

Homework 4 Solutions

BEE 4850/5850

Overview

Instructions

The goal of this homework assignment is to practice simulation-based uncertainty quantification, including the bootstrap and Markov chain Monte Carlo.

- Problem 1 asks you to use the parametric and non-parametric bootstrap to estimate the median of water level data.
- Problem 2 asks you to use the parametric bootstrap to estimate parameter uncertainty in a semi-empirical sea-level rise model.
- Problem 3 (only required for students in BEE 5850) asks you to use the bootstrap to quantify uncertainties in the gender-impact-on-hurricane-damages dataset from Homework 2.

Load Environment

The following code loads the environment and makes sure all needed packages are installed. This should be at the start of most Julia scripts.

```
import Pkg
Pkg.activate(@__DIR__)
Pkg.instantiate()
```

The following packages are included in the environment (to help you find other similar packages in other languages). The code below loads these packages for use in the subsequent notebook (the desired functionality for each package is commented next to the package).

```
using Random # random number generation and seed-setting
using DataFrames # tabular data structure
using DataFramesMeta # API which can simplify chains of DataFrames
    ↪ transformations
using CSV# reads/writes .csv files
using Distributions # interface to work with probability distributions
using Plots # plotting library
using StatsBase # statistical quantities like mean, median, etc
using StatsPlots # some additional statistical plotting tools
using Optim
using LaTeXStrings
using Dates

Random.seed!(1)
```

Problems

Scoring

- Problem 1 is worth 10 points;
- Problem 2 is worth 10 points;
- Problem 3 is worth 5 points.

Problem 1

Let's load the data from 'data/salamanders.csv'.

```
dat = CSV.read(joinpath("data", "salamanders.csv"), DataFrame, delim=";")
```

	SITE	SALAMAN	PCTCOVER	FORESTAGE
	Int64	Int64	Int64	Int64
1	1	13	85	316
2	2	11	86	88
3	3	11	90	548
4	4	9	88	64
5	5	8	89	43
6	6	7	83	368
7	7	6	83	200
8	8	6	91	71
9	9	5	88	42
10	10	5	90	551
11	11	4	87	675
12	12	3	83	217
13	13	3	87	212
14	14	3	89	398
15	15	3	92	357
16	16	3	93	478
17	17	2	2	5
18	18	2	87	30
19	19	2	93	551
20	20	1	7	3
21	21	1	16	15
22	22	1	19	31
23	23	1	29	10
24	24	1	34	49
...

Since we're using a Poisson model, the model specification is

$$y \sim \text{Poisson}(\lambda)$$

$$\log(\lambda) = ax + b,$$

where x is the percent groundcover of the plot and y is the salamander count.

Let's find the maximum likelihood.

```
function salamander_pcover(p, counts, pctcover)
    a, b = p
    λ = exp.(a * pctcover .+ b)
    ll = sum(logpdf.(Poisson.(λ), counts))
    return ll
end
```

```

lb = [-10.0, -50.0]
ub = [10.0, 50.0]
p0 = [0.0, 0.0]

# function to make standardizing the predictor more convenient
stdz(x) = (x .- mean(x)) / std(x)

result = optimize(p → -salamander_pcover(p, dat.SALAMAN,
    ↪ stdz(dat.PCTCOVER)), lb, ub, p0)
θ_mle = result.minimizer

```

```

2-element Vector{Float64}:
 1.1595061318756
 0.42951129115546005

```

Each bootstrap replicate consists of a new dataset formed by resampling plot datum, to which we repeat the above analysis to refit the model.

```

nboot = 1_000
θ_boot = zeros(nboot, 2) # storage for bootstrap replicates
for i = 1:nboot
    idx = sample(1:nrow(dat), nrow(dat), replace=true)
    boot_dat = dat[idx, :]
    result = optimize(p → -salamander_pcover(p, boot_dat.SALAMAN,
    ↪ stdz(boot_dat.PCTCOVER)), lb, ub, p0)
    θ_boot[i, :] = result.minimizer
end
# show the bootstrap mean
θ̂ = mean(θ_boot; dims=1)
@show θ̂;

```

```

θ̂ = [1.1794723089593893 0.38099303881850943]

```

We can now visualize the bootstrapped sampling distributions and compare to the MLE.

```

p1 = histogram(θ_boot[:, 1], xlabel="a", ylabel="Count",
    ↪ fillcolor=:white, label=false)
vline!(p1, [θ_mle[1]], color=:red, lw=2, label="MLE")
vline!(p1, [θ̂[1]], color=:purple, lw=2, label="Bootstrap Mean")
plot!(p1, size=(400, 400))

```

```

p2 = histogram(θ_boot[:, 2], xlabel="b", ylabel="Count",
    ↪ fillcolor=:white, label=false)
vline!(p2, [θ_mle[2]], color=:red, lw=2, label="MLE")
vline!(p2, [θ̂[2]], color=:purple, lw=2, label="Bootstrap Mean")
plot!(p2, size=(400, 400))

display(p1)
display(p2)

```

We can see from Figure ?? that both of the bootstrapped distributions are skewed, but that the estimates only have a small bias (recall that the bootstrap estimate of bias of a statistic t is $\mathbb{E}[\tilde{t}] - \hat{t}$, where \hat{t} is the MLE and \tilde{t} is a bootstrap estimate); a has a bias of 0.02 and b has a bias of -0.05.

To find the 90% confidence intervals, we use the basic bootstrap formula

$$(\hat{t} - (Q_{\tilde{t}}(1 - \alpha/2) - \hat{t}), \hat{t} - (Q_{\tilde{t}}(\alpha/2) - \hat{t})),$$

where $\alpha = 0.1$.

```

a_q = quantile(θ_boot[:, 1], [0.95, 0.05])
b_q = quantile(θ_boot[:, 2], [0.95, 0.05])

a_ci = (2 * θ_mle[1] - a_q[1], 2 * θ_mle[1] - a_q[2])
b_ci = (2 * θ_mle[2] - b_q[1], 2 * θ_mle[2] - b_q[2])
@show round.(a_ci, digits=2);
@show round.(b_ci, digits=2);

```

```

round.(a_ci, digits = 2) = (0.72, 1.49)
round.(b_ci, digits = 2) = (0.1, 0.91)

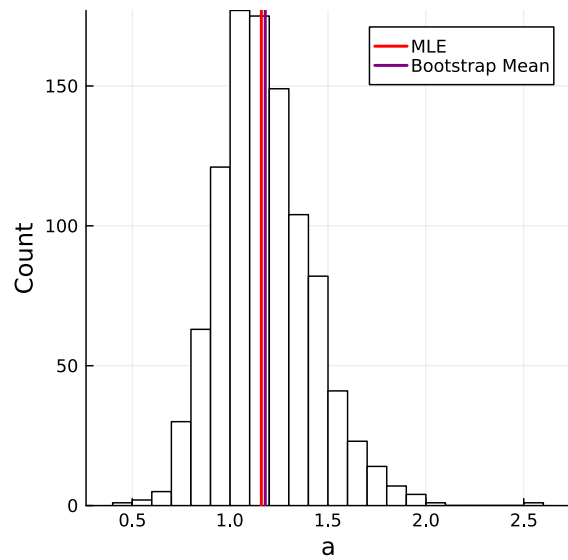
```

One important note is that your answers might differ a bit from these: the confidence interval estimates are a bit sensitive to the random bootstrap replicates with this number of replicates, but the answer should be roughly similar to this.

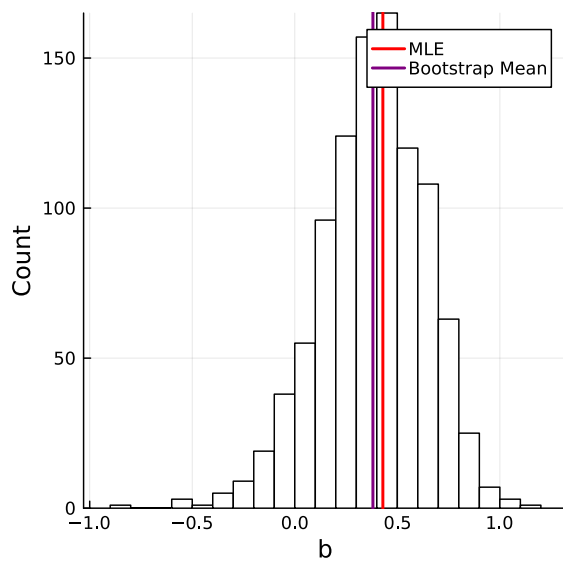
Problem 2

First, load the data

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(a) Bootstrapped estimates of parameter values for Problem 1. The red line is the MLE for the given parameter and the purple line is the bootstrap mean.



(b)

```

norm_yrs = 1980:1999

sl_dat = DataFrame(CSV.File(joinpath("data",
  ↳ "CSIRO_Recons_gmsl_yr_2015.csv")))

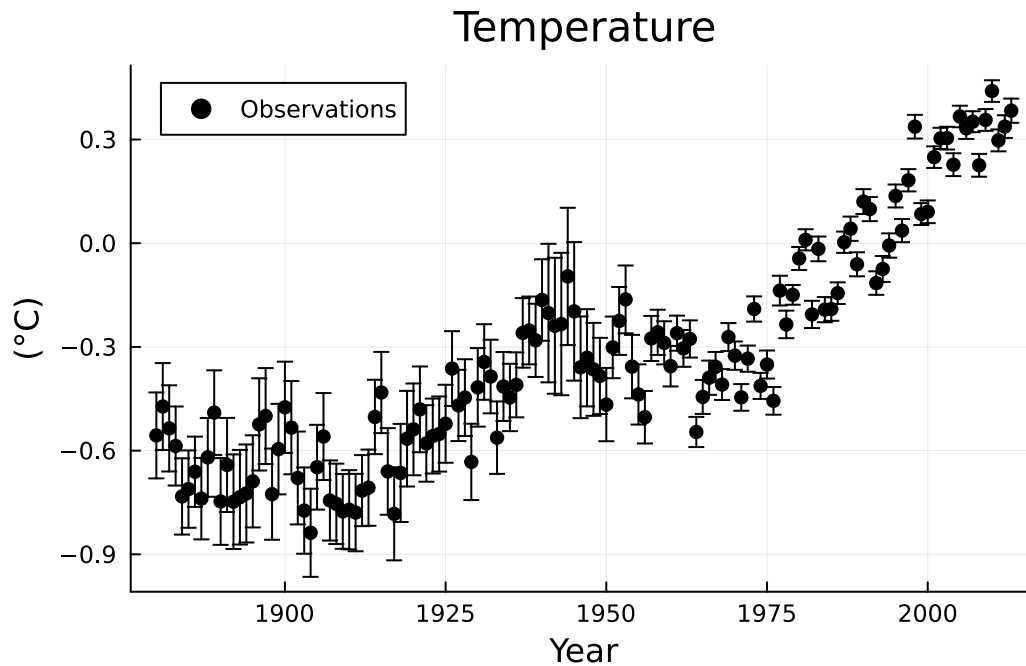
rename!(sl_dat, [:Year, :GMSLR, :SD]) # rename to make columns easier to
  ↳ work with
sl_dat[:, :Year] .-= 0.5 # shift year to line up with years instead of
  ↳ being half-year
sl_dat[:, :GMSLR] .-= mean(filter(row → row.Year ∈ norm_yrs, sl_dat)[!,
  ↳ :GMSLR]) # rescale to be relative to 1880-1900 mean for consistency
  ↳ with temperature anomaly

# load temperature data
temp_dat = DataFrame(CSV.File(joinpath("data",
  ↳ "HadCRUT.5.0.2.0.analysis.summary_series.global.annual.csv")))
rename!(temp_dat, [:Year, :Temp, :Lower, :Upper]) # rename to make
  ↳ columns easier to work with
filter!(row → row.Year ∈ sl_dat[:, :Year], temp_dat) # reduce to the
  ↳ same years that we have SL data for
temp_normalize = mean(filter(row → row.Year ∈ norm_yrs, temp_dat)[!,
  ↳ :Temp]) # get renormalization to rescale temperature to 1880-1900
  ↳ mean
temp_dat[:, :Temp] .-= temp_normalize
temp_dat[:, :Lower] .-= temp_normalize
temp_dat[:, :Upper] .-= temp_normalize

sl_plot = scatter(sl_dat[:, :Year], sl_dat[:, :GMSLR], yerr=sl_dat[:,
  ↳ :SD], color=:black, label="Observations", ylabel="(mm)",
  ↳ xlabel="Year", title="Sea Level Anomaly")

temp_plot = scatter(temp_dat[:, :Year], temp_dat[:, :Temp],
  ↳ yerr=(temp_dat[:, :Temp] - temp_dat[:, :Lower], temp_dat[:, :Upper]
  ↳ - temp_dat[:, :Temp]), color=:black, label="Observations",
  ↳ ylabel="(°C)", xlabel="Year", title="Temperature")

```



The Grinsted model in code (skipping past the discretization since we did this in HW2):

```
function grinsted_slr(params, temps; Δt=1)
    a, b, τ, S₀ = params
    S = zeros(length(temps)) # initialize storage
    Seq = a * temps .+ b
    S[1] = S₀
    for i = 2:length(S)
        S[i] = S[i-1] + Δt * (Seq[i] - S[i-1]) / τ
    end
    return S[1:end]
end
```

grinsted_slr (generic function with 1 method)

The log likelihood function with AR(1) residuals:

```
function ar1_loglik(params, temp_dat, slr_obs, Δt=1.0)
    a, b, τ, S₀, ρ, σ = params
    slr_sim = grinsted_slr((a, b, τ, S₀), temp_dat; Δt = Δt)
    # whiten residuals
    resids = slr_obs .- slr_sim
    ll = 0
end
```



```

    for t = 1:length(temp_dat) - 1
        if t == 1
            ll += logpdf(Normal(0, sqrt( $\sigma^2 / (1 - \rho^2)$ )), resid[s[t]])
        else
            ll += sum(logpdf(Normal( $\rho * resid[s[t]]$ ,  $\sigma$ ), resid[s[t+1]]))
        end
    end
    return ll
end

```

ar1_loglik (generic function with 2 methods)

Now, let's find the MLE by optimizing the ar1_loglik function.

```

low_bds = [-2000.0, -2000.0, 0.1, sl_dat.GMSLR[1] - 1.96 * sl_dat.SD[1],
  ↪ -1.0, 0.1]
up_bds = [20_000.0, 30_000.0, 10_000.0, sl_dat.GMSLR[1] + 1.96 *
  ↪ sl_dat.SD[1], 0.99, 100.0]
p0 = [10.0, 0.0, 1500.0, sl_dat.GMSLR[1], 0.0, 10.0]

mle_optim = optimize(p → -ar1_loglik(p, temp_dat.Temp, sl_dat.GMSLR),
  ↪ low_bds, up_bds, p0)
p_mle = mle_optim.minimizer
@show p_mle;

```

```

p_mle = [546.9549427127922, 410.3873816251446, 179.69518456677608,
-158.18775262248883, 0.5215568507538185, 4.960943964640367]

```

We can plot the fitted model and the residuals:

```

slr_fit = grinsted_slr(p_mle[1:end-2], temp_dat.Temp)
resids = sl_dat.GMSLR - slr_fit

pfit = plot(sl_dat.Year, slr_fit, color=:blue, xlabel="Year",
  ↪ ylabel="Global Mean Sea Level (mm)", label="Model Fit")
scatter!(pfit, sl_dat.Year, sl_dat.GMSLR, color=:black,
  ↪ label="Observations")
plot!(pfit, size=(400, 400))

presids = scatter(sl_dat.Year, resids, color=:black, xlabel="Year",
  ↪ ylabel="Model Residuals (mm)", legend=false)
plot!(presids, size=(400, 400))

```

```
display(pfit)
display(presids)
```

Now we want to generate 1,000 replicates of the residuals using the fitted AR(1) process and add them back to the model hindcast.

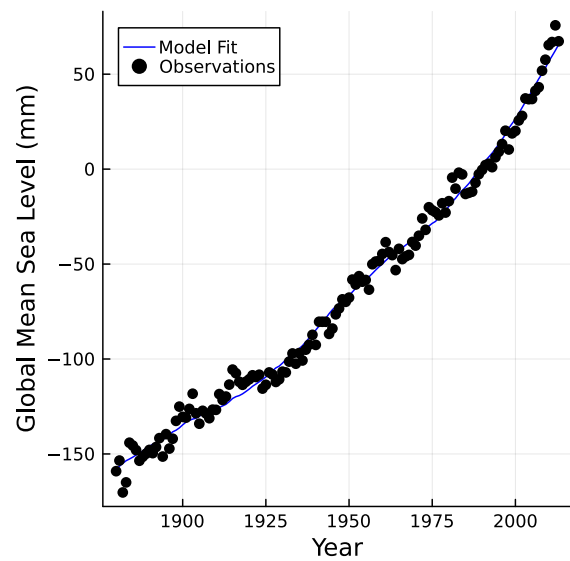
```
function ar1_sim( $\rho$ ,  $\sigma$ , n_sim, T)

    sim_out = zeros(n_sim, T)
    for t = 1:T
        if t == 1
            sim_out[:, t] = rand(Normal(0, sqrt( $\sigma^2$  / (1 -  $\rho^2$ ))), n_sim)
        else
            sim_out[:, t] = rand.(Normal( $\rho$  * sim_out[:, t-1],  $\sigma$ ))
        end
    end
    return sim_out
end

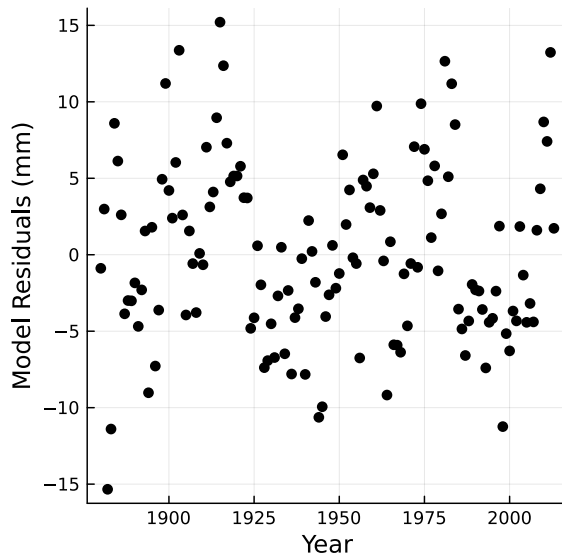
n_sim = 1_000
resids_boot = ar1_sim(p_mle[end-1], p_mle[end], n_sim, nrow(temp_dat))
slr_boot = mapslices(row  $\rightarrow$  row .+ slr_fit, resids_boot; dims=2)
```

```
1000x134 Matrix{Float64}:
-155.578 -160.371 -150.263 ... 65.6277 66.8024 60.9281 66.3092
-166.817 -150.491 -144.177 50.7711 53.8146 58.418 54.321
-159.91 -162.683 -163.252 50.7926 68.1767 61.0413 63.3705
-162.234 -151.419 -150.336 56.3515 66.5042 65.2169 70.7606
-154.785 -153.74 -152.474 53.0952 59.1195 62.5341 61.0469
-158.985 -157.702 -154.061 ... 53.8543 58.2929 60.5728 58.7454
-161.992 -149.346 -157.096 57.0526 65.5743 61.8247 55.8163
-157.933 -162.833 -164.287 52.874 55.6301 63.7023 70.2268
-157.496 -149.485 -152.446 48.1722 47.95 52.4044 63.0504
-145.989 -149.469 -149.711 64.1377 57.1585 53.9137 51.6984
⋮ ⋮ ⋮ ⋮
-152.459 -161.553 -158.747 61.4694 70.3682 68.5933 74.4939
-163.098 -154.776 -146.78 59.051 64.7032 62.6031 66.4458
-156.917 -158.073 -143.377 64.4128 63.236 70.6126 71.2534
-158.449 -159.636 -159.65 62.12 60.2227 61.7398 67.2607
-151.149 -152.926 -151.828 ... 55.7968 62.4473 62.726 64.2052
-159.789 -161.955 -157.092 63.7249 59.6974 61.7888 64.13
```

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(a) Sea-level hindcast using the fitted Grinsted model and the model residuals.



(b)

-159.274	-154.538	-146.59	54.7705	62.0086	53.6441	63.3536
-164.655	-157.275	-162.345	46.6267	49.4417	59.9164	62.2635
-151.977	-153.05	-153.809	50.9142	62.5265	58.7918	65.9182

Now we refit the model to each replicate to get the estimates of the bootstrap parameters.

```

a_boot = zeros(n_sim)
b_boot = zeros(n_sim)
tau_boot = zeros(n_sim)
S0_boot = zeros(n_sim)
rho_boot = zeros(n_sim)
sigma_boot = zeros(n_sim)

for i = 1:n_sim
    mle_optim = optimize(p → -ar1_loglik(p, temp_dat.Temp, slr_boot[i,
↪ :]), low_bds, up_bds, p0)
    a_boot[i], b_boot[i], tau_boot[i], S0_boot[i], rho_boot[i], sigma_boot[i] =
↪ mle_optim.minimizer
end

```

Plotting the histograms:

```

hist1 = histogram(a_boot, xlabel=L"$a$", ylabel="Count", legend=false,
↪ title=L"$a$")
vline!(hist1, [p_mle[1]], color=:red)
plot!(hist1, size=(300, 300))

hist2 = histogram(b_boot, xlabel=L"$b$", ylabel="Count", legend=false,
↪ title=L"$b$")
vline!(hist2, [p_mle[2]], color=:red)
plot!(hist2, size=(300, 300))

hist3 = histogram(tau_boot, xlabel=L"$\tau$", ylabel="Count",
↪ legend=false, title=L"$\tau$")
vline!(hist3, [p_mle[3]], color=:red)
plot!(hist3, size=(300, 300))

hist4 = histogram(S0_boot, xlabel=L"$S_0$", ylabel="Count",
↪ legend=false, title=L"$S_0$")
vline!(hist4, [p_mle[4]], color=:red)
plot!(hist4, size=(300, 300))

```

```

hist5 = histogram(p_boot, xlabel=L"$\rho$", ylabel="Count",
  ↪ legend=false, title=L"$\rho$")
vline!(hist5, [p_mle[5]], color=:red)
plot!(hist5, size=(300, 300))

hist6 = histogram(s_boot, xlabel=L"$\sigma$", ylabel="Count",
  ↪ legend=false, title=L"$\sigma$")
vline!(hist6, [p_mle[6]], color=:red)
plot!(hist6, size=(300, 300))

display(hist1)
display(hist2)
display(hist3)
display(hist4)
display(hist5)
display(hist6)

```

We can see that there are some outlying replicates which result in very high bootstrap samples for the model parameters a , b , and τ . But now we can calculate the sampling distribution of the sea-level sensitivity to temperature, which is a/τ . Plotting this distribution:

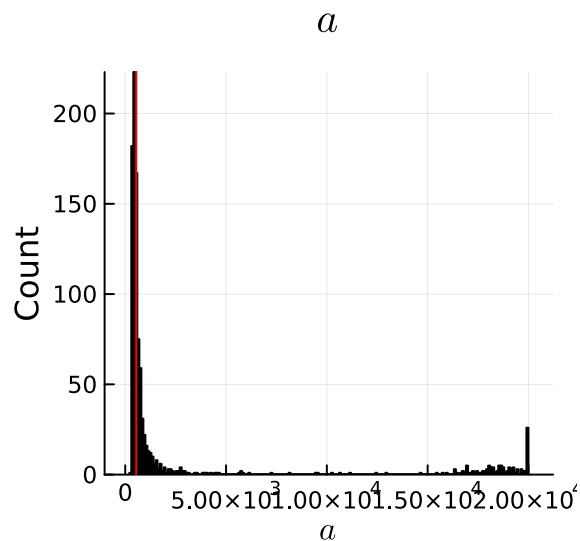
```

sens = a_boot ./ tau_boot

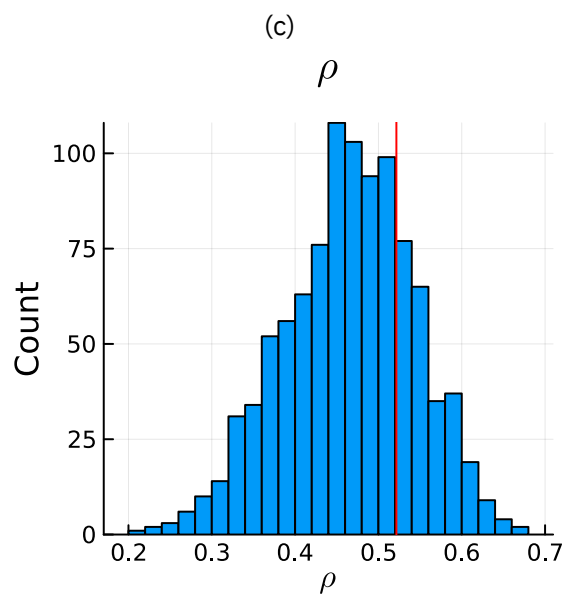
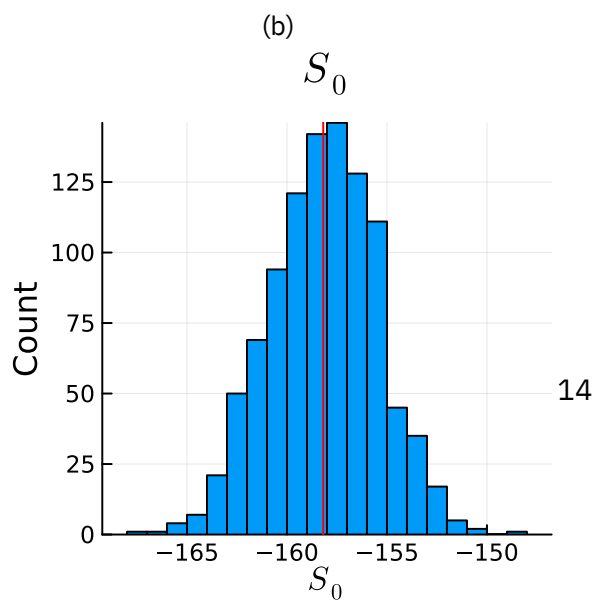
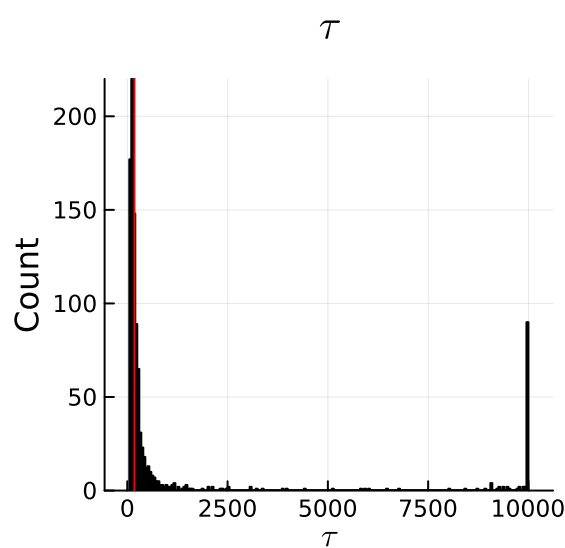
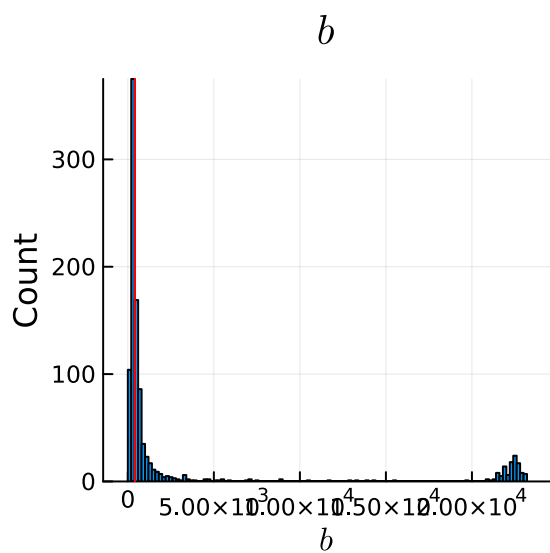
psens = histogram(sens, xlabel="Sea-Level Sensitivity to Temperature
  ↪ (mm/°C)", ylabel="Count", label=false)
vline!(psens, [p_mle[1] / p_mle[3]], color=:red, label="MLE")
plot!(psens, size=(600, 400))

```

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(a) Bootstrap samples for each parameter. The red line is the MLE.



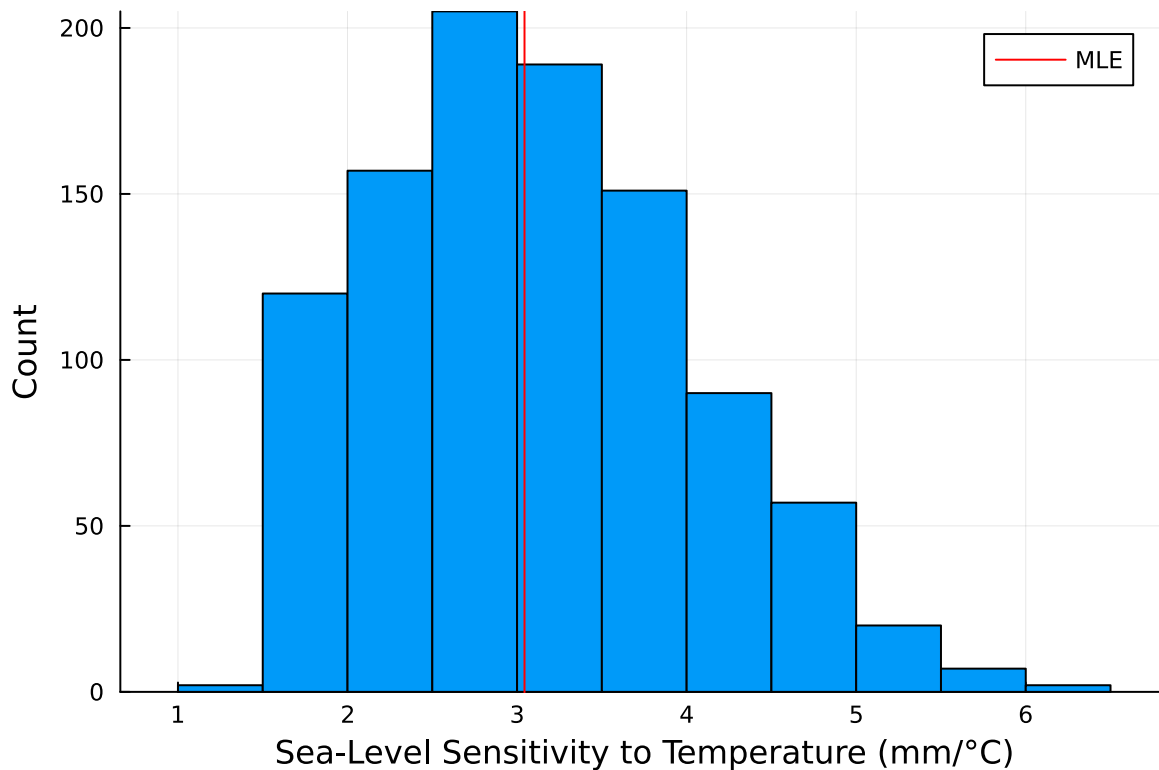


Figure 4: Bootstrap distribution for the sea-level sensitivity to temperature. The red line is the MLE.

We can see from Figure 4 that the large outlying samples cancel out in terms of the numerical effect, which is why the model fit still worked well.¹

The 90% basic bootstrap confidence interval then becomes

```
sens_mle = p_mle[1] / p_mle[3]
q_boot = quantile(sens, [0.95, 0.05])
ci_boot = 2 * sens_mle .- q_boot
vspan!(psens, ci_boot, linecolor=:grey, fillcolor=:grey, alpha=0.3,
  ↪ fillalpha=0.3, label="90% CI")
```

¹This also highlights a key point for model calibration and statistical inference: individual model parameters may not be directly interpretable due to these compensating effects, and it's valuable to look at physically-relevant statistics rather than specific parameters. If the individual model parameters are important for interpretation or extrapolation, they should be appropriately constrained or prior information should be used through a Bayesian analysis.

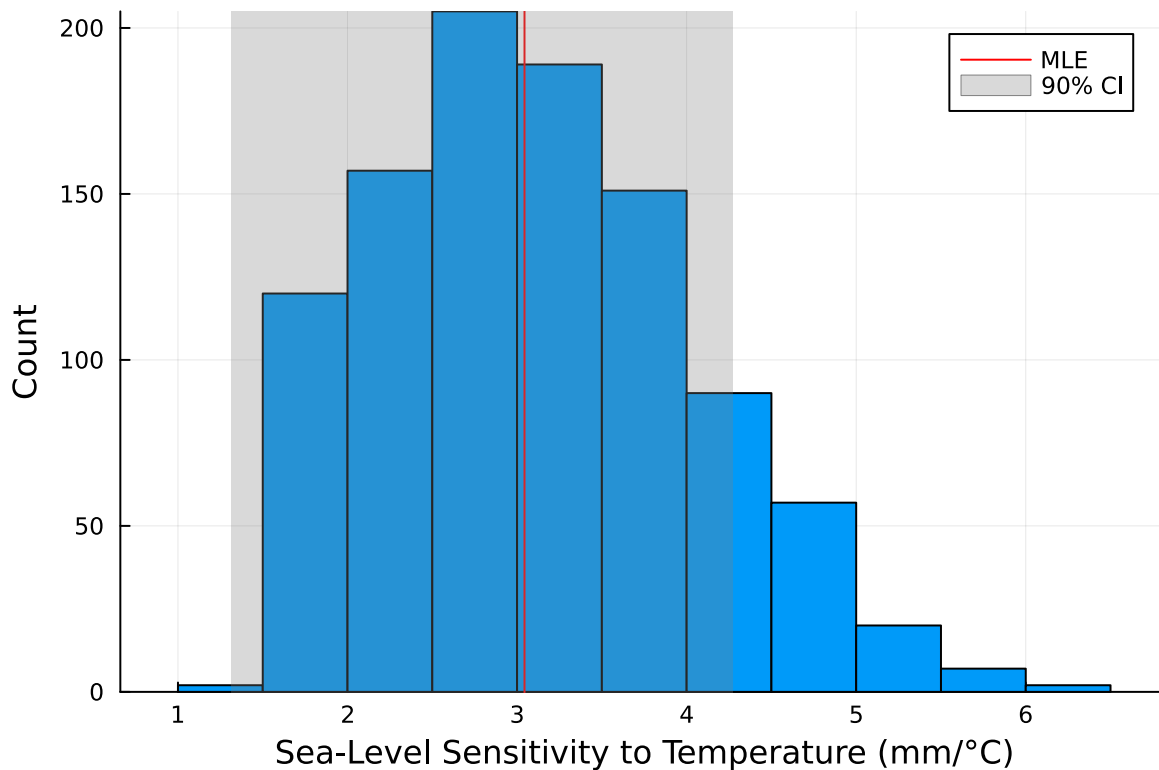


Figure 5: Bootstrap distribution for the sea-level sensitivity to temperature. The red line is the MLE. The grey region is the 90% confidence interval estimated using the basic bootstrap method.

The values for the 90% CI are [1.3, 4.3], and the MLE is 3.0. We can see the impact of the skewed distribution in Figure 5.

Problem 3

Loading the data and separating out the weather-induced variability:

```
function load_data(fname)
  date_format = "yyyy-mm-dd HH:MM"
  # this uses the DataFramesMeta package -- it's pretty cool
  return @chain fname begin
    CSV.File(; dateformat=date_format)
    DataFrame
    rename(
```



```

    "Time (GMT)" => "time", "Predicted (m)" => "harmonic",
    ↪ "Verified (m)" => "gauge"
  )
  @transform :datetime = (Date.(:Date, "yyyy/mm/dd") + Time.(:time))
  select(:datetime, :gauge, :harmonic)
  @transform :weather = :gauge - :harmonic
  @transform :month = (month.(:datetime))
end
end

dat = load_data("data/norfolk-hourly-surge-2015.csv")

plot(dat.datetime, dat.weather; ylabel="Gauge Weather Variability (m)",
  ↪ label="Detrended Data", linewidth=1, legend=:topleft,
  ↪ xlabel="Date/Time")

```

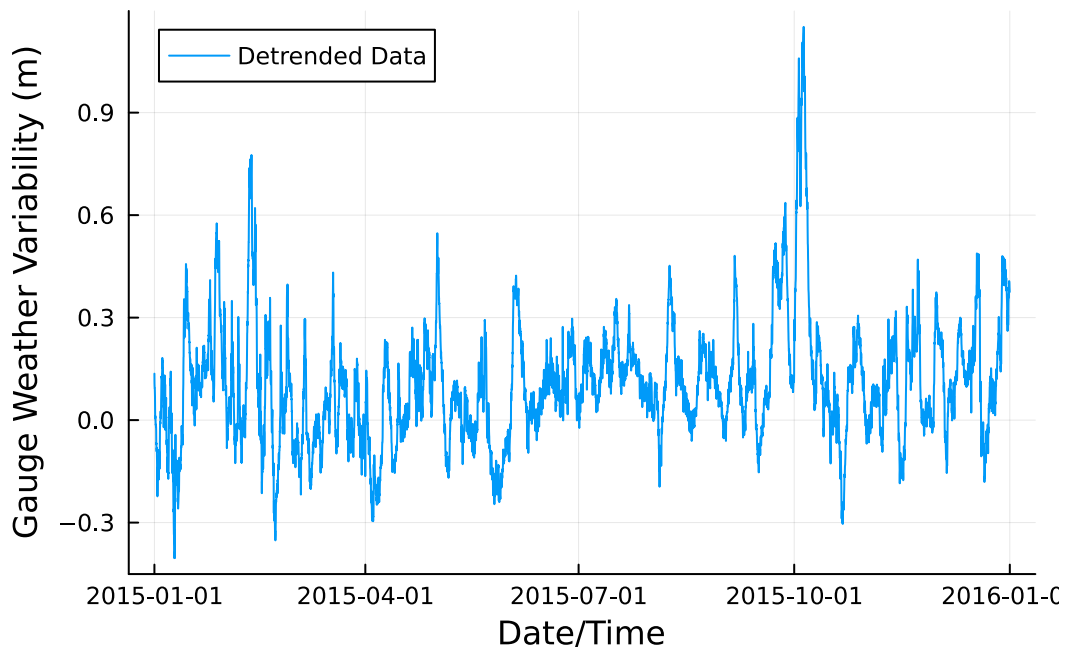


Figure 6: Weather-induced variability data from Problem 3.

Implementing the moving block bootstrap:

```

function generate_blocks(d, k)
    n_blocks = length(d) - k + 1
    blocks = zeros(Float64, (k, n_blocks))
    for i = 1:n_blocks
        blocks[:, i] = d[i:(k+i-1)]
    end
    return blocks
end

n_boot = 1_000
function moving_bootstrap(d, k, n_boot)
    blocks = generate_blocks(d, k)
    m = Int64(floor(length(d) / k))
    l = rem(length(d), k)
    n_blocks = size(blocks)[2]
    surge_bootstrap = zeros(length(d), n_boot)
    for i = 1:n_boot
        block_sample_idx = sample(1:n_blocks, m; replace=true)
        surge_bootstrap[1:length(d)-l, i] = reduce(vcat, blocks[:,
↪ block_sample_idx])
        if l > 0
            surge_bootstrap[length(d)-l+1:end, i] = blocks[:,
↪ sample(1:n_blocks, 1)][1:l]
        end
    end
    return surge_bootstrap
end

k = 20
weather_boot = moving_bootstrap(dat.weather, k, 1_000)

```

```

8760×1000 Matrix{Float64}:
 0.257  0.504  0.128  0.145  0.472  ...  0.069  0.626  0.165  -0.07
-0.172
 0.267  0.5    0.083  0.136  0.467    0.048  0.625  0.159  -0.088
-0.147
 0.294  0.496  0.052  0.119  0.474    0.138  0.641  0.183  -0.094
-0.132
 0.319  0.504  0.052  0.125  0.503    0.171  0.665  0.184  -0.085
-0.091
 0.31   0.511  0.058  0.141  0.498    0.118  0.703  0.183  -0.125
-0.097

```

0.285	0.51	0.07	0.151	0.516	...	0.087	0.744	0.186	-0.153
-0.123									
0.246	0.503	0.084	0.152	0.515		0.078	0.796	0.167	-0.139
-0.128									
0.234	0.487	0.071	0.153	0.518		0.095	0.836	0.164	-0.12
-0.146									
0.223	0.461	0.039	0.162	0.505		0.172	0.874	0.179	-0.108
-0.165									
0.209	0.447	0.009	0.176	0.504		0.19	0.885	0.203	-0.117
-0.177									
⋮					⋮	⋮			
0.188	-0.113	-0.061	0.14	0.038		0.161	0.089	0.181	0.217
0.068									
0.179	-0.11	-0.061	0.138	0.002		0.126	0.099	0.165	0.223
0.057									
0.183	-0.114	-0.056	0.135	-0.001		0.102	0.103	0.129	0.212
0.091									
0.148	-0.115	-0.057	0.128	0.01		0.093	0.102	0.102	0.195
0.127									
0.097	-0.11	-0.054	0.116	0.004	...	0.1	0.095	0.085	0.169
0.155									
0.087	-0.097	-0.04	0.119	-0.013		0.111	0.081	0.062	0.156
0.117									
0.086	-0.102	-0.021	0.091	-0.012		0.137	0.061	0.047	0.161
0.075									
0.091	-0.064	-0.022	0.085	-0.005		0.125	0.064	0.032	0.176
0.084									
0.074	-0.052	-0.01	0.071	-0.024		0.135	0.076	0.03	0.152
0.082									

Now we add these replicates back to the harmonic prediction, then find the distribution of the medians.

```
gauge_boot = mapslices(col → col + dat.harmonic, weather_boot; dims=1)
median_boot = median(gauge_boot, dims=1)'
pmed = histogram(median_boot, xlabel="Median Tide Gauge Reading (m)",
  ↪ ylabel="Count", label=false)
vline!(pmed, [median(dat.gauge)], label="Observation", color=:red)
```

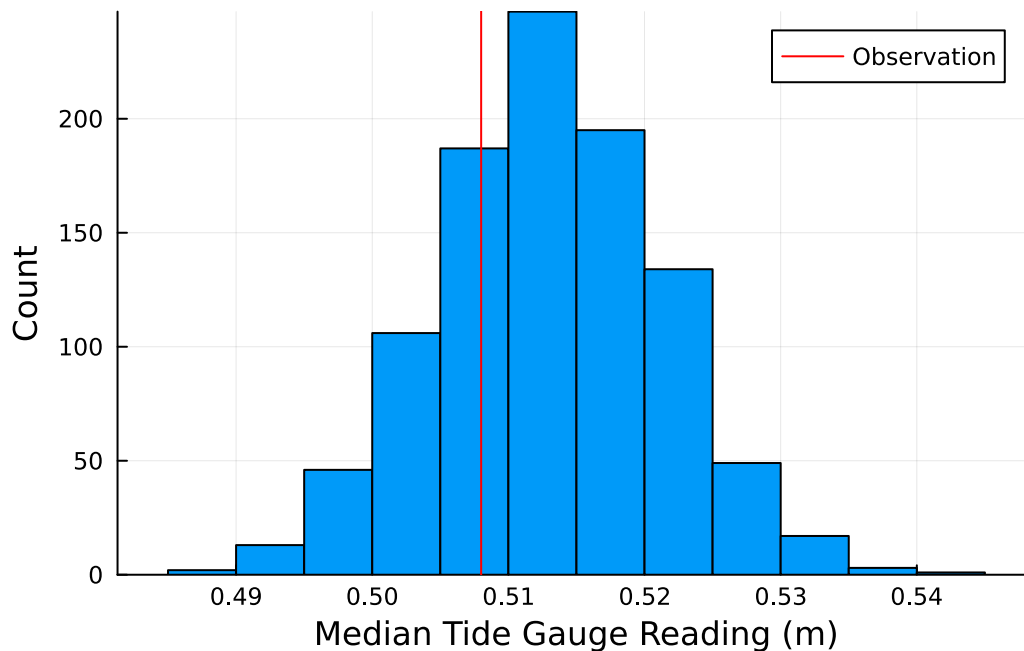


Figure 7: Histogram of the bootstrap distribution of median tide gauge values based on a block size of 20. The red line is the median of the sample.

The bias in the estimator of the median is the difference between the bootstrap mean and the sample estimate, which is about 5 mm. The 90% basic bootstrap confidence interval is [0.49, 0.517] m.

Now, let's repeat the analysis with a block length of 50.

```
k = 50
weather_boot = moving_bootstrap(dat.weather, k, 1_000)
gauge_boot = mapslices(col → col + dat.harmonic, weather_boot; dims=1)
median_boot = median(gauge_boot, dims=1)'
pmed = histogram(median_boot, xlabel="Median Tide Gauge Reading (m)",
  ↪ ylabel="Count", label=false)
vline!(pmed, [median(dat.gauge)], label="Observation", color=:red)
```

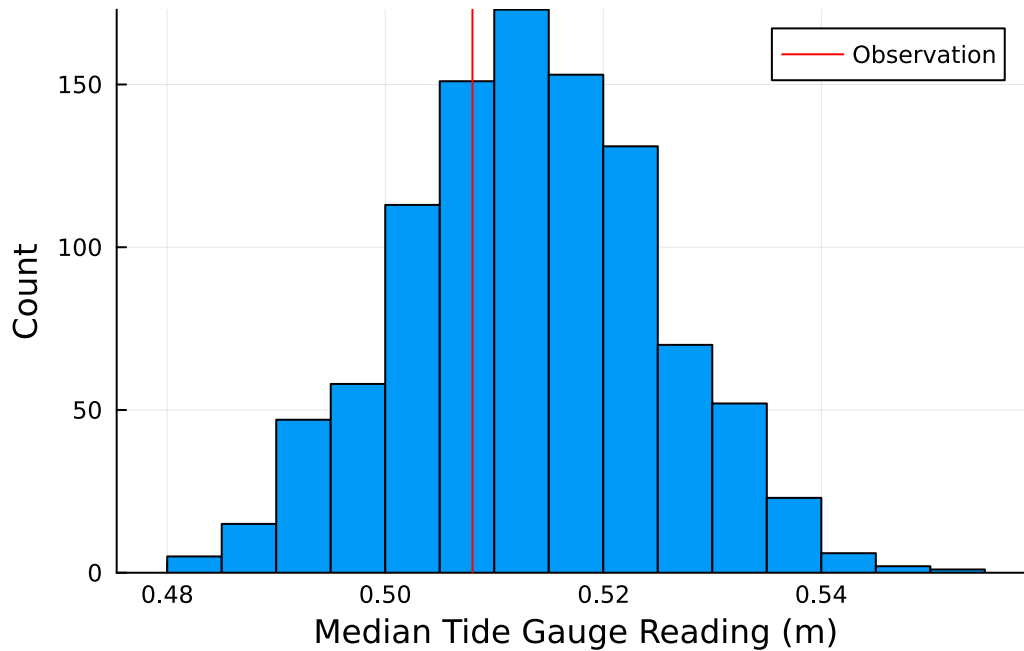


Figure 8: Histogram of the bootstrap distribution of median tide gauge values based on a block size of 50. The red line is the median of the sample.

Now, the bias is about 5 mm and the 90% basic bootstrap confidence interval is [0.483, 0.523] m.

The two different block sizes produced relatively similar results, with some slight variation for the confidence interval estimates, which are shifted slightly higher (but just by a few mm) with the larger block size. Larger blocks mean that there is a greater probability of sampling the (rare) larger values within any given block, which might be expected to produce more occurrences of these values even though the number of re-sampled blocks is smaller. This would have a result of increasing the median. But in general, the median estimator seems to be relatively stable across these block sizes, though even larger blocks might change this.