ARJUNA
for
JEE
MAIN & ADVANCED
MATHEMATICS
FULL COURSE STUDY MATERIAL

Class XI

- Basic Mathematics and Logarithm
- Sets
- Functions
- Trigonometric Ratios and Identities
- Trigonometric Equations
- Principle of Mathematical Induction

Module-1

CONTENTS

Mathematics Module-1 1-158

1. Basic Mathematics and Logarithm 1-28
2. Sets 29-47
3. Functions 48-82
4. Trigonometric Ratios and Identities 83-122
5. Trigonometric Equations 123-150
6. Principle of Mathematical Induction 151-158

CHAPTER

1

Basic Math and Logarithm

NUMBER SYSTEM

(*i*) **Natural numbers:** The counting numbers 1, 2, 3, 4, ... are called Natural Numbers. The set of natural numbers is denoted by N.

Thus $N = \{1, 2, 3, 4, ...\}$.

(*ii*) **Whole numbers:** Natural numbers including zero are called whole numbers. The set of whole numbers is denoted by W.

Thus W = $\{0, 1, 2, ...\}$

(*iii*) **Integers:** The numbers ... – 3, – 2, – 1, 0, 1, 2, 3 .... are called integers and the set is denoted by I or Z. Thus I (or Z) = $\{.. -3, -2, -1, 0, 1, 2, 3...\}$

Note: (*a*) Positive integers $I^+ = \{1, 2, 3 ....\} = N$

(*b*) Negative integers $I^- = \{....., -3, -2, -1\}$.

(*c*) Non-negative integers (whole numbers) = $\{0, 1, 2, ...\}$.

(*d*) Non-positive integers = $\{......, -3, -2, -1, 0\}$.

(*iv*) **Even integers:** Integers which are divisible by 2 are called even integers.

e.g. $0, \pm 2, \pm 4$,.......

(*v*) **Odd integers:** Integers which are not divisible by 2 are called odd integers.

e.g. $\pm 1, \pm 3, \pm 5, \pm 7$......

(*vi*) **Prime numbers:** Natural numbers which are divisible by 1 and itself only are called prime numbers.

e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

(*vii*) **Composite number:** Let 'a' be a natural number, '*a*' is said to be composite if, it has atleast three distinct factors.

e.g. 4, 6, 8, 9, 10, 12, 14, 15 ....

Note: (*a*) 1 is neither a prime number nor a composite number.

(*b*) Numbers which are not prime are composite numbers (except 1).

(*c*) '4' is the smallest composite number.

(*d*) '2' is the only even prime number.

(*viii*) **Co-prime numbers:** Two natural numbers (not necessarily prime) are called coprime, if their H.C.F (Highest common factor) is one.

e.g. (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (5, 6), (7, 8) (15, 16) etc.

These numbers are also called as **relatively prime** numbers.

Note:

(*a*) Two prime number(s) are always co-prime but converse need not be true.

(*b*) Consecutive natural numbers are always co-prime numbers.

(*ix*) **Twin prime numbers :** If the difference between two prime numbers is two, then the numbers are called twin prime numbers.

e.g. $\{3, 5\}, \{5, 7\}, \{11, 13\}, \{17, 19\}, \{29, 31\}$

Note: Number between twin prime numbers is divisible by 6 (except (3, 5)).

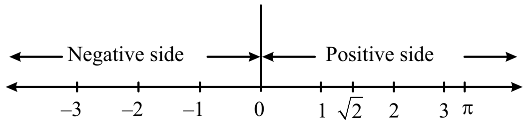
(*x*) **Rational numbers:** All the numbers that can be represented in the form p/q, where p and q are integers and $q \neq 0$, are called rational numbers and their set is denoted by Q. Thus $Q = \{p/q : p, q \in \text{I and } q \neq 0\}$. It may be noted that every integer is a rational number since it can be written as p/q. It may be noted that all recurring decimals are rational numbers.

Note: Maximum number of different decimal digits in $\frac{p}{q}$ is equal to q, i.e. $\frac{11}{9}$ will have maximum of 9 different decimal digits.

(*xi*) **Irrational numbers:** The numbers which can not be expressed in p/q form where $p, q \in$ I and $q \neq 0$ i.e. the numbers which are not rational are called irrational numbers and their set is denoted by Q^c. (i.e. complementary set of Q) e.g. $\sqrt{2}$, $1 + \sqrt{3}$ etc. Irrational numbers can not be expressed as recurring decimals.

Note: $e \approx 2.71$ is called Napier's constant and $\pi \approx 3.14$ are irrational numbers.

(*xii*) **Real numbers:** Numbers which can be expressed on number line are called real numbers. The complete set of rational and irrational numbers is the set of real numbers and is denoted by R. Thus $R = Q \cup Q^C$.



Real line

All real numbers follow the order property i.e. if there are two distinct real numbers a and b then either $a < b$ or $a > b$.

Note:

(*a*) Integers are rational numbers, but converse need not be true.

(*b*) Negative of an irrational number is an irrational number.

(*c*) Sum of a rational number and an irrational number is always an irrational number

e.g. $2 + \sqrt{3}$

(*d*) The product of a non zero rational number and an irrational number will always be an irrational number.

(*e*) If $a \in Q$ and $b \notin Q$, then ab = rational number, only if $a = 0$.

(*f*) Sum, difference, product and quotient of two irrational numbers need not be a irrational number or we can say, result may be a rational number also.

(*xiii*) **Complex number:** A number of the form $a + ib$ is called a complex number, where $a, b \in R$ and $i = \sqrt{-1}$. Complex number is usually denoted by Z and the set of complex number is represented by C. Thus $C = \{a + ib : a, b \in R$ and $i = \sqrt{-1}\}$

Note: It may be noted that $N \subset W \subset I \subset Q \subset R \subset C$.

Train Your Brain

Example 1: The value of $1.\overline{285714} \div 1.\overline{714285} =$ ______.

(*a*) $\frac{3}{4}$ (*b*) $\frac{7}{8}$

(*c*) $\frac{7}{12}$ (*d*) $\frac{3}{7}$

Sol. (*a*)

$1.\overline{285714}$

$= 1 + 0.\overline{285714}$

$= 1 + \frac{2}{7} = \frac{9}{7}$

$1.\overline{714285}$

$= 1 + \frac{5}{7} = \frac{12}{7}$

$\therefore 1.\overline{285714} \div 1.\overline{714285}$

$= \frac{9}{7} \div \frac{2}{7}$

$= \frac{9}{7} \times \frac{7}{12}$

$= \frac{3}{4}$

Example 2: Prove that the difference $10^{25} - 7$ is divisible by 3.

Sol. Write the given difference in the form $10^{25} - 7 = (10^{25} - 1) - 6$. The number $10^{25} - 1 = \underbrace{99..9}_{25\ digits}$ is divisible by 3 (and 9). Since the numbers $(10^{25} - 1)$ and 6 are divisible by 3, the number $10^{25} - 7$, being their difference, is also divisible by 3 without a remainder.

Concept Application

1. The product of $1.\overline{142857}$ and $0.\overline{63}$ = ______.

(*a*) $\frac{8}{11}$ (*b*) $\frac{7}{11}$

(*c*) $\frac{11}{7}$ (*d*) $\frac{8}{7}$

2. If $x = \sqrt{12} - \sqrt{9}, y = \sqrt{13} - \sqrt{10}$, and $z = \sqrt{11} - \sqrt{8}$, then which of the following is true?

(*a*) $z > x > y$

(*b*) $z > y > x$

(*c*) $y > x > z$

(*d*) $y > z > x$

SOME IMPORTANT IDENTITIES

1. $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$
2. $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$
3. $a^2 - b^2 = (a + b)(a - b)$
4. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
5. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
6. $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$
7. $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$
8. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

9. $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}\,[(a-b)^2 + (b-c)^2 + (c-a)^2]$

10. $a^3 + b^3 + c^3 - 3abc = (a + b + c)\,(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a + b + c)\,[(a-b)^2 + (b-c)^2 + (c-a)^2]$

If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

11. $a^4 - b^4 = (a + b)\,(a - b)\,(a^2 + b^2)$

12. $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)\,(1 - a + a^2)$

Train Your Brain

Example 3: Show that the expression, $(x^2 - yz)^3 + (y^2 - zx)^3 + (z^2 - xy)^3 - 3\,(x^2 - yz)\,.\,(y^2 - zx).(z^2 - xy)$ is a perfect square and find its square root.e

Sol. $(x^2 - yz)^3 + (y^2 - zx)^3 + (z^2 - xy)^3 - 3(x^2 - yz)(y^2 - zx)\,(z^2 - xy) = a^3 + b^3 + c^3 - 3abc$

where $a = x^2 - yz,\ b = y^2 - zx,\ c = z^2 - xy$

$= (a + b + c)\,(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a + b + c)\,((a-b)^2 + (b-c)^2 + (c-a)^2)$

$= \frac{1}{2}(x^2 + y^2 + z^2 - xy - yz - zx)\,[(x^2 - yz - y^2 + zx)^2 + (y^2 - zx - z^2 + xy)^2 + (z^2 - xy - x^2 + yz)^2]$

$= \frac{1}{2}(x^2 + y^2 + z^2 - xy - yz - zx)\,[\{x^2 - y^2 + z(x-y)\}^2 + \{y^2 - z^2 + x\,(y-z)\}^2 + \{z^2 - x^2 + y\,(z-x)\}^2]$

$= \frac{1}{2}\,(x^2 + y^2 + z^2 - xy - yz - zx)\,(x + y + z)^2\,[(x-y)^2 + (y-z)^2 + (z-x)^2]$

$= (x + y + z)^2\,(x^2 + y^2 + z^2 - xy - yz - zx)^2$

$= (x^3 + y^3 + z^3 - 3xyz)^2$

(which is a perfect square) its square roots are

$\pm\,(x^3 + y^3 + z^3 - 3xyz)$

Example 4. If $x^2 - 4x + 1 = 0$, then what is the value of $x^3 + \frac{1}{x^3}$?

Sol. $x^2 - 4x + 1 = 0$

$$\Rightarrow x = \frac{4 \pm \sqrt{16 - 4\times1\times1}}{2\times1} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$$\therefore x^3 + \frac{1}{x^3} = \left(2+\sqrt{3}\right)^3 + \frac{1}{\left(2+\sqrt{3}\right)^3} = \left(2+\sqrt{3}\right)^3 + \left[\frac{\left(2-\sqrt{3}\right)\times1}{\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)}\right]^3 = \left(2+\sqrt{3}\right)^3 + \left(2-\sqrt{3}\right)^3$$

$$= 2^3 + \left(\sqrt{3}\right)^3 + 3\times2\times\sqrt{3}\left(2+\sqrt{3}\right) + 2^3 - \left(\sqrt{3}\right)^3 - 3\times2\times\sqrt{3}\left(2-\sqrt{3}\right)$$

$= 8 + 18 + 8 + 18 = -\,50$. Similarly for

$x = 2 - \sqrt{3},\ x^3 + \frac{1}{x^3} = 52$

Example 5. If $x = \frac{1}{x} = a$, then what is the value of $x^3 + x^2 + \frac{1}{x^3} + \frac{1}{x^2}$?

(*a*) $a^3 + a^2$ (*b*) $a^3 + a^2 - 5a$

(*c*) $a^3 + a^2 - 3a - 2$ (*d*) $a^3 + a^2 - 4a - 2$

Sol. (*c*)

Given, $x + \frac{1}{x} = a$

Now, $x^3 + x^2 + \frac{1}{x^3} + \frac{1}{x^2} = \left(x^3 + \frac{1}{x^3}\right) + \left(x^2 + \frac{1}{x^2}\right)$

$= \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) + \left(x + \frac{1}{x}\right)^2 - 2$

$= a^3 - 3a + a^2 - 2 = a^3 + a^2 - 3a - 2.$

Concept Application

3. If $x^{1/3} + y^{1/3} + z^{1/3} = 0$, then what is $(x+y+z)^3$ equal to?

(*a*) 1 (*b*) 3

(*c*) $3xy$ (*d*) $27xyz$

4. If $a + b + c = 0$, then what is the value of

$$\frac{a^2 + b^2 + c^2}{(a-b)^2 + (b-c)^2 + (c-a)^2}$$

(*a*) 1 (*b*) 3 (*c*) $\frac{1}{3}$ (*d*) 0

5. If $x + \frac{1}{x} = p$ then $x^6 + \frac{1}{x^6}$ equals to :

(*a*) $p^6 + 6p$ (*b*) $p^6 - 6p$

(*c*) $p^6 + 6p^4 + 9p^2 + 2$ (*d*) $p^6 - 6p^4 + 9p^2 - 2$

6. If $x + \frac{1}{x} = 4,$ then find values of

(*i*) $x^2 + \frac{1}{x^2}$ (*ii*) $x^3 + \frac{1}{x^3}$

(*iii*) $x^4 + \frac{1}{x^4}$

7. Prove that $(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16}) = \frac{(1 - x^{32})}{(1 - x)}$

8. If x, y, z are all different real numbers, then prove that

$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}=\left(\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}\right)^2$$

9. If $\frac{a}{b}+\frac{b}{a}=-1,$ then find value of a^3-b^3.
10. If $a-b=-8$, $ab=-12$ then a^3-b^3 will be
11. The product $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$ simplifies to
12. Find the real values of p, q, r satisfying $(2p-3)^8+(1-q)^6+(4-3r)^4=0$.

INDICES

If 'a' is any non zero real or imaginary number and 'm' is the positive integer, then $a^m = a \cdot a \cdot a. \ldots a$ (m times). Here a is called the base and m is called the index, power or exponent.

Law of indices:

1. $a^0 = 1, (a \neq 0)$
2. $a^{-m} = \frac{1}{a^m}, (a \neq 0)$
3. $a^{m+n} = a^m \cdot a^n$, where m and n are rational numbers
4. $a^{m-n} = \frac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$
5. $(a^m)^n = a^{mn}$
6. $a^{p/q} = \sqrt[q]{a^p}$
7. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$, where $m, n \in N$ and $(m, n \geq 2)$ and a is positive rational number
8. $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$, $a, b \in R$ and atleast one of a and b should be positive

SURDS

If a is a positive rational number, which is not the nth power (n is any natural number) of any rational number, then the irrational number $\pm\sqrt[n]{a}$ are called simple surds or monomial surds.

Every surd is an irrational number (but every irrational number is not a surd). So, the representation of monomial surd on a number line is same as that of irrational numbers.

Examples:

1. $\sqrt{3}$ is a surd and $\sqrt{3}$ is an irrational number.
2. $\sqrt[3]{5}$ is a surd and $\sqrt[3]{5}$ is an irrational number.
3. π is an irrational number, but it is not a surd.
4. $\sqrt[3]{3+\sqrt{2}}$ is an irrational number. It is not a surd, because $3+\sqrt{2}$ is not a rational number.

Train Your Brain

Example 6: Simplify $\left[\sqrt[3]{\sqrt[6]{a^9}}\right]^4\left[\sqrt[6]{\sqrt[3]{a^9}}\right]^4$

Sol. $a^{9(1/6)(1/3)4} \cdot a^{9(1/3)(1/6)4} = a^2 \cdot a^2 = a^4$.

Example 7: Arrange the following in ascending or descending order of magnitude:

$\sqrt[4]{6}, \sqrt[3]{7}, \sqrt{5}$

Sol. $\sqrt[4]{6}=6^{1/4}, \sqrt[3]{7}=7^{1/3}, \sqrt{5}=5^{1/2}$

LCM of the denominators of the exponents of these three terms, 4, 3 and 2 is12.

Now express the exponent of each term, as a fraction in which then denominator is 12.

$$6^{\frac{1}{4}}=6^{\frac{3}{12}}=\left(6^3\right)^{\frac{1}{12}}=\sqrt[12]{216}$$

$$7^{\frac{1}{3}}=7^{\frac{4}{12}}=\left(7^4\right)^{\frac{1}{12}}=\sqrt[12]{2401}$$

$$5^{\frac{1}{2}}=5^{\frac{6}{12}}=\left(5^6\right)^{\frac{1}{12}}=\sqrt[12]{15625}$$

Now, $\sqrt[4]{6}=\sqrt[12]{216}, \sqrt[3]{7}=\sqrt[12]{2401}, \sqrt{5}=\sqrt[12]{15625}$

Hence, their ascending order is

$\sqrt[12]{216}, \sqrt[12]{2401}, \sqrt[12]{15625}$, *i.e.*, $\sqrt[4]{6}, \sqrt[3]{7}, \sqrt{5}$

$\therefore$ The descending order of magnitude of the given radical is $\sqrt{5}, \sqrt[3]{7}, \sqrt[4]{6}$.

Example 8: Find the square root of

$10+\sqrt{24}+\sqrt{60}+\sqrt{40}$

Sol.

$$=\sqrt{10+\sqrt{24}+\sqrt{60}+\sqrt{40}}$$

$$=\sqrt{10+2\sqrt{6}+2\sqrt{15}+2\sqrt{10}}$$

$$=\sqrt{(2+3+5)+2\sqrt{2(3)}+2\sqrt{3(5)}+2\sqrt{2(5)}}$$

$$=\sqrt{\left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right)^2}$$

$$=\sqrt{2}+\sqrt{3}+\sqrt{5}.$$

Concept Application

13. If the surds $\sqrt[4]{4}, \sqrt[6]{5}, \sqrt[8]{6}$ *and* $\sqrt[12]{8}$ are arranged in ascending order from left to right, then the third surd from the left is

(*a*) $\sqrt[12]{8}$ (*b*) $\sqrt[4]{4}$
(*c*) $\sqrt[8]{6}$ (*d*) $\sqrt[6]{5}$

14. $\sqrt[6]{15-2\sqrt{56}}\,\sqrt[3]{\sqrt{7}+2\sqrt{2}} =$ ______.

(*a*) 0 (*b*) $\sqrt{2}$

(*c*) 1 (*d*) $\sqrt[6]{2}$

15. If $x = 2^{1/3} - 2$, then $x^3 + 6x^2 + 12x =$ _________.

(*a*) 6 (*b*) –6

(*c*) 8 (*d*) –8

16. If $a = 4^{\frac{-3}{2}}$, $b = (125)^{\frac{-2}{3}}$, $c = 81^{\frac{-3}{4}}$ then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} =$

17. If $p = 8^{\frac{2}{3}}$, $q = 9^{\frac{3}{2}}$, $r = 16^{\frac{5}{4}}$, $s = \frac{1}{6^{-2}}$ then $p + q + r + s =$

18. If $u = \left(\frac{1}{27}\right)^{\frac{-4}{3}}$, $v = \left(\frac{1}{216}\right)^{\frac{-2}{3}}$, $w = (8)^{5/3} \cdot (4)^{-5/2}$, $t = (9)^{3/2} \cdot (81)^{-3/4}$ then $\frac{vwt}{\sqrt{u}} =$

19. Which of the following number is greater than 1

(*a*) $\left(\frac{1}{216}\right)^{-2/3} \cdot \left(\frac{1}{6^{-2}}\right)$ (*b*) $(125)^{-2/3} \cdot (625)^{1/2}$

(*c*) $\log_{1/2} 16$ (*d*) $\log_{16} 32$

20. Simplify $a\left(\frac{\sqrt{a}+\sqrt{b}}{2b\sqrt{a}}\right)^{-1} + b\left(\frac{\sqrt{a}+\sqrt{b}}{2a\sqrt{b}}\right)^{-1}$

RATIO

(*i*) If A and B be two quantities of the same kind, then their ratio is $A : B$; which may be denoted by the fraction $\frac{A}{B}$ (This may be an integer or fraction)

(*ii*) A ratio may represented in a number of ways e.g. $\frac{a}{b} = \frac{ma}{mb} = \frac{na}{nb} =$...... where m, n,...... are non-zero numbers.

(*iii*) To compare two or more ratio, reduce them to common denominator.

(*iv*) Ratio between two ratios may be represented as the ratio of two integers

e.g. $\frac{a}{b} : \frac{c}{d} : \frac{a/b}{c/d} = \frac{ad}{bc}$ or $ad : bc$

(*v*) Ratios are compounded by multiplying them together i.e. $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} \ldots\ldots = \frac{ace}{bdf} \ldots\ldots$

(*vi*) If $a : b$ is any ratio then its duplicate ratio is $a^2 : b^2$; triplicate ratio is $a^3 : b^3$..... etc.

(*vii*) If $a : b$ is any ratio, then its sub-duplicate ratio is $a^{1/2} : b^{1/2}$; sub-triplicate ratio is $a^{1/3} : b^{1/3}$ etc.

PROPORTION

When two ratios are equal, then the four quantities compositing them are said to be proportional. If $\frac{a}{b} = \frac{c}{d}$, then it is written as $a : b = c : d$ or $a : b :: c : d$

(*i*) 'a' and 'd' are known as extremes and 'b and c' are known as means.

(*ii*) An important property of proportion Product of extremes = product of means.

(*iii*) If $a : b = c : d$, then $b : a = d : c$ (Invertando)

(*iv*) If $a : b = c : d$, then $a : c = b : d$ (Alternando)

(*v*) If $a : b = c : d$, then $\frac{a+b}{b} = \frac{c+d}{d}$ (Componendo)

(*vi*) If $a : b = c : d$, then $\frac{a-b}{b} = \frac{c-d}{d}$ (Dividendo)

(*vii*) If $a : b = c : d$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (Componendo and dividendo)

(*viii*) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$, then each $\frac{a+c+e+\ldots\ldots}{b+d+f+\ldots\ldots}$

$= \frac{\text{Sum of the numerators}}{\text{Sum of the denominators}}$

(*ix*) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$, then each $= \frac{xa+yc+ze+\ldots\ldots}{xb+yd+zf+\ldots\ldots}$

(*x*) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$, then each $= \left(\frac{xa^n+yc^n+ze^n}{xb^n+yd^n+zf^n}\right)^{1/n}$

Train Your Brain

Example 9: If $\frac{x+y}{2} = \frac{y+z}{3} = \frac{z+x}{4}$, then find $x : y : z$.

Sol. Each $= \frac{\text{Sum of the numerators}}{\text{Sum of the denominators}}$

$= \frac{2(x+y+z)}{9} = \frac{x+y+z}{9/2}$ and therefore each

$= \frac{(x+y+z)-(y+z)}{\frac{9}{2}-3} = \frac{(x+y+z)-(x+z)}{\frac{9}{2}-4}$

$= \frac{(x+y+z)-(x+y)}{\frac{9}{2}-2}$

$= \frac{x}{3/2} = \frac{y}{1/2} = \frac{z}{5/2} \Rightarrow x : y : z = 3 : 1 : 5$

Example 10: If $\frac{a+3b+2c+6d}{-a-3b+2c+6d} = \frac{3a+b+6c+2d}{-3a-b+6c+2d}$, then the correct statement is

(*a*) $ad = bc$ (*b*) $ac = bd$

(*c*) $c = \frac{ab}{d}$ (*d*) $a + d = b + c$

Sol. (*a*) Apply C and D the the given equation

$$\Rightarrow \frac{4c+12d}{2a+6d} = \frac{4d+12c}{2b+6a} \quad \Rightarrow \frac{c+3d}{a+3b} = \frac{d+3c}{b+3a}$$

$\Rightarrow bc + 3ac + 3bd + 9ad = ad + 3ac + 3bd + 9bc$

$\Rightarrow 8ad = 8bc$

$\Rightarrow ad = bc$

Concept Application

21. Solve the equations $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}} = \dfrac{2x^2-1}{2}$

22. Solve: $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = 3$

23. If $(4x - 3y) : (2x + 5y) = 12 : 19$ then $x : y$ is ......

(*a*) 2 : 3 (*b*) 1 : 2

(*c*) 3 : 2 (*d*) 2 : 1

INTERVALS

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers $a, b \in R$ such that $a < b$, we can define four types of intervals as follows :

Open Interval (a, b)

$\{x : a < x < b\}$ i.e. extreme points are not includes

Closed Interval $[a, b]$

$\{x : a \le x \le b\}$ i.e. extreme points are includes

It can possible when a and b are finite

Semi-Open Interval $(a, b]$

$\{x : a < x \le b\}$ i.e. a is not include and b is include

Semi-Closed Interval $[a, b)$

$\{x : a \le x < b\}$ i.e. a is include and b is not include

Note:

1. The infinite intervals are defined as follows :
   (*i*) $(a, \infty) = \{x : x > a\}$
   (*ii*) $[a, \infty) = \{x : x \ge a\}$
   (*iii*) $(-\infty, b) = \{x : x < b\}$
   (*iv*) $(\infty, b] = \{x : x \le b\}$
   (*v*) $(-\infty, \infty) = \{x : x \in R\}$
2. $x \in \{1, 2\}$ denotes some particular values of x, i.e. $x = 1, 2$
3. If there is no value of x, then we say $x \in \phi$ (null set)

GENERAL METHOD TO SOLVE INEQUALITIES

Method of Intervals (Wavy Curve Method)

Let $g(x) = \left(\dfrac{(x-b_1)^{k_1}(x-b_2)^{k_2} --- (x-b_n)^{k_n}}{(x-a_1)^{r_1}(x-a_2)^{r_2} --- (x-a_n)^{r_n}}\right)$ ...(*i*)

Where k_1, k_2 ..... k_n and r_1, r_2 .... $r_n \in N$ and b_1, b_2..... b_n and a_1, a_2 ... a_n are real numbers.

Then to solve the inequality following steps are taken.

Steps: Points where numerator becomes zero are called zeros or roots of the function and where denominator becomes zero are called poles of the function.

(*i*) First we find the zeros and poles of the function.

(*ii*) Then we mark all the zeros and poles on the real line and put a vertical bar to divide the real line in many intervals.

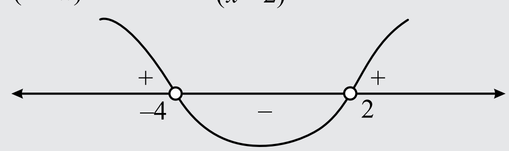
(*iii*) Determine sign of the function in any of the interval and then alternates the sign in the neghbouring interval if the poles or zeros dividing the two interval has appeared odd number of times otherwise retain the sign.

(*iv*) Thus we consider all the intervals. The solution of the $g(x) > 0$ is the union of the intervals in which we have put the plus sign and the solution of $g(x) < 0$ is the union of all intervals in which we have put the minus sign.

Train Your Brain

Example 11: Solution $\dfrac{(x-1)^2(x+4)}{(2-x)} < 0$ is

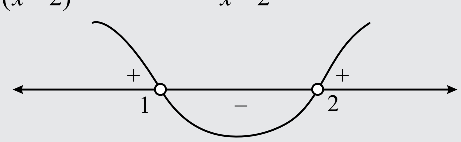
Sol. $\dfrac{(x-1)^2(x+4)}{(2-x)} < 0 \Rightarrow \dfrac{(x+4)}{(x-2)} > 0$



$\Rightarrow (-\infty, -4) \cup (2, \infty)$.

Example 12: The solution of $\dfrac{\sqrt[3]{(x+4)^4}\,(x-1)^3}{(x-2)} > 0$ is

Sol. $\dfrac{(x+4)^{4/3}(x-1)^3}{(x-2)} > 0 \Rightarrow \dfrac{x-1}{x-2} > 0$



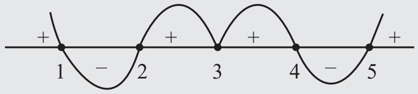
$\Rightarrow (-\infty, 1) \cup (2, \infty)$ Excluding – 4.

Example 13. Find the range of x, so that following expressions are defined.

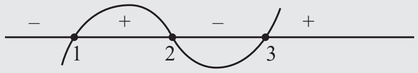
(a) $(x-1)(x-2)(x-3)^2(x-4)^5(x-5) > 0$

(b) $\dfrac{(x-1(x-2)}{(x-3)} \geq 0$

Sol. (a) $\Rightarrow x \in (-\infty, 1) \cup (2, 3) \cup (3, 4) \cup (5, \infty)$



(b) $\Rightarrow x \in [1, 2] \cup (3, \infty)$



Concept Application

24. Solve the inequality $\dfrac{x^3 - 4x^2 - 12x}{x+3} \geq 0$

25. Solve the inequality $\dfrac{(x+1)^4(x-2)(x-3)^3(x-4)^2}{x^3 - 36x} \geq 0$

26. Find the number of integer values of variable x satisfying the following pair of inequalities.

$\dfrac{(x-1)(x+4)}{x-3} < 0 \;\&\; x^2 + 6x - 27 \leq 0$

27. The solution of the inequality $2x - 1 \leq x^2 + 3 \leq x - 1$ is

(a) $x \in R$ (b) $(-2, 2]$

(c) $(-2, 2)$ (d) $x \in \phi$

LOGARITHM FUNCTION

Introduction

The kinetics of some chemical reactions (which rely on the concentration of one or more of the reactants), radioactive decay (used in carbon dating), are examples of exponential decay.

A nautilus's shell develops in such a way that each new chamber is roughly a replica of the previous one but is enlarged by a fixed amount. This results in the creation of the elegant shape known as a logarithmic spiral. The pace of growth is related to the existing size of many objects in nature. So, it is extremely important to comprehend the mathematics that explains the behaviour of entities like birth rates if we are to forecast how they will behave in the future. Logarithmic scales (the Richter scale in the case of an earthquake) are useful in offering a means to handle the numerical values of these measures.

Definition

The logarithm of the number N to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number N.

This number is designated as $\log_a N$.

Hence $\log_a N = x \Leftrightarrow a^x = N, a > 0, a \neq 1$ and $N > 0$

If $a = 10$, then we write $\log b$ rather than $\log_{10} b$

$a = e$, we write $\ln b$ rather than $\log_e b$

The existence and uniqueness of the number $\log_a N$ follows from the properties of an exponential functions.

Domain

The existence and uniqueness of the number $\log_a N$ can be determined with the help of set of conditions, $a > 0$ & $a \neq 1$ & $N > 0$.

FUNDAMENTAL IDENTITY

(i) $\log_a 1 = 0$ $(a > 0,\ a \neq 1)$

(ii) $\log_a a = 1$ $(a > 0,\ a \neq 1)$

(iii) $\log_{1/a} a = -1$ $(a > 0,\ a \neq 1)$

Remember

$\log_{10} 2 \approx 0.3010$; $\log_{10} 3 \approx 0.4771$

$\ln 2 \approx 0.693$; $\ln 10 \approx 2.303$

FUNDAMENTAL LOGARITHMIC IDENTITY

$a^{\log_a N} = N, a > 0, a \neq 1$ & $N > 0$

Proof :

$\log_a N = x$ ... (i)

$N = (a)^x$ ... (ii)

by equation (i) & (ii)

$N = (a)^{\log a N}$

PRINCIPAL PROPERTIES

Let M & N are arbitrary positive numbers, $a > 0$, $a \neq 1$, and x, y are any real numbers, then:

(i) $\log_a (M \cdot N) = \log_a M + \log_a N$;

Proof:

Let $\log_a M = x$ and $\log_a N = y$

$\Rightarrow M = a^x$ and $N = a^y$

Now, $MN = a^x a^y = a^{x+y} \Rightarrow \log_a MN = x + y$

In general

$\log_a (x_1 x_2 \ldots x_n) = \log_a x_1 + \log_a x_2 + \ldots + \log_a x_n$

(ii) $\log_a (M/N) = \log_a M - \log_a N$

Proof:

Let $\log_a M = x$ and $\log_a N = y$

$\Rightarrow M = a^x$ and $N = a^y$

Now, $M/N = a^x/a^y = a^{x-y} \Rightarrow \log_a M/N = x - y$

(*iii*) $\log_a M^\alpha = \alpha \cdot \log_a M$

BASE CHANGING THEOREM

It states that ratio of logarithm of two numbers is independent of their common base

Symbolically

$$\frac{\log_a M}{\log_a b} = \log_b M \ (a > 0, M > 0, b > 0)$$

Proof:

Let $\log_b M = x$

$\Rightarrow M = b^x$

$\Rightarrow \log_a M = \log_a b^x$

$\Rightarrow \log_a M = x \cdot \log_a b$

$\Rightarrow \dfrac{\log_a M}{\log_a b} = x = \log_b M$

Important Results

(*i*) **Base power formula:** $\log_{a^k} M = \dfrac{1}{k} \log_a M$

Proof:

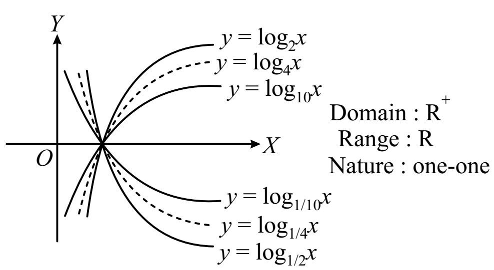
$$\log_{a^k}(M) = \frac{\log_a M}{\log_a a^k} = \frac{\log_a M}{k \log_a a} = \frac{1}{k} \log_a M$$

(*ii*) $a^{\log_b c} = c^{\log_b a}$

Proof: $a^{\log_b c} = a^{\log_a c . \log_b a} = (a^{\log_a c})^{\log_b a} = (c)^{\log_b a}$

GRAPH OF LOGARITHMIC FUNCTIONS

If $a > 0, a \neq 1$, then the function $y = \log_a x, x \in R^+$ (set of positive real numbers) is called the logarithmic Function with base a.

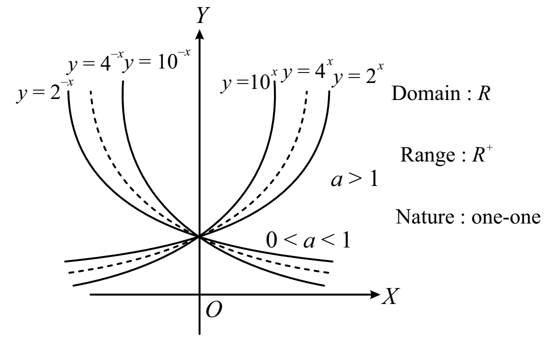


Note: (*i*) If the number and the base are on the same side of the unity, then the logarithm is positive.

(*ii*) If the number and the base are on the opposite sides of unity, then the logarithm is negative.

EXPONENTIAL FUNCTION

If $a > 0, a \neq 1$ then the function defined by $f(x) = a^x, x \in R$ is called an Exponential Function with base a.



LOGARITHMIC EQUATION

The equality $\log_a x = \log_a y$ is possible if and only if $x = y$

i.e. $\boldsymbol{\log_a x = \log_a y \Leftrightarrow x = y}$

Always check validity of given equation, ($x > 0, y > 0, a > 0, a \neq 1$)

LOGARITHMIC INEQUALITY

Let 'a' is a real number such that

(*i*) If $a > 1$, then $\log_a x > \log_a y \Rightarrow x > y$

(*ii*) If $a > 1$, then $\log_a x < \alpha \Rightarrow 0 < x < a^\alpha$

(*iii*) If $a > 1$, then $\log_a x > \alpha \Rightarrow x > a^\alpha$

(*iv*) If $0 < a < 1$, then $\log_a x > \log_a y \Rightarrow 0 < x < y$

(*v*) If $0 < a < 1$, then $\log_a x < \alpha \Rightarrow x > a^\alpha$

Form - I : $f(x) > 0, g(x) > 0, g(x) \neq 1$

Form	Collection of system
(*a*) $\log_{g(x)} f(x) \geq 0 \Leftrightarrow$	$\begin{cases} f(x) \geq 1, & g(x) > 1 \\ 0 < f(x) \leq 1, & 0 < g(x) < 1 \end{cases}$
(*b*) $\log_{g(x)} f(x) \leq 0 \Leftrightarrow$	$\begin{cases} f(x) \geq 1, & 0 < g(x) < 1 \\ 0 < f(x) \leq 1, & g(x) > 1 \end{cases}$
(*c*) $\log_{g(x)} f(x) \geq a \Leftrightarrow$	$\begin{cases} f(x) \geq g(x)^a, & g(x) > 1 \\ 0 < f(x) \leq g(x)^a, & 0 < g(x) < 1 \end{cases}$
(*d*) $\log_{g(x)} f(x) \leq a \Leftrightarrow$	$\begin{cases} 0 < f(x) \leq g(x)^a, & g(x) > 1 \\ f(x) \geq g(x)^a, & 0 < g(x) < 1 \end{cases}$

From - II : $f(x) > 0, g(x) > 0, \phi(x) > 0, \phi(x) \neq 1$

Form	Collection of system
(*a*) $\log_{\phi(x)} f(x) \geq \log_{\phi(x)} g(x) \Leftrightarrow$	$\begin{cases} f(x) \geq g(x), \phi(x) > 1, \\ 0 < f(x) \leq g(x); 0 < \phi(x) < 1 \end{cases}$
(*b*) $\log_{\phi(x)} f(x) \leq \log_{\phi(x)} g(x) \Leftrightarrow$	$\begin{cases} 0 < f(x) \leq g(x), \phi(x) > 1, \\ f(x) \geq g(x) > 0, 0 < \phi(x) < 1 \end{cases}$

COMMON AND NATURAL LOGARITHM

$\log_{10}N$ is referred as a common logarithm and $\log_e N$ is called as natural logarithm of N to the base Napierian and is popularly written as $\ell n\, N$. Note that e is an irrational quantity lying between 2.7 to 2.8 **Note that** $e^{\ell n\, x} = x$.

Train Your Brain

Example 14: How many solutions are there for equation

$\log_4 (x-1) = \log_2 (x-3)$?

Sol. $\log_4 (x-1) = \log_2 (x-3)$

$\Rightarrow \log_{2^2} (x-1) = \log_2 (x-3)$

$\Rightarrow \frac{1}{2} \log_2 (x-1) = \log_2 (x-3)$

$\Rightarrow \log_2 (x-1)^{1/2} = \log_2 (x-3)$

$\Rightarrow (x-1)^{1/2} = (x-3)$

$\Rightarrow x-1 = x^2 - 6x + 9$

$\Rightarrow (x-2)(x-5) = 0$

$\Rightarrow x = 2, 5$

But $x - 1 > 0$ and $x - 3 > 0$

$x > 1$ and $x > 3$

So only one solution $x = 5$

Example 15: Solve the logarithmic inequality $\log_{1/5} \frac{4x+6}{x} \geq 0$.

Sol. Since $\log_{1/5} 1 = 0$, the given inequality can be written as.

$$\log_{1/5} \frac{4x+6}{x} \geq \log_{1/5} 1$$

When the domain of the function is taken into account the inequality is equivalent to the system of inequalities.

$$\begin{cases} \frac{4x+6}{x} > 0, \\ \frac{4x+6}{x} \leq 1 \end{cases}$$

Solving the inequalities by using method of intervals

$$x \in \left[-2, \frac{-3}{2}\right)$$

Example 16: For $x \geq 0$, what is the smallest possible value of the expression $\log(x^3 - 4x^2 + x + 26) - \log(x + 2)$?

Sol. $\log \frac{(x^3 - 4x^2 + x + 26)}{(x+2)}$

$= \log \frac{(x^2 - 6x^2 + 13)(x+2)}{(x+2)}$

$= \log (x^2 - 6x + 13)$ $\quad [\because x \neq -2]$

$= \log\{(x-3)^2 + 4\}$

$\therefore$ Minimum value is log 4 when $x = 3$

Example 17: Given $\log_2 a = s$, $\log_4 b = s^2$ and $\log_{c^2} (8) = \frac{2}{s^3+1}$.

Write $\log_2 \frac{a^2 b^5}{c^4}$ as a function of 's' ($a, b, c > 0, c \neq 1$).

Sol. Given $\log_2 a = s$ ...(i)

$\log_2 b = 2s^2$ ...(ii)

$\log_8 c^2 = \frac{s^3+1}{2}$ ...(iii)

$\Rightarrow \frac{2\log c}{3\log 2} = \frac{s^3+1}{2} \Rightarrow 4\log_2 c = 3(s^3 + 1)$ ...(iv)

to find $2\log_2 a + 5\log_2 b - 4\log_2 c$

$\Rightarrow \quad 2s + 10s^2 - 3(s^3 + 1)$

Example 18. If $\frac{1}{4}\log_4 \text{M} + 4\log_4 \text{N} = 1 + \log_{.008} 5$ then the value of $\text{MN}^{16} = \text{k}.2^{1/3}$, where k is equal to

(*a*) 8 (*b*) 32

(*c*) 36 (*d*) 40

Sol. (*b*) $\frac{1}{8}\log_2 M + 2\log_2 N = 1 + \frac{\log_2 5}{\log_2 (.008)}$

$\Rightarrow \log_2 M^{1/8} + \log_2 N^2 = 1 + \frac{\log_2 10 - 1}{3 - 3\log_2 10}$

$\Rightarrow \log_2 \left(MN^{16}\right)^{1/8} = 1 - \frac{1}{3} = \frac{2}{3}$

$\Rightarrow \left(MN^{16}\right)^{1/8} = 2^{2/3}$

$\Rightarrow MN^{16} = 2^{16/3} = 32(2^{1/3})$

Concept Application

28. Solve the inequality $\log_{1/3} (5x - 1) > 0$.

29. Suppose that a and b are positive real numbers such that $\log_{27} a + \log_9 b = \frac{7}{2}$ and $\log_{27} b + \log_9 a = \frac{2}{3}$. Find the value of the ab.

30. If $m_1 = \log_8 16$, $m_2 = \log_{81} 27$, $m_3 = \log_{1/3} 1/9$, $m_4 = \log_{1/3} 9\sqrt{3}$ then $m_1 \cdot m_2 \cdot m_3 \cdot m_4$

31. If $q = \log_{2\sqrt{3}} 1728$, $q = \log_2 (\cos 45°)$, $r = \log_2 (\log_2 4)$, $s = \log_3 (\tan 30°)$, $t = \log_{625} 125$ then $\frac{prt}{qs} =$

32. If $\log_7(\log_3(\log_2 x)) = 0$, then find $\log_{0.125} x$.

33. Solve for x:

(i) $\log_3 x > 0$ (ii) $\log_5 x \geq 0$

(iii) $\log_6 x < 0$ (iv) $\log_2 x \leq 0$

(v) $\log_{1/7} x > 0$ (vi) $\log_{1/8} x \geq 0$

(vii) $\log_{1/9} x < 0$ (viii) $\log_{1/e} x \leq 0$

(ix) $\log_2(x-1) > 1$ (x) $\log_{1/2}(x-2) \leq 1$

34. Solve for x:

(i) $\log_4(2x-3) < 2$ (ii) $\log_{1/2}(3x-2) \geq 3$

(iii) $\log_{16}(\log_4(x)) > 1$ (iv) $\log_{1/2}(\log_{1/4}(x)) < 1$

35. $\log_2(a^2-5) = 2$

36. $\log_{1/3}(a^2-1) = -1$

37. Prove that : $2^{\sqrt{\log_2 3}} = 3^{\sqrt{\log_3 2}}$

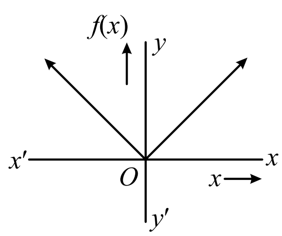
38. If $\log_{12}27 = a$ find the value of $\log_6 16$ in term of a.

ABSOLUTE VALUE FUNCTION / MODULUS FUNCTION

This is also known as absolute value function and denoted by

$$f(x) = |x| \text{ i.e. } f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Domain of this function is set of all real numbers because $f(x)$ exists for all $x \in R$ but $|x| \geq 0$ so range is all non-negative real numbers.



Domain = R; Range = $[0, \infty)$ or $R^+ \cup \{0\}$

Properties of modulus : For any $a, b \in R$

(i) $|a| \geq 0$

(ii) $|a| = |-a|$

(a) $|a|^n = |a^n|$

(b) $|a^n| = a^n$, where n is even and $n \in z$

(iii) $|a| \geq a, |a| \geq -a$

(iv) $|ab| = |a||b|$

(v) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Note: $|f(x)| + |g(x)| = |f(x) + g(x)|$

$\Rightarrow f(x) \cdot g(x) \geq 0$

Train Your Brain

Example 19 The absolute value of sum of real solutions of $\log_2|x^2+5x+4| = \log_2 3 + \log_2|x+1|$ is

(a) 8 (b) 6 (c) 7 (d) 5

Sol. $\log_2 \dfrac{|(x+1)(x+4)|}{|x+1|} = \log_2 3$

$|x+4| = 3$

$x + 4 = -3, +3$

$x = -7, -1$ (rejected); $\Rightarrow x = -7$

Example 20: Solve the following linear equation

(i) $x|x| = 4$

(ii) $|x-3| + 2|x+1| = 4$

Sol. (i) $x|x| = 4$

If $x > 0$

$\therefore x^2 = 4 \Rightarrow x = \pm 2 \quad \therefore x = 2\ (\because x \geq 0)$

If $x < 0 \Rightarrow -x^2 = 4 \Rightarrow x^2 = -4$ which is not possible

(ii) $|x-3| + 2|x+1| = 4$

Case-I If $x \leq -1$

$\therefore -(x-3) - 2(x+1) = 4$

$\Rightarrow -x + 3 - 2x - 2 = 4 \Rightarrow -3x + 1 = 4$

$\Rightarrow -3x = 3 \quad \Rightarrow x = -1$

Case-II If $-1 < x \leq 3$

$\therefore -(x-3) + 2(x+1) = 4$

$\Rightarrow -x + 3 + 2x + 2 = 4$

$\Rightarrow x = -1$ which is not possible

Case-III If $x > 3$

$x - 3 + 2(x+1) = 4$

$3x - 1 = 4 \quad \Rightarrow x = 5/3$ which is not possible

$\therefore x = -1$

Example 21: Number of real solutions of $|x-1| = |x-2| + |x-3|$ is

(a) 0 (b) 1

(c) 2 (d) more than 2

Sol.

Case-I: $x \leq 1$, $1 - x = 2 - x + 3 - x$

$x = 4$ (rejected)

Case-II: $1 < x \leq 2$, $x - 1 = 2 - x + 3 - x$, $x = 2$

Case-III: $2 < x < 3$, $x - 1 = x - 2 + 3 - x$, $x = 2$

Case-IV: $x \geq 3$, $x - 1 = x - 2 + x - 3$

$x = 4 \quad \Rightarrow x = 2, 4$

Concept Application

39. Let $x_1, x_2, x_3, x_4, x_5 \varepsilon R$ and $|x_1 - x_2| = 2$, $|x_2 - x_3| = 4$, $|x_3 - x_4| = 3$, $|x_4 - x_5| = 5$.

Then the sun of all distinct possible values of $|x_5 - x_1|$ is

40. The number of integers which does NOT satisfy $\log_{|2x|}(|x + 2| + |x - 2|) = 1$ is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

41. The real solutions of the equation where $|x|^2 - 3|x| + 2 = 0$ where $x_1 < x_2 < x_3 < x_4$ then

(*a*) $|x_1| = |x_3|$ (*b*) $|x_2| = |x_3|$

(*c*) $x_1 + x_4 = x_2 + x_3$ (*d*) $-x_2 + x_4 = x_1 - x_3$

42. Solve for x:

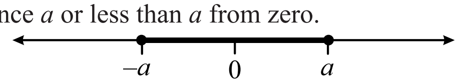
(*i*) $||x - 2| - 1| = 2$ (*ii*) $||x - 3| - 5| = 1$

(*iii*) $|||x - 5| - 4| - 3| = 2$

INEQUALITIES INVOLVING ABSOLUTE VALUE

(*i*) $|x| \le a$ (where $a > 0$)

It implies those values of x on real number line which are at distance a or less than a from zero.



$\Rightarrow -a \le x \le a$

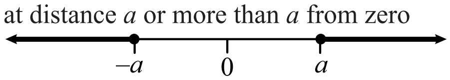
e.g. $|x| \le 2 \Rightarrow -2 \le x \le 2$

$|x| < 3 \Rightarrow -3 < x < 3$

In general, $|f(x)| \le a$ (where $a > 0$) $\Rightarrow -a \le f(x) \le a$.

(*ii*) $|x| \ge a$ (where $a > 0$)

It implies those values of x on real number line which are at distance a or more than a from zero



$\Rightarrow x \le -a$ or $x \ge a$

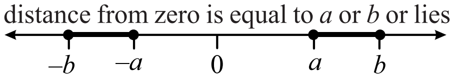
e.g. $|x| \ge 3 \Rightarrow x \le -3$ or $x \ge 3$

$|x| > 2 \Rightarrow x < -2$ or $x > 2$

In general, $|f(x)| \ge a \Rightarrow f(x) \le -a$ or $f(x) \ge a$

(*iii*) $a \le |x| \le b$ (where $a, b > 0$)

It implies those value of x on real number line whose distance from zero is equal to a or b or lies between a and b



$\Rightarrow [-b, -a] \cup [a, b]$

e.g. $2 \le |x| \le 4 \Rightarrow x \in [-4, -2] \cup [2, 4]$

(*iv*) If $|x + y| = |x| + |y|$, $xy \ge 0$

If $|x - y| = |x| + |y|$, $xy \le 0$

If $|x + y| = ||x| - |y||$, $xy \le 0$

If $|x - y| = ||x| - |y||$, $xy \ge 0$

Train Your Brain

Example 22: Solve $x^2 - 4|x| + 3 < 0$.

Sol. $x^2 - 4|x| + 3 < 0$

$\Rightarrow (|x| - 1)(|x| - 3) < 0$

$\Rightarrow 1 < |x| < 3$

$\Rightarrow -3 < x < -1$ or $1 < x < 3$

$\Rightarrow x \in (-3, -1) \cup (1, 3)$

Example 23: Solve $1 \le |x - 2| \le 3$

Sol. $1 \le |x - 2| \le 3$

$\Rightarrow -3 \le x - 2 \le -1$ or $1 \le x - 2 \le 3$

$\Rightarrow -1 \le x \le 1$ or $3 \le x \le 5$

$\Rightarrow x \in [-1, 1] \cup [3, 5]$

Example 24: Solve $|x - 1| + |x - 2| + |x - 3| \ge 6$,

Sol. For $x \le 1$, the given inequation becomes

$1 - x + 2 - x + 3 - x \ge 6 \Rightarrow -3x \ge 0$

$\Rightarrow x \le 0$ and for $x \ge 3$, the given equation becomes

$x - 1 + x - 2 + x - 3 \ge 6 \Rightarrow 3x \ge 12 \Rightarrow x \ge 4$

For $1 < x \le 2$

we get $x - 1 + 2 - x + 3 - x \ge 6$

$\Rightarrow -x + 4 \ge 6$

i.e. $-x \ge 2 \Rightarrow x \le -2$ Not possible

For $2 < x < 3$,

We get $x - 1 + x - 2 + 3 - x \ge 6$

$\Rightarrow x \ge 6$ not possible

Hence solution set is $(-\infty, 0] \cup [4, \infty)$

i.e. $x \le 0$ or $x \ge 4$

Concept Application

43. Solve $||x - 1| - 2| < 5$

44. Number of non-positive integral values of 'x' satisfying the given inequality, $|x^2 - 1| \le |2x - 1|$ is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

45. Solve $|x^2 - 2x| + |x - 4| > |x^2 - 3x + 4|$.

46. Solve $\left|\dfrac{x-3}{x+1}\right| \le 1$.

47. Solve for x:

(*i*) $|x| > 1$ (*ii*) $|x| \ge 5$ (*iii*) $|x| < 7$

(*iv*) $|x| \le 10$ (*v*) $|x| \ge 0$ (*vi*) $|x| < -8$

(*vii*) $|x| > -4$ (*viii*) $|x| \ge -5$ (*ix*) $|x| \le -10$

48. Solve for x:

(*i*) $|x - 1| \ge 1$ (*ii*) $|x - 2| < 1$

(*iii*) $1 < |2x + 1| < 3$ (*iv*) $1 \le |1 - 2x| \le 3$

(*v*) $-1 \le |3x - 1| \le 5$ (*vi*) $-6 \le |1 - 3x| \le -1$

49. Solve for x:

(*i*) $||x - 2| - 1| \le 2$ (*ii*) $||x - 3| - 5| \ge 1$

(*iii*) $|||x - 5| - 4| - 3| \le 2$

Short Notes

Some Important Identities

1. $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$
2. $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$
3. $a^2 - b^2 = (a + b)(a - b)$
4. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
5. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
6. $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$
7. $a^3 - b^3 = (a\ e- b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$
8. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$
9. $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$
10. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
    $= \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$
    If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$
11. $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$
12. $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

Laws of Indices

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $(a^m)^n = (a^n)^m = a^{mn}$
4. $\left(\frac{a}{b}\right)^{\frac{m}{n}} = \left(\frac{b}{a}\right)^{-\frac{m}{n}}$
5. $a^m \div b^{-n} = a^m \times b^n$
6. $(\sqrt[n]{a})^n = a$, where, $n \in$ N, $n \geq 2$ and a is positive rational number
7. $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$, where $n \in$ N, $n \geq 2$ and a, b are rational number
8. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, $a, b \in R$ and atleast one of a or b should be positive.

Ratio and Proportion

a, b, c, d are in proportion. Then,

(*i*) $\frac{a}{b} = \frac{c}{d}$

(*ii*) $\frac{a}{c} = \frac{b}{d}$ ....(alternendo)

(*iii*) $\frac{a+b}{b} = \frac{c+d}{d}$ ....(componendo)

(*iv*) $\frac{a-b}{b} = \frac{c-d}{d}$ ......(dividendo)

(*v*) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ ......(componendo and dividendo)

(*vi*) $\frac{b}{a} = \frac{d}{c}$ ....(invertendo)

(*vii*) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = ...$, then each

$$\frac{a+c+e+......}{b+d+f+......} = \frac{\text{Sum of the numerators}}{\text{Sum of the denominators}}$$

(*viii*) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = ...$, then each $= \frac{xa+yc+ze+...}{xb+yd+zf+...}$

Solving of Inequalities by Wavy Curve Method

Step 1: Obtain critical points by equating all factors to zero.

Step 2: Plot the critical points on the number line in the increasing order.

Step 3: Put plus sign in the right most interval.

Step 4: Now, if a root is repeated even times, the sign of the function will remain the same in the two adjacent sub-intervals of the root. (when we are moving from right to left)

Step 5: If a root is repeated for odd times the sign of the function will be different in the two adjacent sub intervals of the root. (when we are moving from right to left)

Properties of Logarithm

1. $\log_e(ab) = \log_e a + \log_e b$; $(a, b > 0)$
2. $\log_e\left(\frac{a}{b}\right) = \log_e a - \log_e b$; $(a, b > 0)$
3. $\log_e a^m = m\log_e a$; $(a > 0, m \in$ R$)$
4. $\log_a a = 1$; $(a > 0, a \neq 1)$
5. $\log_{b^m} a = \frac{1}{m}\log_b a$; $(a, b > 0, b \neq 1$ and $m \in$ R $- \{0\})$
6. $\log_b a = \frac{1}{\log_a b}$; $(a, b, > 0$ and $a, b \neq 1)$
7. $\log_b a = \frac{\log_m a}{\log_m b}$; $(a, b, m > 0$ and $m, b \neq 1)$
8. $a^{\log_a m} = m$; $(a > 0, a \neq 1\ m > 0)$
9. $a^{\log_c b} = b^{\log_c a}$; $(a, b, c > 0$ and $c \neq 1)$
10. If $\log_m x > \log_m y \Rightarrow \begin{cases} x > y, if\ \ m > 1 \\ x < y, if\ 0 < m < 1 \end{cases}$
    $(m, x, y > 0, m \neq 1)$
11. $\log_m a = b \Rightarrow a = m^b$; $(m, a > 0, m \neq 1 \in$ real number$)$
12. $\log_m a > b \Rightarrow \begin{cases} a > m^b ; if\ \ m > 1 \\ a < m^b ; if\ \ 0 < m < 1 \end{cases}$

13. $\log_m a < b \Rightarrow \begin{cases} a < m^b ; if \ \ m > 1 \\ a > m^b ; if \ \ 0 < m < 1 \end{cases}$

Logarithmic Equations

(i) An equation of the form $\log_a f(x) = b$, $(a > 0)$, $a \neq 1$ is equivalent to the equation. $f(x) = a^b$, $(f(x) > 0)$

(ii) If $\log_a f(x) > \log_a g(x)$ and $a > 1$, then $\Rightarrow \begin{cases} g(x) > 0 \\ f(x) > 0 \\ f(x) > g(x) \end{cases}$

(iii) If $\log_a f(x) > \log_a g(x)$ and $a < 1$, then $\Rightarrow \begin{cases} f(x) > 0 \\ g(x) > 0 \\ f(x) < g(x) \end{cases}$

Definition of Modulus

$$|x| = \begin{cases} x, \ if \ \ x \geq 0 \\ -x, \ \ if \ x < 0 \end{cases}$$

Properties of Modulus

Let 'a', 'b' are positive real number then,

(i) $|x| = a \Rightarrow x = \pm a$

(ii) $|x| \leq a \Rightarrow -a \leq x \leq a \Rightarrow x \in [-a, a]$

(iii) $|x| < a \Rightarrow -a < x < a \Rightarrow x \in (-a, a)$

(iv) $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

(v) $|x| > a \Rightarrow x < -a$ or $x > a \Rightarrow x \in (-\infty, -a) \cup (a, \infty)$

(vi) $a \leq |x| \leq b \Rightarrow x \in [-b, -a] \cup [a, b]$

(vii) $a < |x| < b \Rightarrow x \in (-b, -a) \cup (a, b)$

($viii$) $|x + y| \leq |x| + |y|$; equality holds when $xy \geq 0$

(ix) $|x - y| \geq ||x| - |y||$; equality holds when $xy \geq 0$

(x) $|x + y| \geq ||x| - |y||$; equality holds when $xy \leq 0$

(xi) $|x - y| \leq |x| + |y|$; equality holds when $xy \geq 0$

Solved Examples

1. The value of $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$ is equal to

(a) 49 (b) 625

(c) 216 (d) 890

Sol. (d) $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$

$$= 3^{4\log_3 5} + 3^{3\cdot\frac{1}{2}\log_3 36} + 3^{4\log_9 7}$$

$$= 3^{\log_3 5^4} + 3^{\log_3 36^{3/2}} + 3^{\log_3 7^{4/2}}$$

$$= 5^4 + 36^{3/2} + 7^2 = 890$$

2. Values of x satisfying the equation

$\log_5^2 x + \log_{5x}\left(\frac{5}{x}\right) = 1$ are

(a) 1 (b) 5

(c) $\frac{1}{25}$ (d) 3

Sol. (a,b,c)

$$(\log_5 x)^2 + \log_{5x}\frac{5}{x} = 1$$

$$\Rightarrow (\log_5 x)^2 + \log_{5x} 5 - \log_{5x} x = 1$$

$$\Rightarrow (\log_5 x)^2 + \frac{\log_5 5}{\log_5 5 + \log_5 x} - \frac{\log_5 x}{\log_5 5 + \log_5 x} = 1$$

$$\Rightarrow (\log_5 x)^2 + \frac{1}{1 + \log_5 x} - \frac{\log_5 x}{1 + \log_5 x} = 1$$

Let $\log_5 x = t$

$$\therefore \ t^2 + \frac{1}{1+t} - \frac{t}{1+t} = 1$$

$$\Rightarrow \frac{t^2(1+t) + 1 - t}{1+t} = 1$$

$$\Rightarrow t^3 + t^2 + 1 - t = 1 + t$$

$$t^3 + t^2 - 2t = 0$$

$$t(t^2 + t - 2) = 0$$

$$t(t - 1)(t + 2) = 0$$

$$t = 0, 1, -2$$

$$\therefore \ \log_5 x = 0, 1, -2$$

$$\therefore \ x = 1, 5, \frac{1}{25}$$

3. The equation $\log_{x^2} 16 + \log_{2x} 64 = 3$ has

(a) One irrational solution

(b) No prime solution

(c) Two real solutions

(d) One integral solution

Sol. (a,b,c,d)

$$\frac{4}{2}\log_x 2 + \frac{\log_x 64}{\log_x 2x}$$

$$\Rightarrow \frac{2x\log_x 2}{x} + \frac{6\log_x 2}{1 + \log_x 2} = 3$$

Let $\alpha = \log_x 2$

$$2\alpha + \frac{6\alpha}{1+\alpha} = 3$$

$$2\alpha + 2\alpha^2 + 6\alpha - 3 - 3\alpha = 0$$

$$\Rightarrow 2\alpha^2 + 5\alpha - 3 = 0$$

$\Rightarrow (\alpha + 3)(2\alpha - 1) = 0 \Rightarrow a = -3, 1/2$

$\therefore \log_x 2 = -3 \Rightarrow x = 2^{-1/3}$ (Irrational)

or $\log_x 2 = \dfrac{1}{2} \Rightarrow x = 4$ (Integer)

4. If $\dfrac{1}{2} \le \log_{0.1} x \le 2$, then

(*a*) Maximum value of x is $\dfrac{1}{\sqrt{10}}$

(*b*) x lies between $\dfrac{1}{100}$ and $\dfrac{1}{\sqrt{10}}$

(*c*) Minimum value of x is $\dfrac{1}{10}$

(*d*) Minimum value of x is $\dfrac{1}{100}$

Sol. (*a*,*b*,*d*)

$$\frac{1}{2} \le \log_{0.1} x \le 2 \Rightarrow \left(\frac{1}{10}\right)^{1/2} \ge x \ge \left(\frac{1}{10}\right)^{2}$$

5. Let $N = \dfrac{\log_3 135}{\log_{15} 3} - \dfrac{\log_3 5}{\log_{405} 3}$. Then N is

(*a*) a natural number (*b*) a prime number

(*c*) a rational number (*d*) an integer

Sol. (*a*,*b*,*c*,*d*)

$= \log_3 135 \log_3 15 - \log_3 5 \log_3 405$

$= \log_3(5 \times 3^3) \cdot \log(5 \times 3) - \log_3 5 . \log_3 (5 \times 3^4)$

$= (\log_3 5 + \log_3 3^3)(\log_3 5 + \log_3 3) - \log_3 5 (\log_3 5 + \log_3 3^4)$

$= (x + 3)(x + 1) - x(x + 4)$

{Let $\log_3 5 = x$}

$= x^2 + 4x + 3 - x^2 - 4x = 3$

which is Prime, rational Integer and natural number

6. If $|x - 5| + |x + 5| = 10$, then

(*a*) The number of integral solutions is 10

(*b*) The number of integral solutions is 11

(*c*) The sum of all the integral solutions is 0

(*d*) All the solutions of the equation are rational numbers

Sol. (*b*,*c*)

$|x - 5| + |x + 5| = 10$

Case-I: $x \ge 5$, the equation becomes

$(x - 5) + (x + 5) = 10$

$\Rightarrow 2x = 10$

$\Rightarrow x = 5$ which satisfies the case, therefore accepted.

Case-II: $-5 < x < 5$ The above equation becomes

$-(x - 5) + (x + 5) = 10$

$\Rightarrow -x + 5 + x + 5 = 10$

$\Rightarrow 10 = 10$ which is true.

So, the solution is $x \in (-5, 5)$

Case-III: $x \le -5$, The above equation becomes

$-(x - 5) - (x + 5) = 10$

$\Rightarrow -x + 5 - x - 5 = 10$

$\Rightarrow -2x = 10$

$\Rightarrow x = -5$ which satisfies the above case so, accepted.

$\therefore$ final answer is $x \in [-5, 5]$

7. Let $y = \sqrt{\log_2 3 \cdot \log_2 12 \cdot \log_2 48 \cdot \log_2 192 + 16} - \log_2 12 \cdot \log_2 48 + 10$.

Find $y \in$ N.

Sol. [0006]

$$y = \sqrt{\log_2 3 \cdot \log_2 12 \cdot \log_2 48 \cdot \log_2 192 + 16} - \log_2 12 \cdot \log_2 48 + 10$$

$$= \sqrt{\log_2 3 \cdot (2 + \log_2 3)(4 + \log_2 3)(6 + \log_2 3) + 16} - (2 + \log_2 3)(4 + \log_2 3) + 10$$

Let us put $\log_2 3 = x$

$$= \sqrt{x(2+x)(4+x)(6+x)+16} - (2+x)(4+x) + 10$$

$$= \sqrt{(x^2+6x)(x^2+6x+8)+16} - (x^2+6x+8) + 10$$

Put again $x^2 + 6x = \alpha$

$$= \sqrt{\alpha(\alpha+8)+16} - (\alpha+8) + 10$$

$$= \sqrt{\alpha^2 + 8\alpha + 16} - (\alpha+8) + 10$$

$$= \sqrt{(\alpha+4)^2} - (\alpha+8) + 10$$

$$= (\alpha+4) - (\alpha+8) + 10 = y = 6.$$

8. If 'x' and 'y' are real numbers such that,

$2 \log(2y - 3x) = \log x + \log y$, find $\dfrac{y}{x}$.

Sol. [2.25]

$\log(2y - 3x)^2 = \log xy$

$(2y - 3x)^2 = xy$

$4y^2 - 12xy + 9x^2 = xy$

Dividing the equation by y^2

$$9\left(\frac{x}{y}\right)^2 - 13\frac{x}{y} + 4 = 0$$

$$\left(\frac{x}{y} - 1\right)\left(\frac{9x}{y} - 4\right) = 0$$

$$\frac{x}{y} = 1, \frac{x}{y} = \frac{4}{9}.$$

$x = y$ disregarded as for $x = y$, $2y - 3x$ is negative.

Hence $\dfrac{y}{x} = \dfrac{9}{4}$.

9. Sum of all the solutions of the equation

$\log_6(x^2 - 1) - \log_6 \sqrt{(x-6)^2} = \log_6(x+1)^2$ is $a + \sqrt{b}$, $(a, b \,\varepsilon N)$. Then $a + b$ is equal to

Sol. $\log_6(x^2 - 1) - \log_6 \sqrt{(x-6)^2} = \log_6(x+1)^2$

$$\Rightarrow \log_6 \frac{(x-1)(x+1)}{(x+1)^2} = \log_6 |x - 6|$$

$\Rightarrow \log_6\left[\frac{(x-1)}{(x+1)}\right] = \log_6|x-6|$

$\Rightarrow \frac{x-1}{x+1} = |x-6|$

Case-I: $x \geq 6$

$\Rightarrow x - 1 = x^2 - 5x - 6$

$\Rightarrow x^2 - 6x - 5 = 0$

$\Rightarrow (x-3)^2 = 14$

$\Rightarrow x = 3 \pm \sqrt{14}$

$x = 3 - \sqrt{14} < 1$ rejected

$x = 3 + \sqrt{14}$ accepted

Case-II: $x < 6$

$x - 1 = -(x^2 - 5x - 6)$

$\Rightarrow x^2 - 4x - 7 = 0$

$(x-2)^2 = 11$

$x = 2 \pm \sqrt{11}$

$x = 2 + \sqrt{11}$ (accepted)

$x = 2 - \sqrt{11}$ (accepted)

Sum of roots = $7 + \sqrt{14}$

$\Rightarrow a = 7, b = 14$

$a + b = 21$

10. Match the column

	Column-I		Column-II
A.	The roots of $\log_2(x+e) = \log_2 x + \log_2 e$ is a	p.	Positive Number
B.	The solution of $\log_{1/5}(2x^2 + 5x + 1) < 0$ contains	q.	Rational Number
C.	$\log_{\sin\frac{\pi}{6}} \pi$ is	r.	Irrational Number
D.	$\log_{10}5.\log_{10}20 + \log_{10}^2 2$ simplifies to	s.	Negative Number

Sol. A→(p, r) ; B→(p, q, r, s) ; C→(r, s) ; D→(p, q)

(*A*) $x + e = xe$

$x(e-1) = e$

$x = \frac{e}{e-1}$

(*B*) $2x^2 + 5x + 1 > 1$ and $2x^2 + 5x + 1 > 0$

$\Rightarrow 2x^2 + 5x + 1 > 1$

$\Rightarrow (\underline{x})(2x + 5) > 0$

$\Rightarrow x \in \left(-\infty, \frac{-5}{2}\right) \cup (0, \infty)$

(*D*) $(1 - \log_{10} 2)(1 + \log_{10} 2) + \log_{10}^2 2$

$\Rightarrow 1 - \log_{10}^2 2 + \log_{10}^2 2 = 1$

11. The largest integral value of x satisfying

$\sqrt{18^x - 5} \leq \sqrt{2(18^x + 12)} - \sqrt{18^x + 5}$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) no integral value of x possible

Sol. (*d*) Let $18^x = p$

$\sqrt{p-5} + \sqrt{p+5} \leq \sqrt{2(p+12)}$

$\Rightarrow p - 5 + p + 5 + 2\sqrt{p^2 - 25} \leq 2p + 24$

$\Rightarrow \sqrt{p^2 - 25} \leq 12 \Rightarrow p^2 \leq 169 \Rightarrow p \leq 13$

Also $p \geq 5$

Thus $5 \leq p \leq 13$ $\log_{18}5 \leq x \leq \log_{18}13$

12. If $\log_a x = b$ for permissible values of a and x then identify the statements(s) which can be correct?

(*a*) If a and b are two irrational numbers then x can be rational.

(*b*) If a rational and b irrational then x can be rational.

(*c*) If a irrational and b rational then x can be rational.

(*d*) If a rational and b rational then x can be rational.

Sol. (a,b,c,d)

(*a*) $a = (\sqrt{2})^{\sqrt{2}}$ is irrational

` $b = \sqrt{2}$ is also irrational

but $a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$ which is rational $\Rightarrow$ (*a*) is correct.

(*b*) $a = 2 \in Q$; $b = \log_2 3 \notin Q$

$a^b = 2^{\log_2 3} = 3 \in Q \Rightarrow$ (*b*) is correct

13. Solve if $|x-5| + |x+4| = 9$

Sol. Given equation is of form $|a| + |b| = |a - b|$

It is true for $ab \leq 0$

$(x-5)(x+4) \leq 0$

So $x \in [-4, 5]$

14. Solve $\frac{(e - \sin x)(x-2)}{(x+4)} \geq 0.$

Sol. Zeroes $x = 2$, Pole $x \neq -4$

$e - \sin x > 0$ always positive

$\frac{(e - \sin x)(x-2)}{(x+4)} \geq 0$

Final solution $x \in (-\infty, -4) \cup [2, \infty)$

Exercise-1 (Topicwise)

BASIC CONCEPTS AND NUMBER SYSTEM

1. Let $x \in Q$, $y \in Q^c$, Which of the following statement is always WRONG?

(*a*) $xy \in Q^c$

(*b*) $y/x \in Q$, whenever defined

(*c*) $\sqrt{2}\,x + y \in Q$

(*d*) $x/y \in Q^c$, whenever defined

2. If x and y are two rational numbers such that $(x + y) + (x - 2y)\sqrt{2} = 2x - y + (x - y - 1)\sqrt{6}$, then:

(*a*) $x = 1, y = 1$

(*b*) $x = 2, y = 1$

(*c*) $x = 5, y = 1$

(*d*) x and y can take infinitely many values

3. Which of the following statement is incorrect :

(*a*) rational number + rational number = rational number

(*b*) irrational number + rational number = irrational number

(*c*) integer + rational number = rational number

(*d*) irrational number + irrational number = Irrational number

4. The number of real roots of the equation $(x - 1)^2 + (x - 2)^2 + (x - 3)^2 = 0$ is :

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3

5. If $x - a$ is a factor of $x^3 - a^2x + x + 2$, then 'a' is equal to

(*a*) 0 (*b*) 2

(*c*) –2 (*d*) 1

6. Every irrational number can be expressed on the number line. This statement is

(*a*) Always true

(*b*) Never true

(*c*) True subject to some condition

(*d*) None of these

7. The multiplication of a rational number 'x' and an irrational number 'y' is

(*a*) Always rational

(*b*) Rational except when $y = \pi$

(*c*) Always irrational

(*d*) Irrational except when $x = 0$

8. If x, y are integral solutions of $2x^2 - 3xy - 2y^2 = 7$, then value of $|x + y|$ is

(*a*) 2 (*b*) 4

(*c*) 6 (*d*) 2 or 4 or 6

9. If a, b, c are real, then $a(a - b) + b(b - c) + c(c - a) = 0$, only if

(*a*) $a + b + c = 0$

(*b*) $a = b = c$

(*c*) $a = b$ or $b = c$ or $c = a$

(*d*) $a - b - c = 0$

10. If $2x^3 - 5x^2 + x + 2 = (x - 2)(ax^2 - bx - 1)$, then a & b are respectively

(*a*) 2, 1 (*b*) 2, – 1

(*c*) 1, 2 (*d*) –1, 1/2

11. The value of $[e] - [-\pi]$ is, where [.] denotes greatest integer function.

(*a*) 5 (*b*) 6

(*c*) 7 (*d*) 8

12. If $L = \frac{1}{\sqrt{7}-\sqrt{8}} + \frac{1}{\sqrt{7}-\sqrt{6}} + \frac{1}{3-\sqrt{8}} + \frac{1}{\sqrt{5}+2} + \frac{1}{\sqrt{5}-\sqrt{6}}$ $= 1 + 2\sqrt{a} + 2\sqrt{b}$, then $a.\ b$ is equal to

(*a*) 30 (*b*) 45

(*c*) 8 (*d*) 16

13. If a, b, c are real and distinct numbers, then the value of $\frac{(a-b)^3 + (b-c)^3 + (c-a)^3}{(a-b)(b-c)(c-a)}$ is

(*a*) 1 (*b*) $a\,b\,c$

(*c*) 2 (*d*) 3

14. The remainder obtained when the polynomial $1 + x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ is divided by $x - 1$ is

(*a*) 3 (*b*) 5

(*c*) 7 (*d*) 11

15. $\frac{1}{1+\log_b a + \log_b c} + \frac{1}{1+\log_c a + \log_c b} + \frac{1}{1+\log_a b + \log_a c}$ has the value equal to

(*a*) abc (*b*) $\frac{1}{abc}$

(*c*) 0 (*d*) 1

16. $\log_7 \log_7 \sqrt{7(\sqrt{7\sqrt{7}})} =$

(*a*) $3 \log_2 7$ (*b*) $1 - 3 \log_3 7$

(*c*) $1 - 3\log_7 2$ (*d*) $1 - 10 \log_2 7$

LOGARITHM AND ITS PRINCIPLE PROPERTIES

17. $\dfrac{1}{\log_{\sqrt{bc}} abc}+\dfrac{1}{\log_{\sqrt{ca}} abc}+\dfrac{1}{\log_{\sqrt{ab}} abc}$ has the value equal to

(*a*) 1/2 (*b*) 1
(*c*) 2 (*d*) 4

18. If $\log_x \log_{18}(\sqrt{2}+\sqrt{8})=\dfrac{1}{3}$. Then the value of $1000x$ is equal to

(*a*) 8 (*b*) 1/8
(*c*) 1/125 (*d*) 125

19. Number of real solutions of the equation $\sqrt{\log_{10}(-x)}=\log_{10}\sqrt{x^2}$ is :

(*a*) none (*b*) exactly 1
(*c*) exactly 2 (*d*) 4

20. Greatest integer less than or equal to the number $\log_2 15.\log_{1/6} 2.\log_3 1/6$ is

(*a*) 4 (*b*) 3
(*c*) 2 (*d*) 1

21. The ratio $\dfrac{2^{\log_{2^{1/4}} a}-3^{\log_{27}(a^2+1)^3}-2a}{7^{4\log_{49} a}-a-1}$ simplifies to

(*a*) a^2-a-1 (*b*) a^2+a-1
(*c*) a^2-a+1 (*d*) a^2+a+1

22. If $3^{2\log_3 x}-2x-3=0$, then the number of values of 'x' satisfying the equation is

(*a*) zero (*b*) 1
(*c*) 2 (*d*) More than 2

23. The number $\log_2 7$ is

(*a*) an integer (*b*) a rational number
(*c*) an irrational number (*d*) a prime number

24. Anti logarithm of 0.75 to the base 16 has the value equal to

(*a*) 4 (*b*) 6
(*c*) 8 (*d*) 12

25. The sum of all the solutions to the equation $2\log x-\log(2x-75)=2$:

(*a*) 30 (*b*) 350
(*c*) 75 (*d*) 200

INEQUALITIES

26. If the solution set of the inequality $\log_{\sqrt{0.9}}\log_5(\sqrt{x^2+5+x})>0$ contains 'n' integral values, then n equals to

(*a*) 7 (*b*) 8 (*c*) 6 (*d*) 10

27. If $\log_{0.5}\log_5(x^2-4)>\log_{0.5}1$, then '$x$' lies in the interval

(*a*) $(-3,-\sqrt{5})\cup(\sqrt{5},3)$ (*b*) $(-3,-\sqrt{5})\cup(\sqrt{5},3\sqrt{5})$
(*c*) $(\sqrt{5},3\sqrt{5})$ (*d*) ϕ

28. Solution set of the inequality $2-\log_2(x^2+3x)\geq 0$ is:

(*a*) $[-4,1]$ (*b*) $[-4,-3)\cup(0,1]$
(*c*) $(-\infty,-3)\cup(1,\infty)$ (*d*) $(-\infty,-4)\cup[1,\infty)$

MODULUS FUNCTION

29. Solutions of $|4x+3|+|3x-4|=12$ are

(*a*) $x=-\dfrac{7}{3},\dfrac{3}{7}$ (*b*) $x=-\dfrac{5}{2},\dfrac{2}{5}$
(*c*) $x=-\dfrac{11}{7},\dfrac{13}{7}$ (*d*) $x=-\dfrac{3}{7},\dfrac{7}{5}$

30. If $|x^2-2x-8|+|x^2+x-2|=3|x+2|$, then the set of all real values of x is

(*a*) $[1,4]\cup\{-2\}$ (*b*) $[1,4]$
(*c*) $[-2,1]\cup[4,\infty)$ (*d*) $(-\infty,-2]\cup[1,4]$

31. The complete set of real 'x' satisfying $||x-1|-1|\leq 1$ is:

(*a*) $[0,2]$ (*b*) $[-1,3]$
(*c*) $[-1,1]$ (*d*) $[1,3]$

32. The number of real roots of the equation $|x|^2-3|x|+2=0$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

33. Number of real solution (x) of the equation $|x-3|^{3x^2-10x+3}=1$ is

(*a*) exactly four (*b*) exactly three
(*c*) exactly two (*d*) exactly one

34. Simplify: $7^{\log_3 5}+3^{\log_5 7}-5^{\log_3 7}-7^{\log_5 3}$

(*a*) 0 (*b*) 1 (*c*) 3 (*d*) 4

35. The expression $x^2-y^2-z^2+2yz+x+y-z$ has a factor

(*a*) $x-y+z+1$ (*b*) $-x+y+z$
(*c*) $x+y-z+1$ (*d*) $x-y-z+1$

36. Solve the equation $\dfrac{3x^4+x^2-2x-3}{3x^4-x^2+2x+3}=\dfrac{5x^4+2x^2-7x+3}{5x^4-2x^2+7x-3}$

(*a*) $x=5,2$ (*b*) $x=4,1$
(*c*) $x=3,8$ (*d*) $x=1,5$

37. If x, y, z are positive real number and a, b, c are rational numbers, then the value of $\dfrac{1}{1+x^{b-a}+x^{c-a}}+\dfrac{1}{1+x^{a-b}+x^{c-b}}+\dfrac{1}{1+x^{b-c}+x^{a-c}}$ is

(*a*) –1 (*b*) 1 (*c*) 0 (*d*) 2

38. If $a^x=\sqrt{b}, b^y=\sqrt[3]{c}$ and $c^z=\sqrt{a}$ then the value of xyz is

(*a*) $\dfrac{1}{2}$ (*b*) $\dfrac{1}{3}$ (*c*) $\dfrac{1}{6}$ (*d*) $\dfrac{1}{12}$

39. If $\dfrac{\log a}{b-c}=\dfrac{\log b}{c-a}=\dfrac{\log c}{a-b}$, then $a^a\,.\,b^b\,.\,c^c=$

(*a*) 3 (*b*) 1 (*c*) 4 (*d*) 2

40. The number of prime numbers satisfying the inequality $\dfrac{x^2-1}{2x+5}<3$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

Exercise-2 (Learning Plus)

1. If A & B are two rational numbers and AB, A + B and A – B are rational numbers, then A/B is
(a) Always rational
(b) Never rational
(c) Rational when $B \neq 0$
(d) Rational when $A \neq 0$

2. If $x^{x\sqrt[3]{x}} = \left(x.\sqrt[3]{x}\right)^{x}$, then $x =$
(a) 1
(b) –1
(c) 0
(d) 2

3. The equation $4^{(x^2+2)} - 9.2^{(x^2+2)} + 8 = 0$ has the solution
(a) $x = \pm 1$
(b) $x = 10$
(c) $x = \pm\sqrt{2}$
(d) $x = \sqrt{3}$

4. Logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$ is
(a) 3.6
(b) 5
(c) 5.6
(d) 10

5. If $x = \log_a(bc)$, $y = \log_b(ca)$, $z = \log_c(ab)$, then which of the following is equal to 1
(a) $x + y + z$
(b) $(1 + x)–1 + (1 + y)–1 + (1 + z)–1$
(c) xyz
(d) $x + y – z$

6. The solution of the equation $\log_7 \log_5 \left(\sqrt{x^2+5+x}\right) = 0$.
(a) $x = 2$
(b) $x = 3$
(c) $x = 4$
(d) $x = -2$

7. The value of $(0.05)^{\log_{\sqrt{20}}(0.1+0.01+0.001+........)}$ is
(a) 81
(b) $\frac{1}{81}$
(c) 20
(d) $\frac{1}{20}$

8. The value of $\log_2 .\log_3 ....\log_{100} 100^{99^{98^{\cdot^{\cdot^{\cdot^{2^1}}}}}}$ is
(a) 0
(b) 1
(c) 2
(d) 100!

9. The number of solution of $\log_2(x + 5) = 6 – x$ is
(a) 2
(b) 0
(c) 3
(d) 1

10. If $\log_{10}2 = 0.30103$, $\log_{10}3 = 0.47712$, the number of digits in 312×28 is
(a) 7
(b) 8
(c) 9
(d) 10

11. Exhaustive set of values of x satisfying $\log_{|x|}(x2 + x + 1) \geq 0$ is
(a) $(–1, 0)$
(b) $(-\infty, –1) \cup (1, \infty)$
(c) $(-\infty, \infty) – \{–1, 0, 1\}$
(d) $(-\infty,–1)\cup(–1, 0)\cup(1, \infty)$

12. The set of real values of x satisfying $\log_{1/2}\left(x^2 - 6x + 12\right) \geq -2$ is
(a) $\left(-\infty, 2\right]$
(b) $[2, 4]$
(c) $\left[4, +\infty\right)$
(d) $[3, 8]$

13. If $\log_{0.04}(x – 1) \geq \log_{0.2}(x – 1)$ then x belongs to the interval
(a) $(1, 2]$
(b) $(-\infty, 2]$
(c) $[2, + \infty)$
(d) $[2, 2]$

14. If $\log_{0.3}(x – 1) < \log_{0.09}(x – 1)$, then x lies in the interval
(a) $(2, \infty)$
(b) $(– 2, –1)$
(c) $(1, 2)$
(d) $(–2, 2)$

15. The minimum value of $f(x) = |x – 1| + |x – 2| + |x – 3|$ is equal to
(a) 1
(b) 2
(c) 3
(d) 0

16. The set of real value(s) of p for which the equation $| 2x + 3 | + | 2x – 3 | = px + 6$ has more than two solutions is :
(a) $[0, 4)$
(b) $(– 4, 4)$
(c) $R – \{4, – 4, 0\}$
(d) $\{0\}$

17. Let $3^a = 4$, $4^b = 5$, $5^c = 6$, $6^d = 7$, $7^e = 8$ and $8^f = 9$. The value of the product $(abcdef)$ is :
(a) 1
(b) 2
(c) $\sqrt{6}$
(d) 3

18. There are two positive solutions to the equation $\log_{2x}2 + \log_4 2x = -\frac{3}{2}$. The product of these two solution is:
(a) $\frac{1}{32}$
(b) $\frac{1}{8}$
(c) $\frac{1}{2}$
(d) $\frac{1}{21}$

19. Number of real value of x satisfying the equation $\log_2(2x^2+\sqrt{2}) = \frac{\sqrt{x^2+1}}{x^2+2}$ is :
(a) 0
(b) 1
(c) 2
(d) 3

20. Number of values of x satisfying the equation $x^4 = |x|^{\log_2(x^2+12)}$ is:
(a) 2
(b) 3
(c) 4
(d) 5

21. The number of zeros after decimal before the start of any significant digit in the number $N = (0.15)^{20}$ are :
(a) 15
(b) 16
(c) 17
(d) 18

22. If $n, k \in N$, then the smallest value of k such that $k + 2k + 3k \ldots 24k = n^3$ is

(*a*) 100 (*b*) 90
(*c*) 120 (*d*) 60

23. If $a, b, c \in R$ and $a, b, c \neq 0$ such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=6$ and $\frac{b}{a}+\frac{c}{b}+\frac{a}{c}=8$ then $\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}-3$ is equal to

(*a*) 81 (*b*) 48
(*c*) 72 (*d*) 84

24. If $3^x = 5^y$, then how many ordered pairs (x, y), $x, y \in R$ satisfy the given equation?

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) Infinite

25. The value of $x + y + z$ satisfying the system of equations

$\log_2 x + \log_4 y + \log_4 z = 2$ is
$\log_3 y + \log_9 z + \log_9 x = 2$
$\log_4 z + \log_{16} x + \log_{16} y = 2$

(*a*) $\frac{175}{12}$ (*b*) $\frac{349}{24}$
(*c*) $\frac{353}{24}$ (*d*) $\frac{112}{3}$

26. Which is the correct order for a given number α, $\alpha > 1$

(*a*) $\log_2 \alpha < \log_3 \alpha < \log_e \alpha > \log_{10} \alpha$
(*b*) $\log_{10} \alpha < \log_3 \alpha < \log_e \alpha > \log_2 \alpha$
(*c*) $\log_{10} \alpha < \log_e \alpha < \log_2 \alpha > \log_3 \alpha$
(*d*) $\log_3 \alpha < \log_e \alpha < \log_2 \alpha > \log_{10} \alpha$

27. The smallest integral value of x such that $\sqrt{x+2}-\sqrt{x-2}<\frac{1}{10}$ is

(*a*) 400 (*b*) 20
(*c*) 401 (*d*) 399

28. The ratio $\dfrac{2^{\log_{2^{1/4}} a}-3^{\log_{27}(a^2+1)^3}-2a}{7^{4\log_{49} a}-a-1}$ simplifies to

(*a*) $a^2 - a - 1$ (*b*) $a^2 + a - 1$
(*c*) $a^2 - a + 1$ (*d*) $a^2 + a + 1$

29. $10^{\log_p(\log_q(\log_r x))}=1$ and $\log_q(\log_r(\log_p x))=0$ then 'p' equals

(*a*) $r^{q/r}$ (*b*) rq
(*c*) 1 (*d*) $r^{r/q}$

30. Which one of the following is the smallest?

(*a*) $\log_{10}\pi$ (*b*) $\sqrt{\log_{10}\pi^2}$
(*c*) $\left(\frac{1}{\log_{10}\pi}\right)^3$ (*d*) $\left(\frac{1}{\log_{10}\sqrt{\pi}}\right)$

31. The set of all solutions of the inequality $(1/2)^{x^2-2x}<1/4$ contains the set

(*a*) $(-\infty, 0)$ (*b*) $(-\infty, 1)$
(*c*) $(1, \infty)$ (*d*) $(3, \infty)$

32. The value of b satisfying thee quation, $\log_e 2 \cdot \log_b 625 = \log_{10} 16 \cdot \log_e 10$ is

(*a*) 5 (*b*) 7
(*c*) 9 (*d*) 10

33. The solution set of the system of equation $\log_3 x + \log_3 y = 2 + \log_3 2$ and $\log_{27}(x + y) = \frac{2}{3}$ is:

(*a*) $\{6, 3\}$ (*b*) $\{9, 6\}$
(*c*) $\{6, 12\}$ (*d*) $\{12, 6\}$

34 Consider the quadratic equation, $(\log_{10}8)x^2$ - $(\log_{10}5)x = 2(\log_2 10)^{-1} - x$. Which of the following quantities are irrational.

(*a*) Sum of the roots (*b*) Product of the roots
(*c*) Sum of the coefficients (*d*) Discriminant

35. Which of the following statements are true?

(*a*) $\log_2 3 < \log_{12} 10$ (*b*) $\log_6 5 < \log_7 8$
(*c*) $\log_3 26 > \log_2 9$ (*d*) $\log_{16} 15 > \log_{10} 11 > \log_7 6$

36. If $x + y = a$ and $x^2 + y^2 = b$, then the value of $(x^3 + y^3)$ is

(*a*) ab (*b*) $a^2 + b$
(*c*) $a + b^2$ (*d*) $\frac{3ab-a^3}{2}$

37. If $2x^3 - 5x^2 + x + 2 = (x - 2)(ax^2 - bx - 1)$, then a & b are respectively

(*a*) 2, 1 (*b*) 2, –1 (*c*) 1, 2 (*d*) –1, 1/2

38. If $x + y + z = 0$, then a factor of the expression $(x + y)^3 + (y + z)^3 + (z + x)^3$ is

(*a*) $3(x + y)(y + z)(z + x)$ (*b*) $3xyz$
(*c*) $(x + y - z)$ (*d*) $(x - y + z)$

39. The number of real solution/s of the equation $9^{\log_3(\log_e x)} = \log_e x - (\log_e x)^2 + 1$ is:

(*a*) 0 (*b*) 1
(*c*) 2 (*d*) 3

40. The set of all the solutions of the inequality $\log_{1-x}(x - 2) \geq -1$ is

(*a*) $(-\infty, 0)$ (*b*) $(2, \infty)$
(*c*) $(-\infty, 1)$ (*d*) ϕ

41. The complete solution of $\frac{x^2-1}{x+3}\geq 0$ & $x^2-5x+2\leq 0$ is:

(*a*) $x \in \left[\frac{5-\sqrt{17}}{2},\frac{5+\sqrt{17}}{2}\right]$ (*b*) $x \in \left[1,\frac{5-\sqrt{17}}{2}\right]$
(*c*) $x \in (-3, -1]$ (*d*) $x \in (-3, -1]\ [1, \infty)$

42. The number of the integral solutions of $x^2 + 9 < (x + 3)^2 < 8x + 25$ is:

(*a*) 1 (*b*) 3 (*c*) 4 (*d*) 5

43. Number of non-negative integral values of x satisfying the inequality $\frac{2}{x^2-x+1}-\frac{1}{x+1}-\frac{2x-1}{x^3+1}\geq 0$ is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

44. Match the values of x given in Column-II satisfying the exponential equation given in Column-I (Do not verify). Remember that for $a > 0$, the terms a^x is always greater than zero $\forall\ x \in R$.

	Column-I		Column-II
A.	$5^x - 24 = \dfrac{25}{5^x}$	p.	–3
B.	$(2^{x+1})(5^x) = 200$	q.	–2
C.	$4^{2/x} - 5(4^{1/x}) + 4 = 0$	r.	–1
D.	$2^{2x+1} - 33(2^{x-1}) + 4 = 0$	s.	0
E.	$\dfrac{2^{x-1} \cdot 4^{x+1}}{8^{x-1}} = 16$	t.	1
F.	$3^{2x+1} + 10(3^x) + 3 = 0$	u.	2
G.	$64(9^x) - 84(12^x) + 27(16^x) = 0$	v.	3
H.	$5^{2x} - 7^x - 5^{2x}(35) + 7^x(35) = 0$	w.	None

(*a*) $A \to u$, $B \to u$, $C \to t$, $D \to q, v$, $C \to p,q,r,s,t,u,v$ $F \to w$, $G \to t,u$, $H \to s$

(*b*) $A \to q$, $B \to t$, $C \to u$, $D \to v$, $C \to t,u,v$ $F \to w$, $G \to t, H \to q$

(*c*) $A \to p$, $B \to u$, $C \to s$, $D \to q$, $C \to p,q$, $F \to s$, $G \to u$, $H \to q$

(*d*) $A \to q$, $B \to s$, $C \to r$, $D \to s$, $C \to q,r$, $F \to q$, $G \to u$, $H \to q$

45. Match the columns:

	Column-I		Column-II
A.	$\log_{\sin 30°}(\cos 60°) + 1$	p.	3
B.	$\log_{4/3}(1.\bar{3}) + 3$	q.	5
C.	$\log_{2-\sqrt{3}}(2+\sqrt{3}) + 6$	r.	4
D.	$\log_{\tan 20°} \tan 70° + 4$	s.	2
E.	$\log_{\cot 40°} \tan 50°$	t.	0
F.	$\log_{0.125}(8) + 8$	u.	–1
G.	$\log_{1.5}(0.\bar{6}) + 9$	v.	8
H.	$\log_{2.25}(0.\bar{4})$	w.	7
I.	$\log_{10}(0.\bar{9})$	x.	1

(*a*) $A \to q$, $B \to p$, $C \to s$, $D \to v$, $E \to u$, $F \to u$, $G \to q$, $H \to w$, $I \to x$

(*b*) $A \to s$, $B \to r$, $C \to q$, $D \to p$, $E \to t$, $F \to u$, $G \to v$, $H \to w$, $I \to x$

(*c*) $A \to s$, $B \to v$, $C \to t$, $D \to p$, $E \to t$, $F \to u$, $G \to w$, $H \to x$, $I \to w$

(*d*) $A \to q$, $B \to s$, $C \to r$, $D \to v$, $E \to u$, $F \to v$, $G \to v$, $H \to w$, $I \to x$

46. Match the following columns.

	Column-I		Column-II
A.	If $a = 3\left(\sqrt{8+2\sqrt{7}} - \sqrt{8-2\sqrt{7}}\right)$, $b = \sqrt{(42)(30)+36}$ then the value of $\log_a b$ is equal to	p.	–1
B.	If $a = \sqrt{4+2\sqrt{3}} - \sqrt{4-2\sqrt{3}}$, $b = \sqrt{(42)(30)+36}$ then the value of $\log_a b$ is equal to	q.	1
C.	If $a = \sqrt{3+2\sqrt{2}}, b = \sqrt{3-2\sqrt{2}}$ then the value of $\log_a b$ is equal to	r.	2
D.	If $a = \sqrt{7+\sqrt{7^2-1}}, b = \sqrt{7-\sqrt{7^2-1}}$, then the value of $\log_a b$ is equal to	s.	$2 + 2\log_2 3$

(*a*) $A \to s, B \to p, C \to q, D \to p$

(*b*) $A \to r, B \to p, C \to r, D \to p$

(*c*) $A \to r, B \to s, C \to p, D \to p$

(*d*) $A \to p, B \to q, C \to p, D \to r$

47. Suppose $x, y, z > 0$ and different than one and $\ln x + \ln y + \ln z = 0$. Find the value of $K = x^{\frac{1}{\ln y}+\frac{1}{\ln z}} \cdot y^{\frac{1}{\ln z}+\frac{1}{\ln x}} \cdot z^{\frac{1}{\ln x}+\frac{1}{\ln y}}$.

48. If $\log_2(\log_3(\log_4(x))) = 0$ and $\log_3(\log_4(\log_2(y))) = 0$ and $\log_4(\log_2(\log_3(z))) = 0$ then find the sum of x, y and z is

49. Let $\log_2 x + \log_4 x + \log_8 x = \log_k x$ for all $x \in R^+$. If $k = \sqrt[b]{a}$ where $a, b \in N$ then find the smallest positive value of $(a + b)$.

50. Solve the following inequalities:

(*i*) $\log_x(4x - 3) \geq 2$

(*ii*) $\log_{(3x^2+1)} 2 < \dfrac{1}{2}$

(*iii*) $\log_{x^2}(2+x) < 1$

51. Find the value of the expression $\dfrac{2}{\log_4(2000)^6} + \dfrac{3}{\log_5(2000)^6}$.

52. Solve for x: $\log 4 + \left(1 + \dfrac{1}{2x}\right)\log 3 = \log\left((3)^{1/x} + 27\right)$

53. Simplify:

(*i*) $\dfrac{81^{\frac{1}{\log_5 9}} + 3^{\frac{3}{\log_{\sqrt{6}} 3}}}{409}\left((\sqrt{7})^{\frac{2}{\log_{25} 7}} - (125)^{\log_{25} 6}\right) = ?$

(*ii*) $5^{\log_{1/5}\frac{1}{2}} + \log_{\sqrt{2}}\left(\dfrac{4}{\sqrt{7}+\sqrt{3}}\right) + \log_{1/2}\left(\dfrac{1}{(10+2\sqrt{21})}\right) = ?$

(*iii*) $\sqrt{10^{2+\frac{1}{2}\log(16)}} = ?$

54. $\log_{1/5}\dfrac{4x+6}{x}\ge 0$

55. $\log_7\left(\dfrac{2x-6}{2x-1}\right)>0$

56. $\log_3|3-4x|>2$

57. If $N=7^{\log_{49}900}, A=2^{\log_2 4}+3^{\log_2 4}+4^{\log_2 2}-4\log_2 3,$

$D=(\log_5 49)(\log_7 125)$

Find $P=\log_{\left(A-\frac{N}{10}\right)}|N+A+D+6|-\log_5 2,$

58. Solve for x:

(*i*) $\log_2(\log_{1/2}(x))<2$ (*ii*) $\log_{1/2}(\log_3(x))>3$

(*iii*) $(\log_2(x)-1)(\log_3(x)-2)\le 0$

(*iv*) $(\log_2(x)-1)(\log_{1/2}(x)-2)\le 0$

59. Solve for x:

(*i*) $\dfrac{\log_{10}(x-3)}{\log_{10}(x^2-21)}=\dfrac{1}{2}$ (*ii*) $\log_2(x+1)>\log_3(x+1)$

(*iii*) $\log_{1/2}(x-1)>\log_{1/3}(x-1)$

60. If $a+b+c=1$, $a^2+b^2+c^2=9$, $a^3+b^3+c^3+1$, then find value of $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.

61. If $x^2+\dfrac{1}{x^2}=102,$ then possible values of $x-\dfrac{1}{x}$ will be

62. If $x+\dfrac{1}{x}=\sqrt{5}$ then find

(*i*) $x^2+\dfrac{1}{x^2}$ (*ii*) $x^3+\dfrac{1}{x^3}$

(*iii*) $x^4+\dfrac{1}{x^4}$ (*iv*) $x^6+\dfrac{1}{x^6}$

63. $a+b+c=10$ and $ab+bc+ac=20$ then find the value of $a^3+b^3+c^3-3abc$

64. If a, b, c are real and $a^2+16b^2+25c^2-4ab-20bc-5ac=0$ then find the ratio of a, b and c.

65. Prove that $(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$

66. The value of $\sqrt{5\sqrt{5\sqrt{5\sqrt{5.........}}}}$ - is

67. If $x=\sqrt[3]{7+5\sqrt{2}}-\dfrac{1}{\sqrt[3]{7+5\sqrt{2}}}$, then the value of $x^3+3x-14$ is equal to

68. $\left[\left\{\left(\dfrac{1}{7^2}\right)^{-2}\right\}^{-1/3}\right]^{1/4}=7^x$ then x

69. If $\log_{x-3}(2x-3)$ is a meaningful quantity then find the interval in which x must lie.

70. Number of cyphers after decimal before a significant figure starts in $\left(\dfrac{5}{4}\right)^{-100}$ is equal to [**Use:** $\log_{10}2=0.3010$]

71. Number of real solution of $\log_5[2+\log_3(x+3)]=0$ is

72. Prove that $a^2+b^2+c^2-ab-bc-ac$ is non-negative for all real values of a, b and c. Also prove that equality holds when $a=b=c$.

73. If $4^A+9^B=10^C$, where $A=\log_{16}4$, $B=\log_3 9$ & $C=\log_x 83$, then find x.

74. Prove that $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a)$

75. $a^{\frac{\log_b(\log_b N)}{\log_b a}}$

76. $x>\sqrt{1-x}$

77. $\sqrt{x-2}\ge -1$

78. $(x^2-4)\sqrt{x^2-1}<0$

79. Solve for x:

(*i*) $\dfrac{1}{2}\le|2x-1|\le\dfrac{3}{5}$ (*ii*) $-\dfrac{1}{3}\le|3x-4|\le 2$

(*iii*) $2\le|4-5x|\le\dfrac{10}{3}$ (*iv*) $3\le|x^2-1|\le 8$

80. Solve for x:

(*i*) $|||x+5|-3|-1|=2$

(*ii*) $||||x-5|-7|-3|-2|=1$

81. Solve for x:

(*i*) $2\le|||x-7|-3|+2|\le 5$

(*ii*) $||||x-5|-7|-3|-2|\le 1$

82. Solve for x:

(*i*) $\dfrac{(|x|-1)}{(|x|-2)}\le 0$ (*ii*) $\dfrac{(|x|-1)(|x|-3)}{|x|^2-2|x|}\ge 0$

(*iii*) $\dfrac{(|x|^2-5|x|+6)}{(4-|x|^2)}\le 0$

(*iv*) $(||x-1|-2|-3)(|x-2|-3)\ge 0$

(*v*) $(|||x-1|-2|-1|-2)(|x-2|)\ge 0$

83. Solve for x:

(*i*) $\left|\dfrac{x-3}{x+1}\right|\le 1$ (*ii*) $\left|\dfrac{x+1}{x}\right|+|x+1|=\dfrac{(x+1)^2}{|x|}$

(*iii*) $\left|1+\dfrac{3}{x}\right|>2$ (*iv*) $|2^x-1|+|4-2^x|<3$

(*v*) $\left(\dfrac{1}{3}\right)^{\frac{|x+2|}{2-|x|}}>9$

84. Solve: $||x^2-2x+6|-|x+6||=|x^2-3x|$

85. Solve $\dfrac{5^x(\sin x-\pi)}{(x^2-4)}\ge 0.$

Exercise-3 (JEE Advanced Level)

1. $\log_{10}(\log_2 3)+\log_{10}(\log_3 4)+\log_{10}(\log_4 5)+...+\log_{10}(\log_{1023} 1024)$ simplifies to

(*a*) a composite

(*b*) a prime number

(*c*) rational which is not an integer

(*d*) an integer

2. The expression $\log_p \log_p \underbrace{\sqrt[p]{\sqrt[p]{\sqrt[p]{..........\sqrt[p]{p}}}}}_{n \text{ radical sign}}$, where $p \geq 2$, $p \in N$; $n \in N$ when simplifies is

(*a*) Independent of p

(*b*) Independent of p and of n

(*c*) Dependent on both p and n

(*d*) Positive

3. The solution set of the inequality $\log_{\sin\left(\frac{\pi}{3}\right)}(x^2-3x+2) \geq 2$ is

(*a*) $\left(\frac{1}{2},2\right)$ (*b*) $\left(1,\frac{5}{2}\right)$

(*c*) $\left[\frac{1}{2},1\right)\cup\left(2,\frac{5}{2}\right]$ (*d*) $(1, 2)$

4. If $\log_{0.5}\log_5(x^2-4) > \log_{0.5} 1$, then '$x$' lies in the interval

(*a*) $(-3,-\sqrt{5})\cup(\sqrt{5},3)$ (*b*) $(-3,-\sqrt{5})\cup(\sqrt{5},2)$

(*c*) $(\sqrt{5},3\sqrt{5})$ (*d*) ϕ

5. The number of positive integers not satisfying the inequality $\log_2(4^x - 2.2^x + 17) > 5$.

(*a*) 2 (*b*) 3

(*c*) 4 (*d*) 1

6. If a, b, c are positive real numbers such that

$a^{\log_3 7}=27; b^{\log_7 11}=49$ and $c^{\log_{11} 25}=\sqrt{11}.$

The value of $\left(a^{(\log_3 7)^2}+b^{(\log_7 11)^2}+c^{(\log_{11} 25)^2}\right)$ equals

(*a*) 489 (*b*) 469

(*c*) 464 (*d*) 400

7. If $\sqrt{\log_4\{\log_3\{\log_2(x^2-2x+a)\}\}}$ is defined $\forall\, x \in R$, then the set of values of 'a' is

(*a*) $[9, \infty)$ (*b*) $[10, \infty)$

(*c*) $[15, \infty)$ (*d*) $[2, \infty)$

8. Find the value of x that satisfies the equation

$$\log\left(\frac{x^{1/x}}{x^{1/(x+1)}}\right)=\frac{1}{5050}$$

(*a*) 1 (*b*) 10 (*c*) 100 (*d*) 1000

9. Number of positive solution which satisfy the equation $\log_2 x \cdot \log_4 x \ \log_6 x = \log_2 x \cdot \log_4 x + \log_2 x \cdot \log_6 x + \log_4 x \cdot \log_6 x$?

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) infinite

10. Sum of all the real solutions of the inequality

$\dfrac{(x^2+2)(\sqrt{x^2-16})}{(x^4+2)(x^2-9)} \leq 0$ is

(*a*) 5 (*b*) 4 (*c*) 8 (*d*) 0

11. Number of integers, which satisfy the inequality, $\dfrac{(16)^{1/x}}{(2^{x+3})}>1$ is equal to:

(*a*) Infinite (*b*) Zero

(*c*) 1 (*d*) 4

12. The set of values of x satisfying simultaneously the inequalities $\dfrac{\sqrt{(x-8)(2-x)}}{\log_{0.3}\left(\frac{10}{7}(\log_2 5-1)\right)} \geq 0$ and $2^{x-3}-31>0$ is:

(*a*) a unit set

(*b*) an empty set

(*c*) an infinite set

(*d*) a set consisting of exactly two elements.

13. The solution set of inequality $\dfrac{(3^x-4^x)\cdot\ln(x+2)}{x^2-3x-4} \leq 0$ is

(*a*) $(-\infty, 0] \cup (4, \infty)$ (*b*) $(-2, 0] \cup (4, \infty)$

(*c*) $(-1, 0] \cup (4, \infty)$ (*d*) $(-2, -1] \cup (-1, 0) \cup (4, \infty)$

14. Number of values of x in the interval $(0, 5)$ satisfying the equation $\dfrac{\ln(\sqrt{\sqrt{x^2+1}+x})+\ln(\sqrt{\sqrt{x^2+1}-x})}{\ln x}=x,$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 0

15. The value of x satisfying the equation $(\sqrt{5})^{\log_5 5^{\log_5 5^{\log_5(x)}}}=2$ is

(*a*) 1 (*b*) 2 (*c*) 4 (*d*) 8

16. If $A=\log_{\sqrt{5}}\left(\left(5^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}\cdot^{\cdot^{\cdot} 10 \text{ times}}}$, then value of $\log_{\sqrt{3}}(1024A + 1)$, is equal to

(*a*) 1 (*b*) 3 (*c*) 5 (*d*) 2

17. If $|x-3|^2 - 6\,|x-3| - 23 < 4$, then

(*a*) $x \in (-12, 6)$

(*b*) 17 integers satisfy the inequality

(*c*) 11 non-negative integers satisfy the inequality

(*d*) 6 negative integers satisfy the inequality

18. The roots of the equations $|x| = 49^{\left(\frac{1}{2}+\log_{\frac{1}{7}} 27+\log_{343} 81\right)}$ include

(*a*) One positive number greater than 1 only

(*b*) Two real number

(*c*) Two irrational number

(*d*) One negative rational number

19. If x_1, x_2 are the solution of the inequality $(x-4)^2 + (x-6)^4 = 34$, then

(*a*) $x_1 + x_2 = 10$

(*b*) $x_1 x_2 = 21$

(*c*) $|x_1 - x_2| = 2\sqrt{2}$

(*d*) Both x_1, and x_2 are irrational

20. Which of the following is true?

(*a*) $(\log_{10}2)^2 + 1 > \log_{10}4$

(*b*) $\log_{10}90 > \log_5 50$

(*c*) $\log_4 \log_3 \log_2 16 > \log_{16} 4$

(*d*) $2(\log_{10} 3)^2 - 3(\log_{10} 2)^2 > (\log_{10} 2) \times (\log_{10} 3)$

21. Indicate all correct alternatives, where base of the log is 2.

The equation $x^{(3/4)(\log_2 x)^2 + \log_2 x - (5/4)} = \sqrt{2}$ has:

(*a*) At least one real solution

(*b*) Exactly three real solutions

(*c*) Exactly one irrational solution

(*d*) Imaginary roots

22. The solution set of the system of equations $\log_3 x + \log_3 y = 2 + \log_3 2$ and $\log_{27}(x+y) = \frac{2}{3}$ is

(*a*) (6, 3) (*b*) (3, 6)

(*c*) (6, 12) (*d*) (12, 6)

23. The equation $x^{\left[(\log_3 x)^2 - \frac{9}{2}\log_3 x + 5\right]} = 3\sqrt{3}$ has

(*a*) Exactly three real solution

(*b*) At least one real solution

(*c*) Exactly one irrational solution

(*d*) Complex roots

24. Solution set of the inequality

$$(\log_2 x)^4 - \left(\log_{1/2} \frac{x^3}{8}\right)^2 + 9\log_2\left(\frac{32}{x^2}\right) < 4\left(\log_{1/2} x\right)^2 \text{ is}$$

$(a, b) \cup (c, d)$ then the correct statement is

(*a*) $a = 2b$ and $d = 2c$

(*b*) $b = 2a$ and $d = 2c$

(*c*) $\log_c d = \log_b a$

(*d*) there are 4 integers in (c, d)

COMPREHENSION BASED QUESTIONS

Comprehension (Q. No. 25 to 27): Let α and β are the solutions of the equation $\left(\sqrt{x}\right)^{\log_5 x - 1} = 5$ where $\alpha \in I$ and $\beta \in Q$ Then

[**Use:** $\log_{10}2 = 0.3010$, $\log_{10}3 = 0.4771$]

25. The number of significant digits before decimal in $(\alpha)^{10}$ is

(*a*) 13 (*b*) 14

(*c*) 15 (*d*) None of these

26. Number of zeroes after decimal before a significant digit in $(\beta)^{10}$ is

(*a*) 5 (*b*) 7 (*c*) 8 (*d*) 6

27. The value of $(\beta)^{\log_{25} 9}$ is

(*a*) $\frac{1}{3}$ (*b*) 5

(*c*) $\frac{1}{5}$ (*d*) 9

28. Match the Column:

	Column-I		Column-II
A.	The value(s) of x, which does not satisfy the equation $\log_2^2 (x^2 - x) - 4 \log_2(x-1) \log_2 x = 1$, is (are)	p.	2
B.	The value of x satisfying the equation $2^{\log_2 e^{\ln 5^{\log_5 7^{\log_7 10^{\log_{10}(8x-3)}}}}} = 13$, is	q.	3
C.	The number $N = \left(\frac{1}{\log_2 \pi} + \frac{1}{\log_6 \pi}\right)$ is less than	r.	4
D.	Let $l = (\log_3 4 + \log_2 9)^2 - (\log_3 4 - \log_2 9)^2$ and $m = (0.8)\left(1+9^{\log_3 8}\right)^{\log_{65} 5}$ then $(l + m)$ is divisible by	s.	5
		t.	6

(*a*) A → r, t, s; B → q; C → r, s, q; D → q, r

(*b*) A → q, r, s, t; B → p; C → q, r; D → r, s

(*c*) A → q, r, s, t; B → p; C → q, r, s, t; D → q, r, s

(*d*) A → t; B → s; C → q, t; D → r, s

29. Match the columns:

	Column-I		Column-II
A.	If $p = \frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$ then $\log_{(5+2\sqrt{6})} p$ is	p.	0
B.	If $r = \frac{3\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ then $\log_{9+2\sqrt{15}}(1/r)$ is	q.	2
C.	If $t = \frac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$ then $\log_{\sqrt{3}} t^2$ is	r.	–1
D.	If $k = \frac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}} - \frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}} + \frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$ then $\log_e (k+1)$ is	s.	1

(*a*) A → t, B → s, C → r, D → q

(*b*) A → s, B → r, C → q, D → p

(*c*) A → r, B → p; C → r, D → s

(*d*) A → t, B → q, C → s, D → r

30. Let $f(x)=\dfrac{x^2-6x+5}{x^2-5x+6}$

Column I		Column II	
A.	If $-1<x<1$, then $f(x)$ satisfies	p.	$0<f(x)<1$
B.	If $1<x<2$, then $f(x)$ satisfies	q.	$f(x)<0$
C.	If $3<x<5$, then $f(x)$ satisfies	r.	$f(x)>0$
D.	If $x>5$, then $f(x)$ satisfies	s.	$f(x)<1$

(*a*) A → p, B → q, C → q, D → s
(*b*) A → q, B → s, C → r, D → q
(*c*) A → s, B → r; C → q, D → s
(*d*) A → p, B → q, C → s, D → r

SUBJECTIVE BASED QUESTIONS

31. Find the number of integral solution of the equation $\log_{\sqrt{x}}(x+|x-2|)=\log_x(5x-6+5|x-2|)$.

32. If a, b are co-prime numbers and satisfying

$$\left(2+\sqrt{3}\right)^{\frac{1}{\log_a\left(2-\sqrt{3}\right)}+\frac{1}{\log_b\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)}}=\frac{1}{12},$$ then $(a+b)$ can be is equal to

33. If p is the smallest value of x satisfying the equation $2^x+\dfrac{15}{2^x}=8$ then the value of 4^p is equal to

34. Positive numbers x, y and z satisfy $xyz=10^{81}$ and $(\log_{10}x)(\log_{10}yz)+(\log_{10}y)(\log_{10}z)=468$.

Find the value of

$$\left(\log_{10}x\right)^2+\left(\log_{10}y\right)^2+\left(\log_{10}z\right)^2$$

35. If (x_1, y_1) and (x_2, y_2) are the solution of the system of equation

$\log_{225}(x)+\log_{64}(y)=4$

$\log_x(225)-\log_y(64)=1$,

then find the value of $\log_{30}(x_1y_1x_2y_2)$.

36. Number of ordered pairs (α, β), $\alpha,\beta \in N$ such that $\alpha^2=4002+\beta^2$ is

37. The sum of all integral values of x satisfying the equation $2\log_8(2x)+\log_8(x^2-2x+1)=\dfrac{4}{3}$ is.

38. Suppose n be an integer greater than 1. Let $a_n=\dfrac{1}{\log_n 2002}$.

Suppose $b=a2+a3+a4+a5$ and $c=a10+a11+a12+a13+a14$. Then find the value of $(b-c)$.

39. If $\log_b a.\log_c a+\log_a b.\log_c b+\log_a c.\log_b c=3$ (where a, b, c are different positive real numbers $\neq 1$), then find the value of $a\,b\,c$.

40. If the product of alll solutions of the equation $\dfrac{(2009)x}{2010}=(2009)^{\log_x(2010)}$ can be expressed in the lowest form as $\dfrac{m}{n}$ then the value of $(m-n)$ is

41. If the complete solution set of the inequality $(\log_{10}x)^2\geq\log_{10}x+2$ is $(0, a]\cup[100,\infty)$ then find the value of a.

42. The complete solution set of the inequality

$$\frac{1}{\log_4\dfrac{x+1}{x+2}}<\frac{1}{\log_4(x+3)},$$ is $(-a, \infty)$, then determine 'a'.

43. If complete solution set of inequality $\log_{1/2}(x+5)^2>\log_{1/2}(3x-1)^2$ is $(-\infty, p)\cup(q, r)\cup(s,\infty)$ then find $\dfrac{p^2+q^2+r^2}{s^2}$

44. Find the real solutions to the system of equations

$\log_{10}(2000xy)-\log_{10}x\cdot\log_{10}y=4$

$\log_{10}(2yz)-\log_{10}y\cdot\log_{10}z=1$

and $\log_{10}(zx)-\log_{10}z\cdot\log_{10}x=0$

45. Solve the equation $x^{0.5\log_{\sqrt{x}}(x^2-x)}=3^{\log_9 4}$.

46. Solve for x: $\log^2(4-x)+\log(4-x)\cdot\log\left(x+\dfrac{1}{2}\right)-2\log^2\left(x+\dfrac{1}{2}\right)=0$.

47. Solve for x

(*a*) $\log_{2x+3}(x^2)<\log_{2x+3}(2x+3)$

(*b*) $\log_{x+3}(x^2-x)<1$

48. Solve for x $\log_{0.2}(x^2-x-2)>\log_{0.2}(-x^2+2x+3)$

49. Solve for x $(0.3)^{\log_{1/3}\log_2\frac{3x+6}{x^2+2}}>1$

50. Solve for x $\log_{0.5}\left(\log_6\dfrac{x^2+x}{x+4}\right)<0$

51. Solve for x $\log_3\dfrac{|x^2-4x|+3}{x^2+|x-5|}\geq 0$

52. If x and y satisfying both the equations

$\log_3 x+\log_2 y=2$;

$3^x-2^y=23$

then sum of all the values of x and y is

53. For the equation

$(0.4)^{\log^2 x+1}=(6.25)^{2-p\log x}$

(base 10)

If $p=2$, number of real roots m,

If $p=3$, number of real roots n,

Then $m+n=$

54. Solve for x:

(*i*) $\log_x(2x-1)>0$

(*ii*) $\log_{(1-x)}(2x-1)<0$

(*iii*) $\log(\log x)+\log(\log x^3-2)=0$

55. Solve for x:

(i) $\log_x(2) \cdot \log_{2x} 2 = \log_{4x} 2$

(ii) $5^{\log_a x} + 5x^{\log_a 5} = 3 (a > 0)$

(iii) $x^{(\log x) + 4} = 32$

(iv) $\log_{x+1}(x^2 + x - 6)^2 = 4$

(v) $|x-1|^{(\log x)^2 - \log x^2} = (x-1)^3$

(vi) $x + \log_{10}(1 + 2^x) = x \cdot \log_{10} 5 + \log_{10} 6$

56. Prove that $\log_a N \cdot \log_b N + \log_b N \cdot \log_c N + \log_c N \cdot \log_a N = \dfrac{\log_a N \cdot \log_b N \cdot \log_c N}{\log_{abc} N}$

57. Find the product of the positive roots of the equation $\sqrt{2005}(x)^{\log_{2005} x} = x^2$.

58. If $a = \log_{12} 18$ & $b = \log_{24} 54$ then find the value of $ab + 5(a - b)$.

59. If $x = 1 + \log_a bc, y = 1 + \log_b ca, z = 1 + \log_c(ab)$ then prove that $xyz = xy + yz + zx$.

60. Find the number of real solution of $(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3$

61. Solve: $\log_3(\sqrt{x} + |\sqrt{x} - 1|) = \log_9(4\sqrt{x} - 3 + 4|\sqrt{x} - 1|)$

62. Prove that: $2^{\left(\sqrt{\log_a \sqrt[4]{ab} + \log_b \sqrt[4]{ab}} - \sqrt{\log_a \sqrt[4]{\frac{b}{a}} + \log_b \sqrt[4]{\frac{a}{b}}}\right) \cdot \sqrt{\log_a b}}$

$$= \begin{cases} 2 & \text{if } \; b \geq a > 1 \\ 2^{\log_a b} & \text{if } \; 1 < b < a \end{cases}$$

63. Find all the solutions of the equation $|x-1|^{(\log x)^2 - \log x^2} = |x-1|^3$, where base of logarithm is 10.

64. Solve the following equations for x & y: $\log_{100}|x + y| = \dfrac{1}{2}$

$\log_{10} y - \log_{10}|x| = \log_{100} 4$.

65. Simplify: $\log_{1/3} \sqrt[4]{729 \cdot \sqrt[3]{9^{-1} \cdot 27^{-4/3}}}$

66. Solve for x $-1 \leq \dfrac{1 + x^2}{2x} \leq 1$

67. Solve for x $\dfrac{(x - \sin 1)(x - \sin 2)}{(x - \sin 3)(x - \sin 4)} \leq 0$

68. Solve for x $\dfrac{(|x| - 1)(|x| - 3)}{(|x| - 4)(|x| - 7)} \geq 0$

69. Solve for x $\sqrt{(x-1)}(x-2)x \leq 0$

70. Solve for x $\sqrt{x-5} - \sqrt{9-x} > 1 : x \in Z$

71. Solve for x $\sqrt{x-1} > \sqrt{3-x}$

72. If $\dfrac{(x-3)^{\frac{-|x|}{x}} \sqrt{(x-4)^2}(\pi - x)}{\sqrt{-x}(-x^2 + x - 1)(|x| - 9)} < 0$ then no. of integers x satisfying the inequality is

Exercise-4 (Past Year Questions)

JEE MAIN

1. The sum of the solutions of the equation $\left|\sqrt{x} - 2\right| + \sqrt{x}\left(\sqrt{x} - 4\right) + 2 = 0, (x > 0)$ is equal to: **(2019)**

(a) 4 (b) 9
(c) 10 (d) 12

2. The number of real roots of the equation $5 + |2^x - 1| = 2^x(2^x - 2)$ is **(2019)**

(a) 3 (b) 2
(c) 4 (d) 1

3. The sum of the solutions of the equation **(2019)**
$|\sqrt{x} - 2| + \sqrt{x}(\sqrt{x} - 4) + 2 = 0, (x > 0)$ is equal to

(a) 9 (b) 12
(c) 4 (d) 10

4. Solution set of $3^x(3^x - 1) + 2 = |3^x - 1| + |3^x - 2|$ contains **(2020)**

(a) Singleton set
(b) Two elements
(c) At least four elements
(d) Infinite elements

5. The product of the roots of the equation $9x^2 - 18|x| + 5 = 0$, is : **(2020)**

(a) $\dfrac{25}{9}$ (b) $\dfrac{25}{81}$
(c) $\dfrac{5}{9}$ (d) $\dfrac{5}{27}$

6. The number of solutions of the equation $\log_4(x - 1) = \log_2(x - 3)$ is **(2021)**

7. The number of the real roots of the equation $(x + 1)^2 + |x - 5| = \dfrac{27}{4}$ is **(2021)**

8. The value of $3 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{3 + \ldots\infty}}}}$ is equal to **(2021)**

(a) $1.5+\sqrt{3}$ (b) $2+\sqrt{3}$

(c) $3+2\sqrt{3}$ (d) $4+\sqrt{3}$

9. The number of elements in the set $\{x \in \mathbb{R} : (|x| - 3)\,|x + 4| = 6\}$ is equal to **(2021)**

(a) 3 (b) 2

(c) 4 (d) 1

10. If α, β are the roots of the equation

$$x^2 - \left(5+3^{\sqrt{\log_3 5}} - 5^{\sqrt{\log_5 3}}\right)x + 3\left(3^{(\log_3 5)^{\frac{1}{3}}} - 5^{(\log_5 3)^{\frac{2}{3}}} - 1\right) = 0$$

then the equation, whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$, **(2022)**

(a) $3x^2 - 20x - 12 = 0$ (b) $3x^2 - 10x - 4 = 0$

(c) $3x^2 - 10x + 2 = 0$ (d) $3x^2 - 20x + 16 = 0$

11. Let $S = \left\{x \in [-6, 3] - \{-2, 2\} : \frac{|x+3|-1}{|x|-2} \geq 0\right\}$ and $T = \{x \in Z : x^2 - 7|x| + 9 \leq 0\}$. Then the number of elements is $S \cap T$ is **(2022)**

(a) 7 (b) 5

(c) 4 (d) 3

12. Let p and q be two real numbers such that $p + q = 3$ and $p^4 + q^4 = 369$. Then $\left(\frac{1}{p}+\frac{1}{q}\right)^{-2}$ is equal to _________. **(2022)**

JEE ADVANCED

13. Let (x_0, y_0) be the solution of the following equations

$(2x)^{\ln 2} = (3y)^{\ln 3}$

$3^{\ln x} = 2^{\ln y}$

Then x_0 is **(2011)**

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) 6

14. The value of

$$6+\log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\cdots}}}\right)$$ is

(2012)

15. If $3^x = 4^{x-1}$, then $x =$ **(2013)**

(a) $\frac{2\log_3 2}{2\log_3 2 - 1}$ (b) $\frac{2}{2-\log_2 3}$

(c) $\frac{1}{1-\log_4 3}$ (d) $\frac{2\log_2 3}{2\log_2 3 - 1}$

16. The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$ is ______ **(2018)**

ANSWER KEY

CONCEPT APPLICATION

1. (*a*) **2.** (*b*) **3.** (*d*) **4.** (*c*) **5.** (*d*) **6.** (*i*) 14 (*ii*) 52 (*iii*) 194 **9.** [0] **10.** [–224]

11. $x^6 - y^6$ **12.** $p = 3/2, q = 1, r = 4/3$ **13.** (*d*) **14.** (*c*) **15.** (*b*) **16.** [60] **17.** [99] **18.** [4]

19. (*d*) **20.** [$2ab$] **21.** $x = \pm 4\sqrt{\frac{5}{4}}$ **22.** $x = \pm\sqrt{\frac{3}{5}}$ **23.** (*c*) **24.** $x \in [-\infty, -3] \cup [-2, 0] \cup [6, 0]$

25. $x \in (-6, 0] \cup [2, 3] \cup (6, \infty) \cup \{4\}$ **26.** [6] **27.** (*d*) **28.** $\left(\frac{1}{5}, \frac{2}{5}\right)$ **29.** (243) **30.** [–5] **31.** [18]

32. [–1] **33.** (*i*) $(1, \infty)$ (*ii*) $[1, \infty)$ (*iii*) $(0, 1)$ (*iv*) $(0, 1]$ (*v*) $(0, 1)$ (*vi*) $(0, 1]$ (*vii*) $(1, \infty)$ (*viii*) $[1, \infty)$ (*ix*) $(3, \infty)$ (*x*) $[3/2, \infty)$

34. (*i*) $x \in \left(\frac{3}{2}, \frac{19}{2}\right)$ (*ii*) $x \in \left(\frac{2}{3}, \frac{17}{24}\right]$ (*iii*) $x \in (4^{16}, \infty)$ (*iv*) $x \in \left(0, \frac{1}{2}\right)$ **35.** $\{3, -3\}$ **36.** $\{-2, 2\}$ **38.** $\frac{12-4a}{3+a}$

39. [44] **40.** (*d*) **41.** (*b*, *c*) **42.** (*i*) $\{-1, 5\}$ (*ii*) $\{6, -6, 7, -1\}$ (*iii*) $(14, -4, 0, 10, 2, 8)$ **43.** $(-6, 8)$

44. (*d*) **45.** $(0, 2) \cup (4, \infty)$ **46.** $x \geq 1$ **47.** (*i*) $x \in (-\infty, 1) \cup (1, \infty)$ (*ii*) $x \in (-\infty, -5] \cup [5, \infty)$ (*iii*) $x \in (-7, 7)$ (*iv*) $x \in [-10, 10]$ (*v*) $x \in R$ (*vi*) $x \in \phi$ (*vii*) $x \in R$ (*viii*) $x \in R$ (*ix*) $x \in \phi$

48. (*i*) $x \in (-\infty, 0) \cup (2, \infty)$ (*ii*) $1 < x < 3$ (*iii*) $x \in (-2, -1) \cup (0, 1)$ (*iv*) $x \in (-\infty, 0) \cup (1, \infty)$ (*v*) $-4/3 \leq x \leq 2$ (*vi*) $x \in \phi$

49. (*i*) $-1 \leq x \leq 5$ (*ii*) $x \in (-\infty, -3] \cup [9, \infty) \cup [-1, 7]$ (*iii*) $x \in [-4, 0] \cup [2, 8] \cup [10, 14]$

EXERCISE-1 (TOPICWISE)

1. (*b*)	**2.** (*b*)	**3.** (*d*)	**4.** (*a*)	**5.** (*c*)	**6.** (*a*)	**7.** (*d*)	**8.** (*b*)	**9.** (*b*)	**10.** (*a*)
11. (*b*)	**12.** (*d*)	**13.** (*d*)	**14.** (*c*)	**15.** (*d*)	**16.** (*c*)	**17.** (*b*)	**18.** (*d*)	**19.** (*c*)	**20.** (*c*)
21. (*d*)	**22.** (*b*)	**23.** (*c*)	**24.** (*c*)	**25.** (*d*)	**26.** (*b*)	**27.** (*a*)	**28.** (*b*)	**29.** (*c*)	**30.** (*a*)
31. (*b*)	**32.** (*d*)	**33.** (*b*)	**34.** (*a*)	**35.** (*d*)	**36.** (*c*)	**37.** (*b*)	**38.** (*d*)	**39.** (*b*)	**40.** (*d*)

EXERCISE-2 (LEARNING PLUS)

1. (*c*)	**2.** (*a*)	**3.** (*a*)	**4.** (*a*)	**5.** (*b*)	**6.** (*c*)	**7.** (*a*)	**8.** (*b*)	**9.** (*d*)	**10.** (*c*)
11. (*d*)	**12.** (*b*)	**13.** (*c*)	**14.** (*a*)	**15.** (*b*)	**16.** (*d*)	**17.** (*b*)	**18.** (*a*)	**19.** (*a*)	**20.** (*d*)
21. (*b*)	**22.** (*b*)	**23.** (*c*)	**24.** (*d*)	**25.** (*c*)	**26.** (*b*)	**27.** (*c*)	**28.** (*d*)	**29.** (*a*)	**30.** (*c*)
31. (*c*)	**32.** (*a*)	**33.** (*a*)	**34.** (*d*)	**35.** (*b*)	**36.** (*d*)	**37.** (*a*)	**38.** (*a*)	**39.** (*b*)	**40.** (*d*)
41. (*b*)	**42.** (*d*)	**43.** (*d*)	**44.** (*a*)	**45.** (*b*)	**46.** (*c*)	**47.** (e^{-3})			

48. $x = 64, y = 16, z = 9$ **49.** 75 **50.** (*i*) $x \geq 3$, (*ii*) $|x| > 1$ (*iii*) $x \in (-2, 1)$ *u* $(2, \infty)$ **51.** $\left[\frac{1}{6}\right]$ **52.** (ϕ)

53. (*i*) $\rightarrow$ 1 (*ii*) $\rightarrow$ 6 (*iii*) $\rightarrow$ 20 **54.** $\left[-2, \frac{3}{2}\right)$ **55.** $\left(-\infty, \frac{1}{2}\right)$ **56.** $\left(-\infty, \frac{1}{2}\right)$ **57.** [2]

58. (*i*) $x \in \left(1, 3^{\frac{1}{8}}\right)$, (*ii*) $x \in (1, 3^{1/8})$, (*iii*) $x \in (2, 9)$, (*iv*) $x \in \left(0, \frac{1}{4}\right] \cup [2, \infty)$

59. (*i*) $x \in \{5\}$, (*ii*) $x \in (0, \infty)$, (*iii*) $x \in (1, 2)$ **60.** [1] **61.** [±10] **62.** (*i*) [3], (*ii*) [$2\sqrt{5}$], (*iii*) [7], (*iv*) [18]

63. [$4w$] **64.** (20 : 5 : 4) **66.** [5] **67.** [0] **68.** $\left[-\frac{1}{3}\right]$ **69.** $x \in (3, 4) \cup (4, \infty)$ **70.** [9] **71.** [4]

73. [10] **75.** $\log_b N$ **76.** $x \in \left(\frac{\sqrt{5}-1}{2}, 1\right)$ **77.** $x \in (2, \infty)$ **78.** $x \in (-2, -1) \cup (1, 2)$

79. (*i*) $x \in \left(\frac{1}{5}, \frac{1}{4}\right) \cup \left[\frac{3}{4}, \frac{4}{5}\right]$ (*ii*) $x \in \left[\frac{2}{3}, 2\right]$ (*iii*) $x \in \left[\frac{2}{15}, \frac{2}{5}\right] \cup \left[\frac{6}{5}, \frac{22}{15}\right]$ (*iv*) $x \in (-3, -2) \cup (2, 3)$

80. (*i*) $x \in \{-11, -5, -1\}$ (*ii*) $x \in \{-8, -6, -2, -4, 0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$

81. (*i*) $x \in [1, 13]$ (*ii*) $x \in [-8, -6] \cup [-4, -2] \cup [0, 2] \cup [4, 6] \cup [8, 10] \cup [12, 14] \cup [16, 18]$

82. (*i*) $x \in (-2, -1] \cup [1, 2)$ (*ii*) $x \in (-\infty, -3] \cup (-2, -1] \cup [3, \infty) \cup [1. 2)$ (*iii*) $x \in (-\infty, -3] \cup [3, \infty)$
(*iv*) $x \in (-\infty, -4] \cup [-1, 5] \cup [6, \infty)$ (*v*) $x \in (-\infty, -4] \cup [6, \infty] \cup [6, \infty) \cup \{2\}$

83. (*i*) $x \in [1, \infty]$ (*ii*) $x \in [0, \infty] \cup \{-1\}$ (*iii*) $x \in (-1, 0) \cup (0, 3)$ (*iv*) $x \in \phi$ (*v*) $x \in (2, 6)$

84. $[-6, \infty)$ **85.** $x \in (-2, 2)$

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (*d*) **2.** (*a*) **3.** (*c*) **4.** (*a*) **5.** (*a*) **6.** (*b*) **7.** (*b*) **8.** (*c*) **9.** (*c*) **10.** (*d*)
11. (*a*) **12.** (*a*) **13.** (*d*) **14.** (*a*) **15.** (*c*) **16.** (*d*) **17.** (*b, c*) **18.** (*b, c*) **19.** (*a, c, d*)
20. (*a, d*) **21.** (*a, b, c*) **22.** (*a, b*) **23.** (*a, b, c, d*) **24.** (*b, c*) **25.** (*b*) **26.** (*d*) **27.** (*a*) **28.** (*c*)
29. (*b*) **30.** (*a*) **31.** [1] **32.** [2] **33.** [7] **34.** [3] **35.** [9] **36.** [5625] **37.** [12] **38.** [–1]
39. [1] **40.** [1] **41.** [0.1] **42.** [1] **43.** [17/3] **44.** [$x = 1; y = 5; z = 1$; or $x = 100; y = 20; z = 100$]
45. [2] **46.** [7/4] **47.** (*a*) $\left[x \in \left(\frac{-3}{2}, 3\right) - \{-1, 0\}\right]$ (*b*) [$x \in (-3, -2) \cup (-1, 0) \cup (1, 3)$] **48.** [$x \in (2, 5/2)$]
49. (–1/2, 2) **50.** [$(-4, -3) \cup (8, \infty)$] **51.** $(-\infty, 2/5]$ **52.** [5]
54. (*i*) [$(1/2, \infty) - \{1\}$] (*ii*) [(1/2, 1)] (*iii*) [10] **55.** (*i*) [2] (*ii*) [3/2] (*iii*) [2.22] (*iv*) [2.022] (*v*) [1]
57. [$(2008)^2$] **58.** [1] **61.** [$x \in [0, 1] \cup \{4\}$] **63.** [1000 or 1/10] **64.** $\left[x \in \left(\frac{10}{3}, \frac{20}{3}\right);\ y \in (-10, 20)\right]$
65. [–1] **66.** [1, –1] **67.** [$x \in (\sin 1, \sin 3)$] **68.** $x \in (-\infty, -7] \cup (-4, -3] \cup [-1, 1] \cup [3, 4] \cup [7, \infty]$
69. $x \in (1, 2)$ **70.** $x \in (8.323, 9)$ **71.** $x \in (2, 3)$ **72.** [31]

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (*c*) **2.** (*d*) **3.** (*d*) **4.** (*a*) **5.** (*b*) **6.** [1] **7.** [2] **8.** (*a*) **9.** (*b*) **10.** (*b*)
11. (*d*) **12.** (4)

JEE Advanced

13. (*a*) **14.** (*d*) **15.** (*a, b, c*) **16.** [8]

CHAPTER

2 Set Theory

INTRODUCTION

Fundamental concepts of set theory is a rich and beautiful. It permeates virtually every branch of mathematics. Yet, most mathematics students receive only a cursory overview of the theory of sets.

A set is a collection of objects. The objects in such a collection are called the elements of the set. We write $a \in A$ to assert that a is an element, or a member, of the set A. We write a $a \notin A$ when a is not an element of the set A. A set is merely the result of collecting objects of interest and it is identified by enclosing its elements with braces (curly brackets). For example the collection $A = \{3, 7, 11, \pi\}$ is a set that contains the four elements 3, 7, 11, π. So $7 \in A$, and $8 \notin A$.

DEFINITION

A set is a well defined collection of distinct objects.

		
Set of Fruits in a Basket	Set of Coin in a Bank	Set of Books in a Library

Some examples of Set or not a Set

1.	Consonant in English Alphabet	Set
2.	Difficult topics in Mathematics	Not a Set
3.	Collection of past Presidents of India	Set
4.	Group of Intelligent Students in JEE Batch	Not a Set

Note: Generally, If you can see an adjective like good, difficult intelligent, brave in a sentence then it does not describe a set.

Sets are usually denoted by capital letters $A, B, \ldots, X, Y, Z$. Collection of objects or things in a set called as elements

The elements of the set are denoted by small letters.

REPRESENTATION OF SET

There are two methods for representing a set.

(*i*) **Tabulation method or Roster form**

All the elements belonging to the set are written in curly brackets and separated by commas

If A is the set of days of a week, then

A = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

(*ii*) **Set Builder Method or Set rule method**

In this method, we use the definition, which is satisfied by all the elements of set.

In above example set A may be written as

$A = \{x : x \text{ is a day of week}\}$

NOTATIONS OF SET OF DIFFERENT NUMBERS

(*i*) Set of all natural numbers $N = \{1, 2, 3, \ldots\}$

(*ii*) Set of all integers Z or $I = \{0, \pm1, \pm2, \ldots\}$

(*iii*) Set of non zero integers Z_0 or $I_0 = \{\pm1, \pm2, \pm3, \ldots\}$

(*iv*) Set of all rational numbers

$Q = \{x : x = p/q$, where p and q relatively prime integers and $q \neq 0\}$

Q_0 denotes the set of all non–zero rational numbers.

(*v*) Set of real numbers is denoted by R

(*vi*) Set of complex numbers is denoted by C

TYPES OF SETS

(*i*) **Null set or Void set or Empty set:**

A set having no element is called as null set or empty set or void set. It is denoted by ϕ or { }. The null set is unique and is the subset of every set.

Examples

Set of even prime numbers less than 2.

Set of natural numbers strictly lying between 5 and 6.

$A = \{x : x \in N, 5 < x < 6\} = \phi$

(*ii*) **Finite Set**

A set having contain either no elements or countable elements

Example

Set of all natural numbers less than 6 = {1, 2, 3, 4, 5}

(*iii*) **Disjoint sets**

Two sets A and B are said to be disjoint, if $A \cap B = \phi$. If $A \cap B \neq \phi$, then A and B are said to be intersecting or overlapping sets.

(*iv*) **Singleton Set**

A set having one and only one element is called singleton set or unit set.

Example

Set of all positive integral roots of the equation

$x^2 - 2x - 15 = 0$.

$\Rightarrow (x + 3)(x - 5) = 0$

$\Rightarrow x = -3$ or $x = 5$ i.e., only one positive integral root.

$C = \{x \Rightarrow R : x - 5 = 0\}$

$\Rightarrow x = 5 \Rightarrow C = \{5\}$ i.e., only one element.

Train Your Brain

Example 1: Solve $3x^2 - 12x = 0$.

When

(*i*) $x \in N$

(*ii*) $x \in Z$

(*iii*) $x \in S$ where $S = \{a + ib : b \neq 0;\ a, b \in R\}$

Sol. $3x^2 - 12x = 0 \Leftrightarrow 3x(x - 4) = 0 \Leftrightarrow x = 0$ or $x = 4$

(*i*) when $x \in N \Rightarrow x = 4$

(*ii*) when $x \in Z \Rightarrow x = 0$ or $x = 4$

(*iii*) when $x \in S \Rightarrow$ No solution

Example 2: If $Q = \left\{x : x = \dfrac{1}{y}, \text{where } y \in N\right\}$, then

(*a*) $0 \in Q$ (*b*) $1 \in Q$

(*c*) $2 \in Q$ (*d*) $\dfrac{2}{3} \in Q$

Sol. (*b*) Since $\dfrac{1}{y} \neq 0, \dfrac{1}{y} \neq 2, \dfrac{1}{y} \neq \dfrac{-2}{3}$, $[\because y \in N]$

$\therefore \dfrac{1}{y}$ can be 1, $[\because$ y can be 1].

Concept Application

1. A = $\{x : x \neq x\}$ represents

(*a*) {0} (*b*) { }

(*c*) {1} (*d*) $\{x\}$

2. The set $\{x : x$ is a positive integer less than 6 and $3^x - 1$ is an even number$\}$ in roster form is

(*a*) {1, 2, 3, 4, 5} (*b*) {1, 2, 3, 4, 5, 6}

(*c*) {2, 4, 6} (*d*) {1, 3, 5}

3. Which of the following sets is a finite set?

(*a*) $A = \{x ; x \in Z$ and $x^2 - 5x + 6 = 0\}$

(*b*) $B = \{x ; x \in Z$ and x^2 is even$\}$

(*c*) $D = \{x ; x \in Z$ and $x > -10\}$

(*d*) All of these

CARDINAL NUMBER

The no. of distinct elements in a set A is denoted by $n(A)$ and it is known as cardinal number of the set A.

- Two finite sets A and B are equivalent if their cardinal number are same.

  Example

  $A = \{x \in Z$ and $x^2 - 5x + 6 = 0\} \Rightarrow$ A = {2, 3}

  $\therefore n(A) = 2$

EQUAL AND EQUIVALENT SETS

- Two finite set A and B are said to be Equivalent sets if their cardinalities are same i.e., $n(A) = n(B)$.
- Two sets A and B are said to be Equal sets if not just their cardinalities are same, but also the members in both the sets are the same.
- If A and B are equal sets, we denote it by $A = B$.
- If A and B are unequal sets, we denote it by $A \neq B$.

SUBSET AND SUPERSET

If A and B are two sets such that every element of A is also an element of B, then A is a subset of B and B is superset of A. We write $A \subseteq B$.

Note:

- Every set is subset of itself i.e. $A \subseteq A$ for all A.
- Empty set ϕ is subset of every set
- If $A \subseteq B$ and $B \subseteq A$ then A and B said to be equal sets, i.e. $A = B$.
- If $A \subseteq B$ and $A \neq B$, there A is called as Proper subset of B and denoted by $A \subset B$.
- If a set A has n elements, then the number of subsets of $A = 2^n$.

POWER SET

- Power set of a set A is the collection of all subsets of A and is denoted by $P(A)$ or 2^n.
- Let A be a finite set containing m elements i.e., $n(A) = m$, then

- The number of elements in the power set of A, $n(P(A)) = 2^m$.
- The number of non-void/non-empty subsets of $A = (2^m) - 1$.
- The number of proper subsets of $A = 2^m - 1$.
- The number of non-void proper subsets of $A = 2^m - 2$.

  Example:

  The number of elements in the power set of set $A = \{1, 2\}$ is 2^2.

UNIVERSAL SET

The universal set is the superset for all the sets under the consideration.

The set of complex numbers is the universal set for all possible sets related numbers.

Note:

- In a set, the order in which elements are written makes no difference.
- In a set, the repetition of elements has no etc.

Train Your Brain

Example 3: The number of non-empty subsets of the set $\{1, 2, 3, 4\}$ is

(*a*) 15 (*b*) 14 (*c*) 16 (*d*) 17

Sol. (*a*) The number of non- empty subsets

$= 2^n - 1 = 2^4 - 1 = 16 - 1 = 15.$

Example 4: Consider the following sets.

$A = 0$

$B = \{x : x > 15 \text{ and } x < 5\}$,

$C = \{x : x - 5 = 0\}$,

$D = \{x : x^2 = 25\}$,

$E = \{x : x$ is an integral positive root of the equation $x^2 - 2x - 15 = 0\}$

Choose the pair of equal sets

(*a*) A and B (*b*) C and D

(*c*) C and E (*d*) B and C

Sol. (*c*) Since, $0 \in A$ and 0 does not belong to any of the sets B, C, D and E, it follows that $A \neq B$, $A \neq C$, $A \neq D$, $A \neq E$. Since, $B = \phi$, but none of the other sets are empty. Therefore $B \neq C$, $B \neq D$ and $B \neq E$. Also, $C = \{5\}$ but $-5 \in D$, hence $C \neq D$.

Since, $E = \{5\}$, $C = E$. Further, $D = \{-5, 5\}$ and sets is C and E.

Example 5: If $A = \{x, y\}$ then the power set of A is

(*a*) $\{x^y, y^x\}$ (*b*) $\{\phi, x, y\}$

(*c*) $\{\phi, \{x\}, \{2y\}\}$ (*d*) $\{\phi, \{x\} \{y\}, \{x, y\}\}$

Sol. (*d*) The collection of all the subsets of the set A is called the power set of A. It is denoted by $P(A)$

Given $A = \{x, y\}$; $P(A) = \{\phi, \{x\}, \{y\}, \{x, y\}\}$

Concept Application

4. Consider the following sets.

I. $A = \{1, 2, 3\}$

II. $B = \{x \in R : x^2 - 2x + 1 = 0\}$

III. $C = \{1, 2, 2, 3\}$

IV. $D = \{x \in R : x^3 - 6x^2 + 11x - 6 = 0\}$

Which of the following are equal?

(*a*) $A = B = C$ (*b*) $A = C = D$

(*c*) $A = B = D$ (*d*) $B = C = D$

5. The number of the proper subset of $\{a, b, c\}$ is:

(*a*) 3 (*b*) 8 (*c*) 6 (*d*) 7

6. Given the sets

$A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$ and $C = \{0, 2, 4, 6, 8\}$. Which of the following may be considered as universal set for all the three sets A, B and C?

(*a*) $\{0, 1, 2, 3, 4, 5, 6\}$

(*b*) ϕ

(*c*) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(*d*) $\{1, 2, 3, 4, 5, 6, 7, 8\}$

7. The cardinality of the set $P\{P[P(\phi)]\}$ is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 4

INTERVALS AS SUBSETS OF R

Four type of subsets can be defined on R as given below.

Let $a, b \in R$, such that $a < b$

1. Open Interval

(a, b) or] a, b [$= \{x : a < x < b\}$

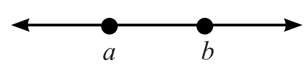
= Set of all real numbers between a and b, not including a and b both.

$]a, b[$ or (a, b)

2. Closed Interval

$[a, b] = \{x : a \leq x \leq b\}$

= Set of all real numbers between a and b as well as including a and b both.



$[a, b]$

3. Open-closed Interval (semi closed or semi open interval)

$(a, b]$ or $]a, b] = \{x : a < x \leq b\}$

= Set of all real numbers between a and b, a not included but b included.

$(a, b]$ or $]a, b]$

4. **Closed-open interval (semi closed or semi open interval)**

$[a, b)$ or $[a, b[\ = \{x : a \le x < b\}$

= Set of all real numbers between a and b including a but excluding b.

$[a, b)$ or $[a, b]$

Some More Representations on Number Line

Infinite open interval

$x > a$ 

$x < b$ 

Infinite close interval

$x \le b$ 

$x \ge a$ 

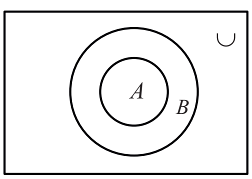
$(0, \infty) = R^+$

$(-\infty, 0) = R^-$

$(-\infty, \infty) = R$

VENN-EULER DIAGRAMS

Venn – diagram is a systematic representation of sets in pictorial form. A set is represented by circle inside the universal set which itself represented by rectangular region.



$A \subset B$

BASIC OPERATIONS

(*a*) **Union of sets**

If A and B are two sets, then the union of two sets is denoted by $A \cup B$ (read as "A union B") and defined as

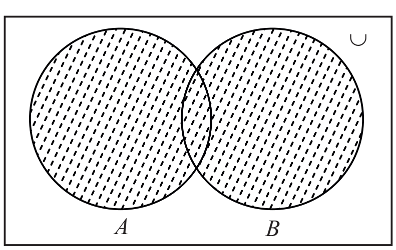
$A \cup B = \{x : x \in A \text{ or } x \in B\}$

Union is also known as join or "logical sum" of A and B.

Example:

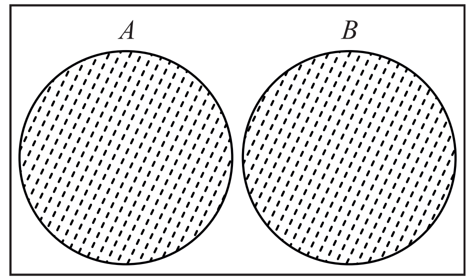
$A = \{1, 2, 3\}$, $B = \{1, 3, 5, 7\}$ then $A \cup B = \{1, 2, 3, 5, 7\}$

Case I: A and B are not equal sets but they have some elements in common.



$A \cup B$

Case II : A and B have no elements in common.



$A \cup B$

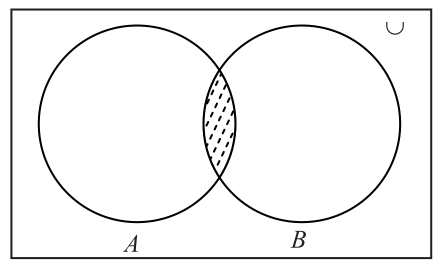
(*b*) **Intersection of sets**

The intersection of sets A and B is denoted by $A \cap B$ (read as "A intersection B") and defined as $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Example:

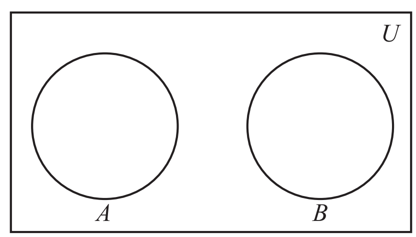
$A = \{1, 2, 3, 5, 6, 9\}$, $B = \{2, 3, 4, 5, 7, 8, 10\}$ then $A \cap B = \{2, 3, 5\}$

Case I: A and B are not equal sets but they have some elements in common.



$A \cap B$

Case II : A and B have no elements in common.



Note:

Disjoint sets

Two sets are said to be disjoint sets if they have no elements in common.

Example: $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{7, 8, 9, 10, 11\}$ then A and B are disjoint sets

(*c*) **Difference of sets**

The difference of A and B, i.e.

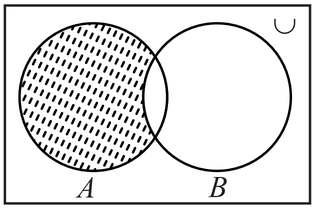
$A - B$ = {all those elements of A which do not belong to B}

$A - B = \{x : x \in A \text{ and } x \notin B\}$

Similarly, $B - A$ = {all those elements of B that do not belong to A}

$B - A = \{x : x \in B \text{ and } x \notin A\}$

Example: $A = \{2, 3, 4, 5, 6, 7\}$, $B = \{3, 5, 7, 9, 11, 13\}$ then $A - B = \{2, 4, 6\}$



A–B

Properties

If A and B are any two sets, then

(*i*) $A - B = A \cap B'$

(*ii*) $B - A = B \cap A'$

(*iii*) $A - B = A \Leftrightarrow A \cap B = \phi$

(*iv*) $(A - B) \cup B = A \cup B$

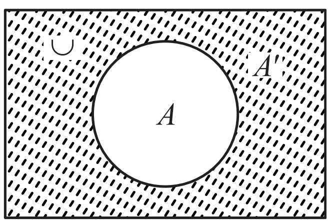
(*v*) $(A - B) \cap B = \phi$

(*vi*) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(*d*) **Complement of Set**

Let U be the universal set and A be a subset of U, then complement of set A represented as A' or A^c is the set of all elements of U which are not belong to set A.

Thus $A' = \{x : x \in \cup \text{ and } x \notin A\}$ OR $A' = \{x : x \notin A\}$



A'

Example: $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{1, 3, 5, 7, 9\}$ then $A^c = \{2, 4, 6, 8\}$

Properties:

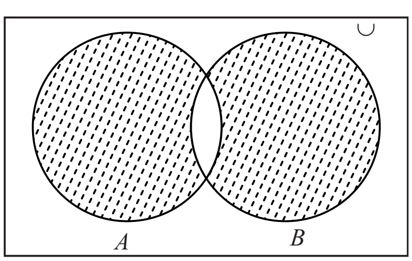
- $A \cap A^c = \phi$ (A and A^c are disjoint set)
- $A \cup A^c = U$
- $(A^c)^c = A$
- $U^c = \phi$
- $\phi^c = U$

(*e*) **Symmetric difference of two sets**

The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$

Thus, $A \Delta B = \{x : x \in (A - B) \cup (B - A)\} = \{x : x \notin A \cap B\}$

Example: $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 3, 5, 6, 7, 8, 9\}$ then $A \Delta B = \{2, 4, 9\}$



AΔB

Train Your Brain

Example 6: If $N = \{ax : x \in N\}$ describe the set $3N \cap 7N$.

Sol. $3N = \{3x : x \in N\} = \{3, 6, 9, \ldots\}$

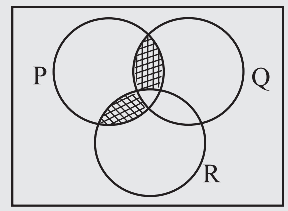
$7N = \{7x : x \in N\} = \{7, 14, 21, 28, \ldots\}$

$\therefore 3N \cap 7N = \{y : y$ is a multiple of 3 and y is a multiple of 7$\}$

$= \{y : y$ is a multiple of 21$\}$

$= \{21, 42, 63, \ldots\} = 21\,N$

Example 7: What does the shaded portion of the Venn diagram given below represent?



(*a*) $(P \cap Q) \cap (P \cap R)$

(*b*) $((P \cap Q) - R) \cup ((P \cap R) - Q)$

(*c*) $((P \cup Q) - R) \cap ((P \cap R) - Q)$

(*d*) $((P \cap Q) \cup R) \cap ((P \cup R) - R)$

Sol. (*b*) In the given Venn diagram, shaded area between sets P and Q is $(P \cap Q) - R$ and shaded area between P and R is $(P \cap R) - Q$. So, both the shaded area is union of these two area and is represented by $((P \cap Q) - R) \cup ((P \cap R) - Q)$.

Example 8: If $A = \{x \in R : 0 < x < 3\}$ and $B = \{x \in R : 1 \le x \le 5\}$ then $A \Delta B$ is

(*a*) $\{x \in R : 0 < x < 1\}$

(*b*) $\{x \in R : 3 \le x \le 5\}$

(*c*) $\{x \in R : 0 \le x \le 1 \text{ or } 3 \le x \le 5\}$

(*d*) ϕ

Sol. (*c*) From the given we have in interval notation $A = (0, 3)$ and $B = [1, 5]$

Clearly $A - B = (0, 1) = \{x \in R : 0 < x < 1\}$ and $B - A = [3, 5] = \{x \in R : 3 \le x \le 5\}$

$\therefore A \Delta B = (A - B) \cup (B - A) = (0, 1) \cup [3, 5]$

$= \{x \in R : 0 < x < 1 \text{ or } 3 \le x \le 5\}$

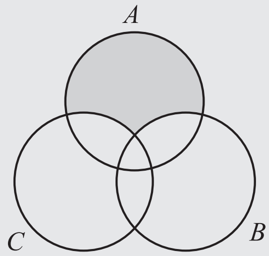
Concept Application

8. Let $A = \{(n, 2n) : n \in N\}$ and $B = \{(2n, 3n) : n \in N\}$. What is $A \cap B$ equal to?

(*a*) $\{(n, 6n) : n \in N\}$ (*b*) $\{(2n, 6n) : n \in N\}$

(*c*) $\{(n, 3n) : n \in N\}$ (*d*) ϕ

9. The shaded region in the given figure is



(*a*) $A \cap B\,(B \cup C)$ (*b*) $A \cup B\,(B \cap C)$
(*c*) $A \cap B\,(B - C)$ (*d*) $A - (B \cup C)$

10. Let X = {Ram, Geeta, Akbar} be the set of students of Class XI, who are in school hockey team and Y = {Geeta, David, Ashok} be the set of students from class XI, who are in the school football team, Then $X \cap Y$ is
(*a*) {Ram, Geeta} (*b*) {Ram}
(*c*) {Geeta} (*d*) {None of these}

11. Let $A = \{3, 6, 9, 12, 15, 18, 21\}$
$B = \{4, 8, 12, 16, 20\}$
$C = \{2, 4, 6, 8, 10, 12, 14, 16\}$
and $D = \{5, 10, 15, 20\}$
(I) $(A \cup B) - C = \{3, 9, 18, 21\}$
(II) $(A - B) \cap (C - D) = \{6\}$
(III) $(A \cap B \cap C) \cup D = \{12, 5, 15, 10, 20\}$
Which of the following is correct?
(*a*) Only *I* and *II* (*b*) Only *II* and *III*
(*c*) Only *III* and *I* (*d*) None of these

SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS

(*i*) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
(*ii*) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A$ and B are disjoint non–void sets.
(*iii*) $n(A - B) = n(A) - n(A \cap B)$
(*iv*) $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
(*v*) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
(*vi*) No. of elements in exactly two of the sets A, B, C
$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
(*vii*) No. of elements in exactly one of the sets A, B, $C = n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$
(*viii*) $n(A' \cup B') = n(U) - n(A \cap B)$
(*ix*) $n(A' \cap B') = n(U) - n(A \cup B)$

4. $n(A \Delta B)$ = Number of elements which belong to exactly one of A or B.
$= n((A - B) \cup (B - A))$
$= n(A - B) + n(B - A)$ [$\therefore$ $(A - B)$ and $(B - A)$ are disjoint]
$= n(A) - n(A \cap B) + n(B) - n(A \cap B) = n(A) + n(B) - 2n(A \cap B)$

5. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
6. n(Number of elements in exactly two of the sets A, B, C)
$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
7. n(Number of elements in exactly one of the sets A, B, C)
$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$
8. $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
9. $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cap B)$

LAWS OF ALGEBRA OF SETS

(*a*) **Idempotent laws**
(*i*) $A \cup A = A$ (*ii*) $A \cap A = A$
(*b*) **Identity laws**
(*i*) $A \cup \phi = A$ (*ii*) $A \cap U = A$
i.e. ϕ and U are identity elements for union and intersection respectively.
(*c*) **Commutative laws**
(*i*) $A \cup B = B \cup A$ (*ii*) $A \cap B = B \cap A$
i.e. union and intersection are commutative
(*d*) **Associative laws**
(*i*) $(A \cup B) \cup C = A \cup (B \cup C)$
(*ii*) $A \cap (B \cap C) = (A \cap B) \cap C$
i.e. union and intersection are associative
(*e*) **Distributive laws**
(*i*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(*ii*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
i.e. union and intersection are distributive over intersection and union respectively.
(*f*) **De–Morgan's laws**
(*i*) $(A \cup B)' = A' \cap B'$ (*ii*) $(A \cap B)' = A' \cup B'$
(*iii*) $(A')' = A$

CARTESIAN PRODUCT

The Cartesian product of two sets A and B is the set $\{(a, b): a \in A$ and $b \in B\}$ and is denoted by $A \times B$. If A has m elements and B has n elements, then $A \times B$ has mn elements

Example: $A = \{1, 2, 3\}$ and $B = \{a, b\}$ then $A \times B = \{(1, a)$, (1, b), $(2, a), (2, b), (3, a), (3, b)\}$
$n(A \times B) = 3 \times 2 = 6$

Properties

For three sets A, B, and C
(*i*) $n(A \times B) = n(A) \times n(B)$
(*ii*) $A \times B = \phi$, if either A or B is an empty set
(*iii*) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(*iv*) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(v) $A \times (B - C) = (A \times B) - (A \times C)$

(vi) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(vii) $A \times B = B \times A \Leftrightarrow A = B$

Train Your Brain

Example 9: If A and B be two sets containing 3 and 6 elements respectively, what can be the minimum number of elements in $A \cup B$? Find also, the maximum number of elements in $A \cup B$.

Sol. We have $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

This shows that $n(A \cup B)$ is minimum or maximum according as $n(A \cap B)$ is maximum or minimum respectively.

Case 1

When $n(A \cap B)$ is minimum, i.e., $n(A \cap B) = 0$

This is possible only when $A \cap B = \phi$. In this case,

$n(A \cup B) = n(A) + n(B) - 0 = n(A) + n(B) = 3 + 6 = 9$

So maximum number of elements in $A \cup B$ is 9.

Case 2

When $n(A \cap B)$ is maximum.

This is possible only when $A \subseteq B$.

In this case $n(A \cap B) = 3$.

$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B) = (3 + 6 - 3) = 6$

So, minimum number of elements in $A \cup B$ is 6.

Example 10: If A, B and C are three sets and U is the universal set such that $n(U) = 700$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$. Find $n(A' \cap B')$.

Sol. We have $A' \cap B' = (A \cup B)'$

$\therefore n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

$= n(U) - [n(A) + n(B) - n(A \cap B)]$

$= 700 - (200 + 300 - 100) = 300$

Example 11: Prove that for non empty sets A and B.

$A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

Sol. Consider

$(A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)'$

$= (A \cap B) \cap (A' \cup B')$

$= [(A \cup B) \cap A'] \cup [(A \cup B) \cap B']$

$= [(A \cap A') \cup (B \cap A')] \cup [(A \cap B') \cup (B \cap B')]$

$= [\phi \cup (B \cap A')] \cup [(A \cap B') \cup \phi]$

$= (B \cap A') \cup (A \cap B')$

$= (B - A) \cup (A - B)$ ...(1)

Now R.H.S. $= (A - B) \cup (B - A) \cup (A \cap B)$

$= [(A \cup B) - (A \cap B)] \cup (A \cap B)$ using (1)

$= [(A \cup B) \cap (A \cap B')] \cup (A \cap B)$

$= [(A \cup B) \cup (A \cap B)] \cap [(A \cap B)' \cup (A \cap B)]$

$= (A \cup B) \cap U$, where U is universal set.

$= A \cup B =$ L.H.S.

Example 12: In a certain city, only two newspapers A and B are published. It is known that 25% of the city population reads A and 20% reads B, while 8% reads both A and B. It is also known that 30% of those who read A but not B, look into advertisements and 40% of those who read B but not A, look into advertisements, while 50% of those who read both A and B, look into advertisements. What % of the population read an advertisement?

Sol. Let L = Set of people who read paper A

M = Set of people who read paper B

Then $n(L) = 25$, $n(M) = 20$, $n(L \cap M) = 8$

$n(L - M) = n(L) - n(L \cap M) = 25 - 8 = 17$

$n(M - L) = n(M) - n(L \cap M) = 20 - 8 = 12$

$\therefore$ % of people reading an advertisement

$= (30\% \text{ of } 17) + (40\% \text{ of } 12) + (50\% \text{ of } 8)$

$= \frac{51}{10} + \frac{24}{5} + 4 = 13.9\%$

Concept Application

12. A market research group conducted a survey of 1000 consumers and reported that 720 consumers liked product A and 450 consumers liked product B. What is the least number that must have liked both products?

(a) 170 (b) 280

(c) 220 (d) None of these

13. A set A has 3 elements and another set B has 6 elements. Then

(a) $3 \le n(A \cup B) \le 6$ (b) $3 \le n(A \cup B) \le 9$

(c) $6 \le n(A \cup B) \le 9$ (d) $0 \le n(A \cup B) \le 9$

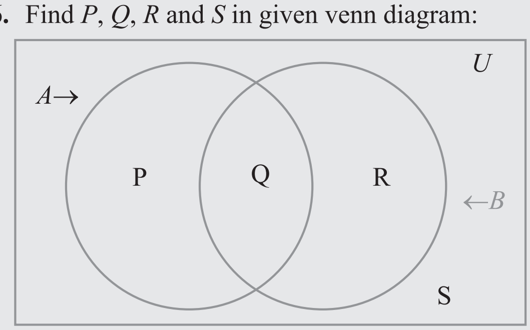
14. At a certain conference of 100 people, there are 29 Indian women and 23 Indian men of these Indian people 4 are doctors and 24 are either men or doctors. There are no foreign doctors. How many foreigners and women doctors are attending the conference?

(a) 48, 1 (b) 34, 3 (c) 46, 4 (d) 42, 2

15. The number of elements in the set $\{(a, b) : 2a^2 + 3b^2 = 35, a, b \in Z\}$ where Z is the set of all integers, is

(a) 2 (b) 4 (c) 8 (d) 12

16. Find P, Q, R and S in given venn diagram:



Short Notes

Set

A set is a well-defined collection of objects. A Collection is said to be well-defined if each object is distinct from all other objects of the set, but all objects have same common properties. Each object of a set is called on element of a set.

Methods of Representing a Set

(*i*) Roster or Tabular Form: In this form, a set is described by listing elements, separated by commas, within braces $\{\}$.

(*ii*) Set-builder Form: In this form, a set is described by a characterizing property $P(x)$ of its elements x. In such a case, the set is described by $\{x{:}P(x)$ holds$\}$ or $\{x{:}\ P(x)$ holds$\}$, which is real as 'the set of all x such that $P(x)$ holds'.

Types of Sets

(*i*) Empty Set: A set having no element in it is called an empty set.

(*ii*) Singleton Set : A set Containing one element is called a singleton set.

(*iii*) Finite Set : A set having fixed no. of elements is called a finite set.

(*iv*) Infinite Set : A set whose elements cannot be counted is called an infinite set.

(*v*) Equal sets: Two sets A and B are said to be equal if every element of A is a member of B and Vice-Versa.

Subsets

A set A is said to be a subset of a set B if every element of A is also an element of B. i.e., $A \subset B$ if $a \in A \Rightarrow a \in B$

Note that:

(*i*) Every set is a subset of itself.

(*ii*) Empty set ϕ is a subset of every set.

Intervals as Subsets of R

Let $a, b \in R$ and $a < b$, then

(*i*) Closed Interval

$[a, b] = \{x \in R : a \leq x \leq b\}$

(*ii*) Open Interval

$(a, b) = \{x \in R : a < x < b\}$

(*iii*) Semi-open or Semi-closed Interval

(and $(a, b] = \{x \in R : a < x \leq b\}$and $[a, b) = \{x \in R : a \leq x < b\}$)

Power Set

The collection of all subsets of set A is called the power set of A. It is denoted by $P(A)$. Every element in $P(A)$ is a set. Note that if A is a finite set having n elements, then $P(A)$ has 2^n elements.

Universal Set

It is a set which includes all the elements of the sets under consideration. i.e., it is a super set of each of the given sets. It is denoted by U. Eg., if $A = \{1, 2, 3\}$, $B = \{3, 4, 7\}$ and $C = \{2, 8, 9\}$, then $U = \{1, 2, 3, 4, 7, 8, 9\}$

Venn Diagrams

Most of the relationships between sets can be represented by means of diagrams which are known as Venn diagrams.

In Venn-diagrams, the universal set U is represented by points within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle.

Operations on Sets

Union of Sets: The union of two sets A and B is the set of all those elements which are either in A or in B. It is denoted by $A \cup B$.

Properties of the Operation of Union

(*i*) $A \cup B = B \cup A$ (Commutative Law)

(*ii*) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative Law)

(*iii*) $A \cup \phi = A$ (Law of identity element, ϕ is the identity of U)

(*iv*) $A \cup A = A$ (Idempotent Law)

(*v*) $U \cup A = U$ (Law of U)

Intersection of Sets

The intersection of two sets A and B is the set of all the elements which are common. It is denoted by $A \cap B$.

Properties of the Operation of Intersection

(*i*) $A \cap B = B \cap A$ (Commutative Law)

(*ii*) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative Law)

(*iii*) $\phi \cap A = \phi$, $U \cap A = A$ (Law of ϕ and $\cup$)

(*iv*) $A \cap A = A$ (Idempotent Law)

(*v*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive Law)

Difference of Sets

The difference of two sets A and B i.e., $A–B$, is the set of all those elements of A which do not belong to B.

Thus, $A - B = \{x{:}x \in A$ and $x \notin B\}$

Similarly, $B - A = \{x{:}x \in B$ and $x \notin A\}$

Some Important Results on Number of Elements in Sets

(*i*) If A and B are finite sets such that $A \cap B = \phi$, then

$n(A \cup B) = n(A) + n(B)$

(*ii*) If $A \cup B \neq \phi$, then

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(*iii*) If A, B and C are finite sets, then

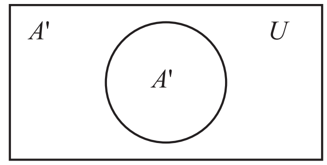
$(n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Complement of a Set

Let U be the universal set and let A be a set such that $A \subset U$. Then, the complement of A with respect to U is denoted by A^c or A' or U – A and is defined as the set of all those elements of U which are not in A.

Therefore, $A' = \{x \in U : x \notin A\}$

Clearly, $x \in A' \Leftrightarrow x \notin A$



Properties of Complement Sets

(1) **Complement Laws**

(*i*) $A \cup A' = U$ (*ii*) $A \cap A' = \phi$

(2) **De Morgan's Law**

(*i*) $(A \cup B)' = A' \cap B'$ (*ii*) $(A \cap B)' = A' \cup B'$

(3) **Law of Double Complementation**

$(A')' = A$

(4) **Laws of ϕ and U**

$\phi' = U$ and $U = 0$

Solved Examples

1. Among employee of a company taking vacations last years, 90% took vacations in the summer, 65% in the winter, 10 % in the spring, 7% in the autumn, 55% in winter and summer, 8% in the spring and summer, 6% in the autumn and summer, 4% in winter and spring, 4% in winter and autumn, 3% in the spring and autumn, 3% in the summer, winter and spring 3% in the summer, winter and autumn, 2% in the summer, autumn and spring, and 2% in the winter, spring and autumn. Percentage of employee that took vacations during every season

(*a*) 4 (*b*) 3

(*c*) 2 (*d*) 8

Sol. (*c*) Suppose that number of employee taking vacations is 100.

S_u – set of employee taking leave in summer

S_w – set of employee taking leave in Winter

S_p – set of employee taking leave in Spring

S_a – set of employee taking leave in Autumn

$n(S_u) = 90$, $n(S_w) = 65$, $n(S_p) = 10$, $n(S_a) = 7$

$n(S_w \cap S_u) = 55$, $n(S_p \cap S_u) = 8$, $n(S_a \cap S_u) = 6$

$n(S_w \cap S_p) = 4$, $n(S_w \cap S_a) = 4$, $n(S_p \cap S_a) = 3$

$n(S_u \cap S_a) = 3$, $n(S_u \cap S_w \cap S_a) = 3$

$n(S_u \cap S_w \cap S_p) = 3$, $n(S_u \cap S_a \cap S_p) = 2$

$n(S_w \cap S_p \cap S_a) = 2$

$n(S_u \cap S_p \cap S_w \cap S_a)$

$= n(S_u) + n(S_p) + n(S_w) + n(S_a) - n(S_u \cap S_p) - n(S_p \cap S_w) - n(S_w \cap S_a) - n(S_u \cap S_a) - n(S_u \cap S_w) - n(S_p \cap S_a) + n(S_u \cap S_p \cap S_w) + n(S_u \cap S_w \cap S_a) + n(S_w \cap S_a \cap S_u) + n(S_u \cap S_p \cap S_a) - n(S_p \cup S_u \cup S_a \cup S_w)$

$= 90 + 65 + 10 + 7 - 55 - 8 - 6 - 4 - 4 - 3 + 3 + 3 + 2 + 2 - 100 = 2$

2. If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$ then the number of elements in $(A \cup B) \times (A \cap B) \times (A \Delta B)$ is

(*a*) 5 (*b*) 30

(*c*) 10 (*d*) 4

Sol. (*b*) $A \cup B = \{1, 2, 3, 4, 5\}$. $n(A \cup B) = 5$

$A \cap B = \{3, 4\}$, $n(A \cap B) = 2$

$A \Delta B = (A \sim B) \cup (B \sim A) = \{1, 2\} \cup \{5\}$

$= \{1, 2, 5\}$

$n(A \Delta B) = 3$. Hence $n((A \cup B) \times (A \cap B) \times (A, \Delta 5))$

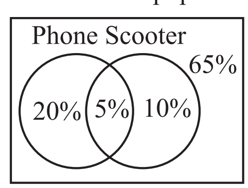
$= 5 \times 2 \times 3 = 30$.

3. In a certain town 25% families own a cell phone, 15% families own a scooter and 65% families own neither a cell phone nor a scooter. If 1500 families own both a cell phone and a scooter. then total number of families in the town is

(*a*) 10000 (*b*) 20000

(*c*) 30000 (*d*) 40000

Sol. (*c*) Let the total population of town be x.



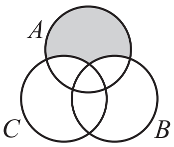
$$\therefore \frac{25x}{100} + \frac{15x}{100} - 1500 + \frac{65x}{100} = x$$

$$\Rightarrow \frac{105x}{100} - x = 1500$$

$\Rightarrow \dfrac{5x}{100} = 1500$

$\Rightarrow x = 30000$

4. The shaded region in the given figure represents



(*a*) $A \cap (B \cup C)$ (*b*) $A \cup (B \cap C)$

(*c*) $A \cap (B - C)$ (*d*) $A - (B \cup C)$

Sol. (*d*) Shaded region contain elements of A not in B and not in C hence it is $A - (B \cup C)$

5. If $A = \{x : x^2 - 5x + 6 = 0\}$, $B = \{2, 4\}$, $C = \{4, 5\}$, then $A \times (B \cap C)$ is

(*a*) $\{(2, 4), (3, 4)\}$ (*b*) $\{(4, 2), (4, 3)\}$

(*c*) $\{(2, 4), (3, 4), (4, 4)\}$ (*d*) $\{(2, 2), (3, 3), (4, 4), (5, 5)\}$

Sol. (*a*) Clearly, $A = \{2, 3\}$, $B = \{2, 4\}$, $C = \{4, 5\}$

$B \cap C = \{4\}$

$\therefore A \times (B \cap C) = \{(2, 4), (3, 4)\}$

6. If A and B are two sets such that $n(A \times B) = 60$ and $n(A) = 12$ also $n(A \cap B) = K$, then the sum of maximum and minimum possible value of K is

(*a*) 17 (*b*) 12

(*c*) 5 (*d*) 7

Sol. (*c*) $n(B) = \dfrac{n(A \times B)}{n(A)} = 5$

$\therefore n(A \cap B)$ has minimum value $= 0$

and $n(A \cap B)$ has maximum value $= n(B) = 5$

7. If the difference between the number of subsets of two sets A and B is 120, then $n(A \times B)$ is equal to

(*a*) 21 (*b*) 25

(*c*) 18 (*d*) 24

Sol. (*a*) Let, $n(A) = \alpha$ and $n(B) = \beta$

Given, $|2^\alpha - 2^\beta| = 120 \Rightarrow (\alpha, \beta) \equiv (7, 3), (3, 7)$

$\therefore n(A \times B) = 7 \times 3 = 21$

8. If the difference between the number of subsets of the sets A and B is 120, then choose the incorrect option.

(*a*) Maximum value of $n(A \cap B) = 3$

(*b*) Minimum value of $n(A \cap B) = 0$

(*c*) Maximum value of $n(A \cup B) = 21$

(*d*) Minimum value of $n(A \cup B) = 7$

Sol. (*c*) Let, $n(A) = \alpha$, $n(B) = \beta(\alpha > \beta)$

Given, $2^\alpha - 2^\beta = 120 \Rightarrow \alpha = 7, \beta = 3$

$\therefore n(A \cap B) \in [0, 3]$

$\because n(A \cup B) = \alpha + \beta - n(A \cap B)$ $n(A \cup B) \in [7, 10]$

9. Let Z be the set of intergers, if $A = \{z \in Z : |x-3|^{(x^2-5x+6)} = 1\}$ and B = $\{x \in Z : 10 < 3x + 1 < 22\}$, then the number of subsets of the set $A \times B$ is

(*a*) 2^6 (*b*) 2^8

(*c*) 2^{15} (*d*) 2^9

Sol. (*a*) For A,

$|x - 3| = 1 \Rightarrow x = 2, 4$ or

$x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$ but $x \neq 3$

$\therefore$ A = {2, 4}

For B,

$B = \{4, 5, 6\}$

$n(A \times B) = 6$

$\therefore$ number of subsets $= 2^6$

10. Let A and B be two sets. The set A has 2016 more subsets than B. If $A \cap B$ has 3 members, then the numbers of members in $A \cup B$ is

(*a*) 10 (*b*) 11 (*c*) 12 (*d*) 13

Sol. (*d*) If A has m members and B has n members, then

$2^m - 2^n = 2016 = 2^5 \cdot 63$

$\Rightarrow 2^n (2^{m-n} - 1) = 2^5 (64 - 1) = 2^5(2^6 - 1)$

$\Rightarrow$ n = 5 and m – n = 6 $\Rightarrow$ m = 11

The number of members in $A \cup B$

= 11 + 5 – 3 = 13

11. If in a class there are 200 students in which 120 take Mathematics, 90 take Physics, 60 take Chemistry, 50 take mathematics and Physics, 50 take Mathematics and Chemistry, 43 take physics and Chemistry and 38 take Mathematics, Physics and Chemistry, then the number of students who have taken exactly one subject is

(*a*) 42 (*b*) 56 (*c*) 270 (*d*) 98

Sol. (*d*) Let, $M \rightarrow$ Mathematics, $P \rightarrow$ Physics, $C \rightarrow$ Chemistry

Given that total students = 200

$n(M) = 120$, $n(P) = 90$, $n(C) = 60$

$n(M \cap P) = 50$, $n(M \cap C) = 50$, $n(P \cap C) = 43$,

$n(M \cap P \cap C) = 38$

Required number of students taking exactly on subject is

$n(M) + n(P) + n(C) - 2n(M \cap P) - 2n(P \cap C) - 2n(M \cap C) + 3n(M \cap P \cap C)$

$= 120 + 90 + 60 - 2(50) - 2(50) - 2(43) + 3(38)$

$= 98$

12. If $A = \{(x, y) : y = e^x; x \in R\}$ U $= \{(x, y) : x, y \in R\}$

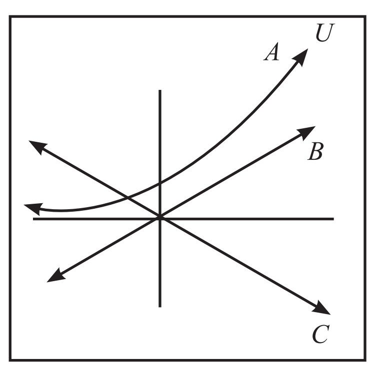
$B = \{(x, y) : y = x; x \in R\}$

$C = \{(x, y) : y = -x; x \in R\}$

Choose the correct statement/s among the following :

(*a*) $(A \cap B)' = \phi$ (*b*) $(A \cap B \cap C)' = \phi$

(*c*) $A - B = \phi$ (*d*) $A \Delta B = A \cup B$



Sol. (*d*) Set A, B and C are the points on the curves as shown in adjacent diagram. Clearly $A \cap B = \phi$ so $(A \cap B)' = U$, option (a) is wrong.

Similarly $A \cap B \cap C = \phi$ as there are no common points to all the three curves. $\therefore$ $(A \cap B \cap C)' = U$. Option (*b*) is wrong.

From figure it is clear that $A - B = A$, since A and B are disjoint sets. Option (*c*) is wrong.

$A \Delta B \quad = (A - B) \cup (B - A)$

$= A \cup B$

13. The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$ is equal to

(*a*) $B \cap C'$ (*b*) $A \cap C$

(*c*) $B' \cap C'$ (*d*) None of these

Sol. (*a*) $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$

$= (A \cup B \cup C) \cap (A' \cap B \cap C) \cap C'$

$= (\phi \cup B \cup C) \cap C'$

$= (B \cap C') \cap C'$

$= (B \cap C') \cup \phi = B \cap C'$

14. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$. Then the number of sets C such that $A \cap B \subseteq C \subseteq A \cup B$ is

(*a*) 6 (*b*) 9

(*c*) 8 (*d*) 10

Sol. (*c*) Here, $A \cap B = \{2, 4\}$

and $A \cup B = \{1, 2, 3, 4, 6\}$

$\because$ $A \cap B \subseteq C \subseteq A \cup B$

$\therefore$ C can be $\{2, 4\}$, $\{1, 2, 4\}$, $\{3, 2, 4\}$, $\{6, 2, 4\}$, $\{1, 6, 2, 4\}$, $\{6, 3, 2, 4\}$, $\{1, 3, 2, 4\}$, $\{1, 2, 3, 4, 6\}$

Thus, number of set C which satisfy the given condition is 8.

15. If $A = \phi$, $P(A)$ denotes power set of A, then number of elements in $P(P(P(P(P(A)))))$ is

(*a*) 1 (*b*) 2^4 (*c*) 2^5 (*d*) 2^{16}

Sol. (*d*) $n(P(A)) = 2^\circ = 1$

$n(P(P(A))) = 2^1 = 2$

$n(P(P(P(A)))) = 2^2 = 4$

$n(P(P(P(P(A))))) = 2^4 = 16$

$n(P(P(P(P(P(A)))))) = 2^{16})$

16. If A and B be two sets such that $n(A) = 15$, $n(B) = 25$, then number of possible values of $n(A \Delta B)$ (symmetric difference of A and B) is

(*a*) 30 (*b*) 16 (*c*) 26 (*d*) 40

Sol. (*b*) $n(A \Delta B) = n(A \cup B) - n(A \cap B)$ for maximum $n(A \Delta B)$, $n(A \cup B)$ should be maximum and $n(A \cap B)$ is minimum

For $n(A \cap B)$ to be minimum, $A \cap B = \phi$

$\Rightarrow n(A \cup B) = 25 + 15 = 40$

$n(A \Delta B) = 40$

For $n(A \cap B)$ to be maximum, $A \subset B$

$n(A \cap B) = 15$

$\Rightarrow n(A \cup B) = 25$

$n(A \Delta B) = 25 - 15 = 10$

$\Rightarrow$ Range of $(A \Delta B) = \{10, 12, 14, 16, \ldots\ldots, 40\}$

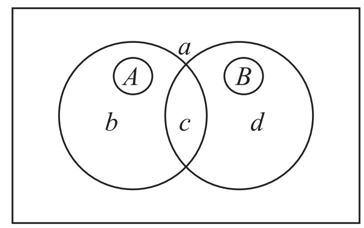
17. $|X|$ represent number of elements in region X. Now the following conditions are given

$|U| = 14$, $|(A - B)^C| = 12$, $|A \cup B| = 9$ and $|A \Delta B| = 7$, where A and B are two subsets of the universal set U and A^C represents complement of set A, then

(*a*) $|A| = 2$ (*b*) $|B| = 5$

(*c*) $y|A| = 4$ (*d*) $|B| = 7$

Sol. (*c,d*)



$a + b + c + d = 14$ ...(*i*)

$a + c + d = 12$ ...(*ii*)

$b + c + d = 9$ ...(*iii*)

$b + d = 7$

$b = 2, a = 5, d = 5, c = 2$

$|A| = b + c = 4$

$|B| = d + c = 7$

18. For any three sets. P, Q and R, S is an element of $(P \Delta Q) \Delta R$, then S may belong to

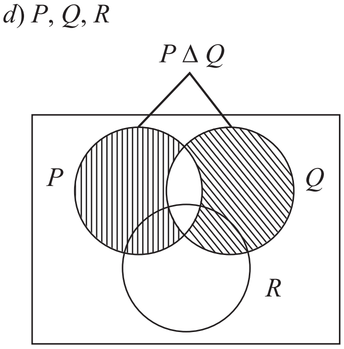
(*a*) Exactly one of P, Q and R, if P, Q and R disjoint sets

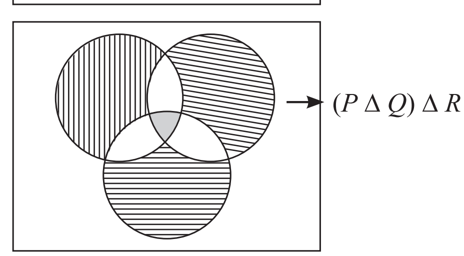
(*b*) At least one of P, Q and R, but not in all three of them at the same time

(*c*) Exactly two of P, Q and R

(*d*) Exactly one of P, Q and R or in all the three of them

Sol. (*a,d*) P, Q, R





Matrix-Match Type Questions

19. $A = \{x : x \in N, \text{G.C.D. } (x, 36) = 1, x < 36\}$, $B = \{y : y \in N, \text{G.C.D. } (y, 40) = 1, y < 40\}$; (G.C.D. stands for greatest common divisors)

Column-I		Column-II	
A.	$n(A \cap B)$	p.	10
B.	$C = \{x : x \in A \cup B, x$ is prime $\}$, $n(C)$=	q.	9
C.	$n(A \Delta B)$	r.	21
D.	$n((A - B) \times (B - A))$	s.	11

Sol. A-(q), B-(s), C-(p), D-(r)

$A = \{1,5,7,11,13,17,19,23,25,29,31,35\}$

$B = \{1,3,7,9,11,13,17,19,21,23,27,29,31,33,37,39\})$

$n(A \cap B) = 9$

$n(A \Delta B) = 10$

$C = \{x : x \in A \cup B, x \text{ is prime}\}$

$= \{3,5,7,11,13,17,19,23,29,31,37\}$

$n(C) = 11$

$n((A - B) \times (B - A)) = n(A - B) \cdot n(B - A)$

$= 3 \times 7 = 21$

20. A, B, C be three sets such that $n(A) = 2$, $n(B) = 3$, $n(C) = 4$. If $P(X)$ denotes power set of X,

$K = \dfrac{n(P(P(C)))}{n(P(P(A))) \times n(P(P(B)))}$. Sum of digits of K is____.

Sol. [7] $K = \dfrac{2^{24}}{2^{22} \times 2^{23}} = \dfrac{2^{16}}{2^4 \times 2^8} = 2^4 = 16$

21. $A_1 \times A_2 \times A_3 \times A_4 = \{(1, 1, 1, 1), (2, 4, 8, 16), (3, 9, 27, 81), ...\}$. Find A_1, A_2, A_3 and A_4.

Sol. Each ordered pair $\{x_1, x_2, x_3, x_4\}$ is of the form $\{x, x^2, x^3, x^4\}$

Hence $x_1 \in A_1 \Rightarrow A_1 = \{x : x \in N\} = \{1, 2, 3, 4, ...\}$

$x_2 \in A_2 \Rightarrow A_2 = \{x^2 : x \in N\} = \{1^2, 2^2, 3^2, 4^2, ...\}$

$x_3 \in A_3 \Rightarrow A_3 = \{x^3 : x \in N\} = \{1^3, 2^3, 3^3, 4^3, ...\}$

$x_4 \in A_4 \Rightarrow A_4 = \{x^4 : x \in N\} = \{1^4, 2^4, 3^4, 4^4, ...\}$

22. Suppose $A_1, A_2....., A_{30}$ are thirty sets each having 5 elements and $B_1, B_2....., B_n$ are n sets each having 3 elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each elements of S belongs to exactly 10 of A_i's and exactly 9 of B_j's. The value of n is equal to

Sol. Number of elements in

$A_1 \cup A_2 \cup A_3 \cup ... \cup A_{30}$ is 30×5 but each element is used 10 times, so

$n(S) = \dfrac{30 \times 5}{10} = 15$ ...(i)

Similarly, number of elements in $B_1 \cup B_2 ... \cup B_n$ is 3 n but each element is repeated 9 times, so

$n(S) = \dfrac{3n}{9}$

$\Rightarrow 15 = \dfrac{3n}{9}$ [from Eq. (i)]

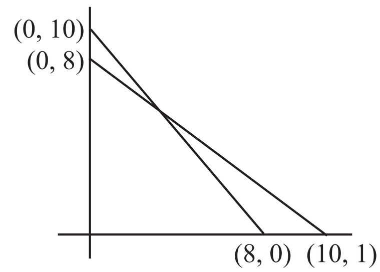
$\Rightarrow n = 45$

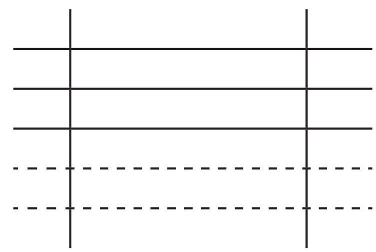
23. $A = \{(x, y) : x, y \in I, x \geq 0, y \geq 0 \text{ and } 4x + 5y \leq 40\}$

$B = \{(x, y) : x, y \in I, x \geq 0, y \geq 0 \text{ and } 5x + 4y \leq 40\}$

where I denotes set of integers, then $n(A \cap B)$=

Sol.





Total $= \dfrac{9 \times 10}{2} = 45$

24. In a survey it was found that 21 persons liked product P_1, 26 liked product P_2 and 29 liked product P_3. If 14 persons liked products P_1 and P_2; 12 persons liked product P_3 and P_1; 14 persons liked products P_2 and P_3, and 8 liked all the three products. Find how many liked product P_3 only.

Sol. Let $n(P_1)$ be a number of people liking product P_1.

Let $n(P_2)$ be a number of people liking product P_2.

Let $n(P_3)$ be a number of people liking product P_3.

Then, According to the questions:

$n(P_1) = 21$, $n(P_2) = 26$, $n(P_3) = 29$, $n(P_1 \cap P_2) = 14$

$n(P_1 \cap P_3) = 12$, $n(P_2 \cap P_3) = 14$, $n(P_1 \cap P_2 \cap P_3) = 8$

$\therefore$ Number of people liking product P_3 only:

$= 29 - (4 + 8 + 6)$

$= 29 - 18 = 11$

Exercise-1 (Topicwise)

DEFINITION AND TYPE OF SETS

1. The set of intelligent students in a class is
(*a*) A null set
(*b*) A singleton set
(*c*) A finite set
(*d*) Not a well defined collection

2. Which of the following is the empty set
(*a*) $\{x : x$ is a real number and $x^2 - 1 = 0\}$
(*b*) $\{x : x$ is a real number and $x^2 + 1 = 0\}$
(*c*) $\{x : x$ is a real number and $x^2 - 9 = 0\}$
(*d*) $\{x : x$ is a real number and $x^2 = x + 2\}$

3. The set $A = \{x : x \in \text{R}, x^2 = 16$ and $2x = 6\}$ equals
(*a*) ϕ (*b*) $[14, 3, 4]$
(*c*) $[3]$ (*d*) $[4]$

4. If a set A has n elements, then the total number of subsets of A is
(*a*) n (*b*) n^2
(*c*) 2^n (*d*) $2n$

5. Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. The values of m and n are
(*a*) 7, 6 (*b*) 6, 3
(*c*) 5, 1 (*d*) 8, 7

6. The number of proper subsets of the set $\{1, 2, 3\}$ is
(*a*) 8 (*b*) 7
(*c*) 6 (*d*) 5

7. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49(n - 1) : n \in N\}$, then
(*a*) $X \subseteq Y$ (*b*) $Y \subseteq X$
(*c*) $X = Y$ (*d*) None of these

OPERATIONS ON SETS

8. Given the sets $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, then $A \cup (B \cap C)$ is
(*a*) $\{3\}$ (*b*) $\{1, 2, 3, 4\}$
(*c*) $\{1, 2, 4, 5\}$ (*d*) $\{1, 2, 3, 4, 5, 6\}$

9. If $A \subseteq B$, then $A \cup B$ is equal to
(*a*) A (*b*) $B \cap A$
(*c*) B (*d*) None of these

10. If A and B are any two sets, then $A \cup (A \cap B)$ is equal to
(*a*) A (*b*) B
(*c*) A^c (*d*) B^c

11. If A and B are two given sets, then $A \cap (A \cap B)^c$ is equal to
(*a*) A (*b*) B
(c) ϕ (*d*) $A \cap B^c$

12. If $N_a = \{an : n \in N\}$ then $N_3 \cap N_4 =$
(*a*) N_7 (*b*) N_{12}
(*c*) $N_3 N_3$ (*d*) $N_4 N_4$

13. If $aN = \{ax : x \in N\}$ and $bN \cap cN = dN$, where $b, c \in N$ are relatively prime, then
(*a*) $d = bc$ (*b*) $c = bd$
(*c*) $b = cd$ (*d*) None of these

14. If the sets A and B are defined as

$A = \{(x, y) : y = \dfrac{1}{x},\ 0 \neq x \in R\}$

$B = \{(x, y) : y = -x, x \in R\}$, then
(*a*) $A \cap B = A$ (*b*) $A \cap B = B$
(*c*) $A \cap B = \phi$ (*d*) None of these

15. Let $A\ [x : x \in R, |x| < 1]$; $B = [x : x \in R, |x - 1| \geq 1]$ and $A \cup B = R - D$, then the set D is
(*a*) $[x : 1 < x \leq 2]$ (*b*) $[x : \leq x < 2]$
(*c*) $[x : 1 \leq x \leq 2]$ (*d*) None of these

16. If the sets A and B are defined as
$A = \{(x , y) : y = e^x, x \in R\}$
$B = \{x, y\} : y = x, x \in R$, then
(*a*) $B \subseteq A$ (*b*) $A \subseteq B$
(*c*) $A \cap B = \phi$ (*d*) $A \cup B = A$

17. If $X = \{4^n - 3n - : n \in N\}$ and $Y = \{9(n - 1) : n \in N\}$ then $X \cup Y$ is equal to
(*a*) X (*b*) Y
(*c*) N (*d*) None of these

SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS

18. Sets A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$
(*a*) 3 (*b*) 6
(*c*) 9 (*d*) 18

19. If A and B are two sets such that $n(A) = 70$, $n(B) = 60$, and $n(A \cup B) = 110$, then $n(A \cap B)$ is equal to
(*a*) 240 (*b*) 50
(*c*) 40 (*d*) 20

20. Let $n(U) = 700$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$ then $n(A^c \cap B^c)$
(*a*) 400 (*b*) 600
(*c*) 300 (*d*) 200

21. If $A = [(x, y) : x^2 + y^2 = 25]$ and $B = [(x, y) : x^2 + 9y^2 = 144]$, then $A \cap B$ contains

(*a*) One point (*b*) Three points

(*c*) Two points (*d*) Four points

22. In a town of 10,000 families it was found that 40% family buy newspaper *A*, 20% buy newspaper *B* and 10% families buy newspaper *C*, 5% families buy *A* and *B*, 3% buy *B* and *C* and 4% buy *A* and *C*. If 2% families buy all the three newspapers, then number of families which buy *A* only is

(*a*) 3100 (*b*) 3300

(*c*) 2900 (*d*) 1400

23. In a city 20 percent of the population travels by car, 50 percent travels by bus and 10 percent travels by both car and bus. Then persons travelling by car or bus is

(*a*) 80 percent (*b*) 40 percent

(*c*) 60 percent (*d*) 70 percent

24. Suppose $A_1, A_2, A_3, ..., A_{30}$ are thirty sets each having 5 elements and $B_1, B_2, ..., B_n$ are *n* sets each with 3 elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$ and each elements of *S* belongs to exactly 10 of the $A'_i s$ and exactly 9 of the $B'_j s$. Then *n* is equal to

(*a*) 15 (*b*) 3

(*c*) 45 (*d*) None of these

25. In a class of 55 students, the number of students studying different subjects are 23 in Mathematics, 24 in Physics, 19 in Chemistry, 12 in Mathematics and Physics, 9 in Mathematics and Chemistry, 7 in Physics and Chemistry and 4 in all the three subjects. The number of students who have taken exactly one subject is

(*a*) 6 (*b*) 9

(*c*) 7 (*d*) All of these

LAWS OF ALGEBRA OF SETS

26. If *A*, *B* and *C* are any three sets, then $A \times (B \cap C)$ is equal to

(*a*) $(A \times B) \cup (A \times C)$ (*b*) $(A \times B) \cap (A \times C)$

(*c*) $(A \cup B) \times (A \cup C)$ (*d*) $(A \cap B) \times (A \cap C)$

27. If *A*, *B* and *C* are any three sets, then $A \times (B \cup C)$ is equal to

(*a*) $(A \times B) \cup (A \times C)$ (*b*) $(A \cup B) \times (A \cup C)$

(*c*) $(A \times B) \cap (A \times C)$ (*d*) None of these

28. If *A*, *B* and *C* are any three sets, then $A - (B \cup C)$ is equal to

(*a*) $(A - B) \cup (A - C)$ (*b*) $(A - B) \cap (A - C)$

(*c*) $(A - B) \cup C$ (*d*) $(A - B) \cap C$

29. If $A = [x : x$ is a multiple of 3] and $B = [x : x$ is a multiple of 5], then $A - B$ is ($\overline{A}$ means complement of *A*)

(*a*) $\overline{A} \cap B$ (*b*) $A \cap \overline{B}$

(*c*) $\overline{A} \cap \overline{B}$ (*d*) $\overline{A \cap B}$

30. If *A*, *B* and *C* are non-empty sets, then $(A - B) \cup (B - A)$ equals

(*a*) $(A \cup B) - B$ (*b*) $A - (A \cap B)$

(*c*) $(A \cup B) - (A \cap B)$ (*d*) $(A \cap B) \cup (A \cup B)$

CARTESIAN PRODUCT OF SETS

31. If $A = \{0, 1)$, and $B = \{1, 0\}$, then $A \times B$ is equal to

(*a*) $\{0, 1, 1, 0\}$ (*b*) $\{(0, 1), (1, 0)\}$

(*c*) $\{0, 0\}$ (*d*) $\{(0,1),(0,0),(1,1),(1,0)\}$

32. If $A = \{2, 4, 5\}$, $B = \{7, 8, 9\}$ then $n(A \times B)$ is equal to

(*a*) 6 (*b*) 9

(*c*) 3 (*d*) 0

33. If the set *A* has *p* elements, *B* has *q* elements, then the number of elements in $A \times B$ is

(*a*) $p + q$ (*b*) $p + q + 1$

(*c*) pq (*d*) p^2

34. If $A = \{a, b\}$, $B = \{c, d\}$, $C = \{d, e\}$ then $\{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$ is equal to

(*a*) $A \cap (B \cup C)$ (*b*) $A \cup (B \cap C)$

(*c*) $A \times (B \cup C)$ (*d*) $A \times (B \cap C)$

35. If $A = \{x : x^2 - 5x + 6 = 0\}$, $B = \{2, 4\}$, $C = \{4, 5\}$ then $A \times (B \cap C)$ is

(*a*) $\{(2, 4), (3, 4)\}$

(*b*) $\{(4, 2), (4, 3)\}$

(*c*) $\{(2, 4), (3, 4), (4, 4)\}$

(*d*) $\{(2,2), (3,3), (4,4), (5,5)\}$

36. If *P*, *Q* and *R* are subsets of a set *A*, then $R \times (P^c \cup Q^c)^c =$

(*a*) $(R \times P) \cap (R \times Q)^c$ (*b*) $(R \times Q) \cap (R \times P)$

(*c*) $(R \times P) \cup (R \times Q)$ (*d*) None of these

37. Let $S = \{x \in R : (x-3)^2 + (x-2)^2 + 5(x-4)^2 = 0\}$. Then *S* is not

(*a*) Singleton set (*b*) Empty set

(*c*) Void set (*d*) null set

38. Let $S = \{x \in R : 4^x + 2^{x+1} - 8 = 0\}$. Then $x =$

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3

39. The set $(A \cap B')' \cup (B \cap C)$ is equal to

(*a*) $A' \cup B \cup C$ (*b*) $A' \cup B$

(*c*) $A' \cap C'$ (*d*) $A' \cap B$

40. If *A* and *B* are two given sets, then $A \cap (A \cap B)^c$ is equal to

(*a*) A (*b*) B

(*c*) ϕ (*d*) $A \cap B^C$

41. Let $A = \{(x, y): x \in R, y \in R, x^3 + y^3 = 1]$, $B = \{(x, y): x \in R, y \in R, x - y = 1\}$ and $C = \{(x, y): x \in R, y \in R, x + y = 1\}$. If $A \cap B$ contains '*p*' elements and $A \cap C$ contains '*q*' elements then find $(q - p)$.

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3

42. In a examination of a certain class, at least 70% of the students failed in Physics, at least 72% failed in Chemistry, at least 80% failed in Mathematics and at least 85% failed in English. How many at least must have failed in all the four subjects?

(*a*) 9%

(*b*) 7%

(*c*) 15%

(*d*) Cannot be determined due to insufficient data

43. **Statement-I:** $N \cup (B \cap Z) = (N \cup B) \cap Z$ for any subset B of R, where N is the set of positive integers, Z is the set of integers, R is the set of real numbers.

Statement-II: Let $A = \{n \in N : 1 \leq n \leq 24, n \text{ is a multiple of } 3\}$. There exists no subset B of N such that the number of elemets in A is equal to the number of elements in B.

Which of the above statements is/are correct?

(*a*) I only (*b*) II only

(*c*) Both I and II (*d*) Neither I nor II

44. In year 10 of a Kuala Lumpur International School, there are 198 students.

C = {students who like chilli}

D = {students who like durian}

90 like durian and 130 like chilli. The number who like both is x and the number who like neither is $212 - 2x$. Find the value of x.

(*a*) 76

(*b*) 78

(*c*) 80

(*d*) None of these

Exercise-2 (Learning Plus)

1. Let $A = \{x : x \in R, x \geq 2\}$ and $B = \{x : x \in R, x < 4\}$. Then $A \cap B =$

(*a*) $\{x : x \in R, 2 < x < 4\}$, (*b*) $\{x : x \in R, 2 \leq x < 4\}$

(*c*) B (*d*) A

2. If $A = \{\phi, \{(\phi)\}\}$, then the power set $P(A)$ of A is

(*a*) A (*b*) $\{\phi, \{\phi\}, A\}$

(*c*) $\{\phi, \{\phi\}, \{\{\phi\}\}, A\}$ (*d*) $\{\phi, \{\phi\}, \{\{\phi\}\}\}$

3. If $A = \{2, 3\}$ and $B = \{x \mid x \in N \text{ and } x < 3\}$, then $A \times B$ is

(*a*) $\{(2, 1), (2, 2), (3, 1), (3, 2)\}$

(*b*) $\{(1, 2), (1, 3), (2, 2), (2, 3)\}$

(*c*) $\{(1, 2), (2, 2), (3, 3), (3, 2)\}$

(*d*) $\{(1, 1), (2, 2), (3, 3), (3, 2)\}$

4. Let U be the universal set containing 700 elements. If A, B are sub–sets of U such that $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$. Then $n(A' \cap B') =$

(*a*) 400 (*b*) 600 (*c*) 300 (*d*) 200

5. Let R be a relation on N defined by $x + 2y = 8$. The domain of R is

(*a*) $\{2, 4, 8\}$ (*b*) $\{2, 4, 6, 8\}$

(*c*) $\{2, 4, 6\}$ (*d*) $\{1, 2, 3, 4\}$

6. If $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{x \in N : 30 < x^2 < 70\}$, $B = \{x : x \text{ is a prime number less than } 10\}$, then which of the following is incorrect?

(*a*) $A \cup B = \{2, 3, 5, 6, 7, 8\}$

(*b*) $A \cap B = \{7, 8\}$

(*c*) $A - B = \{6, 8\}$

(*d*) $A \Delta B = \{2, 3, 5, 6, 8\}$

7. Let X be the universal set for sets A and B, if $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$, then $n(A' \cap B')$ is equal to 300 provided $n(X)$ is equal to

(*a*) 600 (*b*) 700 (*c*) 800 (*d*) 900

8. If two sets A and B are having 80 elements in common, then the number of elements common to each of the sets $A \times B$ and $B \times A$ are

(*a*) 2^{80} (*b*) 80^2 (*c*) 81 (*d*) 79

9. In a town of 10000 families it was found that 40% family buy newspaper A, 20% family buy newspaper B and 10% family buy newspaper C. 5% families buy A and B, 3% families buy B and C and 4% families buy A and C. If 2% families buy all the three newspapers. Then, number of families which buy A only is

(*a*) 3100 (*b*) 3300 (*c*) 2900 (*d*) 1400

10. If A and B be two universal sets and $A \cup B \cup C = U$. Then, $\{(A - B) \cup (B - C) \cup (C - A)\}'$ is equal to

(*a*) $A \cup B \cup C$ (*b*) $A \cup (B \cap C)$

(*c*) $A \cap B \cap C$ (*d*) $A \cap (B \cup C)$

11. The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$ is equal to

(*a*) $B \cup C'$ (*b*) $A \cap C$

(*c*) $B' \cap C'$ (*d*) None of these

12. If there are three athelitic teams in a school 21 are in the basketball team,26 in hockey team and 29 in the football team. 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the games. The total number of members is

(*a*) 42 (*b*) 43

(*c*) 45 (*d*) None of these

13. If $P(A)$ denotes the power set of A and A is the void set, then what is number of elements in $P\{P\{P\{P(A)\}\}\}$?

(*a*) 0 (*b*) 1
(*c*) 4 (*d*) 16

14. There are 4 prime numbers between n and $2n$. Then, possible values of n is

(*a*) $n = 4$ (*b*) $n = 12$
(*c*) $n = 10$ (*d*) $n = 50$

15. Let $S = \{x \in R : 2^{333x-2} + 2^{111x+1} = 2^{222x+2} + 1\}$. Then Sum of all elements of S is

(*a*) $\dfrac{111}{2}$ (*b*) $\dfrac{2}{111}$
(*c*) 2 (*d*) 111

16. Let U be the universal set and $A \cup B \cup C = U$. Then $\{(A - B) \cup (B - C) \cup (C - A)\}'$ is equal to

(*a*) $A \cup B \cup C^C$ (*b*) $A \cup (B \cap C)$
(*c*) $A \cap B \cap C$ (*d*) $A \cap (B \cup C)$

17. Let F_1 be the set of all parallelograms, F_2 the set of all rectangles, F_3 the set of all rhombuses, F_4 the set of all squares and F_5 the set of trapeziums in a plane. Then F_1 may be equal to

(*a*) $F_2 \cap F_2$ (*b*) $F_3 \cap F_4$
(*c*) $F_2 \cap F_3$ (*d*) $F_2 \cup F_3 \cup F_4 \cup F_1$

18. Let $A = \{(x, y) : a^x = a^y;\ a > 0$ and $a \neq 1;\ a, x, y \in R\}$
$B = \{(x, y);\ xy = 1;\ x, y \in R_0\}$
Choose the correct statements amongst the following.

(*a*) $A \cap B = B$ (*b*) $A \cap B = A$
(*c*) $n(B) > n(A)$ (*d*) A and B are non-comparable

19. The value of $(A \cup B \cup C) \cap (A \cap B^C \cap C^C)^C \cap C^C$ is

(*a*) $B \cap C^C$ (*b*) $B^C \cap C^C$
(*c*) $B \cap C$ (*d*) $A \cap B \cap C$

20. Let A_1, A_2 and A_3 be subsets of a set X. Which one of the following is correct?

(*a*) $A_1 \cup A_2 \cup A_3$ is the smallest subset of X containing elements of each of A_1, A_2 and A_3.
(*b*) $A_1 \cup A_2 \cup A_3$ is the smallest subset of X containing either A_1 or $A_2 \cup A_3$ but not both.
(*c*) The smallest subset of X containing $A_1 \cup A_2$ and A_3 equals the smallest subset of X containing both A_1, and $A_2 \cup A_3$ only if $A_2 = A_3$.
(*d*) $A_1 \cup A_2 \cup A_3$ is the largest subset of X containing elements of each of A_1, A_2 and A_3.

21. Match the following sets for all sets A, B and C

	Column-I		Column-II
A.	$((A' \cup B')' - A)'$	p.	$A - B$
B.	$[B' \cup (B' - A)]'$	q.	U
C.	$(A - B) - (B - C)$	r.	B
D.	$(A - B) \cap (C - B)$	s.	$(A \cap C) - B$

(*a*) A → p, B → q, C → r, D → s
(*b*) A → q, B → r, C → p, D → s
(*c*) A → q, B → r, C → s, D → p
(*d*) A → p, B → r, C → q, D → s

22. Match the set P in Column-I with its super set Q in Column-II

	Column-I		Column-II
A.	$[3^{2n} - 8n - 1 : n \in N]$	p.	$\{49\,(n-1) : n \in N\}$
B.	$\{2^{3n} - 1 : n \in N\}$	q.	$\{64\,(n-1) : n \in N\}$
C.	$\{3^{2n} - 1 : n \in N\}$	r.	$\{7n : n \in N\}$
D.	$\{2^{3n} - 7n - 1 : n \in N\}$	s.	$\{8n : n \in N\}$

(*a*) A → p, B → q, C → r, D → s
(*b*) A → q, B → r, C → p, D → s
(*c*) A → q, B → r, C → s, D → p
(*d*) A → p, B → r, C → q, D → s

23. An investigator interviewed 100 students to determine their preferences for the three drinks: milk (M), coffee (C) and tea (T). He reported the following : 10 students had all the three drinks M, C and T; 20 had M and C; 30 had C and T; 25 had M and T; 12 had M only, 5 had C only; and 8 had T only. If number of students who did not take any the three drinks is n, then $\dfrac{n}{5}$ is

24. In a class of 80 students numbered 1 to 80, all odd numbered students opt of Cricket, students whose numbers are divisible by 5 opt for Football and those whose numbers are divisible by 7 opt for Hockey. If the number of students who do not opt any of the three games is n, then $\dfrac{n}{4}$ is equal to

25. A survey shows that 61%, 46% and 29% of the people watched "3 idiots", "Rajneeti" and "Avatar" respectively. 25% people watched exactly two of the three movies and 3% watched none. What percentage of people watched all the three movies?

26. If $n(A) = 4$ and $n(B) = 7$, then the difference between maximum and minimum value of $n(A \cup B)$ is

27. Let $S = \{x \in R : |x-2|^{10x^2-1} = |x-2|^{3x}\}$. Then $n(S)$ is

28. Let $S = \{x \in R : 5^x \sqrt[x]{8^{x-1}} = 500\}$. Then $n(s)$ is

29. Let U be set with number of elements in it is 2009 and A, B are subsets of U with $n(A \cup B) = 280$. If $n(A' \cap B') = x_1^3 + x_2^3 = y_1^3 + y_2^3$ for some positive integers $x_1 < y_1 < y_2 < x_2$, then find value of $\dfrac{x_2 + y_2}{x_1 + y_1}$.

30. Let U be set with number of elements in it is 2009. A is a subset of U with $n(A) = 1681$ and out of these 1681 elements, exactly 1075 elements belong to a subset B of U. If $n(A - B) = m^2 + p_1 p_2 p_3$ for some positive integer m and distinct primes $p_1 < p_2 < p_3$ then for least m find $\dfrac{p_1 p_3}{p_2}$.

31. In a survey of 60 people, it was found that 25 people read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, and 3 read all the three newspapers. Find the number of people who read at least one of the newspapers.

32. In a survey of 100 students, the number of students studying the various languages is found as: English only 18; English but not Hindi 23; English and Sanskrit 8; Sanskrit and Hindi 8; English 26; Sanskrit 48 and no language 24. Find

(*i*) how many students are studying Hindi,

(*ii*) how many students are studying English and Hindi both.

33. Times, Mirror and Sun are three newspapers

(*i*) All readers of the Times read the Sun

(*ii*) Every person either reads the Sun or does not read the Mirror.

(*iii*) 11 people read the Sun but do not read the Mirror

(*iv*) 8 people read either the Times or the Mirror but not both.

(*v*) 10 people read the Sun and either read the Mirror or do not read the Times

(*vi*) 14 people either read the Sun and not the Mirror or read both the Sun and Times

(*vii*) 9 people neither read the Times nor the Mirror

Find the number of people who read Times and Mirror both

34. Let $S = \{x \in R:\ 5^x + \frac{125}{5^x} = 30\}$. Then possible value of x

35. Let $T = \left\{x \,|\, \frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}\right\}$. Is T an empty set? Justify your answer.

36. Given that $n(U) = 40$, $n(A) = 28$, $n(B) = 25$, $n(A \cap B) = x$ and $n(A' \cap B') = y$.

(*i*) Express y in terms of x.

(*ii*) Find the greatest and least values of x and y.

37. In a survey of 25 students, it was found that 12 have taken physics, 11 have taken chemistry and 15 have taken mathematics; 4 have taken physics and chemistry; 9 have taken physics and mathematics; 5 have taken chemistry and mathematics while 3 have taken all the three subjects. Find the number of students who have taken.

(*i*) mathematics only;

(*ii*) physics and mathematics but not chemistry;

(*iii*) only one of the subjects;

(*iv*) at least one of the three subjects;

(*v*) none of the three subjects.

38. The universal set U and the sets O, P and S are given by

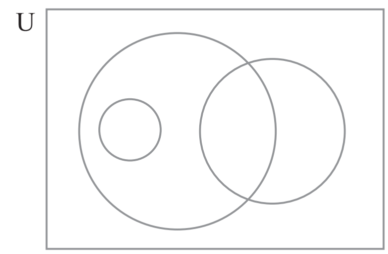
$U = \{x : x \text{ is an integer such that } 3 \le x \le 100\}$,

$O = \{x : x \text{ is an odd number}\}$,

$P = \{x : x \text{ is a prime number}\}$,

$S = \{x : x \text{ is a perfect square}\}$.

In the Venn diagram below, each of the sets O, P and S is represented by a circle.

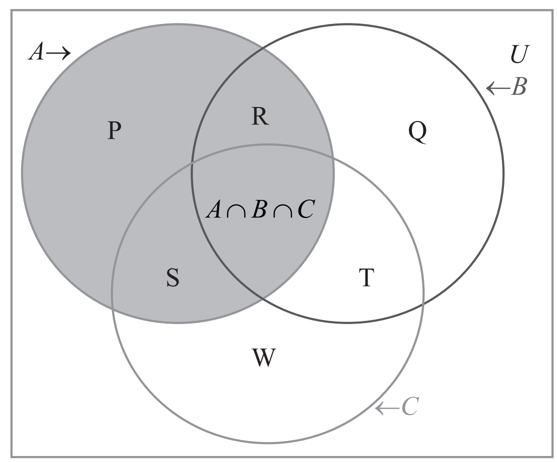


(*i*) Copy the Venn diagram and label each circle with the appropriate letter.

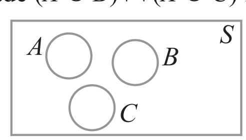
(*ii*) Place each of the numbers 34, 35, 36 and 37 in the appropriate part of your diagram.

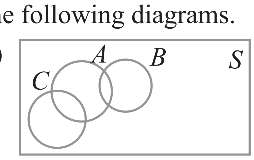
(*iii*) State the value of $n(O \cap S)$ and of $n(O \cup S)$.

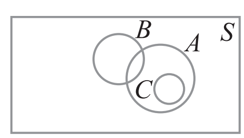
39. Find P, Q, R, S, T, U and V in given venn diagram:

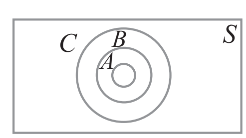


40. Shade $(A \cup B) \cap (A \cup C)$ in the following diagrams.

(*a*)


(*b*)


(*c*)


(*d*)


Exercise-3 (Past Year Questions)

JEE MAIN

1. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then **(2009)**

(a) $A = C$ (b) $B = C$

(c) $A \cap B = \phi$ (d) $A = B$

2. Two sets A and B are as under

$A = \{(a, b) \in R \times R : |a - 5| < 1 \text{ and } |b - 5| < 1\}$;

$B = \{(a, b) \in R \times R : 4(a - 6)^2 + 9(b - 5)^2 \leq 36\}$

Then: **(2018)**

(a) $A \subset B$

(b) $A \not\in B = \phi$ (an empty set)

(c) neither $A \subset B$ nor $B \subset A$

(d) $B \subset A$

3. In a class of 140 students numbered 1 to 140, all even numbered students opted Mathematics course, those whose number is divisible by 3 opted Physics course and those whose number is divisible by 5 opted Chemistry course. Then the number of students who did not opt for any of the three courses is: **(2019)**

(a) 102 (b) 42

(c) 1 (d) 38

4. Let $S = \{1,2,3,......, 100\}$. The number of non-empty subsets A of S such that the product of elements in A is even is: **(2019)**

(a) $2^{100} - 1$ (b) $2^{50}(2^{50} - 1)$

(c) $2^{50} - 1$ (d) $2^{50} + 1$

5. Let Z be the set of integers. If $A = \{X \in Z : 2^{(x+2)(x^2-5x+6)} = 1\}$ and $B = \{x \in Z : -3 < 2x - 1 < 9\}$, then the number of subsets of the set $A \times B$, is: **(2019)**

(a) 2^{15} (b) 2^{18}

(c) 2^{12} (d) 2^{10}

6. Two newspapers A and B are published in a city. It is known that 25% of the city populations reads A and 20% reads B while 8% reads both A and B. Further, 30% of those who read A but not B look into advertisements and 40% of those who read B but not A also look into advertisements, while 50% of those who read both A and B look into advertisements. Then the percentage of the population who look into advertisement is: **(2019)**

(a) 12.8

(b) 13.5

(c) 13.9

(d) 13

7. Let A, B and C be sets such that $\phi \neq A \cap B \subseteq C$. Then which of the following statements is not true? **(2019)**

(a) If $(A - C) \subseteq B$ then $A \subseteq B$

(b) $(C \cup A) \cap (C \cup B) = C$

(c) If $(A - B) \subseteq C$, then $A \subseteq C$

(d) $B \cap C \neq \phi$

8. Let $X = \{n \in N : 1 \leq n \leq 50\}$. If $A = \{n \in X$: n is a multiple of 2); $B = \{n \in X$: n is a multiple of 7$\}$, then the number of elements in the smallest subset of X containing both A and B is _____________. **(2020)**

9. If $A = \{x \in R : |x| < 2\}$ and $B = \{|x - 2| \geq 3\}$: then: **(2020)**

(a) $A - B = [-1, 2)$

(b) $A \cup B = R - (2, 5)$

(c) $B - A = R - (-2, 5)$

(d) $A \cap B = (-2, -1)$

10. Let S be the set of all integer solutions, (x, y, z), of the system of equations **(2020)**

$x - 2y + 5z = 0$

$-2x + 4y + z = 0$

$-7x + 14y + 9z = 0$

such that $15 \leq x^2 + y^2 + z^2 \leq 150$. Then, the number of elements in the set S is equal to ______ .

11. Consider the two sets: **(2020)**

$A = \{m \in R$: both the roots of $x^2 - (m + 1)$ $x = m + 4 = 0$ are real$\}$ and $B = [-3, 5)$.

Which of the following is not true?

(a) $A \cap B = \{-3\}$

(b) $B - A = (-3, 5)$

(c) $A \cup B = R$

(d) $A - B = (-\infty, -3) \cup (5, \infty)$

12. Let $\bigcup_{i=1}^{50} X_i = \bigcup_{i=1}^{n} Y_i = T$ where each X_i contains 10 elements and each Y_i contains 5 elements. If each element of the set T is an element of exactly 20 of sets $X_i's$ and exactly 6 of sets $Y_i's$, then n is equal to **(2020)**

(a) 50 (b) 15

(c) 30 (d) 45

13. A survey shows that 63% of the people in a city read newspaper A whereas 76% read newspaper B. If x% of the people read both the newspapers, then a possible value of x can be: **(2020)**

(a) 37 (b) 55

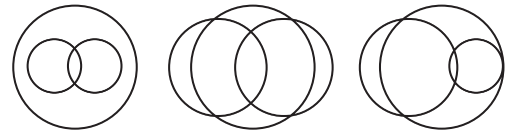
(c) 29 (d) 65

14. A survey shows that 73% of the persons working in an office like coffee, whereas 65% like tea. If x denotes the percentage of them, who like both coffee and tea, then x cannot be: **(2020)**

(*a*) 63
(*b*) 36
(*c*) 38
(*d*) 54

15. Set A has m elements and Set B has n elements. If the total number of subsets of A is 112 more than total number of subsets of B, then the value of $m \cdot n$ is —. **(2020)**

16. In a school, there are three types of games to be played. Some of the students play two types of games, but none play all the three games. Which Venn diagrams can justify the above statement? **(2021)**



P Q R

(*a*) P and Q (*b*) P and R
(*c*) None of these (*d*) Q and R

17. Let $A = \{x \in R : |x+1| < 2\}$ and $B = \{x \in R: |x-1| > 2\}$. Then which one of the following statements is NOT true ? **(2022)**

(*a*) $A - B = (-1, 1)$ (*b*) $B - A = R - (-3, 1)$
(*c*) $A \cap B = (-3, -1]$ (*d*) $A \cup B = R - [1, 3)$

18. Let $A = \{n \in N : \text{H.C.F.}\ (n, 45) = 1\}$ and
Let $B = \{2k : k \in \{1,2,100\}\}$. Then the sum of all the elements of $A \cap B$ is_______. **(2022)**

ANSWER KEY

CONCEPT APPLICATION

1. (*b*)	**2.** (*a*)	**3.** (*a*)	**4.** (*b*)	**5.** (*d*)	**6.** (*c*)	**7.** (*d*)	**8.** (*d*)	**9.** (*d*)	**10.** (*c*)
11. (*b*)	**12.** (*a*)	**13.** (*c*)	**14.** (*a*)	**15.** (*c*)					

16. P $\rightarrow A \cap \overline{B}$, Q $\rightarrow A \cap B$, R $\rightarrow B \cap \overline{A}$ and S $\rightarrow \overline{A} \cap \overline{B}$

EXERCISE-1 (TOPICWISE)

1. (*d*)	**2.** (*b*)	**3.** (*a*)	**4.** (*c*)	**5.** (*b*)	**6.** (*c*)	**7.** (*a*)	**8.** (*b*)	**9.** (*c*)	**10.** (*a*)
11. (*d*)	**12.** (*b*)	**13.** (*a*)	**14.** (*c*)	**15.** (*b*)	**16.** (*c*)	**17.** (*b*)	**18.** (*b*)	**19.** (*d*)	**20.** (*c*)
21. (*d*)	**22.** (*b*)	**23.** (*c*)	**24.** (*c*)	**25.** (*d*)	**26.** (*b*)	**27.** (*a*)	**28.** (*b*)	**29.** (*b*)	**30.** (*c*)
31. (*d*)	**32.** (*b*)	**33.** (*c*)	**34.** (*c*)	**35.** (*a*)	**36.** (*b*)	**37.** (*a*)	**38.** (*b*)	**39.** (*b*)	**40.** (*d*)
41. (*b*)	**42.** (*b*)	**43.** (*a*)	**44.** (*b*)						

EXERCISE-2 (LEARNING PLUS)

1. (*b*)	**2.** (*c*)	**3.** (*a*)	**4.** (*c*)	**5.** (*c*)	**6.** (*d*)	**7.** (*b*)	**8.** (*b*)	**9.** (*b*)	**10.** (*c*)
11. (*a*)	**12.** (*b*)	**13.** (*d*)	**14.** (c)	**15.** (b)	**16.** (c)	**17.** (d)	**18.** (*d*)	**19.** (*a*)	**20.** (*d*)
21. (*b*)	**22.** (*c*)	**23.** [4]	**24.** [7]	**25.** [4]	**26.** (4)	**27.** (4)	**28.** (1)	**29.** (2.2)	**30.** (86/7)
31. [52]	**32.** [18]	**33.** [3]	**34.** [1, 2]	**35.** (No, $T = \{10\}$)					

36. (*i*) $y = x - 13$ (*ii*) max (x) = 25, min (x) = 13, max (y) 12, min (y) = 0 **37.** (*i*) 4 (*ii*) 6 (*iii*) 11 (*iv*) 23 **38.** (*iii*) 4,54

EXERCISE-3 (PAST YEAR QUESTIONS)

JEE Main

1. (*b*)	**2.** (*a*)	**3.** (*d*)	**4.** (*b*)	**5.** (*a*)	**6.** (*c*)	**7.** (*a*)	**8.** [29]	**9.** (*c*)	**10.** [8]
11. (*d*)	**12.** (*c*)	**13.** (*b*)	**14.** (*b*)	**15.** [28.00]	**16.** (*c*)	**17.** (*b*)	**18.** [5264]		

CHAPTER

3 Function

ORDERED PAIRS

An ordered pair consists of two objects or elements in a given fixed order.

For example, if A and B are any two sets, then by an ordered pair of elements we mean a pair (a,b) in that order, where $a \in A$, $b \in B$.

NOTE: An ordered pair is not a set consisting of two elements. The ordering of the two elements in an ordered pair is important and the two elements need not be distinct.

EQUALITY OF ORDERED PAIRS

Two ordered pairs (a_1,b_1) and (a_2,b_2) are equal iff $a_1 = a_2$ and $b_1 = b_2$. i.e. $(a_1,b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$

It is evident from this definition that $(1,2) \neq (2,1)$ and $(1,1) \neq (2,2)$.

CARTESIAN PRODUCT OF SETS

Let A and B be any two non-empty sets. The set of all ordered pairs (a,b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$. Thus, $A \times B = \{(a, b): a \in A \text{ and } b \in B\}$

If $A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.

RELATION

A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product $A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the image of the first element.

DOMAIN OF A RELATION

The set of all first elements of the ordered pairs in a relation R form a set A to a set B is called the domain of the relation R.

RANGE OF A RELATION

The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation R.

Note: Range $\subseteq$ Co-domain.

Note: The total number of relations that can be defined from a A to a set B is the number of possible subsets of $A \times B$. If $n(A)$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relati is 2^{pq}.

Train Your Brain

Example 1: If $\left(\frac{a}{3}+1, b-\frac{2}{3}\right)=\left(\frac{5}{3},\frac{1}{3}\right)$, find the values of a and b.

Sol. Given $\left(\frac{a}{3}+1, b-\frac{2}{3}\right)=\left(\frac{5}{3},\frac{1}{3}\right)$

To find : values of a and b

By the definition of equality of ordered pairs, we have and simultaneously solving for a and b,

$\frac{a}{3}+1=\frac{5}{3}$ and $b-\frac{2}{3}=\frac{1}{3}$

$\Rightarrow \frac{a}{3}=\frac{5}{3}-1$

$\Rightarrow b=\frac{1}{3}+\frac{2}{3}$

$\Rightarrow \frac{a}{3}=\frac{2}{3}$

$\Rightarrow b=1$

$\Rightarrow a=2$

Example 2: If $A=\{1,2,3\}$ and $B=\{2,4\}$, what are $A \times B, B \times A, A \times A, B \times B$, and $(A \times B) \cap (B \times A)$?

Sol. Given : $A = \{1, 2, 3\}$ and $B = \{2, 4\}$

To find : $A \times B, B \times A, A \times A, (A \times B) \cap (B \times A)$

Now,

$A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$

$B \times A = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2), (4, 3)\}$

$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

$B \times B = \{(2, 2), (2, 4), (4, 2), (4, 4)\}$

Intersection of two sets represents common elements of both the sets

So,

$(A \times B) \cap (B \times A) = \{(2, 2)\}$

Example 3: If $A = \{1, 2\}$, form the set $A \times A \times A$.

Sol. Given $A = \{1, 2\}$

To find $A \times A \times A$

Firstly, we will find Cartesian product of A with A

$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

Now, Cartesian product of $A \times A$ with A

$\therefore$ $A \times A \times A = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

Example 4: If A and B are sets having 3 elements in common. If $n(A) = 5$, $n(B) = 4$, find $n(A \times B)$ and $n[(A \times B) \cap (B \times A)]$.

Sol. Given : $(A) = 5$ and $n(B) = 4$

To find : $[(A \times B) \cap (B \times A)]$

$n\ (A \times B) = n(A) \times n(B) = 5 \times 4 = 20$

$n\ (A \cap B) = 3$ (given : A and B has 3 elements in common)

In order to calculate n $[(A \times B) \cap (B \times A)]$, we will assume

$A = (x, x, x, y, z)$ and $B = (x, x, x, p)$

So, we have

$(A \times B) = \{(x,x), (x,x), (x,x), (x,p), (x,x), (x,x), (x, x), (x,p), (x,x), (x,x), (x,x), (x,p), (y,x), (y,x), (y,x), (y, p), (z, x), (z, x), (z, x), (z, p)\}$

$(B \times A) = \{(x, x), (x, x), (x, x), (x, y), (x, z), (x, x), (x, x), (x, x), (x, y), (x, z), (x, x), (x, x), (x, x), (x, y), (x, z), (p, x), (p, x), (p, x), (p, y), (p, z)\}$

$[(A \times B) \cap (B \times A)] = \{(x, x), (x, x), (x, x), (x, x), (x, x), (x, x), (x, x), (x, x), (x, x)\}$

$\therefore$ We can say that n $[(A \times B) \cap (B \times A)] = 9$

Example 5: If $A = \{2, 3\}$, $B = \{4, 5\}$, $C = \{5, 6\}$ find $A \times (B \cup C)$, $(A \times B) \cup (A \times C)$.

Sol. Given : $A = \{2, 3\}$, $B = \{4, 5\}$ and $C = \{5, 6\}$

To find : $(A \times B) \cup (A \times C)$

Since, $(B \cup C) = \{4, 5, 6\}$

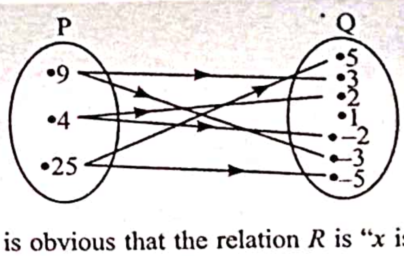
$\therefore$ $A \times (B \cup C) = \{(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$

$(A \times B) = \{(2, 4), (2, 5), (3, 4), (3, 5)\}$

$(A \times C) = \{(2, 5), (2, 6), (3, 5), (3, 6)\}$

$\therefore$ $(A \times B) \cup (A \times C) = \{(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$

Example 6: The Figure shows a relation between the sets P and Q. Write this relation (*i*) in set-builder form, (*ii*) in roster form. What is its domain and range ?



Sol. It is obvious that the relation R is "x is the square of y".

(*i*) In set-builder form, $R = \{(x, y) : x$ is the square of $y, x \in P, y \in Q\}$

(*ii*) In roster form, $R = \{(9, 3), (9, -3), (4, 2), (4, -2), (25, -5) (25, 5)\}$

The domain of this relation is $\{4, 9, 25\}$.

The range of this relation is $\{-2, 2, -3, 3, -5, 5\}$.

Note that the element 1 is not related to any element in set P.

The set Q is the codomain of this relation.

Example 7: Determine the domain and range of the relation R defined by

$R = \{(x, x + 5) : x\ \{0, 1, 2, 3, 4, 5\}$

Sol. Given

$R = \{(x, x + 5) : x\ \{0, 1, 2, 3, 4, 5\}$

$\therefore$ $R = \{(0, 0 + 5), (1, 1 + 5), (2, 2 + 5), (3, 3 + 5), (4, 4 + 5), (5, 5 + 5)\}$

$\Rightarrow$ $R = \{(0, 5), (1, 6), (2, 7), (3, 8), (4, 9), (5, 10)\}$

So,

Domain of relation $R = \{0, 1, 2, 3, 4, 5\}$

Range of relation $R = \{5, 6, 7, 8, 9, 10\}$

Example 8: Determine the domain and range of the relation R defined by $R = \{(x, x^3) : x$ is prime number less than $10\}$

Sol. Given,

$R = \{(x, x^3) : x$ is prime number less than $10\}$

Prime numbers less than 10 are 2, 3, 5 and 7

$\therefore$ $R = \{(2, 2^3), (3, 3^3), (5, 5^3), (7, 7^3)\}$

$\Rightarrow$ $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

So,

Domain of relation $R = \{2, 3, 5, 7\}$

Range of relation $R = \{8, 27, 125, 343\}$

Example 9: If $n(A) = 7$, $n(B) = 8$ and $n(A \cap B) = 4$, then match the following columns.

	Column-I		Column-II
A.	$n(A \cup B)$	p.	56
B.	$n(A \times B)$	q.	16
C.	$n((B \times A) \times A)$	r.	392
D.	$n((A \times B) \cap (B \times A))$	s.	96
E.	$n((A \times B) \cup (B \times A))$	t.	11

Sol. A – t, B – p, C – r, D – q, E – s

A. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 7 + 8 - 4 = 11$

B. $n(A \times B) = n(A)\, n(B) = 7 \times 8 = 56 = n(B \times A)$

C. $n((B \times A) \times A) = n(B \times A).\, n(A) = 56 \times 7 = 392$

D. $n((A \times B) \cap (B \times A)) = (n((A \cap B))^2 = 4^2 = 16$

E. $n((A \times B) \cup (B \times A)) = n(A \times B) + n(B \times A) - n(A \times B) \cap (B \times A)$

$= 56 + 56 - 16 = 96$

Example 10: Find the inverse relation R^{-1} in each of the following cases:

(i) $R = \{(1, 2), (1, 3), (2, 3), (3, 2), (5, 6)\}$

(ii) $R = \{(x, y): x, y\ N; x + 2y = 8\}$

(iii) R is a relation from $\{11, 12, 13\}$ to $(8, 10, 12\}$ defined by $y = x - 3$

Sol. An inverse relation is the set of ordered pairs obtained by interchanging the first and second elements of each pair in the original relation. If the graph of a function contains a point (a, b), then the graph of the inverse relation of this function contains the point (b, a).

(i) Given, $R = \{(1, 2), (1, 3), (2, 3), (3, 2), (5, 6)\}$

$\therefore\ R^{-1} = \{(2, 1), (3, 1), (3, 2), (2, 3), (6, 5)\}$

$\Rightarrow\ R^{-1} = \{(2, 1), (2, 3), (3, 1), (3, 2) (6, 5)\}$

(ii) Given, $R = \{(x, y) : x, y\ N; x + 2y = 8\}$

Here $x + 2y = 8$

$\Rightarrow\ x = 8 - 2y$

As $y\ N$. Put the values or $y = 1, 2, 3, ...$ till xN

On putting $y = 1, x = 8 - 2(1) = 8 - 2 = 6$

On putting $y = 2, x = 8 - 2(2) = 8 - 4 = 4$

On putting $y = 3, x = 8 - 2(3) = 8 - 6 = 2$

On putting $y = 4, x = 8 - 2(4) = 8 - 8 = 0$

Now, y cannot hold value 4 because $x = 0$ for $y = 4$ which is not a natural number.

$\therefore\ R = \{(2, 3), (4, 2), (6, 1)\}$

$R^{-1} = \{(3, 2), (2, 4), (1, 6)\}$

$\Rightarrow\ R^{-1} = \{(1, 6), (2, 4), (3, 2)\}$

Example 11: If $a \in \{-1, 2, 3, 4, 5\}$ and $b \in \{0, 3, 6\}$, write the set of all ordered pairs (a, b) such that $a + b = 5$.

Sol. Given $a \in \{-1, 2, 3, 4, 5\}$ and $b \in \{0, 3, 6\}$.

To find: the ordered pair (a, b) such that $a + b = 5$ then the ordered pair (a, b) such that $a + b = 5$ are as follows $(a, b) \in \{(-1, 6), (2, 3), (5, 0)\}$

Example 12: If $a \in (2, 4, 6, 9\}$ and $b \in (4, 6, 18, 27\}$, then form the set or all ordered pairs (a, b) such that a divides b and $a < b$.

Sol. Given that $a \in \{2, 4, 6, 9\}$ and $b \in \{4, 6, 18, 27\}$.

To find: ordered pairs (a, b) such that a divides b and $a < b$

Here,

2 divides 4, 6, 18 and is also less than all of them

4 divides 4 and is also less than none or them

6 divides 6, 18 and is less than 18 only

9 divides 18, 27 and is less than all or them

Therefore, ordered pairs (a, b) are (2, 4), (2, 6), (2, 18), (6, 18), (9, 18) and (9, 27).

Example 13: A relation R is defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows: $(x, y) R\ x$ is relatively prime to y. Express R as a set of ordered pairs and determine its domain and range.

Sol. Relatively prime numbers are also known as co-prime numbers. If there is no integer greater than one that divides both (that is, their greatest common divisor is one). For example, 12 and 13 are relatively prime. but 12 and 14 are not as their greatest common divisor is two.

Given, $(x, y)\ R\ x$ is relatively prime to y

Here,

2 is co-prime to 3 and 7.

3 is co-prime to 7 and 10.

4 is co-prime to 3 and 7.

5 is co-prime to 3, 6 and 7.

$\therefore\ R = \{(2, 3), (2, 7), (3, 7), (3, 10), (4, 3), (4, 7), (5, 3), (5, 6), (5, 7)\}$

Domain of relation $R = \{2, 3, 4, 5\}$

Range of relation $R = \{3, 6, 7, 10\}$

Example 14: Determine the domain and range of the following relations:

$S = \{a, b): b = |a - 1|, a\ Z$ and $|a| \le 3\}$

Sol. Given,

$S = \{a, b): b = |a - 1|, a\ Z$ and $|a| \le 3\}$

Z denotes integer which can be positive as well as negative

Now, $|a| \le 3$ and $b = |a - 1|$

$\therefore\ a\ \{-3, -2, -1, 0, 1, 2, 3\}$

$S = \{a, b): b = |a - 1|, a\ Z$ and $|a| \le 3\}$

$\Rightarrow\ S = \{a, |a - 1|): b = |a - 1|, a\ Z$ and $|a| \le 3\}$

$\Rightarrow\ S = \{(-3, |-3 - 1|), (-2, |-2 - 1|), (-1, |-1 - 1|), (0, |0 - 1|), (1, |1 - 1|), (2, |2 - 1|), (3, |3 - 1|)\}$

$\Rightarrow\ S = \{(-3, |-4|), (-2, |-3|), (-1, |-2|), (0, |-1|), (1, |0|), (2, |1|), (3, |2|)\}$

$\Rightarrow\ S = \{(-3, 4), (-2, 3), (-1, 2), (0, 1), (1, 0), (2, 1), (3, 2)\}$

So,

Domain of relation $S = \{-3, -2, -1, 0, 1, 2, 3\}$

Range of relation $S = \{0, 1, 2, 3, 4\}$

Concept Application

1. If $(x + 1, 1) = (3y, y - 1)$, find the values of x and y.
2. If $P = \{1, 2\}$, form the set $P \times P \times P$.
3. Given $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, find $(A \times B) \cap (B \times C)$.
4. A relation R is defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows : $(x, y) R x$ is relatively prime to y. Express R as a set of ordered pairs and determine its domain and range.
5. Let $A = \{1, 2, 3, 4, 5, 6\}$. Define a relation R from A to A by $R = \{(x, y) : y = x + 1\}$
   (i) Depict this relation using an arrow diagram
   (ii) Write down the domain, codomain and range of R.

DEFINITION

"A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y", where X and Y are two non empty sets.

Let the correspondence be 'f' then mathematically we write $f : X \to Y$ where $y = f(x)$, $x \in X$ and $y \in Y$. We say that 'y' is the image of 'x' under 'f' (or x is the pre image of y).

So we can say that for a function, following two conditions must be satisfied:

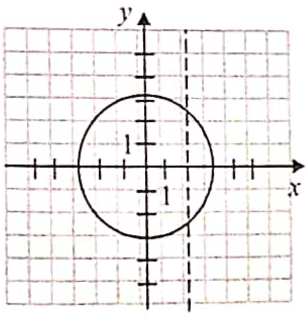
(i) A mapping $f : X \to Y$ is said to be a function if each element in the set X has its image in set Y. It is possible that a few elements in the set Y are present which are not the images of any element in set X.

(ii) Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X. Functions can't be multi-valued.

For $y = f(x)$ (y is a function of x), x is called independent variable and y is called dependent variable.

Note: A relation is a function if no vertical line intersects the graph more than once.

For example: $x^2 + y^2 = 4$ is not a function (as shown in figure)



DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION

Let $f : A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f. If a member 'a' of A is associated to the member 'b' of B, then 'b' is called the **f-image** of 'a' and we write $b = f(a)$. Further 'a' is called a **pre-image** of 'b'. The set $\{f(a) : \forall\ a \in A\}$ is called the **range** of f and is denoted by $f(A)$. Clearly $f(A) \subseteq B$.

If only expression of $f(x)$ is given (domain and codomain are not mentioned), then domain is set of those values of 'x' for which $f(x)$ is real, while codomain is considered to be $(-\infty, \infty)$ (except in ITFs)

A function whose domain and range are both subsets of real numbers is called a **real function.**

Algebraic Operations on Functions : If f & g are real valued functions of x with domains A and B respectively, then both f & g are defined in $A \cap B$. Now we define $f + g$, $f - g$, $(f \cdot g)$ & (f/g) as follows:

(i) $(f \pm g)(x) = f(x) \pm g(x)$
(ii) $(f \cdot g)(x) = f(x) \cdot g(x)$
domain in each case is $A \cap B$

(iii) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B$ and $g(x) \neq 0\}$.

Note: To find the domain of a function, we need to consider what values of the variable make the function undefined.

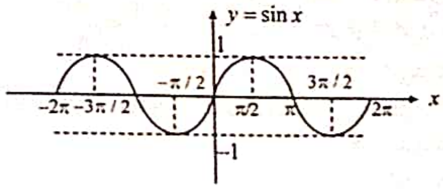
For example:

- the domain of $f(x) = \sqrt{x}$ is $\{x \mid x \geq 0 x \in R\}$, since $\sqrt{x}$ has meaning only when $x \geq 0$.
- the domain of $f(x) = \dfrac{1}{\sqrt{x-1}}$ is $\{x \mid x > 1, x \in R)$, since when $x - 1 = 0$ we are 'dividing by zero' and when $x - 1 < 0$, $\sqrt{x-1}$ is undefined.

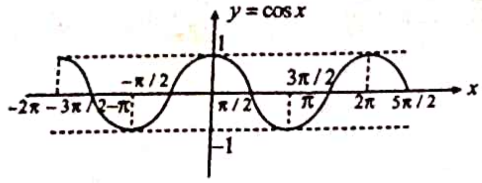
IMPORTANT TYPE OF FUNCTIONS

1. **Trigonometric function:**

Function	Domain
Curve	Range
(i) $f(x) = \sin x$	$x \in R$
$y \in [-1, 1]$	

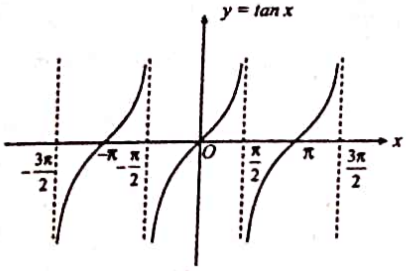


(ii) $f(x) = \cos x$	$x \in R$
$y \in [-1, 1]$	



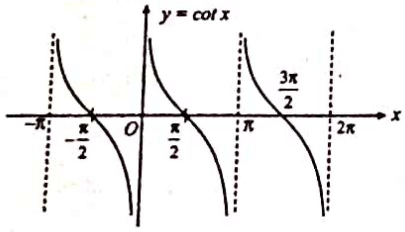
(iii) $f(x) = \tan x$ $\qquad x \in R - (2n+1)\dfrac{\pi}{2}, n \in I$

$y \in R$



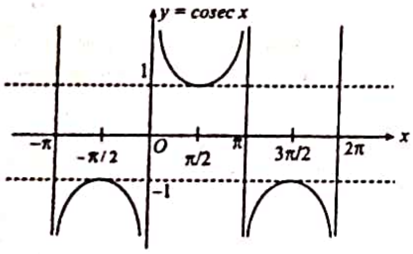
(iv) $f(x) = \cot x$ $\qquad x \in R - n\pi, n \in I$

$y \in R$



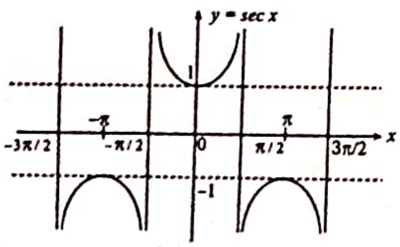
(v) $f(x) = \text{cosec}\, x$ $\qquad x \in R - n\pi, n \in I$

$y \in (-\infty, -1] \cup [1, \infty)$



(vi) $f(x) = \sec x$ $\qquad x \in R - (2n+1)\dfrac{\pi}{2}, n \in I$

$y \in (-\infty, -1] \cup [1, \infty)$



2. Polynomial Function:

If a function f defined by

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$

where $a_0, a_1, a_2 ... a_n \in R, n \in W$

If $a_0 \neq 0$, then $f(x)$ is called n^{th} degree polynomial function and Domain $x \in R$.

Note:

(i) If n is odd, then polynomial is of odd degree. Its range is always R.

(ii) If n is even, then polynomial is of even degree. Its range is never R.

(iii) A polynomial function of degree one with no constant term is called odd linear function.

i.e. $f(x) = ax, a \neq 0$

(iv) $f(x) = ax + b, a \neq 0$ is a linear polynomial function.

(v) $f(x) = c, (c \neq 0)$, is a non linear function (its degree is zero).

(vi) $f(x) = 0$ is a polynomial function whose degree is not defined

(vii) There are only two polynomial functions, satisfying the relation

$$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right). \text{ They are:}$$

(a) $f(x) = x^n + 1$ &

(b) $f(x) = 1 - x^n$, where n is a positive integer.

3. Algebraic Function: A function is called an algebraic function, if it can be constructed using algebraic operations such as additions, subtractions, multiplication, division taking roots etc. e.g.

(i) $f(x) = \sqrt{x^4 + 5x^2} + x + (x^3 + 5)^{3/5}$

$\rightarrow$ algebraic function.

(ii) $f(x) = \dfrac{(x^5 + 5x^2)^{3/5}}{x^3} + 3\sqrt{x^2 + 5x + 6} + \ell nx$

$\rightarrow$ transcendental function

(iii) $f(x) = \sqrt{x^2 + 7} + e^{\ell nx} + \dfrac{x+7}{\sqrt{x^2+7}}$

$\rightarrow$ algebraic function.

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form $P_0(x)\,y^n + P_1(x)y^{n-1} + ... + P_{n-1}(x).y + P_n(x) = 0$. Where n is a positive integer and $P_0(x), P_1(x)$ ... are polynomials in x, e.g. $x^3 + y^3 - 3xy = 0$.

Note:

(i) All polynomial functions are algebraic but converse is not true.

(ii) Function which are not algebraic are called as **transcendental function.**

e.g. exponential function, logarithmic function.

4. Rational Function: It is a function of form $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ & $h(x)$ are polynomial function and $h(x) \neq 0$

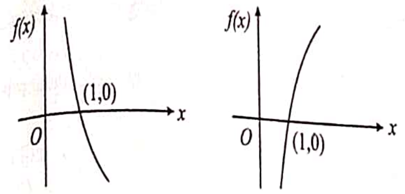
e.g. $f(x) = \dfrac{x^4 - 3x^2 + 2}{x^2 - 4}$

5. Logarithmic function: $f(x) = \log_a x$, where $x > 0, a > 0, a \neq 1$

$a \rightarrow$ base, $x \rightarrow$ number or argument of log.

Case–I : $0 < a < 1$ $\qquad$ **Case–II :** $a > 1$

$f(x) = \log_a x$ $\qquad$ $f(x) = \log_a x$



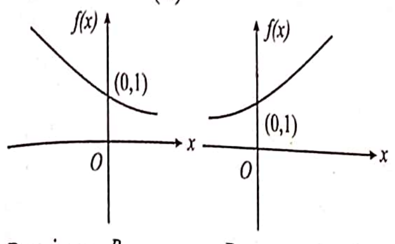
Domain: $x \in (0, \infty)$ Range : $y \in R$

6. **Exponential function:** $f(x) = a^x$, where $a > 0, a \neq 1$

$a \rightarrow$ Base $x \rightarrow$ Exponent

Case–I : $0 < a < 1$; **Case–II :** $a > 1$

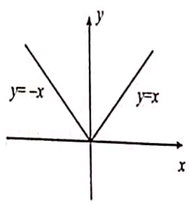
e.g. $a = 1/2, f(x) = \left(\frac{1}{2}\right)^x$ e.g. $a = 2, f(x) = 2^x$



Domain: $x \in R$, Range: $y \in (0, \infty)$

7. **Absolute value function (Modulus function):**

$$y = |x| = \begin{cases} x \ ; \ x \geq 0 \\ -x \ ; \ x < 0 \end{cases}$$



Domain: $x \in R$; Range : $y \in R^+ \cup \{0\}$

Basic properties of $|x|$

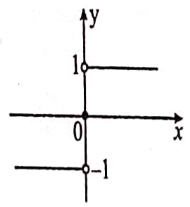
- ❖ $||x|| = |x|$
- ❖ Geometrical meaning of $|x - y|$ is the distance between x and y.
- ❖ $|x| > a \Rightarrow x > a$ or $x < -a$ if $a \in R^+$ and $x \in R$ if $a \in R^-$.
- ❖ $|x| < a \Rightarrow -a < x < a$ if $a \in R^+$ and $x \in \phi$ if $a \in R^- \cup \{0\}$
- ❖ $|xy| = |x||y|$
- ❖ $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \neq 0$
- ❖ $|x + y| \leq |x| + |y|$
- ❖ It is a very useful and interesting property. Here the equality sign holds if x and y either both are non-negative or non-positive (i.e. $x.\ y \geq 0$). $(|x| + |y|)$ represents the sum of distances of numbers x and y from the origin and $|x + y|$ represents the distance of number $x + y$ from the origin (or distance between 'x' and '$-y$' measured along the number line).
- ❖ $|x - y| \geq ||x| - |y||$
- ❖ Here again the equality sign holds if x and y either both are non-negative or non-positive (i.e. $x.\ y \geq 0$). $(|x| - |y|)$ represents the difference of distances of numbers x and y from the origin and $|x - y|$ represents the distance between 'x' and 'y' measured along the number line.

The last two properties can be put in one compact form i.e., $||x| - |y|| \leq |x \pm y| \leq |x| + |y|$.

8. **Signum function :** $y = \text{sgn}(x) = \begin{bmatrix} 1; x > 0 \\ 0; x = 0 \\ -1; x < 0 \end{bmatrix}$

Generally, we can also write

$$\text{sgn}(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



Domain : $x \in R$; Range : $y \in \{-1, 0, 1\}$

9. **Greatest integer function (step-up function):**

The function $f(x) = [x]$ is called the greatest integer function and is defined as follows:

$[x]$ is the greatest integer less than or equal to x.

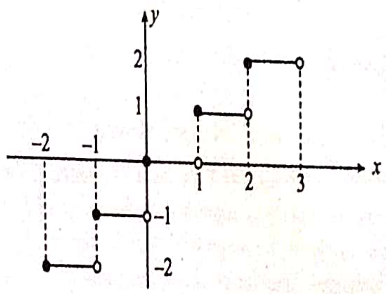
Then $[x] = x$ if x is an integer and integer just less than x if x is not an integer.

Examples: $[3] = 3, [2.7] = 2, [-7.8] = -8, [0.8] = 0$

In other words if we list all the integers less then or equal to x, then the integer greatest among them is called greatest integer of x. Greatest integer of x is also called integral part of x.

$y = f(x) = [x]$

Domain : **R**; Range : **I**;



Properties:

(i) $[x] \leq x < [x] + 1$

(ii) $[x + m] = [x] + m; m \in I$

(iii) $[x] + [-x] = \begin{cases} 0 \ ; \ x \in I \\ -1 \ ; \ x \notin I \end{cases}$

10. **Fractional part function:**

$y = f(x) = \{x\} = x - [x]$

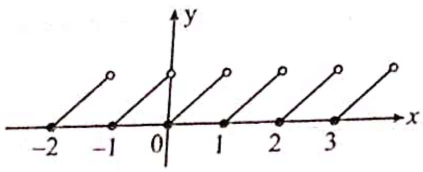
Domain : $x \in R$; Range : $[0, 1)$

Eg.

$2.3 = 2 + 0.3 \rightarrow$ fractional part

$\downarrow$

Integer part



Properties:

(*i*) Fractional part of any integer is zero.

(*ii*) $\{x + n\} = \{x\}, n \in I$

(*iii*) $\{x\} + \{-x\} = \begin{cases} 0\ ; & x \in I \\ 1\ ; & \text{otherwise} \end{cases}$

Important properties of greatest integer function and fractional part of x.

(*i*) $[x] \geq n \Rightarrow x \geq n$ or $x \in [n, \infty)$, $n \in I$

$[x] > n \Rightarrow x \geq n + 1$ or $x \in [n + 1, \infty)$, $n \in I$

$[x] \leq n \Rightarrow x < n + 1$ or $x \in (-\infty, n + 1)$, $n \in I$

$[x] < n \Rightarrow x < n$ or $x \in (-\infty, n)$, $n \in I$

(*ii*) $[[x]] = [x], [\{x\}] = 0, \{[x]\} = 0$

(*iii*) $[n + x] = n + [x]$ where $n \in I$

(*iv*) $[x] + [-x] = \begin{cases} 0, & \text{if } x \in \text{integer} \\ -1, & \text{if } x \notin \text{integer} \end{cases}$

(*v*) $\{x\} + \{-x\} = \begin{cases} 0, & \text{if } x \in \text{integer} \\ 1, & \text{if } x \notin \text{integer} \end{cases}$

(*vi*) $[x + y] = \begin{cases} [x]+[y], & \text{if } \{x\}+\{y\} < 1 \\ [x]+[y]+1, & \text{if } \{x\}+\{y\} \geq 1 \end{cases}$

(*vii*) $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$, $n \in N, x \in \mathbf{R}$

RANGE

It can be evalued using following methods:

1. **Range of Composite Function:** The range of the composed function $f(g(x))$ is contained in the range of f, but it is not the whole of the range of f. And in general, the range of a composed function is either the same as the range of the second function, or else lies inside it. If a value is a possible output from a composed function then it must be a possible output from the second function.

   Key Point: The range of a composed function is either the same as the range of the second function, or else lies inside it.

2. **Range using Elementry Method:** In this method the range of the given function is find using range of sub-function present in function.

3. **Range Completing Perfect Square:** Using this method, range of quadratic function is find out. In this method, quadratic function is converted into perfect square (I constant) and them range is calculated.

4. **Range by converting in the form $x = g(y)$:** To find range, using this method following steps can be performed:

   Step 1: Write the given function in its general representation form, i.e., $y = f(x)$.

   Step 2: Solve it for x and write the obtained function in form of $x = g(y)$.

   Step 3: Now, the domain of the function $x = g(y)$ will be the range of the function $y = f(x)$.

   Thus, the range of a function is calculated.

TYPES OF FUNCTIONS

(1) **Bounded Function:** A function $y = f(x)$ is said to be bounded if it can be express is the form of $a \leq f(x) \leq b$ where a and b are finite quantities.

Ex : $-1 \leq \sin x \leq 1$; $0 \leq \{x\} < 1$; $-1 \leq \text{sgn}(x) \leq 1$ but e^x is not bounded.

Ex : Any function having singleton range like constant function.

(2) **Implicit function & Explicit function :** An explicit function is one where the y can be isolated on the left side of your equation as $y = f(x)$, while the left side of the equation has no other variable except for x.

i.e. this means that the independent variable is written explicitly in terms of the dependent variable.

Functions which are not explicit are called implicit functions: they are functions in which one variable is not defined completely in terms of the other. Some implicit functions can be rewritten as explicit functions. Others cannot.

The function $y - x^2 = 0$ is an implicit function, but it can be rewritten (using basic algebra) as an explicit function as $y = x^2$

(3) **Equal or Identical Functions:** Two functions f & g are said to be equal if :

(*i*) The domain of f = The domain of g $\Rightarrow D_f = D_g$

(*ii*) The range of f = The range of $g \Rightarrow R_f = R_g$

(*iii*) $f(x) = g(x)$, $\forall x \in$ their common domain.

COMPOSITE FUNCTIONS

Let f: $X \rightarrow Y_1$ and g: $Y_2 \rightarrow Z$ be two functions and the set $D = \{x \in X: f(x) \in Y_2\}$. If $D \neq \phi$, then the function h defined on D by $h(x) = g\{f(x)\}$ is called composite function of g and f and is denoted by gof. It is also called function of a function.

Remark: Domain of go f is D which is a subset of X (the domain of f). Range of go f is a subset of the range of g. If $D = X$, then $f(X) \subseteq Y_2$.

fo g means that g converts x to $2x + 3$ and then f converts $(2x + 3)$ to $(2x + 3)^4$. This is illustrated by the two function machines below:

Algebraically, if $f(x) = x^4$ and $g(x) = 2x + 3$, then

$(f \text{ o } g)(x) = f(g(x))$ and $(g \text{ o } f)(x) = g(f(x))$

$= f(2x + 3)$ {g operates on x first} $= g(x^4)$

$= (2x + 3)^4$ {f operates on g(x) next} $= 2(x^4) + 3$

$= 2x^4 + 3$

Algebraically, if $f(x) = x^4$ and $g(x) = 2x + 3$, then

$(f \circ g)(x) = f(g(x))$

$= f(2x + 3)$ {g operates on x first}

$= (2x + 3)^4$ {f operates on $g(x)$ next}

and $(g \circ f)(x) = g(f(x))$

$= g(x^4)$

$= 2(x^4) + 3$

$= 2x^4 + 3$

Train Your Brain

Example 15: Let $A = \{p, q, r, s\}$ and $B = \{1, 2, 3\}$. Which of the following relations from A to B is not a function?

(*i*) $R_1 = \{(p, 1), (q, 2), (r, 1), (s, 2)\}$

(*ii*) $R_2 = \{(p, 1), (q, 1), (r, 1), (s, 1)\}$

(*iii*) $R_3 = \{(p, 1), (q, 2), (p, 2), (s, 3)\}$

(*iv*) $R_4 = \{(p, 2), (q, 3), (r, 2), (s, 2)\}$

Sol. Given $A = \{p, q, r, s\}$ and $B = \{1, 2, 3\}$

(*i*) $R_1 = \{(p, 1), (q, 2), (r, 1), (s, 2)\}$

Every element of set A has an ordered pair in the relation R_1 and no two ordered pairs have the same first component but different second components.

Hence, the given relation R_1 is a function.

(*ii*) $R_2 = \{(p, 1), (q, 1), (r, 1), (s, 1)\}$

Every element of set A has an ordered pair in the relation R_2, and no two ordered pairs have the same first component but different second components.

Hence, the given relation R_2 is a function.

(*iii*) $R_3 = \{(p, 1), (q, 2), (p, 2), (s, 3)\}$

Every element of set A has an ordered pair in the relation R_3. However, two ordered pairs $(p, 1)$ and $(p, 2)$ have the same first component but different second components.

Hence, the given relation R_3 is not a function.

(*iv*) $R_4 = \{(p, 2), (q, 3), (r, 2), (s, 2)\}$

Every element of set A has an ordered pair in the relation R_4, and no two ordered pairs have the same first component but different second components.

Hence, the given relation R_4 is a function.

Example 16: Find the domain of each of the following real valued functions of real variable:

$$f(x) = \frac{1}{\sqrt{x^2 - 1}}$$

Sol. $f(x) = \dfrac{1}{\sqrt{x^2 - 1}}$

We know the square of a real number is never negative.

Clearly, $f(x)$ takes real values only when $x^2 - 1 \ge 0$

$\Rightarrow$ $x^2 - 1^2 \ge 0$

$\Rightarrow$ $(x + 1)(x - 1) \ge 0$

$\Rightarrow$ $x \le -1$ or $x \ge 1$

$\therefore$ $x \in (-\infty, -1] \cup [1, \infty)$

In addition, $f(x)$ is also undefined when $x^2 - 1 = 0$ because denominator will be zero and the result will be indeterminate.

$x^2 - 1 = 0 \Rightarrow x = \pm 1$

Hence, $x \in (-8, -1] \cup [1, \infty) - \{-1, 1\}$

Hence, $x \in (-\infty, -1] \cup [1, \infty) - \{-1, 1\}$

$\therefore$ $x \in (-\infty, -1) \cup (1, \infty)$

Thus, domain of $f = (-\infty, -1) \cup (1, \infty)$

Example 17: Find the domain of each of the following real valued functions of real variable:

(*i*) $f(x) = \dfrac{1}{x}$

(*ii*) $f(x) = \dfrac{1}{x - 7}$

(*iii*) $f(x) = \dfrac{3x - 2}{x + 1}$

(*iv*) $f(x) = \dfrac{2x + 1}{x^2 - 9}$

(*v*) $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12}$

Sol. (*i*) $f(x) = \dfrac{1}{x}$

Clearly, $f(x)$ is defined for all real values of x, except for the case when $x = 0$.

When $x = 0$, $f(x)$ will be undefined as the division result will be indeterminate.

Thus, domain of $f = R - \{0\}$

(*ii*) $f(x) = \dfrac{1}{x - 7}$

Clearly, $f(x)$ is defined for all real values of x, except for the case when $x - 7 = 0$ or $x = 7$.

When $x = 7$, $f(x)$ will be undefined as the division result will be indeterminate.

Thus, domain of $f = R - \{7\}$

(*iii*) $f(x) = \dfrac{3x - 2}{x + 1}$

Clearly, $f(x)$ is defined for all real values of x, except for the case when $x + 1 = 0$ or $x = -1$.

When $x = -1$, $f(x)$ will be undefined as the division result will be indeterminate.

Thus, domain of $f = R - \{-1\}$

(iv) $f(x) = \dfrac{2x+1}{x^2-9}$

Clearly, $f(x)$ is defined for all real values of x, except for the case when $x^2 - 9 = 0$.

$x^2 - 9 = 0$

$\Rightarrow x^2 - 3^2 = 0$

$\Rightarrow (x+3)(x-3) = 0$

$\Rightarrow x + 3 = 0$ or $x - 3 = 0$

$\Rightarrow x = \pm 3$

When $x = \pm 3$, $f(x)$ will be undefined as the division result will be indeterminate.

Thus, domain of $f = R - \{-3, 3\}$

(v) $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12}$

Clearly, $f(x)$ is defined for all real values of x, except for the case when $x^2 - 8x + 12 = 0$.

$x^2 - 8x + 12 = 0$

$\Rightarrow x^2 - 2x - 6x + 12 = 0$

$\Rightarrow x(x-2) - 6(x-2) = 0$

$\Rightarrow (x-2)(x-6) = 0$

$\Rightarrow x - 2 = 0$ or $x - 6 = 0$

$\Rightarrow x = 2$ or 6

When $x = 2$ or 6, $f(x)$ will be undefined as the division result will be indeterminate.

Thus, domain of $f = R - \{2, 6\}$

Example 18: The domain of the function f defined by $f(x) = \sqrt{4-x} + \dfrac{1}{\sqrt{x^2-1}}$ is equal to

(a) $(-\infty, -1) \cup (1, 4]$

(b) $(-\infty, -1] \cup (1, 4]$

(c) $(-\infty, -1) \cup [1, 4]$

(d) $(-\infty, -1) \cup [1, 4)$

Sol. (a) We have, $f(x) = \sqrt{4-x} + \dfrac{1}{\sqrt{x^2-1}}$

$f(x)$ is defined, if $4 - x \geq 0$ or $x^2 - 1 > 0$

$x - 4 \leq 0$ or $(x+1)(x-1) > 0$

$x \leq 4$ or $x < -1$ and $x > 1$

$\therefore$ Domain of $f = (-\infty, -1) \cup (1, 4]$

Example 19: The domain of the function f given by

$$f(x) = \frac{x^2+2x+1}{x^2-x-6}$$

(a) $R - \{3, -2\}$ (b) $R - \{-3, 2\}$

(c) $R - [3, -2]$ (d) $R - (3, -2)$

Sol. (a) We have,

$$f(x) = \frac{x^2+2x+1}{x^2-x-6}$$

$f(x)$ is defined, if $x^2 - x - 6 = 0$

$\Rightarrow x^2 - 3x + 2x - 6 = 0$

$\Rightarrow x(x-3) + 2(x-3) = 0$

$\Rightarrow (x-3)(x+2) = 0$

$\therefore x = -3, -2$

$\therefore$ Domain of $f = R - \{3, -2\}$

Example 20: Let $f(x) = \sqrt{x}, g(x) = \sqrt{2-x}$ find domain, range of

(a) $fog(x)$ (b) $fof(x)$

(c) $gof(x)$ (d) $gog(x)$

Sol. (?) $fog(x) = f(g(x)) = \sqrt{\sqrt{2-x}} = (2-x)^{1/4}$

$2 - x \geq 0 \Rightarrow x \leq 2$

domain $= (-\infty, 2]$

Range $= (0, \infty]$

Example 21: Find range of $3x^2 - 5$ using elementry method.

Sol. $f(x) = 3x^2 - 5$

$0 \leq x^2 < \infty$

$0 \leq 3x^2 < \infty$

$-5 \leq 3x^2 - 5 < \infty$

$-5 \leq f(x) < \infty$,

Range $\equiv [-5, \infty)$

Example 22:

(a) $f(x) = 1 - |x-2|$ (b) $f(x) = |x-5| + 2$

(c) $f(x) = 13 + |2x-3|$ (d) $f(x) = 10 - |x+1|$

Sol.

(a) $(-\infty, 1]$ (b) $[2, \infty)$

(c) $[13, \infty)$ (d) $[-\infty, 10)$

Example 23: $f(x) = x^2 + x + 1$

Sol. $f(x) = x^2 + x + 1 + (1/2)^2 - (1/2)^2$

$$f(x) = \left(x + \frac{1}{2}\right)^2 + 1 - \frac{1}{4}$$

$$f(x) = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$$

Range $= \left[\dfrac{3}{4}, \infty\right)$

Example 24: Find the range of the function $f(x) = 1/(4x-3)$.

Sol. Given: $f(x) = 1/(4x-3)$

Let $y = 1/(4x-3)$

$4xy - 3y = 1$

$4xy = 1 + 3y$

$x = 4y/(1+3y)$

Here, x is defined only when y is not equal to $-1/3$.

So, the range of $f(x) = 1/(4x-3)$ is $(-\infty, -1/3) \cup (1, \infty)$.

Example 25: The range of of function $f(x) = \dfrac{x^2}{1+x^2}$ is

(a) $R - \{1\}$ (b) $R^+ \cup \{0\}$

(c) $[0, 1)$ (d) None of these

Sol. (*d*) Range is containing those real numbers y for which $f(x) = y$ where x is real number.

Now $f(x) = y \Rightarrow \frac{x^2}{1+x^2} = y$

$\Rightarrow x^2 = \frac{y}{1-y}$

$\therefore\ 0 \le y < 1$

Example 26: For real values of x, range of function $y = \frac{1}{2-\sin 3x}$ is

(*a*) $\frac{1}{3} \le y \le 1$ (*b*) $-\frac{1}{3} \le y \le 1$

(*c*) $-\frac{1}{3} > y > -1$ (*d*) $\frac{1}{3} > y > 1$

Sol. (*b*) $2 - \sin 3x = \frac{1}{y}$

$\Rightarrow \sin 3x = 2 - \frac{1}{y}$

So $-1 \le 2 - \frac{1}{y} \le 1$

Example 27: If $f(x+3) = x^2 + 1$ then find $f(x)$

Sol. Let $x + 3 = y \Rightarrow x = y - 3$

$f(y) = (y-3)^2 + 1$

$\Rightarrow f(y) = y^2 - 6y + 10$

Replace $y \leftrightarrow x$ then

$\Rightarrow f(x) = x^2 - 6x + 10$

Example 28: $g(x) + g(1-x) = 1$ find $g(1/2)$

Sol. Put $x = 1/2$ then we get

$\Rightarrow g(1/2) + g(1/2) = 1 \Rightarrow 2g(1/2) = 1$

$\Rightarrow g(1/2) = 1/2$

Example 29: The domain and range of the function f given by $f(x) = 2 - |x-5|$ is

(*a*) Domain $= R^+$, Range $= (-\infty, 1]$

(*b*) Domain $= R$, Range $= (-\infty, 2]$

(*c*) Domain $= R$, Range $= (-\infty, 2)$

(*d*) Domain $= R^+$, Range $= (-\infty, 2]$

Sol. (*b*) We have,

$f(x)$ is defined for all $x \in R$

$\therefore$ Domain of $f = R$

We know that, $|x-5| \ge 0 \Rightarrow -|x-5| \le 0$

$2 - |x-5| \le 2$

Example 30: Find $f+g$, $f-g$, cf $(c \in R, c \ne 0)$, fg, $1/f$ and f/g in each of the following:

$f(x) = x^3 + 1$ and $g(x) = x + 1$

Sol. (*i*) $f(x) = x^3 + 1$ and $g(x) = x + 1$

We have $f(x) : R \to R$ and $g(x) : R \to R$

(*a*) $f + g$

We know $(f+g)(x) = f(x) + g(x)$

$\Rightarrow (f+g)(x) = x^3 + 1 + x + 1$

$\therefore (f+g)(x) = x^3 + x + 2$

Clearly, $(f+g)(x) : R \to R$

Thus, $f+g : R \to R$ is given by $(f+g)(x) = x^3 + x + 2$

(*b*) $f - g$

We know $(f-g)(x) = f(x) - g(x)$

$\Rightarrow (f-g)(x) = x^3 + 1 - (x+1)$

$\Rightarrow (f-g)(x) = x^3 + 1 - x - 1$

$\therefore (f-g)(x) = x^3 - x$

Clearly, $(f-g)(x) : R \to R$

Thus, $f-g : R \to R$ is given by $(f-g)(x) = x^3 - x$

(*c*) cf $(c \in R, c \ne 0)$

We know $(cf)(x) = c \times f(x)$

$\Rightarrow (cf)(x) = c(x^3 + 1)$

$\therefore (cf)(x) = cx^3 + c$

Clearly, $(cf)(x) : R \to R$

Thus, $cf : R \to R$ is given by $(cf)(x) = cx^3 + c$

(*d*) fg

We know $(fg)(x) = f(x)g(x)$

$\Rightarrow (fg)(x) = (x^3 + 1)(x + 1)$

$\Rightarrow (fg)(x) = (x+1)(x^2 - x + 1)(x+1)$

$\therefore (fg)(x) = (x+1)^2(x^2 - x + 1)$

Clearly, $(fg)(x) : R \to R$

Thus, $fg : R \to R$ is given by $(fg)(x) = (x+1)^2(x^2 - x + 1)$

(*e*) $\frac{1}{f}$

We know $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$

$\therefore \left(\frac{1}{f}\right)(x) = \frac{1}{x^3+1}$

Observe that $\frac{1}{f(x)}$ is undefined when $f(x) = 0$ or when $x = -1$.

Thus, $\frac{1}{f} : R - \{-1\} \to R$ is given by

$\left(\frac{1}{f}\right)(x) = \frac{1}{x^3+1}$

(*f*) $\frac{f}{g}$

We know $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$\Rightarrow \left(\frac{f}{g}\right)(x) = \frac{x^3+1}{x+1}$

Observe that $\frac{x^3+1}{x+1}$ is undefined when $g(x) = 0$ or when $x = -1$.

Using $x^3 + 1 = (x+1)(x^2 - x + 1)$, we have

$$\left(\frac{f}{g}\right)(x) = \frac{(x+1)\left(x^2-x+1\right)}{x+1}$$

$$\therefore \left(\frac{f}{g}\right)(x) = x^2 - x + 1$$

Thus, $\frac{f}{g} : R - \{-1\} \to R$ is given by

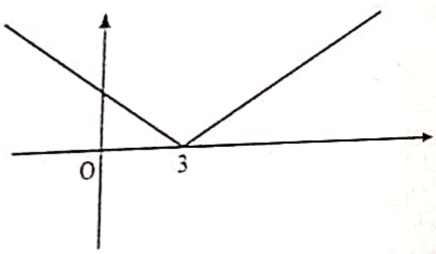
$$\left(\frac{f}{g}\right)(x) = x^2 - x + 1$$

Example 31: Find the range of each of the following functions.

(a) $f(x) = |x-3|$ (b) $f(x) = \frac{x}{1+x^2}$

(c) $f(x) = \sqrt{16-x^2}$ (d) $f(x) = \frac{|x-4|}{x-4}$

Sol: (i) Range $y \in [0, \infty)$



(ii) $y = \frac{x}{x+x^2}$

Method I:

Domain $x \in R$

$yx^2 - x + y = 0$

Quadratic in x has real roots as $x \in R$

$\therefore$ Discriminant $D \geq 0 \Rightarrow 1 - 4y^2 \geq 0$

$\Rightarrow (2y-1)(2y+1) \leq 0 \Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Here at $y = 0$ quadratic vanishes, so we have to check this separately

Put $y = 0$

$\Rightarrow x = 0$ (a point with in domain)

$\Rightarrow y = 0$ point is included in the range

Note: If there is no point of x in the domain for the value of y for which quadratic vanishes, we have to remove that point from range.

Method II:

$f(x) = \frac{x}{1+x^2} = \frac{1}{\left(\frac{1}{x}+x\right)}$ we known that $\left|x+\frac{1}{x}\right| \geq 2$

$$\Rightarrow 0 < \frac{1}{\left|x+\frac{1}{x}\right|} \leq 2$$

$$\Rightarrow \frac{1}{\left(x+\frac{1}{x}\right)} \in \left[-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right]$$

But division by x is done by us

So at $x = 0, y = 0$

$\therefore$ Range $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

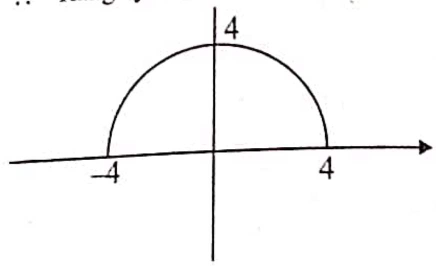
(iii) $f(x) = \sqrt{16-x^2}$, Domain $x \in [-4, 4]$

$\Rightarrow f(x) > 0, y = \sqrt{16-x^2}$

$\Rightarrow x^2 + y^2 = 16$

Equation of semicircle

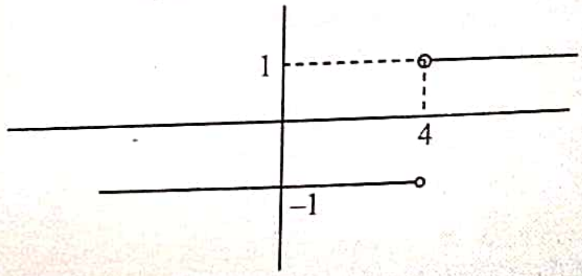
$\therefore$ Range $y \in [0, 4]$



(iv) $f(x) = \frac{|x-4|}{x-4}, x \neq [-4, 4]$

$$\Rightarrow f(x) = \begin{cases} 1 & , x > 4 \\ -1 & , x < 4 \end{cases}$$

$\therefore$ Range $y \in \{-1, 1\}$



Concept Application

6. Which one of the following is not a function?

(a) $\{(x, y) : x, y \in R, x^2 = y\}$

(b) $\{(x, y) : x, y \in R, x^2 = x\}$

(c) $\{(x, y) : x, y \in R, x^2 = y^3\}$

(d) $\{(x, y) : x, y \in R, x^2 = x^3\}$

(e) $\{(x, y) : y^2 = x, y, \in R\}$

(f) $\{(x, y) : y = |x|, x, y, \in R\}$

(h) $\{(x, y) : x^2 + y^2 = 1, x, y \in R\}$

(i) $\{(x, y) : x^2 - y^2 = 1, x, y \in R\}$

7. Find the domain of each of the following real valued functions of real variable:

(i) $f(x) = \dfrac{1}{x}$

(ii) $f(x) = \dfrac{1}{x-7}$

(iii) $f(x) = \dfrac{3x-2}{x+1}$

(iv) $f(x) = \dfrac{2x+1}{x^2-9}$

(v) $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12}$

8. A function $f : R \to R$ is defined by $f(x) = x^2$. Determine

(i) range of f

(ii) $\{x : f(x) = 4\}$

(iii) $\{y : f(y) = -1\}$

9. Find the range of $f(x)$:

(i) $f(x) = \dfrac{x^2-25}{x-5}$ (ii) $f(x) = \dfrac{x^2-3}{x-\sqrt{3}}$

(iii) $f(x) = \dfrac{3x^2-6}{x+\sqrt{2}}$ (iv) $f(x) = \dfrac{1}{\sqrt{2x-1}}$

(v) $f(x) = \dfrac{2}{\sqrt{3x+4}}$ (vi) $f(x) = \sqrt{x-1}$

(vii) $f(x) = \sqrt{2x-5}$ (viii) $f(x) = \sqrt{\dfrac{x-1}{2-x}}$

(ix) $f(x) = \sqrt{\dfrac{4x+1}{2x+5}}$ (x) $f(x) = \dfrac{|x+5|}{x+5}$

(xi) $f(x) = \dfrac{|(x-1)(x-2)|}{(x-1)^2}$ (xii) $f(x) = \dfrac{|x^2+x|}{x}$

(xiii) $f(x) = \dfrac{|2x+3|}{-2x-3}$ (xiv) $f(x) = \dfrac{x^2-1}{x-1}$

(xv) $f(x) = \dfrac{|x^2-1|}{x^2-1}$ (xvi) $f(x) = \dfrac{|x-2|}{x-2}$

(xvii) $f(x) = \dfrac{x-2}{|x-2|}$ (xviii) $f(x) = \dfrac{x^2+1}{|x^2+1|}$

10. Find the range of the following function:

(i) $2x^2 + 12x - 9$ (ii) $3x^2 + 12x + 15$

(iii) $4x^2 + 16x - 3$ (iv) $2x^2 - 8x + 5$

(v) $3x^2 - 18x + 29$ (vi) $4x^2 - 16x + 12$

(vii) $-2x^2 + 4x - 11$ (viii) $-3x^2 + 18x - 20$

(ix) $-5x^2 - 60x - 75$

11. Let $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to Z$ be a function defined by $f(x) = x^2 - 2x - 3$. Find:

(i) range of f i.e. $f(A)$

(ii) pre-images of 6, -3 and 5

12. Which of the following relations are functions? Give reasons. If is a function, determine its domain and range.

(i) $\{(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)\}$

(ii) $\{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)\}$

(iii) $\{(1, 3), (1, 5), (2, 5)\}$.

13. Find the domain of the following functions:

(i) $f(x) = \sqrt{x^2-x-6} + \sqrt{6-x}$

(ii) $f(x) = \sqrt{3x-x^3}$

(iii) $f(x) = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{1+x}}$

(iv) $f(x) = \sqrt{\sin x} + \sqrt{16-x^2}$

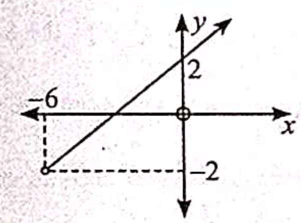
(v) $y = \log_{(x-4)} (x^2 - 11x + 24)$

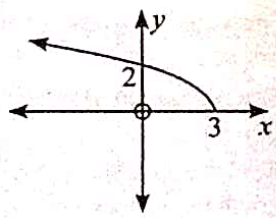
(vi) $y = \sqrt{\log_3(\cos(\sin x))}$

14. Find $f + g$, $f - g$, cf $(c \in R, c \neq 0)$, fg, $1/f$ and f/g in each of the following :

$f(x) = \sqrt{x-1}$ and $g(x) = \sqrt{x+1}$

15. Find the domain and range of the following relations:





16. Evaluate each expression using the given table of values:

x	-2	-1	0	1	2
$f(x)$	1	0	-2	1	2
$g(x)$	2	1	0	-1	0

(i) $f(g(-1))$ (ii) $g(f(0))$

(iii) $f(f(-1))$ (iv) $g(g(2))$

(v) $g(f(-2))$ (vi) $f(g(1))$

17. Evaluate each expression using the functions

$$f(x) = 2 - x, g(x) = \begin{cases} -x, & -2 \le x < 0 \\ x-1, & 0 \le x \le 2 \end{cases}$$

(i) $f(g(0))$ (ii) $g(f(3))$

(iii) $g(g(-1))$ (iv) $f(f(2))$

(v) $g(f(0))$ (vi) $f(g(1/2))$

18. Find the value of the following (where [·] represent greatest integer function):

(i) $[1 \cdot 2]$ (ii) $[\pi]$

(iii) $[-2 \cdot 4]$ (iv) $[\pi^2]$

(v) $[-\pi^2]$ (vi) $[e]$

19. Solve for x:

(i) $[x] = -5$ (ii) $[x] = 0$

(iii) $[x] = \sqrt{3}$ (iv) $[x+3] = 3$

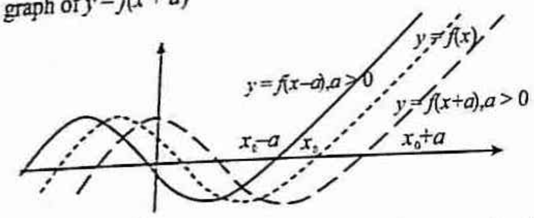
(v) $[x + \frac{3}{4}] = \frac{3}{4}$ (vi) $[2x+3] = 5$

TRANSFORMATIONS OF GRAPHS

1. Drawing the graph of $y = f(x \pm a)$, $a > 0$ from the known graph of $y = f(x)$

Shift the graph of $y = f(x)$ towards R.H.S. by a units to get the graph of $y = f(x-a)$

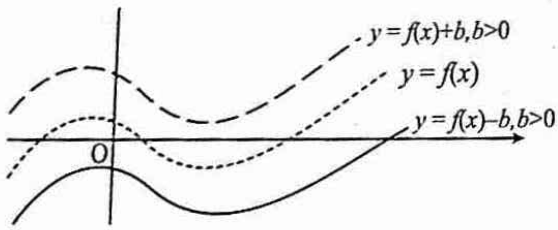
Shift the graph of $y = f(x)$ towards L.H.S. by a units to get the graph of $y = f(x+a)$



2. Drawing the graph of $y = f(x) \pm b$, $b > 0$, from the known graph of $y = f(x)$

Shift the graph of $y = f(x)$ upwards by b units to gel the graph of $y = f(x) + b$

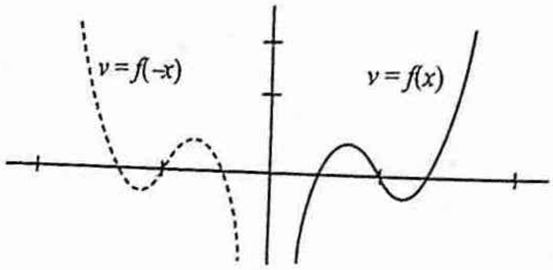
Shift the graph of $y = f(x)$ downwards by b units to get the graph of $y = f(x) - b$



3. Drawing the graph of $y = f(-x)$, from the known graph of $y = f(x)$

Step 1: Draw the graph of f

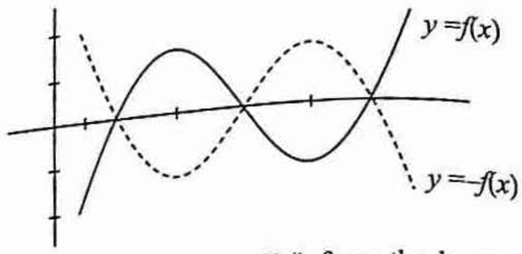
Step 2: Take it's image in y-axis. This image is the graph of $f(-x)$



4. Drawing the graph of $y = -f(x)$, from the known graph of $y = f(x)$

Step 1: Draw the graph of f

Step 2: Take it's image in x-axis. This image is the graph of $y = -f(x)$

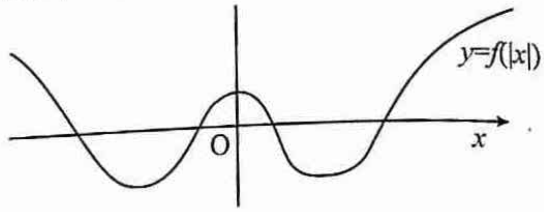


5. Drawing the graph of $y = f(|x|)$ from the known graph of $y = f(x)$

Step 1: Draw the graph of f only on the R.H.S. of y-axis.

Step 2: Take it's image in y-axis. The graph on R.H.S. and the image on L.H.S. together fonn the graph of $y = f(|x|)$

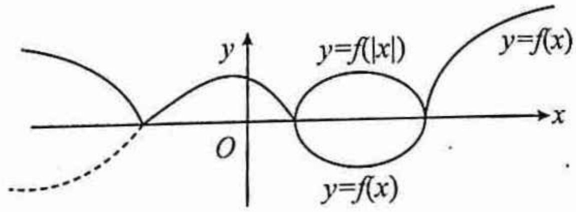
Note: The graph off which lies on L.H.S. of y-axis has no role in the graph of $y = f(|x|)$



6. Drawing the graph of $y = |f(x)|$ from the known graph of $y = f(x)$

Step 1: Draw the graph of f

Step 2: Take the image in x-axis of that portion of the graph which lies below x-axis.

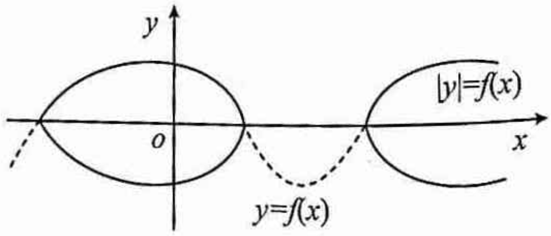


7. Drawing the graph of $|y| = f(x)$ from the known graph of $y = f(x)$

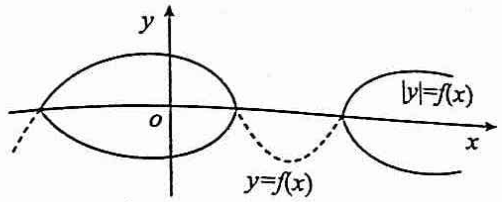
Step 1: Draw the graph of f

Step 2: Delete that portion of the graph which lies below x-axis

Step 3: Take the image in x-axis of the remaining portion of the graph.

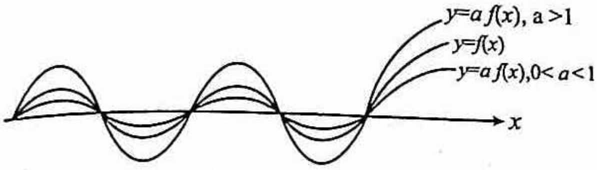


8. Drawing the graph of $y = a. f(x)$ from the known graph $y = f(x)$



It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of $1 : a$.

9. Drawing the graph of $y = f(ax)$ from the known graph of $y = f(x)$



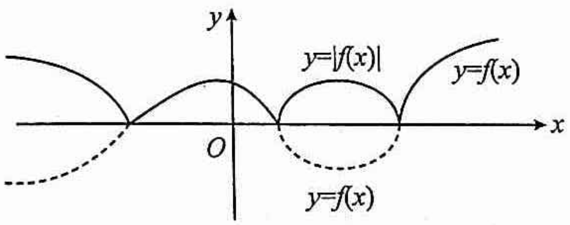
Let us take any point $x_0 \in$ domain of $f(x)$. Let $ax = x_0$ or $x = \frac{x_0}{a}$.

Clearly if $0 < a < 1$ then $x > x_0$ and $f(x)$ will stretch by $1/a$ units against y-axis, and if $a > 1$, $x < x_0$, then $f(x)$ will compress by 'a' units against y-axis.

Note: that the point of maxima and minima are on the line parallel to the x-axis for both the curves above.

10. Drawing the graph of $y = |f(x)|$ from the known graph of $y = f(x)$

$|f(x)| = f(x)$ if $f(x) \geq 0$ and $|f(x)| = -f(x)$ if $f(x) < 0$. It means that the graph of $f(x)$ and $|f(x)|$ would coincide if $f(x) \geq 0$ and the portions where $f(x) < 0$ would get inverted in the upward direction.



The figure would make the procedure clear.

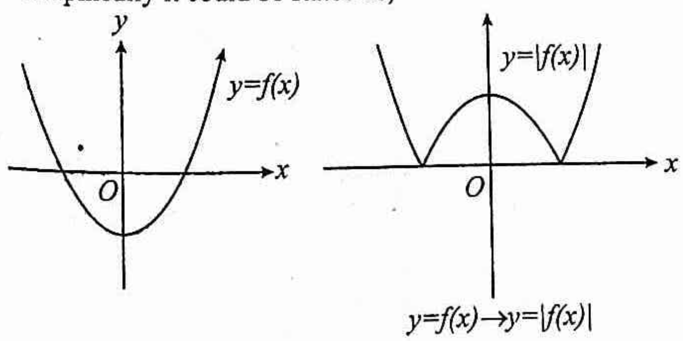
11. $y - f(x)$ transforms to $|y| = |f(x)|$

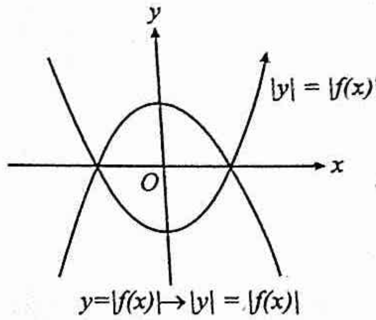
i.e., $y = f(x) \rightarrow |y| = |f(x)|$; is plotted in two steps.

(*i*) $y = f(x) \rightarrow y = |f(x)|$

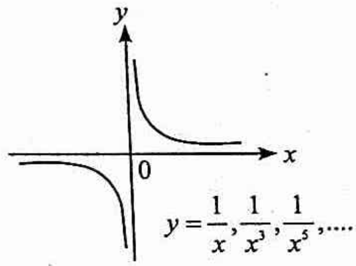
(*ii*) $y = |f(x)| \rightarrow |y| = |f(x)|$

Graphically it could be stated as;



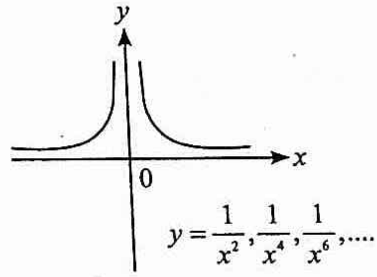


12. (*i*) $y = \frac{1}{x^n}$, n is odd integer ≥ 1



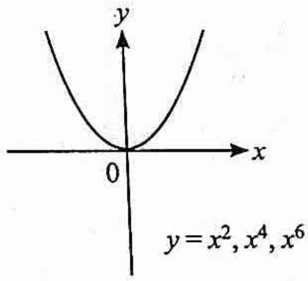
[i.e. graph of $y = \frac{1}{x}, \frac{1}{x^3}, \frac{1}{x^5}, \ldots\ldots$ can be considered same (for sketching)].

(*ii*) $y = \frac{1}{x^n}$, n is even integer > 1



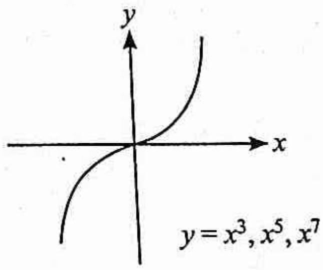
[i.e. graph of $y = \frac{1}{x^2}, \frac{1}{x^4}, \frac{1}{x^6}, \ldots\ldots$ can be considered same (for sketching)].

(*iii*) $y = x^n$ n is even integer ≥ 2



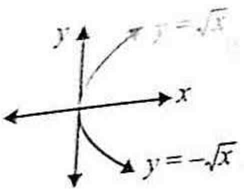
[i.e. graph of x^2, x^4, x^6, ... can be considered same (for sketching)].

(*iv*) $y = x^n$ n is odd integer ≥ 1



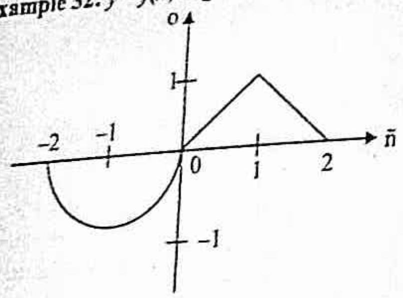
[i.e. graph of x^3, x^5, x^7, ... can be considered same (for sketching)].

Note: $y^2 = x$ represents two separate branches.



Train Your Brain

Example 32: $y = f(x)$ is given as shown



Plot the following:

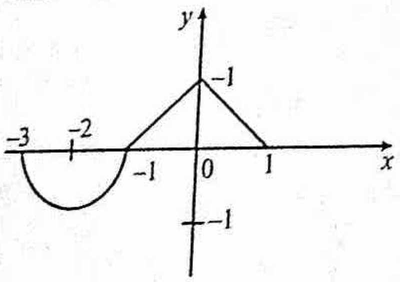
(i) $y = f(x+1)$ (ii) $y = f(x-2)$

(iii) $y = f(2x)$ (iv) $y = 2f(x)$

Sol.

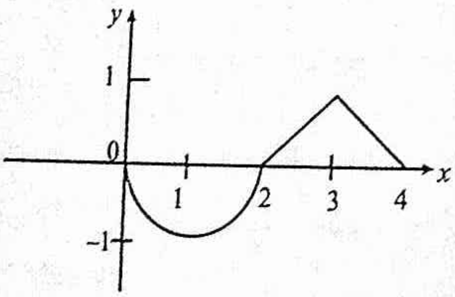
(i) $y = f(x+1)$

Graph of $f(x)$ is shifted by 1 unit on left side along the x-axis.



(ii) $y = f(x-2)$

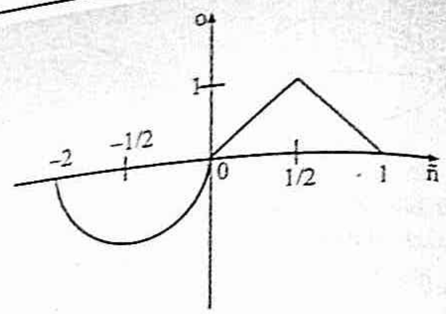
Graph of $f(x)$ is shifted by 2 units on right side along the x-axis.



(iii) $y = f(2x)$

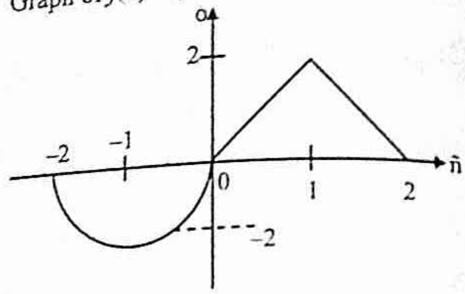
$k = 2$ then $\frac{1}{k} = \frac{1}{2}$ times

Graph of $f(x)$ shrinks by 1/2 times along the x-axis.



(iv) $y = 2f(x)$

Graph of $f(x)$ expands by 2 times along the y-axis.



Example 33: Plot the following expressions:

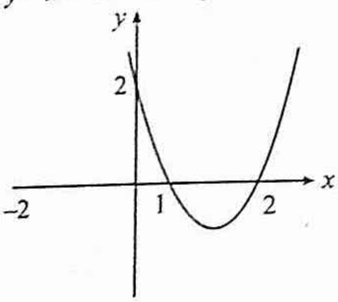
(i) $y = x^2 - 3x + 2$ (ii) $y = x^2 + 3x + 2$

(iii) $y = x^2 - 3|x| + 2$ (iv) $y = x^2 + 3|x| + 2$

(v) $y = |x^2 - 3|x| + 2|$ (vi) $y = |x^2 + 3|x| + 2|$

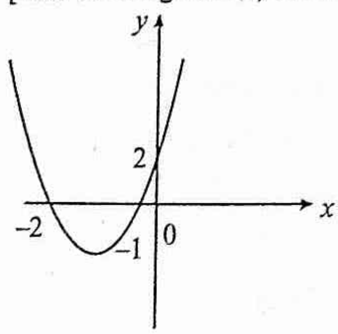
Sol.

(i) $y = x^2 - 3x + 2 \Rightarrow y = (x-1)(x-2) = f(x)$



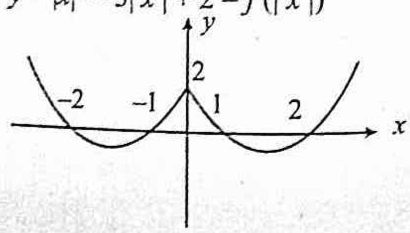
(ii) $y = x^2 + 3x + 2 \Rightarrow y = (x+1)(x+2) = f(-x)$

[Take the image of f(x) about the y-axis]



(iii) $y = x^2 - 3|x| + 2$ $(\because \sqrt{x^2} = |x| \Rightarrow x^2 = |x|^2)$

$\Rightarrow y = |x|^2 - 3|x| + 2 = f(|x|)$



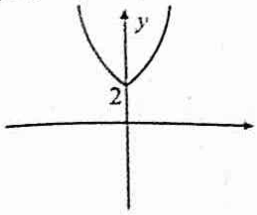
(*iv*) $y = x^2 + 3|x| + 2$

$(f(x) = x^2 - 3x + 2)$

$\Rightarrow \; y = (-|x|)^2 - 3(-|x|) + 2 \Rightarrow y = f(-|x|)$

Sequence of transformation

$f(x) \to f(-|x|)$



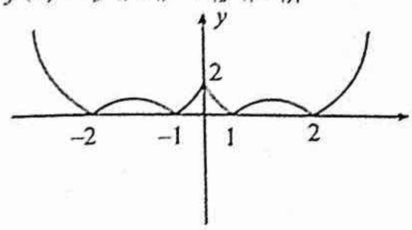
(*v*) $y = |x^2 - 3|x| + 2|$

$f(x) = x^2 - 3x + 2$

$\Rightarrow \quad y = |x^2 - 3|x| + 2| = |f(|x|)|$

Sequence of transformation

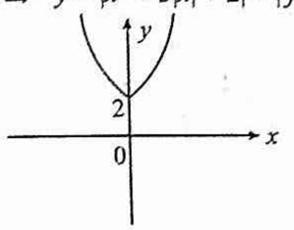
$f(x) \to f(|x|) \to |f(|x|)|$



(*vi*) $y = |x^2 + 3|x| + 2|$

$f(x) = x^2 - 3x + 2$

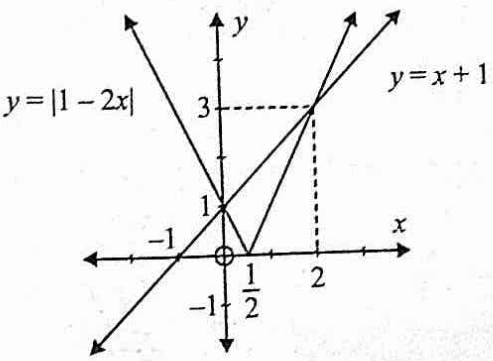
$\Rightarrow \; y = |x^2 + 3|x| + 2| = |f(-|x|)|$



Solve graphically: $|1 - 2x| > x + 1.$

We draw graphs of $y = |1 - 2x|$ and $y = x + 1$ on the same set of axes.

$$y = |1-2x| = \begin{cases} 1-2x & \text{for } 1-2x \geq 0, \text{ i.e., } x \leq \dfrac{1}{2} \\ -1+2x & \text{for } 1-2x < 0, \text{ i.e., } x > \dfrac{1}{2} \end{cases}$$



Now $|1 - 2x| > x + 1$ when the graph of $y = |1 - 2x|$ lies above $y = x + 1$,

$\therefore \quad x < 0$ or $x > 2$,

i.e., $x \in \,]-\infty, 0[$ or $x \in \,]2, \infty[$.

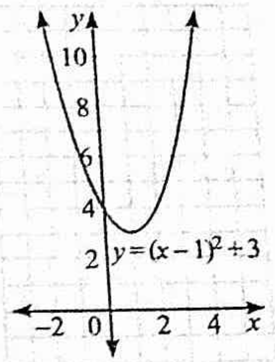
Example 34. Determine the domain and the range for each and sketch graphs also

(*a*) $y = (x - 1)^2 + 3$ (*b*) $y = \dfrac{1}{x + 3}$

(*c*) $y = \sqrt{x - 1} + 3$

Sol.

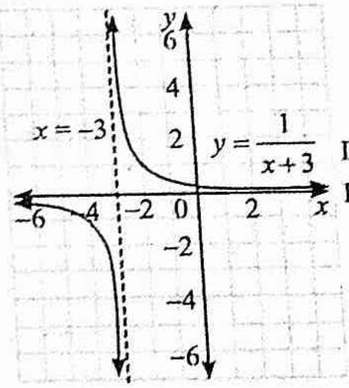
(*a*)



Domain: R

Range: $(3, \infty)$

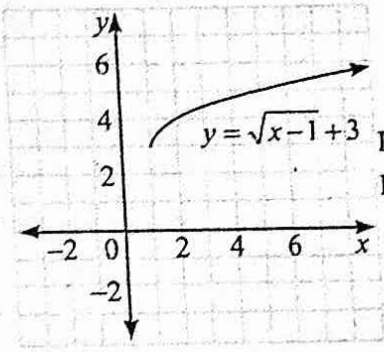
(*b*)



Domain: $R - \{-3\}$

Range: $R - \{0\}$

(*c*)



Domain: $(1, \infty)$

Range: $(3, \infty)$

Example 35. If $|x - 4| - 2|x| + |x + 4| = \lambda$, then find λ when the given equation has

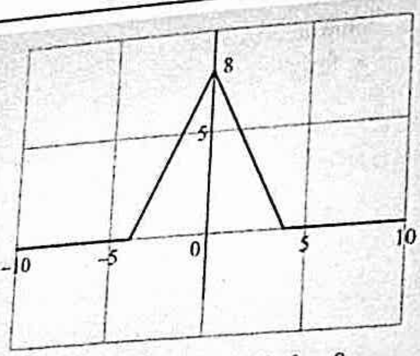
(*i*) no solution

(*ii*) 1 solution

(*iii*) 2 solution

(*iv*) infinite solution

Sol.



(i) $\lambda \notin [0, 8]$ (ii) $\lambda = 8$

(iii) $\lambda \in (0, 8)$ (iv) $\lambda = 0$

Concept Application

20. Find the number of solution $\{3x - 1\} - \frac{1}{2} = 0$, where $\{\cdot\}$ is fractional part function.

21. Solve graphically:
(a) $|2x - 3| < x$ (b) $2x - 3\,|x|$

22. Solve algebraically the equation $|3x - 4| = |5 - 2x|$.

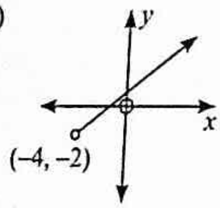
23. Sketch the following graphs:
(a) $y = 2|x| + 5$ (b) $y = 2 - |x|$

24. Solve graphically the inequality $2|x - 2| < |x|$.

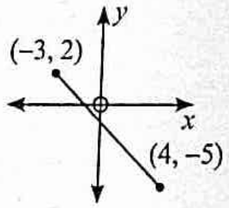
25. Solve graphically the inequality $|2x - 1| < |3x - 4|$.

26. For each of the following graphs, state the domain and range:

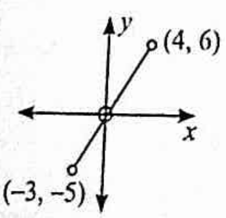
(a)



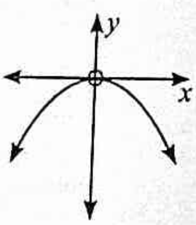
(b)



(c)

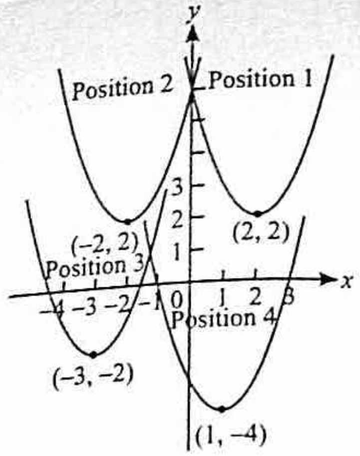


(d)



27. Draw the graph of $y = f(x)$ for:
(a) $f(x) = |x| + 3x$ (b) $f(x) = 2|x| - 4$

28. Match the equation listed in parts (a)–(d) to the graph accompanying figure.
(a) $y = (x - 1)^2 - 4$ (b) $y = (x - 2)^2 + 2$
(c) $y = (x + 2)^2 + 2$ (d) $y = (x + 3)^2 - 2$



Graph the functions in Exercises 25-29.

29. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

30. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 2 \\ 2 - x, & 1 < x \le 2 \end{cases}$

31. $F(x) = \begin{cases} 4 - x^2, & x \le 1 \\ x^2 + 2x, & x > 1 \end{cases}$

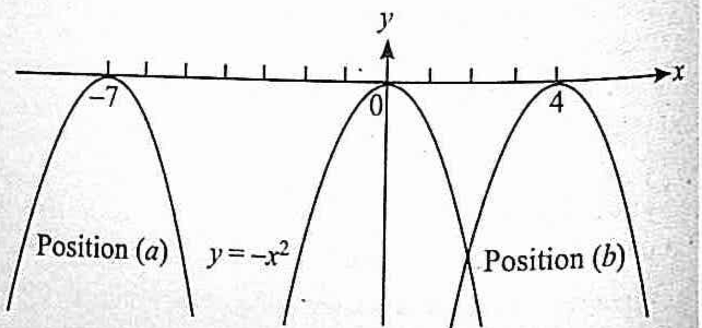
32. $G(x) = \begin{cases} 1/x^2, & x < 0 \\ x, & 0 \le 1 \end{cases}$

33. $f(x) = \begin{cases} 1 - x, x < 0 \\ 1, x = 0 \\ x + 1, x > 0 \end{cases}$

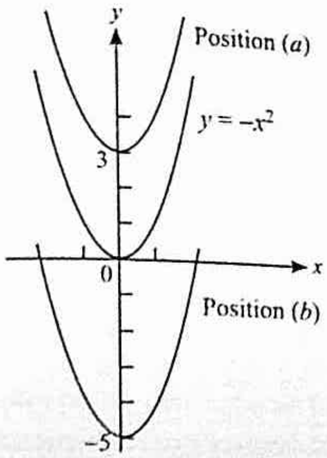
34. Draw graph of following functions (where $[\cdot]$, $\{\cdot\}$ represent greatest integer function and fractional part of x)
(i) $y = [x^2]$ (ii) $y = [\sin x]$
(iii) $y = \left\{\frac{x}{2}\right\}$ (iv) $y = \operatorname{sgn}\{x\}$
(v) $y = sgn\,\{x - |x|\}$

35. Solve for x (where $[\cdot]$ represents greatest integer function)
$[x] + [-x] = x^2 - 3x - 1$

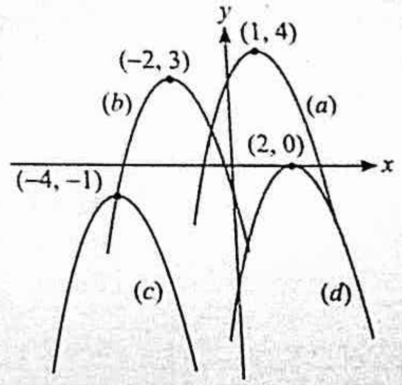
36. The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



37. The accompanying figure shows the graphs of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



38. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



Short Notes

- **Ordered pair** A pair of elements grouped together in a particular order.
- **Cartesian product** $A \times B$ of two sets A and B is given by
  $A \times B = \{(a, b) : a \in A,\ b \in B\}$
  In particular $R \times R = \{(x, y) : x, y \in R\}$
  and $R \times R \times R = \{(x, y, z): x, y, z \in R\}$
- If $(a, b) = (x, y)$, then $a = x$ and $b = y$.
- If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.
- $A \times \phi = \phi$
- In general, $A \times B \neq B \times A$.
- **Relation** A relation R from a set A to a set B is a subset of the cartesian product $A \times B$ obtained by describing a relationship between the first element x and the second element y of the ordered pairs in $A \times B$.
- **The image** of an element x under a relation R is given by y, where $(x,y) \in R$,
- The **domain** of R is the set of all first elements of the ordered pairs in a relation R.
- The **range** of the relation R is the set of all second elements of the ordered pairs in a relation R.
- **Function** A function f from a set A to a set B is a specific type of relation for which every element x of set A has one and only one image y in set B.
  We write $f: A \to B$, where $f(x) = y$.
- A is the domain and B is the codomain of f.
- The range of the function is the set of images.
- A real function has the set of real numbers or one of its subsets both as its domain and as its range.
- Algebra of functions For functions $f: X \to R$ and $g: X \to R$, we have
  $(f + g)(x) = f(x) + g(x), x \in X$
  $(f - g)(x) = f(x) - g(x), x \in X$
  $(f.g)(x) = f(x) \cdot g(x), x \in X)$
  $(kf)(x) = k(f(x)), x \in X$, where k is a real number.
  $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, x \in X, g(x) \neq 0$

Function
- Algebraic
  - Monomial
  - Polynomial
  - Rational
  - Irrational
- Transcendental
  - Logarithmic
  - Exponential
  - Trigonometric
  - Inverse Trigonometric
- Piecewise defined
  - Modulus
  - Greatest Integer
  - Fractional Part
  - Least Integer
  - Signum

S.No.		Transformation	
1.	(a)	$y = f(x) \to y = f(x + a)$	Shift the graph of $y = f(x)$ through 'a' units towards left.
	(b)	$y = f(x) \to y = f(x - a)$	Shift the graph of $y = f(x)$ through 'a' units towards right.
2.	(a)	$y = f(x) \to y + a = f(x)$	Shift graph of $y = f(x)$ by 'a' units downward.
	(b)	$y = f(x) \to y - a = f(x)$	Shift graph of $y = f(x)$ by 'a' units upward.
3.		$y = f(x) \to y = a = f(-x)$	Take the mirror image of $y = f(x)$ in the y-axis.
4.		$y = f(x) \to y = -f(x)$	Take the mirror image of $y = f(x)$ in the x-axis.
5.		$y = f(x) \to y = f = (\lvert x \rvert)$	Remove the left portion of the graph after that take the mirror image of the right portion of the curve in the Y-axis. Also include the right portion of the graph of $y = f(x)$.
6.		$y = f(x) \to y = \lvert f = (x) \rvert$	Take the mirror image of the lower portion of the curve (the curve below x-axis) in x-axis and reject the lower part (or flip lower part into upper).
7.		$y = f(x) \to \lvert y \rvert = f = (x)$	Remove the lower portion of the curve then take the mirror image of upper portion of the curve in the x-axis. Also include the upper portion of the graph of $y = f(x)$.
8.		$y = f(x) \to y = af = (x)$	Stretch ($a > 1$) or squaeeze ($a < 1$) the graph of the given function vertically.
9.		$y = f(x) \to y = f = (ax)$	Stretch ($a > 1$) or squaeeze ($a < 1$) the graph of the given function horizontally.

Solved Examples

1. If $R = \{(x, y) : x, y \in W, x^2 + y^2 = 25\}$, then find the domain and range of R.

Sol. We have,

$R = \{(x, y) : x, y \in W, x^2 + y^2 = 25\}$

$= \{(0,5), (3, 4), (4, 3), (5, 0)\}$

Domain of R = Set of first element of ordered pairs in R

$= \{0, 3, 4, 5\}$

Range of

R = Set of second element of ordered pairs in R

$= \{5, 4, 3, 0,\}$

2. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 5, 9, 11, 15, 16\}$. Determine which of the following sets are functions from X to Y.

(i) $f_1 = \{(1, 1), (2, 11), (3, 1), (4, 15)\}$

(ii) $f_2 = \{(1, 1), (2, 7), (3, 5)\}$

(iii) $f_3 = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$

Sol. Given $X = \{1, 2, 3, 4\}$ and $Y = \{1, 5, 9, 11, 15, 16\}$

(i) $f_1 = \{(1, 1), (2, 11), (3, 1), (4, 15)\}$

Every element of set X has an ordered pair in the relation f_1 and no two ordered pairs have the same first component but different second components.

Hence, the given relation f_1 is a function.

(ii) $f_2 = \{(1, 1), (2, 7), (3, 5)\}$

In the relation f_2, the element 2 of set X does not have any image in set Y.

However, for a relation to be a function, every element of the domain should have an image.

Hence, the given relation f_2 is not a function.

(iii) $f_3 = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$

Every element of set X has an ordered pair in the relation f_3. However, two ordered pairs $(2, 9)$ and $(2, 11)$ have the same first component but different second components.

Hence, the given relation f_3 is not a function.

3. Let $A = \{12, 13, 14, 15, 16, 17\}$ and $f : A \to Z$ be a function given by $f(x)$ = highest prime factor of x. Find range of f.

Sol. Given $A = \{12, 13, 14, 15, 16, 17\}$

$f : A \to Z$ such that $f(x)$ = highest prime factor of x.

A is the domain of the function f. Hence, the range is the set of elements $f(x)$ for all $x \in A$.

We have $f(12)$ = highest prime factor of 12

The prime factorization of $12 = 2^2 \times 3$

Thus, the highest prime factor of 12 is 3.

$\therefore f(12) = 3$

We have $f(13)$ = highest prime factor of 13

We know 13 is a prime number.

$\therefore f(13) = 13$

We have $f(14)$ = highest prime factor of 14

The prime factorization of $14 = 2 \times 7$

Thus, the highest prime factor of 14 is 7.

$\therefore f(14) = 7$

We have $f(15)$ = highest prime factor of 15

The prime factorization of $15 = 3 \times 5$

Thus, the highest prime factor of 15 is 5.

$\therefore f(15) = 5$

We have $f(16)$ = highest prime factor of 16

The prime factorization of $16 = 2^4$

Thus, the highest prime factor of 16 is 2.

$\therefore f(16) = 2$

We have $f(17)$ = highest prime factor of 17

We know 17 is a prime number.

$\therefore f(17) = 17$

Thus, the range of f is $\{3, 13, 7, 5, 2, 17\}$.

4. Find the domain of each of the following real valued functions of real variable:

$$f(x) = \sqrt{\frac{x-2}{3-x}}$$

Sol. $f(x) = \sqrt{\frac{x-2}{3-x}}$

We know the square root of a real number is never negative.

Clearly, $f(x)$ takes real values only when $x - 2$ and $3 - x$ are both positive or negative.

(*a*) Both $x - 2$ and $3 - x$ are positive

$x - 2 \geq 0 \Rightarrow x \geq 2$

$3 - x \geq 0 \Rightarrow x \leq 3$

Hence, $x \geq 2$ and $x \leq 3$

$\therefore x \in [2, 3]$

(*b*) Both $x - 2$ and $3 - x$ are negative

$x - 2 \leq 0 \Rightarrow x \leq 2$

$3 - x$ is $\leq 0 \Rightarrow x \geq 3$

Hence, $x \leq 2$ and $x \geq 3$

However, the intersection of these sets in null set. Thus, this case is not possible.

In addition, $f(x)$ is also undefined when $3 - x = 0$ because the denominator will be zero and the result will be indeterminate.

$3 - x = 0 \Rightarrow x = 3$

Hence, $x \in [2, 3] - \{3\}$

$\therefore x \in [2, 3)$

Thus, domain of $f = [2, 3)$

5. Find the domain and range of each of the following real valued functions:

$$f(x) = \frac{ax+b}{bx-a}$$

Sol. $f(x) = \frac{ax+b}{bx-a}$

Clearly, $f(x)$ is defined for all real values of x, except for the case when $bx - a = 0$ or $x = \frac{a}{b}$.

When $x = \frac{a}{b}$, $f(x)$ will be undefined as the division result will be indeterminate.

Thus, domain of $f = R - \left\{\frac{a}{b}\right\}$

Let $f(x) = y$

$\Rightarrow \frac{ax+b}{bx-a} = y$

$\Rightarrow ax + b = y(bx - a)$

$\Rightarrow ax + b = bxy - ay$

$\Rightarrow ax - bxy = -(ay + b)$

$\therefore x = -\frac{(ay+b)}{a-by}$

Clearly, when $a - by = 0$ or $y = \frac{a}{b}$, x will be undefined as the division result will be indeterminate.

Hence, $f(x)$ cannot take the value $\frac{a}{b}$.

Thus, range of $f = R - \left\{\frac{a}{b}\right\}$

6. Find the domain and range of each of the following real valued functions:

$$f(x) = \frac{ax-b}{cx-d}$$

Sol. $f(x) = \frac{ax-b}{cx-d}$

Clearly, $f(x)$ is defined for all real values of x, except for the case when $cx - d = 0$ or $x = \frac{d}{c}$.

When $x = \frac{d}{c}$, $f(x)$ will be undefined as the division result will be indeterminate.

Thus, domain of $f = R - \left\{\frac{d}{c}\right\}$

Let $f(x) = y$

$\Rightarrow \frac{ax-b}{cx-d} = y$

$\Rightarrow ax - b = y(cx - d)$

$\Rightarrow ax - b = cxy - dy$

$\Rightarrow ax - cxy = b - dy$

$\Rightarrow x(a - cy) = b - dy$

$\therefore x = \frac{b-dy}{a-cy}$

Clearly, when $a - cy = 0$ or $y = \frac{a}{c}$, x will be undefined as the division result will be indeterminate.

Hence, $f(x)$ cannot take the value $\frac{a}{c}$.

7. Find the domain and range of the following

(*i*) $f(x) = \log_{[x-1]} \sin x$, where [.] denotes greatest integer function.

(*ii*) $f(x) = \log_3\{\log_{1/2}(x^2 + 4x + 4)\}$.

Sol. (*i*) $f(x) = \log_{[x-1]}\sin x$

$\sin x > 0 \Rightarrow x \in (2n\pi, (2n + 1)\pi)$

Here, $[x - 1] > 0$ and $[x - 1] \neq 1$

$\Rightarrow x \in [3, \infty)$

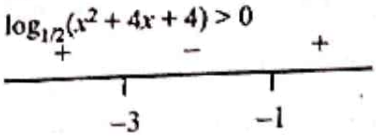
Domain $x \in [3, \pi) \cup \bigcup_{n=1}^{\infty} (2n\pi, (2n+1)\pi)$.

For range, $\sin x \in (0, 1]$ and $[x-1] \in [2, \infty)$

So, range $\in (-\infty, 0]$

(ii) First, to find the domain $\log_3[\log_{1/2}(x^2+4x+4)]$ exists if,

$\log_{1/2}(x^2+4x+4) > 0$



$\Rightarrow x^2+4x+4 < \left(\frac{1}{2}\right)^0$

[using $\log_a x < b \Rightarrow x > a^b$, if $0 < a < 1$]

$\Rightarrow x^2+4x+4 < 1 \Rightarrow x^2+4x+3 < 0$

$\Rightarrow (x+1)(x+3) < 0 \Rightarrow -3 < x < -1$ ...(i)

and $x^2+4x+4 > 0 \Rightarrow (x+2)^2 > 0$ ...(ii)

Which is always true except for $x = -2$

Thus, from (i) and (ii), we have

Hence, Domain : $x \in (-3, -2) \cup (-2, -1)$

Now we find out the range,

Since, $0 < \log_{1/2}(x^2+4x+4) < \infty$

$\Rightarrow -\infty < \log_3[\log_{1/2}(x^2+4x+4)] < \infty$

Thus, range $y \in R$

8. Solve the following inequalities for real values of x:

(i) $|x-1| < 2$

(ii) $|x-3| > 5$

(iii) $0 < |x-1| \le 3$

(iv) $|x-1| + |2x-3| = |3x-4|$

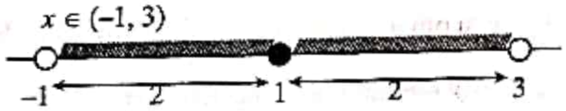
(v) $\left|\frac{x-3}{x^2-4}\right| \le 1$

Sol. (i) $|x-1| < 2$

$-2 < x-1 < 2$

$-1 < x < 3$

$x \in (-1, 3)$



(ii) $|x-3| > 5$

$\Rightarrow x-3 < -5$ or $x-3 > 5$

$\Rightarrow x < -2$ or $x > 8$

$\Rightarrow x \in (-\infty, -2) \cup (8, \infty)$

(iii) $0 < |x-1| \le 3$

Here $|x-1| > 0 \Rightarrow x \ne 1$

Also, $|x-1| \le 3 \Rightarrow -3 \le x-1 \le 3$,

$\Rightarrow -2 \le x \le 4, x \ne 1 \Rightarrow x \in [-2, 1) \cup (1, 4]$.

(iv) Since $3x-4 = x-1+2x-3$, $|3x-4| = |x-1| + |2x-3|$

$\therefore |x| + |y| = |x+y|$, then $(x)(y) \ge 0$

$\Rightarrow (x-1)(2x-3) \ge 0$

$\Rightarrow x \in (-\infty, 1] \cup [3/2, +\infty)$.

(v) $\left|\frac{x-3}{x^2-4}\right| \le 1$ It is clear that $x^2-4 \ne 0$

$\Rightarrow x \ne 2, -2$

Now, $\left|\frac{x-3}{x^2-4}\right| \le 1 \Rightarrow -1 \le \frac{x-3}{x^2-4} \le 1$

Let us take up, $\frac{x-3}{x^2-4} \ge -1$

$$\Rightarrow \frac{(x+2)(x-2)\left(x-\frac{-1-\sqrt{29}}{2}\right)\left(x-\frac{-1+\sqrt{29}}{2}\right)}{(x+2)^2(x-2)^2}$$

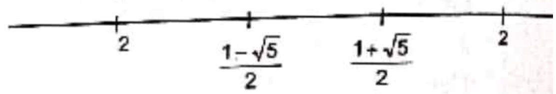
$$\Rightarrow x \in \left(-\infty, \frac{-1-\sqrt{29}}{2}\right] \cup (-2, 2) \cup \left[\frac{-1+\sqrt{29}}{2}, \infty\right)$$

Now let us take up, $\frac{x-3}{x^2-4} \le 1$

$$\Rightarrow \frac{x-3-x^2+4}{(x^2-4)} \le 0 \Rightarrow \frac{-x^2+x+1}{(x^2-4)} \le 0$$

$$\Rightarrow \frac{x^2-x-1}{(x^2-4)} \ge 0$$

$$\Rightarrow \frac{(x+2)(x-2)\left(x-\frac{1+\sqrt{5}}{2}\right)\left(x-\frac{1-\sqrt{5}}{2}\right)}{(x+2)^2(x-2)^2} \ge 0$$



$$\Rightarrow x \in (-\infty, -2) \cup \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right] \cup (2, \infty) \quad ...($$

$$\Rightarrow x \in \left(-\infty, \frac{-1-\sqrt{29}}{2}\right] \cup \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right] \cup \left[\frac{-1+\sqrt{29}}{2}, +\infty\right)$$

9. Find the domain of following functions.

(i) $f(x) = \ln(-2+3x-x^2)$

(ii) $f(x) = e^{-\frac{1}{x^2-1}}$

(iii) $f(x) = \sin(2x+a-b)$, $(a, b \in R)$

Sol. (i) $f(x) = \ln(-2+3x-x^2)$

for $f(x)$ to define, $-2+3x-x^2 > 0$

$\Rightarrow x^2-3x+2 < 0 \Rightarrow (x-1)(x-2) < 0$

$\Rightarrow x \in (1, 2)$

So domain $(f) : x \in (1, 2)$

(ii) $f(x) = e^{-\frac{1}{x^2-1}}$

for $f(x)$ to define, $x^2-1 \ne 0$

$\Rightarrow x \ne \pm 1$

so domain $(f) \to x \in R - \{1, -1\}$

(*iii*) $f(x) = \sin(2x + a - b)$

for $f(x)$ to define, $2x + a - b \in R$

i.e. $x \in R$

So, domain is $x \in R$

10. Solve the inequality $[x]^2 - 3[x] + 2 \le 0$.

Sol. $[x]^2 - 3[x] + 2 \le 0$

$\Rightarrow \quad ([x] - 1)([x] - 2) \le 0$

$\Rightarrow \quad 1 \le [x] \le 2$

$\Rightarrow \quad 1 \le x < 3 \Rightarrow x \in [1, 3)$.

11. Let R be a relation on set of Real numbers defined by

$R = \{(x, y) : y = |x - 1| + |x - 2|, 0 \le x \le 3\}$

By drawing a graph between x and y, find the range of R.

Sol. Clearly domain of $R = \{x : 0 \le x \le 3, x \in$ set of real numbers$\}$

When $\quad 0 \le x \le 1 \quad y = 1 - x + 2 - x = 3 - 2x$

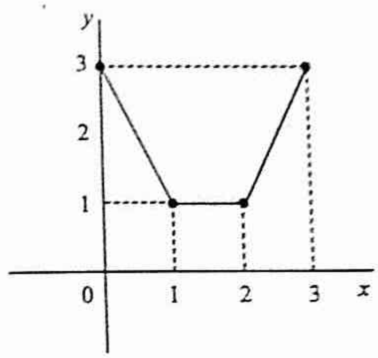
$1 \le x \le 2 \quad y = x - 1 + 2 - x = 1$

$2 \le x \le 3 \quad y = x - 1 + x - 2 = 2x - 3$

Now the graph can be plotted Q

From the graph $1 \le y \le 3$

So Range of R $\in [1, 3]$



12. Determine the domain and range of following relation

$R = \{(x, y) : y = |x - 2|, x, y \in Z, |x| \le 2\}$

Sol. $|x| \le 2 \Rightarrow -2 \le x \le 2, x \in Z$

i.e. $x = -2, -1, 0, 1, 2$

Corresponding values of y are

$y = |-2 - 2| = 4 \qquad$ i.e. $(-2, 4) \in R$

$y = |-1 - 2| = 3 \qquad$ i.e. $(-1, 3) \in R$

$y = |0 - 2| = 2 \qquad$ i.e. $(0, 2) \in R$

$y = |1 - 2| = 1 \qquad$ i.e. $(1, 1) \in R$

$y = |2 - 2| = 0 \qquad$ i.e. $(2, 0) \in R$

$R = \{(-2, 4), (-1, 3), (0, 2), (1, 1), (2, 0)\}$

Hence dom $(R) = \{-2, -1, 0, 1, 2\}$

Range of $R = \{0, 1, 2, 3, 4\}$

13. Let A be the set of first five natural and let R be a relation on A defined as follows: $(x, y)\ Rx \le y$. Express R and R^{-1} as sets of ordered pairs. Determine also

(*i*) the domain of R^{-1}

(*ii*) The Range of R.

Sol. $R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

An inverse relation is the set of ordered pairs obtained by interchanging the first and second elements of each pair in the original relation. If the graph of a function contains a point (a, b), then the graph of the inverse relation of this function contains the point (b, a).

$\therefore R^{-1} = \{(2, 1), (3, 1), (4, 1), (5, 1), (3, 2), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4)\}$

$= R^{-1} = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (5, 4)\}$

(*i*) Domain of $R^{-1} = \{2, 3, 4, 5\}$

(*ii*) Range of $R = \{2, 3, 4, 5\}$

14. If $[x]^2 + [x] - 2 < 0$ and $\{x\} = \dfrac{1}{2}$, then the number of possible real value of x, is

Note: $[x]$ and $\{x\}$ denote the greatest integer less than or equal to x and fractional part of x respectively.

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

Sol. (*b*) $\quad [x]^2 + [x] < 2$

$\Rightarrow ([x] + 2)([x] - 1) < 0$

$\Rightarrow -2 < [x] < 1$

$\therefore [x] = -1, 0$

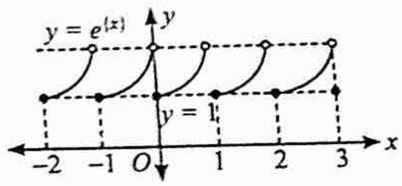
$\Rightarrow x \in [-1, 1)$

So, $\{x\} = \dfrac{1}{2} \Rightarrow x = \dfrac{-1}{2}, \dfrac{1}{2}$

15. Plot: $f(x) = e^{\{x\}}$.

Note: $\{\alpha\}$ denotes the fractional part of α.

Sol.



16. The range of the function $y = \dfrac{x^2}{1 + x^2}$ is $y \in$

Sol. y is defined for all real x.

$\therefore$ Domain of y is $x \in (-\infty, \infty)$

we have, $y = \dfrac{x^2}{1 + x^2} \Rightarrow x^2 y + y = x^2$

$\Rightarrow x^2 = \dfrac{y}{1 - y} \ge 0$

So range of $y = [0, 1)$

17. The range of the function $y = \dfrac{x}{1 + x^2}$ is $y \in$

(*a*) $\left[0, \dfrac{1}{2}\right]$ (*b*) $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

(*c*) $\left[-\dfrac{1}{2}, 0\right]$ (*d*) None of these

Sol. (*b*)

Clearly, y is defined for all real x.

$\therefore$ Domain of y is $x \in (-\infty, \infty)$.

We have, $y = \dfrac{x}{1+x^2} \Rightarrow y + x^2y = x$

$\Rightarrow x^2y - x + y = 0$

For x to be real, discriminant of above quadratic equation ≥ 0

$1 - 4y^2 \geq 0, y \neq 0$

$\Rightarrow (1 - 2y)(1 + 2y) \geq 0, y \neq 0$

$\Rightarrow (2y - 1)(2y + 1) \leq 0, y \neq 0$

$\therefore y \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

Exercise-1 (Topicwise)

ORDERED PAIR

1. In each of the following cases, find a and b.

$$\left(\frac{a}{4}, a-2b\right) = (0, 6+b)$$

(a) (0, –2) (b) (–2, 0)
(c) (0, 2) (d) (2, 0)

2. $A = \{1, 2, 3, 4, 5\}$, $S = \{(x, y) : x \in A, y \in A\}$, then find the ordered which satisfy the conditions given below.
$x + y < 5$
(a) $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$
(b) $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)$
(c) $\{(4, 5), (5, 4), (5, 5)\}$
(d) None of these

RELATIONS

3. If $A = \{x \in Z^+ ; x < 10$ and x, is a multiple of 3 or 4$\}$, where Z^+ is the set of positive integers, then the total number of relations on A is
(a) 2^{25} (b) 2^{20}
(c) 2^{10} (d) 2^{15}

DOMAIN & RANGE

4. If $R_1 = \{(x, y) | y = 2x + 7$, where $x \in R$ and $-5 \le x \le 5\}$ is a relation. Then, find the domain and range of R_1.
(a) domain (–5, 5) range (–3, 17)
(b) domain [–5, 5) range [–3, 17)
(c) domain (–5, 5] range (–3, 17]
(d) domain [–5, 5] range [–3, 17]

5. Express the following functions as set of ordered pairs and determine their range.
$f : x \to R$, $f(x) = x^3 + 1$, where $x = \{-1, 0, 3, 9, 7\}$
(a) {0, 1, 28, 730, 344}
(b) {0, 1, 27, 730, 344}
(c) {0, 1, 28, 729, 344}
(d) {0, 1, 28, 730, 343}

6. Domain of the function $\sqrt{2-x} - \dfrac{1}{\sqrt{9-x^2}}$ is
(a) (–3, 1) (b) [–3, 1]
(c) (–3, 2] (d) [–3, 1)

7. Domain of $f(x) = \log_{(2x-5)}(x^2 - 3x - 10)$
(a) $[5, \infty)$ (b) $(-\infty, 2) \cup (5, \infty)$
(c) (5/2, 3) (d) None of these

8. Domain of $f(x) = \dfrac{1}{\sqrt{[x] - x}}$ is: (Here $[x]$ represents greatest integer function)
(a) $(-\infty, 0)$ (b) $(0, \infty)$
(c) $R - I$ (d) ϕ

9. Domain of the function $f(x) = \dfrac{\sin[x-1]}{[x-1][x+2]}$ where $[x]$ denotes the greatest integer function less than x, is
(a) All real x (b) $[-2, -1)$
(c) $R - [-2, -1) \cup [1, 2)$ (d) $R - [-2, 2]$

10. The midpoint of domain of the function
$g(x) = \sqrt{4 - \sqrt{2x+5}}$
(a) $\dfrac{-2}{5}$ (b) $\dfrac{1}{4}$
(c) $\dfrac{2}{3}$ (d) $\dfrac{3}{2}$

11. The range of the function.
$y = \log_{\sqrt{7}}\left(\sqrt{2}(\sin x - \cos x) + 5\right)$ is
(a) R (b) Z
(c) $[\log_7 4, \log_7 5]$ (d) $[2\log_7 3, 2]$

12. The range of the function $y = \dfrac{1}{2 - \sin 3x}$ is
(a) $\left(\dfrac{1}{3}, 1\right)$ (b) $\left[\dfrac{1}{3}, 1\right)$
(c) $\left[\dfrac{1}{3}, 1\right]$ (d) None of these

13. The range of the function $y = \log_3(5 + 4x - x^2)$ is
(a) (0, 2] (b) $(-\infty, 2]$
(c) (0, 9] (d) None of these

14. Mark the correct alternative in the following :
If $f : R \to R$ be given by $f(x) = \dfrac{4^x}{4^x + 2}$ for all $x \in R$.
Then,
(a) $f(x) = f(1 - x)$ (b) $f(x) + f(1 - x) = 0$
(c) $f(x) + f(1 - x) = 1$ (d) $f(x) + f(x - 1) = 1$

15. Mark the correct alternative in the following:
The domain of definition of the function
$f(x) = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{1+x}}$ is
(a) $(-\infty, -2] \cup [2, \infty)$ (b) $[-1, 1]$
(c) ϕ (d) None of these

16. Find the number of solution of equation $\ln(2-x) = e^x$.

(a) 0
(b) 1
(c) 2
(d) 3

17. Which one of the following is correct?

(a) $A \times (B - C) = (A - B) \times (A - C)$
(b) $A \times (B - C) = (A \times B) - (A \times C)$
(c) $A \cap (B \cup C) = (A \cap B) \cup C$
(d) $A \cup (B \cap C) = (A \cup B) \cap C$

18. Range of $f(x) = 4^x + 2^x + 1$ is

(a) $(0, \infty)$
(b) $(1, \infty)$
(c) $(2, \infty)$
(d) $(3, \infty)$

19. Range of function $f(x) = \dfrac{x-2}{x^2-4x+3}$

(a) $(-\infty, 0)$
(b) (R)
(c) $(0, \infty)$
(d) $R - \{0\}$

20. The maximum value of the function $f(x) = \dfrac{1}{x^2-2|x|+2}$,

(a) 1/4
(b) 1/2
(c) 1
(d) 2

21. $f(2x + 3y, 2x - 7y) = 20x$ then $f(x, y)$ equal to

(a) $7x - 3y$
(b) $7x + 3y$
(c) $3x - 7y$
(d) $x - 10y$

Exercise-2 (Learning Plus)

1. If $A = \{-1, 2, 3\}$ and $B = \{1, 3\}$, then determine $B \times A$

(a) $\{(-1, 1), (-1, 3), (2, 1), (2, 3), (3, 1), (3, 3)\}$
(b) $\{(1, -1), (1, 2), (1, 3), (3, -1), (3, 2), (3, 3)\}$
(c) $\{(1, 1), (1, 3), (3, 1), (3, 3)\}$
(d) $\{(-1, -1), (-1, 2), (-1, 3), (2, -1), (2, 2), (2, 3), (3, -1), (3, 2), (3, 3)\}$

2. The domain and range of the real function f defined by $f(x) = \dfrac{4-x}{x-4}$ is given by

(a) Domain = R, Range = $\{-1, 1\}$
(b) Domain = $R - \{1\}$, Range = R
(c) Domain = $R - \{4\}$, Range = $R - \{-1\}$
(d) Domain = $R - \{-4\}$, Range = $\{-1, 1\}$

3. Domain of $\sqrt{a^2-x^2}\,(a > 0)$ is

(a) $(-a, a)$
(b) $[-a, a]$
(c) $[0, a]$
(d) $(-a, 0]$

4. Let f and g be real functions defined by $f(x) = 2x + 1$ and $g(x) = 4x - 7$.

For what real number x, $f(x) < g(x)$?

(a) $x > 4$
(b) $x < 4$
(c) $x \geq 4$
(d) $x \leq 4$

5. Write the domain and range of function $f(x)$ given by $f(x) = \dfrac{1}{\sqrt{x-[x]}}$.

(a) Domain $\in R$, range $\in R$
(b) Domain $\in R^+$, range $\in R^+$
(c) Domain $\in R^-$, range $\in R^-$
(d) Domain $\in \phi$, range $\in \phi$

6. Write the domain and range of $f(x) = \sqrt{x-[x]}$

(a) Domain $\in R$, range $\in [0, 1)$
(b) Domain $\in R$, range $\in (0, 1)$
(c) Domain $\in R$, range $\in [0, 1]$
(d) Domain $\in R$, range $\in (0, 1]$

7. Write the domain and range of function $f(x)$ given by $f(x) = \sqrt{[x]-x}$.

(a) Domain $\in R$, range $\in \{0\}$
(b) Domain $\in I$, range $\in \{0\}$
(c) Domain $\in R$, range $\in R\,[0, 1]$
(d) Domain $\in R$, range $\in R\,(0, 1]$

8. Write the range of the function $f(x) = \cos [x]$, where $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

(a) $\{1, \cos 1, \cos 2\}$
(b) $\{\cos 1, \cos 2\}$
(c) $\{1, \cos 2\}$
(d) $\{1, \cos 1\}$

9. Write the range of the function $f(x) = e^{x-[x]}$, $x \in R$.

(a) $[1, e]$
(b) $(1, e]$
(c) $(1, e)$
(d) $[1, e)$

10. Let $f(x) = \sqrt{1+x^2}$, then

(a) $f(xy) = f(x) \cdot f(y)$
(b) $f(xy) \geq f(x) \cdot f(y)$
(c) $f(xy) \leq f(x) \cdot f(y)$
(d) None of these

11. If $f(x) = \cos\left[\dfrac{\pi^2}{2}\right]x + \sin\left[-\dfrac{\pi^2}{2}\right]x$, where [.] denotes the greatest integer function, then which of the following is not correct

(a) $f(0) = 1$
(b) $f\left(\dfrac{\pi}{3}\right) = \dfrac{1}{\sqrt{3}+1}$
(c) $f\left(\dfrac{\pi}{2}\right) = 0$
(d) $f(\pi) = 0$

12. The domain of definition of the function $y(x)$ given by $2^x + 2^y = 2$ is

(a) $(0, 1]$ (b) $(0, 1]$

(c) $(-\infty, 0)$ (d) $(-\infty, 1)$

13. Range of the function $f(x)\dfrac{x^2+x+2}{x^2+x+1}; x \in R$ is

(a) $(1, \infty)$ (b) $(1, 11/7)$

(c) $(1, 7/3]$ (d) $(1, 7/5)$

14. If $f(x) = \dfrac{2^x + 2^{-x}}{2}$, the $f(x+y).f(x-y)$ is equal to

(a) $\frac{1}{2}[f(x+y)+f(x-y)]$

(b) $\frac{1}{2}[f(2x)+f(2y)]$

(c) $\frac{1}{2}[f(x+y).f(x-y)]$

(d) none of these

15. If $f(x + ay, x - ay) = axy$ then $f(x, y)$ is equal to

(a) xy (b) $x^2 - a^2y^2$

(c) $\dfrac{x^2-y^2}{4}$ (d) $\dfrac{x^2-y^2}{a^2}$

16. The domain of the function $f(x) = \dfrac{\sqrt{-\log_{0.3}(x-1)}}{\sqrt{x^2+2x+8}}$ is

(a) $(1, 4)$ (b) $(-2, 4)$

(c) $(2, 4)$ (d) $[2, \infty)$

17. Domain of definition of the function $f(x) =$
$= \dfrac{3}{4-x^2} + \log_{10}(x^3 - x)$, is:

(a) $(1, 2)$ (b) $(-1, 0) \cup (1, 2)$

(c) $(1, 2) \cup (2, \infty)$ (d) $(-1, 0) \cup (1, 2) \cup (2, \infty)$

18. Range of $f(x) = \ln(3x^2 - 4x + 5)$ is

(a) $\left[\ln\frac{11}{3}, \infty\right)$ (b) $[\ln 10, \infty)$

(c) $\left[\ln\frac{11}{6}, \infty\right)$ (d) $\left[\ln\frac{11}{12}, \infty\right)$

19. If $f(x) = \cos(\log x)$, then $f(x)f(y) - \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(xy)\right\}$

has the value

(a) -1 (b) $1/2$

(c) -2 (d) None of these

20. If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$ and $(x)\dfrac{3x+x^3}{1+3x^2}$, then f(g(x) is equal to

(a) $f(3x)$ (b) $\{f(x)\}^3$

(c) $3f(x)$ (d) $-f(x)$

21. The range of the function $f(x) = \dfrac{x^2-x}{x^2-2x}$ is

(a) R (b) $R - \{1\}$

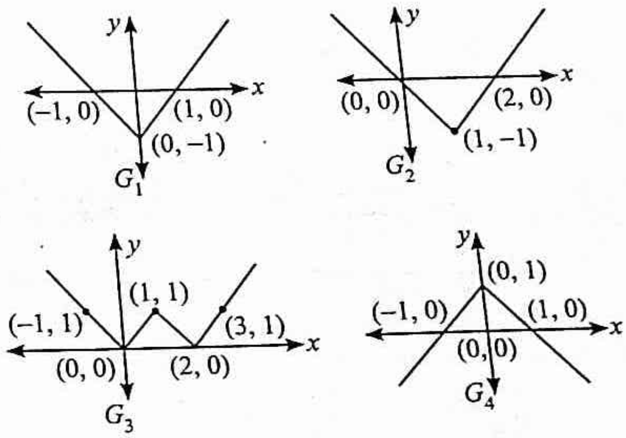
(c) $R - \{-1/2, 1\}$ (d) None of these

22. Four graphs marked G_1, G_2, G_3 and G_4 are given in figure which are graphs of four functions

$f_1(x) = |x - 1| - 1, f_2(x) = |x - 1| - 1|,$

$f_3(x) = |x| - 1, \quad f_4(x) = 1 - |x|,$

not necessarily in the correct order.

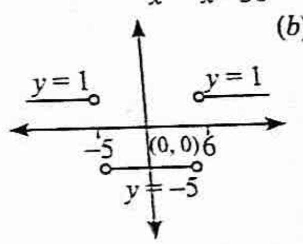


(a) G_2, G_1, G_3, G_4 (b) G_3, G_4, G_1, G_2

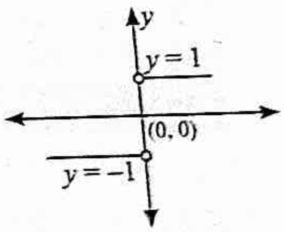
(c) G_2, G_3, G_1, G_4 (d) G_4, G_3, G_1, G_2

23. Which one of the following best represents the graph of function $f(x) = \dfrac{|x^2-x-30|}{x^2-x-30}$?

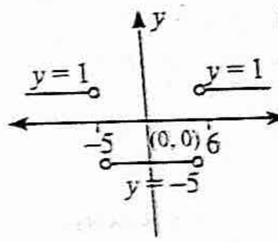
(a)



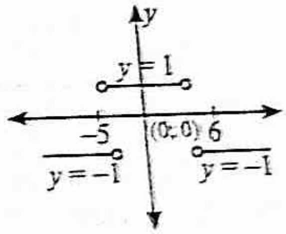
(b)



(c)



(d)



24. The domain of the function $f(x) = \dfrac{\sqrt{x+2}}{x^2-9}$ is

(a) $(-\infty, -3) \cup (2, \infty)$

(b) $[2, 3)$

(c) $(-2, 3) \cup (3, \infty)$

(d) $(-\infty, -3) \cup (3, \infty)$

25. The sum of all real numbers which are not in the range of $f(x) = \dfrac{x^2 - 3x + 2}{x^2 - 4x + 3}$ is equal to

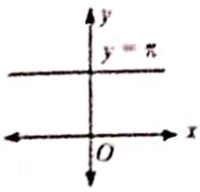
(a) 3/2 (b) 1/2

(c) 1 (d) 5/2

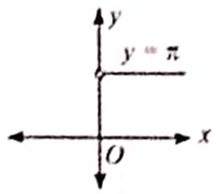
Domain of function $= R - \{1, 3\}$. Range of function $R - \left\{\dfrac{1}{2}, 1\right\}$

26. Which one of function best represent the graph of $y = x^{\log_x \pi}$?

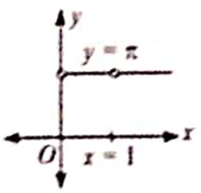
(a)



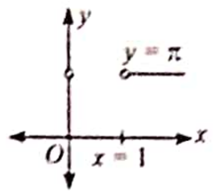
(b)



(c)



(d)



27. If $f(x + ay, x - ay) = axy$, then $f(x, y)$ equals

(a) $\dfrac{x^2 + y^2}{4}$ (b) $\dfrac{x^2 - y^2}{4}$

(c) x^2 (d) y^2

28. Let $f : A_1 \to B_1, g : A_2 \to B_2, p : A_3 \to B_3$ and $h : A_4 \to B_4$ are real valued function $h(x) = g(f(p(x)))$ is a set y then

(a) $y \subseteq B_1$ (b) $y \subseteq B_2$

(c) $y \subseteq B_3$ (d) can't say

29. In Column-I below some relation sets are defined. Column-II enlists the cardinal number of these sets. Match the two columns.

	Column-I		Column-II
A.	$R = \{(x, y) : (x, y) \in Z \times Z, (x + y)(y + 2004) + 1 = 0$	p.	4
B.	$R = \{(x, y) : (x, y) \in Z \times Z, 9x^2 + 4y^2 = 6$	q.	3
C.	$R = \{(x, y) : (x, y) \in N \times N, 2 \le x \le 6, 3 < y < 7, x$ & y are co-prime$\}$	r.	2
D.	$R = \left\{\left(x, \dfrac{1}{x}\right) : 0 < x < 4, x \in N\right\}$	s.	7

30. If the equation $||x + 3| - 2| = p$, has exactly three distinct solution, then find the number of integral values of p, is

31. If $|x^2 - 5|x| + 6| = k$, has 8 distinct solution, then 'k' lies in:

32. Solve the inequality $|x + 3| \ge 2|x - 3|$.

33. Solve the inequality $|x - 2| \le 3|x + 1|$.

34. Let $f: R^+ \to R$, where R^+ is the set of all positive real numbers be such that $f(x) = \log_e x$. Determine

(i) the miage set of the domain of f

(ii) $\{x: f(x) = -2\}$

(iii) whether $f(xy) = f(x) + f(y)$ holds.

35. If for non-zero x, $af(x) + bf\left(\dfrac{1}{x}\right) = \dfrac{1}{x} - 5$, where $a \ne b$, then find $f(x)$.

36. Let $A = \{12, 13, 14, 15, 16, 17\}$ and $f : A \to Z$ be a function given by $f(x)$ = highest prime factor of x. Find range of f.

37. If $f(x)$ be defined on $[-2, 2]$ and is given by $f(x) = \begin{cases} -1, -2 \le x \le 0 \\ x - 1, \le x \le 2 \end{cases}$ and $g(|x|) = f(|x|)$. Find $g(x)$.

38. If f, g, h are real functions defined by $f(x) = \sqrt{x+1}$, $g(x) = \dfrac{1}{x}$ and $h(x) = 2x^2 - 3$, then find the values of $(2f + g - h - h)(1)$ and $(2f + g - h)(0)$.

39. Write the range of the function (where $[\cdot]$ represents greatest integer function)

(i) $f(x) = \sin [x]$ where $\dfrac{-\pi}{4} \le x \le \dfrac{\pi}{4}$.

(ii) $f(x) = e^{x - [x]}, x \in R$.

40. Let f and g be two functions given by

$f = \{(2, 4), (5, 6), (8 -1), (10, -3)$ and $g = \{(2, 5), (7, 1), (8, 4), (10, 13), (11, -5)$.

Find the domain of $f + g$.

41. Write the domain and range of $f(x) = \sqrt{x - [x]}$ (where $[\cdot]$ represents greatest integer function)

42. Let $f(x) = \dfrac{\alpha x}{x+1}, x \ne -1$. Then write the value of a satisfying $f(x) = x$ for all $x \ne -1$.

43. If f, g, h are real functions given by $f(x) = x^2$, $g(x) = \tan x$ and $h(x) = \log_e x$, then write the value of (hogof) $\left(\sqrt{\dfrac{\pi}{4}}\right)$.

44. Solve for x: $x + \sqrt{x} \ge \sqrt{x} - 3$

45. $R = \{(x, y) : y = |x - 1| + |x| + |x + 1|, -2 \le x \le 2\}$

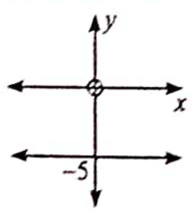
By drawing a graph between x and y, find the range of R.

46. Solve the inequality $2|x - a| > |2x + a|$ where a is a constnat and $a > 0$.

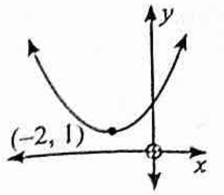
47. If $2f(x) - 3f\left(\dfrac{1}{x}\right) = x^2 (x \ne 0)$. then $f(2)$ is equals to

48. For each of the following graphs, state the domain and range.

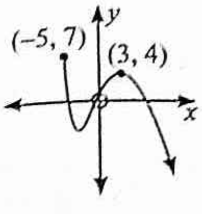
(i)



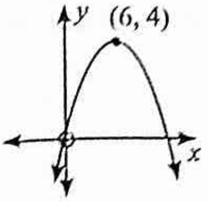
(ii)



(iii)



(iv)



49. Find range of trigonometric functions:

(i) $y = \cos^4 \dfrac{x}{2} - \sin^4 \dfrac{x}{2}$.

(ii) $y = 4$ and $x \cdot \cos x$.

(iii) $y = \sin \sqrt{x}$.

(iv) $y = \cos(2 \sin x)$

(v) $y = \sin^2\left(\dfrac{15\pi}{8} - 4x\right) - \sin^2\left(\dfrac{17\pi}{8} - 4x\right)$.

(vi) $y = \tan^2\left(x - \dfrac{\pi}{4}\right)$.

(vii) $f(x) = \sin^4 x + \cos^4 x$

(viii) $f(x) = \sin^6 x + \cos^6 x$

50. Find the range of the algebraic functions:

(i) $f(x) = x^2 + x + 1$

(ii) $f(x) = 2x^2 + 3x + 5$

(iii) $f(x) = x^2 + 2x + 7$

(iv) $f(x) = (x-1)(x-2)$

(v) $f(x) = \dfrac{(x-1)}{x^2+x+1}$

(vi) $f(x) = \dfrac{2x^2+3x+1}{x^2+x+1}$

(vii) $f(x) = \dfrac{(x-1)(x-2)}{(x-3)}$

(viii) $f(x) = \dfrac{(x-1)(x-2)}{(x-3)(x-4)}$

(ix) $f(x) = \dfrac{(x-1)(x-3)}{(x-2)(x-4)}$

(x) $f(x) = \dfrac{x^2-5}{2x^2+3}$

(xi) $f(x) = x^2 + x + 1; x \geq 0$

(xii) $f(x) = 2x^2 + x + 1; 1 \leq x \leq 5$

51. Solve for x:

(i) $[x] = 3$

(ii) $[3 - 2x] = -4$

(iii) $[x] \geq 2$

(iv) $[2x - 3] \geq \sqrt{2}$

(v) $[3x - 5] > \pi$

(vi) $[x] > 1$

(vii) $[x] < -4$

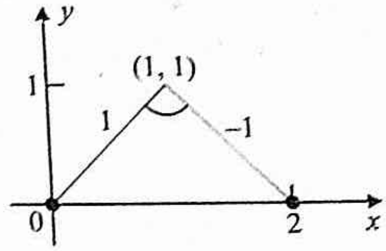
(viii) $[x] \leq 7$

(ix) $[x] = -\sqrt{2}$

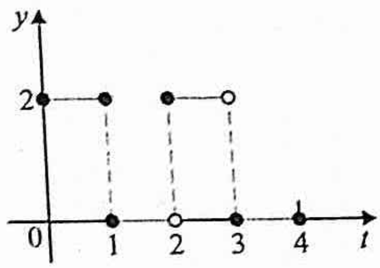
(x) $[x + \frac{1}{2}] = 5$

52. Find a formula for each function graphed in Exercises (a)-(h).

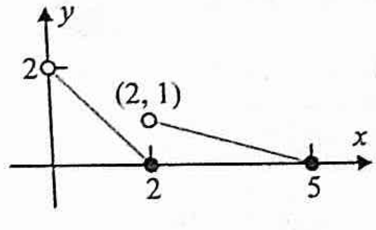
(a)



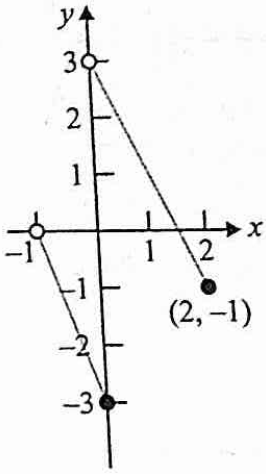
(b)



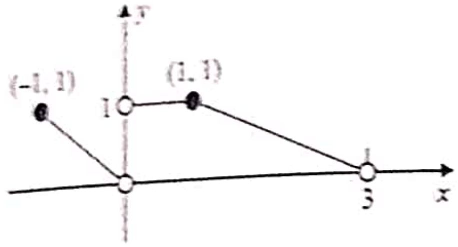
(c)



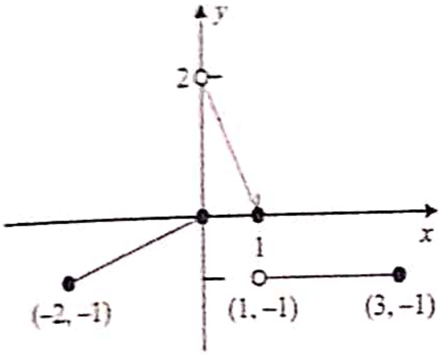
(d)



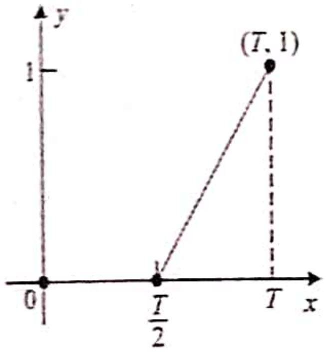
(e)



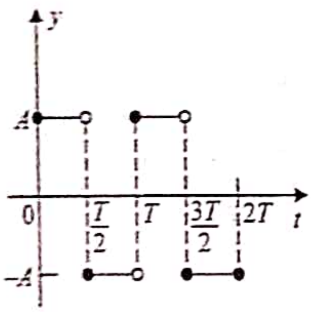
(f)



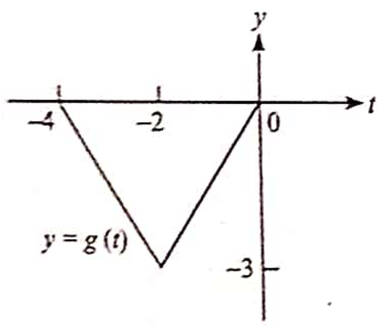
(g)



(h)



53. The accompanying figure shows the graph of a function $f(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



(i) $g(-t)$

(ii) $-g(t)$

(iii) $g(t)+3$

(iv) $1-g(t)$

(v) $g(-t+2)$

(vi) $g(t-2)$

(vii) $g(1-t)$

(viii) $-g(t-1)$

54. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following function, and sketch their graphs.

(i) $f(x)+2$

(ii) $f(x)-1$

(iii) $2f(x)$

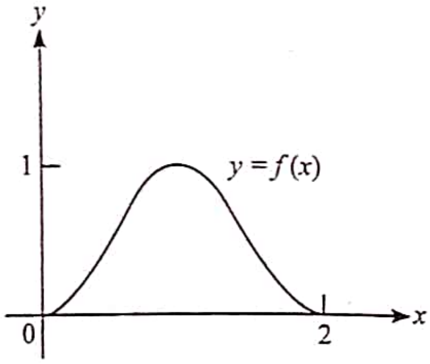
(iv) $-f(x)$

(v) $f(x+2)$

(vi) $f(x-1)$

(vii) $f(-x)$

(viii) $-f(x+1)+1$



55. If $y = f(x) = \dfrac{ax-b}{bx-a}$, show that $x = f(y)$.

56. If $f(x) = \dfrac{2x}{1+x^2}$, show that $f(\tan\theta) = \sin 2\theta$.

57. If $f(x) = (e - x^5)^{1/5}$ then prove that $f([f(x)] = x$ for all x.

58. The function f is defined by $f(x) = \begin{cases} x^2, 0 \le x \le 3 \\ 3x, 3 \le x \le 10 \end{cases}$

The relation g is defined by $g(x) = \begin{cases} x^2, 0 \le x \le 2 \\ 3x, 2 \le x \le 10 \end{cases}$

Show that f is a function and g is not a function.

Exercise-3 (JEE Advanced Level)

1. The domain of the real valued function $f(x)=\sqrt{5-4x-x^2}+x^2\log(x+4)$ is
   (a) $-5\le x\le 1$ (b) $-5\le x$ and $x\ge 1$
   (c) $-4<x\le 1$ (d) ϕ

2. The domain of definition of the function $f(x)\sqrt{1+\log(1-x)}$ is
   (a) $-\infty<x\le 0$ (b) $-\infty<x\le\frac{e-1}{e}$
   (c) $-\infty<x\le 1$ (d) $x\ge 1-e$

3. If $[x]$ stands for the greatest integer function, then the value of
$$\left[\frac{1}{2}+\frac{1}{1000}\right]+\left[\frac{1}{2}+\frac{2}{1000}\right]+\ldots+\left[\frac{1}{2}+\frac{999}{1000}\right]$$
   (a) 498 (b) 499
   (c) 500 (d) 501

4. If $|x^2-12x+32|+|x^2-9x+20|=0$. Then find the value of x.
   (a) 5 (b) 4
   (c) 8 (d) 3

5. If $f(x)=\cos[\pi^2]x+\cos([-\pi^2]x)$, where [.] denotes the step function, then
   (a) $f(0)=1$ (b) $f\left(\frac{\pi}{4}\right)=2$
   (c) $f\left(\frac{\pi}{2}\right)=-1$ (d) $f(\pi)=1$

6. If function $\text{sgn}(\ln(\sin x))=0$, then find the value of x.
   (a) $\frac{\pi}{2}$ (b) π
   (c) $\frac{\pi}{4}$ (d) 0

7. If the function $\text{sgn}\,(x^2-9x+20)=1$, then what will be the value of x.
   (a) $(4, 5)$ (b) $(5, \infty)$
   (c) $(-\infty, 4)\cup(5,\infty)$ (d) None of these

8. Let $f(x)=\text{sgn}\left(\sin\theta-\frac{\sqrt{3}}{2}\right)x^2+\left(\cos\theta\frac{1}{2}\right)x+\left(\tan\theta-\sqrt{3}\right)$

   If $f(x)$ is identically zero for every $x\in R$, then the number of values of θ in $[-2\pi, 2\pi]$, is

   Note: sgn k denotes the signum function of k.
   (a) 0 (b) 1
   (c) 2 (d) 3

9. Let $f(x)=ax+b$, where a and b are integers. If $f(f(0))=0$ and $f(f(f(4))=9$, then the value of $f(f(f(f(10))))$ is equal to
   (a) 0 (b) 4 (c) 9 (d) 10

10. The domain of definition of $f(x)=\log_{(x^2-x-1)}(2x^2-7x+9)$ is
   (a) R (b) $R-\{0\}$
   (c) $R-\{0, 1\}$ (d) $R-\{1\}$

11. Which of the following equation have the same graph
   I. $y=x-2$
   II. $y=\frac{(x^2-4)}{(x+2)}$
   III. $(x+2)y=x^2-4$
   (a) I and II only
   (b) I and II only
   (c) II and III only
   (d) All the equations have different graphs

12. The equation $x^2-12x+35=[x]+[-x]$ has (where $[x]$ denotes largest integer less than or equal to x)
   (a) No sulution (b) 4 solutions
   (c) 3 solutions (d) 2 solutions

13. The range of the function $f(x)=\sqrt{4-x^2}+\sqrt{x^2-1}$ is
   (a) $\left[\sqrt{3},\sqrt{7}\right]$ (b) $\left[\sqrt{3},\sqrt{5}\right]$
   (c) $\left[\sqrt{2},\sqrt{3}\right]$ (d) $\left[\sqrt{3},\sqrt{6}\right]$

14. If x is real, then the expression $\frac{x^2+34x-71}{x^2+2x-7}$
   (a) cannot lie between 5 and 9
   (b) always lies between 5 and 9
   (c) is not real
   (d) none of these

COMPREHENSION BASED QUESTIONS

Comprehension (Q. 15 to 17):

Consider the function $f(x)=\begin{cases}x^2-1, & -1\le x\le 1\\ \ell nx, & 1<x\le e\end{cases}$

Let $f_1(x)=f(|x|)$

$f_2(x)=|f(|x|)|$

$f_3(x)=f(-x)$

Now answer the following questions.

15. Number of positive solution of the equation $2f_2(x) - 1 = 0$ is (are)

(a) 4
(b) 3
(c) 2
(d) 1

16. Number of integral solution of the equation $f_1(x) = f_2(x)$ is (are)

(a) 1
(b) 2
(c) 3
(d) 4

17. If $f_4(x) = \log_{27}(f_3(x) + 2)$, then range of $f_4(x)$ is

(a) $[1, 9]$
(b) $\left[\frac{1}{3}, \infty\right)$
(c) $\left[0, \frac{1}{3}\right]$
(d) $[1, 27]$

NUMERICAL VALUE BASED

18. The function f satisfies the functional e
$3f(x) + 2f\left(\frac{x+59}{x-1}\right) = 10x + 30$ for all real $x \neq 1$. T
of $f(7)$ is equal to

19. The total number of solutions of $[x]^2 = x + 2\{x\}$,
and {.} denote the greatest integer and the fracti
functions, respectively, is equal to

20. If x and y satisfy the equation $y = 2[x] + 3$ a
$[x-2]$ simultaneously, where [.] denotes the greate
function, then $[x + y]$ is equal to

21. If $f(y) = \log y$, then $f(y) + f\left(\frac{1}{y}\right)$ is equal to

22. Find the domain of the explicit form of the
represented implictly by the equation $(1 + x)\cos$

Exercise-4 (Past Year Questions)

EE MAIN

1. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then, $f(\theta)$ **(2000)**

(a) ≥ 0, only when $\theta \geq 0$
(b) ≤ 0, for all real θ
(c) ≥ 0, for all real θ
(d) ≤ 0, only when $\theta \leq 0$

2. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is **(2001)**

(a) [0,1]
(b) [0,1/2]
(c) [1/2,1]
(d) (0,1]

3. The domain of definition of $f(x) = \frac{\log_2(x+3)}{x^2+3x+2}$ is

(a) $R/\{-1, -2\}$
(b) $(-2, \infty)$
(c) $R/\{-1, -2, -3\}$
(d) $(-3,\infty) - \{-1, -2\}$ **(2001)**

4. Let $f(x) = \frac{x^2-6x+5}{x^2-5x+6}$. **(2007)**

	Column-I		Column-II
A.	If $-1<x<1$, then $f(x)$ satisfies	p.	$0 < f(x) < 1$
B.	If $1<x<2$, then $f(x)$ satisfies	q.	$f(x) < 0$
C.	If $3<x<5$, then $f(x)$ satisfies	r.	$f(x) > 0$
D.	If $x>5$, then $f(x)$ satisfies	s.	$f(x) < 1$

(a) $A \to p, B \to q, C \to q, D \to p$
(b) $A \to r, B \to q, C \to q, D \to p$
(c) $A \to p, B \to q, C \to r, D \to p$
(d) $A \to p, B \to q, C \to q, D \to s$

5. If $f(x) + 2f\left(\frac{1}{x}\right) = 3x, x \neq 0$ and
$S = \{x \in R : f(x) = f(-x)\}$; then S:

(a) is an empty set.
(b) contains exactly one element
(c) contains exactly two elements
(d) contains more than two elements

6. Let $S = \{x \in R : x \geq 0$ and
$2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0\}$. Then S:

(a) contains exactly one element.
(b) contains exactly two elements.
(c) contains exactly four elements.
(d) is an empty set.

7. Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ for $k = 1, 2, 3....$ Th
$x \in R$, the value of $f_4(x) - f_6(x)$ is equal to

(a) 1/12
(b) 1/4
(c) $-1/12$
(d) 5/12

8. Let $\sum_{k=1}^{10} f(a+k) = 16(2^{10} - 1)$, where the function
$f(x+y) = f(x)f(y)$ for all natural numbers x, y and $f(1$
the natural number 'a' is

(a) 4
(b) 3
(c) 16
(d) 2

9. The domain of the definition of the function $f(x)=\frac{1}{4-x^2}+\log_{10}(x^3-x)$ is **(2019)**

(a) $(-1,0)\cup(1,2)\cup(3,\infty)$
(b) $(-2,-1)\cup(-1,0)\cup(2,\infty)$
(c) $(-1,0)\cup(1,2)\cup(2,\infty)$
(d) $(1,2)\cup(2,\infty)$

10. If $\{p\}$ denotes the fractional part of the numbe p, then $\left\{\frac{3^{200}}{8}\right\}$, is equal to: **(2020)**

(a) $\frac{5}{8}$ (b) $\frac{1}{8}$
(c) $\frac{7}{8}$ (d) $\frac{3}{8}$

11. If $f(x+y)=f(x)f(y)$ and $\sum_{x=1}^{\infty} f(x)=2,\ x,y\in N$,

Where N is the set of all natural numbers, then the value of $\frac{f(4)}{f(2)}$ is: **(2020)**

(a) $\frac{1}{9}$ (b) $\frac{4}{9}$
(c) $\frac{1}{3}$ (d) $\frac{2}{3}$

12. If $a+\alpha=1$, $\beta+b=2$ and $a f(x)+\alpha f\left(\frac{1}{x}\right)=bx+\frac{\beta}{x}$, $x\neq 0$, then the value of the expression $\frac{f(x)+f\left(\frac{1}{x}\right)}{x+\frac{1}{x}}$ is **(2021)**

13. A function $f(x)$ is given by $f(x)=\frac{5^x}{5^x+5}$, then the sum of the series $f\left(\frac{1}{20}\right)+f\left(\frac{2}{20}\right)+f\left(\frac{3}{20}\right)+\ldots+f\left(\frac{39}{20}\right)$ is equal to: **(2021)**

(a) $\frac{19}{2}$ (b) $\frac{49}{2}$
(c) $\frac{39}{2}$ (d) $\frac{29}{2}$

14. The number of roots of the equation, $(81)^{\sin^2 x}+(81)^{\cos^2 x}=30$ in the interval $(0,\pi]$ is equal to: **(2021)**

(a) 3 (b) 4 (c) 8 (d) 2

15. If the functions are defined as $f(x)=\sqrt{x}$ and $g(x)=\sqrt{1-x}$, then what is the common domain of the following functions:
$f+g$, $f-g$, f/g, g/f, $g-f$ where $(f\pm g)(x)=f(x)\pm g(x)$, $(f/g)(x)=\frac{f(x)}{g(x)}$ **(2021)**

(a) $0\le x\le 1$
(b) $0\le x<1$
(c) $0
(d) $0

JEE ADVANCED

16. Let $y=\sqrt{\frac{(x+1)(x-3)}{(x-2)}}$

Find all the real values of x for which y takes real values. **(1980)**

(a) $x\in[-1,2)\cup[3,\infty)$ (b) $x\in(-1,2)\cup[3,\infty)$
(c) $x\in[-1,2)\cup(3,\infty)$ (d) $x\in[-1,2]\cup[3,\infty)$

17. The domain of definition of the function $y=\frac{1}{\log_{10}(1-x)}+\sqrt{x+2}$ is **(1983)**

(a) $(-3,-2)$ excluding -2.5
(b) $[0,1]$ excluding 0.5
(c) $(-2,1)$ excluding 0
(d) None of these

18. If S is the set of all real x such that $\frac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains **(1986)**

(a) $\left(-\infty,-\frac{3}{2}\right)$ (b) $\left(-\frac{3}{2},-\frac{1}{4}\right)$
(c) $\left(-\frac{1}{4},\frac{1}{2}\right)$ (d) $\left(\frac{1}{2},3\right)$

19. If $f_1(x)$ and $f_2(x)$ are defined on domains D_1 and D_2 respectively, then $f_1(x)+f_2(x)$ is defined on $D_1\cap D_2$.
(a) true (b) false **(1998)**

20. Let the function $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\frac{4^x}{4^x+2}$
Then the value of
$f\left(\frac{1}{40}\right)+f\left(\frac{2}{40}\right)+f\left(\frac{3}{40}\right)+\ldots+f\left(\frac{39}{40}\right)-f\left(\frac{1}{2}\right)$ is ... **(2020)**

21. For $x\in R$, the number of real roots of the equation $3x^2-4|x^2-1|+x-1=0$ is ______. **(2021)**

ANSWER KEY

CONCEPT APPLICATION

1. $[x = 5, y = 2]$ 2. $[x = 5, y = 2]$ 3. $\{3,4\}$ 4. Domain $\equiv \{2, 3, 4, 5\}$, Ray $\equiv \{3, 6, 7, 10\}$

5. Domain $\equiv \{1, 2, 3, 4, 5\}$, Ray $\equiv \{2, 3, 4, 5, 6\}$, Co-domain $\equiv \{1, 2, 3, 4, 5, 6\}$ 6. (b, e, g, h) 7. (i) R – ... (ii) R – {7}, (iii) R – {1}, (iv) R – {–3, 3}, (v) R – {2, 6} 8. (i) $[0, \infty)$, (ii) $\{-2, 2\}$, (iii) $\{\theta\}$ 9. (i) R, (ii) R, (iii) R (iv) $(0, \infty)$, (v) $(0, \infty)$, (vi) $[0, \infty)$ (vii) $[0, \infty)$ (viii) $(0, \infty)$ (xi) $(0, \infty)$, (x) $\{-1, 1\}$, (xi) $[0, \infty)$, (xii) IR – $[0, 1)$ (xiii) {–1, 1} (ivx) R, (xv) $\{1, -1\}$, (xvi) $\{1, -1\}$, (xvii) $\{1, -1\}$, (xviii) $\{1\}$ 10. (i) $[-27, \infty\}$, (ii) $[3, \infty)$, (iii) $[-19, \infty)$, (iv) $[-3, \infty\}$, (v) $[-38, \infty\}$, (vi) $[-9.75, \infty\}$, (vii) $[-\infty, -9\}$, (viii) $[-\infty, 7\}$, (ix) $[-\infty, 105\}$

11. (i) $\{-4, -3, 0, 5\}$, (ii) No pre-image, 0 and 2, –2 12. (i) Domain $\equiv \{2, 5, 8, 11, 14, 17\}$, Range $\equiv \{1\}$, (ii) Domain $\equiv \{2, 4, 6, 8, 10, 12, 14\}$, Range $\equiv \{1, 2, 3, 4, 5, 6, 7\}$, (iii) Not a function

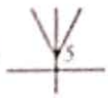
13. (i) $(-\infty, -2] \cup [3, 6]$, (ii) $(-\infty, \sqrt{3}\,] \cup [0, \sqrt{3}\,]$, (iii) 0, (iv) $[-4, -\pi] \cup [0, \pi]$, (v) $[8, \infty]$, (vi) $[n\pi : n \in z]$

14. (i) $\sqrt{x-1}+\sqrt{x+1}$, (ii) $\sqrt{x-1}-\sqrt{x+1}$ (iii) $C\sqrt{x-1}$, (iv) $\sqrt{x^2-1}$, (v) $\dfrac{1}{\sqrt{x-1}}$, (vi) $\sqrt{\dfrac{x-1}{x+1}}$

15. (i) Domain $(-6, \infty)$, Range $(-2, \infty)$, (ii) Domain $(-\infty, 3)$, Range $(-0, \infty)$

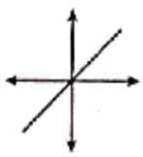
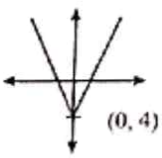
16. (i) 1, (ii) 2, (iii) –2, (iv) 0, (v) –1, (vi) 0 17. (i) 3, (ii) 1, (iii) 0, (iv) 2, (v) 1, (vi) $\dfrac{5}{2}$

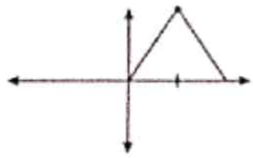
18. (i) 1, (ii) 3, (iii) –3, (iv) 9, (v) 10, (vi) 2 19. (i) $[-5, -4)$, (ii) $[0, 1)$, (iii) 0, (iv) $[0, 1)$, (v) ϕ, (vi) $[1, 15)$

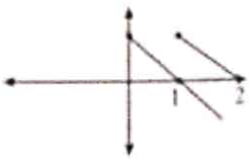
20. (i) $\left\{x; x \in \underset{n \in I}{U}\left\{n+\dfrac{1}{2}, n+\dfrac{1}{6}, n+\dfrac{5}{6}\right\}\right.$ 21. $[1, 3]$ 22. $\left\{-1, \dfrac{9}{5}\right\}$ 23. (i) 

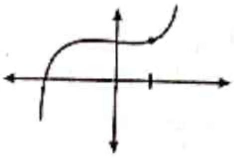
(ii) 

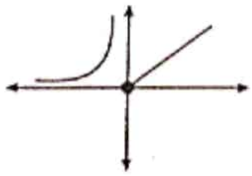
24. $\left(\dfrac{4}{3}, 4\right)$ 25. R – $[1, 3)$ 26. (a) $(-4, \infty)$; $(-2, \infty)$, (b) $(-3, 4)$; $(2, -5)$, (c) $(-3, 4)$; $(-5, 6)$, (d) R – {4} ; (0, –∞)

27. (a)  (b) 

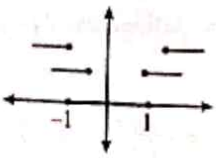
29. 

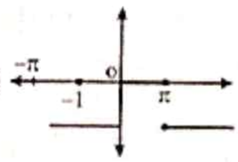
30. 

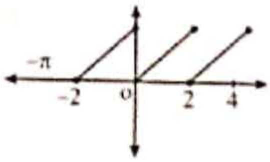
31. 

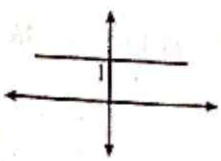
32. 

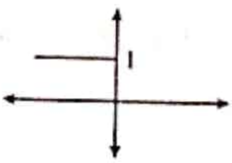
33. 

34. (i) 

(ii) 

(iii) 

(iv) 

(v) 

35. No Solution 36. (a) $y = -(x + 7)^2$ (b) $y = -(x - 4)^2$

37. (a) $y - 3 = x^2$ (b) $y + 5 = x^2$

38. (a) $y - 4 = -x^2$ (b) $y - 3 = -x^2$ (c) $y + 1 = -(x + 4)^2$ (d) $y = -(x - 2)^2$

EXERCISE-1 (TOPICWISE)

1. (*a*) 2. (*b*) 3. (*a*) 4. (*d*) 5. (*a*) 6. (*c*) 7. (*d*) 8. (*d*) 9. (*c*) 10. (*d*)
11. (*d*) 12. (*c*) 13. (*b*) 14. (*c*) 15. (*c*) 16. (*b*) 17. (*b*) 18. (*b*) 19. (*b*) 20. (*c*)
21. (*b*)

EXERCISE-2 (LEARNING PLUS)

1. (*b*) 2. (*c*) 3. (*b*) 4. (*a*) 5. (*d*) 6. (*a*) 7. (*b*) 8. (*a*) 9. (*d*) 10. (*c*)
11. (*d*) 12. (*d*) 13. (*c*) 14. (*b*) 15. (*c*) 16. (*d*) 17. (*d*) 18. (*a*) 19. (*d*) 20. (*c*)
21. (*b*) 22. (*c*) 23. (*a*) 24. (*c*) 25. (*c*) 26. (*c*) 27. (*b*) 28. (*b*) 29. (*a*) 30. [1]

31. $[0, 1/4]$ 32. $[1, 9]$ 33. $(-\infty, -5/2]\cup[-1/4, \infty)$ 34. (*i*) R (*ii*) e^{-2} (iii) $x>0, y>0$

35. $\dfrac{1}{a^2-b^2}\left[a\left(\dfrac{1}{x}-5\right)-b(x-5)\right]$ 36. $\{2, 3, 5, 7, 13, 17\}$ 37. $-x; -2\le x\le 0$
$0; 0<x\le 1$
$2x-2; 1<x\le 2$
38. $(2\sqrt{2}+2, \text{DN}\Sigma)$

39. (*i*) $\{0, -\sin 1\}$, (*ii*) $[1, e]$ 40. $\{2, 8, 10\}$ 41. Domain $\equiv R$, Range $\equiv (0, 1)$ 42. -1 43. 0

44. $[0, \infty]$ 45. $[2, 7]$ 46. $[-\infty, 9/4]$ 47. $[-7/4]$

48. (*i*) Domain $\equiv R$, Range $\equiv (1, \infty)$ (*ii*) Domain $\equiv R$, Range $\equiv (-5, \infty)$ (*iii*) Domain $\equiv (-5, \infty)$, Range $\equiv (-\infty, 7)$
(*iv*) Domain $\equiv R$, Range $\equiv (-\infty, 6)$

49. (*i*) $[-1,1]$ (*ii*) $[-2,2]$ (*iii*) $[-1,1]$ (*iv*) $[\cos 2,1]$ (*v*) $\left[\dfrac{-1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right]$ (*vi*) $[0, \infty]$ (*vii*) $[½, 1]$ (*viii*) $[¼, 1]$

50. (*i*) $\left[\dfrac{3}{4},\infty\right]$ (*ii*) $\left[\dfrac{31}{8},\infty\right]$ (*iii*) $[6, \infty]$ (*iv*) $\left[\dfrac{-1}{4},\infty\right]$ (*v*) $\left[\dfrac{-3-2\sqrt{3}}{3},\dfrac{-3+2\sqrt{3}}{3}\right]$ (*vi*) $\left[\dfrac{3-2\sqrt{3}}{3},\dfrac{3+2\sqrt{3}}{3}\right]$

(*vii*) $R-\left[3-2\sqrt{2},3+2\sqrt{2}\right]$ (*viii*) $R-\left[-7-4\sqrt{3},-7+4\sqrt{3}\right]$ (*xi*) $R-\left[\dfrac{11-\sqrt{117}}{2},\dfrac{11+\sqrt{117}}{2}\right]U\left\{\dfrac{5}{2}\right\}$ (*x*) $\left[\dfrac{-5}{3},\dfrac{1}{2}\right]$

(*xi*) $\left[\dfrac{3}{4},\infty\right]$ (*xii*) $\left[\dfrac{7}{8},\infty\right]$

51. (*i*) $[3, 4]$ (*ii*) $\left(\dfrac{1}{2},3\right)$ (*iii*) $[2, \infty]$ (*iv*) $\left[\dfrac{-5}{2},\infty\right)$ (*v*) $[3, \infty]$ (*vi*) $[2, \infty]$ (*vii*) $[-\infty, -4]$ (*viii*) $[-\infty, -8]$ (*ix*) No Solution
(*x*) $[4.5, 5.5]$

52. (*i*) $y=\begin{cases} x & x\in[0,1) \\ -x+2 & x\in[1,2) \end{cases}$ (*ii*) $y=\begin{cases} 2 & x\in[2n,2n+1) \\ 0 & x\in[2n+1,2n+2) \end{cases}$ (*iii*) $y=\begin{cases} -x+2 & x\in(0,2] \\ \dfrac{-1}{3}x+\dfrac{5}{3} & x\in(2,5] \end{cases}$ (*iv*) $y=\begin{cases} -2x+3 & x\in(0,2] \\ -3x-3 & x\in(-1,0] \end{cases}$

(*v*) $y=\begin{cases} -x & x\in[-1,0) \\ 1 & x\in(0,1] \\ -\dfrac{1}{2}x+\dfrac{3}{2} & x\in[1,3) \end{cases}$ (*vi*) $y=\begin{cases} 2x & x\in[-2,0] \\ -2x+2 & x\in(0,1] \\ -1 & x\in(1,3] \end{cases}$ (*vii*) $y=\dfrac{2}{T}x-1, x\in\left[\dfrac{T}{2},T\right]$

(*viii*) $y=\begin{cases} A & x\in\left[nT,nT+\dfrac{T}{2}\right) \\ -A & x\in\left[nT+\dfrac{T}{2},nT+T\right) \end{cases}$

53. (*i*) Domain ≡ [0, 4], Range ≡ [–3, 0] (*ii*) Domain ≡ [–4, 0], Range ≡ [0, 3] (*iii*) Domain ≡ [–4, 0], Range ≡ [0, 3]
(*iv*) Domain ≡ [–4, 0], Range ≡ [1, 4] (*v*) Domain ≡ [2, 6], Range ≡ [–3, 0] (*vi*) Domain ≡ [–2, 2], Range ≡ [–3, 0]
(*vii*) Domain ≡ [1, 5], Range ≡ [–3, 0] (*viii*) Domain ≡ [–3, 1], Range ≡ [0, 3] (*ix*) Domain ≡ [–4, 0], Range ≡ [–6, 0]
(*x*) Domain ≡ [–2, 0], Range ≡ [–3, 0]

54. (*i*) Domain ≡ [0, 2], Range ≡ [2, 3] (*ii*) Domain ≡ [0, 2], Range ≡ [–1, 0] (*iii*) Domain ≡ [0, 2], Range ≡ [0, 2]
(*iv*) Domain ≡ [0, 2], Range ≡ [–1, 0] (*v*) Domain ≡ [–2, 0], Range ≡ [0, 1] (*vi*) Domain ≡ [1, 3], Range ≡ [0, 1]
(*vii*) Domain ≡ [–2, 0], Range ≡ [0, 1] (*viii*) Domain ≡ [–1, 1], Range ≡ [0, 1]

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (*c*) **2.** (*b*) **3.** (*c*) **4.** (*b*) **5.** (*c*) **6.** (*a*) **7.** (*c*) **8.** (*c*) **9.** (*d*) **10.** (*c*)
11. (*d*) **12.** (*d*) **13.** (*d*) **14.** (*a*) **15.** (*a*) **16.** (*d*) **17.** (*c*) **18.** [4] **19.** [4] **20.** [30]
21. $\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (*c*) **2.** (*d*) **3.** (*d*) **4.** (*a*) **5.** (*c*) **6.** (*b*) **7.** (*a*) **8.** (*b*) **9.** (*c*) **10.** (*b*)
11. (*b*) **12.** (*b*) **13.** (*c*) **14.** (*b*) **15.** (*c*)

JEE Advanced

16. (*a*) **17.** (*c*) **18.** (*a,d*) **19.** (*a*) **20.** [19.00] **21.** (*d*)

CHAPTER

4 Trigonometric Ratios and Identities

INTRODUCTION

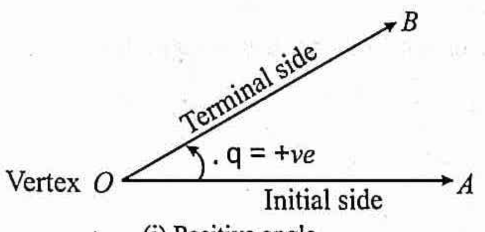
The word trigonometry is derived from three greek words.

Tri	Gon	Metron
↓	↓	↓
three	sides	Measure

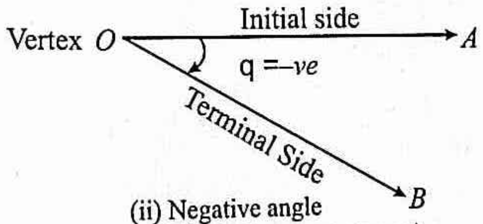
In the ancient time trigonometry defines relations between elements of a triangle. In a triangle there are six basic elements, three sides and three angles. Any three line segments will form a triangle if they satisfy three triangular inequalities i.e. the sum of any two lines segment is greater than third side. In Euclidean geometry the sum of three angles of a triangle is 180°. These requirements impose limitations on the manner in which the relations between the elements are defined.

ANGLE

Angle is a measure of rotation of a given ray about its initial point. The original ray is called the initial side and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is negative.



(i) Positive angle
(anticlockwise measurement)



(ii) Negative angle
(clockwise measurement)

SYSTEMS FOR MEASUREMENT OF ANGLES

An angle can be measured in the following systems.

1. **Sexagesimal System (British System):** In this system $\frac{1}{360}$ of a complete circular turn is called a degree (°), $\frac{1}{60}$ of a degree is called a minute (') and $\frac{1}{60}$ of a minute is called a second ('').
   One right angle = 90°, 1° = 60′, 1′ = 60″

2. **Centesimal System (French System):** In this system $\frac{1}{400}$ of a complete circular turn is called a grade (g), $\frac{1}{100}$ of a grade is called a minute (') and $\frac{1}{100}$ of a minute is called a second ('').
   ∴ One right angle = 100^g; 1^g = 100'; 1' = 100''
   Note: The minutes and seconds in the Sexagesimal system are different with the minutes and seconds respectively in the Centesimal System. Symbols in both systems are also different.

3. **Circular System (Radian Measurement):** The angle subtended by an arc of a circle whose length is equal to the radius of the circle at the centre of the circle is called a radian. In this system the unit of measurement is radian (^{c}) As the circumference of a circle of radius 1 unit is 2π, therefore one complete revolution of the initial side subtends an angle of 2π radian.
   More generally, in a circle of radius r, an arc of length r will subtend an angle of 1 radian. It is well-known that equal arcs of a circle subtend equal angle at the centre. Since in a circle of radius r, an arc of length r subtends an angle whose measure is 1 radian, an arc of length ℓ will subtend an angle whose measure is $\frac{\ell}{r}$ radian. Thus, if in a circle of radius r, arc of length ℓ subtends an angle θ radian at the centre, we have $\theta = \frac{\ell}{r}$ or $\ell = r\theta$.

CONTENTS

Mathematics Module-1 1-158

1. Basic Mathematics and Logarithm 1-28
2. Sets 29-47
3. Functions 48-82
4. Trigonometric Ratios and Identities 83-122
5. Trigonometric Equations 123-150
6. Principle of Mathematical Induction 151-158

CHAPTER 4

Trigonometric Ratios and Identities

INTRODUCTION

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↓	↓	↓
three	sides	Measure

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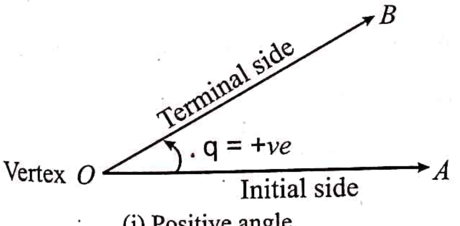
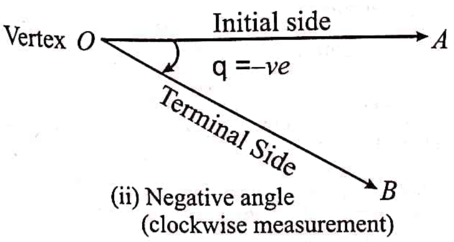


(i) Positive angle (anticlockwise measurement)



(ii) Negative angle (clockwise measurement)

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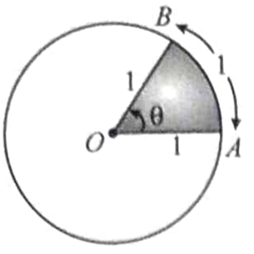
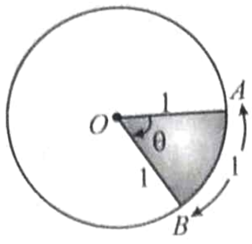
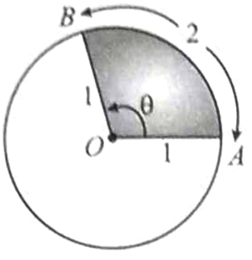


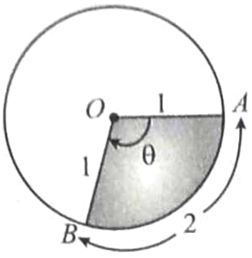
$\theta = 1$ radian
(i)



$\theta = 1$ radian
(ii)

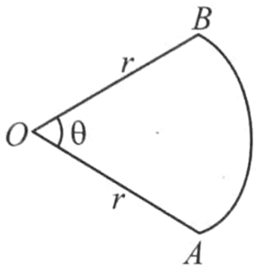


$\theta = 2$ radian



$\theta = 2$ radian

Area of Circular Sector



Area $= \frac{1}{2} r^2\theta$ sq. units

Note: θ radian is written as θ^c or can be written simply as θ. When the unit of angle is not mentioned. it must be taken as radians.

e.g. $\theta = 15$ implies 15 radian

Relation between radian, degree and grade:

$$\frac{\pi}{2} \text{ radian} = 90° = 100^g$$

Train Your Brain

Example 1: Find the radian measure corresponding to $-37° \; 30'$.

Sol. $60' = 1°$

$$\therefore 30' = \left(\frac{1}{2}\right)^{\circ}$$

$$\therefore -37° \; 30' = 37\frac{1}{2}^{\circ} = -\frac{75}{2}^{\circ}$$

$360° = 2\pi$ radians

$$\therefore \quad \frac{-75°}{2} = -\frac{2\pi}{360} \times \frac{75}{2} \text{ radians} = \frac{-5\pi}{24} \text{ radians}$$

Example 2: The minute hand of a clock is 10 cm long. How far does the tip of the hand move in 20 minutes?

Sol. The minute hand moves through 120° in 20 minutes or moves through $\frac{2\pi}{3}$ radians.

Since the length of the minute hand is 10 cm, the distance moved by the tip of the hand is given by the formula

$$l = r\theta = 10 \times \frac{2\pi}{3} = \frac{20\pi}{3} \text{ cm.}$$

Concept Application

1. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.
2. The angles of a triangle are in AP and the number of degrees in the least is to the number of radians in the greatest as 60 to π find the angles in degrees.
3. Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5' at his eye, find the height of the letters that he can read at a distance of 12 metres.

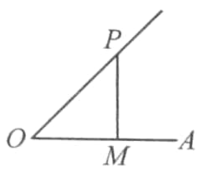
TRIGONOMETRIC RATIOS FOR ACUTE ANGLES

Let a revolving ray OP starts from OA and revolves into the position OP, thus tracing out the angle AOP.

In the revolving ray take any point P and draw PM perpendicular to the initial ray OA.

In the right angle triangle MOP, OP is the hypotenuse, PM is the perpendicular, and OM is the base.

The trigonometrical ratios, or functions, of the angle AOP are defined as follows:



$\frac{MP}{OP}$, i.e., $\frac{\text{Perpendicular}}{\text{Hypotenuse}}$, $\sin \angle AOP$

$\frac{OM}{OP}$, i.e. $\frac{\text{Base}}{\text{Hypotenuse}}$, $\cos \angle AOP$;

$\frac{MP}{OM}$, i.e. $\frac{\text{Perpendicular}}{\text{Base}}$, $\tan \angle AOP$;

$\frac{OM}{MP}$, i.e. $\frac{\text{Base}}{\text{Perpendicular}}$, cot $\angle AOP$;

$\frac{OP}{OM}$, i.e. $\frac{\text{Hypotenuse}}{\text{Base}}$, sec $\angle AOP$;

$\frac{OP}{MP}$, i.e. $\frac{\text{Hypotenuse}}{\text{Perpendicular}}$, cosec $\angle AOP$;

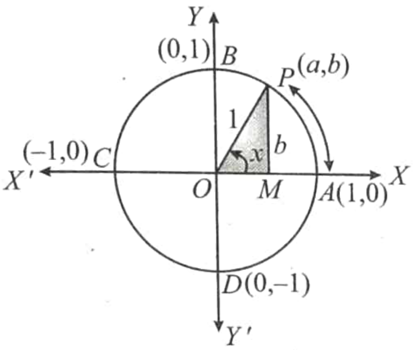
The quantity by which the cosine falls short of unity i.e. $1 - \cos \angle AOP$, is called the Versed Sine of $\angle AOP$; also the quantity $1 - \sin \angle AOP$, by which the sine falls short of unity, is called the Coversed Sine of $\angle AOP$.

It can be noted that the trigonometrical ratios are all real numbers.

The names of these eight ratios are written, for brevity, sin $\angle AOP$, cos $\angle AOP$, tan $\angle AOP$, cot $\angle AOP$, cosec $\angle AOP$, sec $\angle AOP$, vers $\angle AOP$, and covers $\angle AOP$ respectively

TRIGONOMETRIC RATIOS FOR AN ANGLE

We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions. (also called circular functions)



Consider a unit circle (radius 1 unit) with centre at origin of the coordinate axes. Let $P(a, b)$ be any point on the circle with angle $\angle AOP = x$ radian, i.e., length of arcAP $= x$

We define $\cos x = a$ and $\sin x = b$ Since ΔOMP is a right triangle, we have $OM^2 + MP^2 = OP^2$ or $a^2 + b^2 = 1$ Thus, for every point on the unit circle we have $a^2 + b^2 = 1$ or $\cos^2 x + \sin^2 x = 1$

Since one complete revolution subtends an angle of 2π radian at the centre of the circle, $\angle AOB = \frac{\pi}{2}$., $\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$. All angles which are integral multiples of $\frac{\pi}{2}$. are called quadrantal angles. The coordinates of the points A, B, C and D are, respectively, (1, 0), (0, 1), (–1, 0) and (0, –1). Therefore, for quadrantal angles, we have

$\cos 0 = 1$	$\sin 0 = 0$,
$\cos \frac{\pi}{2}. = 0$	$\sin \frac{\pi}{2}. = 1$
$\cos \pi = -1$	$\sin \pi = 0$
$\cos \frac{3\pi}{2} = 0$	$\sin \frac{3\pi}{2} = -1$
$\cos 2\pi = 1$	$\sin 2\pi = 0$

Now if we take one complete revolution from the position OP, we again come back to same position OP. Thus, we also observe that if x increases (or decreases) by any integral multiple of 2π, the values of sine and cosine functions do not change. Thus,

$\sin(2n\pi + x) = \sin x,\ n \in Z$, $\cos(2n\pi + x) = \cos x,\ n \in Z$

Further, $\sin x = 0$, if $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi$ ....., i.e., when x is an integral multiple of π and $\cos x = 0$, if $x = \pm\frac{\pi}{2}., \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, ...i.e., cos x vanishes when x is an odd multiple of $\frac{\pi}{2}$.

Thus $\sin x = 0$ implies $x = n\pi$, where n is any integer

$\cos x = 0$ implies $x = (2n + 1)\frac{\pi}{2}$., where n is any integer.

We now define other trigonometric functions in terms of sine and cosine functions:

$\operatorname{cosec} x = \frac{1}{\sin x}$, $x \neq n\pi$, where n is any integer.

$\sec x = \frac{1}{\cos x}$, $x \neq (2n + 1)\frac{\pi}{2}$., where n is any integer.

$\tan x = \frac{\sin x}{\cos x}$, $x \neq (2n + 1)\frac{\pi}{2}$., where n is any integer.

$\cot x = \frac{\cos x}{\sin x}$, $x \neq n\pi$, where n is any integer.

BASIC TRIGONOMETRIC IDENTITIES

1. $\sin^2\theta + \cos^2\theta = 1$; where $\theta \in R$.
2. $1 + \tan^2\theta = \sec^2\theta$ or $\sec^2\theta - \tan^2\theta = 1$
3. $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ or $\operatorname{cosec}^2\theta - \cot^2\theta = 1$
4. $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cdot\cos^2\theta$
5. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cdot\cos^2\theta$

Note: (*i*) $(\sec\theta - \tan\theta)$ is reciprocal of $(\sec\theta + \tan\theta)$ and vice-versa.

(*ii*) $(\operatorname{cosec}\theta - \cot\theta)$ is reciprocal of $(\operatorname{cosec}\theta + \cot\theta)$ and vice-versa.

Using above identities hundreds of other identities can be proved. While proving identities you can use rationalization, factorization and many other similar mathematical operations.

Train Your Brain

Example 3: Prove that $\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{1 + \sin A}{\cos A}$

Sol. $= \frac{(\tan A + \sec A) - (\sec^2 A - \tan^2 A)}{(1 - \sec A + \tan A)}$

$= \frac{(\tan A + \sec A)(1 - \sec A + \tan A)}{(1 - \sec A + \tan A)} = \frac{1 + \sin A}{\cos A}$

Example 4: If $\operatorname{cosec}\theta - \cot\theta = \frac{1}{5}$ then find the value of $\sin\theta$.

Sol. Given $\operatorname{cosec}\theta - \cot\theta = \frac{1}{5}$ ...(i)

$$\Rightarrow \operatorname{cosec}\theta + \cot\theta = \frac{1}{\operatorname{cosec}\theta - \cot\theta} = 5 \quad ...(ii)$$

Adding (i) and (ii), we get

$$2\operatorname{cosec}\theta = 5 + \frac{1}{5} = \frac{26}{5}$$

$$\Rightarrow \operatorname{cosec}\theta = \frac{13}{5} \Rightarrow \sin\theta = \frac{5}{13}$$

Example 5: If $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1$, Prove that $\sin^4 A + \sin^4 B = 2\sin^2 A \sin^2 B$

Sol. Given, $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1 = \cos^2 A + \sin^2 A$

or, $\frac{\cos^4 A}{\cos^2 B} - \cos^2 A = \sin^2 A - \frac{\sin^4 A}{\sin^2 B}$

or, $\frac{\cos^2 A\left(\cos^2 A - \cos^2 B\right)}{\cos^2 B}$

$$= \frac{\sin^2 A\left(\sin^2 B - \sin^2 A\right)}{\sin^2 B}$$

or, $\frac{\cos^2 A}{\cos^2 B}\left(\cos^2 A - \cos^2 B\right)$

$$= \frac{\sin^2 A}{\sin^2 B}\left[\left(1-\cos^2 B\right) - \left(1-\cos^2 A\right)\right]$$

$$\left(\cos^2 A - \cos^2 B\right)\left(\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B}\right) = 0$$

When $\cos^2 A - \cos^2 B = 0$, $\cos^2 A = \cos^2 B$

when $\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B} = 0$

$\cos^2 A \sin^2 B = \sin^2 A \cos^2 B$

Example 6: If $\sin\theta + \operatorname{cosec}\theta = 2$, the value of $\sin^{10}\theta + \operatorname{cosec}^{10}\theta$ is

(a) 10 (b) 2^{10} (c) 2^9 (d) 2

Sol. (d) We have,

$\sin\theta + \operatorname{cosec}\theta = 2 \Rightarrow \sin^2\theta + 1 = 2\sin\theta$

$\Rightarrow \sin^2\theta - 2\sin\theta + 1 = 0$

$\Rightarrow (\sin\theta - 1)^2 = 0 \Rightarrow \sin\theta = 1$

Required value of $\sin^{10}\theta + \operatorname{cosec}^{10}\theta$

$$= (1)^{10} + \frac{1}{(1)^{10}} = 2.$$

Example 7: The expression $\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A}$ can be written as:

(a) $\sin A\cos A + 1$

(b) $\sec A \operatorname{cosec} A + 1$

(c) $\tan A + \cot A$

(d) $\sec A + \operatorname{cosec} A$

Sol. (b) Given expression can be written as

$$\frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$$

$$\left(\because \tan A = \frac{\sin A}{\cos A} \text{ and } \cot A = \frac{\cos A}{\sin A}\right)$$

$$= \frac{1}{\sin A - \cos A}\left\{\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}\right\}$$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$$= \frac{\sin^2 A + \sin A\cos A + \cos^2 A}{\sin A\cos A}$$

$= 1 + \sec A \operatorname{cosec} A$

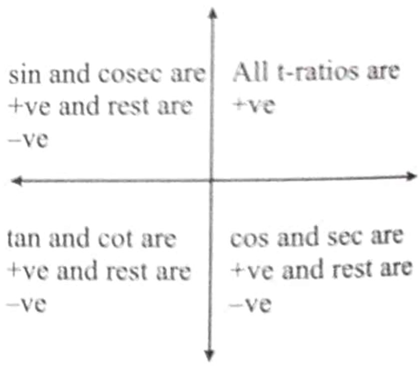
Concept Application

4. If $\sin x + \sin^2 x = 1$, then find the value of $\cos^8 x + 2\cos^6 x + \cos^4 x$.
5. If $\sec\theta + \tan\theta = 4$ then find $\cot\theta$
6. Which of the following reltions is correct

(a) $\sin 1 < \sin 1°$ (b) $\sin 1 > \sin 1°$

(c) $\sin 1 = \sin 1°$ (d) $\frac{\pi}{180}\sin 1 = \sin 1°$

SIGN CONVENTION OF THE TRIGONOMETRIC FUNCTIONS

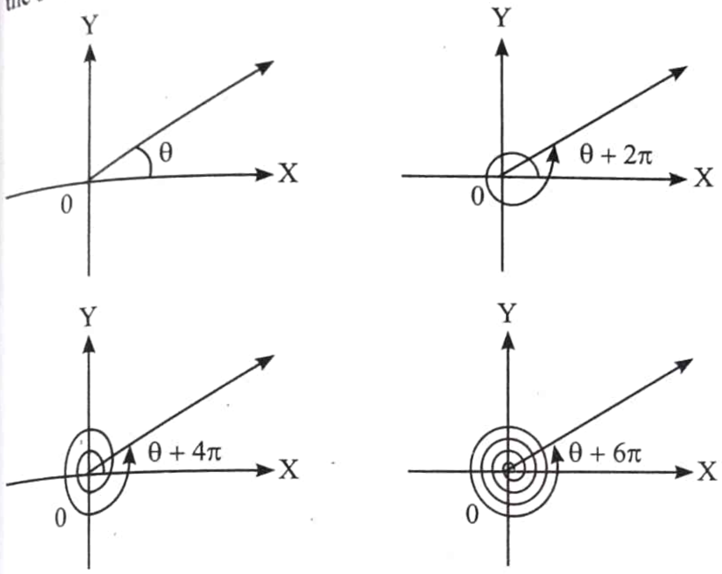


T. Ratio / Angle	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	$\sqrt{\frac{0}{4}}=0$	$\sqrt{\frac{1}{4}}=\frac{1}{2}$	$\sqrt{\frac{2}{4}}=\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$	$\sqrt{\frac{4}{4}}=1$
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N.D.

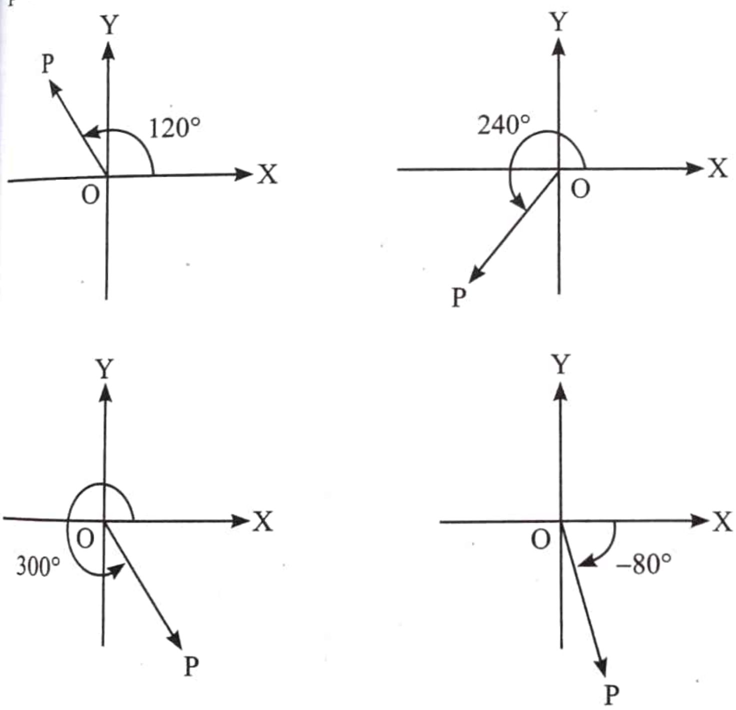
N.D. implies not defined

Note: 1. The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cos x and tan x, respectively.

2. Any trigonometric ratio of $n \cdot 360° + A$ ($n \in I$) is equal to the same trigonometric ratio of A e.g. 420°, –340° are conterminal with 40° and any trigonometric ratio of these angles will equal to the same ratio of 40° (as shown in following diagrams also).



3. An angle can lie in any of four quadrants according to the position of revolving ray for the angle.



TRIGONOMETRIC RATIOS OF ALLIED ANGLES

If θ is any angle, then $-\theta$, $\frac{\pi}{2} \pm \theta$, $\pi \pm \theta$, $\frac{3\pi}{2} \pm \theta$, $2\pi \pm \theta$ etc. are called allied angles.

Trigonometric Ratios of (–θ)

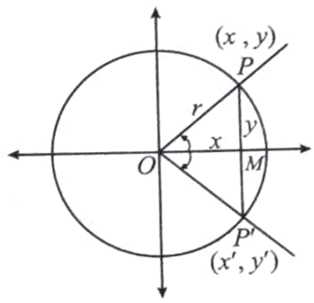
Let θ be an angle in the standard position in the I quadrant. Let its terminal side cuts the circle with centre 'O' and radius r in $P(x, y)$.

Let $P'(x', y')$ be the point of intersection of the terminal side of the angle $-\theta$ in the standard position with the circle.

Now $\angle MOP = \angle MOP'$ (numerically) and P and P' have the same projection M in the x-axis

$\therefore \quad \Delta OMP \cong \Delta OMP' \Rightarrow x = x'$ and $y = -y'$

$\therefore \quad \sin(-\theta) = \frac{y'}{r} = \frac{-y}{r} = -\sin\theta$.;



$\angle$ POM = θ
$\angle$ MOP′ = –θ

$$\cos(-\theta) = \frac{x'}{r} = \frac{x}{r} = \cos\theta.$$

$$\tan(-\theta) = \frac{y'}{x'} = -\frac{y}{x} = -\tan\theta;$$

$$\cot(-\theta) = \frac{x'}{y'} = \frac{x}{-y} = -\cot\theta.$$

$$\sec(-\theta) = \frac{r}{x'} = \frac{r}{x} = \sec\theta;$$

$$\text{cosec}(-\theta) = \frac{r}{y'} = \frac{r}{-y} = -\text{cosec}\,\theta.$$

Similarly if θ is in the other quadrants then the above results can also be proved.

Trigonometric Ratios of (π + θ)

Let θ be an angle in the standard position in the I quadrant. Let its terminal side cuts the circle with centre 'O' and radius r at $P(x, y)$. Let $P'(x', y')$ be the point of intersection of the terminal side of the angle $\pi - \theta$ with the circle. Let M and M' be the projections of P and P' respectively in the x-axis.

Since $\Delta OM'P' \cong \Delta OMP$, $x' = -x$, $y' = y$

$$\therefore \sin(\pi - \theta) = \frac{y'}{r} = \frac{y}{r} = \sin\theta.;$$

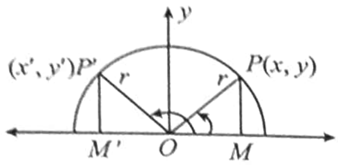
$$\cos(\pi - \theta) = \frac{x'}{r} = -\frac{x}{r} = -\cos\theta.$$

$$\tan(\pi - \theta) = \frac{y'}{x'} = \frac{y}{-x} = -\tan\theta;$$

$$\cot(\pi - \theta) = \frac{x'}{y'} = -\frac{x}{y} = -\cot\theta.$$

$$\sec(\pi - \theta) = \frac{r}{x'} = \frac{r}{-x} = -\sec\theta.;$$

$$\text{cosec}(\pi - \theta) = \frac{r}{y'} = \frac{r}{y} = \text{cosec}\,\theta.$$



$\angle MOP = \theta$

$\angle MOP' = \pi - \theta$

Trigonometric Ratios of $\left(\frac{\pi}{2} - \theta\right)$

Similarly we can easily prove the following results.

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta,\ \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta,$$

$$\text{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta,\ \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta,$$

$$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta,\ \sec\left(\frac{\pi}{2} - \theta\right) = \text{cosec}\,\theta$$

Trigonometric Ratios of $\left(\frac{\pi}{2} + \theta\right)$

Similarly we can easily prove the following results.

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta,\ \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta,$$

$$\text{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec\theta,\ \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta,$$

$$\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta,\ \sec\left(\frac{\pi}{2} + \theta\right) = -\text{cosec}\,\theta$$

Trigonometric Ratios of $(\pi + \theta)$

Similarly we can easily prove the following results.

$\sin(\pi + \theta) = -\sin\theta$, $\tan(\pi + \theta) = \tan\theta$, $\text{cosec}(\pi + \theta) = -\text{cosec}\,\theta$,

$\cos(\pi + \theta) = -\cos\theta$, $\cot(\pi + \theta) = \cot\theta$,

$\sec(\pi + \theta) = -\sec\theta$

Trigonometric Ratios of $\left(\frac{3\pi}{2} - \theta\right)$

Similarly we can easily prove the following results.

$$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta,\ \tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta,$$

$$\text{cosec}\left(\frac{3\pi}{2} - \theta\right) = -\sec\theta,\ \cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta,$$

$$\cot\left(\frac{3\pi}{2} - \theta\right) = \tan\theta,\ \sec\left(\frac{3\pi}{2} - \theta\right) = -\text{cosec}\,\theta,$$

Trigonometric Ratios of $\left(\frac{3\pi}{2} + \theta\right)$

Similarly we can easily prove the following results.

$$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta,\ \cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta,$$

$$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta,\ \cot\left(\frac{3\pi}{2} + \theta\right) = -\tan\theta,$$

$$\sec\left(\frac{3\pi}{2} + \theta\right) = \text{cosec}\,\theta,\ \text{cosec}\left(\frac{3\pi}{2} + \theta\right) = -\sec\theta$$

There is an easy way to remember these formulae. First of all think of θ as an acute angle. Angles like $180° \pm \theta$, $360° \pm \theta$, $-\theta$ can be considered as angles associated with the horizontal line, angles like $90° - \theta$, $90° + \theta$, $270° \mp \theta$ can be considered as angles associated with vertical line. When associated with the horizontal line, the magnitude of the function does not change, whereas when associated with the vertical line the function changes to the corresponding complementary value (sin $\leftrightarrow$ cos, tan $\leftrightarrow$ cot, sec $\leftrightarrow$ cosec). For example $\sin(180° + \theta)$ will be only $\sin\theta$ (in magnitude) plus or minus and $\cos(90° - \theta)$ will be $\sin\theta$ only in magnitude.

To decide upon the sign, consider the quadrant in which the angle falls and decide the sign by the quadrant rule.

For example, $\sin(180° + \theta)$ is $\sin\theta$ (in magnitude) $(180° + \theta)$ lies in third quadrant and hence $\sin(180° + \theta)$ is negative.

$\therefore \sin(180° + \theta) = -\sin\theta$

Again consider $\tan(90° + \theta)$: This should be $\cot\theta$ and must have a negative sign since $(90° + \theta)$ is in II quadrant and hence $\tan(90° + \theta)$ is negative.

$\therefore \tan(90° + \theta) = -\cot\theta$

Train Your Brain

Example 8: Find cos (315°)

Sol. $\cos(315°) = \cos(360° - 45°) = \cos(-45°)$

$= \cos(45°) = \frac{1}{\sqrt{2}}$

Example 9: Find tan (330°)

Sol. $\tan(330°) = \tan(360° - 30°) = \tan(-30°) = \tan 30°$

$= \frac{1}{\sqrt{3}}$

Example 10: Prove that $\sin 420° \cos 390° + \cos(-300°) \sin(-330°) = 1$.

Sol. $= \sin(360° + 60°)\cos(360° + 30°) + \cos(-360° + 60°) \sin(-360° + 30°)$

$= \sin 60° \cos 30° + \cos 60° \sin 30°$

$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = 1$

Example 11: Prove that sin 240° sin 510° – sin330° cos 390° = 0

Sol. sin (270° – 30°) sin (540° – 30°) – sin (360°– 30°) cos (360° + 30°)

$= -\cos 30° \sin 30° + \sin 30° \cos 30° = 0$

Example 12: Prove that $\cos A + \sin(270° + A) - \sin(270° - A) + \cos(180° + A) = 0$

Sol. $\cos A + \sin(270° + A) - \sin(270° - A) + \cos(180° + A)$

$= \cos A - \cos A - (-\cos A) + (-\cos A) = 0$

Example 13: Find the value of tan 35°.tan 40°.tan 45°.tan 50°.tan 55°.

Sol. We have,

tan 35°.tan 40°.tan 45°.tan 50°.tan 55°

= {tan 35° × tan 55°} {tan 40° × tan 50°} × tan 45°

= {tan 35° × cot 35°}.{tan 40° × cot 40°} × tan 45° = 1

Example 14: Find the value of tan(1°).tan(2°).tan(3°)......... tan(89°).

Sol. We have,

tan(1°).tan(2°).tan(3°)......tan(89°)

= tan(1°).tan(2°).tan(3°)........tan(44°)

tan(45°) tan(46°).....tan(87°) tan(88°) tan(89°)

= {tan(1°) × tan(89°)}.{ tan(2°) × tan(88°)}

.....{tan(44°) × tan(46°)}. tan(45°) = 1

Example 15: Find the value of cos(1°).cos(2°).cos(3°)..... cos(189°).

Sol. We have,

cos(1°).cos(2°).cos(3°).......cos(189°)

= cos(1°).cos(2°).cos(3°).......cos(89°)

cos(90°) cos(91°)......cos(189°)

= cos(1°).cos(2°).cos(3°)......cos(89°) × 0 × cos(91°) ..... cos(189°) = 0

Concept Application

7. If $5\cos^2\alpha - 2\sin\alpha - 2 = 0$, $(5\pi/4 < \alpha < 7\pi/4)$, then find the value of $\cot \alpha/2$.

8. Let $a = \tan\left(\frac{-2\pi}{3}\right)$, $b = \sin\left(\frac{-17\pi}{4}\right)$, $c = \cot(-300°)$

$d = \cos(-315°)$, then

(a) $a < b < c < d$
(b) $a < c < b < d$
(c) $b < c < d < a$
(d) $b < c < a < d$

9. Find the value of

$$\cos^2\frac{\pi}{16} + \cos^2\frac{3\pi}{16} + \cos^2\frac{5\pi}{16} + \cos^2\frac{7\pi}{16}$$

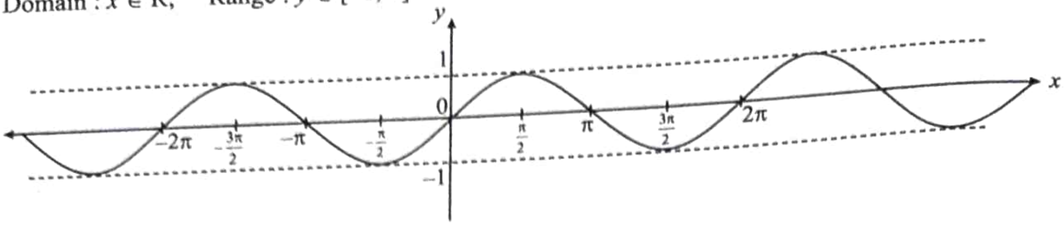
10. Match the columns:

	Column-I		Column-II
A.	sin (1050°)	p.	$-1/\sqrt{3}$
B.	sin (120°)	q.	3/4
C.	$\cos^2$ (120°)	r.	$-\sqrt{3}/2$
D.	tan (120°)	s.	1/2
E.	sin (135°)	t.	$1/\sqrt{2}$
F.	$\sin^2 9\pi/4$	u.	$-\sqrt{3}$
G.	sin 11π/3	v.	1/4
H.	$\cos^2 31\pi/6$	w.	$\sqrt{3}/2$
I.	tan 41π/6	x.	–1/2

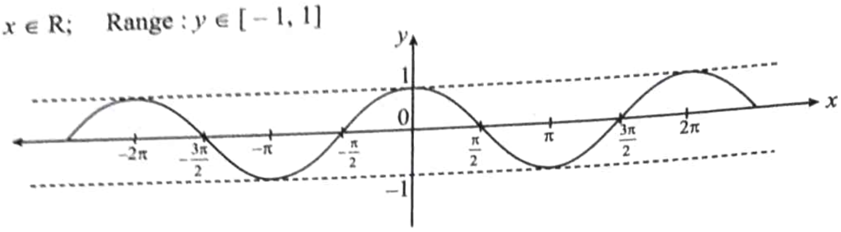
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Degree	0	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0
cot	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	–1	$-\sqrt{3}$	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	–1	$-\sqrt{3}$	ND

TRIGONOMETRIC FUNCTIONS

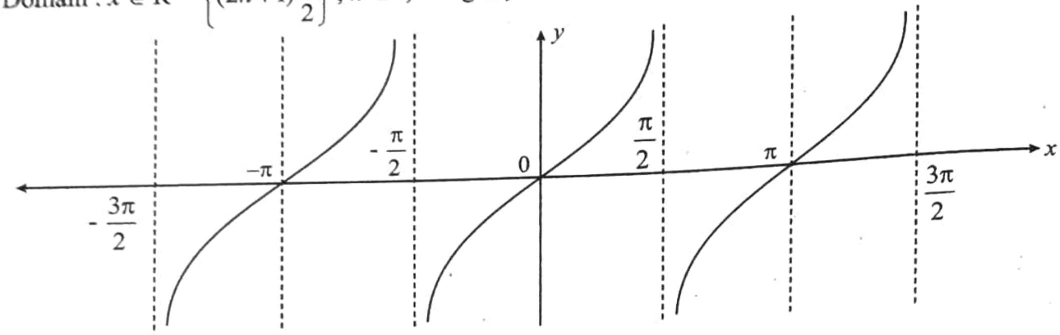
(*i*) $y = \sin x$ Domain : $x \in R$; Range : $y \in [-1, 1]$



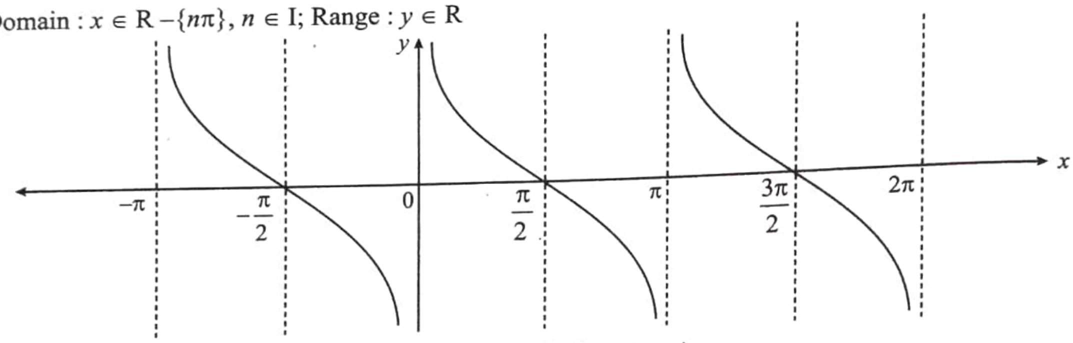
(*ii*) $y = \cos x$ Domain : $x \in R$; Range : $y \in [-1, 1]$



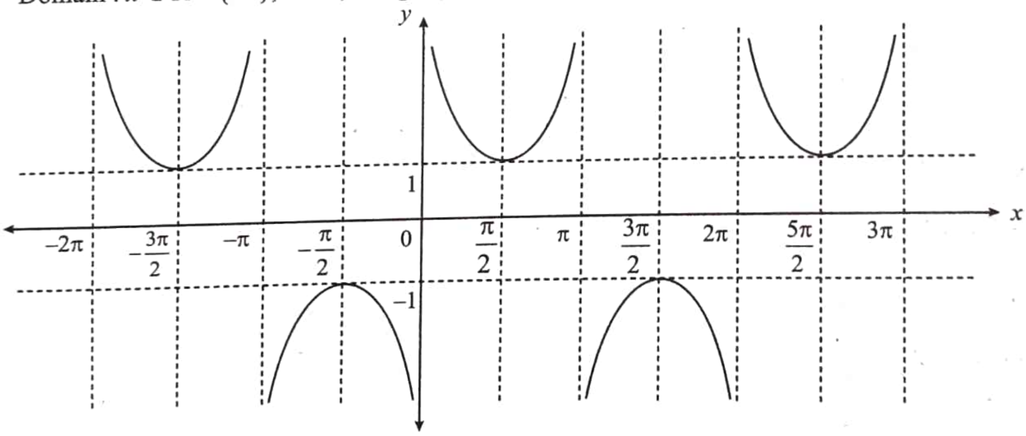
(*iii*) $y = \tan x$ Domain : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$, $n \in I$; Range : $y \in R$



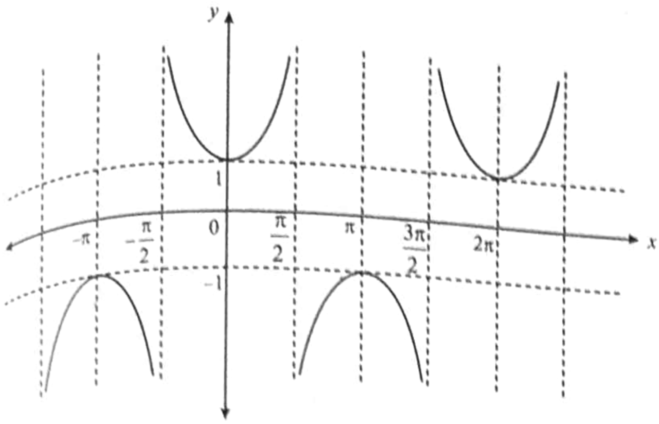
(*iv*) $y = \cot x$ Domain : $x \in R - \{n\pi\}$, $n \in I$; Range : $y \in R$



(*v*) $y = \operatorname{cosec} x$ Domain : $x \in R - \{n\pi\}$, $n \in I$; Range : $y \in (-\infty, -1] \cup [1, \infty)$



(vi) $y = \sec x$ Domain : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}, n \in I$; Range: $y \in (-\infty, -1] \cup [1, \infty)$



TRIGONOMETRIC RATIOS OF SUM OR DIFFERENCE OF AN ANGLES

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B).\sin(A-B)$

(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B).\cos(A-B)$

(e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(f) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

(g) $\sin(A+B+C) = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C$

(h) $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(i) $\tan(A+B+C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$$

(j) $\tan(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n) = \dfrac{S_1 - S_3 + S_5 - ...}{1 - S_2 + S_4 - ...}$

Train Your Brain

Example 16: Prove that $\cos(9^\circ) + \sin(9^\circ) = \sqrt{2}\sin(54^\circ)$

Sol. We have, $\cos(9^\circ) + \sin(9^\circ)$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos(9^\circ) + \frac{1}{\sqrt{2}}\sin(9^\circ)\right)$$

$$= \sqrt{2}\left(\sin(45^\circ)\cos(9^\circ) + \cos(45^\circ)\sin(9^\circ)\right)$$

$$= \sqrt{2}\left(\sin(45^\circ + 9^\circ)\right) = \sqrt{2}\sin(54^\circ)$$

Example 17: If $A + B = 45^\circ$, then find the value of $(1 + \tan A)(1 + \tan B)$

Sol. We have,

$A + B = 45^\circ$

$\Rightarrow \tan(A+B) = \tan(45^\circ) \quad \Rightarrow \tan(A+B) = 1$

$\Rightarrow \dfrac{\tan A + \tan B}{1 - \tan A.\tan B} = 1$

$\Rightarrow \tan A + \tan B = 1 - \tan A . \tan B$

$\Rightarrow \tan A + \tan B + \tan A . \tan B = 1$

$\Rightarrow 1 + \tan A + \tan B + \tan A . \tan B = 1 + 1 = 2$

$\Rightarrow (1 + \tan A) + \tan B(1 + \tan A) = 2$

$\Rightarrow (1 + \tan A)(1 + \tan B) = 2$

Example 18: Find the value of $(1 + \tan 20^\circ)(1 + \tan 24^\circ)(1 + \tan 25^\circ)(1 + \tan 21^\circ)$

Sol. We have,

$(1 + \tan 20^\circ)(1 + \tan 24^\circ)(1 + \tan 25^\circ)(1 + \tan 21^\circ)$

$= \{(1 + \tan 20^\circ)(1 + \tan 25^\circ)\} \times \{(1 + \tan 24^\circ)(1 + \tan 21^\circ)\}$

$= 2 \times 2 = 4$

Example 19: If $2\tan\alpha = 3\tan\beta$, then show that,

$$\tan(\alpha - \beta) = \frac{2\sin\beta\cos\beta}{4 + 2\sin^2\beta}$$

Sol. We have, $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha.\tan\beta}$

$$= \frac{\frac{3}{2}\tan\beta - \tan\beta}{1 + \frac{3}{2}\tan\beta\tan\beta} = \frac{\tan\beta}{2 + 3\tan^2\beta}$$

$$= \frac{\frac{\sin\beta}{\cos\beta}}{2 + 3\frac{\sin^2\beta}{\cos^2\beta}} = \frac{\sin\beta\cos\beta}{2\cos^2\beta + 3\sin^2\beta}$$

$$= \frac{\sin\beta\cos\beta}{2 + \sin^2\beta} = \frac{2\sin\beta\cos\beta}{4 + 2\sin^2\beta}$$

Concept Application

11. Prove that $\tan(70^\circ) = 2\tan(50^\circ) + \tan(20^\circ)$

12. Show that $\cos^2\theta + \cos^2(\alpha + \theta) - 2\cos\alpha\cos\theta\cos(\alpha + \theta)$ is independent of θ.

13. If $\sin(A - B) = \dfrac{1}{\sqrt{10}}$, $\cos(A + B) = \dfrac{2}{\sqrt{29}}$ find the value of $\tan 2A$ where A and R lie between 0 and $\pi/4$.

TRANSFORMATION FORMULAE

(*i*) $\sin(A+B)+\sin(A-B)=2\sin A\cos B$

(*a*) $\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(*ii*) $\sin(A+B)-\sin(A-B)=2\cos A\sin B$

(*b*) $\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(*iii*) $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

(*c*) $\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(*iv*) $\cos(A-B)-\cos(A+B)=2\sin A\sin B$

(*d*) $\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

Train Your Brain

Example 20: Show that sin12° sin 48° . sin 54° = 1/8

Sol. L.H.S. = $\frac{1}{2}$ [cos36°– cos60°]sin 54°

$$\frac{1}{2}\left[\cos 36°\sin 54°-\frac{1}{2}\sin 54°\right]$$

$$=\frac{1}{4}[2\cos36°\sin54°-\sin54°]$$

$$=\frac{1}{4}[\sin 90°+\sin 18°-\sin54°]$$

$$=\frac{1}{4}[1-(\sin 54°-\sin 18°)1$$

$$=\frac{1}{4}[1-2\sin 18°\cos36°]$$

$$\frac{1}{4}\left[1-\frac{2\sin 18°}{\cos 18°}\cos 18°\cos 36°\right]$$

$$\frac{1}{4}\left[1-\frac{\sin 36°\cos 36°}{\cos 18°}\right]$$

$$\frac{1}{4}\left[1-\frac{2\sin 36°\cos 36°}{2\cos 18°}\right]$$

$$\frac{1}{4}\left[1-\frac{\sin 72°}{2\sin 72°}\right]=\frac{1}{8}=\text{R.H.S.}$$

Example 21: Prove that $\frac{\sin 8\theta\cos\theta-\sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta-\sin 3\theta\sin 4\theta}=\tan 2\theta$

Sol.

$$\frac{2\sin 8\theta\cos\theta-2\sin 6\theta\cos 3\theta}{2\cos 2\theta\cos\theta-2\sin 3\theta\sin 4\theta}$$

$$=\frac{\sin 9\theta+\sin 7\theta-\sin 9\theta-\sin 3\theta}{\cos 3\theta+\cos\theta-\cos\theta+\cos 7\theta}$$

$$=\frac{2\sin 2\theta\cos 5\theta}{2\cos 5\theta\cos 2\theta}=\tan 2\theta$$

Concept Application

14. If $\sin A-\sin B=\frac{1}{2}$ and $\cos A-\cos B=\frac{1}{3}$ then find $\tan\left(\frac{A+B}{2}\right)$

15. The product cot 123°. cot 133° . cot 137° . cot 147°, when simplified is equal to

(*a*) –1
(*b*) tan 37°
(*c*) cot 33°
(*d*) 1

TRIGONOMETRIC RATIOS OF MULTIPLE AND SUB-MULTIPLE ANGLES

(*a*) $\sin 2A=2\sin A\cos A$

Note : $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ etc.

(*b*) $\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$

Note : $2\cos^2\frac{\theta}{2}=1+\cos\theta,\ 2\sin^2\frac{\theta}{2}=1-\cos\theta.$

(*c*) $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$

Note: $\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(*d*) $\sin 2A=\frac{2\tan A}{1+\tan^2 A},\ \cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}$

(*e*) $\sin 3A=3\sin A-4\sin^3 A$

(*f*) $\cos 3A=4\cos^3 A-3\cos A$

(*g*) $\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

(*h*) $\sin^2 A=\frac{1}{2}(1-\cos 2A)$

(*i*) $\cos^2 A=\frac{1}{2}(1+\cos 2A)$

(*j*) $\sin^3 A=\frac{1}{4}(3\sin A-\sin 3A)$

(*k*) $\sin A\sin(60-A).\sin(60+A)=\frac{1}{4}\sin 3A$

(*l*) $\cos A.\cos(60-A).\cos(60+A)=\frac{1}{4}\cos 3A$

(*m*) $\tan A\,.\,\tan(60-A)\,.\,\tan(60+A)=\tan 3A$

(*n*) $\cot\theta-\tan\theta=2\cot(2\theta)$

Train Your Brain

Example 22: Prove that

(i) $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$

(ii) $\tan A + \cot A = 2 \operatorname{cosec} 2A$

(iii) $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan\dfrac{A}{2}\cot\dfrac{B}{2}$

(iv) $\dfrac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta} = 4\cos 2\theta\cos 4\theta$

Sol.

(i) L.H.S. $\dfrac{\sin 2A}{1+\cos 2A} = \dfrac{2\sin A\cos A}{2\cos^2 A} = \tan A$

(ii) L.H.S. $\tan A + \cot A = \dfrac{1+\tan^2 A}{\tan A}$

$$= 2\left(\frac{1+\tan^2 A}{2\tan A}\right) = \frac{2}{\sin 2A} = 2\operatorname{cosec} 2A$$

(iii) L.H.S. $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)}$

$$= \frac{2\sin^2\frac{A}{2}+2\sin\frac{A}{2}\sin\left(\frac{A}{2}+B\right)}{2\cos^2\frac{A}{2}-2\cos\frac{A}{2}\cos\left(\frac{A}{2}+B\right)}$$

$$= \tan\frac{A}{2}\left[\frac{\sin\frac{A}{2}+\sin\left(\frac{A}{2}+B\right)}{\cos\frac{A}{2}-\cos\left(\frac{A}{2}+B\right)}\right]$$

$$= \tan\frac{A}{2}\left[\frac{2\sin\frac{A+B}{2}\cos\left(\frac{B}{2}\right)}{2\sin\frac{A+B}{2}\sin\left(\frac{B}{2}\right)}\right] = \tan\frac{A}{2}\cot\frac{B}{2}$$

(iv) $\dfrac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta} = \dfrac{\sin 5\theta\cos 3\theta+\sin 3\theta\cos 5\theta}{\sin 5\theta\cos 3\theta-\sin 3\theta\cos 5\theta}$

$$= \frac{\sin 8\theta}{\sin 2\theta} = 4\cos 2\theta\cos 4\theta$$

Example23: Provethat: $\dfrac{\tan\left(\frac{\pi}{4}+A\right)}{\tan\left(\frac{\pi}{4}-A\right)} = \dfrac{2\cos A+\sin A+\sin 3A}{2\cos A-\sin A-\sin 3A}$

Sol. L.H.S. $= \dfrac{\dfrac{1+\tan A}{1-\tan A}}{\dfrac{1-\tan A}{1+\tan A}} = \dfrac{(1+\tan A)^2}{(1-\tan A)^2} = \dfrac{1+\tan^2 A+2\tan A}{1+\tan^2 A-2\tan A}$

$$= \frac{1+\dfrac{2\tan A}{1+\tan^2 A}}{1-\dfrac{2\tan A}{1+\tan^2 A}} = \frac{1+\sin 2A}{1-\sin 2A} \text{ R.H.S.}$$

$$= \frac{2\cos A+2\sin 2A.\cos A}{2\cos A-2\sin 2A\cos A}$$

$$= \frac{2\cos A(1+\sin 2A)}{2\cos A(1-\sin 2A)} = \frac{1+\sin 2A}{1-\sin 2A}$$

Both sides reduce to the same result.

Example 24: Prove that $\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{8\pi}{7}\right) = \frac{1}{8}$

Sol. We have $\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{8\pi}{7}\right)$

$$= \cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\pi+\frac{\pi}{7}\right)$$

$$= -\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2^2\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{2\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2^3\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{4\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\right)$$

$$= -\frac{\sin\left(\frac{8\pi}{7}\right)}{2^3\sin\left(\frac{\pi}{7}\right)} = -\frac{\sin\left(\pi+\frac{\pi}{7}\right)}{2^3\sin\left(\frac{\pi}{7}\right)} = \frac{\sin\left(\frac{\pi}{7}\right)}{8\sin\left(\frac{\pi}{7}\right)} = \frac{1}{8}$$

Concept Application

16. If $f(\theta) = \sin^2\theta + \sin^2\left(\theta+\frac{2\pi}{3}\right) + \sin^2\left(\theta+\frac{4\pi}{3}\right)$, then $f\left(\frac{\pi}{15}\right)$ is equal to

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

17. Prove that $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)$

$\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)=\frac{1}{8}$

18. Prove that $\sin(20^\circ)\sin(40^\circ)\sin(80^\circ)=\frac{\sqrt{3}}{8}$

TRIGONOMETRIC RATIOS OF STANDARD ANGLES

(*a*) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$, where $n \in I$

(*b*) $\sin 15^\circ$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ$ or $\cos\frac{5\pi}{12}$;

$\cos 15^\circ$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ$ or $\sin\frac{5\pi}{12}$;

$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75^\circ$;

$\tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15^\circ$

To find the trigonometrical functions of an angle of 18°: Let θ stand for 18°, so that 2θ is 36° and 3θ is 54°.
Hence $2\theta = 90^\circ - 3\theta$ and therefore $\sin 2\theta = \sin(90^\circ - 3\theta) = \cos 3\theta$

$\therefore\ 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

Hence, either $\cos\theta = 0$, which gives $\theta = 90^\circ$, or

$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta$

$\therefore\ 4\sin^2\theta + 2\sin\theta = 1$

By solving this quadratic equation, we have (In our case $\sin\theta$ is necessarily a positive quantity. Hence we take the upper sign, and have)

$$\sin\theta = \frac{\sqrt{5}-1}{4} = \sin 18^\circ$$

Hence $\cos 18^\circ = \sqrt{1-\sin^2 18^\circ} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{16}}$

The remaining trigonometrical ratio of 18° may be now found. Since 72° is the complement of 18°, the value of the ratios for 72° may be obtained.

To find the trigonometrical functions of an angle of 36°

Since $\cos 2\theta = 1 - 2\sin^2\theta$,

$\therefore\ \cos 36^\circ = 1 - 2\sin^2 18^\circ = \frac{\sqrt{5}+1}{4}$

The remaining trigonometrical functions of 36° may now be found. Also, since 54° may be found.

Note: 1. We must be careful while determining the square root of trigonometrical function e.g.

$\sin^2 x = |\sin x|^2$ not $\sin x$,

2. $\sqrt{1+\sin A} = \left|\sin\frac{A}{2}+\cos\frac{A}{2}\right|$

Values of standard angles :

Angle → / ↓T. Ratio	$\left(\frac{\pi}{12}\right)$ 15°	$\left(\frac{\pi}{10}\right)$ 18°	$\left(\frac{\pi}{8}\right)$ 22.5°	$\left(\frac{\pi}{5}\right)$ 36°	$\left(\frac{3\pi}{8}\right)$ 67.5°	$\left(\frac{5\pi}{12}\right)$ 75°
sin	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
cos	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
tan	$2-\sqrt{3}$	$\frac{1}{\sqrt{(5+2\sqrt{5})}}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$	$\sqrt{2}+1$	$2+\sqrt{3}$
cot	$2+\sqrt{3}$	$\sqrt{(5+2\sqrt{5})}$	$\sqrt{2}+1$	$\sqrt{\left(1+\frac{2}{\sqrt{5}}\right)}$	$\sqrt{2}-1$	$2-\sqrt{3}$

Train Your Brain

Example 25: Prove that $\sin(12^\circ).\sin(48^\circ).\sin(54^\circ) = \frac{1}{8}$

Sol. We have, $\sin(12^\circ).\sin(48^\circ).\sin(54^\circ)$

$$= \frac{1}{\sin(72^\circ)}(\sin(12^\circ).\sin(48^\circ).\sin(72^\circ))(\sin(54^\circ))$$

$$= \frac{1}{4\sin(72^\circ)}(4\sin(60^\circ - 12^\circ).\sin(12^\circ).$$

$$\sin(60^\circ + 12^\circ))(\cos(36^\circ))$$

$$= \frac{1}{4\sin(72^\circ)}(\sin(36^\circ).\cos(36^\circ))$$

$$= \frac{1}{8\sin(72^\circ)}(2\sin(36^\circ).\cos(36^\circ))$$

$$= \frac{1}{8\sin(72^\circ)}(\sin(72^\circ)) = \frac{1}{8}$$

Example 26: Prove that : $4(\sin(24^\circ) + \cos(6^\circ)) = (1+\sqrt{5})$

Sol. We have, $4(\sin(24^\circ) + \cos(6^\circ))$

$$= 4(\sin(24^\circ) + \sin(84^\circ))$$

$$= 4\left(2\sin\left(\frac{24^\circ + 84^\circ}{2}\right)\cos\left(\frac{24^\circ - 84^\circ}{2}\right)\right)$$

$$= 8(\sin(54^\circ)\cos(30^\circ)) = 8(\cos(36^\circ)\cos(30^\circ))$$

$$= 8\left(\frac{\sqrt{5}+1}{4} \times \frac{1}{2}\right) = (\sqrt{5}+1)$$

Example 27: Prove that $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$

Sol. L.H.S. $= \sin 36^\circ \sin 72^\circ \sin 108^\circ \sin 144^\circ$

$$= \sin 36^\circ \sin 72^\circ \sin(180^\circ - 72^\circ)\sin(108^\circ - 36^\circ)$$

$$= \sin 36^\circ \sin 72^\circ \sin 72^\circ \sin 36^\circ = \sin^2 36^\circ \sin^2 72^\circ$$

$$= \frac{5-\sqrt{5}}{8} \times \frac{5+\sqrt{5}}{8} = \frac{25-5}{64} = \frac{20}{64} = \frac{5}{16}$$

Concept Application

19. If $\alpha = 112^\circ 30'$, find the value of $\sin\alpha$ and $\cos\alpha$.

20. Find the value of $\cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ$.

CONDITIONAL IDENTITIES

If $A + B + C = \pi$ then :

(*i*) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(*ii*) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(*iii*) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

(*iv*) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(*v*) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(*vi*) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(*vii*) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}$

(*viii*) $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$

(*ix*) $A + B + C = \frac{\pi}{2}$ then $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$

Train Your Brain

Example 28: If $A + B + C = 180^\circ$, prove that, $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A\cos B\cos C$.

Sol. Let $S = \sin^2 A + \sin^2 B + \sin^2 C$

so that

$$2S = 2\sin^2 A + 1 - \cos 2B + 1 - \cos 2C$$

$$= 2\sin^2 A + 2 - 2\cos(B + C)\cos(B - C)$$

$$= 2 - 2\cos^2 A + 2 - 2\cos(B + C)\cos(B - C)$$

$$\therefore S = 2 + \cos A[\cos(B - C) + \cos(B + C)]$$

since $\cos A = -\cos(B + C)$

$$\therefore S = 2 + 2\cos A\cos B\cos C$$

Example 29: If $A + B + C = 180^\circ$ prove that $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Sol. $(\cos A + \cos B) + \cos C - 1$

$$= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \cos C - 1$$

$$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\frac{A-B}{2} + \cos C - 1$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2} - 1$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right]$$

$$= 2\sin\frac{C}{2}\,2\sin\frac{A}{2}\sin\frac{B}{2} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Example 30: If A, B, C are the angles of a triangle, show that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Sol. We have

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - (\tan B \tan C + \tan C \tan A + \tan A \tan B)}$$

But $\tan(A + B + C) = \tan 180° = 0$

Hence $0 = \tan A + \tan B + \tan C - \tan A \tan B \tan C$

i.e, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

This may also be proved independently. For

$\tan(A + B) = \tan(180° - C) = -\tan C.$

$$\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$\tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Concept Application

21. In triangle ABC, prove that $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4 \sin A \sin B \sin C.$

22. Prove that in triangle ABC, $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C.$

SUMMATION OF TRIGONOMETRIC SERIES

Type-I

Sum of the sin and cosine series when the angles are in A.P.

(1) $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + \overline{n-1}\,\beta)$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \times \sin\left[\alpha + (n-1)\frac{\beta}{2}\right]$$

Proof : Let $S = \sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + \overline{n-1}\,\beta)$

Here angle are in A.P. and common difference of angles $= \beta$

$\therefore$ multiplying both sides by $2\sin\frac{\beta}{2}$, we get

$$2S\sin\frac{\beta}{2} = 2\sin\alpha\sin\frac{\beta}{2} + 2\sin(\alpha+\beta)\sin\frac{\beta}{2} + ... + 2\sin(\alpha + \overline{n-1}\,\beta)\sin\frac{\beta}{2} \quad ...(i)$$

Now, $2\sin\alpha\sin\frac{\beta}{2} = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)$

$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right)$

... ...

... ...

$2\sin(\alpha + \overline{n-1}\,\beta)\sin\frac{\beta}{2}$

$$= \cos\left[\alpha + (2n-3)\frac{\beta}{2}\right] - \cos\left[\alpha + (2n-1)\frac{\beta}{2}\right]$$

Adding, we get R.H.S. of equation (i)

$$= \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left[\alpha + (2n-1)\frac{\beta}{2}\right];$$

hence, $S = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left[\alpha + (n-1)\frac{\beta}{2}\right]$

(2) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + ... + \cos(\alpha + \overline{n-1}\,\beta)$

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \times \cos\left[\alpha + (n-1)\frac{\beta}{2}\right]$$

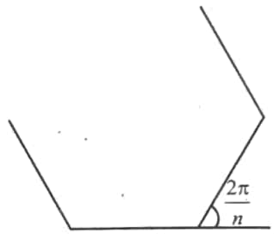
Type-II

For n sided regular polygon

Sum of all exterior angles = 2π

(i) The value of one exterior angle $= \dfrac{2\pi}{n}$

(ii) The value of one interior angle $= \left(\pi - \dfrac{2\pi}{n}\right) = \pi\dfrac{(n-2)}{n}$



(iii) Sum of interior angles $= \pi(n - 2)$

Type-III

Splitting the sum series as difference of 2 terms.

Here each term of trigonometric series is splitted into difference of two so that most of the terms while adding them gets cancelled.

Type-IV Continued Product of Cosine Series

$$\cos A\cos 2A\cos 4A\cos 8A .... \cos 2^{n-1}A = \frac{1}{2^n\sin A}\sin(2^n A)$$

Proof : Multiplying above and below by $2^n \sin A$

$\therefore$ LHS

$$= \frac{2^{n-1}}{2^n\sin A}[2\sin A\cos A\cos 2A\cos 4A .... \cos 2^{n-1}A]$$

$$= \frac{2^{n-2}}{2^n\sin A}[2\sin 2A\cos 2A\cos 4A .... \cos 2^{n-1}A]$$

$$= \frac{2^{n-3}}{2^n \sin A}[2 \sin 4A \cos 4A \ldots \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A}[2 \sin 2^{n-1} A \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A} \sin (2 \cdot 2^{n-1} A) = \frac{\sin(2^n A)}{2^n \sin A}$$

Train Your Brain

Example 31: Find the summation of the following series

(*i*) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

(*ii*) $\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7}$

(*iii*) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$

Sol.

(*i*) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

$$= \frac{\cos\left(\frac{\frac{2\pi}{7} + \frac{6\pi}{7}}{2}\right)\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}$$

$$= \frac{\cos\frac{4\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}} = \frac{-\cos\frac{3\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}} = -\frac{\sin\frac{6\pi}{7}}{2\sin\frac{\pi}{7}} = -\frac{1}{2}$$

(*ii*) $\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7}$

$$= \frac{\cos\left(\frac{\frac{\pi}{7} + \frac{6\pi}{7}}{2}\right)\sin\frac{6\pi}{14}}{\sin\frac{\pi}{14}} = \frac{\cos\frac{\pi}{2}\sin\frac{6\pi}{14}}{\sin\frac{\pi}{14}} = 0$$

(*iii*) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$

$$= \frac{\sin\left(5 \times \frac{2\pi}{11 \times 2}\right)}{\sin\left(\frac{2\pi}{11 \times 2}\right)} \times \cos\left(\frac{\frac{\pi}{11} + \frac{9\pi}{11}}{2}\right)$$

$$= \frac{\sin\left(\frac{5\pi}{11}\right)}{\sin\left(\frac{\pi}{11}\right)} \times \cos\left(\frac{5\pi}{11}\right) = \frac{2\sin\left(\frac{5\pi}{11}\right)\cos\left(\frac{5\pi}{11}\right)}{2\sin\left(\frac{\pi}{11}\right)}$$

$$= \frac{\sin\left(\frac{10\pi}{11}\right)}{2\sin\left(\frac{\pi}{11}\right)} = \frac{1}{2}$$

Example 32: In a regular polygon of n-sides with $A_1, A_2, \ldots A_n$ vertices prove that

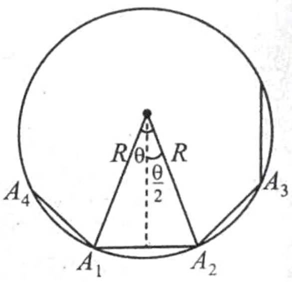
$(A_1A_2)^2 + (A_1A_3)^2 + (A_1A_4)^2 + \ldots + (A_1A_n)^2 = 2nR^2$

where R is the radius of circumcircle circumscribing it.

Sol. $(A_1A_2)^2 + (A_1A_3)^2 + \ldots (A_1A_n)^2 = 2nR^2$

$\theta = \frac{2\pi}{n}$

By trigonometry



$(A_1A_2)^2 = 4R^2 \cdot \sin^2\frac{\pi}{n}$

$(A_1A_3)^2 = 4R^2 \cdot \sin^2\frac{2\pi}{n}$

$(A_1A_4)^2 = 4R^2 \cdot \sin^2\frac{3\pi}{n} \ldots$

$(A_1A_n)^2 = 4R^2 \cdot \sin^2\frac{(n-1)\pi}{n}$

$\therefore (A_1A_2)^2 + (A_1A_3)^2 + \ldots + (A_1A_n)^2$

$$4R^2\left[\sin^2\frac{\pi}{n} + \sin^2\frac{2\pi}{n} + \sin^2\frac{3\pi}{n} + \ldots \sin^2\frac{(n-1)\pi}{n}\right]$$

$\frac{1}{2}4R^2$:

$$\left[1 - \cos\frac{2\pi}{n} + 1 - \cos\frac{4\pi}{n} + 1 - \cos\frac{6\pi}{n} + \ldots 1 - \cos\frac{2(n-1)\pi}{n}\right]$$

$$2R^2(n-1) - 2R^2\left[\frac{\sin\frac{(n-1)\pi}{n}}{\sin\left(\frac{\pi}{n}\right)} \cdot \cos\left(\frac{\frac{2\pi}{n} + \frac{2(n-1)\pi}{n}}{2}\right)\right]$$

$\Rightarrow 2R^2(n-1) - 2R^2(-1)$

$\Rightarrow 2R^2n - 2R^2 + 2R^2 \quad \Rightarrow \quad 2nR^2$

Example 33: S = cosec x + cosec $2x$ + cosec $4x$ + ... + cosec $2^n x = \cot\frac{x}{2} - \cot 2^n x$.

Sol. Let cosec $x = \dfrac{\sin\left(\frac{x}{2}\right)}{\sin\frac{x}{2}\cdot\sin x} = \dfrac{\sin\left(x - \frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)\cdot\sin x}$

$$=\frac{\sin x\cdot\cos\frac{x}{2}-\cos x\cdot\sin\frac{x}{2}}{\sin\frac{x}{2}\cdot\sin x}=\cot\frac{x}{2}-\cot x$$

$$S=\left(\cot\frac{x}{2}-\cot x\right)+(\cot x-\cot 2x)+(\cot 2x-\cot 2^2x)$$

$$+\ldots(\cot 2^{n-1}x-\cot 2^n x)$$

$$\Rightarrow S=\left(\cot\frac{x}{2}-\cot 2^n\cdot x\right)$$

Example 34: Prove that $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{128}$

Sol. $\cos\frac{5\pi}{15}=\cos\frac{\pi}{3}=\frac{1}{2}$, $\cos\frac{7\pi}{15}=-\cos\frac{8\pi}{15}$

$$\text{L.H.S.}=\left[\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\left(-\cos\frac{8\pi}{15}\right)\right]$$

$$\left(\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right)\cdot\frac{1}{2}=-\frac{1}{2}\cdot\frac{\sin 2^4\left(\frac{\pi}{15}\right)}{2^4\sin\frac{\pi}{15}}\cdot\frac{\sin\left(2^2\frac{3\pi}{15}\right)}{2^2\sin\frac{3\pi}{15}}$$

$$=\frac{1}{128},\text{ as }\sin(\pi+\theta)=-\sin\theta$$

Concept Application

23. $\sin\frac{\pi}{n}+\sin\frac{3\pi}{n}+\sin\frac{5\pi}{n}+\ldots$ up to n terms is equal to

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 0

24. The product

$\left(\cos\frac{x}{2}\right)\cdot\left(\cos\frac{x}{4}\right)\cdot\left(\cos\frac{x}{8}\right)\ldots\ldots\left(\cos\frac{x}{256}\right)$ is equal to:

(*a*) $\frac{\sin x}{128\sin\frac{x}{256}}$ (*b*) $\frac{\sin x}{256\sin\frac{x}{256}}$

(*c*) $\frac{\sin x}{128\sin\frac{x}{128}}$ (*d*) $\frac{\sin x}{512\sin\frac{x}{512}}$

25. Find the summation of the following series

$$\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$$

APPLICATION OF TRIGONOMETRY IN MAXIMISING AND MINIMISING I.E. (OPTIMISATION)

Type-I

Maximising and minimising by using the property of boundness of trigonometric functions.

(*a*) Sine and cosine have bounded values between –1 and 1.

(*b*) Tangent and cotangent are unbounded functions.

(*c*) cosec and sec have values greater than 1 and less than –1.

(*d*) $0\le\sin^2x\le 1$, $0\le\cos^2x\le 1$, $\tan^2x\ge 0$, $\sec^2x\ge 1$.

Note: If maximum value of a function is **'b'**, and minimum value **'*a*'** then range is $[a, b]$.

Special Cases

When angle of sine and cosine are same

$E = a\sin\theta + b\cos\theta$

$$\Rightarrow E=\sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\sin\theta+\frac{b}{\sqrt{a^2+b^2}}\cos\theta\right\}$$

Let $\frac{b}{\sqrt{a^2+b^2}}=\sin\alpha$ & $\frac{a}{\sqrt{a^2+b^2}}=\cos\alpha$

$\Rightarrow E=\sqrt{a^2+b^2}\sin(\theta+\alpha)$, where $\tan\alpha=\frac{b}{a}$

Hence for any real value of θ,

$$-\sqrt{a^2+b^2}\le E\le\sqrt{a^2+b^2}$$

Type-II

Argument of sine and cosine are different or a quadratic in sine/cos is given then we make a perfect square in sine/cosine and interpret.

Type-III

Making use of reciprocal relationship between tan and cot, sin/cosec and cos/sec.

Train Your Brain

Example 35: Find the max and min values of $f(\theta)=\sin^6\theta+\cos^6\theta$

Sol. We have, $f(\theta)=\sin^6\theta+\cos^6\theta=(\sin^2\theta)^3+(\cos^2\theta)^3$

$$=(\sin^2\theta+\cos^2\theta)^2-3\sin^2\theta\cos^2\theta(\sin^2\theta+\cos^2\theta)$$

$$=1-3\sin^2\theta\cos^2\theta=1-\frac{3}{4}(4\sin^2\theta\cos^2\theta)$$

$$=1-\frac{3}{4}(\sin^2 2\theta)=1+\frac{3}{4}(-\sin^2 2\theta)$$

As we know that, $-1\le(-\sin^2 2\theta)\le 0$

$$\Rightarrow -\frac{3}{4} \le \frac{3(-\sin^2 2\theta)}{4} \le 0$$

$$\Rightarrow -\frac{3}{4} \le \frac{3(-\sin^2 2\theta)}{4} \le 0 \Rightarrow \frac{1}{4} \le f(\theta) \le 1$$

Hence, the maximum value = 1

and the minimum value = $\frac{1}{4}$

Example 36: Find minimum and maximum value of $7\cos\theta + 24\sin\theta$.

Sol. $y = 7\cos\theta + 24\sin\theta$.

$$-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$$

$$\therefore -\sqrt{7^2+24^2} \le 7\cos\theta + 24\sin\theta \le \sqrt{7^2+24^2}$$

$$-25 \le 7\cos\theta + 24\sin\theta \le 25.$$

Example 37: $y = \cos 2x + 3\sin x$. Find range of y.

Sol. $\Rightarrow 1 - 2\sin^2 x + 3\sin x$

$$\Rightarrow 1-2\left[\sin^2 x - \frac{3}{2}\sin x + \frac{9}{16} - \frac{9}{16}\right]$$

$$\Rightarrow 1-2\left[\sin x - \frac{3}{4}\right]^2 + \frac{9}{8};$$

$$y = \frac{17}{8} - 2\left[\sin x - \frac{3}{4}\right]^2$$

$$y_{max} = \frac{17}{8} \quad \text{at } \sin x = \frac{3}{4}$$

$$y_{min} = -4 \quad \text{at } \sin x = -1$$

$$y \in \left[-4, \frac{17}{8}\right]; y_{max} = \frac{17}{8}; y_{min} = -4$$

Example 38: If $y = a^2\tan^2 x + b^2\cot^2 x$ $(a, b \ge 0)$. Find minimum value of y.

Sol. $y = a^2\tan^2 x + \frac{b^2}{\tan^2 x} - 2ab + 2ab$

$$\Rightarrow \left(a\tan x - \frac{b}{\tan x}\right)^2 + 2ab \ge 2ab$$

$$y_{min.} = 2ab$$

Concept Application

26. Find the range of the function $\frac{1}{3\sin x + 4\cos x + 2}$

27. Find the maximum and minimum value of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

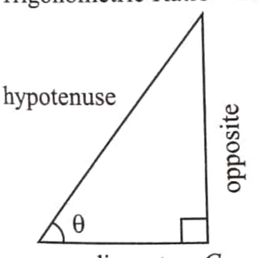
28. If $A = \sin^2\theta + \cos^4\theta$, then for all real values of θ

(a) $1 \le A \le 2$ (b) $\frac{3}{4} \le A \le 1$

(c) $\frac{13}{16} \le A \le 1$ (d) $\frac{3}{4} \le A \le \frac{13}{16}$

Short Notes

1. Trigonometric Ratio – Basic Terminology



The six trigonometric ratios of θ are defined as follows:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}},\quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}},\quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}},\quad \text{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite}},\quad \sec\theta = \frac{\text{hypotenus}}{\text{adjacent}}$$

2. Trigonometric identities:

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $\sec^2\theta - \tan^2\theta = 1$

(iii) $\text{cosec}^2\theta - \cot^2\theta = 1$

3. Allied angles:

Two angles are said to be allied when their sum or difference is either zero or a multiple of $\frac{\pi}{2}$, two angles x, y are allied angles iff $|x \pm y| = 0$ *or* $\frac{n\pi}{2}, n \in N$.

q→	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$	$2\pi+\theta$	$-\theta$
sin	cos θ	cos θ	sin θ	–sin θ	–cos θ	–cos θ	–sin θ	sin θ	–sin θ
cos	sin θ	–sin θ	–cos θ	–cos θ	–sin θ	sin θ	cos θ	cos θ	cos θ
tan	cot θ	–cot θ	– tan θ	tan θ	cot θ	–cot θ	–tan θ	tan θ	–tan θ
cot	tan θ	–tan θ	– cot θ	cot θ	tan θ	–tan θ	–cot θ	cot θ	–cot θ
sec	cosecθ	–cosecθ	–sec θ	–sec θ	–cosec θ	cosecθ	sec θ	sec θ	sec θ
cosec	sec θ	sec θ	cosec θ	–cosec θ	–sec θ	–sec θ	–cosec θ	cosec θ	–cosec θ

4. Sum & Difference Formula

(*i*) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(*ii*) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(*iii*) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(*iv*) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(*v*) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

(*vi*) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

(*vii*) $\cot(A+B) = \frac{\cot B\cot A - 1}{\cot B + \cot A}$

(*viii*) $\cot(A-B) = \frac{\cot B\cot A + 1}{\cot B - \cot A}$

5. Product to sum

(*i*) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(*ii*) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(*iii*) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(*iv*) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

6. Trigonometric transformations:

(*i*) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(*ii*) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(*iii*) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(*iv*) $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$= 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Some useful Formulae:

(*i*) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(*ii*) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(*iii*) $\tan(A+B)\tan(A-B) = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A\tan^2 B}$

(*iv*) $\cot(A+B)\cot(A-B) = \frac{\cot^2 B\cot^2 A - 1}{\cot^2 B - \cot^2 A}$

7. Some standard trigonometric values

Angle → / *Trigonometric Function ↓*	15°	18°	$22\frac{1}{2}°$	36°
sin	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}\sqrt{2-\sqrt{2}}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
cos	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{1}{2}\sqrt{2+\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$
tan	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$

8. Double Angle / Triple Angle

(i) $\sin 2A = 2\sin A\cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Suppose that A is not an odd multiple of $\frac{\pi}{2}$. Then

(iii) $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$

(iv) $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(v) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ (Here $2A$ is also not an odd multiple of $\frac{\pi}{2}$)

(vi) $\sin 3A = 3\sin A - 4\sin^3 A,\ \forall A \in R$

(vii) $\cos 3A = 4\cos^3 A - 3\cos A,\ \forall A \in R$

(viii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$ ($3A$, A are not odd multiples of $\frac{\pi}{2}$)

(ix) $\cot 3A = \dfrac{3\cot A - \cot^3 A}{1-3\cot^2 A}$ ($3A$, A are not multiples of $\frac{\pi}{2}$)

9. Half Angle

(i) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(ii) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$

(iii) $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$ $\left(\text{where } \frac{A}{2} \neq (2n+1)\frac{\pi}{2}, n \in Z\right)$

10. Conditional Identities

In a Triangle If $A + B + C = \pi$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(ii) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

(iv) $\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + 4\cos A\cos B\cos C)$

(v) $\cos^2 A + \cos^2 B + \cos^2 C = (1 - 2\cos A\cos B\cos C)$

11. Domains, Ranges and Periodicity of Trigonometric Functions:

T-Ratio	Domain	Range	Period
$\sin x$	R	$[-1, 1]$	2π
$\cos x$	R	$[-1, 1]$	2π
$\tan x$	$R - \left\{(2n+1)\frac{\pi}{2}\right\}; n \in I$	R	π
$\cot x$	$R - \{n\pi : n \in I\}$	R	π
$\sec x$	$R - \left\{(2n+1)\frac{\pi}{2}\right\}; n \in I$	$(-\infty, -1] \cup [1, \infty)$	2π
$\operatorname{cosec} x$	$R - \{n\pi : n \in I\}$	$(-\infty, -1] \cup [1, \infty)$	2π

12. Maxima-Minima

(i) $a\cos\theta + b\sin\theta$ will always lie in the interval $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$ i.e., the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(ii) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta$ is $2ab$ where $a, b > 0$.

(iii) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} < a\cos(\alpha+\theta) + b\cos(\beta+\theta) \leq \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β are known angles.

(iv) If $\alpha, \beta, \in \left(0, \frac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$ (constant) then

(a) Maximum value of the expression $\cos\alpha\cos\beta$, $\cos\alpha + \cos\beta$, $\sin\alpha\sin\beta$ or $\sin\alpha + \sin\beta$ occurs when $\alpha = \beta = \frac{\sigma}{2}$

(b) Minimum value of $\sec\alpha + \sec\beta$, $\tan\alpha + \tan\beta$, $\operatorname{cosec}\alpha + \operatorname{cosec}\beta$ occurs when $\alpha = \beta = \frac{\sigma}{2}$

(v) If A, B, C are the angles of a triangle then maximum value of $\sin A + \sin B + \sin C$ and $\sin A\sin B\sin C$ occurs when $A = B = C = 60°$

(vi) In case a quadratic in $\sin\theta$ & $\cos\theta$ is given then the maximum or minimum values can be obtained by making perfect square.

13. Sum of Three or More Angles

(i) $\sin(A + B + C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C = \Sigma\sin A\cos B\cos C - \prod\sin A$

(ii) $\cos(A + B + C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C - \sin A\sin B\sin C = \prod\cos A - \Sigma\cos A\sin B\sin C$

(iii) $\tan(A + B + C)$

$$= \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$$

$$= \frac{\Sigma\tan A - \Pi\tan A}{1 - \Sigma\tan A\tan B}$$

14. Summation of Series

$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha + \overline{n-1}\beta)$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha + \overline{n-1}\beta)$

$$= \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

Solved Examples

1. Prove that from the equality $\frac{\sin^4\alpha}{a}+\frac{\cos^4\alpha}{b}=\frac{1}{a+b}$ follows the relation $\frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{1}{(a+b)^3}$;

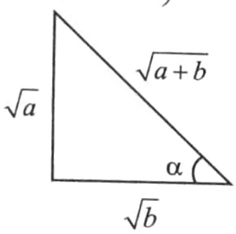
Sol. $$\frac{\sin^4\alpha}{a}+\frac{\cos^4\alpha}{b}=\frac{1}{a+b}$$

$$\frac{(a+b)}{a}\sin^4\alpha+\frac{(a+b)}{b}\cos^4\alpha=1$$

$$\Rightarrow \sin^4\alpha+\frac{b}{a}\sin^4\alpha+\frac{a}{b}\cos^4\alpha+\cos^4\alpha=1$$

$$\Rightarrow 1-2\sin^2\alpha\cos^2\alpha+\frac{b}{a}\sin^4\alpha+\frac{a}{b}\cos^4\alpha=1$$

$$\Rightarrow\left(\sqrt{\frac{b}{a}}\sin^2\alpha-\sqrt{\frac{a}{b}}\cos^2\alpha\right)^2=0$$



$$\Rightarrow \tan^2\alpha=\frac{a}{b}$$

$$\Rightarrow \text{Now, } \frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}$$

$$=\frac{a^4}{(a+b)^4}\frac{1}{a^3}+\frac{b^4}{(a+b)^4}\frac{1}{b^3}=\frac{1}{(a+b)^3}$$

2. If $\alpha+\beta=c$ where $\alpha,\beta>0$ each lying between o and $\pi/2$ and c is a constant, find the maximum or minimum value of

(*a*) $\sin\alpha+\sin\beta$

(*b*) $\sin\alpha\sin\beta$

(*c*) $\tan\alpha+\tan\beta$

Sol. (*a*) $\sin\alpha+\sin\beta=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$=2\sin\frac{c}{2}\cos\left(\frac{\alpha-\beta}{2}\right)$$

Its max. value is $2\sin\frac{c}{2}$

(*c*) $\tan\alpha+\tan\beta=\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$

$$=\frac{2\sin c}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}$$

$$=\frac{\sin c}{\cos^2\frac{c}{2}-\sin^2\left(\frac{\alpha-\beta}{2}\right)}$$

This is minimum when denominator is maximum i.e. when $\sin^2\left(\frac{\alpha-\beta}{2}\right)$ is zero.

$$\frac{2\sin\frac{c}{2}\cos\frac{c}{2}}{\cos^2\frac{c}{2}}=2\tan\frac{c}{2}$$

3. If the equation $\cot^4x-2\text{cosec}^2x+a^2=0$ has at least one solution then possible integral values of a can be:

(*a*) -1 (*b*) 0

(*c*) 1 (*d*) 2

Sol. (*a,b,c*)

$\Rightarrow \cot^4x-2(1+\cot^2x)+a^2=0$

$\Rightarrow \cot^4x-2\cot^2x+a^2-2$

$\Rightarrow (\cot^2x-1)^2=3-a^2$

to have atleast one solution

$\Rightarrow 3-a^2\geq 0$

$a^2-3\leq 0$

$a\in\left[-\sqrt{3},\sqrt{3}\right]$

4. Let α is a root of the equation $(2\sin x-\cos x)(1+\cos x)=\sin^2x$, β is a root of the equation $3\cos^2x-10\cos x+3=0$ and γ is a root of the equation $1-\sin 2x=\cos x-\sin x$. $0\leq\alpha,\beta,\gamma\leq\frac{\pi}{2}$

(1) $\cos\alpha+\cos\beta+\cos\gamma$ can be equal to

(*a*) $\frac{3\sqrt{6}+2\sqrt{2}+6}{6\sqrt{2}}$ (*b*) $\frac{3\sqrt{3}+8}{6}$

(*c*) $\frac{3\sqrt{3}+2}{6}$ (*d*) None of these

Sol. (*b*)

Now, $(2\sin x-\cos x)(1+\cos x)=\sin^2x$

$\Rightarrow (1+\cos x)[2\sin x-\cos x-1+\cos x]=0$

$\Rightarrow (1+\cos x)(2\sin x-1)=0$

$\Rightarrow \cos x = -1$ or $\sin x = \frac{1}{2}$

So, $\sin\alpha = \frac{1}{2}$ $\left[\text{as } 0 \le \alpha \le \frac{\pi}{2}\right]$

$\therefore \cos\alpha = \frac{\sqrt{3}}{2}$

$3\cos^2 x - 10\cos x + 3 = 0$

$\cos x = \frac{1}{3}, \cos x \ne 3$

$\cos\beta = \frac{1}{3}, \sin\beta = \frac{2\sqrt{2}}{3}$

and $1 - \sin 2x = \cos x - \sin x$

$\Rightarrow \sin^2 x + \cos^2 x - 2\sin x\cos x = \cos x - \sin x$

$\Rightarrow (\cos x - \sin x)(\cos x - \sin x - 1) = 0$

$\Rightarrow \sin x = \cos x = \frac{1}{\sqrt{2}}$

or $\cos x - \sin x = 1$

$\Rightarrow \cos x = 1, \sin x = 0$

$\Rightarrow \cos\gamma = 1, \sin\gamma = 0$

$\cos\alpha + \cos\beta + \cos\gamma = \frac{\sqrt{3}}{2} + \frac{1}{3} + 1 = \frac{3\sqrt{3}+8}{6}$

5. $\sin(\alpha - \beta)$ is equal to

(*a*) 1 (*b*) 0

(*c*) $\frac{1-2\sqrt{6}}{6}$ (*d*) $\frac{\sqrt{3}-2\sqrt{2}}{6}$

Sol. (*c*) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \frac{1}{2}\times\frac{1}{3} - \frac{\sqrt{3}}{2}\times\frac{2\sqrt{2}}{3} = \frac{1-2\sqrt{6}}{6}$

6. If L and I are maximum and minimum value of the function $f(x) = (\tan x + 2)^2 + 5)\cos^2$ then, $[L] + [I]$ equal [where, [.] is G.I.F]

(*a*) 9 (*b*) 8

(*c*) 7 (*d*) 10

Sol. (*a*) $f(x) = \frac{\tan^2 x + 2\tan x + 9}{1+\tan^2 x}$

$f(x) = \frac{\tan^2 x}{1+\tan^2 x} + \frac{4\tan x}{1+\tan^2 x} + \frac{9}{1+\tan^2 x}$

$= \sin^2 x + 2\sin 2x + 9\cos^2 x$

$= 1 + 4(1 + \cos 2x) + 2\sin 2x$

$= 5 + 2\sin 2x + 4\cos 2x$

7. Which of the following is/are correct?

(*a*) $(\tan x)^{\ln(\sin x)} > (\cot x)^{\ln(\sin x)}, \forall x \in (0, \pi/4)$

(*b*) $(1/2)^{\ln(\cos x)} < (1/3)^{\ln(\cos x)}, \forall x \in (0, \pi/2)$

(*c*) $4^{\ln \operatorname{cosec} x} < 5^{\ln \operatorname{cosec} x}, \forall x \in (0, \pi/2)$

(*d*) $2^{\ln(\tan x)} > 2^{\ln(\sin x)}, \forall x \in (0, \pi/2)$

Sol. (*a*) For $x \in \left(0, \frac{\pi}{4}\right), \tan x < \cot x$

Also $\ln(\sin x) < 0$

$\Rightarrow (\tan x)^{\ln(\sin x)} > (\cot x)^{\ln(\sin x)}$

(*b*) $x \in \left(0, \frac{\pi}{2}\right), \operatorname{cosec} x \ge 1 \Rightarrow \ln(\operatorname{cosec} x) \ge 0$

$\Rightarrow 4^{\ln \operatorname{cosec} x} < 5^{\ln(\operatorname{cosec} x)}$

(*c*) $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \cos x \in (0,1)$

$\Rightarrow \ln(\cos x) < 0$

Also $\frac{1}{2} > \frac{1}{3}$

$\Rightarrow \left(\frac{1}{2}\right)^{\ln(\cos x)} < \left(\frac{1}{3}\right)^{\ln(\cos x)}$

(*d*) $x \in \left(0, \frac{\pi}{2}\right)$

Since $\sin x < \tan x$, we get $\ln(\sin x) < \ln(\tan x)$ |

$\Rightarrow 2^{\ln(\sin x)} < 2^{\ln(\tan x)}$

8. The value of $f(x) = x^4 + 4x^3 + 2x^2 - 4x + 7$, when $x = \cot\frac{11\pi}{8}$ is

Sol. [6] $: x = \cot\frac{11\pi}{8} = \cot\left(\pi + \frac{3\pi}{8}\right) = \cot\frac{3\pi}{8} = \sqrt{2} - 1$

$(x+1)^2 = 2$

$x^2 + 2x - 1 = 0$

$f(x) = x^4 + 4x^3 + 2x^2 - 4x + 7$

$= x^2(x^2 + 2x - 1) + 2x^3 + 3x^2 - 4x + 7$

$= 2x(x^2 + 2x - 1) - x^2 - 2x + 7 = -x^2 - 2x + 7$

$= -(x^2 + 2x - 1) + 6 = 0 + 6 = 6$

9. Let $f_n(a) = \frac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots + \sin(2n-1)\alpha}{\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots + \cos(2n-1)\alpha}$

Then, the value of $f_4\left(\frac{\pi}{32}\right)$ is equal to

(*a*) $\sqrt{2}+1$ (*b*) $\sqrt{2}-1$

(*c*) $2+\sqrt{3}$ (*d*) $2-\sqrt{3}$

Sol. (*b*) $f_n(\alpha) = \tan n\alpha$ *and* $f_n\left(\frac{\pi}{32}\right) = \tan\frac{\pi}{8} = \sqrt{2} - 1$

10. Let $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where [.] is the greatest integer function, then

(*a*) $f(\pi/2) = -1$ (*b*) $f(\pi) = 1$

(*c*) $f(-\pi) = 0$ (*d*) $f(\pi/4) = 2$

Sol. (*a,c*)

$f(x) = \cos 9x + \cos (-10)x = \cos 9x + \cos 10x$

$f\left(\frac{\pi}{2}\right) = \cos \frac{9\pi}{2} + \cos 5\pi = -1$

$f(\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$

$f(-\pi) = 0$

$f\left(\frac{\pi}{4}\right) = \cos\frac{9\pi}{4} + \cos\frac{5\pi}{2} = \cos\frac{9\pi}{4}$

$= \cos\left(2\pi + \frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

11. $(a + 2) \sin \alpha + (2a - 1) \cos \alpha = (2a + 1)$ if $\tan \alpha =$

(*a*) $\frac{3}{4}$ (*b*) $\frac{4}{3}$

(*c*) $\frac{2a}{a^2+1}$ (*d*) $\frac{2a}{a^2-1}$

Sol. (*b,d*)

Let $\tan\frac{\alpha}{2} = t$

$(a + 2)\, 2t + (2a - 1)(1 - t^2) = (2a + 1)(t^2 + 1)$

$\Rightarrow 2a\,t + 4t + 2a - 2a\,t^2 - 1 + t^2 = 2a + 1 + 2a\,t^2 + t^2$

$\Rightarrow 4a\,t^2 - 2t\,(2 + a) + 2 = 0 \Rightarrow 2a\,t^2 - 2t - a\,t + 1 = 0$

$\Rightarrow 2t(at - 1) - 1\,(at - 1) = 0 \Rightarrow t = 1/2,\ t = 1/a$

$\Rightarrow \tan\alpha = \frac{2\tan\alpha/2}{1-\tan^2\alpha/2}$

$\Rightarrow \tan\alpha = \frac{2\times 1/2}{1-1/4} = 4/3$

or $\tan\alpha = \frac{2/a}{1-1/a^2} = \frac{2a}{a^2-1}$

12. If $3 \sin \beta = \sin (2\alpha+\beta)$, then $\tan (\alpha+\beta) - 2 \tan \alpha$ is

(*a*) Independent of α

(*b*) Independent of β

(*c*) Dependent of both α and β

(*d*) Independent of α but dependent of β

Sol. (*a,b*)

$3 \sin \beta = \sin (2\alpha + \beta)$

$\Rightarrow \frac{\sin(2\alpha+\beta)}{\sin\beta} = \frac{3}{1}$

Applying componendo & dividendo

$\Rightarrow \frac{\sin(2\alpha+\beta)+\sin\beta}{\sin(2\alpha+\beta)-\sin\beta} = \frac{3+1}{3-1} = \frac{2}{1}$

$\Rightarrow \frac{2\sin(\alpha+\beta).\cos\alpha}{2\cos(\alpha+\beta)\sin\alpha} = 2$

$\Rightarrow \tan(\alpha + \beta) = 2 \tan \alpha$

$\Rightarrow \tan (\alpha + \beta) - 2\tan \alpha = 0$

Hence, independent of α & β both.

13.

	Column – I	Column – II	
A.	If for some real x, the equation $x + \frac{1}{x} = 2\cos\theta$ holds, then $\cos\theta$ is equal to	p.	2
B.	If $\sin\theta + \operatorname{cosec}\theta = 2$, then $\sin^{2008}\theta + \operatorname{cosec}^{2008}\theta$ is equal to	q.	1
C.	Maximum value of $\sin^4\theta + \cos^4\theta$ is	r.	0
D.	Least value of $2\sin^2\theta + 3\cos^2\theta$ is	s.	–1

Sol. $A \to q, s, B \to p, C \to q, D \to p$

(*a*) $x + \frac{1}{x} = 2\cos\theta \geq 2$ or ≤ -2

$\Rightarrow \cos\theta = 1$ or -1

(*b*) $\sin\theta + \operatorname{cosec}\theta = 2$

$\because \sin\theta + \frac{1}{\sin\theta} \geq 2$ or ≤ -2

but given that $\sin\theta + \operatorname{cosec}\theta = 2 \Rightarrow \sin\theta + \frac{1}{\sin\theta} = 2$ which is possible only when $\sin\theta = 1$

$\therefore \sin^{2008}\theta + \operatorname{cosec}^{2008}\theta = \sin^{2008} + \frac{1}{\sin^{2008}}$

$= 1 + 1 = 2$

(*c*) $\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta$

$= 1 - \frac{1}{2}\sin^2 2\theta$

$\because 0 \leq \sin^2 2\theta \leq 1$

$\therefore \frac{1}{2} \leq 1 - \frac{1}{2}\sin^2 2\theta \leq 1$

$\therefore$ maximum value $= 1$

(*d*) $2\sin^2\theta + 3\cos^2\theta = 2\sin^2\theta + 3 - 3\sin^2\theta = 3 - \sin^2\theta$

$\because 0 \leq \sin^2\theta \leq 1$

$\therefore 2 \leq 3 - \sin^2\theta \leq 3$

$\therefore$ least value $= 2$

14. If $\cos(\theta - \phi), \cos\theta, \cos(\theta + \phi)$, are in H. P, then the value of $\sqrt{2}\cos\theta.\sec\frac{\phi}{2}$ is $\left(0 < \theta, \phi < \frac{\pi}{2}\right)$

Sol. [2] $\cos\theta = \frac{2\cos(\theta-\phi).\cos(\theta+\phi)}{\cos(\theta-\phi)+\cos(\theta+\phi)} = \frac{2(\cos^2\theta - \sin^2\phi)}{2\cos\theta.\cos\phi}$

$\Rightarrow \cos^2\theta\,.\cos\phi = \cos^2\theta - \sin^2\phi$

$\Rightarrow \sin^2\phi\,.\cos^2\theta\,(1 - \cos\phi)$

$\Rightarrow 4\sin^2\frac{\phi}{2}.\cos^2\frac{\phi}{2} = 2\cos^2\theta\,.\sin^2\frac{\phi}{2}$

$\Rightarrow \cos\theta\sec\frac{\phi}{2} = \sqrt{2}$

15. The maximum value of $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3$ is

Sol. [10] $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right)+3$

$= 5\cos\theta + 3\left(\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right)+3$

$= 5\cos\theta + \frac{3}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$

$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2} = \sin\theta + 3$

$\therefore$ maximum value $= 3 + \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2}$

$= 3 + \sqrt{\frac{196}{4}} = 3 + 7 = 10$

16. If $X = \sin\left(\theta+\frac{7\pi}{12}\right)+\sin\left(\theta-\frac{\pi}{12}\right)+\sin\left(\theta+\frac{3\pi}{12}\right)$,

$Y = \cos\left(\theta+\frac{7\pi}{12}\right)+\cos\left(\theta-\frac{\pi}{12}\right)+\cos\left(\theta+\frac{3\pi}{12}\right)$, then

prove that $\frac{X}{Y}-\frac{Y}{X} = 2\tan 2\theta$.

Sol. $X = \sin\left(\theta+\frac{7\pi}{12}\right)+\sin\left(\theta-\frac{\pi}{12}\right)+\sin\left(\theta+\frac{3\pi}{12}\right)$

$= 2\sin\left(\frac{\theta+\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)+\sin\left(\theta+\frac{\pi}{4}\right)$

$= 2\sin\left(\theta+\frac{\pi}{4}\right)$

$Y = \cos\left(\theta+\frac{7\pi}{12}\right)+\cos\left(\theta-\frac{\pi}{12}\right)+\cos\left(\theta+\frac{3\pi}{12}\right)$

$= 2\cos\left(\theta+\frac{\pi}{4}\right)\cdot\frac{1}{2}+\cos\left(\theta+\frac{\pi}{4}\right)$

$= 2\cos\left(\theta+\frac{\pi}{4}\right)$

Now $\frac{X}{Y}-\frac{Y}{X} = \frac{2\sin(\theta+\pi/4)}{2\cos(\theta+\pi/4)} - \frac{2\cos(\theta+\pi/4)}{2\sin(\theta+\pi/4)}$

$= \left[\frac{-2\cos(2\theta+\pi/2)}{\sin(2\theta+\pi/2)}\right] = 2\tan 2\theta$

17. Evaluate: $\cos^6\frac{\pi}{16}+\cos^6\frac{3\pi}{16}+\cos^6\frac{5\pi}{16}+\cos^6\frac{7\pi}{16}$

Sol. $\left(\cos^6\frac{\pi}{16}+\sin^6\frac{\pi}{16}\right)+\left(\cos^6\frac{3\pi}{16}+\sin^6\frac{3\pi}{16}\right)$

$= 2 - 3\cos^2\frac{\pi}{16}\sin^2\frac{\pi}{16} - 3\cos^2\frac{3\pi}{16}\sin^2\frac{3\pi}{16}$

$= 2 - \frac{3}{4}\left[\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}\right]$

$= 2 - \frac{3}{4}\left[\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right] = \frac{5}{4}$

18. If $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta+\frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta-\frac{2\pi}{3}\right)}$, then $x+y+z$ is equal to:

(*a*) -1 (*b*) 1

(*c*) 0 (*d*) 2

Sol. (*c*) Given $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta+\frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta-\frac{2\pi}{3}\right)} = \lambda$ (say)

$\Rightarrow x+y+z = \lambda\left\{\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta-\frac{2\pi}{3}\right)\right\}$

$= \lambda\left\{\cos\theta + 2\cos\theta\cos\frac{2\pi}{3}\right\} = 0$

19. If $\alpha+\beta+\gamma+\delta = 2\pi$. Prove that $\cos\alpha+\cos\beta+\cos\gamma+\cos\delta + 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\alpha+\delta}{2}\right) = 0$.

Sol. Given $\alpha+\beta+\gamma+\delta = 2\pi$

Tanking L.H.S. we have

$\cos\alpha+\cos\beta+\cos\gamma+\cos\delta +$

$4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\alpha+\delta}{2}\right)$

Using Componendo & Dividendo formula

$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\cos\frac{\gamma+\delta}{2}\cos\frac{\gamma-\delta}{2}+$

$4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\alpha+\delta}{2}\right)$

$\gamma+\delta = 2\pi-(\alpha+\beta)$

so $\cos\frac{\gamma+\delta}{2} = \cos\left(\pi-\frac{\alpha+\beta}{2}\right) = -\cos\left(\frac{\alpha+\beta}{2}\right)$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left[\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\gamma-\delta}{2}\right)\right]$

$+ 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\alpha+\delta}{2}\right) = 2\cos\left(\frac{\alpha+\beta}{2}\right)$

$\left[2\cos\left(\frac{\alpha+\gamma-(\beta+\delta)}{4}\right)\sin\frac{(\gamma+\beta)-(\alpha+\delta)}{4}\right]$

$+ 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\alpha+\delta}{2}\right)$

$\because \beta+\delta+2\pi-(\alpha+\gamma)$ and $\gamma+\beta = 2\pi-(\alpha+\delta)$

so, $\sin\frac{\alpha+\gamma-(\beta+\delta)}{4}$

$$= \sin\frac{2(\alpha+\gamma)-2\pi}{4} = -\sin\left(\frac{\pi}{2}-\frac{\alpha+\gamma}{2}\right)$$

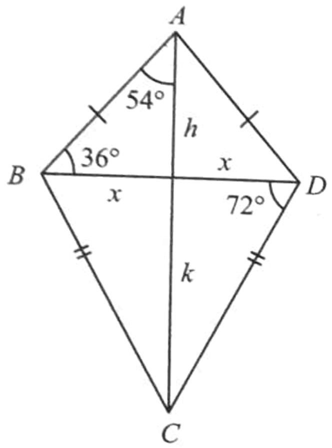
$$= -\cos\frac{\alpha+\gamma}{2} \text{ and } \sin\frac{\gamma+\beta-(\alpha+\delta)}{4}$$

$$= \sin\left(\frac{2\pi-2(\alpha+\delta)}{4}\right) = \cos\left(\frac{\alpha+\delta}{2}\right)$$

Then $\Rightarrow 2\cos\left(\dfrac{\alpha+\beta}{2}\right)$

20. In a kite $ABCD$, $AB = AD$ and $CB = CD$. If $\angle A = 108°$ and $\angle C = 36°$ then the ratio of the area of ΔABD to the area of ΔCBD can be written in the form $\dfrac{a - b\tan^2 36°}{c}$ where a, b and c are relatively prime positive integers. Determine the ordered triple (a, b, c).

Sol. Since the triangles ABD and CBD have a common base, hence the ratio of their areas equals the ratio of their heights.



Since $\tan 36° = \dfrac{h}{x}$, then $h = x\tan 36°$

$\Rightarrow \tan 72° = \dfrac{k}{x}$, then $k = x\tan 72°$

Hence, $\dfrac{h}{k} = \dfrac{x\tan 36°}{x\tan 72°} = \dfrac{\tan 36°}{\dfrac{2\tan 36°}{1-\tan^2 36°}} = \dfrac{1-\tan^2 36°}{2}$

Then ordered triple (a, b, c) is $(1, 1, 2)$.

21. Given the product p of sines of the angles of a triangle & product q of their cosines, find the cubic equation, whose coefficients are functions of p & q & whose roots are the tangents of the angles of the triangle.

Sol. Give $\sin A \sin B \sin C = p$; $\cos A \cos B \cos C = q$

Hence $\tan A \tan B \tan C = \tan A + \tan B + \tan C = p/q$

Hence equation of cubic is

$$x^3 - \frac{p}{q}x^2 + \Sigma \tan A \tan B x - \frac{p}{q} = 0 \qquad ...(i)$$

now

$$\sum \tan A \tan B = \frac{\sin A \sin B \cos C + \sin B \sin C \cos A + \sin C \sin A \cos B}{\cos A \cos B \cos C}$$

We know that $A + B + C = \pi$

$\cos(A + B + C) = -1$; $\cos(A + B)\cos C - \sin(A+B)\sin C = -1$

$(\cos A \cos B - \sin A \sin B)\cos C - \sin C(\sin A \cos B + \cos A \sin B) = -1$

$1 + \cos A \cos B \cos C = \sin A \sin B \cos C + \sin B \sin C \cos A + \sin C \sin A \cos B$ dividing by $\cos A \cos B \cos C$

$$\frac{1+q}{q} = \sum \tan A \tan B$$

Hence (i) becomes $qx^3 - px^2 + (1 + q)x - p = 0$.

22. If $x^2 + y^2 = 4$ and $a^2 + b^2 = 8$. Find minimum and maximum value of $(ax + by)$

Sol. Let $x = r_1 \cos\theta$, $y = r_1 \sin\theta$ and $a = r_2\cos\phi$; $b = r_2 \sin\phi$

$\therefore r_1 = 2, r_2 = 2\sqrt{2}$

Then $(ax + by) = r_1r_2\cos(\theta - \phi)$, $-r_1r_2 \le (ax + by) \le r_1r_2$

$-4\sqrt{2} \le (ax + by) \le 4\sqrt{2}$

$y_{max} = 4\sqrt{2}$ and $y_{min} = -4\sqrt{2}$

23. Eliminate θ from the equations $a\cos\theta + b\sin\theta = c$ and $b\cos\theta - a\sin\theta = d$.

Sol. $a\cos\theta + b\sin\theta = c \qquad ...(i)$

$b\cos\theta - a\sin\theta = d. \qquad ...(ii)$

Square (*i*) and (*ii*) and adding,

$a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = c^2 + d^2$

or $a^2 + b^2 = c^2 + d^2$.

24. Eliminate θ from the equations

$a\sin\alpha - b\cos\alpha = 2b\sin\theta$, and $a\sin 2\alpha - b\cos 2\theta = a$,

Sol. $a\sin\alpha - b\cos\alpha = 2b\sin\theta \qquad ...(i)$

$a\sin 2\alpha - b\cos 2\theta = a \qquad ...(ii)$

From (*i*) $\sin\theta = \dfrac{a\sin\alpha - b\cos\alpha}{2b}$

From (*ii*) $a\sin 2\alpha - b(1 - 2\sin^2\theta) = a$

or $a\sin 2\alpha - b\left\{1 - 2\left(\dfrac{a\sin\alpha - b\cos\alpha}{2b}\right)^2\right\} = a$

or $a^2\sin^2\alpha + b^2\cos^2\alpha + 4ab\sin\alpha\cos\alpha - 2ab\sin\alpha\cos\alpha = 2ab + 2b^2$

or $a\sin\alpha + b\cos\alpha = \sqrt{\{2b(a+b)\}}$

Exercise-1 (Topicwise)

ANGLE AND ITS MEASUREMENT

1. An equilateral triangle has side length 8. The area of the region containing all points outside the triangle but not more than 3 units from a point on the triangle is
(a) $9(8+\pi)$
(b) $8(9+\pi)$
(c) $9\left(8+\frac{\pi}{2}\right)$
(d) $8\left(9+\frac{\pi}{2}\right)$

2. The angles of a triangle are in A.P. and the number of degrees in the least to the number of radians in greatest is 60 to π. The angles in degree are
(a) 60°, 60°, 60°
(b) 30°, 60°, 90°
(c) 45°, 60°, 75°
(d) 15°, 60°, 105°

3. The perimeter of a certain sector of a circle is equal to half that of the circle of which it is a sector. The circular measure of one angle of the sector is
(a) $(\pi-2)$ radian
(b) $(\pi+2)$ radian
(c) π radian
(d) $(\pi-3)$ radian

BASIC TRIGONOMETRIC IDENTITIES

4. Number of values of θ for which $\cos\theta = 0.707$ $(\theta \in (0, 2\pi))$
(a) 1
(b) 3
(c) 4
(d) 2

5. If $\dfrac{2\sin\alpha}{1+\sin\alpha+\cos\alpha} = \lambda$ then $\dfrac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha}$ is equal to
(a) $\dfrac{1}{\lambda}$
(b) λ
(c) $1-\lambda$
(d) $1+\lambda$

6. If $\sin A \tan A = \cos^2 A$ then $\cos^3 A + \cos^2 A$ is equal to
(a) 1
(b) 2
(c) 4
(d) None of these

7. If $\operatorname{cosec} A + \cot A = \dfrac{11}{2}$, then $\tan A$ is
(a) $\dfrac{21}{22}$
(b) $\dfrac{15}{16}$
(c) $\dfrac{44}{117}$
(d) $\dfrac{117}{43}$

8. Which of the following statement is incorrect?
(a) Tangent of odd integral multiple of $\dfrac{\pi}{2}$ is ND
(b) Cotangent of integral multiple of π is not defined.
(c) Tangent of odd integral multiple of π is 0
(d) Cotangent of integral multiple of $\dfrac{\pi}{2}$ is not defined.

9. $\dfrac{\sin^3\theta-\cos^3\theta}{\sin\theta-\cos\theta} - \dfrac{\cos\theta}{\sqrt{1+\cot^2\theta}} - 2\tan\theta\cot\theta = -1$ if
(a) $\theta \in \left(0, \dfrac{\pi}{2}\right)$
(b) $\theta \in \left(\dfrac{\pi}{2}, \pi\right)$
(c) $\theta \in \left(\pi, \dfrac{3\pi}{2}\right)$
(d) $\theta \in \left(\dfrac{3\pi}{2}, 2\pi\right)$

10. If $\dfrac{\tan^3 A}{1+\tan^2 A} + \dfrac{\cot^3 A}{1+\cot^2 A} = p \sec A \operatorname{cosec} A + q \sin A \cos A$, then
(a) $p=2, q=1$
(b) $p=1, q=2$
(c) $p=1, q=-2$
(d) $p=2, q=-1$

SIGN OF TRIGONOMETRIC RATIO AND ALLIED ANGLE AND REDUCTON FORMULAE

11. The ex pression $3\left[\sin^4\left(\dfrac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right] - 2\left[\sin^6\left(\dfrac{\pi}{2}+\alpha\right)+\sin^6(5\pi+\alpha)\right]$ is equal to
(a) 0
(b) 1
(c) 3
(d) $\sin 4\alpha + \sin 6\alpha$

12. $\cos(540° - \theta) - \sin(630° - \theta)$ is equal to
(a) 0
(b) $2\cos\theta$
(c) $2\sin\theta$
(d) $\sin\theta - \cos\theta$

13. The value of $\sin(\pi+\theta)\sin(\pi-\theta)\operatorname{cosec}2\theta$ is equal to
(a) -1
(b) 0
(c) $\sin\theta$
(d) None of these

14. $\cos 0 + \cos\dfrac{\pi}{7} + \cos\dfrac{2\pi}{7} + \cos\dfrac{3\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{5\pi}{7} + \cos\dfrac{6\pi}{7} =$
(a) 1/2
(b) $-1/2$
(c) 0
(d) 1

15. The value of $\cos 10° - \sin 10°$ is
(a) Positive
(b) Negative
(c) 0
(d) 1

16. Angle in 3rd quadrant whose sine and cosine are equal
(a) $3\pi/2$
(b) $5\pi/4$
(c) 2π
(d) $\pi/2$

17. If $\sin\theta = \dfrac{24}{25}$ and θ lies in the second quadrant, then $\sec\theta + \tan\theta =$
(a) -3
(b) -5
(c) -7
(d) -9

18. If A lies in the second quadrant and $3\tan A + 4 = 0$, the value of $2\cot A - 5\cos A + \sin A$ is equal to

(a) $\frac{-53}{10}$ (b) $\frac{-7}{10}$

(c) $\frac{7}{10}$ (d) $\frac{23}{10}$

19. The value of $e^{\log_{10}\tan 1^\circ + \log_{10}\tan 2^\circ + \log_{10}\tan 3^\circ + \ldots\ldots + \log_{10}\tan 89^\circ}$ is

(a) 0 (b) e

(c) $1/e$ (d) 1

20. $\cos A + \sin(270^\circ + A) - \sin(270^\circ - A) + \cos(180^\circ + A) =$

(a) -1 (b) 0

(c) 1 (d) 2

21. $\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{5\pi}{8} + \sin^2\frac{7\pi}{8} =$

(a) 1 (b) -1

(c) 0 (d) 2

22. If ABCD is a cyclic quadrilateral, then the value of $\cos A - \cos B + \cos C - \cos D =$

(a) 0 (b) 1

(c) $2(\cos B - \cos D)$ (d) $2(\cos A - \cos C)$

23. The value of $\sin 1 \cdot \cos 2 \cdot \tan 3 \cdot \cot 4 \cdot \sec 5 . \operatorname{cosec} 6$ is

(a) Positive

(b) Negative

(c) Zero

(d) May be positive and Negative

24. The value of expression $\sum_{\theta=0}^{8} \frac{1}{1+\tan^3(10\theta)^\circ}$ equal is to

(a) 5 (b) $\frac{21}{4}$

(c) $\frac{14}{3}$ (d) $\frac{9}{2}$

25. If $a = \cos(2012\pi)$, $b = \sec(2013\pi)$ and $c = \tan(2014\pi)$, then

(a) $a < b < c$ (b) $b < c < a$

(c) $c < b < a$ (d) $a = b < c$

TRIGONOMETRIC IDENTITIES (SUM TO PRODUCT AND PRODUCT TO SUM)

26. The value of is $\frac{\sin 24^\circ \cos 6^\circ - \sin 6^\circ \sin 66^\circ}{\sin 21^\circ \cos 39^\circ - \cos 51^\circ \sin 69^\circ}$ is

(a) -1 (b) 1

(c) 2 (d) None of these

27. If $3\sin\alpha = 5\sin\beta$, then $\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

28. If $\tan A$ and $\tan B$ are the roots of the quadratic equation $x^2 - ax + b = 0$, then the value of $\sin^2(A+B)$.

(a) $\frac{a^2}{a^2+(1-b)^2}$ (b) $\frac{a^2}{a^2+b^2}$

(c) $\frac{a^2}{(b+c)^2}$ (d) $\frac{a^2}{b^2(1-a)^2}$

29. In a triangle ABC if $\tan A < 0$ then:

(a) $\tan B . \tan C > 1$ (b) $\tan B . \tan C < 1$

(c) $\tan B . \tan C = 1$ (d) None of these

30. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then $\cot(A-B)$ is equal to

(a) $\frac{1}{y} - \frac{1}{x}$ (b) $\frac{1}{x} - \frac{1}{y}$

(c) $\frac{1}{x} + \frac{1}{y}$ (d) None of these

31. If $\tan 25^\circ = x$, then $\frac{\tan 155^\circ - \tan 115^\circ}{1+\tan 155^\circ \tan 115^\circ}$ is equal to

(a) $\frac{1-x^2}{2x}$ (b) $\frac{1+x^2}{2x}$

(c) $\frac{1+x^2}{1-x^2}$ (d) $\frac{1-x^2}{1+x^2}$

32. If $\sin\alpha$ is $\sin\beta - \cos\alpha\cos\beta + 1 = 0$, then the value of $1 + \cot\alpha\tan\beta$ is

(a) 1 (b) -1

(c) 2 (d) None of these

33. If $\tan^2\theta = 2\tan^2\phi + 1$, then the value of $\cos 2\theta + \sin^2\phi$ is

(a) 1 (b) 2

(c) -1 (d) Independent of ϕ

34. The value of the expression $\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)$ is

(a) $\frac{1}{8}$ (b) $\frac{1}{16}$

(c) $\frac{1}{4}$ (d) 0

35. The expression $\frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x}$ is equal to

(a) $\cos 2x$ (b) $2\cos x$

(c) $\cos^2 x$ (d) $1 + \cos x$

36. If $\cos(A-B) = 3/5$ and $\tan A \tan B = 2$,

(a) $\cos A \cos B = -\frac{1}{5}$ (b) $\sin A \sin B = -\frac{2}{5}$

(c) $\cos(A+B) = -\frac{1}{5}$ (d) $\sin A \cos B = \frac{4}{5}$

37. $\dfrac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} =$

(a) $\tan(A - B)$ (b) $\tan(A + B)$
(c) $\cot(A - B)$ (d) $\cot(A + B)$

38. $\tan 3A - \tan 2A - \tan A =$

(a) $\tan 3A \tan 2A \tan A$
(b) $-\tan 3A \tan 2A \tan A$
(c) $\tan A \tan 2A - \tan 2A \tan 3A - \tan 3A \tan A$
(d) 0

39. If $\tan\theta = \dfrac{-4}{3}$, then $\sin\theta$ is

(a) $\dfrac{-4}{5}$ but not $\dfrac{4}{5}$ (b) $\dfrac{-4}{5}$ or $\dfrac{4}{5}$
(c) $\dfrac{4}{5}$ but not $\dfrac{5}{4}$ (d) $\dfrac{4}{5}$ and $\dfrac{5}{4}$ both

40. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 =$

(a) 3 (b) 2
(c) 1 (d) 0

41. The value of $\sin\dfrac{\pi}{10}\sin\dfrac{13\pi}{10}$

(a) $\dfrac{1}{2}$ (b) $\dfrac{-1}{2}$
(c) $\dfrac{-1}{4}$ (d) 1

42. If $\tan\dfrac{\alpha}{2}$ and $\tan\dfrac{\beta}{2}$ are the roots of the equation $8x^2 - 26x + 15 = 0$ then $\cos(\alpha + \beta)$ is equal to

(a) $-\dfrac{627}{725}$ (b) $\dfrac{627}{725}$
(c) $-\dfrac{725}{627}$ (d) -1

43. The value of the expression

$\dfrac{2(\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots + \sin 89^\circ)}{2(\cos 1^\circ + \cos 2^\circ + \cdots + \cos 44^\circ) + 1}$ equals

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$
(c) 1/2 (d) 1

44. Given that $(1+\sqrt{1+x})\tan y = 1+\sqrt{1-x}$. Then $\sin 4y$ is equal to

(a) $4x$ (b) $2x$
(c) x (d) None of these

TRIGONOMETRIC RATIO OF MULTIPLE ANGLES

45. If $\cos A = 3/4$, then the value of $16\cos^2(A/2) - 32\sin(A/2)\sin(5A/2)$ is

(a) -4 (b) -3 (c) 3 (d) 4

46. The numerical value of $\sin 12^\circ . \sin 48^\circ . \sin 54^\circ$ is equal to

(a) 1/2 (b) 1/4
(c) 1/16 (d) 1/8

47. If $A = \tan 6^\circ \tan 42^\circ$ and $B = \cot 66^\circ \cot 78^\circ$, then

(a) $A = 2B$ (b) $A = \dfrac{1}{3}B$
(c) $A = B$ (d) $3A = 2B$

48. The value of $\cos\dfrac{\pi}{10}\cos\dfrac{2\pi}{10}\cos\dfrac{4\pi}{10}\cos\dfrac{8\pi}{10}\cos\dfrac{16\pi}{10}$ is:

(a) $\dfrac{\sqrt{10+2\sqrt{5}}}{64}$ (b) $\dfrac{\cos(\pi/10)}{8}$
(c) $\dfrac{\cos(\pi/10)}{16}$ (d) $-\dfrac{\sqrt{10+2\sqrt{5}}}{64}$

49. Let α, β be such that $\pi < \alpha - \beta < 3\pi$. If $\sin\alpha + \sin\beta = -\dfrac{21}{65}$ and $\cos\alpha + \cos\beta = -\dfrac{27}{65}$, then the value of $\cos\dfrac{\alpha-\beta}{2}$ is

(a) $-\dfrac{3}{\sqrt{130}}$ (b) $\dfrac{3}{\sqrt{130}}$
(c) $\dfrac{6}{65}$ (d) $-\dfrac{6}{65}$

50. If $\tan x + \tan\left(\dfrac{\pi}{3}+x\right) + \tan\left(\dfrac{2\pi}{3}+x\right) = 3$, then

(a) $\tan x = 1$ (b) $\tan 2x = 1$
(c) $\tan 3x = 1$ (d) $\tan x = 2$

51. $\sqrt{2+\sqrt{2+2\cos 4\theta}} =$ [where $\theta \in (0, \pi/4)$]

(a) $\cos\theta$ (b) $\sin\theta$
(c) $2\cos\theta$ (d) $2\sin\theta$

52. If $x + \dfrac{1}{x} = 2\cos\theta$, then $x^3 + \dfrac{1}{x^3} =$

(a) $\cos 3\theta$ (b) $2\cos 3\theta$
(c) $\cos 3\theta$ (d) $\cos 3\theta$

53. If $\tan\theta = \sqrt{n}$ where $n \in N \geq 2$, then $\sec 2\theta$ is always

(a) a rational number (b) an irrational number
(c) a positive integer (d) a negative integer

54. If $\sin 2\theta = \cos 3\theta$ and θ is an acute angle, then $\sin\theta$ equals

(a) $\dfrac{\sqrt{5}-1}{4}$ (b) $-\left(\dfrac{\sqrt{5}-1}{4}\right)$
(c) $\dfrac{\sqrt{5}+1}{4}$ (d) $\dfrac{-\sqrt{5}-1}{4}$

55. In a ΔABC, $\angle B < \angle C$ and the values of B and C satisfy the equation $2\tan x - k(1+\tan^2 x) = 0$, where $(0 < k < 1)$. Then, the measure of $\angle A$ is

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{2\pi}{3}$
(c) $\dfrac{\pi}{2}$ (d) $\dfrac{3\pi}{4}$

TRIGONOMETRIC IDENTIES IN A Δ

56. If $\alpha+\beta+\gamma=2\pi$, then

(a) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(b) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\beta}{2}=\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$

(c) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(d) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=0$

57. If $A+B+C=\frac{3\pi}{2}$, then $\cos 2A+\cos 2B+\cos 2C$ is equal to

(a) $1-4\cos A\cos B\cos C$ (b) $4\sin A\sin B\sin C$
(c) $1+2\cos A\cos B\cos C$ (d) $1-4\sin A\sin B\sin C$

58. If $A+B+C=\pi$ and $\cos A=\cos B\cdot\cos C$, then $\tan B\cdot\tan C$ is equal to

(a) 1 (b) $\frac{1}{2}$
(c) 2 (d) 3

GREATEST AND LEAST VALUE

59. If $f(\theta)=\sin^4\theta+\cos^2\theta$, then range of $f(\theta)$ is

(a) $\left[\frac{1}{2},1\right]$ (b) $\left[\frac{1}{2},\frac{3}{4}\right]$
(c) $\left[\frac{3}{4},1\right]$ (d) None of these

60. Maximum value of $f(x)=\sin x+\cos x$ is

(a) 1 (b) 2
(c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{2}$

61. The number of integral value of k for which the equation $7\cos x+5\sin x=2k+1$ has a solution is

(a) 4 (b) 8
(c) 10 (d) 12

62. If $a\le 3\cos x+5\sin(x-\pi/6)\le b$ for all x, then (a, b) is

(a) $\left(-\sqrt{19},\sqrt{19}\right)$ (b) $(-17, 17)$
(c) $\left(-\sqrt{21},\sqrt{21}\right)$ (d) None of these

MISCELLANEOUS

63. If $\alpha+\beta+\gamma=\pi$ then : $\tan(\beta+\gamma-\alpha)+\tan(\gamma+\alpha-\beta)+\tan(\alpha+\beta-\gamma)$ is :

(a) $\tan(\beta-\gamma-\alpha)\cdot\tan(\gamma+\alpha-\beta)\cdot\tan(\alpha+\beta-\gamma)$
(b) $\cot(\beta+\gamma-\alpha)\cdot\cot(\alpha+\gamma-\beta)\cdot\cot(\alpha+\beta-\gamma)$
(c) $\cot(\beta+\gamma-\alpha)\cdot\cot(\alpha+\gamma-\beta)\cdot\tan(\alpha+\beta-\gamma)$
(d) $\tan(\beta+\gamma-\alpha)\cdot\tan(\gamma-\alpha-\beta)\cdot\tan(\alpha-\beta-\gamma)$

64. If an angle α is divided into two parts A and B such that $A-B=x$ and $\tan A:\tan B=k:1$, then the value of $\sin x$ is

(a) $\left(\frac{k+1}{k-1}\right)\sin\alpha$ (b) $\left(\frac{k}{k+1}\right)\sin\alpha$
(c) $\left(\frac{k-1}{k+1}\right)\sin\alpha$ (d) $\left(\frac{k+1}{k}\right)\sin\alpha$

65. The value of expression: $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}$ is

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$

66. Suppose x and y are real numbers such that $\tan x+\tan y=42$ and $\cot x+\cot y=49$. Find the value of $\tan(x+y)$.

(a) 300 (b) 294
(c) 8 (d) 5

67. If the expression $\cos\left(x-\frac{3\pi}{2}\right)+\sin\left(\frac{3\pi}{2}+x\right)+\sin(32\pi+x)-18\cos(19\pi-x)+\cos(56\pi+x)-9\sin(x+17\pi)$ is expressed in the form of $a\sin x+b\cos x$ find the value of $a+b$.

(a) 5 (b) 28
(c) 10 (d) 50

68. $\cos A\sin(B-C)+\cos B\sin(C-A)+\cos C\sin(A-B)=$

(a) 1 (b) 0
(c) 8 (d) 5

69. If $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{p}{q}$ then $\tan\alpha\cdot\cot\beta$ has the value equal to

(a) $\frac{p+q}{p-q}$ (b) $\frac{p-q}{p+q}$
(c) $\frac{p+q}{q}$ (d) $\frac{p-q}{q}$

Exercise-2 (Learning Plus)

1. If $\sin x + \sin^2 x = 1$, then the value of $\cos^{12}x + 3\cos^{10}x + 3\cos^8 x + \cos^6 x - 1$ is equal to
   (a) 0 (b) 1
   (c) –1 (d) 2

2. The value of $\tan 75° - \cot 75° =$
   (a) $2\sqrt{3}$ (b) $2+\sqrt{3}$
   (c) $2-\sqrt{3}$ (d) $-2\sqrt{3}$

3. If $\tan\alpha = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$, then the expression $\cos 2\alpha + (2+\sqrt{3})\sin 2\alpha$ is
   (a) $2+\sqrt{3}$ (b) –1
   (c) 1 (d) $-(2+\sqrt{3})$

4. The value of $\tan 7\frac{1}{2}°$ is equal to
   (a) $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+1}$ (b) $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$
   (c) $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+1}$ (d) $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{2}-1}$

5. If $a\sin x + b\cos(c+x) + b\cos(c-x) = \alpha$, $a < \alpha$, then the minimum value of $|\cos c|$ is:
   (a) $\sqrt{\dfrac{\alpha^2-a^2}{b^2}}$ (b) $\sqrt{\dfrac{\alpha^2-a^2}{2b^2}}$
   (c) $\sqrt{\dfrac{\alpha^2-a^2}{3b^2}}$ (d) $\sqrt{\dfrac{\alpha^2-a^2}{4b^2}}$

6. If $\dfrac{\sin A}{\sin B} = \dfrac{\sqrt{3}}{2}$ and, $\dfrac{\cos A}{\cos B} = \dfrac{\sqrt{5}}{2}$ $0 < A, B < \dfrac{\pi}{2}$ then $\tan A + \tan B$ is equal to:
   (a) $\sqrt{\dfrac{3}{5}}$ (b) $\sqrt{\dfrac{5}{3}}$
   (c) $\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{5}}$ (d) $\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{3}}$

7. The maximum value of $1+\sin\left(\dfrac{\pi}{4}+\theta\right)+2\cos\left(\dfrac{\pi}{4}-\theta\right)$ for real values of θ is
   (a) 3
   (b) 5
   (c) 4
   (d) 2

8. If $\alpha + \gamma = 2\beta$ then the expression $\dfrac{\sin\alpha - \sin\gamma}{\cos\gamma - \cos\alpha}$ simplifies to:
   (a) $\tan\beta$ (b) $-\tan\beta$
   (c) $\cot\beta$ (d) $-\cot\beta$

9. If $\dfrac{\tan^3 A}{1+\tan^2 A} + \dfrac{\cot^3 A}{1+\cot^2 A} = p\sec A\,\mathrm{cosec}\,A + q\sin A\cos A$, then:
   (a) $p = 2, q = 1$ (b) $p = 1, q = 2$
   (c) $p = 1, q = -2$ (d) $p = 2, q = -1$

10. If $\log_3 \sin x - \log_3 \cos x - \log_3(1-\tan x) - \log_3(1+\tan x) = -1$, then $\tan 2x$ is equal to (wherever defined)
    (a) –2 (b) 3/2
    (c) 2/3 (d) 6

11. $\sum_{r=1}^{n-1} \cos^2 \dfrac{r\pi}{n}$ is equal to
    (a) $\dfrac{n}{2}$ (b) $\dfrac{n-1}{2}$
    (c) $\dfrac{n}{2}-1$ (d) $\dfrac{n+1}{2}$

12. The expression $2\sin 2° + 4\sin 4° + 6\sin 6° + \ldots 180\sin 180°$ equals
    (a) $\cot 1°$ (b) $90\cot 1°$
    (c) $\sin 1°$ (d) $90\cos 1°$

13. The value of $\tan\dfrac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)(1+\sec^2\theta)\ldots(1+\sec 2^n\theta)$ is
    (a) $\tan 2^n\theta$ (b) $\tan 2^{n-1}\theta$
    (c) $\tan 2^{n+1}\theta$ (d) $\tan 2^{n-2}\theta$

14. The minimum value of $27^{\cos 3x}\cdot 81^{\sin 3x}$ is
    (a) 1 (b) $\dfrac{1}{81}$
    (c) $\dfrac{1}{243}$ (d) $\dfrac{1}{27}$

15. If $\cos(\alpha+\beta) + \sin(\alpha-\beta) = 0$ and $\tan\beta = \dfrac{1}{2009}$; then $\tan 3\alpha$ is:
    (a) 2 (b) 1
    (c) 3 (d) 4

16. The value of 'a' for which the equation $\sin x(\sin x + \cos x) = a$ has a real solution are
    (a) $1-\sqrt{2} \le a \le 1+\sqrt{2}$ (b) $2-\sqrt{3} \le a \le 2+\sqrt{3}$
    (c) $0 \le a \le 2+\sqrt{3}$ (d) $\dfrac{1-\sqrt{2}}{2} \le a \le \dfrac{1+\sqrt{2}}{2}$

17. For $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}$ lies in the interval

(a) $(-\infty, \infty)$
(b) $(-2, 2)$
(c) $(0, \infty)$
(d) $(-1, 1)$

18. If $f(\alpha, \beta)=\cos^2\alpha+\sin^2\alpha\cdot\cos 2\beta$, then which of the following is incorrect?

(a) $f\left(\frac{\pi}{5}, \frac{2\pi}{5}\right) \neq f\left(\frac{2\pi}{5}, \frac{\pi}{5}\right)$

(b) $f\left(\frac{\pi}{12}, \frac{\pi}{3}\right) = f\left(\frac{\pi}{3}, \frac{\pi}{12}\right)$

(c) $3f\left(\frac{\pi}{5}, \frac{\pi}{3}\right) \neq f\left(\frac{\pi}{3}, \frac{\pi}{5}\right)$

(d) $f\left(\frac{\pi}{4}, \frac{\pi}{18}\right) \neq 3f\left(\frac{\pi}{18}, \frac{\pi}{4}\right)$

19. Let $f(x) = \cos 10x + \cos 8x + 3\cos 4x + 3\cos 2x$ and $g(x) = 8\cos x \cdot \cos^3 3x$, then for all x we have

(a) $f(x) = g(x)$
(b) $2f(x) = 3g(x)$
(c) $f(x) = 2g(x)$
(d) $2f(x) = g(x)$

20. The value of cot 70°+ 4 cos 70° is

(a) $\frac{1}{\sqrt{3}}$
(b) $\sqrt{3}$
(c) $2\sqrt{3}$
(d) $\frac{1}{2}$

21. The value of tan 9° – tan 27° – tan 63° + tan 81° is

(a) 2
(b) 3
(c) 4
(d) None of these

22. If $\cos^2 A + \cos^2 B + \cos^2 C = 1$, then ΔABC is

(a) Equilateral
(b) Isosceles
(c) Right angled
(d) None of these

23. If $\frac{\sin\theta}{\cos(3\theta)}+\frac{\sin(3\theta)}{\cos(9\theta)}+\frac{\sin(9\theta)}{\cos(27\theta)}+\frac{\sin(27\theta)}{\cos(81\theta)} = \frac{\sin(k\theta)}{2\cos\theta\cos 81\theta}$, then k equals to

(a) 80
(b) 81
(c) 27
(d) 40

24. Let $f_k(\theta) = \sin^k(\theta) + \cos^k(\theta)$, then find the value of $\frac{1}{6}f_6(\theta)-\frac{1}{4}f_4(\theta)$

(a) –1/12
(b) 0
(c) 1/3
(d) 1

25. If 270°< θ < 360°, then $\sqrt{2+\sqrt{2(1+\cos\theta)}}$ find

(a) $-2\sin\left(\frac{\theta}{4}\right)$
(b) $2\sin\left(\frac{\theta}{4}\right)$
(c) $\pm 2\sin\frac{\theta}{4}$
(d) $2\cos\frac{\theta}{4}$

26. If tan α, tan β are the roots of the equation $x^2 + px + q = 0$, then the value of $\sin^2(\alpha+\beta) + p\sin(\alpha+\beta)\cos(\alpha+\beta) + q\cos^2(\alpha+\beta)$ is

(a) Independent of p but dependent on q
(b) Independent of q but dependent on p
(c) Independent of both p and q
(d) Dependent on both p and q

27. The value of $(\cos^4 1° + \cos^4 2° + \cos^4 3° + ... + \cos^4 179°) - (\sin^4 1° + \sin^4 2° + \sin^4 3° + ... + \sin^4 179°)$ equals to

(a) 2cos1°
(b) –1
(c) 2sin1°
(d) 0

28. If in a triangle ABC, cos 3A + cos 3B + cos 3C = 1, then one angle must be exactly equal to

(a) $\frac{\pi}{3}$
(b) $\frac{2\pi}{3}$
(c) π
(d) $\frac{4\pi}{3}$

29. The value of $\cos 2(\theta+\phi) + 4\cos(\theta+\phi)\sin\theta\sin\phi + 2\sin^2\phi$ is

(a) Independent of θ only
(b) Independent of ϕ only
(c) Independent of both θ and ϕ
(d) Dependent on θ and ϕ

30. Which of the following statement is incorrect

(a) The orthocentre of ΔABC with sides 12, 35, 37 is one of the vertex
(b) sin 314° > 0
(c) cos 1 – sin 1 < 0
(d) sin 4 + sin 6 < 0

31. The exact value of cosec 10°+ cosec50°– cosec70° is :

(a) 4
(b) 5
(c) 6
(d) 8

32. Let $t_1 = (\sin\alpha)^{\cos\alpha}, t_2 = (\sin\alpha)^{\sin\alpha}, t_3 = (\cos\alpha)^{\cos\alpha}, t_4 = (\cos\alpha)^{\sin\alpha}$, where $\alpha \in \left(0, \frac{\pi}{4}\right)$ then which of the following is correct

(a) $t_3 > t_1 > t_2$
(b) $t_4 > t_2 > t_1$
(c) $t_4 > t_1 > t_2$
(d) $t_1 > t_3 > t_2$

33. If x_1 and x_2 are two distinct roots of the equation $a\cos x + b\sin x = c$, then $\tan\frac{x_1+x_2}{2}$ is equal to

(a) $\frac{a}{b}$
(b) $\frac{b}{a}$
(c) $\frac{c}{a}$
(d) $\frac{a}{c}$

34. If tanα is equal to the integral solution of the inequality $4x^2 - 16x + 15 < 0$ and cos β is equal to the slope of the bisector of the first quadrant, then sin (α + β) sin (α – β) is equal to

(a) $\frac{3}{5}$
(b) $\frac{2}{5}$
(c) $\frac{2}{\sqrt{5}}$
(d) $\frac{4}{5}$

35. $\cot 16° \cdot \cot 44° + \cot 44° \cdot \cot 76° - \cot 76° \cdot \cot 16° =$

(a) 4 (b) 3 (c) 9 (d) 8

36. $\sin\theta \cdot \sec 3\theta + \sin 3\theta \cdot \sec 3^2\theta + \sin 3^2\theta \cdot \sec 3^3\theta + ...$ upto n terms =

(a) $\frac{1}{2}\left[\tan 3^n\theta - \tan\theta\right]$ (b) $\frac{1}{2}\left[\tan 3^n\theta + \tan\theta\right]$

(c) $\left[\tan 3^n\theta + \tan\theta\right]$ (d) $\left[\tan 3^n\theta - \tan\theta\right]$

37. $\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} =$

(a) $\frac{1}{2}[\tan 27x - \tan x]$

(b) $\frac{1}{2}[\tan 27x + \tan x]$

(c) $[\tan 27x + \tan x]$

(d) $[\tan 27x - \tan x]$

38. The number of all possible 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ such that $a_1 + a_2 \sin x + a_3 \cos x + a_4 \sin 2x + a_5 \cos 2x = 0$ holds for all x is

(a) 4 (b) 3 (c) 1 (d) 0

39. If $\tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\varphi}{2}$, then $\frac{a\cos\varphi + b}{a + b\cos\varphi} =$

(a) $\cos\alpha$ (b) 1

(c) 0 (d) $\sin\alpha$

40. $(1 + \sec 2\theta) . (1 + \sec 2^2\theta)(1 + \sec 2^3\theta) ... (1 + \sec 2^n\theta) =$

(a) $\frac{\tan 2^n\theta}{\tan\theta}$ (b) 1

(c) 0 (d) $\sin\alpha$

41. If $\theta \in R$, the expression $a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$ lies between

(a) $\frac{-1}{2}\sqrt{b^2+(a-c)^2}$ and $\frac{1}{2}\sqrt{b^2+(a-c)^2}$

(b) $\frac{a+c}{2}$ and $\frac{a-c}{2}$

(c) $\frac{a+c}{2} - \frac{1}{2}\sqrt{b^2+(a-c)^2}$ and $\frac{a+c}{2} + \frac{1}{2}\sqrt{b^2+(a-c)^2}$

(d) None of these

42. $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} =$

(a) 1 (b) $-1/2$

(c) 0 (d) 2

43. If m and n are positive integers satisfying $1 + \cos 2\theta + \cos 4\theta + \cos 6\theta + \cos 8\theta + \cos 10\theta = \frac{\cos m\theta \cdot \sin n\theta}{\sin\theta}$ then $m + n$ is equal to

(a) 9 (b) 10

(c) 11 (d) 12

44. $\sum_{r=1}^{n-1}\cos^2\left(\frac{r\pi}{n}\right)$ is equal to

(a) $\frac{n}{2}$ (b) $\frac{n-1}{2}$

(c) $\frac{n}{2}-1$ (d) $\frac{n+1}{2}$

45. Find set of all possible values of α in $[-\pi, \pi]$ such that $\sqrt{\frac{1-\sin a}{1+\sin\alpha}}$ is equal to $(\sec\alpha - \tan\alpha)$

(a) $0 < \alpha < \frac{\pi}{3}$ (b) $-\pi < \alpha < \frac{\pi}{4}$

(c) $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ (d) $-\pi < \alpha < \pi$

46. Find the maximum and minimum value of $\sin^6 x + \cos^6 x$.

(a) 1 and $\frac{1}{4}$ (b) 1 and $\frac{3}{4}$

(c) 0 and $\frac{1}{4}$ (d) 0 and $\frac{3}{4}$

47. $\sin\alpha + \sin 2\alpha + \sin 4\alpha + \sin 5\alpha =$

(a) $4\cos\frac{\alpha}{2}\cos\frac{3\alpha}{2}\sin 3\alpha$ (b) 0

(c) 1 (d) $2\cos\frac{\alpha}{2}\cos\frac{3\alpha}{2}\sin 3\alpha$

48. $\cos(36° - A)\cos(36° + A) + \cos(54° + A)\cos(54° - A) =$

(a) $\cos 2A$ (b) 1

(c) 0 (d) $\sin 2A$

49. $\operatorname{cosec}\theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 2^2\theta + ... + \operatorname{cosec} 2^{n-1}\theta =$

(a) $\cot(\theta/2) - \cot 2^{n-1}\theta$ (b) $\cot(\theta/2) + \cot 2^{n-1}\theta$

(c) 0 (d) $\frac{\tan 2^n\theta}{\tan\theta}$

50. If $A + B + C = \pi$, then $\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}$

(a) ≤ 1 (b) always 0

(c) ≤ 2 (d) ≥ 1

51. Let $P = \cos(A + B + C) + \cos(-A + B + C) + \cos(A - B + C) + \cos(A + B - C)$ and $Q = \sin(A + B + C) + \sin(-A + B + C) - \sin(A - B + C) + \sin(A + B - C)$ then $\frac{P}{Q} =$

(a) $\cos 2B$ (b) $\cot B$

(c) 0 (d) $\sin B$

52. Distance between orthocentre and circumcentre in a triangle with sides 17, 15, 8.

(a) 9 (b) 17/2

(c) 15/2 (d) 4

53. log[sec (2022 π) + sec (2023 π) + cos (12345678 π)] =
(a) 1
(b) –1
(c) 0
(d) 2

54. If cosine of odd integral multiple of $\frac{\pi}{2}$ is a, tangent of integral multiple of π is b, sine of integral multiple of π is c, cosine of odd integral multiple of π is d and even integral multiple of e and let $\sin\left(2n\pi+\frac{\pi}{2}\right)=f, \sin\left(2n\pi-\frac{\pi}{2}\right)=g, (n \in I)$ and then $a+b+c+d+e+f+g=$
(a) 1 (b) –1
(c) 0 (d) 2

55. Value of the expression $\log_{1/2}(\sin 6° \cdot \sin 42° \cdot \sin 45° \cdot \sin 66° \cdot \sin 78°)$
(a) Lies between 4 and 5
(b) Is rational which is not integral
(c) Is irrational which is a simple surd
(d) Is irrational which is a mixed surd

56. Match the column

	Column-I		Column-II
A.	The value of $\frac{\sin 22°}{\sin 56° \sin 34° \cot 68°}$ equals to	p.	5
B.	The value of $\left(\cos 65° + \sqrt{3}\sin 5° + \sin 85°\right)^2 = \mu \cos^2 25°$, then value of μ be	q.	2
C.	If $f(\theta) = 2\sin 2\theta - \sin^2\theta + 3\cos^2\theta$, then number of intergers in the range of $f(\theta)$ equals to	r.	3
D.	If $f(\theta) = \cos(\sin\theta) + \sin(\sin\theta)$, then number of integers in the Range of $f(\theta)$ equals to	s.	1

(a) A → r; B → s; C → p; D → q
(b) A → q; B → r; C → p; D → q
(c) A → q; B → r; C → q; D → p
(d) A → q; B → r; C → s; D → p

57. If α and β are the roots of the equation, $a\cos\theta + b\sin\theta = c$ then match the entries of Column-I with the entries of Column-II.

	Column-I		Column-II
A.	$\sin\alpha + \sin\beta$	p.	$\frac{2b}{a+c}$
B.	$\sin\alpha \cdot \sin\beta$	q.	$\frac{c-a}{c+a}$
C.	$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}$	r.	$\frac{2bc}{a^2+b^2}$
D.	$\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} =$	s.	$\frac{c^2-a^2}{a^2+b^2}$

(a) A → r; B → s; C → p; D → q
(b) A → q; B → r; C → p; D → q
(c) A → q; B → r; C → q; D → p
(d) A → q; B → r; C → s; D → p

58. Match the columns:

	Column-I		Column-II
A.	cot 35° + cot 145° + 3 is equal to	p.	1
B.	$\frac{1}{\sec^2 120°} + \frac{1}{\text{cosec}^2 120°}$ is equal to	q.	3
C.	If $(\sin 2)^x + (\cos 2)^x = 1$ then x is	r.	0
D.	$\tan\frac{\pi}{11} + \tan\frac{2\pi}{11} + \tan\frac{4\pi}{11} + \tan\frac{7\pi}{11}$ $\tan\frac{9\pi}{11} + \tan\frac{10\pi}{11}$	s.	2
E.	cos 10° + cos 20° + cos 30° + ... + cos 80° + cos 100° + cos 150° + cos 160° + cos 170° + 4	t.	4

(a) A → q; B → p; C → s; D → r; E → t
(b) A → r; B → s; C → p; D → q; E → q
(c) A → r; B → s; C → p; D → t; E → q
(d) A → r; B → s; C → p; D → q; E → t

59. Find Trigonometric ratios of

$-300° \,|\, -\frac{3\pi}{4} \,|\, -\frac{5\pi}{6} \,|\, -315° \,|\, -120° \,|\, 450° \,|\, 990° \,|$

$|\, 1080 \,|\, 1800° \,|\, 3600° \,|$

60. Find tan θ if θ is exterior and interior angle of a regular
(a) Hexagon (b) Octagon
(c) Decagon (d) Dodecagon

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. If $\tan\frac{\pi}{9}, x$ and $\tan\frac{5\pi}{18}$ are in A.P. and $\tan\frac{\pi}{9}, y$ and $\tan\frac{7\pi}{18}$ are also in A.P., then
(a) $2x = y$ (b) $x > 2$
(c) $x = y$ (d) None of these

2. If $a = \sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$ and x is the solution the equation $y = 2[x] + 2$ and $y = 3[x-2]$, where $[x]$ denotes the integral part of x, then a is equal to
(a) $[x]$ (b) $\frac{1}{[x]}$
(c) $2[x]$ (d) $[x]^2$

3. It is known that $\sin\beta = \frac{4}{5}$ and $0 < \beta < \pi$ then the value of $\dfrac{\sqrt{3}\sin(\alpha+\beta) - \frac{2}{\cos\pi/6}\cos(\alpha+\beta)}{\sin\alpha}$ is
(a) Independent of α for all β in $(0, \pi)$
(b) $\frac{5}{13}$ for $\tan\beta < 0$
(c) $\frac{3(7+24\cot\alpha)}{15}$ for $\tan\beta > 0$
(d) None of these

4. The sum $\sum_{x=2}^{44}(2\sin x.\sin 1)\ (1 + \sec(x-1)\sec(x+1))$ can be written in the form as $\sum_{n=1}^{4}(-1)^n\frac{\phi^2(\theta_n)}{\psi(\theta_n)}$ where ϕ and ψ are trigonometric functions and $\theta_1, \theta_2, \theta_3, \theta_4$ are in degree $\in [0, 45]$, then the value of $|\theta_1 + \theta_3 - \theta_2 - \theta_4|$ is
(a) 82 (b) 90
(c) 86 (d) 45

5. If $\tan\frac{\pi}{10}\sin\frac{\pi}{5}\cdot\cos\frac{3\pi}{10}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\cos\frac{7\pi}{10}\sin\frac{4\pi}{4}$ $\tan\frac{9\pi}{10}\cos\pi$
$$= \frac{k\sin^2 18^0}{\cos 5^0}\left[\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+2\cos 1360^0}}}}\right]$$
Then k is
(a) Greater than 1 (b) Less than 0
(c) Irrational number (d) Rational number

6. The value of $P = (\sqrt{3}+\tan 1^0)(\sqrt{3}+\tan 2^0)(\sqrt{3}+\tan 3^0)$ ......$(\sqrt{3}+\tan 28^0)(\sqrt{3}+\tan 29^0)$ is
(a) 2^{28} (b) $2^{15}+2-\sqrt{3}$
(c) 2^{29} (d) $2^{14}+2-\sqrt{3}$

7. If $\tan\theta_i$, where $i = 1, 2, 3, 4$ are the roots of equation $x^4 - x^3 \sin 2\beta + x^2\cos 2\beta = x\cos\beta + \sin\beta$ than $\tan\left(\sum_{i=1}^{4}\theta_i\right)$
(a) $-\cot\beta$ (b) $\cot\left(\frac{\beta}{2}\right)$
(c) $\tan\beta$ (d) $\cot\beta$

8. $3\tan^6\frac{\pi}{18} - 27\tan^4\frac{\pi}{18} + 33\tan^2\frac{\pi}{18}$ equals
(a) 0 (b) 1 (c) 2 (d) 3

9. Let $\dfrac{\tan\left(\frac{\pi}{4}+\alpha\right)}{5} = \dfrac{\tan\left(\frac{\pi}{4}+\beta\right)}{3} = \dfrac{\tan\left(\frac{\pi}{4}+\gamma\right)}{2}$. Then $12\sin^2(\alpha-\beta) + 15\sin^2(\beta-\gamma) - 7\sin^2(\gamma-\alpha)$ is equal to
(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) 0

10. Let $f:(-1,1)\to R$ be such that $f(\cos 4\theta) = \frac{2}{2-\sec^2\theta}$. $\theta \in \left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$. Then the values of $f\left(\frac{1}{3}\right)$ is /are
(a) $1-\sqrt{\frac{3}{2}}$ (b) $1+\sqrt{\frac{3}{2}}$
(c) $1-\sqrt{\frac{2}{3}}$ (d) $1+\sqrt{\frac{2}{3}}$

11. Which of the following is/are +ve?
(a) $\log_{\sin 1}\tan 1$ (b) $\log_{\cos 1}(1+\tan 3)$
(c) $\log_{\log_{10}5}(\cos\theta+\sec\theta)$ (d) $\log_{\tan 15^\circ}(2\sin 18^\circ)$

12. If $3\sin\beta = \sin(2\alpha+\beta)$, then :
(a) $(\cot\alpha + \cot(\alpha+\beta))(\cot\beta - 3\cot(2\alpha+\beta)) = 6$
(b) $\sin\beta = \cos(\alpha+\beta)\sin\alpha$
(c) $\tan(\alpha+\beta) = 2\tan\alpha$
(d) $2\sin\beta = \sin(\alpha+\beta)\cos\alpha$

13. Let $P(k)=\left(1+\cos\frac{\pi}{4k}\right)\left(1+\cos\frac{(2k-1)\pi}{4k}\right)$
$\left(1+\cos\frac{(2k+1)\pi}{4k}\right)\left(1+\cos\frac{(4k-1)\pi}{4k}\right)$. Then
(a) $P(3)=\frac{1}{16}$ (b) $P(4)=\frac{2-\sqrt{2}}{16}$
(c) $P(5)=\frac{3-\sqrt{5}}{32}$ (d) $P(6)=\frac{2-\sqrt{3}}{16}$

14. For $0<\theta<\pi/2$, $\tan\theta+\tan 2\theta+\tan 3\theta=0$ if
(a) $\tan\theta=0$ (b) $\tan 2\theta=0$
(c) $\tan 3\theta=0$ (d) $\tan\theta\tan 2\theta=2$

15. If $\tan x=\frac{2b}{a-c}$, $(a\neq c)$ $y=a\cos^2x+2b\sin x\cos x+c\sin^2x$, $z=a\sin^2x-2b\sin x\cos x+c\cos^2x$, then
(a) $y=z$ (b) $y+z=a+c$
(c) $y-z=a-c$ (d) $y-z=(a-c)^2+4b^2$

16. The value of $\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n$ is
(a) $2\tan^n\frac{A-B}{2}$ (b) $2\cot^n\frac{A-B}{2}$: n is even
(c) 0 : n is odd (d) None of these

17. If $\tan A+\tan B+\tan C=\tan A.\tan B.\tan C$, then
(a) A, B, C may be angles of a triangle
(b) $A+B+C$ is an integral multiple of π
(c) Sum of any two of A, B, C is equal to third
(d) None of these

18. In a triangle $\tan A+\tan B+\tan C=6$ and $\tan A\tan B=2$, then the values of $\tan A$, $\tan B$ and $\tan C$ are
(a) 1, 2, 3 (b) 2, 1, 3
(c) 1, 2, 0 (d) None

19. If the sides of a right angled triangle are $\{\cos 2\alpha+\cos 2\beta+2\cos(\alpha+\beta)\}$ and $\{\sin 2\alpha+\sin 2\beta+2\sin(\alpha+\beta)\}$, then the length of the hypotenuse is:
(a) $2[1+\cos(\alpha-\beta)]$ (b) $2[1-\cos(\alpha+\beta)]$
(c) $4\cos^2\frac{\alpha-\beta}{2}$ (d) $4\sin^2\frac{\alpha+\beta}{2}$

20. If $x+y=z$, then $\cos^2 x+\cos^2 y+\cos^2 z-2\cos x\cos y\cos z$ is equal to
(a) $\cos^2 z$ (b) $\sin^2 z$
(c) $\cos(x+y-z)$ (d) 1

21. The equation $\sin^6x+\cos^6x=a^2$ has real solution if
(a) $a\in(-1, 1)$ (b) $a\in\left(-1,-\frac{1}{2}\right)$
(c) $a\in\left(-\frac{1}{2},\frac{1}{2}\right)$ (d) $a\in\left(\frac{1}{2},1\right)$

COMPREHENSION BASED QUESTIONS

Comprehension (Q. 22 to 24): Let p be the product of the sines of the angles of a triangle ABC and q is the product of the cosines of the angles.

22. In this triangle $\tan A+\tan B+\tan C$ is equal to
(a) $p+q$ (b) $p-q$
(c) p/q (d) None of these

23. $\tan A\tan B+\tan B\tan C+\tan C\tan A$ is equal to
(a) $1+q$ (b) $\frac{1+q}{q}$ (c) $1+p$ (d) $\frac{1+p}{p}$

24. The value of $\tan^3 A+\tan^3 B+\tan^3 C$ is
(a) $\frac{p^3-3pq^2}{q^3}$ (b) $\frac{q^3}{p^3}$
(c) $\frac{p^3}{q^3}$ (d) $\frac{p^3-3pq}{q^3}$

Comprehension (Q. 25 to 27): If $\cos\alpha+\cos\beta=a$ and $\sin\alpha+\sin\beta=b$ and $2\theta=\alpha+\beta$, then $\sin 2\theta+\cos 2\theta=1+\frac{nb(a-b)}{a^2+b^2}$ where n is some integer. Also if a, b, c are 3 non-zeor terms in G.P then $b^2=ac$. So answer the following questions:

25. The value of n is
(a) 0 (b) 1 (c) 2 (d) –2

26. If for n obtained in above question, $\sin^nA=x$, then $\sin A\sin 2A\sin 3A\sin 4A$ is a polynomial in x, of degree
(a) 5 (b) 6 (c) 7 (d) 8

27. If degree of polynomial obtained in previous questions is p and $(p-5)+\sin x$, $\cos x$, $\tan x$ are in G.P., then value of $(\cos^9 x+\cos^6 x+3\cos^5 x-1)$ is
(a) –1 (b) 0
(c) 1 (d) None of these

MATCH THE COLUMN

28. Match the column:

	Column-I		Column-II
A.	$\cos^2 52\frac{1^\circ}{2}-\sin^2 22\frac{1^\circ}{2}$	p.	1
B.	$\cos^2\frac{3\pi}{5}+\cos^2\frac{4\pi}{5}$	q.	$\frac{3-\sqrt{3}}{4\sqrt{2}}$
C.	$\sin 24^\circ+\cos 6^\circ$	r.	$\frac{3}{4}$
D.	$\sin^2 50^\circ+\cos^2 130^\circ$	s.	$\frac{\sqrt{15}+\sqrt{3}}{4}$

(a) A → q; B → r; C → s; D → p
(b) A → r; B → s; C → q; D → p
(c) A → s; B → p; C → r; D → q
(d) A → q; B → r; C → s; D → p

NUMERICAL TYPE QUESTIONS

29. $\frac{\cos 5A}{\cos A}+\frac{\sin 5A}{\sin A}=a+b\cos 4A$, then find the value of $(a+b)$.

30. If $\operatorname{cosec}\theta-\sin\theta=a^3$ and $\sec\theta-\cos\theta=b^3$, then $a^2b^2(a^2+b^2)=$

31. If $\sin\theta+\sin^2\theta+\sin^3\theta=1$, then the value of $\cos^6\theta-4\cos^4\theta+8\cos^2\theta$ must be

32. Absolute value of $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}$ is

33. If $\frac{\sin x}{\sin y}=\frac{1}{2}$ and $\frac{\cos x}{\cos y}=\frac{3}{2}$ where $x,y\in\left(0,\frac{\pi}{2}\right)$ then find the value of $\frac{\tan^2(x+y)}{5}$

34. If the expression $\tan(55°)\tan(65°)\tan(75°)$ simplifies to $\cot(x°)$ and m is the numerical value of the expression $\tan(27°)+\tan(18°)+\tan(27°)\tan(18°)$, then find the value of $(m+x+1)$.

35. If $4\sin 54°+\cot\left(7\frac{1°}{2}\right)$

$$=\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\sqrt{a_4}+\sqrt{a_5}+\sqrt{a_6}$$

then, find the value of $\sum_{i=1}^{6}a_i^3$

36. Given that for $a,b,c,d\in R$. if $a\sec(190°)-c\tan(190°)=d$ and $b\sec(190°)+d\tan(190°)=c$, then find the value of $\left(\frac{a^2+b^2+c^2+d^2}{bd-ac}\right)\sin 10°$

37. The absolute value of the expression

$\tan\frac{\pi}{16}+\tan\frac{5\pi}{16}+\tan\frac{9\pi}{16}+\tan\frac{13\pi}{16}$ is

38. The value of $\left(1+\tan\frac{3\pi}{8}\tan\frac{\pi}{8}\right)+\left(1+\tan\frac{5\pi}{8}\tan\frac{3\pi}{8}\right)$

$$+\left(1+\tan\frac{7\pi}{8}\tan\frac{5\pi}{8}\right)+\left(1+\tan\frac{9\pi}{8}\tan\frac{7\pi}{8}\right)$$

39. If $\tan\left(\frac{\pi}{4}+\frac{y}{2}\right)=\tan^3\left(\frac{\pi}{4}+\frac{x}{2}\right)$, if $\frac{\sin y}{\sin x}=\frac{k+\sin^2 x}{1+k\sin^2 x}$. Then find k

40. If $A+B+C=\pi$ and $\sum\left(\frac{\tan A}{\tan B.\tan C}\right)=\sum(\tan A)-k\sum(\cot A)$. Then find k

41. If $P=\cos\frac{\pi}{19}+\cos\frac{3\pi}{19}+\cos\frac{5\pi}{19}+...+\cos\frac{17\pi}{19}$ and $Q=\cos\frac{2\pi}{21}+\cos\frac{4\pi}{21}+\cos\frac{6\pi}{21}+...+\cos\frac{20\pi}{21}$, then find $P-Q$.

42. Let $A_1, A_2, A_3, ..., A_n$ are the vertices of a regular n sided polygon inscribed in a circle of radius R. If $(A_1A_2)^2+(A_1A_3)^2+...+(A_1A_n)^2=14R^2$, find the number of sides in the polygon.

43. If $P=\sec^6\theta-\tan^6\theta-3\sec^2\theta\tan^2\theta$, $Q=\operatorname{cosec}^6\theta-\cot^6\theta-3\operatorname{cosec}^2\theta\cot^2\theta$ and $R=\sin^6\theta+\cos^6\theta+3\sin^2\theta\cos^2\theta$, then find the value of $(P+Q+R)^{(P+Q+R)}$

44. Let $A_1, A_2, ..., A_n$ be the vertices of an n-sided regular polygon such that; $\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$. Find the value of n.

45. Prove that if the angles α & β satisfy the relation $\frac{\sin\beta}{\sin(2\alpha+\beta)}=\frac{n}{m}(|m|>|n|)$ then $\frac{1+\frac{\tan\beta}{\tan\alpha}}{m+n}=\frac{1-\tan\alpha\tan\beta}{m-n}$

46. Let $k=1°$, then prove that $\sum_{n=0}^{88}\sec(nk)\sec(n+1)k=\frac{\cos k}{\sin^2 k}$

47. If $\alpha+\beta+\gamma=\pi$ and $\tan\left(\frac{\beta+\gamma-\alpha}{4}\right)\cdot\tan\left(\frac{\gamma+\alpha-\beta}{4}\right)\cdot\tan\left(\frac{\alpha+\beta-\gamma}{4}\right)=1$, then prove that $1+\cos\alpha+\cos\beta+\cos\gamma=0$.

48. If $\frac{x}{\tan(\theta+\alpha)}=\frac{y}{\tan(\theta+\beta)}=\frac{z}{\tan(\theta+\gamma)}$, show that $\sum\frac{x+y}{x-y}\sin^2(\alpha-\beta)=0$.

49. If $A+B+C=2S$ prove that
    (i) $\sin(S-A)+\sin(S-B)+(S-C)-\sin S=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
    (ii) $\cos^2 S+\cos^2(S-A)+\cos^2(S-B)+\cos^2(S-C)=2+2\cos A\cos B\cos C$.

50. If $A+B+C=2S$, prove that $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1+4\cos S\cos(S-A)\cos(S-B)\cos(S-C)$

51. If $\alpha+\beta+\gamma+\delta=2\pi$, prove that
    (i) $\cos\alpha+\cos\beta+\cos\gamma+\cos\delta-4\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2}=0$
    (ii) $\tan\alpha+\tan\beta+\tan\gamma+\tan\delta=\tan\alpha\tan\beta\tan\gamma\tan\delta(\cot\alpha+\cot\beta+\cot\gamma+\cot\delta)$

52. $\dfrac{\sin A\sin 2A+\sin 3A\sin 6A+\sin 4A\sin 13A}{\sin 4A\sin 3A-\sin 2A\sin 5A+\sin 4A\sin 7A}=\cot 6A\cot 5A$

53. $\cos 6\alpha = 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1$

54. Prove that: $\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)+\cos(A+B+C)=4\cos A\cos B\cos C$

55. Prove that: $\sin(A+B+C+D)+\sin(A+B-C-D)+\sin(A+B-C+D)+\sin(A+B+C-D)=4\sin(A+B)\cos C\cos D$.

56. If $x+y+z=xyz$, prove that

$$\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}$$

57. Prove that a triangle ABC is equilateral if and only if $\tan A+\tan B+\tan C=3\sqrt{3}$.

58. for ΔABC prove that

(i) $\sin\dfrac{A}{2}+\sin\dfrac{B}{2}+\sin\dfrac{C}{2}-1=4\sin\dfrac{\pi-A}{4}\sin\dfrac{\pi-B}{4}\sin\dfrac{\pi-C}{4}$

(ii) $\sin(B+2C)+\sin(C+2A)+\sin(A+2B)=4\sin\dfrac{B-C}{2}\sin\dfrac{C-A}{2}\sin\dfrac{A-B}{2}$

(iii) $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)=4\sin A\sin B\sin C$

(iv) $\cos\dfrac{A}{2}+\cos\dfrac{B}{2}-\cos\dfrac{C}{2}=4\cos\dfrac{\pi-A}{4}\cos\dfrac{\pi-B}{4}\cos\dfrac{\pi-C}{4}$

Exercise-4 (Past Year Questions)

JEE MAINS

1. For any $\theta\in\left(\dfrac{\pi}{4},\dfrac{\pi}{2}\right)$, the expression $3(\sin\theta-\cos\theta)^4+6(\sin\theta+\cos\theta)^2+4\sin^6\theta$ equals **(2001)**

(a) $13-4\cos^2\theta+6\sin^2\theta\cos^2\theta$

(b) $13-4\cos^6\theta$

(c) $13-4\cos^2\theta+6\cos^4\theta$

(d) $13-4\cos^4\theta+2\sin^2\theta\cos^2\theta$

2. Let α and β be the roots of the quadratic equation $x^2\sin\theta-x(\sin\theta\cos\theta+1)+\cos\theta=0$ $(0<\theta<45^0)$, and $\alpha<\beta$. Then $\sum_{n=0}^{\infty}\left(\alpha^n+\dfrac{(-1)^n}{\beta^n}\right)$ is equal to : 20 **(2019)**

(a) $\dfrac{1}{1-\cos\theta}-\dfrac{1}{1+\sin\theta}$ (b) $\dfrac{1}{1+\cos\theta}+\dfrac{1}{1-\sin\theta}$

(c) $\dfrac{1}{1-\cos\theta}+\dfrac{1}{1+\sin\theta}$ (d) $\dfrac{1}{1+\cos\theta}-\dfrac{1}{1-\sin\theta}$

3. The maximum value of $3\cos\theta+5\sin\left(\theta-\dfrac{\pi}{6}\right)$ for any real value of θ is: **(2019)**

(a) $\sqrt{19}$ (b) $\dfrac{\sqrt{79}}{2}$

(c) $\sqrt{34}$ (d) $\sqrt{31}$

4. If $\sin^4\alpha+4\cos^4\beta+2=4\sqrt{2}\sin\alpha\cos\beta$; $\alpha,\beta\in[0,\pi]$, then $\cos(\alpha+\beta)-\cos(\alpha-\beta)$ is equal to: **(2019)**

(a) 0 (b) -1

(c) $\sqrt{2}$ (d) $-\sqrt{2}$

5. If $\cos(\alpha+\beta)=\dfrac{3}{5},\sin(\alpha-\beta)=\dfrac{5}{13}$ and $0<\alpha\ \beta<\dfrac{\pi}{4}$, then $\tan(2\alpha)$ is equal to: **(2019)**

(a) $\dfrac{21}{16}$ (b) $\dfrac{63}{52}$

(c) $\dfrac{33}{52}$ (d) $\dfrac{63}{16}$

6. The value of $\cos^2 10^\circ-\cos 10^\circ\cos 50^\circ+\cos^2 50^\circ$ is **(2019)**

(a) $\dfrac{3}{2}(1+\cos 20^\circ)$ (b) $\dfrac{3}{4}$

(c) $\dfrac{3}{4}+\cos 20^\circ$ (d) $\dfrac{3}{2}$

7. The value of $\sin 10^\circ\sin 30^\circ\sin 50^\circ\sin 70^\circ$ is **(2019)**

(a) $\dfrac{1}{36}$ (b) $\dfrac{1}{32}$

(c) $\dfrac{1}{18}$ (d) $\dfrac{1}{16}$

8. All the pairs (x, y) that satisfy the inequality $2^{\sqrt{\sin^2 x-2\sin x+5}}\cdot\frac{1}{4^{\sin^2 y}}\le 1$ also satisfy the equation. **(2019)**

(a) $\sin x = |\sin y|$ (b) $\sin x = 2\sin y$

(c) $2|\sin x| = 3\sin y$ (d) $2\sin x = \sin y$

9. The equation $y = \sin x \sin(x+2) - \sin^2(x+1)$ represents a straight line lying in: **(2019)**

(a) Second and third quadrants only

(b) Third and fourth quadrants only

(c) First, third and fourth quadrants

(d) First, second and fourth quadrants

10. Let α and β be two real roots of the $(k+1)\tan^2 x - \sqrt{2}\cdot\lambda\tan x = (1-k)$, where $k(\ne -1)$ and λ are equal number. If $\tan^2(\alpha+\beta) = 50$, then a value of λ is **(2020)**

(a) $10\sqrt{2}$ (b) $5\sqrt{2}$

(c) 1 (d) 10

11. If $\frac{\sqrt{2}\sin\alpha}{\sqrt{1+\cos 2\alpha}} = \frac{1}{7}$ = and $\sqrt{\frac{1-\cos 2\beta}{2}} = \frac{1}{\sqrt{10}}$, $\alpha,\beta \in \left(0,\frac{\pi}{2}\right)$

, then $\tan(\alpha+2\beta)$ is equal to **(2020)**

12. The value of $\cos^3\left(\frac{\pi}{8}\right)\cdot\cos\left(\frac{3\pi}{8}\right)+\sin^3\left(\frac{\pi}{8}\right)\cdot\sin\left(\frac{3\pi}{8}\right)$ is: **(2020)**

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\frac{1}{4}$ (d) $\frac{1}{2}$

13. The number of distinct solutions of the equation, $\log_{1/2}|\sin x| = 2 - \log_{1/2}|\cos x|$ in the interval $[0, 2\pi]$, is **(2020)**

14. If $x = \sum_{n=0}^{\infty}(-1)^n \tan^{2n}\theta$ and $y = \sum_{n=0}^{\infty}\cos^{2n}\theta$, for $0 < \theta < \frac{\pi}{4}$, then: **(2020)**

(a) $y(1-x) = 1$ (b) $y(1+x) = 1$

(c) $x(1+y) = 1$ (d) $x(1-y) = 1$

15. Let $a, b, c \in$ R be such that $a^2+b^2+c^2 = 1$. If $a\cos\theta = b\cos\left(\theta+\frac{2\pi}{3}\right) = c\cos\left(\theta+\frac{4\pi}{3}\right)$, where, $\theta+\frac{\pi}{9}$ then the angle between the vectors $a\hat{i}+b\hat{j}+c\hat{k}$ and $b\hat{i}+c\hat{j}+a\hat{k}$ is **(2020)**

(a) 0 (b) $\frac{\pi}{9}$

(c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{2}$

16. The minimum value of $2^{\sin x} + 2^{\cos x}$ is: **(2020)**

(a) $2^{-1+\sqrt{2}}$ (b) $2^{1-\sqrt{2}}$

(c) $2^{1-\frac{1}{\sqrt{2}}}$ (d) $2^{-1+\frac{1}{\sqrt{2}}}$

17. If $L = \sin^2\left(\frac{\pi}{16}\right)-\sin^2\left(\frac{\pi}{8}\right)$ and $M = \cos^2\left(\frac{\pi}{16}\right)-\sin^2\left(\frac{\pi}{8}\right)$, then **(2020)**

(a) $L = -\frac{1}{2\sqrt{2}}+\frac{1}{2}\cos\frac{\pi}{8}$ (b) $M = \frac{1}{2\sqrt{2}}+\frac{1}{2}\cos\frac{\pi}{8}$

(c) $M = \frac{1}{4\sqrt{2}}+\frac{1}{4}\cos\frac{\pi}{8}$ (d) $L = \frac{1}{4\sqrt{2}}-\frac{1}{4}\cos\frac{\pi}{8}$

18. If $\frac{\sqrt{2}\sin\alpha}{\sqrt{1+\cos 2\alpha}} = \frac{1}{7}$ and $\sqrt{\frac{1-\cos 2\beta}{2}} = \frac{1}{\sqrt{10}}$ $\alpha,\beta\in\left(0,\frac{\pi}{2}\right)$

then $\tan(\alpha+2\beta)$ is equal to **(2020)**

19. If $e^{(\cos^2 x+\cos^4 x+\cos^6 x+\ldots\infty)\log_e 2}$ satisfies the equation $t^2-9t+8=0$, then the value of $\frac{2\sin x}{\sin x+\sqrt{3}\cos x}\left(0<x<\frac{\pi}{2}\right)$ is **(2021)**

(a) $\frac{3}{2}$ (b) $2\sqrt{3}$

(c) $\frac{1}{2}$ (d) $\sqrt{3}$

20. The number of integral values of 'k' for which the equation $3\sin x + 4\cos x = k+1$ has a solution, $k\in R$ is **(2021)**

21. If $15\sin^4\alpha + 10\cos^4\alpha = 6$, for some $\alpha\in R$, then the value of $27\sec^6\alpha + 8\,\text{cosec}^6\alpha$ is equal to: **(2021)**

(a) 350 (b) 500

(c) 400 (d) 250

22. If $0 < x, y < \pi$ and $\cos x + \cos y - \cos(x+y) = 3/2$, Then, $\sin x + \cos y =$ **(2021)**

(a) $\frac{1+\sqrt{3}}{2}$ (b) $\frac{1}{2}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1-\sqrt{3}}{2}$

23. If $\tan\left(\frac{\pi}{9}\right)$, x, $\tan\left(\frac{7\pi}{18}\right)$ are in A.P. and $\tan\left(\frac{\pi}{9}\right)$, y, $\tan\left(\frac{5\pi}{18}\right)$ are also in A.P. Then, $|x-2y| =$ **(2021)**

(a) 4 (b) 3

(c) 0 (d) 1

24. The value of $\cot\left(\frac{\pi}{24}\right)$ is: **(2021)**

(a) $\sqrt{2}+\sqrt{3}+2-\sqrt{6}$ (b) $\sqrt{2}+\sqrt{3}+2+\sqrt{6}$

(c) $\sqrt{2}-\sqrt{3}-2+\sqrt{6}$ (d) $3\sqrt{2}-\sqrt{3}-\sqrt{6}$

25. If $\sin\theta + \cos\theta = \frac{1}{2}$ then $16(\sin(2\theta) + \cos(4\theta) + \sin(6\theta)) =$ **(2021)**

(a) 23 (b) –27
(c) –23 (d) 27

26. The value of $2\sin\left(\frac{\pi}{8}\right)\sin\left(\frac{2\pi}{8}\right)\sin\left(\frac{3\pi}{8}\right)\sin\left(\frac{5\pi}{8}\right)\sin\left(\frac{6\pi}{8}\right)\sin\left(\frac{7\pi}{8}\right)$ is **(2021)**

(a) $\frac{1}{4\sqrt{2}}$ (b) $\frac{1}{4}$
(c) $\frac{1}{8}$ (d) $\frac{1}{8\sqrt{2}}$

27. If $0 < \theta, \phi < \frac{\pi}{2}, x = \sum_{n=0}^{\infty}\cos^{2n}\theta, y = \sum_{n=0}^{\infty}\sin^{2n}\phi$ and $z = \sum_{n=0}^{\infty}\cos^{2n}\theta\cdot\sin^{2n}\phi$ then **(2021)**

(a) $xyz = 4$ (b) $xy - z = (x + y)z$
(c) $xy + yz + zx = z$ (d) $xy + z = (x + y)z$

28. The minimum value of α for which the equation $\frac{4}{\sin x} + \frac{1}{1-\sin x} = \alpha$ has at least one solution in $\left(0, \frac{\pi}{2}\right)$ is **(2021)**

29. $a = \sin 36°$ is a root of which of the following equation. **(2022)**

(a) $10x^4 - 10x^2 - 5 = 0$ (b) $16x^4 - 20x^2 - 5 = 0$
(c) $16x^4 - 20x^2 + 5 = 0$ (d) $16x^4 - 10x^2 + 5 = 0$

30. $2\sin\left(\frac{\pi}{22}\right)\sin\left(\frac{3\pi}{22}\right)\sin\left(\frac{5\pi}{22}\right)\sin\left(\frac{7\pi}{22}\right)\sin\left(\frac{9\pi}{22}\right)$ is equal to **(2022)**

(a) $\frac{3}{16}$ (b) $\frac{1}{16}$
(c) $\frac{1}{32}$ (d) $\frac{9}{32}$

31. If $\cot\alpha = 1$ and $\sec\beta = -\frac{5}{3}$, where $\pi < \alpha < \frac{3\pi}{2}$ and, $\frac{\pi}{2} < \beta < \pi$ then the value of $\tan(\alpha+\beta)$ and the quadrant in which $\alpha + \beta$ lies, respectively are **(2022)**

(a) $-\frac{1}{7}$ and IV[th] quadrant
(b) 7 and I[st] quadrant
(c) –7 and IV[th] quadrant
(d) $\frac{1}{7}$ and I[st] quadrant

32. The value of $\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right)$ is equal to: **(2022)**

(a) –1 (b) $-\frac{1}{2}$
(c) $-\frac{1}{3}$ (d) $-\frac{1}{4}$

33. $16\sin(20°)\sin(40°)\sin(80°)$ is equal to: **(2022)**

(a) $\sqrt{3}$ (b) $2\sqrt{3}$
(c) 3 (d) $4\sqrt{3}$

34. The value of $2\sin(12°) - \sin(72°)$ is : **(2022)**

(a) $\frac{\sqrt{5}(1-\sqrt{3})}{4}$ (b) $\frac{1-\sqrt{5}}{8}$
(c) $\frac{\sqrt{3}(1-\sqrt{5})}{2}$ (d) $\frac{\sqrt{3}(1-\sqrt{5})}{4}$

35. For any $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ the expression $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4\sin^6\theta$ equals:

(a) $13 - 4\cos^2\theta + 6\sin^2\theta\cos^2\theta$
(b) $13 - 4\cos^6\theta$
(c) $13 - 4\cos^2\theta + 6\cos^4\theta$
(d) $13 - 4\cos^4\theta + 2\sin^2\theta\cos^2\theta$ **(2022)**

JEE ADVANCED

36. If $\alpha + \beta + \gamma = 2\pi$, then **(1979)**

(a) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(b) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$

(c) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(d) None of these

37. Suppose that $\sin^3 x \sin 3x = \sum_{m=0}^{n} c_m \cos mx$ is an identity in x, where $c_0, c_1, c_2, \ldots, c_n$ are constants and $c_n \neq 0$, find the value of n. **(1981)**

38. Find the value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$. **(1991)**

39. Let $\theta, \phi \in [0, 2\pi]$ be such that

$$2\cos\theta(1-\sin\phi) = \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\phi - 1$$

$\tan(2\pi-\theta)>0$ and $-1<\sin\theta<-\dfrac{\sqrt{3}}{2}$

Then ϕ **cannot** satisfy **(2012)**

(a) $0<\phi<\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{2}<\phi<\dfrac{4\pi}{3}$

(c) $\dfrac{4\pi}{3}<\phi<\dfrac{3\pi}{2}$ (d) $\dfrac{3\pi}{2}<\phi<2\pi$

40. Let α and β be nonzero real numbers such that $2(\cos\beta-\cos\alpha)+\cos\alpha\cos\beta=1$. Then which of the following is/are true? **(2017)**

(a) $\sqrt{3}\tan\left(\dfrac{\alpha}{2}\right)-\tan\left(\dfrac{\beta}{2}\right)=0$

(b) $\tan\left(\dfrac{\alpha}{2}\right)-\sqrt{3}\tan\left(\dfrac{\beta}{2}\right)=0$

(c) $\tan\left(\dfrac{\alpha}{2}\right)+\sqrt{3}\tan\left(\dfrac{\beta}{2}\right)=0$

(d) $\sqrt{3}\tan\left(\dfrac{\alpha}{2}\right)+\tan\left(\dfrac{\beta}{2}\right)=0$

41. In a ∇PQR let $\angle PQR=30$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is(are) TRUE? **(2018)**

(a) $\angle QPR=45°$

(b) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP=120°$

(c) The radius of the in circle of the triangle PQR is $10-15$

(d) The area of the circumcircle of the triangle PQR is 100π.

42. For non-negative integers n, let

$$f(n)=\frac{\sum_{k=0}^{n}\sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n}\sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1}x$ takes values in $[0,\pi]$, which of the following options is/are correct? **(2019)**

(a) $\sin(7\cos^{-1}f(5))=0$

(b) $f(d)=\dfrac{\sqrt{3}}{2}$

(c) $\lim\limits_{x\to\infty}f(n)=\dfrac{1}{2}$

(d) If $\alpha=\tan(\cos^{-1}f(6))$, then $\alpha^2+2\alpha-1=0$

43. In a non-right-angled triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p=\sqrt{3}$, $q=1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct? **(2019)**

(a) Area of $\Delta SOE=\dfrac{\sqrt{3}}{12}$

(b) Radius of incircle of $\Delta PQR\dfrac{\sqrt{3}}{2}(2-\sqrt{3})$

(c) Length of $RS=\dfrac{\sqrt{7}}{2}$.

(d) Length of $OE=\dfrac{1}{6}$

44. Let $f:[0,2]\to R$ be the function defined by

$$f(x)=(3-\sin(2\pi x))\sin\left(\pi x-\frac{\pi}{4}\right)-\sin\left(3\pi x+\frac{\pi}{4}\right)$$

If $\alpha,\beta\in[0,2]$ are such that $\{x\in[0,2];f(x)\geq 0\}=[\alpha,\beta]$, then the value of $\beta-\alpha$ is________. **(2020)**

45. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) TRUE? **(2021)**

(a) $\cos P\geq 1-\dfrac{p^2}{2qr}$

(b) $\cos R\geq\left(\dfrac{q-r}{p+q}\right)\cos P+\left(\dfrac{p-r}{p+q}\right)\cos Q$

(c) $\dfrac{q+r}{p}<2\dfrac{\sqrt{\sin Q\sin R}}{\sin P}$

(d) If $p<q$ and $p<r$, then $\cos Q>\dfrac{p}{r}$ and $\cos R>\dfrac{p}{q}$

46. Let α and β be real numbers such that

$-\dfrac{\pi}{4}<\beta<0<\alpha<\dfrac{\pi}{4}$. If $\sin(\alpha+\beta)=\dfrac{1}{3}$ and $\cos(\alpha-\beta)=\dfrac{2}{3}$ then the greatest integer less than or equal to

$\left(\dfrac{\sin\alpha}{\cos\beta}+\dfrac{\cos\beta}{\sin\alpha}+\dfrac{\cos\alpha}{\sin\beta}+\dfrac{\sin\beta}{\cos\alpha}\right)^2$ is ______. **(2022)**

ANSWER KEY

CONCEPT APPLICATION

1. [20π/3] **2.** [90,60,30] **3.** [1.7] **4.** [1] **5.** [8/15] **6.** *(b)* **7.** [1] **8.** *(c)* **9.** [2]
10. $A \to x; B \to w; C \to v; D \to u; E \to t; F \to s; G \to r; H \to q; I \to p$ **13.** [17] **14.** [–2/3] **15.** *(d)* **16.** *(b)*
19. $\left[\sin\alpha = \frac{\sqrt{2+2\sqrt{2}}}{2}, \cos\alpha = -\sqrt{\frac{2-2\sqrt{2}}{2}}\right]$ **20.** [–1/2] **23.** *(d)* **24.** *(b)* **25.** [1/2] **26.** $\left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{1}{7}, \infty\right)$
27. $\left(4-\sqrt{10}, 4+\sqrt{10}\right)$ **28.** *(b)*

EXERCISE-1 (TOPICWISE)

1. *(a)* **2.** *(b)* **3.** *(a)* **4.** *(d)* **5.** *(b)* **6.** *(a)* **7.** *(c)* **8.** *(d)* **9.** *(b)* **10.** *(c)*
11. *(b)* **12.** *(a)* **13.** *(a)* **14.** *(d)* **15.** *(a)* **16.** *(b)* **17.** *(c)* **18.** *(d)* **19.** *(d)* **20.** *(b)*
21. *(d)* **22.** *(a)* **23.** *(b)* **24.** *(a)* **25.** *(b)* **26.** *(a)* **27.** *(d)* **28.** *(a)* **29.** *(b)* **30.** *(c)*
31. *(a)* **32.** *(d)* **33.** *(d)* **34.** *(b)* **35.** *(b)* **36.** *(c)* **37.** *(b)* **38.** *(a)* **39.** *(b)* **40.** *(d)*
41. *(c)* **42.** *(a)* **43.** *(a)* **44.** *(c)* **45.** *(c)* **46.** *(d)* **47.** *(c)* **48.** *(d)* **49.** *(a)* **50.** *(c)*
51. *(c)* **52.** *(b)* **53.** *(a)* **54.** *(a)* **55.** *(c)* **56.** *(a)* **57.** *(d)* **58.** *(c)* **59.** *(c)* **60.** *(d)*
61. *(b)* **62.** *(a)* **63.** *(c)* **64.** *(c)* **65.** *(b)* **66.** *(b)* **67.** *(b)* **68.** *(b)* **69.** *(a)*

EXERCISE-2 (LEARNING PLUS)

1. *(a)* **2.** *(a)* **3.** *(c)* **4.** *(a)* **5.** *(d)* **6.** *(c)* **7.** *(c)* **8.** *(c)* **9.** *(c)* **10.** *(c)*
11. *(c)* **12.** *(b)* **13.** *(a)* **14.** *(c)* **15.** *(b)* **16.** *(d)* **17.** *(a)* **18.** *(a)* **19.** *(a)* **20.** *(b)*
21. *(c)* **22.** *(c)* **23.** *(a)* **24.** *(a)* **25.** *(b)* **26.** *(a)* **27.** *(b)* **28.** *(b)* **29.** *(b)* **30.** *(b)*
31. *(c)* **32.** *(b)* **33.** *(b)* **34.** *(d)* **35.** *(b)* **36.** *(a)* **37.** *(a)* **38.** *(c)* **39.** *(a)* **40.** *(a)*
41. *(c)* **42.** *(b)* **43.** *(c)* **44.** *(c)* **45.** *(c)* **46.** *(a)* **47.** *(a)* **48.** *(a)* **49.** *(a)* **50.** *(d)*
51. *(b)* **52.** *(b)* **53.** *(c)* **54.** *(c)* **55.** *(a,b)* **56.** *(b)* **57.** *(a)* **58.** *(a)*

EXERCISE-3 (JEE ADVANCED LEVEL)

1. *(a)* **2.** *(b)* **3.** *(d)* **4.** *(c)* **5.** *(d)* **6.** *(c)* **7.** *(d)* **8.** *(b)* **9.** *(d)* **10.** *(a,b)*
11. *(b,d)* **12.** *(a,b,c,d)* **13.** *(a,b,c,d)* **14.** *(c,d)* **15.** *(b,c)* **16.** *(b,c)* **17.** *(a,b)* **18.** *(a,b)* **19.** *(a,c)*
20. *(c,d)* **21.** *(b,d)* **22.** *(c)* **23.** *(b)* **24.** *(d)* **25.** *(c)* **26.** *(a)* **27.** *(b)* **28.** *(a)* **29.** [6]
30. [1] **31.** [4] **32.** [1/16] **33.** [3] **34.** [7] **35.** [441] **36.** [2] **37.** [4] **38.** [0] **39.** [3]
40. [2] **41.** [1] **42.** [7] **43.** [27] **44.** [7]

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. *(b)* **2.** *(c)* **3.** *(a)* **4.** *(d)* **5.** *(d)* **6.** *(b)* **7.** *(d)* **8.** *(a)* **9.** *(b)* **10.** *(d)*
11. [1] **12.** *(a)* **13.** [8] **14.** (a) **15.** *(d)* **16.** *(c)* **17.** *(b)* **18.** [1] **19.** *(d)* **20.** [11]
21. *(d)* **22.** *(a)* **23.** *(b)* **24.** *(c)* **25.** *(c)* **26.** *(c)* **27.** *(d)* **28.** [9] **29.** *(c)* **30.** *(b)*
31. *(c)* **32.** *(b)* **33.** *(b)* **34.** *(d)* **35.** *(b)*

JEE Advanced

36. *(a)* **37.** [6] **38.** [1/64] **39.** *(a, c, d)* **40.** *(b, c)* **41.** *(b, c, d)* **42.** *(a, b, d)* **43.** *(b, c, d)* **44.** [1] **45.** *(a, b)*
46. [1]

CHAPTER

5 Trigonometric Equations

SOLUTION OF TRIGONOMETRIC EQUATIONS

A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g. if $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots\ldots$

Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature) and can be classified as:

1. Principal solution
2. General solution.

PRINCIPAL SOLUTIONS

The solutions of trigonometric equation which lie in the interval $[0, 2\pi)$ are called **principal solutions.**

Example: The Principal solutions of the equation $\sin x = \frac{1}{2}$ is $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

GENERAL SOLUTION

Since trigonometric functions are periodic function, therefore solution of trigonometric equations can be generalised with the help periodicity of a trigonometrical function. The solution consisting of all possible solution of a trigonometrical equation is called the general solution.

The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called **General solution.**

Some Important General Solutions of Trigonometric Equations

1. $\sin\theta = 0 \Rightarrow \theta = n\pi, n \in I$
2. $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}, n \in I$
3. $\tan\theta = 0 \Rightarrow \theta = n\pi, n \in I$
4. $\sin\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{2}, n \in I$
5. $\cos\theta = 1 \Rightarrow \theta = 2n\pi, n \in I$
6. $\tan\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{4}, n \in I$
7. $\sin\theta = -1 \Rightarrow \theta = (4n-1)\frac{\pi}{2}, n \in I$
8. $\cos\theta = -1 \Rightarrow \theta = (2n-1)\pi, n \in I$
9. $\tan\theta = -1 \Rightarrow \theta = (4n-1)\frac{\pi}{4}, n \in I$

General Solution of Some Standard Trigonometric Equations

(*a*) If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$, where, $n \in I$.

(*b*) If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$, where, $n \in I$.

(*c*) If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$, where, $n \in I$.

(*d*) If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$, where, $n \in I$.

(*e*) If $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$, where, $n \in I$.

(*f*) If $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$, where, $n \in I$.

[**Note:** α is called the principal angle]

Proof:

(*i*) $\sin\theta = \sin\alpha \Rightarrow \sin\theta - \sin\alpha = 0$

$$\Rightarrow 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \cos\left(\frac{\theta+\alpha}{2}\right) = 0 \text{ or } \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \frac{\theta+\alpha}{2} = (2m+1)\frac{\pi}{2}$$

or $\frac{\theta-\alpha}{2} = m\pi$, where $m \in I$.

$\Rightarrow \theta = (2m+1)\pi - \alpha$

or $\theta = 2m\pi + \alpha$, where $m \in I$.

$\Rightarrow \theta = (2m+1)\pi + (-1)^{2m+1}\alpha$

or $\theta = 2m\pi + (-1)^{2m}\alpha$

$\Rightarrow \theta = n\pi + (-1)^n\alpha, n \in I$.

(*ii*) $\cos\theta = \cos\alpha$

$\Rightarrow \cos\alpha - \cos\theta = 0$

$\Rightarrow 2\sin\left(\frac{\alpha+\theta}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \sin\frac{\theta+\alpha}{2} = 0$

or $\sin\frac{\theta-\alpha}{2} = 0$, $\frac{\theta+\alpha}{2} = n\pi$ or $\frac{\theta-\alpha}{2} = n\pi$

$\Rightarrow \theta = 2n\pi - \alpha$, or $\theta = 2n\pi + \alpha$ $n \in I$

$\Rightarrow \theta = 2n\pi \pm \alpha$.

(*iii*) $\tan\theta = \tan\alpha \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

$\Rightarrow \sin\theta\cos\theta - \cos\theta.\sin\alpha = 0$

$\Rightarrow \sin(\theta-\alpha) = 0 \Rightarrow \theta - \alpha = n\pi$

$\Rightarrow \theta = n\pi + \alpha$, where $n \in I$

(*iv*) $\sin^2\theta = \sin^2\alpha$

$\sin^2\theta - \sin^2\alpha = \sin(\theta+\alpha)\sin(\theta-\alpha) = 0$

$\sin(\theta+\alpha) = 0$ or $\sin(\theta-\alpha) = 0$

$\theta + \alpha = n\pi$ or $\theta - \alpha = n\pi$, $n \in I$

$\theta = n\pi \pm \alpha$, $n \in I$

(*v*) $\cos^2\theta = \cos^2\alpha \Rightarrow 1 - \sin^2\theta = 1 - \sin^2\alpha$

$\Rightarrow \sin^2\theta = \sin^2\alpha$

$\theta = n\pi \pm \alpha$, $n \in I$

(*vi*) $\tan^2\theta = \tan^2\alpha \Rightarrow \tan\theta = \pm\tan\alpha = \tan(\pm\alpha)$

$\Rightarrow \theta = n\pi \pm \alpha$, where $n \in I$

Train Your Brain

Example 1: Solve $4\sec^2\theta = 5 + \tan^2\theta$

Sol. $4\sec^2\theta = 5 + \tan^2\theta$ ...(*i*)

For equation (*i*) to be defined $\theta \neq (2n+1)\frac{\pi}{2}$, $n \in I$

Equation (*i*) can be written as:

$4(1 + \tan^2\theta) = 5 + \tan^2\theta$

$3\tan^2\theta = \tan^2\pi/6$

$\theta = n\pi \pm \frac{\pi}{6}$, $n \in I$

Example 2: Solve for θ $\sin 2\theta = \sin\theta$

Sol. We have, $\sin 2\theta = \sin\theta$

$\Rightarrow 2\sin\theta\cos\theta = \sin\theta$

$\Rightarrow \sin\theta(2\cos\theta - 1) = 0$

$\Rightarrow \sin\theta = 0$ and $\cos\theta = \frac{1}{2}$

$\Rightarrow \theta = n\pi$ and $\theta = 2n\pi \pm \frac{\pi}{3}$, where $n \in I$.

Example 3: The general solution of the equation $\sin^2 x + \cos^2 3x = 1$ is equal to: (where $n \in I$)

(*a*) $x = \frac{n\pi}{2}$ (*b*) $x = n\pi + \frac{\pi}{4}$

(*c*) $x = \frac{n\pi}{4}$ (*d*) $x = n\pi + \frac{\pi}{2}$

Sol. $\sin^2 x = \sin^2 3x \Rightarrow 3x = n\pi \pm x$

$x = \frac{n\pi}{4}, x = \frac{n\pi}{2}$, hence general solution is $\frac{n\pi}{4}$

Example 4: Solve : $\sqrt{3}\sec 2\theta = 2$.

Sol. $\cos 2\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{6}$

$\theta = n\pi \pm \frac{\pi}{12}$, $n \in I$

Concept Application

1. Find the set of values of x for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x.\tan 2x}$

2. The general solution of the equation $\tan 2\theta\tan\theta = 1$ is

(*a*) $\theta = n\pi \pm \frac{\pi}{6}$ (*b*) $\theta = \frac{n\pi}{3} \pm \frac{\pi}{4}$

(*c*) $\theta = \frac{n\pi}{3} + \frac{\pi}{6}$ (*d*) $\theta = \frac{n\pi}{2} - \frac{\pi}{6}$

3. The number of value of x lying between 0 and 2π satisfying the equation: $\sin x + \sin 3x = 0$ is are:

(*a*) 2 (*b*) 3

(*c*) 4 (*d*) 5

SOLUTIONS OF DIFFERENT TYPES OF TRIGONOMETRIC EQUATION

Type I: Solutions of Equations by Factorising

Solution of trigonometric equation by factorization or equation which are expressed in quadratic form or which can be expressed in quadratic form.

Train Your Brain

Example 5: Solve $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$ in $[0, 2\pi]$.

Sol. $(1 + \cos x)(2\sin x - \cos x) - (1 - \cos^2 x) = 0$

$(1 + \cos x)[(2\sin x - \cos x) - (1 - \cos x)] = 0$

$(1 + \cos x)(2\sin x - 1) = 0$

$\sin x = \frac{1}{2}, \cos x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \pi.$

Example 6: Find the general solutions of equation $1 - \cos x \cot x = \cot x - \cos x$.

Sol. $(1 + \cos x)(1 - \cot x) = 0$

$\Rightarrow \cos x = -1$ or $\cot x = 1$

$\Rightarrow (2n + 1)\pi, n\pi + \pi/4, n \in I$

Example 7: Find the solutions of the equation, $\log_{\sqrt{2}\sin x}(1+\cos x) = 2$ in the interval $x \in [0, 2\pi]$.

Sol. $2\sin^2 x = 1 + \cos x$; $2\cos^2 x + \cos x - 1 = 0$

$\Rightarrow \cos x = \frac{1}{2}$ or -1

$\Rightarrow x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ but $x = \pi$ and $\frac{5\pi}{3}$ are rejected

$\Rightarrow x = \frac{\pi}{3}$

Example 8. If $\frac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in GP, then find the general solution for θ

Sol. Since, $\frac{1}{6}\sin\theta, \cos\theta, \tan\theta$ are in GP.

$\therefore \cos^2\theta = \frac{1}{6}\sin\theta.\tan\theta$

$\Rightarrow 6\cos^3\theta + \cos^2\theta - 1 = 0$

Note that $\cos\theta = \frac{1}{2}$ satisfies the equation (by trial),

$\therefore (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ (other values of $\cos\theta$ are imaginary)

$\Rightarrow \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$

Concept Application

4. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is:
   (a) 0
   (b) 2
   (c) 6
   (d) 8
5. Solve $\cos^3 x + \cos^2 x - 4\cos^2\frac{x}{2} = 0$
6. Solve for x $\frac{\sin^3\frac{x}{2} - \cos^3\frac{x}{2}}{2+\sin x} = \frac{\cos x}{3}$
7. Solve for x : $\sin^3 x - \cos^3 x = 1 + \sin x \cos x$.

Type-II: Solutions of Equations Reducible to Quadratic Equations

Rules:

1. Transform the terms to be a only one trigonometric ratio involving angles in same form.
2. Factorize the equation and express it in $f(x) \times g(x) = 0$
   $\Rightarrow f(x) = 0$ or $g(x) = 0$
3. Solve both the equations one by one to get the general value of the variables.

Train Your Brain

Example 9: Number of values of 'x' in $(-2\pi, 2\pi)$ satisfying the equation $2^{\sin^2 x} + 4.2^{\cos^2 x} = 6$ is:
(a) 8
(b) 6
(c) 4
(d) 2

Sol. (c) $2^{\sin^2 x} + 4.2^{\cos^2 x} = 6$

Let, $2^{\sin^2 x} = t$

$\therefore t + \frac{8}{t} = 6$

$\Rightarrow (t-4)(t-2) = 0$

$2^{\sin^2 x} = 4$ and $2^{\sin^2 x} = 2$

$\Rightarrow \sin x = \pm\sqrt{2}$ (reject) and $\sin x = \pm 1$

$\therefore$ in $(-2\pi, 2\pi)$ there are 4 solutions.

Example 10: Solve the equation $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x.\cos x$.

Sol. We have, $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x.\cos x$.

$\Rightarrow (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = \frac{7}{2}\sin x.\cos x$

$\Rightarrow 1 - \frac{1}{2}(\sin 2x)^2 = \frac{7}{4}(\sin 2x)$

$\Rightarrow 2\sin^2 2x + 7\sin 2x - 4 = 0$

$\Rightarrow (2\sin 2x - 1)(\sin 2x + 4) = 0$

$\Rightarrow \sin 2x = \frac{1}{2}$ or $\sin 2x$

$= -4$ (which is not possible)

$\Rightarrow 2x = n\pi + (-1)^n\frac{\pi}{6}, n \in I$

i.e., $x = \frac{n\pi}{2} + (-1)^n\frac{\pi}{12}, n \in I.$

Concept Application

8. The complete solution of the equation $7\cos^2 x + \sin x \cos x - 3 = 0$ is given by-.

(a) $n\pi + \frac{\pi}{2}; (n \in I)$

(b) $n\pi - \frac{\pi}{4}; (n \in I)$

(c) $n\pi + \tan^{-1}\frac{4}{3}; (n \in I)$

(d) $n\pi + \frac{3\pi}{4}, k\pi + \tan^{-1}\frac{4}{3}; (n, k \in I)$

9. Find the general solution set of the equation $\log_{\tan x}(2 + 4\cos^2 x) = 2$.

TYPE-III: Solving equations by introducing an auxiliary argument

A Trigonometric Equation is of the form

$a\cos\theta \pm b\sin\theta = c$

Rule:

1. Divide by $\sqrt{a^2+b^2}$ on both the sides
2. Reduce the given equation into either $\sin(\theta \pm \alpha)$ or $\cos(\theta \pm \alpha)$
3. Simplify the given equation.

Train Your Brain

Example 11: Solve $\sin x + \cos x = \sqrt{2}$

Sol. $\because \sin x + \cos x = \sqrt{2}$ ...(i)

$\therefore$ Divide both sides of equation (i) by $\sqrt{2}$,

We get $\sin x . \frac{1}{\sqrt{2}} + \cos x . \frac{1}{\sqrt{2}} = 1$

$\Rightarrow \sin x . \sin\frac{\pi}{4} + \cos x \cdot \cos\frac{\pi}{4} = 1$

$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = 1$

$\Rightarrow x - \frac{\pi}{4} = 2n\pi, n \in I$

$\Rightarrow x = 2n\pi + \frac{\pi}{4}, n \in I$

$\therefore$ Solution of given equation is $2n\pi + \frac{\pi}{4}, n \in I$

Example 12: Solve the equation $\cos 7x - \sin 5x = \sqrt{3}(\cos 5x - \sin 7x)$.

Sol. Rewrite the equation in the form

$$\frac{1}{2}\cos 7x + \frac{\sqrt{3}}{2}\sin 7x = \frac{\sqrt{3}}{2}\cos 5x + \frac{1}{2}\sin 5x$$

or $\sin\frac{\pi}{6}\cos 7x + \cos\frac{\pi}{6}\sin 7x = \sin\frac{\pi}{3}\cos 5x + \cos\frac{\pi}{3}\sin 5x$,

i.e. $\sin\left(\frac{\pi}{6} + 7x\right) = \sin\left(\frac{\pi}{3} + 5x\right)$.

But $\sin\alpha = \sin\beta$ if and only if either $\alpha - \beta = 2k\pi$ or $\alpha + \beta = (2m+1)\pi$ $(k, m = 0, \pm 1, \pm 2, ...)$.

Hence $\frac{\pi}{6} + 7x - \frac{\pi}{3} - 5x = 2k\pi$

or $\frac{\pi}{6} + 7x + \frac{\pi}{3} + 5x = (2m+1)\pi$.

Thus, the roots for the equation are

$$\left.\begin{aligned} x &= \frac{\pi}{12}(12k+1), \\ x &= \frac{\pi}{24}(4m+1) \end{aligned}\right\} (k, m = 0, \pm 1, \pm 2, ...).$$

Concept Application

10. Prove that the equation $k\cos x - 3\sin x = k+1$ possesses a solution if $k \in (-\infty, 4]$

11. The number of solution of the equation $\sqrt{3}\sin x + \cos x = 1$ in the interval $0 \le x \le 2\pi$

(a) 3 (b) 2

(c) 4 (d) None of these

12. Solve $\sqrt{3}\cos\theta - 3\sin\theta = 4\sin 2\theta\cos 3\theta$

TYPE-IV:

Solving equations by transforming a sum of trigonometric functions into a product

Train Your Brain

Example 13: General solution of $\sin x + \sin 5x = \sin 2x + \sin 4x$ is

(a) $\frac{n\pi}{3}$ (b) $\frac{2n\pi}{3}$

(c) $2n\pi$ (d) $n\pi$

Sol. $2\sin 3x\cos 2x = 2\sin 3x\cos x$

$\Rightarrow \sin 3x = 0$ or $\cos 2x = \cos x$

$\Rightarrow 3x = n\pi$ or $2x = 2n\pi \pm x$

$\therefore x = \frac{n\pi}{3}, 2n\pi, \frac{2n\pi}{3} \Rightarrow x = \frac{n\pi}{3}$

Example 14: Solve $\cos 3x + \sin 2x - \sin 4x = 0$

Sol. $\cos 3x + \sin 2x - \sin 4x = 0$

$\Rightarrow \cos 3x + 2\cos 3x \cdot \sin(-x) = 0$

$\Rightarrow \cos 3x - 2\cos 3x \cdot \sin x = 0$

$\Rightarrow \cos 3x(1 - 2\sin x) = 0$

$\Rightarrow \cos 3x = 0$ or $1 - 2\sin x = 0$

$\Rightarrow 3x = (2n+1)\dfrac{\pi}{2}, n \in I$ or $\sin x = \dfrac{1}{2}$

$\Rightarrow x = (2n+1)\dfrac{\pi}{6}, n \in I$ or $x = n\pi + (-1)^n \dfrac{\pi}{6}, n \in I$

$\therefore$ Solution of given equation is $(2n+1)\dfrac{\pi}{6}, n \in I$

or $n\pi + (-1)^n \dfrac{\pi}{6}, n \in I$

Example 15: Find the number of solutions of the equations, $\sin x + 2\sin 2x = 3 + \sin 3x$ in $[0, \pi]$

(a) No solution

(b) Infinite solution

(c) Exactly one solution

(d) Two solutions

Sol. We have,

$\Rightarrow \sin 3x - \sin x - 2\sin 2x + 3 = 0$

$\Rightarrow 2\cos 2x \cdot \sin x - 4\sin x \cdot \cos x + 3 = 0$

$\Rightarrow \sin x(2\cos 2x - 4\cos x) + 3 = 0$

$\Rightarrow \sin x(4\cos^2 x - 4\cos x - 2) + 3 = 0$

$\Rightarrow \sin x(4\cos^2 x - 4\cos x + 1) + 3 - 3\sin x = 0$

$\Rightarrow \sin x(2\cos x - 1)^2 + 3(1 - \sin x) = 0$ ...(i)

since $x \in [0, \pi]$,

$\therefore \sin x \geq 0$ and $1 - \sin x \geq 0$

$\therefore$ each part in equation (i) must be zero.

i.e. $\sin x(2\cos x - 1)^2 = 0$ and $3(1 - \sin x) = 0$

from the second equation of system we have, $\sin x = 1$

$\Rightarrow \cos x = 0$ hence $\sin x(2\cos^2 x - 1)^2 \neq 0$

$\therefore$ not a single solution of the second equation is a solution of the first.

Hence the original equation has no real solution.

Concept Application

13. The number of solution of $\cos x + \cos 2x + \cos 4x = 0$, where $0 \leq x \leq \pi$.

(a) 2 (b) 3

(c) 4 (d) None of these

14. Solve $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$

TYPE-V:

Solving equations by transforming a product of trigonometric functions into a sum

Train Your Brain

Example 16: Solve $\cos x \cos 2x \cos 3x = 1/4$.

Sol. $\cos x \cos 2x \cos 3x = 1/4$

$\Rightarrow 2(2\cos x \cos 3x)\cos 2x = 1$

$\Rightarrow 2(\cos 4x + \cos 2x)\cos 2x = 1$

$\Rightarrow 2(2\cos^2 2x - 1 + \cos 2x)\cos 2x = 1$

$\Rightarrow 4\cos^3 2x + 2\cos^2 2x - 2\cos 2x - 1 = 0$

$\Rightarrow (2\cos^2 2x - 1)(2\cos 2x + 1) = 0$

$\Rightarrow \cos 4x(2\cos 2x + 1) = 0$

$\Rightarrow \cos 4x = 0$ or $\cos 2x = -1/2$

$\Rightarrow 4x = (2n+1)\dfrac{\pi}{2}$ or $2x = 2m\pi \pm \dfrac{2\pi}{3} \cdot m, n \in Z$

$\Rightarrow x = (2n+1)\dfrac{\pi}{8}$ or $x = m\pi \pm \dfrac{\pi}{3}$

Example 17: Number of solutions of the trigonometric equation in $[0, \pi]$, $\sin 3\theta = 4\sin\theta \cdot \sin 2\theta \cdot \sin 4\theta$.

(a) 4 (b) 6

(c) 8 (d) 10

Sol. $\sin 3\theta = 4\sin\theta \sin(3\theta - \theta)\sin(3\theta + \theta)$

$= 4\sin\theta(\sin^2 3\theta - \sin^2\theta)$

$\Rightarrow \sin 3\theta + 4\sin^3\theta = 4\sin\theta \sin^2 3\theta$

$\Rightarrow 3\sin\theta = 4\sin\theta \sin^2 3\theta$

$\Rightarrow \sin\theta = 0$ or $\sin^2 3\theta = \dfrac{3}{4}$

$\Rightarrow \theta = n\pi$ or $3\theta = n\pi \pm \dfrac{\pi}{3}$

$\Rightarrow \theta = n\pi$ or $\theta = \dfrac{n\pi}{3} \pm \dfrac{\pi}{9}, n \in I$

$\Rightarrow \theta = 0, \pi, \dfrac{\pi}{9}, \dfrac{2\pi}{9}, \dfrac{4\pi}{9}, \dfrac{5\pi}{9}, \dfrac{7\pi}{9}, \dfrac{8\pi}{9}$

Concept Application

15. Find the number of solutions for,

$\sin 5\theta . \cos 3\theta = \sin 9\theta . \cos 7\theta$ in $\left[0, \dfrac{\pi}{2}\right]$

16. Solve $\sin 3\alpha = 4\sin\alpha \sin(x + \alpha)\sin(x - \alpha)$.

TYPE-VI:

Solving equations by a change of variable

(i) Equations of the form

$P(\sin x \pm \cos x, \sin x.\cos x) = 0$,

where $P(y, z)$ is a polynomial, can be solved by the change $\cos x \pm \sin x = t$

$\Rightarrow 1 \pm 2\sin x \cdot \cos x = t^2$

(ii) Equations of the form of $a.\sin x + b.\cos x + d = 0$, where a, b & d are real numbers & $a, b \neq 0$ can be solved by changing $\sin x$ & $\cos x$ into their corresponding tangent of half the angle.

(iii) Many equations can be solved by introducing a new variable. eg. the equation

$\sin^4 2x + \cos^4 2x = \sin 2x \cdot \cos 2x$ changes to

$2(y+1)\left(y-\frac{1}{2}\right) = 0$ by substituting,

$\sin 2x \cdot \cos 2x = y$.

Train Your Brain

Example 18: Solve $3\cos x + 4\sin x = 5$

Sol. $\because 3\cos x + 4\sin x = 5$ ...(i)

$$\because \cos x = \frac{1-\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}} \text{ and } \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$\therefore$ equation (i) becomes

$$\Rightarrow 3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)=5 \quad ...(ii)$$

Let $\tan\frac{x}{2} = t$ $\therefore$ equation (ii) becomes

$$\left(\frac{1-t^2}{1+t^2}\right)+4\left(\frac{2t}{1+t^2}\right)=5 \Rightarrow 4t^2-4t+1=0$$

$$\Rightarrow (2t-1)^2 = 0 \Rightarrow t = \frac{1}{2}\left(\because t = \tan\frac{x}{2}\right)$$

$$\Rightarrow \tan\frac{x}{2} = \frac{1}{2} \Rightarrow \tan\frac{x}{2} = \tan\alpha, \text{ where } \tan\alpha = \frac{1}{2}$$

$$\Rightarrow \tan\frac{x}{2} = n\pi + a \Rightarrow x = 2n\pi + 2\alpha$$

Where, $\alpha = \tan^{-1}\left(\frac{1}{2}\right), n \in I$

Example 19: Solve the equation
$\sin 2x - 12(\sin x - \cos x) + 12 = 0$

Sol. Putting $\sin x - \cos x = t$ and using the identity $(\sin x - \cos x)^2 = 1 - 2\sin x\cos x$, we rewrite the original equation in the form $t^2 + 12t - 13 = 0$.

This equation has the roots $t_1 = -13$ and $t_2 = 1$. But $t = \sin x - \cos x = \sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$, and thus, $|t| \leq \sqrt{2}$. Consequently, the root $t_1 = -13$ must be discarded. Therefore, the original equation is reduced to the equation $\sin\left(x-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

$$\Rightarrow x_1 = \pi + 2k\pi, x_2 = \frac{\pi}{2} + 2k\pi.$$

Concept Application

17. Solve the equation

$$\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x$$

18. Solve the equation $1 + 2\,\text{cosec}\, x = -\dfrac{\sec^2\frac{x}{2}}{2}$.

TYPE-VII:

Solving equations with the use of the Boundness of the functions $\sin x$ & $\cos x$

Remember:

$-1 \leq \sin x \leq 1, -1 \leq \cos x \leq 1$, $\tan x \in R$, $\cot x \in R$.

$|\text{cosec}\, x| \geq 1$, $|\sec x| \geq 1$.

Train Your Brain

Example 20: Solve for x : $\cos x + \cos 2x + \cos 3x = 3$.

Sol. $\cos x = 1$ and $\cos 2x = 1$ and $\cos 3x = 1$

when $\cos x = 1$

$\Rightarrow x = 2n\pi, n \in I$

when $\cos 2x = 1$

$\Rightarrow x = \frac{2n\pi}{2} = n\pi, n \in I$

when $\cos 3x = 1$

$\Rightarrow 3x = 2n\pi \Rightarrow x = \frac{2n\pi}{3}, n \in I$

$2n\pi, n \in I$

Example 21: Solve for

$x : \sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right)\cos x = 0$

Sol. $\sin\left(\frac{5x}{4}\right) + \cos x = 2$

$\Rightarrow \sin\left(\frac{5x}{4}\right) = 1 \Rightarrow \frac{5x}{4} = (4n+1)\frac{\pi}{2}$

$\Rightarrow x = (4n+1)\frac{2\pi}{5}$ and $\cos x = 1$

$x = 2m\pi$; $x = 0, \pm 2\pi, \pm 4\pi$

Period of the given equation is 8π.

$\therefore$ Consider $x \in [0, 8\pi)$

$\Rightarrow x = \frac{2\pi}{5}, 2\pi, \frac{18\pi}{5}, \frac{26\pi}{5}, \frac{34\pi}{5}$

Common solution $= 2\pi$

$\therefore 8n\pi + 2\pi = 2\pi(4n+1)$

Example 22: If $x, y \in [0, 2\pi]$ then find the total number of order pairs (x, y) satisfying the equation $\sin x \cos y = 1$.

Sol. We have $\sin x \cos y = 1$

$\Rightarrow \sin x = 1, \cos y = 1$ or $\sin x = -1, \cos y = -1$

If $\sin x = 1, \cos y = 1 \Rightarrow x = \pi/2, y = 0, 2\pi$

If $\sin x = -1, \cos y = -1 \Rightarrow x = 3\pi/2, y = \pi$

Then the possible ordered pairs are

$\left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 2\pi\right), \left(\frac{3\pi}{2}, \pi\right)$.

Concept Application

19. If $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$, then the greatest positive solution of $1 + \sin^4 x = \cos^2 3x$ is

(a) π (b) 2π

(c) $\frac{5\pi}{2}$ (d) None of these

20. Solve $2\cos^2 \frac{x}{2} \cdot \sin^2 x = x^2 + \frac{1}{x^2} 0 < x \le \frac{\pi}{2}$.

21. Solve the equation $\frac{1 - \tan x}{1 + \tan x} = 1 + \sin 2x$.

TYPE-VIII:

Solution of trigonometric equation of the form $f(x) = \sqrt{\phi(x)}$.

(i) $f(x) \ge 0, \phi(x) \ge 0$ (ii) $f^2(x) = \phi(x)$

MISCELLANEOUS QUESTIONS

A trigonometric equation is of the form

$b_0\sin^n x + b_1\sin^{n-1} x.\cos x +$
$b_2\sin^{n-2} x.\cos^2 x + \ldots\ldots.. + b_n\cos^n x = 0$

where, $b_0, b_1, b_2, \ldots.., b_n \in R$ is a homogenous equation of $\sin x$ and $\cos x$, where $\cos x$ is non zero.

Rule: 1. Divide both the sides by highest power of $\cos x$.

2. The given equation can be reduced to
$b_0\tan^n x + b_1\tan^{n-1} x + \ldots.. + b_n = 0$

3. and then use the factorization method.

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Example 23: Solve for x, $\sqrt{(1-\cos x)} = \sin x$.

Sol. $\sin x \ge 0$ and $1 - \cos x \ge 0$ ...(i)

$\therefore$ $1 - \cos x = \sin^2 x = 1 - \cos^2 x$

$\Rightarrow$ $(1 - \cos x)[1 - (1 + \cos x)] = 0$

$\Rightarrow$ $\cos x = 0, 1$

when $\cos x = 0, x = 2n\pi \pm \frac{\pi}{2}$ but $\sin x \ge 0$

$\Rightarrow$ $x = 2n\pi + \frac{\pi}{2}$

when $\cos x = 1, x = 2n\pi$

Both are satisfying in equality (i)

Example 24: Solve : $2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -2$

Sol. The given equation can be written as

$2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -2(\sin^2 x + \cos^2 x)$

$\Rightarrow 2\tan^2 x - 5\tan x - 8 = -2(\tan^2 x + 1)$

$\Rightarrow 4\tan^2 x - 5\tan x - 6 = 0 \Rightarrow (\tan x - 2)(4\tan x + 3) = 0$

$\Rightarrow \tan x = 2, \tan x = -\frac{3}{4} \Rightarrow x = n\pi + \alpha, x = m\pi + \beta,$

where $\alpha = \tan^{-1}(2), \beta = \tan^{-1}\left(-\frac{3}{4}\right), (n, m \in Z)$.

Concept Application

22. Values of x between 0 and 2π which satisfy the equation $\sin x\sqrt{8\cos^2 x} = 1$ are in A.P. whose common difference is:

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

23. Solve $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$

TRIGONOMETRIC INEQUALITIES AND SYSTEM OF INEQUALITY

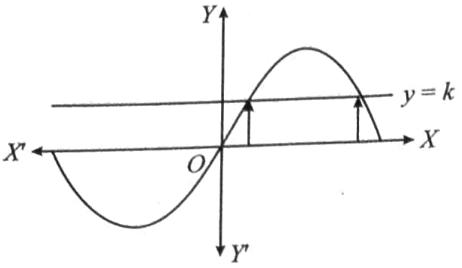
To solve the trigonometric inequalities of the type $f(x) \le a$ or $f(x) \ge a$, where $f(x)$ is some trigonometric ratio we take following steps

1. Draw the graph of $f(x)$ in an interval length equal to the fundamental period of $f(x)$.
2. Draw the line $y = a$.
3. Take the portion of the graph for which the inequality is satisfied.
4. To generalise, add $pn, n \in I$ and in the final solution where p is the fundamental period of $f(x)$.

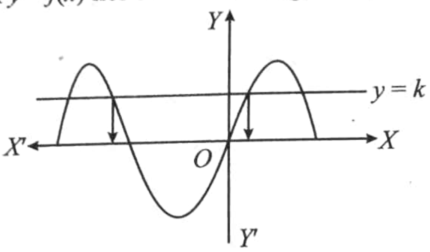
Suppose we have to solve $f(x) > k$ or $f(x) < k$.

When we solve the inequation, we often use the graphs of the functions $y = f(x)$ and $y = k$.

Then, the solution of the inequality $f(x) > k$ is the values of x for which the point $(x, f(x))$ of the graph of $y = f(x)$ lies above the straight line $y = k$.



Similarly, when we solve $f(x) < k$, then the solution of the inequalition $f(x) < k$ is the values of x for which the point $(x, f(x))$ of the graph of $y = f(x)$ lies below the straight line $y = k$.



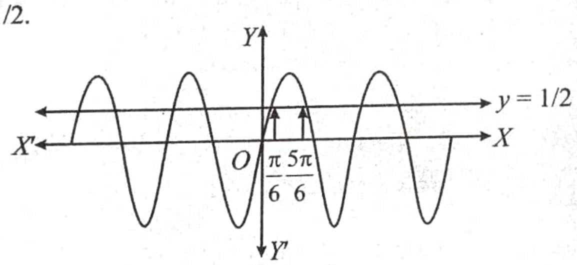
❖ An inequation is of the form $\sin x > k$, $\cos x < k$, $\cos x > k$, $\tan x > k$, $\tan x < k$.

Rule: Find the smallest values of x that satisfies the given inequation and then add $2n\pi$ with that values of x.

Train Your Brain

Example 25: Solve $\sin x > 1/2$.

Sol. Here, we should construct the graph of $y = \sin x$ and $y = 1/2$.

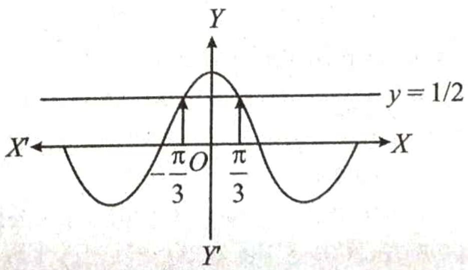


Hence, the solution set is

$$x = \underset{n \in I}{U}\left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$$

Example 26: Solve : $\cos x \geq \dfrac{1}{2}$.

Sol. Here, we should draw the graph of $y = \cos x$ and $y = 1/2$



Hence, the solution set is

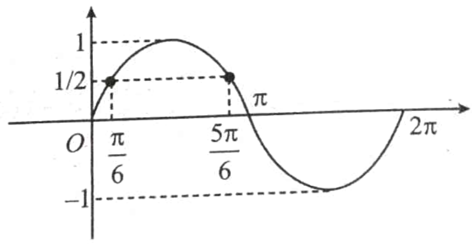
$$x = \underset{n \in I}{U}\left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right)$$

Example 27: Solve : $2\sin^2\theta - \sin\theta \geq 0$, where $\dfrac{\pi}{2} \leq \theta \leq \dfrac{3\pi}{2}$.

Sol. $\sin\theta\,(2\sin\theta - 1) \geq 0$ which is possible. only where

$$\sin\theta \geq \frac{1}{2} \quad \text{or} \quad \sin\theta \leq 0$$

$$\sin\theta \geq \frac{1}{2} \Rightarrow \frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$$



$$\sin\theta \leq 0 \Rightarrow \pi \leq \theta \leq \frac{3\pi}{2}$$

$$\theta \in \left[\frac{\pi}{2}, \frac{5\pi}{6}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$$

Example 28: Solve: $\sin\theta + \sqrt{3}\cos\theta \geq 1$, $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$

Sol. $\dfrac{1}{2}\sin\theta + \dfrac{\sqrt{3}}{2}\cos\theta \geq \dfrac{1}{2}$

$$\sin\left(\theta + \frac{\pi}{3}\right) \geq \frac{1}{2}$$

$$\frac{\pi}{6} \leq \theta + \frac{\pi}{3} \leq \frac{5\pi}{6} \Rightarrow -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$$

Concept Application

24. Find the value of x in the interval $\left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$ for which $\sqrt{2}\sin 2x + 1 \leq 2\sin x + \sqrt{2}\cos x$

25. Solve $\sin 2x > \sqrt{2}\sin^2 x + \left(2 - \sqrt{2}\right)\cos^2 x$

26. Solve : $\tan x \geq \dfrac{1}{\sqrt{3}}$

27. Solve for x:
   (i) $\sin x > 0$, $x \in (\pi, 2\pi)$
   (ii) $\sin x > 1/2$; $x \in (\pi, 2\pi)$
   (iii) $\sin x > 1/2$, $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$
   (iv) $\cos x < \dfrac{-1}{2}$; $x \in (0, 2\pi)$
   (v) $\tan x > 0$

IMPORTANT POINTS

1. Many trigonometrical equations can be solved by different methods. The form of solution obtained in different methods may be different. From these different forms of solutions, the students should not think that the answer obtained by one method are wrong and those obtained by another method are correct. The solutions obtained by different methods may be shown to be equivalent by some supplementary transformations.

   To test the equivalence of two solutions obtained from two methods, the simplest way is to put values of $n = ...-2, -1, 0, 1, 2, 3.......$ etc. and then to find the angles in $[0, 2\pi]$. If all the angles in both solutions are same, the solutions are equivalent.

2. While manipulating the trigonometrical equation, avoid the danger of losing roots. Generally, some roots are lost by cancelling a common factor from the two sides of an equation. For example, suppose we have the equation $\tan x = 2 \sin x$. Here by dividing both sides by $\sin x$, we get $\cos x = \frac{1}{2}$. This is not equivalent to the original equation.

   Here the roots obtained by $\sin x = 0$, are lost. Thus in place of dividing an equation by a common factor, the students are advised to take this factor out as a common factor from all terms of the equation.

3. While equating one of the factors to zero, take care of the other factor that it should not become infinite. For example, if we have the equation $\sin x = 0$, which can be written as $\cos x \tan x = 0$. Here we cannot put $\cos x = 0$, since for $\cos x = 0$, $\tan x = \sin x/ \cos x$ is infinite.

4. **Avoid squaring:** When we square both sides of an equation, some extraneous roots appear. Hence it is necessary to check all the solutions found by substituting them in the given equation and omit the solutions not satisfying the given equation.

   For example: Consider the equation,

   $\sin \theta + \cos \theta = 1$ ...(*i*)

   Squaring we get

   $1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$

   i.e. $2\theta = n\pi$ or $\theta = n\pi/2$, ...(*ii*)

   This gives $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, ...$

   Verification shows that π and $\frac{3\pi}{2}$ do not satisfy the equation

   as $\sin \pi + \cos\pi = -1, \neq 1$

   and $\sin \frac{3\pi}{2} + \cos \frac{3\pi}{2} = -1, \neq 1$.

   The reason for this is simple.

   The equation (*ii*) is not equivalent to (*i*) and (*ii*) contains two equations : $\sin \theta + \cos \theta = 1$ and $\sin\theta + \cos \theta = -1$. Therefore we get extra solutions.

   Thus if squaring is must, verify each of the solution.

5. **Some necessary restrictions:** If the equation involves $\tan x$, $\sec x$, take $\cos x \neq 0$. If $\cot x$ or $\operatorname{cosec} x$ appear, take $\sin x \neq 0$. If log appear in the equation, i.e. $\log [f(\theta)]$ appear in the equation, use $f(\theta) > 0$ and base of $\log > 0, \neq 1$.

   Also note that $\sqrt{[f(\theta)]}$ is always positive, for example $\sqrt{\sin^2\theta} = |\sin \theta|$, not $\pm \sin \theta$.

6. **Verification :** Student are advice to check whether all the roots obtained by them satisfy the equation and lie in the

Short Notes

1. **Solution of Trigonometric Equation:** A solution of a trigonometric equation is the value of an unknown angle that satisfies the equation.

   Thus, the trigonometric equation may has infinite number of solutions and are classified as:

   Principal Solution: The solution of the trigonometric equation lying in the interval $[0, 2\pi)$.

   General Solution: Since all the trigonometric functions are many one & periodic, hence there are infinite values for which trigonometric functions have the same value. All such possible values are given by a general formula. Such general formula is called a general solution of the trigonometric equation.

2. **General Solutions of Some Trigonometric Equations (To be Remembered)**

   (*a*) If $\sin\theta = 0$ then $\theta = n\pi, n \in I$ (set of integers)

   (*b*) If $\cos\theta = 0$ then $\theta = (2n + 1)\pi/2, n \in I$

   (*c*) If $\tan \theta = 0$ then $\theta = n\pi, n \in I$

   (*d*) If $\sin \theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], n \in I$

   (*e*) If $\cos \theta = \cos\alpha$ then $\theta = 2n\pi + \alpha, n \in I, \alpha \in [0, \pi]$

   (*f*) If $\tan \theta = \tan\alpha$ then $\theta = n\pi + \alpha, n \in I, \alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

   (*g*) If $\sin \theta = 1$ then $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}, n \in I$

   (*h*) If $\cos \theta = 1$ then $\theta = 2n\pi, n \in I$

   (*i*) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$ then $\theta = n\pi \pm \alpha, n \in I$

   (*j*) For $n \in I$, $\sin n\pi = 0$

   $\sin(n\pi + \theta) = (-1)^n \sin\theta$

   (*k*) $\cos n\pi = (-1)^n, n \in I$

   $\cos(n\pi + \theta) = (-1)^n \cos\theta$

3. Solution of Different Types of Trigonometric Equations

(*i*) General solution of equation $a\cos\theta + b\sin\theta = c$

Consider $a\sin\theta + b\cos\theta = c$ ...(*i*)

$$\therefore \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

Equation (*i*) has the solution only if $|c| \le \sqrt{a^2+b^2}$

Let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi, \phi = \tan^{-1}\frac{b}{a}$

by introducing this auxiliary argument ϕ, equation (*i*) reduces to $\sin(\theta+\phi) = \frac{c}{\sqrt{a^2+b^2}}$

Now this equation can be solved easily.

(*ii*) **General Solution of Equation of Form**

$a_0\sin^n x + a_1\sin^{n-1}x\cos x + a_2\sin^{n-2}x\cos^2 x + ... + a_n\cos^n x = 0$

$a_0, a_1, ..., a_n$ are real numbers.

Such an equation is solved by dividing equation by $\cos^n x$

(*iii*) **Solving Equations using Boundedness:**

(*a*) Trigonometric equations involving a single variable

Step 1: When LHS and RHS of equation have their ranges say R_1 and R_2 in common domain D and $R_1 \cap R_2 = \phi$, then the equation has no solution.

Step 2: $R_1 \cap R_2$ have finitely many elements and the number of elements are very few, then the individual cases can be analyzed and solved.

(*b*) Trigonometric equations involving more than one variable. To solve an equation involving more than one variable, definite solutions can be obtained if the extreme values (range) of the functions are used.

(*iv*) **Solving equation by changing the variable or by substitution method:**

(*a*) Equation of the form $P(\sin x \pm \cos x, \sin x. \cos x) = 0$, when $P(y, z)$ is a polynomial, can be solved by the substitution: $\cos x \pm \sin x = t$.

(*b*) Equation of the form of $a\sin x + b\cos x + d = 0$, where a, b & d are the real numbers that can be solved by changing $\sin x$ & $\cos x$ into their corresponding tangent of the half-angle.

(*c*) Many equations can be solved by introducing a new variable.

(*v*) **Solution of Trigonometric Equation using Graphs**

Solution of $f(x) - g(x) = 0$ can be find out by application of following steps:

Step 1: Write the equation $f(x) = g(x)$.

Step 2: Draw the graph of $y = f(x)$ and $y = g(x)$ on same $x - y$ plane.

Step 3: The number of points of intersection of $f(x)$ and $g(x)$ are same as the number of solutions of $f(x) - g(x) = 0$

(*vi*) **Solving a System of Trigonometric Equations**

To solve the simultaneous equations in one variable, say x, we observe the following steps

Step 1: Find the values of x satisfying both the equations individually and lying in $[0, 2\pi]$

Step 2: Select the values satisfying both the equations simultaneously.

Step 3: Generalize the values to get the general solution.

(*vii*) **Trigonometric Inequalities:** To solve the trigonometric inequalities of the type $f(x) \le a$ or $f(x) \ge a$ where $f(x)$ is some trigonometric equation, we take the following steps

Step 1: Draw the graph of $f(x)$ in an interval length equal to the fundamental period of $f(x)$.

Step 2: Draw the line $y = a$.

Step 3: Take the portion of the graph for which the inequality is satisfied.

Step 4: To generalize, add $p \times n, n \in I$ in the final solution where p is the fundamental period of $f(x)$.

Solved Examples

1. If $\sin(x-y) = \cos(x+y) = 1/2$ then the values of x & y lying between 0 and π are given by:

(*a*) $x = \pi/4, y = 3\pi/4$

(*b*) $x = \pi/4, y = \pi/12$

(*c*) $x = 5\pi/4, y = 5\pi/12$

(*d*) $x = 11\pi/12, y = 3\pi/4$

Sol. (*b, d*) $\sin(x-y) = \frac{1}{2}$ and $\cos(x+y) = \frac{1}{2}$

$\Rightarrow x - y = \frac{\pi}{6}, \frac{5\pi}{6}$ and $x + y = \frac{\pi}{3}, \frac{5\pi}{3}$

Adding $2x = \frac{\pi}{2}$ or $\frac{7\pi}{6}$ or $\frac{11\pi}{6} \Rightarrow x = \frac{\pi}{4}$ or $\frac{7\pi}{12}$ or $\frac{11\pi}{12}$

when $x = \frac{\pi}{4}, y = \frac{\pi}{12}$

when $x = \frac{7\pi}{12}$ no value of y is possible.

when $x = \frac{11\pi}{12}, y = \frac{3\pi}{4}$.

2. $\cos 15x = \sin 5x$ if

(*a*) $x = -\frac{\pi}{20} + \frac{n\pi}{5}, n \in I$

(*b*) $x = \frac{\pi}{40} + \frac{n\pi}{10}, n \in I$

(c) $x = \frac{3\pi}{20} + \frac{n\pi}{5}, n \in I$

(d) $x = -\frac{3\pi}{40} + \frac{n\pi}{10}, n \in I$

Sol. (a, b, c, d)

$\cos 15x = \sin 5x$

$\cos 15x = \cos\left(\frac{\pi}{2} - 5x\right)$ or $\cos\left(\frac{3\pi}{2} + 5x\right)$

$15x = 2n\pi \pm \left(\frac{\pi}{2} - 5x\right)$ or $15x = 2n\pi \pm \left(\frac{3\pi}{2} + 5x\right)$

$\Rightarrow x = \frac{n\pi}{10} + \frac{\pi}{40}, n \in I, x = \frac{n\pi}{5} + \frac{3\pi}{20}, n \in I$

and $x = \frac{n\pi}{5} - \frac{\pi}{20}, n \in I$ and $x = \frac{n\pi}{10} - \frac{3\pi}{20}, n \in \mathrm{I}$

3. $\sin^2 x + 2 \sin x \cos x - 3\cos^2 x = 0$ if

(a) $\tan x = 3$

(b) $\tan x = -1$

(c) $x = n\pi + \pi/4, n \in I$

(d) $x = n\pi + \tan^{-1}(-3), n \in I$

Sol. (c, d)

$\sin^2 x + 2 \sin x \cos x - 3 \cos^2 x = 0$

Case-I : $\cos x \neq 0 \quad \therefore \tan^2 x + 2 \tan x - 3 = 0$

$\Rightarrow \tan x = 3, 1 \Rightarrow x = n\pi + \tan^{-1}(-3), n\pi + \frac{\pi}{4}$

Case-II : $\cos x = 0 \Rightarrow 1 + 0 - 0 = 0$ not true.

4. $\sin x - \cos^2 x - 1$ assumes the least value for the set of values of x given by:

(a) $x = n\pi + (-1)^{n+1}(\pi/6), n \in I$

(b) $x = n\pi + (-1)^n(\pi/6), n \in I$

(c) $x = n\pi + (-1)^n(\pi/3), n \in I$

(d) $x = n\pi - (-1)^n(\pi/6), n \in I$

Sol. (a, b)

Let $E = \sin x - \cos^2 x - 1$

$\Rightarrow E = \sin x - 1 + \sin^2 x - 1 = \sin^2 x + \sin x - 2$

$= \left(\sin x + \frac{1}{2}\right)^2 - \frac{9}{4}$ assumes least value

when $\sin x = -\frac{1}{2} \Rightarrow x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$.

5. $\sin x, \sin 2x, \sin 3x$ are in A.P if

(a) $x = n\pi/2, n \in I$ (b) $x = n\pi, n \in I$

(c) $x = 2n\pi, n \in I$ (d) $x = (2n+1)\pi, n \in I$

Sol. (b, c, d)

$2\sin 2x = \sin x + \sin 3x$

$\Rightarrow 2\sin 2x = 2\sin 2x \cos x \Rightarrow \sin 2x = 0$ or $\cos x = 1$

$\Rightarrow 2x = n\pi$ or $x = 2m\pi$

$\Rightarrow x = \frac{n\pi}{2}, 2m\pi$

options (a), (b), (c), (d) are all a part of $x = \frac{n\pi}{2}$.

Comprehension – 1 (No. 6 to 8)

Let $a, b, c, d \in R$. Then the cubic equation of the type $ax^3 + bx^2 + cx + d = 0$ has either one root real or all three roots are real. But in case of trigonometric equations of the type $a \sin^3 x + b \sin^2 x + c \sin x + d = 0$ can possess several solutions depending upon the domain of x.

To solve an equation of the type $a \cos\theta + b \sin\theta = c$. The equation can be written as $\cos(\theta - \alpha) = c/\sqrt{(a^2 + b^2)}$.

The solution is $\theta = 2n\pi + \alpha \pm \beta$,

where $\tan \alpha = b/a$, $\cos \beta = c/\sqrt{(a^2 + b^2)}$.

6. On the domain $[-\pi, \pi]$ the equation $4\sin^3 x + 2 \sin^2 x - 2\sin x - 1 = 0$ possess

(a) only one real root (b) three real roots

(c) four real roots (d) six real roots

Sol. (d) $4\sin^3 x + 2 \sin^2 x - 2\sin x - 1$

$= (2\sin x + 1)(2\sin^2 x - 1) = 0$

$\sin x = -\frac{1}{2}, \pm \frac{1}{\sqrt{2}}$

$\therefore$ there are 6 solutions.

7. In the interval $[-\pi/4, \pi/2]$, the equation,

$\cos 4x + \frac{10 \tan x}{1 + \tan^2 x} = 3$ has

(a) no solution (b) one solution

(c) two solutions (d) three solutions

Sol. (c) $3 = \cos 4x + \frac{10 \tan x}{1 + \tan^2 x} = \cos 4x + 5 \sin 2x$

i.e $3 = 1 - 2 \sin^2 2x + 5 \sin 2x$

i.e $\sin 2x = \frac{1}{2} \quad \therefore 2x = \frac{\pi}{6}, \frac{5\pi}{6}$

Thus there are two solutions.

8. $|\tan x| = \tan x + \frac{1}{\cos x}$ $(0 \le x \le 2\pi)$ has

(a) no solution (b) one solution

(c) two solutions (d) three solutions

Sol. (b) (i) when $\tan x \ge 0$, then the equation becomes

$\tan x = \tan x + \frac{1}{\cos x}$ i.e $\frac{1}{\cos x} = 0$ (not possible)

(ii) when $\tan x < 0$, then the equation becomes $-\tan x = \tan x + \frac{1}{\cos x}$ i.e $\sin x = -\frac{1}{2}$

$\therefore x = \frac{11\pi}{6}$ is the only solution.

9. Match the following for number of solutions in $[0, 2\pi]$

Column - I	**Column - II**
(a) $\sin^2\theta - \tan^2\theta = 1$	(p) 2
(b) $\sin\theta + \cos\theta = 1$	(q) 0
(c) $\tan\theta + \sec\theta = 2\cos\theta$	(r) 3
(d) $3\sin^2\theta - 4\sin\theta + 1 = 0$	(s) 1

Sol. $(a) \to (q), (b) \to (r), (c) \to (p), (d) \to (r)$

(a) $\sin^2\theta = \sec^2\theta$

$\Rightarrow \sin^2\theta\cos^2\theta = 1$

$\Rightarrow \sin^2\theta(1 - \sin^2\theta) = 1$

$\Rightarrow \sin^4\theta - \sin^2\theta + 1 = 0$

Let $\sin\theta = t$

$\therefore\ t^2 - t + 1 = 0\ ;\ D = 1 - 4 = -3$

Since $D < 0$, there is no real solution

(b) $\sin\theta + \cos\theta = 1$

If $\sin\theta = 0$, then $\cos\theta = 1$

$\therefore$ In $[0, 2\pi]$, Number of solution $= 2$

and if $\sin\theta = 1$ then $\cos\theta = 0$

$\therefore$ In $[0, 2\pi]$, Number of solution $= 1$

Total number of solution $= 3$

(c) $\tan\theta + \sec\theta = 2\cos\theta$

$\Rightarrow \dfrac{1}{\cos\theta}[\sin\theta + 1] = 2\cos\theta$

$\Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \sin\theta = \dfrac{-1\pm 3}{4}$

$\therefore$ 2 real solutions

(d) $3\sin^2\theta - 4\sin\theta + 1 = 0$

If $\sin\theta = 1$, In $[0, 2\pi]$ one solution

& $\sin\theta = \dfrac{1}{3}$, in $[0, 2\pi]$, there are 2 solutions

Total number of solution $= 3$

10. The number of solution of the equation $1 + \sin x\sin^2 x/2 = 0$ in $[-\pi, \pi]$ is

Sol. [0000]

$$1+\sin x\left(\frac{1-\cos x}{2}\right)=0$$

$$\Rightarrow 4+2\sin x=\sin 2x \Rightarrow \textit{No. sol}$$

11. If $x \in \left[0, \dfrac{\pi}{2}\right]$, then find the number of solutions of the equation $\sin 7x + \sin 4x + \sin x = 0$.

Sol. [5]

$\sin 7x + \sin 4x + \sin x = 0$

$\Rightarrow 2\sin 4x\cos 3x + \sin 4x = 0$

$\Rightarrow \sin 4x = 0$ or $\cos 3x = -\dfrac{1}{2}$

$\Rightarrow 4x = n\pi$ or $3x = 2n\pi \pm \dfrac{2\pi}{3}$

$\Rightarrow x = \dfrac{n\pi}{4}, \dfrac{2n\pi}{3} \pm \dfrac{2\pi}{9} = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{2\pi}{9}, \dfrac{4\pi}{9}.$

12. Solve the following trigonometric equation:
$\sin^2 4x + \cos^2 x = 2\sin 4x\cos^4 x$

Sol. Given equation is
$\sin^2 4x + \cos^2 x = 2\sin 4x\cos^4 x$

$\Rightarrow \sin^2 4x - 2\sin 4x\cos^4 x + \cos^2 x = 0$

$\Rightarrow (\sin^2 4x - \cos^4 x)^2 + \cos^2 x - \cos^8 x = 0$

$\Rightarrow (\sin^2 4x - \cos^4 x)^2 + \cos^2 x(1 - \cos^6 x) = 0$

It is possible only when

$(\sin^2 4x - \cos^4 x) = 0$, $\cos^2 x(1 - \cos^6 x) = 0$

Now, $\cos x = 0$, $\cos^2 x = 1$

when $\cos x = 0$ then $x = (2n + 1)\pi/2$

So, $\sin 4\left(n+\dfrac{1}{2}\right)\pi = 0$

which is true

when $\cos^2 x = 1$, then $x = n\pi$

which will not satisfy the equation

$\sin(4x) - \cos^4 x = 0$

Hence, the solution is $x = (2n+1)\dfrac{\pi}{2}$

13. Solve: $2\sin\left(3x+\dfrac{\pi}{4}\right)=\sqrt{1+8\sin 2x\cdot\cos^2 2x}$

Sol. $2\sin\left(3x+\dfrac{\pi}{4}\right)=\sqrt{1+8\sin 2x\cos^2 2x}$

$$\Rightarrow 2\sin\left(3x+\frac{\pi}{4}\right)=\sqrt{1+4\sin 4x\cos 2x}$$

$$\Rightarrow 4\sin^2\left(3x+\frac{\pi}{4}\right)=1+4\sin 4x\cos 2x$$

$$\Rightarrow 2-2\cos\left(6x+\frac{\pi}{2}\right)=1+2\sin 6x+2\sin 2x$$

$$\Rightarrow \sin 2x=\frac{1}{2}\Rightarrow x=\frac{n\pi}{2}+(-1)^n\frac{\pi}{12}, n\in I$$

$$\Rightarrow x=\frac{\pi}{12},\frac{5\pi}{12},\frac{13\pi}{12},\frac{17\pi}{12}$$

$\because \sin\left(3x+\dfrac{\pi}{4}\right)\ge 0$ so $n=\dfrac{\pi}{12},\dfrac{17\pi}{12}$ satisfies

So general solution is $x = 2n\pi+\dfrac{\pi}{12}, 2n\pi+\dfrac{17\pi}{12}$

14. Find the value(s) of k for which the equation $\sin x + \cos(k + x) + \cos(k - x) = 2$ has real solutions.

Sol. $\sin x + \cos(k + x) + \cos(k - x) = 2$

$2\cos k.\cos x + \sin x = 2$

This equation is of the form $a\cos x + b\sin x = c$

Here $a = 2\cos k$, $b = 1$ and $c = 2$

Since for real solution, $|c|\le\sqrt{a^2+b^2}$

$\therefore |2|\le\sqrt{1+4\cos^2 k}\Rightarrow 2\le\sqrt{1+4\cos^2 k}$

$$\Rightarrow \cos^2 k\ge\frac{3}{4}\Rightarrow\sin^2 k\le\frac{1}{4}$$

$$\Rightarrow \sin^2 k-\frac{1}{4}\le 0\Rightarrow\left(\sin k+\frac{1}{2}\right)\left(\sin k-\frac{1}{2}\right)\le 0$$

$$\Rightarrow -\frac{1}{2}\le\sin k\le\frac{1}{2}\Rightarrow n\pi-\frac{\pi}{6}\le k\le n\pi+\frac{\pi}{6}$$

15. Find the number of solutions of the equations:

(*a*) $2^{\cos x} = |\sin x|$, when $x \in [-2\pi, 2\pi]$

(*b*) $x + 2\tan x = \dfrac{\pi}{2}$, when $x \in [0, 2\pi]$

Sol. (*a*) We have $2^{\cos x}$

It is true only for $\cos x = 0$ and $|\sin x| = 1$

$$\Rightarrow \cos x = \cos\frac{\pi}{2} \text{ and } \sin x = \pm 1 = \sin\left(\pm\frac{\pi}{2}\right)$$

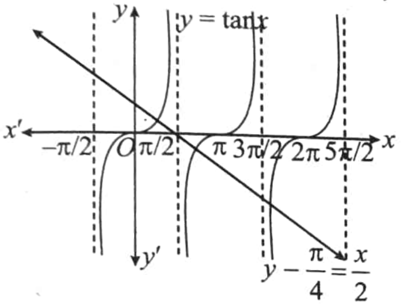
$$\Rightarrow x = 2n\pi \pm \frac{\pi}{2}$$

But, $x \in [-2\pi, 2\pi]$

$$x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}$$

Hence, number of solution = 4.

Sol. (*b*) We have, $x + 2\tan x = \dfrac{\pi}{4}$ or $\tan x = \dfrac{\pi}{4} - \dfrac{x}{2}$



Now the graph of the curve $y = \tan x$ and $y = \dfrac{\pi}{4} - \dfrac{x}{2}$, in the interval $[0, 2\pi]$ intersect at three points.

Hence, number of solution is three.

16. Determine the smallest positive value of x (in degrees) for which $\tan(x + 100°) = \tan(x + 50°)\tan x \tan(x - 50°)$.

Sol. Given $\tan(x + 100°) = \tan(x + 50°)\tan x \tan(x - 50°)$

$$\frac{\tan(x+100°)}{\tan(x-50°)} = \tan(x+50°)\tan x$$

$$\Rightarrow \frac{\sin(x+100°)\cos(x-50°)}{\cos(x+100°)\sin(x-50°)} = \frac{\sin(x+50°)\sin x}{\cos(x+50°)\cos x}$$

Applying componendo and dividendo, we get

$$\frac{\sin(2x+50°)}{\sin 150°} = \frac{-\cos 50°}{\cos(2x+50°)}$$

$\sin(2x + 50°)\cos(2x + 50°) = -\sin 150° \cos 50°$

$\Rightarrow \sin(4x + 100°) = -\sin 40°$

$4x + 100° = n\pi - (-1)^n 40°$

Substitute $n = 1$, smallest positive value of x is given by $x = 30°$

17. Find most general value of θ

(*a*) $\sin 2\theta = \cos 3\theta$

(*b*) $\tan(\pi \cot\theta) = \cot(\pi \tan\theta)$

(c) $\cos(2x + 3y) = \dfrac{1}{2}$ & $\cos(3x + 2y) = \dfrac{\sqrt{3}}{2}$

Sol. (*a*) $\cos 3\theta = \cos(\pi/2 - 2\theta)$

$\Rightarrow 3\theta = 2n\pi + (\pi/2 - 2\theta) \Rightarrow 3\theta = 2n\pi + \pi/2 - 2\theta$

$$0 = \left(2n + \frac{1}{2}\right)\frac{\pi}{5}$$

or $3\theta = 2n\pi - \dfrac{\pi}{2} + 2\theta \Rightarrow \theta = 2n\pi - \dfrac{\pi}{2}\, n \in I$

(*b*) $\tan(\pi\cot\pi) = \tan(\pi/2 - \pi\tan\theta)$

$$\pi\cot\theta = n\pi + \frac{\pi}{2} - \pi\tan\theta;\ \tan\theta + \cot\theta = \frac{2n+1}{2}$$

$$2\tan^2\theta - (2n+1)\tan\theta + 2 = 0$$

$$\tan\theta = \frac{(2n+1) \pm \sqrt{(2n+1)^2 - 16}}{4}$$

$$\Rightarrow \tan\theta = \frac{(2n+1) \pm \sqrt{4n^2 + 4n - 15}}{4}$$

(*c*) $\cos(2x + 3y) = \dfrac{1}{2} = \cos\dfrac{\pi}{3}$

$$\Rightarrow 2x + 3y = 2n\pi \pm \frac{\pi}{3} \quad \text{...}(i)$$

and $\cos(3x + 2y) = \dfrac{\sqrt{3}}{2} = \cos\dfrac{\pi}{6}$

$$\Rightarrow 3x + 2y = 2m\pi \pm \frac{\pi}{6} \quad \text{...}(ii)$$

where $m, n \in I$

Solving (*i*) and (*ii*) we get

$$x = \frac{1}{5}\left[(6m - 4m)\pi \pm \frac{\pi}{2} \mp \frac{2\pi}{3}\right]$$

$$\text{or } y = \frac{1}{5}\left[(6m - 4m)\pi \pm \pi \mp \frac{\pi}{3}\right]$$

18. For what a is the equation $\sin^2 x - \sin x \cos x - 2\cos^2 x = a$ solvable? Find the solutions.

Sol. Multiplying the right member of the equation by $\sin^2 x + \cos^2 x = 1$ we reduce it to the form $(1 - \alpha)\sin^2 x - \sin x \cos x = (a + 2)\cos^2 x = 0$. ...(*i*)

First let us assume that $a \neq 1$. Then form (*i*) is follows that $\cos x = 0$, since otherwise we have $\sin x = \cos x = 0$ which is impossible. Dividing both members of (*i*) by $\cos^2 x$ and putting $\tan x = t$ we get the equation $(1 - a)t^2 - t(a + 2) = 0$. ...(*ii*)

Equation (*i*) is solvable if and only if the roots of equation (*ii*) are real, i.e. if its discriminant is non- negative $D = -4a^2 - 4a + 9 \geq 0$. ....(*iii*)

Solving inequality (3) we find $-\dfrac{\sqrt{10}+1}{2} \leq a \leq \dfrac{\sqrt{10}-1}{2}$ ..(*iv*)

Let t_1, and t_2 be the roots of equation (*ii*). Then the corresponding solutions of equation (*i*) have the form $x_1 = \text{arc}\tan t_1 + k\pi$, $x_2 = \text{arc}\tan t_2 + k\pi$,

Now let us consider the case $a = 1$. In this case equation (*i*) is written in the form $\cos x(\sin x + 3\cos x) = 0$ and has the following solutions : $x_1 = \dfrac{\pi}{2} + k\pi$, $= x_2 = -\text{arc}\tan 3 + k\pi$.

Exercise-1 (Topicwise)

SOLUTION BY FACTORIZATION

1. The equation $2\sin\frac{x}{2}\cos^2 x - 2\sin\frac{x}{2}\sin^2 x = \cos^2 x - \sin^2 x$ has a root for which the false statement is

   (a) $\sin 2x = 1$ (b) $\cos x = \frac{\pi}{6}$

   (c) $\cos 2x = -\frac{\pi}{6}$ (d) $\cos x = 1$

2. If 2 $\tan^2\theta = \sec^2\theta$, then the general value of θ is [where $n \in I$]

   (a) $n\pi + \frac{\pi}{4}$ (b) $n\pi - \frac{\pi}{4}$

   (c) $n\pi \pm \frac{\pi}{4}$ (d) $2n\pi \pm \frac{\pi}{4}$

3. If $\sqrt{3}\tan 2\theta + \sqrt{3}\tan 3\theta + \tan 2\theta \tan 3\theta = 1$, then the general value of θ is [where $n \in$ I]

   (a) $n\pi + \frac{\pi}{5}$ (b) $\left(n + \frac{1}{6}\right)\frac{\pi}{5}$

   (c) $\left(2n \pm \frac{1}{6}\right)\frac{\pi}{5}$ (d) $\left(n + \frac{1}{3}\right)\frac{\pi}{5}$

4. If tan 2θ tan θ = 1, then the general value of θ is [where $n \in I$]

   (a) $\left(n + \frac{1}{2}\right)\frac{\pi}{3}$ (b) $\left(n + \frac{1}{2}\right)\pi$

   (c) $\left(2n \pm \frac{1}{2}\right)\frac{\pi}{3}$ (d) None of these

5. The solution of the equation $4\cos^2 x + 6\sin^2 x = 5$ [where $n \in I$]

   (a) $x = n\pi \pm \frac{\pi}{2}$

   (b) $x = n\pi \pm \frac{\pi}{4}$

   (c) $x = n\pi \pm \frac{3\pi}{2}$

   (d) Both (a) and (c)

6. If $\sin\left(\frac{\pi}{4}\cot\theta\right) = \cos\left(\frac{\pi}{4}\tan\theta\right)$, [where $n \in I$] then θ may be equal to

   (a) $n\pi + \frac{\pi}{4}$ (b) $2n\pi \pm \frac{\pi}{4}$

   (c) $n\pi - \frac{\pi}{4}$ (d) $2n\pi \pm \frac{\pi}{6}$

7. The general value of θ in the equation $2\sqrt{3}\cos\theta = \tan\theta$, is [where $n \in$ I]

   (a) $2n\pi \pm \frac{\pi}{6}$ (b) $2n\pi \pm \frac{\pi}{4}$

   (c) $n\pi + (-1)^n \frac{\pi}{3}$ (d) $n\pi + (-1)^n \frac{\pi}{4}$

8. If $(2\cos x - 1)(3 + 2\cos x) = 0, 0 \le x \le 2\pi$, [where $n \in I$] then $x =$

   (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3}, \frac{5\pi}{3}$

   (c) $\frac{\pi}{2}, \frac{5\pi}{3}, \cos^{-1}\left(-\frac{3}{2}\right)$ (d) $\frac{5\pi}{3}$

9. If tan(cot x) = cot(tan x), [where $n \in$ I] then sin 2x =

   (a) $(2n+1)\frac{\pi}{4}$ (b) $\frac{4}{(2n+1)\pi}$

   (c) $4\pi(2n + 1)$ (d) $\frac{\pi}{2}(2n-1)$

10. The solution set of the equation 4sinθ.cosθ – 2cosθ – 2$\sqrt{3}$ sinθ + $\sqrt{3}$ = 0 in the interval (0, 2π) is

    (a) $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$

    (b) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$

    (c) $\left\{\frac{3\pi}{4}, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$

    (d) $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}\right\}$

11. All solutions of the equation 2 sinθ + tanθ = 0 are obtained by taking all integral values of m and n in:

    (a) $2n\pi + \frac{2\pi}{3}$, $n \in$ I

    (b) $n\pi$ or $2m\pi \pm \frac{2\pi}{3}$ where $n, m \in I$

    (c) $n\pi$ or $m\pi \pm \frac{\pi}{3}$ where $n, m \in I$

    (d) $n\pi$ or $2m\pi \pm \frac{\pi}{3}$ where $n, m \in I$

12. If 2 $\cos^2(\pi + x)$ + 3 sin ($\pi + x$) vanishes then the values of x lying in the interval from 0 to 2π are

    (a) $x = \pi/6$ or $5\pi/6$

    (b) $x = \pi/3$ or $5\pi/3$

    (c) $x = \pi/4$ or $5\pi/4$

    (d) $x = \pi/2$ or $5\pi/2$

13. If $\cos 2\theta + 3\cos\theta = 0$, then

(a) $\theta = 2n\pi \pm \alpha$ where $\alpha = \cos^{-1}\left(\frac{\sqrt{17}-3}{4}\right)$

(b) $\theta = 2n\pi \pm \alpha$ where $\alpha = \cos^{-1}\left(\frac{-\sqrt{17}-3}{4}\right)$

(c) $\theta = 2n\pi \pm \alpha$ where $\alpha = \cos^{-1}\left(\frac{\pm\sqrt{17}-3}{4}\right)$

(d) None of these

14. The general solution of the equation, $2\cos^2 x = 3.2\cos^2 x - 4$ is

(a) $x = 2n\pi,\ n \in I$

(b) $x = n\pi,\ n \in I$

(c) $x = n\pi/4,\ n \in I$

(d) $x = n\pi/2,\ n \in I$

15. If $x \in (0, 1)$, then greatest root of the equation $\sin 2\pi x = \sqrt{2}\cos \pi x$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$

(c) $\frac{3}{4}$ (d) $\frac{1}{3}$

16. The general solution of $\sec\theta + \tan\theta = \sqrt{3}$ is

(a) $\theta = 2n\pi + \frac{\pi}{6}, 2n\pi - \frac{\pi}{2}, n \in I$

(b) $\theta = 2n\pi + \frac{\pi}{6}, n \in I$

(c) $2n\pi - \frac{\pi}{6}, n \in I$

(d) $2n\pi - \frac{\pi}{2}, n \neq 2k+1, n, k \in I$

17. The general solution of $\sin^2\theta\sec\theta + \sqrt{3}\tan\theta = 0$ is

(a) $\theta = n\pi + (-1)^{n+1}\frac{\pi}{3}, \theta = n\pi, n \in I$

(b) $\theta = n\pi, n \in I$

(c) $\theta = n\pi + (-1)^{n+1}\frac{\pi}{3}, n \in I$

(d) $\theta = \frac{n\pi}{2}, n \in I$

SOLVING SIMULTANEOUS EQUATION

18. If $\cos\theta = -\frac{1}{\sqrt{2}}$ and $\tan\theta = 1$, [where $n \in I$] then the general value of θ is

(a) $2n\pi + \frac{\pi}{4}$ (b) $(2n+1)\pi + \frac{\pi}{4}$

(c) $n\pi + \frac{\pi}{4}$ (d) $n\pi \pm \frac{\pi}{4}$

19. If $\tan\theta = -\frac{1}{\sqrt{3}}$ and $\sin\theta = \frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$, [where $n \in I$] then the principal value of θ will be

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$

(c) $\frac{7\pi}{6}$ (d) $-\frac{\pi}{6}$

20. The most general solution of $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$ is

(a) $n\pi + \frac{7\pi}{4}, n \in I$ (b) $n\pi + (-1)^n\frac{7\pi}{4}, n \in I$

(c) $2n\pi + \frac{7\pi}{4}, n \in I$ (d) None of these

21. The most general value of θ which will satisfy both the equations $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ is [where $n \in I$]

(a) $n\pi + (-1)^n\frac{\pi}{6}$ (b) $n\pi + \frac{\pi}{6}$

(c) $2n\pi \pm \frac{\pi}{6}$ (d) $2n\pi + \frac{7\pi}{6}$

22. If $\sin A = \sin B$ and $\cos A = \cos B$, then

(a) $A = B + n\pi, n \in I$

(b) $A = B - n\pi, n \in I$

(c) $A = 2n\pi + B, n \in I$

(d) $A = n\pi - B, n \in I$

SOLUTION OF EQUATION REDUCIBLE TO QUADRATIC EQUATION

23. If $3(\sec^2\theta + \tan^2\theta) = 5$, [where $n \in I$] then the general value of θ is

(a) $2n\pi + \frac{\pi}{6}$ (b) $2n\pi \pm \frac{\pi}{6}$

(c) $n\pi \pm \frac{\pi}{6}$ (d) $n\pi \pm \frac{\pi}{3}$

24. If $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$, [where $n \in I$] the general value of θ is

(a) $n\pi \pm \frac{\pi}{3}$ (b) $n\pi \pm \frac{\pi}{6}$

(c) $2n\pi \pm \frac{\pi}{3}$ (d) $2n\pi \pm \frac{\pi}{6}$

25. The number of values of x for which $\sin 2x + \cos 4x = 2$ is

(a) Zero (b) 1 (c) 2 (d) Infinite

26. If $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$, [where $n \in I$] then the general value of θ is

(a) $n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{3}$ (b) $n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{3}$

(c) $n\pi + \frac{\pi}{4}, n\pi - \frac{\pi}{3}$ (d) $n\pi - \frac{\pi}{4}, n\pi - \frac{\pi}{3}$

27. If $4\sin^2\theta + 2(\sqrt{3}+1)\cos\theta = 4+\sqrt{3}$, [where $n \in I$] then the general value of θ is

(a) $2n\pi \pm \frac{\pi}{3}$ (b) $2n\pi + \frac{\pi}{4}$

(c) $n\pi \pm \frac{\pi}{3}$ (d) $n\pi - \frac{\pi}{3}$

28. The smallest positive angle which satisfies the equation $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$, is

(a) $\frac{5\pi}{6}$ (b) $\frac{2\pi}{3}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

29. If $12\cot^2\theta - 31\operatorname{cosec}\theta + 32 = 0$, then the value of $\sin\theta$ is

(a) $\frac{3}{5}$ or 1 (b) $\frac{2}{3}$ or $\frac{-2}{3}$

(c) $\frac{4}{5}$ or $\frac{3}{4}$ (d) $\pm\frac{1}{2}$

SUM TO PRODUCT AND PRODUCT TO SUM

30. If $\cos 7\theta = \cos\theta - \sin 4\theta$, then the general value of θ is [where $n \in I$]

(a) $\frac{n\pi}{4}, \frac{n\pi}{3} + \frac{\pi}{18}$ (b) $\frac{n\pi}{3}, \frac{n\pi}{3} + (-1)^n\frac{\pi}{18}$

(c) $\frac{n\pi}{4}, \frac{n\pi}{3} + (-1)^n\frac{\pi}{18}$ (d) $\frac{n\pi}{6}, \frac{n\pi}{3} + (-1)^n\frac{\pi}{18}$

31. If $\cos\theta + \cos 2\theta + \cos 3\theta = 0$, then the general value of θ is [where $m \in I$]

(a) $\theta = 2m\pi \pm \frac{2\pi}{3}$ (b) $\theta = m\pi \pm \frac{\pi}{4}$

(c) $\theta = m\pi \pm (-1)^m\frac{2\pi}{3}$ (d) Both (a) and (b)

Equation of the type a sinx + b cos x = C

32. The general solution of equation $\sin x + \sin 5x = \sin 2x + \sin 4x$ is :

(a) $\frac{n\pi}{2}$; $n \in I$ (b) $\frac{n\pi}{5}$; $n \in I$

(c) $\frac{n\pi}{3}$; $n \in I$ (d) $\frac{2n\pi}{3}$; $n \in I$

33. If $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$, then the most general value of θ is [where $n \in I$]

(a) $n\pi + (-1)^n\frac{\pi}{4}$ (b) $(-1)^n\frac{\pi}{4} - \frac{\pi}{3}$

(c) $n\pi + \frac{\pi}{4} - \frac{\pi}{3}$ (d) $n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{3}$

34. The equation $\sqrt{3}\sin x + \cos x = 4$ has

(a) Only one solution

(b) Two solutions

(c) Infinitely many solutions

(d) No solution

35. The equation $3\cos x + 4\sin x = 6$ has

(a) Finite solution (b) Infinite solution

(c) One solution (d) No solution

SOLUTION USING THE BOUNDNESS

36. The number of solutions of $\sin\theta + 2\sin 2\theta + 3\sin 3\theta + 4\sin 4\theta = 10$ in $(0, \pi)$ is

(a) 1 (b) 2

(c) 4 (d) 0

37. If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$, $x \in (0, \pi)$, then

(a) $x = \frac{\pi}{4}, y = 1$ (b) $y = 0$

(c) $y = 2$ (d) $x = \frac{3\pi}{4}$

38. One root of the equation $\cos x - x + \frac{1}{2} = 0$ lies in the interval

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{2}, 0\right]$

(c) $\left[\frac{\pi}{2}, \pi\right]$ (d) $\left[\pi, \frac{3\pi}{2}\right]$

39. If $\sec x \cos 5x + 1 = 0$, where $0 < x < 2\pi$, then $x =$

(a) $\frac{\pi}{5}, \frac{\pi}{5}$ (b) $\frac{\pi}{5}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

40. If the equation $1 + \sin^2 x\theta = \cos\theta$ has a non-zero solution in θ, then x must be

(a) An integer (b) A rational number

(c) An irrational number (d) None of these

41. The number of the solution of the equation $3\sin x + 4\cos x - x^2 - 16 = 0$ is

(a) 0 (b) 1

(c) 2 (d) 3

42. The solution set of $\sin^4 x - \tan^8 x = 1$ is given by

(a) $x = 2n\pi + \frac{\pi}{8}, n \in I$ (b) $x = n\pi + \frac{\pi}{12}, n \in I$

(c) $x = 2n\pi + \frac{\pi}{24}, n \in I$ (d) None of these

TRIGONOMETRIC IN EQUALITIES

43. The set of all x in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ satisfying $|4\sin x - 1| < \sqrt{5}$ is given by

(a) $\left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$ (b) $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$

(c) $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$ (d) None of these

44. If $4\sin^2 x - 12\sin x + 5 \le 0, 0 \le x \le 2\pi$, which of the following is the solution set for x?

(a) $\left[0, \frac{\pi}{6}\right]$

(b) $\left[0, \frac{5\pi}{6}\right]$

(c) $\left[\frac{5\pi}{6}, 2\pi\right]$

(d) $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

45. Which of the following is a set of values x for which $\sin x.\cos^3 x > \cos x.\sin^3 x$ in the interval $0 \le x \le 2\pi$?

(a) $(0, \pi)$ (b) $\left(0, \frac{\pi}{4}\right)$

(c) $\left(\frac{\pi}{4}, \pi\right)$ (d) $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

46. If $[\sin x] + [\sqrt{2}\cos x] = -3$, where $x \in [0, 2\pi]$ then find x (where $[\cdot]$ shows G.I.F.)

(a) $\left(\pi, \frac{5\pi}{4}\right)$ (b) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(c) $\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ (d) None of these

Exercise-2 (Learning Plus)

1. The number of solution of the equation $|\sin x| = |\cos 3x|$ in $[-2\pi, 2\pi]$

(a) 32 (b) 28
(c) 24 (d) 30

2. If $\tan(\cot x) = \cot(\tan x)$, then $\text{cosec } 2x =$

(a) $(2n+1)\frac{\pi}{4}$ (b) $\frac{4}{(2n+1)\pi}$
(c) $4\pi(2n+1)$ (d) None of these

3. The most general solution of $4^{\sin x} + 4^{\cos x} = 2^{1-\sqrt{2}}$ is

(a) $n\pi + \frac{\pi}{4}$ (b) $n\pi - \frac{\pi}{4}$

(c) $n\pi + (-1)^n \frac{\pi}{4}$ (d) $2n\pi + \frac{\pi}{4}$

4. General solution of equation $\sec^2 x + \text{cosec}^2 x + \cot^2 x - 4\cos^2 x = 3$ is

(a) $n\pi \pm \frac{\pi}{4}; n \in Z$ (b) $2n\pi \pm \frac{\pi}{4}; n \in Z$

(c) $n\pi + \frac{\pi}{3}; n \in Z$ (d) $n\pi - \frac{\pi}{6}; n \in Z$

5. Number of solutions of the equation

$$\cot(\theta) + \cot\left(\theta + \frac{\pi}{3}\right) + \cot\left(\theta - \frac{\pi}{3}\right) + \cot(3\theta) = 0$$

where $\theta \in (0, \pi)$.

(a) Infinite (b) 0
(c) 1 (d) 2

6. The total number of solution of the equation max. $(\sin x, \cos x) = 1/2$ for $x \in (-2\pi, 6\pi)$ is equal to:

(a) 3 (b) 6
(c) 7 (d) 8

7. The number of solution of $\sin^4 x - \cos^2 x \sin x + 2\sin^2 x + \sin x = 0$ in $0 \le x \le 3\pi$ is

(a) 3 (b) 4
(c) 6 (d) 5

8. The number of solutions of $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x, 0 \le x \le 2\pi$, is

(a) 7 (b) 5 (c) 4 (d) 6

9. The total number of solutions of $\cos x = \sqrt{1 - \sin 2x}$ in $[0, 2\pi]$ is equal to

(a) 2 (b) 3
(c) 5 (d) None of these

10. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation $(\sqrt{3})^{\sec^2\theta} = \tan^4\theta + 2\tan^2\theta$ is

(a) 2 (b) 4
(c) 0 (d) 1

11. The general solution of $8\tan^2\frac{x}{2} = 1 + \sec x$ is

(a) $x = 2n\pi \pm \cos^{-1}\left(\frac{-1}{3}\right)$ (b) $x = 2n\pi \pm \frac{\pi}{6}$

(c) $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$ (d) None of these

12. The general solution of $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta.\tan 4\theta.\tan 7\theta$

(a) $\theta = \frac{n\pi}{4}$ (b) $\theta = 6\pi$

(c) $\theta = \frac{n\pi}{12}$ (d) None of these

13. Let $2\sin^2 x+3\sin x-2>0$ and $x^2-x-2<0$ (x is measured in radian). Then 'x' lies in the internal.

(a) $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ (b) $\left(-1,\frac{5\pi}{6}\right)$

(c) $(-1,2)$ (d) $\left(\frac{\pi}{6},2\right)$

14. The general solution of the equation $\sin^{100} x-\cos^{100} x=1$ is (where $n\in I$):

(a) $2n\pi+\frac{\pi}{2}$ (b) $n\pi+\frac{\pi}{2}$

(c) $2n\pi-\frac{\pi}{2}$ (d) $n\pi$

15. Number of solutions of $\sum_{r=1}^{5}\cos rx=5$ in the interval $[0, 4\pi]$ is:

(a) 0 (b) 2

(c) 3 (d) 7

16. The general solution of the equation $\tan^2(x+y)+\cot^2(x+y)=1-2x-x^2$ lie on the line is:

(a) $x=-1$ (b) $x=-2$

(c) $y=-1$ (d) $y=-2$

17. If $|k|=5$ and $0°\le\theta\le 360°$, then the number of different solutions of $3\cos\theta+4\sin\theta=k$ is

(a) Zero (b) Two

(c) One (d) Infinite

18. Total number of solutions of $\sin x=\frac{|x|}{10}$ is equal to

(a) 4 (b) 6

(c) 7 (d) None of these

19. $\cos 2x-3\cos x+1=\frac{1}{(\cot 2x-\cot x)\sin(x-\pi)}$ holds, if

(a) $\cos x=0$ (b) $\cos x=1$

(c) $\cos x=5/2$ (d) for no value of x

20. The number of solutions of $[\sin x+\cos x]=3+[-\sin x]+[-\cos x]$ (where [.] denotes the greatest integer function). $x\in[0, 2\pi]$, is

(a) 0 (b) 4

(c) Infinite (d) 1

21. $\sin 3\theta=4\sin\theta\,.\,\sin 2\theta\,.\,\sin 4\theta$ in $0\le\theta\le\pi$ has

(a) 2 real solutions (b) 4 real solutions

(c) 6 real solutions (d) 8 real solutions

22. The arithmetic mean of the roots of the equation $4\cos^3x-4\cos^2x-\cos(\pi+x)-1=0$ in the interval $[0, 315]$ is equal to

(a) 49π (b) 50π

(c) 51π (d) 100π

23. If $\sin\theta+7\cos\theta=5$, then $\tan(\theta/2)$ is a root of the equation

(a) $x^2-6x+1=0$ (b) $6x^2-x-1=0$

(c) $6x^2+x+1=0$ (d) $x^2-x+6=0$

24. $\frac{\sin 3\theta}{2\cos 2\theta+1}=\frac{1}{2}$ if

(a) $\theta=n\pi+\frac{\pi}{6}, n\in I$

(b) $\theta=2n\pi-\frac{\pi}{6}, n\in I$

(c) $\theta=n\pi+(-1)^n\frac{\pi}{6}, n\in I$

(d) $\theta=n\pi-\frac{\pi}{6}, n\in I$

25. The general solution of the equation $\tan x+\tan\left(x+\frac{\pi}{3}\right)+\tan\left(x+\frac{2\pi}{3}\right)=3$ is

(a) $\frac{n\pi}{4}+\frac{\pi}{12}, n\in I$ (b) $\frac{n\pi}{3}+\frac{\pi}{6}, n\in I$

(c) $\frac{n\pi}{3}+\frac{\pi}{12}, n\in I$ (d) None of these

26. A triangle ABC is such that $\sin(2A+B)=\frac{1}{2}$. If A, B, C are in $A.P.$ then the angle A, B, C are respectively

(a) $\frac{5\pi}{12},\frac{\pi}{4},\frac{\pi}{3}$ (b) $\frac{\pi}{4},\frac{\pi}{3},\frac{5\pi}{12}$

(c) $\frac{\pi}{3},\frac{\pi}{4},\frac{5\pi}{12}$ (d) $\frac{\pi}{3},\frac{5\pi}{12},\frac{\pi}{4}$

27. The number of all possible triplets (a_1, a_2, a_3) such that : $a_1+a_2\cos 2x+a_3\sin^2x=0$ for all x is

(a) 0 (b) 1

(c) 2 (d) Infinite

28. The value 'a' for which the equation $4\text{cosec}^2[\pi(a+x)]+a^2-4a=0$ has a real solution is :

(a) $a=1$

(b) $a=2$

(c) $a=3$

(d) None of these

29. The solution of $|\cos x|=\cos x-2\sin x$ is

(a) $x=n\pi, n\in I$

(b) $x=n\pi+\frac{\pi}{4}, n\in I$

(c) $x=n\pi+(-1)^n\frac{\pi}{4}, n\in I$

(d) $(2n+1)\pi+\frac{\pi}{4}, n\in I$

30. If $m, n\in N$ $(n>m)$, then number of solutions of the equation $n\,|\sin x|=m\,|\sin x|$ in $[0, 2\pi]$ is

(a) m (b) n

(c) mn (d) 3

31. The number of solutions of the equation $\cos(\pi\sqrt{x-4})\cos(\pi\sqrt{x})=1$ is

(a) Zero (b) 1

(c) 2 (d) Infinite

32. The number of values of $\theta \in [0, 2\pi]$ satisfying $r \sin\theta = \sqrt{3}$ and $r + 4\sin\theta = 2(\sqrt{3} + 1)$ is

(a) 4 (b) 5
(c) 6 (d) 7

33. The solution set of the equation $4\sin^4 x + \cos^4 x = 1$ is

(a) $x = n\pi + \frac{\pi}{6}$

(b) $x = n\pi \pm \theta$ where $\cos^2\theta = \frac{3}{5}$ and $x = n\pi, n \in I$

(c) $x = (2n+1)\frac{\pi}{2}$

(d) $x = 2n\pi \pm \frac{\pi}{3}$

34. The number of solutions of the equation $\sin x \cdot \cos x (\cos x - \sin x)^2 \cdot (\sin x + \cos x) = \lambda$, where $\lambda > \frac{1}{2\sqrt{2}}$ in the interval $[0, 4\pi]$, is

(a) 0 (b) 2
(c) 4 (d) 8

35. The complete set of values of x, $x \in \left(-\frac{\pi}{2}, \pi\right)$ satisfying the inequality $\cos 2x > |\sin x|$ is:

(a) $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$

(b) $\left(-\frac{\pi}{2}, \frac{\pi}{6}\right) \cup \left(-\frac{\pi}{6}, \frac{5\pi}{6}\right)$

(c) $\left(-\frac{\pi}{2}, -\frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$

(d) $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$

36. Values of x between 0 and 2π which satisfy the equation $\sin x\sqrt{8\cos^2 x} = 1$ are in A.P. whose common difference is:

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

37. If α and β are distinct roots of the equation $a\cos\theta + b\sin\theta = c$, then $\cos(\alpha + \beta) =$

(a) $\frac{2ab}{a^2+b^2}$ (b) $\frac{2ab}{a^2-b^2}$
(c) $\frac{a^2-b^2}{a^2+b^2}$ (d) $\frac{a^2+b^2}{a^2-b^2}$

38. The solution of the equation $\log_{\cos x^2}(3-2x) < \log_{\cos x^2}(2x-1)$ is

(a) (1/2, 1) (b) $(-\infty, 1)$
(c) (1/2, 3) (d) $(1,\infty) - \sqrt{2n\pi}, n \in N$

39. If $\alpha, \beta, \gamma, \delta$ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, then the value of $4\sin\left(\frac{\alpha}{2}\right) + 3\sin\left(\frac{\beta}{2}\right) + 2\sin\left(\frac{\gamma}{2}\right) + \sin\left(\frac{\delta}{2}\right)$ is equal to

(a) $2\sqrt{1-k}$ (b) $2\sqrt{1+k}$
(c) $\frac{\sqrt{1+k}}{2}$ (d) $\frac{\sqrt{1-k}}{2}$

40. The smallest positive value of x (in radians) satisfying the equation $\log_{\cos x}\left(\frac{\sqrt{3}}{2}\sin x\right) = 2 - \log_{\sec x}(\tan x)$ is

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{4}$

41. Suppose $a, b > 0$ and $\log_9(a) = \log_{12}(b) = \log_{16}(a+b)$ and $\frac{b}{a} = 2\sin\theta$, then θ lies is

(a) $\left(0, \frac{\pi}{4}\right)$ (b) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(c) $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ (d) $\left(\frac{3\pi}{4}, \pi\right)$

42. Smallest positive value of θ satisfying the equation $8\sin\theta\cos 2\theta\sin 3\theta\cos 4\theta = \cos 6\theta$ is

(a) $\frac{\pi}{18}$ (b) $\frac{\pi}{22}$
(c) $\frac{\pi}{24}$ (d) None of these

43. If $x \in (0, \pi)$ and $\sin x \cdot \cos^3 x > \cos x \cdot \sin^3 x$, then complete set of values of x is

(a) $x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ (b) $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$
(c) $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) None of these

44. Number of roots of $|\sin|x|| = x + |x|\ \forall\ x \subset [-2\pi, 2\pi]$ is

(a) 0 (b) 1
(c) 2 (d) 3

45. Number of degrees in the smallest positive angle x such that $8\sin x\cos^5 x - 8\sin^5 x\cos x - 1$, is

(a) 5° (b) 7.5°
(c) 10° (d) 15°

46. Number of solution satisfying the equation, $\tan^2 2x = 2\tan 2x \cdot \tan 3x + 1$ in $[0, 2\pi]$ is

(a) 0 (b) 1
(c) 2 (d) 4

47. The value of x, between 0 and 2π, satisfying the equation $\cos 3x + \cos 2x = \sin\frac{3x}{2} + \sin\frac{x}{2}$

(a) $\frac{\pi}{7}$ (b) $\frac{5\pi}{7}$

(c) $\frac{9\pi}{7}$ (d) $\frac{13\pi}{7}$

48. For $0 < \theta < \frac{\pi}{2}$, the solution(s) of $\sum_{m=1}^{6}\operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$ is (are)

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$

49. Let $2\sin x + 3\cos y = 3$ and $3\sin y + 2\cos x = 4$ then
(a) $x + y = (4n+1)\pi/2,\ n \in I$
(b) $x + y = (2n+1)\pi/2,\ n \in I$
(c) x and y can be the two non-right angles of a 3-4-5 triangle with $x > y$
(d) x and y can be the two non-right angles of a 3-4-5 triangle with $y > x$

50. Which of the following set of value of 'x' satisfies the equation $2^{(3\sin^2 x - 3\sin x + \cos^2 x)} + 2^{(2\sin^2 x + 3\sin x)} = 9$.

(a) $x = n\pi \pm \frac{\pi}{6}, n \in I$ (b) $x = \frac{11\pi}{6}$

(c) $x = \frac{9\pi}{2}$ (d) $x = 2n\pi + \frac{\pi}{2}, n \in I$

51. The equation $\cos 7x + \cos 5x = -2$.
(a) Has 50 solutions in $[0, 100\pi]$
(b) Has 3 solutions in $[0, 3\pi]$
(c) Has even number of solutions in $(3\pi, 13\pi)$
(d) Has solutions in $\left[\frac{\pi}{2}, \pi\right]$

52. The general solution of the equation $\cos x \cdot \cos 6x = -1$, is
(a) $x = (2n+1)\pi,\ n \in I$
(b) $x = 2n\pi,\ n \in I$
(c) $x = (2n-1)\pi,\ n \in I$
(d) None of these

53. If the equation $p\sin\theta + \cos 2\theta = 2p - 7$ possesses a solution, then find the value(s) of p.

54. If $4\sin^2 x - 8\sin x + 3 \le 0,\ 0 \le x \le \pi$, then find the solution set for x.

55. Solve for $x : 3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2\sin^2 x} = 28$.

56. Given that $3\sin x + 4\cos x = 5$ where $x \in (0, \pi/2)$. Find the value of $2\sin x + \cos x + 4\tan x$.

57. Express $\cos 5x$ in terms of $\cos x$ and hence find general solution of the equation $\cos 5x = 16\cos^5 x$.

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. For $n \in Z$, the general solution of $(\sqrt{3}-1)\sin\theta + (\sqrt{3}+1)\cos\theta = 2$ is ($n \in Z$)

(a) $\theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$ (b) $\theta = n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{12}$

(c) $\theta = 2n\pi \pm \frac{\pi}{4}$ (d) $\theta = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{12}$

2. One of the general solutions of $\sqrt{3}\cos\theta - 3\sin\theta = 4\sin 2\theta\cos 3\theta$ is
(a) $m\pi + \pi/18,\ m \in Z$ (b) $m\pi/2 + \pi/6,\ \forall\ m \in Z$
(c) $m\pi/3 + \pi/18,\ m \in Z$ (d) None of these

3. Let α, β be two distinct values of x lying in $[0, \pi]$ for which $\sqrt{5}\sin x, 10\sin x, 10(4\sin^2 x + 1)$ are three consecutive terms of G.P. Then sum of minimum and maximum value of $|\alpha - \beta|$ is

(a) $\frac{\pi}{5}$ (b) $\frac{3\pi}{5}$ (c) $\frac{7\pi}{5}$ (d) π

4. The minimum value of the function

$$f(x) = \frac{\sin x}{\sqrt{1-\cos^2 x}} + \frac{\cos x}{\sqrt{1-\sin^2 x}} + \frac{\tan x}{\sqrt{\sec^2 x - 1}} + \frac{\cot x}{\sqrt{\operatorname{cosec}^2 x - 1}}$$

as x varies over all numbers in the largest possible domain of $f(x)$ is
(a) 4 (b) -2
(c) 0 (d) 2

5. Let $[x]$ be the G.I.F., then the equation $\sin x = [1 + \sin x] + [1 - \cos x]$ has
(a) Exactly on solution $\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(b) Exactly one solution in $\left[\frac{\pi}{2}, \pi\right]$
(c) Exactly one solution $\forall x \in R$
(d) No solution

6. The equation $2\sin^3\theta + (2\lambda - 3)\sin^2\theta - (3\lambda + 2)\sin\theta - 2\lambda = 0$ has exactly three roots in $(0, 2\pi)$, then λ can be equal to
(a) 0 (b) 1/2 (c) 1 (d) -1

7. If $4\sin\alpha\times\cos\beta+2\cos\beta-\sqrt{3}=2\sqrt{3}\sin\alpha$, where $\alpha,\beta\in[0,2\pi]$ and $L=\alpha-\beta$ then

(a) Maximum value of L is $\frac{11\pi}{6}$

(b) Minimum value of L is $-\frac{5\pi}{6}$

(c) Maximum value of L is $\frac{5\pi}{3}$

(d) Minimum value of L is $\frac{-11\pi}{6}$

8. The value of θ in $(0, 90°)$ satisfying equation

$$\frac{4\operatorname{cosec}\theta}{\sqrt{5}+1}+\frac{4\sqrt{5}\sec\theta}{\sqrt{10-2\sqrt{5}}}=8$$

(a) 18° (b) 54°
(c) 36° (d) 72°

9. The inequality $4\sin3x+5\geq 4\cos2x+5\sin x$ is true for $x\in$

(a) $\left[-\pi,\frac{3\pi}{2}\right]$ (b) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(c) $\left[\frac{5\pi}{8},\frac{13\pi}{8}\right]$ (d) $\left[\frac{23\pi}{14},\frac{41\pi}{14}\right]$

10. Let $f(\theta)=\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right)$ then

(a) Maximum value of $f(\theta)$ $\forall\ \theta\in R$ is 1/4.
(b) Maximum value of $f(\theta)$ $\forall\ \theta\in R$ is 1/8.
(c) $f(0)=\frac{1}{8}$.
(d) Number of principle solutions of $f(\theta)=0$ is 8.

11. The value of 't' which satisfies $(t-[|\sin x|])!=3!\,5!\,7!$ is/are where [.] is GIF

(a) 9 (b) 10 (c) 11 (d) 12

12. If $\cos\theta+\cot\left(\frac{\pi}{4}+\theta\right)=2$, then the general value of θ is

(a) $2n\pi\pm\frac{\pi}{6}$

(b) $2n\pi\pm\frac{\pi}{3}$

(c) $n\pi\pm\frac{\pi}{3}$

(d) $n\pi\pm\frac{\pi}{6}$

13. Which of the following is/are correct.

(a) $(\tan x)^{\ln(\sin x)}>(\cot x)\ln(\sin x)$, $\forall\ x\in(0,\pi/4)$
(b) $4^{\ln \operatorname{cosin} x}<5^{\ln\operatorname{cosec} x}$, $\forall\ x\in(0,\pi/2)$
(c) $(1/2)^{\ln(\cos x)}<(1/3)^{\ln(\cos x)}$, $\forall\ x\in(0,\pi/2)$
(d) $2^{\ln(\tan x)}>2^{\ln(\tan x)}$, $\forall\ x\in(0,\pi/2)$

14. The equation $2\sin\frac{x}{2}\cdot\cos2x+\sin^2x=2\sin\frac{x}{2}\cdot\sin^2x+\cos^2x$ has a root for which

(a) $\sin2x=1$ (b) $\sin2x=-1$

(c) $\cos x=\frac{1}{2}$ (d) $\cos2x=-\frac{1}{2}$

15. $5\sin^2x+\sqrt{3}\sin x\cos x+6\cos^2x=5$ if

(a) $\tan x=-1/\sqrt{3}$
(b) $\sin x=0$
(c) $x=n\pi+\pi/2,\ n\in I$
(d) $x=n\pi+\pi/6,\ n\in I$

16. $\sin^2x-\cos 2x=2-\sin 2x$ if

(a) $x=n\pi/2,\ n\in I$
(b) $\tan x=3/2$
(c) $x=(2n+1)\pi/2,\ n\in I$
(d) $x=n\pi+(-1)^n\sin^{-1}(2/3),\ n\in I$

17. The equations $2\sin^2\left(\frac{\pi}{2}\cos^2x\right)=1-\cos(\pi\sin 2x)$ is satisfied by

(a) $x=(2n+1)\frac{\pi}{2},n\in Z$ (b) $\tan x=\frac{1}{2},n\in Z$

(c) $\tan x=-\frac{1}{2},n\in Z$ (d) $x=\frac{n\pi}{2},n\in Z$

18. If $(\sin\alpha)x^2-2x+b\geq 2$ for all the real values of $x\leq 1$ and $\alpha\in(0,\pi/2)\cup(\pi/2,\pi)$, then the possible real values of b is/are

(a) 2 (b) 3
(c) 4 (d) 5

19. The value of x in $(0,\pi/2)$ satisfying $\frac{\sqrt{3}-1}{\sin x}+\frac{\sqrt{3}+1}{\cos x}=4\sqrt{2}$ is

(a) $\frac{\pi}{12}$ (b) $\frac{5\pi}{12}$

(c) $\frac{7\pi}{24}$ (d) $\frac{11\pi}{36}$

20. The expression $\cos3\theta+\sin3\theta\,(2\sin2\theta-3)(\sin\theta-\cos\theta)$ is possible for all θ in

(a) $\left(2n\pi-\frac{3\pi}{4},2n\pi+\frac{\pi}{4}\right),n\in Z$

(b) $\left(2n\pi-\frac{\pi}{4},2n\pi+\frac{\pi}{6}\right),n\in Z$

(c) $\left(2n\pi-\frac{\pi}{3},2n\pi+\frac{\pi}{3}\right),n\in Z$

(d) $\left(2n\pi-\frac{\pi}{4},2n\pi+\frac{3\pi}{4}\right),n\in Z$

21. Let the solution set of the equations

$\sec x + \tan x = \sqrt{3}$

$1 + \sin x = \sqrt{3}\cos x$

$\sec 4x + \sec 2x = 2$

$\cos 2x + \cos 4x = 2\cos 2x \cos 4x$

in $[0, 2\pi]$ be S_1, S_2, S_3, S_4 respectively. Then

(a) $S_1 = S_2$ (b) $S_1 \subset S_2$

(c) $S_3 = S_4$ (d) $S_3 \subset S_4$

COMPREHENSION BASED QUESTIONS

Comprehension (Q. 22 to 24): α is a root of the equation $2\sin x + \sin 2x = 1 + \cos x$, β is a root of the equation $3\cos^3 x - 10\cos x + 3 = 0$ and γ is a root of the equation $1 + \sin x = \cos x(1 + 2\sin x)$, $0 \le \alpha, \beta, \gamma \le \pi/2$, then on the basis of above information answer the following questions

22. $\cos\alpha \times \cos\beta \times \cos\gamma$ is equal to

(a) $\frac{1}{2\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

23. $\sin\alpha + \sin\beta + \sin\gamma$ is equal to

(a) $\frac{14-3\sqrt{2}}{6\sqrt{2}}$ (b) $\frac{5}{6}$

(c) $\frac{3+4\sqrt{2}}{6}$ (d) $\frac{1+\sqrt{2}}{2}$

24. $\sin(\alpha - \beta)$ is equal to

(a) 1 (b) 0

(c) $\frac{1-2\sqrt{6}}{6}$ (d) $\frac{\sqrt{3}-2\sqrt{2}}{6}$

Comprehension – 2 (No. 25 to 27): Equation of the form $P(\sin x \pm \cos x, \sin x \cos x) = 0$ where $P(y, z)$ is a polynomial, can be solved by the change : $\cos x \pm \sin x = t$; $1 \pm 2\sin x \cos x = t^2$. Reduce the given equation into $P\left(t, \frac{t^2-1}{2}\right) = 0$

25. General solution of $\sin x + \cos x = 1 + \sin x \cos x$ is

(a) $\frac{\pi}{2} + 2\pi n$ & $2n\pi$ (b) $\frac{\pi}{4} + 2\pi n$ & $(2n+1)\pi$

(c) $2n\pi$ (d) None of these

26. If $(\cos x - \sin x)\left(2\tan x + \frac{1}{\cos x}\right) + 2 = 0$, then x is

(a) $n\pi \pm \frac{\pi}{3}$ (b) $2n\pi \pm \frac{\pi}{3}$

(c) $n\pi \pm \frac{\pi}{6}$ (d) $2n\pi \pm \frac{\pi}{6}$

27. $\sin^4 x + \cos^4 x = \sin x \cos x$ then x is–

(a) $(6n+1)\frac{\pi}{6}$ (b) $n\pi$

(c) $(4n+1)\frac{\pi}{4}$ (d) None of these

Comprehension (28 to 30): Consider the cubic equation $x^3 - (1 + \cos\theta + \sin\theta)x^2 + (\cos\theta\sin\theta + \cos\theta + \sin\theta)x - \sin\theta\cos\theta = 0$ where x_1, x_2, x_3 are roots.

28. The value of $x_1^2 + x_2^2 + x_3^2$ is

(a) 1 (b) $2\cos\theta$

(c) $2\sin\theta$ (d) 2

29. Number of values of θ in $[0, 2\pi]$ for which at least two roots are equal

(a) 3 (b) 4

(c) 5 (d) 6

30. Greatest possible difference between two of roots if $\theta \in [0, 2\pi]$ is

(a) $2\sqrt{2}$ (b) 1

(c) $\sqrt{2}$ (d) 2

Comprehension (31 to 32): Consider the equation $\sec\theta + \operatorname{cosec}\theta = a$, $\theta \in (0, 2\pi) - \{\pi/2, \pi, 3\pi/2\}$

31. If the equation has four distinct real roots, then

(a) $|a| > 2\sqrt{2}$ (b) $|a| < 2\sqrt{2}$

(c) $a \ge -2\sqrt{2}$ (d) $a < -2\sqrt{2}$

32. If the equation has two distinct real roots, then

(a) $|a| \ge 2\sqrt{2}$ (b) $a < 2\sqrt{2}$

(c) $|a| < 2\sqrt{2}$ (d) $|a| \le -2\sqrt{2}$

MATCH THE COLUMN TYPE QUESTIONS

33. Match the following columns

	Column-I		Column-II
A.	If $\cos 3x \cdot \cos^3 x = -\sin 3x . \sin^3 x$ then x is	p.	$\left(n\pi \pm \frac{\pi}{3}\right), n \in I$
B.	$\sin 3\alpha + 2\sin\alpha\,(\cos 2x - \cos 2\alpha) = 0$, then α is where $\alpha \ne n\pi$.	q.	$\left(n\pi + \frac{\pi}{4}\right), n \in I$
C.	If $\|2\tan x - 1\| + \|2\cot x - 1\| = 2$ then x is	r.	$\left(\frac{n\pi}{4} + \frac{\pi}{8}\right), n \in I$
D.	If $\sin^{10}x + \cos^{10}x = \frac{29}{16}\cos^4(2x)$, then x is	s.	$\left(\frac{n\pi}{2} \pm \frac{\pi}{4}\right), n \in I$

(a) A → (q,s); B → (p); C → (q); D → (r)

(b) A → (q); B → (r); C → (q,s); D → (p)

(c) A → (r); B → (p); C → (q); D → (q,s)

(d) A → (p); B → (r); C → (q,s); D → (q)

34. Match the following columns

	Column-I		Column-II
A.	$(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$	p.	$\sin x = 1/2$
B.	$1 + \sin 2x = \cos x + \sin x$	q.	$\tan x = -1$

	Column-I		Column-II
C.	$4x^4 + x^6 + \sin^2 5x = 0$	r.	$x = 0$
D.	$\tan x = 1/\sqrt{3}$	s.	$x = 19\pi/6$

(a) A → (p); B → (q); C → (r); D → (s)
(b) A → (s); B → (q); C → (r); D → (p)
(c) A → (r); B → (q); C → (p); D → (s)
(d) A → (p); B → (s); C → (r); D → (q)

NUMERICAL BASED QUESTIONS

35. Find the number of solutions of the equations, $(1 - \sin x)^3 - (\cos x - 1)^3 - (\sin x)^3 = (2 - \cos x - 2\sin x)^3$ in $[0, 2\pi]$.

36. If the sum of all values of θ, $0 \le \theta \le 3\pi$ satisfying the equation $(8\cos 4\theta - 3)(\cot\theta + \tan\theta - 2)(\cot\theta + \tan\theta + 2) = 12$ is $3p\pi$, then p is equal to

37. The number of solutions for

$$\sin\left(x - \frac{\pi}{4}\right) - \cos\left(x + \frac{3\pi}{4}\right) = 1 \text{ and}$$

$$\frac{2\cos 7x}{\cos 3 + \sin 3} > 2^{\cos 2x} \text{ in } (0, 5\pi/2), \text{ is}$$

38. Find the number of values of y in $[-2\pi, 2\pi]$ for which $|\sin(2x)| + |\cos(2x)| = |\sin(y)|$?

39. The complete set of values of x satisfying $\frac{2\sin 6x}{\sin x - 1} < 0$ and $\sec^2 x - 2\sqrt{2}\tan x \le 0$ in $\left(0, \frac{\pi}{2}\right)$ is $[a, b) \cup (c, d]$, then find the value of $\left(\frac{cd}{ab}\right)$.

40. Find the number of solutions of the equation $2\sin^2 x + \sin^2 2x = 2$; $\sin 2x + \cos 2x = \tan x$ in $[0, 4\pi]$ satisfying the condition $2\cos^2 x + \sin x \le 2$.

41. If $\sin\theta = 3\sin(\theta + 2\alpha)$, then the value of $\tan(\theta + \alpha) + 2\tan\alpha$ is:

42. Number of roots of the equation

$$\cos^2 x + \frac{\sqrt{3}+1}{2}\sin x - \frac{\sqrt{3}}{4} - 1 = 0$$ which lie in the interval $[-\pi, \pi]$ is

43. The number of solutions of $\sin^3 x\cos x + \sin^2 x\cos^2 x + \sin x\cos^3 x = 1$ in $[0, 2\pi]$ is

44. The number of values of x between 0 and 2π that satisfies the equation $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$ must be

45. If $2\tan^2 x - 5\sec x - 1 = 0$ has 7 different roots in $\left[0, \frac{n\pi}{2}\right]$, $n \in N$, Find the greatest value of n.

46. Find the number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is

47. If $\cos^2\theta - 2\cos\theta - 1 = 0$ is to be satisfied for exactly 4 distinct values of $\theta \in [0, n\pi]$, then find the least value of n.

48. The number of values of x satisfying the equation,

$$2^{\tan\left(x - \frac{\pi}{4}\right)} - 2(0.25)^{\frac{\sin^2\left(x - \frac{\pi}{4}\right)}{\cos 2x}} + 1 = 0, x \in [0, 2\pi]$$

49. The number of ordered pair (x, y) satisfying the equation $\sin x + \sin y = \sin(x + y)$ and $|x| + |y| = 1$

50. Find all values of k, x and y for which the system of equations satisfies

$\sin x\cos 2y = (k^2 - 1)2 + 1$

$\cos x\sin 2y = k + 1$

51. If A and B are acute +ve angles satisfying the equations $3\sin^2 A + 2\sin^2 B = 1$ and $3\sin 2A - 2\sin 2B = 0$, then find $A + 2B$.

52. Solve the equation

$$\cos^2\left[\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x)\right] - \tan^2\left(x + \frac{\pi}{4}\tan^2 x\right) = 1.$$

53. Find the sum of all the roots of the equation, $\sin\sqrt{x} = -1$, which are less than $100\pi^2$. Also Find the sum of the square roots of these roots. Now, can we conclude that all the roots $\cos\sqrt{x} = 0$ are also the roots of $\sin\sqrt{x} = -1$? Justify your answer.

54. Solve the equation for x, $5^{\frac{1}{2}} + 5^{\frac{1}{2} + \log_5(\sin x)} = 15^{\frac{1}{2} + \log_{15}(\cos x)}$

55. Find the general solution set of the equation $\log_{\tan x}(2 + 4\cos^2 x) = 2$.

56. Find the value of α and β, $0 < \alpha, \beta < \frac{\pi}{2}$; satisfying the following equations: $\cos\alpha\cos\beta\cos(\alpha + \beta) = -\frac{1}{8}$

57. Find the general solution of the equation

$$\frac{1 + \sin x + \sin^2 x + \sin^3 x + \ldots\ldots + \sin^n x + \ldots\infty}{1 - \sin x + \sin^2 x - \sin^3 x + \ldots(-1)^n x + \ldots\infty} = \frac{4}{1 + \tan^2 x}$$

where $x \ne k\pi + \frac{\pi}{2}, k \in I$.

58. Solve for x: $|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$

59. Solve the system of equations,

$\sin x\sin y = \sqrt{3}/4$

$\cos x\cos y = \sqrt{3}/4$

60. Find the general solution of the equation $|2\tan x - 1| + |2\cot x - 1| = 2$.

61. Prove that $2^{\sin x} + 2^{\cos x} \ge 2^{1 - \frac{1}{\sqrt{2}}}$ for all real x.

Exercise-4 (Past Year Questions)

JEE MAIN

1. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is: **(2016)**

(a) 3 (b) 5
(c) 7 (d) 9

2. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, the value of $\cos 4x$ is: **(2017)**

(a) $-\frac{7}{9}$ (b) $-\frac{3}{5}$
(c) $\frac{1}{3}$ (d) $\frac{2}{9}$

3. If sum of all the solutions of the equation

$8\cos x \cdot \left(\cos\left(\frac{\pi}{6}+x\right)\cdot\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right)=1$ in $[0, \pi]$ is $k\pi$,

then k is equal to : **(2018)**

(a) $\frac{13}{9}$ (b) $\frac{8}{9}$
(c) $\frac{20}{9}$ (d) $\frac{2}{3}$

4. If $0 \le x < \frac{\pi}{2}$, then the number of values of x for which $\sin x - \sin 2x + \sin 3x = 0$, is **(2019)**

(a) 2 (b) 1
(c) 3 (d) 4

5. The sum of all values of $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfying

$\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$ is: **(2019)**

(a) π (b) $\frac{5\pi}{4}$
(c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{8}$

6. The value of $\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot\ldots\cdot\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$ is **(2019)**

(a) $\frac{1}{512}$ (b) $\frac{1}{1024}$
(c) $\frac{1}{256}$ (d) $\frac{1}{2}$

7. Let $S = \{\theta \in [-2\pi, 2\pi] : 2\cos^2\theta + 3\sin\theta = 0\}$. Then the sum of the elements of S is **(2019)**

(a) $\frac{13\pi}{6}$ (b) π
(c) 2π (d) $\frac{5\pi}{3}$

8. The number of solutions of the equation $1 + \sin^4 x = \cos^2 3x$, $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ is : **(2019)**

(a) 5 (b) 4
(c) 7 (d) 3

9. Let S be the set of all $\alpha \in R$ such that the equation, $\cos 2x + \alpha\sin x = 2\alpha - 7$ has a solution. Then S is equal to: **(2019)**

(a) [2, 6] (b) [3, 7]
(c) R (d) [1, 4]

10. If the equation $\cos^4\theta + \sin^4\theta + \lambda = 0$ has real solutions for θ, then λ lies in the interval **(2020)**

(a) $\left[-1, -\frac{1}{2}\right]$ (b) $\left[-\frac{3}{2}, -\frac{5}{4}\right]$
(c) $\left(-\frac{1}{2}, -\frac{1}{4}\right]$ (d) $\left(-\frac{5}{4}, -1\right)$

11. If $\sqrt{3}(\cos^2 x) = (\sqrt{3}-1)\cos x + 1$, the number of solutions of the given equation when $x \in \left[0, \frac{x}{2}\right]$ is **(2021)**

12. If for $x \in \left(0, \frac{\pi}{2}\right)$, $\log_{10}\sin x + \log_{10}\cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, $n > 0$ then the value of n is equal to: **(2021)**

(a) 20 (b) 12 (c) 9 (d) 16

13. The number of solutions of the equation $x + 2\tan x = \frac{\pi}{2}$ in the interval $[0, 2\pi]$ is: **(2021)**

(a) 3 (b) 4 (c) 2 (d) 5

14. The number of solutions of the equation

$\cot x \,| = \cot x + \frac{1}{\sin x}$ in the interval $[0, 2\pi]$ is: **(2021)**

(a) 1 (b) 0 (c) 2 (d) 3

15. Sum of all values of x in $[0, 2\pi]$ for which $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$, is equal to: **(2021)**

(a) 8π (b) 11π (c) 9π (d) 12π

16. The sum of the solution of the equation $\frac{\cos x}{1+\sin x} = |\tan 2x|$, $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) - \left\{\frac{\pi}{4}, -\frac{\pi}{4}\right\}$ is: **(2021)**

(a) $\frac{\pi}{10}$ (b) $-\frac{11\pi}{30}$
(c) $-\frac{7\pi}{30}$ (d) $-\frac{\pi}{15}$

17. Let S be sum of all solution (in radian) of the equation $\sin^4\theta + \cos^4\theta - \sin\theta\cos\theta = 0$ in $[0, 4\pi]$. Then $8s/\pi =$ **(2021)**

18. Number of solutions of the equation

$32^{\tan^2 x} + 32^{\sec^2 x} = 81,\ 0 \le x \le \frac{\pi}{4}$ is **(2021)**

(*a*) 3 (*b*) 1 (*c*) 0 (*d*) 2

19. If n is the number of solutions of the equation

$$2\cos x\left(4\sin\left(\frac{\pi}{4}+x\right)\sin\left(\frac{\pi}{4}-x\right)-1\right)=1, x\in[0,\pi]$$

and S is the sum of all these solutions, then the ordered pair (n, S) is **(2021)**

(*a*) $\left(3, \frac{13\pi}{19}\right)$ (*b*) $\left(2, \frac{2\pi}{3}\right)$

(*c*) $\left(2, \frac{8\pi}{9}\right)$ (*d*) $\left(3, \frac{5\pi}{3}\right)$

20. All possible values of $\theta \in [0, 2\pi]$ for which $\sin 2\theta + \tan 2\theta > 0$ lie in: **(2021)**

(*a*) $\left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$

(*b*) $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$

(*c*) $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

(*d*) $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

21. The number of elements in the set

$$S=\left\{x\in\mathbb{R}:2\cos\left(\frac{x^2+x}{6}\right)=4^x+4^{-x}\right\}$$ **(2022)**

(*a*) 1 (*b*) 3

(*c*) 0 (*d*) Infinite

22. Let

$$S=\left\{\theta\in\left(0,\frac{\pi}{2}\right);\sum_{m=1}^{9}\sec\left(\theta+(m-1)\frac{\pi}{6}\right)\sec\left(\theta+\frac{m\pi}{6}\right)=-\frac{8}{\sqrt{3}}\right\}$$

Then **(2022)**

(*a*) $S=\left\{\frac{\pi}{12}\right\}$ (*b*) $S=\left\{\frac{2\pi}{3}\right\}$

(*c*) $\sum_{\theta\in S}\theta=\frac{\pi}{2}$ (*d*) $\sum_{\theta\in S}\theta=\frac{3\pi}{4}$

23. Let $S=\left\{\theta\in[0,2\pi]:8^{2\sin^2\theta}+8^{2\cos^2\theta}=16\right\}$.

Then $n(S)+\Sigma_{\theta\in S}\left(\sec\left(\frac{\pi}{4}+2\theta\right)\operatorname{cosec}\left(\frac{\pi}{4}+2\theta\right)\right)$ is equal to **(2022)**

(*a*) 0 (*b*) –2

(*c*) –4 (*d*) 12

24. The number of solutions of $|\cos x| = \sin x$, such that $-4\pi \le x \le 4\pi$ is: **(2022)**

(*a*) 4 (*b*) 6

(*c*) 8 (*d*) 12

25. Let $S=\left\{\theta\in[-\pi,\pi]-\left\{\pm\frac{\pi}{2}\right\}:\sin\theta\tan\theta+\tan\theta=\sin 2\theta\right\}$. If

$T=\sum_{\theta\in S}\cos 2\theta$, then $T+n(S)$ is equal **(2022)**

(*a*) $7+\sqrt{3}$ (*b*) 9

(*c*) $8+\sqrt{3}$ (*d*) 10

26. The number of solutions of the equation

$$\cos\left(x+\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}-x\right)=\frac{1}{4}\cos^2 2x, x\in[-3\pi,3\pi]$$ is **(2022)**

(*a*) 8 (*b*) 5

(*c*) 6 (*d*) 7

27. The number of elements in the set

$S=\{\theta\in[-4\pi,4\pi]:3\cos^2 2\theta+6\cos 2\theta-10\cos^2\theta+5=0$ is **(2022)**

28. The number of integral values of 'k' for which the equation $3\sin x + 4\cos x = k + 1$ has a solution, $k \in R$ is **(2022)**

JEE ADVANCED

29. The equation $2\cos^2\frac{x}{2}\sin^2 x = x^2 + x^{-2}; 0 < x \le \frac{\pi}{2}$ has **(1980)**

(*a*) No real solution
(*b*) One real solution
(*c*) More than one solution
(*d*) None of these

30. The general solution of the trigonometric equation is $\sin x + \cos x = 1$ is given by **(1981)**

(*a*) $x = 2n\pi;\ n = 0, \pm1, \pm2 \ldots$
(*b*) $x = 2n\pi + \pi/2;\ n = 0, \pm1, \pm2 \ldots$
(*c*) $x = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4}, n = 0, \pm1, \pm2\ldots$
(*d*) None of these

31. The set of all x in the interval $[0, \pi]$ for which $2\sin^2 x - 3\sin x + 1\ 0$, is _______. **(1987)**

32. For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has **(2014)**

(*a*) Infinitely many solutions
(*b*) Three solutions
(*c*) One solution
(*d*) No solution

33. The number of distinct solutions of the equation $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval $[0, 2\pi]$ is **(2015)**

34. Let $S=\left\{x\in(-\pi,\pi); x\neq 0,\frac{\pi}{2}\right\}$. The sum of all distinct solutions of the equation $\sqrt{3}\sec x+\operatorname{cosec} x+2(\tan x-\cot x)=0$ in the set S is equal to **(2016)**

(a) $-\frac{7\pi}{9}$ (b) $-\frac{2\pi}{9}$

(c) 0 (d) $\frac{5\pi}{9}$

35. Let $-\frac{\pi}{6}<\theta<-\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2-2x\sec\theta+1=0$ and α_1 and β_2 are the roots of the equation $x^2-2x\tan\theta+1=0$. If $\alpha_1>\beta_1$ and $\alpha_2>\beta_2$, then $\alpha_1+\beta_2$ equals **(2016)**

(a) $2(\sec\theta-\tan\theta)$ (b) $2\sec\theta$

(c) $-2\tan\theta$ (d) 0

36. The value of $\sum_{k=1}^{13}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$ is equal to **(2016)**

(a) $3-\sqrt{3}$ (b) $2(3-\sqrt{3})$

(c) $2(\sqrt{3}-1)$ (d) $2(2+\sqrt{3})$

37. Let α and β be non-zero real numbers such that

$2(\cos\beta-\cos\alpha)+\cos\alpha\cos\beta=1$. Then which of the following is/are true? **(2017)**

(a) $\tan\left(\frac{\alpha}{2}\right)+\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(b) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)=0$

(c) $\tan\left(\frac{\alpha}{2}\right)-\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(d) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\beta}{2}\right)=0$

38. Let a, b, c be three non-zero real numbers such that the equation

$\sqrt{3}a\cos x+2b\sin x=c,\ x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

has two distinct real roots α and β with $\alpha+\beta=\frac{\pi}{3}$. Then the value of $\frac{b}{a}$ is______ **(2018)**

Comprehension for 39 and 40

Answer the following by appropriately matching the lists based on the information given in the paragraph

Let $f(x)=\sin(\pi\cos x)$ and $g(x)=\cos(2\pi\sin x)$ be two functions defined for $x>0$. Define the following sets whose elements are written in the increasing order : **(2019)**

$X=\{x: f(x)=0\}$, $Y=\{x: f'(x)=0\}$
$Z=\{x: g(x)=0\}$, $W=\{x: g'(x)=0\}$,

List-I contains the sets X, Y, Z and W. List - II contains some information regarding these sets.

	List-I		List-II
(I)	X	(P)	$\supseteq\left\{\frac{\pi}{2},\frac{3\pi}{2},4\pi,7\pi\right\}$
(II)	Y	(Q)	an arithmetic progression
(III)	Z	(R)	NOT an arithmetic progression
(IV)	W	(S)	$\supseteq\left\{\frac{\pi}{6},\frac{7\pi}{6},\frac{13\pi}{6}\right\}$
		(T)	$\supseteq\left\{\frac{\pi}{3},\frac{2\pi}{3},\pi\right\}$
		(U)	$\supseteq\left\{\frac{\pi}{6},\frac{3\pi}{4}\right\}$

39. Which of the following is the only CORRECT combination ?

(a) (II), (R), (S) (b) (I), (P), (R)

(c) (II), (Q), (T) (d) (I), (Q), (U)

40. Which of the following is the only CORRECT combination ?

(a) (IV), (Q), (T) (b) (IV), (P), (R), (S)

(c) (III), (R), (U) (d) (III), (P), (Q), (U)

41. Let $f:[0,2]\to R$ be the function defined by

$f(x)=(3-\sin(2\pi x))\sin\left(\pi x-\frac{\pi}{4}\right)-\sin\left(3\pi x+\frac{\pi}{4}\right)$

If $\alpha,\beta\in[0,2]$ are such that $\{x\in[0,2]: f(x)\geq 0\}=[\alpha,\beta]$, then the value of $\beta-\alpha$ is .... **(2020)**

42.

	Column-I		Column-II
A.	$\left\{x\in\left[-\frac{2\pi}{3},\frac{2\pi}{3}\right]:\cos x+\sin x=1\right\}$	p.	Has two elements
B.	$\left\{x\in\left[-\frac{5\pi}{18},\frac{5\pi}{18}\right]:\sqrt{3}\tan 3x=1\right\}$	q.	Has three elements
C.	$\left\{x\in\left[-\frac{6\pi}{5},\frac{6\pi}{5}\right]:2\cos(2x)=\sqrt{3}\right\}$	r.	Has four elements
D.	$\left\{x\in\left[-\frac{7\pi}{4},\frac{7\pi}{4}\right]:\sin x-\cos x=1\right\}$	s.	Has five elements
		t.	Has six elements

The correct option is:

(a) A → p; B → s; C → p; D → s

(b) A → p; B → p; C → t; D → r

(c) A → q; B → p; C → t; D → s

(d) A → q; B → s; C → p; D → r

ANSWER KEY

CONCEPT APPLICATION

1. (ϕ) 2. (c) 3. (b) 4. (c) 5. $(2n+1)\pi, n\in I$ 6. $n\pi+\frac{\pi}{2}, n\in I$ 7. (No solution)

8. (d) 9. $n\pi\pm\frac{\pi}{3}, n\in I$ 10. (No solution) 11. (a) 12. $\frac{-n\pi}{2}+\frac{\pi}{6}, n\in I$ 13. (c)

14. $2m\pi, (2n+1)\frac{\pi}{4}, m, n\in I$ 15. [9] 16. $n\pi\pm\frac{\pi}{3}$ 17. $\frac{n\pi}{4}\pm\frac{\pi}{8}, n\in I$ 18. $\frac{-\pi}{2}+2n\pi, n\in I$

19. (b) 20. (No solution) 21. $(n\pi, n\in I)$

22. (a) 23. $n\pi=\frac{\pi}{3}$ 24. $x\in\left[\frac{-\pi}{4},\frac{\pi}{6}\right]\cup\left[\frac{\pi}{4},\frac{5\pi}{6}\right]$ 25. $\left(n\in+\frac{\pi}{8}, n\pi+\frac{\pi}{4}\right), n\in I$

26. $\bigcup_{n\in I}\left[n\pi+\frac{\pi}{6}, n\pi+\frac{\pi}{2}\right)$ 27. (i) $(0,\pi)$ (ii) $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ (iii) $\left(\frac{-7\pi}{6},\frac{5}{6}\pi\right)$ (iv) $\left(\frac{3\pi}{4},\frac{5\pi}{4}\right)$

EXERCISE-1 (TOPICWISE)

1. (d)	2. (c)	3. (b)	4. (a)	5. (b)	6. (a)	7. (c)	8. (b)	9. (b)	10. (d)
11. (b)	12. (a)	13. (a)	14. (a)	15. (c)	16. (b)	17. (b)	18. (b)	19. (b)	20. (c)
21. (d)	22. (c)	23. (c)	24. (c)	25. (a)	26. (a)	27. (a)	28. (a)	29. (c)	30. (c)
31. (d)	32. (c)	33. (d)	34. (d)	35. (d)	36. (d)	37. (a)	38. (a)	39. (c)	40. (b)
41. (a)	42. (d)	43. (a)	44. (d)	45. (b)	46. (a)				

EXERCISE-2 (LEARNING PLUS)

1. (c)	2. (a)	3. (a)	4. (a)	5. (d)	6. (d)	7. (b)	8. (d)	9. (a)	10. (a)
11. (c)	12. (c)	13. (d)	14. (b)	15. (c)	16. (a)	17. (b)	18. (b)	19. (a)	20. (a)
21. (d)	22. (b)	23. (b)	24. (c)	25. (c)	26. (b)	27. (d)	28. (b)	29. (d)	30. (d)
31. (b)	32. (a)	33. (b)	34. (a)	35. (d)	36. (c)	37. (c)	38. (a)	39. (b)	40. (b)
41. (b)	42. (a)	43. (a)	44. (d)	45. (b)	46. (a)	47. (a,b,c,d)		48. (c,d)	49. (a,d)

50. (a,b,c,d) 51. (a,c,d) 52. (a,c) 53. [2,6] 54. $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ 55. $\frac{3\pi}{8}$ 56. [5]

57. $x=(2n+1)\frac{\pi}{2}$ or $x=n\pi\pm\frac{\pi}{3}; n\in I$

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (a)	2. (b)	3. (a)	4. (b)	5. (d)	6. (a,c,d)	7. (a,d)	8. (a,b)	9. (a,b,c,d)	10. (b,c,d)
11. (b,c)	12. (d)	13. (a,b,c,d)	14. (a,b,c,d)	15. (a,c)	16. (b,c)	17. (b,c)	18. (c,d)	19. (a,d)	20. (b,c)
21. (b,c)	22. (a)	23. (c)	24. (c)	25. (a)	26. (b)	27. (c)	28. (d)	29. (c)	30. (d)
31. (a)	32. (c)	33. (a)	34. (a)	35. [5]	36. [6]	37. [1]	38. [4]	39. [6]	40. [8]
41. [0]	42. [4]	43. [0]	44. [4]	45. [15]	46. [2]	47. [4]	48. [0]	49. [6]	50. [–1]

51. $\frac{\pi}{2}$ 52. $n\pi+(-1)^n\frac{\pi}{4}$ 53. $\frac{765\pi^2}{4}, \frac{55\pi}{4}$ 54. $2n\pi+\frac{\pi}{6}$ 55. $\left(2n\pi\pm\frac{\pi}{3}\right)\cup\left(2m\pi\pm\frac{2\pi}{3}\right); n, m\in I$ 56. $\alpha=\beta=\frac{\pi}{3}$

57. $n\pi+(-1)^n\frac{\pi}{6}$ 58. $(n\pi)$ or $\left(n\pi+(-1)^n\frac{\pi}{6}\right)$ 59. $x=\{(2n+k)\frac{\pi}{2}+\frac{\pi}{3}, (2n+k)\frac{\pi}{2}+\frac{\pi}{6}; n, k\in I\}$ 60. $n\pi+\frac{\pi}{4}$

$y=\{(k-2n)\frac{\pi}{2}+\frac{\pi}{6}, (k-2n)\frac{\pi}{2}+\frac{\pi}{3}; n, k\in I\}$

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (c) **2.** (a) **3.** (a) **4.** (a) **5.** (c) **6.** (a) **7.** (c) **8.** (a) **9.** (b) **10.** (a)
11. [1] **12.** (b) **13.** (a) **14.** (a) **15.** (c) **16.** (b) **17.** [56] **18.** (b) **19.** (a) **20.** (b)
21. (a) **22.** (c) **23.** (c) **24.** (c) **25.** (b) **26.** (d) **27.** [32] **28.** [11]

JEE Advanced

29. (d) **30.** (c) **31.** $x = \left[0, \frac{\pi}{6}\right] \cup \left\{\frac{\pi}{2}\right\} \cup \left[\frac{5\pi}{6}, \pi\right]$ **32.** (d) **33.** [8] **34.** (c) **35.** (c) **36.** (c)
37. (a,c) **38.** [0.5] **39.** (c) **40.** (b) **41.** [1.00] **42.** (b)

CHAPTER

6 Principle of Mathematical Induction

INTRODUCTION

In algebra, there are certain results that are formulated with n number of terms in them, where n is a natural number (i.e. a positive integer). Those results can be proved by a specific technique, known as the principle of mathematical induction. We use the symbol $P(n)$ (read as "P of n") to denote some proposition which depends on the positive integer n. For example, $P(n)$ might denote the sum of the first n odd positive numbers, that is $1 + 3 + 5 + 9 + ... + (2n - 1) = n^2$

where $n = 1, 2, 3,......, n$

FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

The proof of the proposition $P(n)$ by mathematical induction for all $n \in N$ consists of the following three steps:

Step-I (Elementary step)

Verify that the proposition $P(n)$ is true for $n = 1$, i.e., the first natural number or the smallest positive integer. This is also called the basic step of the induction and is generally very easy.

Step-II (Assumption step)

Assume that the proposition will also be true for some $n = k \geq 1$, i.e. we assume $P(k)$ to be true. This is called the induction step.

Step-III (Verification step)

If $P(k)$ is true, then prove that the proposition is also true for $n = (k + 1)$, which is the next positive integer (i.e. the next natural number), i.e. we have to prove that $P(k + 1)$ must also be true. In this step we prove that the implication $P(k) \Rightarrow P(k + 1)$ is true.

Hence the proposition will be true for all n belonging to the set of natural numbers.

SECOND PRINCIPLE OF MATHEMATICAL INDUCTION EXTENDED PRINCIPLE

Sometimes, the first principle of mathematical induction does not suffice. In such cases we use the extended principle as below:

Step-I (Elementary step)

We verify that $P(n)$ is true for i and $(i + 1)$ both.

Step-II (Assumption step)

Assume that $P(n)$ is true for $n = k$ and $n = k + 1$, $k \geq i$, and prove that $P(n)$ is true for $n = (k + 2)$.

Step-III (Verification step)

Combining the above two steps leads to the conclusion that $P(n)$ is thus true $\forall\ n \in N$.

Note: The second principle of mathematical induction is useful to prove recurrence relations which involve three successive terms e.g.,

$$\boxed{p\,T_{n+1} = q\,T_n + rT_{n-1}}$$

Train Your Brain

Example 1: By mathematical induction,

$$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + ... + \frac{1}{n(n+1)(n+2)} \text{ is equal to}$$

(a) $\dfrac{n(n+1)}{4(n+2)(n+3)}$ (b) $\dfrac{n(n+2)}{4(n+1)(n+4)}$

(c) $\dfrac{n(n+2)}{4(n+1)(n+3)}$ (d) $\dfrac{n(n+3)}{4(n+1)(n+2)}$

Sol. (d) Let $P(n) : \dfrac{1}{1\cdot2\cdot3} + \dfrac{1}{2\cdot3\cdot4} + ... + \dfrac{1}{n(n+1)(n+2)}$

$$= \frac{n(n+3)}{4(n+1)(n+2)}$$

(i) For $n = 1$

L.H.S. $= \dfrac{1}{1\cdot2\cdot3} = \dfrac{1}{6}$ and

R.H.S. $= \dfrac{1(1+3)}{4(1+1)(1+2)} = \dfrac{1}{6}$

$\therefore$ $P(1)$ is true.

(ii) Let $P(k)$ be true, then

$$P(k) : \frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + ... + \frac{1}{k(k+1)(k+2)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)} \quad ...(i)$$

(*iii*) For $n = k + 1$,

$$P(k+1): \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$\text{L.H.S.} = \frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

[From Eq. (i)]

$$=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

= R.H.S.

Hence, $P(k + 1)$ is true.

Hence, by the principle of mathematical induction for all $n \in N$, $P(n)$ is true.

Example 2: For all $n \in N$, $(3)(5^{2n+1}) + 2^{3n+1}$ is divisible by

(*a*) 17 (*b*) 19

(*c*) 21 (*d*) 23

Sol. (*a*) $P(n) : 3(5^{2n+1}) + 2^{3n+1}$

$P(1) : 3(5^3) + 2^4 = 3(125) + 16 = 375 + 16 = 391 = 17(23)$

So, $P(1)$ is divisible by 17.

Let $P(k) : 3(5^{2k+1}) + 2^{3k+1} = 17\,m$

$\therefore \quad P(k + 1) : 3(5^{2k+3}) + 2^{3k+4} = 17\lambda$

L.H.S. of $P(k+1) = 3(5^{2k+3}) + 2^{3k+4}$

$= 3(5^{2k+1})(5^2) + (2^{3k+1})(2^3)$

$= (17\,m - 2^{3k+1})(25) + 8(2^{3k+1})$

[Assuming $P(k)$ to be true]

$= 17(25\,m) - 25(2^{3k+1}) + 8(2^{3k+1})$

$= 17(25m) - 17(2^{3k+1})$

$= 17(25m - 2^{3k+1}) = 17\lambda$

Then, $P(k + 1)$ is divisible by 17 whenever $P(k)$ is divisible by 17. Hence $P(n)$ is divisible by 17 for all $n \in N$.

Example 3: If $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$, then n is

(*a*) a positive integer

(*b*) an even positive integer

(*c*) an odd positive integer

(*d*) None of these

Sol. (*a*) $P(n) : x^{2n-1} + y^{2n-1}$

$P(1) : x^1 + y^1 = x + y$, which is divisible by $x + y$

Let $P(k) : x^{2k-1} + y^{2k-1} = (x + y)\,m$

$\therefore \quad P(k + 1) : x^{2k+1} + y^{2k+1} = (x + y)\,\lambda$

L.H.S. of $P(k + 1) = x^{2k+1} + y^{2k+1}$

$= x^2(x^{2k-1}) + y^2(y^{2k-1})$

$= x^2[m(x + y) - y^{2k-1}] + y^2(y^{2k-1})$

[Assuming $P(k)$ to be true]

$= (x + y)(mx^2) - y^{2k-1}(x^2 - y^2)$

$= (x + y)[mx^2 - y^{2k-1}(x - y)] = (x + y)\,\lambda$

Thus, $P(k + 1)$ is divisible by $(x + y)$ whenever $P(k)$ is divisible by $(x + y)$.

Hence $P(n)$ is divisible by $x + y$ for all $n \in N$, i.e. for all positive integers.

Example 4: Show that $n! > 3^n$ for $n \geq 7$.

Sol. For any $n \geq 7$, let P_n be the statement that $n! > 3^n$.

Elementary Case: The statement P_7 says that $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040 > 3^7 = 2187$, which is true.

Assumption Step: Fix $k \geq 7$, and suppose that P_k holds, that is, $k! > 3^k$.

It remains to show that P_{k+1} holds, that is, that $(k + 1)! > 3^{k+1}$.

$(k + 1)! = (k + 1)\,k!$

$> (k + 1)\,3^k \geq (7 + 1)\,3^k = 8 \times 3^k$

$> 3 \times 3^k = 3^{k+1}$.

Therefore P_{k+1} holds.

Thus by the principle of mathematical induction, for all $n \geq 7$, P_n holds.

Concept Application

1. For any natural number 'n' prove by M.I. that, $4n + 15n - 1$ is divisible by 9

2. For all $n \geq 1$ natural no. $(3^{2^n} - 1)$ is divisible by

   (*a*) $2n^{+3}$ (*b*) $2n^{+2}$

   (*c*) $2n^{+5}$ (*d*) $3n^{+2}$

3. If 'n' is a natural number than, $(25)n^{+1} - 24\,n + 5735$ is divisible by?

   (*a*) 24^4 (*b*) 24^3

   (*c*) 24^2 (*d*) 25

Some useful Result based on Principle of Mathematical Induction

For any Natural number *n*

(*i*) $1 + 2 + 3 + \ldots + n = \Sigma n = \dfrac{n(n+1)}{2}$

(*ii*) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(*iii*) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \Sigma n^3 = (\Sigma n)^2 = \left\{\dfrac{n(n+1)}{2}\right\}^2$

(iv) $2 + 4 + 6 + \ldots.. + 2n = \Sigma 2n = n(n + 1)$

(v) $1 + 3 + 5 + \ldots.. + (2n - 1) = \Sigma(2n - 1) = n^2$

(vi) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots\ldots + xy^{n-2} + y^{n-1})$

(vii) $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 + \ldots.. - xy^{n-2} + y^{n-1})$

when n is odd positive integer

Train Your Brain

Example 5: If $a_1 = 1$, $a_2 = 5$ and $a_{n+2} = 5a_{n+1} - 6a_n$, $n \in N$. Show by using mathematical induction that $a_n = 3^n - 2^n$.

Sol. Let $P(n) : a_n = 3^n - 2^n$

When $n = 1$, L.H.S. $= a_1 = 1$ (given)

and R.H.S. $= 3^1 - 2^1 = 3 - 2 = 1$.

$\therefore$ L.H.S. = R.H.S.

When $n = 2$, L.H.S. $= a_2 = 5$ (given)

and R.H.S. $= 3^2 - 2^2 = 5$

Thus L.H.S = R.H.S.

Hence $P(1)$ and $P(2)$ are true ... (i)

Let $P(m)$ and $P(m + 1)$ be true

$$\Rightarrow \begin{cases} a_m = 3^m - 2^m \\ a_{m+1} = 3^{m+1} - 2^{m+1} \end{cases} \quad \ldots (ii)$$

To prove $P(m + 2)$ is true *i.e.*, $a_{m+2} = 3^{m+2} - 2^{m+2}$

Given, $a_{n+2} = 5a_{n+1} - 6a_n$

$\therefore \quad a_{m+2} = 5a_{m+1} - 6a_m = 5(3^{m+1} - 2^{m+1}) - 6(3^m - 2^m)$

$= (5.3^{m+1} - 6.3^m) - (5.2^{m+1} - 6.2^m) = 3^m(15 - 6) - 2^m(10 - 6)$

$= 3^m.9 - 2^m.4 = 3^{m+2} - 2^{m+2}$.

Hence $P(m + 2)$ is true whenever $P(m)$ and $P(m + 1)$ are true. By the principle of mathematical induction it follows that $P(n)$ is true for all natural numbers n.

Concept Application

4. If $a_n = 2^{2^n} + 1$, then for $n > 1$, last digit of a_n is

(a) 5

(b) 7

(c) 3

(d) 4

5. If $a_n = \sqrt{7 + \sqrt{7 + \sqrt{7 + \ldots}}}$ having n radical signs then by methods of mathematical induction which of the following is true

(a) $a_n > 7 \ \forall \ n \geq 1$

(b) $a_n > 3 \ \forall \ n \geq 1$

(c) $a_n < 4 \ \forall \ n \geq 1$

(d) $a_n < 3 \ \forall \ n \geq 1$

6. Let $x > -1$, then statement

$P(n) : (1 + x)^n > 1 + nx$ is true for

(a) all $n \in N$

(b) all $n > 1$

(c) all $n > 1$ provided $x \neq 0$

(d) None of these

7. If $a, b, c, \in N$, $a^n + b^n$ is divisible by c when n is odd but not when n is even, then value of c is

(a) $a + b$

(b) $a - b$

(c) $a^3 + b^3$

(d) $a^3 - b^3$

Short Notes

(i) $1 + 2 + 3 + \ldots + n = \Sigma n = \dfrac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \Sigma n^3 = (\Sigma n)^2 = \left\{\dfrac{n(n+1)}{2}\right\}^2$

(iv) $1^4 + 2^4 + 3^4 + \ldots + n^4 = \dfrac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30}$

(v) $2 + 4 + 6 + \ldots.. + 2n = \Sigma 2n = n(n + 1)$

(vi) $1 + 3 + 5 + \ldots.. + (2n - 1) = \Sigma(2n - 1) = n^2$

(vii) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots\ldots + xy^{n-2} + y^{n-1})$, where $n \in N$

(viii) $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 + \ldots.. - xy^{n-2} + y^{n-1})$, where n is odd positive integer

(ix) $a + (a + d) + (a + 2d) + \ldots. + (a + (n - 1)d) = \dfrac{n}{2}[2a + (n - 1)d]$

(x) $a + ar + ar^2 + \ldots + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$, where $r \neq 1$

(xi) $(a + b)^n = \sum_{r=0}^{n} {}^nC_r \, a^{n-r} b^r$

(xii) $\cos(a)\cdot\cos(2a)\cdot\cos(4a)\ldots\cos(2^{n-1}a)=\dfrac{\sin(2^n a)}{2^n\sin a}$

(xiii) $\sin(a)+\sin(a+b)+\sin(a+2b)+\ldots+\sin(a+(n-1)b)$

$=\dfrac{\sin(nb/2)}{\sin(b/2)}\sin(a+(n-1)\dfrac{b}{2})$

(xiv) $\cos(a)+\cos(a+b)+\cos(a+2b)+\ldots+\cos(a+(n-1)b)$

$=\dfrac{\sin(nb/2)}{\sin(b/2)}\cos(a+(n-1)\dfrac{b}{2})$

Solved Examples

1. Let $P(n): n^2+n$ is an odd integer. It is seen that truth of $P(n)$ $\Rightarrow$ the truth of $P(n+1)$. Therefore, $P(n)$ is true for all

(a) $n>1$

(b) $n\in N$

(c) $n>2$

(d) None of these

Sol. (d) $\because p(1), p(2), p(3)$ is not true

$\therefore$ none of these

2. If $n\in N$, then $3^{4n+2}+5^{2n+1}$ is a multiple of

(a) 14 (b) 16

(c) 18 (d) 20

Sol. (a) Let $p(n)=3^{4n+2}+5^{2n+1}$

Here $P(1)=3^6+5^3=9^3+5^3=14\times 61$

Which is multiple of 14 but not of 16, 18 and 20.

3. If P is a prime number then n^P-n is divisible by P when n is a

(a) Natural number greater than 1

(b) Odd number

(c) Even number

(d) None of these

Sol. (a) Let $P(n): n^P-n$, P is prime

$P(1)$: $1^P-1=0$ divisible by all prime

Let $P(k)$ is divisible by P(any prime)

Then $k^P-k=\lambda P$

Now, for $n=k+1$

$P(k+1)$: $(k+1)^P-(k+1)$

$=(k^P+{}^PC_1k^{P-1}+\ldots+{}^PC_{P-1}K+{}^PC_PK^0)-(k+1)$

$=(k^P-k)+[{}^PC_1k^{P-1}+\ldots+{}^PC_{P-1}k]+{}^PC_Pk^0-1$

$\{\because {}^PC_1k^{P-1}+\ldots+{}^PC_{P-1}k$ is divisible by P. and ${}^PC_Pk^0=1\}$

$=\lambda P+\mu P=(\lambda+\mu)P=\delta P$

Hence, $P(k+1)$ is divisible by prime P.

Hence, by the principle of mathematical induction for all $n\in N$, $P(n)$ is true.

4. The n^{th} term of the series $4+14+30+52+80+114+\ldots$ is

(a) $5n-1$

(b) $2n^2+2n$

(c) $3n^2+n$

(d) $2n^2+2$

Sol. (c) Let n^{th} term of the series is T_n and

$S_n=4+14+30+52+80+114+\ldots+T_n$ ...(i)

$S_n=4+14+30+52+80+\ldots+T_n$ ...(ii)

Subtract (ii) from (i)

$0=(4+10+16+22+28+34+\ldots n \text{ terms})-T_n$

$T_n=4+10+16+22+\ldots n$ terms

$=\dfrac{n}{2}[2\times 4+(n-1)6]=n(3n+1)=3n^2+n$

5. $n^3+(n+1)^3+(n+2)^3$ is divisible for all $n\in N$ by

(a) 3 (b) 9

(c) 27 (d) 81

Sol. (b) Let $p(n)=n^3+(n+1)^3+(n+2)^3$,

$p(1)=36$, $p(2)=99$ both are divisible by 9

Let it is true for $n=k$

$k^3+(k+1)^3+(k+2)^3=9q$; $q\in I$

adding $9k^2+27k+27$ both sides

$k^3+(k+1)^3+(k+2)^3+9k^2+27k+27=9q+9k^2+27k+27$

$(k+1)^3+(k+2)^3+(k+3)^3$

$=9r$; $r\in I$

Exercise-1 (Topicwise)

EXISTENCE OF LIMIT

1. For all $n \in N$, $a^{2n-1} + b^{2n-1}$ is divisible by
   (a) $a + b$ (b) $(a + b)^2$
   (c) $a^3 + b^3$ (d) None of these

2. The greatest positive integer which divides $(n + 1)(n + 2)(n + 3) \ldots (n + r)$ for all $n \in N$ is
   (a) r (b) $r!$
   (c) $n + r$ (d) $(r + 1)!$

3. $49^n + 16n - 1$ is divisible by: $(n \in N)$
   (a) 3 (b) 64
   (c) 19 (d) 29

4. The statement $P(n) : 1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n + 1)! - 1$ is
   (a) True for no n (b) True for all $n > 1$
   (c) True for all $n \in N$ (d) True for all $n > 2$

5. For all $n \in N$, $\cos\alpha \cos 2\alpha \cos 4\alpha \ldots \cos(2^{n-1}\alpha)$ is equal to $(\alpha \neq n\pi)$
   (a) $\dfrac{\sin(2^n\alpha)}{2\sin\alpha}$ (b) $\dfrac{\sin(2^n\alpha)}{2^n\sin\alpha}$
   (c) $\dfrac{\cos(2^n\alpha)}{2^n\cos\alpha}$ (d) $\dfrac{\cos(2^n\alpha)}{2^{n-1}\cos\alpha}$

6. $2^{3n} - 7n - 1$ is divisible by, $(n \in N)$
   (a) 64 (b) 36
   (c) 49 (d) 25

7. If $10^n + 3 \cdot 4^{n+2} + k$ is divisible by 9 for all $n \in N$, then the least positive integral value of k is
   (a) 5 (b) 3
   (c) 7 (d) 1

8. For all $n \in N$, $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by
   (a) 23 (b) 3
   (c) 7 (d) 1

9. If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is: $(n \in N)$
   (a) 1 (b) 2
   (c) 3 (d) 4

10. The sum of n terms of the series $\dfrac{1^3}{1} + \dfrac{1^3 + 2^3}{1+3} + \dfrac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots$
    (a) $\dfrac{n(n+1)^3(n+2)}{24}$
    (b) $\dfrac{n(2n^2+9n+13)}{24}$
    (c) $\dfrac{4n^2+1}{5}$
    (d) None of these

11. For all $n \in N$, $\displaystyle\int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\,dx$ is equal to
    (a) $1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}$ (b) $1 + \dfrac{1}{3} + \dfrac{1}{5} + \ldots + \dfrac{1}{2n-1}$
    (c) 0 (d) None of these

12. **Statement-1:** $3^{2n}\ \forall\ n \in N$ leaves the remainder 1 when divided by 8.
    Statement-2: $9^n = 1 + 8\lambda$. $(n \in N)$
    (a) Statement-I is true, statement-II is true and statement-II is **correct** explanation for statement-I
    (b) Statement-I is true, statement-II is true but statement-II is **NOT** the correct explanation for statement-I
    (c) Statement-I is true, statement-II is false
    (d) Statement-I is false, statement-II is true

13. **Statement-1 :** $P(n) = n^2 + n + 1$ is an odd natural number $\forall\ n \in N$.
    Statement-2: If 1 is added to an even number, then it becomes an odd number.
    (a) Statement-1 is true, statement-2 is true and statement-2 is **correct** explanation for statement-1
    (b) Statement-1 is true, statement-2 is true but statement-2 is **NOT** the correct explanation for statement-1
    (c) Statement-1 is true, statement-2 is false
    (d) Statement-1 is false, statement-2 is true

14. **Statement-1:** The greatest positive integer which divides $(n + 11)(n + 12)(n + 13)(n + 14)\ \forall\ n \in N$ is 24.
    Statement-2: The product of any n consecutive positive integers is divisible by $n!$.
    (a) Statement-1 is true, statement-2 is true and statement-2 is **correct** explanation for statement-1
    (b) Statement-1 is true, statement-2 is true but statement-2 is **NOT** the correct explanation for statement-1
    (c) Statement-1 is true, statement-2 is false
    (d) Statement-1 is false, statement-2 is true

15. **Statement-1:** The digit in the unit place of $183! + 3^{183}$ is 7.
    Statement-2: 183! has unit place 0 and 3^{4k+3} ends with 7.
    (a) Statement-1 is true, statement-2 is true and statement-2 is **correct** explanation for statement-1
    (b) Statement-1 is true, statement-2 is true but statement-2 is **NOT** the correct explanation for statement-1
    (c) Statement-1 is true, statement-2 is false
    (d) Statement-1 is false, statement-2 is true

16. Let $P(n)$ be statement $2n < n!$. Where n is a natural number, then $P(n)$ is true for:
    (a) All $n > 3$ (b) All n
    (c) All $n > 2$ (d) None of these

17. The statement $P(n) : (n+3)^2 > 2^{n+3}$ is true for:
(a) All $n \geq 3$
(b) All n.
(c) All $n \geq 2$
(d) No $n \in N$

18. The greatest positive integer. which divides $(n+16)(n+17)(n+18)(n+19)$, for all $n \in N$, is-
(a) 2
(b) 4
(c) 24
(d) 120

19. The sum of the cubes of three consecutive natural numbers is divisible by-
(a) 2
(b) 5
(c) 7
(d) 9

20. For every positive integer n, $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is
(a) an integer
(b) a rational number
(c) a negative real number
(d) an odd integer

21. For positive integer n, $3n < n!$ when–
(a) $n \geq 6$
(b) $n > 7$
(c) $n \geq 4$
(d) $n \leq 7$

22. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$, $n \in N$, is true for
(a) $n \geq 3$
(b) $n \geq 2$
(c) $n \geq 4$
(d) all n

23. For every natural number n, $n(n+3)$ is always–
(a) Multiple of 4
(b) Multiple of 5
(c) Even
(d) Odd

24. If $p(n) : n^2 > 100$ then
(a) $p(1)$ is true
(b) $p(4)$ is true
(c) $p(k)$ is true $\forall\, k \geq 5, k \in N$
(d) $p(k+1)$ is true whenever $p(k)$ is true where $k \in N$

25. The inequality $n! > 2n - 1$ is true
(a) For all $n > 1$
(b) For all $n > 2$
(c) For all $n \in N$
(d) None of these

26. $1 + 2 + 3 + \ldots\ldots + n < \frac{(n+2)^2}{8}$, $n \in N$, is true for
(a) $n \geq 1$
(b) $n \geq 2$
(c) all n
(d) None of these

27. A student was asked to prove a statement by induction. He proved
(i) $P(5)$ is true and
(ii) Truth of $P(n) \Rightarrow$ truth of $p(n+1)$, $n \in N$
On the basis of this, he could conclude that $P(n)$ is true for
(a) No $n \in N$
(b) All $n \in N$
(c) All $n \geq 5$
(d) None of these

28. The difference between an + ve integer and its cube is divisible by–
(a) 4
(b) 6
(c) 9
(d) None of these

29. For all $n \in N$, $\sum n$
(a) $< \frac{(2n+1)^2}{8}$
(b) $> \frac{(2n+1)^2}{8}$
(c) $= \frac{(2n+1)^2}{8}$
(d) None of these

30. If n is a natural number then $\left(\frac{n+1}{2}\right)^n \geq n!$ is true when–
(a) $n > 1$
(b) $n \geq 1$
(c) $n > 2$
(d) Never

31. If $n \in N$, then $11^{n+2} + 12^{2n+1}$ is divisible by-
(a) 113
(b) 123
(c) 133
(d) None of these

Exercise-2 (Learning Plus)

NUMERICAL BASED QUESTIONS

1. If $a_1 = 1$ and $a_n = na_{n-1}$ for all positive integer $n \geq 2$, then a_5 is equal to

2. $(2^{3n} - 1)$ will be divisible by $(\forall n \in N)$

3. The product of any three consecutive natural numbers is divisible by

4. The greatest positive integer, which divides $(n+2)(n+3)(n+4)(n+5)(n+6)$ for all $n \in N$ is

5. If inequality $n! > 2^{n-1}$ is true then n is greater than

6. Prove the following by the principle of mathematical induction:
$$a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{r^n - 1}{r - 1}\right), r \neq 1$$

7. Prove that $\frac{(2n)!}{2^{2n}(n!)^2} \leq \frac{1}{\sqrt{3n+1}}$ for all $n \in N$

8. Prove that $7\sin x + \sin 3x + \ldots + \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}$
for all $n \in N$.

9. Prove that $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos$
$$(\alpha + (n-1)\beta) = \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$
for all $n \in N$.

10. Given $a_1 = \frac{1}{2}\left(a_0 + \frac{A}{a_0}\right), a_2 = \frac{1}{2}\left(a_1 + \frac{A}{a_1}\right)$ and

$a_{n+1} = \frac{1}{2}\left(a_n + \frac{A}{a_n}\right)$ for $n \geq 2$, where $a > 0, A > 0$.

11. Prove that $\frac{a_n - \sqrt{A}}{a_n + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^{n-1}}$.

12. Show by the Principle of Mathematical induction that the sum S_n of the n terms of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + 7^2 + \ldots$ is given by

$$S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \frac{n^2(n+1)^2}{2}, & \text{if } n \text{ is odd} \end{cases}$$

13. A sequence $x_0, x_1, x_2, x_3, \ldots$ is defined by letting $x_0 = 5$ and $x_k = 4 + x_{k-1}$ for all natural numbers k. Show that $x_n = 5$ for all $n \in N$ using mathematical induction.

14. Mark the correction alternative in the following: A student was asked to prove a statement $P(n)$ by induction. He proved $P(k + 1)$ is true whenever $P(k)$ is true for all $k > 5 \in N$ and also $P(5)$ is true. On the basis of this he could conclude that $P(n)$ is true.

(*a*) for all $n \in N$
(*b*) for all $n > 5$
(*c*) for all $n \geq 5$
(*d*) for all $n < 5$

15. Prove that

$$\frac{1}{2}\tan\left(\frac{x}{2}\right) + \frac{1}{4}\tan\left(\frac{x}{4}\right) + \ldots + \frac{1}{2^n}\tan\left(\frac{x}{2^n}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right) - \cot x$$

for all $n \in N$ and $0 < x < \frac{\pi}{2}$.

Exercise-3 (Past Year Questions)

JEE MAIN

1. Let $S(k) = 1 + 3 + 5 + \ldots\ldots + (2k - 1) = 3 + k^2$, then which of the following is true? **(2004)**

(*a*) $S(1)$ is true
(*b*) $S(k) \Rightarrow S(k + 1)$
(*c*) $S(k) \nRightarrow S(k + 1)$
(*d*) Principle of mathematical Induction can be used to prove that formula

2. **Statement-I**: For every natural number $n \geq 2$
$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$

Statement-II: For every natural number $n \geq 2$, $\sqrt{n(n+1)} < n + 1$ **(2007)**

(*a*) Statement–I is false, Statement–II is true
(*b*) Statement–I is true, Statement–II is false
(*c*) Statement–I is true, Statement–II is true; Statement–II is a correct explanation for Statement–I
(*d*) Statement–I is true, Statement–II is true; Statement–II is not a correct explanation for Statement–I

3. **Statement-I**: For each natural number $n, (n + 1)^7 - n^7 - 1$ is divisible by 7.

Statement-II: For each natural number n, $n^7 - n$ is divisible by 7. **(2011)**

(*a*) Statement–I is false, Statement –II is true
(*b*) Statement–I is true, Statement–II is false
(*c*) Statement–I is true, Statement–II is true; Statement–II is a correct explanation for Statement–I
(*d*) Statement–I is true, Statement–II is true; Statement–II is not a correct explanation for Statement–I

4. Consider the statement: "$P(n) : n^2 - n + 41$ is prime." Then which one of the following is true? **(2019)**

(*a*) Both $P(3)$ and $P(5)$ are true
(*b*) $P(3)$ is false but $P(5)$ are true
(*c*) Both $P(3)$ and $P(5)$ are false
(*d*) $P(5)$ is false but $P(3)$ are true

ANSWER KEY

CONCEPT APPLICATION

2. (*b*) **3.** (*c*) **4.** (*b*) **5.** (*c*) **6.** (*c*) **7.** (*a*)

EXERCISE-1 (TOPICWISE)

1. (*a*)	**2.** (*b*)	**3.** (*b*)	**4.** (*c*)	**5.** (*b*)	**6.** (*c*)	**7.** (*a*)	**8.** (*b*)	**9.** (*a*)	**10.** (*b*)
11. (*b*)	**12.** (*a*)	**13.** (*a*)	**14.** (*a*)	**15.** (*a*)	**16.** (*a*)	**17.** (*d*)	**18.** (*c*)	**19.** (*d*)	**20.** (*a*)
21. (*c*)	**22.** (*d*)	**23.** (*c*)	**24.** (*d*)	**25.** (*b*)	**26.** (*d*)	**27.** (*c*)	**28.** (*b*)	**29.** (*a*)	**30.** (*b*)
31. (*c*)									

EXERCISE-2 (LEARNING PLUS)

1. [120] **2.** [7] **3.** [6] **4.** [120] **5.** [2]

EXERCISE-3 (PAST YEAR QUESTIONS)

1. (*c*) **2.** (*a*) **3.** (*b*) **4.** (*c*)