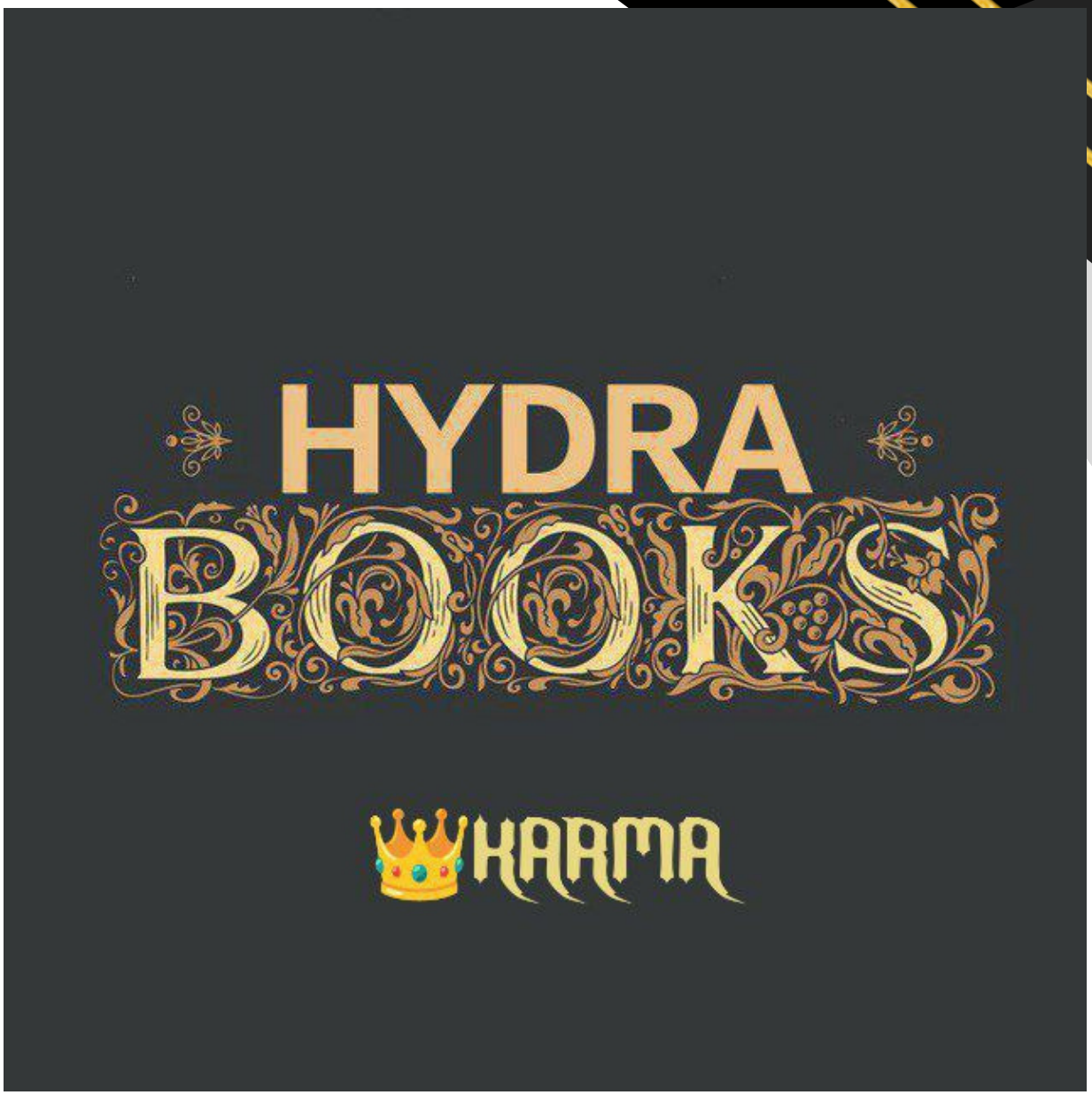

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ARJUNA

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MAIN & ADVANCED

MATHEMATICS

FULL COURSE STUDY MATERIAL

Class XI

- Quadratic Equations
- Complex Number-I
- Linear Inequalities
- Permutations and Combinations
- Binomial Theorem
- Sequences and Series

Module-2

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CHAPTER

7 Quadratic Equations

GENERAL POLYNOMIAL

An expression written as, $a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0$, where $a_0, a_1, a_2, \ldots\ldots, a_n \in R$ is called a **polynomial** of degree n where ($a_n \neq 0$, $n \in W$).

If $a_0, a_1, a_2, \ldots, a_n \in C$, then it is called **complex coefficient polynomial**.

QUADRATIC POLYNOMIAL

$ax^2 + bx + c$, is a polynomial of degree two in one variable, where $a \neq 0$ & $a, b, c \in R$. ($a \rightarrow$ leading coefficient, $c \rightarrow$ absolute term / constant term)

If $a = 0$ then above expression redues to, $bx + c$, which kown as **linear polynomial** ($b \neq 0$). If $c = 0$ in linear polynomial it reduces to, bx which is **odd linear polynomial**.

QUADRATIC EQUATIONS WITH REAL COEFFICIENTS

The expression, $ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$ is called as a quadratic expression in x. The quadratic expression when equated to zero, $ax^2 + bx + c = 0$, is called as a quadratic equation in x. Where the numbers a, b, c are called the coefficients of the equation.

1. The values of x which satisfy the quadratic equation is called roots (also called solutions or zeros) of the quadratic equation.
2. This equation has two roots which are given by $x = \dfrac{-b \pm \sqrt{D}}{2a}$,

   where D (or Δ) $= b^2 - 4ac$ is called as discriminant of the quadratic equation.

Train Your Brain

Example 1: The roots of $3x^2 - 5x - 12 = 0$ are

Sol. $x = \dfrac{5 \pm \sqrt{25 - 4(3)(-12)}}{2 \times 3}$ i.e. $x = -\dfrac{4}{3}, 3$

or (using factorization method)

$3x^2 - 9x + 4x - 12 = 0$

$3x(x - 3) + 4(x - 3) = 0$

$(3x + 4)(x - 3) = 0, x = -4/3, 3$

Example 2: The quadratic equation whose roots are $1+\sqrt{2}, 1-\sqrt{2}$ is

Sol. $x^2 - \left(1+\sqrt{2}+1-\sqrt{2}\right)x + \left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right) = 0$

i.e. $x^2 - 2x + (-1) = 0$

$x^2 - 2x - 1 = 0$ (or) $x = 1 + \sqrt{2}$

$x = 1 = \sqrt{2}(x-1)^2 = 2$

$x^2 - 2x - 1 = 0$

Concept Application

1. If $3^{1+x} + 3^{1-x} = 10$ then the values of x are
   (a) 1, –1 (b) 1, 0
   (c) 1, 2 (d) –1, –2
2. If a, b, c are positive real numbers then the number of real roots of the equation $ax^2 + b|x| + c = 0$ is
   (a) 2 (b) 4
   (c) 0 (d) –1
3. Number of real roots of the equation $3x^2 + 4|x| + 5 = 0$ are
   (a) 2 (b) 4
   (c) 0 (d) –1

RELATION BETWEEN ROOTS AND CO-EFFICIENTS

(*i*) The solutions of quadratic equation, $a x^2 + b x + c = 0$, ($a \neq 0$) are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(*ii*) If α, β are the roots of quadratic equation,

$a x^2 + b x + c = 0$ ...(*i*)

then equation (*i*) can be written as

$a(x - \alpha)(x - \beta) = 0$

or $ax^2 - a(\alpha + \beta)x + a\alpha\beta = 0$ ...(*ii*)

equations (*i*) and (*ii*) are identical,

$\therefore$ by comparing the coefficients sum of the roots,

$$\alpha + \beta = -\frac{b}{a} = -\frac{\text{constant term}}{\text{coefficient of } x^2}$$

and product of the roots,

$$\alpha\beta = \frac{c}{a} = \frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

(*iii*) Dividing the equation (i) by a,

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Rightarrow x^2 - \left(\frac{-b}{a}\right)x + \frac{c}{a} = 0$$

$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow x^2 -$ (sum of the roots) x + (product of the roots) $= 0$

Hence we conclude that the quadratic equation whose roots are α & β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Train Your Brain

Example 3: If α and β are the roots of $ax^2 + bx + c = 0$ and γ, δ are those of $lx^2 + mx + n = 0$, find the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$.

Sol. $ax^2 + bx + c = 0$ has roots a and b

$$\therefore \alpha + \beta = -\frac{b}{a}; \quad \alpha\beta = \frac{c}{a}$$

$lx^2 + mx + n = 0$ has roots γ and δ

$$\therefore \gamma + \delta = -\frac{m}{l}; \quad \gamma\delta = \frac{n}{l}$$

The required quadratic equation is

$x^2 - (\alpha\gamma + \beta\delta + \alpha\delta + \beta\gamma)x + (\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma) = 0$

where $\alpha\gamma + \beta\delta + \alpha\delta + \beta\gamma = (\alpha + \beta)(\gamma + \delta) = \frac{b}{a}\cdot\frac{m}{l}$

$$(\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma) = \frac{n}{l}\left(\frac{b^2 - 2ac}{a^2}\right) + \frac{c}{a}\left(\frac{m^2 - 2nl}{l^2}\right)$$

$$= \frac{ln(b^2 - 2ac) + ac(m^2 - 2nl)}{a^2l^2}$$

$$\therefore a^2l^2 \cdot x^2 - abl m\, x + ln(b^2 - 2ac) + ac(m^2 - 2nl) = 0$$

Example 4: The quadratic equation whose roots are $\frac{3 \pm i\sqrt{5}}{2}$ is

Sol. $$x^2 - \left(\frac{3 + i\sqrt{5}}{2} + \frac{3 - i\sqrt{5}}{2}\right)x + \left(\frac{3 + i\sqrt{5}}{2}\cdot\frac{3 - i\sqrt{5}}{2}\right) = 0$$

$$x^2 - 3x + \frac{7}{2} = 0 \Rightarrow 2x^2 - 6x + 7 = 0$$

or $x = \frac{3 + i\sqrt{5}}{2} \Rightarrow (2x - 3)^2 = -5$

$\Rightarrow 2x^2 - 6x + 7 = 0$

Example 5: If α, β are the roots of $ax^2 + bx + c = 0$ and $c \neq 0$ then the value of $\frac{1}{(a\alpha + b)^2} + \frac{1}{(a\beta + b)^2}$ in terms of a, b, c is

Sol. $\because$ α is a root of $ax^2 + bx + c = 0$

$\Rightarrow a\alpha^2 + b\alpha = -c$

$\Rightarrow a\alpha + b = \frac{-c}{\alpha}$, similarly $a\beta + b = \frac{-c}{\beta}$

$$\therefore \frac{1}{(a\alpha + b)^2} + \frac{1}{(a\beta + b)^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{c^2} = \frac{b^2 - 2ac}{a^2c^2}$$

Concept Application

4. If α, β are roots of the equation $2x^2 + 6x + b = 0 (b < 0)$ then $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ is less than

   (*a*) 2 (*b*) -2
   (*c*) 18 (*d*) 0

5. If α, β are roots of $x^2 + p(x + 1) + C = 0$ then $(\alpha + 1)(\beta + 1)$ than $(\alpha + 1)(\beta + 1) =$

   (*a*) $1 - C$ (*b*) $1 + C$
   (*c*) $C - 1$ (*d*) C

6. The quadratic equation whose roots are $\frac{\sqrt{2} + \sqrt{3}i}{\sqrt{2} - \sqrt{3}i}$ and $\frac{\sqrt{2} - \sqrt{3}i}{\sqrt{2} + \sqrt{3}i}$

   (*a*) $5x^2 - 2x + 5 = 0$ (*b*) $5x^2 + 2x + 5 = 0$
   (*c*) $5x^2 + 2x - 5 = 0$ (*d*) $5x^2 - 2x - 5 = 0$

7. Let two numbers have arithmetic means 9 and geometric mean 4. Then these numbers are the roots of the quadratic equation

   (*a*) $x^2 + 18x + 16 = 0$
   (*b*) $x^2 - 18x - 16 = 0$
   (*c*) $x^2 + 18x - 16 = 0$
   (*d*) $x^2 - 18x + 16 = 0$

SYMMETRIC FUNCTION OF THE ROOTS

A function of α and β is said to be a symmetric function if it remains unchanged when α and β are interchanged.

For example, $\alpha^2 + \beta^2 + 2\alpha\beta$ is a symmetric function of α and β whereas $\alpha^2 - \beta^2 + 3\alpha\beta$ is not a symmetric function of α and β.

In order to find the value of a symmetric function of α and β, express the given function in terms of $\alpha + \beta$ and $\alpha\beta$. The following results may be useful.

(*i*) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(*ii*) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

(*iii*) $\alpha^4 + \beta^4 = (\alpha^3 + \beta^3)(\alpha + \beta) - \alpha\beta(\alpha^2 + \beta^2)$

(*iv*) $\alpha^5 + \beta^5 = (\alpha^3 + \beta^3)(\alpha^2 + \beta^2) - \alpha^2\beta^2(\alpha + \beta)$

(*v*) $|\alpha - \beta| = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

(*vi*) $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

(*vii*) $\alpha^3 - \beta^3 = (\alpha - \beta)[(\alpha + \beta)^2 - \alpha\beta]$

(*viii*) $\alpha^4 - \beta^4 = (\alpha + \beta)(\alpha - \beta)(\alpha^2 + \beta^2)$

Train Your Brain

Example 6: If α, β are the root of $ax^2 + bx + c = 0$, then the equation whose roots are $2 + \alpha$ and $2 + \beta$ is

Sol. $f(x-2) = 0 \Rightarrow a(x-2)^2 + b(x-2) + c = 0$

$\Rightarrow ax^2 + x(b - 4a) + 4a - 2b + c = 0$

Example 7: If each root of the equation $x^2 + 11x + 13 = 0$ is diminished by 4, then the transfered equation is

Sol. $f(x+4) = 0 \Rightarrow (x-4)^2 + 11(x+4) + 13 = 0$

$\Rightarrow x^2 - 19x + 73 = 0$

Example 8: If α, β are the roots of $4x^2 + 7x + 2 = 0$, then the equation whose roots are α^2, β^2 is

Sol. $f(x) = 4x^2 + 7x + 2 = 0$

The required equation is $f\left(\sqrt{x}\right) = 0$.

$\Rightarrow 4\left(\sqrt{x}\right)^2 + 7\sqrt{x} + 2 = 0 \Rightarrow 4x + 2 = -7\sqrt{x}$

$\Rightarrow (4x+2)^2 = \left(-7\sqrt{x}\right)^2 \Rightarrow 16x^2 - 33x + 4 = 0$

Concept Application

8. If α, β are root of $ax^2 + bx + c = 0$ then $\frac{1}{\alpha^3} + \frac{1}{\beta^3} =$

(*a*) $\frac{3abc - b^3}{a^3}$ (*b*) $\frac{3abc - b^3}{a^3c}$

(*c*) $\frac{3abc - b^3}{c^3}$ (*d*) $\frac{b^3 - 2ac}{ac}$

9. If α, β are the roots of the equation $x^2 + 3x + 2$. Then $\alpha^5 + \beta^5 =$

(*a*) 33 (*b*) –32 (*c*) 35 (*d*) –33

NATURE OF ROOTS

❖ Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, $c \in R$ & $a \neq 0$ then;

(*i*) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal)

(*ii*) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(*iii*) $D < 0 \Leftrightarrow$ roots are imaginary

(*iv*) If $p + iq$ is one root of a quadratic equation, then the other must be it's conjugate $p - iq$ & vice versa. ($p, q \in R$ & $i = \sqrt{-1}$)

❖ Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, $c \in Q$ & $a \neq 0$ then ;

(*i*) If $D > 0$ & is a perfect square , then roots are rational & unequal.

(*ii*) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then the other root must be the conjugate of it i.e. $\beta = p - \sqrt{q}$ & vice versa.

If $\Delta = b^2 - 4ac > 0$.

Now consider the following cases.

Case		Nature of roots
Case-I	$a > 0, b > 0, c > 0$ $\Rightarrow \alpha + \beta < 0, \alpha\beta > 0$	Both roots are negative.
Case-II	$a > 0, b > 0, c < 0$ $\Rightarrow \alpha + \beta < 0, \alpha\beta < 0$	Both roots are opposite in sign; Magnitude of negative root is more than the magnitude of positive root.
Case-III	$a > 0, b < 0, c > 0$ $\Rightarrow \alpha + \beta > 0, \alpha\beta > 0$	Both roots are positive.
Case-IV	$a > 0, b < 0, c < 0$ $\Rightarrow \alpha + \beta > 0, \alpha\beta < 0$	Roots are opposite in sign. Magnitude of positive root is more than magnitude of negative root.

Remember

(*i*) Roots are rational $\Leftrightarrow$ D is a perfect square.

(*ii*) Roots are irrational $\Leftrightarrow$ D is positive but not a perfect square.

(*iii*) If $a + b + c = 0$, then 1 is a root of the equation $ax^2 + bx + c = 0$.

(*iv*) If a and c are of opposite sign, the roots must be of opposite sign.

(*v*) If the roots are equal in magnitude but opposite in sign, then $b = 0$, $ac < 0$.

(*vi*) If the roots are reciprocal of each other, then $c = a$.

(*vii*) The quadratic equation whose roots are reciprocals of the roots of $ax^2 + bx + c = 0$ is $cx^2 + bx + a = 0$ (*i.e.*, the coefficients are written in reverse order).

(*viii*) If $a = 1$, $b, c \in Z$ and the roots are rational numbers, then these roots must be integers.

(*ix*) If $ax^2 + bx + c = 0$ is satisfied by more than two values, it is an identity and $a = b = c = 0$ and vice-versa.

(*x*) If $P(x) = a_1x^2 + b_1x + c_1$ and $Q(x) = a_2x^2 + b_2x + c_2$, then $P(x).\ Q(x) = 0$ have atleast two real roots if $D_1 + D_2 > O$.

Condition for a Root of a Quadratic Equation

$ax^2 + bx + c = 0$ to be reciprocal of the root of $a'x^2 + b'x + c' = 0$ is

$$(cc' - aa')^2 = (ba' - b'c)(ab' - bc')$$

Train Your Brain

Example 9: Determine the values of m for which the equation $5x^2 - 4x + 2 + m(4x^2 - 2x - 1) = 0$ will have

(*a*) equal roots

(*b*) product of the roots as 2

Sol. $5x^2 - 4x + 2 + m(4x^2 - 2x - 1) = 0$

$(5 + 4m)x^2 - 2(m + 2)x + (2 - m) = 0$ ...(i)

(*a*) For equal roots, the discriminant of (i) should be equal to zero.

$\therefore\ 4(m + 2)^2 - 4.(5 + 4m)(2 - m) = 0$

$\Rightarrow\ m^2 + 4m + 4 - (10 + 8m - 5m - 4m^2) = 0$

$\Rightarrow\ 5m^2 + m - 6 = 0$

$\Rightarrow\ (5m + 6)(m - 1) = 0$

$\Rightarrow\ m = -\frac{6}{5}$ or $m = 1$

(*b*) $\frac{2 - m}{5 + 4m} = 2$

$\therefore\ 2 - m = 10 + 8m \Rightarrow m = -\frac{8}{9}$

Example 10: The nature of the roots of $x^2 + x - 1 = 0$ is

Sol. $\Delta = (1)^2 - 4(1)(-1) = 5$ is not a perfect square.

$\Rightarrow$ roots are irrational and more over conjugate

Example 11: The nature of the roots of $2x^2 + x + 6 = 0$ is

Sol. $\Delta = 1 - 4(1)(-1) = 5$ is not a perfect square.

$\Rightarrow$ roots are irrational and more over conjugate complex number

Example 12: If the roots of the equation $x^2 - 15 - m(2x - 8) = 0$ are equal then the value of m is

Sol. $x^2 - 15 - m(2x - 8) = 0$

$x^2 - 2x + 8m - 15 = 0$

$\Delta = 0 \Rightarrow (-2m)^2 - 4 \times 1 \times (8m - 15) = 0$

$m^2 - 8m + 15 = 0$

$(m - 3)(m - 5) = 0,\ m = 3;\ m = 5$

Example 13: If a, b, c are real, the roots of $a^2x^2 + b^2x^2 - 2abx - 2bcx + b^2 + c^2 = 0$ are real and equal, then the progression in which a, b, c lies is

Sol. $a^2x^2 + b^2x^2 - 2abx - 2bcx + b^2 + c^2 = 0$

$a^2x^2 - 2abx + b^2 + b^2x^2 - 2bcx + c^2 = 0$

$(ax - b)^2 + (bx - c)^2 = 0$

$ax - b = 0;\ bx - c = 0$

$x = \frac{b}{a};\ x = \frac{c}{b}$

$\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac$

$\therefore$ a, b, c are in G.P.

Example 14: If the roots $(a^2 - bc)x^2 + 2(b^2 - ca)x + (c^2 - ab) = 0$ are equal then the condition (s) is (are)

Sol. $(a^2 - bc)x^2 + 2(b^2 - ca)x + (c^2 - ab) = 0$

roots are equal $\Rightarrow \Delta = 0$

$\Rightarrow\ 4(b^2 - ca)^2 - 4(a^2 - bc)(c^2 - ab) = 0$

$\Rightarrow\ b^4 - 2ab^2c + a^3b + bc^3 - ab^2c) = 0$

$\Rightarrow\ b = 0$ (or) $a^3 + b^3 + c^3 - 3abc = 0$

Example 15: A quadratic equation $p(x) = 0$ has $1 + \sqrt{5}$ and $1 - \sqrt{5}$ as roots and it satisfies $p(1) = 2$. Find the quadratic polynomial.

Sol. Sum of the roots = 2, product of the roots = – 4

$\therefore$ let $p(x) = a(x^2 - 2x - 4) \Rightarrow p(1) = 2$

$\Rightarrow\ 2 = a(1^2 - 2 \cdot 1 - 4) \Rightarrow a = -2/5$

$\therefore\ p(x) = -2/5\ (x^2 - 2x - 4)$

Example 16: The quadratic equation $x^2 + mx + n = 0$ has roots which are as twice as those of $x^2 + px + m = 0$ and m, n and $p \neq 0$. Find the value of $\frac{n}{p}$.

Sol. $x^2 + mx + n = 0 \begin{cases} 2\alpha \\ 2\beta \end{cases}$

and $x^2 + px + m = 0 \begin{cases} \alpha \\ \beta \end{cases}$

$2(\alpha + \beta) = -m$ ...(i)

$4\alpha\beta = n$ ...(ii)

and $\alpha + \beta = -p$ ...(iii)

$\alpha\beta = m$ ...(iv)

$\therefore$ from (i) and (iii)

$\Rightarrow 2p = m$

and from (ii) and (iv)

$\Rightarrow 4m = n \Rightarrow \frac{n}{p} = \frac{4m}{m/2} = 8$

Example 17: For what values of m the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has equal roots.

Sol. Given equation is

$(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$

Roots of equation will be equal if, $D = 0$.

$\Rightarrow 4(1 + 3m)^2 - 4(1 + m)(1 + 8m) = 0$

$\Rightarrow 4(1 + 9m^2 + 6m - 1 - 9m - 8m^2) = 0$

$\Rightarrow m^2 - 3m = 0$ or, $m(m - 3) = 0$

$\therefore m = 0, 3.$

Example 18: Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

Sol. Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$. For integral roots D should be a perfect square.

$D = a^2 - 20a + 96.$

$\Rightarrow D = (a - 10)^2 - 4$

$\Rightarrow 4 = (a - 10)^2 - D$

If D is a perfect square means difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and $D = 0$.

$\Rightarrow (a - 10) = \pm 2; \Rightarrow a = 12, 8$

Concept Application

10. If one roots of $x^2 - (3 + 2i)x + (1 + 3i) = 0$ is $1 + i$ then the other roots is

(a) $1 - i$ (b) $2 + i$
(c) $3 + i$ (d) $1 + 3i$

11. If the expression $x^2 - (5m - 2)x + (4m^2 + 10m + 25)$ can be expressed as a perfect square, then $m =$

(a) $\frac{8}{3}$ or 4 (b) $-\frac{8}{3}$ or 4
(c) $\frac{4}{3}$ or 8 (d) $-\frac{4}{3}$ or 8

12. If the roots of the quadratic equation $x^2 - 4x - \log_3 a = 0$ are real, then the least value of a is

(a) 81 (b) $\frac{1}{81}$ (c) $\frac{1}{64}$ (d) 9

13. If the roots of the equation $(a^2 + b^2)x^2 + 2(bc + ad)x + (c^2 + d^2) = 0$ are real then a^2, bd, c^2 are in

(a) A.P. (b) G.P.
(c) H.P (d) A.G.P.

14. The roots of the equation $x^2 - 2\sqrt{3x} + 3 = 0$

(a) rational and equal (b) rational and not equal
(c) irrational (d) imaginary

15. If (1) is one of the roots of $ax^2 + 3x + 5 = 0$ then the second root is

(a) $\frac{5}{3}$ (b) $\frac{-5}{3}$ (c) $\frac{5}{16}$ (d) $\frac{-5}{8}$

QUADRATIC EXPRESSION

Let $y = ax^2 + bx + c = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right)$ ...(i)

1. Represents a parabola with vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ and axis of the parabola is $x = \frac{-b}{2a}$

If $a > 0$, the parabola opens upward while if $a < 0$, the parabola opens downward. The parabola cuts the x-axis at points corresponding to roots of $ax^2 + bx + c = 0$. If this equation has

(i) $D > 0$, the parabola cuts x-axis at two real and distinct points.

(ii) $D = 0$, the parabola touches x-axis at $x = \frac{-b}{2a}$.

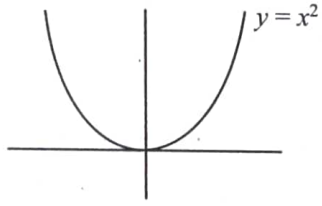
(iii) $D < 0$, then;

if $a > 0$, parabola lies above x-axis.

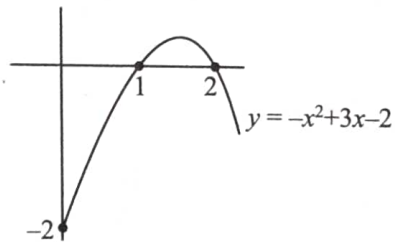
if $a < 0$, parabola lies below x-axis.

e.g.

1. $y = x^2$



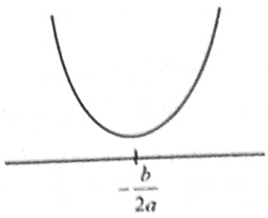
2. $y = -x^2 + 3x - 2$



GRAPHS OF QUADRATIC EXPRESSIONS

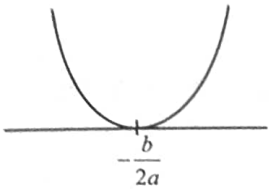
Let $f(x) = ax^2 + bx + c$ and $\alpha, \beta : \alpha < \beta$, be its roots

(i) $a > 0$ and $D < 0 \Leftrightarrow f(x) > 0 \quad \forall \quad x \in R$



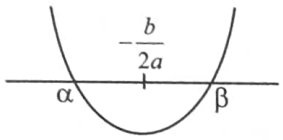
α and β are complex conjugates

(*ii*) $a > 0$ and $D = 0 \Leftrightarrow f(x) \geq 0 \quad \forall \quad x \in R$



$f(x) = 0$ at $x = \frac{-b}{2a}$

(*iii*) $a > 0$ and $D > 0$; then,

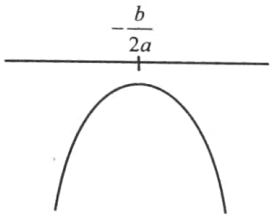


$f(x) > 0 \; \forall \; x \in (-\infty, \alpha) \cup (\beta, \infty)$

$f(x) < 0 \; \forall \; x \in (\alpha, \beta)$

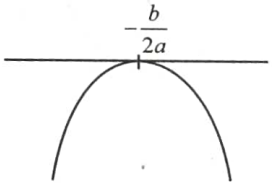
$f(x)_{min} = -\frac{D}{4a}$ at $x = -\frac{b}{2a}$

(*iv*) $a < 0$ and $D < 0 \Leftrightarrow f(x) < 0 \quad \forall \quad x \in R$



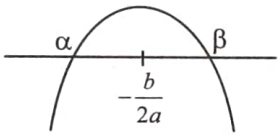
α and β are complex conjugates

(*v*) $a < 0$ and $D = 0 \Leftrightarrow f(x) \leq 0 \quad \forall \quad x \in R$



$f(x) = 0$ at $x = \frac{-b}{2a}$

(*vi*) $a < 0$ and $D > 0$; then



$f(x) < 0 \quad \forall \quad x \in (-\infty, \alpha) \cup (\beta, \infty)$

$f(x) > 0 \quad \forall \quad x \in (\alpha, \beta)$

$f(x)_{max} = -\frac{D}{4a}$ *at* $x = -\frac{b}{2a}$

EQUATION V/S IDENTITY

A quadratic equation is satisfied by exactly two values of 'x' which may be real or imaginary. The equation, $a x^2 + b x + c = 0$ is: A **quadratic equation** if $a \neq 0$ and An **identity** if $a = b = c = 0$ Infinite Roots

If a equation is satisfied by three distinct values of 'x', then it is an identity.

$(x + 1)^2 = x^2 + 2x + 1$ is an identity in x.

Here highest power of x in the given relation is 2 and this relation is satisfied by three different values $x = 0$, $x = 1$ and $x = -1$ and hence it is an identity because a polynomial equation of nth degree cannot have more than n distinct roots.

THEORY OF EQUATIONS

If $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ are the roots of the equation; $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ where $a_0, a_1, ... a_n$ are all real & $a_0 \neq 0$ then,

$$\Sigma \alpha_1 = -\frac{a_1}{a_0}, \Sigma \alpha_1 \alpha_2 = +\frac{a_2}{a_0}, \Sigma \alpha_1 \alpha_2 \alpha_3 = -\frac{a_3}{a_0}, ..., \alpha_1 \alpha_2 \alpha_3$$

$$.....\alpha_n = (-1)^n \frac{a_n}{a_0}$$

Note:

(*i*) If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of $f(x)$ and conversely.

(*ii*) Every equation of nth degree $(n \geq 1)$ has exactly n roots & if the equation has more than n roots, it is an identity.

(*iii*) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs.**

(*iv*) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.

(*v*) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then $f(x) = 0$ must have atleast one real root between 'a' and 'b'.

(*vi*) Every equation $f(x) = 0$ of degree odd has atleast one real root of a sign opposite to that of its last term.

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Example 19: If $a + b + c = 0$; $a n^2 + b n + c = 0$ and $a + b n + c n^2 = 0$ where $n \neq 0, 1$, then prove that $a = b = c = 0$.

Sol. Note that $a x^2 + b x + c = 0$ is satisfied by $x = 1$;

$x = n$ & $x = \frac{1}{n}$ where $n \neq \frac{1}{n}$

$\Rightarrow$ Q.E. has 3 distinct real roots which implies that it must be an identity.

Example 20: If $x = 1$ and $x = 2$ are solutions of the equation $x^3 + ax^2 + bx + c = 0$ and $a + b = 1$, then find the value of b.

Sol. $a + b + c = -1$

$\Rightarrow c = -2$ & $8 + 4a + 2b + c = 0$

$\Rightarrow 4a + 2b = -6$

$\Rightarrow 2a + b = -3$

$\Rightarrow a = -4, b = 5$

Example 21: A polynomial in x of degree greater than 3 leaves the remainder 2, 1 and –1 when divided by $(x - 1)$; $(x + 2)$ & $(x + 1)$ respectively. Find the remainder, if the polynomial is divided by, $(x^2 - 1)(x + 2)$.

Sol. $f(x) = Q_1(x - 1) + 2 = Q_2(x + 2) + 1 = Q_3(x + 1) - 1$

$\Rightarrow f(1) = 2$; $f(-2) = 1$; $f(-1) = -1$

Let $f(x) = Q_r(x^2 - 1)(x + 2) + ax^2 + bx + c$

Hence $a + b + c = 2$; $4a - 2b + c = 1$

and $a - b + c = -1$

Example 22: The roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ $(a \neq b \neq c \neq 0)$ are

Sol. $a(b - c) + b(c - a) + c(a - b) = 0$

$\therefore$ the roots are $1, \dfrac{c(a-b)}{a(b-c)}$

Example 23: The roots of $ax^2 + 3bx + c = 0$ if $3b = a + c$ are

Sol. $ax^2 + 3bx + c = 0$ and $3b = a + c$

$ax^2 + (a + c)x + c = 0$; $ax^2 + ax + cx + c = 0$

$ax(x + 1) + c(x + 1) = 0$; $(x + 1)(ax + c) = 0$

$x = -1; x = \dfrac{-c}{a}$

$\therefore$ roots are $-1, \dfrac{-c}{a}$

Concept Application

16. If ω is an imaginary cube root of unity. Then the equation whose roots are $2\omega + 3\omega^2$, $2\omega^2 + 3\omega$ is

(a) $x^2 + 5x + 7 = 0$ (b) $x^2 + 5x - 7 = 0$

(c) $x^2 - 5x + 7 = 0$ (d) $x^2 - 5x - 7 = 0$

17. If p and q are the roots of $x^2 + px + q = 0$ then

(a) $p = 1$ (b) $p = 1$ or 0

(c) $p = -2$ (d) $p = -2$ or 0

COMMON ROOTS

One Root Common

If $\alpha \neq 0$ is a common root of the equation

$a_1x^2 + b_1x + c_1 = 0$ ...(i)

and $a_2x^2 + b_2x + c_2 = 0$ ...(ii)

Then we have

$a_1\alpha^2 + b_1\alpha + c_1 = 0$ and

$a_2\alpha^2 + b_2\alpha + c_2 = 0$

These give $\dfrac{\alpha^2}{b_1c_2 - b_2c_1} = \dfrac{\alpha}{c_1a_2 - c_2a_1}$

$= \dfrac{1}{a_1b_2 - a_2b_1}(a_1b_2 - a_2b_1 \neq 0)$.

Thus, the required condition for one common root is $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$ and the value of the common root is

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \text{ or } \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}.$$

Both Roots Common

If the equations (i) and (ii) have both roots common, then these equations will be identical. Thus the required condition for both roots common is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \text{ (If no root is equal to zero)}$$

Remember

(*i*) To find the common root of two equations, make the coefficient of second degree terms in two equations equal and subtract. The value of x so obtained is the required common root.

(*ii*) If two quadratic equations with real and rational coefficients have a common imaginary or irrational root, then both roots will be common and the two equations will be identical. The required condition is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

(*iii*) If α is a repeated root of the quadratic equation

$f(x) = ax^2 + bx + c = 0$,

Then α is also a root of the equation $f'(x) = 0$.

Note: If $f(x) = 0$ & $g(x) = 0$ are two polynomial equation having some common root(s) then those common root(s) is/are also the root(s) of $h(x) \equiv a\,f(x) + bg(x) = 0$.

Train Your Brain

Example 24: Find m and n in order that the equations $mx^2 + 5x + 2 = 0$ and $3x^2 + 10x + n = 0$ may have both the roots common.

Sol. The equations are $mx^2 + 5x + 2 = 0$ and $3x^2 + 10x + n = 0$. Since they have both the roots common,

$$\frac{m}{3} = \frac{5}{10} = \frac{2}{n} \because \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_1}$$

From the first-relation, $m = \dfrac{15}{10} = \dfrac{3}{2}$.

From the last-relation, $n = 4$.

Example 25: If the quadratic equations, $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then prove that either $b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$.

Sol. $\alpha^2 + b\alpha + c = 0$ ...(*i*)

$b\alpha^2 + c\alpha + 1 = 0$ ...(*ii*)

$$\therefore \frac{\alpha^2}{b - c^2} = \frac{\alpha}{bc - 1} = \frac{1}{c - b^2}$$

$$\Rightarrow \alpha^2 = \frac{b - c^2}{c - b^2} \text{ or } \alpha = \frac{bc - 1}{c - b^2}$$

$\Rightarrow (bc - 1)^2 = (b - c^2)(c - b^2)$

or $b^3 + c^3 + 1 - 3bc = 0$

$\Rightarrow (b + c + 1)(b^2 + c^2 + 1 - bc - c - b) = 0$

$\Rightarrow b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$

Example 26: If $x^2 + 4ax + 3 = 0$ and $2x^2 + 3ax - 9 = 0$ have a common root then the value of a is

Sol. $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$

$(6 + 9)^2 = (3a - 8a)(-36a) \quad \Rightarrow \quad a = \pm 1$

Example 27: If $x^2 + bx + c = 0$, $x^2 + cx + b = 0$ $(b \neq c)$, have a common root then the condition is

Sol. Let α be the common root

$\Rightarrow \alpha^2 + b\alpha + c = 0$ ...(i)

$\alpha^2 + c\alpha + b = 0$ ...(ii)

(i) – (ii) $\Rightarrow \alpha = 0$

Substitute in (1)

$\Rightarrow b + c + 1 = 0$

Example 28: If the equation $k(6x^2 + 3) + rx + (2x^2 - 1) = 0$ and $6k(2x^2 + 1) + px + (4x^2 - 2) = 0$ have both roots common, then the value of p/r is

Sol. $(6k + 2)x^2 + rs + 3k - 1 = 0$ ...(i)

$(12k + 4)x^2 + px + 6k - 2 = 0$ ...(ii)

Since the roots of (i) and (ii) and equal,

$$\frac{6k + 2}{12k + 4} = \frac{r}{p} = \frac{3k - 1}{6k - 2} \quad \therefore \frac{r}{p} = \frac{1}{2}$$

Concept Application

18. If the equation $x^2 + ax + b = 0$ and $x^2 + bx + a = 0 (a \neq b)$ have a common root then $a + b =$

(*a*) –1 (*b*) 0

(*c*) 1 (*d*) 2

19. If $(x - 2)$ is a common factor of the expressions $x^2 + ax + b$ and $x^2 + cx + d$, then $\frac{b - d}{c - a}$

(*a*) –2 (*b*) –1 (*c*) 1 (*d*) 2

RANGE OF QUADRATIC EXPRESSION

$$f(x) = ax^2 + bx + c$$

(*i*) **Range when $x \in R$:** If $a > 0 \Rightarrow f(x) \in \left[-\frac{D}{4a}, \infty\right)$

$$a < 0 \Rightarrow f(x) \in \left(-\infty, \frac{D}{4a}\right]$$

Maximum & Minimum Value of $y = ax^2 + bx + c$ occurs $x = -(b/2a)$ according as $a < 0$ or $a > 0$ respectively

(*ii*) **Range in restricted domain :** Given $x \in [x_1, x_2]$

(a) If $-\frac{b}{2a} \notin [x_1, x_2]$ then,

$$f(x) \in [\min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\}]$$

(b) If $-\frac{b}{2a} \in [x_1, x_2]$ then,

$$f(x) \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$$

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Example 29: Find the minimum value of $f(x) = x^2 - 5x + 6$.

Sol. Minimum value of $f(x) = -\frac{D}{4a} = -\left(\frac{25 - 24}{4}\right)$

$= -\frac{1}{4}\left(\text{at } x = -\frac{b}{2a} = \frac{5}{2}\right)$. Hence range is $\left[-\frac{1}{4}, \infty\right)$.

Example 30: Find the range of $y = \frac{x + 2}{2x^2 + 3x + 6}$, if x is real.

Sol. $y = \frac{x + 2}{2x^2 + 3x + 6}$

$\Rightarrow 2yx^2 + 3yx + 6y = x + 2$

$\Rightarrow 2yx^2 + (3y - 1)x + 6y - 2 = 0$ ...(*i*)

Case-I: If $y \neq 0$, then equation (*i*) is quadratic in x

$\because$ x is real

$\therefore D \geq 0$

$\Rightarrow (3y - 1)^2 - 8y(6y - 2) \geq 0$

$\Rightarrow (3y - 1)(13y + 1) \leq 0$

$$y \in \left[-\frac{1}{13}, \frac{1}{3}\right] - \{0\}$$

Case-I: if $y = 0$, then equation becomes

$x = -2$

Which is possible as x is real

$$\therefore \text{Range } y \in \left[-\frac{1}{13}, \frac{1}{3}\right]$$

Example 31: The maximum or minimum values of $2x + 5 - 3x^2$ on R is

Sol. Given expression $2x + 5 - 3x^2$,

$a = -3 < 0, b = 2, c = 5$

$\Rightarrow$ given expression has maximum value

$\therefore$ maximum value $= \dfrac{4ac - b^2}{4a} = \dfrac{4(-3)(5) - 4}{4(-3)} = \dfrac{16}{3}$

Concept Application

20. Maximum value of $-5x^2 + 2x + 3$ is

(*a*) $\dfrac{14}{5}$ (*b*) $\dfrac{2}{3}$

(*c*) 1 (*d*) $\dfrac{3}{4}$

21. The maximum value of $\dfrac{1}{4x^2 + 2x + 1}$ is

(*a*) $\dfrac{4}{3}$ (*b*) $\dfrac{2}{3}$

(*c*) 1 (*d*) $\dfrac{3}{4}$

22. If $x \in [2, 4]$ then for the expression $x^2 - 6x + 5$

(*a*) The least value = 4

(*b*) The greatest value = 4

(*c*) The least value = 3

(*b*) The greatest value = –5

23. If $2x - 7 - 5x^2$ has maximum value at $x = a$ then $a =$

(*a*) $\dfrac{-1}{5}$ (*b*) $\dfrac{1}{5}$ (*c*) $\dfrac{34}{5}$ (*d*) $\dfrac{-34}{5}$

24. Find the range of the following functions:

(*i*) $f(x) = x^2 + x + 1$ (*ii*) $f(x) = 2x^2 + 3x + 5$

(*iii*) $f(x) = x^2 + 2x + 7$ (*iv*) $f(x) = (x - 1)(x - 2)$

(*v*) $f(x) = \dfrac{(x-1)}{x^2 + x + 1}$ (*vi*) $f(x) = \dfrac{2x^2 + 3x + 1}{x^2 + x + 1}$

(*vii*) $f(x) = \dfrac{(x-1)(x-2)}{(x-3)}$ (*viii*) $f(x) = \dfrac{(x-1)(x-2)}{(x-3)(x-4)}$

(*ix*) $f(x) = \dfrac{(x-1)(x-3)}{(x-2)(x-4)}$ (*x*) $f(x) = \dfrac{x^2 - 5}{2x^2 + 3}$

(*xi*) $f(x) = x^2 + x + 1; x \geq 0$ (*xii*) $f(x) = 2x^2 + x + 1; 1 \leq x \leq 5$

(*xiii*) $f(x) = \sin 4x + \cos^4 x$ (*xiv*) $f(x) = \sin^6 x + \cos^6 x$

SOLUTION OF QUADRATIC INEQUALITIES

The values of 'x' satisfying the inequality, $ax^2 + bx + c > 0$ $(a \neq 0)$ are:

(*i*) If $D > 0$, i.e. the equation $ax^2 + bx + c = 0$ has two different roots $\alpha < \beta$.

$a > 0 \Rightarrow x \in (\infty, \alpha) \cup (\beta, \infty)$

$a < 0 \Rightarrow x \in (\alpha, \beta)$

(*ii*) If $D = 0$, i.e. roots are equal, i.e. $\alpha = \beta$.

$a > 0 \Rightarrow x \in (-\infty, \alpha) \cup (\alpha, \infty)$

$a < 0 \Rightarrow x \in \phi$

(*iii*) If $D < 0$, i.e. the equation $ax^2 + bx + c = 0$ has no real roots.

$a > 0 \Rightarrow x \in R$

$a < 0 \Rightarrow x \in \phi$

(*iv*) Inequalities of the form $\dfrac{P(x)}{Q(x)} \lesseqgtr 0$ can be solved using the method of intervals.

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Example 32: Find the solution set of k so that $y = kx$ is secant to the curve $y = x^2 + k$.

Sol. Put $y = kx$ in $y = x^2 + k$

$\Rightarrow kx = x^2 + k = 0$

$\Rightarrow x^2 - kx + k = 0$ for line to be secant, $D > 0$

$\Rightarrow k^2 - 4k > 0$ $k(k - 4) > 0$

hence $k > 4$ or $k < 0$

$\Rightarrow k \in (-\infty, 0) \cup (4, \infty)$

Example 33: Solve inequality $\dfrac{x^2 + x + 1}{|x + 1|} > 0$.

Sol. $\dfrac{x^2 + x + 1}{|x + 1|} > 0$ ...(*i*)

$\therefore |x + 1| > 0, \forall x \in R - \{-1\}$

$\therefore$ (i) becomes $x^2 + x + 1 > 0$ $x \neq -1$

$\because x^2 + x + 1 > 0, \forall x \in R$ (as its $a > 0$ and $D < 0$)

$\therefore$ Solution of (i) is $x \in R - \{-1\}$

Example 34: Solve $\left|\dfrac{x^2 - 3x - 1}{x^2 + x + 1}\right| < 3$.

Sol. $\left|\dfrac{x^2 - 3x - 1}{x^2 + x + 1}\right| < 3$.

$\because (x^2 + x + 1) > 0, \forall x \in R$

$\therefore |x^2 - 3x - 1| < 3(x^2 + x + 1)$

$\Rightarrow (x^2 - 3x - 1)^2 - \{3(x^2 + x + 1)\}^2 < 0$

$\Rightarrow (4x^2 + 2)(-2x^2 - 6x - 4) < 0$

$\Rightarrow (2x^2 + 1)(x + 2)(x + 1) > 0$

$\Rightarrow x \in (-\infty, -2) \cup (-1, \infty)$

Concept Application

25. The set of values of x for which the inequality $[x]^2 - 5[x] + 6 \le 0$ (where [.] denote the greatest function) hold good is

(*a*) $2 \le [x] < 3$ (*b*) $2 \le x < 4$

(*c*) $2 \le x < 3$ (*d*) $2 \le x \le 4$

26. Let $f(x) = ax^2 + bx + c$, if $f(-1) < 1, f(1) > -1, f(3) < -4$ and $a \ne 0$ then

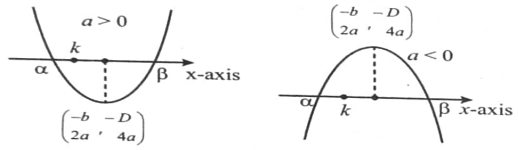
(*a*) $a > 0$ (*b*) $a > 1$

(*c*) $a < -\frac{1}{8}$ (*d*) $-\frac{1}{8} < a < 0$

LOCATION OF ROOTS

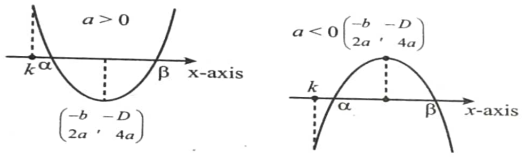
Let $f(x) = ax^2 + bx + c, a, b, c \in R, a \ne 0$ and α, β be roots of $f(x) = 0$

1. A real number k lies between the roots of $f(x) = 0$



(*i*) $D > 0$, (*ii*) $af(k) < 0$, where $\alpha < \beta$

2. If both roots of quadratic equation $f(x) = 0$ are greater than k, then

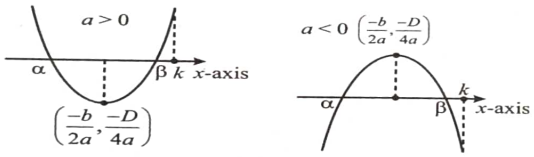


(*i*) $D \ge 0$,

(*ii*) $af(k) > 0$

(*iii*) $k < \frac{-b}{2a}$, where $\alpha \le \beta$

3. If both roots are less than real number k, then

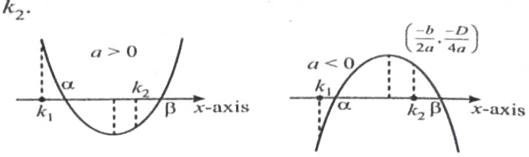


(*i*) $D \ge 0$,

(*ii*) $af(k) > 0$

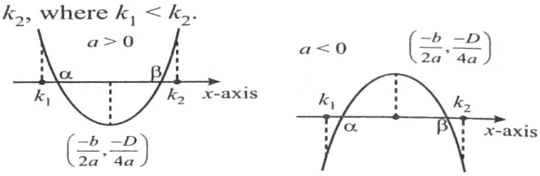
(*iii*) $k > \frac{-b}{2a}$, where $\alpha \le \beta$

4. Exactly one root lies between real numbers k_1 and k_2, where $k_1 < k_2$.



(*i*) $D > 0$, (*ii*) $f(k_1).f(k_2) < 0$, where $\alpha < \beta$

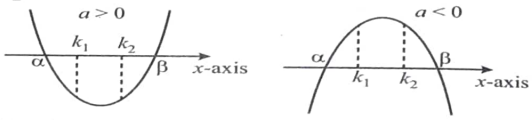
5. If both roots of $f(x) = 0$ are confined between real numbers k_1 and k_2, where $k_1 < k_2$.



(*i*) $D \ge 0$, (*ii*) $af(k_1) > 0$

(*iii*) $af(k_2) > 0$, (*iv*) $k_1 < \frac{-b}{2a} < k_2$, where $\alpha \le \beta$

6. If real numbers k_1 and k_2 lie between roots of $f(x) = 0$, where $k_1 < k_2$.



(*i*) $D > 0$,

(*ii*) $af(k_1) < 0$

(*iii*) $af(k_2) < 0$, where $\alpha < \beta$

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Example 35: If α & β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always $(p, q, r, s \in R)$

Sol. $\alpha + \beta = -p$; $\alpha\beta = q$; $\alpha^4 + \beta^4 = r$; $\alpha^4\beta^4 = s$

Now for equation $x^2 - 4q^2 + 2q^2 - r = 0$

$D = 16q^2 - 4(2q^2 - r)$

$= 8q^2 + 4r = 4(2q^2 + r) = 4[2\alpha^2\beta^2 + \alpha^4 + \beta^4]$

$= 4[(\alpha^2 + \beta^2)^2] = 4[(\alpha + \beta)^2 - 2\alpha\beta]^2 = 4(p^2 - 2q)^2 > 0$

Again consider the product of the roots $= 2q^2 - r$ which can be either positive or negative.

Example 36: Find the range of values of a for which the equation $x^2 - (a - 5)x + \left(a - \frac{15}{4}\right) = 0$ has at least one positive root.

Sol. $D \ge 0 \Rightarrow (a - 5)^2 - 4\left(a - \frac{15}{4}\right) \ge 0$

$\Rightarrow a^2 - 10a + 25 - 4a + 15 \ge 0 \Rightarrow a^2 - 14a + 40 \ge 0$

$\Rightarrow (a - 4)(a - 10) \ge 0$

$\Rightarrow a \in (-\infty, 4] \cup [10, \infty)$

Case I: When both roots are positive

$$D \geq 0, a - 5 > 0, a - \frac{15}{4} > 0$$

$$\Rightarrow D \geq 0, a > 5, a > \frac{15}{4} \Rightarrow a \in [10, \infty)$$

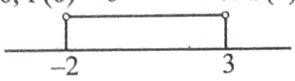
Case II: when exactly one root is positive

$$\Rightarrow a - \frac{15}{4} \leq 0 \Rightarrow a \leq \frac{15}{4}$$

then finally $a \in (-\infty, 15/4] \cup [10, \infty)$

Example 37: Find the value of k for which one root of the equation of $x^2 - (k + 1)x + k^2 + k - 8 = 0$ exceed 2 and other is smaller than 2.

Sol. Since $a > 0$, $f(0) < 0$ $\therefore f(2) < 0$



$\therefore 4 - 2(k + 1) + k^2 + k - 8 < 0$

$\Rightarrow k^2 - K + 6 < 0$

$\Rightarrow (k + 2)(k - 3) < 0 \quad \therefore k \in (-2, 3)$

Example 38: Let $x^2 - (m - 3)x + m = 0$ $(m \in R)$ be a quadratic equation, then find the values of 'm' for which

(*a*) both the roots are greater than 2.

(*b*) both roots are positive.

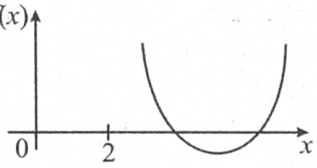
(*c*) one root is positive and other is negative.

(*d*) One root is greater than 2 and other smaller than 1

(*e*) Roots are equal in magnitude and opposite in sign.

(*f*) both roots lie in the interval (1, 2)

Sol. (*a*)



Condition I: $D \geq 0$

$\Rightarrow (m - 3)^2 - 4m \geq 0$

$\Rightarrow m^2 - 10m + 9 \geq 0$

$\Rightarrow (m - 1)(m - 9) \geq 0$

$\Rightarrow m \in (-\infty, 1] \cup [9, \infty)$ ...(*i*)

Condition II: $f(2) > 0$

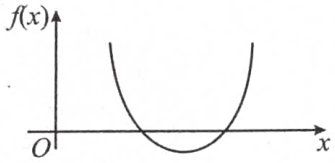
$\Rightarrow 4 - (m - 3)2 + m > 0$

$\Rightarrow m < 10$ ...(*ii*)

Condition III: $-\frac{b}{2a} > 2 \Rightarrow m > 7$ ...(*iii*)

Intersection of (*i*), (*ii*) and (*iii*) gives $m \in [9, 10)$

(*b*)



Condition I: $D \geq 0$

$\Rightarrow m \in (-\infty, 1] \cup [9, \infty)$

Condition II: $f(0) > 0$

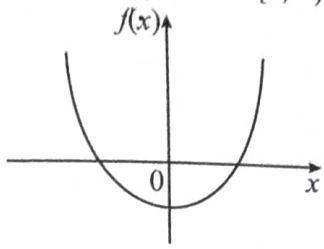
$\Rightarrow m > 0$

Condition III: $-\frac{b}{2a} > 0 \Rightarrow \frac{m-3}{2} > 0$

$\Rightarrow m > 3$

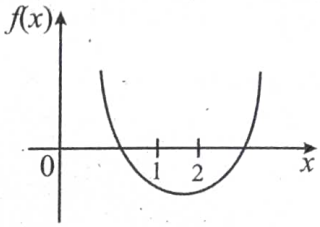
intersection gives $m \in [9, \infty)$ **Ans.**

(*c*)



Condition I: $f(0) < 0 \Rightarrow m < 0$ **Ans.**

(*d*)



Condition I: $f(1) < 0 \Rightarrow 4 < 0 \Rightarrow m \in \phi$

Condition II: $f(2) < 0 \Rightarrow m > 10$

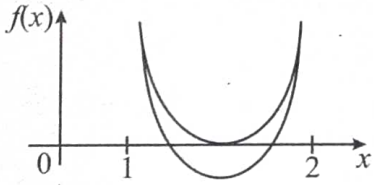
Intersection gives $m \in \phi$

(*e*) sum of roots $= 0 \Rightarrow m = 3$

and $f(0) < 0 \Rightarrow m < 0$

$\therefore m \in \phi$.

(*f*)



Condition I: $D \geq 0 \Rightarrow m \in (-\infty, 1] \cup [9, \infty)$

Condition II: $f(1) > 0 \Rightarrow 1 - (m - 3) + m > 0 \Rightarrow 4 > 0$ which is true $\forall m \in R$

Condition III: $f(2) > 0$

$\Rightarrow m < 10$

Condition IV: $1 < -\frac{b}{2a} < 2$

$\Rightarrow 1 < \frac{m-3}{2} < 2$

$\Rightarrow 5 < m < 7$ intersection gives $m \in \phi$

Example 39: If both roots of the quadratic equation $x^2 - 2kx + k^2 + K - 5 = 0$ are less than 5, then find the interval in which k lies

Sol. $\Delta \geq 0 \Rightarrow 4k^2 - 4(k^2 + k - 5) \geq 0 \Rightarrow k \leq 5$ ...(i)

$af(k) \geq 0 \Rightarrow 1.f(5) > 0$

$\Rightarrow k^2 - 9k + 20 > 0$

$\Rightarrow k < 4$ or $k > 5$ ...(ii)

$$k > \frac{-b}{2a} \Rightarrow 5 > \frac{-(-2k)}{2} \Rightarrow k < 5 \quad \text{....(iii)}$$

from (i), (ii) and (iii)

$k \in (-\infty, 4)$.

Example 40: The interval in which 'a' lies when the graph of the function $y = 16x^2 + 8(a + 5) - 7a - 5$ is strictly above the x-axis is

Sol. $\Delta < 0 \Rightarrow 64(a + 5)^2 + 64(7a + 5) < 0$

$\Rightarrow 64(a + 2)(a + 2) < 0$

$\therefore -15 < a < -2$.

Concept Application

27. Both the roots of the equation $(x - b)(x - c) + (x - c)(x - a) + (x - a)(x - b) = 0$ are

(*a*) Positive (*b*) Negative

(*c*) Real (*d*) Imaginary

28. The value of 'a' for which one root of the quadratic equation $(\alpha^2 - 5a + 3)x2 - (3a - 1)x + 2 = 0$ is twice as large as other is

(*a*) –2/3 (*b*) 1/3 (*c*) –1/3 (*d*) 2/3

EQUATIONS REDUCIBLE TO QUADRATIC EQUATIONS

1. **Reciprocal equation** of the standard form can be reduced to an equation of half its dimension.
2. **Equations of the form**
   (a) $(x - a)(x - b)(x - c)(x - d) = A$ where $a < b < c < d$ can be solved by change of variable.
   $$y = x - \left(\frac{a + b + c + d}{4}\right)$$
   (b) $(x - a)(x - b)(x - c)(x - d) = Ax^2$ where
   $ab = cd$ can be solved by assumption $y = x + \frac{ab}{x}$.
3. **For the equation of the type $(x - a)^4 + (x - b)^4 = A$**
   Substitute $y = \frac{(x - a) + (x - b)}{2}$

IMPORTANT THEOREMS AND RESULTS

1. If α is a root of the equation $f(x) = 0$, then $f(x)$ is exactly divisible by $(x - \alpha)$ and conversely, if $f(x)$ is exactly divisible by $(x - \alpha)$ then α is a root of the equation $f(x) = 0$.
2. Every equation of an odd degree has atleast one real root, whose sign is opposite to that of its last term, provided that the coefficient of the first term is positive.
3. Every equation of an even degree whose last term is negative has at least two real roots, one positive and one negative, provided that the coefficient of the first term is positive.
4. If an equation has no odd powers of x, then all roots of the equation are complex provided all the coefficients of the equation are having positive sign.
5. If $x = \alpha$ is root repeated m times in $f(x) = 0$,
   ($f(x) = 0$ is an nth degree equation in x) then
   $f(x) = (x - \alpha)^m g(x)$
   when $g(x)$ is a polynomial of degree $(n - m)$ and the root $x = \alpha$ is repeated $(m - 1)$ times in $f'(x) = 0$, $(m - 2)$ times in $f''(x) = 0$,...$(m - (m - 1))$ times in $f^{m-1}(x) = 0$.
6. The condition that a quadratic function
   $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factor if
   $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$
   or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ is 0.

Train Your Brain

Example 41: Solve $2x^4 + x^3 - 11x^2 + x + 2 = 0$

Sol. Since $x = 0$ is not a solution hence, divide by x^2

We get,

$$2x^2 + x - 11 + \frac{1}{x} + \frac{2}{x^2} = 0$$

or $2\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 11 = 0$

Let $x + \frac{1}{x} = y \Rightarrow 2(y^2 - 2) + y - 11 = 0$

or $2y^2 + y - 15 = 0 \Rightarrow y = -3, \frac{5}{2}$

Corresponding values of x are $\frac{1}{2}, 2, \frac{-3 \pm \sqrt{5}}{2}$

Example 42: If the equations $4x^2 - 11x + 2k = 0$ and $x^2 - 3x - k = 0$ have a common root, then the value of k and common root is

(*a*) $0, \frac{17}{6}$ (*b*) $\frac{-17}{36}, \frac{17}{6}$

(*c*) $0, \frac{-17}{6}$ (*d*) $\frac{-17}{6}, 0$

Sol. The condition for the equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ to have a common root α is

$(ca' - c'a)2 = (bc' - b'c)(ab' - a'b)$ and $\alpha = \frac{ca' - c'a}{ab' - a'b}$

$4x^2 - 11x + 2k = 0 \Rightarrow x^2 - 3x - k = 0$

$\therefore (2k + 4k)^2 = (11k + 6k)(-12 + 11)$

$\therefore (6k)^2 = -17k \Rightarrow k = 0$ or $k = -\frac{17}{36}$

The common root α is given by $\alpha = \frac{6k}{-1}$

$\therefore$ If $k = 0$, then $\alpha = 0$ and if $k = -\frac{17}{36}$ then $\alpha = \frac{17}{6}$.

Hence answer is (b)

Example 43: The value of $\sqrt{7+\sqrt{7-\sqrt{7+\sqrt{7}...\infty}}}$ is

(*a*) 5 (*b*) 4

(*c*) 3 (*d*) 2

Sol. Let $y = \sqrt{7+\sqrt{7-\sqrt{7+\sqrt{7-}}}}$

or $y = \sqrt{7+\sqrt{7-y}}$

or $y^2 - 7 = \sqrt{7-y}$ ...(*i*)

Clearly $y^2 - 7 \geq 0 \Rightarrow y^2 \geq 7$

$\Rightarrow y \geq \sqrt{7}$ or $y \leq -\sqrt{7}$

$y \leq -\sqrt{7}$ is not acceptable as $y \geq 0$ so only solution may be in the interval $y \geq \sqrt{7}$

Since $\sqrt{7} > 2$ so $y = 2$ is not possible

Keeping $y = 3$, 4 and 5 in equation (i) we see

That only $y = 3$ satisfies it and not $y = 4$ or $y = 5$.

So only solution is $y = 3$.

Example 44: If $0 \leq x \leq \pi$, then the solution of the equation $16^{\sin^2 x} + 16^{\cos^2 x} = 10$ is given by x equal to

(*a*) $\frac{\pi}{6}, \frac{\pi}{3}$ (*b*) $\frac{\pi}{3}, \frac{\pi}{2}$

(*c*) $\frac{\pi}{6}, \frac{\pi}{2}$ (*d*) None of these

Sol. Let $16^{\sin^2 x} = y$, then $16^{\cos^2 x} = 16^{1-\sin^2 x} = \frac{16}{y}$

Hence, $y + \frac{16}{y} = 10$

$\Rightarrow y^2 - 10y + 16 = 0$ or $y = 2, 8$

Now $16^{\sin^2 x} = 2 \Rightarrow 4\sin^2 x = 1$

$\therefore \sin x = \pm\frac{1}{2} \Rightarrow x = \frac{\pi}{6}$

Also $16^{\sin^2 x} = 8 \Rightarrow 4\sin^2 x = 3$

$\therefore \sin x = \pm\frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3}$

Hence (*a*) is the correct answer.

Example 45: The root of $x^2 + 5|x| + 4 = 0$ are

Sol. Write $x^2 + 5|x| + 4 = 0$ as $|x|^2 + 5|x| + 4 = 0$

$\Rightarrow$ $|x| = -1$, $|x| = -4$ (which is not possible if $x \in R$)

$\therefore$ The given equation has no real roots.

Concept Application

29. The condition for $ax^2 + 2cxy + by^2 + 2bx + 2ay + c$ is resolvable into two linear factors is

(*a*) $a^3 + b^3 + c^3 = 3abc$

(*b*) $a^3 + b^3 + c^3 = abc$

(*c*) $a^3 + b^3 + c^3 = ab + bc + ca$

(*b*) $a^3 + b^3 + c^3 + abc = 0$

30. If $x^4 + 4xy + 4y^2 + 4x + cy + 3$ can be written as the product of two linear factors, then $c =$

(*a*) 2 (*b*) 4

(*c*) 6 (*d*) 8

31. $(x + 4)(x + 6)(x + 8)(x + 12) = 1680x^2$

32. If $\left(7+4\sqrt{3}\right)^{x^2-8} + \left(7-4\sqrt{3}\right)^{x^2-8} = 14$ then the value of x is

Short Notes

Solution of Quadratic Equation & Relation Between Roots & Co-Efficients

(*a*) The solutions of the quadratic equation, $ax^2 + bx + c = 0$ is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(*b*) The expression $b^2 - 4ac = D$ is called the discriminant of the quadratic equation.

(*c*) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then;

(*i*) $\alpha + \beta = -b/a$

(*ii*) $\alpha\beta = c/a$

(*iii*) $|\alpha - \beta| = \sqrt{D}/|a|$

(*d*) Quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ *i.e.*, $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ *i.e.* x^2 –(sum of roots)x + product of roots = 0.

Nature of Roots

(*a*) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then;

(*i*) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(*ii*) $D = 0 \Leftrightarrow$ roots are real & coincident (equal).

(*iii*) $D < 0 \Leftrightarrow$ roots are imaginary.

(*iv*) If $p + i\,q$ is one root of a quadratic equation, then the other root must be the conjugate $p - i\,q$ & vice versa. ($p, q \in R$ & $i = \sqrt{-1}$).

(*b*) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in Q$ & $a \neq 0$ then;

(*i*) If D is a perfect square, then roots are rational.

(*ii*) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$. (if a, b, c are rational)

Common Roots of Two Quadratic Equations

(*a*) Only one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then $a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$.

By Cramer's Rule

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

Therefore, $\alpha = \dfrac{ca' - c'a}{ab' - a'b} = \dfrac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is

$(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$

(*b*) If both roots are same then $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

Roots Under Particular Cases

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(*a*) If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign

(*b*) If $c = 0 \Rightarrow$ one roots is zero other is $-b/a$

(*c*) If $a = c \Rightarrow$ roots are reciprocal to each other

(*d*) If $\left.\begin{matrix} a > 0\ c < 0 \\ a < 0\ c > 0 \end{matrix}\right\} \Rightarrow$ roots are of opposite signs.

(*e*) If $\left.\begin{matrix} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are negative.

(*f*) If $\left.\begin{matrix} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are positive.

(*g*) If sign of a = sign of $b \neq$ sign of $c \Rightarrow$ Greater root in magnitude is negative.

(*h*) If sign of b = sign of $c \neq$ sign of $a \Rightarrow$ Greater root in magnitude is positive.

(*i*) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a.

Maximum & Minimum Values of Quadratic Expression

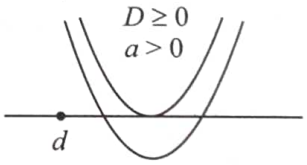
Maximum & Minimum Values of expression $y = ax^2 + bx + c$ is $\dfrac{-D}{4a}$ which occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

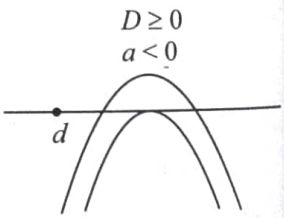
$y \in \left[\dfrac{-D}{4a}, \infty\right)$ if $a > 0$ & $y \in \left(-\infty, \dfrac{-D}{4a}\right]$ if $a < 0$.

location of roots

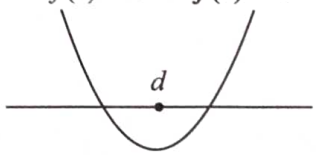
Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$

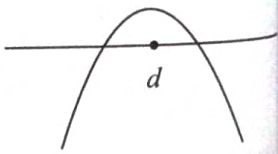
(*a*) Conditions for both the roots of f(x) = 0 to be greater than a specified number 'd' are $\boldsymbol{D \geq 0;\ a.f(d) > 0\ \&\ (-b/2a) > d}$.



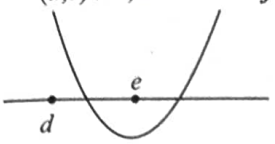


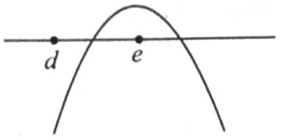
(*b*) Conditions for the both roots of $f(x) = 0$ to lie on either side of the number 'd' in other words the number 'd' lies between the roots of $f(x) = 0$ is $\boldsymbol{a.f(d) < 0}$.





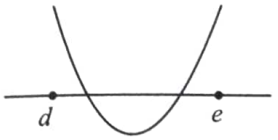
(*c*) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d,e) *i.e.*, $d < x < e$ is $f(d).f(e) < 0$.

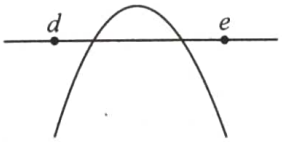




(*d*) Conditions that both roots of $f(x) = 0$ to be confined between the numbers d & e are (here $d < e$).

$D \geq 0$; $a.f(d) > 0$ & $af(e) > 0$; $d < (-b/2a) < e$





General Quadratic Expression in two Variables

$f(x, y) = ax^2 + 2\,hxy + by^2 + 2gx + 2\,fy + c$ may be resolved into two linear factors if;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ OR } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Theory of Equations

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\alpha_n$ are the roots of the equation; $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1} x + a_n = 0$ where $a_0, a_1, \ldots\ldots a_n$ are constants $a_0 \neq 0$ then,

$$\sum \alpha_1 = -\frac{a_1}{a_0}, \sum \alpha_1\alpha_2 = +\frac{a_2}{a_0}, \sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0},$$

$$\alpha_1\alpha_2\alpha_3\ldots\ldots\alpha_n = (-1)^n \frac{a_n}{a_0}$$

Note:

(*i*) Every odd degree equation has at least one real root whose sign is opposite to that of its last term, when coefficient of highest degree term is (+)ve {If not then make it (+)ve}.

Ex. $x^3 - x^2 + x - 1 = 0$

(*ii*) Even degree polynomial whose last term is (–)ve & coefficient of highest degree term is (+)ve has atleast two real roots, one (+)ve & one (–)ve.

(*iii*) If equation contains only even power of x & all coefficient are (+)ve, then all roots are imaginary.

Solved Examples

1. $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$. Solve for x.

(*a*) $\{5, 10\}$ (*b*) $[1, \infty)$

(*c*) $[5, 10]$ (*d*) $(5, 10)$

Sol. (*c*) Put $x - 1 = t^2$ in the given equation, we get

$\sqrt{t^2+1+3-4t} + \sqrt{t^2+1+8-6t} = 1$

$\Rightarrow |t-2| + |t-3| = 1$

$\Rightarrow t \in [2,3]$

$\Rightarrow x \in [5,10]$

2. If α & β are the two distinct roots of $x^2 + 2(K-3)x + 9 = 0$, then find the values of K such that $\alpha, \beta \in (-6, 1)$.

Sol. $x^2 + 2(k-3)x + 9 = 0$ ...(i)

Roots α, β of equation (i) are distinct and lies between -6 and 1

$D > 0$

$\Rightarrow 4(k-3)^2 - 36 > 0$

$\Rightarrow k(k-6) > 0$

$\Rightarrow k \in (-\infty, 0) \cup (6, \infty)$ ...(ii)

$f(1) > 0$

$\Rightarrow 1 + 2(k-3) + 9 > 0$

$\Rightarrow k \in (-2, \infty)$ ...(iii)

$f(-6) > 0$

$\Rightarrow 36 - 12(k-3) + 9 > 0$

$\Rightarrow 4k - 27 < 0 \Rightarrow k \in \left(-\infty, \frac{27}{4}\right)$ ...(iv)

$-6 < -\frac{b}{2a} < 1 \Rightarrow -6 < \frac{-2(k-3)}{2} < 1$

$\Rightarrow -1 < k - 3 < 6 \Rightarrow 2 < k < 9$ ...(v)

(ii) $\cap$ (iii) $\cap$ (iv) $\cap$ (v) $\Rightarrow k \in \left(6, \frac{27}{4}\right)$

3. If α is a root of the equation $4x^2 + 2x - 1 = 0$ then the other root must be

(*a*) $3\alpha^3 - 4\alpha$ (*b*) $4\alpha^3 - 3\alpha$

(*c*) $2\alpha 2 - 1$ (*d*) all above

Sol. (*c*) $4x^2 + 2x - 1 = 0$

$x = \frac{-2 \pm \sqrt{4+16}}{8}$

$$x = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

$$\text{If } \alpha = \frac{\sqrt{5}-1}{4} = \sin 18°$$

$$\beta = -\left[\frac{\sqrt{5}+1}{4}\right] = -\cos 36°$$

$\beta = -\cos 36° = -\sin [90° - 36°] = -\sin 54° = 4 \sin^3 18° - 3 \sin 18° = 4\alpha^3 - 3\alpha$

$\beta = -\cos 36° = 2 \cos^2 18° - 1 = 2\alpha^2 - 1$

4. Solve the equation $x(x + 1)^2(x + 2) = 72$.

Sol. We have $x(x + 1)^2(x + 2) = 72$

$\Rightarrow (x + 0)(x + 1)(x + 1)(x + 2) = 72$ ...(i)

(i) $[(x + 0)(x + 2)][(x + 1)(x + 1) = 72$ $(\because 0 + 2 = 1 + 1)$

$\Rightarrow (x^2 + 2x)(x^2 + 2x + 1) = 72$ ...(ii)

Let $y = x^2 + 2x$

(ii) $y(y + 1) = 72 \Rightarrow y^2 + y - 72 = 0$

$$\Rightarrow y = \frac{-1 \pm \sqrt{1+288}}{2} = \frac{-1 \pm 17}{2} = 8, -9$$

$\therefore$ Either $y = 8 \Rightarrow x^2 + 2x = 8 \Rightarrow x^2 + 2x - 8 = 0$

$$\Rightarrow x = \frac{-2 \pm \sqrt{4+32}}{2} = \frac{-2 \pm 6}{2} = 2, -4$$

or $y = -9 \Rightarrow x^2 + 2x = -9 \Rightarrow x^2 + 2x + 9 = 0$

$$\Rightarrow x = \frac{-2 \pm \sqrt{4-36}}{2} = \frac{-2 \pm 4\sqrt{2}i}{2} = -1 \pm 2\sqrt{2}i$$

5. If α, β are roots of the $ax^2 + bx + c = 0$, then the value of $\frac{1}{(a\alpha+b)^2} + \frac{1}{(a\beta+b)^2}$ is

(a) $\frac{b^2 - 2ac}{ac}$ (b) $\frac{2ac - b^2}{ac}$

(c) $\frac{b^2 - 2ac}{a^2c^2}$ (d) $\frac{b^2}{a^2c}$

Sol. Since α, β are the roots of the $ax^2 + bx + c$ then $a\alpha^2 + b\alpha + c = 0$ $\alpha(a\alpha + b) + c = 0$

$$(a\alpha + b) = \frac{-c}{\alpha} \quad ...(i)$$

Similarly

$$(a\beta + b) = \frac{-c}{\beta} \quad ...(ii)$$

$$\therefore \frac{1}{(a\alpha+b)^2} + \frac{1}{(a\beta+b)^2} = \frac{1}{(-c/\alpha)^2} + \frac{1}{(-c/\beta)^2}$$

$$\Rightarrow \frac{\alpha^2}{c^2} + \frac{\beta^2}{c^2} = \frac{\alpha^2+\beta^2}{c^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{c^2}$$

$$= \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{c^2} = \frac{b^2 - 2ac}{a^2c^2}$$

6. If the roots of the quadratic equation $x^2 + mx + n = 0$ tan 30° and tan 15°, respectively, then find the value of $n - m$.

Sol. The equation $x^2 + mx + n = 0$ has roots tan 30° and tan 1

Therefore,

$\tan 30° + \tan 15° = -m$ ...

$\tan 30° \tan 15° = n$ ...(

$$\text{Now, } \tan 45° = \tan (30° + 15°) \Rightarrow 1 = \frac{\tan 30° + \tan 15°}{1 - \tan 30° \tan 15°}$$

$\tan 30° \tan 15° = n$ ...(i

$$\text{Now, } \tan 45° = \tan (30° + 15°) \Rightarrow 1 = \frac{\tan 30° + \tan 15°}{1 - \tan 30° \tan 15°}$$

$$\Rightarrow 1 = \frac{-m}{1-n} \quad \text{[Using (i) and (ii)]}$$

$\Rightarrow 1 - n = -m \Rightarrow n - m = 1$

$\Rightarrow 2 + n - m = 3$

7. If the difference between the roots of the equation $x^2 + a + 1 = 0$ is less than $\sqrt{5}$, then find the set of possible value of a.

Sol. If α, β are roots of $x^2 + ax + 1 = 0$, then

$|\alpha - \beta| < \sqrt{5}$

$$\Rightarrow \left|\sqrt{(\alpha-\beta)^2 - 4\alpha\beta}\right| < \sqrt{5}$$

$$\Rightarrow \left|\sqrt{a^2 - 4}\right| < \sqrt{5}$$

$\Rightarrow a^2 - 4 < 5$

$\Rightarrow a^2 - 9 < 0$

$\therefore a \in (-3, 3)$

8. Find the values of a for which one root of equation $(a - 5)x^2 - 2ax + a - 4 = 0$ is smaller than 1 and the other greater than 2.

Sol. Let $f(x) = (a-5)x^2 - 2ax + a - 4$ $(a \neq 5)$ as 1 and 2 lie between the roots of $f(x) = 0$

Consider $(a - 5) f(1) < 0$: $(a - 5)(a - 5 - 2a + a - 4) < 0$

$\Rightarrow (a - 5)(-9) < 0$

$\Rightarrow a - 5 > 0$

$\Rightarrow a \in (5, \infty)$ ...(i)

Consider $(a - 5) f(2) < 0$: $(a - 5)(4 (a - 5) - 4a + a - 4) < 0$

$\Rightarrow (a - 5)(a - 24) < 0$

$\Rightarrow a \in (5, 24)$ ...(ii)

Hence, the values of a satisfying (i) and (ii) at the same time are $a \in (5, 24)$

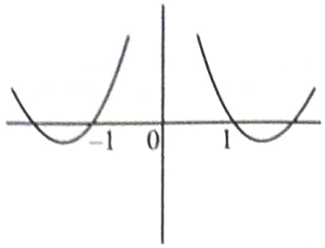
9. Find all values of k for which the inequality, $2x^2 - 4k^2x - k^2 + 1 > 0$ is valid for all real x which do not exceed unity in the absolute value.

Sol. **Case 1:** When $d < 0$ which gives the result

Case 2: $f(1) \geq 0$; $D \geq 0$; $-\frac{b}{2a} > 1$

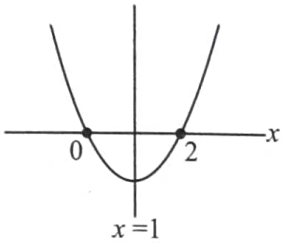
or $f(-1) \geq 0$; $d \geq 0$; $-\frac{b}{2a} < -1$

No solution in case-II; $-\frac{1}{\sqrt{2}} < k < \frac{1}{\sqrt{2}}$



10. Let $f(x) = ax^2 + bx + c$ be a polynomial. If $f(1-x) = f(1+x)$, $f(2) = 0$ and $f(3) = 6$, then find the value of a, b and c.

Sol. $f(1-x) = f(1+x) \Rightarrow$ the graph of the function $y = f(x)$ will be symmetric about the line $x = 1$



$f(2) = 0 \Rightarrow f(0) = 0$

$f(x) = ax(x-2)$

Now $f(3) = 6 \Rightarrow a = 2$

$f(x) = 2x^2 - 4x$

11. Find all values of a for which the equation $2x^2 - 2(2a+1)x + a(a+1) = 0$ has two roots, one of which is greater than a and the other is smaller than a.

Sol. Here coefficient of x^2 is positive

Let $f(x) = 2x^2 - 2(2a+1)x + a(a+1)$

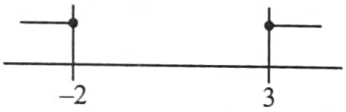
$\therefore f(x) = 2a^2 - 2(2a+1)a + a(a+1) = -a^2 - a < 0$

$\Rightarrow -a(a+1) < 0 \Rightarrow a(a+1) > 0$

$a \in (-\infty, -1) \cup (0, \infty)$

12. If x is real and $4y^2 + 4xy + x + 6 = 0$, then find the complete set of values of x for which y is real.

Sol. $4y^2 + 4xy + x + 6 = 0 \quad y \in R$



$\therefore D \geq 0$

$16x^2 - 16(x+6) \geq 0$

$\Rightarrow x^2 - x - 6 \geq 0 \Rightarrow (x-3)(x+2) \geq 0$

$\therefore x \in (-\infty, -2] \cup [3, \infty)$

13. If the equation $x^4 + x^3 - 2ax^2 - ax + a^2 = 0$ has 4 real roots than find the least possible value of 'a'.

Sol. To solve this type of equation try to make it a quadratic equation in 'a'.

$a^2 - a(2x^2 + x) + x^4 + x^3 = 0$

$\Rightarrow a^2 - a(x^2 + x^2 + x) + x^4 + x^3 = 0$

$\Rightarrow (x^2 - a)(x^2 + x - a) = 0$

If all roots are real then both factors must have real roots.

$x^2 - a = 0$ and $x^2 + x - a = 0$

$a \geq 0$ and $1 + 4a \geq 0 \geq a \geq -\frac{1}{4}$

$\Rightarrow$ the least possible value of 'a' is 0.

14. If $Q_1(x) = x^2 + (k-29)x - k$ and $Q_2(x) = 2x^2 + (2k-43)x + k$ both are factors of a cubic polynomial $P(x)$, then the largest value of k is

(*a*) 0 (*b*) 33 (*c*) 23 (*d*) 30

Sol. Two quadratic polynomials can be a factor of cubic polynomial only when they have atleast one root common

$\Rightarrow x^2 + (b - 2a)x - k = 0$ ...(i)

and $2x^2 + (2k-43)x + k = 0$ ...(ii)

Must have a common roots

Multiple equation (i) by (ii) and subtracting, we get

$15x + 3k = 0 \Rightarrow x = \frac{-k}{5}$ is the common root and it must satisfy equation (i)

$\Rightarrow \frac{k^2}{25} + (k-29)\left(-\frac{k}{5}\right) - k = 0$

$\Rightarrow \left(-\frac{k}{5}\right)\left[-\frac{k}{5} + k - 29 + 5\right] = 0 \quad \Rightarrow \quad k = 0$ or $k = 30$

15. Prove that the expression $2x^2 + 3xy + y^2 + 2y + 3x + 1$ can be factorized into two linear factors. Find them

Sol. $2x^2 + 3xy + y^2 + 2y + 3x + 1$

Comparing given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, we get

$a = 2, h = \frac{3}{2}, b = 1 g = \frac{3}{2}, f = 1, c = 1$

Clearly, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow$ Given expression can be factorized into two linear factor

To find the factors, form the quadratic in x

$2x^2 + 3x(y+1) + (y+1)^2$

$\Rightarrow x = \frac{-3(y+1) \pm \sqrt{(y+1)^2}}{y}$

$\Rightarrow 4x = -3(y+1) \pm (y+1)$

$\Rightarrow 4x = -2(y+1)$ or $4x = -4(y+1)$

$\Rightarrow 2x + y + 1 = 0$ or $x + y + 1 = 0$

Exercise-1 (Topicwise)

QUADRATIC EQUATION & NATURE OF ROOTS

1. If $a + b + c = 0$, and $a, b, c \in R$, then the roots of the equation $4ax^2 + 3bx + 2c = 0$ are
   (*a*) Equal (*b*) Imaginary
   (*c*) Real (*d*) Both (*a*) and (*b*)

2. The roots of the given equation $2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$ are
   [Where $a \neq b$]
   (*a*) Rational (*b*) Irrational
   (*c*) Real (*d*) Imaginary

3. If $a, b, c \in Q$, then roots of the equation $(b + c - 2a)x^2 + (c + a - 2b)x + (a + b - 2c) = 0$ are
   (*a*) Rational (*b*) Non-real
   (*c*) Irrational (*d*) Equal

4. The value of k for which the quadratic equation, $kx^2 + 1 = kx + 3x - 11x^2$ has real and equal roots are
   (*a*) $-11, -3$ (*b*) $5, 7$
   (*c*) $5, -7$ (*d*) $-7, 25$

5. $x^2 + x + 1 + 2k(x^2 - x - 1) = 0$ is a perfect square for how many values of k
   (*a*) 2 (*b*) 0 (*c*) 1 (*d*) 3

6. If $x^2 - 3x + 2$ be a factor of $x^4 - px^2 + q$, then (pq) =
   (*a*) (3, 4) (*b*) (4, 5)
   (*c*) (4, 3) (*d*) (5, 4)

7. The roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ are
   (*a*) $\frac{c-a}{b-c}, 1$ (*b*) $\frac{a-b}{b-c}, 1$
   (*c*) $\frac{b-c}{a-b}, 1$ (*d*) $\frac{c-a}{a-b}, 1$

8. If a, b, c are integers and $b^2 = 4(ac + 5d^2)$, $d \in N$, then roots of the equation $ax^2 + bx + c = 0$ are
   (*a*) Irrational (*b*) Rational & different
   (*c*) Complex conjugate (*d*) Rational & equal

9. Let a, b and c be real numbers such that $4a + 2b + c = 0$ and $ab > 0$. Then the equation $ax^2 + bx + c = 0$ has
   (*a*) real roots (*b*) imaginary roots
   (*c*) exactly one root (*d*) none of these

10. Consider the equation $x^2 + 2x - n = 0$, where $n \in N$ and $n \in [5, 100]$. Total number of different values of 'n' so that the given equation has integral roots, is
    (*a*) 4 (*b*) 6 (*c*) 8 (*d*) 3

11. The entire graph of the expression $y = x^2 + kx - x + 9$ strictly above the x-axis if and only if
    (*a*) $k < 7$ (*b*) $-5 < k < 7$
    (*c*) $k > -5$ (*d*) None

12. If $a, b \in R$, $a \neq 0$ and the quadratic equation $ax^2 - bx + 1$ 0 has imaginary roots then $a + b + 1$ is:
    (*a*) positive
    (*b*) negative
    (*c*) zero
    (*d*) depends on the sign of b

13. If a and b are the non-zero distinct roots of $x^2 + ax + b =$ then the least value of $x^2 + ax + b$ is
    (*a*) $\frac{3}{2}$ (*b*) $\frac{9}{4}$
    (*c*) $-\frac{9}{4}$ (*d*) 1

14. If both roots of the quadratic equation $(2 - x)(x + 1) = p$ distinct & positive, then p must lie in the interval
    (*a*) $(2, \infty)$ (*b*) $(2, 9/4)$
    (*c*) $(-\infty, -2)$ (*d*) $(-\infty, \infty)$

15. If $(1 - p)$ is root of quadratic equation $x^2 + px + (1 - p) =$ then its roots are
    (*a*) 0, 1 (*b*) $-1, 1$
    (*c*) $0, -1$ (*d*) $-1, 2$

16. If $a + b + c = 0$ then the equation $3ax^2 + 2bx + c = 0$ has
    (*a*) Imaginary roots (*b*) Real and equal roots
    (*c*) Real and different roots (*d*) Rational roots

17. If the roots of the equation
    $(a^2 + b^2)x^2 + 2(bc + ad)x + (c^2 + d^2) = 0$
    are real then a^2, bd, c^2 are in
    (*a*) AP (*b*) GP (*c*) HP (*d*) AGP

18. If a, b, c are in A.P. the roots of the equation $ax^2 + 2bx$ $= 0$ are
    (*a*) Real and distinct (*b*) Real and equal
    (*c*) Real (*d*) Imaginary

19. If the roots of $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x$ $= 0$ are equal then which of the following is not possib
    (*a*) $a + b + c = 0$ (*b*) $a = b = c$
    (*c*) $a + bw + cw^2 = 0$ (*d*) $a + bw^2 + cw = 0$

20. If the roots of the equation $bx^2 + cx + a = 0$ be imagi then for all real values of x, the expression $3b^2x^2 + 6b$ $2c^2$ is
    (*a*) Greater than $4ab$ (*b*) Less than $4ab$
    (*c*) Greater than $-4ab$ (*d*) Less than $-4ab$

Sum and Product of Roots

21. If one root of $5x^2 + 13x + k = 0$ is reciprocal of the other, then k =

(*a*) 0 (*b*) 5 (*c*) 1/6 (*d*) 6

22. If α and β are the roots of the equation $4x^2 + 3x + 7 = 0$, then $\frac{1}{\alpha}+\frac{1}{\beta}=$

(*a*) $-\frac{3}{7}$ (*b*) $\frac{3}{7}$

(*c*) $-\frac{3}{5}$ (*d*) $\frac{3}{5}$

23. If the sum of the roots of the equation $ax^2 + bx + c = 0$ be equal to the sum of their squares, then

(*a*) $a(a + b) = 2bc$ (*b*) $c(a + c) = 2ab$

(*c*) $b(a + b) = 2ac$ (*d*) $b(a + b) = a$

24. If α, β be the roots of the equation $x^2 - 2x + 3 = 0$, then the equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ is

(*a*) $x^2 + 2x + 1 = 0$ (*b*) $9x^2 + 2x + 1 = 0$

(*c*) $9x^2 - 2x + 1 = 0$ (*d*) $9x^2 + 2x - 1 = 0$

25. If the product of roots of the equation, $mx^2 + 6x + (2m - 1) = 0$ is –1, then the value of m will be

(*a*) 1 (*b*) – 1

(*c*) $\frac{1}{3}$ (*d*) $-\frac{1}{3}$

26. If α and β are the roots of the equation $x^2 - 4x + 1 = 0$ the value of $\alpha^3 + \beta^3$ is

(*a*) 76 (*b*) 52 (*c*) –52 (*d*) –76

27. If α, β, γ are the roots of the equation $x^3 + x + 1 = 0$, then the value of $\alpha^3 \beta^3\gamma^3$

(*a*) 0 (*b*) – 3 (*c*) 3 (*d*) – 1

28. If a, b are the roots of quadratic equation $x^2 + px + q = 0$ and g, d are the roots of $x^2 + px - r = 0$, then (a – g). (a – d) is equal to:

(*a*) q + r (*b*) q – r

(*c*) – (*q* + r) (*d*) – (p + q + r)

29. Two real numbers a & b are such that a + b = 3 & $|a - b| = 4$, then a & b are the roots of the quadratic equation:

(*a*) $4x^2 - 12x - 7 = 0$ (*b*) $4x^2 - 12x + 7 = 0$

(*c*) $4x^2 - 12x + 25 = 0$ (*d*) None of these

30. Let conditions C_1 and C_2 be defined as follows:

$C_1 : b^2 - 4ac \geq 0$, C_2 : a, – b, c are of same sign. The roots of $ax^2 + bx + c = 0$ are real and positive, if

(*a*) both C_1 and C_2 are satisfied

(*b*) only C_2 is satisfied

(*c*) only C_1 is satisfied

(*d*) None of these

31. If α, β are roots of the equation $ax^2 + bx + c = 0$, then the value of $\alpha^3 + \beta^3$ is

(*a*) $\frac{3abc + b^3}{a}$ (*b*) $\frac{a^3 + b^3}{3abc}$

(*c*) $\frac{3abc - b^3}{a^3}$ (*d*) $\frac{-(3abc + b^3)}{a^3}$

32. If $\alpha + \beta = -2$ and $\alpha^2 + \beta^2 = -56$ then the quadratic equation whose roots are α, β is

(*a*) $x^2 + 2x - 16 = 0$ (*b*) $x^2 + 2x - 15 = 0$

(*c*) $x^2 + 2x - 12 = 0$ (*d*) $x^2 + 2x - 8 = 0$

33. tan 22° and tan 23° are roots of $x^2 + ax + b = 0$ then

(*a*) $a + b + 1 = 0$ (*b*) $a - b + 1 = 0$

(*c*) $b - a + 1 = 0$ (*d*) $a + b = 1$

34. If the difference of the squares of the roots of the equation $x^2 - 6x + q = 0$ is 24, then the value of *q* is

(*a*) –7 (*b*) 8 (*c*) 5 (*d*) 4

35. Let α, β be the roots of the equation $(x - a)(x - b) = c$, $c \neq 0$ then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are

(*a*) *a*,*c* (*b*) *b*,*c* (*c*) *a*,*b* (*d*) $a + b, b + c$

36. Let α, β are the roots of the equation $x^2 - 15x + 1 = 0$, then the value of $\left(\frac{1}{\alpha}-15\right)^{-2}+\left(\frac{1}{\beta}-15\right)^{-2}$ is

(*a*) 225 (*b*) 900 (*c*) 223 (*d*) 0

37. The quadratic equation whose roots are the *x* and *y* intercepts of the line passing through (1, 1) and making a triangle of area *A* with the axes, may be

(*a*) $x^2 + Ax + 2A = 0$ (*b*) $x^2 - 2Ax + 2A = 0$

(*c*) $x^2 - Ax + 2A = 0$ (*d*) $x^2 - 2Ax - 2A = 0$

38. If one root of $x^2 + px + 12 = 0$ is 4 while the equation $x^2 + px + q = 0$ has equal roots then *q* =

(*a*) –7 (*b*) 4

(*c*) 42 (*d*) $\frac{49}{4}$

39. If $k > 0$ and the product of the roots of the equation $x^2 - 3kx + 2e^{2\log k} - 1 = 0$ is 7 then the sum of the roots is

(*a*) 2 (*b*) 4 (*c*) 6 (*d*) 8

40. Two students while solving a quadratic equation in x, one copied the constant term incorrectly and got the roots as 3 and 2. The other copied the constant terms and coefficient of x^2 as –6 and 1 respectively. The correct roots are

(*a*) 3, –2 (*b*) –3, 2

(*c*) –6, –1 (*d*) 6, –1

41. Let A, G and H be the A.M., G.M. and H.M, of two positive number *a* and *b*. The quadratic equation whose roots are *A* and *H* is

(*a*) $Ax^2 - (A^2 + G^2)x + AG^2 = 0$

(*b*) $Ax^2 - (A^2 + G^2)x + AH^2 = 0$

(*c*) $Hx^2 - (H^2 + G^2)x + HG^2 = 0$

(*d*) $Gx^2 - (H^2 + G^2)x + GH^2 = 0$

42. If one root of the equation $ax^2 + bx + c = 0$ is the square of the other, then

(a) $b^2 + ac^2 + a^2c = 3abc$
(b) $b^3 + ac^2 + a^2c = 3abc$
(c) $b^2 + ac^2 + a^2c + 3abc = 0$
(d) $b^3 + ac^2 + a^2c + 3abc = 0$

43. The roots of the equation $ax^2 + bx + c = 0$, $a \in R^+$, are two consecutive odd positive integers. Then

(a) $|b| \le 4a$ (b) $|b| \ge 4a$
(c) $|b| \ge 2a$ (d) $|b| \le a$

44. The number of real solution of the equation $(9/10)^x = -3 + x - x^2$ is

(a) 2 (b) 0
(c) 1 (d) 3

45. If the roots of $x^2 - bx + c = 0$ are two consecutive integers then $b^2 - 4c =$

(a) 0 (b) 1
(c) 2 (d) 3

46. If the roots of the quadratic equation $x^2 + px + q$ are tan 30° and tan 15° respectively then the value of $2 + q - p$ is

(a) 1 (b) 2
(c) 3 (d) 0

COMMON ROOTS

47. If both the roots of $k(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6k(2x^2 + 1) + px + 4x^2 - 2 = 0$ are common, then $2r - p$ is equal to

(a) –1 (b) 0
(c) 1 (d) 2

48. If the equation $x^2 + px + q = 0$ and $x^2 + qx + p = 0$, have a common root,(Where $p \ne q$) then $p + q + 1 =$

(a) 0 (b) 1
(c) 2 (d) – 1

49. If a, b, p, q are non-zero real numbers, then the equations $2a^2x^2 - 2abx + b^2 = 0$ and $p^2x^2 + 2pqx + q^2 = 0$ have:

(a) no common root
(b) one common root if $2a^2 + b^2 = p^2 + q^2$
(c) two common roots if $3pq = 2ab$
(d) two common roots if $3qb = 2ap$

50. If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root and $a \ne 0$ then $\dfrac{a^3 + b^3 + c^3}{abc} =$

(a) 1 (b) 2
(c) 3 (d) 9

51. The equations $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have a common root. If $a, b, c \in N$ then the least possible values of $a + b + c$ is equal to

(a) 3 (b) 6
(c) 9 (d) 12

52. If the equation $ax^2 + 2bx + 3c = 0$ and $3x^2 + 8x + 15 = 0$ h a common root where a, b, c are the length of the side ΔABC then $\sin^2 A + \sin^2 B + \sin^2 C$ is equal to

(a) 1 (b) $\dfrac{3}{2}$
(c) $\sqrt{2}$ (d) 2

Theory of Equation and Identity, Inequalities

53. If x is real and satisfies $x + 2 > \sqrt{x+4}$, then

(a) $x < -2$ (b) $x > 0$
(c) $-3 < x < 0$ (d) $-3 < x < 4$

54. The set of all real numbers x for which $x^2 - |x+2| + x > 0$ is

(a) $(-\infty, -2) \cup (2, \infty)$ (b) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$
(c) $(-\infty, -1) \cup (1, \infty)$ (d) $(\sqrt{2}, \infty)$

55. Number of values of ' p ' for which the equation $(p^2 - + 2)x^2 - (p^2 - 5p + 4)x + p - p^2 = 0$ possess more than roots, is:

(a) 0 (b) 1
(c) 2 (d) None

56. The number of the integer solutions of $x^2 + 9 < (x + 3)^2 < 8x + 25$ is

(a) 1 (b) 2
(c) 3 (d) None of these

57. The complete set of values of 'x' which satisfy th equations : $5x + 2 < 3x + 8$ and $\dfrac{x+2}{x-1} < 4$ is

(a) $(-\infty, 1)$ (b) $(2, 3)$
(c) $(-\infty, 3)$ (d) $(-\infty, 1) \cup (2, 3)$

58. The complete solution set of the inequality $\dfrac{x^4 - 3x^3 + }{x^2 - x - 3} \ge 0$ is:

(a) $(-\infty, -5) \cup (1, 2) \cup (6, \infty) \cup \{0\}$
(b) $(-\infty, -5) \cup [1, 2] \cup (6, \infty) \cup \{0\}$
(c) $(-\infty, -5] \cup [1, 2] \cup [6, \infty) \cup \{0\}$
(d) none of these

59. If the inequality $(m - 2)x^2 + 8x + m + 4 > 0$ is satisfied all $x \in R$, then the least integral 'm' is:

(a) 4 (b) 5
(c) 6 (d) none

60. The complete set of real ' x ' satisfying $||x - 1| - 1| \le 1$ i

(a) $[0, 2]$ (b) $[-1, 3]$
(c) $[-1, 1]$ (d) $[1, 3]$

61. If $\log_{1/3} \dfrac{3x-1}{x+2}$ is less than unity, then 'x' must lie in interval:

(a) $(-\infty, -2) \cup (5/8, \infty)$ (b) $(-2, 5/8)$
(c) $(-\infty, -2) \cup (1/3, 5/8)$ (d) $(-2, 1/3)$

62. Solution set of the inequality $2 - \log_2(x^2 + 3x) \geq 0$ is:

(*a*) $[-4, 1]$ (*b*) $[-4, -3) \cup (0, 1]$

(*c*) $(-\infty, -3) \cup (1, \infty)$ (*d*) $(-\infty, -4) \cup [1, \infty)$

63. The set of all the solutions of the inequality $\log_{1-x}(x-2) \geq 0$ is

(*a*) $(-\infty, 0)$ (*b*) $(2, \infty)$

(*c*) $(-\infty, 1)$ (*d*) ϕ

64. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval

(*a*) $(2, \infty)$ (*b*) $(1, 2)$

(*c*) $(-2, -1)$ (*d*) None of these

65. If $\log_{0.5}\log_5(x^2-4) > \log_{0.5}1$, then 'x' lies in the interval

(*a*) $(-3, -\sqrt{5}) \cup (\sqrt{5}, 3)$

(*b*) $(-3, -\sqrt{5}) \cup (\sqrt{5}, 3\sqrt{5})$

(*c*) $(\sqrt{5}, 3\sqrt{5})$

(*d*) ϕ

66. The set of all solutions of the inequality $(1/2)^{x^2-2x} < 1/4$ contains the set

(*a*) $(-\infty, 0)$ (*b*) $(-\infty, 1)$

(*c*) $(1, \infty)$ (*d*) $(3, \infty)$

67. If $\frac{6x^2-5x-3}{x^2-2x+6} \leq 4$, then least and the highest values of $4x^2$ are

(*a*) 0 & 81 (*b*) 9 & 81

(*c*) 36 & 81 (*d*) None of these

68. If two roots of the equation $x^3 - px^2 + qx - r = 0$ are equal in magnitude but opposite in sign, then:

(*a*) $pr = q$ (*b*) $qr = p$

(*c*) $pq = r$ (*d*) None

69. If α, β & γ are the roots of the equation $x^3 - x - 1 = 0$ then, $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$ has the value equal to:

(*a*) zero (*b*) -1

(*c*) -7 (*d*) 1

70. For what value of a and b the equation $x^4 - 4x^3 + ax^2 + bx + 1 = 0$ has four real positive roots?

(*a*) $(-6, -4)$ (*b*) $(-6, 5)$

(*c*) $(-6, 4)$ (*d*) $(6, -4)$

71. If α, β are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are $2\alpha + 3\beta$ and $3\alpha + 2\beta$ is

(*a*) $ab\,x^2 - (a+b)\,cx + (a+b)^2 = 0$

(*b*) $ac\,x^2 - (a+c)\,bx + (a+c)^2 = 0$

(*c*) $ac\,x^2\,(a+c)\,bx - (a+c)^2 = 0$

(*d*) None of these

72. If S is the set of all real x such that $\frac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains

(*a*) $(-\infty, -3/2)$ (*b*) $(-3/2, 1/4)$

(*c*) $(-1/4, 1/2)$ (*d*) $(-1/2, 3)$

73. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a is

(*a*) $(-3, 3)$ (*b*) $(-3, \infty)$

(*c*) $(3, \infty)$ (*d*) $(\infty, -3)$

74. The greatest negative integer satisfying $x^2 + 4x - 77 < 0$ and $x^2 > 4$ is

(*a*) -1 (*b*) -2

(*c*) -3 (*d*) -10

75. The solution set contained in R of the in equation $3^x + 3^{1-x} - 4 < 0$ is

(*a*) $(1, 3)$ (*b*) $(0, 1)$ (*c*) $(1, 2)$ (*d*) $(0, 2)$

76. If the equation $ax^2 + 2bx - 3c = 0$ has non real roots and $\frac{3c}{4} < a + b$ then c is always

(*a*) < 0 (*b*) > 0 (*c*) ≥ 0 (*d*) zero

77. The equation $(\cos p - 1)x^2 + (\cos p)x + (\sin p) = 0$ in the variable x has real roots. Then p can take any value in the interval

(*a*) $(0, 2\pi)$ (*b*) $(-\pi, 0)$

(*c*) $(-\pi/2, \pi/2)$ (*d*) $(0, \pi)$

78. If a and $b(\neq 0)$ are the root of the equation $x^2 + ax + b = 0$ then the least value of $x^2 + ax + b (x \in R)$ is

(*a*) $\frac{9}{4}$ (*b*) $\frac{-9}{4}$

(*c*) $\frac{-1}{4}$ (*d*) $\frac{1}{4}$

79. If $x^2 + \frac{1}{x^2} = A$ and $x - \frac{1}{x} = B$ $(|x| > 1)$ then least value of $\frac{A}{B}$ is

(*a*) 2 (*b*) $\sqrt{2}$

(*c*) $-\sqrt{2}$ (*d*) $-2\sqrt{2}$

80. If α, β are the roots of $x^2 - (a-2)x - (a+1) = 0$ where 'a' is a variable then the least value of $\alpha^2 + \beta^2$ is

(*a*) 2 (*b*) 3

(*c*) 5 (*d*) 7

81. The minimum value of $|x| + \left|x + \frac{1}{2}\right| + |x-3| + \left|x - \frac{5}{2}\right|$ is

(*a*) 2 (*b*) 4 (*c*) 6 (*d*) 8

82. Sum of the roots of the equation $(x-2)^2 - 2|x-2| - 15 = 0$ is

(*a*) 4 (*b*) 0 (*c*) -4 (*d*) 8

83. The real roots of the equation $|x^2 + 4x + 3| + 2x + 5 = 0$ are

(*a*) $4, -1+\sqrt{3}$ (*b*) $-4, -1-\sqrt{3}$

(*c*) $-6, -1$ (*d*) $6, -1$

84. If x satisfies $|x-1| + |x-2| + |x-3| \geq 6$ then

(*a*) $0 \leq x \leq 4$ (*b*) $x \leq -2$ (or) $x \geq 4$

(*c*) $x \leq 0$ (or) $x \geq 4$ (*d*) R

Max and Min Value, Factorization

85. If x is real, then the maximum and minimum values of the expression $\frac{x^2-3x+4}{x^2+3x+4}$ will be

(*a*) 2, 1 (*b*) $5, \frac{1}{5}$

(*c*) $7, \frac{1}{7}$ (*d*) 2, 7

86. The smallest value of $x^2 - 3x + 3$ in the interval (–3, 3/2) is

(*a*) 3/4 (*b*) 5.

(*c*) –15 (*d*) –20

87. If $y = -2x^2 - 6x + 9$, (for $x \in R$) then

(*a*) maximum value of y is –11 and it occurs at x = 2

(*b*) minimum value of y is –11 and it occurs at x = 2

(*c*) maximum value of y is 13.5 and it occurs at x = –1.5

(*d*) minimum value of y is 13.5 and it occurs at x = –1.5

88. If 'x' is real and $k = \frac{x^2-x+1}{x^2+x+1}$, then:

(*a*) $\frac{1}{3} \le k \le 3$ (*b*) $k \ge 5$

(*c*) $k \le 0$ (*d*) None of these

89. Consider $y = \frac{2x}{1+x^2}$, where x is real, then the range of expression $y^2 + y - 2$ is

(*a*) [–1, 1] (*b*) [0, 1]

(*c*) [–9/4, 0] (*d*) [–9/4, 1]

90. The values of x and y besides y can satisfy the equation (x, y ∈ real numbers) $x^2 - xy + y^2 - 4x - 4y + 16 = 0$

(*a*) 2, 2 (*b*) 4, 4

(*c*) 3, 3 (*d*) None of these

91. If x is real, then $\frac{x^2-x+c}{x^2+x+2c}$ can take all real values if

(*a*) $c \in [0, 6]$ (*b*) $c \in [-6, 0]$

(*c*) $c \in (-\infty, -6) \cup (0, \infty)$ (*d*) $c \in (-6, 0)$

LOCATION OF ROOTS

92. The real values of 'a' for which the quadratic equation $2x^2 - (a^3 + 8a - 1)x + a^2 - 4a = 0$ possesses roots of opposite sign is given by:

(*a*) a > 5 (*b*) 0 < a < 4

(*c*) a > 0 (*d*) a > 7

93. If a, b are the roots of the quadratic equation $x^2 - 2p(x-4) - 15 = 0$, then the set of values of 'p' for which one root is less than 1 & the other root is greater than 2 is:

(*a*) (7/3, ∞) (*b*) (–∞, 7/3)

(*c*) x ∈ R (*d*) None of these

94. The values of k, for which the equation $x^2 + 2(k-1)$ + 5 = 0 possess atleast one positive root, are

(*a*) [4, ∞) (*b*) (–∞, – 1] ∪ [4, ∞)

(*c*) [–1, 4] (*d*) (–∞, – 1]

95. The set of values of *p* for which 6 lies between the roo the equation $x^2 + 2(p-3)x + 9 = 0$ is

(*a*) $\left(\frac{3}{4}, \infty\right)$ (*b*) $\left(-\infty, -\frac{3}{4}\right)$

(*c*) $\left(-\frac{3}{4}, \frac{3}{4}\right)$ (*d*) $\left[-\frac{3}{4}, \frac{3}{4}\right]$

96. All the values of *m* for which both roots of the equ $x^2 - 2mx + m^2 - 1 = 0$ are greater than 2 but less than in the interval

(*a*) $-1 < m < 3$ (*b*) $1 < m < 4$

(*c*) $-2 < m < 0$ (*d*) $m > 3$

97. If $a \in R$ and the roots of $x^2 - 2x - a^2 + 1 = 0$ lies bet the roots of $x^2 - 2(a+1)x + a(a-1) = 0$ then '*a*' belon

(*a*) (*1*, ∞) (*b*) $\left(-\frac{1}{4}, 1\right)$

(*c*) (– ∞, 0) (*d*) $\left(-\infty, -\frac{1}{4}\right)$

98. Consider the equation $(1 + a + b)^2 = 3(1 + a^2 + b^2)$, w *a*, *b* are real numbers. Then

(*a*) There is no solution pair (*a*, *b*).

(*b*) There are infinitely many solution pairs (*a*, *b*).

(*c*) There are exactly two solution pairs (*a*, *b*).

(*d*) There is exactly one solution pair (*a*, *b*).

99. Let *S* be the set of all ordered pairs (*x*, *y*) of positive inte satisfying the condition $x^2 - y^2 = 12345678$. Then

(*a*) *S* is an infinite set

(*b*) *S* is the empty set

(*c*) *S* has exactly one element

(*d*) *S* is a finite set and has at least two elements

100. Consider $f(x) = 4x^4 - 24x^3 + 31x^2 + 6x - 8$ be a polyno function and α, β, γ, σ are the roots (α < β < γ < σ). So

$\sigma^p + \frac{1}{\sigma^\alpha} + \alpha^\gamma + \gamma^\sigma$

(*a*) 36 (*b*) 18 (*c*) 9 (*d*) 10

101. Solve $\left(\frac{x-1}{x+2}\right)^3 = \frac{x-4}{x+5}$.

(*a*) $-\frac{1}{2}$ (*b*) $\frac{1}{2}$ (*c*) $-\frac{1}{4}$ (*d*) $\frac{1}{4}$

102. If α, β, γ are the roots of the equation $x^3 + 4x + 1 = 0$, find the value of $\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha}$.

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

103. If α, β, γ are the roots of the equation, $x^3 + px^2 + qx + r = 0$, then find the value of $\left(\alpha - \frac{1}{\beta\gamma}\right)\left(\beta - \frac{1}{\gamma\alpha}\right)\left(\gamma - \frac{1}{\alpha\beta}\right)$.

(*a*) $-\frac{(r-1)^3}{r^2}$ (*b*) $-\frac{(r-2)^3}{r^2}$

(*c*) $-\frac{(r+1)^3}{r^2}$ (*d*) $-\frac{(r-2)^3}{(r-1)}$

104. Solve the equation $x^3 - 13x^2 + 15x + 189 = 0$, if one root exceeds the other by 2.

(*a*) 7, 9, –3 (*b*) 2, –5, –4 (*c*) 1, 2, 6 (*d*) –1, –8, 9

105. If the inequality $\frac{mx^2 + 3x + 4}{x^2 + 2x + 2} < 5$ is satisfied for all $x \in R$, then find the values of m.

(*a*) $1 < m < 5$ (*b*) $-1 < m < 5$

(*c*) $1 < m < 6$ (*d*) $m < 71/24$

106. Sum of all the roots of the equation $x^2 - 2x + |x - 1| - 5 = 0$ is

(*a*) 0 (*b*) 2

(*c*) 1 (*d*) 5

107. If α and β are the distinct roots of the equation $x^2 + 3^{1/4}x + 3^{1/2} = 0$ then the value of $\alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1)$ is equal to

(*a*) 56×3^{24} (*b*) 52×3^{24}

(*c*) 56×3^{25} (*d*) 28×3^{25}

108. For what integral k is the inequality $x^2 - 2(4k - 1)x + 15k^2 - 2k - 7 > 0$ valid for any real x?

(*a*) 2 (*b*) 3

(*c*) 4 (*d*) 5

109. Number of real roots of the equation $x^4 - 4x - 1 = 0$ has

(*a*) atmost two positive real root

(*b*) atmost one negative real root

(*c*) atmost two real roots

(*d*) 4 real roots

Exercise-2 (Learning Plus)

1. If a > 0, b > 0, c > 0 then both the roots of the equation $ax^2 + bx + c = 0$

(*a*) Are real and negative

(*b*) Have negative real parts

(*c*) Are rational numbers

(*d*) Both (*a*) and (*c*)

2. The value of k for which the equation $(k-2)x^2 + 8x + k + 4 = 0$ has both real, distinct and negative is

(*a*) 0 (*b*) 2 (*c*) 3 (*d*) – 4

3. If α, β are the roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are and $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$, is

(*a*) $acx^2 + (a + c)bx + (a + c)^2 = 0$

(*b*) $abx^2 + (a + c)bx + (a + c)^2 = 0$

(*c*) $acx^2 + (a + b)cx + (a + c)^2 = 0$

(*d*) $acx^3 + (a + c)bx + (a + c)^3$

4. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then $\frac{\alpha}{a\beta + b} + \frac{\beta}{a\alpha + b} =$

(*a*) $\frac{2}{a}$ (*b*) $\frac{2}{b}$ (*c*) $\frac{2}{c}$ (*d*) $-\frac{2}{a}$

5. If the roots of the equation $12x^2 - mx + 5 = 0$ are in the ratio 2 : 3, then m =

(*a*) $5\sqrt{10}$ (*b*) $3\sqrt{10}$

(*c*) $2\sqrt{10}$ (*d*) $10\sqrt{5}$

6. $x^2 - 11x + a$ and $x^2 - 14x + 2a$ will have a common factor, if $a =$

(*a*) 24 (*b*) 0, 24 (*c*) 3, 24 (*d*) 0, 3

7. If α, β, γ are the roots of the equation $x^3 + 4x + 1 = 0$, then $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1}$

(*a*) 2 (*b*) 3 (*c*) 4 (*d*) 5

8. If the roots of $x^2 + x + a = 0$ exceed a , then

(*a*) $2 < a < 3$ (*b*) $a > 3$

(*c*) $-3 < a < 3$ (*d*) $a < -2$

9. The value of p for which both the roots of the equation $4x^2 - 20px + (25p^2 + 15p - 66) = 0$ are less than 2, lies in

(*a*) (4/5, 2) (*b*) $(2, \infty)$

(*c*) $(-1, -4/5)$ (*d*) $(-\infty, -1)$

10. The equation $x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$ has

(*a*) No root (*b*) One root

(*c*) Two equal root (*d*) Infinitely many roots

11. The number of quadratic equations which are unchanged by squaring their roots

(*a*) 2 (*b*) 3

(*c*) 4 (*d*) None of these

12. For what value of the curve $y = x^2 + ax + 25$ touches the x-axis

(*a*) 0 (*b*) ± 5

(*c*) ± 10 (*d*) None of these

13. If $b^2 < 2ac$, then equation $ax^3 + bx^2 + cx + d = 0$ has

(*a*) exactly one real roots (*b*) Has three real roots

(*c*) at least two roots (*d*) None of these

14. Let α and β are the roots of the equation $x^2 + x + 1 = 0$ then

(*a*) $\alpha^2 + \beta^2 = 4$ (*b*) $(\alpha - \beta)^2 = 3$

(*c*) $\alpha^3 + \beta^3 = 2$ (*d*) $\alpha^4 + \beta^4 = 1$

15. If $\sec\alpha$, $\tan\alpha$ are roots of $ax^2 + bx + c = 0$, then

(*a*) $a^4 - b^4 + 4ab^2c = 0$ (*b*) $a^4 + b^4 - 4ab^2c = 0$

(*c*) $a^2 - b^2 = 4ac$ (*d*) $a^2 + b^2 = ac$

16. Let α, β be the roots of $ax^2 + bx + c = 0$, γ, δ be the roots of $px^2 + qx + r = 0$ and D_1 and D_2 be their respective discriminant. If α, β, γ, δ , are in A.P., then the ratio $D_1 : D_2$ is equal to

(*a*) $\frac{a^2}{b^2}$ (*b*) $\frac{a^2}{p^2}$ (*c*) $\frac{b^2}{q^2}$ (*d*) $\frac{c^2}{r^2}$

17. If one root of the equation $(l - m)x^2 + lx + 1 = 0$ is double the other and if l is real, then the greatest value of m is

(*a*) $\frac{9}{8}$ (*b*) $\frac{7}{8}$ (*c*) $\frac{8}{9}$ (*d*) $\frac{8}{9}$

18. If p, q, r are real numbers satisfying the condition $p + q + r = 0$, then the roots of the quadratic equation $3px^2 + 5qx + 7r = 0$ are

(*a*) Positive (*b*) Negative

(*c*) Real and distinct (*d*) Imaginary

19. If a, b, c are in G.P., then the equation $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in

(*a*) A.P. (*b*) G.P. (*c*) H.P. (*d*) $ab = cd$

20. If $a < b < c < d$ and $K > 0$, then the quadratic equation $(x - a)(x - c) + k(x - b)(x - d) = 0$ has

(*a*) All roots real and distinct

(*b*) All roots real but not necessarily distinct

(*c*) All root real and negative

(*d*) May be imaginary

21. The number of integers satisfying the inequality $\frac{x}{x+6} \le \frac{1}{x}$ is:

(*a*) 7 (*b*) 8 (*c*) 9 (*d*) 3

22. Sum of values of x and y satisfying the equation $3^x - 4^y = 77$; $3^{x/2} - 2^y = 7$ is:

(*a*) 2 (*b*) 3 (*c*) 4 (*d*) 5

23. If the value of $m^4 + \frac{1}{m^4} = 119$, then the value of $\left|m^3 - \frac{1}{m^3}\right| =$

(*a*) 11 (*b*) 18 (*c*) 24 (*d*) 36

24. The number of integral roots of the equation $x^8 - 24x^7 - 18x^5 + 39x^2 + 1155 = 0$ is:

(*a*) 0 (*b*) 2 (*c*) 4 (*d*) 6

25. If the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude but opposite in sign, then the product of the roots is:

(*a*) $-2(p^2 + q^2)$ (*b*) $-(p^2 + q^2)$

(*c*) $-\frac{(p^2+q^2)}{2}$ (*d*) $-pq$

26. If $\alpha \ne \beta$ but $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$, then the equation with roots $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ is:

(*a*) $3x^2 - 25x + 3 = 0$ (*b*) $x^2 + 5x - 3 = 0$

(*c*) $x^2 - 5x + 3 = 0$ (*d*) $3x^2 - 19x + 3 = 0$

27. Minimum possible number of positive root of the quadratic equation $x^2 - (1 + \lambda)x + \lambda - 2 = 0$, $\lambda \in R$:

(*a*) 2 (*b*) 0

(*c*) 1 (*d*) Can not be determined

28. If $\alpha, \beta, \gamma, \delta \in R$ satisfy

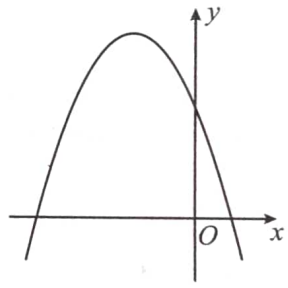
$$\frac{(\alpha+1)^2 + (\beta+1)^2 + (\gamma+1)^2 + (\delta+1)^2}{\alpha+\beta+\gamma+\delta} = 4$$

If biquadratic equation $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ has the roots $\left(\alpha + \frac{1}{\beta} - 1\right), \left(\beta + \frac{1}{\gamma} - 1\right), \left(\gamma + \frac{1}{\delta} - 1\right), \left(\delta + \frac{1}{\alpha} - 1\right)$. Then the value of a_2/a_0 is:

(*a*) 4 (*b*) –4

(*c*) 6 (*d*) None the these

29. If graph of the quadratic $y = ax^2 + bx + c$ is given below:



(*a*) $a > 0, b > 0, c > 0$ (*b*) $a < 0, b > 0, c < 0$

(*c*) $a < 0, b < 0, c > 0$ (*d*) $a < 0, b < 0, c < 0$

30. Both the roots of the equation $(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b) = 0$ are always

(*a*) positive (*b*) real

(*c*) negative (*d*) None of these.

31. The least value of the expression $2\log_{10}x - \log_x(0.01)$ for $x > 1$ is

(*a*) 10 (*b*) 2

(*c*) –0.01 (*d*) None of these

32. If $(x^2 + px + 1)$ is a factor of $(ax^3 + bx + c)$, then

(*a*) $a^2 + c^2 = -ab$ (*b*) $a^2 - c^2 = -ab$

(*c*) $a^2 - c^2 = ab$ (*d*) none of these

33. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is

(a) 4 (b) 1 (c) 3 (d) 2

34. Two towns A and B are 60km apart. A school is to be built to serve 150 students in town A and 50 students in town B. If the total distance to be travelled by all 200 student is to be as small as possible, then the school should be built at

(a) town B (b) 45 km from town A

(c) town A (d) 45 km from town B

35. If α and β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then equation $x^2 - 4qx + 2q^2 - r = 0$ has always

(a) two real roots

(b) two positive roots

(c) two negative roots

(d) one positive and one negative roots

36. Let a, b, c be real numbers, $a = 0$, if a is roots of $a^2x^2 + bx + c = 0$. β is the root of $a^2x^2 - bx - c = 0$ and $0<\alpha<\beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies

(a) $\gamma = \frac{\alpha+\beta}{2}$ (b) $\gamma = \alpha + \frac{\beta}{2}$

(c) $\gamma = \alpha$ (d) $\alpha < \gamma < \beta$

37. The number of solutions of the equation $\sin(e)^x = 5x + 5^{-x}$ is

(a) 0 (b) 1

(c) 2 (d) Infinitely many

38. The number of points of intersection of two curves $y = 2\sin x$ and $y = 5x^2 + 2x + 3$ is

(a) 0 (b) b, c (c) 2 (d) $a + c, b + c$

39. If p, q, r are +ve and are in $A.P.$, the roots of quadratic equation $px^2 + qx + r = 0$ are all real for

(a) $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$ (b) $\left|\frac{p}{r} - 7\right| \geq 4\sqrt{3}$

(c) all p and r (d) no p *and* r

40. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots is

(a) 15 (b) 9 (c) 7 (d) 8

41. If α and β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$

(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$

42. If a, b, c, d are positive real numbers such that $a + b + c + d = 2$, then $M = (a + b)(c + d)$ satisfies the relation

(a) $0 \leq M \leq 1$ (b) $1 \leq M \leq 2$

(c) $2 \leq M \leq 3$ (d) $3 \leq M \leq 4$

43. If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$ has

(a) both roots in (a, b)

(b) both roots in $(-\infty, a)$

(c) both roots in $(b, +\infty)$

(d) one roots in $(-\infty, a)$ and the other in $(b, +\infty)$

44. For the equations $3x^2 + px + 3 = 0, p > 0$, if one of the roots is square of the order, then p is equal to

(a) 1/3 (b) 1 (c) 3 (d) 2/3

45. If $a_1, a_2 \ldots\ldots a_n$ are positive real numbers whose product is a fixed number c, then the minimum value of $a_1 + a_2 + \ldots\ldots + a_{n-1} + 2a_n$ is

(a) $n(2c)^{1/n}$ (b) $(n+1)c^{1/n}$

(c) $2nc^{1/n}$ (d) $(n+1)(2c)^{1/n}$

46. If $\alpha \in \left(0, \frac{\pi}{2}\right)$ then $\sqrt{x^2 + x} + \frac{\tan^2\alpha}{\sqrt{x^2 + x}}$ is always greater then or equal to

(a) $2\tan\alpha$ (b) 1 (c) 2 (d) $\sec^2\alpha$

47. Number of positive integers n for which $n^2 + 96$ is a perfect square is

(a) 4 (b) 8 (c) 12 (d) infinite

48. If the equation $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0$, $a \neq 0$, $b \neq c$, have a common root then their other roots are the roots of the quadratic equation

(a) $a^2x(x + 1) + 4bc = 0$ (b) $a^2x(x + 1) + 8bc = 0$

(c) $a^2x(x + 2) + 8bc = 0$ (d) $a^2x(1 + 2x) + 8bc = 0$

49. If $\alpha, \beta\ \gamma$ are the roots of the equation $x^3 + ax + b = 0$ then value of $\frac{\alpha^3 + \beta^3 + \gamma^3}{\alpha^2 + \beta^2 + \gamma^2}$ is equal to:

(a) $\frac{3b}{2a}$ (b) $\frac{-3b}{2a}$

(c) $3b$ (d) $2b$

50. If $x^2 + 5 = 2x - 4\cos(a + bx)$, where $a, b \in (0, 5)$, is satisfied for at least one real x, then the maximum value of $a + b$ is equal to

(a) 3π (b) 2π (c) π (d) $-\pi$

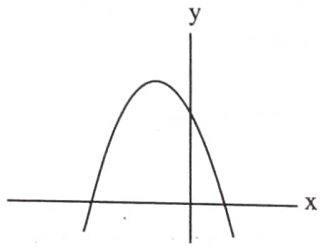
51. If α, β and γ are 3 real roots of the equation $x^3 - 6x^2 + 5x - 1 = 0$, then the value of $\alpha^4 + \beta^4 + \gamma^4$ is:

(a) 250 (b) 650 (c) 150 (d) 950

52. Let r, s, t be the roots of the equation $x^3 + ax^2 + bx + c = 0$, such that $(rs)^2 + (st)^2 + (rt)^2 = b^2 - kac$, then $k =$

(a) 1 (b) 2 (c) 3 (d) 4

53. If the graph of quadratic $y = ax^2 + bx + c$ is given below:



(a) $a < 0, b > 0\ c > 0$ (b) $a < 0, b > 0\ c < 0$

(c) $a < 0, b < 0\ c > 0$ (d) $a < 0, b < 0\ c < 0$

54. The complete set of values of 'a' for which the inequality $(a-1)x^2-(a+1)x+(a-1)\geq 0$

(a) $\left(\frac{3}{7},1\right]$ (b) $(-\infty,1)$

(c) $\left(-\infty,\frac{7}{3}\right]$ (d) $\left[\frac{7}{3},\infty\right)$

55. The number of integral values of k for which the inequality $x^2-2(4k-1)x+15k^2-2k-7\geq 0$ hold for all $x\in R$ is:

(a) 2 (b) 3

(c) 4 (d) infinity

56. The complete set of values of a for which the roots of the equation $x^2-2|a+1|x+1=0$ are real is given by:

(a) $(-\infty,-2]\cup[0,\infty)$ (b) $(-\infty,-1]\cup[0,\infty)$

(c) $(-\infty,-1]\cup[1,\infty)$ (d) $(-\infty,-2]\cup[1,\infty)$

57. For the quadratic equation $x^2-(2+\lambda)x+\lambda-2=0, \lambda\in R$

(a) Minimum possible number of positive root of the equation is 2

(b) Minimum possible number of positive root of the equation is 1

(c) Minimum possible number of negative roots of the equation is 1

(d) Minimum possible number of negative roots of the equation is 2

58. If α, β are the roots of $3x^2+2x+1=0$, then the quadratic equation whose en of the root is $\left(\alpha+\frac{1}{\beta}\right)$

(a) $3x^2+8x+16=0$ (b) $3x^2-8x+16=0$

(c) $3x^2-8x-16=0$ (d) $3x^2+8x-16=0$

59. Let a, b, c, d be distinct real numbers such that a, b are roots or $x^2-5cx-6d=0$, and c, d are roots of $x^2-5ax-6b=0$. Then $b+d$ is

(a) 180 (b) 162 (c) 144 (d) 126

60. Let $S=\{1, 2, 3, ..., 100\}$. Suppose b and c are chosen at random from the set S. The probability that $4x^2+bx+c$ has equal roots is

(a) 0.001 (b) 0.004 (c) 0.007 (d) 0.01

61. Consider the quadratic equation $nx^2+7\sqrt{(nx)}+n=0$, where n is a positive integer. Which of the following statements are necessarily correct?

I. For any n, the roots are distinct.

II. There are infinite many values of n for which both roots are real.

III. The product of the roots is necessarily an integer.

(a) III only (b) I and III only

(c) II and III only (d) I, II and III

62. Let a_1, a_2, a_3, a_4 be real numbers such that $a_1+a_2+a_3+a_4=0$ and $a_1^2+a_2^2+a_3^2+a_4^2=1$. Then the smallest possible value of the expression $(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_4)^2+(a_4-a_1)^2$ lies in the interval

(a) (0, 1.5) (b) (1.5, 2.5) (c) (2.5, 3) (d) (3, 3.5)

63. Let $r(x)$ be the remainder when the polynomial $x^{135}+x^{125}-x^{115}+x^5+1$ is divided by x^3-x. Then

(a) $r(x)$ is the zero polynomial

(b) $r(x)$ is a non-zero constant

(c) degree of $r(x)$ is one

(d) degree or $r(x)$ is two

64. The sum of all non-integer roots of the equation $x^5-6x^4+11x^3-5x^2-3x+2=0$ is

(a) 6 (b) −11 (c) −5 (d) 3

65. If $a+b+c=5$, $a^2+b^2+c^2=12$ and $a^3+b^3+c^3=25$. Then the value of $a^4+b^4+c^4$ is

(a) $\frac{251}{6}$ (b) $\frac{253}{6}$

(c) $\frac{255}{6}$ (d) $\frac{257}{6}$

66. If $x=2+2^{\frac{2}{3}}+2^{\frac{1}{3}}$, then the value of x^3-6x^2+6x is

(a) 3 (b) 2

(c) 1 (d) −2

67. The least value of the expression $x^2+4y^2+3z^2-2x-12y-6z+14$ is

(a) 1 (b) no least value

(c) 0 (d) None of these

68. If $P(x)=ax^2+bx+c$ leaves a remainder of 4 when divided by x, a remainder of 3 when divided by $(x+1)$ and a remainder of 1 when divided by $(x-1)$ then $P(2)$ is

(a) 3 (b) 6 (c) −3 (d) −6

69. If the equations $px^2+qx+r=0$ and $rx^2+qx+p=0$ $(p\neq r\neq 0)$ have a negative common root then the value of $(p-q+r)$ is equal to

(a) −1 (b) 1 (c) 2 (d) 0

70. Let α and β two real roots of the equation $(k+1)\tan^2 x-\sqrt{2}.\lambda\tan x=(1-k)$, where $k(\neq -1)$ and λ are real numbers. If $\tan^2(\alpha+\beta)=50$, then a value of λ is

(a) 5 (b) 10 (c) $10\sqrt{2}$ (d) $5\sqrt{2}$

71. Let α and β be the roots of the equation, $5x^2+6x-2=0$. If $S_n=\alpha^n+\beta^n$, $n=1, 2, 3, ...$, then

(a) $5S_6+6S_5+2S_4=0$ (b) $6S_6+5S_5=2S_4$

(c) $6S_6+5S_5+2S_4=0$ (d) $5S_6+6S_5=2S_4$

72. Let $\lambda\neq 0$ be in R. If α and β are the roots of the equation, $x^2-x+2\lambda=0$ and α and γ are the roots of the equation, $3x^2-10x+27\lambda=0$, then $\frac{\beta\gamma}{\lambda}$ is equal to

(a) 27 (b) 9 (c) 18 (d) 36

73. Let p and q be two positive numbers such that $p + q = 2$ and $p^4 + q^4 = 272$. Then p and q are roots of the equation

(a) $x^2 - 2x + 2 = 0$ (b) $x^2 - 2x + 8 = 0$

(c) $x^2 - 2x + 136 = 0$ (d) $x^2 - 2x + 16 = 0$

74. Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{3a_9}$

(a) 2 (b) 1 (c) 4 (d) 3

75. If $\alpha, \beta, \gamma, \delta \in R$ satisfy $\frac{(\alpha-1)^2 + (\beta+1)^2 + (\gamma+1)^2 + (\delta+1)^2}{\alpha+\beta+\gamma+\delta} = 4$ if biquadratic equation $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ has the roots $\left(\alpha + \frac{1}{\beta} - 1\right), \left(\beta + \frac{1}{\gamma} - 1\right), \left(\gamma + \frac{1}{\delta} - 1\right), \left(\delta + \frac{1}{\alpha} - 1\right)$.

Then the value of $\frac{a_2}{a_0}$ is

(a) 4 (b) –4

(c) 6 (d) None of these

76. The number of all 3-digit integers abc for which $(a \times b \times c) + (a \times b)6 + (c \times a) + a + b + c = 29$ is

(a) 6 (b) 10 (c) 14 (d) 18

Subjective Type Question

77. If the ratio of the roots of, $lx^2 + nx + n = 0$ is $p : q$, then prove that $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} - \sqrt{\frac{n}{l}} = 0$, where $l, n, p, q \in R^+$.

78. Solve for x: $x^{\frac{1}{3}} + (2x-3)^{\frac{1}{3}} = \{12(x-1)\}^{\frac{1}{3}}$.

79. Find two positive numbers such that their sum multiplied by the sum of their squares is 5500, and their difference multiplied by the difference of their squares is 352.

80. Solve: $\sqrt[3]{6(5x+6)} - \sqrt[3]{5(6x-11)} = 1$.

81. Find two numbers such that their sum is 9, and the sum of their fourth powers 2417.

82. If α, β, γ are the roots of the equation $x^3 - px^2 + r = 0$, find the equation whose roots are $\frac{\beta+\gamma}{\alpha}, \frac{\gamma+\alpha}{\beta}, \frac{\alpha+\beta}{\gamma}$.

83. Solve the equations: $x + y + z = x^2 + y^2 + x^2 = \frac{1}{2}(x^3 + y^3 + x^3) = 3$.

84. If a, b, c are the roots of the equation $x^2 + px + q = 0$, find the equation whose roots are $\frac{b^2+c^2}{a}, \frac{c^2+a^2}{b}, \frac{a^2+b^2}{c}$.

85. Solve

(i) $\sqrt{4x^2 - 7x - 15} - \sqrt{x^2 - 3x} = \sqrt{x^2 - 9}$.

(ii) $\sqrt{2x^2 - 9x - 4} + 3\sqrt{2x - 1} = \sqrt{2x^2 + 21x - 11}$.

(iii) $\sqrt{2x^2 + 5x - 7} + \sqrt{3(x^2 - 7x + 6)} - \sqrt{7x^2 - 6x - 1} = 0$.

(iv) $\sqrt{a^2 + 2ax - 3x^2} - \sqrt{a^2 + ax - 6x^2} = \sqrt{2a^2 + 3ax - 9x^2}$.

86. Let p and q be the two roots of the equation $mx^2 + x(2-m) + 3 = 0$. If m_1, m_2 are the two values of m satisfying $\frac{p}{q} + \frac{q}{p} = \frac{2}{3}$ the value of $\frac{m_1}{m_2^2} + \frac{m_2}{m_1^2}$ is

87. If $c > 0$ and $2ax^2 + 3bx + 5c = 0$ does not have any real roots, then prove that $2a - 3b + 5c > 0$.

88. If $ax^2 + bx + 6 = 0$ does not have distinct real roots, then find the least value of $3a + b$.

89. The minimum number of real roots of $(x^2 + 3x + a)(x^2 + ax + 1) = 0$ equals

90. The value of k for which $kx^2 - 2kx + 3x - 6$ is positive for exactly two integral values of x is

91. The number of order pairs of integers (x, y) satisfying the equation $x^2 + 6x + y^2 = 4$ is

92. Suppose that $2 - \sqrt{95}$ is a root of $x^2 + ax + b = 0$ where b is negative real number and a is a integer. The largest possible value of a, is

93. Given that α, β, γ are the roots of cubic equation $x^3 = 3x^2 + 2x + \frac{2}{3} = 0$. The value of $\alpha^4 + \beta^4 + \gamma^4$ is equal to _____

94. Let $P(x)$ be a polynomial whose coefficient are integer satisfies $P(1) = 5$ and $P(2) = 7$. The smallest possible positive value of $P(10)$ is equal to _____

95. Let α and β be two real numbers such that $\alpha + \beta = 1$ and $\alpha\beta = -1$. Let $p_n = (\alpha)^n + (\beta)^n, p_{n-1} = 11$ and $p_{n+1} = 29$ for some integer $n \geq 1$. Then, the value of is p^2_n is _____.

96. Let the quadratic equation $x^2 + 3x - k = 0$ has roots a, b and $x^2 + 3x - 10 = 0$ has roots c, d such that modulus of difference of the roots of the first equation is equal to twice the modulus of the difference of the roots of the second equation. If the value of 'k' can be expressed as rational number in the lowest form as m/n then find the value of $(m + n)$.

97. When $y^2 + my + 2$ is divided by $(y - 1)$ then the quotient is $f(y)$ and the remainder is R_1. When $y^2 + my + 2$ is divided by $(y + 1)$ then quotient is $g(y)$ and the remainder is R_2. If $R_1 = R_2$ then find the value of m.

98. Let $f(x)$ be the remainder when $p(x) = x^{73} - 2x^6 + 3x^3$ is divided by $x^4(x^2 - 1)$ then $f(1) =$

99. Find all the values of 'a', so that exactly one root of the equation $x^2 - 2ax + a^2 - 1 = 0$, lies between the numbers 2 and 4, and no root of the equation is either equal to 2 or equal to 4.

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. The equation $x-\frac{2}{x-1}=1-\frac{2}{x-1}$ has
   (a) no root
   (b) one root
   (c) two equal roots
   (d) infinitely many roots

2. If $a^2+b^2+c^2=1$, then $ab+bc+ca$ lies in the interval
   (a) $[\frac{1}{2},2]$ (b) $[-1,2]$
   (c) $[-\frac{1}{2},1]$ (d) $[-1,\frac{1}{2}]$

3. $\log_{0.3}(x-1)<\log_{0.09}(x-1)$, then x lies in the interval–
   (a) $(2,\infty)$ (b) $(1,2)$
   (c) $(-2,-1)$ (d) None of these

4. Let a, b, c be the sides of a triangle where $a\neq b\neq c$ and $\lambda\in R$. If the roots of the equation $x^2+2(a+b+c)x+3\lambda(ab+bc+ca)=0$ are real, then
   (a) $\lambda<\frac{4}{3}$ (b) $\lambda<\frac{5}{3}$
   (c) $\lambda\in\left(\frac{1}{3},\frac{5}{3}\right)$ (d) $\lambda\in\left(\frac{4}{3},\frac{5}{3}\right)$

5. Let α, β be the roots of the equation $x^2-px+r=0$ and $\frac{\alpha}{2}$, 2β be the roots of the equation $x^2-qx+r=0$. Then the value of r is
   (a) $\frac{2}{9}(p-q)(2q-p)$ (b) $\frac{2}{9}(q-p)(2p-q)$
   (c) $\frac{2}{9}(q-2p)(2q-q)$ (d) $\frac{2}{9}(2p-q)(2q-q)$

6. Let p and q be real numbers such that $p=0$, $p^3=q$ and $p^3=-q$. If α and β are nonzero complex numbers satisfying $\alpha+\beta=-p$ and $\alpha^3+\beta^3=q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is
   (a) $(p^3+q)x^2-(p^3+2q)x+(p^3+q)=0$
   (b) $(p^3+q)x^2-(p^3-2q)x+(p^3+q)=0$
   (c) $(p^3-q)x^2-(5p^3-2q)x+(p^3-q)=0$
   (d) $(p^3-q)x^2-(5p^3+2q)x+(p^3-q)=0$

7. Let (x_0, y_0) be the solution of the following equations (2x)
   $\ell^{n2}=(3y)\ell^{n3}$
   $3\ell^{nx}=2\ell^{ny}$
   Then x_0 is
   (a) $\frac{1}{6}$ (b) $\frac{1}{3}$
   (c) $\frac{1}{2}$ (d) 6

8. Let α and β be the roots of $x^2-6x-2=0$, with $\alpha>\beta$. If $=a^n-\beta^n$ for $n\geq 1$, then the value of $\frac{a_{10}-2a_8}{2a_9}$ is
   (a) 1 (b) 2
   (c) 3 (d) 4

9. A value of b for which the equations
   $x^2+bx-1=0$
   $x^2+x+b=0$
   have one root in common is
   (a) $-\sqrt{2}$ (b) $-i\sqrt{3}$
   (c) $i\sqrt{5}$ (d) $\sqrt{2}$

10. The number of real roots of equation $x^8-x^5+x^2-x+2$ 0 is
    (a) 2 (b) 4
    (c) 6 (d) 0

11. The set of values of $K(K\in R)$ for which the equation $x^2-|x|+3-|K^2-1|=0$ will have exactly four real roots is
    (a) $(-2,4)$ (b) $(-4,4)$
    (c) $(-2,2)$ (d) $(-1,0)$

12. $A=\left\{x:\frac{x+2}{x-4}\leq 0\right\}, B=\{x/x^2-ax-4\leq 0\}$.
    If B is the subset of A, then a CAN NOT take the integr value
    (a) 0 (b) 1 (c) 2 (d) 3

13. Let S be the set of all non-zero real numbers α such that tl quadratic equation $\alpha x^2-x+\alpha=0$ has two distinct real roo x_1 and x_2 satisfying the inequality $|x_1-x_2|<2$.
    Which of the following interval(s) is a subset of S?
    (a) $\left(-\frac{1}{2},-\frac{1}{3}\right)$ (b) $\left(\frac{1}{\sqrt{10}},\frac{1}{2}\right)$
    (c) $\left(\frac{1}{\sqrt{7}},\frac{1}{2}\right)$ (d) None of these

14. If r, s, t and u are roots of $x^4-5x+3=0$, then for the equatic whose roots are $\frac{r+s+t}{u^2}\cdot\frac{r+s+u}{t^2}\cdot\frac{r+t+u}{s^2}\cdot\frac{s+t+u}{r^2}$ is
    (a) Sum of roots is 0 (b) Leading coefficient is 2
    (c) Product of roots is 1/3 (d) None of these

15. Polynomials $f(x)$ and $g(x)$ are defined as follows:

$f(x) = x^2 + ax + 2$; $g(x) = x^2 + 2x + a$. $(a \neq 2)$

If the equation $f(x) = 0$ and $g(x) = 0$ have a common root then the sum of the roots of the equation $f(x) + g(x) = 0$ is.

(*a*) $-\frac{1}{2}$ (*b*) 0 (*c*) $\frac{1}{2}$ (*d*) 1

16. For the equation

$\sqrt[3]{(a+x)^2} + 4\sqrt[3]{(a-x)^2} = 5\sqrt[3]{a^2 - x^2}$ incorrect statement is

(*a*) Sum of real roots = (63/65)a

(*b*) Product of real roots = 0

(*c*) No. of real roots = 2

(*d*) No. of real roots = 3

17. If $a, b, c \in R$, $a \neq 0$ and $(b-1)^2 < 4ac$, then the number of real roots of the system of equation (in three unknowns x_1, x_2, x_3)

$ax_1^2 + bx_1 + c = x_2$, $ax_2^2 + bx_2 + c = x_3$, $ax_3^2 + bx_3 + c = x_1$ is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

18. The values of a and b so that $x^4 + 12x^3 + 46x^2 + ax + b$ is square of quadratic expression are respectively

(*a*) 60, 25 (*b*) 45, 25

(*c*) 60, 30 (*d*) 25, 45

19. If the roots of the equation $ax^2 - bx + c = 0$ are α, β, then the roots of the equation $b^2cx^2 - ab^2x + a^3 = 0$ are

(*a*) $\frac{1}{\alpha^3+\alpha\beta}, \frac{1}{\beta^3+\alpha\beta}$ (*b*) $\frac{1}{\alpha^2+\alpha\beta}, \frac{1}{\beta^2+\alpha\beta}$

(*c*) $\frac{1}{\alpha^4+\alpha\beta}, \frac{1}{\beta^4+\alpha\beta}$ (*d*) $\frac{1}{\alpha+\alpha\beta}, \frac{1}{\beta+\alpha\beta}$

20. Values of k for which the inequality $k\sin^2 x - k\sin x + 1 \geq 0$ is true $\forall x \in R$ is

(*a*) $k > -\frac{1}{2}$ (*b*) $k > 4$

(*c*) $-\frac{1}{2} \leq k \leq 4$ (*d*) $\frac{1}{2} \leq k \leq 5$

21. Consider the following statements:

S_1 : If the roots of $x^2 - bx + c = 0$ are two consecutive integers, then value of $b^2 - 4c$ is equal to 1.

S_2 : if α, β are roots of $x^2 - x + 3 = 0$ then value of $\alpha^4 + \beta^4$ is equal 7.

S_3 : If $\alpha, \beta\ \gamma$ are the roots of $x^3 - 7x^2 + 16x - 12 = 0$ then value of $\alpha^2 + \beta^2 + \gamma^2$ is equal to 17.

State, in order, whether S_1, S_2, S_3 are true or false

(*a*) TTT (*b*) FTF (*c*) TFT (*d*) FTT

22. The equation $2^{2x} + (a-1)2^{x+1} + a = 0$ has roots of opposite signs then exhaustive set of values of a is

(*a*) $a \in (-1, 0)$ (*b*) $a < 0$

(*c*) $a \in (-\infty, 1/3)$ (*d*) $a \in (0, 1/3)$

23. The complete set of values of 'b' for which the equation $2\log_{1/25}(bx+28) = -\log_5(12 - 4x - x^2)$ has only one solution.

(*a*) $(-\infty, -14) \cup \left[\frac{14}{3}, \infty\right) \cup \{4\}$

(*b*) $(-\infty, -14) \cup \{4\} \cup \left[\frac{14}{3}, \infty\right) \cup \{-12\}$

(*c*) $(-\infty, -14) \cup \left[\frac{14}{3}, \infty\right) \cup \{-12\}$

(*d*) $(-\infty, -14) \cup \left[\frac{14}{3}, \infty\right)$

24. Suppose the quadratic polynomial $P(x) = ax^2 + bx + c$ has positive coefficients a, b, c in arithmetic progression in that order. If $P(x) = 0$ has integer roots α and β, then $\alpha + \beta + \alpha\beta$ equals

(*a*) 3 (*b*) 5 (*c*) 7 (*d*) 14

25. Number of solutions of the equation

$\sqrt{3x-2} + \sqrt{4x+4} = \sqrt{5x+1} + \sqrt{4x-5}$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 4

26. If $6x^5 + 5x^3 + x^2 + 4x - 1 = a(x-1)^5 + b(x-1)^4 + c(x-1)^3 + d(x-1)^2 + e(x-1) + f$ then what is the value of $a + 2b + 3c + 4d + 5e + 5f$?

(*a*) 780 (*b*) 820

(*c*) 870 (*d*) 910

27. $f(x) = \dfrac{x^4 - 14x^2 + 25}{x - \frac{5}{x} + 2}$, zeroes are in the form $a \pm \sqrt{b}$, $a, b \in z$ then $a + b$ is equal to

(*a*) 5 (*b*) 6

(*c*) 7 (*d*) 8

28. If $\left|x^2 - \frac{\pi}{4} + \sin x - 2\right| = |x^2 - 2| + |\sin x| + \frac{\pi}{4}$ then

(*a*) $x \in (0, \sqrt{2})$ (*b*) $x \in (-\sqrt{2}, \sqrt{2})$

(*c*) $x \in R$ (*d*) $x \in (-\sqrt{2}, 0)$

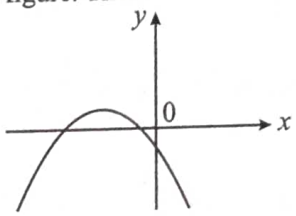
29. The number of positive integral solution of $x^4 - y^4 = 3789108$ is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 4

30. Let $\alpha = \max_{x \in R}\{8^{2\sin 3x} \cdot 4^{4\cos 3x}\}$ and $\min_{x \in R}\{8^{2\sin 3x} \cdot 4^{4\cos 3x}\}$. If $8x^2 + bx + c = 0$ is a quadratic equation whose roots are $\alpha^{1/5}$ and $\beta^{1/5}$, then the value of $c - b$ is equal to

(*a*) 42 (*b*) 43 (*c*) 47 (*d*) 50

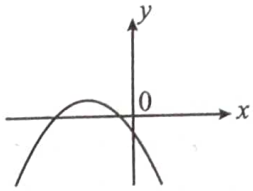
31. The graph of the quadratic polynomial $= ax^2 + bx + c$ is as shown in the figure. Then:



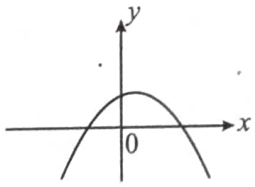
(*a*) $b^2 - 4ac > 0$ (*b*) $b < 0$
(*c*) $a > 0$ (*d*) $c < 0$

32. For which of the following graphs of the quadratic expression $y = ax^2 + bx + c$, the product a b c is negative?

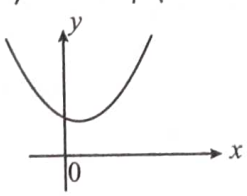
(*a*)



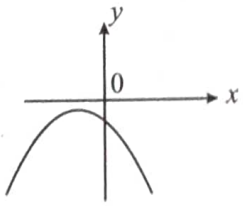
(*b*)



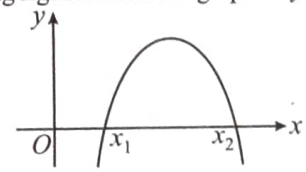
(*c*)



(*d*)



33. The adjoining figure shows the graph of $y = ax^2 + bx + c$. Then



(*a*) $a < 0$
(*b*) $b^2 < 4ac$
(*c*) $c > 0$
(*d*) a and b are of opposite sign

34. For the equation $|x|^2 + |x| - 6 = 0$, the correct statement (s) is (are):

(*a*) sum of roots is 0 (*b*) product of roots is -4
(*c*) there are 4 roots (*d*) there are only 2 roots

35. If a, b are the roots of $ax^2 + bx + c = 0$, and a + h, b + h are the roots of $px^2 + qx + r = 0$, (where h ¹ 0), then

(*a*) $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$ (*b*) $h = \frac{1}{2}\left(\frac{b}{a} - \frac{q}{p}\right)$

(*c*) $h = \frac{1}{2}\left(\frac{b}{a} + \frac{q}{p}\right)$ (*d*) $\frac{b^2 - 4ac}{a^2} = \frac{q^2 - 4pr}{p^2}$

36. If a, b are non-zero real numbers and α, β the roots of $x^2 + ax + b = 0$, then

(*a*) α^2, β^2 are the roots of $x^2 - (2b - a^2)x + a^2 = 0$

(*b*) $\frac{1}{\alpha}, \frac{1}{\beta}$ are the roots of $bx^2 + ax + 1 = 0$

(*c*) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of $bx^2 + (2b - a^2)x + b = 0$

(*d*) $(\alpha - 1), (\beta - 1)$ are the roots of the equation $x^2 + x(a + 2) + 1 + a + b = 0$

37. $x^2 + x + 1$ is a factor of $ax^3 + bx^2 + cx + d = 0$, then the rea root of above equation is
$(a, b, c, d \in R)$

(*a*) $-d/a$ (*b*) d/a (*c*) $(b - a)/a$ (*d*) $(a - b)/a$

38. If $\frac{1}{2} \le \log_{0.1} x \le 2$, then

(*a*) maximum value of x is $\frac{1}{\sqrt{10}}$

(*b*) x lies between $\frac{1}{100}$ and $\frac{1}{\sqrt{10}}$

(*c*) minimum value of x is $\frac{1}{\sqrt{10}}$

(*d*) minimum value of x is $\frac{1}{100}$

39. If the quadratic equations $x^2 + abx + c = 0$ and $x^2 + acx + = 0$ have a common root, then the equation containing thei other roots is/are:

(*a*) $x^2 + a(b + c)x - a^2bc = 0$
(*b*) $x^2 - a(b + c)x + a^2bc = 0$
(*c*) $a(b + c)x^2 - (b + c)x + abc = 0$
(*d*) $a(b + c)x^2 + (b + c)x - abc = 0$

40. If the quadratic equations $ax^2 + bx + c = 0$ (a, b, c ∈ R, $a \neq 0$) and $x^2 + 4x + 5 = 0$ have a common root then a, b, c must satisfy the relations:

(*a*) $a > b > c$
(*b*) $a < b < c$
(*c*) $a = k; b = 4k; c = 5k$ ($k \in R, k \neq 0$)
(*d*) $b^2 - 4ac$ is negative.

41. If α, β are the real and distinct roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always

(*a*) two real roots
(*b*) two negative roots
(*c*) two positive roots
(*d*) one positive root and one negative root

42. If $(x^2 + x + 1) + (x^2 + 2x + 3) + (x^2 + 3x + 5) + \ldots\ldots + (x^2 + 20x + 39) = 4500$, then x is equal to

(*a*) 10 (*b*) -10 (*c*) 20.5 (*d*) -20.5

43. If roots of equation, $x^3 + bx^2 + cx - 1 = 0$ forms an increasing G.P., then

(*a*) $b + c = 0$
(*b*) $b \in (-\infty, -3)$
(*c*) one of the roots = 1
(*d*) one root is smaller than 1 & other > 1

44. Let $f(x) = \frac{3}{x-2} + \frac{4}{x-3} + \frac{5}{x-4}$, then $f(x) = 0$ has

(*a*) exactly one real root in (2, 3)
(*b*) exactly one real root in (3, 4)
(*c*) at least one real root in (2, 3)
(*d*) None of these

Paragraph for question no. 45 & 46: A quadratic polynomial $f(x) = px^2 + qx + r$ has two distinct roots x_1 & x_2. If its vertex (of parabola) is V and x_1, x_3, x_2 are in A.P., then answer the following

45. If $p, q\ r$ are any real number, then

(a) $\max(p, q) < \max(p, q, r)$

(b) $\min(p, q) = \frac{1}{2}(p + q - |p - q|)$

(c) $\max(p, q) < \min(p, q, r)$

(d) none of these

46. If $x, y \in R$ and $2x^2 + 6xy + 5y^2 = 1$, then

(a) $|x| \le \sqrt{5}$ (b) $|x| \ge \sqrt{5}$

(c) $y^2 \le 2$ (d) $y^2 \le 4$

47. If the equation whose roots are the squares of the roots the cubic $x^3 - ax^2 + bx - 1 = 0$ is identical with the give cubic equation, then

(a) $a = 0, b = 3$

(b) $a = b = 0$

(c) $a = b = 3$

(d) a, b are roots of $x^2 + x + 2 = 0$

48. If the equation $ax^2 + bx + c = 0, a, b, c \in R$ have non-re roots, then

(a) $c(a - b + c) > 0$ (b) $c(a + b + c) > 0$

(c) $c(4a - 2b + c) > 0$ (d) none of these

49. If $c \ne 0$ and the equation $p/(2x) = a/(x + c) + b/(x - c)h$ two equal roots, then p can be

(a) $(\sqrt{a} - \sqrt{b})^2$ (b) $(\sqrt{a} + \sqrt{b})^2$

(c) $a + b$ (d) $a - b$

50. If the equations $4x^2 - x - 1 = 0$ and $3x^2 + (\lambda + \mu)x + \lambda = 0$ have a root common, then the rational values of and μ are

(a) $\lambda = \frac{-3}{4}$ (b) $\lambda = 0$

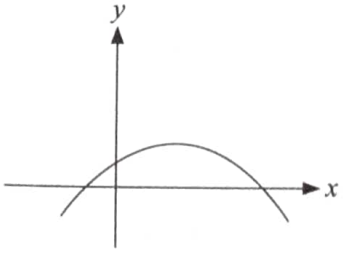
(c) $\mu = \frac{3}{4}$ (d) $\mu = 0$

51. If the equation $ax^2 + bx + c = 0\ (a > 0)$ has two real roots α and β such that $\alpha < -2$ and $b > 2$, then which of the following statements is/are true?

(a) $a - |b| + c < 0$ (b) $c < 0, b^2 - 4ac > 0$

(c) $4a - 2|b| + c < 0$ (d) $9a - 3|b| + c < 0$

52. If shown the graph of $f(x) = ax^2 + bx + c$, then



(a) $ac < 0$ (b) $bc > 0$

(c) $ab - 0$ (d) $abc < 0$

53. If $\cos x - y^2 - \sqrt{y - x^2 - 1} \ge 0$, then

(a) $y \ge 1$ (b) $x \in R$

(c) $y = 1$ (d) $x = 0$

54. The value of x satisfying the equation $2^{2x} - 8 \times 2^x = 0$ is

(a) $1 + \frac{\log 3}{\log 2}$ (b) $\frac{1}{2}\log 6$

(c) $1 + \log\frac{3}{2}$ (d) 1

55. If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then which of the following expression will be the symmetric function of roots

(a) $\left|\log\frac{\alpha}{\beta}\right|$ (b) $\alpha^2\beta^5 + \beta^2\alpha^5$

(c) $\tan(\alpha - \beta)$ (d) $\left(\log\frac{1}{\alpha}\right)^2 + (\log\beta)^2$

56. If the quadratic equation $ax^2 + bx + c = 0\ (a > 0)$ has $\sec^2$ and $\operatorname{cosec}^2\theta$ as its roots, then which of the following must hold good?

(a) $b + c = 0$ (b) $b^2 - 4ac \ge 0$

(c) $c \ge 4a$ (d) $4a + b \ge 0$

57. If $f(x) = (\sin^{-1}x)^2 + (\cos^{-1}x)^2$, then

(a) $f(x)$ has the least value of $\frac{\pi^2}{8}$

(b) $f(x)$ has the greatest value of $\frac{5\pi^2}{8}$

(c) $f(x)$ has the least value of $\frac{\pi^2}{16}$

(d) $f(x)$ has the greatest value of $\frac{5\pi^2}{4}$

58. If $\tan^{-1}(\sin^2\theta + 2\sin\theta + 2) + \cot^{-1}(4^{\sec^2 q} + 1) = \frac{\pi}{2}$ has solution for some θ and ϕ then

(a) $\sin\theta = -1$ (b) $\sin\theta = 1$

(c) $\cos\theta = 1$ (d) $\cos\theta = -1$

59. If α, β are the roots of the equation $6x^2 + 11x + 3 = 0$, then –

(a) Both $\cos^{-1}\alpha$ and $\cos^{-1}\beta$ are real

(b) Both $\operatorname{cosec}^{-1}\alpha$ and $\operatorname{cosec}^{-1}\beta$ are real

(c) Both $\cot^{-1}\alpha$ and $\tan^{-1}\beta$ are real

(d) Both $\tan^{-1}\alpha$ and $\tan^{-1}\beta$ are real

60. If $x \in R$, then the roots of the equation $x^4 + 4x^3 - 8x^2 + k = 0$ with respect to the values of k, ($k \in [0, 2014]$), are

(a) for $k \in [0, 3]$: 4 real and distinct roots

(b) for $k \in (3, 128)$: two real and two imaginary roots

(c) for $k \in (128, 583)$: no real root

(d) for $k \in (0, 3)$: four real and distinct roots

61. For the quadratic polynomial $f(x) = 4x^2 - 8kx + k$, the statements which hold good are

(*a*) there is only one integral k for which $f(x)$ is non-negative $\forall\, x \in R$

(*b*) for $k < 0$ the number zero lies between the zeros of the polynomial

(*c*) $f(x) = 0$ has two distinct solutions in (0, 1) for $k \in (1/4, 4/7)$

(*d*) Minimum value of $f(x)\ \forall\, k \in R$ is $k(1 + 12k)$

62. Let α and β be theroots of the equation $x^2 - x - 1 = 0$. If $p_k = (\alpha)^k + (\beta)^k$, $k \geq 1$,then which one of the following statements is not true?

(*a*) $p_5 = p_2 \cdot p_3$

(*b*) $p_3 = p_5 - p_4$

(*c*) $(p_1 + p_2 + p_3 + p_4 + p_5) = 26$

(*d*) $p_5 = 11$

63. If 'a' and 'b' are two numbers such that $a^2 + b^2 = 7$ and $a^3 + b^3 = 10$, then

(*a*) greatest value of $|a + b|$ is 5

(*b*) greatest value of $(a + b)$ is 4

(*c*) least value of $(a + b)$ is 1

(*d*) least value of $|a + b|$ is 1

COMPREHENSION BASED QUESTIONS

Comprehension (Q. No. 64 to 66)

Consider the equation $x^4 - \lambda x^2 + 9 = 0$. This can be solved by substituting $x^2 = t$ such equations are called as pseudo quadratic equations.

64. If the equation has four real and distinct roots, then λ lies in the interval

(*a*) $(-\infty, -6) \cup (6, \infty)$ (*b*) $(0, \infty)$

(*c*) $(6, \infty)$ (*d*) $(-\infty, -6)$

65. If the equation has no real root, then λ lies in the interval

(*a*) $(-\infty, 0)$ (*b*) $(-\infty, 6)$

(*c*) $(6, \infty)$ (*d*) $(0, \infty)$

66. If the equation has only two real roots, then set of values of λ is

(*a*) $(-\infty, -6)$ (*b*) $(-6, 6)$ (*c*) $\{6\}$ (*d*) ϕ

Comprehension (Q. No. 67 to 69)

Consider an unknown polynomial which when divided by $(x - 3)$ and $(x - 4)$ leaves remainders 2 and 1, respectively. Let $R(x)$ be the remainder when this polynomial is divided by $(x - 3)(x - 4)$

67. If equation $R(x) = x^2 + ax + 1$ has two distinct real roots, then exhaustive values of a are

(*a*) $(-2, 2)$ (*b*) $(-\infty, -2) \cup (2, \infty)$

(*c*) $(-2, \infty)$ (*d*) all real numbers

68. If $R(x) = px^2 + (q - 1)x + 6$ has no distinct real roots and $p > 0$, then the least value of $3p + q$ is

(*a*) -2 (*b*) $2/3$

(*c*) $-1/3$ (*d*) None of these

69. Range of $f(x) = [R(x)] / (x^2 - 3x + 2)$ is

(*a*) $[-2, 2]$

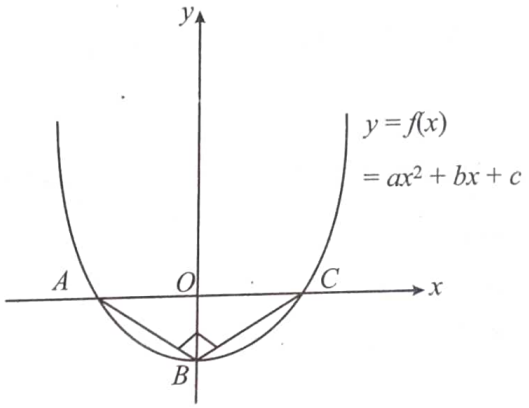
(*b*) $(-\infty, -2 - \sqrt{3}] \cup [-2 + \sqrt{3}, \infty)$

(*c*) $(-\infty, -7 - 4\sqrt{3}] \cup [-7 + 4\sqrt{3}, \infty)$

(*d*) none of these

Comprehension (Q. No. 70 to 72)

In vertices of ΔABC lie on $y = f(x) = ax^2 + bx + c$. The ΔABC is right-angled isosceles triangle whose hypotenuse $AC = 4\sqrt{2}$ units.



70. $y = f(x)$ is given by

(*a*) $y = x^2 - 2\sqrt{2}$ (*b*) $y = x^2 - 12$

(*c*) $y = \dfrac{x^2}{2} - 2$ (*d*) $y = \dfrac{x^2}{2\sqrt{2}} - 2\sqrt{2}$

71. Minimum value of $y = f(x)$ is

(*a*) -4 (*b*) -2

(*c*) $-2\sqrt{2}$ (*d*) none of these

72. Number of integral values of k for which one roots of $f(x) = 0$ is more than k and other less than k

(*a*) 6 (*b*) 5 (*c*) 5 (*d*) 7

Comprehension (Q. No. 73 to 75) Consider the inequality $9^x - a3^x - a + 3 \leq 0$, where a is real parameter.

73. The given inequality has atleast one negative solution for $a \in$

(*a*) $(-\infty, 2)$ (*b*) $(3, \infty)$ (*c*) $(-2, \infty)$ (*d*) $(2, 3)$

74. The given inequality has atleast one positive solution for $a \in$

(*a*) $(-\infty, 2)$ (*b*) $[3, \infty)$ (*c*) $[2, \infty)$ (*d*) $[-2, \infty)$

75. The given inequality has atleast one real solution for $a \in$

(*a*) $(-\infty, 3)$ (*b*) $[2, \infty)$ (*c*) $(3, \infty)$ (*d*) $[-2, \infty)$

Comprehension (Q. No. 76 to 78) Consider the equation $x^4 - \lambda x^2 + 9 = 0$.

76. If the equation has 4 real and distinct roots, then λ lies in the interval.

(*a*) $(-\infty, -6) \cup (6, \infty)$ (*b*) $(0, \infty)$

(*c*) $(6, \infty)$ (*d*) $(-\infty, -6)$

77. If the equation has no real root, then λ lies in the interval

(*a*) $(-\infty, 0)$ (*b*) $(-\infty, 6)$ (*c*) $(6, \infty)$ (*d*) $(0, \infty)$

78. If the equation has only 2 real roots, then the set of values of λ is

(*a*) $(-\infty, -6)$ (*b*) $(-6, 6)$ (*c*) $\{6\}$ (*d*) ϕ

Comprehension (Q. No. 79 to 81) Consider the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$, where $a \in R$. Also, range of $f(x) = x + \frac{1}{x}$ is $(-\infty, -2] \cup [2, \infty)$

79. If equation has at least two distinct positive real roots, then all possible value of 'a' are

(*a*) $\left(-\infty, -\frac{1}{4}\right)$ (*b*) $\left(\frac{5}{4}, \infty\right)$

(*c*) $\left(-\infty, -\frac{3}{4}\right)$ (*d*) $\left(\frac{-5}{4}, \infty\right)$

80. If equation has at least two distinct negative real roots, then all possible values of 'a' are

(*a*) $\left(\frac{3}{4}, \infty\right)$ (*b*) $\left(-\frac{5}{4}, \infty\right)$

(*c*) $\left(-\infty, \frac{1}{4}\right)$ (*d*) All of above

81. If exactly two roots are positive and two roots are negative, then number of integral values of 'a' is

(*a*) 2 (*b*) 1 (*c*) 0 (*d*) 3

MATCH THE COLUMN TYPE QUESTIONS

82. Match the column.

	Column-I		Column-II
A.	If α, $\alpha + 4$ are two roots of $x^2 - 8x + k = 0$, then possible value of k is	p.	2
B.	Number of real roots of equation $x^2 - 5\|x\| + 6 = 0$ are 'n', then value of n/2 is	q.	3
C.	If $3 - i$ is a root of $x^2 + ax + b = 0$ ($a, b \in R$), then b is	r.	12
D.	If both roots of $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then 'k' may be equal to	s.	10

(*a*) A → r, B → p, C → s, D → p,q

(*b*) A → r, B → p, C → s, D → p,r

(*c*) A → r, B → q, C → s, D → p

(*d*) A → p, B → r, C → s, D → q

83. Match the following for the equation $x^2 + a|x| + 1 = 0$, where a is a parameter.

	Column-I		Column-II
A.	No real roots	p.	$a < -2$
B.	Two real roots	q.	ϕ
C.	Three real roots	r.	12
D.	Four distinct real roots	s.	10

(*a*) A → p, B → r, C → q, D → s

(*b*) A → s, B → r, C → q, D → p

(*c*) A → s, B → q, C → r, D → p

(*d*) A → p, B → q, C → r, D → s

84. Match the column.

	Column-I		Column-II
A.	If a, b, c, and d are foure zero real number such that $(d + a - b)^2 + (d + b - c)^2 = 0$ and the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are real and equal. Then	p.	$a + b + c = 0$
B.	If the roots of the equation $(a^2 + b^2)x^2 - 2b(a + c)x + (b^2 + c^2) = 0$ are real and equal, then	q.	a,b,c are in $A.P$
C.	If the equation $ax^2 + bx + c = 0$ and $x^3 - 3x^2 + 3x - 1 = 0$ have a common real root, then	r.	a,b,c are in $G.P$
D.	Let a, b, c be positive real number such that the expression $bx^2 + (\sqrt{(a+c)^2 + 4b^2})\,x + (a + c)$ is non-negative $\forall\, x \in R$, then	s.	a,b,c are in $H.P$

(*a*) A → s, B → r, C → p, D → q

(*b*) A → q,s, B → p, C → r, D → p

(*c*) A → q,s, B → r, C → p, D → p

(*d*) A → p, B → r, C → p, D → q,s

85. Match the column.

	Column-I		Column-II
A.	one roots is positive and the other is negative for the equation $(m-2)x^2 - (8-2m)x - (8-3m) = 0$, then number of Integral values of m	p.	0
B.	exactly one root of equation $x^2 - m(2x - 8) - 15 = 0$ lies in interval (0, 1)	q.	infinity solution
C.	the equation $x^2 + 2(m + 1)x + 9m - 5 = 0$ has both roots negative	r.	1
D.	the equation $x^2 + 2(m - 1)x + m + 5 = 0$ has both roots lying on either sides of 1	s.	2

(*a*) A → q, B → r, C → q, D → p

(*b*) A → p, B → r, C → q, D → q

(*c*) A → p, B → r, C → q, D → s

(*d*) A → p, B → q, C → r, D → q

NUMERICAL BASED QUESTIONS

86. Find number of integer roots of equation $x(x+1)(x+2)(x+3) = 120$.

87. Find product of all real values of x satisfying $(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10$

88. The least prime integral value of '2a' such that the roots α, β of the equation $2x^2 + 6x + a = 0$ satisfy the inequality $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$ is

89. If a, b are the roots of $x^2 + px + 1 = 0$ and c, d are the roots of $x^2 + qx + 1 = 0$. Then find the value of $(a-c)(b-c)(a+d)(b+d)/(q^2-p^2)$.

90. α, β are roots of the equation $\lambda(x^2 - x) + x + 5 = 0$. If λ_1 and λ_2 are the two values of λ for which the roots α, β are connected by the relation $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = 4$, then the value of $\left(\frac{\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}}{14}\right)$ is

91. Let α, β be the roots of the equation $x^2 + ax + b = 0$ and γ, δ be the roots of $x^2 - ax + b - 2 = 0$. If $\alpha\beta\gamma\delta = 24$ and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{5}{6}$, then find the value of a.

92. The least value of expression $x^2 + 2xy + 2y^2 + 4y + 7$ is:

93. If $a > b > 0$ and $a^3 + b^3 + 27ab = 729$ then the quadratic equation $ax^2 + bx - 9 = 0$ has roots α, β $(\alpha < \beta)$. Find the value of $4\beta - a\alpha$.

94. Let α and β be roots of $x^2 - 6(t^2 - 2t + 2)x - 2 = 0$ with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then find the minimum value of $\frac{a_{100} - 2a_{98}}{a_{99}}$ (where $t \in R$)

95. If roots of the equation $x^2 - 10ax - 11b = 0$ are c and d and those of $x^2 - 10cx - 11d = 0$ are a and b, then find the value of $\frac{a+b+c+d}{110}$. (where a, b, c, d are all distinct numbers)

NUMERICAL BASED QUESTIONS

96. If $\sqrt{\sqrt{\sqrt{x}}} = \sqrt[4]{\sqrt[4]{\sqrt[4]{3x^4+4}}}$, then the value of x^4 is _____.

97. All the value of k for which the quadratic polynomial $f(x) = -2x^3 + kx + k^2 + 5$ has two distinct zeroes and or one of them satisfying $0 < x < 2$, lie in the interval (a, b). The value of $(a + 10b)$ is _____.

98. The quadratic equation $x^2 + mx + n = 0$ has roots which are twice those of $x^3 + px + m = 0$ and m, n and $p = q$. Then the value of n/p is _____.

99. a, b, c are real such that $a+b+c=3$ and $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{10}{3}$. The value of $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$ is _____.

100. Let a, b and be real numbers which satisfy the equation $a + \frac{1}{bc} = \frac{1}{5}$, $b + \frac{1}{ac} = \frac{-1}{15}$, and $c + \frac{1}{ab} =$. The value of $\frac{c-b}{c-a}$ is equal ot _____.

101. If a, b, c are non-zero real number, then the minimum value of the expression

$$\left(\frac{(a^4+3a^2+1)(b^4+5b^2+1)(c^4+7c^2+1)}{a^2b^2c^2}\right)$$

102. If $a, b \in R$ such that $a + b = 1$ and $(1 - 2ab)(a^3 + b^3) = 12$. The value of $(a^2 + b^2)$ is equal

103. If the cubic $2x^3 - 9x^2 + 12x + k = 0$ has two equal roots then maximum value of $|k|$ is _____.

104. Let a, b and c be distinct non-zero real numbers such that $\frac{1-a^3}{a} = \frac{1-b^3}{b} = \frac{1-c^3}{c}$. The value of $(a^3 + b^3 + c^3)$ is _____.

105. If the fraction $\frac{x^3 + (a-10)x^2 - x + a - 6}{x^3 + (a-6)x^2 - x + a - 10}$ reduces to quotient of two linear functions, then a equals _____.

106. If $a + b + c = 0$, $a^2 + b^2 + c^2 = 4$, then $a^4 + b^4 + c^4$.

107. Let $f(x)$ be the remainder when $p(x) = x^{73} - 2x^6 + 3x^3$ is divided by $x^4(x^2 - 1)$ then find the value of $f(1)$

108. If one root of the equation $ax^2 + bc + c = 0$ is equal to n^{th} power of the other root, then show that $(ac^n)^{1/n(n+1)} + (a^n c)^{1/(n+1)} + b = 0$

109. If a, b, c, be the roots of the equation $x^3 + mx + n = 0$, then

$$\frac{6(a^5+b^5+c^5)}{(a^3+b^3+c^3)(a^2+b^2+c^2)}$$

110. Let p & q are real numbers such that $(\tan p - 1)^3 + 2019(\tan p - 1) = 1$ & $(1 - \cot q)^3 + 2019(1 - \cot q) = -1$ $\tan p \neq \cot q$, then number of possible values of r which satisfy the equation $\tan p + \cot q + \sin r + \cos r = 3$ in $x \in [-2\pi, 2\pi]$ is

111. If the quadratic equation $x^2 + (2 - \tan\theta)x - (1 + \tan\theta) = 0$ has two integral roots, then sum of all possible values of θ in the interval $(0, 2\pi)$ is $k\pi$, then the value of k is equal to__

112. Solve for x, y, z

(i) $x^3 + y^3 + z^3 = a^3$, $x^2 + y^2 + z^2 = a^2$, $x + y + z = a$.

(ii) $x^2y^2z^2u = 12$, $x^2y^2zu^2 = 8$, $x^2yz^2u^2 = 1$, $3xy^2z^2u^2 = 4$.

(iii) $(y-z)(z+x) = 22$, $(z+x)(x-y) = 33$, $(x-y)(y-z) = 6$.

(iv) $x^2 + y^2 + z^2 = yz + zx + xy = a^2$, $3x - y + z = a\sqrt{3}$.

(v) $x^3y^2z = 12$, $x^3yz^3 = 54$, $x^7y^3z^2 = 72$.

(vi) $xy + x + y = 23$, $xz + x + z = 41$, $yz + y + z = 27$.

Exercise-4 (Past Year Questions)

JEE MAIN

1. The sum of all real values of x satisfying the equation **(2016)**

$(x^2-5x+5)^{x^2+4x-60} = 1$ is:

(a) 3 (b) –4
(c) 6 (d) 5

2. If, for a positive integer n, the quadratic equation, $x(x + 1) + (x + 1)(x + 2) + \ldots\ldots + (x + n - 1)(x + n) = 10n$ has two consecutive integral solutions, then n is equal to: **(2017)**

(a) 11 (b) 12 (c) 9 (d) 10

3. If $\alpha, \beta \in C$ are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to **(2018)**

(a) 0 (b) 1
(c) 2 (d) –1

4. Let α and β be two roots of the equation $x^2 + 2x + 2 = 0$, then $\alpha^{15} + \beta^{15}$ is equal to: **(2019)**

(a) –256 (b) 512
(c) –512 (d) 256

5. If both the roots of the quadratic equation $x^2 - mx + 4 = 0$ are real and distinct and they lie in the interval [1, 5], then m lies in the interval **(2019)**

(a) (4, 5) (b) (3, 4)
(c) (5, 6) (d) (–5, –4)

6. The number of all possible positive integral values of α for which the roots of the quadratic equation, $6x^2 - 11x + \alpha = 0$ are rational numbers is: **(2019)**

(a) 2 (b) 5
(c) 3 (d) 4

7. Consider the quadratic equation $(c-5)x^2 - 2cx + (c-4) = 0$, $c \neq 5$. Let S be the set of all integral values of c for which one root of the equation lies in the interval (0, 2) and its other root lies in the interval (2, 3). Then the number of elements in S is: **(2019)**

(a) 18 (b) 12
(c) 10 (d) 11

8. If one real root of the quadratic equation $81x^2 + kx + 256 = 0$ is cube of the other root, then a value of k is: **(2019)**

(a) –81 (b) 100
(c) 144 (d) –300

9. The number of integral values of m for which the quadratic expression, $(1 + 2m)x^2 - 2(1 + 3m)x + 4(1 + m)$, $x \in R$, is always positive, is: **(2019)**

(a) 3 (b) 8
(c) 7 (d) 6

10. If λ be the ratio of the roots of the quadratic equation in x, $3m^2x^2 + m(m - 4)x + 2 = 0$, then the least value of m for which $\lambda + \frac{1}{\lambda} = 1$, is **(2019)**

(a) $2-\sqrt{3}$ (b) $4-3\sqrt{2}$
(c) $-2+\sqrt{2}$ (d) $4-2\sqrt{3}$

11. If α and β be the roots of the equation $x^2 - 2x + 2 = 0$, then the least value of n for which $\left(\frac{\alpha}{\beta}\right)^n = 1$ is **(2019)**

(a) 2 (b) 3
(c) 4 (d) 5

12. If three distinct numbers a,b,c are in G.P. and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then which one of the following statements is correct? **(2019)**

(a) d, e, f are in A.P. (b) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in G.P.
(c) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P. (d) d, e, f are in G.P.

13. The number of integral values of m for which the equation $(1 + m^2)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has no real root is: **(2019)**

(a) infinitely many (b) 2
(c) 3 (d) 1

14. Let $p, q \in R$. If $2-\sqrt{3}$ is a root of the quadratic equation, $x^2 + px + q = 0$, then: **(2019)**

(a) $q^2 + 4p + 14 = 0$ (b) $p^2 - 4q - 12 = 0$
(c) $q^2 - 4p - 16 = 0$ (d) $p^2 - 4q + 12 = 0$

15. If m is chosen in the quadratic equation $(m^2 + 1)x^2 - 3x + (m^2 + 1)^2 = 0$ such that the sum of its roots is greatest, then the absolute difference of the cubes of its roots is: **(2019)**

(a) $8\sqrt{3}$ (b) $4\sqrt{3}$
(c) $10\sqrt{5}$ (d) $8\sqrt{5}$

16. If α and β are the roots of the quadratic equation, $x^2 + x\sin\theta - 2\sin\theta = 0$, $\theta \in \left(0, \frac{\pi}{2}\right)$, then $\frac{\alpha^{12} + \beta^{12}}{(\alpha^{-12} + \beta^{-12})(\alpha - \beta)^{24}}$ is equal to: **(2019)**

(*a*) $\frac{2^6}{(\sin\theta + 8)^{12}}$ (*b*) $\frac{2^{12}}{(\sin\theta - 8)^6}$

(*c*) $\frac{2^{12}}{(\sin\theta - 4)^{12}}$ (*d*) $\frac{2^{12}}{(\sin\theta + 8)^{12}}$

17. The number of real roots of the equation $5 + |2^x - 1| = 2^x(2^x - 2)$ is: **(2019)**

(*a*) 2 (*b*) 3

(*c*) 4 (*d*) 1

18. If α and β are the roots of the equation $375x^2 - 25x - 2 = 0$, then $\lim_{n\to\infty}\sum_{r=1}^{n}\alpha^r + \lim_{n\to\infty}\sum_{r=1}^{n}\beta^r$ is equal to: **(2019)**

(*a*) $\frac{21}{346}$ (*b*) $\frac{29}{358}$

(*c*) $\frac{1}{12}$ (*d*) $\frac{7}{116}$

19. If α, β and γ are three consecutive terms of a non-constant G.P. such that the equations $\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common root, then $\alpha(\beta + \gamma)$ is equal to: **(2019)**

(*a*) $\beta\gamma$ (*b*) 0

(*c*) $\alpha\gamma$ (*d*) $\alpha\beta$

20. Let α and β be the roots of the equation $x^2 - x - 1 = 0$. If $p_k = (\alpha)^k + (\beta)^k$, $k \geq 1$, then which of the following statements is not true? **(2020)**

(*a*) $p_5 = p_2 \cdot p_3$

(*b*) $(p_1 + p_2 + p_3 + p_4 + p_5) = 26$

(*c*) $p_3 = p_5 - p_4$

(*d*) $p_5 = 11$

21. The least positive value of 'a' for which the equation, $2x^2 + (a - 10)x + \frac{33}{2} = 2a$ has real roots is _____. **(2020)**

22. Let S be the set of all real roots of the equation, $3^x(3^x - 1) + 2 = |3^x - 1| + |3^x - 2|$. Then S: **(2020)**

(*a*) contains exactly two elements

(*b*) is a singleton

(*c*) contains at least four elements

(*d*) is an empty set

23. The number of real roots of the equation, $e^{4x} + e^{3x} - 4e^{2x} + e^x + 1 = 0$ **(2020)**

(*a*) 3 (*b*) 1

(*c*) 4 (*d*) 2

24. Let $a, b \in R$, $a \neq 0$ be such that the equation, $ax^2 - 2bx + 5 = 0$ has a repeated root α, which is also a root of the equation, $x^2 - 2bx - 10 = 0$. If β is the other root of this equation, $\alpha^2 + \beta^2$ is equal to: **(2020)**

(*a*) 25 (*b*) 26

(*c*) 24 (*d*) 28

25. Let $f(x)$ be a quadratic polynomial such that $f(-1) + f(b) = 0$. If one of the roots of $f(x) = 0$ is 3, then its other root lies in: **(2020)**

(*a*) $(-1, 0)$ (*b*) $(-3, -1)$

(*c*) $(0, 1)$ (*d*) $(1, 3)$

26. The set of all real values of λ for which the quadratic equations, $(\lambda^2 + 1)x^2 - 4\lambda x + 2 = 0$ always have exactly one root in the interval (0,1) is: **(2020)**

(*a*) $(-3, -1)$ (*b*) $(2, 4]$

(*c*) $(0, 2)$ (*d*) $(1, 3]$

27. If α and β are the roots of the equation $x^2 + px + 2 = 0$ and $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of the equation $2x^2 + 2qx + 1 = 0$, then $\left(\alpha - \frac{1}{\alpha}\right)\left(\beta - \frac{1}{\beta}\right)\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$ is equal to: **(2020)**

(*a*) $\frac{9}{4}(9 - q^2)$ (*b*) $\frac{9}{4}(9 + p^2)$

(*c*) $\frac{9}{4}(9 + q^2)$ (*d*) $\frac{9}{4}(9 - p^2)$

28. Let α and β be the roots of $x^2 - 3x + p = 0$ and γ and δ be the roots of $x - 6x + q = 0$. If $\alpha, \beta, \gamma, \delta$ form a geometric progression. Then ratio $(2q + p) : (2q - p)$ is: **(2020)**

(*a*) 3 : 1 (*b*) 5 : 3

(*c*) 9 : 7 (*d*) 33 : 31

29. If α and β are the roots of the equation, $7x^2 - 3x - 2 = 0$, then the value of $\frac{\alpha}{1 - \alpha^2} + \frac{\beta}{1 - \beta^2}$ is equal to: **(2020)**

(*a*) $\frac{1}{24}$ (*b*) $\frac{27}{32}$

(*c*) $\frac{3}{8}$ (*d*) $\frac{27}{16}$

30. The product of the roots of the equation $9x^2 - 18|x| + 5 = 0$, is: **(2020)**

(*a*) $\frac{25}{9}$ (*b*) $\frac{25}{81}$

(*c*) $\frac{5}{9}$ (*d*) $\frac{5}{27}$

31. If α and β be two roots of the equation $x^2 - 64x + 256 = 0$. Then the value of $\left(\frac{\alpha^3}{\beta^5}\right)^{1/8} + \left(\frac{\beta^3}{\alpha^5}\right)^{1/8}$ is: **(2020)**

(*a*) 3 (*b*) 2
(*c*) 4 (*d*) 1

32. Let a, b, c be in arithmetic progression. Let the centroid of the triangle with vertices (a, c), $(2, b)$ and (a, b) be $\left(\frac{10}{3}, \frac{7}{3}\right)$. If α, β are the roots of the equation $ax^2 + bx + 1 = 0$, then the value of $\alpha^2 + \beta^2 - \alpha\beta$ is: **(2021)**

(*a*) $\frac{71}{256}$ (*b*) $-\frac{69}{256}$
(*c*) $\frac{69}{256}$ (*d*) $-\frac{71}{256}$

33. The number of the real roots of the equation $(x + 1)^2 + |x - 5| = \frac{27}{4}$ is **(2021)**

34. The coefficients a, b and c of the quadratic equation, $ax^2 + bx + c = 0$ are obtained by throwing a dice three times. The probability that this equation has equal roots is **(2021)**

(*a*) $\frac{1}{54}$ (*b*) $\frac{1}{72}$
(*c*) $\frac{1}{36}$ (*d*) $\frac{5}{216}$

35. The integer 'k'. for which the inequality $x^2 - 2(3k - 1)x + 8k^2 - 7 > 0$ is valid for every x in R is: **(2021)**

(*a*) 3 (*b*) 2
(*c*) 4 (*d*) 0

36. If $\alpha, \beta \in R$ are such that $1 - 2i$ (here $i^2 = -1$) is a root of $z^2 + az + \beta = 0$, then $(\alpha - \beta)$ is equal to: **(2021)**

(*a*) 7 (*b*) -3
(*c*) 3 (*d*) -7

37. Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = a^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{3a_9}$ is: **(2021)**

(*a*) 4 (*b*) 1 (*c*) 2 (*d*) 3

38. The number of solutions of the equation $\log_4(x - 1) = \log_2(x - 3)$ is **(2021)**

39. The number of elements in the set $\{x \in R: (|x| - 3)|x + 4| = 6\}$ is equal to **(2021)**

(*a*) 3 (*b*) 2 (*c*) 4 (*d*) 1

40. Let $P(x) = x^2 + bx + c$ be a quadratic polynomial with real coefficients such that $\int_0^1 P(x)dx = 1$ and $P(x)$ leaves remainder 5 when it is divided by $(x - 2)$. Then the value of $9(b + c)$ is equal to: **(2021)**

(*a*) 9 (*b*) 15
(*c*) 7 (*d*) 11

41. The value of $4 + \cfrac{1}{5 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{4 + \ldots \infty}}}}$ **(2021)**

(*a*) $2 + \frac{2}{5}\sqrt{30}$ (*b*) $2 + \frac{4}{\sqrt{5}}\sqrt{30}$
(*c*) $4 + \frac{4}{\sqrt{5}}\sqrt{30}$ (*d*) $5 + \frac{2}{5}\sqrt{30}$

42. If $f(x)$ and $g(x)$ are two polynomials such that the polynomial $P(x) = f(x^3) + xg(x^3)$ is divisible by $x^2 + x + 1$, then $P(1)$ is equal to— **(2021)**

43. If the sum of the squares of the reciprocals of the roots α and β of the equation $3x^2 + \lambda x - 1 = 0$ is 15, then $6(\alpha^3 + \beta^3)^2$ is equal to **(2022)**

(*a*) 18 (*b*) 24
(*c*) 36 (*d*) 96

44. Let $a, b \in R$ be such that the equation $ax^2 - 2bx + 15 = 0$ has a repeated root α. If α and β are the roots of the equation $x^2 - 2bx + 21 = 0$, then $\alpha^2 + \beta^2$ is equal to: **(2022)**

(*a*) 37 (*b*) 58
(*c*) 68 (*d*) 92

45. If the sum of all the roots of the equation $e^{2x} - 11e^x - 45e^{-x} + \frac{81}{2} = 0$ is $\log_e P$, then p is equal to _____. **(2022)**

46. Let α, β be the roots of the equation $x^2 - 4\lambda x + 5 = 0$ and α, γ be the roots of the equation $x^2 - (3\sqrt{2} + 2\sqrt{3})x + 7 + 3\lambda\sqrt{3} = 0$. If $\beta + \gamma = 3\sqrt{2}$, then $(\alpha + 2\beta + \gamma)^2$ is equal to **(2022)**

JEE ADVANCED

47. The largest interval for which $x^{12} - x^9 + x^4 - x + 1 > 0$ is **(1982)**

(*a*) $-4 < x \leq 0$ (*b*) $0 < x < 1$
(*c*) $-100 < x < 100$ (*d*) $-\infty < x < \infty$

48. The equation $x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$ has **(1989)**

(*a*) atleast one real solution
(*b*) exactly three real solutions
(*c*) exactly one irrational solution
(*d*) complex roots

49. The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then the equation $p(p(x)) = 0$ has **(2014)**

(*a*) only purely imaginary roots

(*b*) all real roots

(*c*) two real and two purely imaginary roots

(*d*) neither real nor purely imaginary roots

50. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is(are) a subset(s) of S? **(2015)**

(*a*) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (*b*) $\left(-\frac{1}{\sqrt{5}}, 0\right)$

(*c*) $\left(0, \frac{1}{\sqrt{5}}\right)$ (*d*) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

Comprehension (51 and 52)

Let p, q_n be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$ where $\alpha \neq \beta$. For $n = 0, 1, 2, \ldots\ldots$ let $a_n = p\alpha^n + q\beta^n$.

FACT: If a and b are rational numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$. **(2017)**

51. $a_{12} =$

(*a*) $a_{11} + 2a_{10}$ (*b*) $2a_{11} + a_{10}$

(*c*) $a_{11} - a_{10}$ (*d*) $a_{11} + a_{10}$

52. If $a_4 = 28$, then $p + 2q =$

(*a*) 14 (*b*) 7

(*c*) 21 (*d*) 12

53. Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define $a^n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, $n \geq 1$ **(2019)**

$b_1 = 1$ and $b_n = a_{n-1} + a_{n+1}$, $n \geq 2$.

Then which of the following options is/are correct?

(*a*) $a_1 + a_2 + a_3 + \ldots\ldots + a_n = a_{n+2} - 1$ for all $n \geq 1$

(*b*) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(*c*) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

(*d*) $b_n = \gamma^n + \beta^n$ for all $n \geq 1$

54. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of $ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$ is **(2021)**

(*a*) 0 (*b*) 8000

(*c*) 8080 (*d*) 16000

ANSWER KEY

CONCEPT APPLICATION

1. (*a*)	**2.** (*c*)	**3.** (*c*)	**4.** (*b*)	**5.** (*b*)	**6.** (*b*)	**7.** (*d*)	**8.** (*c*)	**9.** (*d*)	**10.** (*d*)
11. (*c*)	**12.** (*b*)	**13.** (*b*)	**14.** (*c*)	**15.** (*d*)	**16.** (*a*)	**17.** (*b*)	**18.** (*a*)	**19.** (*d*)	**20.** (*d*)
21. (*a*)	**22.** (*a*)	**23.** (*b*)							

24. (*i*) $\left[\frac{3}{4},\infty\right)$ (*ii*) $\left[\frac{31}{8},\infty\right)$ (*iii*) $[-\infty,8)$ (*iv*) $\left[-\frac{1}{4},8\right)$

(*v*) $\left[-1-\frac{2}{\sqrt{3}},-1+\frac{2}{\sqrt{3}}\right]$ (vi) $\left[1-\frac{2}{\sqrt{3}},1+\frac{2}{\sqrt{3}}\right]$ (*vii*) $\left(-\infty,3-\sqrt{8}\right]\cup\left[3+\sqrt{8},\infty\right)$

(*viii*) $\left(-\infty,-7-\sqrt{48}\right]\cup\left[-7+\sqrt{48},\infty\right)$ (*ix*) $(-\infty,\infty)$ (*x*) $\left[-\frac{5}{3},\frac{1}{2}\right)$

(*xi*) $[1,\infty)$ (*xii*) $[4,56]$ (*xiii*) $\left[\frac{1}{2},1\right]$ (*xiv*) $\left[\frac{1}{4},1\right]$

25. (*b*) **26.** (*c*) **27.** (*c*) **28.** (*d*) **29.** (*a*) **30.** (*d*)

31. $x=\left\{2,24,-28-4\sqrt{46},-28+4\sqrt{46}\right\}$ **32.** $x=\pm\sqrt{7},\ x=\pm3$

EXERCISE-1 (TOPICWISE)

1. (*c*)	**2.** (*d*)	**3.** (*a*)	**4.** (*c*)	**5.** (*a*)	**6.** (*d*)	**7.** (*b*)	**8.** (*a*)	**9.** (*a*)	**10.** (*c*)
11. (*b*)	**12.** (*a*)	**13.** (*c*)	**14.** (*b*)	**15.** (*c*)	**16.** (*c*)	**17.** (*b*)	**18.** (*a*)	**19.** (*a*)	**20.** (*c*)
21. (*b*)	**22.** (*a*)	**23.** (*c*)	**24.** (*b*)	**25.** (*c*)	**26.** (*b*)	**27.** (*d*)	**28.** (*c*)	**29.** (*a*)	**30.** (*a*)
31. (*c*)	**32.** (*d*)	**33.** (*b*)	**34.** (*c*)	**35.** (*c*)	**36.** (*c*)	**37.** (*b*)	**38.** (*d*)	**39.** (*c*)	**40.** (*d*)
41. (*a*)	**42.** (*b*)	**43.** (*b*)	**44.** (*b*)	**45.** (*b*)	**46.** (*c*)	**47.** (*a*)	**48.** (*a*)	**49.** (*a*)	**50.** (*c*)
51. (*c*)	**52.** (*b*)	**53.** (*b*)	**54.** (*b*)	**55.** (*b*)	**56.** (*d*)	**57.** (*d*)	**58.** (*b*)	**59.** (*b*)	**60.** (*b*)
61. (*a*)	**62.** (*b*)	**63.** (*d*)	**64.** (*a*)	**65.** (*a*)	**66.** (*d*)	**67.** (*a*)	**68.** (*c*)	**69.** (*c*)	**70.** (*b*)
71. (*d*)	**72.** (*a*)	**73.** (*a*)	**74.** (*c*)	**75.** (*b*)	**76.** (*a*)	**77.** (*d*)	**78.** (*b*)	**79.** (*d*)	**80.** (*c*)
81. (*c*)	**82.** (*a*)	**83.** (*b*)	**84.** (*c*)	**85.** (*c*)	**86.** (*a*)	**87.** (*c*)	**88.** (*a*)	**89.** (*c*)	**90.** (*b*)
91. (*d*)	**92.** (*b*)	**93.** (*b*)	**94.** (*d*)	**95.** (*b*)	**96.** (*a*)	**97.** (*b*)	**98.** (*a*)	**99.** (*b*)	**100.** (*a*)
101. (*a*)	**102.** (*d*)	**103.** (*c*)	**104.** (*a*)	**105.** (*d*)	**106.** (*b*)	**107.** (*a*)	**108.** (*b*)	**109.** (*b*)	

EXERCISE-2 (LEARNING PLUS)

1. (*b*)	**2.** (*c*)	**3.** (*a*)	**4.** (*d*)	**5.** (*a*)	**6.** (*b*)	**7.** (*c*)	**8.** (*d*)	**9.** (*d*)	**10.** (*a*)
11. (*c*)	**12.** (*c*)	**13.** (*a*)	**14.** (*c*)	**15.** (*a*)	**16.** (*b*)	**17.** (*a*)	**18.** (*c*)	**19.** (*a*)	**20.** (*a*)
21. (*a*)	**22.** (*d*)	**23.** (*d*)	**24.** (*a*)	**25.** (*c*)	**26.** (*d*)	**27.** (*c*)	**28.** (*c*)	**29.** (*c*)	**30.** (*b*)
31. (*d*)	**32.** (*c*)	**33.** (*a*)	**34.** (*c*)	**35.** (*a*)	**36.** (*d*)	**37.** (*a*)	**38.** (*a*)	**39.** (*a*)	**40.** (*c*)
41. (*b*)	**42.** (*a*)	**43.** (*d*)	**44.** (*c*)	**45.** (*a*)	**46.** (*a*)	**47.** (*a*)	**48.** (*d*)	**49.** (*a*)	**50.** (*a*)
51. (*b*)	**52.** (*b*)	**53.** (*c*)	**54.** (*d*)	**55.** (*d*)	**56.** (*a*)	**57.** (*b*)	**58.** (*a*)	**59.** (*c*)	**60.** (*a*)
61. (*b*)	**62.** (*b*)	**63.** (*c*)	**64.** (*d*)	**65.** (*d*)	**66.** (*b*)	**67.** (*a*)	**68.** (*b*)	**69.** (*d*)	**70.** (*b*)
71. (*d*)	**72.** (*c*)	**73.** (*d*)	**74.** (*d*)	**75.** (*c*)	**76.** (*c*)	**78.** [1, 3]	**79.** [13, 9]	**80.** [6, –161/30]	

81. [7, 2] **82.** $y^3+3y^2-\frac{p^3}{r}y+3y+1=0$ **83.** $x=2;\ y=\frac{1}{2}+\frac{\sqrt{3}}{2};\ z=\frac{1}{2}-\frac{\sqrt{3}}{2}$

85. [(*i*)1, 3, (*ii*) 1/2, 5 (*iii*) –3 (*iv*) *a*, *a*/2] **86.** [99] **88.** [–2] **89.** [2] **90.** [–3/4, –3/5] **91.** [8]

92. [7] **93.** [9] **94.** [23] **95.** [324] **96.** [191] **97.** [0]

98. [2] **99.** $(0, 3) \cup (3, 5)$

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (*a*)	**2.** (*c*)	**3.** (*a*)	**4.** (*a*)	**5.** (*a*)	**6.** (*b*)	**7.** (*a*)	**8.** (*c*)	**9.** (*b*)	**10.** (*d*)
11. (*c*)	**12.** (*d*)	**13.** (*c*)	**14.** (*c*)	**15.** (*c*)	**16.** (*d*)	**17.** (*a*)	**18.** (*a*)	**19.** (*b*)	**20.** (*c*)
21. (*a*)	**22.** (*c*)	**23.** (*a*)	**24.** (*c*)	**25.** (*b*)	**26.** (*d*)	**27.** (*c*)	**28.** (*d*)	**29.** (*a*)	**30.** (*a*)
31. (*a,b,d*)	**32.** (*a,b,c,d*)	**33.** (*a,d*)	**34.** (*a,b,d*)	**35.** (*b,d*)	**36.** (*b,c,d*)	**37.** (*a,d*)	**38.** (*a,b,d*)	**39.** (*b,d*)	**40.** (*c,d*)
41. (*a,d*)	**42.** (*a,d*)	**43.** (*a,b,c,d*)	**44.** (*a,b*)	**45.** (*a,b*)	**46.** (*a,c*)	**47.** (*b,c,d*)	**48.** (*a,b,c*)	**49.** (*a,b*)	**50.** (*a,d*)
51. (*b*)	**52.** (*a,b,d*)	**53.** (*c,d*)	**54.** (*a,d*)	**55.** (*a,b,d*)	**56.** (*a,b,c*)	**57.** (*a,d*)	**58.** (*a,c,d*)	**59.** (*c,d*)	**60.** (*b,c,d*)
61. (*b,c*)	**62.** (*b,c,d*)	**63.** (*a,d*)	**64.** (*c*)	**65.** (*b*)	**66.** (*d*)	**67.** (*d*)	**68.** (*c*)	**69.** (*c*)	**70.** (*d*)
71. (*c*)	**72.** (*b*)	**73.** (*d*)	**74.** (*c*)	**75.** (*b*)	**76.** (*c*)	**77.** (*b*)	**78.** (*c*)	**79.** (*c*)	**80.** (*a*)
81. (*c*)	**82.** (*a*)	**83.** (*b*)	**84.** (*c*)	**85.** (*b*)	**86.** [2]	**87.** [8]	**88.** [11]	**89.** [1]	**90.** [73]
91. [10]	**92.** [3]	**93.** [13]	**94.** [6]	**95.** [11]	**96.** [4]	**97.** [7]	**98.** [8]	**99.** [917]	**100.** [2]
101. [315]	**102.** [3]	**103.** [5]	**104.** [3]	**105.** [8]	**106.** [16]	**107.** [2]	**109.** [8]	**110.** [5]	**111.** [4]

112. (*i*) $(a, 0, 0); (0, a, 0); (0, 0, a)$

(*ii*) $x = 3, y = 4, z = \frac{1}{2}, 4 = \frac{1}{3}$

(*iii*) $x = 8; y = 5; z = 3$

(*iv*) $x = y = z = \frac{a}{\sqrt{3}}$

(*v*) $x = 1; y = 2; z = 3$

(*vi*) $x = -7, 5, y = -5, 3, z = -8, 6$

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (*a*)	**2.** (*a*)	**3.** (*b*)	**4.** (*a*)	**5.** (Bonus)	**6.** (*c*)	**7.** (*d*)	**8.** (*d*)	**9.** (*c*)	**10.** (*b*)
11. (*c*)	**12.** (*c*)	**13.** (*a*)	**14.** (*b*)	**15.** (*d*)	**16.** (*d*)	**17.** (*d*)	**18.** (*c*)	**19.** (*a*)	**20.** (*a*)
21. [8]	**22.** (*b*)	**23.** (*b*)	**24.** (*a*)	**25.** (*a*)	**26.** (*d*)	**27.** (*d*)	**28.** (*c*)	**29.** (*d*)	**30.** (*b*)
31. (*b*)	**32.** (*d*)	**33.** (*b*)	**34.** (*d*)	**35.** (*a*)	**36.** (*d*)	**37.** (*c*)	**38.** (*a*)	**39.** (*b*)	**40.** (*c*)
41. (*a*)	**42.** [0]	**43.** (*b*)	**44.** (*b*)	**45.** [45]	**46.** [98]				

JEE Advanced

47. (*d*)	**48.** (*a,b,c*)	**49.** (*d*)	**50.** (*a, d*)	**51.** (*d*)	**52.** (*d*)	**53.** (*a,b,c*)	**54.** (*d*)

CHAPTER

8 Complex Number-I

INTRODUCTION

Indian mathematician Mahavira (850 A.D.) was first to mention in his work 'Ganitasara Sangraha', as in nature of things a negative (quantity) is not a square (quantity), it has therefore no square root. Hence there is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$.

Whenever $\sqrt{x}$ is thought of to give a real value, it has been, till now, insisted that $x \geq 0$. In other words, in the set of real numbers it is not possible to provide for the existence of a value for $\sqrt{x}$ when $x < 0$. To make this possible we extend the number system so as to include and cover yet another class of numbers called imaginary numbers.

Every complex number z can be written as $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$. x is called real part of z and y is the imaginary part of z.

Let us take the quadratic equation $x^2 - 2x + 10 = 0$. The formal solution of this equation is $\frac{2 \pm \sqrt{4-40}}{2}$ i.e, $1 \pm 6\sqrt{-1}$ which is not meaningful in the set of real numbers.

It is therefore, the symbol i is thought of to possess the following properties:

(*i*) It combines with itself and with real numbers satisfying the laws of algebra.

(*ii*) Whenever we come across -1 we may substitute i^2.

In the light of the foregoing the roots of the equation discussed earlier may be taken as $1 + 6i$, $1 - 6i$.

It is taken that 1 is real part and 6(or – 6) is the imaginary part of this complex number $1 + 6i$ or $1 - 6i$ respectively.

It has now to be mentioned that, + symbol standing between 1 and $6i$ does not appear to be meaningful, though the following are true.

$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2)$ ...(*i*)

i.e., the real parts are added (or subtracted) separately and so in fact the imaginary parts.

Note:

(*a*) The set R of real numbers is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(*b*) For $z = x + iy$, if $y = 0$, z is purely real and if, $x = 0$ then z is purely imaginary.

(*c*) Zero is purely real as well as purely imaginary.

(*d*) $i = \sqrt{-1}$ is called the imaginary unit.

Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(*e*) $\sqrt{a}\sqrt{b} = \sqrt{ab}$ only if atleast one of a or b is non-negative.

(*f*) If $z = a + ib$, then $a - ib$ is called complex conjugate of z and written as $\bar{z} = a - ib$

(*g*) Real numbers satisfy order relations where as imaginary numbers do not satisfy order relations i.e., $i > 0$, $3 + i < 2$ are meaningless.

Algebraic Operations

Fundamental operations with complex numbers

In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by -1 when it occurs.

1. **Addition:**

$(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)\, i$

2. **Subtraction:**

$(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)\, i$

3. **Multiplication:**

$(a + bi)(c + di) = ac + adi + bci + bdi^2$

$= (ac - bd) + (ad + bc)i$

4. **Division:**

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2 i^2}$$

$$= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$$

Inequalities in imaginary numbers are not defined. There is no validity if we say that imaginary number is positive or negative.

e.g., $z > 0$, $4 + 2i < 2 + 4i$ are meaningless.

In real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$, however in complex numbers $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

Equality in Complex Number

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real and imaginary parts are equal respectively.

i.e., $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $I_m(z_1) = I_m(z_2)$.

Train Your Brain

Example 1: If $x = -5 + 2\sqrt{-4}$

find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Sol. $x = -5 + 4i$ $(i = \sqrt{-1})$

$x + 5 = 4i$

Squaring, $x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$

Now $(x^2 - x + 4)(x^2 + 10x + 41) - 160$

and $x^2 + 10x + 41 = 0$

Hence given expression $= 0 - 160 = -160$

Example 2: Find the sum and product of the two complex numbers $Z_1 = 2 + 3i$ and $Z_2 = -1 + 5i$

Sol. $Z_1 + Z_2 = 2 + 3i + (-1 + 5i) = 2 - 1 + 8i = 1 + 8i$

$Z_1Z_2 = (2 + 3i)(-1 + 5i) = -2 + 15i^2 - 3i + 10i$

$= -17 + 7i$ $(\because i^2 = -1)$

Example 3: Find the value of x and y for which $(2 + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i$ where $x, y \in R$.

Sol. $(2 + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i$

$\Rightarrow 2x^2 - 3y = 2x - 3y \Rightarrow x^2 - x = 0$

$\Rightarrow x = 0, 1$ and $3x^2 + 2y = 5$

$\Rightarrow$ if $x = 0, y = \frac{5}{2}$ and if $x = 1, y = 1$

$\therefore x = 0, y = \frac{5}{2}$ and $x = 1, y = 1$

are two solutions of the given equation which can also be represented as $\left(0, \frac{5}{2}\right)$ & $(1, 1)$

Concept Application

1. Number of integral values of n for which the quantity $(n + i)^4$ where $i^2 = -1$. is an integer is
   (a) 1 (b) 2
   (c) 3 (d) 4

2. The value of the sum in $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals
   (a) i (b) $i - 1$
   (c) $-i$ (d) 0

3. In the quadratic equation $x^2 + (p + iq)x + 3i = 0$. p & q are real. If the sum of the squares of the roots is 8 then
   (a) $p = 3, q = -1$ (b) $p = -3, q = -1$
   (c) $p = \pm 3, q = \pm 1$ (d) $p = -3, q = 1$

4. If $\frac{x-3}{3+i} + \frac{y-3}{3-i} = i$ where $x, y \in R$ then
   (a) $x = 2$ & $y = -8$ (b) $x = -2$ & $y = 8$
   (c) $x = -2$ & $y = -6$ (d) $x = 2$ & $y = 8$

5. Let $z = 9 + bi$ where b is non zero real and $i^2 = -1$. If the imaginary part of z^2 and z^3 are equal, then b^2 equals
   (a) 261 (b) 225
   (c) 125 (d) 361

6. The value of $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} - 1 =$
   (a) -1 (b) -2 (c) -3 (d) -4

Representation of a Complex Number

Early in the 19th century, Karl Friedrich Gauss (1777 – 1855) and William Rowan Hamilton (1805 – 1865) independently and almost simultaneously proposed idea of defining complex number as ordered pair of real numbers. i.e., $a + ib = (a, b)$

To each complex number there corresponds one and only one point in plane, and conversely to each point in the plane there corresponds one and only one complex number. Because of this we often refer to the complex number z as the point z.

GEOMETRICAL REPRESENTATION

It is known, from coordinate geometry, that the ordered pair (x, y) represents a point in the Cartesian plane.

It is now seen that the ordered pair (x, y) taken as Z represents a complex number.

It is therefore, that to every complex number $Z \equiv (x, y)$, one can associate, a point $P \equiv (x, y)$ in the Cartesian plane. The point may be said to be geometrical representation of Z. This association is a bijection – in the mapping language – whereby, this correspondence between Z and P is ONE-ONE and ONTO. It is therefore possible to go over to a point from Z or reversing the roles come back to Z from the point.

ARGAND DIAGRAM

The graphical representation of a complex number $Z = (x, y)$ by a point $P(x, y)$ is called representation in the Argand's Diagram also called Gaussian plane. In this representation, all complex numbers like (2, 0), (3, 0), (–1, 0), (α, 0) with imaginary part 0 will be represented by points on the x-axis. Since the real number α is represented as a complex number (α, 0), all real numbers will get marked on the x-axis.

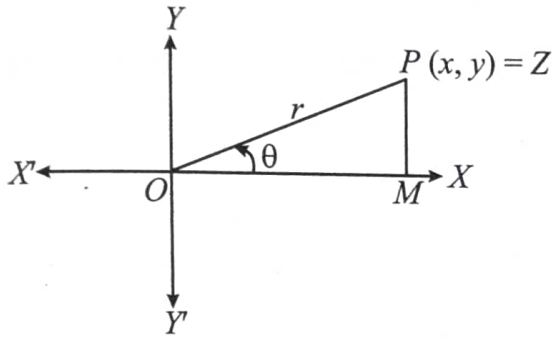
For this reason, the x-axis is called the real axis. Similarly all purely imaginary numbers (with real part 0) like (0, 1), (0, 2), (0, – 3), (0,β) will be marked on the y-axis. Hence the y-axis is also called the imaginary axis in this context. The Cartesian plane (two dimensional plane) is also called the complex plane.

POLAR REPRESENTATION

Let $P(x, y)$ be any point on the complex plane representing the complex number $z = (x, y)$, with $X'OX$ and $Y'OY$ as the axis of coordinates.

Let $OP = r$ and $\angle XOP = \theta$ (measured in anticlockwise).

Then from ΔOMP, we find that $x = OM = r\cos\theta$ and $y = MP = r\sin\theta$



Thus $z = (x, y) = x + iy = r\cos\theta + ir\sin\theta$
$= r(\cos\theta + i\sin\theta)$
where $e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$ by Euler's formula
Thus $z = r(\cos\theta + i\sin\theta)$ can be written as
$\boldsymbol{z = re^{i\theta}}$

This form of representation of Z is called the **trigonometric form** or the **polar form** or the **modulus amplitude form**.

When z is written in the form $r(\cos\theta + i\sin\theta)$, r is called the modulus of z and is written as $|z|$; $|z| = r = \sqrt{x^2 + y^2}$, a non-negative number. $|z| = 0$ for the only number (0, 0).

Euler's Form:

$z = re^{i\theta}$, $|z| = r$, $\arg z = \theta$

$\bar{z} = re^{-i\theta}$

Proof of this formula is beyond scope of present discussion. A heuristic proof serving as motivation for this formula is by considering expansion.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

put $x = i\theta$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

$$= \cos\theta + i\sin\theta.$$

Note: If θ is real then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Train Your Brain

Example 4: Represent the given complex numbers in polar form:

(*i*) $(1 + i\sqrt{3})^2 / 4i(1 - i\sqrt{3})$

(*ii*) $\sin\alpha - i\cos\alpha$ (α acute)

(*iii*) $1 + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

Sol. (*i*) $i(1 - i\sqrt{3}) = i - i^2\sqrt{3} = \sqrt{3} + i$

$$\therefore \frac{(1 + i\sqrt{3})^2}{4i(1 - i\sqrt{3})} = \frac{(1 + i\sqrt{3})^2}{4(\sqrt{3} + i)} = \frac{-2 + 2i\sqrt{3}}{4(\sqrt{3} + i)}$$

$$= \frac{(-1 + i\sqrt{3})(\sqrt{3} - i)}{2(\sqrt{3} + i)(\sqrt{3} - i)} = \frac{-\sqrt{3} + \sqrt{3} + 4i}{2(3 + 1)} = \frac{i}{2}$$

and $\frac{i}{2} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

Hence $\frac{(1 + i\sqrt{3})^2}{4i(1 - i\sqrt{3})} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{1}{2}e^{i\pi/2}$

(*ii*) Real part > 0, Imaginary part < 0 argument of $\sin\alpha - i\cos\alpha$ is in the nature of a negative acute angle.

$$\therefore \sin\alpha - i\cos\alpha = \cos\left(\alpha - \frac{\pi}{2}\right) + i\sin\left(\alpha - \frac{\pi}{2}\right) = e^{i\left(\alpha - \frac{\pi}{2}\right)}$$

(*iii*) $1 + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = 2\cos^2\frac{\pi}{6} + i \cdot 2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$

$$= 2\cos\frac{\pi}{6}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6}e^{i\pi/6}$$

Concept Application

7. Express the fol towing complex numbers in polar and exponential form :

(*i*) $\frac{1 + 3i}{1 - 2i}$ (*ii*) $\frac{i - 1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

CONJUGATE OF A COMPLEX NUMBER

The complex numbers $z = (a, b) = a + ib$ and $\bar{z} = (a, -b) = a - ib$, where a and b are real numbers, $i = \sqrt{-1}$ and $b \neq 0$ are said to be complex conjugate of each other. (Here the complex conjugate is obtained by just changing the sign of i).

Note that, sum $= (a + ib) + (a - ib) = 2a$ which is real and product $= (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 - i^2b^2 = a^2 - (-1)b^2 = a^2 + b^2$ which is real.

Properties of Conjugate

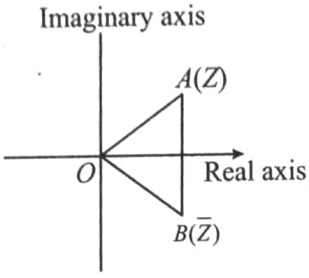
(*i*) $\overline{(\overline{Z})} = Z$

(*ii*) $Z = \overline{Z} \Leftrightarrow z$ is real

(*iii*) $Z = -\overline{Z} \Leftrightarrow z$ is purely imaginary

(*iv*) $\text{Re}(z) = \text{Re}(\overline{z}) = \dfrac{z+\overline{z}}{2}$

(*v*) $\text{Im}(z) = \dfrac{z-\overline{z}}{2i}$



(*vi*) $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$

(*vii*) $\overline{Z_1 - Z_2} = \overline{Z_1} - \overline{Z_2}$

(*viii*) $\overline{\left(\dfrac{Z_1}{Z_2}\right)} = \dfrac{\overline{Z_1}}{\overline{Z_2}}$ $(Z_2 \neq 0)$

(*ix*) $Z_1\overline{Z_2} + \overline{Z_1}Z_2 = 2\,\text{Re}(\overline{Z_1}Z_2) = 2\,\text{Re}(Z_1\overline{Z_2})$

(*x*) $\overline{Z^n} = (\overline{Z})^n$

(*xi*) If $z = f(z_1)$, then $\overline{z} = f(\overline{z_1})$

Modulus of a Complex Number

Modulus of a complex number $z = x + iy$ is a real number given by $|z| = \sqrt{x^2 + y^2}$. It is always non-negative and $|z| = 0$ only for $z = 0$ i.e. origin of Argand plane. Geometrically it represents the distance of the point complex number from its origin.

Properties of Modulus

(*i*) $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$.

(*ii*) $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$.

(*iii*) $|z| = |\overline{z}| = |-z| = |-\overline{z}|$

(*iv*) $z\overline{z} = |z|^2$

(*v*) $|z_1z_2| = |z_1||z_2|$.
In general $|z_1z_2z_3 \ldots z_n| = |z_1||z_2||z_3| \ldots |z_n|$

(*vi*) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$.

(*vii*) $|z_1 \pm z_2| \leq |z_1| + |z_2|$
In particular, if $|z_1 + z_2| = |z_1| + |z_2|$, then origin, z_1 and z_2 are collinear with origin at one of the ends.

(*viii*) $|z_1 \pm z_2| \geq ||z_1| - |z_2||$
In particular, if $|z_1 - z_2| = ||z_1| - |z_2||$, then origin, z_1 and z_2 are collinear with origin at one of the ends.

(*ix*) $|z^n| = |z|^n$

(*x*) $||z_1| - |z_2|| \leq |z_1| + |z_2|$
Thus $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 + z_2|$

(*xi*) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\overline{z_1} \pm \overline{z_2}) = |z_1|^2 + |z_2|^2 \pm (z_1\overline{z_2} + \overline{z_1}z_2)$
or $|z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1\overline{z_2})$

(*xii*) $z_1\overline{z_2} + \overline{z_1}z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$
where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.

(*xiii*) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.

(*xiv*) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$
$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ where $a, b \in R$.
Unimodular i.e., unit modulus
If z is unimodular then $|z| = 1$. A unimodular complex number can always be expressed as $\cos\theta + i\sin\theta$, $\theta \in R$.

Note: $\dfrac{z}{|z|}$ is always a unimodular complex number if $z \neq 0$.

Note:

(*a*) $||z_1| - |z_2|| = |z_1 + z_2|$, $|z_1 - z_2| = |z_1| + |z_2|$ if origin, z_1 and z_2 are collinear and origin lies between z_1 and z_2.

(*b*) $|z_1 + z_2| = |z_1| + |z_2|$, $||z_1| - |z_2|| = |z_1 - z_2|$ if origin, z_1 and z_2 are collinear and z_1 and z_2 lies on the same side of origin.

Train Your Brain

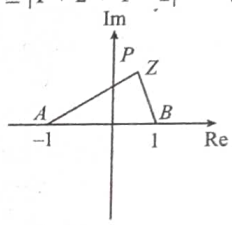
Example 5: If $|z - 5 - 7i| = 9$, then find the greatest and least values of $|z - 2 - 3i|$.

Sol. We have $9 = |z - (5 + 7i)| =$ distance between z and $5 + 7i$.

Thus locus of z is the circle of radius 9 and centre at $5 + 7i$. For such a z (on the circle), we have to find its greatest and least distance as from $2 + 3i$, which obviously 14 and 4.

Example 6: Find the minimum value of $|1 + z| + |1 - z|$.

Sol. $|1 + z| + |1 - z| \geq |1 + z + 1 - z|$ (triangle inequality)



$\Rightarrow |1 + z| + |1 - z| \geq 2$

$\therefore$ minimum value of $(|1 + z| + |1 - z|) = 2$

Geometrically $|z + 1| + |1 - z| = |z + 1| + |z - 1|$

which represents sum of distances of z from 1 and –1

it can be seen easily that minimum $(PA + PB) = AB = 2$

Example 7: $\left|z - \dfrac{2}{z}\right| = 1$ then find the maximum and minimum value of $|z|$

Sol. $\left|z - \dfrac{2}{z}\right| = 1 \Rightarrow \left||z| - \left|\dfrac{2}{z}\right|\right| \leq \left|z - \dfrac{2}{z}\right| \leq |z| + \left|-\dfrac{2}{z}\right|$

Let $|z| = r \Rightarrow \left|r - \dfrac{2}{r}\right| \leq 1 \leq r + \dfrac{2}{r}$

$r + \dfrac{2}{r} \geq 1 \Rightarrow r \in R^+$ ... (*i*)

and $\left|r - \dfrac{2}{r}\right| \leq 1 \Rightarrow -1 \leq r - \dfrac{2}{r} \leq 1 \Rightarrow r \in [1, 2]$ ...(*ii*)

$\therefore$ from (*i*) and (*ii*) $r \in [1, 2]$

$|z|_{max} = 2$, $|z|_{min} = 1$

Example 8: If $|z-2+i| \le 2$ then find the greatest and least value of $|z|$.

Sol. Given that

$|z-2+i| \le 2$ ... (*i*)

$\because \; |z-2+i| \ge ||z|-|2-i||$

$\therefore \; |z-2+i| \ge ||z|-\sqrt{5}|$ ... (*ii*)

From (*i*) and (*ii*)

$||z|-\sqrt{5}| \le |z-2+i| \le 2 \qquad \therefore ||z|-\sqrt{5}| \le 2$

$\Rightarrow -2 \le |z|-\sqrt{5} \le 2 \Rightarrow \sqrt{5}-2 \le |z| \le \sqrt{5}+2$

Hence greatest value of $|z|$ is $\sqrt{5}+2$ and least value of $|z|$ is $\sqrt{5}-2$.

Concept Application

8. If $z = (3+7i)(p+iq)$ where $p, q \in I - \{0\}$, is purely imaginary then minimum value of $|z|^2$ is

(*a*) 0 (*b*) 58

(*c*) $\frac{3364}{3}$ (*d*) 3364

9. If $z = x+iy$ & $\omega = \frac{1-iz}{z-i}$ then $|\omega|$ co I = 1 implies that, in the complex plane

10. On the complex plane locus of a point z satisfying the inequality $2 < |z-1| < 3$ denotes

(*a*) region between the concentric circles of radii 3 and 1 centered at (1. 0)

(*b*) region between the concentric circles of radii 3 and 2 centered at (1. 0) excluding the inner and outer boundaries.

(*c*) region between the concentric circles of radii 3 and 2 centered at (1. 0) including the inner and outer boundaries.

(d) region between the concentric circles of radii 3 and 2 centered at (1. 0) including the inner boundary and excluding the outer boundary.

11. The complex number z satisfies $z + |z| = 2 + 8i$. The value of $|z|$ is

(*a*) 10 (*b*) 13

(*c*) 17 (*d*) 23

12. If the complex number z satisfies the condition $|z| > 3$. then the least value of $\left|z+\frac{1}{z}\right|$ is equal to

(*a*) 5/3 (*b*) 8/3

(*c*) 11/3 (*d*) none of these

13. Modulus of $\left(\frac{3+2i}{3-2i}\right)$ is

(*a*) 1 (*b*) 1/2

(*c*) 2 (*d*) $\sqrt{2}$

14. If $z_1 = 1+2i$ and $z_2 = 3+5i$, and then $\text{Re}\left(\frac{\bar{z}_2 z_1}{z_2}\right)$ is equal to

(*a*) $\frac{-31}{17}$ (*b*) $\frac{17}{22}$ (*c*) $\frac{-17}{31}$ (*d*) $\frac{22}{17}$

ARGUMENTS OF A COMPLEX NUMBER

If $Z = x+iy = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2+y^2}$.

θ is called the argument of Z or amplitude of Z. Since $x = r\cos\theta$ and $y = r\sin\theta$, θ is such that $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$. Since there can be many values of θ satisfying these conditions, by convention, θ such that $-\pi < \theta \le \pi$ is defined as the principal argument of Z and is denoted by arg Z. The argument of a complex number $a+ib$ is given by α, $\pi-\alpha$, $-\pi+\alpha$, or $-\alpha$ if $a+ib$ is in first, second, third or fourth quadrant respectively, where $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$.

For example $Z = 1+i = (1, 1)$ and is marked by point $P(1, 1)$ lies in first quadrant.

$\therefore |Z| = \sqrt{2}$ and arg $Z = \pi/4$.

If $Z = 1-i = (1, -1)$, then P lies in the fourth quadrant and $|Z| = \sqrt{2}$ and arg $Z = -\pi/4$.

If $Z = -1+i = (-1, 1)$, then P lies in the second quadrant and arg $Z = \frac{3\pi}{4}$.

If $Z = -1-i$, arg $Z = -\frac{3\pi}{4}$

Argument of all positive real numbers like 1, 2, 3, 1/2, ... is 0 since they are marked on the positive x-axis. Argument of all negative real numbers like $-1, -2, -3, \ldots$ is π since they are marked on OX'. Argument of purely imaginary numbers like $i, 2i, 3i, \ldots$ is $\pi/2$ since these are marked on the positive y-axis. Argument of purely imaginary numbers like $-i, -2i, -3i, \ldots$ is $-\pi/2$.

Properties of Arguments

(*i*) $\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi$ ($k = 0$ or 1 or -1)
In general $\text{Arg}(z_1z_2z_3 \ldots z_n) = \text{Arg}(z_1) + \text{Arg}(z_2) + \text{Arg}(z_3) + \ldots + \text{Arg}(z_n) + 2k\pi$ (where $k \in I$)

(*ii*) $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}\, z_1 - \text{Arg}\, z_2 + 2k\pi$ ($k = 0$ or 1 or -1)

(*iii*) $\text{Arg}(z^2) = 2\,\text{Arg}\, z + 2k\pi$ ($k = 0$ or 1 or -1)

(*iv*) $\text{Arg}(z^n) = n\,\text{Arg}\, z + 2k\pi$ ($k \in I$)

(*v*) If $\text{Arg}\left(\frac{z_2}{z_1}\right) = \theta$, then $\text{Arg}\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$ where $k \in I$.

(*vi*) $\text{Arg}\,\bar{z} = -\text{Arg}\, z$

(*vii*) If $\arg(z) = 0 \Rightarrow z$ is real.

Note: Proper value of k must be chosen so, that R.H.S. of (*i*), (*ii*), (*iii*), (*iv*), (*v*) lies in $[-\pi, \pi]$.

All the above formulae are written on the basis of principal argument.

Train Your Brain

Example 9: Among the complex numbers z which satisfies $|z - 25i| \le 15$, find the complex numbers z having

(*i*) least positive argument

(*ii*) maximum positive argument

(*iii*) least modulus

(*iv*) maximum modulus

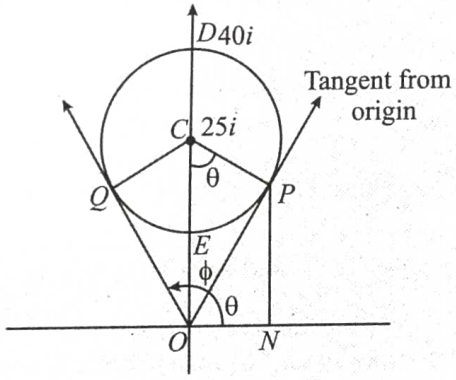
Sol. The complex numbers z satisfying the condition $|z - 25i| \le 15$ are represented by the points inside and on the circle of radius 15 and centre at the point $C(0, 25)$.

The complex number having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle.

Here θ = least positive argument and ϕ = maximum positive argument

$\therefore$ In ΔOCP,

$$OP = \sqrt{(OC)^2 - (CP)^2} = \sqrt{(25)^2 - (15)^2} = 20$$



and $\sin\theta = \dfrac{OP}{OC} = \dfrac{20}{25} = \dfrac{4}{5}$

$\therefore \quad \tan\theta = \dfrac{4}{3} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

Thus, complex number at P has modulus 20 and argument $\theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

$\therefore \quad Z_P = 20(\cos\theta + i\sin\theta) = 20\left(\dfrac{3}{5} + i\dfrac{4}{5}\right)$

$\therefore \quad Z_P = 12 + 16i$

Similarly $Z_Q = -12 + 16i$

From the figure, E is the point with least modulus and D is the point with maximum modulus.

Hence, $Z_E = \overrightarrow{OE} = \overrightarrow{OC} - \overrightarrow{EC} = 25i - 15i = 10\,i$

and $Z_D = \overrightarrow{OD} = \overrightarrow{OC} - \overrightarrow{CD} = 25i - 15i = 40\,i$

Example 10: Let z_1 and z_2 be complex numbers such that $z_1 \ne z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then show that $\dfrac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary.

Sol. $z_1 = r(\cos\theta + i\sin\theta), -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

$z_2 = r(\cos\phi + i\sin\phi), -\pi < \phi < 0$

$$\Rightarrow \quad \frac{z_1 + z_2}{z_1 - z_2} = -i\cot\left(\frac{\theta - \phi}{2}\right), -\frac{\pi}{4} < \frac{\theta - \phi}{2} < \frac{3\pi}{4}$$

Hence, purely imaginary

Example 11: If $z^4 + 1 = \sqrt{3}i$

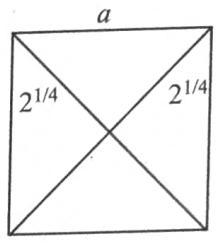
(*a*) z^3 is purely real

(*b*) z represents the vertices of a square of side $2^{1/4}$

(*c*) z^9 is purely imaginary

(*d*) z^9 represents the vertices of a square of side $2^{3/4}$.

Sol. (*d*) $z^4 = -1 + \sqrt{3}i = 2\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$



$z^4 = 2w \Rightarrow A, C$ are not possible

root are $z_1 = 2^{1/4}\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$;

$z_2 = 2^{1/4}\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$ etc.

$\therefore a = \sqrt{2^{1/2} + 2^{1/2}} = (2^{3/2})^{1/2} = 2^{3/4} \Rightarrow (D)$

Concept Application

15. If $z_1 = -3 + 5$; $z_2 = 5 - 3z$ and z is a complex number lying on the line segment joining z_1 & z then arg z can be :

(*a*) $-\dfrac{3\pi}{4}$ (*b*) $-\dfrac{\pi}{4}$

(*c*) $\dfrac{\pi}{6}$ (*d*) $\dfrac{5\pi}{6}$

16. Find amp z and $|z|$ if $z = \left[\dfrac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)}\right]^2$

Concept Application

17. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$, and Arg (z)–Arg $(\omega) = \pi/2$, then $z\omega$ co is equal to

(*a*) 1 (*b*) -1 (*c*) i (*d*) $-i$

18. If $|z| = 4$ and $\arg z = \frac{5\pi}{6}$, then $z =$

(*a*) $2\sqrt{3} - 2i$ (*b*) $2\sqrt{3} + 2i$

(*c*) $-2\sqrt{3} + 2i$ (*d*) $-\sqrt{3} + i$

19. The amplitude of $\frac{1+\sqrt{3}i}{\sqrt{3}+1}$ is

(*a*) $\frac{\pi}{3}$ (*b*) $-\frac{\pi}{3}$ (*c*) $\frac{\pi}{6}$ (*d*) $-\frac{\pi}{6}$

20. Let z be a purely imaginary number such that $\text{Im}(z) < 0$. Then $\arg(z)$ is equal to

(*a*) π (*b*) $\frac{\pi}{2}$

(*c*) 0 (*d*) $-\frac{\pi}{2}$

21. If z is a purely real number such that $\text{Re}(z) < 0$, then $\arg(z)$ is equal to

(*a*) π (*b*) $\frac{\pi}{2}$

(*c*) 0 (*d*) $-\frac{\pi}{2}$

22. The modulus and amplitude of are $\frac{1+2i}{1-(1-i)^2}$

(*a*) $\sqrt{2}$ and $\frac{\pi}{6}$ (*b*) 1 and 0

(*c*) 1 and $\frac{\pi}{3}$ (*d*) 1 and $\frac{\pi}{4}$

De Moivre's Theorem

Case I

Statement: If n is any integer then

(*i*) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(*ii*) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)(\cos\theta_4 + i\sin\theta_4)\ldots(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1+\theta_2+\theta_3+\ldots\theta_n) + i\sin(\theta_1+\theta_2+\theta_3+\ldots+\theta_n)$

Case II

Statement: If $p, q \in Z$ and $q \neq 0$ then

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$$

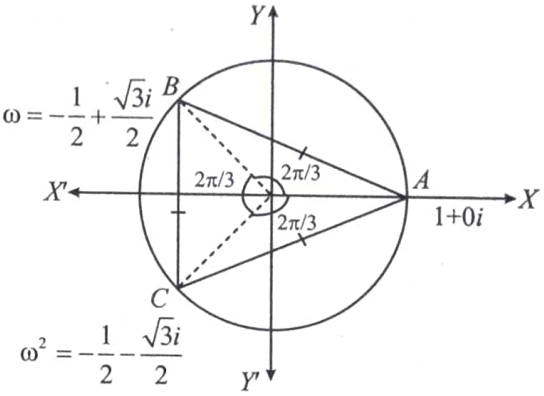
where $k = 0, 1, 2, 3, \ldots, q-1$

Note: Continued product of the roots of a complex quantity should be determined using theory of equations.

Cube Root of Unity

Consider the cubic (3rd degree) equation

$$x^3 = 1 = \cos 0 + i\sin 0$$
$$= \cos 2k\pi + i\sin 2k\pi$$
$$\therefore x = \sqrt[3]{1} = (\cos 2k\pi + i\sin 2k\pi)^{1/3}$$
$$= \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$$



To get the three roots of the cubic equation we give

$k = 0$, giving the real root, $\cos 0 + i\sin 0 = 1$

$k = 1$, giving one imaginary root,

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$$

$k = 2$, giving the other imaginary root,

It is said that 1, ω, ω^2 are the three cubic roots of unity satisfying

(*i*) $1 + \omega + \omega^2 = 0$

(*ii*) $\omega^3 = 1$

(*iii*) 1, ω, ω^2 are represented respectively by points A, B, C lying on the unit circle $|Z| = 1$ and forming the corners of an equilateral triangle of sides $\sqrt{3}$.

(*iv*) The following factorisation should be remembered:

($a, b, c \in R$ & ω is the cube root of unity)

$a^3 - b^3 = (a-b)(a-\omega b)(a-\omega^2 b)$;

$x^2 + x + 1 = (x-\omega)(x-\omega^2)$;

$a^3 + b^3 = (a+b)(a+\omega b)(a+\omega^2 b)$;

$a^2 + ab + b^2 = (a-b\omega)(a-b\omega^2)$

$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)$

n^{th} ROOTS OF UNITY

More generally the nth degree equation $x^n = 1$ has 'n' n^{th} roots of unity given by

$$1, \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \ldots, \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}$$

i.e. 1, α, α^2 .... α^{n-1} satisfying

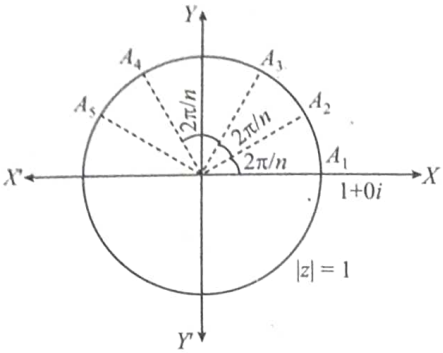
(*i*) $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$

(*ii*) $\alpha^n = 1$

(*iii*) $1, \alpha, \ldots, \alpha^{n-1}$ represent n points in the Argand's plane situated on the unit circle $|Z| = 1$ and forming the corners of a regular n sides polygon.

As sum of nth roots of unity = 0

$$\Rightarrow \sum_{k=0}^{n-1} \alpha^k = 0 \Rightarrow \sum_{k=0}^{n-1}\left(\cos\left(\frac{2k\pi}{n}\right) + i\left(\sin\frac{2k\pi}{n}\right)\right)$$



$$\Rightarrow \sum_{k=0}^{n-1}\cos\left(\frac{2k\pi}{n}\right) = 0 \text{ and } \sum_{k=0}^{n-1}\sin\left(\frac{2k\pi}{n}\right) = 0$$

More general equation like $x^n = a + ib$ can be solved by using this method.

First write $a + ib = r\,[\cos\theta + i\sin\theta] = r\,[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$ and hence the n n^{th} roots of $x^n = a + ib$ are

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right); k = 0, 1, 2 \ldots, (n-1)$$

Train Your Brain

Example 12: If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, prove the following:

(*i*) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

(*ii*) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$

Sol. (*i*) Given $x + \frac{1}{x} = 2\cos\theta \Rightarrow x^2\cos\theta + 1 = 0$.

Solving this, $x = \cos\theta \pm i\sin\theta$

In fact if $x = \cos\theta + i\sin\theta$; $\frac{1}{x} = \cos\theta - i\sin\theta$.

It may be noted also that $x + \frac{1}{x} = 2\cos\theta$ is symmetrical w.r.t. $\frac{1}{x}$ and hence if one root is the value for x, the other root is $\frac{1}{x}$ and vice-versa.

Similarly, given that $2\cos\phi = y + \frac{1}{y}$, we have $y = \cos\phi + i\sin\phi$

$\therefore x^m = (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

and $y^n = (\cos\theta + i\sin\theta)^n = \cos n\phi + i\sin n\phi$

$x^m y^n = (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$

$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi)$

and $\frac{1}{x^m y^n} = \cos(m\theta + n\theta) - i\sin(m\theta + n\theta)$

Adding we get

$$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$$

(*ii*) By similar reasoning $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$

Example 13: Find the value of $\omega^{192} + \omega^{194}$

Sol. $\omega^{192} + \omega^{194} = 1 + \omega^2 = -\omega$

Example 14: If $1, \omega, \omega^2$ are cube roots of unity, then prove that

(*i*) $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$

(*ii*) $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$

(*iii*) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8) \ldots$ to $2n$ factors $= 2^{2n}$.

Sol. (*i*) $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = (-2\omega)(-2\omega^2) = 4$

(*ii*) $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5$

$= (1 + \omega^2 - \omega)^5 + (1 + \omega - \omega^2)^5$

$= (-\omega - \omega)^5 + (-\omega^2 - \omega^2)^5 = (-2\omega)^5 + (-2\omega^2)^5$

$= -32\omega^5 - 32\omega^{10} = -32(\omega^2)(\omega^3) - 32(\omega^3)^3 \cdot \omega$

$= -32\omega^2 - 32\omega = -32(\omega^2 + \omega) - 32(-1) = 32$

(*iii*) $\omega^4 = \omega$, $\omega^8 = \omega^2$ etc. 3rd, 5th, 7th factors are each equal to 1st and 4th, 6th, 8th factors are each equal to 2nd.

L.H.S. $= (-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2) \ldots$ to $2n$ factors

$= (2^2\omega^3)(2^2\omega^3) \ldots$ to n factors $= (2^2)^n = 2^{2n}$

Example 15: Find the roots of the equation $z^6 + 64 = 0$ where real part is positive.

Sol. $z^6 = -64$

$z^6 = 2^6 \cdot e^{i(2n+1)\pi}$

$n = 0, 1, 2, 3, 4, 5 \Rightarrow z = 2e^{i(2n+1)\frac{\pi}{6}}$

$\therefore 2e^{i\frac{\pi}{6}}, 2e^{i\frac{\pi}{2}}, 2e^{i\frac{5\pi}{6}}, 2e^{i\frac{7\pi}{6}}, 2e^{i\frac{3\pi}{2}}, 2e^{i\frac{11\pi}{6}}$

$\therefore$ roots with +ve real part are $= 2e^{i\frac{\pi}{6}}, 2e^{i\frac{11\pi}{6}}$

Example 16: Find the value $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - \cos\frac{2\pi k}{7}\right)$

Sol. $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}\right) - \sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7}\right)$

$\sum_{k=0}^{6}\sin\frac{2\pi k}{7} - \sum_{k=0}^{6}\cos\frac{2\pi k}{7} + 1 = \sum_{k=0}^{6}$

(Sum of imaginary part of seven seventh roots of unity)

$-\sum_{k=0}^{6}$ (Sum of real part of seven seventh roots of unity)

$+ 1 = 0 - 0 + 1 = 1$

Concept Application

23. Number of values of z (real or complex) simultaneously satisfying the system of equations
$1 + z + z^2 + z3 + \ldots\ldots\ldots + z^{17} = 0$ and $1 + z + z^2 + z^3 + \ldots\ldots\ldots + z^{13} = 0$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

24. Least positive argument of the 4th root of the complex number $2 - i\sqrt{12}$ is

(*a*) $\pi/6$ (*b*) $5\pi/12$
(*c*) $7\pi/12$ (*d*) $11\pi/12$

25. If $(w \neq 1)$ is a cube root of unity then

$$\begin{vmatrix} 1 & 1+i+w^2 & w^2 \\ 1-i & -1 & w^2-1 \\ -i & -i+w-1 & -1 \end{vmatrix} =$$

(*a*) 0 (*b*) 1 (*c*) i (*d*) w

26. z is a complex number such that $z + \frac{1}{2} = 2\cos 3°$, then the value of $z^{2000} + \frac{1}{z^{2000}} + 1$ is equal to

(*a*) 0 (*b*) -1
(*c*) $\sqrt{3}+1$ (*d*) $1-\sqrt{3}$

27. If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then

(*a*) $\text{Re}(z) = 0$ (*b*) $\text{Im}(z) = 0$
(*c*) $\text{Re}(z) > 0, \text{Im}(z) > 0$ (*d*) $\text{Re}(z) > 0, \text{Im}(z) < 0$

28. If $z = \frac{\sqrt{3}+i}{2}$, then the value of z^{69} is

(*a*) $-i$ (*b*) i (*c*) 1 (*d*) -1

29. The value of $(1 - \omega + \omega^2)(1 - \omega^2 + \omega)^6$, where ω, ω^2 are cube roots of unity

(*a*) $128\,\omega$ (*b*) $-128\,\omega^2$
(*c*) -128ω (*d*) $128\omega^2$

30. If $x = \frac{1+\sqrt{3}i}{2}$ then the value of the expression, $y = x^4 - x^2 + 6x - 4$, equals

(*a*) $-1 + 2\sqrt{3}\,i$ (*b*) $2 - 2\sqrt{3}\,i$
(*c*) $2 + 2\sqrt{3}\,i$ (*d*) $-1 - 2\sqrt{3}\,i$

The sum of the following series should be remembered :

(*i*) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta$

$$= \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\frac{n+1}{2}\right)\theta.$$

(*ii*) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$

$$= \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\frac{n+1}{2}\right)\theta.$$

Note : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

Logarithm Of A Complex Quantity

$$\text{Log}_e(\alpha + i\beta) = \frac{1}{2}\,\text{Log}_e(\alpha^2 + \beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$$

where $n \in I$.

Train Your Brain

Example 17: Find the value of

(*i*) $\log(1 + \sqrt{3}\,i)$
(*ii*) 2^i

Sol. (*i*) $\log(1 + \sqrt{3}\,i) = \log\left(2\,e^{i\left(\frac{\pi}{3}+2n\pi\right)}\right)$

$$= \log 2 + i\left(\frac{\pi}{3} + 2n\pi\right)$$

(*ii*) $2^i = e^{i\ln 2} = [\cos(\ln 2) + i\sin(\ln 2)]$

Concept Application

31. (*i*) i^i

(*ii*) $|(1+i)^i|$

(*iii*) $\arg((1+i)^i)$

Short Notes

IOTA

$$i = \sqrt{-1},\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$$

So, $$i^{4n+1} = i,\ i^{4n+2} = -1,\ i^{4n+3} = -i,\ i^{4n+4} = 1$$

In other words, $i^n = \begin{cases} (-1)^{n/2}, \text{ if } n \text{ is an even integer} \\ (-1)^{\frac{n-1}{2}} \cdot i, \text{ if } n \text{ is an odd integer} \end{cases}$.

The Complex Number System

$z = a + ib$, then $a - ib$ is called conjugate of z and is denoted by $\bar{z}$.

Equality in Complex Number

$z_1 = z_2 \Rightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $I_m(z_1) = I_m(z_2)$.

Conjugate Complex

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z}. = a - ib$.

Note:

(*i*) $z + \bar{z} = 2\,\text{Re}(z)$

(*ii*) $z - \bar{z} = 2i\,\text{Im}(z)$

(*iii*) $z\bar{z} = a^2 + b^2$ which is real

(*iv*) If z is purely real then $z - \bar{z} = 0$

(*v*) If z is purely imaginary then $z + \bar{z} = 0$

important Properties of Conjugate

(*a*) $\overline{(\bar{z})} = z$ (*b*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(*c*) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (*d*) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(*e*) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}; z_2 \neq 0$

(*f*) If $f(\alpha + i\beta) = x + iy \Rightarrow f(\alpha - i\beta) = x - iy$

Important Properties of Modulus

(*a*) $|z| \geq 0$ (*b*) $|z| \geq \text{Re}(z)$

(*c*) $|z| \geq \text{Im}(z)$ (*d*) $|z| = |\bar{z}.| = |-z| = |-\bar{z}.|$

(*e*) $z\bar{z}. = |z|^2$ (*f*) $|z_1 z_2| = |z_1| \cdot |z_2|$

(*g*) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$ (*h*) $|z^n| = |z|^n$

(*i*) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1 \bar{z}_2)$
or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

(*j*) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[|z_1|^2 + |z_2|^2]$

(*k*) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
[Triangle Inequality]

(*l*) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$
[Triangle Inequality]

(*m*) If $\left|z + \frac{1}{z}\right| = a\ (a > 0)$, then $\max|z| = \frac{a + \sqrt{a^2 + 4}}{2}$

and $\min|z| = \frac{1}{2}\left(\sqrt{a^2 + 4} - a\right)$.

Important Properties of Amplitude

(*a*) $\text{amp}(z_1 \cdot z_2) = \text{amp}\, z_1 + \text{amp}\, z_2 + 2k\pi;\ k \in I$.

(*b*) $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp}\, z_1 - \text{amp}\, z_2 + 2k\pi;\ k \in I$.

(*c*) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$, where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(*d*) $\log(z) = \log(re^{i\theta}) = \log r + i\theta = \log|z| + i\,\text{amp}(z)$.

Demoivre's Theorem

Case I: If n is any integer then

(*i*) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(*ii*) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta$
$+ i\sin\theta_3) \ldots (\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + \theta_3 + \ldots \theta_n)$
$i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots \theta_n)$

Case II: If $p, q \in Z$ and $q \neq 0$ then $(\cos\theta + i\sin\theta)^{p/q}$

$$= \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where $k = 0, 1, 2, 3 \ldots q - 1$.

Cube Root of Unity

(*i*) The cube roots of unity are $1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$.

(*ii*) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

(*c*) $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$
$a^3 + b^3 = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$
$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$
$x^2 + x + 1 = (x - \omega)(x - \omega^2)$

Square root of Complex Number

$$\sqrt{a + ib} = \pm\left\{\frac{\sqrt{|z| + a}}{2} + i\frac{\sqrt{|z| - a}}{2}\right\} \text{ for } b > 0$$

and $\pm\left\{\frac{\sqrt{|z| + a}}{2} - i\frac{\sqrt{|z| - a}}{2}\right\}$ for $b < 0$ where $|z| = \sqrt{a^2 + b^2}$.

Solved Examples

1. If z_1 and z_2 are two complex numbers and $c > 0$, then prove that $|z_1 + z_2|^2 \le (1 + c)\,|z_1|^2 + (1 + c^{-1})\,|z_2|^2$

Sol. We have to prove :

$|z_1 + z_2|^2 \le (1 + c)\,|z_1|^2 + (1 + c^{-1})\,|z_2|^2$

i.e. $|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \le (1 + c)\,|z_1|^2 + (1 + c^{-1})\,|z_2|^2$

or $z_1\bar{z}_2 + \bar{z}_1 z_2 \le c|z_1|^2 + c^{-1}|z_2|^2$

or $c|z_1|^2 + \dfrac{1}{c}\,|z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2 \ge 0$

or $\left|\sqrt{c}\,z_1 - \dfrac{1}{\sqrt{c}}\,z_2\right|^2 \ge 0$ which is always true.

2. If $\theta_i \in [\pi/6, \pi/3]$, $i = 1, 2, 3, 4, 5$, and $z^4 \cos\theta_1 + z^3 \cos\theta_2 + z^2 \cos\theta_3 + z\cos\theta_4 + \cos\theta_5 = 2\sqrt{3}$, then show that $|z| > \dfrac{3}{4}$

Sol. Given that

$\cos\theta_1 \cdot z^4 + \cos\theta_2 \cdot z^3 + \cos\theta_3 \cdot z^2 + \cos\theta_4 \cdot z + \cos\theta_5 = 2\sqrt{3}$

or $|\cos\theta_1 \cdot z^4 + \cos\theta_2 \cdot z^3 + \cos\theta_3 \cdot z^2 + \cos\theta_4 \cdot z + \cos\theta_5| = 2\sqrt{3}$

$2\sqrt{3} \le |\cos\theta_1 \cdot z^4| + |\cos\theta_2 \cdot z^3| + |\cos\theta_3 \cdot z^2| + \cos\theta_4 \cdot z| + |\cos\theta_5|$

$\because\ \theta_i \in [\pi/6, \pi/3]$

$\therefore\ \dfrac{1}{2} \le \cos\theta_i \le \dfrac{\sqrt{3}}{2}$

$2\sqrt{3} \le \dfrac{\sqrt{3}}{2}|z|^4 + \dfrac{\sqrt{3}}{2}|z|^3 + \dfrac{\sqrt{3}}{2}|z|^2 + \dfrac{\sqrt{3}}{2}|z| + \dfrac{\sqrt{3}}{2}$

$3 \le |z|^4 + |z|^3 + |z|^2 + |z|$

Case I: If $|z| \ge 1$, then above result is automatically true

Case II: If $|z| < 1$, then

$3 < |z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + \ldots \infty$

$3 < \dfrac{|z|}{1-|z|} \Rightarrow 3 - 3|z| < |z| \Rightarrow |z| > \dfrac{3}{4}$

Hence by both cases, $|z| > \dfrac{3}{4}$

3. If $z_1 = a + ib$ and $z_2 = c + id$ are complex number such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1\bar{z}_2) = 0$, then show that the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies the following

(*i*) $|w_1| = 1$

(*ii*) $|w_2| = 1$

(*iii*) $\text{Re}\,(w_1\bar{w}_2) = 0$

Sol. $a = \cos\theta$, $b = \sin\theta$, $c = \cos\phi$, $d = \sin\phi$

$\text{Re}(z_1\bar{z}_2) = 0$

$\Rightarrow\ \theta - \phi = \dfrac{n\pi}{2} \qquad n = \pm 1$

$\Rightarrow\ c = \sin\theta,\ d = -\cos\theta \qquad$ (for n = 1)

$\Rightarrow\ w_1 = \cos\theta + i\sin\theta$

$w_2 = \sin\theta - i\cos\theta$

$\Rightarrow\ |w_1| = 1,\ |w_2| = 1$

$w_1\bar{w}_2 = \cos\theta\sin\theta - \sin\theta\cos\theta + i(\sin^2\theta - \cos^2\theta)$

$= -i\cos 2\theta$

$\Rightarrow\ \text{Re}\,(w_1\bar{w}_2) = 0$

similarly for $n = -1$, we get $\text{Re}\,(w_1\bar{w}_2) = 0$

4. Number of imaginary complex numbers satisfying the equation, $z^2 = \bar{z}\,2^{1-|z|}$ is

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3

Sol. (*c*) $z^2 = z \cdot 2^{1-|z|}$

$z^3 = |z|^2\,2^{1-|z|} \Rightarrow |z| = 2^{1-|z|}$

hence z^3 is purely +ve real (as $z \ne 0$) $\Rightarrow$ z is + ve real

hence $z = re^{i\frac{2k\pi}{3}}\quad k = 0, 1, 2$

$\therefore\ z = e^{i\frac{2k\pi}{3}}$

hence $z = 1, \omega, \omega^2$

but 1 is not imaginary

hence $z = w$ or $w^2 \Rightarrow (c)$

5. If z & w are two complex numbers simultaneously satisfying the equations, $z^3 + w^5 = 0$ and $z^2 \cdot \bar{w}^4 = 1$, then:

(*a*) z and w both are purely real

(*b*) z is purely real and w is purely imaginary

(*c*) w is purely real and z is purely imaginarly

(*d*) z and w both are imaginary.

Sol. (*a*) $z^3 = -\omega^5 \Rightarrow |z|^3 = |\omega|^5 \Rightarrow |z|^6 = |\omega|^{10}$ ....(i)

and $z^2 = \dfrac{1}{\bar{\omega}^4} \Rightarrow |z|^2 = \dfrac{1}{|\omega|^4} \Rightarrow |z|^6 = \dfrac{1}{|\omega|^{12}}$ ...(ii)

From (i) and (ii) $|\omega| = 1$ and $|z| = 1 \Rightarrow z\bar{z} = w\bar{\omega} = 1$

Again $z^6 = \omega^{10}$ ...(iii)

and $z^6 \cdot \bar{\omega}^{12} = 1$

$z^6 = \dfrac{1}{\bar{\omega}^{12}} = \omega^{10}$ {from (iii)} $\Rightarrow (\omega\bar{\omega})^{10}(\bar{\omega})^2 = 1 \Rightarrow (\bar{\omega})^2 = 1$

$\Rightarrow \bar{\omega} = 1$ or $-1 \Rightarrow \omega = 1$ or -1

if $\omega = 1$ then $z^3 + 1 = 0$ and $z^2 = 1 \Rightarrow z = -1$

if $\omega = -1$ then $z^3 - 1 = 0$ and $z^2 = 1 \Rightarrow z = 1$

Hence $z = 1$ & $\omega = -1$ or $z = -1$ & $\omega = 1$

6. A root of unity is a complex number that is a solution to the equation, $z^n = 1$ for some positive integer n. Number of non real complex roots of unity that are also the roots of the equation $z^2 + az + b = 0$, for some integer a and b is

(*a*) 6 (*b*) 8 (*c*) 9 (*d*) 10

Sol. (*b*) Let α is a non real complex root of unity that is also a root of the equation $z^2 + az + b = 0$, then $\bar{\alpha}$ will also be its root. ($|\alpha| = 1$)

Hence $\alpha + \bar{\alpha} = -a$

$\therefore |a| = |\alpha + \bar{\alpha}| \le |\alpha| + |\bar{\alpha}| = 2$

and $b = \alpha\bar{\alpha} = 1$

Hence we must check those equation for which $-2 \le a \le 2$ and $b = 1$

i.e. $z^2 + 2z + 1 = 0$; $z^2 + z + 1 = 0$; $z^2 + 1 = 0$

$z^2 - 2z + 1 = 0$; $z^2 - z + 1 = 0$

hence roots are $\pm 1, \pm i$; $\frac{-1 \pm \sqrt{-3}}{2}, \frac{1 \pm \sqrt{-3}}{2}$ i.e. 8 Ans.

7. Show that,

$$\frac{\cos A_1}{2} + \frac{\cos A_2}{2^2} + \frac{\cos A_3}{2^3} + \ldots + \frac{\cos A}{2^n} = 0,$$

and $$\frac{\sin A_1}{2} + \frac{\sin A_2}{2^2} + \frac{\sin A_3}{2^3} + \ldots + \frac{\sin A_n}{2^n} = 0$$

cannot occur simultaneously for any real numbers $A_1, A_2 \ldots A_n$

Sol. Let $A = \sum_{r=1}^{n} \frac{\cos A_r}{2^r}$ and $B = \sum_{r=1}^{n} \frac{\sin A_r}{2^r}$

Now $A + iB = \sum_{r=1}^{n} \frac{\cos A_r + i \sin A_r}{2r} = \sum_{r=1}^{n} \frac{e^{iAr}}{2^r}$

if $A = B = 0 \Rightarrow A + iB = 0$

$$\Rightarrow \sum_{r=1}^{n} \frac{e^{iAr}}{2^r} = 0$$

$$\left|\sum_{r=1}^{n} \frac{e^{iA_r}}{2^r}\right| = \left|\frac{e^{iA_1}}{2} + \frac{e^{iA_2}}{2^2} + \ldots + \frac{e^{iAn}}{2^n}\right|$$

$$\le \left|\frac{e^{iA_1}}{2}\right| + \left|\frac{e^{iA_1}}{2}\right| + \left|\frac{e^{iA_2}}{2^2}\right| + \ldots + \left|\frac{e^{iAn}}{2^n}\right|$$

$$\le \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$$

$$\le \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \infty \le 1$$

Clearly, $\left|\frac{e^{iA_1}}{2}\right| = \frac{1}{2}$

By the some Analogy, we can show

$$\left|\frac{e^{iA_2}}{2^2} + \frac{e^{iA_3}}{2^3} + \ldots + \frac{e^{iAn}}{2^n}\right| < \frac{1}{2}$$

Therefore, $\left|\frac{e^{iA_1}}{2}\right| > \left|\frac{e^{iA_2}}{2^2} + \frac{e^{iA_3}}{2^3} + \ldots + \frac{e^{iA_n}}{2^n}\right|$

$$\therefore \left|\frac{e^{iA_1}}{2}\right| - \left|\frac{e^{iA_2}}{2^2} + \frac{e^{iA_3}}{2^2} + \ldots + \frac{e^{iA_n}}{2^n}\right| > 0$$

Therefore, $\left|\sum_{r=1}^{n} \frac{e^{iA_r}}{2^r}\right| \ne 0$ (Hence Proved).

8. If $\cos\alpha + \cos\beta + \cos\gamma = 0$, $\sin\alpha + \sin\beta + \sin\gamma = 0$, then show that

(1) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

(2) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(3) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

Sol. Let $z_1 = \cos\alpha + i\sin\alpha = e^{i\alpha}$

$z_2 = \cos\beta + i\sin\beta = e^{i\alpha}$

$z_3 = \cos\gamma + i\sin\gamma = e^{i\gamma}$

Given, $z_1 + z_2 + z_3 = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$

if $z_1 + z_2 + z_3 = 0 \Rightarrow z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$

$z_1^3 + z_2^3 + z_3^3 = e^{ib\alpha} + e^{i3\beta} + e^{i3\beta} + e^{ib\gamma} = (\cos 3\alpha + \cos 3\beta + \sin 3\gamma)$

again, $z_1z_2z_3\ e^{i\alpha} \cdot e^{i\beta} \cdot e^{i\gamma}$

$= e^{i(\alpha+\beta+\gamma)} = \cos(\alpha+\beta+\gamma)\ i\sin(\alpha + \beta + \gamma)$

Therefore,

$(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma) = 3\cos(\alpha + \beta + \gamma)\ 3i\sin(\alpha + \beta + \gamma)$

equating real and imaginary parts, we get

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$, $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

we know $z_1 + z_2 + z_3 = 0$

we know, $(z_1 + z_2 + z_3)^2 = (z_1^2 + z_2^2 + z_3^2) + 2(z_1z_2 + z_1z_3 + z_2z_3)$

$$\therefore z_1^2 + z_3^2 = 2z_1z_2z_3\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) (\because z_1 + z_2 + z_3 = 0)$$

$= -z_1z_2z_3[(\cos\alpha + \cos\beta + \cos\gamma)\ i(\sin\alpha + \sin\beta + \sin\gamma)]$

$z_1^2 + z_2^2 + z_3^2 = 0$

$\therefore e^{2i\alpha} + e^{2i\beta} + e^{i2\gamma} = 0$

$(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0 + i.0$

9. If $1, \alpha, \alpha^2 \ldots\ldots \alpha^{n-1}$ are the nth roots of unity, then prove that

(1) $(1 - \alpha)(1 - \alpha^2)\ldots(1 - \alpha^{n-1}) = n$

(2) $\sin\left(\frac{\pi}{n}\right).\sin\left(\frac{2\pi}{n}\right).\sin\left(\frac{3\pi}{n}\right)\ldots\sin\left(\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}$

Sol. $1, \alpha, \alpha^2 \ldots \alpha^{n-1}$ are the roots of unity, then

$x^n - 1 = (x - 1)(x - \alpha)(x - \alpha^2)\ldots(x - \alpha^{n-1})$

$$\Rightarrow \frac{x^n - 1}{x - 1} = (x - \alpha)(x - \alpha^2)\ldots(x - \alpha^{n-1})$$

$$\Rightarrow (x - \alpha)(x - \alpha^2)\ldots(x - a^{n-1}) = \lim_{x \to 1} \frac{x^n - 1}{x - 1}$$

$$\Rightarrow \lim_{x \to 1} \frac{x^n - 1}{x - 1} = (x^{n-1} + x^{n-2} + x^{n-3} + \ldots + x^2 + x + 1)$$

Putting x = 1, in both sides, we get

$|(1 - \alpha)(1 - \alpha^2)\ldots(1 - \alpha^{n-1})| = n$

$$\left|2^{n-1}\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\ldots\sin\left(\frac{(n-1)\pi}{n}\right)\right| = n$$

$$\therefore \sin\left(\frac{\pi}{n}\right).\sin\left(\frac{2\pi}{n}\right)\ldots\sin\left(\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}$$

10. Quadratic $az^2 + bz + c = 0$; (a, b, c ∈ C). Find the condition that equation has

(*a*) One purely imaginary root.
(*b*) two purely imaginary roots.
(*c*) one purely real root.
(*d*) two purely real roots.

Sol. (*a*) Let z1 be the purely imaginary root of $az^2 + bz + c = 0$, so z_1 satisfies the equation.

$z_1 = -\bar{z}_1 \Rightarrow$ Purely imaginary.

$az_1^2 + bz_1 + c = 0$ ...(i)

Taking conjugate on both sides.

$$\overline{az_1^2 + bz_1 + c} = \bar{0} \Rightarrow \bar{a}\bar{z}_1^2 + \bar{b}\bar{z}_1 + \bar{c} = 0$$

$$\therefore \bar{a}z_1^2 - \bar{b}z_1 + \bar{c} = 0 \quad ...(ii)$$

Equations (i) and (ii) must have one common root,

$$\frac{z_1^2}{b\bar{c} + c\bar{b}} = \frac{z_1}{c\bar{a} - \bar{c}a} = \frac{-1}{a\bar{b} + \bar{a}b} \Rightarrow z_1 = \frac{(\bar{c}a - \bar{c}a)}{(a\bar{b} + \bar{a}b)}$$

$$z_1^2 = \frac{-(b\bar{c} + c\bar{b})}{(a\bar{b} + \bar{a}b)}$$

eliminating z_1 we get the required condition

$$(c\bar{a} - \bar{c}a)^2 + (a\bar{b} + \bar{a}b)(b\bar{c} + c\bar{b}) = 0$$

(*b*) If both the roots are purely imaginary then equation (i) and (ii) have both the roots common.

$$\Rightarrow \frac{a}{\bar{a}} = \frac{-b}{b} = \frac{c}{c}$$

(*c*) Let z1 be purely real root of $az^2 + bz + c = 0$

$z_1 = \bar{z}_1 \Rightarrow$ Real

$az_1^2 + bz_1 + c = 0$ ...(i)

$\Rightarrow \bar{a}\bar{z}_1^2 + \bar{b}\bar{z}_1 + \bar{c} = 0 \Rightarrow \bar{a}\bar{z}_1^2 + \bar{b}z_1^2 + \bar{c} = 0$ ...(ii)

equation (i) and equation (ii) have one root common

$$\frac{z_1^2}{b\bar{c} - \bar{b}c} = \frac{-z_1}{\bar{a}c - \bar{a}c} = \frac{1}{a\bar{b} - \bar{a}b}$$

$$\Rightarrow z_1 = \frac{-(a\bar{c} - \bar{a}c)}{a\bar{b} - \bar{a}b}, z_1^2 = \frac{b\bar{c} - \bar{b}c}{(a\bar{b} - \bar{a}b)}$$

eliminating z_1 we get :

$$(a\bar{c} - \bar{a}c)^2 = (a\bar{b} - \bar{a}b)(b\bar{c} - \bar{b}c)$$

(*d*) If both the roots are purely real then equation (i) and (ii) have both roots common $\Rightarrow \frac{a}{\bar{a}} = \frac{b}{\bar{b}} = \frac{c}{\bar{c}}$

11. If $\left|z + \frac{1}{z}\right| = a$, where z is a complex number, find the least and the greatest values of |z| , also find a for which the greatest and the least values of |z| are equal.

Sol. $\left|z + \frac{1}{z}\right| = a \Rightarrow a \geq 0$ ...(i)

$$\left|z + \frac{1}{z}\right| \geq \left||Z| - \left|\frac{1}{Z}\right|\right| = \left||Z| - \frac{1}{|Z|}\right|$$

$$\Rightarrow a \geq \left||z| - \frac{1}{|z|}\right| \text{ or } -a \leq |z| - \frac{1}{|z|} \leq a$$

$$-a|z| \leq |z|^2 - 1 \leq a|z|$$

Taking last two relations we get,

$$\frac{a - \sqrt{a^2 + 4}}{2} \leq |z| \leq \frac{a + \sqrt{a^2 + 4}}{2}$$

$$0 \leq |z| \leq \frac{a + \sqrt{a^2 + 4}}{2} \quad ...(ii) \quad (\because |z| \geq 0)$$

and taking last two relations we get

$$|z| \geq \frac{\sqrt{a^2 + 4} - a}{2} \quad ...(iii)$$

and $|z| \leq \frac{-a - \sqrt{a^2 + 4}}{2}$ (impossible)

Hence from (ii) and (iii), we get,

$$\frac{\sqrt{(a^2 + 4)} - a}{2} \leq |z| \leq \frac{a + \sqrt{a^2 + 4}}{2}$$

$$\therefore |z|_{max} = \frac{a + \sqrt{a^2 + 4}}{2}$$

$$|z|_{min} = \frac{-a + \sqrt{a^2 + 4}}{2}.$$

12. If $(a + \omega)^{-1} + (b + \omega)^{-1} + (c + \omega)^{-1} + (d + \omega)^{-1} = 2\omega^{-1}$, $(a + \omega')^{-1} + (b + \omega')^{-1} + (c + \omega')^{-1} (d + \omega') = 2(\omega')^{-1}$ where ω and ω' are the imaginary cube roots of unity, prove that $(a + 1)^{-1} + (b + 1)^{-1} + (c + 1)^{-1} + (d + 1)^{-1} = 2$

Sol. ω and ω' are the imaginary cube roots of unity,

i.e., $\omega = \frac{-1 + i\sqrt{3}}{2}$ and $\omega' = \frac{-1 - i\sqrt{3}}{2}$

$\therefore \omega + \omega' = -1$ and $\omega\omega' = 1$

$\Rightarrow$ ω and ω' are the roots of equation.

$(a + x)^{-1} + (b + x)^{-1} + (c + x)^{-1} + (d + x)^{-1} = 2x^{-1}$ ...(i)

$$\Rightarrow 2x^4 + \sum ax^3 + 0.\, x^2 - \sum abc\, x - 2\, abcd = 0$$

Let the other roots of the equation are α and β then

$$\alpha + \beta + \omega + \omega' = -\frac{\sum a}{2}$$

$$\Rightarrow \alpha + \beta - 1 = -\frac{\sum a}{2}$$

and $\sum \alpha\beta = 0$

$(\alpha + \beta)(\omega + \omega') + \alpha\beta + \omega\omega' = 0$

$\Rightarrow (\alpha + \beta)(-1) + \alpha\beta + 1 = 0$

$\Rightarrow (\alpha - 1)(\beta - 1) = 0$

$\therefore \alpha = 1$ or $\beta = 1$

If a = 1, then $\beta = \frac{\sum a}{2}$ and if β = 1 then $\alpha = \frac{\sum a}{2}$

Hence 1 is the root of the equation (i)

$\therefore (a + 1)^{-1} + (b + 1)^{-1} + (c + 1)^{-1} + (d + 1)^{-1} = 2$ (Proved)

Exercise-1 (Topicwise)

IOTA AND BASIC PROPERTIES

1. If n is a positive integer, then which of the following relations is false
   (a) $i^{4n} = 1$ (b) $i^{4n-1} = i$
   (c) $i^{4n+1} = i$ (d) $i^{-4n} = 1$

2. The real part of $\frac{1}{1-\cos\theta + i\sin\theta}$ is equal to
   (a) 1/4 (b) 1/2 (c) $\tan\theta/2$ (d) $1/1-\cos\theta$

3. Which of the following is equal to $\sqrt[3]{-1}$?
   (a) $\frac{\sqrt{3}+\sqrt{-1}}{2}$ (b) $\frac{-\sqrt{3}+\sqrt{-1}}{\sqrt{-4}}$
   (c) $\frac{\sqrt{3}-\sqrt{-1}}{\sqrt{-4}}$ (d) $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$

4. The least value of $n (n \in N)$, for which $\left(\frac{1+i}{1-i}\right)^n$ is real, is
   (a) 1 (b) 2 (c) 3 (d) 4

REPRESENTATION OF COMPLEX NUMBER

5. If $(-7-24i)^{1/2} = x - iy$, then $x^2 + y^2 =$
   (a) 15 (b) 25 (c) -25 (d) -15

6. The square root of $3 - 4i$ is
   (a) $\pm(2+i)$ (b) $\pm(2-i)$
   (c) $\pm(1-2i)$ (d) $\pm(1+2i)$

7. If $(a+ib)^5 = \alpha + i\beta$, then $(b+ia)^5$ is equal to
   (a) $\beta + i\alpha$ (b) $\alpha - i\beta$
   (c) $\beta - i\alpha$ (d) $-\alpha - i\beta$

8. If $(2+i)(2+2i)(2+3i) \ldots (2+ni) = x + iy$, then the value of $5.\ 8.\ 13. \ldots (4+n^2)$
   (a) (x^2+y^2) (b) $\sqrt{(x^2+y^2)}$
   (c) $2(x^2+y^2)$ (d) $(x+y)$

9. If $z = x + iy$ and $z^{1/3} = a - ib$ then $\frac{x}{a} - \frac{y}{b} = k\,(a^2 - b^2)$ where $k =$
   (a) 1 (b) 2 (c) 3 (d) 4

ALGEBRAIC OPERATION, MODULUS ARGUMENT AND CONJUGATE

10. The conjugate of $\frac{(2+i)^2}{3+i}$, in the form of $a + ib$, is
    (a) $\frac{13}{2} + i\left(\frac{15}{2}\right)$ (b) $\frac{13}{10} + i\left(\frac{-15}{2}\right)$
    (c) $\frac{13}{10} + i\left(\frac{-9}{10}\right)$ (d) $\frac{13}{10} + i\left(\frac{9}{10}\right)$

11. If z_1 and z_2 are two complex numbers satisfying the equation $\left|\frac{z_1+z_2}{z_1-z_2}\right| = 1$, then $\frac{z_1}{z_2}$ is a number which is
    (a) Positive real
    (b) Negative real
    (c) Zero or purely imaginary
    (d) Both (a) and (b)

12. Let z_1 be a complex number with $|z_1| = 1$ and z_2 be any complex number, then $\left|\frac{z_1 - z_2}{1 - z_1\bar{z}_2}\right| =$
    (a) 0 (b) 1 (c) -1 (d) 2

13. The argument of the complex number $\frac{13-5i}{4-9i}$ is
    (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{5}$ (d) $\frac{\pi}{6}$

14. If z and ω are two non–zero complex numbers such that $|z\omega| = 1$ and $\arg(z) - \arg(\omega) = \frac{\pi}{2}$, then $\bar{z}\omega$ is equal to
    (a) 1 (b) -1 (c) i (d) $-i$

15. If $a^2 + b^2 = 1$, then $\frac{1+b+ia}{1+b-ia} =$
    (a) 1 (b) 2 (c) $b + ia$ (d) $a + ib$

16. Total number of complex numbers 'z', satisfying $\text{Re}(z^2) = 0$, $|z| = \sqrt{3}$, is equal to
    (a) 2 (b) 4 (c) 6 (d) 8

17. The number of solutions of the system of equations $\text{Re}(z^2) = 0$, $|z| = 2$ is
    (a) 4 (b) 3 (c) 2 (d) 1

18. If $|z_1| = |z_2|$ and $\arg(z_1/z_2) = \pi$, then $z_1 + z_2$ is equal to
    (a) 1 (b) 3 (c) 0 (d) 2

19. If z_1 and $\bar{z}_1$ represent adjacent vertices of a regular polygon of n sides with centre at the origin and if $\frac{\text{Im}\, z_1}{\text{Re}\, z_1} = \sqrt{2} - 1$ then the value of n is equal to
    (a) 8 (b) 12 (c) 16 (d) 24

20. If $|z| \le 4$, then the maximum value of $|iz + 3 - 4i|$ is equal to
    (a) 2 (b) 4 (c) 3 (d) 9

21. The minimum value of $|z - 1 + 2i| + |4i - 3 - z|$ is
    (a) $\sqrt{5}$ (b) 5 (c) $2\sqrt{13}$ (d) $\sqrt{15}$

22. If $(1+i)z = (1-i)\bar{z}$ then z is

(a) $t(1-i), t \in R$ (b) $t(1+i), t \in R$

(c) $\frac{t}{1+i}, t \in R^+$ (d) $\frac{t}{1-i}, t \in R^+$

23. If z_1 and z_2 are two non-zero complex numbers such that $\left|\frac{z_1}{z_2}\right| = 2$ and $\arg(z_1 z_2) = \frac{3\pi}{2}$, then $\frac{\bar{z}_1}{z_2}$ is equal to

(a) 2 (b) -2 (c) $-2i$ (d) $2i$

24. The inequality $|z-4| < |z-2|$ represents:

(a) $\text{Re}(z) > 0$ (b) $\text{Re}(z) < 0$

(c) $\text{Re}(z) > 2$ (d) $\text{Re}(z) > 3$

25. If $|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$, then

(a) $\text{Im}(z) = 2$ (b) $\text{Re}(z) = 2$

(c) $\text{Re}(z) + \text{Im}(z) = 2$ (d) None of these

26. If $|z-i| \le 2$ and $z_0 = 5+3i$ then the maximum value of $||iz + z_0||$ is

(a) $2+\sqrt{31}$ (b) 7 (c) $\sqrt{31}-2$ (d) 2

27. If amp $(z_1 z_2) = 0$ and $|z_1| = |z_2| = 1$, then

(a) $z_1 + z_2 = 0$ (b) $z_1 z_2 \ne 1$

(c) $z_1 = \bar{z}_2$ (d) $z_1 = z_2$

28. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ ($a, b, c, d \in R$) has 4 non real roots, two with sum $3+4i$ and the other two with product $13+i$.

(a) $b = 51$ (b) $a = -5$

(c) $c = -60$ (d) $d = 160$

29. The number of complex numbers satisfying $|z+2| + |z-2| = 8$ and $|z-1| + |z+1| = 2$ is

(a) 4 (b) 2 (c) 0 (d) 6

30. If z_1, z_2, z_3 are 3 distinct complex numbers such that $\frac{3}{|z_2 - z_3|} = \frac{4}{|z_3 - z_1|} = \frac{5}{|z_1 - z_2|}$, then the value of $\frac{9}{z_2 - z_3} + \frac{16}{z_3 - z_1} + \frac{25}{z_1 - z_2}$ equals

(a) 4 (b) 0 (c) 1 (d) 6

DE MOIVER'S THEOREM & ROOTS OF UNITY

31. If $1, \omega, \omega^2$ are the cube roots of unity, then

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$$ is equal to:

(a) 0 (b) 1 (c) ω (d) ω^2

32. The complex number ω satisfying the equation $\omega^3 = 8i$ and lying in the second quadrant on the complex plane is

(a) $-\sqrt{3}+i$ (b) $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$

(c) $-2\sqrt{3}+i$ (d) $-\sqrt{3}+2i$

33. Let z and ω be two non-zero complex numbers such that $|z| = |\omega|$ and $\arg z = \pi - \arg \omega$, then z equals

(a) ω (b) $-\omega$

(c) $\bar{\omega}$ (d) $-\bar{\omega}$

34. Let $A(z_1)$, $A_2(\bar{z}_1)$ are the adjacent vertices of a regular polygon. If $\frac{\text{Im}(\bar{z}_1)}{\text{Re}(z_1)} = 1-\sqrt{2}$, then number of sides of the polygon is equal to

(a) 6 (b) 8

(c) 12 (d) 16

35. If $x_n = \cos\frac{\pi}{2^n} + i\sin\frac{\pi}{2^n}$, then $x_1 x_2 x_3 \ldots\ldots$ (to infinity) is equal to

(a) 0 (b) 1

(c) -1 (d) ∞

36. If $\alpha = \cos\left(\frac{8\pi}{11}\right) + i\sin\left(\frac{8\pi}{11}\right)$ then $\text{Re}(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) 0 (d) 1

37. If $x^2 - x + 1 = 0$, $\sum_{n=1}^{5}\left(x^n + \frac{1}{x^n}\right)^2$ equal

(a) 8 (b) 10

(c) 12 (d) 14

38. If Z_r; r = 1, 2, 3,..., 50 are the roots of the equation $\sum_{r=0}^{50}(Z)^r = 0$, then find the value of $\sum_{r=1}^{50}\frac{1}{Z_r - 1}$?

(a) -25 (b) 25

(c) 50 (d) -50

39. Find the sum of the series. $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta$ for $0 < \theta < 2\pi$.

(a) $\frac{\cos\left(\frac{n\theta}{2}\right)\cos\left(\frac{(n-1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$ (b) $\frac{\cos\left(\frac{n\theta}{2}\right)\cos\left(\frac{(n+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$

(c) $\frac{\sin\left(\frac{n\theta}{2}\right)\cos\left(\frac{(n+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$ (d) $\frac{\sin\left(\frac{n\theta}{2}\right)\cos\left(\frac{(n-1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$

40. Let $z = \left(\frac{\sqrt{3}}{2}\right) - \left(\frac{i}{2}\right)$. Then the smallest positive integer n such that $(z^{95} + i^{67})^{94} = z^n$ is

(a) 30 (b) 20

(c) 50 (d) 10

Exercise-2 (Learning Plus)

1. If $z = (3 + 7i)(p + iq)$, where $p, q \in I - \{0\}$, is purely imaginary, then minimum value of $|z|^2$ is

(a) 0 (b) 58

(c) $\frac{3364}{3}$ (d) 3364

2. Let z is complex number satisfying the equation $z^2 - (3+i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root.

The additive inverse of non real root, is

(a) $1 - i$ (b) $1 + i$ (c) $-1 - i$ (d) -2

3. In G.P. the first term & common ratio are both $\frac{1}{2}(\sqrt{3}+i)$, then the modulus of nth term is:

(a) 1 (b) 2^n (c) 4^n (d) 3^n

4. The set of values of $a \in R$ for which $x^2 + i(a-1)x + 5 = 0$ will have a pair of conjugate imaginary roots is

(a) R (b) $\{1\}$

(c) $\{a : a^2 - 2a + 21 > 0\}$ (d) $\{0\}$

5. If $z = x + iy$ satisfies amp $(z-1)$ = amp $(z+3)$ then the value of $(x-1) : y$ is equal to

(a) 2 : 1 (b) 1 : 3

(b) –1 : 3 (d) does not exist

6. The system of equations $\left.\begin{array}{ll} |z+1-i| & = 2 \\ \text{Re}(z) & \geq 1 \end{array}\right\}$, where z is a complex number has :

(a) no solution (b) exactly one solution

(c) two distinct solutions (d) infinite solution

7. If $|z| = 1$ and $\omega = \frac{z-1}{z+1}$ (where $z \neq -1$), the Re(ω) is

(a) 0 (b) $-\frac{1}{|z+1|^2}$

(c) $\left|\frac{z}{z+1}\right| \cdot \frac{1}{|z+1|^2}$ (d) $\frac{\sqrt{2}}{|z+1|^2}$

8. The expression $\left(\frac{1+i\tan\alpha}{1-i\tan\alpha}\right)^n - \frac{1+i\tan n\alpha}{1-i\tan n\alpha}$ when simpli-fied reduces to:

(a) zero (b) $2\sin n\alpha$

(c) $2\cos n\alpha$ (d) $\sin n\alpha$

9. If $\alpha = -1 + i\sqrt{3}$ and n is a positive integer which is not a multiple of 3, then $\alpha^{2n} + 2^n \alpha^n + 2^{2n} =$

(a) 1 (b) –1 (c) 0 (d) α^2

10. Let $z_1 \neq z_2$ are two points in the argand plane. If $a|z_1| = b|z_2|$; then point $\frac{az_1 - bz_2}{az_1 + bz_2}$ lies

(a) in 1st quadrant (b) in 2nd quadrant

(c) on real axis (d) on Imaginary axis

11. If $x^6 = (4 - 3i)^5$, then the product of all of its roots $\left(\text{where } \theta = -\tan^{-1}\left(\frac{3}{4}\right)\right)$

(a) $5^5(\cos 5\theta + i \sin 5\theta)$ (b) $-5^5(\cos 5\theta + i \sin 5\theta)$

(c) $5^5(\cos 5\theta - i \sin 5\theta)$ (d) $-5^5(\cos 5\theta - i \sin 5\theta)$

12. If ω is an imaginary x^n root of unit then $\sum_{r-1}^{n}(ar+b)\omega^{r-1}$ is

(a) $\frac{n(n+1)a}{2\omega}$ (b) $\frac{nb}{1-n}$

(c) $\frac{na}{\omega - 1}$ (d) $\frac{na}{1-n}$

13. The least positive integer n for which

$\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}\sin^{-1}\left(\frac{1+x^2}{2x}\right), (x \geq 0)$ is

(a) 0 (b) 2 (c) 4 (d) 8

14. If $|z| = 1$ and $z - 1 = ik(z + 1)$, then find range of k?

(a) $(-\infty, \infty)$ (b) $(-\infty, 0]$ (c) $[0, \infty)$ (d) $\{0\}$

15. Find the area bounded by the curves Arg $z = \pi/3$, Arg $z = 2\pi/3$ and Arg $(z - 2 - 2\sqrt{3}i) = \pi$ on the complex plane.

(a) $8\sqrt{3}$ (b) $4\sqrt{3}$ (c) $2\sqrt{3}$ (d) $\sqrt{3}$

16. Given f(z)= the real part of a complex number z. For example, f(3 – 4i) = 3. If a ∈ N, n ∈ N then find the value of $\sum_{n=1}^{6a}\log_2\left|f\left(1+i\sqrt{3}\right)^n\right|$?

(a) $18a^2 - a$ (b) $18 - a^2$

(c) $18a - a^2$ (d) $18a^2 + a$

17. A root of unity is a complex number that is a solution to the equation $z^n = 1$ for some positive integer n. Find the Number of roots of unity that are also the roots of the equation $z^2 + az + b = 0$, for some integer a and b?

(a) 8 (b) 2 (c) 4 (d) 0

18. If $|z + 1| + |z-3| = 6$ and $|z| = |z-2|$, then find z?

(a) $1 \pm \sqrt{5}i$ (b) $\sqrt{5} \pm i$ (c) $\sqrt{3} \pm i$ (d) $1 \pm \sqrt{3}i$

19. If α is non real and $\alpha = \sqrt[5]{1}$ then the value of $2^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$ is equal to

(a) 4 (b) 2 (c) 1 (d) 8

20. If $x = 9^{1/3}9^{1/9}9^{1/27} \ldots\ldots\infty$, $y = 4^{1/3}4^{1/9}4^{1/27} \ldots\ldots \infty$, and $z = \sum_{r=1}^{\infty}(1+i)^{-r}$, then arg $(x + yz)$ is equal to

(a) 0 (b) $\pi - \tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$

(c) $-\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$ (d) $-\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

21. If z_1, z_2, z_3 are three complex numbers and $A = \begin{vmatrix} \arg z_1 & \arg z_2 & \arg z_3 \\ \arg z_2 & \arg z_3 & \arg z_1 \\ \arg z_3 & \arg z_1 & \arg z_2 \end{vmatrix}$ then A is divisible by

(a) $\arg(z_1 + z_2 + z_3)$ (b) $\arg(z_1 z_2 z_3)$
(c) all numbers (d) cannot say

22. The polynomial $x^6 + 4x^5 + 3x^4 + 2x^3 + x + 1$ is divisible by
(a) $x + \omega$ (b) $x + \omega^2$
(c) $(x + \omega)(x + w^2)$ (d) $(x - \omega)(x - \omega^2)$
Where ω is one of the imaginary cube roots of unity.

23. If α is the nth root of unity, then $1 + 2\alpha + 2\alpha^2 + \ldots$ to n terms equal to

(a) $\frac{-n}{(1-\alpha)^2}$ (b) $\frac{-n}{1-\alpha}$ (c) $\frac{-2n}{1-\alpha}$ (d) $\frac{-2n}{(1-\alpha)^2}$

24. Dividing $f(z)$ by $z - i$, we obtain the remainder $1 - i$ and dividing it by it by $z + i$ we get the remainder $1 + i$. The remainder upon the division of $f(z)$ by $z^2 + 1$ is
(a) $i + z$ (b) $1 + z$ (c) $1 - z$ (d) $i - z$

25. If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ then

(a) $\frac{z_1}{z_2}$ is purely real (b) $\frac{z_1}{z_2}$ is purely imaginary

(c) $z_1\bar{z}_2 + z_2\bar{z}_1 = 0$ (d) amp $\frac{z_1}{z_2}$ maybe equal to $\frac{\pi}{2}$

26. Let $i = \sqrt{-1}$. Define a sequence of complex number by $z_1 = 0, z_{n+1} = z_n^2 + i$ for $n \geq 1$. Then which of the following are true.
(a) $|z_{2050}| = \sqrt{3}$ (b) $|z_{2017}| = \sqrt{2}$
(c) $|z_{2016}| = 1$ (d) $|z_{2111}| = \sqrt{2}$

27. a, b, c are real numbers in the polynomial, $P(z) = 2z^4 + az^3 + bz^2 + cz + 3$. If two roots of the equation $P(z) = 0$ are 2 and i. Then which of the following are true.

(a) $a = -\frac{11}{2}$ (b) $b = 5$ (c) $c = -\frac{11}{2}$ (d) $a = -11$

28. If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\varphi = y + \frac{1}{y}$, then

(a) $x^n + \frac{1}{x^n} = 2\cos(n\theta), n \in z$

(b) $\frac{x}{y} + \frac{y}{x} = 2\cos(\theta - \varphi)$

(c) $xy + \frac{1}{xy} = 2\cos(\theta + \varphi)$

(d) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n), m, n \in z$

29. If $a^3 + b^3 + 6abc = 8c^3$ & ω is a cube root of unity then :
(a) a, c, b are in $A.P.$ (b) a, c, b are in $H.P.$
(c) $a + b\omega - 2c\omega^2 = 0$ (d) $a + b\omega^2 - 2c\omega = 0$

30. If $|z_1 + z_2| = |z_1 - z_2|$ then

(a) $|\text{amp } z_1 - \text{amp } z_2| = \frac{\pi}{2}$ (b) $|\text{amp } z_1 - \text{amp}_2| = \pi$

(c) $\frac{z_1}{z_2}$ is purely real (d) $\frac{z_1}{z_2}$ is purely imaginary

31. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2| = 1$, if z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1 + z_2}{z_1 - z_2}$ may be

(a) zero (b) Real and positive
(c) Real and negative (d) Purely imaginary

32. If z is a complex number then the equation $z^2 + z|z| + |z^2| = 0$ is satisfied by (ω and ω^2 are imaginary cube roots of unity)
(a) $z = k\omega$ where $k \in R$
(b) $z = k\omega^2$ where k is non negative real
(c) $z = k\omega$ where k is positive real
(d) $z = k\omega^2$ where $k \in R$

33. Let z represents complex number satisfying $|z - 2| = 1$, then

(a) Least value of $\arg(z)$ is $\tan^{-1}\frac{1}{\sqrt{3}}$

(b) Maximum value of $|z|$ is 3

(c) Maximum value $\arg(z)$ is π-$\tan^{-1}\frac{1}{\sqrt{3}}$

(d) Least value of $|z|$ is 1

Comprehension (Q. No. 34 to 36)

Let a, b and c be three real numbers satisfying

$$[a\ b\ c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0]$$

34. If the point $P(a, b, c)$, with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$, is
(a) 0 (b) 12
(c) 7 (d) 6

35. Let ω be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If $a = 2$ with b and c satisfying (E), then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to
(a) -2 (b) 2
(c) 3 (d) -3

36. Let $b = 6$, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$, is

(a) 6 (b) 7 (c) 6/7 (d) ∞

Comprehension (Q. No. 37 to 38)

Let $p = x + y + z + a(y + z - 2x)$

$q = x + y + z + a(z + x - 2y)$

$r = x + y + z + a(x + y - 2z)$

37. $p^2 + q^2 + r^2 - pq - qr - rp$ is equal to

(a) $a^2(x^2 + y^2 + z^2 - xy - yz - zx)$

(b) $3a^2(x^2 + y^2 + z^2 - xy - yz - zx)$

(c) $9a^2(x^2 + y^2 + z^2 - xy - yz - zx)$

(d) 0

38. $p^3 + q^3 + r^3 - 3pqr$ is equal 10

(a) $3a^2(x^3 + y^3 + z^3 - 3xyz)$

(b) $9a^2(x^3 + y^3 + z^3 - 3xyz)$

(c) $27a^2(x^3 + y^3 + z^3 - 3xyz)$

(d) 0

39. Match the items in Column-I with those in Column-II

	Column-I		Column-II
A.	If $\lvert z-(3+4i)\rvert = 2$ then the maximum value of $\lvert z\rvert$ is	p.	10
B.	If $\lvert z - i\rvert < 1$, then the value of $\lvert z + 12 - 6i\rvert$ is less than	q.	14
C.	If $\lvert z_1\rvert = 1$ and $\lvert z_2\rvert = 2$, then $\lvert z_1 + z_2\rvert^2 + \lvert z_1 - z_2\rvert^2$ is equal to	r.	1
D.	If $z = 1 + i$, then $4(z^4 - 4z^3 + 7z^2 - 6z + 3)$ is equal to	s.	7
		t.	4

Choose the correct answer from the options given below

(a) A → s; B → p; C → q; D → r

(b) A → s; B → q; C → p; D → t

(c) A → q; B → t; C → r; D → s

(d) A → t; B → p; C → r; D → q

40. If z is any complex number satisfying $|z - k| = k$, where k is positive real number, then which of the following is correct?

	Column-I		Column-II
A.	$\lvert z\rvert_{max}$	p.	0
B.	$\lvert z\rvert_{min}$	q.	$k+\sqrt{k^2+1}$
C.	$\lvert z-i\rvert_{max}$	r	$\sqrt{k^2+1}-k$
D.	$\lvert z-i\rvert_{min}$	s.	$2k$

(a) A → s; B → p; C → q; D → r

(b) A → r; B → s; C → q; D → p

(c) A → q; B → p; C → r; D → s

(d) A → s; B → p; C → r; D → q

41. Find the number of distinct complex numbers z such that | = 1 and z^3 is purely imaginary.

42. If a and b are positive integer such that $N = (a + ib)^3 - 10$ is a positive integer then find the value of $\frac{N}{2}$

43. Let z is a complex number satisfying the equatior $z^3 - (3 + i)z + m + 2i = 0$, where $m \in R$. Suppos the equation has a real root α, then find the value o $\alpha^4 + m^4$

44. Let $F(z) = \frac{z+i}{z-i}$ for all complex numbers $z \neq i$ and z_n = $F(z_{n-1})$. Given that $z_0 = \frac{1}{37} + i$ and $z_{2020} = a + ib$, where a $b \in R$, find $a + b$.

45. For how many positive integers 'n' less than or equal to 100(is $(\sin t + i\cos t)^n = \sin(nt) + i\cos(nt)$ true for all real t?

46. The $z^{10} + (13z - 1)^{10} = 0$ has 10 complex roots $z_1, \bar{z}_1, z_2, \bar{z}_2$,....,$z_5, \bar{z}_5$. The the value of

$\frac{1}{|z_1|^2} + \frac{1}{|z_2|^2} + \frac{1}{|z_3|^2} + \frac{1}{|z_4|^2} + \frac{1}{|z_5|^2}$ is equal to

47. If z_1, z_2 are complex numbers such that $|z_1| = \frac{5}{3}$, then $\left|\frac{75 - 27z_1\bar{z}_2}{z_1 - z_2}\right|$ is equal to

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. Let z_1, z_2, z_3 be three points on $|z| = 1$ Let θ_1 θ_2, and θ_3 be the argumets of z_1, z_2, z_3 then $\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_2)$

(a) $\geq -\frac{3}{2}$ (b) $\leq -\frac{3}{2}$ (c) $\geq \frac{3}{2}$ (d) $\frac{3}{2}$

2. $\text{Sin}^{-1}\left[\frac{(z-1)}{i}\right]$, z is non real, can be the angle of triangle if

(a) $\text{Re}(z) = 1, \text{lm}(z) = 2$

(b) $\text{Re}(z) = 1, -1 \leq \text{lm}(z) \leq 1$

(c) $\text{Re}(z) + \text{lm}(z) = 0$

(d) $\text{Re}(z) = \text{lm}(z)$

3. If $z = x + iy(x, y \in R, x \neq -1/2)$, the number of values of z satisfying $|z|^n = z^2|z|^{n-2} + z|z|^{n-2} + 1$. $(n \in N, n > 1)$ is

(a) 0 (b) 1 (c) 2 (d) 3

4. If α, β be the roots of the equation $u^2 - 2u + 2 = 0$ and if $\cot\theta = x + 1$, then $[(x+\alpha)^n - (x+\beta)^n]/[\alpha - \beta]$ is equal to

(a) $\frac{\sin n\theta}{\sin^n\theta}$ (b) $\frac{\cos n\theta}{\cos^n\theta}$ (c) $\frac{\sin n\theta}{\cos^n\theta}$ (d) $\frac{\cos n\theta}{\sin^n\theta}$

5. If $\arg\left(\frac{z_1 - \frac{z}{|z|}}{\frac{z}{|z|}}\right) = \frac{\pi}{2}$ and $\left|\frac{z}{|z|} - z_1\right| = 3$ then $|z_1|$ equals to

(a) $\sqrt{26}$ (b) $\sqrt{10}$ (c) $\sqrt{3}$ (d) $2\sqrt{2}$

6. If 'p' and 'q' are distinct prime numbers, then the number of distinct imaginary numbers which are p^{th} as well as q^{th} roots of unity are

(a) Min (p, q) (b) Max (p, q)

(c) 1 (d) Zero

7. Consider $az^2 + bz + c = 0$, where $a, b, c \in R$ and $4ac > b^2$. If z_1 and z_2 are the roots of the equation given above, then which of the following complex number is purely real?

(a) $z_1\bar{z}_2$ (b) $\bar{z}_1 z_2$ (c) $z_1 - z_2$ (d) $(z_1 - z_2)i$

8. Let $\alpha + i\beta$; $\alpha, \beta \in R$, be a root of the equation $x^3 + qx + r = 0$; $q, r \in R$. Find a real cubic equation, independent of α and β, whose one root is 2α.

(a) $x^3 + qx - r = 0$ (b) $x^3 - qx + r = 0$

(c) $x^3 + qx + r = 0$ (d) $x^3 - qx - r = 0$

9. Let a,b,c be distinct complex numbers such that $\frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k$. Find the value of k.

(a) $-\omega, -\omega^2$ (b) $-1, \omega$ (c) $\omega^2, 1$ (d) ω, ω^2

10. Use complex numbers to find the sum $\sum_{r=0}^{n-1}\cos^2\left(\alpha + \frac{r\pi}{n}\right) = \frac{n}{2}$ where $n \in N$; $n \geq 2$.

(a) n/2 (b) n (c) 2n (d) 2n–1

11. The value of $i^n + i^{-n}$, for $i = \sqrt{-1}$ and $n \in I$ is:

(a) $\frac{2^n}{(1-i)^{2n}} + \frac{(1+i)^{2n}}{2^n}$ (b) $\frac{(1+i)^{2n}}{2^n} + \frac{(1-i)^{2n}}{2^n}$

(c) $\frac{(1+i)^{2n}}{2^n} + \frac{2^n}{(1-i)^{2n}}$ (d) $\frac{2^n}{(1+i)^{2n}} + \frac{2^n}{(1-i)^{2n}}$

12. For complex numbes z and w, if $|z|^2 w - |w|^2 z = z - w$. Which of the following are true:

(a) $z = w$ (b) $z\bar{w} = 1$ (c) $z = w + 2$ (d) $\bar{z}w = 1$

13. If z is a complex number then the equation $z^2 + z|z| + |z^2| = 0$ is satisfied by (ω and ω^2 are imaginary cube roots of unity)

(a) $z = k\omega$ where $k \in R$

(b) $z = k\omega^2$ where k is non negative real

(c) $z = k\omega$ where k is positive real

(d) $z = k\omega^2$ where $k \in R$.

14. If z is a complex number which simultaneously satisfies the equations $3|z - 12| = 5|z - 8i|$ and $|z - 4| = |z - 8|$ then the Im(z) can be

(a) 15 (b) 16 (c) 17 (d) 8

15. If z_1, z_2, z_3, z_4 are root of the equation $a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$; where a_0, a_1, a_2, a_3 and a_4 are real, then

(a) $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$ are also roots of the equation

(b) z_1 is equal to at least one of $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$

(c) $-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -\bar{z}_4$ are also roots of the equation

(d) None of these

16. If $z_1 = a + ib$ and $z_2 = c + id$ are complex number such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1\bar{z}_2) = 0$, then pair of complex numbers $\omega_1 + a + ic$ and $\omega_2 = b + id$ satisfy

(a) $|\omega_1| = 1$ (b) $|\omega_2| = 1$

(c) $\text{Re}(\omega_1\bar{\omega}_2) = 0 = 0$ (d) $|\omega_2| = 2$

17. If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are complex numbers such that $|z_1| = 1$, $|z_2| = 2$ and $\text{Re}(z_1z_2) = 0$, then the pair of complex numbers $\omega_1 = a_1 + \frac{ia_2}{2}$ and $\omega_2 = 2b_1 + ib_2$ satisfy

(a) $|\omega_1| = 1$ (b) $|\omega_2| = 2$

(c) $\text{Re}(\omega_1\omega_2) = 0$ (d) $\text{Im}(\omega_1\omega_2) = 2$

18. Let a,b,c be distinct non zero complex numbers with $|a| = |b| = |c|$ if each of equations $az^2 + bz + c = 0$ and $bz^2 + cz + a = 0$ has a root having modulus 1, then

(a) $b^2 = ac$ (b) $c^2 = ab$

(c) $|a - b| = |b - c| = |c - a|$ (d) $a^2 = bc$

19. If $1, z_1, z_2, z_3, \ldots, z_{n-1}$, are '$n$' n^{th} roots of unity, then the value of $\frac{1}{3-z_1} + \frac{1}{3-z_2} + \frac{1}{3-z_3} + \ldots + \frac{1}{3-z_{n-1}}$ is equal to

(a) $\frac{\sum_{r=1}^{n} r\,3^{r-1}}{\sum_{r=1}^{n} 3^{r-1}}$ (b) $\frac{\sum_{r=1}^{n-1} r\,3^{r-1}}{\sum_{r=1}^{n} 3^{r-1}}$

(c) $\frac{n\,3^{n-1}}{3^n - 1} - \frac{1}{2}$ (d) $\frac{n3^n}{3^n - 1} - \frac{1}{2}$

20. If z is any complex number satisfying $|z - k| = k$, where k is +ve real number, then which of the following is correct?

Column-I	Column-II
(A) 2arg z	(p) $\pi/2$
(B) $\lvert z\rvert_{max}$	(q) arg $(z - k)$
(C) $(\arg z)_{max}$ tends to	(r) arg $(z + k)$
	(s) $2k$

(a) A → q; B → s; C → p

(b) A → r; B → s; C → q

(c) A → q; B → p; C → r

(d) A → s; B → p; C → r

NUMERICAL BASED QUESTIONS

21. Total number of complex numbers 'z', satisfying $\text{Re}(z^2) = 0, |z| =$, is equal to

22. $z_1, z_2 \in C$ and $z_1^2 + z_2^2 \in R$, $z_1(z_1^2 - 3z_2^2) = 2$, $z_2(3z_1^2 - z_2^2) = 11$ If $z_1^2 + z_2^2 = \lambda$ then determine λ^2

23. If z_1 and z_2 both satisfy the relation, $z + \bar{z} = 2|z-1|$ and $\arg(z_1 - z_2) = \frac{\pi}{4}$, then find the imaginary part of $(z_1 + z_2)$.

24. The polynomial $f(z) = az^{2018} + bz^{2017} + cz^{2016}$ has real coefficients not exceeds 2019 and $f\left(\frac{1+\sqrt{3}i}{2}\right) = 2015 + 2019\sqrt{3}i$. Find the remainder when $f(1)$ is divided by 1000.

25. Let $a = \sum_{r=1}^{8} \tan^2\left(\frac{r\pi}{17}\right)$ and $b = \prod_{r=1}^{8} \tan^2\left(\frac{r\pi}{17}\right)$. then $\frac{a}{b}$ is equal to:

Exercise-4 (Past Year Questions)

JEE MAIN

1. A value of θ for which $\frac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$ is purely imaginary, is **(2019)**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$

(c) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (d) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

2. Let $A = \left\{\theta \in \left(-\frac{\pi}{2}, \pi\right) : \frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} \text{ purely imaginary}\right\}$. Then the sum of the elements in A is: **(2019)**

(a) $\frac{5\pi}{6}$ (b) π (c) $\frac{3\pi}{4}$ (d) $\frac{2\pi}{3}$

3. Let Z_0 be a root of the quadratic equation, $x^2 + x + 1 = 0$. If $z = 3 + 6iz_0^{81} - 3iz_0^{93}$, then arg z is equal to **(2019)**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) 0 (d) $\frac{\pi}{6}$

4. Let z_1 and z_2 be any two non-zero complex numbers such that $3|z_1| = 4|z_2|$. If $z = \frac{3z_1}{2z_2} + \frac{2z_2}{3z_1}$ then; **(2019)**

(a) $\text{Re}(z) = 0$ (b) $|z| = \sqrt{\frac{5}{2}}$

(c) $|z| = \frac{1}{2}\sqrt{\frac{17}{2}}$ (d) $\text{Im}(z) = 0$

5. Let z be a complex number such that $|z| + z = 3 + i$ (where $i = \sqrt{-1}$). Then $|Z|$ is equal to: **(2019)**

(a) $\frac{\sqrt{34}}{3}$ (b) $\frac{5}{3}$ (c) $\frac{\sqrt{41}}{4}$ (d) $\frac{5}{4}$

6. Let $\left(-2 - \frac{1}{3}i\right)^3 = \frac{x + iy}{27} (i = \sqrt{-1})$, where x and y are real numbers, then $y - x$ equals: **(2019)**

(a) 91 (b) -85 (c) 85 (d) -91

7. If $\frac{z - \alpha}{z + \alpha}$ $(\alpha \in R)$ is a purely imaginary number and $|z| = 2$, then a value of α is: **(2019)**

(a) 2 (b) 1 (c) 1/2 (d) $\sqrt{2}$

8. Let $z \in C$ be such that $|z| < 1$. If $\omega = \frac{5 + 3z}{5(1 - z)}$, then: **(2019)**

(a) $5\,\text{Im}(\omega) < 1$ (b) $4\,\text{Im}(\omega) > 5$

(c) $5\,\text{Re}(\omega) > 1$ (d) $5\,\text{Re}(\omega) > 4$

9. If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$ then: **(2019)**

(a) $\bar{z}w = i$ (b) $\bar{z}w = -i$

(c) $z\bar{w} = \frac{1 - i}{\sqrt{2}}$ (d) $z\bar{w} = \frac{-1 + i}{\sqrt{2}}$

10. If $a > 0$ and $z = \frac{(1+i)^2}{a - i}$, has magnitude $\sqrt{\frac{2}{5}}$, then $\bar{z}$ is equal to: **(2019)**

(a) $-\frac{3}{5} - \frac{1}{5}i$ (b) $-\frac{1}{5} + \frac{3}{5}i$

(c) $-\frac{1}{5} - \frac{3}{5}i$ (d) $\frac{1}{5} - \frac{3}{5}i$

11. Let $z \in C$ with $\text{Im}(z) = 10$ and it satisfies $\frac{2z - n}{2z + n} = 2i - 1$ for some natural number n. Then: **(2019)**

(a) $n = 20$ and $\text{Re}(z) = -10$

(b) $n = 20$ and $\text{Re}(z) = 10$

(c) $n = 40$ and $\text{Re}(z) = -10$

(d) $n = 40$ and $\text{Re}(z) = 10$

12. If $z = \frac{\sqrt{3}}{2} + \frac{i}{2} (i = \sqrt{-1})$, then $(1 + iz + iz^5 + iz^8)^9$ is equal to **(2019)**

(a) -1 (b) 1 (c) 0 (d) $(-1 + 2i)^9$

13. If $\frac{3+i\sin\theta}{4-i\cos\theta}, \theta \in [0, 2\pi]$ is a real number, then an argument of $\sin\theta + i\cos\theta$ is: **(2020)**

(a) $\pi - \tan^{-1}\left(\frac{3}{4}\right)$ (b) $\tan^{-1}\left(\frac{4}{3}\right)$

(c) $\pi - \tan^{-1}\left(\frac{4}{3}\right)$ (d) $-\tan^{-1}\left(\frac{3}{4}\right)$

14. If the equation, $x^2 + bx + 45 = 0$ $(b \in R)$ has conjugate complex roots and they satisfy $|z+1| = 2\sqrt{10}$, then

(a) $b^2 - b = 30$ (b) $b^2 + b = 72$

(c) $b^2 - b = 42$ (d) $b^2 + b = 12$

15. Let $\alpha = \frac{-1+i\sqrt{3}}{2}$ If $a = (1+\alpha)\sum_{k=0}^{100}\alpha^{2k}$ and $b = \sum_{k=0}^{100}\alpha^{3k}$, then a and b are the roots of the quadratic equation: **(2020)**

(a) $x^2 + 101x + 100 = 0$ (b) $x^2 - 102x + 101 = 0$

(c) $x^2 - 101x + 100 = 0$ (d) $x^2 + 102x + 101 = 0$

16. Let z be a complex number such that $\left|\frac{z-i}{z+2i}\right| = 1$ and $|z| = \frac{5}{2}$. Then the value of $|z+3i|$ is: **(2020)**

(a) $\sqrt{10}$ (b) $2\sqrt{3}$ (c) $\frac{7}{2}$ (d) $\frac{15}{4}$

17. If z be a complex number satisfying $|\text{Re}(z)| + |\text{Im}(z)| = 4$, then $|z|$ cannot be: **(2020)**

(a) $\sqrt{10}$ (b) $\sqrt{8}$ (c) $\sqrt{7}$ (d) $\frac{\sqrt{17}}{2}$

18. The imaginary part of $(3+2\sqrt{-54})^{1/2} - (3-2\sqrt{-54})^{1/2}$ can be: **(2020)**

(a) $\sqrt{6}$ (b) $-\sqrt{6}$ (c) $-2\sqrt{6}$ (d) 6

19. The value of $\left(\frac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$ is: **(2020)**

(a) $-\frac{1}{2}(1-i\sqrt{3})$ (b) $\frac{1}{2}(1-i\sqrt{3})$

(c) $\frac{1}{2}(\sqrt{3}-i)$ (d) $-\frac{1}{2}(\sqrt{3}-i)$

20. If z_1, z_2 are complex numbers such that $\text{Re}(z_1) = |z_1 - 1|$, $\text{Re}(z_2) = |z_2 - 1|$, and $\arg(z_1 - z_2) = \frac{\pi}{6}$, then $\text{Im}(z_1 + z_2)$ is equal to **(2020)**

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) $2\sqrt{3}$

21. If $\left(\frac{1+i}{1-i}\right)^{m/2} = \left(\frac{1+i}{1-i}\right)^{n/3} = 1\ (m, n \in N)$ then the greatest common divisor of the least values of m and n is_____. **(2020)**

22. If a and b are real numbers such that $(2+\alpha)^4 = a + b\alpha$, where $\alpha = \frac{-1+i\sqrt{3}}{2}$, then $a + b$ is equal to **(2020)**

(a) 33 (b) 9 (c) 24 (d) 57

23. Let $u = \frac{2z+i}{z-ki}$, $z = x + iy$ and $k > 0$. If the curve represented by $\text{Re}(u) + \text{Im}(u) = 1$ intersects the y-axis at the point P and Q where $PQ = 5$, then the value of K is: **(2020)**

(a) 1/2 (b) 3/2 (c) 2 (d) 4

24. The value of $\left(\frac{-1+i\sqrt{3}}{1-i}\right)^{30}$ is : **(2020)**

(a) $-2^{15}i$ (b) -2^{15} (c) $2^{15}i$ (d) 6^5

25. Let $z = x + iy$ be a non-zero complex number such that $z^2 = i|z|^2$, where $i = \sqrt{-1}$, then z lies on the: **(2020)**

(a) line, $y = x$ (b) imaginary axis

(c) real axis (d) line, $y = -x$

26. The region represented by $\{z = x + iy \in C : |z| - \text{Re}(z) \le 1\}$ is also given by the inequality: **(2020)**

(a) $y^2 \le x + \frac{1}{2}$ (b) $y^2 \le 2\left(x + \frac{1}{2}\right)$

(c) $y^2 \ge x + 1$ (d) $y^2 \ge 2(x+1)$

27. If are $\alpha, \beta \in R$ such that $1 - 2i$ (here $i^2 = -1$) is a root of $z^2 + \alpha z + \beta = 0$, then $(\alpha - \beta)$ is equal to: **(2021)**

(a) -7 (b) 7 (c) 3 (d) -3

28. Let a complex number $z, |z| \ne 1$, satisfy $\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \le 2$. Then, the largest value of $|z|$ is equal to **(2021)**

(a) 5 (b) 8 (c) 6 (d) 7

29. The least value of $|z|$ where z is complex number which satisfies the inequality

$\exp\left(\frac{(|z|+3)(|z|-1)}{||z|+1|}\log_e 2\right) \ge \log_{\sqrt{2}}|5\sqrt{7}+9i|, i = \sqrt{-1}$, is equal to: **(2021)**

(a) 8 (b) 3 (c) 2 (d) $\sqrt{5}$

30. If $z \ne 0$ be a complex number such that $\left|z - \frac{1}{z}\right| = 2$, then the maximum $|z|$ value of $|z|$ is **(2022)**

(a) $\sqrt{2}$ (b) 1 (c) $\sqrt{2}-1$ (d) $\sqrt{2}+1$

31. If $z = 2 + 3i$, then $z^5 + (\bar{z})^5$ is equal to

(a) 244 (b) 224 (c) 245 (d) 265

32. Let S be the set of all (α, β), π, $\alpha, \beta < 2\pi$ for which the complex number $\frac{1-i\sin\alpha}{1+2i\sin\alpha\beta}$ is purely imaginary and $\frac{1+i\cos\beta}{1-2i\cos\beta}$ is purely real. Let $z_{\alpha\beta} = \sin 2\alpha + i\cos 2\beta$,

$(\alpha, \beta) \in S$. Then $\sum_{(\alpha,\beta)\in S}\left(iZ_{\alpha\beta}+\frac{1}{i\bar{Z}_{\alpha\beta}}\right)$ is equal to **(2022)**

(a) 3 (b) $3i$ (c) 1 (d) $2-i$

33. Let $S=\{z\in C: z^2+\bar{z}=0\}$. Then, $\sum_{z\in S}(\text{Re}(z)+\text{Im}(z))$ is equal **(2022)**

34. If $z = x + iy$ satisfies $|z| - 2 = 0$ and $|z - i| - |z + 5i| = 0$, then **(2022)**

(a) $x+2y-4=0$ (b) $x^2+y-4=0$
(c) $x+2y+4=0$ (d) $x^2-y+3=0$

35. For n ∈ N, let $S_n\left\{z\in C:|z-3+2i|=\frac{n}{4}\right\}$ and $T_n\left\{z\in C:|z-2+3i|=\frac{1}{n}\right\}$. Then, the number of elements in the set $\{n\in N: S_n\cap T_n=\phi\}$ is **(2022)**

(a) 0 (b) 2 (c) 3 (d) 4

36. Let $S=\{z\in C:|z-2|<1, z(1+i)+\bar{z}(1-i)\le 2\}$. Let $|z-4i|$ attains minimum and maximum values, respectively, at $z_1\in S$ and $z_2\in S$. If $5(|z_1|^2+|z_2|^2)=\alpha+\beta\sqrt{5}$, where α and β are integers, then the value of α + β is equal to **(2022)**

37. Sum of squares of modulus of all the complex numbers z satisfying $\bar{z}=iz^2+z^2-z$ is equal to **(2022)**

38. Let z_1 and z_2 be two complex numbers such that $\bar{z}_1=i\bar{z}_2$ and $\arg\left(\frac{z_1}{\bar{z}_2}\right)=\pi$. Then, **(2022)**

(a) $\arg z_2=\left(\frac{\pi}{4}\right)$ (b) $\arg z_2=-\frac{3\pi}{4}$
(c) $\arg z_1=\frac{\pi}{4}$ (d) $\arg z_1=\frac{3\pi}{4}$

39. Let $A=\{z\in C:1<|z-(1+i)|<2\}$ and $B=\{z\in A:|z-(1-i)|=1\}$. Then, B **(2022)**

(a) is an empty set
(b) contains exactly two elements'
(c) contains exactly three elements
(d) is an infinite set

40. Let $S=\{z=x+iy:|z-1+i|\ge|z|, |z|<2, |z+i|=|z-1|\}$. Then, the set of all values of x, for which $w=2x+iy\in S$ for some $y\in R$, is **(2022)**

(a) $\left(-\sqrt{2},\frac{1}{2\sqrt{2}}\right]$ (b) $\left(-\frac{1}{\sqrt{2}},\frac{1}{4}\right]$
(c) $\left(-\sqrt{2},\frac{1}{2}\right]$ (d) $\left(-\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}}\right]$

41. Let the minimum value of $v=|z|^2+|z-3|^2+|z-6i|^2, z\in C$ is attained at $z=z_0$. Then, $|2z_0^2-z_0^3+3|^2+v_0^2$ is equal to **(2022)**

(a) 1000 (b) 1064 (c) 1105 (d) 1196

42. For $z\in C$ if the minimum value of $\left(|z-3\sqrt{2}|+|z-p\sqrt{2}i|\right)$ is $5\sqrt{2}$, then a value of p is **(2022)**

(a) 3 (b) $\frac{7}{2}$ (c) 4 (d) $\frac{9}{2}$

43. The area of the polygon, whose vertices are the non-real roots of the equation $z = iz^2$ is **(2022)**

(a) $\frac{3\sqrt{3}}{4}$ (b) $\frac{3\sqrt{3}}{2}$ (c) $\frac{3}{2}$ (d) $\frac{3}{4}$

JEE ADVANCED

44. Let z be a complex number such that the imaginary part of z is non zero and $a = z^2 + z + 1$ is real. Then a cannot take the value **(2012)**

(a) -1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

45. Let $w=\frac{\sqrt{3}+i}{2}$ and $P=\{w^n: n=1,2,3,...\}$. Further $H_1=\left\{z\in C:\text{Re } z>\frac{1}{2}\right\}$ and $H_2=\left\{z\in C:\text{Re } z<-\frac{1}{2}\right\}$, where C is the set of all complex numbers. If $z_1\in P\cap H_1, z_2\in P\cap H_2$ and O represents the origin, then $\angle z_1Oz_2=$ **(2013)**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$
(c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

46. Let ω be a complex cube root of unity with $\omega\ne 1$ and $P=[p_{ij}]$ be $n\times n$ matrix with $p_{ij}=\omega^{i+j}$. Then $P^2\ne 0$, when $n=$ **(2013)**

(a) 57 (b) 55
(c) 58 (d) 56

Comprehension (Q. No. 47 to 48)

Let $S=S_1\cap S_2\cap S_3$, where
$S_1=\{z\in C:|z|<4\}$,
$S_2=\left\{z\in C:\text{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right]>0\right\}$ and
$S_3:\{z\in C:\text{Re } z>0\}$. **(2013)**

47. Area of $S=$

(a) $\frac{10\pi}{3}$ (b) $\frac{20\pi}{3}$
(c) $\frac{16\pi}{3}$ (d) $\frac{32\pi}{3}$

48. $\min_{z\in S}|1-3i-z|=$

(a) $\frac{2-\sqrt{3}}{2}$ (b) $\frac{2+\sqrt{3}}{2}$
(c) $\frac{3-\sqrt{3}}{2}$ (d) $\frac{3+\sqrt{3}}{2}$

49. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, \ldots 9$. **(2014)**

	List-I		List-II
P.	For each z_k there exists a z_j such that $z_k \cdot z_j = 1$	1.	True
Q.	There exists a $k \in \{1, 2, \ldots, 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.	2.	False
R.	$\frac{\lvert 1-z_1\rvert\lvert 1-z_2\rvert \ldots \lvert 1-z_9\rvert}{10}$ equals	3.	1
S.	$1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, \ldots 9$.	4.	2

	P	Q	R	S
(a)	1	2	4	3
(b)	2	1	3	4
(c)	1	2	3	4
(d)	2	1	4	3

50. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression $\frac{\sum_{k=1}^{12}\lvert\alpha_{k+1} - \alpha_k\rvert}{\sum_{k=1}^{3}\lvert\alpha_{4k-1} - \alpha_{4k-2}\rvert}$ is **(2015)**

51. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is **(2016)**

52. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is (are) possible value(s) of x? **(2017)**

(a) $1-\sqrt{1+y^2}$ (b) $-1-\sqrt{1-y^2}$

(c) $1+\sqrt{1+y^2}$ (d) $-1+\sqrt{1-y^2}$

53. Let s, t, r be the non-zero complex numbers and L be the set of solutions $z = x + iy$ $(x, y \in \mathbb{R}, i = \sqrt{-1})$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) True? **(2018)**

(a) If L has exactly one element, then $|s| \neq |t|$

(b) If $|s| = |t|$, then L has infinitely many elements

(c) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(d) If L has more than one element, then L has infinitely many elements

54. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are True? **(2020)**

(a) $\left|z + \frac{1}{2}\right| \leq \frac{1}{2}$ for all $z \in S$

(b) $|z| \leq 2$ for all $z \in S$

(c) $\left|z + \frac{1}{2}\right| \geq \frac{1}{2}$ for all $z \in S$

(d) The set S has exactly four elements

55. For a complex number z, let Re(z) denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with Re(z_1) > 0 and Re(z_1) < 0, is ... **(2020)**

56. Let $\theta_1, \theta_2, \ldots, \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + \ldots + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}, z_k = z_{k-1}e^{i\theta_k}$ for $k = 2, 3, \ldots, 10$, where $i = \sqrt{-1}$.

Consider the statements P and Q given below:

$P: |z_2 - z_1| + |z_3 - z_2| + \ldots + |z_{10} - z_9| + |z_1 - z_{10}| \leq 2\pi$

$Q: |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \ldots + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \leq 4\pi$

Then, **(2021)**

(a) P is **TRUE** and Q is **FALSE**

(b) Q is **TRUE** and P is **FALSE**

(c) both P and Q are **TRUE**

(d) both P and Q are **FALSE**

57. Let z be a complex number with non-zero imaginary part. If $\frac{2+3z+4z^2}{2-3z+4z^2}$ is a real number, then the value of $|z|^2$ is_____. **(2022)**

58. Let z denote the complex conjugate of a complex number z and let $i = \sqrt{-1}$ in the set of complex numbers, the number of distinct roots of the equation $\bar{z} - z^2 = i(\bar{z} + z^2)$ is_____ **(2022)**

59. Lei $\bar{z}$ denote the complex conjugate of a complex number z. If z is a non-zero complex number for which both real and imaginary parts of $(\bar{z})^2 + \frac{1}{z^2}$ are integers, then which of the following is/are possible value(s) of $|z|$? **(2022)**

(a) $\left(\frac{43+3\sqrt{205}}{2}\right)^{\frac{1}{4}}$ (b) $\left(\frac{7+\sqrt{33}}{4}\right)^{\frac{1}{4}}$

(c) $\left(\frac{9+\sqrt{65}}{4}\right)^{\frac{1}{4}}$ (d) $\left(\frac{7+\sqrt{13}}{6}\right)^{\frac{1}{4}}$

ANSWER KEY

CONCEPT APPLICATION

1. (*c*) 2. (*b*) 3. (*c*) 4. (*b*) 5. (*b*) 6. (*b*)

7. (*i*) $z=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right); z=\sqrt{2}e^{3\pi/4}$ (*ii*) $z=\sqrt{2}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right); z=\sqrt{2}e^{5\pi/12}$

8. (*d*) 9. (z lies on real axis) 10. (*d*) 11. (*c*) 12. (*b*) 13. (*a*) 14. (*d*) 15. (*d*)

16. $|z|=1$, amp $(z)=-\frac{\pi}{3}$ 17. (*d*) 18. (*c*) 19. (*c*) 20. (*d*) 21. (*a*) 22. (*b*) 23. (*a*) 24. (*b*)

25. (*a*) 26. (*a*) 27. (*b*) 28. (*a*) 29. (*c*) 30. (*a*) 31. (*i*) $e^{-(4n+1).\frac{\pi}{2}}$ (*ii*) $e^{-(8n+1).\frac{\pi}{4}}$ (*iii*) $\frac{1}{2}\ln(2)$.

EXERCISE-1 (TOPICWISE)

1. (*b*) 2. (*b*) 3. (*b*) 4. (*b*) 5. (*b*) 6. (*b*) 7. (*a*) 8. (*a*) 9. (*d*) 10. (*c*)
11. (*c*) 12. (*b*) 13. (*b*) 14. (*d*) 15. (*c*) 16. (*b*) 17. (*a*) 18. (*c*) 19. (*a*) 20. (*d*)
21. (*c*) 22. (*a*) 23. (*d*) 24. (*d*) 25. (*d*) 26. (*b*) 27. (*c*) 28. (*a*) 29. (*c*) 30. (*b*)
31. (*a*) 32. (*a*) 33. (*d*) 34. (*b*) 35. (*c*) 36. (*b*) 37. (*a*) 38. (*a*) 39. (*c*) 40. (*d*)

EXERCISE-2 (LEARNING PLUS)

1. (*d*) 2. (*c*) 3. (*a*) 4. (*b*) 5. (*d*) 6. (*b*) 7. (*a*) 8. (*a*) 9. (*c*) 10. (*d*)
11. (*b*) 12. (*c*) 13. (*c*) 14. (*a*) 15. (*b*) 16. (*a*) 17. (*a*) 18. (*a*) 19. (*a*) 20. (*c*)
21. (*b*) 22. (*d*) 23. (*b*) 24. (*c*) 25. (*b,c,d*) 26. (*b,c,d*) 27. (*a,b,c*) 28. (*a,b,c,d*) 29. (*a,c,d*) 30. (*a,c*)
31. (*a,d*) 32. (*b,c*) 33. (*a,b,c,d*) 34. (*d*) 35. (*a*) 36. (*b*) 37. (*c*) 38. (*c*) 39. (*b*)
40. (*a*) 41. [6] 42. [99] 43. [32] 44. [275] 45. [250] 46. [850] 47. [45]

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (*a*) 2. (*b*) 3. (*b*) 4. (*a*) 5. (*b*) 6. (*d*) 7. (*d*) 8. (*a*) 9. (*a*) 10. (*a*)
11. (*b,d*) 12. (*a,b,d*) 13. (*b,c*) 14. (*c,d*) 15. (*a,b*) 16. (*a,b,c*) 17. (*a,b,c,d*) 18. (*a,b,c,d*) 19. (*a,d*) 20. (*a*)
21. [4] 22. [25] 23. [2] 24. [53] 25. [8]

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Mains

1. (*d*) 2. (*d*) 3. (*a*) 4. (*) 5. (*b*) 6. (*a*) 7. (*a*) 8. (*c*) 9. (*b*) 10. (*c*)
11. (*c*) 12. (*a*) 13. (*c*) 14. (*a*) 15. (*b*) 16. (*c*) 17. (*c*) 18. (*c*) 19. (*d*) 20. (*d*)
21. (*d*) 22. (*b*) 23. (*c*) 24. (*a*) 25. (*a*) 26. (*b*) 27. (*a*) 28. (*d*) 29. (*b*) 30. (*d*)
31. (*a*) 32. (*c*) 33. [0] 34. (*c*) 35. (*) 36. [26] 37. [2] 38. (*c*) 39. (*d*) 40. (*b*)
41. (*b*) 42. (*c*) 43. (*a*)

JEE Advanced

44. (*d*) 45. (*c,d*) 46. (*b,c,d*) 47. (*b*) 48. (*c*) 49. (*c*) 50. [4] 51. [1] 52. (*b,d*) 53. (*a,c,d*)
54. (*b,c*) 55. [8] 56. (*b*) 57. [0.5] 58. [4] 59. (*a*)

CHAPTER

9 Linear Inequalities

INTRODUCTION

The mathematical expressions ($a > b$) or ($a < b$) are used to compare the two real numbers a and b i.e., it implies that 'a is greater than b' or 'a is smaller than b' respectively.

Suppose that if we say that salary of Mr. A is more than salary of Mr. B. Let Mr. B earns Rs. 20,000 per month, and the salary of Mr. A is x, then we can say that $x > 20000$. This is a linear in equation involving one variable.

Similarly, if we say that father's age is 25 years more than the twice of the age of his child, then if father's age is x and child's age is y, then we can say that, $x > 2y + 25$. This is linear in equation in two variables.

RULES FOR SOLVING LINEAR INEQUALITY

The following list shows all the rules you need to know when performing operations on inequalities. Note that although only the < symbol is shown in this list, these same rules apply to any inequality, including >, ≤ and ≥.

- **If $a < b$, then $a + c < b + c$.** Adding the same number to each side of an inequality does not change the direction of the inequality symbol.
- **If $a < b$, then $a - c < b - c$.** Subtracting the same number from each side of an inequality does not change the direction of the inequality symbol.
- **If $a < b$, and if c is a positive number, then $a \cdot c < b \cdot c$.** Multiplying each side of an inequality by a positive number does not change the direction of the inequality symbol.
- **If $a < b$, and if c is positive number, then $\frac{a}{c} < \frac{b}{c}$.** Dividing each side of an inequality by a positive number does not change the direction of the inequality symbol.
- **If $a < b$, and if c is negative number, then $a \cdot c > b \cdot c$.** Multiplying each side of an inequality by a negative number reverses the direction of the inequality symbol.
- **If $a < b$, and if c is negative number, then $\frac{a}{c} > \frac{b}{c}$.** Dividing each side of an inequality by a negative number reverses the direction of the inequality symbol.

INTERVALS

A subset of number line is called an interval and it contains all the real numbers lying between every pair of its elements.

Let $a, b \in R$, then the set

(*i*) $\{x : x \in R, a \le x \le b\}$ is called a closed interval and is denoted by $[a, b]$.

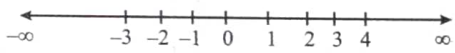
(*ii*) $\{x : x \in R, a < x < b\}$ is called an open interval and is denoted by (a, b), or $]a, b[$.

(*iii*) $\{x : x \in R, a \le x < b\}$ is called left closed and right open interval and is denoted by $[a, b)$ or $[a, b[$.

(*iv*) $\{x : x \in R, a < x \le b\}$ is called left open and right closed interval and is denoted by $(a, b]$ or $]a, b]$.

(*v*) Discrete number are represented by curly { } brackets, *e.g.*, $\{a, b\}$.

NUMBER LINE



The extension of number line is from $-\infty$ to $+\infty$ and any real number can be represented on this number line.

The solution of the in equations are shown on this number line as illustrated in the following examples

Train Your Brain

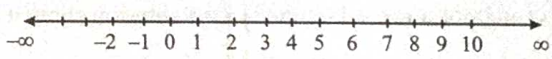
Example 1: Solve $x < 7$ for the following cases

(*a*) $x \in I$ (*b*) $x \in N$

(*c*) $x \in R$

Sol. (*a*) $x \in I$

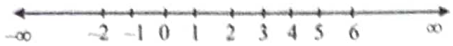
In this case $x = \{...-2, -1, 0, 1, 2, 3, ...6\}$



The infinite solutions are represented by dots

(*b*) $x < 7, x \in N$

In this case $x = \{1, 2, 3, 4, 5, 6\}$



(*c*) $x < 7, x \in R$

The infinite solutions are represented by a dark line as shown. A circle over 7 indicates that point 7 is not included in the solution $x \in (-\infty, 7)$.

Example 2: Solve $30x < 200$ when

(*i*) x is a natural number

(*ii*) x is an integer.

Sol. We are given $30x < 200$ or $\frac{30x}{30} < \frac{200}{30}$, i.e., $x < 20/3$.

(*i*) When x is a natural number, in this case the following values of x make the statement true.

1, 2, 3, 4, 5, 6.

The solution set of the inequality is $\{1,2,3,4,5,6\}$.

(*ii*) When x is an integer, the solutions of the given inequality are ..., –3, –2, –1, 0, 1, 2, 3, 4, 5, 6

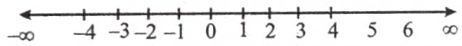
The solution set of the inequality is {..., –3, –2, –1, 0, 1, 2, 3, 4, 5, 6}

Example 3: Show the solution of the linear in equation $-\frac{3}{2} \le x < 5$ on the number line:

(*i*) when $x \in N$ (*ii*) when $x \in I$

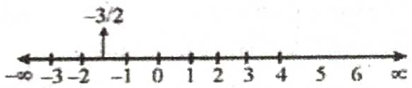
(*iii*) when $x \in R$

Sol. (*i*)



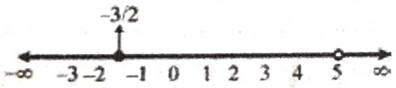
$x = \{1, 2, 3, 4\}$

(*ii*)



$x = \{-1, 0, 1, 2, 3, 4\}$

(*iii*)



$x \in \left[\frac{-3}{2}, 5\right)$

Note: While representing the solution on the number line as an internal.

- If the end point is not included in the solution show it by hole i.e., o.
- If the end point is included in the solution show it by dot i.e.,

Concept Application

1. Solve the following linear in equations for $x \in I$:

   (*a*) $2x - 4 \le 0$ (*b*) $-3x + 12 < 0$

2. Solve the following linear in equations for $x \in R$:

   (*a*) $4x - 12 \ge 0$ (*b*) $7x + 9 > 30$

SOLUTION OF LINEAR IN EQUATIONS

The linear in equations are solved for set of real numbers. The method is to collect all terms involving the variable on one side of the in equation and the constant terms on the other side of the in equation, then use number line system to represent the solution. Consider the following examples, where in every case $x \in R$.

Train Your Brain

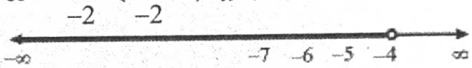
Example 4: Solve $2x - 3 > 4x + 5$.

Sol. $2x - 3 > 4x + 5 \Rightarrow (2x - 3) - 4x > (4x + 5) - 4x$

$\Rightarrow -2x - 3 > 5$

or $(-2x - 3) + 3 > 5 + 3 \Rightarrow -2x > 8$

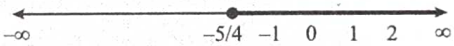
or $\frac{-2x}{-2} < \frac{8}{-2} \Rightarrow x < -4$



Example 5: Solve : $x - 3 \le 5x + 2$.

Sol. Collecting terms involving variables on one side and terms involving constants on other side, we get

$-4x \le 5$ or $x \ge \frac{-5}{4}$ (dividing by –4)



Example 6: Solve $5x - 3 < 3x + 1$ when

(*i*) x is an integer, (*ii*) x is a real number.

Sol. We have, $5x - 3 < 3x + 1$

or $5x - 3 + 3 < 3x + 1 + 3$

or $5x < 3x + 4$

or $5x - 3x < 3x + 4 - 3x$

or $2x < 4$

or $x < 2$

(*i*) When x is an integer, the solutions of the inequality are {..., –4, – 3, – 2, –1, 0, 1}

(*ii*) When x is a real number, the solutions of the inequality are given by $x < 2$, i.e., all real numbers x which are less 2. Therefore, the solution set of the inequality is $x \in (-\infty, 2)$.

We have considered solutions of inequalities in the set of natural number, set of integers and in the set of real numbers. Henceforth, unless stated otherwise, we shall solve the inequalities in this Chapter in the set of real numbers.

Example 7: Solve $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$. Show the solutions on number line.

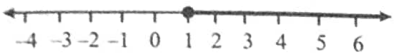
Sol. We have $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$ or $\frac{3x-4}{2} \geq \frac{x-3}{4}$

or $2(3x-4) \geq (x-3)$

or $6x - 8 \geq x - 3$

or $5x \geq 5$ or $x \geq 1$

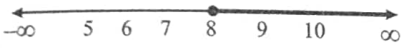
The graphical representation of solutions is given following



Example 8: Solve: $\frac{5-2x}{3} \leq \frac{x}{6} - 5$.

Sol. Multiplying both sides by the *LCM* of 3 and 6, i.e. 6, we get $2(5-2x) \leq x - 30$

$\Rightarrow 10 - 4x \leq x - 30 \Rightarrow 5x \geq 40 \Rightarrow x \geq 8$



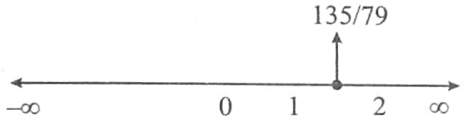
Example 9: Solve: $7 - \left(\frac{3x+5}{2}\right) > \frac{x}{3} + \frac{4x}{5}$.

Sol. Multiplying by 30 on both sides, we get

$210 - 15(3x+5) > 10x + 24x$

$\Rightarrow 210 - 45x - 75 > 34x$

$\Rightarrow 79x < 135$ or $x < \frac{135}{79}$



Example 10: Solve the following in equation : $\frac{1}{x-2} < 0$

Sol. We have, $\frac{1}{x-2} < 0$

$\Rightarrow x - 2 < 0 \Rightarrow x < 2 \Rightarrow x \in (-\infty, 2)$

Hence, the solution set of the given in equation is $(-\infty, 2)$

Example 11: Solve the following in equation : $\frac{x-2}{x+5} > 2$

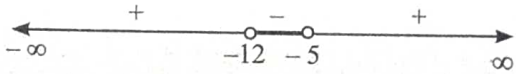
Sol. We have, $\frac{x-2}{x+5} > 2$

$\Rightarrow \frac{x-2}{x+5} - 2 > 0 \Rightarrow \frac{x-2-2(x+5)}{x+5} > 0$

$\Rightarrow \frac{x-2-2x-10}{x+5} > 0 \Rightarrow \frac{-x-12}{x+5} > 0$

$\Rightarrow \frac{x+12}{x+5} < 0$ [Multiplying by -1 to make coefficient of x positive in the expression in numerator] ...(i)

On equating $x + 12$ and $x + 5$ to zero, we obtain $x = -12, -5$ as critical points. These points are on number line as shown in figure. The real line is divided into three regions and the signs of L.H.S of in equation (i) are marked. Since the in equation in (i) possesses less than sign plotted which means that L.H.S of the in equation is negative. So, the solution set of the given in equation is the union of the regions obtaining negative sign in figure. Hence, the solution set of the given in equation is $(-12, -5)$.



Concept Application

3. Solve $4x + 3 < 6x + 7$.
4. Solve $7x + 3 < 5x + 9$. Show the solutions on number line.
5. Solve the following in equations:

(i) $\frac{2x-3}{4} + 9 \geq 3 + \frac{4x}{3}$

(ii) $\frac{5x-2}{3} - \frac{7x-3}{5} > \frac{x}{4}$

(iii) $\frac{1}{2}\left(\frac{3}{5}x + 4\right) \geq \frac{1}{3}(x-6)$

(iv) $\frac{3(x-2)}{5} \geq \frac{5(2-x)}{3}$

(v) $\frac{x+1}{x+2} \geq 1$

SOLUTION OF SYSTEM OF LINEAR IN EQUATIONS IN ONE VARIABLE

System of linear in equations consists of more than one linear in equations. To solve a system, we first solve each linear in equation separately and then look for the common values of the variable satisfying each of the linear in equations.

Train Your Brain

Example 12: Solve the following system of in equations:

$2x - 3 > x + 1, 3x + 5 \geq 2$

Sol. $2x - 3 > x + 1$

$\Rightarrow x > 4$ ... (*i*)

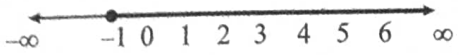
Also, $3x+5 \geq 2$

$\Rightarrow \quad 3x \geq -3$

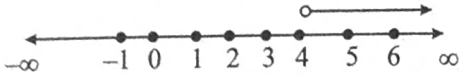
$\Rightarrow \quad x \geq -1$ ... (*ii*)

The common solution of equations (*i*) and (*ii*) is $x > 4$ as required x is such that it is greater than or equal to -1 as well as greater than $4 \Rightarrow x > 4$

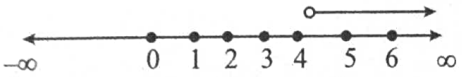
Graphically: $x \geq -1$



$x > 4$



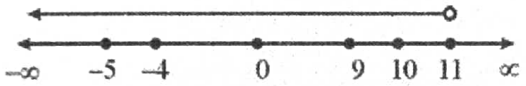
$\Rightarrow x > 4$



Example 13: Solve $5(2x-7)-3(2x+3) \leq 0$; $2x+19 < 6x+47$

Sol. Solving in equation

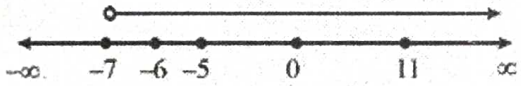
(*i*) $10x-35-6x-9 \leq 0$



$\Rightarrow 4x-44 \leq 0 \Rightarrow x \leq 11$

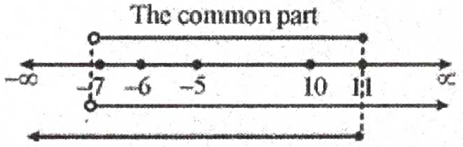
Solving in equation

(*ii*) $2x+19 < 6x+47 \Rightarrow -4x < 28 \Rightarrow x > -7$



The solution is the common values of x such $x \leq 11$ and $x > -7$

$\Rightarrow -7 < x \leq 11$

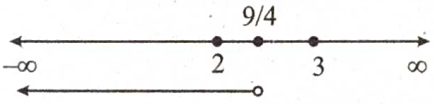


Example 14: Solve: $x+\frac{1}{2} > 3x-4$ and $\frac{5x}{2}+\frac{3x-14}{3} \geq 2x$

Sol. Solving $x+\frac{1}{2} > 3x-4$

$\Rightarrow 2x+1 > 2(3x-4) \Rightarrow 2x+1 > 6x-8$

$\Rightarrow 4x < 9 \Rightarrow x < \frac{9}{4}$



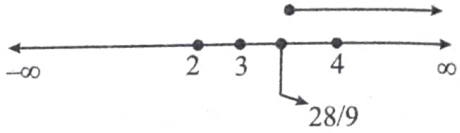
Solving $\frac{5x}{2}+\frac{3x-14}{3} \geq 2x$

Multiplying both sides by 6, we get

$15x+2(3x-14) \geq 12x$

$\Rightarrow 15x+6x-28 \geq 12x \Rightarrow 21x-28 \geq 12x$

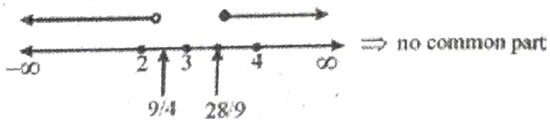
$\Rightarrow 9x \geq 28 \Rightarrow x \geq \frac{28}{9}$



The solution is the common values of x, such that $x < \frac{9}{4}$ and $x \geq \frac{28}{9}$

We note that there is no value of x, which satisfies both in equation $\Rightarrow$ no solution

Graphically:



Example 15: Solve $\frac{5}{4} < 2x-3 \leq \frac{20}{3}$.

Sol. Consider $\frac{5}{4} < 2x-3$ and $2x-3 \leq \frac{20}{3}$

Separately, $\frac{5}{4} < 2x-3 \Rightarrow 5 < 8x-12$

$\Rightarrow 8x > 17 \Rightarrow x > \frac{17}{8}$

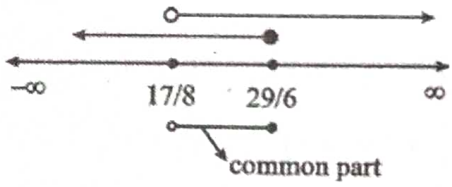
Similarly, $2x-3 \leq \frac{20}{3} \Rightarrow 6x-9 \leq 20$

$\Rightarrow 6x \leq 29 \Rightarrow x \leq \frac{29}{6}$

Common values of x are such that

$x > \frac{17}{8}$ and $x \leq \frac{29}{6} \Rightarrow \frac{17}{8} < x \leq \frac{29}{6}$

Graphically,



Alternate:

$\frac{5}{4} < 2x-3 \leq \frac{20}{3}$

Multiplying the in equation by 12,

$15 < 12(2x-3) \leq 80 \Rightarrow 15 < 24x-36 \leq 80$

$\Rightarrow 15+36 < 24x-36+36 \leq 80+36$

$\Rightarrow 51 < 24x \leq 116 \Rightarrow \frac{51}{24} < x \leq \frac{116}{24} \Rightarrow \frac{17}{8} < x \leq \frac{29}{6}$

Example 16: Solve the following system of in equations:

$$\frac{x+3}{x-2} \le 2, \quad \frac{2x+5}{x+7} \ge 3.$$

Sol. The given in equations are

$$\frac{x+3}{x-2} \le 2 \qquad \dots (i)$$

and $$\frac{2x+5}{x+7} \ge 3 \qquad \dots (ii)$$

(i) $\Rightarrow \frac{x+3}{x-2} - 2 \le 0 \Rightarrow \frac{x+3-2x+4}{x-2} \le 0 \Rightarrow \frac{7-x}{x-2} \le 0$

$\therefore\ 7-x \ge 0,\ x-2 < 0$

$\Rightarrow x \le 7,\ x < 2 \Rightarrow x < 2 \Rightarrow x \in (-\infty, 2)$

or $7-x \le 0,\ x-2 > 0$

$\Rightarrow 7-x \le 0,\ x-2 > 0 \Rightarrow x \ge 7 \Rightarrow x \in [7, \infty)$

$\therefore\ x \in (-\infty, 2) \cup [7, \infty)$

(ii) $\frac{2x+5}{x+7} - 3 \ge 0$

$\Rightarrow \frac{2x+5-3x-21}{x+7} \ge 0 \Rightarrow \frac{-x-16}{x+7} \ge 0$

$\therefore\ -x-16 \ge 0,\ x+7 > 0 \Rightarrow x \le -16,\ x > -7$

This is impossible

or $-x-16 \le 0,\ x+7 < 0$

$\Rightarrow x \ge -16,\ x < -7 \Rightarrow x \in [-16, -7)$

$\therefore\ x \in [-16, -7)$

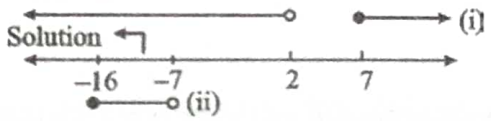
$\therefore$ Solution set of given system

$= ((-\infty, 2) \cup [7, \infty)) \cap [-16, -7)$

$= [(-\infty, 2) \cap [-16, -7)] \cup [[7, \infty) \cap [-16, -7)]$

$= [-16, -7) \cup \phi = [-16, -7)$

The solution set can also be shown on the number line as follows:



Concept Application

6. Solve the following system of linear in equations: $3x - 6 \ge 0,\ 4x - 10 \le 6$
7. Solve the following system of in equations:

$$\frac{5x}{4} + \frac{3x}{8} > \frac{39}{8}, \quad \frac{2x-1}{12} - \frac{x-1}{3} < \frac{3x+1}{4}$$

8. Solve the system of in equations: $\frac{x}{2x+1} \ge \frac{1}{4}, \frac{6x}{4x-1} < \frac{1}{2}$

GRAPHICAL SOLUTION OF LINEAR IN EQUATIONS IN TWO VARIABLES

Any linear in equation in two variable is of the form $ax + by > c$, $ax + by \ge c$, $ax + by < c$

or $ax + by \le c$, $(a \ne 0, b \ne 0)$.

The graphical solution of the in equations will be discussed assuming x, y to be real numbers. Now consider the following examples.

Train Your Brain

Example 17: Solve $2x - 3y > 6$ graphically.

Sol. First consider $2x - 3y = 6$. This is equation of a straight line which divides the coordinate plane in two half planes.

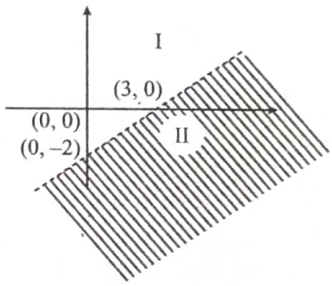
Since in equation doesn't involve the sign of equality, so we will draw this line dotted or broken. Now, consider any point which doesn't lie on this line, i.e. doesn't satisfy the equation $2x - 3y = 6$ for example take (0, 0) and check whether this point satisfies the in equation $2x - 3y > 6$ or not.

$\Rightarrow 2 \times 0 - 3 \times 0 > 6$

$\Rightarrow 0 > 6$

$\Rightarrow$ false statement.

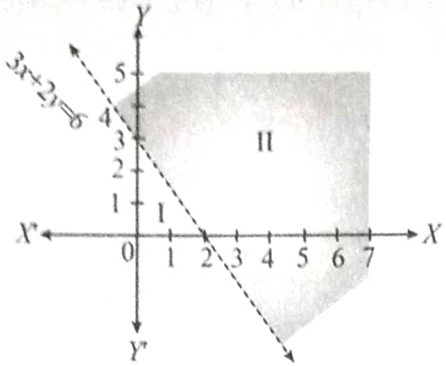
Since (0, 0) is a point lying in the plane I and it doesn't satisfy the in equation, hence half plane II will be the required solution. This region is called solution region and it is shaded as shown:



Example 18: Solve $3x + 2y > 6$ graphically.

Sol. Graph of $3x + 2y = 6$ is given as dotted line in the Figure. This line divides the xy-plane in two half planes I and II. We select a point (not on the line), say (0, 0), which lies in one of the half planes (Figure) and determine if this point satisfies the given inequality, we note that $3(0) + 2(0) > 6$ or $0 > 6$, which is false.

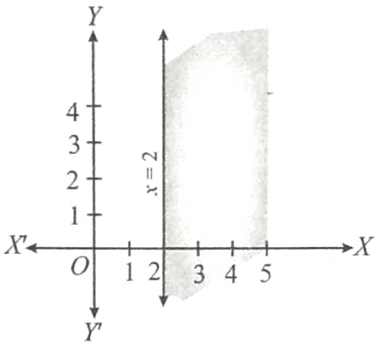
Hence, half plane I is not the solution region of the given inequality. Clearly, any point on the line does not satisfy the given strict inequality. In other words, the shaded half plane II excluding the points on the line is the solution region of the inequality.



Example 19: Solve $3x - 6 \geq 0$ graphically in two dimensional plane.

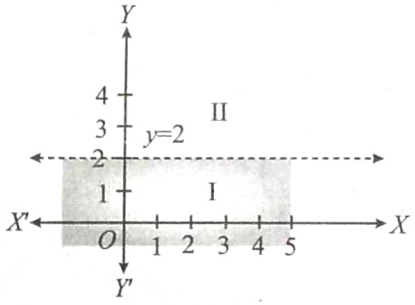
Sol. Graph of $3x - 6 = 0$ is given in the Figure.

We select a point, say (0, 0) and substituting it in given inequality, we see that: $3(0) - 6 \geq 0$ or $-6 \geq 0$ which is false. Thus, the solution region is the shaded region on the right hand side of the line $x = 2$.



Example 20: Solve $y < 2$ graphically.

Sol. Graph of $y = 2$ is given in the Figure.



Let us select a point, (0, 0) in lower half plane I and putting $y = 0$ in the given inequality,

We see that $1 \times 0 < 2$ or $0 < 2$ which is true.

Thus, the solution region is the shaded region below the line $y = 2$. Hence, every point below the line (excluding all the points on the line) determines the solution of the given inequality.

Concept Application

9. Solve the following in equations graphically:

(*i*) $2x + 3y \leq 6$ (*ii*) $2x - y \geq 1$

(*iii*) $y \leq -3$

SOLUTIONS OF SYSTEM OF LINEAR IN EQUATIONS IN TWO VARIABLES

In previous Section, you have learnt how to solve linear inequality in one or two variables graphically. We will now illustrate the method for solving a system of linear inequalities in two variables graphically through some examples.

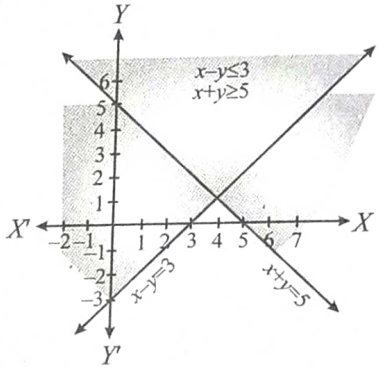
Train Your Brain

Example 21: Solve the following system of linear inequalities graphically.

$x + y \geq 5$ ...(i)

$x - y \leq 3$ ...(ii)

Sol. The graph of linear equation $x + y = 5$ is drawn in Figure.



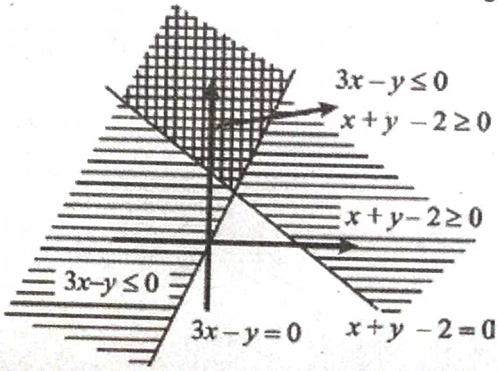
We note that solution of inequality (i) is represented by the shaded region above the line $x + y = 5$, including the points on the line.

On the same set of axes, we draw the graph of the equation $x - y = 3$ as shown in Figure. Then we note that inequality (ii) represents the shaded region above the line $x - y = 3$, including the points on the line.

Clearly, the double shaded region, common to the above two shaded regions is the required solution region of the given system of inequalities.

Example 22: Solve the following system of linear in equations graphically $x + y - 2 \geq 0$, $3x - y \leq 0$

Sol. Obtain the graphical solution for each of the in equations on the same set of axis and shade them. Thus obtained double shaded region is the required solution region.

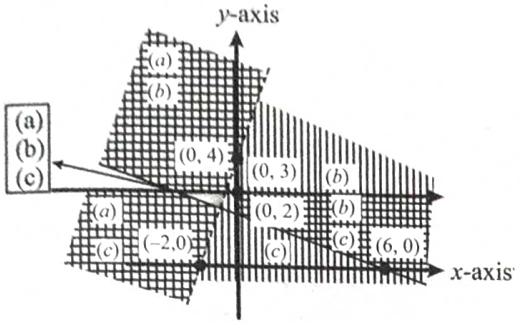


Example 23: Solve the following system of linear in equations graphically.

(*a*) $y-2x-4>0$ (*b*) $x+3y-6\geq 0$

(*c*) $y\leq 3$

Sol. Obtain the graphical solution for each of the in equations on the same set of axis and shade them. Thus obtained black shaded region is the required solution region.



Note: $x\geq 0,\ y\geq 0$ represent the first quadrant, so if present entire discussion will be limited only to first quadrant.

Concept Application

10. Exhibit graphically the solution set of the linear is in equations $3x+4y\leq 12,\ 4x+3y\leq 12,\ x\geq 0,\ y\geq 0$

11. Draw the diagram of the solution set of the linear in equations $3x+4y\geq 12,\ y\leq 2,\ x\geq 0$.

12. Exhibit graphically the solution set of the in equations $x+y\leq 5,\ 4x+y\geq 4,\ x+5y\geq 5,\ x\leq 4,\ y\leq 3$

INEQUALITIES RELATED TO MODULUS FUNCTION

In general $|x|=\begin{cases} x, & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$

Now consider the following inequalities and their solutions:

(*i*) $|x|\leq k,\ k>0$

Solution: We have $|x|\leq k$ ... (*i*)

Case I: $x\geq 0$

$\therefore$ (*i*) $\Rightarrow x\leq k$ $(\because |x|=x)$

$\therefore\ 0\leq x\leq k$ i.e. $x\in[0,\ k]$

Case II: $x<0$

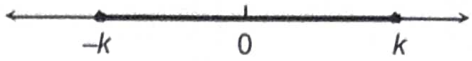
$\therefore$ (*i*) $\Rightarrow -x\leq k$ $(\because |x|=-x)$

$\Rightarrow (-1)(-x)\geq(-1)k \Rightarrow x\geq -k$

$\Rightarrow -k\leq x$

$\therefore -k\leq x<0$ i.e., $x\in[-k,\ 0)$

$\therefore$ The solution is $[-k,\ 0)\cup[0,\ k]$ which is same as $[-k,\ k]$. This solution can also be shown on the number line as follows:



Note: In particular, if $k=0$, then the solution of the in equation $|x|\leq 0$ is $\{0\}$.

(*ii*) $|x|\geq k,\ k>0$

Solution: We have $|x|\geq k$ ... (*i*)

Case-I : $x\geq 0$

(*i*) $\Rightarrow x\geq k$ $(\because |x|=x)$

$\therefore\ x\in[k,\ \infty)$

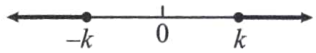
Case-II : $x<0$

(*i*) $\Rightarrow -x\geq k$ $(\because |x|=-x)$

$\Rightarrow (-1)(-x)\leq(-1)k \Rightarrow x\leq -k$

$\therefore\ x\in(-\infty,\ -k]$

$\therefore$ The solution is $(-\infty,\ -k]\cup[k,\ \infty)$. This solution can also be shown on the number line as follows:



Note: In particular, if $k=0$, then the solution of the in equation $|x|\geq 0$ is $(-\infty,\ \infty)$.

Important formulae/points

In general,

- If $|x|\geq k,\ k>0 \Rightarrow x\in(-\infty,\ -k]\cup[k,\ \infty)$
- If $|x|>k,\ k>0 \Rightarrow x\in(-\infty,\ -k)\cup(k,\ \infty)$
- If $|x|\leq k,\ k>0 \Rightarrow x\in[-k,\ k]$
- If $|x|<k,\ k>0 \Rightarrow x\in(-k,\ k)$

Solving $|x-x_0|\geq k,\ k>0$

- Under the case $x-x_0\geq 0$, take $|x-x_0|=x-x_0$ and solve $x-x_0\geq k$
- When $x-x_0<0$, *take* $|x-x_0|=-(x-x_0)$ *and solve* $-(x-x_0)\geq k$
- Combined the solutions obtained for the two cases.

Train Your Brain

Example 24: Solve the in equations:

(*i*) $|7x-2|\leq 11$ (*ii*) $|5-4x|>8$

Sol. (*i*) $|7x-2|\leq 11$ ... (*i*)

Case-I: $7x-2\geq 0$ i.e. $7x\geq 2$ or $x\geq\frac{2}{7}$

$\Rightarrow 7x-2\leq 11$ $(\because |7x-2|=7x-2)$

$\Rightarrow 7x\leq 13 \Rightarrow x\leq\frac{13}{7}$

$\therefore \ \frac{2}{7} \le x \le \frac{13}{7}$ i.e. $x \in \left[\frac{2}{7}, \frac{13}{7}\right]$

Case-II: $7x-2<0$ i.e. $7x<2$ or $x<\frac{2}{7}$

$\Rightarrow -(7x-2) \le 11 \ (\because |7x-2| = -(7x-2))$

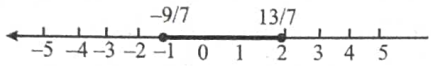
$\Rightarrow -7x+2 \le 11 \Rightarrow -7x \le 9 \Rightarrow x \ge -\frac{9}{7}$

$\therefore \ -\frac{9}{7} \le x < \frac{2}{7}$ i.e. $x \in \left[-\frac{9}{7}, \frac{2}{7}\right)$

$\therefore$ The solution is $\left[-\frac{9}{7}, \frac{2}{7}\right) \cup \left[\frac{2}{7}, \frac{13}{7}\right]$,

which is same as $\left[-\frac{9}{7}, \frac{13}{7}\right]$.

This solution can also be shown on the number line as follows:



(*ii*) $|5-4x| > 8$

Case-I: i.e. $5 \ge 4x$ or $x \le \frac{5}{4}$

$\Rightarrow 5-4x > 8 \ (\because |5-4x| = 5-4x)$

$\Rightarrow 4x < -3 \Rightarrow x < -\frac{3}{4}$

$\therefore \ x < -\frac{3}{4}$ i.e. $x \in \left(-\infty, -\frac{3}{4}\right) \left(\because x < -\frac{3}{4} \Rightarrow x \le \frac{5}{4}\right)$

Case-II: $5-4x<0$ i.e. $5<4x$ or $x>\frac{5}{4}$

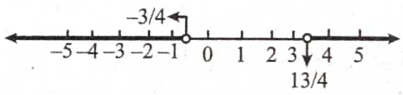
$\Rightarrow -(5-4x) > 8 \ (\because |5-4x| = -(5-4x))$

$\Rightarrow -5+4x > 8 \Rightarrow 4x > 13 \Rightarrow x > \frac{13}{4}$

$\therefore \ x > \frac{13}{4}$ i.e. $x \in \left(\frac{13}{4}, \infty\right) \left(\because x > \frac{13}{4} \Rightarrow x > \frac{5}{4}\right)$

$\therefore$ The solution is $\left(-\infty, -\frac{3}{4}\right) \cup \left(\frac{13}{4}, \infty\right)$.

This solution can also be shown on the number line as follows:



Example 25: Solve : $1 \le |x-2| \le 3$

Sol. We know that:

$a \le |x-c| \le b \Leftrightarrow x \in [-b+c, -a+c] \cup [a+c, b+c]$

$\therefore \ 1 \le |x-2| \le 3 \Leftrightarrow x \in [-3+2, -1+2] \cup [1+2, 3+2]$

$\Rightarrow x \in [-1, 1] \cup [3, 5]$

Hence, the solution set of the given in equations is $[-1, 1] \cup [3, 5]$.

Example 26: Solve: $\frac{|x|-1}{|x|-2} \ge 0, \ x \in R, x \ne 2.$

Sol. We have, $\frac{|x|-1}{|x|-2} \ge 0$

$\Rightarrow \frac{y-1}{y-2} \ge 0$, where $y = |x|$

$\Rightarrow y \le 1$ or $y > 2$

$\Rightarrow |x| \le 1$ or $|x| > 2$

$\Rightarrow (-1 \le x \le 1)$ or $(x < -2$ or $x > 2$

$\Rightarrow x \in [-1, 1]$ or $x \in (-\infty, -2) \cup (2, \infty)$

$\Rightarrow x \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$

Hence, the solution set on the given in equation is $[-1, 1] \cup (-\infty, -2) \cup (2, \infty)$.

Concept Application

13. Solve $|3x-2| \le \frac{1}{2}$

14. Solve $|x-2| \ge 5$

15. Solve the following system of in equations :

$|x-1| \le 5, |x| \ge 2$

16. Solve: $\frac{-1}{|x|-2} \ge 1$, where $x \in R, x \ne \pm 2$

MISCELLANEOUS EXAMPLES

Train Your Brain

Example 27: In an experiment, a solution of hydrochloric acid is to be kept between 30° and 35° Celsius. What is the range of temperature in degree Fahrenheit if conversion formula is given by $C = \frac{5}{9}(F-32)$, where C and F represent temperature in degree Celsius and degree Fahrenheit, respectively.

Sol. It is given that $30 < C < 35$.

Putting $C = \frac{5}{9}(F-32)$, we get

$30 < \frac{5}{9}(F-32) < 35$,

or $\left(\frac{9}{5}\right)(30) < (F-32) < \left(\frac{9}{5}\right)(35)$

or $54 < (F - 32) < 63$

or $86 < F < 95$.

Thus, the required range of temperature is between 86° F and 95° F.

Example 28: A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%?

Sol. Let x litres of 30% acid solution is required to be added.

Then Total mixture = $(x + 600)$ litres

Therefore 30% x + 12% of 600 > 15% of $(x + 600)$

and 30% x + 12% of 600 < 18% of $(x + 600)$

or $\frac{30x}{100} + \frac{12}{100}(600) > \frac{15}{100}(x + 600)$

and $\frac{30x}{100} + \frac{12}{100}(600) > \frac{18}{100}(x + 600)$

or $30x + 7200 > 15x + 9000$

and $30x + 7200 < 18x + 10800$

or $15x > 1800$ and $12x < 3600$

or $x > 120$ and $x < 300$,

i.e. $120 < x < 300$

Thus, the number of litres of the 30% solution of acid will have to be more than 120 litres but less than 300 litres.

Example 29: A plumber can be paid under two schemes as given below:

I : Rs. 600 and Rs. 50 per hour

II : Rs. 170 per hour

If the job takes n hours, for what values of n does the scheme I give the plumber the better wages ?

Sol. Number of hours = n

Under scheme I, wage of plumber = Rs. $(600 + 50\,n)$

Under scheme II, wage of plumber = Rs. $170\,n$

Let $600 + 50n > 170\,n \Rightarrow 600 > 120n \Rightarrow n < \frac{600}{120} = 5$

$\therefore$ Number of hours should be less than 5 hours.

Concept Application

17. In drilling world's deepest hole, it was found that the temperature T in degree Celsius, x km below the surface of earth, was given by

$T = 30 + 25(x - 3),\ 3 < x < 15$. At what depth, will the temperature be between 200°C and 300°C ?

18. The water acidity in a pool is considered normal when the average pH reading of three daily measurements is between 7.2 and 7.8. If the first two pH readings are 7.48 and 7.85, find the range of pH value for the third reading that will result in the acidity level being normal.

Short Notes

1. A Two real numbers or two algebraic expressions related by the symbols <, >, ≤ or ≥ form an inequality.
2. Equal numbers may be added to (or subtracted from) both sides of an inequality.
3. Both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied (or divided) by a negative number, then the inequality is reversed.
4. The values of x, which make an inequality a true statement, are called *solutions of the inequality*.
5. To represent $x < a$ (or $x > a$) on a number line, put a circle on the number a and dark line to the left (or right) of the number a.
6. To represent $x \le a$ (or $x \ge a$) on a number line, put a dark circle on the number a and dark the line to the left (or right) of the number x.
7. If an inequality is having ≤ or ≥ symbol, then the points on the line are also included in the solutions of the inequality and the graph of the inequality lies left (below) or right (above) of the graph of the equality represented by dark line that satisfies an arbitrary point in that part.
8. If an inequality is having < or > symbol, then the points on the line are not included in the solutions of the inequality and the graph of the inequality lies to the left (below) or right (above) of the graph of the corresponding equality represented by dotted line that satisfies an arbitrary point in that part.
9. The solution region of a system of inequalities is the region which satisfies all the given inequalities in the system simultaneously.

Solved Examples

1. Solve $\frac{x-3}{x+4} > 0, \frac{2x-1}{x+1} < 1.$

Sol. Taking $\frac{x-3}{x+4} > 0$

Multiplying both sides by $(x+4)^2$, we get $(x-3)(x+4) > 0$

Now, we have the product of $(x-3)$ and $(x+4)$

It will be positive, if both of the values are positive or both of the values are negative.

$\Rightarrow x-3>0$ and $x+4>0$

$\Rightarrow x>3$ and $x>-4 \Rightarrow x>3$

or

$\Rightarrow x-3<0$ and $x+4<0$

$\Rightarrow x<3$ and $x<-4 \Rightarrow x<-4$

$\Rightarrow x>3$ or $x<-4 \Rightarrow x \in (-\infty, -4) \cup (3, \infty)$

Taking, $\Rightarrow \frac{2x-1}{x+1} - 1 < 0$

$\Rightarrow \frac{2x-1-(x+1)}{x+1} < 0 \Rightarrow \frac{x-2}{x+1} < 0$

Multiplying both sides by $(x+1)^2$,

we get $(x+1)(x-2) < 0$

The product of $(x + 1)$ and $(x - 2)$ will be negative if one of them is positive and other is a negative quantity i.e., $x + 1 > 0$ and $x - 2 < 0$

$\Rightarrow x > -1$ and $x < 2$

$\Rightarrow -1 < x < 2$

or $x + 1 < 0$ and $x - 2 > 0$

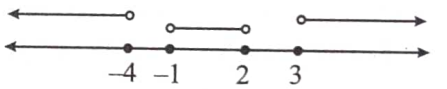
$\Rightarrow x < -1$ and $x > 2$

$\Rightarrow$ not possible.

Hence solution is $-1 < x < 2$ for the second equation.

We note that for $x \in (-\infty, -4) \cup (3, \infty)$ and $-1 < x < 2$, there is no common value of x and hence no solution.

Graphically,



Obviously, there is no common part, hence no solution.

2. Graphically solve the system of linear in equations:

$3x - 4y + 12 \ge 0, \quad 2x - y + 2 \ge 0$

$2x + 3y - 12 \ge 0, \quad x \le 4, \quad y \ge 2, \quad x, y \ge 0$

Sol. The linear in equations in the given system are

$3x - 4y + 12 \ge 0$ ...(i)

$2x - y + 2 \ge 0$ ...(ii)

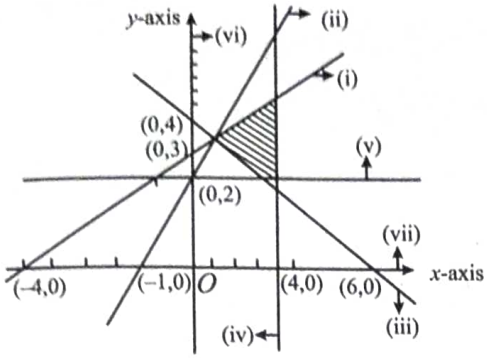
$2x + 3y - 12 \ge 0$ ...(iii)

$x \le 4$ ...(iv)

$y \ge 2$ ...(v)

$x \ge 0$ ...(vi)

$y \ge 0$ ...(vii)



The line corresponding to

(*i*) is $3x-4y+12=0$ ... (*viii*)

(0, 3) and (–4, 0) are on (*viii*)

(0, 0) is not on (*viii*) and it lies in the half-plane of (*i*), if $3(0)-4(0)+12 \geq 0$, which is true.

$\therefore$ The closed half-plane containing the origin is the graph of (*i*)

The line corresponding to

(*ii*) is $2x-y+2=0$ ... (*ix*)

(0, 2) and (–1, 0) are on (*ix*)

(0, 0) is not on the line and it lies in the half-plane of (*ii*), if $2(0)-0+2 \geq 0$, which is true.

$\therefore$ The closed half-plane containing the origin is the graph of (*ii*)

The line corresponding to

(*iii*) is $2x+3y-12=0$ ... (*x*)

(0, 4) and (6, 0) are on (*x*), (0, 0) is not on this line and it lies in the half-plane of (*iii*),

if $2(0)+3(0)-12 \geq 0$, which is not true.

$\therefore$ The closed half-plane not containing the origin is the graph of (*iii*)

The in equation $x \leq 4$ represents the closed half-plane on the left of $x=4$

The in equation $y \geq 2$ represents the closed half-plane above $y=2$.

The in equation $x \geq 0$ and $y \geq 0$ represents the closed half-planes on the right of *y*-axis and above *x*-axis respectively.

The shaded region is the required graph of the given system of in equations.

3. Solve: $\dfrac{|x+3|+x}{x+2}>1$

Sol. We have, $\dfrac{|x+3|+x}{x+2}>1.$

Clearly, L.H.S of this in equation is meaningful for $x \neq -2$.

Now, $\dfrac{|x+3|+x}{x+2}>1$

$\Rightarrow \dfrac{|x+3|+x}{x+2}-1>0$

$\Rightarrow \dfrac{|x+3|+x-x-2}{x+2}>0$

$\Rightarrow \dfrac{|x+3|-2}{x+2}>0.$

Now two cases arise:

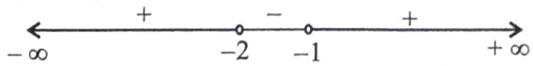
Case: When $x+3 \geq 0$ i.e. $x \geq -3$: In this case, $|x+3|=x+3$.

$\therefore \dfrac{|x+3|-2}{x+2}>0$

$\Rightarrow \dfrac{x+3-2}{x+2}>0$

$\Rightarrow \dfrac{x+1}{x+2}>0$

$\Rightarrow x \in (-\infty, -2) \cup (-1, \infty)$



But, $X \geq -3$. Therefore, the solution set of the given in equation in this case is $[-3, -2) \cup (-1, \infty)$.

Case: When $x+3<0$ i.e. $x<-3$: In this case, $|x+3|=-(x+3)$.

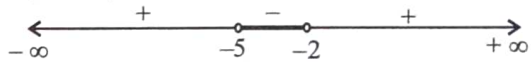
$\therefore \dfrac{|x+3|-2}{x+2}>0$

$\Rightarrow \dfrac{-(x+3)-2}{x+2}>0$

$\Rightarrow \dfrac{-(x+5)}{x+2}>0$

$\Rightarrow \dfrac{x+5}{x+2}<0$

$\Rightarrow x \in (-5-2)$



But, $x<-3$. Therefore, the solutions set of the given in equation (–5, –3).

From Case I and Case II, we obtain that the solution set of the given in equation is

$[-3, -2) \cup (-1, \infty) \cup (-5, -3) = (-5, -2) \cup (-1, \infty)$.

4. Solve $|x-1|+|x-2| \geq 4$

Sol. On the LHS of the given in equation there are two terms both containing modules. By equating the expressions within the modules to zero, we get $x=1, 2$ as critical points. These points divide real line in three parts viz. $(-\infty, 1]$, $[1, 2]$ and $[2, \infty)$. So, we consider the following three cases.

Case I : When $-\infty < x < 1$: In this case, we have $|x-1|=-(x-1)$ and $|x-2|=-(x-2)$

$\therefore |x-1|+|x-2| \geq 4$

$\Rightarrow -(x-1)-(x-2) \geq 4$

$\Rightarrow -2x+3 \geq 4 \Rightarrow -2x \geq 1$

$\Rightarrow x \leq -\dfrac{1}{2}$

But, $-\infty < x < 1$. Therefore, in this case the solution set of the given in equation is $(-\infty, -1/2]$

Case II : When $1 \le x < 2$. In this case, we have $|x-1| = (x-1)$ and $|x-2| = -(x-2)$

$\therefore\ |x-1| + |x-2| \ge 4$

$\Rightarrow\ x - 1 - (x-2) \ge 4$

$\Rightarrow\ 1 \ge 4$, which is an absurd result.

So, the given inequation has no solution for $x \in [1, 2)$

Case III : When $x \ge 2$: In this case, we have $|x-1| = x-1$ and $|x-1| = x-1$ and $|x-2| = x-2$

$\therefore\ |x-1| + |x-2| \ge 4$

$\Rightarrow\ x - 1 + x - 2 \ge 4$

$\Rightarrow\ 2x - 3 \ge 4$

$\Rightarrow\ 2x \ge 7$

$\Rightarrow\ x \ge \dfrac{7}{2}$

But, $x > 2$. Therefore, in this case the solution set of the given in equation is $[7/2, \infty)$.

Combining Case I and Case II, we obtain that the solution set of the given in equation is $(-\infty, -1/2] \cup [7/2, \infty)$

5. Solve: $\dfrac{|x-1|}{x+2} < 1$.

Sol. We have,

$$\frac{|x-1|}{x+2} < 1 \Rightarrow \frac{|x-1|}{x+2} - 1 < 0 \Rightarrow \frac{|x-1| - (x+2)}{x+2} < 0$$

Now the following cases arise.

Case I: When $x - 1 \ge 0$ i.e. $x \ge 1$: In this case, we have $|x-1| = x-1$

$\Rightarrow\ \dfrac{|x-1| - (x+2)}{x+2} < 0$

$\Rightarrow\ \dfrac{(x-1) - (x+2)}{x+2} < 0$

$\Rightarrow\ \dfrac{-3}{x+2} < 0$

$\Rightarrow\ x + 2 > 0$ $\quad\left[\because \dfrac{a}{b} < 0 \text{ and } a < 0 \Rightarrow b > 0\right]$

$\Rightarrow\ x > -2$

But, $x \ge 1$. Therefore, $x > -2$ and $x \ge 1$, implies that $x \ge 1$. Thus, in this case the solution set of the given inequation is $[1, \infty)$.

Case II: When $x - 1 < 0$ i.e. $x < 1$: In this case, we have $|x-1| = -(x-1)$.

$\dfrac{|x-1| - (x+2)}{x+2} < 0$

$\Rightarrow\ \dfrac{-(x-1) - (x+2)}{x+2} < 0$

$\Rightarrow\ -\dfrac{2x+1}{x+2} < 0$

$\Rightarrow\ \dfrac{2x+1}{x+2} > 0$

$\Rightarrow\ x \in (-\infty, -2) \cup (-1/2, \infty)$

But $x < 1$. Therefore, $x \in (-\infty, -2) \cup (-1/2, \infty)$ and $x < 1$ implies that $x \in (-\infty, -2) \cup \left(-\dfrac{1}{2}, 1\right)$.

Thus, in this case the solution set of the given inequation is $x \in (-\infty, -2) \cup \left(-\dfrac{1}{2}, 1\right)$

Combining Case I and Case II, we obtain that the solution set of the given in equation is $(-\infty, -2) \cup (-1/2, \infty)$

6. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution, how many litres of the 2% solution will have to be added?

Sol. Let required quantity of 2% boric acid = x litres

$\therefore$ Quantity of mixture = $(640 + x)$ litres

By the given conditions,

8% of 640 + 2% of x > 4% of $(640 + x)$ ... (*i*)

and 8% of 640 + 2% of x < 6% of $(640 + x)$ ... (*ii*)

(*i*) $\left(\dfrac{8}{100}\right)640 + \left(\dfrac{2}{100}\right)x > \left(\dfrac{4}{100}\right)(640 + x)$

$\Rightarrow\ 5120 + 2x > 2560 + 4x \Rightarrow 2560 > 2x$

$\Rightarrow\ x < 1280$ ... (*iii*)

(*ii*) $\left(\dfrac{8}{100}\right)640 + \left(\dfrac{2}{100}\right)x < \left(\dfrac{6}{100}\right)(640 + x)$

$\Rightarrow\ 5120 + 2x < 3840 + 6x \Rightarrow 1280 < 4x$

$\Rightarrow\ x > 320$ ... (*iv*)

(*iii*) and (*iv*) $\Rightarrow 320 < x < 1280$

$\therefore$ Quantity of 2% boric acid lies between 320 litres and 1280 litres.

Exercise-1 (Topicwise)

1. Ravi goes to market with Rs. 200 to buy rice, which is available in packets of 1 kg. The price of one packet of rice is Rs. 30. If x denotes the number of packets of rice which he buys, then the total amount spent by him is Rs. $30x$. The mathematical formulation of given problem is
   (*a*) $30x > 200$ (*b*) $30x < 200$
   (*c*) $30x = 200$ (*d*) $30x \geq 200$

2. Reshma has Rs. 120 and wants to buy some registers and pens. The cost of one register is Rs. 40 and that of a pen is Rs. 20. The mathematical formulation of given problem is
   (*a*) $40x + 20y = 120$ (*b*) $40x + 20y > 120$
   (*c*) $40x + 20y \leq 120$ (*d*) $40x + 20y \geq 120$

3. Two real numbers or two algebraic expressions related by the symbol '<', '>', '≤' or '≥' forms an ... X ... . Here, X refers to
   (*a*) equation (*b*) inequality
   (*c*) set (*d*) None of these

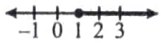
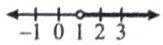
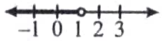
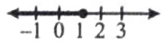
4. Which of the following is/are the examples of numerical inequalities ?
   I. $3 < 5$ II. $7 > 5$
   III. $x < 5$ IV. $y > 2$
   V. $x \geq 3$
   (*a*) I, II (*b*) III, IV
   (*c*) IV (*d*) All of these

5. Which of the following is/are double inequalities?
   (*a*) $3 < 5 < 7$ (*b*) $3 \leq x < 5$
   (*c*) $2 < y \leq 4$ (*d*) All of these

6. The rules for solving an inequality is/are
   (*a*) different mumber may be added to both sides of an inequality without affecting the sign of inequality
   (*b*) both sides of an inequality can be multiplied (or divided) by any number without affecting the sign of inequality
   (*c*) Both (*a*) and (*b*)
   (*d*) None of the above

7. The solution set of the inequality $4x + 3 < 6x + 7$ is
   (*a*) $[-2, \infty)$ (*b*) $(-\infty, -2)$
   (*c*) $(-2, \infty)$ (*d*) None of these

8. The set of real x satisfying the inequality $\frac{5-2x}{3} \leq \frac{x}{6} - 5$ is
   (*a*) $(-\infty, 8)$ (*b*) $(8, \infty)$
   (*c*) $[8, \infty)$ (*d*) $(-\infty, 8]$

9. The graph of the solutions of inequality $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$ on number line is
   (*a*) 
   (*b*) 
   (*c*) 
   (*d*) 

10. The marks obtained by a student of class *XI* in first and second terminal examinations are 62 and 48, respectively. The minimum marks he should get in the annual examination to have an average of atleast 60 marks, are
    (*a*) 70 (*b*) 50
    (*c*) 74 (*d*) 48

11. The number of pairs of consecutive odd natural numbers both of which are larger than 10, such that their sum is less than 40, is
    (*a*) 8 (*b*) 6
    (*c*) 4 (*d*) 3

12. If $3x + 8 > 2$, then which of the following is ture ?
    (*a*) $x \in \{-1, 0, 1, 2, ...\}$, when x is an integer
    (*b*) $x \in [-2, \infty)$, when x is a real number
    (*c*) Both (*a*) and (*b*)
    (*d*) None of the above

13. The solution set of the inequality
    $4x + 3 < 5x + 7 \ \forall \ x \in R$ is
    (*a*) $(-4, \infty)$ (*b*) $[-4, \infty)$
    (*c*) $(4, \infty)$ (*d*) $[4, \infty)$

14. Which of the following is the solution set of
    $3x - 7 > 5x - 1 \forall \ x \in R$?
    (*a*) $(-\infty, -3)$ (*b*) $(-\infty, -3]$
    (*c*) $(-3, \infty)$ (*d*) $(-3, 3)$

15. The solution set of the inequality $3(x - 1) \leq 2(x - 3)$ is "A". Here, A refers to
    (*a*) $(-\infty, 3)$ (*b*) $(-\infty, -3)$
    (*c*) $(-\infty, 3]$ (*d*) $(-\infty, -3]$

16. The solution set of the inequality $3(2 - x) \geq 2(1 - x)$ is "B". Here, B refers to
    (*a*) $(-\infty, 4)$ (*b*) $(-\infty, 4]$ (*c*) $(-\infty, -4)$ (*d*) $(-\infty, -4]$

17. The solution set of the inequality
    $2(2x + 3) - 10 < 6(x - 2)$ is
    (*a*) $x \geq 4$ (*b*) $x \leq 4$ (*c*) $x > 4$ (*d*) $x < 4$

18. The solution set of the inequality
    $37 - (3x + 5) \geq 9x - 8(x - 3)$ is
    (*a*) $(-\infty, 2)$ (*b*) $(-\infty, -2)$ (*c*) $(-\infty, 2]$ (*d*) $(-\infty, -2]$

19. Which of the following is the solution set of the in equality $\frac{x}{4} < \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$?

(*a*) $(4, \infty)$ (*b*) $(-\infty, 4)$ (*c*) $(4, \infty)$ (*d*) $(-\infty, 4]$

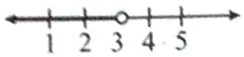
20. The solution set of $\frac{2x-1}{3} \geq \left(\frac{3x-2}{4}\right) - \left(\frac{2-x}{5}\right)$ is ... *A*....

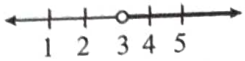
Here, *A* refers to

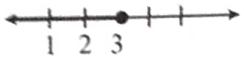
(*a*) $(-\infty, 2)$ (*b*) $(-\infty, 2]$

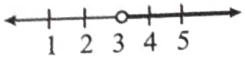
(*c*) $\left(-\infty - \frac{1}{2}\right)$ (*d*) $\left(-\infty - \frac{1}{2}\right]$

21. The graph of the solution on number line of the inequality $3x - 2 < 2x + 1$ is

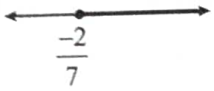
(*a*) 

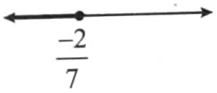
(*b*) 

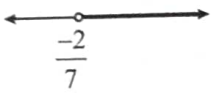
(*c*) 

(*d*) 

22. The graph of solution of the inequality $\frac{x}{2} \geq \frac{5x-2}{3} - \frac{7x-3}{5}$ on the number line is

(*a*) 

(*b*) 

(*c*) 

(*d*) None of the above

23. Ravi obtained 70 and 75 marks in first two unit tests. Then, the minimum marks he should get in the third test to have an average of atleast 60 marks, are

(*a*) 45 (*b*) 35

(*c*) 25 (*d*) None of these

24. To receive grade *A* in a course, one must obtain an average of 90 marks or more in five examinations (each of 100 marks). If Sunita's marks in first four examinations are 87, 92, 94, and 95, the minimum marks that Sunita must obtain in fifth examination to get grade. *A* in the course, are

(*a*) 72 (*b*) 82

(*c*) 90 (*d*) None of these

25. The number of pairs of consecutive odd positive integers both of which are smaller than 10 such that their sum is more than 11, are

(*a*) 2 (*b*) 3 (*c*) 4 (*d*) 0

26. The pairs of consecutive even positive integers, both of which are large than 5 such that their sum is less than 23, are

(*a*) (4, 6), (6, 8), (8, 10), (10, 12)

(*b*) (6, 8), (8, 10), (10, 12)

(*c*) (6, 8), (8, 10), (10, 12), (12, 14)

(*d*) (8, 10), (10, 12)

27. The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is atleast 61 cm, then minimum length of the shortest side is

(*a*) 7 cm (*b*) 8 cm (*c*) 9 cm (*d*) 10 cm

28. A man wants to cut three lengths from a single piece of board of length 91 cm. The second length is to be 3 cm longer than the shortest and the third length is to be twice as long as the shortest. The possible length of the shortest board, if the third piece is to be atleast 5 cm longer than the second is

(*a*) less than 8 cm

(*b*) greater than or equal to 8 cm but less than or equal to 22 cm

(*c*) less than 22 cm

(*d*) greater than 22 cm

29. A line divides the carteian plane in ... *X*... part(s). Here, *X* refers to

(*a*) One (*b*) Two

(*c*) Three (*d*) None of these

30. Each part in which a line divides the cartesian plane, is known as ...*Y*... plane. Here, *Y* refers to

(*a*) half (*b*) quarter

(*c*) double (*d*) None of these

31. Which of the following is /are true ?

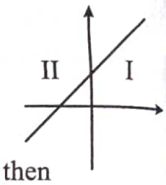
(*a*) A vertical line will divide the plane in left and right half planes

(*b*) A non-vertical line will divide the plane into left and right half planes

(*c*) Both (*a*) and (*b*)

(*d*) None of the above

32. A point in the cartesian plane will either lie on a ...*A* ... or will lie in ...*B* ... Here, *A* and *B* refer to



(*a*) line and half plane I

(*b*) line and half plane II

(*c*) line and either of half plane I or II

(*d*) None of the above

33. If a point is satisfying the line $ax + by = c$, then

(*a*) it will lie on the line

(*b*) it will lie in upper half plane

(*c*) it will lie in lower half plane

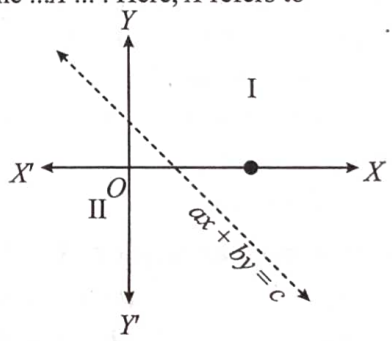
(*d*) None of the above

34. If a point $P(\alpha, \beta)$ lies on the line $ax + by = c$, then

(*a*) $a\alpha + b\beta > c$ (*b*) $a\alpha + b\beta < c$

(*c*) $a\alpha + b\beta = c$ (*d*) None of these

35. The graph of the inequality $ax + by > c$ is represented by shading in the ...*A* ... . Here, *A* refers to



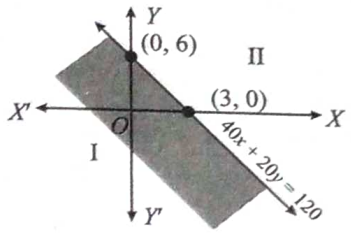
(*a*) half plane I including the points on $ax + by = c$

(*b*) half plane II including the points on $ax + by = c$

(*c*) half plane I not including the points on $ax + by = c$

(*d*) half plane II not including the points on $ax + by = c$

36. Consider the inequality $40x + 20y \le 120$, where x and y are whole numbers. Then,



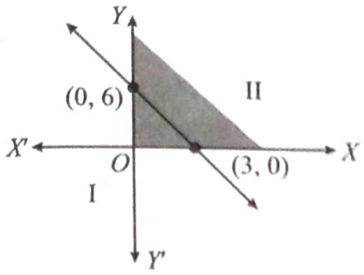
(*a*) (0, 0), (0, 5), (1, 1), (2, 2), (3, 0) are some solutions of given inequality

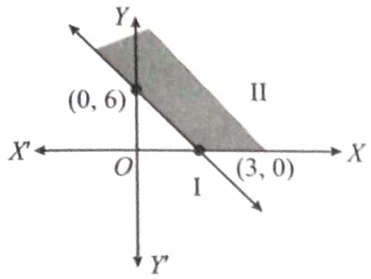
(*b*) The graphical representation of given inequality is II

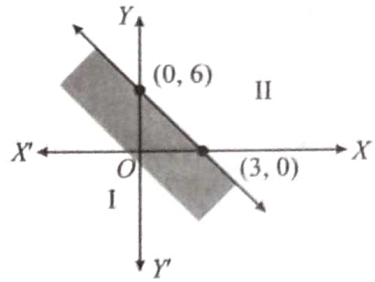
(*c*) Both (*a*) and (*b*)

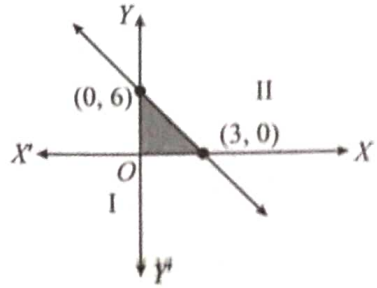
(*d*) None of these

37. The graphs of the inequality $40x + 20y \le 120$, $x \ge 0$, $y \ge 0$ is

(*a*)


(*b*)


(*c*)


(*d*)


38. The solution of the inequality $3x + 2y > 6$ in the graphs is represented by shading the

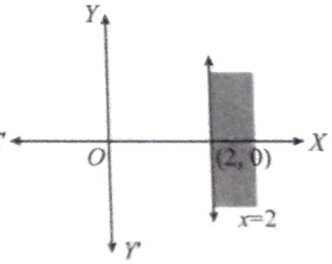
(*a*) half plane, which contains origin including the points on the line

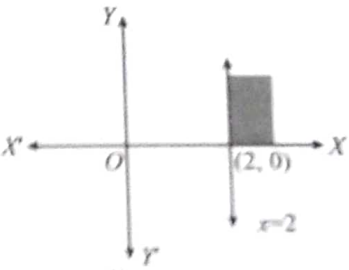
(*b*) half plane, which does not contain origin excluding the points on the line

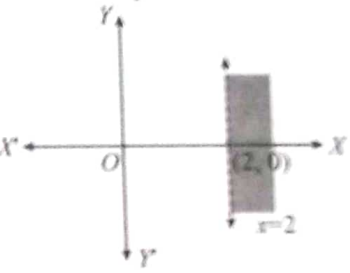
(*c*) half plane, which does not contain origin including the points on the line

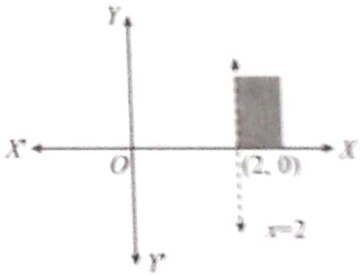
(*d*) half plane, which contain origin excluding the points on the line

39. The graphical solution of $3x - 6 \ge 0$ is

(*a*)


(*b*)


(*c*)


(*d*)


40. The region represented by $2x + y \ge 6$ is

(*a*) bounded (*b*) unbounded

(*c*) Does not exist (*d*) None of these

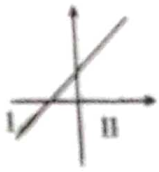
41. The graphical solution of $-3x + 2y \ge 6$ is represented by shading the

(*a*) half plane II including the points on the line $-3x + 2y = 6$

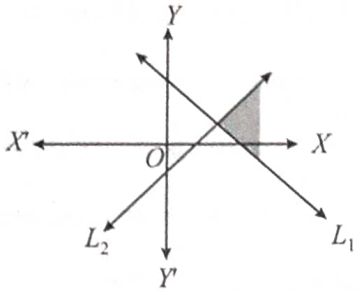
(*b*) half plane II excluding the points on the line $-3x + 2y = 6$

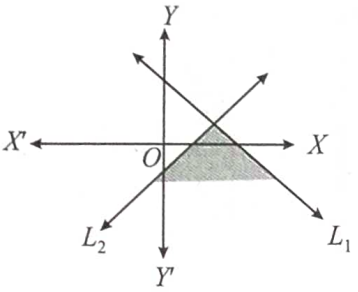
(*c*) half plane I including the points on the line $-3x + 2y = 6$

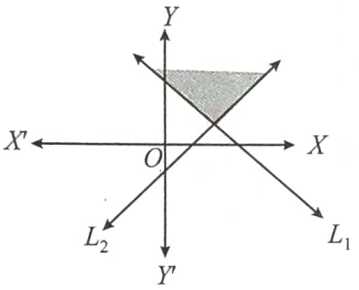
(*d*) half plane I excluding the points on the line $-3x + 2y = 6$

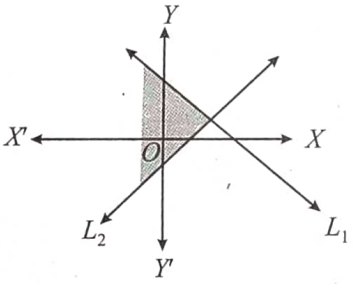


42. The graphical solution of linear inequalities $x + y \geq 5$ and $x - y \leq 3$, where $L_1 \equiv x + y = 5$ and $L_2 \equiv x - y = 3$, is

(*a*)


(*b*)


(*c*)


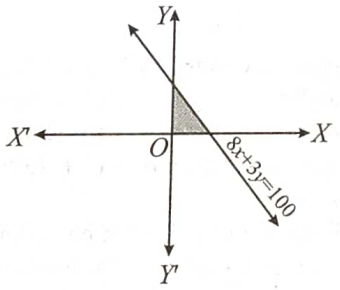
(*d*)


43. The region represented by the solution of system of inequalities $5x + 4y \leq 40$, $x \geq 2$, $y \geq 3$ is

(*a*) bounded (*b*) unbounded

(*c*) Does not exist (*d*) None of the above

44. Which of the following represent the region in the figure?



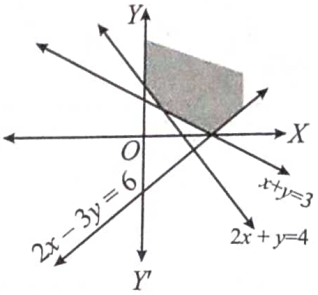
(*a*) $8x + 3y \geq 100$, $x \geq 0$, $y \geq 0$

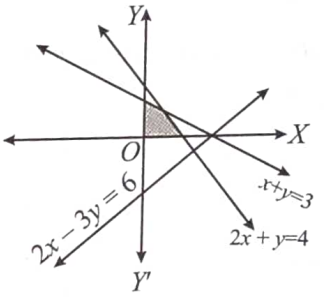
(*b*) $8x + 3y \leq 100$, $x \geq 0$, $y \geq 0$

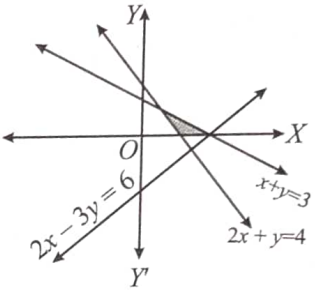
(*c*) $8x + 3y \geq 100$

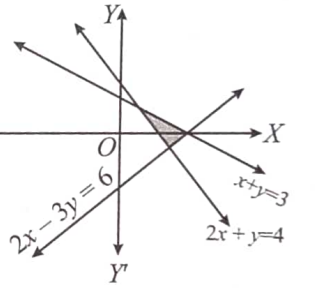
(*d*) $8x + 3y \leq 100$

45. The solution set of the inequalities $2x + y \geq 4$, $x + y \leq 3$, $2x - 3y \leq 6$ is

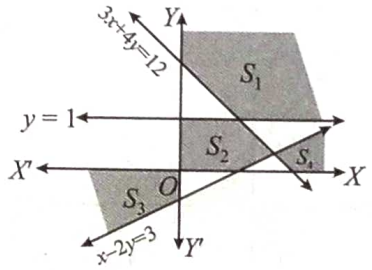
(*a*)


(*b*)


(*c*)


(*d*)


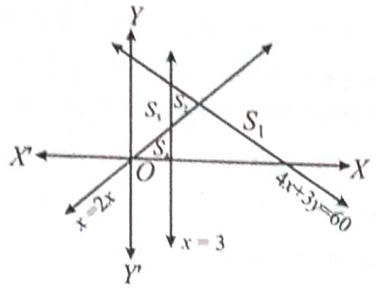
46. The shaded region representing the solution set of the system of in equalities $x - 2y \leq 3$, $3x + 4y \geq 12$, $x \geq 0$, $y \geq 1$ in the following figure is:



(*a*) S_1 (*b*) S_2

(*c*) S_3 (*d*) S_4

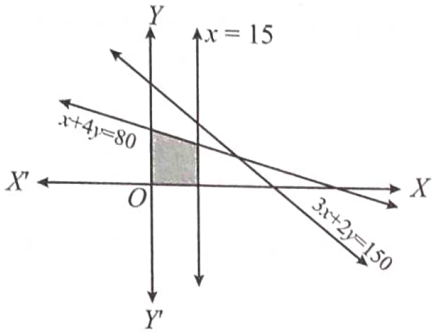
47. The solution set of the inequalities $4x + 3y \le 60$, $y \ge 2x$, $x \ge 3$, $x, y \ge 0$ in the figure is ... A... . Here, A refers to



(*a*) S_1 (*b*) S_2

(*c*) S_3 (*d*) S_4

48. Which of the following represent the solution set in the following figure?



(*a*) $3x + 2y \ge 150, x + 4y \le 80, x \le 15, x, y \ge 0$

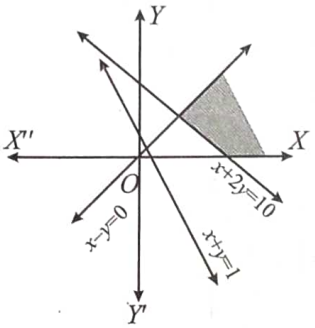
(*b*) $3x + 2y \le 150, x + 4y \ge 80, x \ge 15, x, y \ge 0$

(*c*) $3x + 2y \le 150, x + 4y \le 80, x \le 15, x, y \ge 0$

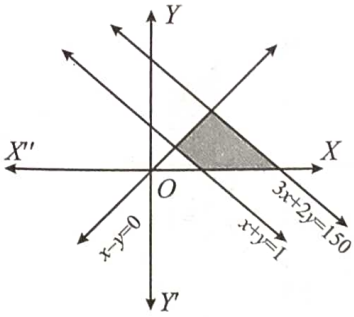
(*d*) None of the above

49. The graphical solution of the inequalities $x + 2y \le 10, x + y \ge 1, x - y \le 0, x \ge 0, y \ge 0$ is

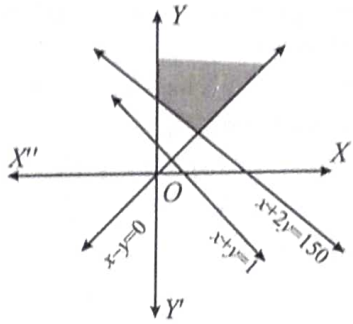
(*a*)



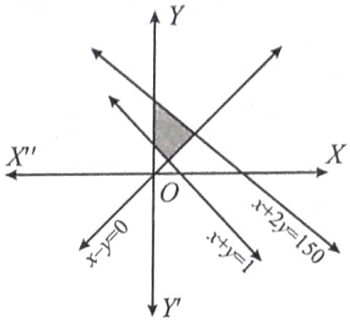
(*b*)



(*c*)



(*d*)



50. A shopkeeper sell two types of computer - a desktop model and portable model. He knows that total monthly demand will not exceed 250 units. If x denotes the number of computers of type desktop model and y denotes the number of computers of type portable model, then

(*a*) $x + y > 250$ (*b*) $x + y < 250$

(*c*) $x + y \le 250$ (*d*) $x + y \ge 250$

51. If $3 \le 3t - 18 \le 18$, then which one of the following is true ?

(*a*) $15 \le 2t + 1 \le 20$ (*b*) $8 \le t < 12$

(*c*) $8 \le t + 1 \le 13$ (*d*) $21 \le 3t \le 24$

52. The solution set of the inequalities $6 \le -3(2x - 4) < 12$ is

(*a*) $(-\infty, 1]$ (*b*) $(0, 1]$

(*c*) $(0, 1] \cup [1, \infty)$ (*d*) $[1, \infty)$

53. If $-3 \le 4 - \frac{7x}{2} \le 18$, then

(*a*) $x \in (-\infty, -4]$ (*b*) $x \in [-4, \infty)$

(*c*) $x \in [-4, 2]$ (*d*) $x \in [2, \infty)$

54. Consider the following statements

Statement I: The solution set of $2 \le 3x - 4 \le 5$ is $(2, 3)$.

Statement II: $(-80, -10)$ is the solution set of $-12 < 4 - \frac{3x}{-5} \le 2$

Choose the correct option.

(*a*) Statement I is true (*b*) Statement II is true

(*c*) Both are true (*d*) Both are false

55. If $5x + 1 > -24$ and $5x - 1 < 24$, then $x \in$ .... A ... .Here, A refers to

(*a*) $(-\infty, -5)$ (*b*) $(-5, 5)$

(*c*) $(5, \infty)$ (*d*) $(-\infty, 5)$

56. Which of the following is the solution set of linear inequalities $2(x - 1) < x + 5$ and $3(x + 2) > 2 - x$?

(*a*) $(-\infty, -1)$ (*b*) $(-1, 1)$

(*c*) $(-1, 7)$ (*d*) $(1, 7)$

57. If x satisfies the in equations $2x - 7 < 11$ and $3x + 4 < -5$, then x lines in the interval

(a) $(-\infty, 3)$ (b) $(-\infty, 2)$ (c) $(-\infty, -3)$ (d) $(-\infty, \infty)$

58. If $|x| < 3$ and x is a real number, then

(a) $x \geq 3$ (b) $-3 < x < 3$

(c) $x \leq -3$ (d) $-3 \leq x \leq 3$

59. x and b are real numbers. If $b > 0$ and $|x| > b$, then

(a) $x \in (-b, \infty)$ (b) $x \in [-\infty, b)$

(c) $x \in (-b, b)$ (d) $x \in (-\infty, -b) \cup (b, \infty)$

60. The solution set of $\frac{x-2}{x+5} > 2$ is

(a) $(-\infty, 12)$ (b) $[-12, -5]$

(c) $(-12, -5)$ (d) $(-5, \infty)$

61. If $|3 - 4x| \geq 9$, then

(a) $x \in \left(-\infty, -\frac{3}{2}\right)$ (b) $x \in [3, \infty)$

(c) $x \in \left(-\infty, -\frac{3}{2}\right] \cup [3, \infty)$ (d) $x \in (-\infty, \infty)$

62. If $\frac{|x-2|}{x-2} \geq 0$, then

(a) $x \in [2, \infty)$ (b) $x \in (2, \infty)$

(c) $x \in (-\infty, 2)$ (d) $x \in (-\infty, 2]$

63. If $a < b$ and $c < 0$, then

(a) $\frac{a}{c} = \frac{b}{c}$ (b) $\frac{a}{c} > \frac{b}{c}$

(c) $\frac{a}{c} < \frac{b}{c}$ (d) None of these

64. If $p > 0$ and $q < 0$, then $p + q$ ...A...p. Here, A refers to

(a) $>$ (b) $<$ (c) $\geq$ (d) $\leq$

65. If $-3x + 17 < -13$, then

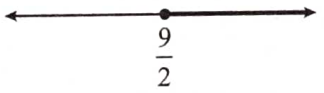
(a) $x \in (10, \infty)$

(b) $x \in (-\infty, 10)$

(c) $10x \in (-10, 10)$

(d) $x \in (-\infty, -10) \cup (10, \infty)$

66. The solution set representing by following figure is



(a) $x \in \left(\frac{9}{2}, \infty\right)$ (b) $x \in \left[\frac{9}{2}, \infty\right)$

(c) $x \in \left(-\infty, \frac{9}{2}\right)$ (d) $x \in \left(-\infty, \frac{9}{2}\right]$

67. The cost and revenue function of a product are given by $C(x) = 20x + 4000$ and $R(x) = 60x + 2000$ respectively, where x is the number of items produced and sold. Then, the number of items must be sold to realise some profit is given by

(a) $x = 50$ (b) $x > 50$

(c) $x < 50$ (d) None of these

68. If $|x + 3| \geq 10$, then

(a) $x \in (-13, 6]$ (b) $x \in (-13, 7)$

(c) $x \in (-\infty, -13] \cup [7, \infty)$ (d) None of These

69. The length of a rectangle is three times the breadth. If the minimum perimeter of the rectangle is 160 cm, then

(a) breadth > 20 cm (b) length < 20 cm

(c) breadth ≥ 20 cm (d) length ≤ 20 cm

70. The minimum value of $f(x) = |3 - x| + 7$ is

(a) 0 (b) 6

(c) 7 (d) 8

71. The set of values of x satisfying $2 \leq |x - 3| < 4$ is

(a) $(-1, 1] \cup [5, 7)$ (b) $-4 \leq x \leq 2$

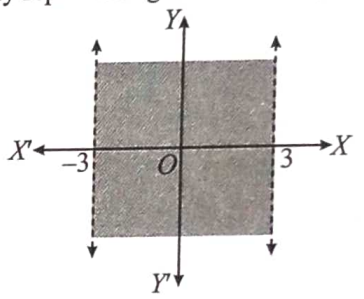
(c) $-1 < x < 7$ or $x \geq 5$ (d) $x < 7$ or $x \geq 5$

72. The number of pairs of consecutive even positive integers, both of which are larger than 8, such that their sum is less than 25, is

(a) 1 (b) 2

(c) 3 (d) 4

73. The inequality representing the following graphs is

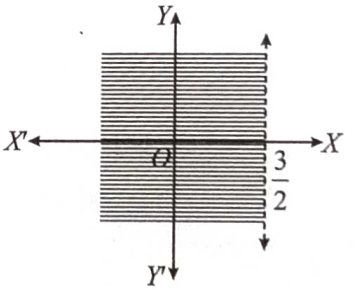


(a) $|x| < 3$ (b) $|x| \leq 3$

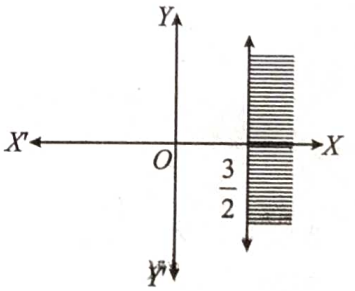
(c) $|x| > 3$ (d) $|x| \geq 3$

74. The graph of $2x - 3 \geq 0$ is

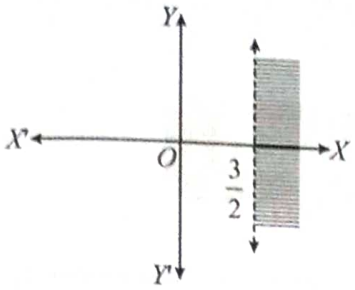
(a)



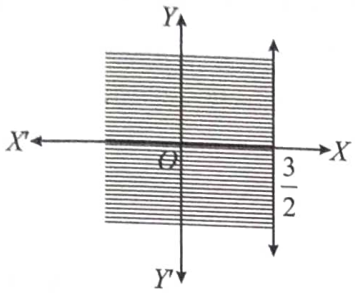
(b)



(c)

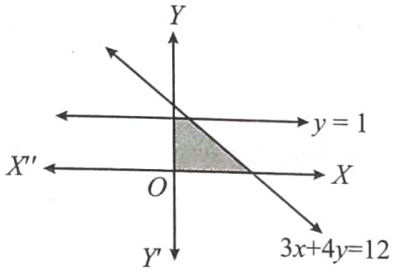


(d)

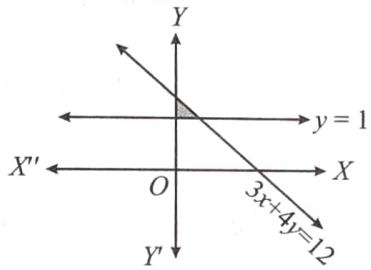


75. The gaphical solution of the system of linear inequalities $3x + 4y \geq 12, y \geq 1, x \geq 0$ is

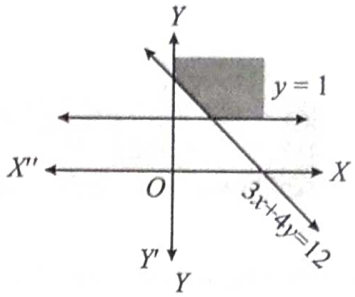
(a)



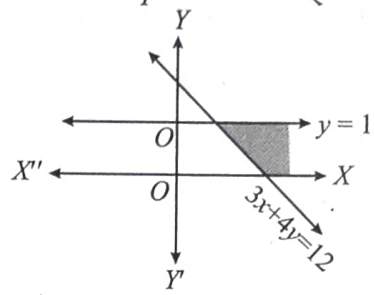
(b)



(c)



(d)

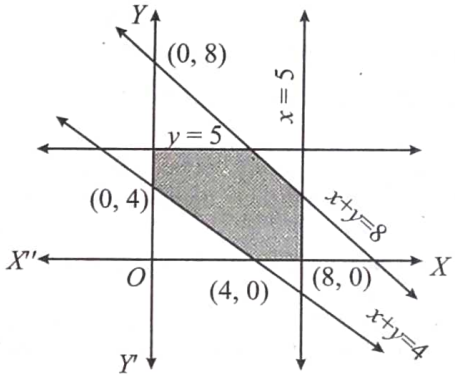


76. The region represented by the solution set of the system of inequalities $x - y \leq 1, x + 2y \leq 8, 2x + y \geq 2, x \geq 0, y \geq 0$ is

(a) bounded (b) unbounded

(c) Does not exist (d) None of these

77. Linear inequalities for which the shaded region for the given figure is the solution set, are



(a) $x + y \leq 8, x + y \leq 4, x \leq 5, y \leq 5, x \geq 0, y \geq 0$

(b) $x + y \leq 8, x + y \geq 4, x \leq 5, y \leq 5, x \geq 0, y \geq 0$

(c) $x + y \geq 8, x + y \geq 4, x \geq 5, y \geq 5, x \geq 0, y \geq 0$

(d) None of the above

Exercise-2 (Learning Plus)

1. Solve each of the following system of equation in R.

$5x-7<3(x+3),\ 1-\frac{3x}{2}\geq x-4$

2. Solve each of the following system of equation in R.

$\frac{2x-3}{4}-2\geq\frac{4x}{3}-6,\ 2(2x+3)<6(x-2)+10$

3. Solve each of the following system of equation in R.

$\frac{7x-1}{2}<-3,\frac{3x+8}{5}+11<0$

4. Solve each of the following system of equation in R.

$\frac{2x+1}{7x-1}>5,\frac{x+7}{x-8}>2$

5. A solution is to be kept between 86° and 95°F. What is the range of temperature in degree Celsius, if the Celsius (C)/ Fahrenheit (F) conversion formula is given by $F=\frac{9}{5}C+32$

6. A solution is to be kept between 30°C and 35°C. What is the range of temperature in degree Fahrenheit?

7. IQ of a person is given by the formula : $IQ=\frac{MA}{CA}\times 100$, where MA is mental age and CA is chromological age. If $80\leq IQ\leq 140$ for a group of 12 year children find the range of their mental age.

8. The marks scored by Rohit in two tests were 65 and 70. Find the minimum marks he should score in the third test to have an average of at least 65 marks.

9. A company manufactures cassettes and its cost and revenue functions for a week are $C=300+\frac{3}{2}x$ and $R=2x$ respectively, where is the number of cassettes produced and sold in a week. How many cases trust be sold for the company to realize a profit?

10. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?

11. Solve each of the following system of equation in R.

$\left|x+\frac{1}{3}\right|>\frac{8}{3}$

12. Solve each of the following system of equation in R.

$|4-x|+1<3$

13. Solve each of the following system of equation in R.

$\left|\frac{3x-4}{2}\right|\leq\frac{5}{12}$

14. Represent to solution set of each of the following in equation graphically in two dimensional plane:

$3x-2y\leq x+y-8$

15. Solve the following in equation graphically:

$|y-x|\leq 3$

16. Solve the following in equation graphically:

$|x-y|\geq 1$

17. Solve each of the following system of equations in R.

$\frac{1}{|x|-3}<\frac{1}{2}$

18. Solve each of the following system of equations in R.

$\frac{|x+2|-x}{x}<2$

19. Solve each of the following system of equations in R.

$\left|\frac{2x-1}{x-1}\right|>2$

20. Solve each of the following system of equations in R.

$|x-1|+|x-2|+|x-3|\geq 6$

21. Solve each of the following system of equations in R.

$\frac{|x-2|-1}{|x-2|-2}\leq 0$

22. Solve each of the following system of equations in R.

$\frac{1}{|x|-3}\leq\frac{1}{2}$

23. Solve each of the following system of equations in R.

$|x+1|+|x|>3$

24. Solve the following system of in equations graphically:

$2x+y\geq 8,\ x+2y\geq 8,\ x+y\leq 6$

25. Solve the following system of in equations graphically:

$12x+12y\leq 840,\ 3x+6y\leq 300,\ 8x+4y\leq 480,\ x\geq 0,\ y\geq 0$

26. Solve the following system of in equations graphically:

$x+2y\leq 40,\ 3x+y\geq 30,\ 4x+3y\geq 60,\ x\geq 0,\ y\geq 0$

27. Solve the following system of in equations graphically:

$5x+y\geq 10,\ 2x+2y\geq 12,\ x+4y\geq 12,\ x\geq 0,\ y\geq 0$

28. Show that the following system of linear equations has no solution:

$x+2y\leq 3,\ 3x+4y\geq 12,\ x\geq 0,\ y\geq 1.$

ANSWER KEY

CONCEPT APPLICATION

1. (*a*) $x = \{....., -3, -2, -1, 0, 1, 2\}$ (*b*) $x = \{5, 6, 7, 8, ...\}$
2. (*a*) $x \in [3, \infty)$ (*b*) $x \in (3, \infty)$
3. $x \in (-2, \infty)$ 4. $x \in (-\infty, 3)$
5. (i) $x \in (-\infty, 63/10]$ (ii) $x \in (4, \infty)$
   (iii) $x \in (-\infty, 120]$
   (iv) $x \in [2, \infty)$
   (v) $x \in (-\infty, -2)$
6. $x \in [2, 4]$ 7. $x \in (3, \infty)$, 8. No solution. 13. $x \in \left[\frac{1}{2}, \frac{5}{6}\right]$ 14. $x \in (-\infty, -3] \cup [7, \infty)$
15. $x \in [-4, -2] \cup [2, 6]$ 16. $x \in (-2, -1] \cup [1, 2)$ 17. (9.8, 13.8) 18. (6.27, 8.07)

EXERCISE-1 (TOPICWISE)

1. (*b*)	2. (*c*)	3. (*b*)	4. (*a*)	5. (*d*)	6. (*d*)	7. (*c*)	8. (*c*)	9. (*a*)	10. (*a*)
11. (*c*)	12. (*a*)	13. (*a*)	14. (*a*)	15. (*d*)	16. (*b*)	17. (*c*)	18. (*c*)	19. (*a*)	20. (*b*)
21. (*a*)	22. (*a*)	23. (*b*)	24. (*b*)	25. (*a*)	26. (*b*)	27. (*c*)	28. (*b*)	29. (*b*)	30. (*a*)
31. (*a*)	32. (*c*)	33. (*a*)	34. (*c*)	35. (*c*)	36. (*a*)	37. (*d*)	38. (*b*)	39. (*a*)	40. (*b*)
41. (*c*)	42. (*c*)	43. (*a*)	44. (*b*)	45. (*d*)	46. (*a*)	47. (*b*)	48. (*c*)	49. (*d*)	50. (*c*)
51. (*c*)	52. (*b*)	53. (*c*)	54. (*d*)	55. (*b*)	56. (*c*)	57. (*c*)	58. (*b*)	59. (*d*)	60. (*c*)
61. (*c*)	62. (*b*)	63. (*b*)	64. (*b*)	65. (*a*)	66. (*b*)	67. (*b*)	68. (*c*)	69. (*c*)	70. (*c*)
71. (*a*)	72. (*a*)	73. (*a*)	74. (*b*)	75. (*c*)	76. (*a*)	77. (*b*)			

EXERCISE-2 (LEARNING PLUS)

1. $(-\infty, 2]$ 2. No solution 3. $(-\infty, 21)$ 4. No solution 5. Between 30°C and 35°C
6. Between 86°F and 95°F 7. $9.6 \le MA \le 16.8$ 8. 60 9. More than 600
10. More than 562.5 litres but less than 90 litres 11. $(-\infty, 3) \cup (7/3, \infty)$ 12. (2, 6) 13. [19/18, 29/18]
17. $(-\infty, -5) \cup (-3, 3) \cup (5, \infty)$ 18. $(-\infty, 0) \cup (1, \infty)$ 19. $(3/4, 1) \cup (1, \infty)$ 20. $(-\infty, 0] \cup [4, \infty)$
21. $(0, 1] \cup [3, 4)$ 22. $(-\infty, -5] \cup (-3, 3) \cup [5, \infty)$ 23. $(-\infty, -2) \cup (1, \infty)$

CHAPTER

10 Permutations and Combinations

INTRODUCTION

A branch of mathematics where we count number of objects or number of ways of doing a particular job without actually counting them, is known as combinatorics and in this chapter we will deal with elementary combinatorics.

For example, if in a room there are five rows of chairs and each rows contains seven chairs, then without counting them we can say, total number of chairs is 35. We start this chapter with fundamental principle of counting.

FUNDAMENTAL PRINCIPLE OF COUNTING

Additional Principle

If a work can be done in m different ways and another work which is independent of first can be done in n different ways, then either of the two operations can be preformed in $(m + n)$ ways.

Illustrating the Concepts

There are 15 gates to enter a city from north and 10 gates to enter the city from east. In how many ways a person can enter the city?

Number of ways to enter the city from north = 15

Number of ways to enter the city from east = 10

A person can enter the city from north or from east.

Hence, the number of ways to enter the city = 15 + 10 = 25

Multiplication Principle

If one operation (I) can be done in m different ways and another Operation (II) can be performed in n different ways, then total number of ways in which both of these can be performed together is $m \times n$. If there are more than two operations to be done, then the total number of different ways to do all of them together will be $m \times n \times p \times \ldots$

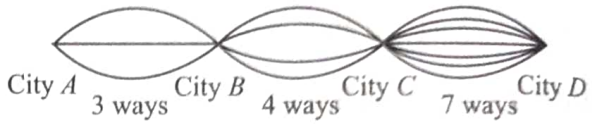
For example: Operation A can be performed in "a, b, c" different ways and operation B can be perfomred in p, q different ways, then possible ways to perform operation A and operation B together are a-p, a-q, b-p, b-q, c-p, c-q i.e., each way of operation A is combined by 2 ways operation B.

i.e., Total ways = 2 + 2 + 2 = 2 added 3 times = 2 × 3 = 6.

In general, total ways = $m \times n$ where m and n are individual ways of performing the operations.

Illustrating the Concepts

There are 3 routes to travel from city A to city B and 4 routes travel from city B to city C and 7 routes from C to D. In how many different ways (routes) a man can travel from city A to city D via city B and city C.



The man can perform the task of travelling from city A to city B in ways = 3.

The man can perform task of travelling from city B to city C in ways = 4.

Similarly from city C to city D in ways = 7.

Using fundamental principle of counting,

Total routes to travel from A to D via B and via $C = m \times n \times p$

$= 3 \times 4 \times 7 = 84$ routes.

Train Your Brain

Example 1: There are 20 boys and 10 girls in a group. In how many ways can a group leader be selected if the group leader can either be a boy or a girl.

Sol. If it is clear by above sum rule that to select leader, we have 20 boys or 10 girls

⇒ Total number of ways to do the same is 20 + 10 = 30 ways.

Example 2: How many possible out comes are there if a coin is tossed 5 times?

Sol. Each time there are two possible outcomes

∴ Number of outcomes when coin is tossed 5 times = 2 × 2 × 2 × 2 × 2 = 32

Example 3: In how many ways 2 different prizes be awarded to 15 students, without giving both to the same student?

Sol. First prize can be awarded in 15 ways and corresponding to each way of doing so, the second prize can be awarded in 14 ways. Hence, total ways = 15 × 14.

Concept Application

1. There are nine students (5 boys and 4 girls) in the class. In how many ways
   (*i*) One student (either girl or boy) can be selected to represent the class.
   (*ii*) A team of two students (one girl and one boy) can be selected.
   (*iii*) Two medals can be distributed. (No one get both)
   (*iv*) One prize for maths, two prizes for physics and three prizes for chemistry can be distributed.
   (No student can get more than one prize in same subject and prizes are distinct)

NOTATION OF FACTORIAL

Notation of Factorial and its Algebra

The continued product of first n, natural number is called as "n factorial" and dentoed by $n!$ or $\lfloor n$

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots (n-1)n$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$3! = 1 \cdot 2 \cdot 3 = 6$

$5! = 120;\ 6! = 720;\ 7! = 5040$

Special Results

- $0! = 1$ i.e. factorial of zero is 1

  Proof: $n! = (n-1)! \cdot n$

  Putting $n = 1$

  $1! = (1-1)!\ 1 \Rightarrow 0! = 1$
- Factorial of negative number is underfined

  Illustrating the concept:

  (*i*) Find n if $(n+1)! = 12 \times (n-1)!$

  (*ii*) $(n+2)! = 2550n!$

  Sol.

  (*i*) $(n+1)\, n(n-1)! = 12 \times (n-1)!$

  $n^2 + n - 12 = 0;$

  $(n+4)(n-3) = 0 \therefore n = 3$

  (*ii*) $(n+2)(n+1) = 2550;$

  $(n+52)(n-49) = 0 \therefore n = 49$
- $(2n!) = 2^n \cdot n!\,[1 \cdot 3 \cdot 5 \ldots (2n-1)]$

  Proof: $(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \ldots (2n-1)(2n)$

  Take 2 common each n even terms

  $= 2^n (1 \cdot 3 \cdot 5 \ldots (2n-1))(1 \cdot 2 \cdot 3 \ldots n)$

  $= 2^n (n!)(1 \cdot 3 \cdot 5 \ldots (2n-1))$

PERMUTATIONS

Each of different arrangement which can be made by taking some or all of a number of things is called a permutation. It is assumed that

- The given things let us say there are n of them are all distinct, that is, no two are alike.
- The arrangement is placing one thing next to another as in a straight line; hence it is also known as linear permutation.
- In any arrangement, any one thing is used only once. In other words, there is no repetition.

Counting Formulae for Permutation

The meaning of ${}^nP_r$

Suppose there are r blank spaces in a row and n different letters. The number of ways of filling up the blank spaces with n different letters is the number of ways of arranging n things r at a time, i.e. ${}^nP_r$. It must be noted that each space has to be filled up with only one letter.

The first space can be filled in n ways. Having filled it, there are $n-1$ letters left and therefore the second space can be filled in $n-1$ ways. Hence the first two spaces can be filled in $n(n-1)$ ways. When the first two spaces are filled, there are $n-2$ letters left, so that the third space can be filled in $n-2$ ways. Therefore the first three spaces can be filled in $n(n-1)(n-2)$ ways; proceeding like this, the r spaces can be filled in $n(n-1)(n-2) \ldots [n-(r-1)]$ ways.

The number of permutations of n things taken r at a time is denoted as ${}^nP_r$ and its value is equal to

$${}^nP_r = n(n-1)(n-2) \ldots (n-r+1)$$

$$= \frac{n!}{(n-r)!} \text{ (using factorial notation)}$$

$n! = n(n-1) \ldots 3.2.1.$ where $0 \le r \le n$.

In particular

- The number of permutations of n different things taken all at a time $= {}^nP_n = n!$

  ${}^nP_0 = 1$, ${}^nP_1 = n$ and ${}^nP_{n-1} = {}^nP_n = n!$

  ${}^nP_r = n({}^{n-1}P_{r-1})$ where $r = 1, 2, \ldots n$.

Important Results

Number of permutations under certain conditions:

- Number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement is $r.{}^{(n-1)}P_{r-1}$.
- Number of permutations of n different things, taken r at a time, when a particular thing is never taken in any arrangement is ${}^{n-1}P_r$.
- Number of permutations of n different things, taken r at a time, when m particular things are never taken in any arrangement is ${}^{n-m}P_r$.
- Number of permutations of n different things, taken all at a time, when m specified things always come together is $(m!)(n-m+1)!$.

- Number of permutations of n different things, taken all at a time, when m specified things never come together is $n! - m!(n-m+1)!$

PERMUTATION OF N DISTINCT OBJECT WHEN REPETITION IS ALLOWED

The number of permutations of n different things taken r at time when each thing may be repeated any number of times is n^r.

In other words if a job can be completed in n ways and it is to be repeated r number of times then the total number of ways is n^r.

As an example the number of 5 digit numbers, in which digits are not repeated is $9({}^9P_4)$ while when repetition is allowed is 9×10^4.

Also if n distinct things are arranged at r places when repetition is allowed, then the number of arrangements are $\underbrace{n \times n \times \ldots \times n}_{r\ times} = n^r$.

Train Your Brain

Example 4: In a steamer there are stalls for 12 animals, and there are cows, horses, and calves (not less than 12 of each) ready to be shipped; in how many ways can the shipload be made?

Sol. The first stall can be occupied by a cow, a horse or a calf, i.e., it can be occupied in 3 ways. The second stall can also be occupied in 3 ways. Therefore the first 2 stalls can be occupied in 3^2 ways. Proceeding like this, it is clear that the 12 stalls can be occupied in 3^{12} ways, i.e., in 531441 ways.

Example 5: Show that ${}^nP_r = {}^{(n-1)}P_r + r.{}^{(n-1)}P_{(r-1)}$

Sol. ${}^{(n-1)}P_r + r.{}^{(n-1)}P_{(r-1)} = \dfrac{(n-1)!}{(n-1-r)!} + \dfrac{r(n-1)!}{(n-r)!}$

$$= (n-1)!\left[\frac{1}{(n-1-r)!} + \frac{r}{(n-r)!}\right]$$

$$= (n-1)!\,\frac{(n-r+r)}{(n-r)!}$$

$(\because (n-r)! = (n-r)(n-r-1)!)$

$$= \frac{n(n-1)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

A common sense interpretation of the identity above is possible. The number of permutations of r things which may be made from n things, $r.{}^{(n-1)}P_{r-1}$ contain one specified thing and ${}^{(n-1)}P_r$ do not contain that specified thing and these two together give ${}^nP_r$.

Example 6: How many three digit numbers can be formed using the digits 1, 2, 3, 4, 5, without repetition of digits? How many of these are even?

Sol. Three places are to be filled with 5 different objects.

$\therefore$ Number of ways $= {}^5P_3 = 5 \times 4 \times 3 = 60$

For the 2nd part, unit digit can be filled in two ways & the remaining two digits can be filled in ${}^4P_2$ ways.

$\therefore$ Number of even numbers $= 2 \times {}^4P_2 = 24$.

Example 7: If all the letters of the word 'QUEST' are arranged in all possible ways and put in dictionary order, then find the rank of the word QUEST.

Sol. Number of words beginning with $E = {}^4P_4 = 24$

Number of words beginning with $QE = {}^3P_3 = 6$

Number of words beginning with $QS = 6$

Number of words beginning with $QT = 6$.

Next word is 'QUEST'

$\therefore$ its rank is $24 + 6 + 6 + 6 + 1 = 43$.

Example 8: There are m men and n monkeys ($n > m$). If a man can have any number of monkeys, in how many ways may every monkey have a master?

Sol. The first monkey can select his master by m ways, and after that the second monkey can select his master again by m ways, so can the third. And so on, hence all monkeys can select master by $= m \times m \times m \ldots$ up to n times $= (m)^n$ ways.

Example 9: How many numbers between 10 and 10,000 can be formed by using the digits 1, 2, 3, 4, 5 if

(*i*) No digit is repeated in any number.

(*ii*) Digits can be repeated.

Sol.

(*i*) Number of two digit numbers $= 5 \times 4 = 20$

Number of three digit numbers $= 5 \times 4 \times 3 = 60$

Number of four digit numbers $= 5 \times 4 \times 3 \times 2 = 120$

Total $= 200$

(*ii*) Number of two digit numbers $= 5 \times 5 = 25$

Number of three digit numbers $= 5 \times 5 \times 5 = 125$

Number of four digit numbers $= 5 \times 5 \times 5 \times 5 = 625$

Total $= 775$

Concept Application

2. If ${}^{(m+n)}P_2 = 56$ and ${}^{m-n}P_2 = 12$ then (m, n) equals

(*a*) (5, 1) (*b*) (6, 2) (*c*) (7, 3) (*d*) (9, 6)

3. How many numbers of four digits greater than 2300 can be formed with the digits 0, 1, 2, 3, 4, 5 and 6; no digit being repeated in any number?

(*a*) 560 (*b*) 590 (*c*) 90 (*d*) 360

4. The letters of the word SURITI are written in all possible orders and these words are written out as in a dictionary.

Then the rank of the words SURITI, is

(a) 236 (b) 263

(c) 326 (d) 260

5. There are $(n + 1)$ white and $(n + 1)$ black balls each set numbered 1 to $(n + 1)$. The number of ways in which the balls can be arranged in row so that the adjacent balls are of different colours is-

(a) $(2n + 2)!$ (b) $(2n + 2)! \times 2$

(c) $(n + 1)! \times 2$ (d) $2\{(n + 1)!\}^2$

6. The total number of words which can be formed using 'FAILURE' letters of the word that consonants always occupy odd places, is-

(a) 144 (b) 576

(c) 5040 (d) 625

7. In how many ways four friends can put up in 8 hotels of a town if

(i) There is no restriction

(a) 4096 (b) 4609

(c) 4926 (d) 4962

(ii) No two friends can stay together

(a) 1680 (b) 1608

(c) 1660 (d) 1672

(iii) All the friends do not stay in same hotel?

(a) 4088 (b) 4808

(c) 4880 (d) 4888

Arrangement of n Things When All are Not Distinct

- If given n things are not all distinct, then it is possible that few many be of one kind, and few others may be of second kind, etc. In such case, the number of permutations of n things taken all at a time, where p are alike of one kind, q are alike of second kind and r are alike of third kind and the rest $n - (p + q + r)$ are all distinct is given by

$$\frac{n!}{p!q!r!} \qquad (p + q + r \le n)$$

Train Your Brain

Example 10: Find total number of word's formed by using all letters of the word "IITJEE"

Sol. Ways are $= \dfrac{6!}{2!(2!)}$

Example 11: Consider word ASSASSINATION, find number of ways of arranging the letters of word using all.

Sol. (i) ASSASSINATION contains four S, three A, two N and two I.

$$\text{Total ways} = \frac{13!}{(4!)\,(3!)\,(2!)\,(2!)}$$

Example 12: In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative positions of vowels & consonants?

Sol. The consonants in their positions can be arranged in $\dfrac{4!}{2!} = 12$ ways.

The vowels in their positions can be arranged in $\dfrac{3!}{2!} = 3$ ways.

$\therefore$ Total number of arrangements $= 12 \times 3 = 36$

Concept Application

8. In how many ways can 21 identical white and 19 identical black balls be put in a row so that no two black balls are together?

(a) 1470 (b) 1540

(c) 735 (d) 745

9. If out of 8 flags, 5 flags are white (identical) and 3 flags are of red colour (identical), then how many signals can be given by using all of them at a time?

(a) 15 (b) 28

(c) 56 (d) 126

10. How many six letter words be made out of the letters of 'ASSIST? In how many words the alphabet S alternates with other letters?

(a) 120, 6 (b) 720, 12

(c) 120, 12 (d) 720, 24

11. In how many ways the alphabets of the word 'MULTIPLE can be ordered other than itself, when the order of vowels is not changed?

(a) 3360 (b) 3359

(c) 6720 (d) 20160

Exponent of Prime p in n!

Exponent of prime p in n! is denoted by $E_p(n!)$, where p is a prime number and n is a natural number. Then the last integer amongst 1, 2, 3, ..., $(n - 1)$, n which is divisible by p is $[n/p]p$, where $[n/p]$ denotes the greatest integer. ($\because [x] \le x$)

$$E_p(n!) = E_p(1, 2, 3 \ldots (n-1).n)$$
$$= E_p\,(p.2p.3p. \ldots (n-1)p.[n/p]p)$$

Because the remaining natural numbers from 1 to n are not divisible by p.

Thus, $\left[\frac{n}{p}\right] + E_p\left(1.2.3...\left[\frac{n}{p}\right]\right)$

Now the last integer among 1, 2, 3, ..., $\left[\frac{n}{p}\right]$ which is divisible by p is

$$\left[\frac{n/p}{p}\right] = \left[\frac{n}{p^2}\right] = \left[\frac{n}{p}\right] + E_p\left(p.2p.3p...\left[\frac{n}{p^2}\right]p\right)$$

Because the remaining natural numbers from 1 to $\left[\frac{n}{p}\right]$ are not divisible by p.

Thus, $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1.2.3...\left[\frac{n}{p^2}\right]\right)$

Similarly, we get

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ... + \left[\frac{n}{p^s}\right]$$

Where s is the largest natural number such that $p^s \le n < p^{s+1}$

Note: $E_{rp}(n!) = E_r(n!)$ (If $r > p$)

Where r and p are prime numbers.

Train Your Brain

Example 13: Find the number of zeros at the end of 100! .

Sol. In terms of prime factors 100! can be written as $2^a.3^b.5^c.7^d$ ...

Now

$$E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

$= 50 + 25 + 12 + 6 + 3 + 1 = 97$

and $E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$

$\therefore 100! = 2^{97} . 3^b . 5^{24} . 7^d .....$

$= 2^{73} . 3^b . (2 \times 5)^{24} . 7^d ....$

$= 2^{73} . 3^b . (10)^{24} . 7^d ....$

Hence, number of zeros at the end of 100 ! is 24.

Exponent of 10 in 100 ! = min (97, 24) = 24.

Example 14: Find the highest power of 3 in 98!

Sol. $E_3(98!) = \left[\frac{98}{3}\right] + \left[\frac{98}{9}\right] + \left[\frac{98}{27}\right] = 32 + 10 + 3 = 45$

Example 15: For how many values of n, $n!$ ends in 15 zeros?

Sol. Since $[x] \le x$,

$$\therefore E_5(n!) = \left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right] + ... < \frac{n}{5} + \frac{n}{5^2} + ...\infty$$

$$\Rightarrow 15 < \frac{\frac{n}{5}}{1 - \frac{1}{5}} \Rightarrow n > 60$$

Let us check from 61 onwards.

$E_5(64!) = 14$, $E_5(65!) = 15$, $E_5(66!) = 15$

$E_5(67!) = 15$, $E_5(68!) = 15$, $E_5(69!) = 15$,

n can be 65, 66, 67, 68, 69.

Concept Application

12. If $(500)! = k.(2^a 3^b 5^c)$, (where $a, b, c \in N$, k does not contain any factor of 2, 3 and 5) then which one is/are correct

(*a*) $a > c$

(*b*) $c = 124$

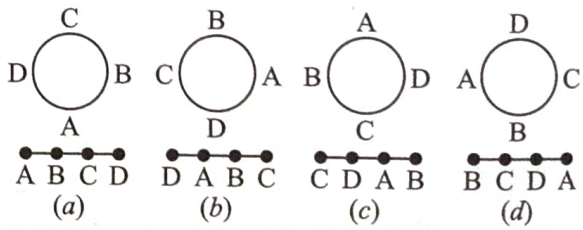
(*c*) Number of zeros at the end of (500)! is equal to 124

(*d*) (500)! is an even number

CIRCULAR PERMUTATIONS

In the event of the given n things arranged in a circular or even elliptical permutation the first and the last thing in the arrangement are indistinguishable.

Arrangements Around a Circular Table



Let us consider that persons A, B, C, D are sitting around a round table. If all of them (A, B, C, D) are shifted by one place in anticlockwise order, then we will get Fig. (*b*) from Fig.(*a*). Now, if we shift A, B, C, D in anticlockwise order, we will get Fig.(*c*). Again, if we shift them, we will get Fig. (*d*) and in the next time, Fig. (*a*).

Thus, we see that if 4 persons are sitting at a round table, they an be shifted four times and the four different arrangements, thus btained will be the same, because anticlockwise order of A, B, C,) does not change.

But if A, B, C, D are sitting in a row and they are shifted in uch an order that the last occupies the place of first, then the four rrangements will be different.

Thus, if there are 4 things, then for each circular arrangement umber of linear arrangements is 4.

Similarly, if n different things are arranged along a circle, for ach circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different ings is n (number of circular arrangements of different things). Hence, the number of circular arrangements of n different things s-

$$\frac{1}{n}\text{ (number of linear arrangements of } n \text{ different things)} = \frac{n!}{n}$$

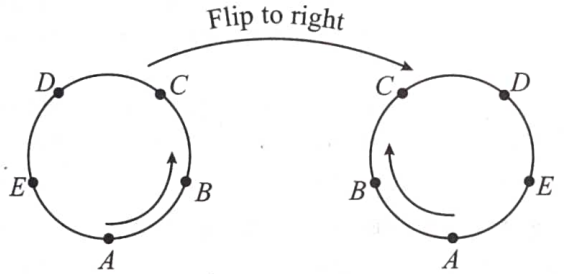
$= (n-1)!$

Therefore note that:

The number of circular permutations of n different things aken all at a time is : $(n-1)!$.

Arrangements of beads or flowers (all different) around a ircular necklace or garland

Consider five beads A, B, C, D, E in a necklace or five flowers A, B, C, D, E in a garland etc. If the necklace or garland on the eft is turned over we obtain the arrangement on the right, i.e., nticlockwise and clockwise order of arrangement is not different we will get arrangements as follows:



We see that arrangements in above figures are not different.

Then the number of circular permutations of n different things aken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are taken as not different.

Number of circular permutations of n different things taken r at a time

CASE-I: If clockwise and anticlockwise orders are taken as different, then the required number of circular permutations $= \frac{{}^nP_r}{r}$.

CASE-II: If clockwise and anticlockwise orders are taken as not different, then the required number of circular permutations $= \frac{{}^nP_r}{2r}$

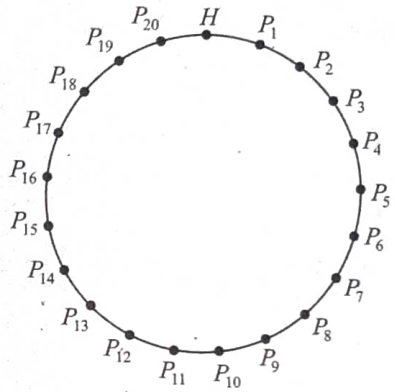
Train Your Brain

Example 16: 20 persons were invited to a party. In how many ways can they and the host be seated at a circular table? In how many of these ways will two particular persons be seated on either side of the host?

Sol. 1st part: Total persons on the circular table = 20 guest + 1 host = 21 they can be seated in $(21-1)! = 20!$ ways.

2nd part: After fixing the places of three persons (1 host + 2 persons). Treating (1 host + 2 person) = 1 unit, so we have now {(remaining 18 persons + 1 unit) = 19} and the number of arrangement will be $(19-1)! = 18!$ also these two particular person can be seated on either side of the host in 2! ways.

Hence, the number of ways of seating 21 persons on the circular table such that two particular persons be seated on either side of the host = $18! \times 2! = 2 \times 18!$



Example 17: Consider 21 different pearls on a necklace. How many ways can the pearls be placed in on this necklace such that 3 specific pearls always remain together?

Sol. After fixing the places of three pearls. Treating 3 specific pearls = 1 units, so we have now 18 pearls + 1 unit = 19 and the number of arrangement will be $(19-1)! = 18!$ also, the number of ways of 3 pearls can be arranged between themselves is $3! = 6$. Since, there is no distinction between the clockwise and anticlockwise arrangements. So, the required number of arrangements $= \frac{1}{2}\ 18!\ .\ 6 = 3\ (18!)$.

Example 18: In how many ways can 24 persons be seated round a table, if there are 13 seats?

Sol. In case of circular table the clockwise and anticlockwise order are different, then the required number of circular permutations $= \frac{{}^{24}P_{13}}{13} = \frac{24!}{13 \times 11!}$.

Example 19: How many necklace of 12 beads each can be made from 18 beads of various colours?

Sol. In the case of necklace there is not distinction between the clockwise and anticlockwise arrangements, then the required number of circular permutations

$$= \frac{{}^{18}P_{12}}{2\times 12} = \frac{18!}{6!\times 24}$$

$$= \frac{18\times 17\times 16\times 15\times 14\times 13!}{6\times 5\times 4\times 3\times 2\times 1\times 24} = \frac{119\times 13!}{2}$$

Example 20: In how many ways 10 boys and 5 girls can sit around a circular table so that no two girls sit together.

Sol. 10 boys can be seated in a circle in 9! ways. There are 10 spaces in between the boys, which can be occupied by 5 girls in ${}^{10}P_5$ ways. Hence total numbers of ways are 9! ${}^{10}P_5$.

Example 21: In how many ways can we arrange 6 different flowers in a circle? In how many ways we can form a garland using these flowers?

Sol. The number of circular arrangements of 6 different flowers = (6 – 1)! = 120

When we form a garland, clockwise and anticlockwise arrangements are similar. Therefore, the number of ways of forming garland $= \frac{1}{2}(6-1)! = 60$.

Concept Application

13. 20 persons were invited to a party. In how many ways can they and the host be seated at a circular table? In how many of these ways will two particular persons be seated on either side of the host.

(*a*) 20!, 2! 18! (*b*) 18!, 2! 20!
(*c*) 19!, 18! (*d*) 19!, 19!

14. The number of ways in which 4 boys and 4 girls can stand in a circle so that each boy and each girl is one after the other is-

(*a*) 3!. 4! (*b*) 4!
(*c*) 8!. 4! (*d*) 7!

15. The number of ways in which 6 red roses and 3 white roses (all roses different) can form a garland so that all the white roses come together is-

(*a*) 2170 (*b*) 2165
(*c*) 2160 (*d*) 2155

COMBINATIONS

Each of different grouping or selections that can be made by some or all of a number of given things without considering the order in which things are placed in each group, is called combinations.

Counting Formulae for Combinations

The number of combinations (selections or groupings) that can b formed from n different objects taken *r* at a time is denoted by ${}^nC$ and its value is equal to

$${}^nC_r = \frac{n!}{(n-r)!r!} \quad (0 \le r \le n)$$

as $${}^nC_r = \frac{{}^nP_r}{r!}$$

As in a permutation the arrangement of r selected objects o of n, is done in r! ways and in combination arrangement in a grou is not considered.

In particular

- ${}^nC_0 = {}^nC_n = 1$ i.e. there is only one way to select none or t select all objects out of n distinct objects.
- ${}^nC_1 = n$ There are *n* ways to select one thing out of *n* distinc things.
- ${}^nC_r = {}^nC_{n-r}$

  Therefore ${}^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$.
- If *n* is odd then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$ or ${}^nC_{\frac{n-1}{2}}$.
- If *n* is even then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$.

Important Results of Combinations (Selections)

- The number of ways in which *r* objects can be selected from *n* distinct objects if a particular object is always included is ${}^{n-1}C_{r-1}$.
- The number of ways in which *r* objects can be selected from *n* distinct objects if a particular object is always excluded is ${}^{n-1}C_r$.
- The number of ways in which *r* objects can be selected from *n* distinct objects if *m* particular objects are always included is ${}^{n-m}C_{r-m}$.
- The number of ways in which *r* objects can be selected from *n* distinct objects if *m* particular objects are always excluded is ${}^{n-m}C_r$.

Some Important Results About Points

If there are *n* points in a plane which *m* (< *n*) are collinear, then

(*a*) The number of different straight lines obtained by joining these *n* points is

${}^nC_2 - {}^mC_2 + 1$

(*b*) Total number of different triangles formed by joining these *n* points is

${}^nC_3 - {}^mC_3$

(*c*) Number of diagonals in polygon of *n* sides is

${}^nC_2 - n$ i.e. $\frac{n(n-3)}{2}$

(*d*) If *m* parallel lines in a plane are intersected by a family of other *n* parallel lines. Then total number of parallelograms so formed is

${}^mC_2 \times {}^nC_2$ i.e. $\frac{mn(m-1)(n-1)}{4}$

(e) Number of triangles formed by joining vertices of convex polygon of n sides is ${}^nC_3$ of which

(i) Number of triangles having exactly two sides common to the polygon $= n$

(ii) Number of triangles having exactly one side common to the polygon $= n(n-4)$

(iii) Number of triangles having no side common to the polygon $= \dfrac{n(n-4)(n-5)}{6}$

Train Your Brain

Example 22: If ${}^{49}C_{3r-2} = {}^{49}C_{2r+1}$, find '$r$'.

Sol. ${}^nC_r = {}^nC_s$ if either $r = s$ or $r + s = n$.

Thus $3r - 2 = 2r + 1 \Rightarrow r = 3$

or $3r - 2 + 2r + 1 = 49 \Rightarrow 5r - 1 = 49 \Rightarrow r = 10$

$\therefore r = 3, 10$

Example 23: Fifteen players are there for selection for a cricket match.

(i) In how many ways the playing 11 can be selected?

(ii) In how many ways the playing 11 can be selected including a particular player?

(iii) In how many ways the playing 11 can be selected excluding two particular players?

Sol.

(i) 11 players are to be selected from 15

Number of ways $= {}^{15}C_{11} = 1365$.

(ii) Since one player is already included, we have to select 10 from the remaining 14

Number of ways $= {}^{14}C_{10} = 1001$.

(iii) Since two players are to be excluded, we have to select 11 from the remaining 13.

Number of ways $= {}^{13}C_{11} = 78$.

Example 24: A committee of 5 members is to be formed from 6 men and 4 women so that

(i) At least two women and atleast one man

(ii) Two particular opposite sex always included.

(iii) Two particular opposite sex refuse to serve in the same committee

(iv) Majority of women serve in the committee.

(v) M_1, M_2, and W are particular men and woman respectively, M_1 refuses M_2, M_2 refuses W to serve in the same committee.

(vi) (M_1, M_2) (W_1, W_2) are two particular men and, women respectively. Whenever M_1, serve in the same committee, W_1, W_2 do not and vice-versa.

Sol. Before we attempt the question, let us analyse all possible cases and put them in tabular form

(i) $W(4)$ $M(6)$

2 3

Hence, total selections $= {}^4C_2 \cdot {}^6C_3 + {}^4C_3 \cdot {}^6C_2 + {}^4C_4, {}^5C_1$

(ii) After including two particular opposite sex, remaining 3, from 8 can be selected in ${}^8C_3$ ways.

(iii) Total ways $= {}^{10}C_5 - {}^8C_3$

(iv) Required ways $= {}^4C_4 \cdot {}^6C_1 + {}^4C_3 \cdot {}^6C_2$

(v) The following cases to be excluded from the total

(a) Committee includes all three M_1, M_2, W

(b) Committee includes M_1, M_2 but not W

(c) Committee includes M_2, W but not M_1.

Hence, total no. of committees

$= {}^{10}C_5 - {}^7C_2 - {}^7C_3 \times 2 = 161$

[(b) and (c) have same number]

(vi) Required ways $= {}^{10}C_5 - {}^6C_1$ (M_1, M_2, W_1, W_2 included)

Example 25: A man has 7 relatives 4 of them ladies and 3 gentlemen, his wife also 7 relatives, 3 of them ladies and 4 gentlemen.

(i) In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that 3 of man relatives and 3 of his wife relatives?

(ii) In how many can they invite a dinner party of 6 members so that 3 of man relatives and 3 of his wifes relative with atleast one lady, one gentleman from each of man and his wife.

Sol.

Man		Wife	
4L	3G	3L	4G
3	0	0	3
1	2	2	1
2	1	1	2

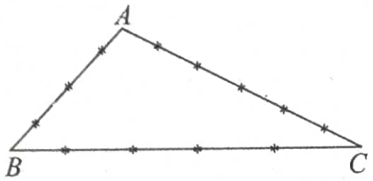
Total $= {}^4C_3 \cdot {}^3C_0 \cdot {}^3C_0 \cdot {}^4C_3 + {}^4C_1 \cdot {}^3C_2 \cdot {}^3C_2 \cdot {}^4C_1 + {}^4C_2 \cdot {}^3C_1 \cdot {}^3C_1 \cdot {}^4C_2 = 485$

Man		Wife	
4L	3G	3L	4G
1	2	1	2
1	2	2	1
2	1	1	2
2	1	2	1

Total $= {}^4C_1 \cdot {}^3C_2 \cdot {}^3C_1 \cdot {}^4C_2 + {}^4C_1 \cdot {}^3C_2 \cdot {}^3C_2 \cdot {}^4C_1 + {}^4C_2 \cdot {}^3C_1 \cdot {}^3C_1 \cdot {}^4C_2 + {}^4C_2 \cdot {}^3C_1 \cdot {}^3C_2 \cdot {}^4C_1 = 900$

Concept Application

16. The sides AB, BC, CA of ΔABC contains 3, 4, 5 points excluding angular angular points. By joining these points excluding A, B, C. Find



(*i*) The number of straight lines

(*ii*) The number of triangles

(*iii*) The number of triangles so that no two points on the same side

(*iv*) The number of quadrilaterals

(*v*) The number of quadrilaterals whose vertices lie on all sides joining the points.

17. If ${}^{28}C_{2r} : {}^{24}C_{2r} - 4 = 225 : 111$ then the value of r is

(*a*) 14 (*b*) 8

(*c*) 5 (*d*) 7

18. In how many ways can a committee of 6 be formed out of 6 men and 4 women so that committees include at the most 2 women?

(*a*) 90 (*b*) 185

(*c*) 115 (*d*) 210

19. A van can accomodate 8 persons, 4 in the front and 4 at the back. If heights of 8 persons are all different, then in how many ways they can be seated in the van so that the person who is sitting in the front is shorter than just at back?

20. An 8 oared boat to be manned from a crew of eleven of whom 3 can only steer but can not row. 8 can row but can not steer. In how many ways the staff can be arranged if 2 of the men can only row on bow side?

TOTAL NUMBER OF SELECTION

Selection From Distinct Objects

The number of ways (or combinations) of selection from n distinct objects, taken at least one of them is

$${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - 1$$

Logically it can be explained in two ways, as one can be selected in ${}^nC_1$ ways, two in ${}^nC_2$ ways and so on ... and by addition principle of counting the total number of ways of doing either of the job is

$${}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$$

Also, for every object, there are two choices, either selection or non-selection. Hence total choices are 2^n. But this also includes the case when none of them is selected. Therefore the number of selections, when atleast one is selected $= 2^n - 1$.

Selection From Identical Objects

1. The number of selections of $r (r \le n)$ objects out of n identical objects is 1.
2. The number of ways of selections of at least one object out of n identical object is n.
3. The number of ways of selections of at least one out of $a_1 + a_2 \ldots + a_n$ objects, where a_1 are alike of one kind, a_2 are alike of second kind, and so on ... a_n are alike of nth kind, is $(a_1 + 1)(a_2 + 1) \ldots (a_n + 1) - 1$.
4. The number of ways of selections of atleast one out of $a_1 + a_2 + a_3 + \ldots + a_n + k$ objects, where a_1 are alike of one kind, ... a_n are alike of nth kind and k are distinct is $(a_1 + 1)(a_2 + 1) \ldots (a_n + 1)\, 2^k - 1$.

Train Your Brain

Example 26: Given five different green dyes, four different blue dyes and three different red dyes, how many combination of dyes can be chosen taking at least one green, one blue dye?

Sol. Any one dye of a particular colour can be either chosen or not; and, thus there are 2 ways in which each one may be dealt with.

Number of ways of selection so that at least one green dye is included $= 2^5 - 1 = 31$

(1 is subtracted to correspond to the case when none of the green dyes is chosen.)

A similar argument may be advanced in respect of other two colours also.

Number of combinations $= (2^5 - 1)(2^4 - 1)(2^3)$.

$= 31 \times 15 \times 8 = 3720$

Example 27: Find the number of combinations that can be formed with 5 oranges, 4 mangoes and 3 bananas when it is essential to take

(*i*) At least one fruit

(*ii*) One fruit of each kind.

Sol. Here 5 oranges are alike of one kind, 4 mangoes are alike of second kind and 3 bananas are alike of third kind

(*i*) The required number of combinations (when at least one fruit)

$= (5 + 1)(4 + 1)(3 + 1)\, 2^0 - 1 = 120 - 1 = 119$

(*ii*) The required number of combinations (when one fruit of each kind)

$= {}^5C_1 \times {}^4C_1 \times {}^3C_1 = 5 \times 4 \times 3 = 60.$

Concept Application

21. A basket contains 4 identical apples, 5 identical mangoes, 3 distinct oranges.

(*i*) Find the total number of ways of selecting fruits from the basket.

(*ii*) Find the total number of ways of selecting fruits from the basket

- With at least one fruit
- At least one from each kind

22. There are 12 fruits in a basket of which 5 are apples, 4 mangoes and 3 bananas (fruits of same species: identical).

How many ways are there to select atleast one are fruit?

23. In how many ways we can select 4 letters from the letters of the word MISSISSIPPI?

24. The number of words which can be formed using 4 letters of the word 'EXAMINATION' is-

(*a*) 1896 (*b*) 2136

(*c*) 2454 (*d*) 1986

DIVISORS OF A GIVEN NATURAL NUMBER

Let $n \in \mathbf{N}$ and $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \ldots p_k^{\alpha_k}$, where $p_1, p_2, p_3, \ldots p_k$ are different prime numbers and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are natural numbers then:

- The total number of divisors of N including 1 and n is
  $= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1)$
- The total number of divisors of n excluding 1 and n is
  $= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1) - 2$ (Proper Divisor)
- The sum of these divisors is
  $= (p_1^0 + p_1^1 + p_1^2 + \ldots + p_1^{\alpha_1})(p_2^0 + p_2^1 + p_2^2 + \ldots + p_2^{\alpha_2})$
  $(p_k^0 + p_k^1 + p_k^2 + \ldots + p_k^{\alpha_k})$
  (Use sum of G.P. in each bracket)
- The number of ways in which n can be resolved as a product of two factors is
  $= \frac{1}{2}(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1)$, if n is not a perfect square
  or
  $= \frac{1}{2}[(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1) + 1]$, if n is a perfect square
- The number of ways in which composite number n can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{k-1} where k is the number of different factors (or different primes) in n.

Train Your Brain

Example 28: If $n = 10800$, then find the

(*i*) Total number of divisors of n

(*ii*) The number of even divisors

(*iii*) The number of divisors of the form $4m + 2$

(*iv*) The number of divisors which are multiples of 15

Sol. $n = 10800 = 2^4 \times 3^3 \times 5^2$

Any divisor of n will be of the form $2^a \times 3^b \times 5^c$ where $0 \le a \le 4, 0 \le b \le 3, 0 \le c \le 2$.

For any distinct choices of a, b and c, we get a divisor of n

(*i*) Total number of divisors $= (4 + 1)(3 + 1)(2 + 1) = 60$

(*ii*) For a divisor to be even, a should be at least one. So total number of even divisors $= 4(3 + 1)(2 + 1) = 48$.

(*iii*) $4m + 2 = 2(2m + 1)$. In any divisor of the form $4m + 2$, a should be exactly 1. So number of divisors of the form $4m + 2 = 1(3 + 1)(2 + 1) = 12$.

(*iv*) A divisor of n will be a multiple of 15 if b is at least one and c is at least one. So number of such divisors $= (4 + 1) \times 3 \times 2 = 30$.

Example 29: Find the number of divisors of 1350. Also find the sum of all divisors.

Sol. $1350 = 2 \times 3^3 \times 5^2$

$\therefore$ Number of divisors $= (1 + 1)(3 + 1)(2 + 1) = 24$

Sum of divisors $= (1 + 2)(1 + 3 + 3^2 + 3^3)(1 + 5 + 5^2) = 3720$.

Example 30: In how many ways 8100 can be resolved into product of two factors?

Sol. $8100 = 2^2 \times 3^4 \times 5^2$

Number of ways $= \frac{1}{2}[(2 + 1)(4 + 1)(2 + 1) + 1] = 23$

Example 31: How many functions can be defined from a set A containing 5 elements to a set B having 3 elements? How many of these are surjective functions?

Sol. Image of each element of A can be taken in 3 ways.

$\therefore$ Number of functions from A to $B = 3^5 = 243$.

Number of into functions from A to B

$= 2^5 + 2^5 + 2^5 - 3 = 93$.

$\therefore$ Number of onto functions $= 150$.

Concept Application

25. Consider the number $n = 2^2 \cdot 3^3 \cdot 5^5 \cdot 7^4 \cdot 11^3$

(*i*) The total number of divisors of n is

(*ii*) The number of proper divisors of n is

(*iii*) The number of odd divisors of n is

(*iv*) The number of even divisors of n is

(*v*) The number of ordered pairs (a, b) so that LCM $(a, b) = n$ is

(*vi*) The sum of divisors of n is

(*vii*) The number of unordered pairs (x, y) so that $n = xy$ is

(*viii*) The number of unordered pairs (x, y) so that $n = xy$ and $gcd(x, y) = 1$ is

(*ix*) The number of divisors of n which are not multiple of 15 is

(*x*) The number of divisors of n in the form $4k + 2$, $k \geq 0, k \in z$ is

(*xi*) Then number of divisors of the form $6k + 1$ is

DIVISION OF DISTINCT OBJECT INTO GROUPS

In the case of grouping we have the following. If $m + n + p$ things are divided into 3 groups one containing m, the second n and the third p things; number of groupings is

$${}^{(m+n+p)}C_m \cdot {}^{(n+p)}C_n \cdot {}^{p}C_p$$

$$= \frac{(m+n+p)!}{m!n!p!}$$ where m, n, p are distinct natural numbers.

If $m = n = p$ (say) then the number of groupings (unmindful of the order of grouping) is

$$\frac{3m!}{(m!)^3 3!}$$

Thus if 52 cards be divided into four groups of 13 each, the number of groupings is

$$\frac{52!}{(13!)^4 . 4!}$$

On the other hand, when 52 cards are dealt 13 each to four persons, the number of ways in which this can be done is

$$\frac{52!}{(13!)^4}$$

As another example, if we consider the division of 52 cards into four groups, three groups containing each 16 and the fourth cards, the number of ways in which this can be done is

$$\frac{52!}{3!(16!)^3 4!}$$

Note the 3! factor in the denominator. This is for the reason that there are only 3 equal groups.

In general, the number of ways in which mn different things can be divided equally into m distinct groups is $\frac{(mn)!}{(n!)^m}$ when order of groups is important.

Whereas when order of groups is not important, then division into m equal group is done in $\frac{(mn)!}{m!(n!)^m}$ ways.

In general, we can write formula for grouping as factorial of total number of elements divided by product of factorial of number of elements in each group and product of factorial of number of groups having same number of elements, if any. Also formula for number of distribution is number of grouping multiplied with factorial of number of persons in which objects are distributed, divided by factorial of number of persons who got nothing, if any.

Train Your Brain

Example 32: 12 different toys are to be distributed to three children equally. In how many ways this can be done?

Sol. The problem is to divide 12 different things into three different groups.

Number of ways $= \frac{12!}{4!4!4!3!} \times 3! = 34650.$

Example 33: In how many ways can 15 students be divided into 3 groups of 5 students each such that 2 particular students are always together? Also find the number of ways if these groups are to be sent to three different colleges.

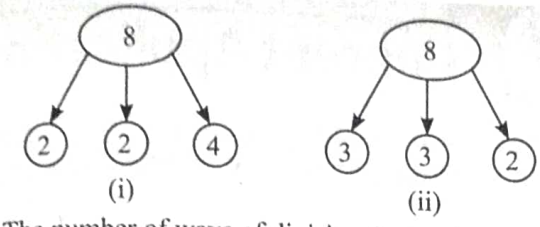
Sol. Assuming two particular students as one student (as they are always together), we have to make groups of 5 + 5 + 4 students out of 14 students.

Therefore total number of ways $= \frac{14!}{5!5!4!2!}$

Now if these groups are to be sent to three different colleges, the total number of ways $= \frac{14!}{5!5!4!2!} \times 3!$

Example 34: In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books?

Sol. If each receives at least two books, then the division trees would be as shown below:



The number of ways of division for tree in figure (i) is $\left[\frac{8!}{(2!)^2 4!2!}\right]$.

The number of ways of division for tree in figure (ii) is $\left[\frac{8!}{(3!)^2 4!2!}\right]$.

The total number of ways of distribution of these groups among 3 students is $\left[\frac{8!}{(2!)^2 4!2!}+\frac{8!}{(3!)^2 4!2!}\right]\times 3!$

Concept Application

26. The number of ways in which six different prizes can be distributed among three children each receiving at least one prize is

(*a*) 270 (*b*) 540

(*c*) 1080 (*d*) 2160

27. The number of ways in which mn students can be distributed equally among n sections is

(*a*) $(mn)^n$ (*b*) $\frac{(mn)!}{(m!)^n}$

(*c*) $\frac{(mn)!}{m!}$ (*d*) $\frac{(mn)!}{m!n!}$

28. Number of ways in which a pack of 52 playing cards be distributed equally amount four players, so that each may have the ace, king, queen and jack of the same suit is-

(*a*) $\frac{36!}{(9!)^4}$ (*b*) $\frac{36!4!}{(9!)^4}$

(*c*) $\frac{36!}{(9!)^4 4!}$ (*d*) $\frac{36!}{(9!)^4(4!)^4}$

29. Note that number of ways in which 30 Jawans can be divided into 3 equal groups

30. The number of ways in which 30 Jawans can be deputed equally on 3 borders, (10 on each border) is

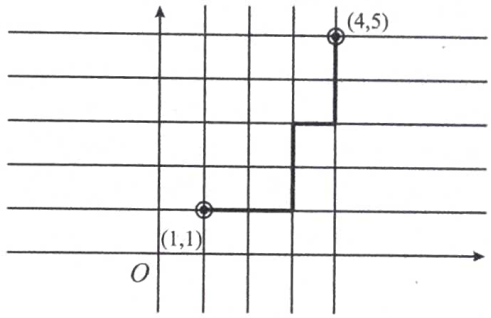
GRID PROBLEM

Train Your Brain

Example 35: Number of ways in which an ant can reach from (1, 1) to (4, 5) via shortest path.

Sol. Whatever may be the mode of travel of the ant; it has to traverse 3H (Horizontal) and 4V (Vertical) paths.

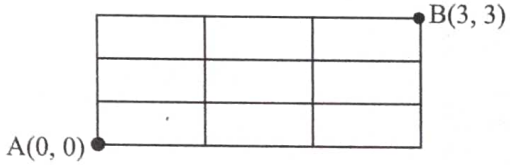
Hence required number of ways $= \frac{7!}{4!3!} = {}^7C_3$



Note: If there are n vertical and m horizontal lines then there will be $(n-1)$ horizontal and $(m-1)$ vertical paths.

Concept Application

31. In how many ways can a person go from point A to point B if he can travel only to the right or upward along the lines (Grid Problem)?



32. The streets of a city are arranged like the lines of a chess board. There are m streets running North to South and 'n' streets running East to West. The number of ways in which a man can travel from NW to SE corner going the shortest possible distance is-

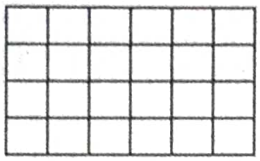
(*a*) $\sqrt{m^2+n^2}$

(*b*) $\sqrt{(m-n)^2\cdot(n-1)^2}$

(*c*) $\frac{(m+1)!}{m!\cdot n!}$

(*d*) $\frac{(m+n-2)!}{(m-1)!\cdot(n-1)!}$

33. Number of rectangles in the grid shown which are not squares is



(*a*) 160 (*b*) 162
(*c*) 170 (*d*) 185

SUMMATION OF NUMBERS (3 DIFFERENT WAYS)

(*a*) Sum of all the numbers greater then 10000 formed by the digits 1, 3, 5, 7, 9 if no digit being repeated.

Method 1: All possible numbers = 5! = 120

If one occupies the units place then total number = 24.

Hence 1 enjoys units place 24 times

				1

Similarly 1 enjoys each place 24 times

Sum due to 1 = $1 \times 24\,(1 + 10 + 10^2 + 10^3 + 10^4)$

Similarly

Sum due to the digit 3 = $3 \times 24(1 + 10 + 10^2 + 10^3 + 10^4)$

⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

Required total sum = $24\,(1 + 10 + 10^2 + 10^3 + 10^4)$

$(1 + 3 + 5 + 7 + 9)$

Method 2: In 1st column there are twenty four 1's, Twenty four 3's and so on and their sum is = 24 × 25 = 600

Hence add. in vertical column normally we get = 6666600

	5th			2nd	1st
	X	X	X	X	X
	X	X	X	X	X
120 Number	⋮	⋮	⋮	⋮	⋮
	⋮	⋮	⋮	⋮	⋮
	X	X	X	X	X
	666	6	6	0	0 = 6666600

Method 3: Applicable only if the digits used are such that they have the same common difference. (valid even if the digits are repeating) Writing all the numbers in ascending order of magnitude

$S = (13579 + 13597 + \ldots + 97513 + 97531)$

$S = (13579 + 99531) + (13597 + 97513) + \ldots$

$= (111110)$ 60 time = 6666600 Ans.

$S = \frac{n}{2}(l + L)$ where n = number of numbers, l = smallest, L = Largest

Concept Application

34. The sum of all numbers greater than 1000 formed by using the digits 1, 3, 5, 7 no digit is repeated in any number is

(*a*) 106656 (*b*) 101276
(*c*) 117312 (*d*) 811273

35. The sum of the digits in the unit place of all the numbers formed with the help of 3, 4, 5, 6 taken all at a time is

(*a*) 18 (*b*) 108 (*c*) 432 (*d*) 180

USE OF MULTIONOMIALS

- If there are l objects of one kind, m objects of second kind, n objects of third kind and so on; then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + \ldots$) is the coefficient of x^r in the expansion of $(1 + x + x^2 + x^3 + \ldots x^l)\,(1 + x + x^2 + \ldots + x^m)\,(1 + x + x^2 + \ldots + x^n)$

  Further if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + \ldots$) is the coefficient of x^r in the expansion of $(x + x^2 + x^3 + \ldots x^l)\,(x + x^2 + x^3 + \ldots + x^m)\,(x + x^2 + x^3 + \ldots + x^n)$

- If there are l objects of one kind, m object of second kind, n object of third kind and so on; then the number of possible arrangements/permutations of r objects out of these objects (i.e., $l + m + n + \ldots$) is the coefficient of x^r in the expansion of

$$r!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\ldots+\frac{x^\ell}{\ell!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\ldots+\frac{x^m}{m!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\ldots+\frac{x^n}{n!}\right)\ldots$$

Note: For use in problems of the above type the following Binomial expansions may be noted

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + \ldots + x^r + \ldots$$

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + \ldots + (r+1)\,x^r + \ldots$$

$$\frac{1}{(1-x)^3} = (1-x)^{-3} = 1 + 3x + 6x^2 + \ldots + \frac{(r+1)(r+2)}{1.2}x^r + \ldots$$

$$\frac{1}{(1-x)^4} = (1-x)^{-4} = 1 + 4x + 10x^2 + \ldots + \frac{(r+1)(r+2)(r+3)}{1.2.3}x^r + \ldots$$

and more generally $\frac{1}{(1-x)^p} = (1-x)^{-p} = 1 + px + \ldots + \frac{(r+1)(r+2)\ldots(r+\overline{p-1})}{1.2.3\ldots(p-1)}x^4 + \ldots$

and the coefficient of x^r in this general case is easily seen to be ${}^{(r+p-1)}C_{p-1}$

An Alternative Method for the General Problem

In this case, we consider additionally $(r-1)$ things (this number is one less than r, the number of persons). It may be seen that the number of ways of dividing the n things among the r persons as per the condition of the problem is ${}^{(n+r-1)}C_{(r-1)}$.

Use of Multinomials in Solving Linear Equation

Multinomials can be used to solve linear equations. The method is illustrated with the help of following illustrations.

***e.g.* Find the number of non negative integral solutions of $x+y+z+w=20$.**

Any one of the four variables can take from the value zero to the value 20; and hence we construct a polynomial in a variable (say x) with x raised to different powers which would constitute the values that any one variable can take when the equation is solved in the manner indicated. We, thus, consider the product expression,

$(1+x+x^2+\ldots+x^{20})(1+x+\ldots+x^{20})(1+x+\ldots+x^{20})(1+x+\ldots+x^{20})$

Four factors since there are four variables. If we take x^4 in the first factor; x^5 in the second; x^8 in the third; then we take the term x^3 in the fourth so that the sum of the powers $(4+5+8+3=20)$ is 20. It is then we say that there is a solution corresponding to $x=4; y=5; z=8; w=3$. Hence the number of solutions in the manner required is the coefficient of x^{20} in

$(1+x+\ldots+x^{20})^4$

i.e. in $\left(\frac{1-x^{21}}{1-x}\right)^4$

i.e. in $(1-x^{21})^4(1-x)^{-4}$

i.e. in $(1-x)^{-4}$ and this is $= {}^{23}C_3$

(**Note:** In $(1-x)^{-4}$ coefficient of x^n is ${}^{(n+3)}C_3$

Method of Fictious Partition:

Number of ways in which n identical things may be distributed among p persons if each person may receive none, one or more things is ${}^{n+p-1}C_n$.

Train Your Brain

Example 36: Let us consider the more general problem of distributing n identical things given among r persons, each one whom, can receive 0, 1, 2 or more things ($\leq n$).

Sol. Consider r brackets corresponding to the r persons.

In each bracket take an expression given by $1+x+x^2+\ldots+x^n$ (the various powers of x; namely, 0, 1, 2, ..., n correspond to the number of things each person can have in the distribution)

In the continued product $(1+x+x^2+\ldots+x^n)(\,)(\,)\ldots$ repeated r times, collect the coefficient of x^n. This coefficient gives the required number of ways of distribution.

$\therefore$ The number of ways

$=$ Coefficient of x^n in $(1+x+x^2+\ldots+x^n)(\,)(\,)\ldots$ repeated r times

$=$ Coefficient of x^n in $\left(\frac{1-x^{n+1}}{1-x}\right)^r$

$=$ Coefficient of x^n in $(1-x^{n+1})^r(1-x)^{-r}$

$=$ Coefficient of x^n in $(1-x)^{-r}$

$= \frac{(r+1)(r+2)\ldots(r+n-1)}{1.2\ldots(r-1)} = {}^{(n+r-1)}C_{r-1}$

Example 37: An unlimited number of Red, White, Blue and Green balls are given. Ten balls are drawn. Find the number of ways of selection.

Sol. In this the expression in x to be considered is $1+x+x^2+\ldots$ to correspond to the unlimited number available in each colour. There are four such colours.

The required number of ways

$=$ Coefficient of x^{10} in $[1+x+x^2+\ldots]^4$

$=$ Coefficient of x^{10} in $\left(\frac{1}{1-x}\right)^4 = \frac{11\times12\times13}{6} = 286$

Example 38: Find the number of ways in which 11 identical apples can be distributed among 6 children, so that each child receives at least one apple.

Sol. First give one apple to each child. There are remaining 5 apples to be distributed among 6 children (so that each may receive any number of apples not exceeding five).

The number of ways required

$= {}^{(11-1)}C_{6-1} = {}^{10}C_5 = 252$

Alternatively, the number of ways of distribution is given by the coefficient of x^{11} in $(x+x^2+\ldots+x^{11})^6$.

Example 39: Find the number of solutions of the equation $x+y+z=6$, where $x, y, z \in W$.

Sol. Number of solutions = coefficient of x^6 in $(1+x+x^2+\ldots x^6)^3$

$=$ coefficient of x^6 in $(1-x^7)^3(1-x)^{-3}$

$=$ coefficient of x^6 in $(1-x)^{-3}$

$= {}^{3+6-1}C_2 = {}^8C_2 = 28.$

Example 40: Let n and k be positive integers such that $n \geq \frac{k(k+1)}{2}$. Find the number of solutions $(x_1, x_2, \ldots x_k)$, $x_1 \geq 1, x_2 \geq 2, \ldots, x_k \geq k$, all integers satisfying $x_1 + x_2 + \ldots + x_k = n$

Sol. The number of solutions is the coefficient of x^n in $(x + x^2 + \ldots + x^n)(x^2 + x^3 + \ldots + x^n)(x^k + x^{k+1} + \ldots + x^n)$

i.e. in

$$x^{\frac{k(k+1)}{2}} \frac{(1-x^n)}{1-x} \frac{(1-x^{n-1})}{1-x} \ldots \frac{(1-x^{n-k+1})}{1-x}$$

i.e in $x^{\frac{k(k+1)}{2}} (1-x^n)(1-x^{n-1}) \ldots (1-x^{n-k+1})(1-x)^{-k}$

$= \text{coefficient of } x^{n-\frac{k(k+1)}{2}} \text{ in } (1-x)^{-k}$

$= {}^{\left(n-\frac{k(k+1)}{2}+k-1\right)}C_{k-1}$

(**Note:** Coefficient of x^n in $(1-x)^{-k}$ is ${}^{(n+k-1)}C_{k-1}$)

Concept Application

36. For the equation $x + y + z + w = 19$, the number of positive integral solutions is equal to-

(*a*) The number of ways in which 15 identical things among 4 persons can be distributed

(*b*) The number of ways in which 19 identical things can be distributed among 4 persons

(*c*) The number of ways in which 16 identical things among 3 persons can be distributed

(*d*) The number of ways in which 14 identical things among 4 persons can be distributed

37. Three distinguishable dice are rolled. In how many ways we can get a total 15?

38. In how many ways we can give 5 apples, 4 mangoes and 3 oranges (fruits of same species are similar) to three persons if each may receive none, one or more?

39. The number of non-negative integral solutions of $x_1 + x_2 + x_3 + x_4 \leq n$ (where n is a positive integer) is-

(*a*) ${}^{n+3}C_3$ (*b*) ${}^{n+4}C_4$

(*c*) ${}^{n+5}C_5$ (*d*) ${}^{n+4}C_n$

40. Find the number of integral solutions of $x + y + z = 20$, if $x \geq -4, y \geq 1, z \geq 2$.

METHOD OF INCLUSION EXCLUSION

If $A_1, A_2, \ldots, A_m$ are finite sets and $A = A_1 \cup A_2 \cup \ldots \cup A_m$, then

$$n(A) = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{m+1} a_m$$

where $a_1 = n(A_1) + n(A_2) + \ldots + n(A_m)$

$$a_2 = \sum_{1 \leq i < j < m} n(A_i \cap A_j)$$

$$a_3 = \sum_{1 \leq i < j < k \leq m} n(A_i \cap A_j \cap A_k)$$

and so on.

Corollary (Sieve-Formula)

If $A_1, A_2, \ldots A_m$ are m subsets of a set A containing N eler

then

$$n(A'_1 \cap A'_2 \cap \ldots \cap A'_m) = N - \sum_i n(A_i) + \sum_{1 \leq i < j \leq m} n(A_i \cap A$$

$$- \sum_{1 \leq i < j < k \leq m} n(A_i \cap A_j \cap A_k) + \ldots + (-1)^m n\ (A_1 \cap A_2 \cap \ldots \ldots \cap$$

DEARRANGEMENTS

It is rearrangement of objects such that no one goes to its orig place.

If n things are arranged in a row, the number of ways in w they can be rearranged so that none of them occupies its orig placed is

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} \ldots + (-1)^n \frac{1}{n!}\right)$$

It is denoted by $D_n = n! \sum_{r=0}^{n} (-1)^r \frac{1}{r!}$

Train Your Brain

Example 41: A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) at least two of them are in the wrong envelopes. (ii) all the letters are in the wrong envelopes.

Sol. (*i*) The number of ways in which at least two of them are in the wrong envelopes

$$= \sum_{r=2}^{6} {}^nC_{n-r} D_r$$

$$= {}^nC_{n-2}D_2 + {}^nC_{n-3}D_3 + {}^nC_{n-4}D_4 + {}^nC_{n-5}D_5 + {}^nC_{n-6}D_6$$

Here $n = 6$

$$= {}^6C_4.2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3.3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right)$$

$$+ {}^6C_2.4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$+ {}^6C_1.5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$

$$+ {}^6C_0.6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$$= 15 + 40 + 135 + 264 + 265 = 719.$$

(*ii*) The number of ways in which all letters be placed in wrong envelopes

$$= 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$$= 720\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)$$

$$= 360 - 120 + 30 - 6 + 1 = 265.$$

Example 42: Four slip of papers with the numbers 1, 2, 3, 4 written on them are put in a box. They are drawn one by one (without replacement) at random. In how many ways it can happen that the ordinal number of atleast one slip coincide with its own number?

Sol. Total number of ways = 4 ! = 24.

The number of ways in which ordinal number of any slip does not coincide with its own number is the number of dearrangements of 4 objects $= 4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$

Thus the required number of ways. = 24 – 9 = 15

Concept Application

41. The number of ways to put five letters in five envelopes when any one letter is kept in right envelope and four letters in wrong envelopes are

(*a*) 40

(*b*) 45

(*c*) 30

(*d*) 70

42. There are four balls of different colours and four boxes of colours same as those of the balls. The number of ways in which the balls, one in each box, could be placed such that a ball does not go to box of its own colour is

(*a*) 8 (*b*) 7

(*c*) 9 (*d*) 6

Short Notes

Fundamental Principle of Counting (Counting without actually counting)

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then the total number of different ways of

(*a*) Simultaneous occurrence of both events in a definite order is $m \times n$. This can be extended to any number of events (known as multiplication principle).

(*b*) Happening of exactly one of the events is $m + n$ (known as addition principle).

Factorial

A Useful Notation : $n! = n(n-1)(n-2) \ldots\ldots 3 \cdot 2 \cdot 1$;

$n! = n\,.\,(n-1)!$ where $n \in W$

$0! = 1! = 1$

$(2n)! = 2^n\,.\,n!\,[1.\,3.\,5.\,7 \ldots (2n-1)]$

Permutation

(*a*) ${}^nP_r$ denotes the number of permutations of n different things, taken r at a time ($n \in N, r \in W, n \geq r$)

$${}^nP_r = n(n-1)(n-2) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

(*b*) The number of permutations of n things taken all at a time when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining $n - (p + q + r)$ are all different is : $\frac{n!}{p!\,q!\,r!}$.

(*c*) The number of permutation of n different objects taken r at a time, when a particular object is always to be included is $r! \cdot {}^{n-1}C_{r-1}$.

(*d*) The number of permutation of n different object taken r at a time, when repetition be allowed any number of times is $n \times n \times n \ldots r$ times $= n^r$.

(*e*) (*i*) The number of circular permutations of n different things taken all at a time is ; $(n-1)! = \frac{{}^nP_n}{n}$.

If clockwise & anti-clockwise circular permutations are considered to be same, then it is $\frac{(n-1)!}{2}$.

(*ii*) The number of circular permutation of n different things taking r at a time distinguishing clockwise & anticlockwise arrangement is $\frac{{}^nP_r}{r}$.

Combination

(*a*) ${}^nC_r$ denotes the number of combinations of n different things taken r at a time, and ${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{{}^nP_r}{r!}$ where $r \leq n$; $n \in N$ and $r \in W$. ${}^nC_r$ is also denoted by $\binom{n}{r}$ or A_r^n or $C(n, r)$.

(*b*) The number of combination of n different things taking r at a time.

(*i*) When p particular things are always to be included $= {}^{n-p}C_{r-p}$.

(*ii*) When p particular things are always to be excluded $= {}^{n-p}C_r$.

(*iii*) When p particular things are always to be included and q particular things are to be excluded $= {}^{n-p-q}C_{r-p}$.

(*c*) Given n different objects, the number of ways of selecting atleast one of them is, ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(*d*) (*i*) Total number of ways in which it is possible to make a selection by taking some or all out of $p + q + r + \ldots$ things, where p are alike of one kind, q a like of a second kind, r alike of third kind and so on is given by : $(p + 1)(q + 1)(r + 1) \ldots - 1$.

(*ii*) The total number of ways of selecting one or more things from p identical of one kind, q identical things of second kind, r identical things of third kind and n different things is $(p + 1)(q + 1)(r + 1)\, 2^n - 1$.

Divisors

Let $N = p^a \cdot q^b \cdot r^c \ldots$ where $p, q, r \ldots$ are distinct primes and $a, b, c \ldots$ are natural numbers then :

(*a*) The total numbers of divisors of N including 1 & N is $= (a + 1)\,(b + 1)\,(c + 1) \ldots$.

(*b*) The sum of these divisors is $= (p^0 + p^1 + p^2 + \ldots + p^a)\,(q^0 + q^1 + q^2 + \ldots + q^b)\,(r^0 + r^1 + r^2 + \ldots + r^c) \ldots$

(*c*) Number of ways in which N can be resolved as a product of two factor is =

$\frac{1}{2}(a + 1)\,(b + 1)\,(c + 1) \ldots$ if N is not a perfect square

$\frac{1}{2}[(a + 1)\,(b + 1)\,(c + 1) \ldots + 1]$ if N is a perfect square

(*d*) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

Division and Distribution

(*a*) (*i*) The number of ways in which $(m + n)$ different things can be divided into two groups containing m and n things respectively is :

$\frac{(m+n)!}{m!\,n!}\ (m \neq n)$.

(*ii*) If $m = n$, it means the groups are equal & in this case the number of subdivision is $\frac{(2n)!}{n!\,n!\,2!}$; for in any one way it is possible to inter change the two groups without obtaining a new distribution.

(*iii*) If $2n$ things are to be divided equally between two persons then the number of ways $= \frac{(2n)!}{n!\,n!\,(2!)} \times 2!$.

(*b*) (*i*) Number of ways in which $(m + n + p)$ different things can be divided into three groups containing m, n and p things respectively is $\frac{(m+n+p)!}{m!\,n!\,p!}$, $m \neq n \neq p$.

(*ii*) If $m = n = p$ then the number of groups $= \frac{(3n)!}{n!\,n!\,n!\,3!}$.

(*iii*) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

(*c*) In general, the number of ways of dividing n distinct object into l groups containing p objects each, m groups containing q objects each is equal to $\frac{n!\,(l+m)!}{(p!)^l\,(q!)^m\,l!\,m!}$.

Here $lp + mq = n$.

(*d*) Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them $= p^n$.

(*e*) Number of ways in which n identical may be distributed among p persons if each person may receive one, one or more things is; ${}^{n+p-1}C_n$.

Dearrangement

Number of ways in which n letters can be placed in n directed envelopes so that no letter goes into its own envelope is

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}\right]$$

Important Result

(*a*) Number of rectangle of any size in a square of size $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of square of any size is $\sum_{r=1}^{n} r^2$.

(*b*) Number of rectangle of any size in a rectangle of size $n \times p$ $(n < p)$ is $\frac{np}{4}(n + 1)(p + 1)$ and number of squares of any size is $\sum_{r=1}^{n}(n + 1 - r)\,(p + 1 - r)$.

(*c*) If there are n points in a plane of which m $(< n)$ are collinear:

(*i*) Total number of lines obtained by joining these points is ${}^nC_2 - {}^mC_2 + 1$.

(*ii*) Total number of different triangle ${}^nC_3 - {}^mC_3$.

(*d*) Maximum number of point of intersection of n circles is ${}^nP_2$ and n lines is ${}^nC_2$.

Solved Examples

1. India and South Africa play one day international series until one team wins 4 matches. No match ends in a draw. Find in how many ways the series can be won.

Sol. Let I for India and S for South Africa. We can arrange I and S to show the wins for India and South Africa respectively. Suppose S. Africa wins the series, then last match is always won by S. Africa.

Wins	Wins of S	No. of ways
(*i*) O	4	1
(*ii*) 1	4	$4!/3! = 4$
(*iii*) 2	4	$\frac{5!}{2!3!} = 10$
(*iv*) 3	4	$\frac{6!}{3!3!} = 20$

$\therefore$ Total no. of ways = 35

In the same number of ways India can win the series

$\therefore$ Total no. of ways in which the series can be won = $35 \times 2 = 70$

2. There are m points on one straight line AB and n points on another straight line AC, none of them being A. How many triangles can be formed with these points as vertices? How many can be formed if point A is also included?

Sol. To get a triangle, we either take two points on AB and one point on AC, or one point on AB and two points on AC. Therefore, the number of triangles we obtain

$= ({}^mC_2)\,({}^nC_1) + ({}^mC_1)\,({}^nC_2)$

$= \frac{m(m-1)}{2}n + m\frac{n(n-1)}{2}$

$= \frac{1}{2}mn(m-1+n-1) = \frac{1}{2}mn(m+n-2)$

If the point A is included, additional triangles. We get mn additional triangles.

Thus, in this case we get $= \frac{mn}{2}(m+n-2) + mn$

$= \frac{mn(m+n)}{2}$ triangles

3. Find the total number of selections of 5 letters from five As, four Bs three Cs and two Ds. Also find the number of 5-letter words that can be made.

Sol. There is one letter A to take 5 identical letters, two letters A, B to take 4 identical letters, three letters A, B, C to take 3 identical letters and four letters A, B, C, D to take pairs. There are only four different letters.

Then following possibilities occurs

Possibilities	Combinations	Permutations
1 pair, 3 different	${}^4C_1 \times {}^3C_3$	${}^4C_1 \times {}^3C_3 \times \frac{5!}{2!}$
2 pair, 1 different	${}^4C_2 \times {}^2C_1$	${}^4C_1 \times {}^2C_1 \times \frac{5!}{2!2!}$
3 identical, 2 different	${}^3C_1 \times {}^3C_2$	${}^3C_1 \times {}^3C_2 \times \frac{5!}{3!}$
3 identical, 1 pair	${}^3C_1 \times {}^3C_1$	${}^3C_1 \times {}^3C_1 \times \frac{5!}{3!2!}$
4 identical, 1 differenet	${}^2C_1 \times {}^3C_1$	${}^2C_1 \times {}^3C_1 \times \frac{5!}{4!}$
5 identical	${}^1C_1$	${}^1C_1 \times \frac{5!}{5!}$

The required number of selections

$= {}^1C_1 + {}^2C_1 \times {}^3C_1 + {}^3C_1 \times {}^3C_1 + {}^3C_1 \times {}^3C_2 + {}^4C_2 \times {}^2C_1 + {}^4C_1 \times {}^3C_3$

$= 1 + 2 \times 3 + 3 \times 3 + 3 \times 3 + 6 \times 2 + 4 \times 1$

$= 1 + 6 + 9 + 9 + 12 + 4 = 41$

The required number of words (i.e., permutations)

$= {}^1C_1 \times \frac{5!}{5!} + {}^2C_1 \times {}^3C_1 \times \frac{5!}{4!} + {}^3C_1 \times {}^3C_1 \times \frac{5!}{3!2!} + {}^3C_1 \times {}^3C_2 \times \frac{5!}{3!} + {}^4C_2 \times {}^2C_1 \times {}^2C_1 \times \frac{5!}{2!2!} + {}^4C_1 \times {}^3C_3 \times \frac{5!}{2!}$

$= 1 + 30 + 90 + 180 + 360 + 240 = 901.$

4. In an examination, the maximum marks for each of three papers is n and that for the fourth paper is $2n$. Prove that the number of ways in which a candidate can get $3n$ marks is $\frac{1}{6}(n+1)(5n^2 + 10n + 6)$.

Sol. The number of ways of getting $3n$ marks

= coefficient of x^{3n} in $(1 + x + x^2 + \ldots + x^n)^3 \cdot (1 + x + \ldots + x^{2n})$

= coefficient of x^{3n} in $(1 - x^n + 1)^3\,(1 - x^{2n} + 1) \cdot (1 - x)^{-4}$

= coefficient of x^{3n} in

$(1 - 3x^{n+1} + 3x^{2n+2} - x^{3n+3})\,(1 - x^{2n+1}) \times (1 + {}^4C_1 x + {}^5C_2 x^2 + {}^6C_3 x^3 + \ldots)$

$= \text{coefficient of } x^{3n} \text{ in}$

$(1 - 3x^{n+1} - x^{2n+1} + 3x^{2n+2})(1 + {}^4C_1 x + {}^5C_2 x^2 + \ldots)$

$= {}^{3n+3}C_{3n} - 3 \cdot {}^{2n+2}C_{2n-1} + 3 \cdot {}^{n+1}C_{n-2} - {}^{n+2}C_{n-1}$

$= \dfrac{(3n+3)!}{3!(3n)!} - 3 \cdot \dfrac{(2n+2)!}{3!(2n-1)!} + 3\dfrac{(n+1)!}{3!(n-2)!} - \dfrac{(n+2)!}{3!(n-1)!}$

$= \dfrac{1}{6}(n+1)(27n^2 + 27n + 6 - 24n^2 - 12n + 3n^2 - 3n - n^2 - 2n)$

$= \dfrac{1}{6}(n+1)(5n^2 + 10n + 6)$

5. Find the number of times of the digits 4 will be written when listing the integers from 1 to 1000.

Sol. Since 4 does not occur in 1000, we have to count the number of times 4 occurs when we list the integers from 1 to 999. Any number between 1 and 999 is of the form xyz where $0 \le x, y, z, \le 9$.

Let us first count the numbers in which 4 occurs exactly once. i.e. $({}^3C_1)(9)(9) = 243$ and 4 can occurs exactly at two places in $({}^3C_2)(9) = 3 \times 9$ such number.

Lastly, 4 can occur in all three digits in one number only. Hence the number of times 4 occurs is $3(9)(9) + 3 \times 9 + 1 = 271$.

6. Ten persons numbered 1, 2, ..., 10 play a chess tournament, each player playing against every other player exactly one game. It is known that no game ends in a draw. If $w_1, w_2, \ldots, w_{10}$ are the number of games won by players 1, 2, 3, ...10 respectively, and l_1, l_2, l_{10} are the number of games lost by the players 1, 2, ...10, respectively, then show that

(*i*) $\Sigma w_i = \Sigma l_i = 45$ (*ii*) $w_i + l_i = 9$ (*iii*) $\Sigma w_i^2 = \Sigma l_i^2$

Sol. Clearly, each player will play 9 games total number of games is ${}^{10}C_2 = 45$. Clearly,

$w_i + l_i = 9$ and $\Sigma w_i = \Sigma l_i = 45$

$\Rightarrow w_i = 9 - l_i \Rightarrow w_i^2 = 81 + l_i^2 - 18l_i$

$\Rightarrow \Sigma w_i^2 = 81 \times 10 + \Sigma l_i^2 - 18\Sigma l_i$

$= 810 + \Sigma l_i^2 - 18 \times 45 = \Sigma l_i^2$

7. How many integral solutions are there to the system of equations $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 + x_3 = 5$ when $x_k \ge 0$?

Sol. We have: $x_1 + x_2 + x_3 + x_4 + x_5 = 20$

and $x_1 + x_2 + x_3 = 5$

These two equations reduce to

and, $x_1 + x_2 + x_3 = 5$ ...(2)

Since corresponding to each solution of (1) there are solutions of equation (2). So, total number of solutions of the given system of equations

= No. of solutions of (1) × No. of solutions of (2)

$= {}^{15+2-1}C_{2-1} \times {}^{5+3-1}C_{3-1} = {}^{16}C_1 \times {}^7C_2 = 336.$

8. How many integers between 1 and 10° have the sum of their digits equal to 18?

Sol. Let $n = x_6 x_5 x_4 x_3 x_2 x_1$ be a number between 1 and 10, where $x_1, x_2, x_3 \ldots x_6$ are digits at one's, tens, hundreds... places. Clearly $0 \le x_1 \le 9$, $i = 1, 2, \ldots, 9$.

The total number of values of n is the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$, where $0 \le x \le 9$; $i = 1, 2, \ldots\ldots 9$

The number of solutions of the above equation is

Coefficient of x^{18} in $(x^0 + x^1 + x^2 + \ldots + x^9)^6$

$= \text{Coefficient of } x^{18} \text{ in } \left(\dfrac{1-x^{10}}{1-x}\right)^6$

$= \text{Coefficient of } x^{18} \text{ in } (1 - x^{10})^6 (1 - x)^{-6}$

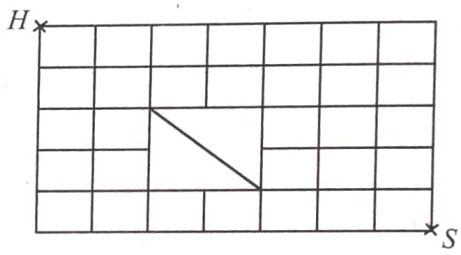
Coefficient of x^{18} in $(1 - {}^6C_1 x^{10} + {}^6C_2 x^{20} \ldots)(1 - x)^{-6}$

$= \text{Coefficient of } x^{18} \text{ in } (1 - x)^{-6} - {}^6C_1 \times \text{Coefficient of } x^8 \text{ in } (1 - x)^{-6}$

$= {}^{18+6-1}C_{6-1} - C_1 \times {}^{8+6-1}C_{6-1} = {}^{23}C_5 - 6 \times {}^{13}C_5$

$= 33649 - 7722 = 25927$

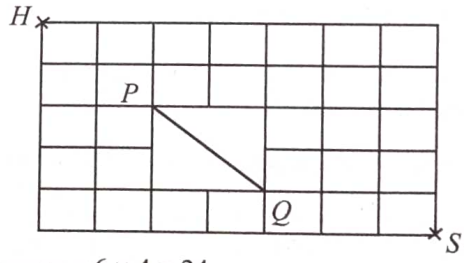
9. The figure shows the plan of a town where all the street blocks are square. In the middle of the town is a park with a diagonal road through it. Madhuri walks every day from her house at H to her school at S, always taking one of the shortest routes. Find the number of different shortest routes that she can choose.



Sol. Number of way from H to $P = \dfrac{4!}{2!2!} = 6$

Number of ways from P to $Q = 1$ (obvious)

Number of ways from Q to $S = \dfrac{4!}{1!3!} = 4$



Total ways $= 6 \times 4 = 24$.

10. A palindrome is a word or number that reads the same backwards as forwards; e.g., LEVEL or 9889 or 121. If the sum of all the three digit palindromes are 100 K, then find the value of K.

Sol. Total 3 digit palindromes are $9 \times 10 \times 1$

We have to dind sum of these 90 numbers

Now sum due to unit place

$1 \times 10(1+2+3+4+5+6+7+8+9) = 45 \times 10$

Sum due to 10th place

$9 \times 10(1+2+3+4+4+5+6+7+8+9) = 45 \times 10 \times 9$

Sum due to 100th place

$10 \times 102\,(1+2+3+4+5+6+7+8+9) = 45 \times 10 \times 102$

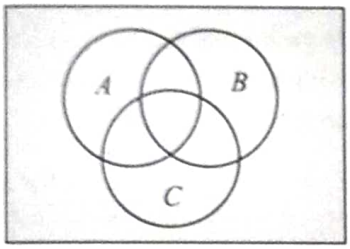
$\therefore S = SU + ST + SH$

$(1) + (2) + (3) = 45 \times 10(1 + 9 + 10^2)$

$= 45 \times 10 \times 110 = 495 \times 10^2$

11. The total numnber of 7 digit numbers that can be formed using two 3's, two 4's, two 5's and one 6 so that no two consecutive digits are identicals is:

Sol. In such of type of examples we can proceed through Venn diagram: Let A denotes region in which two 3's are together Let B denotes region in which two 4's are together Let C denotes region in which two 5's are together



$n(A \cap B \cap C)$ = Total number of ways in which 33, 44 and 55 are together = 4! = 24

$n(A \cap B) = \dfrac{5!}{2} = 60$

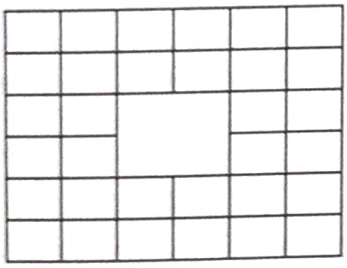
Also $n(A) = \dfrac{6!}{2!2!} = 180$

$\therefore$ Total number of ways in which no two consecutive digits are identical = Total – Invalid

$= \dfrac{7!}{2!2!2!} - (3 \times 180 - 3 \times 60 + 24)$

$= 630 - 384 = 246$

12. The number of rectangles whose edges lie completely on the grid lines of tollowing figure is:



Sol. Total number of rectangles $= {}^7C_2 \times {}^7C_2 - 4(27 + 9)$

$= 441 - 144 = 297$

13. Let A_i (where i = 1, 2, 3, ... 12) are the vertices of regular dodecagon and G is its centre. Let S denotes number of straight lines that can be formed with these 13 points and T denotes the n number of triangles formed with these points and D denotes the number of diagonals in the dodecagon, then $\left[\dfrac{T+D+S}{T-D-S}\right]$ is (where $[\cdot]$ denotes greatest integer).

Sol. $S = {}^{12}C_1 = 66$; $T = {}^{13}C_3 - 6 = 286 - 6 = 280$

and $D = {}^{12}C_2 - 12 = 55$

$\therefore T + D + S = 400$ and $T - D - S = 160$

14. Let N be the number of ordered 6-tuples $(x_1, x_2, x_3, x_4, x_5, x_6)$ of positive integers satisfying $x_1 + x_2 + x_3 + 3x_4 + 3x_5 + 5x_6 = 21$, then $\sqrt{N}$ is equal to

Sol. $x_1 + x_2 + x_3 = x$, $\quad x_4 + x_5 = y$, $\quad z = x_6$

$x + 3y = 21 \quad -5x_6 \quad x_6 = 1, 2$

$x + 3y = 16, 11$

$(x, y, x_6) = (4, 4, 1), (7, 3, 1), (10, 2, 1), (5, 2, 2)$

Number of 6-tuples

$= {}^{4-3+2}C_2\,{}^{4-2+1}C_1 + {}^{7-3+2}C_2\,{}^{3-2+1}C_1 + {}^{10-3+2}C_2\,{}^{2-2+1}C_1 + {}^{5-3+2}C_2\,{}^{2-2+1}C_1 = 9 + 30 + 36 + 6 = 81$

15. Let N is the number of 4 element subsets $\{a, b, c, d\}$ of $\{1, 2, 3, 4, ..., 20\}$ such that $a + b + c + d$ is divisible by 3. Then the sum of the digits of N is equal to

Sol. Number of ways $= {}^6C_4 + {}^6C_2\,{}^7C_1\,{}^7C_1 + {}^6C_1({}^7C_3 + {}^7C_3) + {}^7C_2\,{}^7C_2$

all of form $3k$ $\quad$ two of form $3k$, $\quad$ one of form $3k$, two of form $3k + 1$

one $3k + 1$, $\quad$ one $3k + 2$ $\quad$ 3 of $3k + 1$ or & two of $3k + 2$

$3k + 2$

$= 15 + 735 + 420 + 441 = 1611$

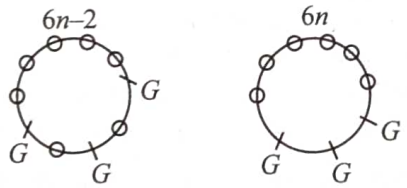
16. There are $6n$ flowers of one kind and 3 flowers of another kind. How many differen garlands can be made using all the flowers?

when '0' is always taken	'1' is always taken	'2' is always taken	$(2n-1)$ is always taken	'$2n$' is always taken
0 0 $6n$ 0 1 $6n-1$ 0 2 $6n-2$ ⋮ ⋮ ⋮ 0 $3n$ $3n$	1 1 $6n-2$ 1 2 $6n-3$ 1 3 $6n-4$ ⋮ ⋮ ⋮ 1 $3n-1$ $3n$	2 2 $6n-4$ 2 3 $6n-5$ ⋮ ⋮ ⋮ 2 $3n-2$ $3n$ 1 $3n-1$ $3n+1$	$2n-1$ $2n-1$ $2n+2$ $2n-1$ $2n$ $2n+1$	$2n$ $2n$ $2n$
$(3n+1)$ ways	$(3n-1)$ ways	$(3n-2)$ ways	2 ways	1 way

Sol. Total $= 1 + (2+4) + (5+7) + (8+10) \ldots ((3n-1) + (3n+1))$

$= 1 + 6\,[1 + 2 + 3 + \ldots + n)$

$= 1 + 6 + \dfrac{n(n+1)}{2} = 3n^2 + 3n + 1$



Note: 'O' is always taken means all $6n$ flowers on the same side of the extreme G.

|||ly 1 is always taken means at least 1 different flower between two successive G's.

17. Consider the equation $x^4 - ax^3 - bx^2 - cx - d = 0$, $a, b, c, d \in I^+$, $a \geq b \geq c \geq d$ then, the number of integral solutions are

(*a*) 0 (*b*) 4

(*c*) 2 (*d*) 3

Sol. (*a*) Suppose m is an integer root of

$x^4 - ax^3 - bx^2 - cx - d = 0$ as $d \neq 0$

$\Rightarrow m \neq 0$

(i) $m > 0$

$m^4 - am^3 - bm^2 - cm = d$

$\Rightarrow d = m(m^3 - am^2 - bm - c)$

$\Rightarrow d \geq m$

Also, $m^4 - am^3 = bm^2 + cm + d$

$m^3(m - a) = bm^2 + cm + d$

$\Rightarrow m > a$ contradiction

(ii) $m < 0$

$m = -n$

$\Rightarrow n^4 + an^3 - bn^2 + cn - cn - d = 0$

$n^4 + n^2(an - b) + (cn - d) > 0$

So, contradiction hence, equation has no integral solution.

18. We have 21 identical balls available with us which we need to be distributed amongst 3 boys A, B and C such that A always gets an even number of balls. The number of possible ways of doing this is

(*a*) 112 (*b*) 120

(*c*) 126 (*d*) 132

Sol. (*d*) We have to find non negative integral solution of equation $2x + y + z = 21$

Note that x can take a maximum value of 10 and a minimum value of 0.

We rearrange equation so that we get an integer equation with y and z as variables, x as a constant

$y + z = 21 - 2x$

The number of non negative integer solution is

${}^{21-2x+2-1}C_1 = {}^{22-2x}C_1 = 22 - 2x$

We now add the number of solutions so obtained for all the possible values of x.

The total number of solution is therefore

$\sum_{x=0}^{10}(22 - 2x) = 132$

19. All the 7 digit numbers containing each of the digits 2, 3, 5, 6, 7, 8, 9 exactly once and not divisible by 5 are arranged in increasing order. The 2017 number in the list is

(*a*) 6298753 (*b*) 6325789

(*c*) 6352789 (*d*) 6252789

Sol. (*b*) Number beginning with 2 and not divisible by 5 is

$6! - 120 = 600$

Number beginning with 3 and not divisible by 5 = 600

Number beginning with 5 and not divisible by 5 = 720

Total = 1920

Number beginning with 6 2 and not divisible by 5 is

$120 - 24 = 96$

$\therefore$ Total = 2016

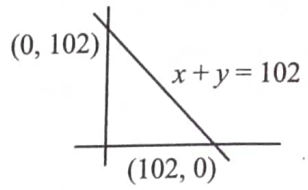
$\therefore$ 2017th number is 6325789

20. Number of points with integral co-ordinates that lie inside a triangle whose co-ordinates are (0, 0), (0, 102) and (102,0) is

Sol. [5050]

$x + y < 102 \quad x \in N, y \in N$

i.e. $x + y \leq 101$



give 1 to each x and y

hence $x + y \leq 99$

equivalent to $x + y + z = 99$

$\Rightarrow {}^{101}C_2 = \frac{101.100}{2} \underbrace{0000.....00}_{99}\underbrace{\emptyset\emptyset}_{2} = 5050$

21. The figure 4, 5, 6, 7, 8 are wirtten in ever possible order. The number of numbers greater than 56000 is 15K then K is equal to _____.

Sol. [90]

Case-I: When we use 6, 7 or 8 at ten thousand place then number of numbers = $3 \times {}^4P_4 = 72$

Case-II: When we use 5 at ten thousand place and 6, 7 or 8 at thousand place then number of numbers is

$= 1 \times 3 \times {}^3P_3 = 18$

Hence the required numbers of numbers is 72 + 18 = 90

22. 6 different books to be distributed between R | S | G if each child gets at least one book. (Do it in two different ways)

Sol. **Method 1:** $3^6 - \{{}^3C_1(2^6 - 2) + {}^3C_2\}$ (note that: ${}^6C_3 . 3! . 3^3$ is wrong)

Method 2: 6 different things (123, 114 and 222) – broad groups

$= \left[\frac{6!}{1!\,2!\,3!} + \frac{6!}{1!\,1!\,4!\,2!} + \frac{6!}{(2!)^3\,3!}\right].3!$

23. Seven digit numbers are formed using digits 1, 2, 3, 4, 5, 6, 7, 8, 9 without repetition. Then find number of ways of selecting a number such that product of any 5 consecutive digits is divisible by either 5 or 7 is p. Then 12p is equal to

(*a*) $\frac{7}{24}(9!)$ (*b*) $\frac{7}{8}(9!)$

(*c*) $\frac{7}{12}(9!)$ (*d*) 9!

Sol. (*a*) ${}^3C_1 \times {}^8P_6 + {}^3C_1 \times {}^8P_6 - {}^3P_2 \times {}^7P_5$

Exercise-1 (Topicwise)

FUNDAMENTAL PRINCIPLE OF COUNTING

1. The number of 10-digit numbers such that the product of any two consecutive digits in the number is a prime number, is
(a) 1024 (b) 2048
(c) 512 (d) 64

2. The number of six digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6 & 7 so that digits do not repeat and the terminal digits are even is:
(a) 144 (b) 72
(c) 288 (d) 720

3. Number of positive integers which have no two digits having the same value with sum of their digits being 45, is
(a) 10 ! (b) 9 !
(c) 9 · 9 ! (d) 17·8!

4. The number of signals that can be made with 3 flags each of different colour by hoisting 1 or 2 or 3 above the other, is:
(a) 3 (b) 7
(c) 15 (d) 16

LINEAR PERMUTATION AND PERMUTATION UNDER VARIOUS CONDITIONS

5. If all the letters of the word "QUEUE" are arranged in all possible manner as they are in a dictionary, then the rank of the word QUEUE is:
(a) 15th (b) 16th
(c) 17th (d) 18th

6. 8 chairs are numbered from 1 to 8. Two women & 3 men wish to occupy one chair each. First the women choose the chairs from amongst the chairs marked 1 to 4, then the men select the chairs from among the remaining. The number of possible arrangements is:
(a) ${}^6C_3 \cdot {}^4C_4$ (b) ${}^6P_2 \cdot {}^4P_3$
(c) ${}^4C_3 \cdot {}^4P_3$ (d) ${}^4P_2 \cdot {}^6P_3$

7. Number of words that can be made with the letters of the word "GENIUS" if each word neither begins with G nor ends in S, is:
(a) 24 (b) 240
(c) 480 (d) 504

8. 5 boys & 3 girls are sitting in a row of 8 seats. Number of ways in which they can be seated so that not all the girls sit side by side, is:
(a) 36000 (b) 9080
(c) 3960 (d) 11600

9. 10 different letters of an alphabet are given. Words with 5 letters are formed from these given letters, then the number of words which have atleast one letter repeated is:
(a) 69760 (b) 30240
(c) 99748 (d) 69670

10. In a conference 10 speakers are present. If S_1 wants to speak before S_2 & S_2 wants to speak after S_3, then the number of ways all the 10 speakers can give their speeches with the above restriction if the remaining seven speakers have no objection to speak at any number is:
(a) ${}^{10}C_3$ (b) ${}^{10}P_8$
(c) ${}^{10}P_3$ (d) $\frac{10!}{3!}$

11. 20 Passengers are to travel by a double decked bus which can accommodate 13 in the upper deck and 7 in the lower deck. The number of ways that they can be divided if 5 refuse to sit in the upper deck and 8 refuse to sit in the lower deck, is
(a) 25 (b) 21
(c) 18 (d) 15

12. How many nine digit numbers can be formed using the digits 2, 2, 3, 3, 5, 5, 8, 8, 8 so that the odd digits occupy even positions?
(a) 7560 (b) 180
(c) 16 (d) 60

13. The number of words that an be formed by using the letters of the word 'MATHEMATICS' that start as well as end with T, is
(a) 80720 (b) 90720
(c) 20860 (d) 37528

14. The number of permutations that can be formed by arranging all the letters of the word 'NINETEEN' in which no two E's occur together. is
(a) $\frac{8!}{3!3!}$ (b) $\frac{5!}{3! \times {}^6C_2}$
(c) $\frac{5!}{3!} \times {}^6C_3$ (d) $\frac{8!}{5!} \times {}^6C_3$

15. There are 2 identical white balls, 3 identical red balls and 4 green balls of different shades. The number of ways in which they can be arranged in a row so that atleast one ball is separated from the balls of the same colour, is:
(a) 6 (7! – 4!) (b) 7 (6! – 4!)
(c) 8! – 5! (d) 7 (8! – 6!)

CIRCULAR PERMUTATION

16. The number of ways in which 6 red roses and 3 white roses (all roses different) can form a garland so that all the white roses come together, is

(*a*) 2170 (*b*) 2165
(*c*) 2160 (*d*) 2155

17. The number of ways in which 5 beads, chosen from 8 different beads be threaded on to a ring, is:

(*a*) 672 (*b*) 1344
(*c*) 336 (*d*) 1433

18. How many ways are there to seat n married couples ($n \geq 3$) around a table such that men and woman alternate and each woman is not adjacent to her husband.

(*a*) $n!(n-1)! - 2(n-1)!$
(*b*) $2(n-1)! - n!(n-1)!$
(*c*) $n!(n-1)!$
(*d*) $2n!(n-1)!$

19. The number of ways in which 6 men and 5 woman can dine at a round table if no two woman are to sit together is given by

(*a*) $6! \times 5!$ (*b*) 6×5
(*c*) 30 (*d*) 3×4

COMBINATION AND IT'S GEOMETRIAL APPLICATION

20. Out of 7 consonants and 3 vowels, how many words can be make each containing 3 consonants and 2 vowels?

(*a*) 120 (*b*) 25200
(*c*) 4200 (*d*) 42000

21. The straight lines L_1, L_2, L_3 are parallel and lie in the same plane. A total number of m points are taken on L_1; n points on L_2, k points on L_3. The maximum number of triangles formed with vertices at these points are

(*a*) ${}^{m+n+k}C_3 -$
(*b*) ${}^{m+n+k}C_3 - {}^{m}C_3 - {}^{n}C_3 - {}^{k}C_3$
(*c*) ${}^{m}C_3 + {}^{n}C_3 + {}^{k}C_3$
(*d*) ${}^{m+n+k}C_3 + 1$

22. In a place these are 37 straight lines of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Then the number of intersection points the lines have is equal to

(*a*) 535 (*b*) 601
(*c*) 728 (*d*) 730

23. From six gentlemen and four radies, a committee of five is to be formed. Number of ways in which this can be done if the committee is to include at least one lady is

(*a*) 252 (*b*) 246
(*c*) 248 (*d*) 250

24. Which of the following is correct

(*a*) The number of diagonals that can be drawn by joining the vertices of an octagon is 28.
(*b*) If a polygon has 54–diagonals, the number of its sides is 9.
(*c*) In a polygon no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon be 70 then the number of diagonals of the polygon is 20.
(*d*) All above are correct.

25. The number of ways in which one of more balls can be selected out of 10 white, 9 green and 7 blue balls is

(*a*) 892 (*b*) 881
(*c*) 891 (*d*) 879

26. A lady gives a dinner party to 5 guests to be selected from nine friends. The number of ways of forming the party of 5, given that two of the friends will not attend the party together is

(*a*) 56 (*b*) 126
(*c*) 91 (*d*) 92

27. The number of words of 5 letters that can be made with the letters of the word "PROPOSITION'.

(*a*) 6890 (*b*) 7000
(*c*) 6800 (*d*) 6900

DIVISORS OF A GIVEN NATURAL NUMBER

28. The number of proper divisors of $a^p b^q c^r d^s$ where a, b, c, d are primes & $p, q, r, s \in N$, is

(*a*) $pqrs$
(*b*) $(p+1)(q+1)(r+1)(s+1) - 4$
(*c*) $pqrs - 2$
(*d*) $(p+1)(q+1)(r+1)(s+1) - 2$

29. N is a least natural number having 24 divisors. Then the number of ways N can be resolved into two factors is

(*a*) 12 (*b*) 24
(*c*) 6 (*d*) 18

30. How many divisors of 21600 are divisible by 10 but not by 15?

(*a*) 10 (*b*) 30
(*c*) 40 (*d*) 20

GROUPING

31. A team of 8 students goes on an excursion, in two cars, of which one can seat 5 and the other only 4. If internal arrangement inside the car does not matter then the number of ways in which they can travel, is

(*a*) 91 (*b*) 182
(*c*) 126 (*d*) 3920

32. 9 students of class 12th of whom 2 from section A, 3 from section B and 4 from section C are available for inter school debate competition. Number of ways in which 4 students can be selected so that there is atleast 1 from each section, is

(a) 144 (b) 120
(c) 72 (d) 71

33. There are five different peaches and three different apples. Number of ways they can be divided into two packs of four fruits if each pack must contain atleast one apple, is

(a) 95 (b) 65
(c) 60 (d) 30

MULTINOMIAL THEOREM

34. The number of ways in which 10 identical apples can be distributed among 6 children so that each child receives atleast one apple is:

(a) 126 (b) 252
(c) 378 (d) 152

35. The number of terms in a complete homogeneous expression of degree n in x, y and z is

(a) n (b) $\frac{n(n+1)}{2}$
(c) $\frac{(n+1)(n+2)}{2}$ (d) $\frac{n(n-1)}{2}$

36. Number of ways of selecting 5 coins from coins three each of Rs. 1, Rs. 2 and Rs. 5 if coins of the same denomination are alike, is:

(a) 9 (b) 12
(c) 21 (d) 15

37. The number of ways in which 16 identical things can be distributed among 4 persons if each person gets at least 3 things, is

(a) 33 (b) 35
(c) 38 (d) 36

DERANGEMENTS AND MISCELLENOUS

38. There are six letters L_1, L_2, L_3, L_4, L_5, L_6 and their corresponding six envelopes E_1, E_2, E_3, E_4, E_5, E_6. Letters having odd value can be put into odd value envelopes and even value letters can be put into even value envelopes, so that no letter go into the right envelopes, then number of arrangement equals.

(a) 6 (b) 9
(c) 44 (d) 4

39. The sum of all the numbers which can be formed by using the digits 1, 3, 5, 7 all at a time and which have no digit repeated, is

(a) $16 \times 4!$ (b) $1111 \times 3!$
(c) $16 \times 1111 \times 3!$ (d) $16 \times 1111 \times 4!$.

40. The sum of all the four digit numbers which can be formed using the digits 6,7,8,9 (repetition is allowed)

(a) 2133120 (b) 2133140
(c) 2133150 (d) 2133122

41. The number of ways in which 10 identical apples can be distributed among 6 children so that each child receives atleast one apple is.

(a) 126 (b) 125
(c) 120 (d) 130

42. Number of numbers greater than a million using the digits 2, 3, 0, 3, 4, 2, 5 of these how many are div. by 5.

(a) 330, 1080 (b) 1080, 330
(c) 1000, 300 (d) 300, 1000

43. Coeffecient of $x^p y^q z^r$ in the expansion of $(x + y + z)^{p+q+r}$ equal to.

(a) $(p+q+r)!$ (b) $\frac{(p+q-r)!}{p!q!r!}$
(c) $\frac{2(p+q+r)!}{p!q!r!}$ (d) $\frac{(p+q+r)!}{p!q!r!}$

44. In case these 6 toys are to be distributed equally between R/S/G then our answer will be

(a) 75 (b) 90
(c) 100 (d) 80

45. Number of ways in which 12 different books can be tied up in 3 bundles one containing 2 and the other two bundles containing 5 different books.

(a) $\frac{12!}{5!5!2!2!}$ (b) $\frac{12!}{5!2!}$
(c) $\frac{12!}{5!5!2!}$ (d) None of these

46. Maximum number of points of intersection of six circles in the plane

(a) 20 (b) 25
(c) 30 (d) 35

47. Nine hundred distinct N-digit numbers are to be formed by using 6, 8 and 9 only. The smallest value of N for which this is possible, is

(a) 6 (b) 7
(c) 8 (d) 9

48. Number of distinct terms in the expansions of $(x + y + z + w)^{50}$ is equal to.

(a) ${}^{53}C_3$ (b) 51
(c) 51^2 (d) ${}^{51}C_3$

49. Number of natural solution of the equation $x + y + z = 102$.

(a) 5050 (b) 4050
(c) 4950 (d) None of these

50. Number of ways in which K identical balls can be distributed in p different boxes if no box remains empty.

(a) $\frac{(K-I)!}{(K-P)!(P-I)!}$ (b) $\frac{(K-P)!}{(K-I)!(P-I)!}$

(c) $\frac{(P-I)!}{(K-P)!(K-I)!}$ (d) None of these

51. AA, BB, CC, D, E, G, F can be seated if persons of the same nationality are to be separated.

(a) $47 \times 8!$ (b) $48 \times 8!$

(c) $43 \times 9!$ (d) None of these

52. A set contains $(2n + 1)$ elements. If the number of subsets of this which contain at most n elements is 4096, then the value of n is-

(a) 6 (b) 15

(c) 21 (d) 12

53. There are n points in a plane of which p points are collinear. How many lines can be formed from these points

(a) ${}^{(n-p)}C_2$ (b) ${}^{n}C_2 - {}^{p}C_2$

(c) ${}^{n}C_2 - {}^{p}C_2 + 1$ (d) ${}^{n}C_2 - {}^{p}C_{2-1}$

54. In how many ways a garland can be made from exactly 10 flowers?

(a) 10! (b) 9!

(c) 2(9!) (d) $\frac{9!}{2}$

55. Number of divisors of $n = 38808$ (except 1 and n) is

(a) 70 (b) 68

(c) 72 (d) 74

56. The number of positive integral solutions of $abc = 30$ is

(a) 30 (b) 27

(c) 8 (d) 16

57. In how many ways 3 letters can be posted in 4 letter-boxes, if all the letters are not posted in the same letter-box

(a) 63 (b) 60

(c) 77 (d) 81

58. The exponent of 3 in 100! is

(a) 33 (b) 44

(c) 48 (d) 52

Exercise-2 (Learning Plus)

1. The number of 4 digit numbers that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contain digit 1 is (Repetition is not allowed)

(a) 1225 (b) 1252

(c) 750 (d) 480

2. The number of word groups by taking at least 1 letters of each words 'PATH', 'GROW' and 'SKIN' are-

(a) $15^3 - 1$ (b) 15^3

(c) $16^3 - 1$ (d) 16^3

3. The total number of seven digit numbers the sum of whose digits is even is-

(a) 9000000

(b) 4500000

(c) 8100000

(d) 450000

4. ${}^{1}P_1 + 2.\,{}^{2}P_2 + 3.\,{}^{3}P_3 + \ldots\ldots + n.\,{}^{n}P_n =$

(a) ${}^{n+1}P_{n+1}$

(b) ${}^{n+1}P_{n+1} - 1$

(c) ${}^{n+1}P_{n+1} - 2$

(d) ${}^{n+1}P_n$

5. The number of 10-digit numbers such that the product of any two consecutive digits in the number is a prime number, is

(a) 1024 (b) 2048

(c) 512 (d) 64

6. There are 9 children comprising of 3 boys and 6 girls. Let m denotes the number of ways when the 3 boys are in succession and n denotes the corresponding figure when the two end position are occupied by boys. If $m = kn$ then k is equal to

(a) 3 (b) 2

(c) 1 (d) (1/3)

7. In a football championship, there were played 153 matches. Every team played one match with each other. The number of teams participating in the championship is

(a) 17 (b) 18

(c) 9 (d) 13

8. In how many ways a team of 11 players can be formed out of 25 players, if 6 out of them are always to be included and 5 are always to be excluded

(a) 2020

(b) 2002

(c) 2008

(d) 8002

9. In how many ways can a committee consisting of one or more members be formed out of 12 members of the Municipal Corporation

(a) 4095 (b) 5095

(c) 4905 (d) 4090

10. The number of ways in which we can select three numbers from 1 to 30 so as to exclude every selection of all even numbers is

(a) 4060 (b) 3605
(c) 455 (d) 446

11. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2+k-1}{(k+1)!}$ is equal to

(a) 1 (b) 2
(c) 3 (d) 4

12. The number of numbers between 1 and 10^{10} which contain the digit 1 is-

(a) $10^{10}-9^{10}-1$ (b) 9^{10}
(c) $10^{10}-8^{10}$ (d) 9^9

13. Number of ways of selection of 4 letters of the word "PERFECT" is

(a) 15 (b) 21
(c) 25 (d) 31

14. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1}-T_n=21$, then n equals

(a) 5 (b) 7
(c) 6 (d) 4

15. Between two junction stations A and B there are 12 intermediate stations. The number of ways in which a train can be made to stop at 4 of these stations so that no two of these halting stations are consecutive is-

(a) ${}^8C_4$ (b) ${}^9C_4$
(c) ${}^{12}C_4-4$ (d) ${}^9C_3$

16. The number of ways of selecting n things out of $3n$ things, of which n are of one kind and n are of a second kind and alike and the rest are unlike, is.

(a) $(n+2)2^{2n}$ (b) $(n+2)2^{n-1}$
(c) $(n+2)2^{n+1}$ (d) $n+2^n$

17. To fill 12 vacancies there are 25 candidates of which five are from scheduled caste. If 3 of the vacancies are reserved for scheduled caste candidates while the rest areopen to all, then the number of ways in which the selection can be made

(a) ${}^{20}C_7+5\times{}^{20}C_8+10\times{}^{20}C_9$
(b) ${}^{22}C_9-{}^5C_3$
(c) ${}^{22}C_3+{}^5C_3$
(d) ${}^5C_3-{}^{22}C_9$

18. All possible 120 permutations of WDSMC are arranged in dictionary order, as if each were an ordinary five-letter word. The last letter of the 86th word in the list, is

(a) W (b) D
(c) M (d) C

19. The number of ways in which 5 male and 2 female members of a committee can be seated around a round table so that the two female are not seated together is

(a) 480 (b) 600
(c) 720 (d) 840

20. The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by

(a) $6!\times5!$ (b) 30
(c) $5!\times4!$ (d) $7!\times5!$

21. The number of ways in which four letters of the word 'MATHEMATICS' can be arranged is given by

(a) 136 (b) 192
(c) 1680 (d) 2454

22. The total number of ways of selecting six coins out of 20 one rupee coins, 10 fifty paise coins and 7 twenty five paise coins is

(a) 28 (b) 56
(c) ${}^{37}C_6$ (d) ${}^{37}P_6$

23. The number of six digit numbers between 700000 and 800000 such that the first digit is equal to the sum of the other five digits is

(a) 1512 (b) 330
(c) 290 (d) 130

24. In an eleven storey building 9 people enter a lift cabin. It is known that they will leave the lift in groups of 2, 3 and 4 at different residential storeys. Number of ways in which they can get down is :

(a) $\frac{(12)!}{2!(3!)^2.4!}$ (b) $\frac{10!}{4}$
(c) $\frac{10!}{4!}$ (d) $\frac{(12)!}{(3!)^2\cdot4!}$

25. Sixteen players $S_1, S_2, S_3, \ldots, S_{16}$ play in a tournament. Number of ways in which they can be grouped into eight pairs so that S_1 and S_2 are in different groups, is equal to

(a) $\frac{(14)!}{2^6\cdot6!}$ (b) $\frac{(15)!}{2^7\cdot7!}$
(c) $\frac{(14)!}{2^7\cdot6!}$ (d) $\frac{(14)!}{2^6\cdot7!}$

26. If $A_1A_2\ldots A_{18}$ is a regular polygon, then number of ways in which a quadrilateral can be formed with the help of vertices of given polygon, such that no side of quadrilateral is a side of given polygon,

(a) 1365 (b) 1287
(c) 3060 (d) 715

27. The number of odd proper divisors of $3^p\,.\,6^m\,.\,21^n$ is

(a) $(p+1)(m+1)(n+1)-2$
(b) $(p+m+n+1)(n+1)-1$
(c) $(p+1)(m+1)(n+1)-1$
(d) $(p+1)(m+1)(n+1)$

28. Let there be $n \geq 3$ circles in a plane. The value of n for which the number of radical centres is equal to the number of radical axes is (assume that all radical axes and radical centre exist and are different)

(*a*) 7 (*b*) 6
(*c*) 5 (*d*) None of these

29. If three one by one squres are selected at random from the chessboard, then the probability that they form the letter '*L*' is

(*a*) $\frac{196}{{}^{64}C_3}$ (*b*) $\frac{49}{{}^{64}C_3}$
(*c*) $\frac{36}{{}^{64}C_3}$ (*d*) $\frac{98}{{}^{64}C_3}$

30. If a cricket team of 11 players is to be selected from 8 bastman, 6 bowlers , 4 all rounder and 2 wicket keepers, then number of selection when 2 particular bastmen don't wan to play when a particular bowled will play is

(*a*) ${}^{17}C_{10} + {}^{19}C_{11}$ (*b*) ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$
(*c*) ${}^{17}C_{10} + {}^{20}C_{11}$ (*d*) ${}^{19}C_{10} + {}^{19}C_{11}$

31. Number of words can be formed using all the letters of the work 'HIPHIPHURRAY' In which all *H*'s lie somewhere between *R*'s is

(*a*) (198)7! (*b*) (99)7!
(*c*) (99)8! (*d*) (198)8!

32. Let $A = \{1, 2, 3, 4\}$. The number of different ordered pairs (B, C) that can be formed such that can be formed such that $B \subseteq A, C \subseteq A$ and $B \cap C = \phi$

(*a*) 3 (*b*) 9
(*c*) 27 (*d*) 81

33. There were two women participating in a chess tournament. Every participant played two games with every other participant. The number of games that the men played among themselves exceeds by 66 that of the games which the man played with the two women then total number of participants is

(*a*) 11 (*b*) 13
(*c*) 10 (*d*) 12

34. Rajdhani express stops at six intermediate stations between Kota and Mumbai. Five passengers board at Kota. Each passenger can get down at any station till Mumbai. Then find number of ways so that all five passengers will get down at different stations, is

(*a*) ${}^6P_5$ (*b*) ${}^6C_5$
(*c*) ${}^7P_5$ (*d*) ${}^7C_5$

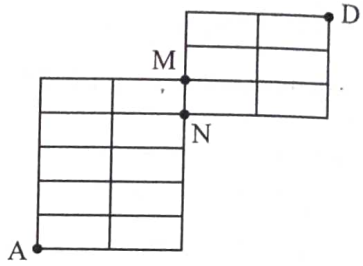
35. A bag contains 17 markers with numbers 1 to 17. A marker is drawn at random and then replaced; a second marker is drawn then find number of ways so that first number is even and second is odd, is

(*a*) 36 (*b*) 128
(*c*) 256 (*d*) 72

36. If all numbers starting from 1 are listed in increasing order by using the digits 0, 1, 2, 3, 4, 5, 6 then the number whose rank is 8000 is

(*a*) 32605 (*b*) 32116
(*c*) 32216 (*d*) 33316

37. Let a person has to go from A to D moving along horizontal and vertical grids using shortest path (as shown in figure). Then



(*a*) The number of paths is equal to 276.
(*b*) The number of paths is equal to 186.
(*c*) The number of paths in which he doesn't cross point N is equal to 36.
(*d*) The number of paths in which he doesn't cross point M is equal to 60.

38. Number of ways in which 6 different toys can be distributed equally among 3 children is also equal to

(*a*) number of ways in which all the letters of the word KUMKUM can be arranged in a definite order.
(*b*) number of different signals which can be obtained from 10 different flags taken two at a time keeping one above the other.
(*c*) number of ways in which five different marbles can be placed in three different boxes so that no box remains empty.
(*d*) number of ways in which all the letters of the word CALCULUS can be arranged without changing the order and position of non alike letters.

39. There are 10 seats in the first row of a theatre of which 4 are to be occupied. The number of ways of arranging 4 persons so that no two persons sit side by side is

(*a*) ${}^7C_4$ (*b*) $4 \cdot {}^7C_3$
(*c*) ${}^7C_3 \times 4!$ (*d*) 840

40. Let $S = x_1 + x_2 + x_3 + x_4 + x_5$ and $K = x_1 + x_2 + x_3$ then

(*a*) Number of positive integral solutions of the equation $S = 21$ is 4845
(*b*) Number of integral solutions of the equation $S = 21$ such that $x_i \geq 2\ \forall_i = 1, 2, 3, 4, 5$ is 1365.
(*c*) Number of integral solutions of the equation $S = 20$ is 42404
(*d*) Number of integral solutions $(x_1, x_2, x_3, x_4, x_5)$ such that $S = 20$ and $K = 5$ is 336

Comprehension-1 (Q. 41 to 43) : In how many ways can the letters of the word ARRANGE be arranged so that

41. The two R' are never together

(*a*) 900 (*b*) 872

(*c*) 960 (*d*) 980

42. The two A's are together but not the two R's are together.

(*a*) 240 (*b*) 260

(*c*) 300 (*d*) 340

43. Neither two A's nor the two R's are together.

(*a*) 660 (*b*) 600

(*c*) 640 (*d*) 690

Comprehension-2 (Qs. 44 to 46)

We have to choose 11 players for cricket team from 8 batsmen, 6 bowlers, 4 all-rounders, and 2 wicket keepers.

44. The number of selections when almost one all-rounder and one wicket keeper will play is

(*a*) ${}^4C_1 \times {}^{14}C_{10} + {}^2C_1 \times {}^{14}C_{10} + {}^4C_1 \times {}^2C_1 \times {}^{14}C_9 + {}^{14}C_{11}$

(*b*) ${}^4C_1 \times {}^{15}C_{11} + {}^{15}C_{11}$

(*c*) ${}^4C_1 \times 15C_{10} + {}^{15}C_{11}$

(*d*) ${}^4C_1 \times {}^{15}C_9 + {}^{15}C_{10}$

45. The number of selections when two particular batsmen do not want to play when a particular bowler will play is

(*a*) ${}^{17}C_{10} + {}^{19}C_{11}$ (*b*) ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$

(*c*) ${}^{17}C_{10} + {}^{20}C_{11}$ (*d*) ${}^{19}C_{10} + {}^{19}C_{11}$

46. The number of selection when a particular batsman and a particular wicket keeper do not want to play together is

(*a*) $2{}^{18}C_{10}$ (*b*) ${}^{19}C_{11} + {}^{18}C_{10}$

(*c*) ${}^{19}C_{10} + {}^{19}C_{11}$ (*d*) ${}^{19}C_6 + {}^{18}C_9$

Comprehension-3 (Qs. 47 to 48)

A is a set containing n elements. A subset S_1 of A is chosen. The set A is reconstructed by replacing the elements of S_1. Again, a subset S of A is chosen and again the set is reconstructed by replacing the elements of S_2. The number of ways of choosing S_1 or S_2 where

47. S_1 and S_2 have one element common is

(*a*) 3^{n-1} (*b*) $n \cdot 3^{n-1}$

(*c*) 2^{n-1} (*d*) n

48. $S_1 \cup S_2 = A$ is

(*a*) 3^n (*b*) $n \cdot 3^n$

(*c*) 4^n (*d*) 4^{n-1}

49. Match the column

	Column – I		Column – II
A.	There are 12 points in a plane of which 5 are collinear. The number of distinct convex quadrilaterals which can be formed with vertices at these points is:	p.	185
B.	If 7 points out of 12 are in the same straight line, then the number of triangles formed is	q.	420
C.	If AB and AC be two line segemets and there are 5, 4 points on AB and AC (other than A), then the number of quadrilateral, with vertices on these points equals	r.	126
D.	The maximum number of points of intersection of 8 unequal circles and 4 straight lines.	s.	60

(*a*) A → r; B → s; C → p; D → q

(*b*) A → q; B → p; C → s; D → r

(*c*) A → q; B → s; C → p; D → r

(*d*) A → p; B → q; C → r; D → s

50. Match the following:

	Column – I		Column – II
A.	If n be the numbers between 500 and 4000 can be formed with the digits 2, 3, 4, 5, 6 when repetition is not allowed, then n is divisible by	p.	2
B.	If n be the number of even numbers between 200 and 3000 can be formed with the digit 0, 1, 2, 3, 4 when repetition is not allowed, then n is divisible by	q.	3
C.	If n be the number of words that can be made by arranging the letters of the word ROORKEE that neither begin with R nor end with E, then n is divisible by	r.	5
D.	In a class tournament when the participants where to play one game with another, two class players fell ill, having played 3 games each. If the total number of games played is 84, the number of participants at the beginning was k then k is divisible by	s.	11
		t.	17

(*a*) A → q,r; B → q; C → p,q,r,s; D → p,q

(*b*) A → p,q; B → q,r,s; C → p; D → q,r

(*c*) A → p,q,r,s; B → q; C → p,q; D → q,r

(*d*) A → p,q; B → q; C → p,q,r,s; D → q,r

51. Match the following Column-I with Column-II.

	Column-I		Column-II
A.	Four dice (six faced) are rolled. The number of possible outcomes in which at least one dice shows 2 is	p.	210

B.	Let A be the set of 4-digit number $a_1a_2a_3a_4$ where $a_1 > a_2 > a_3 > a_4$. Then $n(A)$ is equal to	q.	750
C.	The total number of three-digit numbers, the sum of whose digits is even, is equal to	r.	671
D.	The number of four-digit numbers without repetition that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contains digit 1 is	s.	450

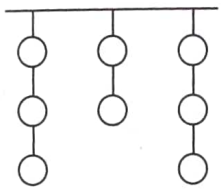
(*a*) A → s; B → q; C → p; D → r

(*b*) A → r; B → p; C → s; D → q

(*c*) A → p; B → r; C → q; D → s

(*d*) A → r; B → s; C → p; D → q

52. 8 clay targets are arranged as shown. If N be the number of different ways they can be shot (one at a time) if no target can be shot until the target(s) below it have been shot. Find the ten's digit of N.



53. Find number of positive integral solutions satisfying the equation $(x_1 + x_2 + x_3)(y_1 + y_2) = 77$.

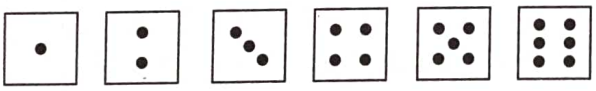
54. Number of five digits numbers divisible by 3 that can be formed using the digits 0, 1, 2, 3, 4, 7 and 8 if, each digit is to be used atmost one is N where 'N' is three digit number then sum of the digit of 'N'

55. Find number of positive integral solutions satisfying the equation $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 = 240$.

56. Sum of all the numbers that can be formed using all the digits 2, 3, 3, 4, 4, 4, is N then sum of the digit of the number 'N' equals

57. The number of polynomials of the form $x^3 + ax^2 + bx + c$ which are divisible by $x^2 + 1$ where $a, b, c \in \{1, 2, 3, ..., 10\}$ is $10K$, then K is _____ .

58. Number of distinct throws in a throw of n indistinguishable (alike) ordinary dice.



treat these six faces as beggars.

59. Number of numbers greater than 1000 from the digits 1, 1, 2, 3, 4, 0 taken 4 at a time.

60. How many 8 digit numbers can be formed using two 1's, two 2's, two 3's, one 4 and one 5 so that no two consecutive digits are identical.

61. A cricket team comprising of A, B, C, ..., J, K is to be sent for batting. If 'A' wants to bat before 'J' and 'J' wants to bat after 'G'. Then the number of batting orders if other players could go in any order.

62. Rajdhani express bound from Delhi to Bombay stops at 9 intermediate stations. 6 passengers enter the train during the journey holding 6 different tickets of the same class (K classes). How many different sets of tickets they may have had.

63. Number of non negative integral solution of the inequality $x + y + z + t \le 30$

64. Find the number of integral solutions of $x_1 + x_2 + x_3 = 24$, subjected to the condition that $1 \le x_1 \le 5$, $12 \le x_2 \le 18$ and $-1 \le x_3$.

65. The total number of five-digit numbers of different digits in which the digit in the middle is the largest is

66. Maximum number of points of intersection of six parabolas

67. The number of odd proper divisor of 24300 is $N + 11$, then N is equal to _____.

68. There are 15 seats in the first row of a theatre of which 6 are to be occupied. The number of ways of arranging 15 person so that no two persons are side by side is $k + 5$, then digit in unit's place in k is equal to

69. Number of ways in which 2 apples, 3 oranges and 4 mangoes can be distributed between R|S|G if each child receives none, one or more number of fruits. (Fruits of the same species are alike).

70. Number of different collections of 7 letters that can be formed from the letters of the word ACCOMMODATION.

71. A train having 12 stations enroute has to be stopped at 4 stations. Number of ways it can be stopped if no two of the stopping stations are consecutive.

72. How many words can be formed using the letters of the word "CIVILISATION" without changing the relative order of vowels and consonants.

73. A shelf contains 6 separate compartments. Number of ways in which 12 indistinguishable (like) marbles can be placed in the compartments so that no compartment is empty.

74. The number of five digit numbers that can be formed using its digits 1, 2, 3, 4, 5, 6, 7, 8, 9 in which one digit appears once and two digits appear twice (e.g. 44533) is k then the value of $k/1890$ is equal to ____.

75. In how many ways we can make a 5 letter word form MISSISSIPPI.

76. A supermarket offers ice creams in 10 different varieties. Ice creams of each variety are identical. In how many ways 4 ice creams can be selected?

(i) without any restriction ; (ii) at least two ice creams of the same variety are taken.

77. For the number 35700, find
   (*i*) The total number of proper factors and sum of all these factors,
   (*ii*) The total number and sum of the odd proper divisors,
   (*iii*) The number of proper divisors divisible by 10 and the sum of these divisors.

78. Find the number of positive integral solutions of $x + y + z + w = 20$ under the following conditions:
   (*i*) Zero values of x, y, z, w are include
   (*ii*) Zero values are excluded
   (*iii*) No variable may exceed 10; Zero values excluded
   (*iv*) Each variable is an odd number
   (*v*) x, y, z, w have different values (zero excluded).

79. Consider the word ASSASSINATION (Vowels-6; Consonants-7; | A | A | I | A | I | O |)
   (*i*) No two vowels are consecutive
   (*ii*) S's are separated/no two 'S' occur together
   (*iii*) Without changing the order of vowels (AAIAIO only)
   (*iv*) At least one 'S' is separated from the rest of the S's (Total—all four together)
   (*v*) Keeping the positions of each vowel fixed (each V must remain at its own place)
   (*vi*) Without changing the relative order of vowels and consonants.

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. Two teams are to play a series of five matches between them. A match ends in a win, loss or draw for a team. A number of people forecast the result of each match and no two people make the same forecast for the series of matches. The smallest group of people in which one person forecasts correctly for all the matches will contain n people, where n is
   (*a*) 81 (*b*) 243
   (*c*) 486 (*d*) None of these

2. In a certain test, there are n questions. In the test 2^{n-i} students gave wrong answers to at least i questions, where $i = 1, 2, ..., n$. If the total number of wrong answers given is 2047, then n is equal to
   (*a*) 10 (*b*) 11
   (*c*) 12 (*d*) 13

3. n is selected from the set {1, 2, 3, ..., 10) and the number $2^n + 3^n + 5^n$ is formed. Total number of ways of selecting n so that the formed number is divisible by 4 is equal to
   (*a*) 50 (*b*) 49
   (*c*) 48 (*d*) None of these

4. The number 916238457 is an example of nine digit number which contain each of the digits 1 to 9 exactly one, it also has the property that the digit 1 to 5 occur in their nautral order, while the digit 1 to 6 do not. Number of such numbers are
   (*a*) 2268 (*b*) 2520
   (*c*) 2975 (*d*) 1560

5. Consider a set {1, 2, 3, ..., 100). The number of ways in which a number can be selected from the set so that it is of the form x^y, (where $x, y \in N$ and ≥ 2), is
   (*a*) 12 (*b*) 16 (*c*) 5 (*d*) 11

6. Consider a dodecagon (12 sides) then:
   (*a*) Number of ways of selecting two triangles which can be made using the vertices of dodecagon so that they do not have a common side is 14692
   (*b*) Number of ways of selecting two triangle which can be made using the vertices of th dodecagon so that they do not have a common side is 21120
   (*c*) Number of hexagons which can be formed using the 12 vertices such that none of sides of hexagon is also a side of dodecagon is 2
   (*d*) Assume the dodecagon to be regular and also consider its centre then the number of straight line formed using 13 vertices is 66

7. Which of the following statements are correct?
   (*a*) Number of words that can be formed with 6 of the letters of the word "CENTRIFUGAL" if each word must contain all the vowels is $3 \cdot 7!$
   (*b*) There are 15 balls of which some are white and the rest black. If the number of ways in which the balls can be arranged in a row, is maximum then the number of white balls must be equal to 7 or 8. Assume balls of the same colour to be alike.
   (*c*) There are 12 things, 4 alike of one kind, 5 alike of another kind and the rest are all different. The total number of combinations is 240.
   (*d*) Number of selections that can be made of 6 letters from the word "COMMITTEE" is 35.

8. The sequence of all positive palindromes are written in ascending order 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22,...
   Then

(*a*) The 2018 positive palindrome is 1019101

(*b*) The 1998 positive palindrome is 1001001

(*c*) The rank of 2140412 is 3139

(*d*) The rank of 1999999 is 2998

9. Consider the word D = F R E E W H E E L. Which of the following statement(s) is/are correct

(*a*) Number of other ways in which the letters of the word D can be arranged is ${}^9P_5$.

(*b*) Number of ways is which the letters of the word *D* can be an arranged in a circle is ${}^8P_4$ distisguishing between clockwise and anticlockwise.

(*c*) Number of ways in which the letters of the word *D* can be arranged if vowels and consonants both are in alphabetical order is $\frac{1}{2}{}^0C_5$

(*d*) If as many more words are formed as possible using the letters of the word *D* the number of words which contain the word FEEL is ${}^6P_4$.

10. The number of ways of arranging the letters *AAAAA*, *BBB*, *CCC*, *D*, *EE* & *F* in a row if the letter *C* are separated from one another is:

(*a*) ${}^{13}C_3 \cdot \frac{12!}{5!\,3!\,2!}$ (*b*) $\frac{13!}{5!\,3!\,3!\,2!}$

(*c*) $\frac{14!}{3!\,3!\,2!}$ (*d*) $11 \cdot \frac{13!}{6!}$

11. Number of ways in which the letters of the word "TOMATO" can be arranged if two alike vowels are separated, is not equal to

(*a*) Number of ways in which letters of the word "RAITHATHA" can be arranged if no two alike letters are separated.

(*b*) Number of 3 digit numbers with atleast one 3 and atleast one 2.

(*c*) Number of ways in which Ram and Rama can exchange their maps if Ram has 3 and Rama has 7 maps, all maps being different, maintaining their original number of maps at the end.

(*d*) Number of ways in which 2 alike apples and 4 alike oranges can be distributed in three children if each child get none, one or more fruits.

12. A forecast is to be made of the results of five cricket matches, each of which can be a win, draw or a loss for Indian team. If *p* denotes the number of forecast with exactly 1 error, *q* denotes the number of forecast with exactly 3 error and '*r*' denotes the number of forecast with all error, then the correct statements are:

(*a*) $2q = 5r$

(*b*) $8p = q$

(*c*) $8p = 5r$

(*d*) $2(p + r) > q$

13. Coefficient of $x^2 y^3 z^4$ in the expansion of $(x + y + z)^9$ is equal to

(*a*) The number of ways in which 9 things of which 2 alike of one kind, 3 alike of 2nd kind, and 4 alike of 3rd kind can be arranged.

(*b*) The number of ways in which 9 identical things can be distributed in 3 persons each receiving atleast two things.

(*c*) The number of ways in which 9 identical things can be distributed in 3 persons each receiving none one or more.

(*d*) The number of ways in which 9 different books can be tied up in to three bundles one containing 2, other 3 and third containing 4 books.

14. The number of ways of choosing triplet (x, y, z) such that $z \geq \max\{x, y\}$ and $x, y, z \in \{1, 2, ..., n, n + 1\}$ is

(*a*) ${}^{n+1}C_3 + {}^{n+2}C_3$ (*b*) ${}^{n+1}C_2 + 2({}^{n+1}C_3)$

(*c*) $1^2 + 2^2 + ... + n^2$ (*d*) $2({}^{n+2}C_3) - {}^{n+1}C_2$

15. Number of ways in which 30 identical things are distributed among 6 persons is

(*a*) ${}^{17}C_5$ if each gets odd number of things

(*b*) ${}^{16}C_{11}$ if each gets odd number of things if each gets even number of

(*c*) ${}^{14}C_5$ things (excluding 0)

(*d*) ${}^{15}C_{10}$ if each gets even number of things (excluding 0)

16. Number of ways in which three numbers in A.P. can be selected from 1, 2, 3, ..., *n* is

(*a*) $\left(\frac{n-1}{2}\right)^2$ if *n* is even (*b*) $\frac{n(n-1)}{4}$ if *n* is even

(*c*) $\frac{(n-1)^2}{4}$ if *n* is odd (*d*) $\frac{(n-1)^2}{2}$ if *n* is odd

17. The number of ways of choosing triplet (x, y, z) such that $z \geq \max\{x, y\}$ and $x, y, z \in \{1,2, ..., n, n + 1\}$ is

(*a*) ${}^{n+1}C_3 + {}^{n+2}C_3$ (*b*) $\frac{1}{6}n(n+1)(2n+1)$

(*c*) $1^2 + 2^2 + ... + n^2$ (*d*) $2({}^{n+2}C_3) - {}^{n+1}C_2$

COMPREHENSION BASED QUESTIONS

Comprehension-1 (Qs. 18 and 19)

Consider a polygon of sides '*n*' which satisfies the equation, $3 \cdot {}^nP_4 = {}^{n-1}P_5$

18. Number of quadrilaterals that can be made using the vertices of the polygon of sides '*n*' if exactly two adjacent sides of the quadrilateral are common to the sides of the polygon, is

(*a*) 50 (*b*) 60 (*c*) 70 (*d*) 84

19. Number of quadrilaterals that can be formed using the vertices of a polygon of sides '*n*' if exactly 1 side of the quadrilateral is common with the side of the polygon, is

(*a*) 150 (*b*) 100 (*c*) 96 (*d*) 104

Comprehension 4 (Qs. 20 to 22)

When letters of any word are written in all possible ways using all letters of given word and these words are arranged as in a dictionary then the position of the word in that list in called the rank of the word.

20. What is the rank of word 'PARALLEL

(*a*) 2353 (*b*) 2629

(*c*) 2593 (*d*) 2623

21. If list of all words is made by using the letters of the word IITJEE in dictionary order then the word at 141th position is

(*a*) JEE IIT (*b*) JII TEE

(*c*) JIE TIE (*d*) JII EE

MATCH THE COLUMN TYPE QUESTIONS

22. Match the column

	Column – I		Column – II
A.	The total number of selections of fruits which can be made from, 3 bananas, 4 apples and 2 oranges is, it is given that fruits of one kind are identical.	p.	120
B.	There are 10 true-false statements in a question paper. How many sequences of answers are possible in which exactly three are correct?	q.	286
C.	The number of ways of selecting 10 balls from unlimited number of red, black, white and green balls is, it is given that balls of same colours are identical.	r.	59
D.	The number of words which can be made from the letters of the word 'MATHEMATICS' so that consonants occur together?	s.	75600

(*a*) A → r; B → p; C → q; D → s

(*b*) A → s; B → p; C → q; D → r

(*c*) A → r; B → s; C → q; D → p

(*d*) A → s; B → q; C → p; D → r

23. PHYSICS WALLAH, Total 12 letters. Set of repeating letters = {H, A, L}

	Column -I		Column-II
A.	The two A's are not together	p.	$\frac{5}{4}(11!)$
B.	Neither two A's nor two L are together	q.	$57\frac{10!}{2!2!}$
C.	The number of words taking all letters of the given word such that atleast one repeating letter is at odd position is	r.	$\frac{12!}{2!2!2!}\left[1-\frac{6\times 6!}{12!}\right]$ OR 83070×61
D.	The number of words formed taking all letters of the given word such that in each word both H are together and both L's are together but both A's are not together is	s.	$4\times 9!$

(*a*) A → r; B → q; C → p; D → s

(*b*) A → p; B → q; C → s; D → r

(*c*) A → p; B → q; C → r; D → s

(*d*) A → p; B → r; C → q; D → s

NUMERICAL BASED QUESTIONS

24. The number of ways to choose 7 distinct natural numbers from the first 100 natural numbers such that any two chosen numbers differ atleast by 7 can be expressed as ${}^nC_7$. Find the number of divisors of n.

25. A box contains 6 balls which may be all of different colours or three each of two colours or two each of three different colours. The number of ways of selecting 3 balls from the box (if ball of same colour are identical), is N then sum of the digits in the number 'N' equals

26. Six married couple are sitting in a room. Number of ways in which 4 people can be selected so that there is exactly one married couple among the four is N then number of divisors of N equals.

27. A condolence meeting is being held in a hall which has 7 doors, by which mourners enter the hall. One can use any of the 7 doors to enter and can come at any time during the meeting. At each door, a register is kept in which a mourner has to affix his signature while entering the hall. If 200 people attend the meeting, how many different sequences of 7 lists of signatures can arise?

28. There are p intermediate railway stations on a railway line from one terminal to another. In how many ways can a train stop at three of these intermediate stations if no two of these stations (where it stops) are to be consecutive?

29. How many hexagons can be constructed by joining the vertices of a quindecagon (15 sides) if none of the sides of the hexagon is also the side of the 15-gon.

30. Number of 7 digit numbers if the sum of their digits is 59, how many of these are divisible by 11 (Digits to be used can be 5, 6, 7, 8, 9)

31. How many different words can be formed using all the letters of the words HONOLULU if (no two alike letters are together).

32. Number of ways in which $5B$ and $5G$ can be seated on a circle alternately if a particular B_1 and G_1 are never adjacent to each other in any arrangement.

33. Let $x_1, x_2, ..., x_k$ be the divisors of positive integer 'n' (including 1 and n). If $x_1 + x_2 + ... + x_k = 75$, then $\sum_{i=1} 1/x_i$ is equal to

34. The total number of ways in which n^2 number of identical balls can be put in n numbered boxes (1, 2, 3, ..., n) such that i^{th} box contains at least i number of balls is

35. If a seven-digit number made up of all distinct digits 8, 7, 6, 4, 3, x and y is divisible by 3, then maximum value of $x - y$ + minimum value of xy is

36. Four faces of a tetrahedral dice are marked with 2, 3, 4, 5. The lowest face being considered as the outcome. In how many ways a total of 30 can occur in 7 throws.

37. Ten persons numbered 1, 2, ..., 10 play a chess tournament, each player playing against every other player exactly one game. It is known that no game ends in a draw. If $w_1, w_2, ..., w_{10}$ are the number of games won by players 1, 2, 3, ..., 10, respectively, and $l_1, l_2, ..., l_{10}$ are the number of games lost by the players 1, 2, ..., 10, respectively, then, $\Sigma w_i - \Sigma l_i = ?$

38. Five balls are to be placed in three boxes. Each box should hold all the five balls so that no box remains empty, then (i) find P, Q, R, S.

Balls	Boxes	Number of ways
Different	Identical	P
Identical	Identical	Q
Identical	Different	R
Different	Different	S

(ii) If there are six balls instead of five balls then find P, Q, R, S

Exercise-4 (Past Year Questions)

JEE MAIN

1. Total number of 6-digit numbers in which only one digit is repeated and all the five digits 1, 3, 5, 7 and 9 appear, is **(2020)**

(a) 5^6 (b) $\frac{1}{2}$ 6!

(c) 6! (d) $\frac{5}{2}(6!)$

2. An urn contains 5 red marbles, 4 black marbles and 3 white marbles. Then the number of ways in which 4 marbles can be drawn so that at the most three of them are red is _____. **(2020)**

3. The number of 4 letter words (with or without meaning) that can be formed from the eleven letters of the word **'EXAMINATION'** is **(2020)**

4. If the number of five digit numbers with distinct digits and 2 at the 10th place is 336 k, then k is equal to: **(2020)**

(a) 7 (b) 4

(c) 6 (d) 8

5. Let $n > 2$ be an integer. Suppose that there are n Metro stations in a city located along a circular path. Each pair of stations is connected by a straight track only. Further, each pair of nearest stations is connected by blueline, whereas all remaining pairs of stations are connected by redline. If the number of red lines is 99 times the number of blue lines, then the value of n is: **(2020)**

(a) 199 (b) 201

(c) 101 (d) 200

6. If the letters of the word 'MOTHER' be permuted and all the words so formed (with or without meaning) be listed as in a dictionary, then the position of the word ' MOTHER' is _____ **(2020)**

7. A test consists of 6 multiple choice questions, each having 4 alternative answers of which only one is correct. The number of ways, in which a candidate answers all six questions such that exactly four of the answers are correct, is _____. **(2020)**

8. There are 3 sections in a question paper and each section contains 5 questions. A candidate has to answer a total of 5 questions, choosing at least one question from each section. Then the number of ways, in which the candidate can choose the questions, is: **(2020)**

9. The number of words, with or without meaning, that can be formed by taking 4 letters at a time from the letters of the word 'SYLLABUS' such that two letters are distinct and two letters are alike, is _____. **(2020)**

10. The number of words (with or without meaning) that can be formed from all the letters of the word "LETTER" in which vowels never come together is _____ . **(2020)**

11. Two families with three members each and one family with four members are to be seated in a row. In how many ways can they be seated so that the same family members are not separated? **(2020)**

(a) 2! 3! 4! (b) $(3!)^3 \cdot (4!)$

(c) $(3!)^2 \cdot (4!)$ (d) $3!\,(4!)^3$

12. A scientific committee is to formed from 6 Indians and 8 foreigners, which includes at least 2 Indians and double the number of foreigners as Indians. Then the number of ways, the committee can be formed is: **(2020)**

(*a*) 560 (*b*) 1050
(*c*) 1625 (*d*) 575

13. The students $S_1, S_2, \ldots, S_{10}$ are to be divided into 3 groups A, B and C such that each group has at least one student and the group C has at most 3 students. Then the total number of possibilities of forming such groups is **(2021)**

14. The total number of positive integral solutions (x, y, z) such that $xyz = 24$ is **(2021)**

(*a*) 36 (*b*) 45
(*c*) 24 (*d*) 30

15. The total number of numbers, lying between 100 and 1000 that can be formed with the digits 1, 2, 3, 4, 5, if the repetition of digits is not allowed and numbers are divisible by either 3 or 5 is **(2021)**

16. The total number of two digit numbers 'n', such that $3^n + 7^n$ is a multiple of 10, is **(2021)**

17. The number of seven digit integers with sum of the digits equal to 10 and formed by using the digits 1,2 and 3 only is **(2021)**

(*a*) 77 (*b*) 42
(*c*) 35 (*d*) 82

18. A natural number has prime factorization given by $n = 2^x 3^y 5^z$, where y and z are such that $y + z = 5$ and $y^{-1} + z^{-1} = \frac{5}{6}$, $y > z$. Then the numberof odd divisors of n, including 1, is: **(2021)**

(*a*) 11 (*b*) $6x$
(*c*) 12 (*d*) 6

19. The total number of 4-digit numbers whose greatest common divisor with 18 is 3, is **(2021)**

20. Consider a rectangle $ABCD$ having 5, 7, 6, 9 points in the interior of the line segments AB, CD, BC, DA respectively. Let α be the numberof triangles having these points from different sides as vertices and β be the number of quadrilaterals having these points from different sides as vertices. Then $(\beta - \alpha)$ is equal to: **(2021)**

(*a*) 795 (*b*) 1173
(*c*) 1890 (*d*) 717

21. Team 'A' consists of 7 boys and girls and Team 'B' has 4 boys and 6 girls. If a total of 52 single matches can be arranged between these two teams when a boy plays against a boy and a girl plays against a girl, then n is equal to : **(2021)**

(*a*) 5 (*b*) 2
(*c*) 4 (*d*) 6

22. If the sides AB, BC and CA of a triangle ABC have 3, 5 and 6 interior points respectively, then the total number of triangles that can be constructed using these points as vertices, is equal to: **(2021)**

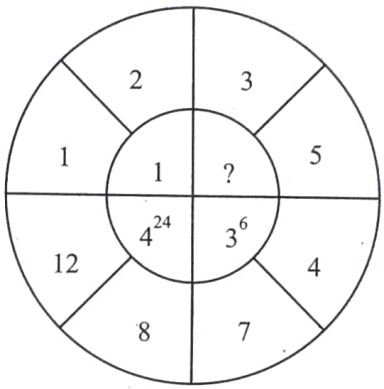
(*a*) 364 (*b*) 240
(*c*) 333 (*d*) 360

23. The sum of all the 4 digit distinct numbers that can be formed with the digits 1,2,2 and 3 is: **(2021)**

(*a*) 26664 (*b*) 122664
(*c*) 122234 (*d*) 22264

24. The number of times the digit 3 will be written when listing the integers from 1 to 1000 is **(2021)**

25. The missing value in the following figure is



Use the logic which gives answer in single digit. **(2021)**

26. The number of one-one functions $f : \{a, b, c, d\} \to \{0, 1, 2, \ldots, 10\}$ such that $2f(a) - f(b) + 3f(c) + f(d) = 0$ is _____. **(2022)**

27. In an examination, there are 5 multiple choice questions with 3 choices, out which exactly one is correct. There are 3 marks for each correct answer, –2 marks for each wrong answer and 0 mark if the question is not attempted. Then, the number of ways a student appearing in the examination gets 5 marks is ____. **(2022)**

28. The number of 7-digit numbers which are multiples of 11 and are formed using all the digits 1, 2, 3, 4, 5, 7 and 9 is ____. **(2022)**

29. The number of 3-digit odd number, whose sum of digits is a multiple of 7, is _____. **(2022)**

30. Let A be a 3×3 matrix having entries from the set $\{-1, 0, 1\}$. The number of all such matrices A having sum of all the entries equal to 5, is ____. **(2022)**

31. The total number of three-digit numbers, with one digit Repeated exactly two times, is ____. **(2022)**

32. There are ten boys $B_1, B_2, \ldots, B_{10}$ and five girls $G_1, G_2, \ldots\ldots, G_5$ in a class. Then the number of forming a group consisting of three boys and three girls, if both B_1 and B_2 together should not be the members of a group, is _____. **(2022)**

33. The total number of 3-digit numbers, whose greatest common divisor with 36 is 2, is ____. **(2022)**

34. The number of ways, 16 identical cubes, of which 11 are blue and rest are red, can placed in a row so that between any two red cubes there should be at least 2 blue cubes, is _____. **(2022)**

35. Let A be a matrix of order 2×2, whose entries are from the set $\{0, 1, 2, 3, 4, 5\}$. If the sum of all the entries of A is a prime number p, $2 < p < 8$, then the number of such matrices A is **(2022)**

36. Let $b_1b_2b_3b_4$ be a 4-element permutation with bi $\in \{1, 2, 3, \ldots, 100\}$ for $1 \le i \le 4$ and $b_i \ne j$, such that either b_1, b_2, b_3 are consecutive integers or b_2, b_3, b_4 are consecutive integers. Then the number of such permutations $b_1b_2b_3b_4$ is equal to _____. **(2022)**

37. The total number of four digit numbers such that each of the first three digits is divisible by the l digit, tis equal to ____. **(2022)**

38. The letters of the word 'MANKIND' are written in all possible orders and arranged in serial order as in an English dictionary. Then the serial number of the word 'MANKIND' is _____. **(2022)**

39. The number of 5-digit natural numbers, such that the product of their digits is 36, is ____. **(2022)**

40. Number are to formed between 1000 and 3000, which are divisible by 4, using the digits 1, 2, 3, 4, 5 and 6 without repetition of digits. Then the total number of such numbers is ____. **(2022)**

41. Let S be the set of all passwords which are six to eight characters long, where each character is either an alphabet from $\{A, B, C, D, E\}$ or a number from $\{1,2,3,4,5\}$ with the repetition of characters allowed. If the number of passwords in S whose at least one character is a number from $\{1, 2, 3, 4, 5\}$ is $\alpha \times 56$ then α is equal to ____. (2022)

42. A class contains b boys and g girls. If the number of ways of selecting 3 boys and 2 girls from the class is 168, then $b + 3g$ is equal to _____. **(2022)**

43. The number of natural numbers lying between 1012 and 23421 that can be formed using the digits 2, 3, 4, 5, 6 (repletion of digits is not allowed) and divisible by 55 is _____. **(2022)**

JEE ADVANCED

44. Let n and k be positive integers such that $n \ge \frac{k(k+1)}{2}$. The number of solutions $(x_1, x_2, \ldots, x_k)$, $x_1 \ge 1, x_2 \ge 2, \ldots, x_k \ge k$ for all integers satisfying $x_1 + x_2 + \ldots + x_k = n$ is **(1996)**

45. Number of divisors of the form $(4n + 2)$, $n \ge 0$ of the integer 240 is **(1998)**

(a) 4 (b) 8
(c) 10 (d) 3

46. How many different nine-digit numbers can be formed from the number 22 33 55 888 by rearranging its digits so that the odd digits occupy even positions? **(2000)**

(a) 16 (b) 36
(c) 60 (d) 180

47. The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that each person gets at least one ball is **(2012)**

(a) 75 (b) 150
(c) 210 (d) 243

Paragraph for Question Nos. 48 & 49

Let a_n denote the number of all n-digit positive integers formed by the digits 0,1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n-digit integers ending with digit 1 and c_n = the number of such n-digit integers ending with digit 0.

48. Which of the following is correct? **(2012)**

(a) $a_{17} = a_{16} + a_{15}$ (b) $c_{17} \ne c_{16} + c_{15}$
(c) $b_{17} \ne b_{16} + c_{16}$ (d) $a_{17} = c_{17} + b_{16}$

49. The value of b_6 is

(a) 7 (b) 8
(c) 9 (d) 11

50. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positivie integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is **(2014)**

51. Let $n \ge 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is **(2014)**

52. Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 is always placed in envelope numbered 2. Then the number of ways it can be done is **(2014)**

(a) 264 (b) 265
(c) 53 (d) 67

53. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then the value of m/n is **(2015)**

54. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is **(2016)**

(a) 380 (b) 320
(c) 260 (d) 95

55. Let $S = \{1, 2, 3, ..., 9)$. For $k = 1, 2, ... 5$, let N_k be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$ **(2017)**

(*a*) 210 (*b*) 252

(*c*) 126 (*d*) 125

56. Words of length 10 are formed using the letters *A, B, C, D, E, F, G, H, I, J*. Let x be the number of such words where no letter is repeated; and let y be the number of such words where exactly one letter is repeated twice and no other letter is repeated. Then, $\frac{y}{9x} =$ **(2017)**

57. The number of 5 digit numbers which are divisible by 4, with digits from the set {1, 2, 3, 4, 5} and the repetition of digits is allowed, is _____ **(2018)**

58. Let X be a set with exactly 5 element and Y be a set with exactly 7 elements. If α is the number of one-one functions from X and Y and β is the number of onto functions from Y to X, then the value of is ..... **(2018)**

59. In a high school, a committee has to be formed from a group of 6 boys $M_1, M_2, M_3, M_4, M_5, M_6$ and 5 girls G_1, G_2, G_3, G_4, G_5.

(*i*) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 boys and 2 girls.

(*ii*) Let. a_2 be the total number of ways in whcih the committe can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.

(*iii*) Let α_3 be te total number of ways in which the committe can be formed such that the committee has 5 members, at elst 2 of them being girls.

(iv) Let α_4 be the total number of ways in which the committeee can be formed such that the commitee has 4 members, having at least 2 girls and such that both M_1 and G_1 are **NOT** in the committee together.

(2018)

	Column -I		Column-II
A.	The value of α_1 is	p.	136
B.	The value of α_2 is	q.	186
C.	The value of α_3 is	r.	192
D.	The value of α_4 is	s.	200
		t.	381
		u.	461

(*a*) A → s; B → u; C → q; D → p

(*b*) A → p; B → s; C → q; D → r

(*c*) A → s; B → u; C → t; D → q

(*d*) A → s; B → q; C → r; D → p

60. Five person *A, B, C, D* and *E* are seated in a circular arrangement. If each of them is given a hat of one of the three colours red, blue and green ,then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is **(2019)**

61. An engineer is required to visit a factory for exactly four days during the first 15 days of every month and it is mandatory that **no** two visits take place on consecutive days. Then the number of all possible ways in which such visits to the factory can be made by the engineer during 1-15 June 2021 is _____ **(2020)**

62. In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is _____ **(2020)**

63. Let $S_1 = \{(i, j, k) : i, j, k \in \{1, 2,...,10\}\}$

$S_2 = \{(i, j): 1 \le i < j + 2 \le 10, i, j \in \{1, 2,...,10\}\}$,

$S_3 = \{(i, j, k, l) : 1 \le i < j < k < l, i, j, k, l \in \{1, 2,..,10\}\}$.

and $S_4 = \{(i, j, k, l)$: i, j, k and I are distinct elements in $\{1, 2,....,10\}\}$.

If the total number of elements in the set S_r is n_r, r = 1, 2, 3, 4, then which of the following statements is (are) TRUE? **(2021)**

(*a*) $n_1 = 1000$

(*b*) $n_2 = 44$

(*c*) $n_3 = 220$

(*d*) $n_4/12 = 420$

64. The number of 4-digit integers in the closed interval [2022, 4482] formed by using the digits 0, 2, 3, 4, 6, 7 is _____ **(2022)**

65. Consider 4 boxes, where each box contains 3 red balls and 2 blue balls. Assume that all 20 balls are distinct. In how many different ways can 10 balls be chosen from these 4 boxes so that from each box at least one red ball and one blue ball are chosen? **(2022)**

(*a*) 21816 (*b*) 85536

(*c*) 12096 (*d*) 156816

ANSWER KEY

CONCEPT APPLICATION

1. (*i*) 9, (*ii*) 20, (*iii*) 72, (*iv*) 326592 **2.** (*b*) **3.** (*a*) **4.** (*a*) **5.** (*d*) **6.** (*b*)
7. (*i*) (*a*), (*ii*) (*a*), (*iii*) (*a*) **8.** (*b*) **9.** (*c*) **10.** (*c*) **11.** (*b*) **12.** (*a,b,c,d*) **13.** (*a*)
14. (*a*) **15.** (*c*) **16.** (*i*) 50, (*ii*) 205, (*iii*) 60, (*iv*) 378, (*v*) 270 **17.** (*d*) **18.** (*c*) **19.** [2520]
20. [25920] **21.** (*i*) $5 \times 6 \times 2^3$, (*ii*) $5 \times 6 \times 2^3 - 1$, (*iii*) $4 \times 5 \times (2^3 - 1)$ **22.** (*i*) 60, (*ii*) 3255, (*iii*) 3968
23. [21] **24.** (*c*) **25.** (*i*) 1440, (*ii*) 1438, (*iii*) 480, (*iv*) 960, (*v*) $5 \cdot 7 \cdot 11 \cdot 9 \cdot 7$, (*vi*) $\frac{2^3-1}{2-1} \cdot \frac{3^4-1}{3-1} \cdot \frac{5^6-1}{5-1} \cdot \frac{7^5-1}{7-1} \cdot \frac{11^4-1}{11-1}$
(*vii*) 720, (*viii*) 16, (*ix*) 540, (*x*) 480, (*xi*) 50 **26.** (*b*) **27.** (*b*) **28.** (*b*) **29.** $\frac{30!}{10!10!10!} \frac{1}{3!}$
30. $\frac{30!}{10!10!10!3!} 3!$ **31.** [20] **32.** (*d*) **33.** (*a*) **34.** (*a*) **35.** (*b*) **36.** (*a*) **37.** [10] **38.** [3150]
39. (*b,d*) **40.** $({}^{23}C_2)$ **41.** (*b*) **42.** (*c*)

EXERCISE-1 (TOPICWISE)

1. (*b*)	**2.** (*d*)	**3.** (*a*)	**4.** (*c*)	**5.** (*c*)	**6.** (*d*)	**7.** (*d*)	**8.** (*a*)	**9.** (*a*)	**10.** (*d*)
11. (*b*)	**12.** (*d*)	**13.** (*b*)	**14.** (*c*)	**15.** (*a*)	**16.** (*c*)	**17.** (*a*)	**18.** (*a*)	**19.** (*a*)	**20.** (*b*)
21. (*b*)	**22.** (*a*)	**23.** (*b*)	**24.** (*c*)	**25.** (*d*)	**26.** (*c*)	**27.** (*a*)	**28.** (*d*)	**29.** (*a*)	**30.** (*a*)
31. (*c*)	**32.** (*c*)	**33.** (*d*)	**34.** (*a*)	**35.** (*c*)	**36.** (*b*)	**37.** (*b*)	**38.** (*d*)	**39.** (*c*)	**40.** (*a*)
41. (*a*)	**42.** (*b*)	**43.** (*d*)	**44.** (*b*)	**45.** (*a*)	**46.** (*c*)	**47.** (*b*)	**48.** (*a*)	**49.** (*a*)	**50.** (*a*)
51. (*a*)	**52.** (*a*)	**53.** (*c*)	**54.** (*d*)	**55.** (*a*)	**56.** (*b*)	**57.** (*b*)	**58.** (*c*)		

EXERCISE-2 (LEARNING PLUS)

1. (*c*)	**2.** (*b*)	**3.** (*b*)	**4.** (*b*)	**5.** (*b*)	**6.** (*c*)	**7.** (*b*)	**8.** (*b*)	**9.** (*a*)	**10.** (*b*)
11. (*b*)	**12.** (*a*)	**13.** (*c*)	**14.** (*b*)	**15.** (*b*)	**16.** (*b*)	**17.** (*a*)	**18.** (*b*)	**19.** (*a*)	**20.** (*a*)
21. (*d*)	**22.** (*a*)	**23.** (*a*)	**24.** (*b*)	**25.** (*a*)	**26.** (*b*)	**27.** (*b*)	**28.** (*c*)	**29.** (*a*)	**30.** (*a*)
31. (*a*)	**32.** (*d*)	**33.** (*a*)	**34.** (*c*)	**35.** (*d*)	**36.** (*c*)	**37.** (*b,c,d*)	**38.** (*a,b,d*)	**39.** (*b,c,d*)	**40.** (*b,d*)
41. (*a*)	**42.** (*a*)	**43.** (*a*)	**44.** (*a*)	**45.** (*b*)	**46.** (*b*)	**47.** (*b*)	**48.** (*a*)	**49.** (*b*)	**50.** (*d*)
51. (*b*)	**52.** [6]	**53.** [420]	**54.** [15]	**55.** [1750]	**56.** [12]	**57.** [1]	**58.** $[{}^{n+5}C_n]$	**59.** [159]	**60.** [2220]

61. $[{}^{11}C_3 \times 2 \times 8!]$ **62.** $[{}^{45k}C_6]$ **63.** $[{}^{34}C_4]$ **64.** $[{}^{14}C_{12} + 1 - {}^{9}C_2 - {}^{7}C_2]$ **65.** [5292] **66.** [60] **67.** [17]
68. [5] **69.** $[{}^{4}C_2 \times {}^{5}C_2 \times {}^{6}C_2)]$ **70.** [315] **71.** $[{}^{9}C_4]$ **72.** [21599] **73.** [462] **74.** [4] **75.** [25]
76. [(*i*) 10^4, (*ii*) 10^3] **77.** (*i*) 89291, (*ii*) 17855, (*iii*) 67980 **78.** (*i*) 1771, (*ii*) 969, (*iii*) 633, (*iv*) 165, (*v*) 552
79. (*i*) 176400, (*ii*) $\frac{70 \times 9!}{8}$, (*iii*) $\frac{28 \times 7!}{2! \times 4!}$, (*iv*) $\frac{10 \times 10!}{3! \times 2! \times 2!}$, (*v*) $\frac{7!}{3!2!}$

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (*b*)	**2.** (*b*)	**3.** (*b*)	**4.** (*b*)	**5.** (*a*)	**6.** (*b,d*)	**7.** (a,*b,c,d*)	**8.** (*a, c, d*)	**9.** (*b,c,d*)	**10.** (*a,d*)
11. (*b,c,d*)	**12.** (*a,b,d*)	**13.** (*a,d*)	**14.** (*b,c,d*)	**15.** (*a,c*)	**16.** (*b,c*)	**17.** (*b,c,d*)	**18.** (*a*)	**19.** (*b*)	**20.** (*d*)

21. (*b*) **22.** (*a*) **23.** (*c*) **24.** [7] **25.** [4] **26.** [20] **27.** $\frac{206!}{6!}$ **28.** ${}^{p-2}C_3$ **29.** [140] **30.** [40]
31. [2220] **32.** [5×6!] **33.** $\left[\frac{75}{n}\right]$ **34.** $C_{n-1}^{\left(\frac{n^2+n-2}{2}\right)}$ **35.** [9] **36.** [413] **37.** [0]
38. (*i*) P = 15, Q = 2, R = 6, S = 150, (*ii*) P = 75, Q = 2, R = 10, S = 450

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Mains

1. (*d*)	**2.** [490]	**3.** [2454]	**4.** (*d*)	**5.** (*b*)	**6.** [309]	**7.** [135]	**8.** [2250]	**9.** [240]	**10.** [120]
11. (*b*)	**12.** (*c*)	**13.** [31650]	**14.** (*d*)	**15.** [32]	**16.** [45]	**17.** (*a*)	**18.** (*c*)	**19.** [1000]	**20.** (*d*)
21. (*c*)	**22.** (*c*)	**23.** (*a*)	**24.** [300]	**25.** [4]	**26.** [31]	**27.** [40]	**28.** [576]	**29.** [63]	**30.** [414]
31. [243]	**32.** [1120]	**33.** [150]	**34.** [56]	**35.** [180]	**36.** [18915]	**37.** [1086]	**38.** [1492]	**39.** [180)]	**40.** [30]
41. [7073]	**42.** [17]	**43.** [6]							

JEE Advanced

44. $\frac{1}{2}\left(2n-k^2+k-2\right)$	**45.** (*a*)	**46.** (*c*)	**47.** (*b*)	**48.** (*a*)	**49.** (*b*)	**50.** [7]	**51.** [5]	**52.** (*c*)	
53. [5]	**54.** (*a*)	**55.** (*c*)	**56.** [5]	**57.** [625]	**58.** [119]	**59.** (*c*)	**60.** [30]	**61.** [495]	**62.** [1080]
63. (*d*)	**64.** [569]	**65.** (*a*)							

CHAPTER

11 Binomial Theorem

INTRODUCTION

Any algebraic expression consisting of only two different terms is known as binomial expression. The terms may consist of variables x, y etc. or constants or their mixed combinations.

For example: $2x + 3y$, $4xy + 5$ etc.

Terminology used in Binomial Theorem

Factorial notation: $\lfloor n$ or $n!$ is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2).....3.2.1 & ; \text{ if } n \in N \\ 1 & ; \text{ if } n = 0 \end{cases}$$

Note: $n! = n \cdot (n-1)!$; $n \in N$

Mathematical meaning of ^nC_r: The term ^nC_r denotes number of combinations of r things choosen from n distinct things

mathematically, $^nC_r = \dfrac{n!}{(n-r)!\,r!}$, $n \in N, r \in W, 0 \le r \le n$

Note: Other symbols of ^nC_r are $\begin{pmatrix} n \\ r \end{pmatrix}$ and $C(n, r)$.

Properties Related to ^nC_r

(*i*) $^nC_r = {}^nC_{n-r}$

Note: If $^nC_x = {}^nC_y$

$\Rightarrow$ Either $x = y$ or $x + y = n$

(*ii*) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(*iii*) $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

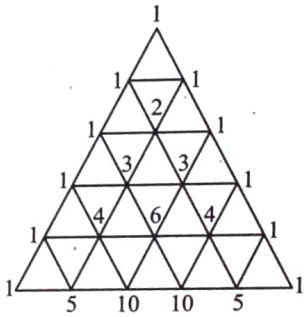
(*iv*) $^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n(n-1)}{r(r-1)}\,{}^{n-2}C_{r-2} = ...$

$$= \frac{n(n-1)(n-2).....(n-(r-1))}{r\,(r-1)(r-2).....2.1}$$

(*v*) If n and r are relatively prime, then ^nC_r is divisible by n. But converse is not necessarily true.

HISTORIAL DEVELOPMENT

Earlier people used to multiply the brackets to expand the given binomial of known index. Then came the Pascal triangle.



Note that

(*a*) The powers of x go down until it reaches zero, starting values is n.

(*b*) The power of y goes up from zero untill it reaches n.

(*c*) The nth row of the Pascals triangle will be the coefficients of the expanded binomial of index n.

BINOMIAL THEOREM FOR POSITIVE INDEX

Binomial theorem gives a formula for the expansion of a binomial expression raised to any positive integral power. In general for a positive integer n

$(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + ... + {}^nC_n x^0 y^n$, where

$$^nC_r = \frac{n!}{(n-r)!r!}$$

for $r = 0, 1, 2, ..., n$ is called binomial coefficient.

Observations

(*a*) The number of terms in the expansion is $(n + 1)$ i.e. one more than the index.

(*b*) The sum of the indices of x & y in each term is n.

(*c*) The binomial coefficients of the terms ($^nC_0$, $^nC_1$...) equidistant from the beginning and the end are equal.

Proof of Binomial Theorem

The Binomial theorem can be proved by mathematical induction

Let $P(n)$ stands for the mathematical statement

$$(x+a)^n = x^n + {}^nC_1\, x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + a^n \quad \ldots(i)$$

Note: That there are $(n + 1)$ terms in R.H.S. and all the terms are of the same degree in x and a together.

When $n = 1$, L.H.S. $= x + a$ and R.H.S. $= x + a$ (there are only 2 terms)

$\therefore P(1)$ is verified to be true

Assume $P(m)$ to be true

i.e., $(x+a)^m = x^m + {}^mC_1\, x^{m-1}\, a + {}^mC_2\, x^{m-2}\, a_2 + \ldots + {}^mC_r\, x^{m-r}\, a^r + \ldots + a^m \quad \ldots(ii)$

Multiplying equation (ii) by $(x + a)$, we have

$$(x+a)^m\,(x+a) = (x+a)\,\{x^m + {}^mC_1\, x^{m-1}\, a + {}^mC_2\, x^{m-2}\, a^2 + \ldots + {}^mC_r\, x^{m-r}\, a^r + \ldots + a^m\}$$

i.e., $(x+a)^{m+1} = x^{m+1} + ({}^mC_1 + 1)x^m a + ({}^mC_2 + {}^mC_1)x^{m-1}a^2 + \ldots + ({}^mC_r + {}^mC_{r-1})x^{m-r+1}a^r + \ldots + a^{m+1}$

$$= x^{m+1} + {}^{(m+1)}C_1\, x^m\, a + {}^{(m+1)}C_2\, x^{m-1}\, a^2 + \ldots + {}^{m+1}C_r\, x^{m+1-r}\, a^r + \ldots a^{m+1} \quad \ldots(iii)$$

(using the formula ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$)

Equation (iii) implies that $P(m + 1)$ is true and hence by induction $P(n)$ is true.

Alternative Method

By choosing x from all the brackets we get the term x^n. Choosing x from $(n - 1)$ factors and 'a' from the remaining factor we get $x^{n-1}a$. But the number of ways of doing this is equal to the number of ways of choosing one factor from n factors (i.e.,) ${}^nC_1$. Choosing x from $(n - 2)$ factor and a from the remaining two factors, we get $x^{n-2}a^2$. But the number of ways of doing this is equal to the number of ways of choosing two factors from n factors. i.e., ${}^nC_2$. Finally choosing 'a' from all the factors we get the term a^n.

$$\therefore (x+a)^n = x^n + {}^nC_1\, x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + a^n$$

Train Your Brain

Example 1: The value of

$$\frac{(18^3 + 7^3 + 3.18.7.25)}{3^6 + 6.243.2 + 15.81.4 + 20.27.8 + 15.9.16 + 6.3.32 + 64} \text{ is}$$

Sol. The numerator is of the form

$a^3 + b^3 + 3ab\,(a + b) = (a + b)^3$ where $a = 18$ and $b = 7$

$\therefore Nr = (18 + 7)^3 = (25)^3$,

Denominator can be written as

$3^6 + {}^6C_1.3^5.2^1 + {}^6C_2.3^4.2^2 + {}^6C_3.3^3.2^3 + {}^6C_4 3^2.2^4 + {}^6C_5 3.2^5 + {}^6C_6 2^6 = (3 + 2)^6 = 5^6 = (25)^3$

$$\therefore \frac{N_r}{D_r} = \frac{(25)^3}{(25)^3} = 1$$

Example 2: Expand $\left(x - \frac{1}{x}\right)^6$

Sol. $\left(x - \frac{1}{x}\right)^6 = {}^6C_0x^6 + {}^6C_1x^5\left(\frac{-1}{x}\right) + {}^6C_2x^4\left(\frac{-1}{x}\right)^2 + {}^6C_3x^3\left(\frac{-1}{x}\right)^3 + {}^6C_4x^2\left(\frac{-1}{x}\right)^4 + + {}^6C_5x\left(\frac{-1}{x}\right)^5 + {}^6C_6x^0\left(\frac{-1}{x}\right)^6$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

Concept Application

1. $(2n + 1)(2n + 3)(2n + 5)\ldots(4n - 1)$ is equal to

   (a) $\dfrac{(4n)!}{2^n\cdot(2n)!\,(2n)!}$ (b) $\dfrac{(4n)!n!}{2^n\cdot(2n)!\,(2n)!}$

   (c) $\dfrac{(4n)!n!}{(2n)!\,(2n)!}$ (d) $\dfrac{(4n)!n!}{2^n!\,(2n)!}$

2. If $n \in N$ and is even, then

$$\frac{1}{1-(n-1)!} + \frac{1}{3!(n-3)!} + \frac{1}{5!(n-5)!} + \ldots + \frac{1}{(n-1)!1!} =$$

   (a) 2^n (b) $\dfrac{2^{n-1}}{n!}$

   (c) $2^n\, n!$ (d) $\dfrac{2^{n+1}}{n!}$

GENERAL TERM IN THE BINOMIAL EXPANSION

The general term in the expansion of $(x + y)^n$ is $(r + 1)^{th}$ term, given by $T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$ where $r = 0, 1, 2 \ldots n$.

- Every term in the expansion is of nth degree in variables x and y.
- The total number of terms in the expansion is $n + 1$.
- Binomial expansion can also be expressed as $(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$.
- The binomial coefficients of the expansion equidistant from the beginning and the end are equal. In other words ${}^nC_r = {}^nC_{n-r}$.
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- **Term independent of x:** Term independent of x contains no x; Hence find the value of r for which the exponent of x is zero.

Train Your Brain

Example 3: Find the 11th term in the expansion of $\left(3x-\frac{1}{x\sqrt{3}}\right)^{20}$.

Sol. The general term

$$= T_{r+1} = (-1)^r \, {}^{20}C_r (3x)^{20-r}\left(\frac{1}{x\sqrt{3}}\right)^r$$

For the 11th term, we must take $r = 10$ ($\therefore r + 1 = 11$)

$$\therefore T_{11} = T_{10+1} = (-1)^{10} \, {}^{20}C_{10}(3x)^{20-10}\left(\frac{1}{x\sqrt{3}}\right)^{10}$$

$$= {}^{20}C_{10}3^{10}x^{10}\frac{1}{x^{10}\left(\sqrt{3}\right)^{10}} = {}^{20}C_{10}\left(\sqrt{3}\right)^{10} = {}^{20}C_{10}\,3^5$$

Example 4: Find the term independent of x in $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

Sol. The general term $= {}^9C_r\left(\frac{3x^2}{2}\right)^{9-r}\left(\frac{-1}{3x}\right)^r$

$$= (-1)^r \, {}^9C_r\frac{3^{9-2r}}{2^{9-r}}.x^{18-3r}$$

The term independent of x, (or the constant term) corresponds to x^{18-3r} being

x^0 or $18 - 3r = 0$

$\Rightarrow r = 6$

$\therefore$ The term independent of x is the 7th term and its value is

$$(-1)^6 \, {}^9C_6\frac{3^{9-12}}{2^{9-6}} = {}^9C_3\frac{3^{-3}}{2^3} = \frac{9.8.7}{3.2.1}\cdot\frac{1}{(6)^3} = \frac{7}{18}$$

Example 5: Find the coefficient of x^7 in the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$.

Sol. In the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$, the general terms is

$$T_{r+1} = {}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r.\frac{a^{11-r}}{b^r}.x^{22-3r}$$

Putting $22 - 3r = 7$

$\Rightarrow 3r = 15$

$\Rightarrow r = 5$

$$\therefore T_6 = {}^{11}C_5\frac{a^6}{b^5}.x^7$$

Hence, the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ is ${}^{11}C_5\,a^6b^{-5}$.

Example 6: Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Sol. The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r\left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r 3^{\frac{1000-r}{2}}2^{\frac{r}{2}}$$

The above term will be rational if exponent of 3 and 2 are integers. i.e. $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is $\{0, 2, 4, ...1000\}$.
Hence, number of rational terms is 501

Concept Application

3. If the constant term of the binomial expansion $\left(2x-\frac{1}{x}\right)^n$ is -160, then n is equal to

(a) 4 (b) 6
(c) 8 (d) 10

4. If the coefficients of x^4 and x^8 in the expansion of $\left[2+\frac{x}{3}\right]^n$ are equal, then the value of n is:

(a) 15 (b) 45
(c) 55 (d) 56

5. Number of rational terms in the expansion of $(2^{1/7} + 3^{1/4})^{198}$ is equal to

(a) 7 (b) 8
(c) 9 (d) 10

6. What is the sum of the coefficients of the polynomial after expansion of $(25 - 60x + 22x^2 + 18x^3 - 6x^8)^{2013}$?

(a) 1 (b) -1
(c) 5262 (d) 0

7. Find the sum of following series:

(i) $\sum_{r=20}^{80} {}^{100-r}C_{20}$ (ii) $\sum_{r=10}^{100} {}^{r}C_{10}$

(iii) $\sum_{r=20}^{100} {}^{r}C_{10}$

8. Find the sum of following series

(i) $\sum_{r=0}^{n} {}^{n}C_r(2^r+3^r+4^r)$

(ii) $\sum_{r=0}^{n} {}^{n}C_r(-1)^r\left(\frac{1}{2^r}+\frac{1}{3^r}+\frac{1}{4^r}\right)$

MIDDLE TERMS OF THE EXPANSION

In the binomial expansion of $(x + y)^n$

When n is Odd

There are $(n + 1)$ i.e. even terms in the expansion and hence two middle terms are given by

$$t_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}} x^{\frac{n+1}{2}} y^{\frac{n-1}{2}} \text{ for } r = \frac{n-1}{2}$$

and $t_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}} x^{\frac{n-1}{2}} y^{\frac{n+1}{2}}$ for $r = \frac{n+1}{2}$

When n is Even

There are odd terms in the expansion and hence, only one middle term is given by:

$$t_{\frac{n}{2}+1} = {}^nC_{n/2}\, x^{n/2}\, y^{n/2} \text{ for } r = \frac{n}{2}$$

Note: The greatest binomial coefficient in the expansion is always the binomial coefficient of middle term/terms.

Train Your Brain

Example 7: Find the middle term in the expression of $(1 - 2x + x^2)^n$.

Sol. $(1 - 2x + x^2)^n = [(1 - x)^2]^n = (1 - x)^{2n}$

Here, $2n$ is even integer, therefore, $\left(\frac{2n}{2}+1\right)^{\text{th}}$

i.e., $(n + 1)^{\text{th}}$ term will be the middle term.

Now $(n + 1)^{\text{th}}$ term in $(1 - x)^{2n} = {}^{2n}C_n\,(1)^{2n-n}\,(-x)^n$

$= {}^{2n}C_n\,(-x)^n = \frac{(2n)!}{n!n!}(-x)^n$

Example 8: Find the middle term in the expansion of $\left(3x - \frac{x^3}{6}\right)^9$

Sol. The number of terms in the expansion of $\left(3x - \frac{x^3}{6}\right)^9$ is 10 (even). So there are two middle terms.

i.e., $\left(\frac{9+1}{2}\right)^{\text{th}}$ and $\left(\frac{9+3}{2}\right)^{\text{th}}$ two middle terms. They are given by T_5 and T_6

$$\therefore T_5 = T_{4+1} = {}^9C_4(3x)^5\left(-\frac{x^3}{6}\right)^4$$

$${}^9C_4 3^5 x^5 . \frac{x^{12}}{6^4} = \frac{9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4}\cdot\frac{3^5}{2^4 3^4}x^{17} = \frac{189}{8}x^{17}$$

and $T_6 = T_{5+1} = {}^9C_5(3x)^4\left(-\frac{x^3}{6}\right)^5 = -{}^9C_4 3^4 . x^4 . \frac{x^{15}}{6^5}$

$$= \frac{-9.8.7.6}{1.2.3.4}.\frac{3^4}{2^5.3^5}x^{19} = -\frac{21}{16}x^{19}$$

Example 9: The term independent of x in $\left[\sqrt{\left(\frac{3}{2x^2}\right)} + \sqrt{\frac{x}{3}}\right]^{10}$ is

Sol. General term in the expansion is

$${}^{10}C_r\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10}.\frac{3^{5-r}}{2^{\frac{10-r}{2}}}$$

For constant term, $\frac{3r}{2} = 10$

$\Rightarrow r = \frac{20}{3}$ which is not an integer. Therefore, there will be no constant term.

Example 10: If the coefficient of $(2r + 4)^{\text{th}}$ term and $(r - 2)^{\text{th}}$ term in the expansion of $(1 + x)^{18}$ are equal, find r.

Sol. Since coefficient of $(2r + 4)^{\text{th}}$ term in $(1 + x)^{18} = {}^{18}C_{2r+3}$.

Coefficient of $(r - 2)^{\text{th}}$ term $= {}^{18}C_{r-3}$

$\Rightarrow {}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$\Rightarrow 2r + 3 + r - 3 = 18$

$\Rightarrow 3r = 18$

$\Rightarrow r = 6.$

Concept Application

9. The coefficient of x^{49} in the expansion of $(x - 1)\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2^2}\right)\ldots\left(x - \frac{1}{2^{49}}\right)$ is equal to

(a) $-2\left(1 - \frac{2}{2^{50}}\right)$

(b) Positive coefficient of x

(c) $2\left(1 - \frac{1}{2^{49}}\right)$

(d) $-2\left(1 - \frac{1}{2^{49}}\right)$

GREATEST TERM IN THE EXPANSION

The numerically greatest term (absolute value) in the expansion of $(1 + x)^n$ is determined by the following process:

Let t_{r+1} be the numerically greatest term, then $|t_r| \le |t_{r+1}| \ge |t_{r+2}|$

Considering, $\left|\frac{t_{r+1}}{t_r}\right| \ge 1 \Rightarrow \left|\frac{n-r+1}{r}x\right| \ge 1$

$\Rightarrow \frac{n-r+1}{r}|x| \ge 1 \Rightarrow (n - r + 1)|x| \ge r$

$\Rightarrow r \le (n+1)\frac{|x|}{1+|x|}$ ...(i)

Also, $\left|\frac{t_{r+1}}{t_{r+2}}\right| \geq 1 \Rightarrow \left|\frac{r+1}{n-r}\frac{1}{x}\right| \geq 1 \Rightarrow r+1 \geq (n-r)|x|$

$$\Rightarrow \quad r \geq \frac{n|x|-1}{1+|x|}$$

$$\Rightarrow \quad r \geq (n+1)\frac{|x|}{1+|x|} - 1 \qquad \ldots(ii)$$

Thus, from (*i*) and (*ii*) for t_{r+1} to be the greatest term

$r \leq m$ and $r \geq m-1$ where, $m = (n+1)\dfrac{|x|}{1+|x|}$

For expansion of $(a+b)^n$

Case I: When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is an integer (say m), then

(*i*) $T_{r+1} > T_r$ when $r < m$ $(r = 1, 2, 3 \ldots, m-1)$
i.e., $T_2 > T_1$, $T_3 > T_2$, ..., $T_m > T_{m-1}$

(*ii*) $T_{r+1} = T_r$ when
$r = m$
i.e. $T_{m+1} = T_m$

(*iii*) $T_{r+1} < T_r$ when
$r > m$ $(r = m+1, m+2, \ldots n)$
i.e. $T_{m+2} < T_{m+1}$, $T_{m+3} < T_{m+2}$, ... $T_{n+1} < T_n$

CONCLUSION

When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude).

Case II: When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is not an integer (Let its integral part be m), then

(*i*) $T_{r+1} > T_r$ when $r < \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ $(r = 1, 2, 3, \ldots, m-1, m)$

i.e. $T_2 > T_1$, $T_3 > T_2$, ..., $T_{m+1} > T_m$

(*ii*) $T_{r+1} < T_r$ when $r > \dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ $(r = m+1, m+2, \ldots\ldots n)$

i.e., $T_{m+2} < T_{m+1}$, $T_{m+3} < T_{m+2}$, ..., $T_{n+1} < T_n$

Conclusion

When $\dfrac{n+1}{1+\left|\dfrac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Note: (*i*) In any binomial expansion, the middle term(s) has greatest binomial coefficient.

In the expansion of $(a+b)^n$

If n	No. of greatest binomial coefficient	Greatest binomial coefficient
Even →	1	${}^nC_{n/2}$
Odd →	2	${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$

(Values of both these coefficients are equal).

(*ii*) In order to obtain the term having numerically greatest coefficient, put $a = b = 1$, and proceed as discussed above.

In general, for the expansion $(a+x)^n$ or $a^n\left(1+\dfrac{x}{a}\right)^n$ we can consider $m = (n+1)\dfrac{|x/a|}{1+|x/a|}$.

Train Your Brain

Example 11: Find the greatest term in the expansion of $(4+3x)^7$ when $x = \dfrac{2}{3}$.

Sol. Here the greatest term means the numerically greatest term.

$$\left|\frac{t_{r+1}}{t_r}\right| = \frac{{}^7C_r 4^{7-r}(3x)^r}{{}^7C_{r-1}4^{8-r}(3x)^{r-1}} = \frac{8-r}{r}\frac{3x}{4} = \frac{8-r}{2r} \text{ since } x = 2/3$$

Now $|t_{r+1}| \geq |t_r|$ if $8-r \geq 2r$ or $\dfrac{8}{3} \geq r$

This inequality is valid only for $r = 1$ or 2

Thus, for $r = 1, 2$; $|t_{r+1}| > |t_r|$ and

for $r = 3, 4$; $|t_{r+1}| < |t_r|$

$\therefore$ $|t_1| < |t_2| < |t_3| > |t_4| > |t_5| > \ldots$

greatest term $= |t_3| = {}^7C_2 4^5 \cdot (3x)^2$ where $x = \dfrac{2}{3}$
$= 21 \times 4^5 \times 2^2 = 86016$

Concept Application

10. The greatest value of the terms independent of x in the expansion of $\left(x\sin\theta + \dfrac{\cos\theta}{x}\right)^{10}$ is

(*a*) ${}^{10}C_5$ (*b*) 2^5

(*c*) $2^5 \cdot {}^{10}C_5$ (*d*) $\dfrac{{}^{10}C_5}{2^5}$

11. In the expansion of $(3^{x/4} + 3^{5x/4})^n$ the sum of the binomial coefficients is 64 and the term with the greatest binomial coefficient exceeds the third term by $(n-1)$, then the value of x must be

(a) 1 (b) 2 (c) 0 (d) –1

12. Greatest term in the binomial expansion of $(a + 2x)^9$ when $a = 1$ and $x = \frac{1}{3}$ is

(a) 3rd and 4th (b) 4th and 5th
(c) only 4th (d) only 5th

13. Greatest term in the expansion of $(7 - 4x)^{10}$ is (when $x = 1$)

(a) T_4 and T_5 (b) T_4 only
(c) T_5 only (d) T_5 and T_6

SUMMATION OF SERIES INCLUDING BINOMIAL COEFFICIENTS

In the binomial expansion of $(1 + x)^n$, let us denote the coefficients ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_r, ..., {}^nC_n$ by $C_0, C_1, C_2, ..., C_r, ..., C_n$ respectively.

- The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n:

$\because$ $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_n x^n$

Putting $x = 1$,

$\therefore$ $2^n = C_0 + C_1 + C_2 + ... + C_n$ ... (i)

or $\sum_{r=0}^{n} C_r = 2^n$.

- The sum of the coefficients of the odd terms in the expansion of $(1 + x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1}.

Since, $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_nx^n$

Putting $x = -1$,

$0 = C_0 - C_1 + C_2 - C_3 + ... + (-1)^n C_n$

and $2^n = C_0 + C_1 + C_2 + C_3 + ... + C_n$ {from (i)}

Adding and subtracting these two equations, we get

$2^n = 2 (C_0 + C_2 + C_4 + ...)$

and $2^n = 2(C_1 + C_3 + C_5 + ...)$

$\therefore C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

sum of coefficients of odd terms = sum of coefficients of even terms = 2^{n-1}

Some Results on Binomial Coefficients

(a) $C_0 + C_1 + C_2 + ... + C_n = 2^n$

(b) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

(c) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!n!}$

(d) $C_0 . C_r + C_1 . C_{r+1} + C_2 . C_{r+2} + ... + C_{n-r}$

$$C_n = \frac{(2n)!}{(n+r)!(n-r)!} = {}^{2n}C_{n-r}$$

Remember: $(2n)! = 2^n . n! [1.3.5... (2n-1)]$

Train Your Brain

Example 12: Prove that the sum of the coefficients in the expansion of $(1 + x - 3x^2)^{2163}$ is –1.

Sol. Putting $x = 1$ in $(1 + x - 3x^2)^{2163}$, the required sum of coefficients $= (1 + 1 - 3)^{2163} = (-1)^{2163} = -1$.

Example 13: If the sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$ is equal to the sum of the coefficients in the expansion of $(x - \alpha y)^{35}$, then find the value of α.

Sol. Sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$

= Sum of the coefficients in the expansion of $(x - \alpha y)^{35}$

Putting $x = y = 1$

$\therefore$ $(\alpha - 1)^{35} = (1 - \alpha)^{35}$

$\Rightarrow$ $(\alpha - 1)^{35} = -(\alpha - 1)^{35}$

$\Rightarrow$ $2(\alpha - 1)^{35} = 0$

$\therefore$ $\alpha - 1 = 0$ $\therefore \alpha = 1$

Differentiation can be used to solve series in which each term is a product of an integer and a binomial coefficient i.e.. in the form k . ${}^nC_r$.

Example 14: Show that $3 . C_0 + 7 . C_1 + 11 . C_2 + ... + (4n + 3) C_n = (2n + 3) 2^n$.

Sol. This problem can be done be differentiating the expansion of $x^3(1 + x^4)^n$ and putting $x = 1$.

$x^3 (1 + x^4)^n = x^3 (C_0 + C_1x^4 + C_2x^8 + ... + C_nx^{4n})$

$= C_0x^3 + C_1x^7 + C_2x^{11} + ... + C_nx^{4n+3}$

Differentiating we get,

$\Rightarrow$ $3x^2 (1 + x^4)^n + x^3n (1 + x^4)^{n-1} 4x^3$

$= 3x^2C_0 + 7x^6C_1 + 11x^{10}C_2 + ... + (4n + 3)x^{4n+2} C_n$

Now substituting $x = 1$ in both sides.

$\Rightarrow$ $3C_0 + 7C_1 + 11C_2 + ... + (4n + 3)C_n$

$= 3(2^n) + 4n (2)^{n-1} = (3 + 2n)2^n$

Example 15: Show that $C_1 - 2 . C_2 + 3 . C_3 - 4 . C_4 + ... + (-1)^{n-1} n . C_n = 0$.

Sol. The problem can be done by differentiating the expansion of $(1 + x)^n$ and then putting $x = -1$.

2nd Method :

L.H.S. $= {}^nC_1 - 2. {}^nC_2 + 3. {}^nC_3 - .....+ (-1)^{n-1} . n . {}^nC_n$

$= n \{1 - {}^{(n-1)}C_1 + {}^{(n-1)}C_2 - {}^{n-1}C_3 + ...+ (-1)^{n-1} {}^{(n-1)}C_{n-1}\}$

$= n (1 - 1)^{n-1} = n \times 0 = 0$

Integration can be used to solve series in which each term is a binomial coefficient divided by an integer i.e. in the form $\frac{{}^nC_r}{k}$.

Example 16: Show that

$$2.C_0+2^2.\frac{C_1}{2}+2^3.\frac{C_2}{3}+\ldots+2^{n+1}\frac{C_n}{n+1}=\frac{3^{(n+1)}-1}{n+1}$$

Sol. Integrating the expansion of $(1+x)^n$ between the limits 0 to 2.

$$\int_0^2(1+x)^n\,dx=\int_0^2(C_0+C_1x+\ldots+C_nx^n)\,dx$$

$$\Rightarrow \left.\frac{(1+x)^{n+1}}{n+1}\right|_0^2=\left.C_0x+C_1\frac{x^2}{2}+\ldots+C_n\frac{x^{n+1}}{n+1}\right|_0^2$$

$$\Rightarrow 2.C_0+2^2.\frac{C_1}{2}+2^3.\frac{C_2}{3}+\ldots+2^{n+1}\frac{C_n}{n+1}=\frac{3^{(n+1)}-1}{n+1}$$

2nd Method: L.H.S.

$$=\frac{1}{n+1}\left\{{}^{(n+1)}C_1.2+{}^{(n+1)}C_2.2^2+{}^{(n+1)}C_3.2^3+\ldots+{}^{(n+1)}C_{n+1}.2^{(n+1)}\right\}$$

$$=\frac{1}{n+1}\left\{1+{}^{(n+1)}C_1.2+{}^{(n+1)}C_2.2^2+\ldots+{}^{(n+1)}C_{n+1}.2^{(n+1)}-1\right\}$$

$$=\frac{1}{(n+1)}\left\{(1+2)^{n+1}-1\right\}=\frac{3^{n+1}-1}{n+1}$$

Product of two expansions can be used to solve some problems related to series of binomial coefficients in which each term is a product of two binomial coefficients.

Example 17: If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$ then prove that $\sum\sum_{0\le i<j\le n}(C_i+C_j)^2=(n-1)^{2n}C_n+2^{2n}$

Sol. L.H.S $\sum\sum_{0\le i<j\le n}(C_i+C_j)^2=(C_0+C_1)^2+(C_0+C_2)^2+\ldots$

$$+(C_0+C_n)^2+(C_1+C_2)^2+(C_1+C_3)^2+\ldots+(C_1+C_n)^2$$
$$+(C_2+C_3)^2+(C_2+C_4)^2+\ldots+(C_2+C_n)^2+\ldots+(C_{n-1}+C_n)^2$$

$$=n(C_0^2+C_1^2+C_2^2+\ldots+C_n^2)+2\sum\sum_{0\le i<j\le n}C_iC_j$$

$$=n.{}^{2n}C_n+2.\left\{2^{2n-1}-\frac{2n!}{2.n!n!}\right\}$$

$$=n.{}^{2n}C_n+2^{2n}-{}^{2n}C_n=(n-1).{}^{2n}C_n+2^{2n}=\text{R.H.S.}$$

Concept Application

14. The sum of the series

$$\frac{{}^{101}C_1}{{}^{101}C_0}+\frac{2\cdot{}^{101}C_2}{{}^{101}C_1}+\frac{3\cdot{}^{101}C_3}{{}^{101}C_2}+\ldots+\frac{101\cdot{}^{101}C_{101}}{{}^{101}C_{100}}\text{ equals}$$

(a) 5050 (b) 4950 (c) 5151 (d) 5051

15. The value of $\binom{50}{1}^2+2\binom{50}{2}^2+3\binom{50}{3}^2+\ldots+50\binom{50}{50}^2$ is equal to (where $={}^nC_r$)

(a) $\binom{100}{50}$ (b) $50\binom{100}{50}$

(c) $100\binom{100}{50}$ (d) $\frac{2^{49}}{49!}(1.3.5\ldots99)$

16. We have $3m$ balls, out of which m are identical and the rest are all different. The number of ways of selecting m objects from the group is

(a) $\frac{{}^{2m}C_m}{2}$ (b) $\frac{1}{2}\,{}^{2m}C_{m-1}+2^{2m-1}$

(c) $\frac{1}{2}\,{}^{2m}C_m+2^{2m+1}$ (d) $\frac{1}{2}\,{}^{2m}C_m+2^{2m-1}$

17. Let $C_r's$ denote the combinational coefficient in the expansion of the binomial $(1+x)^{14}$, then the value of the sum $\Sigma_{r=1}^{15}r\cdot C_{r-1}$ is

(a) 2^{18} (b) 2^{17} (c) 2^{16} (d) 2^{15}

18. The value of $\Sigma_{r=1}^{15}r^2.\left(\frac{{}^{26}C_r}{{}^{26}C_{r-1}}\right)$ is equal to

(a) 900 (b) 1100 (c) 1300 (d) 1800

19. Coefficient of α^t in the expansion of,
$(\alpha+p)^{(m-1)}+(\alpha+p)^{(m-2)}(\alpha+q)+(\alpha+p)^{(m-3)}(\alpha+q)^2+\ldots(\alpha+q)^{(m-1)}$ where $\alpha\ne-q$ and $p\ne q$ is

(a) $\frac{{}^mC_t(p^t-q^t)}{p-q}$ (b) $\frac{{}^mC_1(p^{m-t}-q^{m-t})}{p-q}$

(c) $\frac{{}^mC_t(p^t+q^t)}{p-q}$ (d) $\frac{{}^mC_t(p^{m-t}+q^{m-t})}{p-q}$

20. Sum of the series $\frac{1\cdot{}^{10}C_1}{2^9}-\frac{2\cdot{}^{10}C_2\cdot3^1}{2^8}+\frac{3\cdot{}^{10}C_3 3^2}{2^7}-\ldots-10-{}^{10}C_{10}3^9$ is equal to $\frac{k}{2^8}$, then k is equal to

(a) 5^8 (b) -5^9 (c) -5^{10} (d) 5^{11}

21. The value of expression ${}^{(n+1)}C_2+3({}^2C_2+{}^3C_2+{}^4C_2+\ldots+{}^nC_2)$ is

(a) Σn^2 (b) Σn^2

(c) Σn (d) $\frac{n^2(n+1)}{2}$

22. Find the sum of following series

(i) $\sum_{r=0}^{n}{}^{2n}C_{2r}$ (ii) $\sum_{r=0}^{25}{}^{100}C_{4r}$

(iii) $\sum_{r=0}^{50}{}^{100}C_r$ (iv) $\sum_{r=0}^{24}{}^{100}C_{4r+3}$

Important Results

If $(\sqrt{P}+Q)^n = I + f$ where I and n are positive integers, n being odd, and $0 \le f < 1$, then show that $(I+f)f = k^n$ where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.

Proof. Given $\sqrt{P} - Q < 1$

$\therefore\ 0 < (\sqrt{P} - Q)^n < 1$

Now let $\left(\sqrt{P} - Q\right)^n = f'$ where $0 < f' < 1$

$\therefore\ I + f - f' = (\sqrt{P} + Q)^n - \left(\sqrt{P} - Q\right)^n$

$\because$ RHS contains even powers of $\sqrt{P}$ ($\because n$ is odd)

$\Rightarrow$ RHS is an integer

Since RHS and I are integers,

$\therefore\ f - f'$ is also integer.

$\Rightarrow\ f - f' = 0$

$\because\ -1 < f - f' < 1$

or $f = f'$

$\therefore\ (I+f)f = (I+f)f' = (\sqrt{P}+Q)^n(\sqrt{P}-Q)^n$

$= (P - Q^2)^n = k^n$

If $(\sqrt{P}+Q)^n = I + f$ where I and n are positive integer, n being even, and $0 \le f < 1$, then show that $(I+f)(1-f) = k^n$ where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.

Proof: If n is an even integer then

$(\sqrt{P}+Q)^n + (\sqrt{P}-Q)^n = I + f + f'$

Hence, LHS and I are integer.

$\therefore\ f + f'$ is also integer.

$\Rightarrow\ f + f' = 1$

$\because\ 0 < f + f' < 2$

$\therefore\ f' = (1 - f)$

Hence, $(I+f)(1-f) = (I+f)f' = (\sqrt{P}+Q)^n(\sqrt{P}-Q)^n$

$= (P - Q^2)^n = k^n$.

Train Your Brain

Example 18: If $(2+\sqrt{3})^n = I + f$ where I and n are positive integers and $0 < f < 1$, show that

(*i*) I is an odd integer and

(*ii*) $(I+f)(1-f) = 1$.

Sol. (*i*) Now $0 < 2 - \sqrt{3} < 1$, since $2 - \sqrt{3} = 0.268$ (approx.)

$\therefore\ 0 < (2-\sqrt{3})^n < 1$;

we can take $(2-\sqrt{3})^n$ as f'.

Now $(2+\sqrt{3})^n + (2-\sqrt{3})^n = I + f + f'$

But

L.H.S. $= 2\{2^n + {}^nC_2 2^{n-2}(\sqrt{3})^2 + {}^nC_4 2^{n-4}(\sqrt{3})^4 + \ldots\}$

= an integer

(in fact an even integer)

$\therefore$ RHS. $= I + f + f'$ = an even integer

Also $f + f' = 1$, since f and f' are both positive proper fractions.

$\therefore\ I$ = an even integer -1 = an odd integer.

(*ii*) $(I+f)(1-f) = (I+f)(f') = (2+\sqrt{3})^n.(2-\sqrt{3})^n$

$= (4-3)^n = 1^n = 1$

Concept Application

23. If $(9+\sqrt{10})^n = I + f$ where I, n are integers and $0 < f < 1$, then

(*a*) I is an odd integer (*b*) I is an even integer

(*c*) $(I+f)(1-f) = 1$ (*d*) $1 - f = (9-\sqrt{80})^n$

DIVISIBILITY PROBLEMS

Working Rule

(*i*) First of all write down the given expression in such a way that there is a term containing n^{th} power of an integer a.

(*ii*) If an occurs, then go on subtracting 1, 2, 3, ... from a and find integer r such that some power of $(a - r)$ is divisible by the number k from which the given expression is to be shown to be divisible.

(*iii*) Now write $a_n = [r + (a - r)]^n$ and expand and then collect the terms containing $(a - r)^m$ and higher powers of $(a - r)$ in one bracket if $(a - r)^m$.

Train Your Brain

Example 19: Show that $3^{2n+2} - 8n - 9$ is divisible by 64 if $n \in N$.

Sol. $3^{2n+2} - 8n - 9 = (1+8)^{n+1} - 8n - 9$

$= [1 + (n+1)8 + {}^{n+1}C_2 8^2 + \ldots] - 8n - 9$

$= {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + {}^{n+1}C_4 8^4 + \ldots$

$= 64[{}^{n+1}C_2 + {}^{n+1}C_3 8 + {}^{n+1}C_4 8^2 + \ldots]$

which is clearly divisible by 64.

Concept Application

24. The remainder, when $(15^{23} + 23^{23})$ is divided by 19, is

(*a*) 4 (*b*) 15 (*c*) 0 (*d*) 18

25. The remainder, if $1 + 2 + 2^2 + 2^3 + ... + 2^{1999}$ is divided by 5 is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

26. Last two digits of 21 to the $(100)^{th}$ power, is

(*a*) 000 (*b*) 01 (*c*) 11 (*d*) 21

27. Last three digits of the number $N = 7^{100} - 3^{100}$ are .

(*a*) 100 (*b*) 300 (*c*) 500 (*d*) 000

28. If $6^{83} + 8^{83}$ is divided by 49, then the remainder is

(*a*) 35 (*b*) 5 (*c*) 1 (*d*) 0

MULTINOMIAL EXPANSION

If $n \in N$, then the general term of the multinomial expansion $(x_1 + x_2 + x_3 + ... + x_k)^n$ is

$$\frac{n!}{a_1!a_2!a_3!.....a_k!} x_1^{a_1}.x_2^{a_2}.x_3^{a_3}.....x_k^{a_k}$$

where $a_1 + a_2 + a_3 + ... + a_k = n$

and $0 \le a_i \le n$, $i = 1, 2, 3, ... k$.

and the number of terms in the expansion are ${}^{n+k-1}C_{k-1}$.

The greatest coefficient in the expansion of $(a_1 + a_2 + a_3 + ... + a_m)^n$ is $\frac{n!}{(q!)^{m-r}((q+1)!)^r}$

where q is the quotient and r is the remainder when n is divided by m.

Train Your Brain

Example 20: Find the co-efficient of $a^2 b^3 c^5$ in the expansion of $(a + b + c)^{10}$.

Sol. General term is $\frac{10!}{a_1!a_2!a_3!}(a)^{a_1}(b)^{a_2}(c)^{a_3}$

$\therefore$ $a_1 = 2, a_2 = 3, a_3 = 5$

$\therefore$ Co-efficient of at $a^2b^3c^5 = \frac{10!}{2!3!5!}$.

Concept Application

29. The number of terms with rational coefficients in the expansion of $(\sqrt[3]{5x} + \sqrt[2]{3y} + z)^6$ is

(*a*) 7 (*b*) 6 (*c*) 8 (*d*) 9

30. The coefficient of term that contains $x^{10}y^3z^2$ expansion of the binomial $(x^2 - 2y + z)^{10}$ is

(*a*) $-\frac{8\cdot(10)!}{2!\cdot 3!\cdot 5!}$ (*b*) $\frac{8\cdot(10)!}{2!\cdot 3!\cdot 5!}$

(*c*) $\frac{(10)!}{2!\cdot 3!\cdot 5!}$ (*d*) $-\frac{(10)!}{2!\cdot 3!\cdot 5!}$

BINOMIAL THEOREM FOR ANY INDEX

If $n \in R$, $-1 < x < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + \frac{n(n-1)(n-2)....(n-r+1)}{r!}x^r + ...\infty$$

Note: (*i*) In the expansion if $n \in R$ and $n > 0$ then $-1 < x < 1$.

(*ii*) ${}^nC_r$ can not be used because it is defined only for natural number.

(*iii*) If x be so small then its square and higher powers may be neglected, then approximate value of $(1 + x)^n$ is $1 + nx$.

Important Results

- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ... + (-1)^r x^r + ...$
- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...$
- $(1 + x)^{-2} = 1 - {}^2C_1x + {}^3C_2X^2 - {}^4C_3x^3 + ... + (-1)^r\,{}^{r+1}C_rx^r + ...$
- $(1 - x)^{-2} = 1 + {}^2C_1x + {}^3C_2x^2 + {}^4C_3x^3 + ... + {}^{r+1}C_rx^r + ...$
- $(1 + x)^{-3} = 1 - {}^3C_1x + {}^4C_2x^2 - {}^5C_3x^3 + ... + (-1)^r\,{}^{r+2}C_rx^r + ...$
- $(1 - x)^{-3} = 1 + {}^3C_1x + {}^4C_2x^2 + {}^5C_3x^3 + ...+ {}^{r+2}C_rx^r + ...$

  In general, the coefficient of x^n in $(1 - x)^{-k}$ is ${}^{n+k-1}C_{k-1}$

APPROXIMATIONS

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + ...$$

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its squares and higher powers may be neglected then $(1 + x)^n = 1 + nx$, approximately,

This is an approximate value of $(1 + x)^n$

EXPONENTIAL SERIES

(*a*) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(*b*) Logarithms to the base 'e' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm.**

(c) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\infty$; where x may be any real or complex number & $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(d) $a^x = 1+\frac{x}{1!}\ell na+\frac{x^2}{2!}\ell n^2 a+\frac{x^3}{3!}\ell n^3 a+\ldots\infty$

where $a > 0$

(e) $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\infty$

LOGARITHMIC SERIES

(a) $ln\,(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty$ where $-1 < x \le 1$

(b) $ln\,(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty$ where $-1 \le x < 1$

Remember:

(i) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\infty = ln\,2$

(ii) $e^{ln\,x} = x$

(iii) $ln 2 = 0.693$

(iv) $ln\,10 = 2.303$

Train Your Brain

Example 21: If x is so small such that its square and higher powers may be neglected then find the approximate value of $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$.

Sol. $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right)$

$= \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right) = 1-\frac{x}{8}-\frac{19}{12}x = 1-\frac{41}{24}x$

Example 22: The value of cube root of 1001 upto five decimal places is

Sol. $(1001)^{1/3} = (1000+1)^{1/3} = 10\left(1+\frac{1}{1000}\right)^{1/3} = 10\left\{1+\frac{1}{3}.\frac{1}{1000}+\frac{1/3(1/3-1)}{2!}\frac{1}{1000^2}+\ldots\right\}$

$= 10\{1 + 0.0003333 - 0.00000011 + \ldots\} = 10.00333$

Concept Application

31. If the value of x is so small that and higher powers of x can be neglected, then $\frac{\sqrt{1+x}+\sqrt[3]{(1-x)^2}}{(1+x)+\sqrt{1+x}}$ is equal to:

(a) $1+\frac{5}{6}x$ (b) $1-\frac{5}{6}x$ (c) $1+\frac{2}{3}x$ (d) $1-\frac{2}{3}x$

32. If x is positive, the first negative term in the expansion of $(1+x)^{27/5}$, is:

(a) 7th term (b) 5th term (c) 8th term (d) 6th term

Short Notes

Important terms in the binomial expansion are

(*a*) **General term:** The general term or the $(r+1)^{th}$ term in the expansion of $(x+y)^n$ is given by

$$T_{r+1} = {}^nC_r x^{n-r} . y^r$$

(*b*) **Middle term:** The middle term (s) is the expansion of $(x+y)^n$ is (are):

(*i*) If n is even, there is only one middle term which is given by

$$T_{(n+2)/2} = {}^nC_{n/2} . x^{n/2} . y^{n/2}$$

(*ii*) If n is odd, there are two middle terms which are

$$T_{(n+1)/2} \text{ and } T_{[(n+1)/2]+1}$$

(*c*) **Term independent of x:** Term independent of x contains no x; Hence find the value of r for which the exponent of x is zero.

If $(\sqrt{A}+B)^n = I + f$, where I & n are positive integers and $0 \le f < 1$, then

(*a*) $(I+f) \cdot f = K^n$ if n is odd & $A - B^2 = K > 0$

(*b*) $(I+f)(1-f) = k^n$ if n is even & $\sqrt{A} - B < 1$

Some results on binomial coefficients

(*a*) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(*b*) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

(*c*) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(*d*) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} \ldots + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

(*e*) $C_0 + C_1 + C_2 + \ldots = C_n = 2^n$

(*f*) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

(*g*) $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!n!}$

(*h*) $C_0 \cdot C_r + C_1 . C_{r+1} + C_2 \cdot C_{r+2} + \ldots + C_{n-r} C_n = \frac{(2n)!}{(n+r)!(n-r)}$

Greatest coefficient and Greatest Term in Expansion of $(x + a)^n$

(*a*) If n is even greatest coefficient is ${}^nC_{n/2}$.

If n is odd greatest coefficient is ${}^nC_{\left(\frac{n-1}{2}\right)}$ or ${}^nC_{\left(\frac{n+1}{2}\right)}$

(*b*) **For greatest term:** Greatest term

$$= \begin{cases} T_p \text{ and } T_{p+1} & \text{if } \dfrac{n+1}{\left|\dfrac{x}{a}\right|+1} \text{ is an integer} \\ T_{q+1} & \text{if } \dfrac{n+1}{\left|\dfrac{x}{a}\right|+1} \text{ is non integer and} \in (q, q+1), q \in I \end{cases}$$

Multinomial Theorem

For any $n \in N$,

(*i*) $(x_1 + x_2 + \ldots + x_k)^n = \sum_{r_1 + r_2 + \ldots + r_k = n} \frac{n!}{r_1! r_2! \ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$

(*ii*) The general term in the above expansion is

$$\frac{n!}{r_1! r_2! \ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$$

Here total number of terms in the expansion $= {}^{n+k-1}C_{k-1}$.

Binomial Theorem for Negative or Fractional Indices

If $n \in Q$, then $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots \infty$ provided $|x| < 1$.

Notes

(*i*) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots \infty$

(*ii*) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots \infty$

(*iii*) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \infty$

(*iv*) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \infty$

Exponential series

(*a*) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty$; where x may be any real or complex number and $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

(*b*) $a^x = 1 + \frac{x}{1!} \ell n a + \frac{x^2}{2!} \ell n^2 a + \frac{x^3}{3!} \ell n^3 a + \ldots \infty$, where $a > 0$.

Logarithmic Series

(*a*) $ln\,(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty$, where $-1 < x \le 1$.

(*b*) $ln\,(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \infty$, where $-1 \le x < 1$.

(*c*) $ln \frac{(1+x)}{(1-x)} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \infty\right), |x| < 1$.

Solved Examples

1. If $(5+2\sqrt{6})^n = m+f$, where n and m are positive integers and $0 \le f < 1$, then $\left(\alpha + \frac{1}{\beta}\right)$ is equal to

(*a*) $\frac{1}{m}$ (*b*) m (*c*) $m+\frac{1}{m}$ (*d*) $m-\frac{1}{m}$

Sol. (*b*) We have $5-2\sqrt{6} = 1/\left(5+2\sqrt{6}\right)$.

Therefore, $0 < 5-2\sqrt{6} < 1$. Let $F = (5-2\sqrt{6})^n$. Then $0 < F < 1$. Also,

$$m+f+F = (5+2\sqrt{6})^n + (5-2\sqrt{6})^n$$

$$= 2\left[C_0 5^n + C_2 5^{n-2}(2\sqrt{6})^2 + C_4 5^{n-4}(2\sqrt{6})^4 + \cdots\right] = 2k$$

where k is some positive integer.

Hence, $f+F = 2k-m$ is a positive integer. Also, $0 < f+F < 2$. Therefore, $f+F = 1$. Now,

$$\frac{1}{1-f} - f = \frac{1}{F} - (1-F)$$

$$= (5+2\sqrt{6})^n - \left\{1-(5-2\sqrt{6})^n\right\}$$

$$= (5+2\sqrt{6})^n + (5-2\sqrt{6})^n - 1$$

$$= m+f+F-1 = m$$

2. If $S_1 = \sum_{k=0}^{n} {}^nC_k \cos(kx)\cos(n-k)x$ and

$S_2 = \sum_{k=1}^{n-1} {}^nC_k \sin(kx)\sin(n-k)x$, then $S_1 - S_2$ equals

(*a*) $2^{n-1}\cos(nx)$ (*b*) $2^n\cos(nx)$

(*c*) 0 (*d*) $2+2^n\cos nx$

Sol. (*b*)

$$S_1 - S_2 = \sum_{k=0}^{n} {}^nC_k[\cos(kx)\cos(n-k)x - \sin(kx)\sin(n-k)x]$$

$$= \sum_{k=0}^{n} {}^nC_k \cos(k+n-k)x$$

$$= 2^n\cos(nx)$$

3. If $\left(1+x+x^2\right)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, then value of $N = a_0a_1 - a_1a_2 + a_2a_3 - a_3a_4 + \cdots$ is equal to

(*a*) 0 (*b*) n (*c*) $-n$ (*d*) 1

Sol. (*a*) $N =$ Coefficient of $\frac{1}{x}$ in

$$\left(a_0 - a_1x + a_2x^2 \cdots + a_{2n}x^{2n}\right)$$

$$\times\left(a_0 + a_1\left(\frac{1}{x}\right) + a_2\left(\frac{1}{x}\right)^2 + \cdots a_{2n}\left(\frac{1}{x}\right)^{2n}\right.$$

$$= \text{coefficient of } \frac{1}{x} \text{ in } \left(1-x+x^2\right)^n$$

$$\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n \text{ or in } \frac{1}{x^{2n}}\left[\left(1+x^2\right)^2 - x^2\right] = 0$$

4. Sum of the first 20 terms of the series

$$\frac{1}{(2)(4)} + \frac{(1)(3)}{(2)(4)(6)} + \frac{(1)(3)(5)}{(2)(4)(6)(8)} + \cdots \text{ is}$$

(*a*) $\frac{1}{2} - \frac{1}{2^{40}}\left({}^{40}C_{20}\right)$ (*b*) $\frac{1}{2} - \frac{1}{2^{41}}\left({}^{42}C_{21}\right)$

(*c*) $\frac{1}{2} - \frac{1}{2^{42}}\left({}^{42}C_{21}\right)$ (*d*) $\frac{1}{2} - \frac{1}{2^{43}}\left({}^{40}C_{20}\right)$

Sol. (*c*) Let a_k denote the kth term of the series, then

$$a_k = \frac{(1)(3)(5)\cdots(2k-1)}{(2)(4)(6)\cdots(2k)(2k+2)}$$

$$= \frac{(1)(3)(5)\cdots(2k-1)(2k+2-2k-1)}{(2)(4)(6)\cdots(2k)(2k+2)} = b_k - b_{k+1}$$

where

$$b_k = \frac{(1)(3)(5)\cdots(2k-1)}{(2)(4)(6)\cdots(2k)}$$

$$= \frac{(1)(2)(3)(4)\cdots(2k-1)(2k)}{[(2)(4)(6)\cdots(2k)]^2}$$

$$= \frac{1}{2^{2k}}\left({}^{2k}C_k\right)$$

Thus, $\sum_{k=1}^{20} a_k = \sum_{k=1}^{20}(b_k - b_{k+1}) = b_1 - b_{21}$

$$= \frac{1}{2} - \frac{1}{2^{42}}\left({}^{42}C_{21}\right)$$

5. Value of $S = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{1}{3^k}\left({}^kC_r\right)$ is

(a) 2 (b) $\frac{2}{3}$

(c) $\frac{1}{3}$ (d) None of these

Sol. (a) $\sum_{k=0}^{\infty}\frac{1}{3^k}\left({}^kC_r\right) = \frac{1}{3^k}\sum_{r=0}^{k}{}^kC_r = \frac{2^k}{3^k}$

Thus, $S = \frac{2/3}{1-2/3} = 2$

6. Which of the following statement(s) is/are correct?

(a) The coefficient of x^{10} in $(1+x^2-x^3)^8$ is 476

(b) The remainder when (9^7+7^9) is divided by 32 is 0.

(c) The coefficient of the term independent of 'x' in the expansion of $\left(2x^2-\frac{1}{3x}\right)^{11}$ is $\frac{-{}^{11}C_7\cdot 2^4}{3^7}$.

(d) The only value of 'r' for which the coefficient of $(2r+3)^{th}$ term is equal to the coefficient of $(r+5)^{th}$ term in the expansion of $(1+x)^{15}$ is 2

Sol. (a, b)

(a) $\left[(1+x^2)-x^3\right]^8 = {}^8C_0(1+x^2)^8 + {}^8C_1(1+x^2)^7(-x^3)$

$+ {}^8C_2(1+x^2)^6(-x^3)^2 + {}^8C_3(1+x^2)^5(-x^3)^3$

Coefficient of $x^{10} = {}^8C_0\,{}^8C_5 + {}^8C_2\,{}^6C_2$

(b) Consider $(1+8)^7-(1-8)^9 = 128+8^2k$ remainder is 0

(c) not possible

(d) $r=2$ or $r=3$

7. The coefficient of term that contains $x^{10}y^3z^2$ in the expansion of the binomial $(x^2-2y+z)^{10}$ is

(a) $-\frac{8\cdot(10)!}{2!\,3!\,5!}$ (b) $\frac{8\cdot(10)!}{2!\,3!\,5!}$

(c) $\frac{(10)!}{2!\,3!\,5!}$ (d) $-\frac{(10)!}{2!\,3!\,5!}$

Sol. (a) Consider $(x^2+(z-2y))^{10}$ and expand binomially.

8. If $2a+2b+3c=\frac{1}{5}$ and $a, b, c>0$ then maximum value of term independent of x in the expansion of $\left(abx^{\frac{1}{2}}+cx^{\frac{-1}{3}}\right)^{25}$ is:

(a) ${}^{25}C_{10}$ (b) ${}^{25}C_{15}(25)^{25}$

(c) ${}^{25}C_{15}\left(\frac{1}{35}\right)^{35}$ (d) ${}^{25}C_{10}\left(\frac{1}{25}\right)^{35}$

Sol. (c) $T_{r+1} = {}^{25}C_r(ab)^{(25-r)}\cdot x^{\left(\frac{25-r}{2}\right)}\cdot c^r\cdot x^{\left(\frac{-r}{3}\right)}$

$\Rightarrow r=15 = {}^{25}C_{15}\,a^{10}\cdot b^{10}\cdot c^{15}$

$10a+10b+15c=1$

$$\frac{(a+a+\ldots+a)+(b+b+\ldots+b)+(c+c+\ldots+c)}{35}$$

$$\geq\left(a^{10}b^{10}c^{15}\right)^{\frac{1}{35}} \Rightarrow \left(\frac{1}{35}\right)^{35}\geq a^{10}b^{10}c^{15}$$

9. Let $a_n=\left(\frac{3+\sqrt{5}}{2}\right)^n+\left(\frac{2}{3+\sqrt{5}}\right)^n \forall n\in N$,

(a) a_1, a_2 are primes

(b) If a_k, a_{k+1} are integers then a_{k+2} is an integer

(c) a_n is an integer for each $n\in N$

(d) ${}^{2n}C_0 5^n+{}^{2n}C_2 5^{n-1}+\ldots+{}^{2n}C_{2n}$ is divisible by 2^{2n-1}

Sol. (a,b,c,d)

$$\frac{2}{3+\sqrt{5}}=\frac{2(3-\sqrt{5})}{9-5}=\frac{3-\sqrt{5}}{2}$$

$$\therefore\ a_n=\left(\frac{3+\sqrt{5}}{2}\right)^n+\left(\frac{3-\sqrt{5}}{2}\right)^n$$

$$a_1=\frac{1}{2}(3+\sqrt{5})+\frac{1}{2}(3-\sqrt{5})=3$$

$$a_2=\frac{1}{4}\left\{2(3^2)+2(\sqrt{5})^2\right\}=7$$

$\Rightarrow$ a_1, a_2 are primes.

Also $3a_{k+1}-a_k = 3\left(\frac{3+\sqrt{5}}{2}\right)^{k+1}+3\left(\frac{3-\sqrt{5}}{2}\right)^{k+1}$

$$-\left(\frac{3+\sqrt{5}}{2}\right)^k-\left(\frac{3-\sqrt{5}}{2}\right)^k$$

$$=\left(\frac{3+\sqrt{5}}{2}\right)^k\left(3\left(\frac{3+\sqrt{5}}{2}\right)-1\right)$$

$$+\left(\frac{3-\sqrt{5}}{2}\right)^k\left(3\left(\frac{3-\sqrt{5}}{2}\right)-1\right)$$

$$=\left(\frac{3+\sqrt{5}}{2}\right)^k\left(\frac{14+6\sqrt{5}}{4}\right)+\left(\frac{3-\sqrt{5}}{2}\right)^k\left(\frac{14-6\sqrt{5}}{4}\right)$$

$$=\left(\frac{3+\sqrt{5}}{2}\right)^k\left(\frac{3+\sqrt{5}}{2}\right)^2+\left(\frac{3-\sqrt{5}}{2}\right)^k\left(\frac{3-\sqrt{5}}{2}\right)^2$$

$$=\left(\frac{3+\sqrt{5}}{2}\right)^{k+2}+\left(\frac{3-\sqrt{5}}{2}\right)^{k+2}=a_{k+2}\,m$$

As a_k, a_{k+1} are integer $a_{k+2}=3a_{k+1}-a_k$ is also an integer.

By the principle of mathematical induction $a_n \in N \forall n \in N$.

Also, $\frac{3+\sqrt{5}}{2}=\frac{6+2\sqrt{5}}{2}=\left(\frac{\sqrt{5}+1}{2}\right)^2 0$

and $\frac{3-\sqrt{5}}{2}=\left(\frac{\sqrt{5}-1}{2}\right)^2$

$$\therefore a_n=\left(\frac{\sqrt{5}+1}{2}\right)^{2n}+\left(\frac{\sqrt{5}-1}{2}\right)^{2n}$$

$$\Rightarrow 2^{2n}a_n=2\left[{}^{2n}C_0 5^n+{}^{2n}C_1 5^{n-1}+\cdots+{}^{2n}C_{2n}\right]$$

$$\Rightarrow 2^{2n-1}a_n=({}^{2n}C_0 5^n+{}^{2n}C_2 5^{n-1}+\cdots+{}^{2n}C_{2n})$$

10. Suppose $\alpha,\beta>0$ and $x_n={}^nC_0\alpha^n+{}^nC_2\alpha^{n-2}\beta^2+\cdots$ $y_n={}^nC_1\alpha^{n-1}\beta+{}^nC_3\alpha^{n-3}\beta^3+\cdots$ then

(*a*) $2x_n=(\alpha+\beta)^n+(\alpha-\beta)^n$

(*b*) $2y_n=(\alpha+\beta)^n-(\alpha-\beta)^n$

(*c*) $\lim_{n\to\infty}\frac{x_n}{y_n}=1$

(*d*) $\lim_{n\to\infty}e^{x_n/y_n}=e$

Sol. (*a,b,c,d*)

$$x_n+y_n=(\alpha+\beta)^n$$

$$x_n-y_n=(\alpha-\beta)^n \Rightarrow 2x_n=(\alpha+\beta)^n+(\alpha-\beta)^n$$

and $2y_n=(\alpha+\beta)^n-(\alpha-\beta)^n$

$$\Rightarrow \frac{x_n}{y_n}=\frac{(\alpha+\beta)^n+(\alpha-\beta)^n}{(\alpha+\beta)^n-(\alpha-\beta)^n}=\frac{1+\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^n}{1-\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^n}$$

As $-(\alpha+\beta)<\alpha-\beta<\alpha+\beta$

$$\Rightarrow -1<\frac{\alpha-\beta}{\alpha+\beta}<1$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^n=0$$

$$\therefore \lim_{n\to\infty}\frac{x_n}{y_n}=1\Rightarrow \lim_{n\to\infty}e^{x_n/y_n}=e$$

11. If $(1+2x+3x^2)^{10}=a_0+a_1x+a_2x^2+\cdots+a_{20}x^{20}$, then

(*a*) $a_1=20$ (*b*) $a_2=210$

(*c*) $a_4=8085$ (*d*) $a_{20}=2^2\times 3^7\times 7$

Sol.(*a, b, c*) $(1+2x+3x^2)^{10}$

$=1+10(2x+3x^2)+{}^{10}C_2(2x+3x^2)^2+\ldots+{}^{10}C_{10}(2x+3x^2)^{10}$

$\Rightarrow a_1=20, a_2=30+(45)(4)=210$

$a_4={}^{10}C_2(3^2)+{}^{10}C_3({}^3C_1)(2^2)(3)+{}^{10}C_4(2^4)=8085$

$a_{20}=3^{10}$

12. The coefficient of x^{324} in the expansion of $(x+1)(x-2)^2(x+3)^3(x-4)^4\ldots\ldots(x-24)^{24}(x+25)^{25}$ is

(*a*) 325

(*b*) $\left(\sum_{k=0}^{8}{}^8C_k+4\cdot\sum_{s=0}^{4}{}^4C_s+\sum_{r=0}^{2}{}^2C_r+{}^5C_0\right)$

(*c*) $1^2-2^2+3^2-4^2+\ldots\ldots-24^2+25^2$

(*d*) $1^1+2^2+3^3+4^4+\ldots.+25^{25}$.

Sol. (*a, b, c*)

Coefficient of x^{324} is

$=1^2-2^2+3^2-4^2+\ldots.24^2+25^2=325$

$\because \sum_{k=0}^{8}{}^8C_k=2^8=256,$

$\sum_{s=0}^{4}{}^4C_s=2^4=16\ \sum_{r=0}^{2}{}^2C_r=4$

$\therefore$ Option (*b*) $=256+4\times16+4+1=325$

13. Let *m* be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2+(1+x)^3+\ldots\ldots+(1+x)^{49}+(1+mx)^{50}$ is $(3n+1)\,{}^{51}C_3$ for some positive integer *n*. Then the value of *n* is:

(*a*) 1 (*b*) 2 (*c*) 4 (*d*) 5

Sol. (*d*) ${}^2C_2+{}^3C_2+{}^4C_2+\ldots.+{}^{49}C_2+{}^{50}C_2\,m^2=(3n+1)\,{}^{51}C_3$

${}^3C_3+{}^3C2+{}^4C_2+\ldots.+{}^{49}C_2+{}^{50}C_2\,m^2=(3n+1)\,{}^{51}C_3$

${}^nC_r+{}^nC_{r-1}={}^{n+1}C_r\Rightarrow{}^{50}C_3+{}^{50}C_2\cdot m^2=(3n+1)\,{}^{51}C_3$

${}^{50}C_3+{}^{50}C_2+(m^2-1)\,{}^{50}C_2=3n.\ \frac{51}{3}.\ {}^{50}C_2+{}^{51}C_3$

$\Rightarrow {}^{51}C_3+(m^2-1)\,{}^{50}C_2=51n\cdot{}^{50}C_2+{}^{51}C_3$

$m^2-1=51n\Rightarrow\ m^2=51n+1$

min value of m^2 for $51n+1$ is integer for $n=5$

Comprehension (14 to 16)

If $C_r={}^nC_r$, then to evaluate the expression

$P=\sum\sum_{0\le r<s\le n}C_rC_s$ we make use of

$C_0^2+C_1^2+\ldots+C_n^2={}^{2n}C_n$

and expansion of $(C_0+C_1+\ldots+C_n)^2$

14. Value of $P = \sum\sum_{0 \le r < s \le n} C_r C_s$ is

(a) $2^{2n} - \frac{1}{2}\left({}^{2n}C_n\right)$ (b) $2^{2n-1} - \frac{1}{2}\left({}^{2n}C_n\right)$

(c) $2^{2n} - {}^{2n}C_n$ (d) 2^{2n-1}

15. Value of $Q = \sum\sum_{0 \le r < s \le n} (C_r - C_s)^2$ is

(a) $(n+1)\,({}^{2n}C_n) - 2n$ (b) $(n+1)\,({}^{2n}C_n) - 2^{2n}$

(c) $(n+1)^n - 2^{2n}$ (d) $2(n^{2n})$

16. Value of the expression $R = \sum\sum_{0 \le r < s \le n} (r+s)C_r C_s$ is

(a) $n(2^{2n-1})$ (b) $n(2^{2n-1}) + {}^{2n}C_n$

(c) $n\left(2^{2n-1}\right) - \frac{1}{2}\left({}^{2n}C_n\right)$ (d) $n/2[2^{2n} - {}^{2n}C_n]$

Sol.14 to 16

14. (b) **15.** (b)

16. (d) We have

$$\left(C_0 + C_1 + C_2 + \cdots + C_n\right)^2 = \sum_{r=0} C_r^2 + 2P$$

$$\Rightarrow 2P = (2^n)^2 - \sum_{r=0} C_r^2$$

But $C_0^2 + C_1^2 + \cdots + C_n^2 = {}^{2n}C_n$

Thus, $P = 2^{2n-1} - \frac{1}{2}\left({}^{2n}C_n\right)$

Next, $Q = \left(C_0 - C_1\right)^2 + \left(C_0 - C_2\right)^2 + \cdots + \left(C_0 - C_n\right)^2$

$+\left(C_1 - C_2\right)^2 + \cdots + \left(C_1 - C_n\right)^2 + \left(C_2 - C_3\right)^2 + \cdots$

$+\left(C_2 - C_n\right)^2 + \cdots + \left(C_{n-1} - C_n\right)^2$

$= n\left(C_0^2 + C_1^2 + \cdots + C_n^2\right) - 2P$

But $C_0^2 + C_1^2 + \cdots + C_n^2 = {}^{2n}C_n$ and

$P = 2^{2n-1} - \frac{1}{2}\left({}^{2n}C_n\right)$

Next, note that R can also be written as

$$R = \sum\sum_{r<s}(r+s)C_r C_s$$

$$= \sum\sum_{r<s}(n-r+n-s)C_{n-r}C_{n-s}$$

$$= \sum\sum_{r<s}(2n)C_r C_s - \sum_{r<s}(r+s)C_r C_s$$

$$\Rightarrow 2R = 2n\sum\sum_{r<s} C_r C_s = 2nP$$

$$\Rightarrow R = \frac{n}{2}\left[2^{2n} - {}^{2n}C_n\right]$$

Comprehension (17-19)

We know that if ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are binomial co-efficient, then

Various relations among binomial co-efficient can be derived by putting $x = \pm 1, \pm i, \omega, \omega^2$;

where $i = \sqrt{-1}$, $\omega = \frac{-1 + i\sqrt{3}}{2}$.

Some other identities can be derived by adding and subtracting two such identities. The expression $(a + ib)^n$ can be evaluated by using De-Moivre's theorem by putting $a = r\cos\theta$, $b = r\sin\theta$. Now answer the following questions

17. The value of the expression $({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \ldots)^2 + ({}^nC_1 - {}^nC_3 + {}^nC_5 \ldots)^2$ must be

(a) 2^{2n} (b) 2^n (c) $2^{n/2}$ (d) $n/2$

18. The value of ${}^nC_0 + {}^nC_4 + {}^nC_8 + \ldots$ must be

(a) $2^{n/2}\cos\frac{n\pi}{8}$ (b) $2^{n/2}\sin\frac{n\pi}{8}$

(c) $2^n + 2^{n/2}\cos\frac{n\pi}{4}$ (d) $2^{n-2} + 2^{\frac{n-2}{2}}\cos\frac{n\pi}{4}$

19. The value of $\sum_{t=1}^{k}(-3)^{t-13n}\,C_{2t-1}$ where, n is an even positive integer and $k = \frac{3n}{2}$, must be

(a) $3n$ (b) 3^{n-1}

(c) $6n$ (d) 0

Sol.17 to 19

17. (b) $(C_0 - C_2 + C_4 - C_6 + \ldots)^2 + (C_1 - C_3 + C_5 \ldots)^2$

$= A^2 + B^2$ (say)

Now, $A = C_0 - C_2 + C_4 - C_6 + \ldots$

$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots$

Put $x = (-1)^{1/2} = i, -i$

$\therefore (1+i)^n = C_0 i + C_1 i - C_2 - iC_3 + C_4 + \ldots$

$(1-i)^n = C_0 - C_1 i - C_2 + iC_3 + C_4 + \ldots$

$\Rightarrow (1+i)^n + (1-i)^n = 2[C_0 - C_2 + C_4 + \ldots]$

$$\Rightarrow A = \frac{(1+i)^n + (1-i)^n}{2}$$

$$= \frac{(\sqrt{2})^n}{2}\left[\left(\cos\frac{\pi}{4}\right)^n + \left(\cos\left(\frac{-\pi}{4}\right)\right)^n\right]$$

$$= \frac{(\sqrt{2})^n}{2}\left[2\cos\frac{n\pi}{4}\right] = (2)^{n/2}\cos\frac{n\pi}{4}$$

Again, $(1+x)^n x = C_0x + C_1x^2 + C_2x^3 + C_3x^4 + C_4x^5 + \ldots$

Put $x = (-1)^{1/2} = i, -i$

$\Rightarrow (1+i)^n i = C_0 i + C_1(-1) + C_2(-i) + C_3 + C_4 i + \ldots$

And $(1-i)^n(-i)\ C_0(-i) + C_1(-1) + C_2(i) + C_3 - C_4 i + \ldots$

$\Rightarrow i((1+i)^n - (1-i)^n) = -2[C_1 - C_3 + \ldots]$

$$\Rightarrow \frac{i\left[(\sqrt{2})^n\left(\text{cis}\frac{n\pi}{4} - \text{cis}\left(-\frac{n\pi}{4}\right)\right)\right]}{-2} = B$$

$$\Rightarrow A = \frac{i}{-2}\left[(2)^{n/2}\left(2i\sin\frac{n\pi}{4}\right)\right] = (2)^{n/2}\sin\frac{n\pi}{4}$$

$$\therefore A^2 + B^2 = (2)^n\left[\cos^2\frac{n\pi}{4} + \sin^2\frac{n\pi}{4}\right] = (2)^n$$

18. (*d*) ${}^nC_0 + {}^nC_4 + {}^nC_8 + ...$

Put $x = (1)^{1/4} = 1, -1, i, -i$ in

$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + ...$

$(2)^n = C_0 + C_1 + C_2 + C_3 + C_4 + ...$

$0 = C_0 - C_1 + C_2 - C_3 + C_4 - ....$

$(1 + i)^n = C_0 + C_1i - C_2 - iC_3 + C_4 + ...$

$(1 - i)^n = C_0 - C_1i - C_2 + iC_3 + C_4 + ...$

Adding, $\frac{(2)^n + (1+i)^n + (1-i)^n}{4} = [C_0 + C_4 + ...]$

$$\therefore \text{Sum} = \frac{(2)^n + \left(\sqrt{2}\right)^n\left[2\cos\frac{n\pi}{4}\right]}{4}$$

$$= (2)^{n-2} + (2)^{\frac{n-2}{2}}\cos\frac{n\pi}{4}$$

19. (*d*) $\sum_{r=1}^{k}(-3)^{r-13n}C_{2r-1} = {}^{3n}C_1 - 3^{13n}C_3 + 3^{23n}C_6 - ....$

In the expression $(1 + x)^{3n}$ $x = {}^{3n}C_0\,x + {}^{3n}C_1\,x^2 + {}^{3n}C_2\,x^3 + {}^{3n}C_3\,x^4 + {}^{3n}C_4\,x^5 + ...$

Put $x = (-3)^{1/2} = \sqrt{3}\,i, -\sqrt{3}\,i$

$$\text{Gives} = \frac{\sqrt{3}i\left[\left(1-\sqrt{3}i\right)^{3n} - \left(1+\sqrt{3}i\right)^{3n}\right]}{3}$$

$$= {}^{3n}C_1 - 3\cdot{}^{3n}C_3 + 3^2\cdot{}^{3n}C_5 + ...$$

i.e., $\sum_{r=1}^{k}(-3)^{r-1}\cdot{}^{3n}C_{2r-1}$; $\left(k = \frac{3n}{2}\right)$

$$= \frac{-i}{\sqrt{3}}\left[(1 + \sqrt{3}\,i)^{3n} - (1 - \sqrt{3}\,i)^{3n}\right]$$

$$= \frac{-i}{\sqrt{3}}\left[(2)^{3n}\left\{\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{3n} - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right\}^{3n}\right]$$

$$= \frac{-i}{\sqrt{3}}(2)^{3n}\left[\text{cis}3n\cdot\frac{\pi}{3} - \text{cis}\left(3n\cdot\frac{\pi}{3}\right)\right]$$

$$= \frac{-i}{\sqrt{3}}(2)^{3n}[2i\sin n\pi] = 0$$

20. For $n \geq 2$, let $C_r = \binom{n}{r}$ and $a_n = \sum_{r=0}^{n}\frac{1}{C_r}$

	Column-I		Column-II
A.	$\sum_{r=0}^{n}\frac{r}{C_r}$	p.	na_n
B.	$\sum_{r=0}^{n}\frac{n-r}{C_r}$	q.	$\frac{1}{2}na_n$
C.	$\sum_{r=1}^{n}\frac{1}{rC_r}$	r.	$\frac{1}{2}na_n$
D.	$\sum_{r=0}^{n-1}\frac{1}{(n-r)C_r}$	s.	$\frac{1}{2n}a_{n-1}$

Sol. (*a*) → (R); (*b*) → (R); (*c*) → (Q); (*d*) → (Q)

	p	q	r	s
a	p	q	**r**	s
b	p	q	**r**	s
c	p	**q**	r	s
d	p	**q**	r	s

Using $C_r = C_{n-r}$, we get

$$\sum_{r=0}^{n}\frac{r}{C_r} = \sum_{r=0}^{n}\frac{r}{C_{n-r}} = \sum_{r=0}^{n}\frac{n-r}{C_r}$$

$$\Rightarrow 2\sum_{r=0}^{n}\frac{r}{C_r} = na_n \Rightarrow \sum_{r=0}^{n}\frac{r}{C_r} = \frac{1}{2}na_n$$

Similarly, $\sum_{r=0}^{n}\frac{n-r}{C_r} = \frac{1}{2}na_n$

Using $rC_r = nC_{r-1}$, we get

$$\sum_{r=1}^{n}\frac{1}{rC_r} = \frac{1}{n}\sum_{r=1}^{n}\frac{1}{C_{r-1}} = \frac{1}{n}\sum_{r=0}^{n}\frac{1}{C_r} = \frac{1}{n}a_{n-1}$$

Similarly, $\sum_{r=0}^{n-1}\frac{1}{(n-r)C_r} = \frac{1}{n}a_{n-1}$

21. Let $f(n) = \sum_{k=1}^{n}k^2C_k^2$ then $\frac{f(5)}{875}$ is

Sol. [2] $k(C_k) = k\frac{n!}{k!(n-k)!} = n\left({}^{n-1}C_{k-1}\right)$

Thus, $\sum_{k=1}^{n}k^2C_k^2 = n^2\sum_{k=1}^{n}\left({}^{n-1}C_{k-1}\right)^2$

$$= n^2\sum_{k=0}^{n-1}\left({}^{n-1}C_k\right)^2$$

But $\Sigma_{k-0}^{n}C_k^2 = {}^{2n}C_n$, therefore,

$$\sum_{k=1}^{n}k^2C_k^2 = n^2\left({}^{2n-2}C_{n-1}\right)$$

$$\Rightarrow \frac{f(5)}{875} = \frac{5^2\left({}^8C_4\right)}{875}$$

22. For $n > 2$, $a, b \in R$, let $S(n, a, b)$

$$= \sum_{r=0}^{n}(-1)^r \frac{n(n-1)\cdots(n-r+1)}{r!}(a-r)(b-r),$$

then $S\left(5, \frac{1}{2}, \frac{1}{7}\right)$ is

Sol. [0] For $0 \le r \le n$,

$$t_r = (-1)^r \frac{n(n-1)\cdots(n-r+1)}{r!}(a-r)(b-r)$$

$$= (-1)^r\left({}^nC_r\right)\left[ab - r(a+b) + r^2\right]$$

$$\Rightarrow \sum_{r=0}^{n} t_r = ab\sum_{r=0}^{n}(-1)^r\left({}^nC_r\right)$$

$$-(a+b)\sum_{r=0}^{n}(-1)^r r\left({}^nC_r\right) + \sum_{r=0}^{n}(-1)^r r^2\left({}^nC_r\right)$$

But $\sum_{r=0}^{n}(-1)^r\left({}^nC_r\right) = (1-1)^n = 0$,

$$\sum_{r=0}^{n}(-1)^r r\left({}^nC_r\right) = \frac{d}{dt}(1-t)^n\Big]_{t=1} = 0,$$

and $\sum_{r=0}^{n}(-1)^r r^2\left({}^nC_r\right) = \sum_{r=0}^{n}(-1)^r r(r-1)\left({}^nC_r\right)$

$$+\sum_{r=0}^{n}(-1)^r r\left({}^nC_r\right) = \frac{d^2}{dt^2}(1-t)^n\Big]_{t=1} = 0$$

23. For $n > 2$, let $C_r = {}^nC_r$, and $f(n) = {}^3C_2 + {}^5C_4 + {}^7C_6 + \ldots$, then $2^{-5}f(6)$ equals

Sol. [4] We have

$$C_0 + C_2x^2 + C_4x^4 + \ldots = \frac{1}{2}\left[(1+x)^n + (1-x)^n\right]$$

$$\Rightarrow C_0x + C_2x^3 + C_4x^5 + \ldots = \frac{1}{2}\left[(1+x)^n x + (1-x)^n x\right]$$

Differentiating both the sides w.r.t. x, we get

$$C_0 + 3C_2x^2 + 5C_4x^4 + \ldots$$

$$= \frac{1}{2}\left[(1+x)^n + n(1+x)^{n-1}x + (1-x)^n - nx(1-x)^{n-1}\right]$$

Putting $x = 1$, we get

$$f(n) = C_0 + 3C_2 + 5C_4 + \ldots = 2^{n-1} + n\left(2^{n-2}\right)$$

$$\therefore 2^{-5}f(6) = 4$$

24. If $\left(x + \frac{1}{x} + 1\right)^6 = a_0 + \left(a_1x + \frac{b_1}{x}\right) + \left(a_2x^2 + \frac{b_2}{x^2}\right) + \ldots\ldots + \left(a_6x^6 + \frac{b_6}{x^6}\right)$, then find the value of a_0.

Sol. [141]

$$\left(x + \frac{1}{x} + 1\right)^6 = \sum_{r=1}^{6} {}^6C_r\left(x + \frac{1}{x}\right)^r$$

$$\therefore a_0 = 1 + {}^6C_2 \cdot {}^2C_1 + {}^6C_4 \cdot {}^4C_2 + {}^6C_6 \cdot {}^6C_3$$

$$= 1 + 30 + 90 + 20 = 141$$

25. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time represented by : $\sum_{0 \le i < j \le n}\sum C_iC_j$ is equal to $2^{2n-1} - \frac{2n!}{2 \cdot n!n!}$

Sol. Since $(C_0 + C_1 + C_2 + \ldots\ldots + C_{n-1} + C_n)^2$

$= C_0^2 + C_1^2 + C_2^2 + \ldots. + C_{n-1}^2 + C_n^2 + 2(C_0C_1 + C_0C_2 + C_0C_3 + \ldots + C_0C_n + C_1C_2 + C_1C_3 + \ldots + C_1C_n + C_2C_3 + C_2C_4 + \ldots + C_2C_n + \ldots + C_{n-1}C_n)$

$$(2^n)^2 = {}^{2n}C_n + 2\sum_{0 \le i < j \le n}\sum C_iC_j$$

Hence $\sum_{0 \le i < j \le n}\sum C_iC_j = 2^{2n-1} - \frac{2n!}{2 \cdot n!n!}$

26. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots. + C_nx_n$ then prove that $\sum_{0 \le i < j \le n}(C_i + C_j)^2 = (n-1)\,{}^{2n}C_n + 2^{2n}$

Sol. L.H.S $\sum_{0 \le i < j \le n}\sum (C_1 + C_1)^2$

$(C_0 + C_1)^2 + (C_0 + C_2)^2 + \ldots + (C_0 + C_n)^2 + (C_1 + C_3)^2 + \ldots + (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 + \ldots + (C_2 + C_n)^2 + \ldots + (C^{n-1} + C_n)^2$

$$= n\left(C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2\right) + 2\sum_{0 \le i < j \le n}\sum C_1C_j$$

$$= n.{}^{2n}C_n + 2.\left\{2^{2n-1} - \frac{2n!}{2.n!n!}\right\}$$

$n.\,{}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n-1).\,{}^{2n}C_n + 2^{2n} =$ R.H.S

27. Prove that : ${}^{25}C_{10} + {}^{24}C_{10} + \ldots\ldots\ldots + {}^{10}C_{10} = {}^{26}C_{11}$

Sol. L.H.S. $= {}^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + \ldots\ldots\ldots + {}^{25}C_{10}$

$\Rightarrow {}^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + \ldots\ldots\ldots + {}^{25}C_{10}$

$\Rightarrow {}^{12}C_{11} + {}^{12}C_{10} + \ldots\ldots\ldots + {}^{25}C_{10}$

$\Rightarrow {}^{13}C_{11} + {}^{13}C_{10} + \ldots\ldots\ldots + {}^{25}C_{10}$

and so on $\therefore$ L.H.S. $= {}^{26}C_{11}$

Alternate: L.H.S. = coefficient of x^{10} in $\{(1+x)^{10} + (1+x)^{11} + \ldots (1+x)^{25}\}$

$\Rightarrow$ coefficient of x^{10} in $\left[(1+x)^{10}\frac{\{1+x\}^{16} - 1}{1+x-1}\right]$

$\Rightarrow$ coefficient of x^{10} in $\frac{\left[(1+x)^{26} - (1+x)^{10}\right]}{x}$

$\Rightarrow$ coefficient of x^{11} in $[(1+x)^{26} - (1+x)^{10}] = {}^{26}C_{11} - 0 = {}^{26}C_{11}$

28. Find the

(i) last digit

(ii) last two digit

(iii) last three digit of 17^{256}

Sol. $17^{256} = 289^{128} = (290-1)^{128}$

$= {}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + \ldots + {}^{128}C_{126}(290)^2$
$- {}^{128}C_{127}(290) + 1$

$= 1000m + {}^{128}C_2(290)^2 - {}^{128}C_1(290) + 1$

$= 1000m + \frac{128\times127}{2}\times(290)^2 - \frac{128\times290}{1} + 1 = 1000m$

$+ 683527680 + 1$

Hence the last digit is 1. Last two digits is 81. Last three digit is 681.

29. Coefficient of $x^5 y^4 z^3$ in $(x+y+z)^{12}$ is $\frac{12!}{5!\,4!\,3!}$ and coeff. of $x^6 y^3 z^3$ is $\frac{12!}{6!\,3!\,3!}$.

Sol. $[x+y+z]^{12} = [x+(y+z)]^{12}$

$T_{r+1} = {}^{12}C_r \cdot x^{12-r} \cdot (y+z)^r$

put $r = 7$

$T_8 = {}^{12}C_7 \cdot x^5 \cdot (y+z)^7 = {}^{12}C_7 \cdot x^5 \cdot {}^7C_p \cdot y^{7-p} \cdot z^p$

put $p = 3 = {}^{12}C_7 \cdot {}^7C_3 \cdot x^5 \cdot y^4 \cdot z^3$

Hence coefficient $x^5 y^4 z^3$ is ${}^{12}C_7 \cdot {}^7C_3 = \frac{(12)!}{7!\cdot5!}\frac{7!}{3!\cdot4!} = \frac{(12)!}{3!\cdot4!\cdot5!}$.

Exercise-1 (Topicwise)

GENERAL TERM

1. 6th term in expansion of $\left(2x^2-\frac{1}{3x^2}\right)^{10}$ is

(a) $\frac{4580}{17}$ (b) $-\frac{896}{27}$

(c) $\frac{5580}{17}$ (d) $\frac{-764}{25}$

2. If the coefficients of rth term and $(r+4)^{th}$ term are equal in the expansion of $(1+x)^{20}$, then the value of r will be

(a) 7 (b) 8

(c) 9 (d) 10

3. 16th term in the expansion of $\left(\sqrt{x}-\sqrt{y}\right)^{17}$ is

(a) $136xy^7$ (b) $136\,xy$

(c) $-136\,xy^{15/2}$ (d) $-136\,xy^2$

4. In $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$ if the ratio of 7th term from the beginning to the 7th term from the end is 1/6, then n is equal to

(a) 7 (b) 8 (c) 9 (d) 5

5. If x^4 occurs in the rth term in the expansion of $\left(x^4+\frac{1}{x^3}\right)^{15}$, then $r=$

(a) 7 (b) 8 (c) 9 (d) 10

6. If the third term in the binomial expansion of $(1+x)^m$ is $-\frac{1}{8}x^2$, then the rational value of m is

(a) 2 (b) 1/2 (c) 3 (d) 4

7. Coefficient of x in the expansion of $\left(x^2+\frac{a}{x}\right)^5$ is

(a) $9a^2$ (b) $10a^3$ (c) $10a^2$ (d) $10a$

8. In the expansion of $\left(x-\frac{1}{x}\right)^6$, the constant term is

(a) -20 (b) 20 (c) 30 (d) -30

9. In the expansion of $(x^2-2x)^{10}$, the coefficient of x^{16} is

(a) -1680 (b) 1680 (c) 3360 (d) 6720

10. The coefficient of x^{32} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$ is

(a) ${}^{15}C_5$ (b) ${}^{15}C_6$ (c) ${}^{15}C_4$ (d) ${}^{15}C_7$

11. If the coefficients of x^7 and x^8 in $\left(2+\frac{x}{3}\right)^n$ are equal, then n is

(a) 56 (b) 55

(c) 45 (d) 15

12. In the expansion of $(1+3x+2x^2)^6$ the coefficient of x^{11} is

(a) 144 (b) 288

(c) 216 (d) 576

13. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$, which of the following statements is false.

(a) The number of irrational terms is 19.

(b) Middle term is rational.

(c) The number of rational terms is 2 .

(d) 9th term is rational.

14. In the expansion of $\left(x^{2/3}-\frac{1}{\sqrt{x}}\right)^{30}$, a term containing the power x^{14}

(a) Does not exist

(b) Exists & the co-efficient is divisible by 29

(c) Exists & the co-efficient is divisible by 63

(d) Exists & the co-efficient is divisible by 65

15. In the expansion of $\left(x^3+3.2^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^{11}$, which of the following statements is false.

(a) There appears a term with the power x^2

(b) There does not appear a term with the power x^2

(c) There appears a term with the power x^{-3}

(d) The ratio of the co-efficient of x^3 to that of x^{-3} is 1/3

16. If the 6th term in the expansion of $\left(\frac{3}{2}+\frac{x}{3}\right)^n$ when $x=3$ is numerically greatest then the possible integral values of n can not be

(a) 11 (b) 12 (c) 13 (d) 14

TERM INDEPENDENT OF 'X'

17. In the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$, the term independent of x is

(a) ${}^9C_3\cdot\frac{1}{6^3}$ (b) ${}^9C_3\left(\frac{3}{2}\right)^3$

(c) ${}^9C_3$ (d) ${}^9P_2\left(\frac{3}{2}\right)^3$

18. The term independent of x in the expansion of $\left(x^2-\frac{3\sqrt{3}}{x^3}\right)^{10}$ is

(a) 153090 (b) 150000
(c) 150090 (d) 153180

19. The term independent of x in the expansion of $(1-x)^n\left(1+\frac{1}{x}\right)^n$ is

(a) $C_0^2+2C_1^2+\ldots+(n+1)C_n^2$
(b) $(C_0+C_1+\ldots+C_n)^2$
(c) $C_0^2+C_1^2+\ldots+C_n^2$
(d) $C_0^2+2C_1^2+\ldots+nC_n^2$

MIDDLE TERM/NUMERICALLY GREATEST TERM AND ALGEBRICALLY GREATEST & LEAST TERM

20. If the middle term in the expansion of $\left(x^2+\frac{1}{x}\right)^n$ is $924\,x^6$, then $n=$

(a) 10 (b) 12 (c) 14 (d) 16

21. The middle term in the expansion of $(1+x)^{2n}$ is

(a) $\frac{(2n)!}{n!}x^2$ (b) $\frac{(2n)!}{n!(n-1)!}x^{x+1}$
(c) $\frac{(2n)!}{(n!)^2}x^n$ (d) $\frac{(2n)!}{(n+1)!(n-1)!}x^n$

22. The middle term in the expansion of $\left(x+\frac{1}{2x}\right)^{2n}$ is

(a) $\frac{1.3.5....(2n-3)}{n!}$ (b) $\frac{1.3.5....(2n-1)}{n!}$
(c) $\frac{1.3.5....(2n+1)}{n!}$ (d) $\frac{1.3.5....(2n+3)}{n!}$

PROPERTIES OF BINOMIAL COEFFICIENT

23. $C_0C_r+C_1C_{r+1}+C_2C_{r+2}+\ldots+C_{n-r}C_n=$

(a) $\frac{(2n)!}{(n-r)!(n+r)!}$ (b) $\frac{n!}{(-r)!(n+r)!}$
(c) $\frac{n!}{(n-r)!}$ (d) $\frac{2n!}{(n-r)!}$

24. $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}+\ldots+\frac{C_n}{n+1}=$

(a) $\frac{2^n}{n+1}$ (b) $\frac{2^n-1}{n+1}$
(c) $\frac{2^{n+1}-1}{n+1}$ (d) $\frac{2^{n+1}+1}{n+1}$

25. The sum of all the coefficients in the binomial expansion of $(x^2+x-3)^{319}$ is

(a) 1 (b) 2
(c) −1 (d) 0

26. The sum of the coefficients of even power of x in the expansion of $(1+x+x^2+x^3)^5$ is

(a) 256 (b) 128
(c) 512 (d) 64

27. The value of ${}^{15}C_0^2-{}^{15}C_1^2+{}^{15}C_2^2-\ldots-{}^{15}C_{15}^2$ is

(a) 15 (b) −15
(c) 0 (d) 51

APPLICATION OF BINOMIAL THEOREM

28. The greatest integer which divides the number $101^{100}-1$, is

(a) 100 (b) 1000
(c) 10000 (d) 100000

29. The last digit in 7^{300} is

(a) 7 (b) 9 (c) 1 (d) 3

30. The number 111 ... 1 (91 times) is

(a) Not a prime (b) An even number
(c) Not an odd number (d) Prime Number

31. The number $101^{100}-1$ is divisible by

(a) 100 (b) 1000
(c) 10000 (d) 100000

32. 7^9+9^7 is divisible by:

(a) 16 (b) 24 (c) 65 (d) 72

MULTINOMIAL THEOREM AND OTHER INDEX

33. In the expansion of $\left(\frac{1+x}{1-x}\right)^2$, the coefficient of x^n will be [where $|x|<1$]

(a) $4n$ (b) $4n-3$
(c) $4n+1$ (d) $4n-1$

34. The coefficient of two consecutive terms in the expansion of $(1+x)^n$ will be equal, if

(a) n is any integer (b) n is an odd integer
(c) n is an even integer (d) n is prime number

NUMERICAL FRACTION

35. Let $x=(5\sqrt{2}+7)^{19}$, then x{x}{.}.denotes the fractional part of x) is equal to

(a) 2^{19} (b) 3^{19} (c) 0 (d) 1

36. If $(4+\sqrt{15})^n = I + f$, where n is an odd natural number, I is an integer and $0 < f < 1$, then which of the following statements is false.

(*a*) I is natural number (*b*) I is an even integer

(*c*) $(I + f)(1 - f) = 1$ (*d*) $1 - f = (4-\sqrt{15})^n$

37. The total number of terms in the expansion of, $(x+a)^{100} + (x-a)^{100}$ after simplification is:

(*a*) 50 (*b*) 202 (*c*) 51 (*d*) 49

38. If the second term of the expansion $\left[a^{1/13} + \frac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$, then the value of $\frac{{}^nC_3}{{}^nC_2}$ is:

(*a*) 4 (*b*) 3 (*c*) 12 (*d*) 6

39. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$, the number of terms free from radicals is:

(*a*) 730 (*b*) 729 (*c*) 725 (*d*) 750

40. The value of m, for which the coefficients of the $(2m+1)^{th}$ and $(4m+5)^{th}$ terms in the expansion of $(1+x)^{10}$ are equal, is

(*a*) 3 (*b*) 1 (*c*) 5 (*d*) 8

41. The greatest terms of the expansion $(2x+5y)^{13}$ when $x = 10, y = 2$ is

(*a*) ${}^{13}C_5 . 20^8 . 10^5$ (*b*) ${}^{13}C_6 . 20^7 . 10^4$

(*c*) ${}^{13}C_4 . 20^9 . 10^4$ (*d*) ${}^{13}C_2 . 20^6 . 10^5$

42. Find numerically greatest term in the expansion of $(2+3x)^9$, when $x = 3/2$.

(*a*) ${}^9C_6 . 2^9 . (3/2)^{12}$ (*b*) ${}^9C_3 . 2^9 . (3/2)^6$

(*c*) ${}^9C_5 . 2^9 . (3/2)^{10}$ (*d*) ${}^9C_4 . 2^9 . (3/2)^8$

43. The remainder when 2^{2003} is divided by 17 is:

(*a*) 1 (*b*) 2 (*c*) 8 (*d*) 11

44. The greatest integer less than or equal to $(\sqrt{2}+1)^6$ is

(*a*) 196 (*b*) 197 (*c*) 198 (*d*) 199

Exercise-2 (Learning Plus)

1. The $(m+1)^{th}$ term of $\left(\frac{x}{y}+\frac{y}{x}\right)^{2m+1}$ is:

(*a*) Independent of x

(*b*) A constant

(*c*) Depends on the ratio x/y and m

(*d*) Independent of y

2. If the 6th term in the expansion of the binomial $\left[\frac{1}{x^{8/3}} + x^2 \log_{10} x\right]^8$ is 5600, then $x =$

(*a*) 10 (*b*) 8

(*c*) 11 (*d*) 9

3. The co-efficient of x in the expansion of $(1 - 2x^3 + 3x^5)\left(1+\frac{1}{x}\right)^8$ is:

(*a*) 56 (*b*) 65

(*c*) 154 (*d*) 62

4. The term independent of x in the expansion of $\left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$ is:

(*a*) –3 (*b*) 0

(*c*) 1 (*d*) 3

5. If the coefficients of x^7 & x^8 in the expansion of $\left[2+\frac{x}{3}\right]^n$ are equal, then the value of n is

(*a*) 15 (*b*) 45

(*c*) 55 (*d*) 56

6. Number of rational terms in the expansion of $(\sqrt{2}+\sqrt[4]{3})^{100}$ is

(*a*) 25 (*b*) 26 (*c*) 27 (*d*) 28

7. If $k \in R$ and the middle term of $\left(\frac{k}{2}+2\right)^8$ is 1120, then value of k is:

(*a*) 3 (*b*) 2 (*c*) – 3 (*d*) – 4

8. The sum of the binomial coefficients of $\left[2x+\frac{1}{x}\right]^n$ is equal to 256. The constant term in the expansion is

(*a*) 1120 (*b*) 2110 (*c*) 1210 (*d*) 1102

9. The sum of the co-efficients in the expansion of $(1 - 2x + 5x^2)^n$ is 'a' and the sum of the co-efficients in the expansion of $(1+x)^{2n}$ is b. Then

(*a*) $a = b$ (*b*) $a = b^2$ (*c*) $a^2 = b$ (*d*) $ab = 1$

10. The binomial expansion of $\left(x^k + \frac{1}{x^{2k}}\right)^{3n}$, $n \in N$ contains a term independent of x

(*a*) Only if k is an integer

(*b*) Only if k is a natural number

(*c*) Only if k is rational

(*d*) For any real k

11. If the expansion of $(3x+2)^{-1/2}$ is valid in ascending powers of x, then x lies in the interval.

(a) $(-2/3, 2/3)$ (b) $(-3/2, 3/2)$
(c) $(-1/3, 2/3)$ (d) $(-\infty, -3/2) \cup (3/2, \infty)$

12. Coefficient of α^t in the expansion of $(\alpha+p)^{m-1} + (\alpha+p)^{m-2}(\alpha+q) + (\alpha+p)^{m-3}(\alpha+q)^2 + \ldots (\alpha+q)^{m-1}$ where $\alpha \neq -q$ and $p \neq q$ is

(a) $\dfrac{{}^mC_t(p^t-q^t)}{p-q}$ (b) $\dfrac{{}^mC_t(p^{m-t}-q^{m-t})}{p-q}$
(c) $\dfrac{{}^mC_t(p^t+q^t)}{p-q}$ (d) $\dfrac{{}^mC_t(p^{m-t}+q^{m-t})}{p-q}$

13. The numerically greatest term in the expansion of $(2x+5y)^{34}$, when $x=3$ & $y=2$ is :

(a) T_{21} (b) T_{22}
(c) T_{23} (d) T_{24}

14. $\displaystyle\sum_{r=0}^{n-1} \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} =$

(a) $\dfrac{n}{2}$ (b) $\dfrac{n+1}{2}$
(c) $(n+1)\dfrac{n}{2}$ (d) $\dfrac{n(n-1)}{2(n+1)}$

15. Set of values of r for which, ${}^{18}C_{r-2} + 2.{}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ contains

(a) 4 elements (b) 5 elements
(c) 7 elements (d) 10 elements

16. $\dfrac{C_0}{1} + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots\ldots + \dfrac{C_{10}}{11} = (n=10)$

(a) $\dfrac{2^{11}}{11}$ (b) $\dfrac{2^{11}-1}{11}$
(c) $\dfrac{3^{11}}{11}$ (d) $\dfrac{3^{11}-1}{11}$

17. The value of the expression ${}^{47}C_4 + \displaystyle\sum_{j=1}^{5} {}^{52-j}C_3$ is equal to:

(a) ${}^{47}C_5$ (b) ${}^{52}C_5$
(c) ${}^{52}C_4$ (d) ${}^{49}C_4$

18. The value of $\dbinom{50}{0}\dbinom{50}{1} + \dbinom{50}{1}\dbinom{50}{2} + \ldots + \dbinom{50}{49}\dbinom{50}{50}$ is,

where ${}^nC_r = \dbinom{n}{r}$

(a) $\dbinom{100}{50}$ (b) $\dbinom{100}{51}$
(c) $\dbinom{50}{25}$ (d) $\dbinom{50}{25}^2$

19. The last two digits of the number 3^{400} are:

(a) 81 (b) 43 (c) 29 (d) 01

20. The co-efficient of x^4 in the expansion of $(1-x+2x^2)^{12}$ is

(a) ${}^{12}C_3$ (b) ${}^{13}C_3$
(c) ${}^{14}C_4$ (d) ${}^{12}C_3 + 3\,{}^{13}C_3 + {}^{14}C_4$

21. The value of ${}^nC_0 \cdot {}^nC_n + {}^nC_1 \cdot {}^nC_{n-1} + \ldots + {}^nC_n \cdot {}^nC_0$ is

(a) ${}^{2n}C_{n-1}$ (b) ${}^{2n}C_n$
(c) ${}^{2n}C_{n+1}$ (d) ${}^{2n}C_2$

22. The value of r for which ${}^{30}C_r \cdot {}^{20}C_0 + {}^{30}C_{r-1} \cdot {}^{20}C_1 + \ldots + {}^{30}C_0 \cdot {}^{20}C_r$ is maximum, is

(a) 10 (b) 15
(c) 20 (d) 25

23. The value of p, for which coefficient of x^{50} in the expression $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \ldots + 1001\,x^{1000}$ is equal to ${}^{1002}C_p$, is:

(a) 10 (b) 50
(c) 20 (d) 25

24. The number of values of 'r' satisfying the equation, ${}^{39}C_{3r-1} - {}^{39}C_{r2} = {}^{39}C_{r2-1} - {}^{39}C_{3r}$ is

(a) 1 (b) 2
(c) 3 (d) 4

25. 2^{60} when divided by 7 leaves the remainder equal to

(a) 0 (b) 1
(c) 2 (d) 3

26. The last digit of the number 3^{400} is equal to ...

(a) 0 (b) 1
(c) 2 (d) 3

27. The sum $\displaystyle\sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i}$, (where $\binom{p}{q} = 0$ if $p<q$) is maximum when m is

(a) 10 (b) 15
(c) 20 (d) 25

28. How many terms in the expansion of $\left(y^{\frac{1}{3}} + x^{\frac{1}{10}}\right)^{55}$ are free from radical sign?

(a) 1 (b) 2 (c) 3 (d) 6

29. Let k be the coefficient of x^{18} in the polynomial $f(x) = (1+x)^{20} + x(1+x)^{19} + x^2(1+x)^{18} + . + x^{18}(1+x)^2$ then the value of $\dfrac{k}{190}$ is equal to

(a) 1300 (b) 1310 (c) 1330 (d) 1340

30. If $\displaystyle\sum_{r-1}^{10} r(r-1)\,{}^{10}C_r = k \cdot 2^9$, then k is equal to

(a) 10 (b). 45 (c) 90 (d) 100

31. The sum of the coefficients of even power of x in the expansion of $(1+x+x^2+x^3)^5$ is

(a) 256 (b) 128 (c) 512 (d) 64

32. Let $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2} + \ldots x, n \in R$. Then the sum of the series $3 + \frac{8}{3} + \frac{80}{3^3} + \frac{240}{3^4} + \ldots$ is

(a) 9 (b) 27 (c) 12 (d) 101

33. $13^{99} - 19^{93}$ is divisible by 162, then the remainder is

(a) 1 (b) 0 (c) 3 (d) 4

34. The remainder when $6^{83} + 8^{83}$ is divided by 49 is equal to

(a) 1 (b) 2 (c) 5 (d) 4

35. If $x + y = 1$, then $\sum_{k=0}^{n} k^2\, {}^nC_k \cdot x^k \cdot y^{n-k} =$

(a) $nxy - n^2x^2$ (b) $nxy + n^2x^2$

(c) 0 (d) None of these

36. $C_1 - C_3 + C_5 - \ldots = 2^{n/2} \sin\frac{n\pi}{4}$

(a) $2^{n/2}\cos\frac{n\pi}{4}$ (b) $2^{n/2}\sin\frac{n\pi}{4}$

(c) 0 (d) None of these

37. Find the value of x for which the fourth term in the expansion $\left(5^{\frac{2}{5}\log_{\sqrt{4^x}}+44} + \frac{1}{5^{\log_5 \sqrt[3]{2^{x-1}+7}}}\right)^8$ is 336.

(a) 4 (b) 5 (c) 6 (d) 1

38. Number of terms free from radical sign in the expansion o $(1 + 3^{1/3} + 7^{1/7})^{10}$ is

(a) 4 (b) 5 (c) 6 (d) 8

39. Find last three digits in 19^{100}

(a) 004 (b) 001 (c) 002 (d) 011

40. If m is a positive integer next above $(\sqrt{3+1})^{2m}$ contains a a factor

(a) 2^{m+1} (b) 2^m (c) $2^{(m+1)/2}$ (d) $2^{m/2}$

41. Let $S_n = \sum_{r=0}^{n}(-2)^r\left(\frac{{}^nC_r}{{}^{r+2}C_r}\right)$, then

(a) $S_n = \frac{1}{n+1}$ if n is odd (b) $S_n = \frac{1}{n+2}$ if n is odd

(c) $S_n = \frac{1}{n+1}$ if n is even (d) $S_n = \frac{1}{n+2}$ if n is even

42. Let $(1 + x^2)^2 (1 + x)^n = A_0 + A_1x + A_2x^2 + \ldots$. If A_0, A_1, A are in $A.P.$ then the value of n is

(a) 2 (b) 3 (c) 5 (d) 7

43. If $(9 + \sqrt{80})^n = I + f$, where I, n are positive integers and $< f < 1$, then:

(a) I is an odd integer (b) I is an even integer

(c) $(I + f)(1 - f) = 1$ (d) $1 - f = (9 - \sqrt{80})^n$

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. The largest real value for x such that

$$\sum_{k=0}^{4}\left(\frac{5^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right) = \frac{8}{3} \text{ is}$$

(a) $2\sqrt{2} - 5$ (b) $2\sqrt{2} + 5$

(c) $-2\sqrt{2} - 5$ (d) $-2\sqrt{2} + 5$

2. Sum of the series $S = \sum_{r=0}^{n}\frac{3^{r+4}\left({}^nC_r\right)}{{}^{n+4}C_4} + \sum_{r=0}^{3}\frac{{}^{n+4}C_r 3^r}{{}^{n+4}C_4}$ is

(a) 4^{n+4} (b) $4^{n+4}\left({}^{2n}C_4\right)$

(c) $\frac{4^{n+4}}{{}^{n+4}C_4}$ (d) $\frac{3^{n+4} + 2^{n+4}}{{}^{n+4}C_4}$

3. Suppose m and n are positive integers and let

$$S = \sum_{k=0}^{n}(-1)^k \frac{1}{k+m+1}\left({}^nC_k\right) \text{ and}$$

$$T = \sum_{k=0}^{m}(-1)^k \frac{1}{k+n+1}\left({}^mC_k\right), \text{ then } S - T \text{ is equal to}$$

(a) 0 (b) $n^m - m^n$

(c) $(n+1)^m - (m+1)^n$ (d) $(1-n)^m - (1-m)^n$

4. Let $S = \sum_{r=0}^{n}\frac{{}^nC_r}{\sum_{s=0}^{r}\frac{{}^sC_r}{\sum_{t=0}^{s}{}^sC_t\left(2\sin\frac{\pi}{10}\right)^t}}$

Then $S^{1/n}$ is equal to

(a) $\frac{1}{\alpha^2+\alpha\beta}, \frac{1}{\beta^2+\alpha\beta}$ (b) $2\cos(\pi/10)$

(c) $2\sin(\pi/10)$ (d) $2\sin(\pi/5)$

5. Suppose $[x]$ denote the greatest integer $\leq$, and $n \in N$, then $\lim_{n\to\infty} \frac{\left[{}^nC_0x^2\right]+\left[{}^nC_1x^2\right]+\cdots+\left[{}^nC_nx^2\right]}{2^n-2}$ is equal to-

(a) $\frac{1}{2}x^2$ (b) x^2 (c) $2x^2$ (d) $4x^2$

6. The remainder on dividing $1234^{567} + 89^{1011}$ by 12 is

(a) 1 (b) 5
(c) 9 (d) None of these

7. The value of $\sum_{r=0}^{s}\sum_{\substack{s=1\\ r\leq s}}^{n} {}^nC_s\,{}^sC_r$ is

(a) $3^n - 1$ (b) $3^n + 1$
(c) 3^n (d) $3(3^n - 1)$

8. Let $N = 2^{1224} - 1$, $\alpha = 2^{153} + 2^{77} + 1$ and $\beta = 2^{408} - 2^{204} + 1$. Then which of the following statement is correct ?

(a) α divides N but β does not
(b) β divides N but α does not
(c) α and β both divide N
(d) Neither α nor β divides N

9. The remainder when $2^{30}.3^{20}$ is divided by 7 is:

(a) 1 (b) 2 (c) 4 (d) 6

10. The coefficient of x^{83} in $(1 + x + x^2 + x^3 + x^4)^n (x - 1)^{n+3}$ is equal to

(a) ${}^nC_7(-1)^n$ (b) $(-1)^n\,{}^nC_{16}$
(c) ${}^nC_{13}$ (d) ${}^nC_9$

11. If recursion polynomials $P_k(x)$ are defined as $P_1(x) = (x-2)^2$, $P_2(x) = ((x-2)^2 - 2)^2$

$P_3(x) = ((x-2)^2 - 2)^2 - 2)^2$ ... (In general $P_k(x) = (P_{k-1}(x) - 2)^2$, then the constant term in $P_k(x)$ is

(a) 4 (b) 2
(c) 16 (d) A perfect square

12. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20}x^{20}$, then:

(a) $a_1 = 20$ (b) $a_2 = 210$
(c) $a_4 = 8085$ (d) $a_{20} = 2^2.\,3^7.\,7$

13. The value of $r(0 \leq r \leq 30)$ for which ${}^{20}C_r.\,{}^{10}C_0 + {}^{20}C_{r-1}.\,{}^0C_1 + ... + {}^{20}C_0.\,{}^{10}C_r$ is minimum, is

(a) 0 (b) 1 (c) 5 (d) 15

COMPREHENSION BASED QUESTIONS

Comprehension (No. 14 to 16)

We know that of ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ be binomial coefficients, then $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ...+ C_nx^n$. Various relations among binomial coefficients can be derived by putting $x = 1, -1, i, \omega$, $\left(\text{where } i = \sqrt{-1},\ \omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$.

14. The value of ${}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + ...$ must be

(a) $2^{n/2}\cos\frac{n\pi}{2}$ (b) $2^{n/2}\sin\frac{n\pi}{2}$
(c) $2^{n/2}\cos\frac{n\pi}{4}$ (d) $2^{n/2}\sin\frac{n\pi}{4}$

15. The value of expression $({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + ...)^2 + ({}^nC_1 - {}^nC_3 + {}^nC_5 ...)^2$ must be

(a) 2^{2n} (b) 2^n (c) 2^{n^2} (d) $2^{n/2}$

16. The value of ${}^nC_0 + {}^nC_3 + {}^nC_6 + ...$ must be

(a) $\frac{2^n}{3}$ (b) $\frac{1}{3}\left(2^n + \cos\frac{n\pi}{3}\right)$
(c) $\frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$ (d) $\frac{1}{3}\left(2^n + 2\sin\frac{n\pi}{3}\right)$

MATCH THE COLUMN TYPE QUESTIONS

17. Match the column:

	Column-I		Column-II
A.	The sum may be $\sum_{0\leq i<j\leq n}\sum({}^nC_i + {}^nC_j)^2$	p.	$\frac{1}{2}\left[2^{2n} - {}^{2n}C_n\right]$
B.	The sum may be $\sum_{0\leq i<j\leq n}\sum({}^nC_i \cdot {}^nC_j)$	q.	$n^2 \cdot 2^{n-1}$
C.	The sum may be $\frac{1}{n}\sum_{0\leq i<j\leq n}\sum(i+j)({}^nC_i \cdot {}^nC_j)$	r.	$\sum_{r=0}^{n-1} {}^{2n}C_r$
D.	The sum $\sum_{0\leq i<j\leq n}\sum(i.{}^nC_i \cdot + j \cdot {}^nC_j)$ may be	s.	$(n-1)^{2n}Cn + 2^{2n}$

(a) A → s, B → p, C → p, D → q
(b) A → r, B → q, C → p, D → s
(c) A → p, B → s, C → r, D → q
(d) A → s, B → q, C → r, D → p

18. Match the columns:

	Column-I		Column-II
A.	If $f(n) = \sum_{r=1}^{n} r \cdot {}^nC_r$ than f(1) =	p.	1
B.	If $f(n) = \sum_{r=1}^{n} r^2 \cdot {}^nC_r$ then $f(2) =$	q.	6
C.	If $f(n) = \sum_{r=1}^{n} (-1)^r r^2 \cdot C_r$ then $f(3)$	r.	0
D.	If $f(n) = \sum_{r=1}^{n} r^3 \cdot {}^nC_r$ then $\frac{f(4)}{32} =$	s.	7

(a) A → p, B → q, C → r, D → s
(b) A → r, B → q, C → p, D → s
(c) A → p, B → s, C → r, D → q
(d) A → s, B → q, C → r, D → p

19. Match the Column:

	Column-I		Column-II
A.	If $(r+1)^{th}$ term is the first negative term in the expansion of $(1+x)^{7/2}$, then the value of r (where $\|x\|<1$) is	p.	Divisible by 2
B.	The coefficient of y in the expansion of $(y^2+1/y)^5$ is	q.	Divisible by 5
C.	${}^nC_r$ is divisible by n, $(1<r<n)$ if n is	r.	Divisible by 10
D.	The coefficient of x^4 in the expansion of $(1+2x+3x^2+4x^3+\ldots$ up to $\infty)^{1/2}$ is c, $(c \in N)$, then $c+1$ (where $\|x\|<1$) is	s.	A prime number

(a) A → p, B → q, C → r, D → p, q, r
(b) A → q, s, B → p, q, r, C → s, d, D → p, s
(c) A → p, B → p, q, r, C → r, D → q
(d) A → p, q, r, B → q, C → r, D → p

20. Match the Column:

	Column-I		Column-II
A.	If $x=(7+4\sqrt{3})^{2n}=[x]+f$, then $x(1-f)=$	p.	6
B.	If second, third and fourth terms in the expansion of $(x+a)^n$ are 240, 720 and 1080 respectively, then n is equal to	q	1
C.	Value of ${}^4C_0{}^4C_4-{}^4C_1{}^4C_3+{}^4C_2{}^4C_2-{}^4C_3{}^4C_1+{}^4C_4{}^4C_0$ is	r.	2
D.	If x is very large as compare to y, then the value of k in $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}}=1+\frac{y^2}{kx^2}$	s.	5

(a) A → p, B → q, C → r, D → s
(b) A → r, B → q, C → p, D → s
(c) A → q, B → s, C → p, D → r
(d) A → s, B → q, C → r, D → p

21. Match the columns:

	Column-I		Column-II
A.	The coefficient of two consecutive terms in the expansion of $(1+x)^n$ will be equal then n can be	p.	9
B.	If 15^n+23^n is divided by 19, then n can be	q.	10
C.	${}^{10}C_0\,{}^{20}C_{10}-{}^{10}C_1\,{}^{18}C_{10}+{}^{10}C_2\,{}^{16}C_{10}-\ldots$ is divisible by 2^n, then n can be	r.	11
D.	If the coefficients of T_r, T_{r+1}, T_{r+2} terms of $(1+x)^{14}$ are in A.P., then r is less than	s.	12
		t.	14

(a) A → p, B → q, C → r, D → p, q, r
(b) A → q, s, B → p, q, r, C → s, d, D → p, s
(c) A → p, B → p, q, r, C → r, D → q
(d) A → p, r, B → p, r, C → p, q, D → q, r, s, t

NUMERICAL BASED QUESTIONS

22. The index 'n' of the binomial $\left(\frac{x}{5}+\frac{2}{5}\right)^n$ if for $x=1$ the 9th term of the expansion has numerically the greatest coefficient $(n \in N)$, is:

23. If $(6\sqrt{6}+14)^{2n+1}=N$, and if F be the fractional part of N, NF is

24. Let $(x^3+px^2+2x-5)^{19}(x^2+qx-41)^8(x^4-x^3+x-7)^6=x^{97}+391x^{96}+a^{95}x^{95}+a^{94}x^{94}+\ldots a_1x+a_0$ be an identity, where $p, q, a_{95} \ldots a_0$ are integers. Compute the smallest positive value of p.

25. Show that $\sum_{r=0}^{n}\frac{C_r\cdot 3^{r+4}}{(r+1)(r+2)(r+3)(r+4)}$

$$=\frac{1}{(n+1)(n+2)(n+3)(n+4)}\left[4^{n+4}-\sum_{s=0}^{3}{}^{n+4}C_s\cdot 3^s\right]$$

26. If C_r are the binomial co-efficient in the expansion of $(1+x)^n$, the value of $\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)C_iC_j$ is

27. Prove that $2222^{5555}+5555^{2222}$ is divisible by 7.

28. Prove that $\sum_{m=0}^{n}{}^nC_m\cdot\sin(mx)\cdot\cos(n-m)x=2^{n-1}\sin(nx)$

29. Find the remainder when 32^{3232} is divided by 7.

30. Let $N_1=2^{55}+1$ and $N_2=165$. Then sum of digits of the HCF of N_1 and N_2 is

31. Real value for x such that $\sum_{k=0}^{4}\left(\frac{3^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)=\frac{32}{3}$ is

32. Find the sum of possible real values of x for which the term of $\left(3^{\log_3\sqrt{9|x-2|}}+7^{\frac{1}{5}\log_7(3^{|x-2|-9})}\right)$ equal 567

33. Prove that : ${}^nC_0\cdot{}^{2n}C_n-{}^nC_1\cdot{}^{2n-2}C_n+{}^nC_2\cdot{}^{2n-4}C_n+\ldots=2^n$

34. Find the number of divisors of the number
$N={}^{2000}C_1+2\cdot{}^{2000}C_2+3\cdot{}^{2000}C_3+\ldots\ldots+2000\cdot{}^{2000}C_{2000}$

35. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \geq n$, then show that $b_n = {}^{2n+1}C_{n+1}$.

36. Prove that $\sum_{r=1}^{k}(-3)^{r-1}3n_{C_{2r-1}} = 0$, where $k = (3n)/2$ and n is an even positive integer.

37. If ${}^nJ_r = \frac{(1-x^n)(1-x^{n-1})...(1-x^{n-r+1})}{(1-x)(1-x^2)(1-x^3)...(1-x^r)}$, prove that ${}^nJ_{n-r} = {}^nJ_r$

38. Find the sum of following series:

(i) $\sum_{r=0}^{49} {}^{100}C_r$ (ii) $\sum_{r=0}^{24} {}^{100}C_{4r+1}$

(iii) $\sum_{r=0}^{49} {}^{99}C_r$ (iv) $\sum_{r=0}^{n} {}^{2n+1}C_{2r+1}$

39. Find the sum of following series:

(i) $\sum_{r=1}^{51} r\left({}^{(1001-r)}C_{950}\right)$ (ii) $\sum_{r=0}^{10} {}^{100+r}C_r$

(iii) $\sum_{r=0}^{20} {}^{m+n+r}C_r$

40. Find the sum of following series:

(i) $\sum_{r=0}^{n} {}^nC_r(-1)^r\left(\frac{1}{2^r}+\frac{1}{3^r}+\frac{1}{4^r}\right)$

(ii) $\sum_{r=0}^{n} {}^nC_r(-1)^r\left(\frac{1}{2^r}+\frac{3^r}{3^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+....\text{ upto } m \text{ terms}\right)$

41. Prove That: $\sqrt{C_1}+\sqrt{C_2}+\sqrt{C_3}+...+\sqrt{C_n} \leq 2^{n-1}+\frac{n-1}{2}$

42. If then $\sum_{r=0}^{n}\frac{2r}{{}^nC_r} = \sum_{r=0}^{n}\frac{n^2-3n+3}{{}^nC_r}$ then find n.

Exercise-4 (Past Year Questions)

JEE MAIN

1. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n, x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is: **(2016)**

(a) 64 (b) 2187 (c) 243 (d) 729

2. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + ...+ ({}^{21}C_{10} - {}^{10}C_{10})$ is **(2017)**

(a) $2^{20} - 2^{10}$ (b) $2^{21} - 2^{11}$

(c) $2^{21} - 2^{10}$ (d) $2^{20} - 2^9$

3. The sum of the co-efficients of all odd degree terms in the expansion of $(x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5$, $(x > 1)$ is **(2018)**

(a) 0 (b) 1

(c) 2 (d) –1

4. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to: **(2019)**

(a) 6 (b) 8

(c) 4 (d) 14

5. The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is **(2019)**

(a) 12 (b) 15 (c) 10 (d) 14

6. If $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i + {}^{20}C_{i-1}}\right)^3 = \frac{k}{21}$, then k equals: **(2019)**

(a) 400 (b) 50 (c) 200 (d) 100

7. If the third term in the binomial expansions of $(1 + x^{\log 2x})^5$ equals 2560, then a possible value of x is: **(2019)**

(a) $\frac{1}{4}$ (b) $4\sqrt{2}$ (c) $\frac{1}{8}$ (d) $2\sqrt{2}$

8. The positive value of λ for which the co-efficient of x^2 in the expression $x^2\left(\sqrt{x}+\frac{\lambda}{x^2}\right)^{10}$ is 720, is: **(2019)**

(a) 4 (b) $2\sqrt{2}$ (c) $\sqrt{5}$ (d) 3

9. If $\sum_{r=0}^{25}\left\{{}^{50}C_r.{}^{50-r}C_{25-r}\right\} = K\left({}^{50}C_{25}\right)$, then K is equal to **(2019)**

(a) $(25)^2$ (b) $2^{25} - 1$ (c) 2^{24} (d) 2^{25}

10. Let $S_n = 1 + q + q^2 + ...+q^n$ and $T_n = 1+\left(\frac{q+1}{2}\right)+\left(\frac{q+1}{2}\right)^2 +...+\left(\frac{q+1}{2}\right)^n$ where q is a real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2.S_1 + ...+{}^{101}C_{101}.S_{100} = \alpha T_{100}$ then α is equal to: **(2019)**

(a) 2^{99} (b) 202 (c) 200 (d) 2^{100}

11. Let $(x+10)^{50}+(x-10)^{50}=a_0+a_1x+a_2x^2+\ldots+a_{50}x^{50}$, for $x \in R$: then $\frac{a_2}{a_0}$ is equal to: **(2019)**

(*a*) 12.50 (*b*) 12.00 (*c*) 12.25 (*d*) 12.75

12. The value of r for which ${}^{20}C_r\,{}^{20}C_0+{}^{20}C_{r-1}\,{}^{20}C_1+{}^{20}C_{r-2}\,{}^{20}C_2+\ldots\ldots{}^{20}C_0\,{}^{20}C_r$ is maximum is: **(2019)**

(*a*) 15 (*b*) 20 (*c*) 11 (*d*) 10

13. The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3}+\frac{3}{x}\right)^8$ equals 5670 is: **(2019)**

(*a*) 0 (*b*) 6 (*c*) 4 (*d*) 8

14. A ratio of the 5th term from the beginning to the 5th term from the end in the binomial expansion of $\left(2^{\frac{1}{3}}+\frac{1}{2(3)^{\frac{1}{3}}}\right)^{10}$ is: **(2019)**

(*a*) $1:2(6)^{\frac{1}{3}}$ (*b*) $1:4(16)^{\frac{1}{3}}$ (*c*) $4(36)^{\frac{1}{3}}:1$ (*d*) $2(36)^{\frac{1}{3}}:1$

15. The total number of irrational terms in the binomial expansion of $(7^{1/5}-3^{1/10})^{60}$ is: **(2019)**

(*a*) 55 (*b*) 49 (*c*) 48 (*d*) 54

16. If ${}^nC_4, {}^nC_5, and\ {}^nC_6$ are in *A.P.*, then n can be: **(2019)**

(*a*) 9 (*b*) 14 (*c*) 11 (*d*) 12

17. The sum of the series $2.{}^{20}C_0+5.{}^{20}C_1+8.{}^{20}C_2+11.{}^{20}C_3+\ldots+62.{}^{20}C_{20}$ is equal to: **(2019)**

(*a*) 2^{24} (*b*) 2^{25} (*c*) 2^{26} (*d*) 2^{23}

18. The sum of the co-efficients of all even degree terms in x in the expansion of $(x+\sqrt{x^3-1})^6+(x-\sqrt{x^3-1})^6$, $(x>1)$ is equal to: **(2019)**

(*a*) 32 (*b*) 26 (*c*) 29 (*d*) 24

19. If the fourth term in the binomial expansion of $\left(\sqrt{\frac{1}{x^{1+\log_{10}x}}}+x^{\frac{1}{12}}\right)^6$ is equal to 200, and $x>1$, then the value of x is: **(2019)**

(*a*) 10^3 (*b*) 100 (*c*) 10^4 (*d*) 10

20. If the fourth term in the binomial expansion of $\left(\frac{2}{x}+x^{\log_8 x}\right)^6 (x>0)$ is 20×8^7, then *a* value of x **(2019)**

(*a*) 8 (*b*) 8^2 (*c*) 8^{-2} (*d*) 8^3

21. If coefficient of some three consecutive terms in the binomial expansion of $(x+1)^n$ are in the ratio 2 : 15 : 70, then the average of these three coefficient is: **(2019)**

(*a*) 964 (*b*) 625 (*c*) 227 (*d*) 232

22. The smallest natural number n, such that the coefficient of x in the expansion of $\left(x^2+\frac{1}{x^3}\right)^n$ is ${}^nC_{23}$, is: **(2019)**

(*a*) 35 (*b*) 38 (*c*) 23 (*d*) 58

23. If the coefficients of x^2 and x^3 are both zero, in the expansion of the expression $(1+ax+bx^2)(1-3x)^{15}$ in powers of x, then the ordered pair (a, b) is equal to: **(2019)**

(*a*) (28, 315) (*b*) (– 54, 315)
(*c*) (–21, 714) (*d*) (24, 861)

24. The coefficient of x^{18} in the product $(1+x)(1-x)^{10}(1+x+x^2)^9$ is: **(2019)**

(*a*) – 84 (*b*) 84 (*c*) 126 (*d*) – 126

25. The term independent of x in the expansion of $\left(\frac{1}{60}-\frac{x^8}{81}\right)\cdot\left(2x^2-\frac{3}{x^2}\right)^6$ is equal to: **(2019)**

(*a*) 36 (*b*) – 108 (*c*) – 72 (*d*) – 36

26. If the sum of the coefficients of all even powers of x in the product $(1+x+x^2+\ldots+x^{2n})(1-x+x^2-x^3+\ldots+x^{2n})$ is 61, then n is equal to **(2020)**

27. The number of ordered pairs (r, k) for which $6.{}^{35}C_r=(k^2-3).{}^{36}C_{r+1}$, where k is an integer, is: **(2020)**

(*a*) 3 (*b*) 6 (*c*) 4 (*d*) 2

28. The coefficient of x^7 in the expression $(1+x)^{10}+x(1+x)^9+x^2(1+x)^8+\ldots\ldots+x^{10}$ is:

(*a*) 210 (*b*) 330 (*c*) 420 (*d*) 120

29. If a, b and c are the greatest values of ${}^{19}C_p$, ${}^{20}C_q$ and ${}^{21}C_r$ respectively, then **(2020)**

(*a*) $\frac{a}{10}=\frac{b}{11}=\frac{c}{42}$ (*b*) $\frac{a}{10}=\frac{b}{11}=\frac{c}{21}$

(*c*) $\frac{a}{11}=\frac{b}{22}=\frac{c}{42}$ (*d*) $\frac{a}{11}=\frac{b}{22}=\frac{c}{21}$

30. The coefficient of x^4 in the expansion of $(1+x+x^2)^{10}$ is____ **(2020)**

31. In the expansion of $\left(\frac{x}{\cos\theta}+\frac{1}{x\sin\theta}\right)^{16}$, if I_1 is the least value of the term independent of x when $\frac{\pi}{8}\le\theta\le\frac{\pi}{4}$ and I_2 is the least value of the term independent of x when $\frac{\pi}{16}\le\theta\le\frac{\pi}{8}$, then the ratio $I_2:I_1$ is equal to: **(2020)**

(*a*) 1 : 8 (*b*) 1 : 16 (*c*) 8 : 1 (*d*) 16 : 1

32. If $C_r = {}^{25}C_r$ and $C_0 + 5.C_1 + 9.C_2 + ...+ (101).C_{25} = 2^{25}.k$, then k is equal to _____ **(2020)**

33. For a positive integer n, $\left(1+\frac{1}{x}\right)^n$ is expanded in increasing powers of x. If three consecutive coefficients in this expansion are in the ratio, 2 : 5 : 12, then n is equal to _____. **(2020)**

34. Let $\alpha > 0$, $\beta > 0$ be such that $\alpha^3 + \beta^2 = 4$. If the maximum value of the term independent of x in the binomial expansion of $(\alpha x^{1/9} + \beta x^{-1/6})^{10}$ is $10K$, then K is equal to: **(2020)**

(a) 84 (b) 176 (c) 336 (d) 352

35. If the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is k, then $18k$ is equal to:

(a) 9 (b) 11 (c) 5 (d) 7

36. If the number of integral terms in the expansion of $(3^{1/2} + 5^{1/8})^n$ is exactly 33, then the least value of n is: **(2020)**

(a) 264 (b) 128 (c) 256 (d) 248

37. The value of $(2.{}^1P_0 - 3.{}^2P_1 + 4.{}^3P_2 - ...$ up to 51th term) + $(1! - 2! + 3! - ...$ up to 51th term) is equal to: **(2020)**

(a) 1 (b) 1 + (52)!
(c) 1 – 51(51)! (d) 1 + (51)!

38. If for some positive iteger n, the coefficients of three consecutive terms in the binomial expansion of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14, then the largest coefficient in this expansion is: **(2020)**

(a) 252 (b) 462 (c) 792 (d) 330

39. The value of $\sum_{r=0}^{20} {}^{50-r}C_6$ is equal to: **(2020)**

(a) ${}^{51}C_7 - {}^{30}C_7$ (b) ${}^{50}C_7 - {}^{30}C_7$
(c) ${}^{51}C_7 + {}^{30}C_7$ (d) ${}^{50}C_6 - {}^{30}C_6$

40. The coef ficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^6$ in powers of x, is _____. **(2020)**

41. The natural number m, for which the coefficient of x in the binomial expansion of $\left(x^m + \frac{1}{x^2}\right)^{22}$ is 1540, is _____. **(2020)**

42. If the constant term in the binomial expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, then $|k|$ equals: **(2020)**

(a) 2 (b) 3 (c) 9 (d) 1

43. The value of $-{}^5C_1 + 2\cdot{}^{15}C_2 - 3\cdot{}^{15}C_3 + ... -15.{}^{15}C_{15} + {}^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + ... + {}^{14}C_{11}$ is **(2021)**

(a) 2^{14} (b) $2^{13} - 13$ (c) $2^{16} - 1$ (d) $2^{13} - 14$

44. If $n \geq 2$ is a positive integer, then the sum of the series ${}^{n+1}C_2 + 2({}^2C_2 + {}^3C_2 + {}^4C_2 + ... + {}^nC_2)$ is: **(2021)**

(a) $\frac{n(n+1)^2(n+2)}{12}$ (b) $\frac{n(n-1)(2n+1)}{6}$

(c) $\frac{n(n+1)(2n+1)}{6}$ (d) $\frac{n(2n+1)(3n+1)}{6}$

45. For integers n and r, let $\binom{n}{r} = \begin{cases} {}^nC_r, & \text{if } n \geq r \geq 0 \\ 0, & \text{otherwise} \end{cases}$

The maximum value of k for which the sum $\sum_{i=0}^{k}\binom{10}{i}\binom{15}{k-i} + \sum_{i=0}^{k}\binom{12}{i}\binom{13}{k+1-i}$ exists, is equal to

Note: NTA has dropped this question in the final official answer key. **(2021)**

46. If the remainder when x is divided by 4 is 3, then the remainder when $(2020 + x)^{2022}$ is divided by 8 is **(2021)**

47. The maximum value of the term independent of 't' in the expansion of $\left(tx^{\frac{1}{5}} + \frac{(1-x)^{\frac{1}{10}}}{t}\right)^{10}$ where $x \in (0, 1)$ is: **(2021)**

(a) $\frac{10!}{\sqrt{3}(5!)^2}$ (b) $\frac{2.10!}{3(5!)^2}$

(c) $\frac{10!}{3(5!)^2}$ (d) $\frac{2.10!}{3\sqrt{3}(5!)^2}$

48. Let $m, n \in N$ and $\gcd(2, n) = 1$, If $30\binom{30}{0} + 29\binom{30}{1} + ... + 2\binom{30}{28} + 1\binom{30}{29} - n \cdot 2^m$, then $n + m$ is equal to **(2021)**

49. If n is the number of irrational terms in the expansion of $(3^{1/4} + 5^{1/8})^{60}$, then $(n - 1)$ is divisible by: **(2021)**

(a) 26 (b) 30 (c) 8 (d) 7

50. Let $[x]$ denote greatest integer less than or equal to x. If for $n \in \mathbb{N}$, $(1 - x + x^3)^n = \sum_{j=0}^{3n} a_j x^j$, then $\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} + 4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j} + 1$ is equal to: **(2021)**

(a) 2 (b) 2^{n+1} (c) 1 (d) n

51. Let n be a positive integer. Let

$$A = \sum_{k=0}^{n}(-1)^k n_{C_k}\left[\left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{7}{8}\right)^k + \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k\right]$$

If $63A = 1 - \frac{1}{2^{30}}$, then n is equal to _____ **(2021)**

52. If the fourth term in the expansion of $(x + x^{\log_2 x})^7$ is 4480, then the value of x where $x \in N$ is equal to: **(2021)**

(a) 2 (b) 4 (c) 3 (d) 1

53. If $(2021)^{3762}$ is divided by 17, then the remainder is ____. **(2021)**

54. Let the coefficients of third, fourth and fifth terms in the expansion of $\left(x + \frac{a}{x^2}\right)^n$, $x \neq 0$, be in the ratio 12 : 8 : 3. Then the term independent of x in the expansion, is equal to ____. **(2021)**

55. Let $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + ... + a_{40}x^{40}$ then $a_1 + a_3 + a_5 + ... + a_{37}$ is equal to **(2021)**

(a) $2^{20}(2^{20} - 21)$ (b) $2^{19}(2^{20} - 21)$
(c) $2^{19}(2^{20} + 21)$ (d) $2^{20}(2^{20} + 21)$

56. The term independent of x in the expansion of $\left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right]^{10}$, $x \neq 1$, is equal to ____ **(2021)**

57. Let ${}^nC_r$ denote the binomial coefficient of x^r in the expansion of $(1 + x)^n$.

If $\sum_{k=0}^{10}(2^2 + 2k)^n C_k = \alpha \cdot 3^{10} + \beta \cdot 2^{10}$, $\alpha, \beta \in R$, then $\alpha + \beta$ is equal to ____ **(2021)**

58. The coefficient of x^{101} in the expression $(5 + x)^{500} + x(5 + x)^{499} + x^2(5 + x)^{498} + ... + x^{500}$, $x > 0$, is **(2022)**

(a) ${}^{501}C_{101}(5)^{339}$ (b) ${}^{501}C_{101}(5)^{400}$
(c) ${}^{501}C_{100}(5)^{400}$ (d) ${}^{500}C_{101}(5)^{309}$

59. The remainder when $(2021)^{2023}$ is divided by 7 is **(2022)**

(a) 1 (b) 2 (c) 5 (d) 6

60. If $\sum_{k=1}^{31}({}^{31}C_k)({}^{31}C_{k-1}) - \sum_{k=1}^{30}({}^{30}C_k)({}^{30}C_{k-1}) = \frac{\alpha(60!)}{(30!)(31!)}$, where $\alpha \in R$, then the value of 16α is equal to **(2022)**

(a) 1411 (b) 1320 (c) 1615 (d) 1855

61. Let $n \geq 5$ be an integer. If $9^n - 8n - 1 = 64\alpha$ and $6^n - 5n - 1 = 25\beta$, then $\alpha - \beta$ is equal to **(2022)**

(a) $1 + {}^nC_2(8 - 5) + {}^nC_3(8^2 - 5^2) + ... + {}^nC_n(8^{n-1} - 5^{n-1})$
(b) $1 + {}^nC_3(8 - 5) + {}^nC_4(8^2 - 5^2) + ... + {}^nC_n(8^{n-2} - 5^{n-2})$
(c) ${}^nC_3(8 - 5) + {}^nC_4(8^2 - 5^2) + ... + {}^nC_n(8^{n-2} - 5^{n-2})$
(d) ${}^nC_4(8 - 5) + {}^nC_5(8^2 - 5^2) + ... + {}^nC_n(8^{n-3} - 5^{n-3})$

62. The remainder when $(11)^{1011} + (1011)^{11}$ is divided by 9 is **(2022)**

(a) 1 (b) 4 (c) 6 (d) 8

63. $\sum_{\substack{i,j=0 \\ i \neq j}}^{n} {}^nC_i \, {}^nC_j$ is equal to **(2022)**

(a) $2^{2n} - {}^{2n}C_n$ (b) $2^{2n-1} - {}^{2n-1}C_{n-1}$
(c) $2^n - \frac{1}{2}\,{}^{2n}C_n$ (d) $2^{n-1} + {}^{2n-1}C_n$

64. The remainder when $(2021)^{2022} + (2022)^{2021}$ is divided by 7 is **(2022)**

(a) 0 (b) 1 (c) 2 (d) 6

65. Let C_r denote the binomial coefficient of x^r in the expansion of $(1 + x)^{10}$. If $\alpha, \beta \in R$, $C_1 + {}^{3.2}C_2 + {}^{5.3}C_3 + ...$ up to 10 terms $= \frac{\alpha \times 2^{11}}{2^\beta - 1}\left(C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... \text{ upto 10 terms}\right)$, then the value of $\alpha + \beta$ is equal to ____. **(2022)**

66. If $({}^{40}C_0) + ({}^{41}C_1) + ({}^{42}C_2) + ... + ({}^{60}C_{20}) = \frac{m}{n}\,{}^{60}C_{20}$, m and n are coprime, then $m + n$ is equal to ____. **(2022)**

67. If the coefficient of x^{10} in the binomial expansion of $\left(\frac{\sqrt{x}}{5^{1/4}} + \frac{\sqrt{5}}{x^{1/3}}\right)^{60}$ is $5kl$, where $l, k \in N$ and l is co-prime to 5, then k is equal to ____. **(2022)**

68. Let the coefficients of the middle terms in the expansion of $\left(\frac{1}{\sqrt{6}} + \beta x\right)^4$, $(1 - 3\beta x)^2$ and $\left(1 - \frac{\beta}{2}x\right)^6$, $\beta > 0$, respectively form the first three terms of an A.P. If d is the common difference of this A.P., then $50 - \frac{2d}{\beta^2}$ is equal to ____. **(2022)**

69. If $1 + (2 + {}^{49}C_1 + {}^{49}C_2 + ... + {}^{49}C_{49})({}^{50}C_2 + {}^{50}C_4 + ... {}^{50}C_{50})$ is equal to $2n \cdot m$, where m is odd, then $n + m$ is equal to ____. **(2022)**

70. If $\sum_{K=1}^{10} K^2({}^{10}C_k)^2 = 22000L$, then L is equal to ____. **(2022)**

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71. If C_r stands for ${}^nC_r$, then the sum of the series $\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 - ... + (-1)^n(n+1)C_n^2]$, where n is an even positive integer is **(1986)**

(a) $(-1)^{n/2}(n + 2)$ (b) $(-1)^n(n + 1)$
(c) $(-1)^{n/2}(n + 1)$ (d) None of these

72. For any odd integer $n \geq 1$, $n^3 - (n - 1)^3 + ... + (-1)^{n-1}1^3 =$ **(1988)**

73. The expression $[x + (x^3 - 1)^{1/2}]^5 + [x - (x^3 - 1)^{1/2}]^5$ is a polynomial of degree **(1992)**

(*a*) 5 (*b*) 6

(*c*) 7 (*d*) 8

74. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then $n =$ **(2013)**

75. Coefficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$ is **(2014)**

(*a*) 1051 (*b*) 1106

(*c*) 1113 (*d*) 1120

76. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3) \ldots\ldots (1 + x^{100})$ is **(2015)**

77. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \ldots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)^{51}C_3$ for some positive integer n. Then the value of n is **(2016)**

78. Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \ldots + 10({}^{10}C_{10})^2$, where ${}^{10}C_r, r \in \{1, 2, \ldots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is____ **(2018)**

79. Suppose det $\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$, holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals **(2019)**

80. For non-negative integers s and r,

let $\binom{s}{r} = \begin{cases} \frac{s!}{r!(s-r)!} & \text{if } r \le s, \\ 0 & \text{if } r > s. \end{cases}$. For positive integers m and n, let $(m,n)\sum_{p=0}^{m+n} \frac{f(m,n,p)}{\binom{n+p}{p}}$

where for any nonnegative integer p,

$$f(m,n,p) = \sum_{i=0}^{p} \binom{m}{i}\binom{n+i}{p}\binom{p+n}{p-i}$$

Then which of the following statements is/are TRUE? **(2020)**

(*a*) $g(m, n) = g(n, m)$ for all positive integers m, n

(*b*) $g(m, n+1) = g(m+1, n)$ for all positive integers m, n

(*c*) $g(2m, 2n) = 2g(m, n)$ for all positive integers m, n

(*d*) $g(2m, 2n) = (g(m, n))^2$ for all positive integers m, n

ANSWER KEY

CONCEPT APPLICATION

1. (*b*) **2.** (*b*) **3.** (*b*) **4.** (*c*) **5.** (*a*) **6.** (*b*)

7. (*i*) ${}^{81}C_{21}$

(*ii*) ${}^{100}C_{11}$

(*iii*) ${}^{101}C_{11} - {}^{20}C_9$

8. (*i*) $3^n + 4^n + 5^n$

(*ii*) $\left(\frac{1}{2}\right)^2 + \left(\frac{2}{3}\right)^n + \left(\frac{3}{4}\right)^n$

9. (*a*) **10.** (*d*) **11.** (*c*) **12.** (*b*) **13.** (*c*) **14.** (*c*) **15.** (*d*) **16.** (*d*) **17.** (*b*) **18.** (*b*)

19. (*b*) **20.** (*c*) **21.** (*d*)

22. (*i*) 2^{2n}

(*ii*) $2^{98} - 2^{49}$

(*iii*) $\frac{2^{100} - {}^{100}C_{50}}{2}$

(*iv*) 2^{98}

23. (*a, c, d*) **24.** (*c*) **25.** (*a*) **26.** (*b*) **27.** (*d*) **28.** (*a*) **29.** (*a*) **30.** (*a*) **31.** (*b*) **32.** (*c*)

EXERCISE-1 (TOPICWISE)

1. (*b*)	2. (*c*)	3. (*c*)	4. (*c*)	5. (*c*)	6. (*b*)	7. (*b*)	8. (*a*)	9. (*c*)	10. (*c*)
11. (*b*)	12. (*d*)	13. (*b*)	14. (*a*)	15. (*a*)	16. (*a*)	17. (*a*)	18. (*a*)	19. (*c*)	20. (*b*)
21. (*c*)	22. (*b*)	23. (*a*)	24. (*c*)	25. (*c*)	26. (*c*)	27. (*c*)	28. (*c*)	29. (*c*)	30. (*a*)
31. (*d*)	32. (*a*)	33. (*a*)	34. (*b*)	35. (*d*)	36. (*b*)	37. (*c*)	38. (*a*)	39. (*a*)	40. (*b*)
41. (*c*)	42. (*a*)	43. (*c*)	44. (*b*)						

EXERCISE-2 (LEARNING PLUS)

1. (*c*)	2. (*a*)	3. (*c*)	4. (*b*)	5. (*c*)	6. (*b*)	7. (*b*)	8. (*a*)	9. (*a*)	10. (*d*)
11. (*a*)	12. (*b*)	13. (*b*)	14. (*a*)	15. (*c*)	16. (*b*)	17. (*c*)	18. (*b*)	19. (*d*)	20. (*d*)
21. (*b*)	22. (*d*)	23. (*b*)	24. (*b*)	25. (*b*)	26. (*b*)	27. (*b*)	28. (*d*)	29. (*c*)	30. (*b*)
31. (*c*)	32. (*b*)	33. (*b*)	34. (*c*)	35. (*b*)	36. (*b*)	37. (*b*)	38. (*c*)	39. (*b*)	40. (*a*)
41. (*b*, *c*)	42. (*a*, *b*)	43. (*a*, *c*, *d*)							

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (*a*)	2. (*c*)	3. (*a*)	4. (*a*)	5. (*b*)	6. (*c*)	7. (*a*)	8. (*b*)	9. (*a*)	10. (*b*)
11. (*a*, *d*)	12. (*a*, *b*, *c*)	13. (*a*)	14. (*c*)	15. (*b*)	16. (*c*)	17. (*a*)	18. (*a*)	19. (*b*)	20. (*c*)
21. (*d*)	22. [12]	23. (20^{2n+1})	24. [7]	26. ($n.2^{2n}$)	29. [4]	30. [33]	31. (*a*)		
32. [4]	34. [8016]								

38. (*i*) $\dfrac{2^{100}-100_{C_{50}}}{2}$ (*ii*) 2^{98} (*iii*) 2^{98} (*iv*) 2^{2n}

39. (*i*) ${}^{1001}C_{951}$ (*ii*) ${}^{111}C_{10}$ (*iii*) ${}^{n+m+21}C_{20}$

40. (*i*) $\left(\frac{1}{2}\right)^n+\left(\frac{2}{3}\right)^n+\left(\frac{3}{4}\right)^n$ (*ii*) $\dfrac{2^{mn}-1}{2^{mn}(2^n-1)}$

42. [{1, 3}]

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (Bonus)	2. (*a*)	3. (*c*)	4. (*b*)	5. (*b*)	6. (*d*)	7. (*a*)	8. (*a*)	9. (*d*)	10. (*d*)
11. (*c*)	12. (*b*)	13. (*a*)	14. (*c*)	15. (*d*)	16. (*b*)	17. (*b*)	18. (*d*)	19. (*d*)	20. (*b*)
21. (*d*)	22. (*b*)	23. (*a*)	24. (*b*)	25. (*d*)	26. [30]	27. (*c*)	28. (*b*)	29. (*c*)	30. [615]
31. (*d*)	32. [51]	33. [118]	34. (*c*)	35. (*d*)	36. (*c*)	37. (*b*)	38. (*b*)	39. (*a*)	40. [120]
41. [13]	42. (*b*)	43. (*d*)	44. (*c*)	45. [12]	46. (*a*)	47. (*d*)	48. [45]	49. (*a*)	50. (*c*)
51. [6]	52. (*a*)	53. [2]	54. $\frac{15}{4}$	55. (*b*)	56. [210]	57. [19]	58. (*a*)	59. (*c*)	60. (*a*)
61. (*c*)	62. (*d*)	63. (*a*)	64. (*a*)	65. (Bonus)	66. [102]	67. [5]	68. [57]	69. [99]	70. [221]

JEE Advanced

71. (*a*)	72. $\frac{1}{4}(n+1)^2(2n-1)$	73. [7]	74. [6]	75. (*c*)	76. [8]	77. [5]	78. [646]	79. [6.20]
80. (*a*,*b*,*d*)								

CHAPTER

12 Sequences and Series

SEQUENCE

A sequence is a set of terms arranged in a definite order with a rule for obtaining the terms.

e.g. 1 , 1/2 , 1/3 , ..., 1/n , ... is a sequence .

A sequence is a function whose domain is the set N of natural numbers. Since the domain for every sequence is the set of N natural numbers, therefore a sequence is represented by its range. $f: N \to R$, then $f(n) = t_n$, $n \in N$ is called a sequence and is denoted by

$\{f(1), f(2), f(3), ...\}$

$= \{t_1, t_2, t_3, ...\} = \{t_n\}$

Real Sequence: A sequence whose all terms are real is called a real sequence.

Examples:

(*i*) 2, 5, 8, 11, ...

(*ii*) 4, 1, –2, –5, ...

(*iii*) 3, –9, 27, –81, ...

Types of Sequence: On the basis of the number of terms there are two types of sequence.

(*i*) Finite sequence: A sequence is said to be finite if it has finite number of terms.

(*ii*) Infinite sequence: A sequence is said to be infinite if it has infinite number of terms.

Series: By adding or substracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, ..., a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + ... + a_n$ is a series.

Example:

(*i*) $1 + 2 + 3 + 4 + ... + n$

(*ii*) $2 + 4 + 8 + 16 + ...$

Progression: It is not necessary that the terms of a sequence always described by some explicit formula of the nth term. Those sequences whose terms follow certain patterns are called progressions.

ARITHMETIC PROGRESSION (AP)

AP is a sequence whose consecutive terms increase or decrease by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then AP can be written as $a, a+d, a+2d, ... a+(n-1)d, ...n^{th}$ term of this AP, $t_n = a + (n-1)d$, where $d = a_n - a_{n-1}$.

The sum of the first n terms of the AP is given by;

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l].$$

where l is the last term.

Remarks

(*i*) Three numbers in AP can be taken as $a - d$, a, $a + d$; four numbers in AP can be taken as $a - 3d$, $a - d$, $a + d$, $a + 3d$; five numbers in AP are $a - 2d$, $a - d$, a, $a + d$, $a + 2d$ & six terms in AP are $a - 5d$, $a - 3d$, $a - d$, $a + d$, $a + 3d$, $a + 5d$ etc.

(*ii*) The common difference can be zero, positive or negative.

(*iii*) The sum of the terms in an AP which are equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(*iv*) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it.

$a_n = 1/2\,(a_{n-k} + a_{n+k}), k < n.$

For $k = 1$, $a_n = (1/2)(a_{n-1} + a_{n+1})$;

For $k = 2$, $a_n = (1/2)(a_{n-2} + a_{n+2})$ and so on.

(*v*) $t_r = S_r - S_{r-1}$

(*vi*) If $a_1, a_2, a_3, ..., a_n$ are in A.P., then

(*a*) $a_1 + k, a_2 + k, a_3, ..,., a_n + k$ are also in A.P.

(*b*) $ka_1, ka_2, ..., ka_n$ are also in A.P.

(*c*) $\frac{a_1}{k}, \frac{a_2}{k}, ..., \frac{a_n}{k}, k \neq 0$ are also in A.P.

(*vii*) If $a_1, a_2, a_3, ...,$ and $b_1, b_2, b_3, ...,$ are two A.Ps., then

(*a*) $a_1 + b_1, a_2 + b_2, a_3 + b_3, ...,$ are also in A.P.

(*b*) $a_1, -b_1, a_2, -b_2, a_3, -b_3, ...,$ are also in A.P.

(*viii*) If a_1, a_2, a_3, a_n, are in A.P., then

(*a*) $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = ... = 2a_1 + (n-1)d$

(*b*) $a_r = \frac{a_{r-k} + a_{r+k}}{2}, 0 \le k \le n - r$

(*ix*) If n^{th} term of a sequence is a linear expression in n then the sequence is an A.P.

(x) If the sum of first n terms of a sequence is a quadratic expression in n, then the sequence is an A.P.

(xi) Three numbers a, b, c are in A.P. if and only if $b - a = c - b$, i.e., if and only if $a + c = 2b$.

Train Your Brain

Example 1: If t_{54} of an A.P. is –61 and $t_4 = 64$, find t_{10}.

Sol. Let a be the first term and d be the common difference

so $t_{54} = a + 53d = -61$ ... (i)

and $t_4 = a + 3d = 64$ ...(ii)

equation (i) – (ii) we get

$\Rightarrow 50d = -125$

$d = -\frac{5}{2}$

$\Rightarrow a = \frac{143}{2}$ so $t_{10} = \frac{143}{2} + 9\left(-\frac{5}{2}\right) = 49$

Example 2: Find the number of terms in the sequence 4, 12, 20, ..., 108.

Sol. $a = 4, d = 8$ so $108 = 4 + (n - 1)8$

$\Rightarrow n = 14$

Example 3: Find the sum of all the three digit natural numbers which on division by 7 leaves remainder 3.

Sol. All these numbers are 101, 108, 115,..... 997, to find n.

$997 = 101 + (n - 1)\,7$

$\Rightarrow n = 129$ so $S = \frac{129}{2}[101 + 997]$

$= 70821.$

Example 4: The sum of n terms of two A.Ps. are in ratio $\frac{7n+1}{4n+27}$. Find the ratio of their 11th terms.

Sol. Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.P.s respectively, then

$$\frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

For ratio of 11th terms

Put $\frac{n-1}{2} = 10 \Rightarrow n = 21$ on both sides

so, $\frac{a_1 + 10d_1}{a_2 + 10d_2} = \frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}$

Example 5: The sum of first three terms of an A.P. is 27 and the sum of their squares is 293 . Find the sum to 'n' terms of the A.P.

Sol. Let $a - d, a, a + d$ be the numbers $\Rightarrow a = 9$

Also $(a - d)^2 + a^2 + (a + d)^2 = 293.$

$\Rightarrow 3a^2 + 2d^2 = 293$

$\Rightarrow d^2 = 25 \Rightarrow d = \pm 5$

therefore numbers are 14, 9, 4.

Hence the A.P. is 4, 9, 14, ... or 14, 9, 4, ...

$\Rightarrow s_n = \frac{n}{2}[5n + 3]$ or $s_n = \frac{n}{2}[33 - 5n]$

Example 6: If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P., prove that a^2, b^2, c^2 are also in A.P.

Sol. $\because \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$$\Rightarrow \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$$\Rightarrow \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$$

$$\Rightarrow \frac{b-a}{b+c} = \frac{c-b}{a+b} \Rightarrow b^2 - a^2 = c^2 - b^2$$

$\Rightarrow a^2, b^2, c^2$ are in A.P.

Concept Application

1 Find the sum of all odd numbers between 1 and 1000 which are divisible by 3

2. The ratio between the sum of n term of two A.P.'s is $3n + 8 : 7n + 15$. Then find the ratio between their 12 th term

3 If sum of n terms of a sequence is given by $S_n = 3n^2 - 4n$, find its 50th term.

4 If $a(b + c)$. $b(c + a)$, $c(a + b)$ are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

5. If a_1, a_2, a_3, a_4, a_5 are in A.P. with common difference $\neq 0$. then find the value of $\sum_{i=1}^{5} a_i$, when $a_3 = 2$.

6. Write down the sequence whose n^{th} term is

(i) $\frac{2^n}{n}$

(ii) $\frac{3+(-1)^n}{3^n}$

GEOMETRIC PROGRESSION (GP)

GP is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the proceeding terms multiplied by same constant . Thus in a GP the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the G.P. Therefore $a, ar, ar^2, ar^3, ar^4, \ldots$ is a GP with a as the first term & r as common ratio.

(*i*) n^{th} term of a G.P. $= a\,r^{n-1}$

(*ii*) Sum of the Ist n terms i.e. $S_n = \dfrac{a(r^n-1)}{r-1}$, if $r \neq 1$.

(*iii*) Sum of an infinite GP when $|r| < 1$ when $n \to \infty\ r^n \to 0$ if $|r| < 1$ therefore, $S_\infty = \dfrac{a}{1-r}\ (|r| < 1)$.

(*vi*) If a, b, c are in GP $\Rightarrow$ $b^2 = ac$.

Sum to *n* terms of a G.P. is given by

Let a is first term and r is common ratio of G.P. then sum is given by

$$S_n = \frac{a(1-r^n)}{1-r} \text{ provided } r \neq 1$$

Properties of Geometrical Progression

(i) $a_1, a_2, a_3, \ldots$ are in *G.P.* then $a_1k, a_2k, a_3k, \ldots$ and $a_1/k, a_2/k, a_3/k \ldots$ are also in G.P. ($k \neq 0$)

(ii) If $a_1, a_2, a_3, \ldots$ are in G.P. Then $1/a_1, 1/a_2, 1/a_3, \ldots$ are also in G.P.

(iii) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two G.P.s, then $a_1b_1, a_2b_2, a_2b_2, a_3b_3, \ldots$ and $a_1/b_1, a_2/b_2, a_3/b_3, \ldots$ are also in G.P.

(iv) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are tow G.P.s, then $a_1 + b_1, a_1 + b_2, a_3 + b_3, \ldots$ are not in G.P.

(v) If $a_1, a_2, a_3, \ldots$ are in G.P. ($a_i > 0, \forall i$), then $\log a_1, \log a_2, \log a_3, \ldots$ are in A.P. In this case the converse also holds good.

(vi) If $a_1, a_2, a_3, \ldots a_n$ are in G.P., then

(*a*) $a_2a_n = a_2a_{n-1} = a_3a_{n-2} = \ldots = a^2r^{n-1}$

(*b*) $a_r = \sqrt{a_{r-k}a_{r+k}}, \quad 0 \le k \le n-r$

(vii) If $a_1, a_2, a_3, a_4 \ldots a_{n-1}\ a_n$ are in G.P., then

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots = \frac{a_n}{a_{n-1}}$$

$\Rightarrow a_2^2 = a_3a_1, a_3^2 = a_2a_4, \ldots$

also $a_2 = a_1r, a_3 = a_1r^2, a_4 = a_1r^3, \ldots, a_n = a_1r^{n-1}$

where r is the common ration.

(viii) Three numbers in G.P. can be taken as $\dfrac{a}{r}, a, ar$; Five numbers in G.P. can be taken as $\dfrac{a}{r^2}, \dfrac{a}{r}, a, ar, ar^2$ etc.

In general: $(2m+1)$ numbers in G.P. can be written as ($m \in N$)

$$\frac{a}{r^m}, \frac{a}{r^{m-1}}, \ldots, \frac{a}{r}, a\ \ ar, \ldots, ar^{m-1}, ar^m$$

(ix) Four numbers in G.P. can be taken as $\dfrac{a}{r^3}, \dfrac{a}{r}, ar, ar^3$; Six numbers in G.P. can be taken as $\dfrac{a}{r^5}, \dfrac{a}{r^3}, \dfrac{a}{r}, ar, ar^3, ar^5$; etc.

In **general:** $(2m)$ numbers in G.P. can be written as ($m \in N$)

$$\frac{a}{r^{2m-1}}, \frac{a}{r^{2m-3}}, \ldots, \frac{a}{r^3}, \frac{a}{r}\ ar, ar^3, \ldots, ar^{2m-3}, ar^{2m-1}$$

Train Your Brain

Example 7: If the first term of G.P. is 7, its nth term is 448 and sum of first n terms is 889, then find the fifth term of G.P.

Sol. Given a = 7 the first term, $t_n = ar^{n-1} = 7(r)^{n-1} = 448$

$\Rightarrow 7r^n = 448\,r$

Also $S_n = \dfrac{a(r^n-1)}{r-1} = \dfrac{7(r^n-1)}{r-1}$

$\Rightarrow 889 = \dfrac{448r-7}{r-1}$

$\Rightarrow r = 2.$

Hence $T_5 = ar^4 = 7(2)^4 = 112$

Example 8: Find three numbers in G.P. having sum 19 and product 216.

Sol. Let the three numbers be $\dfrac{a}{r}, a, ar$

so $a\left[\dfrac{1}{r} + 1 + r\right] = 19$ ... (*i*)

and $a^3 = 216 \quad \Rightarrow \quad a = 6$

so from (*i*) $6r^2 - 13r + 6 = 0. \quad \Rightarrow \quad r = \dfrac{3}{2}, \dfrac{2}{3}$

Hence the three numbers are 4, 6, 9.

Example 9: Evaluate 7 + 77 + 777 + ... upto n terms.

Sol. Let $S = 7 + 77 + 777 + \ldots$ upto n terms.

$= \dfrac{7}{9}\ [9 + 99 + 999 + \ldots]$

$= \dfrac{7}{9}\ [(10-1) + (10^2-1) + (10^3-1)$

$+ \ldots\ldots + \text{upto n terms}]$

$= \dfrac{7}{9}\ [10 + 10^2 + 10^3 + \ldots\ldots + 10^n - n]$

$= \dfrac{7}{9}\left(\dfrac{10(10^n-1)}{9} - n\right) = \dfrac{7}{81}\ [10^{n+1} - 9n - 10]$

Example 10: If the third and fourth terms of an arithmetic sequence are increased by 3 and 8 respectively, then the first four terms form a geometric sequence. Find

(*i*) the sum of the first four terms of A.P.

(*ii*) second term of the G.P.

Sol. $a, (a+d), (a+2d), (a+3d)$ in A.P.

$a, a+d, (a+2d+3), (a+3d+8)$ are in G.P.

hence $a+d = ar$

also $r = \frac{a+d}{a} = \frac{a+2d+3}{a+d} = \frac{a+3d+8}{a+2d+3}$ $\therefore$ $\frac{d+3}{d} = \frac{a+5}{d+3}$

$\Rightarrow d^2 + 6d + 9 = d^2 + 5d$

$\Rightarrow d = -9$

$\therefore \frac{a-9}{a} = \frac{a-15}{a-9}$

$\Rightarrow a^2 - 18a + 81 = a^2 - 15a$

$\Rightarrow 3a = 81 \Rightarrow a = 27$

hence A.P. is 27, 18, 9, 0, G.P. is 27, 18, 12, 8

(i) sum of the first four terms of A.P. = 54

(ii) 2nd term of G.P. = 18

Concept Application

7. Find the G.P. if the common ratio of G.P. is 3, n^{th} term is 486 and sum of first n terms is 728.
8. If x, $2y$, $3z$ are in A.P. where the distinct numbers x, y, z are in G.P. Then find the common ratio of G.P
9. Three numbers form an increasing G.P. If the middle number is doubled, then the new numbers are in A.P. The common ratio of G.P. is:
10. Evaluate 9 + 99 + 999 + ...upto n terms.

ARITHMETICO-GEOMETRIC SERIES

A series each term of which is formed by multiplying the corresponding term of an AP & GP is called the Arithmetico-Geometric Series . e.g. $1 + 3x + 5x^2 + 7x^3 + ...$

Here 1, 3, 5, ... are in AP & 1, x, x^2, x^3 ..... are in GP.

Sum of n terms of an Arithmetico-Geometric Series

Let $S_n = a + (a+d)r + (a+2d)r^2 + ...+ [a+(n-1)d]\, r^{n-1}$

then $S_n = \frac{a}{1-r} + \frac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}, r \neq 1.$

Sum to Infinity: If $|r| < 1$ & $n \to \infty$ then $\underset{n\to\infty}{\text{Limit}}\, r^n = 0.$

$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}.$

Train Your Brain

Example 11: Find the sum to n terms of the series $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+.....$ Also find the sum if it exist if $n \to \infty$.

Sol. $S = \frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+.....+\frac{n}{2^n}$ ... (i)

$\frac{S}{2} = \frac{1}{4}+\frac{2}{8}+\frac{3}{16}+.....+\frac{n-1}{2^n}+\frac{n}{2^{n+1}}$ ...(ii)

By (i) – (ii)

$\frac{S}{2} = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+.....+\frac{1}{2^n}-\frac{n}{2^{n+1}}$

$= \frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} - \frac{n}{2^{n+1}} = 1 - \frac{1}{2^n} - \frac{n}{2^{n+1}}$

$S_n = 2\left[\frac{2^{n+1}-2-n}{2^{n+1}}\right] = \frac{2^{n+1}-2-n}{2^n}$

If $n \to \infty$ then $S_\infty = \underset{n\to\infty}{Lim}\left[2 - \frac{1}{2^{n-1}} - \frac{n}{2^n}\right] = 2$

Example 12: Find the sum of the series.

$1 + 3.2 + 5.22 + 7.23 + ...+(2n-1)\, 2^{n-1}$

Sol. $Sn = 1 + 3.2 + 5.22 + 7.23 + ...+ (2n-1)\, 2^{n-1}$ ....(i)

$2Sn = 2 + 3.2^2 + 5.2^3 + ...+ (2n-3).\, 2^{n-1} + (2n-1)2n$ ...(ii)

Subtracting (ii) from (i) we get

$-Sn = 1 + 2.2 + 2.2^2 + 2.2^3 .....+ 2.2^{n-1} - (2n-3)$

$-Sn = 1 + \frac{4\left(2^{n-1}-1\right)}{2-1} - (2n-1)2^n$

$Sn = (2n-1)\, 2^n - 2n + 1 + 3 = 2^{n+1}(n-1) - 2^n + 3$

Concept Application

11. Find the sum to n terms of the series $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+.....$ Also find the sum if it exist if n approaches ∞
12. If positive square root of, $a^{\frac{1}{a}} \cdot (2a)^{\frac{1}{2a}} \cdot (4a)^{\frac{1}{4a}} \cdot (8a)^{\frac{1}{8a}} .....\infty$. is $\frac{8}{27}$, find the value of 'a'.
13. Find sum to infinite terms of the series $1 + 2x + 3x^2 + 4x^3 + ...,-1 < x < 1$

HARMONIC PROGRESSION (HP)

A sequence is said to HP if the reciprocals of its terms are in AP.

If the sequence $a_1, a_2, a_3, ..., a_n$ is an HP then $1/a_1$, $1/a_2$, ..., $1/a_n$ is an AP & converse. Here we do not have

the formula for the sum of the n terms of an HP. For HP whose first term is a & second term is b, the nth term is $t_n = \frac{ab}{b+(n-1)(a-b)}$. If a, b, c are in HP

$$\Rightarrow b = \frac{2ac}{a+c} \quad \text{or} \quad \frac{a}{c} = \frac{a-b}{b-c}.$$

Train Your Brain

Example 13: If $\frac{1}{a}+\frac{1}{c}+\frac{1}{a-b}+\frac{1}{c-b}=0$ prove that a, b, c are in HP unless $b = a + c$.

Sol. $\Rightarrow \frac{a+c}{ac}+\frac{c-b+a-b}{(a-b)(c-b)}=0$

$$\Rightarrow \frac{a+c}{ac}+\frac{(a+c)-2b}{ac-b(a+c)+b^2}=0$$

Let $a+c=\lambda \quad \therefore \frac{\lambda}{ac}+\frac{\lambda-2b}{ac-b\lambda+b^2}=0$

$$\Rightarrow \frac{ac\lambda-b\lambda^2+b^2\lambda+ac\lambda-2abc}{ac(ac-b\lambda+b^2)}=0$$

$\Rightarrow ac\lambda - b\lambda^2 + b^2\lambda + ac\lambda - 2abc = 0$

$\Rightarrow 2ac(\lambda - b) - b\lambda(\lambda - b) = 0$

$\Rightarrow (2ac - b\lambda)(\lambda - b) = 0$

$\Rightarrow \lambda = b$ or $\lambda = \frac{2ac}{b}$

$\Rightarrow a + c = b$ or $a + c = \frac{2ac}{b} \quad (\because a + c = \lambda)$

$\Rightarrow a + c = b \quad$ or $\quad b = \frac{2ac}{a+c}$

$\Rightarrow a, b, c$ are in H.P. or $a + c = b$.

Example 14: If p^{th}, q^{th}, r^{th} terms of an H.P. be a, b, c respectively, prove that $(q-r)bc + (r-p)ac + (p-q)ab = 0$

Sol. Let 'x' be the first term and 'd' be the common difference of the corresponding A.P..

so $\frac{1}{a} = x + (p-1)d \quad \ldots (i)$

$\frac{1}{b} = x + (q-1)d \quad \ldots (ii)$

$\frac{1}{c} = x + (r-1)d \quad \ldots (iii)$

$(i) - (ii) \Rightarrow ab(p-q)d = b - a \quad \ldots (iv)$

$(ii) - (iii) \Rightarrow bc(q-r)d = c - b \quad \ldots (v)$

$(iii) - (i) \Rightarrow ac(r-p)d = a - c \quad \ldots (vi)$

$(iv) + (v) + (vi)$ gives

$bc(q-r) + ac(r-p) + ab(p-q) = 0.$

Concept Application

14. a, b, c are in H.P. then prove that $\frac{b+a}{b-a}+\frac{b+c}{b-c}=2$

15. The 7th term of a H.P. is $\frac{1}{10}$ and 12th term is $\frac{1}{25}$, find the 20th term of H.P.

16. If a, b, c, d are in H.P., then find the value of $ab + be + cd$.

ARITHMETIC, GEOMETRIC & HARMONIC MEANS (AM, GM & HM)

Arithmetic Mean: If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c.

So, A.M. of two numbers a and b is $\frac{a+b}{2}$

AM for any n positive number $a_1, a_2, \ldots, a_n$ is;

$$A = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}.$$

n - Arithmetic Means Between Two Numbers:

If a, b are any two given numbers & $a, A_1, A_2, \ldots, A_n, b$ are in AP then $A_1, A_2, \ldots A_n$ are the n AM's between a & b.

$$A_1 = a + d = a + \frac{b-a}{n+1},$$

$$A_2 = a + 2d = a + \frac{2(b-a)}{n+1}$$

$, \ldots\ldots,$

$$A_n = a + nd = a + \frac{n(b-a)}{n+1}$$

Note: Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

Geometric Means: If a, b, c are in GP, b is the GM between a & c.

$b^2 = ac$, therefore $b = \sqrt{ac}$; $a > 0, c > 0$.

Geometric means of n positive numbers $a_1, a_2, \ldots a_n$ is $(a_1.a_2 \ldots a_n)^{1/n}$

n-Geometric Means Between a, b: If a, b are two given numbers & $a, G_1, G_2, \ldots, G_n, b$ are in GP. Then $G_1, G_2, G_3, \ldots, G_n$ are n GMs between a & b.

$G_1 = ar = a(b/a)^{1/n+1}$, $G_2 = ar^2 = a(b/a)^{2/n+1}$, ..., $G_n = ar^n = a(b/a)^{n/n+1}$

Note: The product of n GMs between a & b is equal to the nth power of the single GM between a & b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a & b.

Harmonic Mean: If a, b, c are in HP, b is the HM between a & c, then $b = 2ac/[a+c]$.

If a, b are two given number such that

$a, H_1, H_2, H_3 \ldots H_n, b$ are in H.P.

then $H_1, H_2, H_3 \ldots H_n$, are n harmonic means between a and b.

$\because \quad a, H_1, H_2, H_3 \ldots H_n, b \rightarrow$ H.P.

$\therefore \quad \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3} \ldots \frac{1}{H_n}, \frac{1}{b} \rightarrow$ A.P.

Common difference $(d) = \dfrac{\frac{1}{b} - \frac{1}{a}}{n+1}$

So, $\frac{1}{H_1} = \frac{1}{a} + d, \frac{1}{H_2} = \frac{1}{a} + 2d \ldots \frac{1}{H_n} = \frac{1}{a} + nd$

Note: $x + \frac{1}{x} \geq 2$, if $x > 0$ and $x + \frac{1}{x} \leq -2$, if $x < 0$.

ARITHMETIC MEAN OF Mth POWER

Let $a_1, a_2 \ldots, a_n$ be n positive real numbers and let m be a real number, then

$$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} \geq \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m,$$

if $m \in R - [0, 1]$

However if $m \in (0, 1)$, then

$$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} = \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$$

Obviously if $m \in \{0, 1\}$, then

$$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} = \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$$

INEQUALITIES

(*i*) **A.M. ≥ G.M. ≥ H.M.**

Let $a_1, a_2, \ldots\ldots\ldots, a_n$ be n positive real numbers, then we define their arithmetic mean

(A), geometric mean (G) and harmonic mean (H) as

$$A = \frac{a_1 + a_2 + \ldots + a_n}{n},$$

$$G = (a_1 a_2 \ldots\ldots a_n)^{1/n} \text{ and } H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots\ldots \frac{1}{a_n}\right)}$$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if

$a_1 = a_2 = \ldots\ldots = a_n$

(*ii*) **Weighted Means**

Let $a_1, a_2, \ldots\ldots, a_n$ be n positive real numbers and $w_1, w_2, \ldots\ldots, w_n$ be n positive rational numbers. Then we define weighted Arithmetic mean (A^*), weighted Geometric mean (G^*) and weighted harmonic mean (H^*) as

$$A^* = \frac{a_1 w_1 + a_2 w_2 + \ldots + a_n w_n}{w_1 + w_2 + \ldots + w_n},$$

$$G^* = (a_1^{w_1} . a_2^{w_2} \ldots a_n^{w_n})^{\frac{1}{w_1 + w_2 + \ldots w_n}}$$

and $H^* = \dfrac{w_1 + w_2 + \ldots + w_n}{\frac{w_1}{a_1} + \frac{w_2}{a_2} + \ldots + \frac{w_n}{a_n}}$.

$A^* \geq G^* \geq H^*$ More over equality holds at either place if & only if $a_1 = a_2 i \ldots\ldots = .a_n$

(*iii*) **Cauchy's Schwartz Inequality:**

If $a_1, a_2, \ldots\ldots a_n$ and $b_1, b_2, \ldots\ldots, b_n$ are $2n$ real numbers, then $(a_1 b_1 + a_2 b_2 + \ldots\ldots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \ldots\ldots + a_n^2)(b_1^2 + b_2^2 + \ldots\ldots\ldots + b_n^2)$ with the equality holding if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots\ldots = \frac{a_n}{b_n}$.

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Example 15: Insert 4 H.M between $\frac{2}{3}$ and $\frac{2}{13}$.

Sol. Let 'd' be the common difference of corresponding A.P..

So $d = \dfrac{\frac{13}{2} - \frac{3}{2}}{5} = 1.$

$\therefore \quad \frac{1}{H_1} = \frac{3}{2} + 1 = \frac{5}{2}$ or $H_1 = \frac{2}{5}$

$\frac{1}{H_2} = \frac{3}{2} + 2 = \frac{7}{2}$ or $H_1 = \frac{2}{7}$

$\frac{1}{H_3} = \frac{3}{2} + 3 = \frac{9}{2}$ or $H_3 = \frac{2}{9}$

$\frac{1}{H_4} = \frac{3}{2} + 4 = \frac{11}{2}$ or $H_4 = \frac{2}{11}$

Example 16: If $a, b, c > 0$, prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$

Sol. Using the relation A.M. ≥ G.M. we have

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \geq \left(\frac{a}{b} . \frac{b}{c} . \frac{c}{a}\right)^{\frac{1}{3}} \Rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$$

Example 17: If one AM 'a' and two GMN;s p and q be inserted between any two given numbers then show that $p^3 + q^3 = 2\,apq$.

Sol. Let two numbers are A and B.

If one AM 'a' is inserted between A & B then A, a, B .....are in A.P.

therefore $\alpha = \frac{A+B}{2}$ or $\boxed{A+B=2a}$ ... (i)

If two GM's p & q are inserted between A & B then A, p, q, B ..... are in G.P.

therefore $p^2 = Aq$ & $q^2 = BP$

or $\boxed{A = \frac{p^2}{q}}$ ... (ii)

or $\boxed{B = \frac{q^2}{p}}$ ... (iii)

Adding equation (ii) & (ii)

$A + B = \frac{p^2}{q} + \frac{q^2}{p}$ or $2a = \frac{p^3+q^3}{pq}$

(using equation (i))

or $\boxed{p^3 + q^3 = 2apq}$

Example 18: Insert 10 A.M. between 3 and 80.

Sol Here 3 is the first term and 80 is the 12th term of A.P. so $80 = 3\ 4 - (11\ )d$

$\Rightarrow d = 7$

So the series is 3,10,17,24,.....,73. 80

$\therefore$ Required means are 10, 17, 24,.....,73.

Example 19: If $2x^3 + ax^2 + bx + 4 = 0$ (a and b are positive real numbers) has 3 real roots, then prove that $a + b \geq 6(2^{1/3} + 4^{1/3})$.

Sol Let α, β, γ be the roots of $2x^3 + ax^2 + bx + 4 = 0$. Given that all the coefficients are positive, so all the roots will be negative.

Let $\alpha = -\alpha, \alpha_2. = -\beta, \alpha_2 = -\gamma$

$\Rightarrow \alpha, \alpha_2\ \alpha_3 = \frac{a}{2}$

$\alpha_1\ \alpha_2\ \alpha_3 = 2$

Applying $AM \geq GM$, we have

$$\frac{\alpha_1+\alpha_2+\alpha_3}{3} \geq (\alpha_1\alpha_2\alpha_3)^{1/3} \Rightarrow a \geq 6 \times 2^{1/3}$$

Also $\frac{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3}{3} > (\alpha_1\alpha_2\alpha_3)^{2/3} \Rightarrow b \geq 64^{1/3}$

Therefore $a + b \geq 6(2^{1/3} + 4^{1/3})$.

Example 20: Prove that $a^4 + b^4 + c^4 \geq abc(a + b + c)$, $[a, b, c > 0]$

Sol. Using m^th power inequality, we get

$$\frac{a^4+b^4+c^4}{3} \geq \left(\frac{a+b+c}{3}\right)^4$$

$$= \left(\frac{a+b+c}{3}\right)\left(\frac{a+b+c}{3}\right)^3 \geq \left(\frac{a+b+c}{3}\right)[(abc)^{1/3}]^3$$

($\because$ A.M $\geq$ G.M)

or $\frac{a^4+b^4+c^4}{3} \geq \left(\frac{a+b+c}{3}\right)abc$

$\therefore\ a^4 + b^4 + c^4 \geq abc(a + b + c)$.

Example21: Provethat $\frac{s}{s-a} + \frac{s}{s-b} + \frac{s}{s-c} \geq \frac{9}{2}$,if$s=a+b+c$, $[a, b, c > 0]$

Sol. A.M. $\geq$ H.M.

$$\Rightarrow \frac{(a+b)+(b+c)+(c+a)}{3} \geq \frac{3}{\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}}$$

$$\Rightarrow \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{9}{2(a+b+c)}$$

Concept Application

17. For what value of n, $\frac{a^{n+3}+b^{n+3}}{a^{n+2}+b^{n+2}}$, $a \neq b$ is the A.M. of a and b

18. If $x^6 - 12x^5 + ax^4 + bx^3 + cx^2 + dx + 64 = 0$ has positive roots then find a,b,c,d.

19. If a, b, c, d, e are positive real numbers, such that $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$, find the range of e.

20. f x, y, z are positive then prove that $(x + y)(y + z)(z + x)$

$$\left(\frac{1}{x}+\frac{1}{y}\right)\left(\frac{1}{y}+\frac{1}{z}\right)\left(\frac{1}{z}+\frac{1}{x}\right) \geq 64.$$

SIGMA NOTATIONS

Sigma (Σ) Notation

Σ indicates sum i.e., $\sum_{i=1}^{n} i = \sum n = 1 + 2 + 3 + \ldots\ldots + n$

(i) $\sum_{i=1}^{n} \frac{i+1}{i+2} = \frac{1+1}{1+2} + \frac{2+1}{2+2} + \frac{3+1}{3+2} + \ldots + \frac{n+1}{n+2}$

(ii) $\sum_{i=1}^{m} a = a + a + \ldots\ldots + a$ m times

$= am$ where a is constant

(iii) $\sum_{i=1}^{m} ai = a\sum_{i=1}^{m} i = a(1 + 2 + \ldots + m)$

(iv) $\sum_{i=1}^{m}(i^3 - 2i^2 + i) = \sum_{i=1}^{m} i^3 - 2\sum_{i=1}^{m} i^2 + \sum_{i=1}^{m} i$

Therorems

(a) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(b) $\sum_{r=1}^{n} ka_r = \sum_{r=1}^{n} a_r$

(c) $\sum_{r=1}^{n} k = nk$: where k is a constant.

Important Results

(i) Sum of first n natural numbers

$\Sigma n = 1+2+....+n$

$= \frac{n(n+1)}{2}$

(ii) Sum of squares of first n natural numbers

$\Sigma n^2 = 1^2 + 2^2 + ... + n^2$

$= \frac{n(n+1)(2n+1)}{6}$

(iii) Sum of cubes of first n natural numbers

$\Sigma n^3 = 1^3 + 2^3 + 3^3 + ... + n^3$

$= \left(\frac{n(n+1)}{2}\right)^2 = (\Sigma n)^2$

(iv) Sum of n terms of a sequence $T_n = an^3 + bn^2 + cn + d$

$\boxed{S_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + dn}$

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Example 22: Find the sum of the series 3.5 + 6.8 + 9.11 + ... + upto n terms

Sol. n^{th} term of 3, 6, 9, ... is $3n$

n^{th} term of 5, 8, 11, ... is $(3n + 2)$

$\therefore\ T_n = 3n\,(3n+2) = 9n^2 + 6n$

$\therefore\ S_n = 9\Sigma n^2 + 6\Sigma n$

$= \frac{9n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2}$

$= \frac{3}{2}n(n+1)[2n+1+2] = \frac{3n(n+1)(2n+3)}{2}$

Example 23: Sum up to 16 terms of the series

$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots$ is

Sol. $t_n = \frac{1^3+2^3+3^3+\ldots+n^3}{1+3+5+\ldots.(2n-1)}$

$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\{2+2(n-1)\}} = \frac{\frac{n^2(n+1)^2}{4}}{n^2}$

$= \frac{(n+1)^2}{4} = \frac{n^2}{4} + \frac{n}{2} + \frac{1}{4}$

$= \frac{1}{4}\Sigma n^2 + \frac{1}{2}\Sigma n + \frac{1}{4}\Sigma 1$

$= \frac{1}{4}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\cdot\frac{n(n+1)}{2} + \frac{1}{4}\cdot n$

$\therefore S_{16} = \frac{16.17.33}{24} + \frac{16.17}{4} + \frac{16}{4} = 446$

Concept Application

21. Find the sum of n-terms of the series 2.5 + 5.8 + 8.11 +...

22. Evaluate $\sum_{k=1}^{11}(2+3^k)$.

METHOD OF DIFFERENCES

Let u_1, u_2, u_3..... be a sequence, such that $u_2 - u_1$, $u_3 - u_2$, ... is either an A.P. or a G.P. then n^{th} term u_n of this sequence is obtained as follows

$S = u_1 + u_2 + u_3 + ... + u_n$ ... (i)

$S = u_1 + u_2 + ... + u_{n-1} + u_n$ ... (ii)

(i) – (ii)

$\Rightarrow\ u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$

Where the series $(u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$ is either in A.P. or in G.P. then we can find u_n and hence sum of this series as $S = \sum_{r=1}^{k} u_r$

Remark: It is not always necessary that the series of first order of differences i.e. $u_2 - u_1$, $u_3 - u_2$ ... $u_n - u_{n-1}$ is always either in A.P. or in G.P. in such case let $u_1 = T_1$, $u_2 - u_1 = T_2$, $u_3 - u_2 = T_3$ ....., $u_n - u_{n-1} = T_n$.

So $u_n = T_1 + T_2 + ... + T_n$ ... (i)

$u_n = T_1 + T_2 + ... + T_{n-1} + T_n$ ... (ii)

(i) – (ii)

$\Rightarrow T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + ... + (T_n + T_{n-1})$

Now, the series $(T_2 - T_1) + (T_3 - T_2) + ... + (T_n - T_{n-1})$ is series of second order of differences and when it is either in A.P. or in G.P., then $u_n = u_1 + \Sigma\, T_r$

Otherwise in the similar way we find series of higher order of differences and the nth term of the series.

If possible express r^{th} term as difference of two terms as $t_r = f(r) - f(r \pm 1)$. This can be explained with help of examples given below.

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Example 24: Find the sum to n terms of the series, 0.7 + 7.7 + 0.77 + 77.7 + 0.777 + 777.7 + 0.7777 + ... where n is even.

Sol. $n = 2m$

$$s = (0.7 + 0.77 + 0.777 + ... m \text{ term})$$
$$+ (7.7 + 77.7 + 777.7 + ... m \text{ terms})$$
$$= \frac{7}{9}(0.9 + 0.99 + 0.999 + ... m \text{ terms})$$
$$+ \frac{7}{90}((10^2 - 1) + (10^3 - 1) + ... + (10^{m+1} - 1))$$
$$= \frac{7}{9}\left\{\frac{n}{2} - \frac{1}{9}\left(1 - \frac{1}{10^{n/2}}\right)\right\} + \frac{7}{90}\left\{\frac{100}{9}\left(10^{n/2} - 1\right) - \frac{n}{2}\right\}$$

Example 25: Find the sum to n-terms 3 + 7 + 13 + 21 + ...

Sol. Let $S = 3 + 7 + 13 + 21 + ... + T_n$ ... (i)

$S = 3 + 7 + 13 + ... + T_{n-1} + T_n$ ... (ii)

(i) – (ii) $\Rightarrow$ $T_n = 3 + 4 + 6 + 8 + ... + (T_n - T_{n-1})$

$$= 3 + \frac{n-1}{2}[8 + (n-2)2] = 3 + (n-1)(n+2) = n^2 + n + 1$$

Hence $S = \Sigma(n^2 + n + 1) = \Sigma n^2 + \Sigma n + \Sigma 1$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n = \frac{n}{3}(n^2 + 3n + 5)$$

Example 26: Find the nth term and the sum of n term of the series 2, 12, 36, 80, 150, 252

Sol. $S = 2 + 12 + 36 + 80 + 150 + 252 + ... + T_n$ ... (i)

$S = 2 + 12 + 36 + 80 + 150 + ... + T_{n-1} + T_n$ ... (ii)

(i) – (ii)

$\Rightarrow T_n = 2 + 10 + 24 + 44 + 70 + 102 + ... + (T_n - T_{n-1})$ ...(iii)

$T_n = 2 + 10 + 24 + 44 + 70 + 102$
$+ ... + (T_{n-1} - T_{n-2}) + (T_n - T_{n-1})$ ... (iv)

(iii) – (iv)

$$\Rightarrow T_n - T_{n-1} = 2 + 8 + 14 + 20 + 26 + ... = \frac{n}{2}[4 + (n-1)6]$$

$= n[3n - 1]$ $\Rightarrow$ $T_n - T_{n-1} = 3n^2 - n$

$\therefore$ general term of given series is

$\Sigma T_n - T_{n-1} = \Sigma 3n^2 - n = n^3 + n^2.$

Hence sum of this sereis is

$$S = \Sigma n^3 + \Sigma n^2 = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{12}(3n^2 + 7n + 2) = \frac{1}{12}n(n+1)(n+2)(3n+1)$$

Example 27: Find the sum of the series to n terms whose general term is $2n + 1$.

Sol. $S_n = \Sigma T_n = \Sigma(2n+1) = 2\Sigma n + \Sigma 1 = \frac{2(n+1)n}{2} + n = n^2 + 2n$

Example 28:

(a) Sum the following series to infinity

$$\frac{1}{1\cdot4\cdot7} + \frac{1}{4\cdot7\cdot10} + \frac{1}{7\cdot10\cdot13} + ...$$

(b) Sum the following series upto n-terms

$1\cdot2\cdot3\cdot4 + 2\cdot3\cdot4\cdot5 + 3\cdot4\cdot5\cdot6 + ...$

Sol. (a) $T_n = \frac{1}{[1+(n-1)3][1+3n][4+3n]}$

$$= \frac{1}{(3n-2)(2n+1)(3n+4)}$$

$$= \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$T_1 = \frac{1}{6}\left[\frac{1}{1\cdot4} - \frac{1}{4\cdot7}\right]$$

$$T_2 = \frac{1}{6}\left[\frac{1}{4\cdot7} - \frac{1}{7\cdot10}\right]$$

$\vdots \quad \vdots \quad \vdots$

$$T_n = \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$\therefore S_n = \sum T_n = \frac{1}{6}\left[\frac{1}{1\cdot4} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$= \left[\frac{1}{24} - \frac{1}{6(3n+1)(3n+4)}\right] \text{ as } n \to \infty$$

$$\therefore S_\infty = \frac{1}{24}$$

(b) $1\cdot2\cdot3\cdot4 + 2\cdot3\cdot4\cdot5 + 3\cdot4\cdot5\cdot6 + ...$

$$T_n = \frac{1}{5}n(n+1)(n+2)(n+3)[(n+4) - (n-1)]$$

$$\therefore T_1 = \frac{1}{5}[1\cdot2\cdot3\cdot4\cdot5 - 0]$$

$$T_2 = \frac{1}{5}[2\cdot3\cdot4\cdot5\cdot6 - 1\cdot2\cdot3\cdot4\cdot5]$$

$\vdots \quad \vdots \quad \vdots$

$$T_n = \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4) - (n-1)n(n+1)(n+2)(n+3)]$$

$$\therefore S_n = \sum T_n$$

$$= \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4)]$$

$$= \frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$$

Example 29: Sun to n terms of the series

$$\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+.....$$

Sol. Let

$$T_r=\frac{r+3}{r(r+1)(r+2)}=\frac{1}{(r+1)(r+2)}+\frac{3}{r(r+1)(r+2)}$$

$$=\left[\frac{1}{r+1}-\frac{1}{r+2}\right]+\frac{3}{2}\left[\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right]$$

$$\therefore\ S=\left[\frac{1}{2}-\frac{1}{n+2}\right]+\frac{3}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]$$

$$=\frac{5}{4}-\frac{1}{n+2}\left[1+\frac{3}{2(n+1)}\right]$$

$$=\frac{5}{4}-\frac{1}{2(n+1)(n+2)}[2n+5]$$

Example 30: Find the sum of the infinite series

$$\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+.....$$

Sol. Here $u_n=\dfrac{2n+1}{n^2(n+1)^2}=\dfrac{(n+1)^2-n^2}{n^2(n+1)^2}=\dfrac{1}{n^2}-\dfrac{1}{(n+1)^2}$

$$\therefore\ u_1=\frac{1}{1^2}-\frac{1}{2^2},$$

$$u_2=\frac{1}{2^2}-\frac{1}{3^2}.....u_n=\frac{1}{n^2}-\frac{1}{(n+1)^2}$$

By addition, $S_n=1-\dfrac{1}{(n+1)^2}$

If $n\to\infty$, then $S_n\to 1$. Hence the sum of the infinite series is 1.

Example 31: If $1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+...=\log_e 2$, then sum

$$\frac{1}{1.3}+\frac{1}{2.5}+\frac{1}{3.7}+\frac{1}{4.9}+.....$$

Sol. If S denote the sum of the given series, then

$$S=\frac{2}{2.3}+\frac{2}{4.5}+\frac{2}{6.7}+\frac{2}{8.9}+...$$

$$=2\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\left(\frac{1}{8}-\frac{1}{9}\right)+.....\right]$$

$$=2\left[1-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-...\right)\right]$$

$$=2[1-\log_e 2]=2-2\log_e 2.$$

Example 32: If a, b, c, d be four distinct positive quantities in G.P., then show that

(*a*) $a+d>b+c$

(*b*) $\dfrac{1}{cd}+\dfrac{1}{ab}>2\left(\dfrac{1}{bd}+\dfrac{1}{ac}-\dfrac{1}{ad}\right)$

Sol. Since a, b, c, d are in G.P., therefore

(*a*) using A.M. > G.M., we have for the first three terms

$$\frac{a+c}{2}>b \text{ i.e. } a+c>2b \qquad ...(1)$$

$$\frac{b+d}{2}>c \text{ i.e. } b+d>2c \qquad ...(2)$$

From results (1) and (2), we have $a+c+b+d>2b+2c$ i.e. $a+d>b+c$ which is the desired result.

(*b*) using G.M. > H.M., we have for the first three terms

$$b>\frac{2ac}{a+c} \text{ i.e. } ab+bc>2ac \qquad ...(3)$$

and for the last three terms $c>\dfrac{2ac}{a+c}$

i.e. $bc+cd>2bd$ ...(4)

From results (3) and (4), we have

$ab+cd+2bc>2ac+2bd$ i.e. $ab+cd>2(ac+bd-bc)$

i.e. $\dfrac{1}{cd}+\dfrac{1}{ab}>2\left(\dfrac{1}{bd}+\dfrac{1}{ac}-\dfrac{1}{ad}\right)$

[dividing both sides by abcd] which is the desired result.

Concept Application

23. If $\sum_{r=1}^{n}T_r=\dfrac{n}{8}(n+1)(n+2)(n+3)$ then find $\sum_{r=1}^{n}\dfrac{1}{T_r}$.

24. The series of natural numbers is divided into groups (1), (2, 3, 4), (5, 6, 7. 8, 9)..... and so on. Show that the sum of the numbers in n* group is $n^3+(n-1)^3$.

Short Notes

An arithmetic progression (A.P.) : a, $a + d$, $a + 2d$.....$a + (n-1)d$ is an A.P.

Let a be the first term and d be the common difference of an A.P., then n^{th} term $= t_n = a + (n-1)d$

The sum of first n terms of are A.P.

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right] = \frac{n}{2}\left[a+\ell\right]$$

r^{th} term of an A.P. when sum of first r terms is given is $t_r = S_r - S_{r-1}$.

Properties of A.P.

(i) If a, b, c are in A.P

$\Rightarrow 2b = a + c$ & if a, b, c, d are in A.P.

$\Rightarrow a + d = b + c$.

(ii) Three numbers in A.P. can be taken as $a-d, a, a+d$; four numbers in A.P. can be taken as $a-3d, a-d, a+d, a+3d$; five numbers in A.P. are $a-2d, a-d, a, a+d, a+2d$ & six terms in A.P. are $a-5d, a-3d, a-d, a+d, a+3d, a+5d$ etc.

(iii) Sum of the terms of an A.P. equidistant from the beginning & end = sum of first & last term.

Arithmetic Mean (Mean or Average) (A.M.):

If three terms are in A.P. then the middle term is called the A M. between the other two, so if a, b, c are in A.P., b isA.M. of a & c.

n–Arithmetic Means Between Two Numbers:

If a, b are any two given numbers & a, $A_1\ A_2$.....A_n, b are in A.P. then $A_1\ A_2$,... A_n are the A.M .

$$A_1 = a + \frac{b-a}{n+1}, A_2 = a + \frac{2(b-a)}{n+1}, \ldots, A_n = a + \frac{n(b-a)}{n+1}$$

$\sum_{r=1}^{n} A_r = nA$ where A is the single A M. between a & b.

Geometric Progression: a_1 ar. ar_2, ar_3, ar_4.....Is a G.P. with a as the first term & r as common ratio.

(i) n^{th} terms $= ar^{n-1}$

(ii) Sum of the first n terms i.e. $S_n = \begin{cases} \dfrac{a(r^n-1)}{r-1}, & r \neq 1 \\ na & r = 1 \end{cases}$

Geometric Means (Mean Proportional) (GM.):

If $a, b, c > 0$ are in G P . b is the G M between a & c, then $b^2 = ac$

n Geometric Means Between Positive Number a, b: If a, b are two given numbers & a. G_1, G_2.....,G_3 b are in G.P Then G_1, G_2, G_3....., Gn are n GM.s between a & b. $G_1 = a(b/a)^{1/n+1}$ Gn", $G_2 = a\,(b/a)^{2/n+1}$..$G_n = a(b/a)^{n/n+1}$

Harmonic Mean (H.M.):

If a, b, c are in HP., b is the H.M. between a & c, then $b = \dfrac{2ac}{a+c}$

H.M, H oi $a_1, a_2, ..a_n$ is given by $\dfrac{1}{H} = \dfrac{1}{n}\left[\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots\ldots + \dfrac{1}{a_n}\right]$

Relation between Means:

$G^2 = AH$, A.M. > G.M. ≥ H.M. and A.M. = G.M. = H.M.

if $a_1 = a_2 = a_3 = \ldots\ldots = a_n$

Important Results

(i) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(ii) $\sum_{r=1}^{n} ka_r = k\sum_{r=1}^{n} a_r$

(iii) $\sum_{r=1}^{n} k = nk$ where k is a constant,

(iv) $\sum_{r=1}^{n} r = 1+2+3+\ldots\ldots\ldots+n = \dfrac{n(n+1)}{2}$

(v) $\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+\ldots\ldots\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(vi) $\sum_{r=1}^{n} r^3 = 1^3+2^3+3^3+\ldots\ldots\ldots+n^3 = \dfrac{n^2(n+1)^2}{4}$

(vii) $\sum_{i<j=1}^{n} a_i a_j = (a_1+a_2+\ldots\ldots+a_n)^2 - (a_1{}^2+a_2{}^2+\ldots\ldots+a_n^2)$

Solved Examples

1. If the sum of n terms of an A.P. is $(pn + qn^2)$. where p and q are constants, find the common difference.

Sol. Given $S_n = pn + qn^2$

Changing n to n 1

$S_n - S_{n-1} = P(n-1) + q(n-1)^2$

Substracting (ii) from (i), we obtain

$S_n - S_{n-1} = pn - p(n-1) + qn^2 - q(n-1)^2$

$\Rightarrow T_{n-1} = pn - pn + p + qn^2 - qn^2 - q + 2qn$

$\Rightarrow T_n = p - q + 2qn$

Changing n to $n-1$, we have

$T_{n-1} = p - q + 2q(n-1)$

Subtracting (iv) from (i), we get

$T_n - T_{n-1} = 2qn - 2q(n-1) = 2q$,

which is constant independent of n.

Hence, the series in reference is an A.P., with common difference = 2q.

2. The difference any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120°, Find the number of the sides of the polygon.

Sol. Let the number of sides of the polygon be n, then the sum of all exterior angles = 360° and sum of all interior angles = 180°n – 360°.

Also, smallest angle = 120° and the angles form the A. P. 120°, 125°, 130°,....., 120° + $(n-1)5°$

(There are n interior angles)

Since the sum of all the angles = 180°n – 360°

120° + 125° + 130° +.....upto n terms = 180° $(n-2)$

$\Rightarrow 2x\ 120° + (n-1)5°\} = 180°\ (n-2)$

$\Rightarrow 240\,n + 5n(n-1) = 360\,n - 720$

$\Rightarrow 48 + n(n-1)$

$= 72n - 720$

$\Rightarrow n^2 - 25n + 144 = 0$ (Dividing throughout by 5)

$$n = \frac{25 \pm \sqrt{625 - 4\times1\times144}}{2\times1} = \frac{25 \pm \sqrt{49}}{2}$$

$$= \frac{25 \pm 7}{2} = 16, 9.$$

If $n = 16$, then the greatest angle becomes 120° +(16 – 1)5 = 195° > 180°, which is not possible.

Hence, $n = 9$.

($\therefore$ An interior angle of a polygon is always < 180°)

3. Between 1 and 31, m numbers have been inserted in such a way that the ratio of 7th and (m – 1)th numbers is 5 :9. Find the value of m.

Sol. Let the m numbers between 1 and 31 be $A_1, A_2, A_3, \ldots A_m$, then $1, A_1, A_2, A_3, \ldots A_m, 31$ are in A.P.

Here, 31 is $(m+2)^{th}$ term of this A.P.

$31 = 1 + \{(m+2) - 1\}\,d$ where d is the common difference

$$\Rightarrow 30 = (m+1)d \Rightarrow d = \frac{30}{m+1} \quad \text{...(i)}$$

It is given that $\dfrac{7 \text{ th number}}{(m-1)\text{th number}}$

$$\Rightarrow \frac{A_7}{A_{m-1}} = \frac{5}{9}$$

$$\Rightarrow \frac{1+7d}{31-2d} = \frac{5}{9}$$

$\Rightarrow 9 + 63d = 155 - 10d$

$\Rightarrow 73d \Rightarrow 146$

($\because A_m = A_{m-1} + d$ and $31 = Am + d = \Rightarrow 31 - A_{m-1} + d + d$)

$$\Rightarrow d = \frac{146}{73} = 2$$

From (i) and (ii), we get

$$2 = \frac{30}{m+1} \Rightarrow 2m + 2 = 30 \Rightarrow 2m = 28 \Rightarrow m = 14.$$

4. If the 4th, 10th and 16th terms of a GP. are x, y and z respectively. Prove that x, y, z are in GP.

Sol. Let the G.P. be a, ar, ar^2

4th term = $ar^3 \Rightarrow x = ar^3$

then 10th term = $ar^3 \Rightarrow x = ar^9$ (ii)

16th term = $ar^{15} \Rightarrow z = ar^{15}$ (iii)

Multiplying (i) and (iii), we get

$xz = a^2 r^3 r^{15} \Rightarrow xz = a^2 r^{18} = (ar^9)^2$

x, y, z are in G.P. (using (iii))

5. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter, find the amount spent on the postage when 8th set ol letter is mailed.

Sol. Total number of letters posted = $4 + 4\times4 + 4\times4 - 1 - \ldots\ldots$ upto 8 terms

$= 4 + 4^2 + 4^3 + \ldots\ldots$upto 8 terms

$$= \frac{4\{4^8 - 1\}}{4-1} = \frac{4}{3}(65536 - 1) = 4 \times 21845 = 87380$$

$\therefore$ Total amount spent on the postage = Rs $\frac{1}{2}\times87380$

= 87380 = Rs. 43690

6. Let a, b and c be three positive numbers than prove that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$

Sol. We know that for positive numbers AM>HM

$$\Rightarrow \frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

$$\Rightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9.$$

7. Let $f(x) = x^5 + x^3 + 5x^2 + \frac{1}{x^7} + \frac{1}{x^8} + \frac{3}{x} + 1$ be a function defined for positive values of x, then find the least value of $f(x)$.

Sol. x is positive thats why we can think about AM ≥ GM

$$f(x) = x^5 + x^3 + 5x^2 + \frac{1}{x^7} + \frac{1}{x^8} + \frac{3}{x} + 1$$

$$= x^5 + x^3 + x^2 + x^2 + x^2 + x^2 + x^2 + \frac{1}{x^7} + \frac{1}{x^8} + \frac{1}{x} + \frac{1}{x} + \frac{1}{x} + 1$$

(split always in equal parts)

$$\Rightarrow \frac{f(x)}{3} \geq (1)^{1/13} \quad \text{(using AM > GM)}$$

$f(x) > 13$

8. Find the sum of the series

$1+3+7+13+21+31+$.....up to n terms

Sol. $Sn = 1+3+7+13+21+31+.....+T_n$

$Sn = 1+3+7+13+21.....+T_{n-1}+T_n$

$0=1+(2+4+6+.....\text{up to } n-1)-T_n$

$\Rightarrow T_n = 1+n(n-1) = n^2-n+1$

$Sn = \Sigma T_n = \Sigma n^2 - \Sigma n + \Sigma 1$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n = \frac{n^3+2n}{3}$$

9. If $S_n = \sum_{r=1}^{n} \frac{r^2-2r-1}{2^{r+1}}$ then find the value of S_∞

Sol. $Sn = \sum_{r=1}^{n} \frac{r^2-2r-1}{2^{+1}}$; $Tr = \frac{r^2-2r-1}{2^{r+1}}$

Try to split Tr in two parts such that

$Tr = f(r) - f(r+1)$

$$\Rightarrow Tr = \frac{r^2}{2^r} - \frac{(r+1)^2}{2^{r+1}}$$

$$T_1 = \frac{1^2}{2} - \frac{2^2}{2^2}$$

$$T_2 = \frac{2^2}{2^2} - \frac{3^2}{2^3}$$

$$T_3 = \frac{3^2}{2^3} - \frac{4^2}{2^4}$$

$$T_n = \frac{n^2}{2^n} - \frac{(n+1)^2}{2^{n+1}}$$

$$Sn = T_1 + T_2 + ... + T_n = \frac{1}{2} - \frac{(n+1)^2}{2^{n+1}}$$

$$S_\infty = \frac{1}{2}$$

10. Let, $A_1, A_2, ..., A_m$ be non-empty subsets of $\{1, 2, 3, ..., 100\}$ satisfying the following conditions:

(1) The numbers $|A_1|, |A_2|, ..., |A_m|$ are distinct.

(2) $A_1, A_2, ..., A_m$ are pairwise disjoint.

(Here, $|A|$ denotes the number of elements in the set A. Then the maximum possible value of m is

(*a*) 13 (*b*) 14 (*c*) 15 (*d*) 16

Sol. (*a*) Each subset $A_1, A_2, ..., A_m$ has distinct no. of elements and are pair-wise disjoint.

Therefore, for maximum value of 'm'

$|A_1| = 1, |A_2| = 2, ..., |A_m| = m$

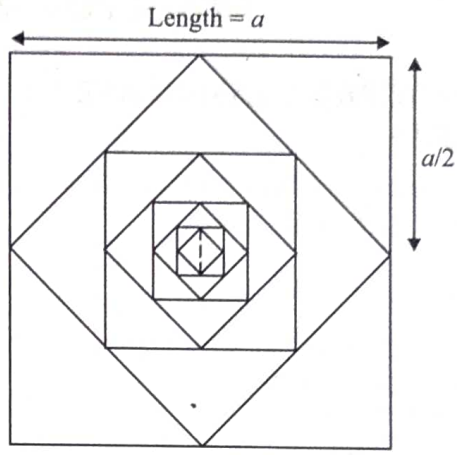
Also, $1+2+3+...+m \le 100$

So, $m(m+1)/2 \le 100$

$m < 14$

Therefore, $m = 13$

11. Let S_1 be the sum of areas of the squares whose sides are parallel to coordinate axes. Let S_2 be the sum of areas of the slanted squares as shown in the figure. Then S_1/S_2 is



(*a*) 2 (*b*) $\sqrt{2}$ (*c*) 1 (*d*) $1/\sqrt{2}$

Sol. (*a*) $S_1 = a^2 + a^2/4 + a^2/16 + ... \infty$

{Infinite G.P. First Term = a^2, Common ratio= 1/4}

$= a^2/(1-1/4)$

$= 4a^2/3$

$S_2 = a^2/2 + a^2/8 + a^2/32 + ... \infty$

{Infinite G.P. First Term = $a^2/2$, Common ratio = 1/4}

$= (a^2/2)/(1-1/4)$

$= 2a^2/3$

Therefore, $S_1/S_2 = 2$

12. Let A, G and H be the arithmetic mean. geometric mean and harmonic mean, respectively of two distinct positive real numbers. If α is the smallest of the two roots of the equation $A(G-H)x^2 + G(H-A)x + H(A-G) = 0$, then

(*a*) $-2 < \alpha < -1$ (*b*) $0 < \alpha < 1$

(*c*) $-1 < \alpha < 0$ (*d*) $1 < \alpha < 2$

Sol. (*b*) Given, $A(G-H)x^2 + G(H-A)x + H(A-G) = 0$

Put $x = 1$ in above equation

$A(G-H) + G(H-A) + H(A-G) = 0$

$0 = 0$

$\Rightarrow$ $x = 1$ is one root (let it be 'α' and other root be 'β')

Now using product of roots,

$\alpha.\beta = A(A-G)/A(G-H)$

$1.\beta = (H.A - H.G)/A(G-H)$

$\beta = (G^2 - H.G)/A(G-H)\{A.H = G^2\}$

$\beta = G(G-H)/A(G-H)$

$\beta = G/A < 1$ {as $A > G$}

Exercise-1 (Topicwise)

ARITHMETIC PROGRESSION AND ITS PROPERTIES

1. The sum of 24 terms of the following series $\sqrt{2}+\sqrt{8}+\sqrt{18}+\sqrt{32}+...$ is

(*a*) 300 (*b*) $300\sqrt{2}$

(*c*) $200\sqrt{2}$ (*d*) $100\sqrt{2}$

2. If p^{th}, the qth and r^{th} term of an arithmetic sequence are a, b and c respectively, then the value of $[a(q-r)+b(r-p)+c(p-q)]=$

(*a*) 1 (*b*) –1 (*c*) 0 (*d*) 1/2

3. Let T_r be the rth term of an A.P. for r = 1, 2, 3, ... If for some positive integers m, n we have $T_m=\frac{1}{n}$ and $T_n=\frac{1}{m}$, then T_{mn} equal equals

(*a*) $\frac{1}{mn}$ (*b*) $\frac{1}{m}+\frac{1}{n}$

(*c*) 1 (*d*) 0

4. If 1, $\log_9(3^{1-x}+2)$, $\log_3(4.3^x-1)$ are in A.P. then x equals

(*a*) $\log_3 4$ (*b*) $1-\log_3 4$

(*c*) $1-\log_4 3$ (*d*) $\log_4 3$

5. If the ratio of the sum of n terms of two A.P.'s be $(7n+1):(4n+27)$ then the ratio of their 11th terms will be

(*a*) 2 : 3 (*b*) 3 : 4

(*c*) 4 : 3 (*d*) 5 : 6

6. The ratio of the sums of first n even numbers and odd numbers will be

(*a*) $1:n$ (*b*) $(n+1):1$

(*c*) $(n+1):n$ (*d*) $(n-1):1$

7. If S_n denotes the sum of n terms of an arithmetic progression, then the value of $(S_{2n}-S_n)$ is equal to

(*a*) $2S_n$ (*b*) S_{3n}

(*c*) $\frac{1}{3}S_{3n}$ (*d*) $\frac{1}{2}S_n$

8. The sum of the integers from 1 to 100 which are not divisible by 3 or 5 is

(*a*) 2489 (*b*) 4735

(*c*) 2317 (*d*) 2632

9. Let S_n denotes the sum of n terms of an A.P.

If $S_{2n}=3_{Sn}$, then ratio $\frac{S_{3n}}{S_n}=$

(*a*) 4 (*b*) 6 (*c*) 8 (*d*) 10

10. The sum of integers from 1 to 100 that are divisible by 2 or 5 is

(*a*) 2550 (*b*) 1050

(*c*) 3050 (*d*) None of these

ARITHMETIC MEAN

11. A number is the reciprocal of the other. If the arithmetic mean of the two numbers be $\frac{13}{12}$, then the numbers are

(*a*) $\frac{1}{4},\frac{4}{1}$ (*b*) $\frac{3}{4},\frac{4}{3}$ (*c*) $\frac{2}{5},\frac{5}{2}$ (*d*) $\frac{3}{2},\frac{2}{3}$

12. If n A-M's are inserted between two quantities a and b, then their sum is equal to

(*a*) $n(a+b)$ (*b*) $\frac{n}{2}(a+b)$

(*c*) $2n(a+b)$ (*d*) $\frac{n}{2}(a-b)$

13. If a, b, c, d, e,fare AM.'s between 2 and 12, then $a+b+c+d+e+f$ is equal to

(*a*) 14 (*b*) 42

(*c*) 84 (*d*) None of these

GEOMETRIC PROGRESSION AND ITS PROPERTIES

14. If x, $2x+2$, $3x+3$, are in G.P., then the fourth term is

(*a*) 27 (*b*) –27 (*c*) 13.5 (*d*) –13.5

15. The 6th term of a G.P. is 32 and its 8th term is 128, then the common ratio of the G.P. is

(*a*) –1 (*b*) 2 (*c*) 4 (*d*) –4

16. The sum of 100 terms of the series .9 + .09 + .009 ... will be

(*a*) $1-\left(\frac{1}{10}\right)^{100}$ (*b*) $1+\left(\frac{1}{10}\right)^{106}$

(*c*) $1-\left(\frac{1}{10}\right)^{106}$ (*d*) $1+\left(\frac{1}{10}\right)^{100}$

17. The sum of the series 6 + 66 + 666 + ... upto n terms is

(*a*) $(10^{n-1}-9n+10)/81$ (*b*) $2(10^{n+1}-9n-10)/27$

(*c*) $2(10^{n}-9n-10)/27$ (*d*) $(10^{n+1}-9n-10)/27$

18. Consider an infinite G.P. with first term a and common ratio r, its sum is 4 and the second term is 3/4, then

(*a*) $a=\frac{7}{4},r=\frac{3}{7}$ (*b*) $a=\frac{3}{2},r=\frac{1}{2}$

(*c*) $a=2,r=\frac{3}{8}$ (*d*) $a=3,r=\frac{1}{4}$

19. If are a, b, c are in G.P., then

(*a*) a^2, b^2, c^2 are in G.P.

(*b*) $a^2(b+c), c^2(a+b), b^2(a+c)$ are in G.P.

(*c*) $\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}$ are in G.P.

(*d*) a^2, b^2, c^2 are in A.P.

20. The third term of a G.P. is 4. The product of the first five terms is

(*a*) 4^3 (*b*) 4^5

(*c*) 4^4 (*d*) None of these

21. If sum of the infinite G.P., $p, 1, \frac{1}{p}, \frac{1}{p^2}, \frac{1}{p^3}, \ldots$ is $\frac{9}{2}$, then value of p is

(*a*) $3, \frac{3}{2}$ (*b*) $\frac{2}{3}, 2$ (*c*) $\frac{3}{2}, 1$ (*d*) $\frac{1}{3}, 3$

GEOMETRIC MEAN

22. If G be the geometric mean of x and y, then $\frac{1}{G^2-x^2}+\frac{1}{G^2-y^2}=$

(*a*) G^2 (*b*) $\frac{1}{G^2}$

(*c*) $\frac{2}{G^2}$ (*d*) $3G^2$

23. The two geometric mean between the number 1 and 64 are

(*a*) 1 and 64 (*b*) 4 and 16

(*c*) 2 and 16 (*d*) 8 and 16

24. If a, b, c are in A.P. $b-a, c-b$ and a are in G.P., then a : b : c is

(*a*) 1 : 2 : 3 (*b*) 1 : 3 : 5

(c) 2 : 3 : 4 (*d*) 1 : 2 : 4

A.G.P.

25. $1+\frac{3}{2}+\frac{5}{2^2}+\frac{7}{2^3}+\ldots\infty$ is equal to

(*a*) 3 (*b*) 6

(*c*) 9 (*d*) 12

26. The sum of the first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots$ is

(*a*) 2^n-n-1 (*b*) $1-2^{-n}$

(*c*) $n+2^{-n}-1$ (*d*) 2^n-1

27. The sum of the series $3+33+333+\ldots+n$ terms is

(*a*) $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$ (*b*) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

(*c*) $\frac{(\alpha+\beta)^2}{\alpha\beta}-2=\frac{18}{b}-2<-2$ (*d*) None of these

28 The sum to n terms of the series $1+3+7+15+31+\ldots$ is

(*a*) $2^{n+1}-n$ (*b*) $2^{n+1}-n-2$

(*c*) 2^n-n-2 (*d*) None of these

HARMONIC PROGRESSION AND HARMONIC MEAN

29. Which number should be added to the numbers 13, 15, 19 so that the resulting numbers be the consecutive terms of H.P.

(*a*) 7 (*b*) 6

(*c*) –6 (*d*) –7

30. If 5th term of a H.P. is $\frac{1}{45}$ and 11th term is $\frac{1}{69}$, then its 16th term will be

(*a*) 1/89 (*b*) 1/85

(*c*) 1/80 (*d*) 1/79

31. If the 7th term of a H.P. is $\frac{1}{10}$ and the 12th term is $\frac{1}{25}$, then the 20th term is

(*a*) $\frac{1}{37}$ (*b*) $\frac{1}{41}$

(*c*) $\frac{1}{45}$ (*d*) $\frac{1}{49}$

32. If H is the harmonic mean between p and q, then the value of $\frac{H}{p}+\frac{H}{q}$ is

(*a*) 2 (*b*) $\frac{pq}{p+q}$

(*c*) $\frac{p+q}{pq}$ (*d*) $\frac{2pq}{p+q}$

RELATION BETWEEN A.M., G.M., H.M.

33. If the arithmetic mean of two numbers be A and geometric mean be G, then the numbers will be

(*a*) $A\pm(A^2-G^2)$

(*b*) $\sqrt{A}\pm\sqrt{A^2-G^2}$

(*c*) $A\pm\sqrt{(A+G)(A-G)}$

(*d*) $\frac{A\pm\sqrt{(A+G)(A-G)}}{2}$

34. If the arithmetic, geometric and harmonic means between two positive real numbers be A, G and H, then

(*a*) $A^2=GH$ (*b*) $H^2=AG$

(*c*) $G=AH$ (*d*) $G^2=AH$

35. If a^2, b^2, c^2 be in A.P., then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ will be in

(*a*) A.P. (*b*) G.P.

(*c*) H.P. (*d*) A.G.P.

36. If the A.M., G.M. and H.M. between two positive numbers a and b are equal, then

(*a*) $a=b$ (*b*) $ab=1$

(*c*) $a>b$ (*d*) $a<b$

37. If $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ are in H.P., then

(*a*) a^2b, c^2a, b^2c are in A.P.

(*b*) a^2b, b^2c, c^2a are in H.P.

(*c*) a^2b, b^2c, c^2a are in G.P.

(*d*) None of these

38. If a, b, c are in G.P., then $\log_a x, \log_b x, \log_c x$ are in

(*a*) A.P. (*b*) G.P.

(*c*) H.P. (*d*) None of these

MISCELLANEOUS SERIES

39. The sum of the series $1 + (1 + 2) + (1 + 2 + 3) + ...$ upto n terms, will be

(a) $n^2 - 2n + 6$ (b) $\frac{n(n+1)(2n-1)}{6}$

(c) $n^2 + 2n + 6$ (d) $\frac{n(n+1)(n+2)}{6}$

40. $\frac{1^3+2^3+3^3+4^3+.....12^3}{1^2+2^2+3^2+4^2+.....12^2} =$

(a) $\frac{234}{25}$ (b) $\frac{243}{35}$

(c) $\frac{263}{27}$ (d) $\frac{236}{35}$

41. If the nth term of a series be $3 + n(n - 1)$, then the sum of n terms of the series is

(a) $\frac{n^2+n}{3}$ (b) $\frac{n^3+8n}{3}$

(c) $\frac{n^2+8n}{5}$ (d) $\frac{n^2-8n}{3}$

42. If the sum of $1+\frac{1+2}{2}+\frac{1+2+3}{3}+.....$ to n terms is S, then S is equal to

(a) $\frac{n(n+3)}{4}$ (b) $\frac{n(n+2)}{4}$

(c) $\frac{n(n+1)(n+2)}{6}$ (d) n^2

43 The sum of the series $\frac{1}{3\times7}+\frac{1}{7\times11}+\frac{1}{11\times15}+....$ is

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$

(c) $\frac{1}{9}$ (d) $\frac{1}{12}$

44 The sum of the series $1.2\cdot3 + 2.3\cdot4 + 3.4\cdot5 + ...$ to n terms is

(a) $n(n+1)(n+2)$

(b) $(n+1)(n+2)(n+3)$

(c) $\frac{1}{4}n(n+1)(n+2)(n+3)$

(d) $\frac{1}{4}(n+1)(n+2)(n+3)$

45. The first term of an A.P. of consecutive integer is $p^2 + 1$. The sum of $(2p + 1)$ terms of this series can be expressed as

(a) $(p + 1)^2$ (b) $(2p + 1)(p + 1)^2$

(c) $(p + 1)^3$ (d) $p^3 + (p + 1)^3$

46. The sum of the series $\frac{1}{\log_2 4}+\frac{1}{\log_4 4}+\frac{1}{\log_8 4}+$

(a) $\frac{1}{2}n(n + 1)$ (b) $\frac{1}{12}n(n + 1)(2n + 1)$

(c) $\frac{1}{n(n+1)}$ (d) $\frac{1}{4}n(n + 1)$

47. Sum of the series $S = 1^2 - 2^2 + 3^2 - 4^2 + ... - 2002^2 + 2003^2$ is

(a) 2007006

(b) 1005004

(c) 2000506

(d) None of these

48. If a an b are p^{th} and q^{th} terms of an AP, then the sum of its $(p + q)$ terms is

(a) $\frac{p+q}{2}\left[a-b+\frac{a+b}{p-q}\right]$

(b) $\frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right]$

(c) $\frac{p-q}{2}\left[a+b+\frac{a+b}{p+q}\right]$

(d) None of these

49. If S is the sum of infinity of a G.P. whose first term is 'a', then the sum of the first n terms is

(a) $S\left(1-\frac{a}{S}\right)^n$ (b) $S\left[1-\left(1-\frac{a}{S}\right)^n\right]$

(c) $a\left[1-\left(1-\frac{a}{S}\right)^n\right]$ (d) None of these

50. 2. If a, b, c are in A.P. and x, y, z are in G.P., then find the value of $x^{b-c}, y^{c-a}, z^{a-b}$.

(a) 0 (b) 1

(c) 4 (d) 2

51. If $a_1, a_2, ..., a_n$ are distinct terms of an A.P., then

(a) $a_1 + 2a_2 + a_3 = 0$

(b) $a_1 + 2a_2 + a_3 = 0$

(c) $a_1 + 3a_2 - 3a_3 - a_4 = 0$

(d) $a_1 - 4a_2 + 6a_3 - 4a_4 + a_5 = 0$

52. The value of $2^{1/4} \times 4^{1/8} \times 8^{1/6} ... \infty$ is

(a) 2 (b) 1

(c) 3 (d) 0

53. If x, y, z be respectively the pth, q^{th} and r^{th} terms of G.P., then $(q - r)\log x + (r - p)\log y + (p - q)\log z =$

(a) 1 (b) 2

(c) 0 (d) None of these

54. If H be the $H.M.$ between a & b, then

(a) $(H - 2a)(H - 2b) = H^2$

(b) $(H - a)(H - b) = H^2$

(c) $(H - 2a)(H - 2b) = H$

(d) None of these

55. If x, y, z are positive then the minimum value of $x^{\log y - \log z} + y^{\log z - \log x} + z^{\log x - \log y}$ is

(a) 3 (b) 1

(c) 9 (d) 16

Exercise-2 (Learning Plus)

1. If $a_1, a_2, a_3, \ldots$ are in A.P. such that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$, then $a_1 + a_2 + a_3 + \ldots + a_{23} + a_{24}$ is equal to
   (a) 909 (b) 75
   (c) 750 (d) 900

2. The interior angles of a polygon are in A.P. If the smallest angle is 120° & the common difference is 5°, then the number of sides in the polygon is:
   (a) 7 (b) 9
   (c) 16 (d) None of these

3. The rational number, which equals the number $2.\overline{357}$ with recurring decimal is
   (a) $\frac{2355}{1001}$ (b) $\frac{2379}{997}$
   (c) $\frac{2355}{999}$ (d) None of these

4. Consider an A.P. with first term 'a' and the common difference 'd' Let S_k denote the sum of its first K terms. If $\frac{S_{kx}}{S_x}$ is independent of x, then
   (a) $a = d/2$ (b) $a = d$
   (c) $a = 2d$ (d) None of these

5. If $x \in R$, the numbers $5^{1+x} + 5^{1-x}$, $a/2$, $25^x + 25^{-x}$ form an A.P. then 'a' must lie in the interval:
   (a) [1, 5] (b) [2, 5]
   (c) [5, 12] (d) [12, ∞)

6. There are n A.M's between 3 and 54, such that the 8th mean: $(n-2)$th mean :: 3: 5. The value of n is.
   (a) 12 (b) 16
   (c) 18 (d) 20

7. The sum to 10 terms of the series is $\sqrt{2} + \sqrt{6} + \sqrt{18} + \sqrt{54} + \ldots$ is
   (a) $121\ (\sqrt{6} + \sqrt{2})$ (b) $\frac{121}{2}(\sqrt{3}+1)$
   (c) $243\ (\sqrt{3}+1)$ (d) $243\ (\sqrt{3}-1)$

8. For a sequence $\{a_n\}$, $a_1 = 2$ and $\frac{a_{n+1}}{a_n} = \frac{1}{3}$. Then $\sum_{r=1}^{20} a_r$ is
   (a) $\frac{20}{2}[4 + 19 \times 3]$ (b) $3\left(1 - \frac{1}{3^{20}}\right)$
   (c) $2(1 - 3^{20})$ (d) None of these

9. α, β be the roots of the equation $x^2 - 3x + a = 0$ and γ, δ the roots of $x^2 - 12x + b = 0$ and numbers α, β, γ, δ (in this order) form an increasing G.P., then
   (a) $a = 3, b = 12$ (b) $a = 12, b = 3$
   (c) $a = 2, b = 32$ (d) $a = 4, b = 16$

10. In a G.P. of positive terms, any term is equal to the sum of the next two terms. The common ratio of the G.P. is
    (a) 2 cos 18° (b) sin 18°
    (c) cos 18° (d) 2 sin 18°

11. If p is positive, then the sum to infinity of the series, $\frac{1}{1+p} - \frac{1-p}{(1+p)^2} + \frac{(1-p)^2}{(1+p)^3} - \ldots\ldots$ is
    (a) 1/2 (b) 3/4
    (c) 1 (d) None of these

12. If $x > 0$, and $\log_2 x + \log_2 (\sqrt{x}) + \log_2 (\sqrt[4]{x}) + \log_2 (\sqrt[8]{x}) + \log_2 (\sqrt[16]{x}) + \ldots = 4$, then x equals
    (a) 2 (b) 3
    (c) 4 (d) 5

13. If $3 + \frac{1}{4}(3 + d) + \frac{1}{4^2}(3 + 2d) + \ldots +$ upto $\infty = 8$, then the value of d is
    (a) 9 (b) 5
    (c) 1 (d) None of these

14. If the sum of the roots of the quadratic equation, $ax^2 + bx + c = 0$ is equal to sum of the squares of their reciprocals, then $\frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in
    (a) A.P. (b) G.P.
    (c) H.P. (d) None of these

15. If $x = \sum_{n=0}^{\infty} a^n, y = \sum_{n=0}^{\infty} b^n, z = \sum_{n=0}^{\infty} c^n$ where a, b, c are in AP and $|a| < 1, |b| < 1, |c| < 1$, then x, y, z are in
    (a) HP
    (b) Arithmetic-Geometric Progression
    (c) AP
    (d) GP

16. If a, b, c are in A.P. p, q, r are in H.P. and ap, bq, cr are in G.P., then $\frac{p}{r} + \frac{r}{p}$ is equal to
    (a) $\frac{a}{c} + \frac{c}{a}$ (b) $\frac{a}{c} - \frac{c}{a}$
    (c) $\frac{b}{q} + \frac{q}{b}$ (d) $\frac{b}{q} - \frac{a}{p}$

17. If $a^x = b^y = c^z = d^t$ and a, b, c, d are in G.P., then x, y, z, t are in
    (a) A.P. (b) G.P.
    (c) H.P. (d) None of these

18. If A, G & H are respectively the A.M., G.M. & H.M. of three positive numbers a, b, & c, then the equation whose roots are a, b & c is given by:

(a) $x^3 - 3Ax^2 + 3G^3x - G^3 = 0$
(b) $x^3 - 3Ax^2 + 3(G^3/H)x - G^3 = 0$
(c) $x^3 + 3Ax^2 + 3(G^3/H)x - G^3 = 0$
(d) $x^3 - 3Ax^2 - 3(G^3/H)x + G^3 = 0$

19. The sum $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$ is equal to

(a) 1 (b) 3/4
(c) 4/3 (d) None of these

20. If $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+ \ldots$ upto $\infty = \frac{\pi^2}{6}$, then $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+ \ldots =$

(a) $\pi^2/12$ (b) $\pi^2/24$
(c) $\pi^2/8$ (d) None of these

21. The sum of the first n-terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the sum is

(a) $\frac{n(n+1)^2}{4}$ (b) $\frac{n^2(n+2)}{4}$
(c) $\frac{n^2(n+1)}{2}$ (d) $\frac{n(n+2)^2}{4}$

22. $1^2 + 2^2 + \ldots + n^2 = 1015$, then value of n is

(a) 15 (b) 14
(c) 13 (d) None of these

23. $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)$ is equal to:

(a) –1 (b) 0
(c) 1 (d) None of these

24. If $[a_n]$ is an arithmetic sequence, then $\Delta = \begin{vmatrix} a_m & a_n & a_p \\ m & n & p \\ 1 & 1 & 1 \end{vmatrix}$ equal

(a) 1 (b) –1
(c) 0 (d) None of these

25. If the sum of first p terms, first q terms and first r terms of an A.P. be x, y and z respectively, then $\frac{x}{p}(q-r)+\frac{y}{q}(r-p)+\frac{z}{r}(p-q)$ is

(a) 0 (b) 2
(c) pqr (d) $\frac{8xyz}{pqr}$

26. If a, b, c are in A.P. and $|a|, |b|, |c| < 1$, and

$x = 1 + a + a^2 + \ldots\infty$

$y = 1 + b + b^2 + \ldots\ldots \infty$

Then x, y, z shall be in

(a) A.P. (b) G.P.
(c) H.P. (d) None of these

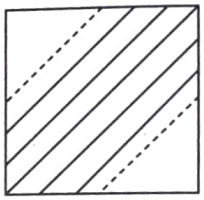
27. If A_1, A_2; G_1, G_2 and H_1, H_2 be two A.M.'s, G.M.'s and H.M.'s between two quantities, then the value of $\frac{G_1G_2}{H_1H_2}$ is

(a) $\frac{A_1+A_2}{H_1+H_2}$ (b) $\frac{A_1-A_2}{H_1+H_2}$
(c) $\frac{A_1+A_2}{H_1-H_2}$ (d) $\frac{A_1-A_2}{H_1-H_2}$

28. If the ratio of A.M. between two positive real numbers a and b to their H.M. is $m : n$, then $a : b$ is

(a) $\frac{\sqrt{m-n}+\sqrt{n}}{\sqrt{m-n}-\sqrt{n}}$ (b) $\frac{\sqrt{n}+\sqrt{m-n}}{\sqrt{n}-\sqrt{m-n}}$
(c) $\frac{\sqrt{m}+\sqrt{m-n}}{\sqrt{m}-\sqrt{m-n}}$ (d) None of these

29. In the given square, a diagonal is drawn and parallel line segments joining points on the adjacent sides are drawn on both sides of the diagonal. The length of the diagonal is $n\sqrt{2}\,cm$. If the distance between consecutive line segments be $\frac{1}{\sqrt{2}}cm$ then the sum of the lengths of all possible line segments and the diagonal is



(a) $n(n+1)\sqrt{2}\,cm$ (b) n^2cm
(c) $n(n+2)cm$ (d) $n^2\sqrt{2}\,cm$

30. Greatest integer by which $1+\sum_{r=1}^{30} r\times r!$ is divisible is

(a) Composite number (b) Odd number
(c) Divisible by 3 (d) None of these

31. If $\sum_{r=1}^{n} r^4 = I(n)$ then $\sum_{r=1}^{n}(2r-1)^4$ is equal to

(a) $I(2n) - I(n)$ (b) $I(2n) - 16I(n)$
(c) $I(2n) - 16I(n)$ (d) $I(2n) - 4I(n)$

32. If $x_1, x_2, \ldots, x_{20}$ are in H.P. and $x_1, 2, x_{20}$ are in G.P.. then $\sum_{r=1}^{19} x_r x_{r+1} =$

(a) 76 (b) 80
(c) 84 (d) None of these

33. The sum of the series $\frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}+\cdots$, to infinite terms, if $|x| < 1$, is

(a) $\frac{x}{1-x}$ (b) $\frac{1}{1-x}$
(c) $\frac{1+x}{1-x}$ (d) 1

34. If $1^2 + 2^2 + 3^2 + ... + 2003^2 = (2003)(4007)(334)$ and $(1)(2003) + (2)(2002) + (3)(2001) + ... + (2003)(1) = (2003)(334)(x)$, then x equals

(a) 2005 (b) 2004
(c) 2003 (d) 2001

35. If x, $|x + 1|$, $|x - 1|$ are three terms of an A.P., then its sum upto 20 terms is:

(a) 180 (b) 350
(c) 90 (d) 720

36. Let $t_r = 2^{r/2} + 2^{-r/2}$. Then $\sum_{r=1}^{10} t_r^2$ is equal to

(a) $\frac{2^{21}-1}{2^{10}} + 20$ (b) $\frac{2^{21}-1}{2^{10}} + 19$
(c) $\frac{2^{21}-1}{2^{10}} - 1$ (d) $\frac{2^{21}-1}{2^{10}} - 19$

37. If $a_1\ a_2\ a_3 \ldots a_n$ are in H. P. then $a_1a_2 + a_2a_3 + a_3a_4 + ... + a_{n-1}\ a_n$ is equal to

(a) $(n-1)\ a_1a_n$ (b) na_1a_n
(c) $\frac{n}{2}(a_1 + a_n)$ (d) $(n+1)\ a_1a_n$

38. Sides of a triangle ABC; a, b, c are in $G.P.$ If 'r' be the common ratio of this of this $G.P.$, then

(a) $r \in \left(\frac{\sqrt{5}-1}{2}, \infty\right)$ (b) $r \in \left(\frac{\sqrt{5}+1}{2}, \infty\right)$
(c) $r \in \left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right)$ (d) $r \in \left(\frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+3}{2}\right)$

39. If $a, b, c, d \in R^+$ and a, b, c, d are in $H.P.$, then

(a) $a + d > b + c$ (b) $a + b > c + d$
(c) $a + c > b + d$ (d) $a + b < c + d$

40. Let r_1, r_2, r_3, r_4 be the roots of the equation, $x^4 - 9x^3 + ax^2 + bx + 16 = 0$ where a, b are constants. Then the difference between the arithmetic mean $\frac{r_1 + r_2 + r_3 + r_4}{4}$ and the geometric mean $\sqrt[4]{r_1r_2r_3r_4}$ of the roots, is equal to

(a) 5/2 (b) 1/2
(c) 17/4 (d) 1/4

41. If $a_1, a_2, a_3, \ldots a_{2n+1}$, are in A.P., then $\frac{a_{2n+1} - a_1}{a_{2n+1} + a_1} + \frac{a_{2n} - a_2}{a_{2n} + a_2} + ... + \frac{a_{n+2} - a_n}{a_{n+2} + a_n}$ is equal to

(a) $\frac{n(n+1)}{2}\frac{(a_2 - a_1)}{a_{n+1}}$ (b) $\frac{n(n+1)}{2}$
(c) $\frac{(n-1)n(a_2 - a_1)}{a_{n+1}}$ (d) $\frac{n(n-1)(n+1)(a_2 - a_1)}{a_{n+1}}$

42. Let x_n, y_n, z_n, w_n denotes nth terms of four different arithmetic progressions with positive terms. If $x_4 + y_4 + z_4 + w_4 = 8$ and $x_{10} + y_{10} + z_{10} + w_{10} = 20$. then maximum value of $x_{20} \cdot y_{20} \cdot z_{20} \cdot w_{20}$ is

(a) 10^4 (b) 10^6
(c) 10^8 (d) 10^{10}

43. The sum of all positive integers n for which $\frac{1^3 + 2^3 + ... + (2n)^3}{1^2 + 2^2 + ... + n^2}$ is also an integer is.

(a) 8 (b) 9
(c) 15 (d) Infinite

44. Let x and y be two 2-digit numbers such that y is obtained by reversing the digits of x. Suppose they also satisfy $x^2 - y^2 = m^2$ for some positive integerm. The value of $x + y + m$ is.

(a) 88 (b) 112
(c) 144 (d) 154

45. The value of $\sum_{n=0}^{1947} \frac{1}{2^n + \sqrt{2^{1947}}}$ is equal to

(a) $\frac{487}{\sqrt{2^{1945}}}$ (b) $\frac{1946}{\sqrt{2^{1947}}}$
(c) $\frac{1947}{\sqrt{2^{1947}}}$ (d) $\frac{1948}{\sqrt{2^{1947}}}$

46. If n is the smallest natural number such that $n + 2n + 3n + ... + 99n$ is a perfect square, then the number of digits in n^2 is

(a) 1 (b) 2
(c) 3 (d) more than 3

47. Let a_n, $n \geq 1$, be an arithmetic progression with first term 2 and common difference 4, Let M_n be the average of the first n terms. Then the sum $\sum_{n=1}^{10} M_n$ is

(a) 110 (b) 335
(c) 770 (d) 1100

48. The number of three digit numbers $\overline{abc}$ such that the arithmetic mean of b and c and the square of their geometric mean are equal is :

(a) 9 (b) 18
(c) 36 (d) 54

49. If $a, b, c \in R^+$ then the minimum value of $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2)$ is equal to

(a) abc (b) $2abc$
(c) $3abc$ (d) $6abc$

50. If $T_r = \frac{r}{r^4 + 4}$ and $S_n = \sum_{r=1}^{n} t_r$ then the value of $37S_5 - \frac{7}{26}$ is equal to

(a) 13 (b) 25/26
(c) 1/2 (d) 7/26

51. If S denote the sum to infinity and S_n the sum of n terms of the series $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$ such that $S-S_n<\frac{1}{300}$, then the least value of n is

(a) 4 (b) 5
(c) 6 (d) 7

52. If a_1, a_2, a_3..... are in A.P. and $b_k = a_k + a_{k+1} + \ldots\ldots + a_{k+n-1}$ ($k = 1, 2, 3, \ldots$), then $b_1 + b_2 + \ldots + b_n$ is equal to

(a) $n(n+1)a_n$ (b) $(n-1)na_n$
(c) n^2a_n (d) $(n+1)^2a_n$

53. The first two terms of a geometric progression add up to 12. The sum of third and fourth terms is 48. If the terms are alternately positive and negative, then

(a) Common ratio of G.P. is –2
(b) Common ratio of G.P. is –3
(c) Sixth term of G.P. is 128
(d) None of these

54. The sides of a right triangle form a G.P. The tangent of the smallest angle is

(a) $\sqrt{\frac{\sqrt{5}+1}{2}}$ (b) $\sqrt{\frac{\sqrt{5}-1}{2}}$
(c) $\sqrt{\frac{2}{\sqrt{5}+1}}$ (d) $\sqrt{\frac{2}{\sqrt{5}-1}}$

55. If b_1, b_2, b_3 ($b_i > 0$) are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds is given by

(a) $r > 3$ (b) $0 < r < 1$
(c) $r = 3.5$ (d) $r = 5.2$

56. If the arithmetic mean of two positive numbers a & b ($a > b$) is twice their geometric mean, then $a : b$ is:

(a) $2+\sqrt{3} : 2-\sqrt{3}$ (b) $7+4\sqrt{3} : 1$
(c) $1 : 7-4\sqrt{3}$ (d) $2 : \sqrt{3}$

57. If $S_n = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots$, then

(a) $S_{40} = -820$ (b) $S_{2n} > S_{2n-1}$
(c) $S_{51} = 1326$ (d) $S_{2n+1} > S_{2n-1}$

58. Given that $x + y + z = 15$ when a, x, y, z, b are in A.P. and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{3}$ when a, x, v, z, b are in H.P. Then

(a) G.M. of a and b is 3
(b) one possible value of $a + 2b$ is 11
(c) A.M. of a and b is 6
(d) greatest value of $a - b$ is 8

59. Let $a_1, a_2, a_3, \ldots, a_n$ be in G.P. such that $3a_1 + 7a_2 + 3a_3 - 4a_5 = 0$. Then common ratio of G.P. can be

(a) 2 (b) $\frac{3}{2}$
(c) $\frac{5}{2}$ (d) $-\frac{1}{2}$

60. Let $E = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ Then,

(a) $E < 3$ (b) $E > 3/2$
(c) $E > 2$ (d) $E < 2$

61. If A_1, A_2; G_1, G_2; and H_1, H_2 are two arithmetic, geometric and harmonic means respectively, between two quantities a and b, then ab is equal to

(a) A_1H_2 (b) A_2H_1
(c) $G_1 G_2$ (d) None of these

62. Let $S_1 S_2$,.... be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1} If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 sq. cm?

(a) 7 (b) 8
(c) 9 (d) 10

63. $\frac{1}{\sqrt{2}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{8}}+\frac{1}{\sqrt{8}+\sqrt{11}}+\cdots n$ terms value of the abovbe expression is

(a) $\frac{\sqrt{3n+2}-\sqrt{2}}{3}$ (b) $\frac{n}{\sqrt{2+3n}+\sqrt{2}}$
(c) Less than n (d) Less than $\sqrt{\frac{n}{3}}$

64. If the sum of the first 100 terms of an AP is –1 and the sum of the even terms lying in the first 100 terms is 1, then

(a) Common difference is $\frac{3}{50}$
(b) First term is $-\frac{149}{50}$
(c) Common difference is $\frac{7}{50}$
(d) First term is $-\frac{151}{50}$

Comprehension (No. 65 to 67): We know that $1 + 2 + 3 + \ldots\ldots = \frac{n(n+1)}{2} = f(n)$,

$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6} = g(n)$,

$1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = h(n)$

65. $g(n) - g(n-1)$ must be equal to

(a) n^2 (b) $(n-1)^2$
(c) $n-1$ (d) n^3

66. Greatest even natural number which divides $g(n) - f(n)$, for every $n \geq 2$, is

(a) 2 (b) 4
(c) 6 (d) None of these

67. $f(n) + 3g(n) + h(n)$ is divisible by $1 + 2 + 3 + \ldots + n$

(a) Only if $n = 1$ (b) Only if n is odd
(c) Only if n is even (d) For all $n \in N$

68. Match the column

	Column-I		Column-II
A.	If $\log_x y$, $\log_z x$, $\log_y z$ are in G.P., $xyz = 64$ and x^3, y^3, z^3 are in A.P., then $\frac{3x}{y}$ is equal to	p.	2
B.	The value of $2^{\frac{1}{4}} \cdot 4^{\frac{1}{8}} \cdot 8^{\frac{1}{16}} \ldots \infty$ is equal to	q.	1
C.	If x, y, z are in A.P., then $(x + 2y - z)(2y + z - x)(z + x - y) = kxyz$, where $k \in N$, then k is equal to	r.	3
D.	There are m A.M. between 1 and 31. If the ratio of the 7th and $(m-1)$th means is 5 : 9, then $\frac{m}{7}$ is equal to	s.	4

(*a*) A → r, B → p, C → s, D → q

(*b*) A → q, B → p, C → s, D → r

(*c*) A → s, B → p, C → r, D → q

(*d*) A → p, B → r, C → s, D → q

69. An infinite G.P. is selected from $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ to converge to 1/7. If $1/2^a$ is the first term of such a G.P., find a. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$

70. Find the value of the expression $\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1$

71. Solve the following equations for x and y

$\log_{10}x + \log_{10}x^{1/2} + \log_{10}x^{1/4} + \ldots\ldots\ldots\infty = y$

$$\frac{1+3+5+\ldots\ldots\ldots+(2y-1)}{4+7+10+\ldots\ldots\ldots+(3y+1)} = \frac{20}{7\log_{10}x}$$

72. Find the nth term and the sum of n terms of the series 1.2.4 + 2.3.5 + 3.4.6 + ...

73. Find the sum to n-terms 1 + 4 + 10 + 22 + ...

74. Find the sum of the products of the integers 1, 2, 3, ..., n taken two at a time, and show that it is equal to half the excess of the sum of the cubes of the given integers over the sum of their squares.

75. After travelling for 30 minutes a train meets with an accident, due to which it has to stop for 45 minutes. Due to the accident its speed is also reduced to 2/3 of its former value and the train reaches its destination 1 hour 30 minutes late. Had the accident occurred 60 km later, the train would have reached 30 minutes earlier. The length of journey is .....

76. Let $a_1, a_2, a_3, \ldots a_n, \ldots$ be in A.P. If $a_3 + a_7 + a_{11} + a_{15} = 72$, then the sum of its first 17 terms is equal to:

77. If 8 harmonic means are inserted between two numbers a and b $(a < b)$ such that arithmetic mean of a and b is 5/4 times equal to geometric mean of a and b then $\left(\frac{H_8 - a}{b - H_8}\right)$ is equal to.

78. If $a = \sum_{r=1}^{\infty} \frac{1}{r^2}, b = \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}$, then $3a/b$ is equal to

79. If positive number x, y, z are in A.P. then the minimum value of $\frac{x+y}{2y-x} + \frac{y+z}{2y-z}$ is equal to ____.

80. Given $(2 + 6 + 10 + 14 + \ldots + 2a) + (2 + 6 + 10 + 14 + \ldots + 2b) = (2 + 6 + 10 + 14 + \ldots + 2c)$, where $a + b + c = 21$ and $a > 6$. Then $a - b$ is equal to

81. Show that number $\underbrace{1111.....1}_{91 \text{ digits}}$ is a composite number.

82. If A, B and C are the angles of a triangle, prove that:

(*i*) $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \le \frac{1}{8}$

(*ii*) $\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) \le \frac{3\sqrt{3}}{8}$

(*iii*) $\cos A + \cos B + \cos C \le \frac{3}{2}$.

(*iv*) $\tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) \ge 1$.

83. If a, b, c are positive real numbers representing the sides of a scalene triangle. prove that $ab + bc + ca < a^2 + b^2 + c^2 < 2(ab + bc + ca)$ or, $1 < \frac{a^2+b^2+c^2}{ab+bc+ca} < 2$ and hence prove that $3(ab + bc + ca) < (a + b + c)^2 < 4(ab + bc + ca)$ or $3 < \frac{(a+b+c)^2}{ab+bc+ca} < 4$.

84. Prove that $\left[\frac{a^2+b^2}{a+b}\right]^{a+b} > a^a b^b$

85. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P. then prove $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ in A.P.

86. If $a = \underbrace{111.....1}_{55}$, $b = 1 + 10 + 10^2 + 10^3 + 10^4$ and $c = 1 + 10^5 + 10^{10} + \ldots\ldots 10^{50}$, then prove that

(i) 'a' is a composite number

(ii) $a = bc$.

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. "The sum of 20 terms of the series whose r^{th} term is given by $T(n) = (-1)^n \frac{n^2+n+1}{n!}$

(a) $\frac{20}{19!} - 2$ (b) $\frac{21}{20!} - 1$

(c) $\frac{21}{20!}$ (d) None of these

2. In a geometric series, the first term is a and common ratio is r.

If S_n denotes the sum of the n terms and $U_n = \sum_{n=1}^{n} S_n$ then $rS_n + (1-r)U_n$ equals

(a) 0 (b) n

(c) na (d) nar

3. The line $x + y = 1$ meets x-axis at A and y-axis at B, P is the mid-point of AB;

P is the foot of die perpendicular from P to OA;

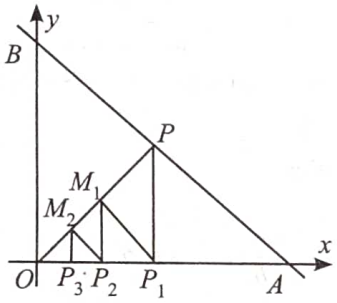
M_1 is that of P_1 from OP;

P_2 is that of M_1 from OA;

M_3 is that of P_2 from OP;

P_3 is that of M_2. from OA; and so on.

If P_n denotes the nth' foot of the perpendicular on OA; then OP is



(a) $\left(\frac{1}{2}\right)^{n-1}$ (b) $\left(\frac{1}{2}\right)^{n}$

(c) $\left(\frac{1}{2}\right)^{n+1}$ (d) None of these

4. If x, y, and z are distinct prime numbers, then

(a) x, y and z may be in A.P. but not in G.P.

(b) x, y, and z may be in G.P. but not in A.P.

(c) x, y, and z can neither be in A.P. nor in G.P.

(d) none of these

5. Let x, y, z be positive reals. Which of the following implies $x = y = z$?

I. $x^3 + y^3 + z^3 = 3xyz$

II. $x^3 + y^2z + yz^2 = 3xyz$

III. $x^3 + y^2z + z^2x = 3xyz$

IV. $(x + y + z)^3 = 27xyz$

(a) I, IV only (b) I, II, IV only

(c) I, II and III only (d) All of them

6. Let $a_1, a_2, \ldots\ldots\ldots a_{100}$ be non-zero real numbers such that $a_1 + a_2 + \ldots\ldots + a_{100} = 0$, Then

(a) $\sum_{i=1}^{100} a_i 2^{a_1} > 0$ and $\sum_{i=1}^{100} a_i 2^{-a_1} < 0$

(b) $\sum_{i=1}^{100} a_i 2^{a_1} \geq 0$ and $\sum_{i=1}^{100} a_i 2^{-a_1} \geq 0$

(c) $\sum_{i=1}^{100} a_i 2^{a_1} \leq 0$ and $\sum_{i=1}^{100} a_i 2^{-a_1} \geq 0$

(d) $\sum_{i=1}^{100} a_i 2^{a_1}$ or $\sum_{i=1}^{100} a_i 2^{-a_1}$ depends on the choice of a_i's

7. Let a and b be two positive real numbers such that $a + 2b \leq 1$. Let A_1 and A_2 be, respectively, the areas of circles with radil ab^3 and b^2. Then the maximum possible value of A_1/A_2 is

(a) 1/16 (b) 1/64

(c) $1/16\sqrt{2}$ (d) 1/32

8. Let $a_1, a_2, \ldots, a_n$ be the terms of a G.P. whose common ratio is r. set S_k denotes the sum of first k terms of the G.P. then the value of $\sum_{1 \leq i < j \leq m} a_i a_j$ in terms of S_{m-1} and S_m is

(a) $rS_{m-1}S_m$ (b) $\frac{r}{1+r} S_{m-1}S_m$

(c) $\frac{S_{m-1} S_m}{1+r}$ (d) $S_{m-1} S_m$

9. Let a, b, c be the sides of a triangle whose perimeter is P and area is A, then

(a) $P^3 \leq 27(b+c-a)(c+a-b)(a+b-c)$

(b) $P^2 \leq 3(a^2+b^2+c^2)$

(c) $a^2+b^2+c^2 \geq 4\sqrt{3}A$

(d) $P^4 < 256A^2$

10. Consider a cuboid all of whose edges are integers and whose base is square. Suppose the sum of all its edges are numerically equal to the sum of the areas of all its six faces. Then the sum of all its edges is

(a) 12 (b) 18 (c) 24 (d) 36

11. If positive numbers a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then

(a) $a = b = c$ (b) $2b = a + c$

(c) $b^2 = \sqrt{\frac{ac}{8}}$ (d) None of these

12. Let a, x, b be in A.P; a, y, b be in G.P. and a, z, b be in H.P. If $x = y + 2$ and $a = 5z$ then

(a) $y^2 = xz$ (b) $x > y > z$

(c) $a = 9, b = 1$ (d) $a = 1/4, b = 9/4$

13. If $\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$, then

(a) $a + c = b + d$

(b) $e = 0$

(c) $a, b - 2/3, c - 1$ are in A.P.

(d) c/a is an integer

14. The value of $\sum_{r=1}^{n} \frac{1}{\sqrt{a+rx} + \sqrt{a+(r-1)x}}$ is

(a) $\frac{n}{\sqrt{a} + \sqrt{a+nx}}$ (b) $\frac{n}{\sqrt{a} - \sqrt{a+nx}}$

(c) $\frac{\sqrt{a+nx} - \sqrt{a}}{x}$ (d) $\frac{\sqrt{a} + \sqrt{a+nx}}{x}$

15. For the series, $S = 1 + \frac{1}{(1+3)}(1+2)^2 + \frac{1}{(1+3+5)}(1+2+3)^2 + \frac{1}{(1+3+5+7)}(1+2+3+4)^2 + \cdots$

(a) 7th term is 16

(b) 7th term is 18

(c) sum of first 10 terms is $\frac{505}{4}$

(d) sum of first 10 terms is $\frac{405}{4}$

16. 5. The consecutive digits of a three digit number are in G.P. If the middle digit be increased by 2, then they form an A.P. If 792 is subtracted from this, then we get the number constituting of same three digits but in reverse order. Then number is divisible by

(a) 7 (b) 49

(c) 49 (d) None of these

17. Consider a sequence $\{a_n\}$ with $a_1 = 2$ and $a_n = \frac{a_{n-1}^2}{a_{n-2}}$ for all $n \geq 3$, terms of the sequence being distinct. Given that a_2 and a_5 are positive integers and $a_5 \leq 162$ then the possible value(s) of a_5 can be

(a) 162 (b) 64

(c) 32 (d) 2

COMPREHENSION BASED QUESTIONS

Comprehension (Q. No. 18 to 20): The sum of squares of three distinct real numbers which form an increasing GP is S^2 (common ratio is r). If sum of numbers is αS, then

18. If $r = 3$ then α^2 cannot lie in

(a) $\left(\frac{1}{3}, 1\right)$ (b) $(1, 2)$

(c) $\left(\frac{1}{3}, 3\right)$ (d) $(1, 3)$

19. If $\alpha^2 = 2$, then the value of common ratio r is greater than

(a) 9 (b) 4

(c) 2 (d) 3

20. If $r = 2$, then the value of α^2 is $\frac{a}{b}$ (where a and b are coprime) then $a + b$ is

(a) 3 (b) 5

(c) 8 (d) 10

Comprehension (Q. No. 21 to 23): Consider three distinct real numbers a, b, c in a G.P. with $a^2 + b^2 + c^2 = t^2$ and $a + b + c = \alpha t$. Sum of the common ratio and its reciprocal is denoted by S.

21. Complete set of α^2 is

(a) $\left(\frac{1}{3}, 3\right)$ (b) $\left[\frac{1}{3}, 3\right]$

(c) $\left(\frac{1}{3}, 3\right) - \{1\}$ (d) $\left(-\infty, \frac{1}{3}\right) \cup (3, \infty)$

22. Complete set of S is

(a) $(-2, 2)$ (b) $(-\infty, -2) \cup (2, \infty)$

(c) $(-1, 1)$ (d) $(-\infty, -1) \cup (1, \infty)$

23. If a, b, and c also represent the sides of a triangle, then the complete set of α^2 is

(a) $\left(\frac{1}{3}, 3\right)$ (b) $(2, 3)$

(c) $\left[\frac{1}{3}, 2\right]$ (d) $\left(\frac{\sqrt{5}+3}{2}, 3\right)$

Comprehension (Q. No. 24 to 26): Let $A_1, A_2, \ldots, A_m$ be the arithmetic means between -2 and 1027

and $G_1\ G_2\ G_3, \ldots\ldots G_n$ be the geometric means between 1 and 1024. The product of geometric means is 245 and sum of arithmetic means is 1025×171.

24. The value of $\sum_{r=1}^{n} G_r$ is

(a) 512 (b) 2046

(c) 1022 (d) None of these

25. The number of arithmetic means is

(a) 442 (b) 342

(c) 378 (d) None of these

26. The numbers $2\ A_{|?|}\ G^2{}_5 + \mathrm{I}$, $2A_{172}$, are in

(a) A.P. (b) G.P.

(c) H.P. (d) None of these

Comprehension (Q. No. 27 to 29): The numbers a, b, and c are between 2 and 18, such that

(i) their sum is 25

(ii) the numbers 2, a, and b are consecutive terms of an A.P.

(iii) the numbers b, c, 18 are consecutive terms of a G.P.

27. The value of abc is

(*a*) 500 (*b*) 450

(*c*) 720 (*d*) None of these

28. Roots of the equation $ax^2 + bx + c = 0$ are

(*a*) Real and positive

(*b*) Real and negative

(*c*) Imaginary

(*d*) Real ans of opposite sign

29. If a,b and c are roots of the equation $x^3 + qx^2 + rx + s = 0$, then the value of r is

(*a*) 184 (*b*) 196

(*c*) 224 (*d*) None of these

MATCH THE COLUMN TYPE QUESTIONS

Column-I		Column-II		Column-III	
I.	a, b, c, d are in A.P.	i.	$ab+cd>2bc$	P.	$ac < bd$
II.	a, b, c, d are in G.P.	ii.	$bc > ad$	Q.	$ab+cd>2ad$
III.	ab, bc, cd are in G.P.	iii.	$ab+bc<ad$	R.	$ab+bc+cd>3ad$
IV.	ab, bc, cd are in H.P.	iv.	$a+d>b+c$	S.	$a+c<b+d$

30. Which of the following options is correct?

(*a*) (I) – (iii) – (Q) (*b*) (I) – (ii) – (R)

(*c*) (IV) – (iii) – (P) (*d*) (III) – (ii) – (R)

31. Which of the following options is correct?

(*a*) (II) - (iv) - (R) (*b*) (I) - (ii) - (P)

(*c*) (III) - (ii) - (R) (*d*) (IV) - (iii) - (Q)

32. Which of the following options is correct?

(*a*) (I) - (ii) - (S) (*b*) (II) - (iv) - (P)

(*c*) (III) - (i) - (Q) (*d*) (III) - (ii) - (R)

33. Match the column

	Column-I		Column-II
A.	If $\log_5 2$, $\log_5(2^x - 5)$ and $\log_5(2^x - 7/2)$ are in A.P., then value of $2x$ is equal to	p.	2
B.	Sum of infinite series $4+\frac{8}{3}+\frac{12}{3^2}+\frac{16}{3^3}+\ldots$ is	q.	5
C.	The length, breadth, height of a rectangular box are in G.P. The volume is 27, the total surface area is 78. Then the length is	r.	3
D.	Between 1 and 31, m numbers have been inserted in such a way that the resulting sequence is an A.P. and the ratio of 7th and $(m-1)^{th}$ numbers is 5 : 9. Find the value of m.	s.	4

(*a*) A → r, B → p, C → s, D → q

(*b*) A → r, B → p, C → s, D → p

(*c*) A → s, B → p, C → r, D → q

(*d*) A → p, B → r, C → s, D → q

34.

	Column -I		Column -I
A.	If a, b, c are in G.P., then $\log_c 10$, $\log_b 10$. $\log_c 10$ are in.	p.	A. P
B.	If $\frac{a+be^x}{a-be^x} = \frac{b+ce^x}{b-ce^x} \frac{c+de^x}{c-de^x}$, then a,b,c,d are in	q.	H. P
C.	If a, b, c are in A.P.; a, x, b are in G.P. and b, y, c are in G.P., then x^2, b^2, y^2 are in	r.	G.P
D.	If x, y, z are in G.P., $a^x = b^x = c^z$ then $\log a$, $\log b$, $\log c$ are in	s.	None of these

(*a*) A → r, B → p, C → s, D → q

(*b*) A → q, B → r, C → p, D → r

(*c*) A → s, B → p, C → r, D → q

(*d*) A → p, B → r, C → s, D → q

35.

	Column -I		Column -I
A.	If $\Sigma n = 210$, then Σn^2 is divisible by the greatest prime number which is greater than	p.	16
B.	Between 4 and 2916 is inserted odd number $(2n + 1)$ G.M's. Then the $(n + 1)$ th G.M. is divisible by greatest odd integer which is less titan	q.	10
C.	In a certain progression, four consecutive terms are 40, 30, 24, 20. Then the integral part of the next term of the progression is more than	r.	34
D.	$1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\cdots$ where H.C.F.$(a, b) = 1$, then $a - b$ is less than	s.	30

(*a*) A → q, r, s; B → r, s; C → p, q; D → r, s

(*b*) A → p, q, r, s; B → r, s; C → p, q; D → r, s

(*c*) A → p, q, r, s; B → r, p; C → p, q; D → r, s

(*d*) A → p, q, r, s; B → r; C → p, q; D → r, s

NUMERICAL BASED QUESTIONS

36. First term of an A.P. of non-constant terms is 3 and its second, tenth and thirty-fourth terms form a G.P., find the common difference.

37. Consider an infinitely decreasing geometric progression b_1, $b_2, b_3, \ldots, b_n, \ldots$ whose sum is 3 and sum of its cubes is $\frac{108}{13}$, find the value of b_2.

38. $\frac{1}{2.4}+\frac{1.3}{2.4.6}+\frac{1.3.5}{2.4.6.8}+\frac{1.3.5.7}{2.4.6.8.10}+\ldots\infty$ is equal to

39. If the roots of $10r^3 - nx^2 - 54x - 27 = 0$ are in harmonic progression, then 'n' equals.

40. The value of the $\sum_{n=0}^{\infty}\frac{2n+3}{3^n}$ is equal to.

41. The coefficient of the quadratic equation $ax^2 + (a + d)x + (a + 2d) = 0$ are consecutive terms of a positively valued, increasing arithmetic sequence. Then the least integral value of $\frac{d}{a}$ such that the equation has real solutions is.

42. Given a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in H.P. If $a = 2$ and $e = 18$, then the sum of all possible value of 'c' is.

43. Let sum of first three terms of G.P. with real terms is $\frac{13}{12}$ and 12 their product is -1. If the absolute value of the sum of their infinite terms is S, then find S

44. The rth, s^{th} and t^{th} terms of a certain G.P. are R, S and T respectively. Prove that $R^{s-t}\cdot S^{t-r}\cdot T^{r-s} = 1$.

45. (i) If n is a positive integer, prove that

$$\{(n+1)!\}^{1/(n+1)} < 1+\frac{n}{n+1}(n!)^{1/n}.$$

(ii) If n is a positive integer, show that

$$\left(1-\frac{1}{n}\right)^n < \left(1-\frac{1}{n+1}\right)^{n+1}.$$

46. If $a^x = b^y = c^z$ and x, y, z are in G.P., show that $\log_b a = \log_c b$

Exercise-4 (Past Year Questions)

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1. If the 2nd, 5th and 9th terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is: **(2016)**
   (a) 8/5 (b) 4/3
   (c) 1 (d) 7/4

2. If the sum of th∈ first ten terms of the series $\left(1\frac{3}{5}\right)^2+\left(2\frac{2}{5}\right)^2+\left(3\frac{1}{5}\right)^2+4^2+\left(4\frac{4}{5}\right)^2+\ldots$, is $\frac{16}{5}m$, then m is equal to: **(2016)**
   (a) 102 (b) 101
   (c) 100 (d) 99

3. For any three positive real numbers a, b and c, then **(2017)**
   (a) a, b and c are in G.P.
   (b) b, c and a are in G.P.
   (c) b, c and a are in A.P.
   (d) a, b and c are in A.P.

4. Let $a, b, c \in R$. If $f(x) = ax^2 + bx + c$ is such that $a + b + c = 3$ and $f(x + y) = f(x) + f(y) + xy$, $\forall x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is equal to: **(2017)**
   (a) 255 (b) 330
   (c) 165 (d) 190

5. Let $a_1, a_2, a_3, \ldots, a_{49}$ be in A.P. such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + \ldots a_{17}^2 = 140m$, then m is equal to **(2018)**
   (a) 68 (b) 34
   (c) 33 (d) 66

6. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series **(2018)**
   $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots$ If $B - 2A = 100\lambda$, then λ is equal to:
   (a) 248 (b) 464
   (c) 496 (d) 232

7. If a, b and c be three distinct numbers in G.P. and $a + b + c = xb$ then x can not be **(2019)**
   (a) -2 (b) -3
   (c) 4 (d) 2

8. Let $a_1, a_2, \ldots a_{30}$ be an A.P., $S = \sum_{i=1}^{30} a_i$ and $T = \sum_{i=1}^{15} a_{2i-1}$. If $a_5 = 27$ and $S - 2T = 75$, then a_{10} is equal to **(2019)**
   (a) 52
   (b) 57
   (c) 47
   (d) 42

9. The sum of the following series $1+6+\frac{9(1^2+2^2+3^2)}{7}+\frac{12(1^2+2^2+3^2+4^2)}{9}+\frac{15(1^2+2^2+....+5^2)}{11}+....$ upto 15 terms, is: **(2019)**

(a) 7820 (b) 7830 (c) 7520 (d) 7510

10. Let a, b and c be the 7th, 11th and 13th terms respectively of a non-constant A.P. If these are also the three consecutive terms of a G.P. then is equal to: **(2019)**

(a) 1/2 (b) 4 (c) 2 (d) 7/13

11. The sum of all two digit positive numbers which when divided by 7 yield 2 or 5 as remainder is: **(2019)**

(a) 1256 (b) 1465 (c) 1365 (d) 1356

12. If 19th terms of non-zero A. P is zero, then its (49th term) : (29th term) is: **(2019)**

(a) 4 : 1 (b) 1 : 3 (c) 3 : 1 (d) 2 : 1

13. The sum of an infinite geometric series with positive terms is 3 and the sum of the cubes of its terms is $\frac{27}{19}$. Then the common ratio of this series is: **(2019)**

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{2}{9}$ (d) $\frac{4}{9}$

14. Let $a_1, a_2, ... a_{10}$ be a G. P. If $\frac{a_3}{a_1}=25$, then $\frac{a_9}{a_5}$. equals: **(2019)**

(a) 5^4 (b) $4(5^2)$

(c) 5^3 (d) $2(5^2)$

15. The product of three consecutive terms of a G.P. is 512. If 4 is added to each of the first and the second of these terms, the three terms now form an A.P. Then the sum of the original three terms of the given G.P. is: **(2019)**

(a) 36 (b) 32

(c) 24 (d) 28

16. Let $S_k=\frac{1+2+3+...+k}{k}$. If $S_1^2+S_2^2+.....+S_{10}^2=\frac{5}{12}A$, then A equal to: **(2019)**

(a) 283 (b) 301

(c) 303 (d) 156

17. If the sum of the first 15 terms of the series $\left(\frac{3}{4}\right)^3+\left(1\frac{1}{2}\right)^3+\left(2\frac{1}{4}\right)^3+3^3+\left(3\frac{3}{4}\right)^3+....$ is equal to 225 k, then k is equal to: **(2019)**

(a) 108 (b) 27 (c) 54 (d) 9

18. The sum of all natural numbers 'h' such that $100<n<200$ and H.C.F. $(91, n)>1$ is: **(2019)**

(a) 3221 (b) 3121

(c) 3203 (d) 3303

19. The sum $\sum_{k=1}^{20} k\frac{1}{2^k}$ is equal to: **(2019)**

(a) $2-\frac{3}{2^{17}}$ (b) $2-\frac{11}{2^{19}}$

(c) $1-\frac{11}{2^{20}}$ (d) $2-\frac{21}{2^{20}}$

20. Let the sum of the first n terms of a non-constant A.P., $a_1, a_2, a_3, ...$ be $50n+(-1)^n\left(\frac{3n+2}{5}\right)A$, where A is a constant. If d is the common difference of this A.P., then the ordered pair (d, a_{50}) is equal to **(2019)**

(a) (A, 50 + 46A) (b) (A, 50 + 45A)

(c) (50, 50 + 46A) (d) (50, 50 + 45A)

21. If the sum and product of the first three term in an A.P. are 33 and 1155, respectively, then a value of its 11th term is: **(2019)**

(a) – 25 (b) 25 (c) – 36 (d) – 35

22. The sum of the series $1+2\times3+3\times5+4\times7+...$ upto 11th term is: **(2019)**

(a) 915 (b) 946 (c) 945 (d) 916

23. The sum $1+\frac{1^3+2^3}{1+2}+\frac{1^3+2^3+3^3}{1+2+3}+...+\frac{1^3+2^3+3^3+....+15^3}{1+2+3+....+15}-\frac{1}{2}(1+2+3+...+15)]$ **(2019)**

(a) 1240 (b) 1860

(c) 660 (d) 620

24. Let a, b and c be in G.P. with common ratio r, where $a\neq$ and $0<r\leq\frac{1}{2}$. If $3a$, $7b$ and $15c$ are the first three terms of an A.P. then the 4th term of this A.P. is: **(2019)**

(a) $\frac{7}{3}$ (b) a

(c) $\frac{2}{3}a$ (d) $5a$

25. The sum $\frac{3\times1^3}{1^2}+\frac{5\times(1^3+2^3)}{1^2+2^2}+\frac{7\times(1^3+2^3+3^3)}{1^2+2^2+3^2}+...$ up to 10 terms **(2019)**

(a) 660 (b) 620

(c) 680 (d) 600

26. If $a_1, a_2, a_3, ... a_n$ are in A.P. and $a_1+a_4+a_7+...+a_{16}=114$, then $a_1+a_6+a_{11}+a_{16}$ is equal to: **(2019)**

(a) 38 (b) 98

(c) 76 (d) 64

27. Let S_n denote the sum of the first n terms of an A.P. If $S_4=16$ and $S_6=-48$, then S_{10} is equal to: **(2019)**

(a) –320 (b) –260

(c) –380 (d) –410

28. For $x \in R$, let $[x]$ denote the greatest integer $\le x$, then the sum of the series

$$\left[-\frac{1}{3}\right]+\left[-\frac{1}{3}-\frac{1}{100}\right]+\left[-\frac{1}{3}-\frac{2}{100}\right]+.....+\left[-\frac{1}{3}-\frac{99}{100}\right] \text{ is}$$

(2019)

(a) -153 (b) -133 (c) -131 (d) -135

29. If $a_1, a_2, a_3, ...$ are in A.P. such that $a_1 + a_7 + a_{16} = 40$, then the sum of the first 15 terms of this A.P. is **(2019)**

(a) 200 (b) 280 (c) 120 (d) 150

30. The greatest positive integer k, for which $49^k + 1$ is a factor of the sum $49^{125} + 49^{124} + ... + 49^2 + 49 + 1$, is **(2020)**

(a) 32 (b) 63 (c) 65 (d) 60

31. Five numbers are in A.P., whose sum is 25 and product is 2520. If one of these five numbers is $\frac{-1}{2}$, then the greatest number amongst them is **(2020)**

(a) $\frac{21}{2}$ (b) 16 (c) 27 (d) 7

32. If the sum of the first 40 terms of the series, $3 + 4 + 8 + 9 + 13 + 14 + 18 + 19 + ...$ is $(102)m$, then m is equal to: **(2020)**

(a) 25 (b) 20 (c) 10 (d) 5

33. Let $a_1, a_2, a_3, ...$ be a G.P. such that $a_1 < 0$, $a_1 + a_2 = 4$ and $a_3 + a_4 = 16$. If $\sum_{i=1}^{9} a_i = 4\lambda$, then λ is equal to: **(2020)**

(a) 171 (b) -513

(c) $\frac{511}{3}$ (d) -171

34. Let $f : R \to R$ be such that for all $x \in R$ $(2^{1+x} + 2^{1-x})$, $f(x)$ and $(3^x + 3^{-x})$ are in A.P., then the minimum value of $f(x)$ is **(2020)**

(a) 0 (b) 2 (c) 4 (d) 3

35. The sum $\sum_{k=1}^{20}(1+2+3+.....+k)$ is **(2020)**

36. If the 10th term of an A.P. is $\frac{1}{20}$ and its 20th term is $\frac{1}{10}$, then the sum of its first 200 term is **(2020)**

(a) 100 (b) $50\frac{1}{4}$

(c) $100\frac{1}{2}$ (d) 50

37. The sum, $\sum_{n=1}^{7} \frac{n(n+1)(2n+1)}{4}$ is equal to **(2020)**

38. The product $2^{\frac{1}{4}}, 4^{\frac{1}{16}}, 8^{\frac{1}{48}}, 16^{\frac{1}{128}}$ ... to ∞ is equal to **(2020)**

(a) $2^{\frac{1}{4}}$ (b) $2^{\frac{1}{2}}$ (c) 1 (d) 2

39. Let a_n be the nth term of G.P. of positive terms. If $\sum_{n=1}^{100} a_{2n+1} = 200$ and $\sum_{n=1}^{100} a_{2n} = 100$, then $\sum_{n=1}^{200} a_n$ is equal to: **(2020)**

(a) 175 (b) 150 (c) 300 (d) 225

40. The number of terms common the two A.P. s 3, 7, 11, ..., 407 and 2, 9, 16, ..., 709 is _____. **(2020)**

41. Let S be the sum of the first 9 terms of the series: $\{x + ka\} + \{x^2 + (k + 2)a\} + \{x^3 + (k + 4)a\} + \{x^4 + (k + 6)a\} + ...$ where $a \neq 0$ and $x \neq 1$.

If $S = \frac{x^{10} - x + 45a(x-1)}{x-1}$, then k is equal to **(2020)**

(a) -3 (b) 1 (c) -5 (d) 3

42. If the sum of first 11 terms of an A.P., $a_1, a_2, a_3 ...$ is $0 (a_1 \neq 0)$, then the sum of the A.P., $a_1, a_3, a_5 ..., a_{23}$ is ka_1, where k is equal to: **(2020)**

(a) $-\frac{121}{10}$ (b) $-\frac{72}{5}$ (c) $\frac{72}{5}$ (d) $\frac{121}{10}$

43. The sum of the first three terms of a G.P. is S and their product is 27. Then all such S lie in: **(2020)**

(a) $(-\infty, -9] \cup [3, \infty)$ (b) $[-3, \infty)$

(c) $(-\infty, -3] \cup [9, \infty)$ (d) $(-\infty, 9]$

44. If $|x| < 1$, $|y| < 1$ and $x \neq y$, then the sum to infinity of the following series $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ...$ is: **(2020)**

(a) $\frac{x+y+xy}{(1+x)(1+y)}$ (b) $\frac{x+y-xy}{(1-x)(1-y)}$

(c) $\frac{x+y-xy}{(1+x)(1+y)}$ (d) $\frac{x+y+xy}{(1-x)(1-y)}$

45. If m arithmetic means (A. Ms) and three geometric means (G.Ms) are inserted between 3 and 243 such that 4th A.M. is equal to 2nd G.M., then m is equal to ____. **(2020)**

46. If the sum of the series $20 + 19\frac{3}{5} + 19\frac{1}{5} + 18\frac{4}{5} + ...$ upto n^{th} term is 488 and the nth term is negative, then **(2020)**

(a) $n = 41$ (b) n^{th} term is $-4\frac{2}{5}$

(c) $n = 60$ (d) n^{th} term is -4

47. The value of $(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+....\infty\right)}$ equal to _____. **(2020)**

48. If the first term of an A.P. is 3 and the sum of its first 25 terms is equal to the sum of its next 15 terms, then the common difference of this A.P. is: **(2020)**

(a) $\frac{1}{6}$ (b) $\frac{1}{4}$

(c) $\frac{1}{7}$ (d) $\frac{1}{5}$

49. Let $a_1, a_2, ... a_n$ be a given A.P. whose common difference is an integer and $S_n = a_1 + a_2 + ... + a_n$. If $a_1 = 1$, $a_n = 300$ and $15 \le n \le 50$, then the ordered pair (S_{n-4}, a_{n-4}) is equal to **(2020)**

(a) (2490, 249) (b) (2480, 249)

(c) (2490, 248) (d) (480, 248)

50. If $1 + (1 - 2^2 \cdot 1) + (1 - 4^2 \cdot 3) + (1 - 6^2 \cdot 5) + ... + (1 - 20^2 \cdot 19) = \alpha - 220\beta$, then an ordered pair (α, β) is equal to: **(2020)**

(*a*) (10, 103) (*b*) (10, 97)
(*c*) (11, 97) (*d*) (11, 103)

51. If the sum of the first 20 terms of the series $\log_{(7^{1/2})} x + \log_{(7^{1/3})} x + \log_{(7^{1/4})} x + ...$ is 460, then x is equal to **(2020)**

(*a*) 7^2 (*b*) e^2 (*c*) $7^{1/2}$ (*d*) $7^{46/21}$

52. If the sum of the second, third and fourth terms of a positive term G.P. is 3 and the sum of its sixth, seventh and eighth terms is 243, then the sum of the first 50 terms of this G.P. is: **(2020)**

(*a*) $\frac{2}{13}(3^{50} - 1)$ (*b*) $\frac{1}{13}(3^{50} - 1)$
(*c*) $\frac{1}{26}(3^{49} - 1)$ (*d*) $\frac{1}{26}(3^{50} - 1)$

53. If $2^{10} + 2^9 \cdot 3^1 + 2^8 \cdot 3^2 + ... + 2 \cdot 3^9 + 3^{10} = S - 2^{11}$, then S is equal to: **(2020)**

(*a*) $2 \cdot 3^{11}$ (*b*) $3^{11} - 2^{12}$
(*c*) $\frac{3^{11}}{2} + 2^{10}$ (*d*) 3^{11}

54. If $3^{2\sin 2\alpha - 1}$, 14 and $3^{4 - 2\sin 2\alpha}$ are the first three terms of an A.P. for some α, then the sixth term of this A.P is: **(2020)**

(*a*) 65 (*b*) 78 (*c*) 81 (*d*) 66

55. Suppose that a function $f : R \to R$ satisfies $f(x + y) = f(x) f(y)$ for all $x, y \in R$ and $f(a) = 3$. If $\sum_{i=1}^{n} f(i) = 363$, then n is equal to _____. **(2020)**

56. The common difference of the A.P. $b_1, b_2, ..., b_m$ is 2 more than the common difference of A.P. $a_1, a_2, ..., a_n$. If $a_{40} = -159$, $a_{100} = -399$ and $b_{100} = a_{70}$, then b_1 is equal to: **(2020)**

(*a*) –127 (*b*) –81 (*c*) 127 (*d*) 81

57. Let a, b, c, d and p be any non zero distinct real numbers such that $(a^2 + b^2 + c^2) p^2 - 2 (ab + bc + cd)p + (b^2 + c^2 + d^2) = 0$. Then: **(2020)**

(*a*) a, c, p are in G. P. (*b*) a, b, c, d are in A. P.
(*c*) a, c, p are in A.P. (*d*) a, b, c, d are in G.P.

58. If $0 < \theta, \phi < \frac{\pi}{2}, x = \sum_{n=0}^{\infty} \cos^{2n}\theta, y = \sum_{n=0}^{\infty} \sin^{2n}\phi$ and $z = \sum_{n=0}^{\infty} \cos^{2n}\theta . \sin^{2n}\phi$ then **(2021)**

(*a*) $xy - z = (x + y)z$ (*b*) $xyz = 4$
(*c*) $xy + z = (x + y)z$ (*d*) $xy + yz + zx = z$

59. The sum of the infinite series

$1 + \frac{2}{3} + \frac{7}{3^2} + \frac{12}{3^3} + \frac{17}{3^4} + \frac{22}{3^5} + ...$ is equal to: **(2021)**

(*a*) $\frac{9}{4}$ (*c*) $\frac{13}{4}$ (*c*) $\frac{15}{4}$ (*d*) $\frac{11}{4}$

60. In an increasing geometric series, the sum of the second and the sixth term is $\frac{25}{2}$ and the product of the third and fifth term is 25. Then, the sum of 4th, 6th and 8th terms is equal to: **(2021)**

(*a*) 30 (*b*) 32 (*c*) 26 (*d*) 35

61. The sum of the series $\sum_{n=1}^{\infty} \frac{n^2 + 6n + 10}{(2n+1)!}$ is equal to: **(2021)**

(*a*) $\frac{41}{8}e + \frac{19}{8}e^{-1} - 10$ (*b*) $\frac{41}{8}e - \frac{19}{8}e^{-1} - 10$
(*c*) $\frac{41}{8}e + \frac{19}{8}e^{-1} + 10$ (*d*) $-\frac{41}{8}e + \frac{19}{8}e^{-1} - 10$

62. If α, β are natural numbers such that

$$100^{\alpha} - 199\beta = (100)(100) + (99)(101) + (98)(102) + ... + (1)(199)$$

then the slope of the line passing through (α, β) and origin is: **(2021)**

(*a*) 540 (*b*) 550 (*c*) 530 (*d*) 510

63. $\frac{1}{3^2 - 1} + \frac{1}{5^2 - 1} + \frac{1}{7^2 - 1} + ... + \frac{1}{(201)^2 - 1}$ is equal to **(2021)**

(*a*) $\frac{101}{404}$ (*b*) $\frac{25}{101}$ (*c*) $\frac{101}{408}$ (*d*) $\frac{99}{400}$

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64. If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \le 0$, then a, b, c, d **(1987)**

(*a*) are in AP (*b*) are in GP
(*c*) are in HP (*d*) satisfy $ab = cd$

65. Let a_n be the nth term of the G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \ne \beta$, then the common ratio is **(1992)**

(a) $\frac{\alpha}{\beta}$ (b) $\frac{\beta}{\alpha}$ (c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{\beta}{\alpha}}$

66. For a positive integer n let

$a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{(2^n) - 1}$, then **(1999)**

(*a*) $a(100) \le 100$ (*b*) $a(100) > 100$
(*c*) $a(200) \le 100$ (*d*) $a(200) > 100$

67. If a, b, c, d are positive real numbers such that $a + b + c + d = 2$, then $M = (a + b)(c + d)$ satisfies the relation **(2000)**

(*a*) $0 < M \le 1$ (*b*) $1 \le M \le 2$
(*c*) $2 \le M \le 3$ (*d*) $3 \le M \le 4$

68. Let $a_1, a_2, a_3, ...$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is **(2012)**

(*a*) 22 (*b*) 23
(*c*) 24 (*d*) 25

69. Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(s) **(2013)**

(*a*) 1056 (*b*) 1088 (*c*) 1120 (*d*) 1332

70. A pack contains n card numbered from 1 to n. Two consecutive numbered card are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then k – 20 = **(2013)**

71. Let a, b, c be positive integers such that $\frac{b}{a}$ is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c is $b + 2$, then the value of $\frac{a^2+a-14}{a+1}$ is **(2014)**

72. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is **(2015)**

73. Let $b_i > 1$ for $i = 1, 2, ..., 101$. Suppose $\log_e b_1, \log_e b_2, ... \log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, ..., a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + ... + b_{51}$ and $s = a_1 + a_2 + ... + a_{51}$, then **(2016)**

(*a*) $s > t$ and $a_{101} > b_{101}$ (*b*) $s > t$ and $a_{101} < b_{101}$

(*c*) $s < t$ and $a_{101} > b_{101}$ (*d*) $s < t$ and $a_{101} < b_{101}$

74. The sides of the right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side? **(2017)**

75. Let AP(a; d) denote the set of all the terms of an infinite arithmetic progression with first term a and common difference $d > 0$. If AP(1; 3) ∩ AP(2; 5) ∩ AP(3; 7) = AP(a; d) then $a + d$ equals ____ **(2019)**

76. Let m be the minimum possible value of $\log_3 (3^{y_1} + 3^{y_2} + 3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2 (m^3) + \log_3 (M^2)$ is ____. **(2020)**

77. Let $a_1, a_2, a_3, ...$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, ...$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality $2(a_1 + a_2 + ... + a_n) = b_1 + b_2 + ... + b_n$ holds for some positive integer n, is ____ **(2020)**

78. For any positive integer n, let $S_n : (0, \infty) \to \mathbb{R}$ be defined by

$$S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right)$$

Where for any $x \in \mathbb{R}$, $\cot^{-1} x \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) **TRUE?** **(2021)**

(*a*) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$

(*b*) $\lim_{n\to\infty} \cot(S_n(x)) = x$, for all $x > 0$

(*c*) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$

(*d*) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

79. Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then **(2021)**

(*a*) $k + 21 = 22$ (*b*) $2k + l = 26$

(*c*) $2k + 3l = 34$ (*d*) $3k + 2l = 40$

80. Consider M with $r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is **(2021)**

(*a*) 198 (*b*) 199 (*c*) 200 (*d*) 201

ANSWER KEY

CONCEPT APPLICATION

1. 83667 2. $\frac{7}{16}$ 3. 293 5. 10 6. (*i*) 2, 4/2, 8/3, 16/4,...(*ii*) 2/3, 4/9, 2/27, 4/81,... 7. 2,6,18,54,162...

8. $\frac{1}{3}$ 9. $2+\sqrt{3}$ 10. $\left(\frac{10(10^n-1)}{9} - n\right)$ 11. 2 12. $\frac{1}{3}$ 13. $\frac{1}{(1-x)^2}$ 15. $\frac{1}{49}$ 16. 3 *ad*

17. –2 18. $a = 60, b = -160, c = 240, d = -192$ 19. $\left[0, \frac{16}{5}\right]$ 21. $3n(n+1)^2 - 2n$ 22. $= \frac{41}{2} + \frac{3^{12}}{2}$

23. $\frac{n^2+3n}{2(n+1)(n+2)}$

EXERCISE-1 (TOPICWISE)

1. (*b*)	**2.** (*c*)	**3.** (*c*)	**4.** (*b*)	**5.** (*c*)	**6.** (*c*)	**7.** (*c*)	**8.** (*d*)	**9.** (*b*)	**10.** (*c*)
11. (*d*)	**12.** (*b*)	**13.** (*b*)	**14.** (*d*)	**15.** (*b*)	**16.** (*a*)	**17.** (*a*)	**18.** (*d*)	**19.** (*a*)	**20.** (*b*)
21. (*a*)	**22.** (*b*)	**23.** (*b*)	**24.** (*a*)	**25.** (*b*)	**26.** (*c*)	**27.** (*b*)	**28.** (*b*)	**29.** (*d*)	**30.** (*a*)
31. (*d*)	**32.** (*a*)	**33.** (*c*)	**34.** (*b*)	**35.** (*a*)	**36.** (*a*)	**37.** (*a*)	**38.** (*c*)	**39.** (*d*)	**40.** (*a*)
41. (*b*)	**42.** (*a*)	**43.** (*d*)	**44.** (*c*)	**45.** (*d*)	**46.** (*d*)	**47.** (*a*)	**48.** (*b*)	**49.** (*b*)	**50.** (*b*)
51. (*d*)	**52.** (*a*)	**53.** (*c*)	**54.** (*a*)	**55.** (*a*)					

EXERCISE-2 (LEARNING PLUS)

1. (*d*)	**2.** (*b*)	**3.** (*c*)	**4.** (*a*)	**5.** (*d*)	**6.** (*b*)	**7.** (*a*)	**8.** (*b*)	**9.** (*c*)	**10.** (*d*)
11. (*a*)	**12.** (*c*)	**13.** (*a*)	**14.** (*c*)	**15.** (*a*)	**16.** (*a*)	**17.** (*c*)	**18.** (*b*)	**19.** (*b*)	**20.** (*c*)
21. (*c*)	**22.** (*b*)	**23.** (*b*)	**24.** (*c*)	**25.** (*a*)	**26.** (*c*)	**27.** (*a*)	**28.** (*c*)	**29.** (*d*)	**30.** (*d*)
31. (*b*)	**32.** (*a*)	**33.** (*b*)	**34.** (*a*)	**35.** (*a*)	**36.** (*b*)	**37.** (*a*)	**38.** (*c*)	**39.** (*a*)	**40.** (*d*)
41. (*a*)	**42.** (*a*)	**43.** (*a*)	**44.** (*d*)	**45.** (*a*)	**46.** (*c*)	**47.** (*a*)	**48.** (*b*)	**49.** (*d*)	**50.** (*a*)
51. (*c*)	**52.** (*c*)	**53.** (*a*)	**54.** (*b,c*)	**55.** (*a,b,c,d*)	**56.** (*a,b,c*)	**57.** (*a,b,c,d*)	**58.** (*a,b,d*)	**59.** (*b,d*)	**60.** (*a,b,d*)
61. (*a,b,c*)	**62.** (*b,c,d*)	**63.** (*c,d*)	**64.** (*a,b*)	**65.** (*a*)	**66.** (*a*)	**67.** (*d*)	**68.** (*a*)	**69.** [3]	

70. $\left[\dfrac{n(n+1)(n+2)}{6}\right]$ **71.** $x = 10^5; y = 10$ **72.** $\left[\dfrac{n(n+1)(3n^2+19n+26)}{12}\right]$ **73.** $3.2^n - 2n - 3$

74. $\dfrac{(n-1)n(n+1)(3n+2)}{24}$ **75.** [120] **76.** [306] **77.** [2] **78.** [4/3] **79.** [4] **80.** [2]

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (*b*)	**2.** (*c*)	**3.** (*b*)	**4.** (*a*)	**5.** (*b*)	**6.** (*a*)	**7.** (*b*)	**8.** (*b*)	**9.** (*c*)	**10.** (*c*)
11. (*a,b*)	**12.** (*a,b,c*)	**13.** (*a,b,c,d*)	**14.** (*a,c*)	**15.** (*a,d*)	**16.** (*a,b,c*)	**17.** (*a,c*)	**18.** (*a*)	**19.** (*c*)	**20.** (*d*)
21. (*c*)	**22.** (*c*)	**23.** (*d*)	**24.** (*c*)	**25.** (*b*)	**26.** (*a*)	**27.** (*d*)	**28.** (*c*)	**29.** (*b*)	**30.** (*b*)
31. (*a*)	**32.** (*c*)	**33.** (*b*)	**34.** (*b*)	**35.** (*b*)	**36.** [1]	**37.** [2/3]	**38.** [1/2]	**39.** [0]	**40.** [6]
41. [7]	**42.** {6,–6}	**43.** [6]							

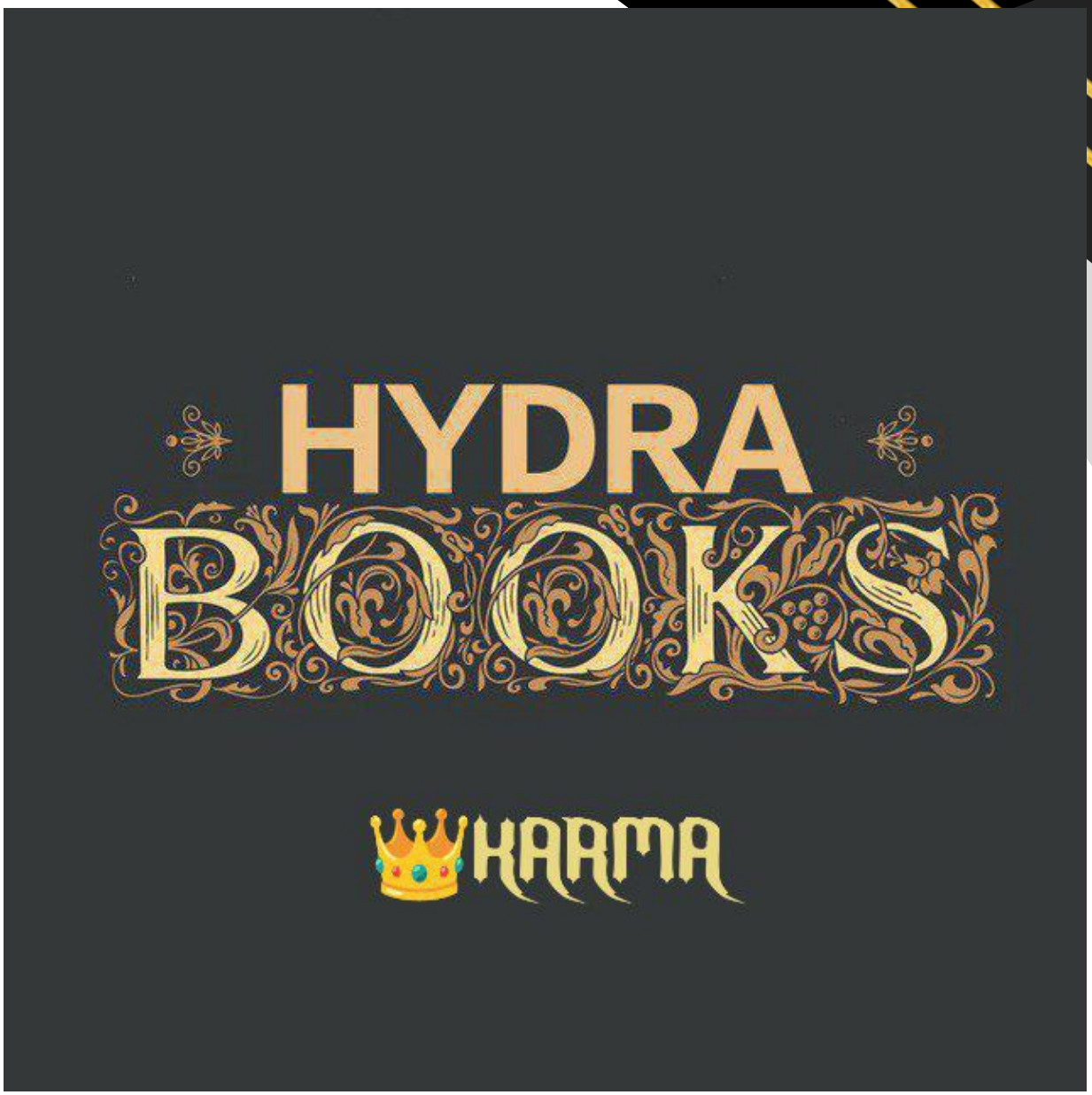
EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (*b*)	**2.** (*b*)	**3.** (*c*)	**4.** (*b*)	**5.** (*b*)	**6.** (*a*)	**7.** (*d*)	**8.** (*a*)	**9.** (*a*)	**10.** (*b*)
11. (*d*)	**12.** (*c*)	**13.** (*b*)	**14.** (*a*)	**15.** (*d*)	**16.** (*c*)	**17.** (*b*)	**18.** (*b*)	**19.** (*b*)	**20.** (*a*)
21. (*a*)	**22.** (*b*)	**23.** (*d*)	**24.** (*b*)	**25.** (*a*)	**26.** (*c*)	**27.** (*a*)	**28.** (*b*)	**29.** (*a*)	**30.** (*b*)
31. (*b*)	**32.** (*b*)	**33.** (*d*)	**34.** (*d*)	**35.** [1540]	**36.** (*c*)	**37.** [504]	**38.** (*b*)	**39.** (*b*)	**40.** [14]
41. (*a*)	**42.** (*b*)	**43.** (*c*)	**44.** (*b*)	**45.** [39]	**46.** (*d*)	**47.** [4]	**48.** (*a*)	**49.** (*c*)	**50.** (*d*)
51. (*a*)	**52.** (*d*)	**53.** (*d*)	**54.** (*d*)	**55.** [05.00]	**56.** (*b*)	**57.** (*d*)	**58.** (*c*)	**59.** (*b*)	**60.** (*d*)
61. (*b*)	**62.** (*b*)	**63.** (*b*)							

JEE Advanced

64. (*b*)	**65.** (*a*)	**66.** (*a,d*)	**67.** (*a*)	**68.** (*d*)	**69.** (*a,d*)	**70.** [5]	**71.** [4]	**72.** [9]	**73.** (*b*)
74. [6]	**75.** [157]	**76.** [8]	**77.** [1]	**78.** (*a,b*)	**79.** (*d*)	**80.** (*b*)			



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