Probability and Statistics for Engineers

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Lecture 5 :
2.7 Bayes' Rule

Outline

1. Bayes' Rule
   - Total Probability
   - Bayes' Rule

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Total Probability

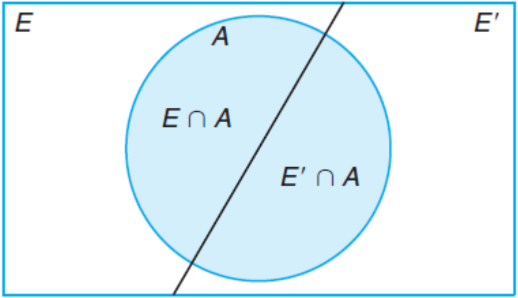

Figure 2.12: Venn diagram for the events A, E, and E'

Total Probability

We can write A as the union of the two mutually exclusive events $E \cap A$ and $E' \cap A$.
Hence,

$$A = (E \cap A) \cup (E' \cap A),$$

then, we can write

$$P(A) = P[(E \cap A) \cup (E' \cap A)] = P(E \cap A) + P(E' \cap A)$$

$$= P(E)P(A|E) + P(E')P(A|E').$$

Total Probability

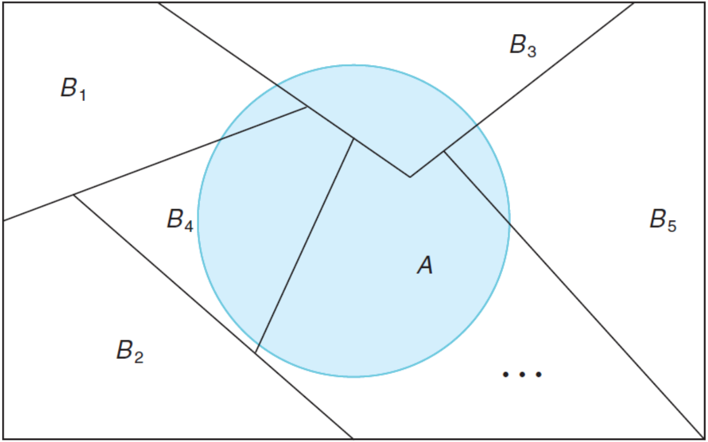

Figure 2.14: Partitioning the sample space S.

Total Probability

Theorem (Total probability)

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i) \cdot P(A|B_i).$$

Total Probability

Example

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Total Probability

Consider the following events :

$$A : \text{the product is defective,}$$

$$B_1 : \text{the product is made by machine B1,}$$

$$B_2 : \text{the product is made by machine B2,}$$

$$B_3 : \text{the product is made by machine B3.}$$

The three events B_1, B_2, and B_3 constitute a partition Applying the Theorem of total probability, we can write

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) + P(B_3) \cdot P(A|B_3)$$

Total Probability

We have

$$P(B_1) \cdot P(A|B_1) = (0.3)(0.02) = 0.006,$$
$$P(B_2) \cdot P(A|B_2) = (0.45)(0.03) = 0.0135,$$
$$P(B_3) \cdot P(A|B_3) = (0.25)(0.02) = 0.005,$$

and hence

$$P(A) = 0.006 + 0.0135 + 0.005 = 0.0245.$$

Bayes' Rule

Theorem (Bayes' Rule)

If the events $B_1, B_2, \ldots, B_k$ *constitute a partition of the sample space* S *such that* $P(B_i) \neq 0$ *for* $i = 1, 2, \ldots, k$*, then for any event* A *in* S *such that* $P(A) \neq 0$*,*

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r) \cdot P(A|B_r)}{\sum_{i=1}^{k} P(B_i) \cdot P(A|B_i)}$$

for $r = 1, 2, \ldots, k$

Bayes' Rule

Example

With reference to the previous example, if a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3 ?

Bayes' Rule

Using Bayes' rule to write
$P(B_3|A) = \frac{P(B_3)\cdot P(A|B_3)}{P(B_1)\cdot P(A|B_1)+P(B_2)\cdot P(A|B_2)+(B_3)\cdot P(A|B_3)}$,
and then substituting the probabilities calculated in the previous example, we have

$$P(B_3|A) = \frac{0.005}{0.006 + 0.0135 + 0.005} = \frac{0.005}{0.0245} = \frac{10}{49}.$$

Exercise 2

A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively. The defect rate is different for the three procedures as follows :

$$P(D|P_1) = 0.01, \quad P(D|P_2) = 0.03, \quad P(D|P_3) = 0.02,$$

where $P(D|P_j)$ is the probability of a defective product, given plan j. If a random product was observed and found to be defective, which plan was most likely used and thus responsible ?

Bayes' Rule

From the statement of the problem

$$P(P_1) = 0.30, \quad P(P_2) = 0.20, \quad \text{and} \quad P(P_3) = 0.50,$$

we must find $P(P_j|D)$ for $j = 1, 2, 3$.The theorem of Bayes'rule shows

$$P(P_1|D) = \frac{P(P_1)P(D|P_1)}{P(P_1)P(D|P_1)+P(P_2)P(D|P_2)+P(P_3)P(D|P_3)}$$

$$P(P_1|D) = \frac{(0.30)(0.01)}{(0.3)(0.01)+(0.20)(0.03)+(0.50)(0.02)}$$

$$P(P_1|D) = \frac{0.003}{0.019} = 0.158.$$

Bayes' Rule

Similarly,

$$P(P_2|D) = \frac{(0.03)(0.20)}{0.019} = 0.316$$

and

$$P(P_3|D) = \frac{(0.02)(0.50)}{0.019} = 0.526.$$

The conditional probability of a defect given plan 3 is the largest of the three ; thus a defective for a random product is most likely the result of the use of plan 3.

Exercise 3

A paint-store chain produces and sells latex and semigloss paint. Based on long-range sales, the probability that a customer will purchase latex paint is 0.75. Of those that purchase latex paint, 60% also purchase rollers. But only 30% of semigloss paint buyers purchase rollers. A randomly selected buyer purchases a roller and a can of paint. What is the probability that the paint is latex ?

Exercise 3

Consider the following events :

$$L : \text{The customer will purchase latex paint,}$$
$$G : \text{The customer will purchase semigloss paint,}$$
$$R : \text{The customer will purchase a roller}$$

The two events L, and G constitute a partition

$P(L|R) = \frac{P(L)P(R|L)}{P(L)P(R|L)+P(G)P(R|G)}$

$P(L|R) = \frac{P(L)P(R|L)}{P(L)P(R|L)+(1-P(L'))P(R|G)}$

$P(L|R) = \frac{(0.75)(0.6)}{(0.75)(0.6)+(1-0.75)(0.3)}$

$P(L|R) = 0.857$