figure?
   Use these facts to show that the angle sum of a triangle is 180˚.

5. A line of an object is at 35˚ to the line of the mirror. What is the size of the angle between the corresponding image line and the mirror?

6. An object line and its image make an angle of 90˚. Describe the position of the mirror line. Draw a diagram to illustrate this.
7. Draw the figure with vertices at (-1, 1), (2, 3), (5, 1) and (2, -1).
   a) What kind of figure is it?
   b) How many lines of symmetry does it have? Mark them out clearly.
   c) What are the equations of the lines of symmetry?

8. Find the equation of the mediator of the line segment joining each of the following pairs of points.
   a) A(-2, 4), B(5, 4)
   b) C(-5, 3), D(-1, 3)
   c) E(3, 8), F(3, 5)
   d) G(-1, 1), H(-1, -5)

9. Points A(-4, 2), B(-3, -3) and C(-1, -1) are the vertices of a triangle. Plot and join them to form the triangle.
   a) What type of triangle is ABC?
   b) If points B and C are images of each other, draw the mirror line.

10. P (-1, 3), Q (-3, -1) and R (3, -1) are the vertices of a triangle.

a) What type of triangle is ΔPQR?
b) Draw the mediators of the sides.
c) What are the coordinates of the point of intersection of the mediators?