Probability and Statistics for Engineers

Dr. Sanae KOUISMI

AUI, School of Science and Engineering
Ifrane, Summer 25

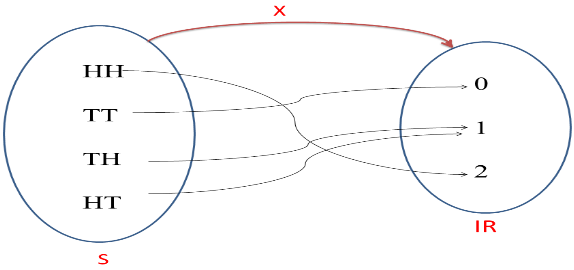
Lecture 6 :
3.1 Concept of a Random Variable and 3.2 Discrete Probability Distributions

Outline

Concept of a Random Variable

Example

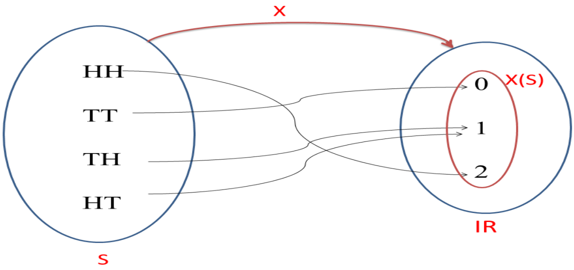
We toss two coins. The sample space S includes 4 possible outcomes noted TT ; TH ; HT ; HH.
Let X be the function that counts the number of heads obtained. X takes values 0 ; 1 ; 2.



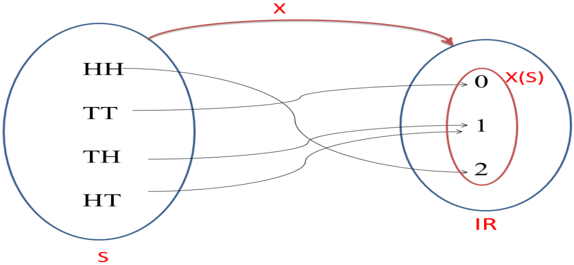
Concept of a Random Variable

Example

We toss two coins. The sample space S includes 4 possible outcomes noted TT ; TH ; HT ; HH.
Let X be the function that counts the number of heads obtained. X takes values 0 ; 1 ; 2.



Concept of a Random Variable



So, $\begin{array}{rcccc} X & : & S & \to & \mathbb{R} \\ & & \omega & \to & X(\omega) \end{array}$

is a function associated with a random experiment such that each possible outcome (or elementary event) associates with a real number.

X is called a **random variable**.

Concept of a Random Variable

Definition (Random Variable)

Let S be a sample space associated with a random experiment.
A random variable is any function

$$\begin{array}{cccccc} X & : & S & \to & \mathbb{R} \\ & & \omega & \to & X(\omega) \end{array}$$

such that each possible outcome ω (element of the sample space S) associates with a real number $X(\omega)$.

Concept of a Random Variable

Example 2

We roll two distinct dice and focus on the sum of the points. Let X be this random variable, defined by :

$$\begin{array}{cccc} X & : & S & \to & \mathbb{R} \\ & & (\omega_1, \omega_2) & \to & \omega_1 + \omega_2 \end{array}$$

The set of possible values taken by X is $X(S) = \{2, 3, ..., 12\}$.

Concept of a Random Variable

Example

Two balls are drawn from an urn containing 4 red balls and 3 black balls. The possible outcomes and the values x of the random variable X, where X is the number of red balls, are

Sample Space	x
RR	2
RB	1
BR	1
BB	0

Concept of a Random Variable

Example

Consider the simple condition in which components are arriving from the production line and they are stipulated to be defective or not defective. Define the random variable X by

$$X = \begin{cases} 1, & \text{if the component is defective,} \\ 0, & \text{if the component is not defective.} \end{cases}$$

Concept of a Random Variable

Definition (Discrete Random Variable)

*If X takes a finite number of values, then X is called a **discrete random variable**.*

Example 3

Let S be the set of students in a class, we can associate with each student ω the number $X(\omega)$ of their siblings.
X takes a finite number of values.
Then X is a discrete random variable.

Concept of a Random Variable

Definition (Continuous Random Variable)

*If X can take an infinite number of values within a given interval (bounded or unbounded), then X is called a **continuous random variable**.*

Example

Examples : height, weight, volume,..

Concept of a Random Variable

Definition : Writing Convention

Let X be a discrete random variable on S, and $x, y \in \mathbb{R}$ with $A \subset \mathbb{R}$. We denote the following events :

1. $[X = x] = \{\omega \in S / X(\omega) = x\}$
2. $[X < x] = \{\omega \in S / X(\omega) < x\}$
3. $[x \leq X \leq y] = \{\omega \in S / x \leq X(\omega) \leq y\}$
4. $[X \in A] = \{\omega \in S / X(\omega) \in A\}$

Concept of a Random Variable

Example :
We consider the roll of a die associated with the sample space of possibilities $S = \{1, 2, 3, 4, 5, 6\}$ and consider the random variable that associates the square of each outcome. We define the following events :

1. $[X = 5] = \{\omega \in S / X(\omega) = 5\} = \emptyset$
2. $[X < 5] = \{\omega \in S / X(\omega) < 5\} = \{1, 2\}$
3. $[3 \leq X \leq 5] = \{\omega \in S / 3 \leq X(\omega) \leq 5\} = \{2\}$
4. $[X \in \{1, 4, 36\}] = \{\omega \in S / X(\omega) \in \{1, 4, 36\}\} = \{1, 2, 6\}$

Exercise 1

Classify the following random variables as discrete or continuous :

- X : the number of automobile accidents per year in Virginia.
- Y : the length of time to play 18 holes of golf.
- M : the amount of milk produced yearly by a particular cow.
- N : the number of eggs laid each month by a hen.
- P : the number of building permits issued each month in a certain city.
- Q : the weight of grain produced per acre.

Exercise 1

Solution :

- X : the number of automobile accidents per year in Virginia. (Discrete)
- Y : the length of time to play 18 holes of golf. (Continuous)
- M : the amount of milk produced yearly by a particular cow. (Continuous)
- N : the number of eggs laid each month by a hen. (Discrete)

Exercise 1

Solution :

- P : the number of building permits issued each month in a certain city. (Discrete)
- Q : the weight of grain produced per acre. (Continuous)

Probability distribution

Definition (Probability distribution)

The set of ordered pairs $(x, f(x))$ is a ***probability function***, ***probability mass function***, *or* ***probability distribution*** *of the discrete random variable X if, for each possible outcome x,*

1. $f(x) \geq 0$,
2. $\sum_{x} f(x) = 1$,
3. $P(X = x) = f(x)$.

Probability distribution

Example 3

We toss two coins. The sample space S includes 4 elementary events noted TT ; TH ; HT ; HH.
Let X be the random variable that counts the number of heads obtained. X takes values 0 ; 1 ; 2.
The probability distribution of the random variable X is :
$P(X = 0) = \frac{1}{4}$, $P(X = 1) = \frac{1}{2}$, and $P(X = 2) = \frac{1}{4}$

Probability distribution

Example 4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school. Then x can only take the numbers 0, 1, and 2.

Probability distribution

$$f(0) = P(X = 0) = \frac{\binom{17}{2}}{\binom{20}{2}} = \frac{68}{95},$$

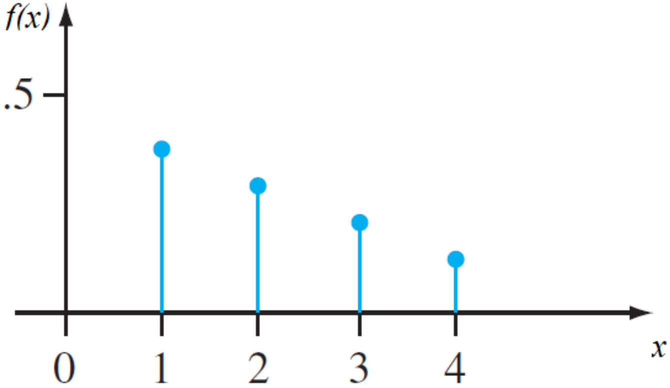
$$f(1) = P(X = 1) = \frac{\binom{17}{1}\binom{3}{1}}{\binom{20}{2}} = \frac{51}{190},$$

$$f(2) = P(X = 2) = \frac{\binom{3}{2}}{\binom{20}{2}} = \frac{3}{190}.$$

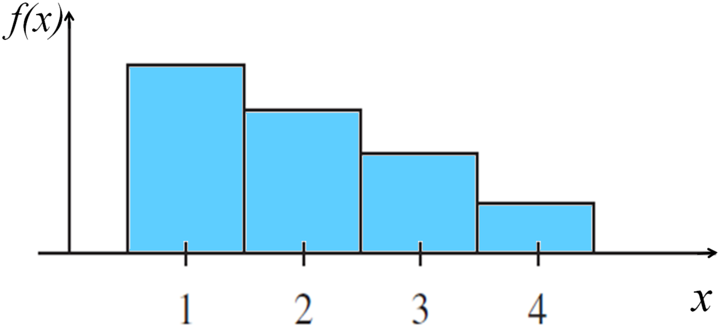
Thus, the probability distribution of X is

x	0	1	2
$f(x)$	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

The probability mass function plot or line graph



The probability histogram



Discrete Probability Distributions

Definition (Cumulative distribution function)

*The **cumulative distribution function** $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is*

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t),$$

for $x \in \mathbb{R}$

Discrete Probability Distributions

Proposition

For all $1 \leq i \leq n$, $P(X = a_i) = F(a_i) - F(a_{i-1})$

Discrete Probability Distributions

Example

Let X be the random variable that characterizes the outcome of the random experiment "rolling a standard die."

X is a discrete random variable, and it can take the integer values 1, 2, 3, 4, 5, and 6.

1. Determine the probability distribution of the random variable X.
2. Determine the values of the cumulative distribution function.

Discrete Probability Distributions

Solution

1. The probability distribution of the random variable X is :

x_i	1	2	3	4	5	6	$Total$
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

2. The cumulative distribution function :

x_i	1	2	3	4	5	6	$Total$
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1
$F(x_i)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1	

Exercise 2

Determine the value c so that each of the following functions can serve as a probability distribution of the discrete random variable X :

(a) $f(x) = c(x^2 + 4)$, for $x = 0, 1, 2, 3$.

(b) $f(x) = c\binom{2}{x}\binom{3}{3-x}$, for $x = 0, 1, 2$.