

Floating Point Numbers

- C has three floating point types
 - `float` ... typically 32-bit (lower precision, narrower range)
 - `double` ... typically 64-bit (higher precision, wider range)
 - `long double` ... typically 128-bits (but maybe only 80 bits used)
- Floating point constants, e.g : `3.14159` `1.0e-9` are `double`
- Reminder: division of 2 ints in C yields an int.
 - but division of double and int in C yields a double.

Floating Point Number - Output

```
double d = 4/7.0;
// prints in decimal with (default) 6 decimal places
printf("%lf\n", d);           // prints 0.571429
// prints in scientific notation
printf("%le\n", d);           // prints 5.714286e-01
// picks best of decimal and scientific notation
printf("%lg\n", d);           // prints 0.571429
// prints in decimal with 9 decimal places
printf("%.9lf\n", d);         // prints 0.571428571
// prints in decimal with 1 decimal place and field width of 5
printf("%10.1lf\n", d);       // prints          0.6
```

source code for float_output.c

Floating Point Numbers

- if we represent floating point numbers with a fixed small number of bits
 - there are only a finite number of bit patterns
 - can only represent a finite subset of reals
- almost all real values will have no exact representation
- value of arithmetic operations may be real with no exactly representation
- we must use closest value which can be exactly represented
- this approximation introduces an error into our calculations
- often does not matter
- sometime can be disastrous

Fixed Point Representation

- can have fractional numbers in other bases, e.g.: $110.101_2 == 6.625_{10}$
- could represent fractional numbers similarly to integers by assuming decimal point is in *fixed* position
- for example with 32 bits:
 - 16 bits could be used for integer part
 - 16 bits could be used for the fraction
 - equivalent to storing values as integers after multiplying (*scaling*) by 2^{16}
 - major limitation is only small range of values can be represented
 - minimum $2_{-16} \approx 0.000015$
 - maximum $2_{15} \approx 32768$
- usable for some problems, but not ideal
- used on small embedded processors without silicon floating point

floating_types.c - print characteristics of floating point types

```
float f;  
double d;  
long double l;  
printf("float          %2lu bytes  min=%-12g  max=%g\n", sizeof f, FLT_MIN, FLT_MAX);  
printf("double          %2lu bytes  min=%-12g  max=%g\n", sizeof d, DBL_MIN, DBL_MAX);  
printf("long double %2lu bytes  min=%-12Lg  max=%Lg\n", sizeof l, LDBL_MIN, LDBL_MAX);
```

source code for floating_types.c

```
$ ./floating_types  
float          4 bytes  min=1.17549e-38  max=3.40282e+38  
double         8 bytes  min=2.22507e-308  max=1.79769e+308  
long double 16 bytes  min=3.3621e-4932  max=1.18973e+4932
```

IEEE 754 standard

- C floats almost always IEEE 754 single precision (binary32)
- C double almost always IEEE 754 double precision (binary64)
- C long double might be IEEE 754 (binary128)
- IEEE 754 representation has 3 parts: *sign*, *fraction* and *exponent*
- numbers have form $\text{sign fraction} \times 2^{\text{exponent}}$, where *sign* is $+/-$
- ***fraction*** always has 1 digit before decimal point (***normalized***)
 - as a consequence only 1 representation for any value
- ***exponent*** is stored as positive number by adding constant value (***bias***)
- numbers close to zero have higher precision (more accurate)

Floating Point Numbers

Example of normalising the fraction part in binary:

- 1010.1011 is normalized as 1.0101011×2^{011}
- $1010.1011 = 10 + 11/16 = 10.6875$
- $1.0101011 \times 2^{011} = (1 + 43/128) \times 2^3 = 1.3359375 \times 8 = 10.6875$

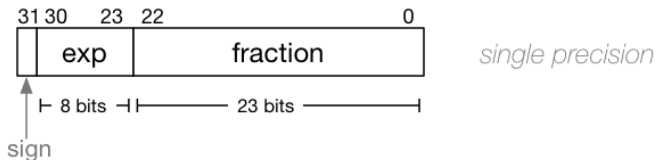
The normalised fraction part always has 1 before the decimal point.

Example of determining the exponent in binary:

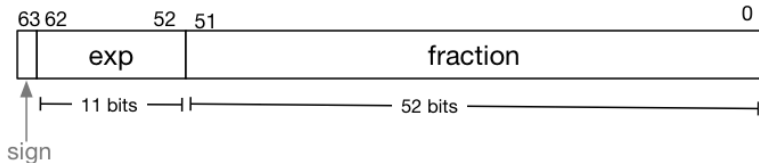
- if exponent is 8-bits, then the bias = $2^{8-1} - 1 = 127$
- valid bit patterns for exponent 00000001 .. 11111110
- correspond to B exponent values -126 .. 127

Floating Point Numbers

Internal structure of floating point values



double precision



IEEE-754 Single Precision example: 0.15625

0.15625 is represented in IEEE-754 single-precision by these bits:

00111111000100000000000000000000

sign | exponent | fraction

0 | 01111100 | 010000000000000000000000

sign bit = 0

sign = +

raw exponent = 01111100 binary

= 124 decimal

actual exponent = 124 - exponent_bias

= 124 - 127

= -3

number = +1.010000000000000000000000 binary * 2**-3

= 1.25 decimal * 2**-3

= 1.25 * 0.125

= 0.15625

IEEE-754 Single Precision example: -0.125

```
$ ./explain_float_representation -0.125
```

-0.125 is represented as a float (IEEE-754 single-precision) by the

```
10111110000000000000000000000000
```

```
sign | exponent | fraction
```

```
1 | 01111100 | 000000000000000000000000
```

```
sign bit = 1
```

```
sign = -
```

```
raw exponent    = 01111100 binary
```

```
                = 124 decimal
```

```
actual exponent = 124 - exponent_bias
```

```
                = 124 - 127
```

```
                = -3
```

```
number = -1.000000000000000000000000 binary * 2**-3
```

```
        = -1 decimal * 2**-3
```

```
        = -1 * 0.125
```

```
        = -0.125
```

IEEE-754 Single Precision example: 150.75

```
$ ./explain_float_representation 150.75
```

150.75 is represented in IEEE-754 single-precision by these bits:

```
01000011000101101100000000000000
```

```
sign | exponent | fraction
```

```
0 | 10000110 | 001011011000000000000000
```

```
sign bit = 0
```

```
sign = +
```

```
raw exponent    = 10000110 binary
```

```
                = 134 decimal
```

```
actual exponent = 134 - exponent_bias
```

```
                = 134 - 127
```

```
                = 7
```

```
number = +1.001011011000000000000000 binary * 2**7
```

```
        = 1.17773 decimal * 2**7
```

```
        = 1.17773 * 128
```

```
        = 150.75
```

IEEE-754 Single Precision example: -96.125

```
$ ./explain_float_representation -96.125
```

-96.125 is represented in IEEE-754 single-precision by these bits:

```
11000010110000000100000000000000
```

```
sign | exponent | fraction
```

```
1 | 10000101 | 100000001000000000000000
```

```
sign bit = 1
```

```
sign = -
```

```
raw exponent    = 10000101 binary
```

```
                = 133 decimal
```

```
actual exponent = 133 - exponent_bias
```

```
                = 133 - 127
```

```
                = 6
```

```
number = -1.100000001000000000000000 binary * 2**6
```

```
        = -1.50195 decimal * 2**6
```

```
        = -1.50195 * 64
```

```
        = -96.125
```

IEEE-754 Single Precision exploring bit patterns #1

```
$ ./explain_float_representation 00111101110011001100110011001101
sign bit = 0
sign = +
raw exponent    = 01111011 binary
                 = 123 decimal
actual exponent = 123 - exponent_bias
                 = 123 - 127
                 = -4
number = +1.10011001100110011001101 binary * 2**-4
        = 1.6 decimal * 2**-4
        = 1.6 * 0.0625
        = 0.1
```

infinity.c - exploring infinity

- C (IEEE-754) has a representation for $+/-$ infinity
- propagates sensibly through calculations

```
double x = 1.0/0.0;
printf("%lf\n", x); //prints inf
printf("%lf\n", -x); //prints -inf
printf("%lf\n", x - 1); // prints inf
printf("%lf\n", 2 * atan(x)); // prints 3.141593
printf("%d\n", 42 < x); // prints 1 (true)
printf("%d\n", x == INFINITY); // prints 1 (true)
```

source code for infinity.c

nan.c - handling errors robustly

- C (IEEE-754) has a representation for invalid results:
 - NaN (not a number)
- ensures errors propagates sensibly through calculations

```
double x = 0.0/0.0;
printf("%lf\n", x); //prints nan
printf("%lf\n", x - 1); // prints nan
printf("%d\n", x == x); // prints 0 (false)
printf("%d\n", isnan(x)); // prints 1 (true)
```

source code for nan.c

IEEE-754 Single Precision example: inf

```
$ ./explain_float_representation inf
inf is represented in IEEE-754 single-precision by these bits:
01111111100000000000000000000000
sign | exponent | fraction
  0  | 11111111 | 000000000000000000000000
sign bit = 0
sign = +
raw exponent      = 11111111 binary
                  = 255 decimal
number = +inf
```


IEEE-754 Single Precision exploring bit patterns #2

```
$ ./explain_float_representation 01111111110000000000000000000000
sign bit = 0
sign = +
raw exponent    = 11111111 binary
                 = 255 decimal

number = NaN
```

source code for `explain_float_representation.c`

Consequences of most reals not having exact representations

```
double a, b;  
a = 0.1;  
b = 1 - (a + a + a + a + a + a + a + a + a + a);  
if (b != 0) { // better would be fabs(b) > 0.000001  
    printf("1 != 0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1\n");  
}  
printf("b = %g\n", b); // prints 1.11022e-16
```

source code for double_imprecision.c

- do not use == and != with floating point values
- instead check if values are close

Consequences of most reals not having exact representations

```
double x = 0.000000011;
double y = (1 - cos(x)) / (x * x);
// correct answer y = ~0.5
// prints y = 0.917540
printf("y = %lf\n", y);
// division of similar approximate value
// produces large error
// sometimes called catastrophic cancellation
printf("%g\n", 1 - cos(x)); // prints 1.11022e-16
printf("%g\n", x * x); // prints 1.21e-16
```

source code for double_catastrophe.c

Another reason not to use == with floating point values

```
if (d == d) {  
    printf("d == d is true\n");  
} else {  
    // will be executed if d is a NaN  
    printf("d == d is not true\n");  
}  
  
if (d == d + 1) {  
    // may be executed if d is large  
    // because closest possible representation for d + 1  
    // is also closest possible representation for d  
    printf("d == d + 1 is true\n");  
} else {  
    printf("d == d + 1 is false\n");  
}
```

source code for double_not_always.c

Another reason not to use == with floating point values

```
$ gcc double_not_always.c -o double_not_always
$ ./double_not_always 42.3
d = 42.3
d == d is true
d == d + 1 is false
$ ./double_not_always 4200000000000000000
d = 4.2e+18
d == d is true
d == d + 1 is true
$ ./double_not_always NaN
d = nan
d == d is not true
d == d + 1 is false
```

because closest possible representation for $d + 1$ is also closest possible representation for d

Consequences of most reals not having exact representations

```
double d = 9007199254740992;  
// loop never terminates  
while (d < 9007199254740999) {  
    printf("%lf\n", d); // always prints 9007199254740992.000000  
    // 9007199254740993 can not be represented as a double  
    // closest double is 9007199254740992.0  
    // so 9007199254740992.0 + 1 = 9007199254740992.0  
    d = d + 1;  
}
```

source code for double_disaster.c

- 9007199254740993 is $2^{53} + 1$
it is smallest integer which can not be represented exactly as a double
- The closest double to 9007199254740993 is 9007199254740992.0
- aside: 9007199254740993 can not be represented by a `int32_t`
it can be represented by `int64_t`

Exercise: Floating point → Decimal

Convert the following floating point numbers to decimal.

Assume that they are in IEEE 754 single-precision format.

0 10000000 110000000000000000000000

1 01111110 100000000000000000000000