Probability and Statistics for Engineers

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Ifrane, Summer 25

Lecture 7 :
3.3 Continuous Probability Distributions

Outline

1. Continuous Probability Distributions
   - Probability density function
   - Exercise 3
   - Cumulative distribution function
   - Exercise 4
   - Exercise 5

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Continuous Probability Distributions

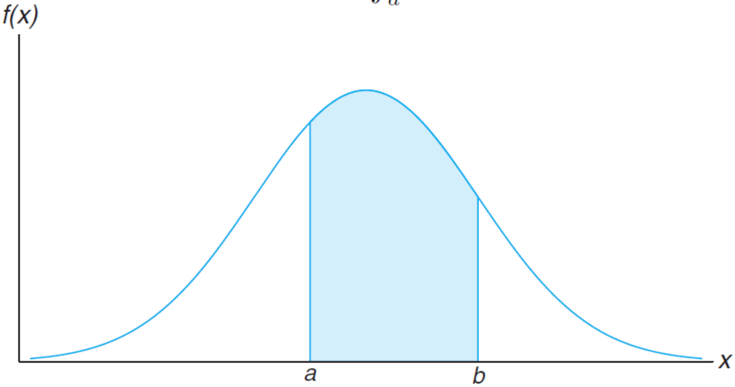
Definition (Probability density function)

The function $f(x)$ is a ***probability density function*** *(pdf) for the continuous random variable X, defined over the set of real numbers, if*

1. $f(x) \geq 0$ *for all* $x \in \mathbb{R}$,
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$,
3. $P(a < X < b) = \int_{a}^{b} f(x)\, dx$.

Continuous probability density function

$$P(a < X < b) = \int_a^b f(x)\, dx.$$



Continuous Probability Distributions

Example

Suppose that the error in the reaction temperature, in °C, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify that $f(x)$ is a density function.
(b) Find $P(0 < X \leq 1)$.

Continuous Probability Distributions

Solution

(a) To verify that $f(x)$ is a density function, we need to check two conditions :

1. $f(x) \geq 0$ for all x,
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Condition 1 : $f(x) \geq 0$ for all x since $x^2 \geq 0$ for all x.

Condition 2 :

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-1}^{2} \frac{x^2}{3}\, dx = \left[\frac{x^3}{9}\right]_{-1}^{2} = \frac{8}{9} - \left(-\frac{1}{9}\right) = 1.$$

So, $f(x)$ satisfies both conditions and is a valid probability density function.

Continuous Probability Distributions

Solution (b)
To find $P(0 < X \leq 1)$, we need to calculate the integral of $f(x)$ over the interval $(0, 1)$:

$$P(0 < X \leq 1) = \int_0^1 \frac{x^2}{3}\,dx = \left[\frac{x^3}{9}\right]_0^1 = \frac{1}{9}.$$

Exercise 3

The total number of hours, measured in units of 100 hours, that a family runs a vacuum cleaner over a period of one year is a continuous random variable X that has the density function

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 \leq x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Probability of running the vacuum cleaner less than 120 hours :
(b) Probability of running the vacuum cleaner between 50 and 100 hours :

Exercise 3

(a) Probability of running the vacuum cleaner less than 120 hours $(x < 1.2)$:

$$P(X < 1.2) = \int_0^{1.2} f(x)\,dx$$

$$P(X < 1.2) = \int_0^{1} x\,dx + \int_1^{1.2} (2 - x)\,dx$$

$$P(X < 1.2) = \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^{1.2}$$

Exercise 3

$$P(X < 1.2) = \frac{1}{2} + \left[2(1.2) - \frac{(1.2)^2}{2}\right] - \left[2(1) - \frac{1^2}{2}\right]$$

$$P(X < 1.2) = 0.68$$

Exercise 3

(b) Probability of running the vacuum cleaner between 50 and 100 hours $(0.5 < x < 1)$:

$$P(0.5 < X < 1) = \int_{0.5}^{1} x\,dx$$

$$P(0.5 < X < 1) = \left[\frac{x^2}{2}\right]_{0.5}^{1}$$

$$P(0.5 < X < 1) = \frac{1}{2} - \frac{0.5^2}{2}$$

$$P(0.5 < X < 1) = \frac{1}{2} - \frac{0.25}{2} = \frac{3}{8} = 0.375$$

Continuous Probability Distributions

Definition (Cumulative distribution function)

*The **cumulative distribution function** $F(x)$ of a continuous random variable X with density function $f(x)$ is*

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt, \quad \text{for } x \in \mathbb{R}.$$

Cumulative distribution function

Proposition

If X is a continuous random variable, then :

- *At any point x, we have*
  $P(X < a) = P(X \leq a) = F(a)$
  $P(X > a) = 1 - P(X \leq a) = 1 - F(a)$
- *For any $a < b$, we have :*
  $P(a \leq X \leq b) = P(a < X \leq b) = P(a < X < b) = F(b) - F(a)$

Continuous Probability Distributions

Example

For the density function of the previous example, find $F(x)$, and use it to evaluate $P(0 < X \leq 1)$.

Continuous Probability Distributions

Solution

For $-1 < x < 2$, the cumulative distribution function $F(x)$ is given by

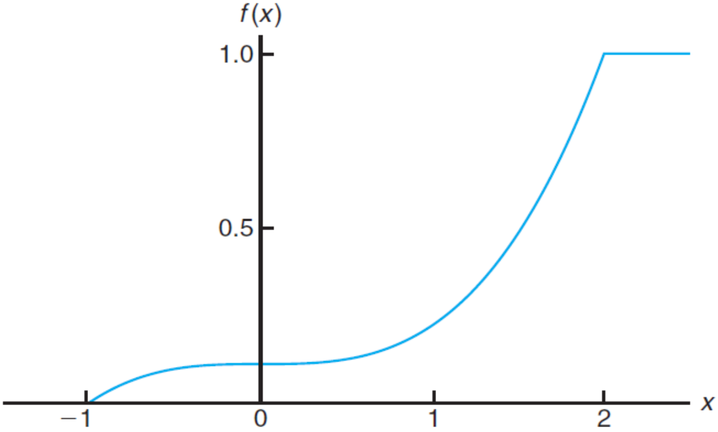
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-1}^{x} \frac{t^2}{3}\,dt = \frac{t^3}{9}\bigg|_{-1}^{x} = \frac{x^3+1}{9}.$$

Therefore,

$$F(x) = \begin{cases} 0, & \text{if } x < -1, \\ \frac{x^3+1}{9}, & \text{if } -1 \le x < 2, \\ 1, & \text{if } x \ge 2. \end{cases}$$

$P(0 < X \le 1) = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}.$

Continuous cumulative distribution function



Exercise 4

An investment firm offers its customers municipal bonds that mature after varying numbers of years. Given that the cumulative distribution function of T, the number of years to maturity for a randomly selected bond, is

$$F(t) = \begin{cases} 0, & t < 1, \\ \frac{1}{4}, & 1 \leq t < 3, \\ \frac{1}{2}, & 3 \leq t < 5, \\ \frac{3}{4}, & 5 \leq t < 7, \\ 1, & t \geq 7. \end{cases}$$

Exercise 4

Find

(a) $P(T > 3)$,

(b) $P(1.4 < T < 6)$,

(c) $P(T \leq 5 \,|\, T \geq 2)$.

Exercise 5

The waiting time, in hours, between successive speeders spotted by a radar unit is a continuous random variable with a cumulative distribution function $F(x)$:

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-8x}, & x \geq 0. \end{cases}$$

Find the probability of waiting less than 12 minutes between successive speeders :

(a) Using the cumulative distribution function of X

(b) Using the probability density function of X.