CHAPTER 1

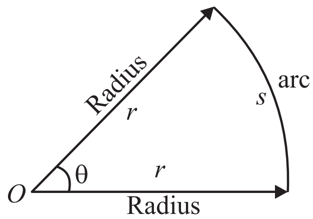
Mathematical Tools

TRIGONOMETRY

Angle

It is a measure of change in direction.

$$\text{Angle } (\theta) = \frac{\text{Arc}(s)}{\text{Radius}(r)}$$



Angles measured in anticlockwise and clockwise direction are usually taken as positive and negative respectively.

System of measurement of an angle

(A) Sexagesimal system:

In this system, angle is measured in degrees.

In this system, 1 right angle = 90°, 1° = 60′ (arc minutes), 1′ = 60″ (arc seconds)

(B) Circular system:

In this system, angle is measured in radians. If arc = radius then $\theta = 1$ rad

Relation between degrees and radians.

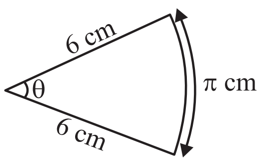
$$2\pi \text{ rad} = 360° \Rightarrow \pi \text{ rad} = 180° \Rightarrow 1 \text{ rad} = \frac{180°}{\pi} = 57.3°$$

To convert from degree to radian multiply by $\frac{\pi}{180°}$

To convert from radian to degree multiply by $\frac{180°}{\pi}$

Train Your Brain

Example 1: A circular arc of length π cm. Find angle subtended by it at the centre in radians and degrees.



Sol. $\theta = \frac{s}{r} = \frac{\pi \text{ cm}}{6 \text{ cm}} = \frac{\pi}{6} \text{rad} = 30°$

Example 2: The moon's distance from the earth is 360000 km and its diameter subtends an angle of 42' at the eye of the observer. The diameter of the moon in kilometers is

(*a*) 4400 (*b*) 1000

(*c*) 3600 (*d*) 8800

Sol. Here angle is very small so diameter ≈ arc

$$\theta = 42' = \left(42 \times \frac{1}{60}\right) = 42 \times \frac{1}{60} \times \frac{\pi}{180} = \frac{7\pi}{1800} \text{ rad}$$

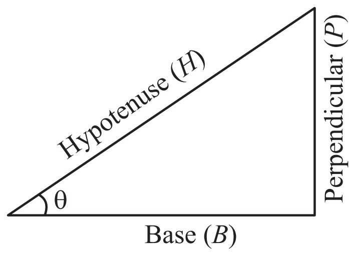
$$\text{Diameter} = R\theta = 360000 \times \frac{7}{1800} \times \frac{22}{7} = 4400 \text{ km}$$

Concept Application

1. Convert 15° into radians
2. Convert $\frac{5\pi}{6}$ radians into degrees

Trigonometric Ratios (T-ratios)

Following ratios of the sides of a right angled triangle are known as trigonometrical ratios.



$$\sin\theta = \frac{P}{H}$$

$$\cos\theta = \frac{B}{H}$$

$$\tan\theta = \frac{P}{B} \qquad \text{cosec}\theta = \frac{1}{\sin\theta} = \frac{H}{P}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{H}{B} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{B}{P}$$

Trigonometric Identities

In figure, $P^2 + B^2 = H^2$ Divide by H^2, $\left(\frac{P}{H}\right)^2 + \left(\frac{B}{H}\right)^2 = 1$

$\Rightarrow \sin^2\theta + \cos^2\theta = 1$

Divide by B^2, $\left(\frac{P}{B}\right)^2 + 1 = \left(\frac{H}{B}\right)^2 \Rightarrow 1 + \tan^2\theta = \sec^2\theta$

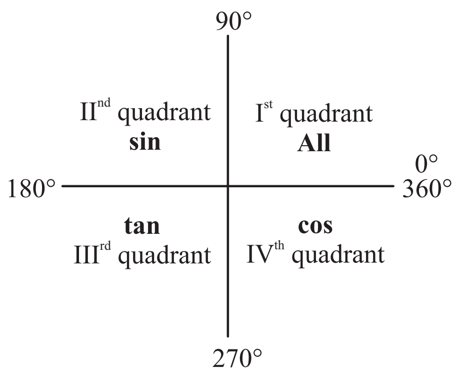
Divide by P^2, $1 + \left(\frac{B}{P}\right)^2 = \left(\frac{H}{P}\right)^2 \Rightarrow 1 + \cot^2\theta = \text{cosec}^2\theta$

Commonly Used Values of Trigonometric Functions

Angle (θ)	0°	30°	37°	45°	53°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{4}{5}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{4}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{3}{5}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	$\frac{3}{4}$	1	$\frac{4}{3}$	$\sqrt{3}$	∞

Four Quadrants and ASTC Rule

In first quadrant, all trigonometric ratios are positive. In second quadrant, only sinθ and cosecθ are positive. In third quadrant, only tanθ and cotθ are positive.



In fourth quadrant, only cosθ and secθ are positive

Remember as 'Add Sugar To Coffee' or 'After School To College'.

Trigonometrical Ratios of General Angles (Reduction Formulae)

(*i*) The value of a trignometric function does not change if you add $2n\pi$ to its argument where n is an integer.

$\sin(2n\pi + \theta) = \sin\theta$ $\cos(2n\pi + \theta) = \cos\theta$

$\tan(2n\pi + \theta) = \tan\theta$

(*ii*) Trigonometric function of an angle $\left(\frac{n\pi}{2} + \theta\right)$ will remain same if n is even and sign of trigonometric function will be according to value of that function in quadrant.

$\sin(\pi - \theta) = +\sin\theta$ $\cos(\pi - \theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$

$\sin(\pi + \theta) = -\sin\theta$ $\cos(\pi + \theta) = -\cos\theta$

$\tan(\pi + \theta) = +\tan\theta$

$\sin(2\pi - \theta) = -\sin\theta$ $\cos(2\pi - \theta) = +\cos\theta$

$\tan(2\pi - \theta) = -\tan\theta$

(*iii*) Trigonometric function of an angle $\left(\frac{n\pi}{2} + \theta\right)$ will be changed into co-function if n is odd and sign of trigonometric function will be according to value of that function in quadrant.

$\sin\left(\frac{\pi}{2} + \theta\right) = +\cos\theta$ $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$

$\sin\left(\frac{\pi}{2} - \theta\right) = +\cos\theta$ $\cos\left(\frac{\pi}{2} - \theta\right) = +\sin\theta$

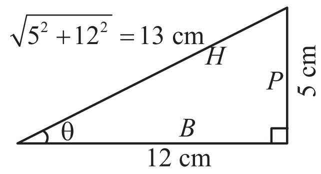
$\tan\left(\frac{\pi}{2} - \theta\right) = +\cot\theta$

(iv) Trigonometric function of an angle –θ (negative angles)
$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = +\cos\theta$, $\tan(-\theta) = -\tan\theta$

Train Your Brain

Example 3: The two shorter sides of right angled triangle are 5 cm and 12 cm. Let θ denote the angle opposite to the 5 cm side. Find sinθ, cosθ and tanθ.

Sol.

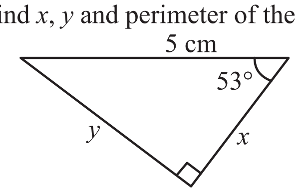


$\sin\theta = \frac{P}{H} = \frac{5\text{cm}}{13\text{cm}} = \frac{5}{13}$

$\cos\theta = \frac{B}{H} = \frac{12\text{cm}}{13\text{cm}} = \frac{12}{13}$

$\tan\theta = \frac{P}{B} = \frac{5\text{cm}}{12\text{cm}} = \frac{5}{12}$

Example 4: Find x, y and perimeter of the triangle shown



Sol.

$\frac{y}{5} = \sin 53° = \frac{4}{5} \Rightarrow y = 4$ cm

and $\frac{x}{5} = \cos 53° = \frac{3}{5} \Rightarrow x = 3$ cm

Perimeter of the triangle $= x + y + 5 = 3 + 4 + 5 = 12$ cm

Example 5: Find the value of :

(*i*) sin30° + cos60° (*ii*) sin 0° – cos 0°

(*iii*) tan 45°–tan 37° (*iv*) sin 390°

(*v*) cos 405° (*vi*) tan 420°

(*vii*) sin 150° (*viii*) cos 120°

(*ix*) tan 135° (*x*) sin 330°

(*xi*) cos 300° (*xii*) sin(–30°)

(*xiii*) cos(–60°) (*xiv*) tan(–45°)

(*xv*) sin(–150°)

Sol.

(*i*) $\sin 30° + \cos 60° = \frac{1}{2}+\frac{1}{2}=1$

(*ii*) $\sin 0° - \cos 0° = 0 - 1 = -1$

(*iii*) $\tan 45° - \tan 37° = 1-\frac{3}{4}=\frac{1}{4}$

(*iv*) $\sin 390° = \sin(360° + 30°) = \sin 30° = \frac{1}{2}$

(*v*) $\cos 405° = \cos(360° + 45°) = \cos 45° = \frac{1}{\sqrt{2}}$

(*vi*) $\tan 420° = \tan (360° + 60°) = \tan 60° = \sqrt{3}$

(*vii*) $\sin 150° = \sin (90° + 60°) = \cos 60° = \frac{1}{2}$ or

$\sin 150° = \sin(180° - 30°) = \sin 30° = \frac{1}{2}$

(*viii*) $\cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}$

(*ix*) $\tan 135° = \tan(180° - 45°) = -\tan 45° = -1$

(*x*) $\sin 330° = \sin (360° - 30°) = -\sin 30° = -\frac{1}{2}$

(*xi*) $\cos 300° = \cos (360° - 60°) = \cos 60° = \frac{1}{2}$

(*xii*) $\sin (-30°) = -\sin 30° = -\frac{1}{2}$

(*xiii*) $\cos (-60°) = +\cos 60° = \frac{1}{2}$

(*xiv*) $\tan (-45°) = -\tan 45° = -1$

(*xv*) $\sin (-150°) = -\sin (150°) = -\sin(180°-30°)$

$= -\sin 30° = -\frac{1}{2}$

Concept Application

3. Evaluate

(*i*) sin(210°) (*ii*) cos120°

(*iii*) sin(330°) (*iv*) tan(300°)

4. If $\tan\theta$ is $\frac{5}{12}$, find

(*i*) $\sin\theta$ (*ii*) $\cos\theta$

Addition / Subtraction Formulae for Trigonometrical Ratios

(*i*) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(*ii*) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(*iii*) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(*iv*) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Small Angle Approximation

If θ is small in radians then $\sin \theta \approx \theta$, $\cos \theta \approx 1$ and $\tan\theta \approx \theta$

Note : here θ must be in radian.

Smaller the value of θ, better will be approximation

Maximum and Minimum Values of Some useful Trigonometric Functions

(*i*) $-1 \le \sin\theta \le 1$

(*ii*) $-1 \le \cos\theta \le 1$

(*iii*) $-\sqrt{a^2+b^2} \le a\cos\theta \pm b\sin\theta \le \sqrt{a^2+b^2}$

Train Your Brain

Example 6: Find the value of

(*i*) sin 74° (*ii*) cos 106°

(*iii*) sin 15° (*iv*) cos 75°

Sol. (*i*) $\sin 74° = \sin (2 \times 37°) = 2 \sin 37° \cos 37°$

$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)=\frac{24}{25}$

(*ii*) $\cos 106° = \cos (2 \times 53°) = \cos^2 53° - \sin^2 53°$

$= \left(\frac{3}{5}\right)^2-\left(\frac{4}{5}\right)^2=\frac{9-16}{25}=-\frac{7}{25}$

(*iii*) $\sin 15° = \sin (45° - 30°) = \sin 45° \cos 30° - \cos 45°$

$\sin 30° = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}$

(*iv*) $\cos 75° = \cos (45° + 30°) = \cos 45° \cos 30° - \sin 45°$

$\sin 30° = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}$

Example 7: Find the approximate values of (*i*) sin 10° (*ii*) tan 20° (*iii*) cos 10°.

Sol. (*i*) $\sin 10° = \sin\left(10°\times\frac{\pi}{180°}\right)=\sin\frac{\pi}{18}\approx\frac{\pi}{18}$

(*ii*) $\tan 20° = \tan\left(20°\times\frac{\pi}{180°}\right)=\tan\frac{\pi}{9}\approx\frac{\pi}{9}$

(*iii*) $\cos 10° = \cos\left(10°\times\frac{\pi}{180°}\right)=\cos\frac{\pi}{18}=1$

Example 8: Find maximum and minimum values of y:

(*i*) $y = 2\sin x$

(*ii*) $y = 4 - \cos x$

(*iii*) $y = 3\sin x + 4\cos x$

Sol. (*i*) $y_{max} = 2(1) = 2$ and $y_{min} = 2(-1) = -2$

(*ii*) $y_{max} = 4 - (-1) = 4 + 1 = 5$ and $y_{min} = 4 - (1) = 3$

(*iii*) $y_{max} = \sqrt{3^2+4^2} = 5$ and $y_{min} = -\sqrt{3^2+4^2} = -5$

Example 9: A ball is projected with speed u at an angle θ to the horizontal. The range R of the projectile is given by

$$R = \frac{u^2 \sin 2\theta}{g}$$

For which value of θ will the range be maximum for a given speed of projection? (Here g = constant)

(*a*) $\frac{\pi}{2}$ rad (*b*) $\frac{\pi}{4}$ rad

(*c*) $\frac{\pi}{3}$ rad (*d*) $\frac{\pi}{6}$ rad

Sol. As $\sin 2\theta \le 1$ so range will be maximum if $\sin 2\theta = 1$.

Therefore $2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$ rad.

Example 10: The position of a particle moving along x-axis varies with time t according to equation $x = \sqrt{3}\sin\omega t - \cos\omega t$ where ω is constant. Find the region in which the particle is confined.

Sol. $\because x = \sqrt{3}\sin\omega t - \cos\omega t$

$\therefore x_{max} = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$ and

$x_{min} = -\sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = -2$

Thus, the particle is confined in the region $-2 \le x \le 2$

Concept Application

5. Find the maximum and the minimum value of
   (*i*) $8 - 6\cos x$
   (*ii*) $3\sin x - 4\cos x$
   (*iii*) $5\sin(x) + 12\cos x + 4$
6. Find the approximate value of
   (*i*) sin(19.1°) (*ii*) cos(5°)
   (*iii*) tan(9°)
7. Find the value of $\left[\sin 37° \approx \frac{3}{5}\right]$
   (*i*) cos(74°) (*ii*) sin(106°)

QUADRATIC EQUATION

An algebraic equation of second order (highest power of the variable is equal to 2) is called a quadratic equation. The equation $ax^2 + bx + c = 0$....(i) is the general form of quadratic equation where $a \neq 0$. The general solution of above equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If values of x be x_1 and x_2 then $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Here x_1 and x_2 are called roots of equation (i). We can easily see that

sum of roots $= x_1 + x_2 = -\frac{b}{a}$

and

product of roots $= x_1 x_2 = \frac{c}{a}$

Train Your Brain

Example 11: Find roots of equation $2x^2 - x - 3 = 0$.

Sol. Compare this equation with standard quadratic equation $ax^2 + bx + c = 0$, we have $a = 2$, $b = -1$, $c = -3$.

Now from

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}; x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{1 \pm \sqrt{1+24}}{4} = \frac{1 \pm 5}{4} \Rightarrow x = \frac{6}{4}, x = \frac{-4}{4} \Rightarrow x = \frac{3}{2}$$

$$\Rightarrow x = \frac{3}{2} \text{ or } x = -1$$

Concept Application

8. Find the root of quadratic equation
   (*i*) $x^2 + \frac{2}{3}x - \frac{5}{3} = 0$ (*ii*) $4x^2 + 6x - 12 = 0$
9. Find the sum and product of roots of equation
   (*i*) $\frac{5x^2}{3} - \frac{2}{3}x + 3 = 0$ (*ii*) $2x^2 - \frac{3}{2}x + 4 = 0$

ARITHMETIC PROGRESSION (AP)

General form : $a, a + d, a + 2d, ..., a + (n - 1)d$

Here a = first term, d = common difference

Sum of n terms $S_n = \frac{n}{2}[a + a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$

GEOMETRICAL PROGRESSION (GP)

General form : $a, ar, ar^2, ..., ar^{n-1}$ Here a = first term, r = common ratio

Sum of n terms $S_n = \frac{a(1 - r^n)}{1 - r}$

Sum of ∞ term $S_\infty = \frac{a}{1 - r}$

Train Your Brain

Example 12: Find the sum of first n natural numbers

Sol. $S_n = 1 + 2 + 3 + 4 + ... + n = \frac{n}{2}(1+n) = \frac{n(n+1)}{2}$

Example 13: Find $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....$upto ∞

Sol. Here, $a = 1$, $r = \frac{1}{2}$ So, $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

Concept Application

10. Find the sum of $1 + 3 + 5 + 7 + ... + 51$

11. Find the sum of the series $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}.......\infty$

12. Find the 20th term of series $1 + 5 + 9 + 13 + ...$

BINOMIAL EXPRESSION

An algebraic expression containing two terms is called a binomial expression.

For example $(a + b)$, $(a + b)^3$, $(2x - 3y)^{-1}$, $\left(x+\frac{1}{y}\right)^2$ etc. are binomial expressions.

Binomial Theorem

$$(a + b)^n = a^n + na^{n-1} b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + ...$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + ...$$

Binomial Approximation

If x is very small, then terms containing higher powers of x can be neglected so

$(1 + x)^n \approx 1 + nx \qquad (x << 1)$

$\frac{(1+x)^n}{(1+y)^n} \approx 1+nx-my \qquad (x, y << 1)$

Train Your Brain

Example 14: Calculate the value of

(i) $\sqrt{0.99}$ (ii) $(1030)^{1/3}$ (iii) $\sqrt{\frac{101}{98}}$

Sol. (i) $\sqrt{0.99} = \left(1-\frac{1}{100}\right)^{\frac{1}{2}} \approx \left(1-\frac{1}{200}\right) \approx 0.995$

(ii) $(1030)^{\frac{1}{3}} = 10\left[1+\frac{30}{1000}\right]^{\frac{1}{3}}$

$\approx 10\left[1+\frac{10}{1000}\right] \approx 10.01$

(iii) $\sqrt{\frac{101}{98}} = \frac{(1.01)^{1/2}}{(1-0.02)^{1/2}} \approx 1+\frac{1}{2}\times 0.01+\frac{1}{2}\times 0.02 \approx 1.015$

Example 15: The mass m of a body moving with a velocity v is given by $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ where m_0 = rest mass of body = 10 kg and c = speed of light = 3×10^8 m/s. Find the value of m at $v = 3 \times 10^7$ m/s.

Sol.

$$m = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 10\left[1-\left(\frac{3\times 10^7}{3\times 10^8}\right)^2\right]^{-1/2} = 10\left[1-\frac{1}{100}\right]^{-1/2}$$

$$\approx 10\left[1-\left(-\frac{1}{2}\right)\left(\frac{1}{100}\right)\right] \approx 10+\frac{10}{200} \approx 10.05 \text{ kg}$$

Concept Application

13. Calculate the approximate value of

(i) $\sqrt{52}$ (ii) $(130)^{\frac{1}{3}}$ (iii) $\sqrt{\frac{49}{47}}$

14. The value of acceleration due to gravity at height h above the surface of earth is given by $g' = g\left(\frac{R}{R+h}\right)^2$ where $g = 9.81$ m/s^2 and $R = 6400$ km.

Find the acceleration due to gravity at height of 64 km above the surface of earth.

CO-ORDINATE GEOMETRY

To specify the position of a point in space, we use right handed rectangular axes coordinate system. This system consists of (i) origin (ii) axis or axes. If point is known to be on a given line or in a particular direction, only one coordinate is necessary to specify its position. If it is in a plane, two coordinates are required. If it is in space three coordinates are needed.

Origin

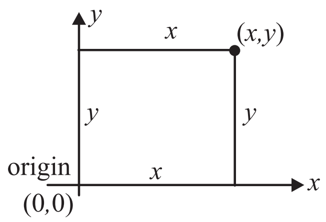
This is any fixed point which is convenient to you. All measurement are taken w.r.t. this fixed point.

Axis or Axes

Any fixed direction passing through origin and convenient to you can be taken as an axis. If the position of a point or position of all the points under consideration always happen to be in a particular direction, then only one axis is required. This is generally called the x-axis. If the positions of all the points under consideration are always in a plane, two perpendicular axes are required. These are generally called x and y-axis. If the points are distributed in a space, three perpendicular axes are taken which are called x, y and z-axis.

Position of a point in xy plane

The position of a point is specified by its distances from origin along (or parallel to) x and y-axis as shown in figure. Here x-coordinate and y-coordinate is called abscissa and ordinate respectively.



Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Train Your Brain

Example 16: Find the value of a if distance between the points (–9 cm, a cm) and (3 cm, 3cm) is 13 cm.

Sol. By using distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$\Rightarrow 13 = \sqrt{[3-(-9)]^2 + [3-a]^2}$$

$$\Rightarrow 13^2 = 12^2 + (3-a)^2 \Rightarrow (3-a)^2 = 13^2 - 12^2 = 5^2$$

$$\Rightarrow (3-a) = \pm 5 \Rightarrow a = -2 \text{ cm or } 8 \text{ cm}$$

Example 17: A dog wants to catch a cat. The dog follows the path whose equation is $y - x = 0$ while the cat follows the path whose equation is $x^2 + y^2 = 8$. The coordinates of possible points of catching the cat are:

(*a*) (2, –2) (*b*) (2, 2)
(*c*) (–2, 2) (*d*) (–2, –2)

Sol. Let catching point be (x_1, y_1) then, $y_1 - x_1 = 0$ and $x_1^2 + y_1^2 = 8$

Therefore, $2x_1^2 = 8 \Rightarrow x_1^2 = 4 \Rightarrow x_1 = \pm 2$; So possible points are (2, 2) and (–2, –2).

Example 18: Distance between two points (8, – 4) and (0, a) is 10. All the values are in the same unit of length. Find the positive value of a.

Sol. From distance formula $(8-0)^2 + (-4-a)^2 = 100$

$$\Rightarrow (4+a)^2 = 36 \Rightarrow a = 2 \text{ or } -10$$

Concept Application

15. Find the distance of point (–12, 5) from the origin.

16. Find the distance between points

(*i*) (5, 4) and (6, 7)

(*ii*) (–7, 3) and (7, –6)

17. Find the value of a if distance between (a, 3) cm and (–2, 6) cm is 5 cm.

FUNCTION AND GRAPH

Function

Physics involves study of natural phenomena and describes them in terms of several physical quantities. A mathematical formulation of interdependence of these physical quantities is necessary for a concise and precise description of the phenomena. These mathematical formulae are expressed in form of equations and known as function.

Thus, a function describing a physical process expresses an unknown physical quantity in terms of one or more known physical quantities. We call the unknown physical quantity as dependent variable and the known physical quantities as independent variables. For the sake of simplicity, we consider a function that involves a dependent variable y and only one independent variable x. It is denoted $y = f(x)$ and is read as y equals to f of x. Here $f(x)$ is the value of y for a given x. Following are some examples of functions.

$$y = 2x + 1,\ y = 2x^2 + 3x + 1,\ y = \sin x,\ y = \ell n\,(2x + 1)$$

Knowledge of the dependent variable for different values of the independent variable, and how it changes when the independent variable varies in an interval is collectively known as behavior of the function.

GRAPH OF A FUNCTION

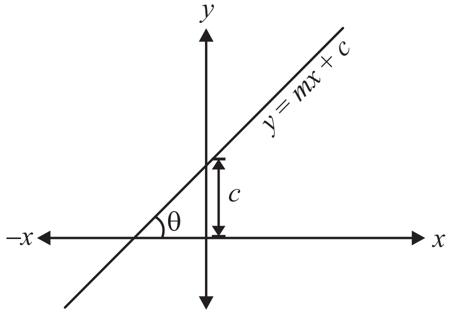
Graph of a function is the diagrammatic representation of a function and allows us to visualize it. To plot a graph the dependent variable (here y) is usually taken on the ordinate and the independent variable (here x) on the abscissa. Graph being an alternative way to represent a function does not require elaborate calculations and explicitly shows behavior of the function in a concerned interval.

Graphs of some commonly used Functions

Linear, parabolic, trigonometric and exponential functions are the most common in use.

(i) Straight line Equation and its Graph

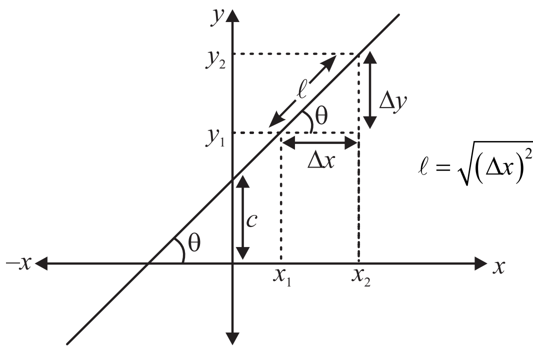
When the dependent variable y varies linearly with the independent variable x, the relationship between them is represented by a linear equation of the type given below $y = mx + c$. The equation is also shown in graph by an arbitrary line.



Here m & c are known as slope of the line and intercept on the y-axis, respectively.

Slope:

Slope of a line is a quantitative measure to express the inclination of the line. It is expressed by ratio of change in ordinate to change in abscissa.



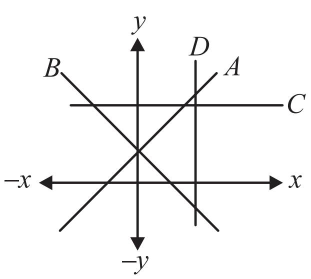
$$\ell = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Slope of a line $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x}$ = slope of tangent

When the x and the y axes are scaled identically, slope equals to tangent of the angle, which the line makes with the positive x-axis.

$$m = \tan\theta$$

It is positive if y increases with increase in x, negative if y decreases with increase in x, zero if y remains unchanged with change in x and infinite if y changes but x remains unchanged. For these cases the line is inclined up, inclined down, parallel to x-axis and parallel to y-axis respectively as shown in the adjoining figure by lines A, B, C and D respectively.

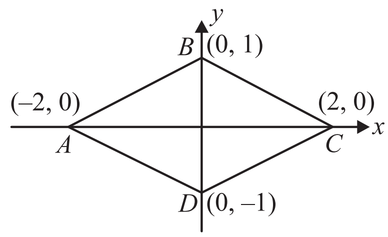


Intercept

It is equal to the value of ordinate y, where the line cuts the y-axis. It may be positive, negative or zero for lines crossing the positive y-axis, negative y-axis and passing through the origin respectively.

Train Your Brain

Example 19: A parallelogram $ABCD$ is shown in figure.



	Column-I		Column-II
(A)	Equation of side AB	(P)	$2y + x = 2$
(B)	Equation of side BC	(Q)	$2y - x = 2$
(C)	Equation of side CD	(R)	$2y + x = -2$
(D)	Equation of side DA	(S)	$2y - x = -2$
		(T)	$y + 2x = 2$

Sol. (A)-Q, (B)-P, (C)-S, (D)-R

For side AB : $m = \dfrac{1-0}{0-(-2)} = \dfrac{1}{2}, \quad c = 1 \Rightarrow y = \dfrac{1}{2}x + 1$

For side BC : $m = \dfrac{0-1}{2-0} = \dfrac{-1}{2}, \quad c = 1 \Rightarrow y = \dfrac{-1}{2}x + 1$

For side CD : $m = \dfrac{-1-0}{0-2} = \dfrac{1}{2}, \quad c = -1 \Rightarrow y = \dfrac{1}{2}x - 1$

For side DA : $m = \dfrac{0-(-1)}{-2-0} = -\dfrac{1}{2}, \quad c = -1 \Rightarrow y = \dfrac{1}{2}x - 1$

Example 20: Frequency f of a simple pendulum depends on its length ℓ and acceleration g due to gravity according to the following equation $f = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{\ell}}$

Graph between which of the following quantities is a straight line?

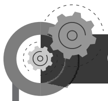
(*a*) f on the ordinate and ℓ on the abscissa

(*b*) f on the ordinate and $\sqrt{\ell}$ on the abscissa

(*c*) f^2 on the ordinate and ℓ on the abscissa

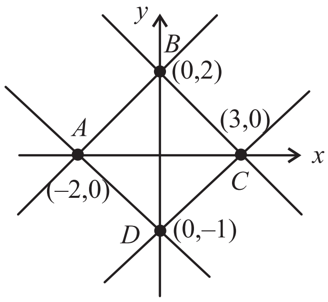
(*d*) f^2 on the ordinate and $1/\ell$ on the abscissa

Sol. $f^2 \propto 1/\ell$



Concept Application

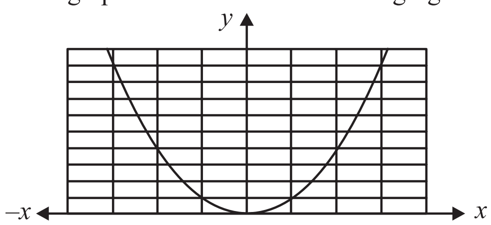
18. Four lines are drawn in the figure below. Match the entries in Column-I with entries in Column-II



Column-I		Column-II	
A.	Equation of line *AB*	P.	$3y - x + 3 = 0$
B.	Equation of line *BC*	Q.	$2y + x + 2 = 0$
C.	Equation of line *CD*	R.	$y = x + 2$
D.	Equation of line *AD*	S.	$3y + x = 2$
		T.	$3y + 2x = 6$

(ii) Quadratic equation and its graph

A function of the form $y = ax^2 + bx + c$ is known as quadratic equation. The graph of the quadratic function is a U-shaped curve is called a parabola. The simplest parabola has the form $y = ax^2$. Its graph is shown in the following figure.

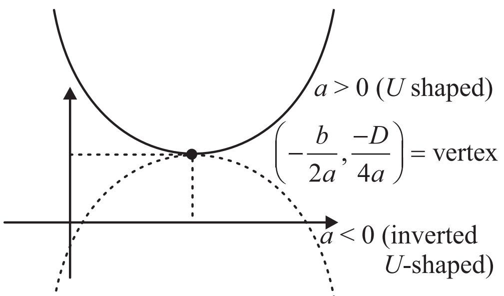


In general

$y = ax^2 + bx + c$

$$= a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{-D}{4a}\right)$$

where D = $b^2 - 4ac$



$a > 0$ ☺ ⇒ smiling parabola. (parabola opens upwards)

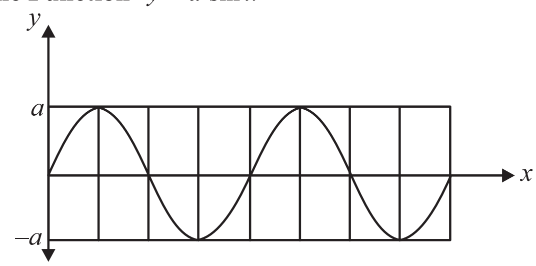
$a < 0$ ☹ ⇒ sad parabola. (parabola opens downwards)

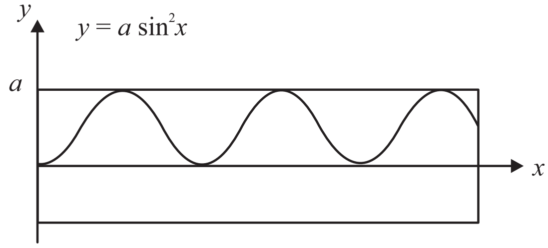
Vertex can lie in any quadrant depending on value of *a*, *b* and *c*.

Graph of some trigonometric functions

Among all the trigonometric functions, sinusoidal function,which includes both sine and cosine is most commonly in use.

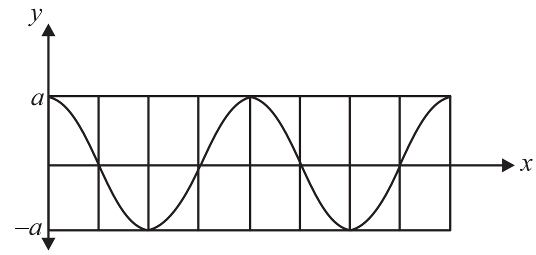
Sine Function $y = a \sin x$

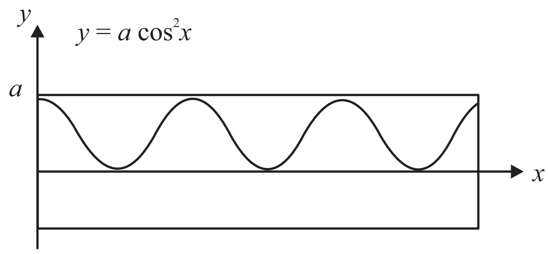




Here, a is known as the amplitude and equals the maximum magnitude of *y*. In the adjoining figure graph of a sine function is shown, which has amplitude a units.

Cosine Function $y = a \cos x$

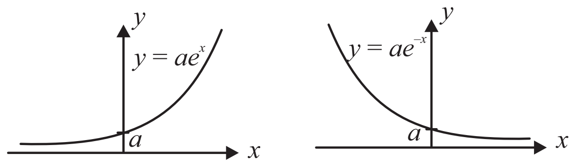




Here, a is known as the amplitude and equals the maximum magnitude of *y*. In the adjoining figure graph of a cosine function is shown, which has amplitude a units.

(iii) Exponential function and its graph

Behavior of several physical phenomena is described by exponential function to the base e. Here *e* is known as Euler's Number. $e = 2.718218$

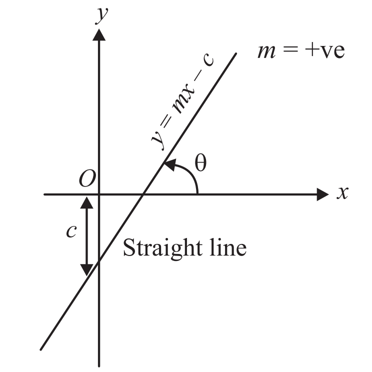


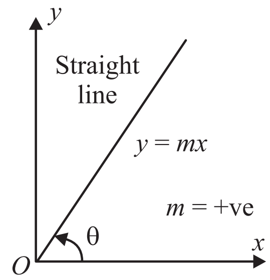
Most commonly used exponential function has the form $y = ae^x$, $y = ae^{-x}$. In the adjoining figure graph of this function is shown.

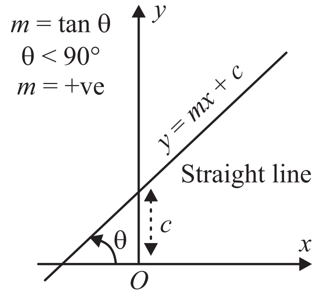
(iv) Circle and Ellipse

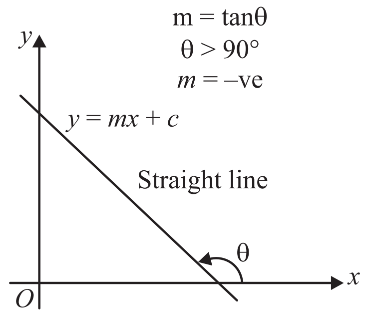
Circle : $x^2 + y^2 = a^2$ Ellipse : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

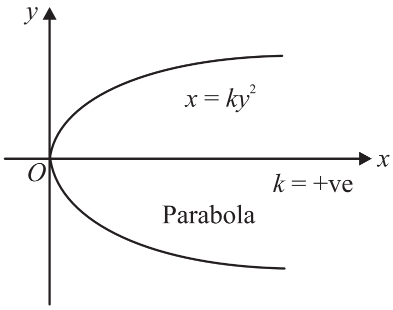
Some standard graphs & their equations

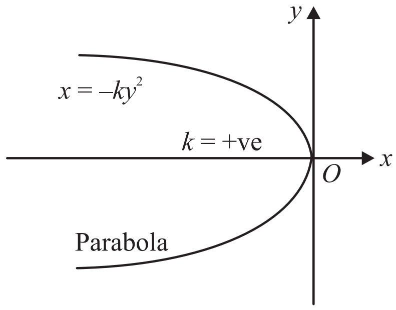


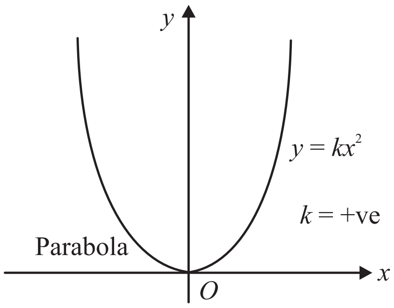


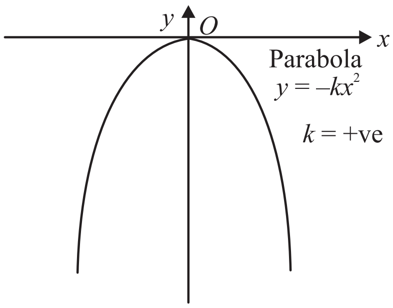


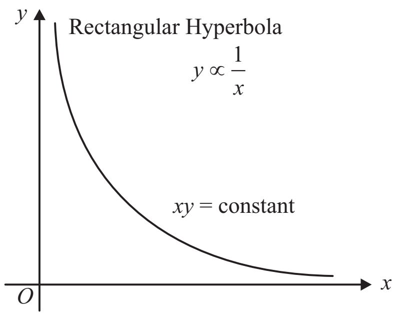


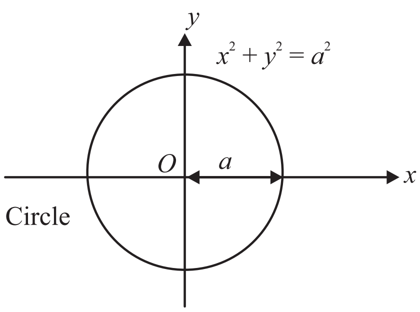


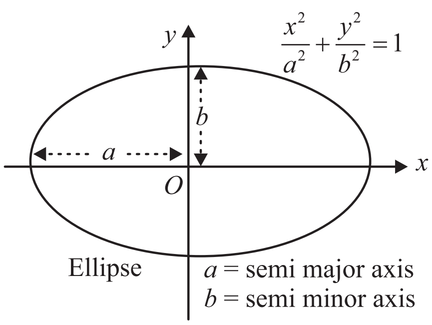


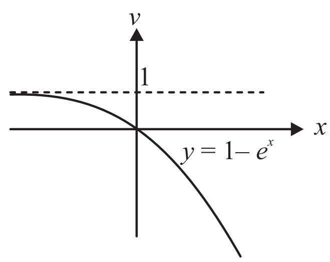


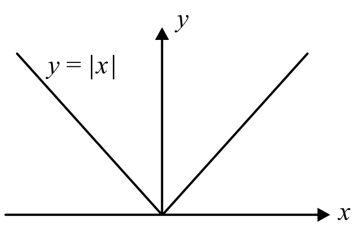












Logarithm

Logarithm is the exponent or power to which a base must be raised to yield a given number. Expressed mathematically, x is the logarithm of n to the base b if $b^x = n$ in which case one writes $x = \log_b n$. For example $2^5 = 32$, therefore 5 is the logarithm of 32 to base 2 or $5 = \log_2 32$. Similarly since $10^3 = 1000$ then $3 = \log_{10} 1000$.

Logarithms with base 10 are called common and are written simply as log

Logarithm laws

- $\log_a a = 1$
- $\log mn = \log m + \log n$
- $\log\left(\dfrac{m}{n}\right) = \log m - \log n$
- $\log m^n = n \log m$

Logarithms with base e denoted by ln are called natural logarithms where e (Euler's number) is an irrational number approximately equal to 2.71828. The relation between natural logarithm and common logarithm is given by

$\log_e m = 2.303 \log_{10} m$

$\ln 2 = 0.69$

$\log 2 = 0.3010$

$\ln e = 1$

$\ln 1 = 0$

Train Your Brain

Example 21: Calculate the value of $\log_{10} 12$

(Given $\log_{10} 2 = 0.301$, $\log_{10} 3 = 0.477$)

Sol. $\log_{10} 12 = \log_{10}(4 \times 3) = \log_{10} 4 + \log_{10} 3$

$= \log_{10} 2^2 + \log_{10} 3 = 2\log_{10} 2 + \log_{10} 3$

$= 2 \times 0.301 + 0.477 = 1.079$

Concept Application

19. Calculate the value of

(*i*) $\log_{10} 16$ (*ii*) $\log_{10} 4 + \log_{10} 12$

(*iii*) $\log_{10} 64$ (*iv*) $\log_{10} 3 - \log_{10} 24$

DIFFERENTIATION

(*i*) **Finite difference:** The finite difference between two values of a physical quantity is represented by Δ notation.

For example: Difference in two values of y is written as Δy as given in the table below

y_2	100	100	100
y_1	50	99	99.5
$\Delta y = y_2 - y_1$	50	1	0.5

(*ii*) **Infinitely small difference:** The infinitely small difference means very-very small difference. And this difference is represented by 'd' notation instead of 'Δ'.

For example infinitely small difference in the values of y is written as 'dy' if $y_2 = 100$ and $y_1 = 99.99999999$........

then $dy = 0.000000$..................00001

DEFINITION OF DIFFERENTIATION

Another name for differentiation is derivative. Suppose y is a function of x or $y = f(x)$

Differentiation of y with respect to x is denoted by symbol $f'(x)$ where $f'(x) = \dfrac{dy}{dx}$

dx is very small change in x and dy is corresponding very small change in y.

NOTATION: There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are these:

y'	"y prime" or "y dash"	Nice and brief but does not name the independent variable
$\dfrac{dy}{dx}$	"dy by dx"	Names the variables and used d for derivative
$\dfrac{df}{dx}$	"df by dx"	Emphasizes the function's name.
$\dfrac{d}{dx}f(x)$	"d by dx of f"	Emphasizes the idea that differentiation is an operation performed on f.
$d(f)$	"d of f"	A common operator notation.
$\dot{y}$	"y dot"	One of Newton's notations, now common for time derivatives i.e. $\dfrac{dy}{dt}$.

SLOPE OF A LINE

It is the tan of angle made by a line with the positive direction of x-axis, measured in anticlockwise direction.

Slope = $\tan\theta$

(In 1st quadrant $\tan\theta$ is +ve and 2nd quadrant $\tan\theta$ is –ve)

In figure (*a*)slope is positive

In figure (*b*)slope is negative

$\theta < 90°$ (1st quadrant)

$\theta > 90°$ (2nd quadrant)

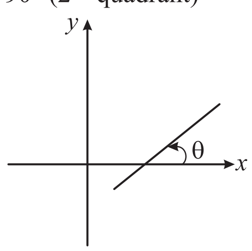


figure (*a*)

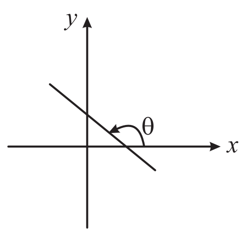
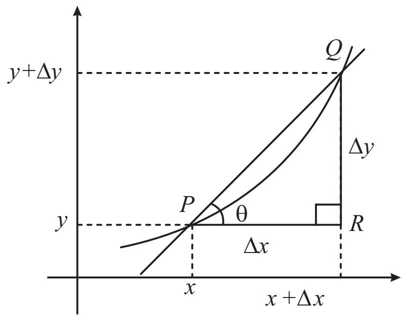


figure (*b*)

AVERAGE RATE OF CHANGE

Given an arbitrary function $y = f(x)$ we calculate the average rate of change of y with respect to x over the interval $(x, x + \Delta x)$ by dividing the change in value of y, i.e. $\Delta y = f(x + \Delta x) - f(x)$, by length of interval Δx over which the change occurred.



The average rate of change of y with respect to x over the interval $[x, x + \Delta x] = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Geometrically, $\dfrac{\Delta y}{\Delta x} = \dfrac{QR}{PR} = \tan\theta$ = Slope of the line PQ therefore we can say that average rate of change of y with respect to x is equal to slope of the line joining P and Q.

THE DERIVATIVE OF A FUNCTION

We know that, average rate of change of y w.r.t. x is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

If the limit of this ratio exists as $\Delta x \to 0$, then it is called the derivative of given function $f(x)$ and is denoted as

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Derivatives with respect to time

In physics, we are often looking at how things change over time:

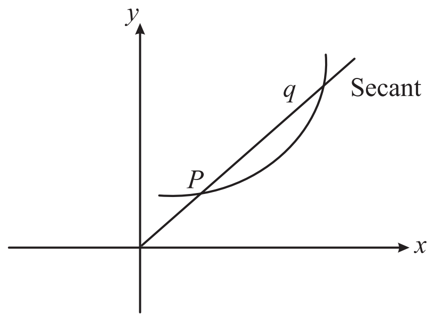
1. Velocity is the derivative of position with respect to time $v(t) = \dfrac{d}{dt}\big(x(t)\big)$.
2. Acceleration is the derivative of velocity with respect to time: $a(t) = \dfrac{d}{dt}\big(v(t)\big) = \dfrac{d^2}{dt^2}\big(x(t)\big)$.
3. Momentum (usually denoted p) is mass times velocity, and force (F) is mass times acceleration, so the derivative of momentum is $\dfrac{dp}{dt} = \dfrac{d}{dt}(mv) = m\dfrac{dv}{dt} = ma = F$.

GEOMETRICAL MEANING OF DIFFERENTIATION

The geometrical meaning of differentiation is very much useful in the analysis of graphs in physics. To understand the geometrical meaning of derivatives we should have knowledge of secant and tangent to a curve

Secant and tangent to a curve

A secant to a curve is a straight line, which intersects the curve at any two points.



TANGENT

A tangent is a straight line, which touches the curve at a particular point. Tangent is a limiting case of secant which intersects the curve at two overlapping points.

In the figure (a) shown, if value of Δx is gradually reduced then the point Q will move nearer to the point P. If the process is continuously repeated (figure (b)) value of Δx will be infinitely small and secant PQ to the given curve will become a tangent at point P.

Therefore $\displaystyle \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx} = \tan\theta$

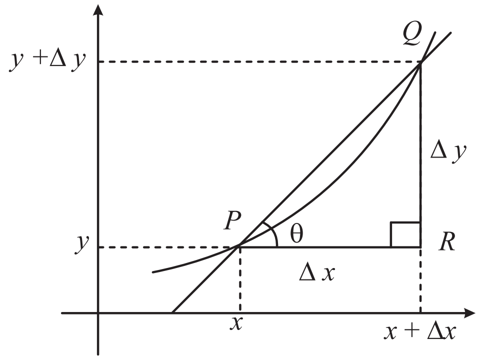


figure (a)

we can say that differentiation of y with respect to x, i.e. $\left(\dfrac{dy}{dx}\right)$ is equal to slope of the tangent at point P (x, y) or $\tan\theta = \dfrac{dy}{dx}$

(From fig. a, the average rate of change of y from x to $x + \Delta x$ is identical with the slope of secant PQ.)

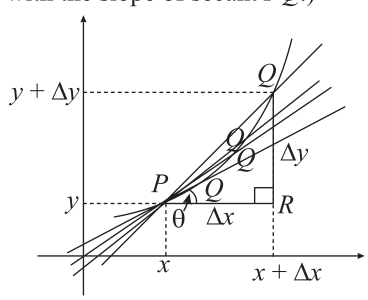


Figure (b)

THEOREMS OF DIFFERENTIATION

1. If c is constant, then $\frac{d}{dx}(c)=0$
2. If $y = cu$, where c is a constant and u is a function of x, then
$$\frac{dy}{dx}=\frac{d}{dx}(cu)=c\frac{du}{dx}$$
3. If $y = u \pm v \pm w$, where, u, v and w are functions of x, then
$$\frac{dy}{dx}=\frac{d}{dx}(u\pm v\pm w)$$
$$=\frac{du}{dx}\pm\frac{dv}{dx}\pm\frac{dw}{dx}$$
4. If $y = u\,v$ where u and v are function of x, then
$$\frac{dy}{dx}=\frac{d}{dx}(uv)$$
$$=u\frac{dv}{dx}+v\frac{du}{dx}$$
5. If $y=\frac{u}{v}$, where u and v are functions of x, then
$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{u}{v}\right)$$
$$=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
6. If $y = x^n$ where n is a real number, then
$$\frac{dy}{dx}=\frac{d}{dx}(x^n)=nx^{n-1}$$

Formulae for differential coefficients of trigonometric, logarithmic and exponential functions

1. $\frac{d}{dx}(\sin x)=\cos x$ **2.** $\frac{d}{dx}(\cos x)=-\sin x$

3. $\frac{d}{dx}(\tan x)=\sec^2 x$ **4.** $\frac{d}{dx}(\cot x)=-\mathrm{cosec}^2 x$

5. $\frac{d}{dx}(\sec x)=\sec x\tan x$ **6.** $\frac{d}{dx}(\mathrm{cosec}\, x)=-\mathrm{cosec}\, x\cot x$

7. $\frac{d}{dx}(\log_e x)=\frac{1}{x}$ **8.** $\frac{d}{dx}(e^x)=e^x$

Train Your Brain

Example 22: Find $\frac{dy}{dx}$, when

(i) $y=\sqrt{x}$ (ii) $y = x^5 + x^4 + 7$

(iii) $y = x^2 + 4x^{-1/2} - 3x^{-2}$

Sol. (i) Here, $y=\sqrt{x}$
$$\frac{dy}{dx}=\frac{d}{dx}(\sqrt{x})=\frac{d}{dx}(x^{1/2})=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$$

(ii) Here, $y = x^5 + x^4 + 7$
$$\frac{dy}{dx}=\frac{d}{dx}(x^5+x^4+7)=\frac{d}{dx}(x^5)+\frac{d}{dx}(x^4)+\frac{d}{dx}(7)$$
$$= 5x^4 + 4x^3 + 0 = 5x^4 + 4x^3$$

(iii) Here, $y = x^2 + 4x^{-1/2} - 3x^{-2}$
$$\frac{dy}{dx}=\frac{d}{dx}(x^2+4x^{-1/2}-3x^{-2})$$
$$=\frac{d}{dx}(x^2)+\frac{d}{dx}(4x^{-1/2})-\frac{d}{dx}(3x^{-2})$$
$$=\frac{d}{dx}(x^2)+\frac{d}{dx}(4x^{-1/2})-\frac{d}{dx}(3x^{-2})$$
$$=\frac{d}{dx}(x^2)+4\frac{d}{dx}(x^{-1/2})-3\frac{d}{dx}(x^{-2})$$
$$=2x+4\left(-\frac{1}{2}\right)x^{-3/2}-3(-2)x^{-3}=2x-2x^{-3/2}+6x^{-3}$$

Example 23: If $3y = 4x^2 - 5$ find $\frac{dy}{dx}$

Sol. $y=\frac{4}{3}x^2-\frac{5}{3}\Rightarrow\frac{dy}{dx}=\frac{8x}{3}$

Example 24: $\sqrt{y}=x-1$, find $\frac{dy}{dx}$

Sol. $y = x^2 - 2x + 1 \Rightarrow \frac{dy}{dx}=2x-2$

Concept Application

20. If acceleration $=\frac{dv}{dt}$. Find acceleration at $t = 1$ sec from $v = 3t^2 - 1$

21. $y=\sqrt{x}-3x^2$, find $\frac{dy}{dx}$ **22.** $y=\frac{1}{x^2}-2x$, find $\frac{dy}{dx}$

23. $y=\frac{3x-5}{x^2}$ find $\frac{dy}{dx}$ **24.** $x = 9y^2$ find $\frac{dy}{dx}$

Find $\frac{dy}{dx}$ for the following

25. $y = x^{7/2}$ **26.** $y = x^{-3}$

27. $y = x$ **28.** $y = x^5 + x^3 + 4x^{1/2} + 7$

29. $y = 5x^4 + 6x^{3/2} + 9x$ **30.** $y = ax^2 + bx + c$

31. $y=3x^5-3x-\frac{1}{x}$

32. Given $S = t^2 + 5t + 3$, than $\frac{dS}{dt}$

33. Given $S=ut+\frac{1}{2}at^2$, where u and a are constants. Obtain the value of $\frac{dS}{dt}$.

34. The area of a blot of ink is growing such that after t seconds, its area is given by $A = (3t^2 + 7)$ cm^2. Calculate the rate of increase of area at $t = 5$ seconds.

35. The area of a circle is given by $A = \pi r^2$, where r is the radius. What is the rate of increase of area w.r.t. rate of change of radius.

Obtain the differential coefficient (differentiation) of the following:

36. $(x - 1)(2x + 5)$

37. $(9x^3 - 8x + 7)(3x^5 + 5)$

38. If $t = \sqrt{s} - 1$, then the velocity at $t = 2$ sec is

39. If $S = 3t^2$, then double differentiation of s with. respect to t. is

40. Velocity of a body is given by $v = 3t^2 - 4t$, then rate of change of velocity w.r.t. to time at $t = 1$ sec is

41. Acceleration is given by $a = \frac{dv}{dt}$. then acceleration at $t = 10$ sec from $v = 3t^2 + t$

42. $\frac{1}{2x+1}$

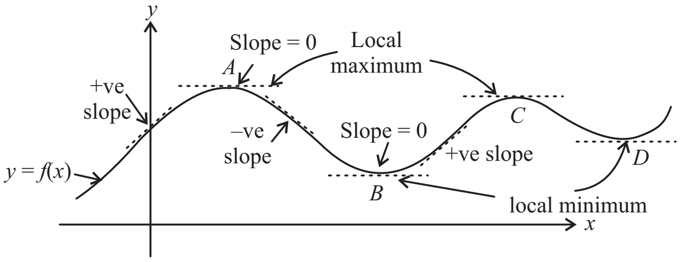
43. $\frac{3x+4}{4x+5}$

44. $\frac{x^2}{x^3+1}$

MAXIMA AND MINIMA

Finding maxima and minima of a function using derivatives:-

A maximum is a high point and minimum is low point of a function (see figure below)



In a smoothly changing function a maximum or a minimum is always where function flattens out or where slope of tangent line is zero. We know slope = $\frac{dy}{dx}$. So a function reaches its maximum or minimum value when $\frac{dy}{dx} = 0$.

In the neighbourhood of maximum (point A), slope changes from positive to zero at point A and then becomes negative as x increases which means $\frac{d}{dx}(\text{slope}) < 0 \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} < 0$

In the neigbourhood of minimum (point B), slope changes from negative to zero and then becomes positive as x increases which means

$$\frac{d}{dx}(\text{slope}) > 0 \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} > 0$$

Second Derivative Test:

When a function's slope $\left(\frac{dy}{dx}\right) = 0$ at a point and its second derivative at that point is

(*i*) less than zero, it is a local maximum

(*ii*) greater than zero, it is a local minimum

Train Your Brain

Example 25: What is the minimum value of y for the curve $y = -8x^3 + x^4$.

Sol. $y = -8x^2 + x^4$

$$\frac{dy}{dx} = -16x + 4x^3 = -x(16 - 4x^2)$$

The function will have a maximum or minimum value when $\frac{dy}{dx} = 0$

$\Rightarrow x(16 - 4x^2) = 0$

$\Rightarrow x = 0$ or $x = \pm 2$

Now

$$\frac{d^2y}{dx^2} = 16 - 12x^2$$

At $x = 0$, $\frac{d^2y}{dx^2} = -16$ (maximum)

At $x = \pm 2$, $\frac{d^2y}{dx^2} = -16 + 48 = +32$ (minimum)

So function has minimum value at $x = \pm 2$

$y_{\min} = -8 \times 4 + 16 = -16$

Example 26: A ball is thrown vertically upward in the air. Its height y at any time t is given by $y = 10t - 5t^2$ where y is in meters and t is in seconds. What is the maximum height attained by the ball?

Sol. $y = 10t - 5t^2$

$$\frac{dy}{dt} = 10 - 10t = 0$$

$\Rightarrow t = 1$ sec

$$\frac{d^2y}{dt} = -10 \text{ (maximum)}$$

ball attains maximum height at $t = 1s$

$y_{max} = 10 \times 1 - 5 \times 1^2 = 5$m.

Concept Application

45. Find the minimum value of

(*i*) $y = 1 + x^2 - 2x$ (*ii*) $y = 5x^2 - 2x + 1$

46. The position of a particle moving along the y-axis is given by $y = 3t^2 - t^3$, where y is in meters and t is in sec. The time when particle attains its maximum positive y position is

(*a*) 1.5 sec (*b*) 4 sec

(*c*) 2 sec (*d*) 3 sec

INTEGRATION

(*i*) Integration is the inverse process of differentiation: Integration is the process of finding the function, whose derivative is given. For this reason, the process of integration is the inverse process of differentiation.

Consider a function $f(x)$, whose derivative w.r.t. x is another function $f'(x)$ i.e. $\frac{d}{dx}(f(x)) = f'(x)$

If differentiation of $f(x)$ w.r.t. x is equal to $f'(x)$, then $f(x) + c$ is called the integration of $f'(x)$, where c is called the constant of integration.

Symbolically, it is written as $\int f'(x)dx = f(x) + c$

Here, $f'(x)\ dx$ is called element of integration and $\int$ is called indefinite integral operator. Let us proceed to obtain integral of x^n w.r.t. x, $\frac{d}{dx}(x^{n+1}) = (n+1)x^n$

Since the process of integration is the inverse process of differentiation,

$\int (n+1)x^n dx$ or $(n+1)\int x^n dx$

or $\int x^n dx = \frac{x^{n+1}}{n+1}$

The above formula holds for all values of n, except $n = -1$.

It is because, for $n = -1$, $\int x^n dx = \int x^{-1} dx = \int \frac{1}{x} dx$

Since $1/x$ is differential coefficient of $\log_e x$

i.e. $\frac{d}{dx}(\log_e x) = \frac{1}{x} \Rightarrow \therefore \int \frac{1}{x} dx = \log_e x$

Similarly, the formulae for integration of some other functions can be obtained if we know the differential coefficients of various functions.

(*ii*) Few basic formulae of integration: Following are a few basic formulae of integration:

1. $\int 1 dx = x + C$
2. $\int a\ dx = ax + C$
3. $\int x^n\ dx = \left(\left(x^{n+1}\right)/\left(n+1\right)\right) + C; n \neq -1$
4. $\int \sin x\ dx = -\cos x + C$
5. $\int \cos x\ dx = \sin x + C$
6. $\int \sec^2 x\ dx = \tan x + C$
7. $\int \csc^2 x\ dx = -\cot x + C$
8. $\int \sec x\left(\tan x\right) dx = \sec x + C$
9. $\int \csc x\left(\cot x\right) dx = -\csc x + C$
10. $\int (1/x) dx = \ln|x| + C$
11. $\int e^x dx = e^x + C$

Train Your Brain

Example 27: Integrate w.r.t. x:

(*i*) $x^{11/2}$ (*ii*) x^{-7} (*iii*) $x^{p/q}$

Sol. (*i*) $\int x^{11/2} dx = \frac{x^{\frac{11}{2}+1}}{\frac{11}{2}+1} + c = \frac{2}{13} x^{13/2} + c$

(*ii*) $\int x^{-7} dx = \frac{x^{-7+1}}{-7+1} + c = -\frac{1}{6} x^{-6} + c$

(*iii*) $\int x^{\frac{p}{q}} dx = \frac{x^{\frac{p}{q}+1}}{\frac{p}{q}+1} + c = \frac{q}{p+q} x^{(p+q)/q} + c$

Example 28: Find integer w.r.t. x, for $\left(\frac{\sqrt{x}}{x^2}\right)$

Sol. $\int \frac{\sqrt{x}}{x^2} dx = \int x^{-2+\frac{1}{2}} dx = \int x^{\frac{-3}{2}} dx = \int \frac{x^{\frac{-3}{2}+1}}{-\frac{3}{2}+1}$

$= \frac{-2}{\sqrt{x}} + c$

Example 29: $\int (x+1)^2 dx$

Sol. $\int x^2 dx + 2\int x dx + 1\int dx$

$\frac{x^3}{3} + 2\frac{x^2}{2} + x + c = \frac{x^3}{3} + x^2 + x + c$

Concept Application

47. $\int (1 - 3x^3) dx$

48. $\int \frac{6x^2 + 4}{x^2} dx$

49. $\int (\sin x + \cos x) dx$

50. $\int \left(\frac{1}{x^3} - \cos x\right) dx$

51. $\int \cos 3x\ dx$

52. $\left(\frac{1}{x^2} + 3\sin 2x\right) dx$

(*iii*) Definite integrals: When a function is integrated between a lower limit and an upper limit, it is called a definite integral. If $\frac{d}{dx}\left(f(x)\right) = f'\left(x\right)$, then $\int f'(x)dx$ is called indefinite integral and $\int_a^b f'(x)dx$ is called definite integral

Here, a and b are called lower and upper limits of the variable x. After carrying out integration, the result is evaluated between upper and lower limits as explained below:

$$\int_a^b f'(x)dx = |\, f(x)|_a^b = f(b) - f(a)$$

Train Your Brain

Example 30: Evaluate the integral: $\int_1^5 x^2 dx$

Sol. $\int_1^5 x^2 dx = \left|\frac{x^3}{3}\right|_1^5 = \frac{1}{3}|x^3|_1^5 = \frac{1}{3}((5)^3-(1)^3)$

$= \frac{1}{3}(125-1) = \frac{124}{3}$

Example 31: $\int_0^{\pi}(\sin x + \cos x)\,dx$

Sol. $\int_0^{\pi} \sin x dx + \int_0^{\pi} \cos x dx$

$= |-\cos x|_0^{\pi} + |\sin x|_0^{\pi}$

$= -[\cos\pi - \cos 0] + [\sin\pi - \sin 0] = -(-2) + 0 = 2$

Concept Application

53. $\int_{x=-1}^{x=1}(ax^2+b)\,dx$

54. $\int_{x=0}^{x=2}\frac{2x}{4x^2+1}dx$

55. $\int_{x=0}^{\pi/2}\sin(3x-1)\,dx$

56. $\int_{x=A}^{B}\cos(2x-30°)\,dx$

57. $\int_R^{\infty}\frac{GMm}{r^2}dr$

58. $\int_{r_1}^{r_2} -k\frac{q_1 q_2}{r^2}dr$

59. $\int_u^v M\upsilon d\upsilon$

60. $\int_0^{\infty} x^{-1/2}dx$

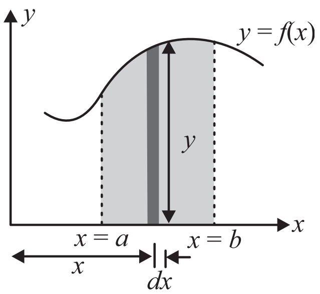
61. $\int_0^{\pi/2}\sin x\,dx$

62. $\int_0^{\pi/2}\cos x\,dx$

63. $\int_{-\pi/2}^{\pi/2}\cos x\,dx$

Application of integration

❖ Area under the curve :



Area of small element = $ydx = f(x)dx$

If we sum up all areas between $x = a$ and $x = b$ then $\int_a^b f(x)\,dx$ = shaded area between curve and x-axis.

❖ Average value

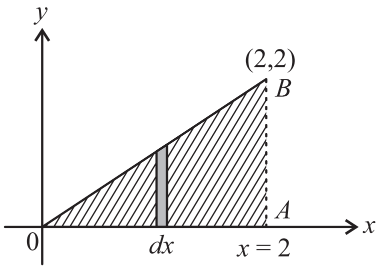
if $y = f(x)$ than $\bar{y}$ or $y_{\text{average}} = \frac{\int_{x=a}^{x=b} f(x)dx}{b-a}$

$= \frac{\text{Area under the curve}}{\text{interval}}$

Train Your Brain

Example 32: Find the area bounded by $y = x$ and x-axis between $x = 0$ and $x = 2$.

Sol.



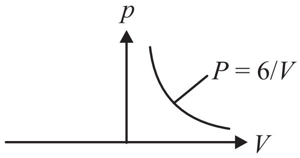
$\int y dx = \int_0^2 x dx$

$\left.\frac{1}{2}x^2\right|_0^2 = 2$ unit

This result can also be obtained by using geometry

$\Delta OAB = \frac{1}{2}\times 2\times 2 = 2$ units

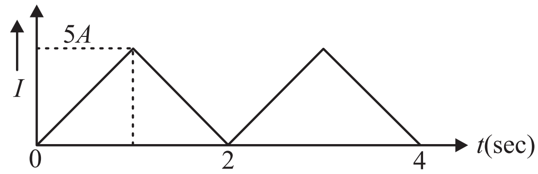
Example 33: A gas expands its volume from V to 3V as shown in figure. Calculate the work in this process if $W = \int pdv$.



(*a*) 6.6 (*b*) 6.9 (*c*) 3 (*d*) 3.5

Sol. $W = \int_V^{3V} pdv \Rightarrow \int_V^{3V}\frac{6}{v}dv = [6\log_e v]_V^{3V} = 6.6\,\text{J}$

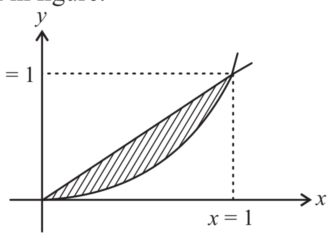
Example 34: Calculate average value of current from $t = 0$ to $t = 4$ seconds.



(*a*) 2.5 (*b*) 3 (*c*) 3.5 (*d*) 2

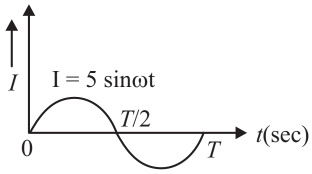
Sol. $I_{\text{average}} = \frac{\int_a^b Idt}{\int_a^b dt} = \frac{\text{total area}}{\text{total time}} = \frac{5\times 2}{4} = 2.5$ amp

Example 35: Find the shaded area bounded by line $y = x$ and $y = x^2$ as shown in figure.



Sol. $A = \int_0^1 x\,dx - \int_0^1 x^2\,dx = \left.\frac{x^2}{2}\right|_0^1 - \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ units

Example 36: Calculate average value of current from $t = 0$ to $t = T$ seconds ($T = 2\pi/\omega$).



(*a*) 0 (*b*) 1 (*c*) 5 (*d*) 2

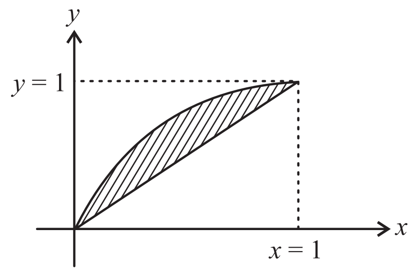
Sol. $I_{\text{average}} = \frac{\int_0^T 5\sin\omega t}{\int_0^T dt} = \frac{5}{\omega T}\left[-\cos\omega t\right]_0^T$

$$= -\frac{5}{\omega T}\left[\cos\omega T - \cos 0\right] = 0 \qquad \left[\omega = \frac{2\pi}{T}\right]$$

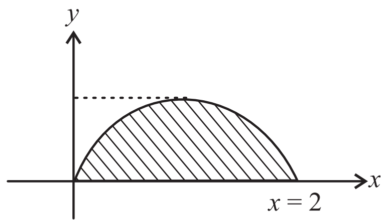
Concept Application

64. Find the area bounded by the curve $y = 4 - x^2$ from $x = -2$ to $x = 2$

65. Find the shaded area bounded by the curve $y = \sqrt{x}$ and $y = x$ as shown in figure?



66. Find the area bounded by $y = 2x - x^2$ and x-axis between $x = 0$ to $x = 2$ as shown in figure.



EXERCISE

TROGNOMETRY

1. Change degree into radian

(*a*) 160° (*b*) 135° (*c*) 75° (*d*) 65°

(*e*) 225° (*f*) 250° (*g*) 310°

2. Change radian into degree-

(*a*) $\frac{\pi}{4}$ (*b*) $\frac{7\pi}{2}$

(*c*) $\frac{3\pi}{5}$ (*d*) $\frac{2\pi}{3}$

(*e*) $\frac{3\pi}{4}$

3. Evaluate:-

(*a*) cos 15° (*b*) tan 15° (*c*) sin 53° (*d*) cos 53°

(*e*) tan 37° (*f*) tan 53° (*g*) sin53° – cos 37°

4. Evaluate:-

(*a*) $\frac{\sin 135°}{\cos 120°}$ (*b*) $\frac{\sin 120°}{\cos 15°}$

(*c*) sin 105° (*d*) sin300°

(*e*) cos 240° (*f*) $\sin^2(20°) + \sin^2(70°)$

(*g*) sin225° (*h*) sin 315°

(*i*) cos 270°

5. Evaluate:-

(*a*) 2 sin 15° cos 15° (*b*) sin 22.5° cos 22.5°

(*c*) tan 75° (*d*) $\sin^2 22.5°$

6. Evaluate:-

(*a*) $\cos\left(\frac{5\pi}{4}\right)$ (*b*) $\sin\left(\frac{2\pi}{3}\right)$

(*c*) $\sin\left(\frac{3\pi}{4}\right)$ (*d*) $\tan\left(\frac{7\pi}{6}\right)$

7. $\sqrt{1+\sin\theta}$ is equal to

(*a*) $(\sin\theta + \cos\theta)$ (*b*) $\sin\theta - \cos\theta$

(*c*) $\sin\frac{\theta}{2}+\cos\frac{\theta}{2}$ (*d*) $\sin\frac{\theta}{2}-\cos\frac{\theta}{2}$

8. $\sqrt{1+\cos\theta}$ is equal to

(*a*) $\sqrt{2}\sin\frac{\theta}{2}$ (*b*) $\sqrt{2}\cos\frac{\theta}{2}$

(*c*) $\frac{1}{\sqrt{2}}\sin\frac{\theta}{2}$ (*d*) $\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}$

9. If $\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}=\frac{7}{3}$, , then $\tan\theta =$

(*a*) $\frac{3}{5}$ (*b*) $\frac{5}{2}$ (*c*) $\frac{3}{5}$ (*d*) $\frac{2}{5}$

ALGEBRA

10. The equation whose roots are the squares of the roots of the equation $ax^2 + bx + c = 0$ is

(*a*) $a^2x^2 + b^2x + c^2 = 0$

(*b*) $a^2x^2 - (b^2 - 4ac)x + c^2 = 0$

(*c*) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$

(*d*) $a^2x^2 + (b^2 - ac)x + c^2 = 0$

11. The value of 'a' for which one root of quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as other is:

(*a*) $\frac{2}{3}$ (*b*) $\frac{-2}{3}$ (*c*) $\frac{1}{3}$ (*d*) $\frac{-1}{3}$

12. If the roots of the equation $ax^2 + bx + c = 0$ are in the ratio $m : n$, then

(*a*) $mn\, b^2 = ac\,(m+n)^2$ (*b*) $b^2\,(m+n) = mn$

(*c*) $m + n = b^2\, mn$ (*d*) $mnc^2 = ab\,(m+n)^2$

13. The quadratic equation whose roots are the *x* and *y* intercepts of the line passing through (1, 1) and making a triangle of area *A* with axes may be-

(*a*) $x^2 + Ax + 2A = 0$ (*b*) $x^2 - 2Ax + 2A = 0$

(*c*) $x^2 - Ax + 2A = 0$ (*d*) None of these

AP AND GP SERIES

14. Find the sum of first 20 natural Number.

(*a*) 210 (*b*) 200 (*c*) 220 (*d*) 230

15. 30th term of the A.P: 10, 7, 4, …, is

(*a*) 97 (*b*) 77 (*c*) –77 (*d*) –87

16. The sum of the first five multiples of 3 is:

(*a*) 45 (*b*) 55 (*c*) 65 (*d*) 75

17. Find sum of $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$ ...... ∞.

(*a*) $\frac{3}{4}$ (*b*) $\frac{4}{3}$ (*c*) $\frac{2}{3}$ (*d*) $\frac{3}{2}$

18. Find sum of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$ .... up to ∞.

(*a*) $\frac{3}{2}$ (*b*) $\frac{2}{3}$ (*c*) $\frac{4}{3}$ (*d*) $\frac{3}{4}$

19. Find the sum of the geometric series:

4 – 12 + 36 – 108 + ………….. to 10 terms

(*a*) 59048 (*b*) –30421

(*c*) –59048 (*d*) 30421

BINOMIAL APPROXIMATION

20. Find approximate value of the $\sqrt{0.95}$:

(*a*) 1 (*b*) 0.595 (*c*) 0.60 (*d*) 0.975

21. Find approximate value of the $\sqrt{104}$:

(*a*) 10.2 (*b*) 12 (*c*) 13.5 (*d*) 15

22. Find approximate value of the $(102)^2$

(*a*) 11600 (*b*) 9200 (*c*) 10400 (*d*) 12400

23. Find approximate value of the $(4.04)^3$

(*a*) 60.05 (*b*) 75.63 (*c*) 65.92 (*d*) 55.72

24. Find approximate value of the $(9.6)^4$

(*a*) 4200 (*b*) 3600 (*c*) 2100 (*d*) 8400

FUNCTIONS & GRAPHS

25. The equation of straight line having slope $\sqrt{3}$ and y intercept of –2 will be :

(*a*) $y=\sqrt{3}x+2$ (*b*) $y=\sqrt{3}x-2$

(*c*) $y=-\sqrt{3}x-2$ (*d*) $y=-\sqrt{3}x+2$

26. The equation of line is $2y = 3x - 6$ its x and y intercepts are

(*a*) –3, 2 (*b*) 3, –2 (*c*) 2, –3 (*d*) –2, 3

27. The equation of line making an angle 135º with the positive x-axis and passing through a point (2, 3) will be :

(*a*) $y - x = 5$ (*b*) $y = x - 5$

(*c*) $y - x = 1$ (*d*) $y = -x + 5$

28. The minimum distance of origin from line $3x - 4y + 5 = 0$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

29. The distance between points (2, 3, –7) and (–2, 0, 5) is

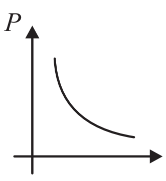
(*a*) 5 (*b*) 13 (*c*) $\sqrt{145}$ (*d*) $\sqrt{119}$

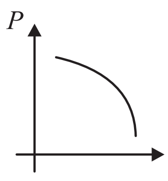
30. The points where the line $y = x$ intersect the curve $x^2 + y^2 = 32$ is

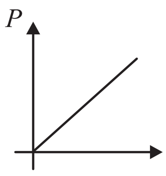
(*a*) $4\sqrt{2}, 4\sqrt{2}$ (*b*) 4, 4

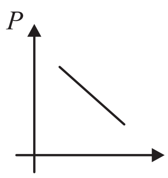
(*c*) $-4\sqrt{2}, -4\sqrt{2}$ (*d*) –2, –2

31. P – V graph for ideal gas at constant temperature is {ideal gas eq^n $PV = nRT$}

(*a*)


(*b*)


(*c*)


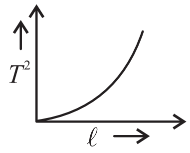
(*d*)


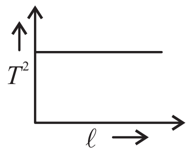
32. Time period of oscillations of a pendulum is given by $T = 2\pi\sqrt{\frac{\ell}{g}}$ then graph between T & ℓ is

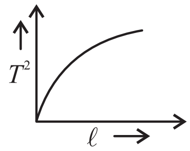
(*a*) Straight line (*b*) Parabola

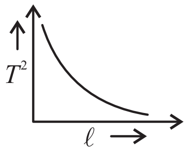
(*c*) Ellipse (*d*) Rectangular hyperbola

33. Which of the following curve is related to function $T^2 \propto \ell^3$

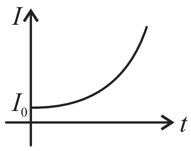
(*a*)


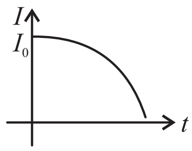
(*b*)


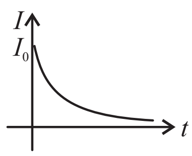
(*c*)


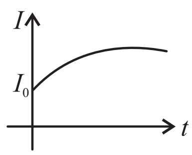
(*d*)


34. The variation of current flow in a circuit is given as $I = I_0\, e^{-t/RC}$. The graph representing I vs t will be

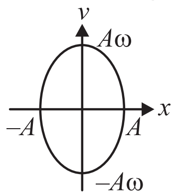
(*a*)


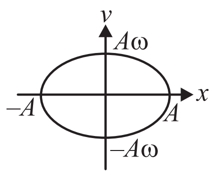
(*b*)


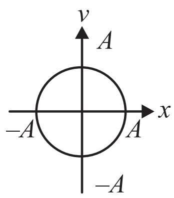
(*c*)


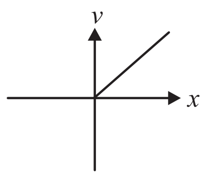
(*d*)


35. Velocity of a particle undergoing SHM is $v = \omega\sqrt{A^2 - x^2}$ where $\omega > 1$. The graph representing v versus x will be

(*a*)


(*b*)


(*c*)


(*d*)


LOGARITHMS

36. log 25 + log 4 – log 5 is equal to

(*a*) log 20 (*b*) log 25 (*c*) log 15 (*d*) log 10

37. $\log_e 8$ is equal to

(*a*) $\log_e 2$ (*b*) $2\log_e 2$ (*c*) $3\log_e 2$ (*d*) $4\log_e 2$

38. $\log_e 15$ is equal to

(*a*) $\log_e 3 + \log_e 5$ (*b*) $\log_e 5 - \log_e 3$

(*c*) $\log_e 10 + \log_e 5$ (*d*) $\log_e 10 - \log_e 5$

39. $\log_3 x^2 = 4$, find the value of x

(*a*) 3 (*b*) 5 (*c*) 7 (*d*) 9

40. $\log_5 x - \log_5(y) = 2$, find the value of $\frac{x}{y}$

(*a*) 100 (*b*) 25 (*c*) 50 (*d*) 75

DIFFERENTIATION

41. $f(x) = \cos x + \sin x$ then $f(\pi/2)$ will be

(*a*) 2 (*b*) 1 (*c*) 3 (*d*) 0

Direction (No. 42 to 44): Derivative of given function w.r.t. corresponding independent variable is.

42. $y = x^2 + x + 8$

(*a*) $\frac{dy}{dx} = 2x - 1$ (*b*) $\frac{dy}{dx} = -x + 1$

(*c*) $\frac{dy}{dx} = 2x + 1$ (*d*) $\frac{dy}{dx} = x - 1$

43. $s = 5t^3 - 3t^5$

(*a*) $\frac{ds}{dt} = 15t^2 + 15t^4$ (*b*) $\frac{ds}{dt} = 15t^4 + 15t^2$

(*c*) $\frac{ds}{dt} = 15t^4 - 15t^2$ (*d*) $\frac{ds}{dt} = 15t^2 - 15t^4$

44. $y = 5\sin x$

(*a*) $\frac{dy}{dx} = 3\cos x$ (*b*) $\frac{dy}{dx} = 5\cos x$

(*c*) $\frac{dy}{dx} = 5\sin x$ (*d*) $\frac{dy}{dx} = 3\sin x$

Direction (No. 45 to 48): First derivative and second derivative of given functions w.r.t. Corresponding independent variable is:

45. $y = 6x^2 - 10x - 5x^{-2}$

(*a*) $12x - 10 + 10x^{-3}$, $12 - 30x^{-4}$

(*b*) $10x - 12 + 20x^{-3}$, $15 - 30x^{-4}$

(*c*) $12x - 10 + 15x^{-3}$, $12 - 30x^{-4}$

(*d*) $10x - 15 + 12x^{-3}$, $12 - 30x^{-4}$

46. $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

(*a*) $12\theta^{-2} - 12\theta^{-4} + 4\theta^{-5}$, $24\theta^{-3} + 48\theta^{-5} + 20\theta^{-6}$

(*b*) $-12\theta^{-2} + 12\theta^{-4} - 4\theta^{-5}$, $24\theta^{-3} - 48\theta^{-5} + 20\theta^{-6}$

(*c*) $-6\theta^{-2} + 12\theta^{-4} - 8\theta^{-5}$, $12\theta^{-3} - 24\theta^{-5} + 10\theta^{-6}$

(*d*) $-8\theta^{-2} + 12\theta^{-4} - 6\theta^{-5}$, $24\theta^{-3} - 24\theta^{-5} + 10\theta^{-6}$

47. $\omega = 3z^7 - 7z^3 + 21z^2$

(*a*) $21z^6 + 21z^2 - 42z$, $126z^5 + 42z - 42$.

(*b*) $14z^6 - 28z^2 + 22z$, $120z^5 - 21z + 42$.

(*c*) $28z^6 - 14z^2 + 42z$, $122z^5 - 42z + 21$

(*d*) $21z^6 - 21z^2 + 42z$, $126z^5 - 42z + 42$

48. $y = \sin x + \cos x$

(*a*) $\cos x - \cos x$, $-\sin x - \sin x$

(*b*) $\sin x - \sin x$, $-\sin x - \cos x$

(*c*) $\cos x - \sin x$, $-\sin x - \cos x$

(*d*) $\sin x + \cos x$, $-\cos x - \cos x$

Direction (No. 49 to 51): Derivative of given functions w.r.t. the independent variable x is.

49. $y = x\sin x$

(*a*) $\sin x + x\cos x$ (*b*) $\sin x - x\cos x$

(*c*) $\cos^2 x - x\sin^2 x$ (*d*) $\sin^2 x - x\cos^2 x$

50. $y = e^x \ell n\, x$

(*a*) $e^x \ell nx - \frac{e^x}{x}$ (*b*) $e^x \ell nx - \frac{e^x}{x^2}$

(*c*) $e^x \ell nx + \frac{e^x}{x^2}$ (*d*) $e^x \ell nx + \frac{e^x}{x}$

51. $y = (x-1)(x^2 + x + 1)$

(*a*) $\frac{dy}{dx} = 3x$ (*b*) $\frac{dy}{dx} = 3x^2$

(*c*) $\frac{dy}{dx} = 2x^2$ (*d*) $\frac{dy}{dx} = 2x$

Direction (No. 52 to 54): Derivative of given function w.r.t. the independent variable is

52. $y = \frac{\sin x}{\cos x}$

(*a*) $\sec^2 x$ (*b*) $\sec x$ (*c*) $\sec^2 2x$ (*d*) $\sec^3 2x$

53. $y = \frac{2x+5}{3x-2}$

(*a*) $y' = \frac{-19}{(3x-2)^2}$ (*b*) $y' = \frac{19}{(3x-2)^2}$

(*c*) $y' = \frac{19}{(3x-2)}$ (*d*) $y' = \frac{-19}{(3x+2)^2}$

54. $z = \frac{2x+1}{x^2-1}$

(*a*) $\frac{-2x^2-2x+2}{(x^2+1)^2}$ (*b*) $\frac{-2x^2-2x-2}{(x^2-1)^2}$

(*c*) $\frac{-2x^2+2x+2}{(x+1)^2}$ (*d*) $\frac{-2x^2-2x-2}{(x^2-1)}$

Direction (No. 55 to 59): $\frac{dy}{dx}$ for following functions is:

55. $y = (2x+1)^5$

(*a*) $10(2x+1)^3$ (*b*) $10(2x+1)^4$

(*c*) $10(2x-1)^3$ (*d*) $10(2x-1)^4$

56. $y = (4-3x)^9$

(*a*) $-8(4-3x)^8$ (*b*) $-27(4-3x)^9$

(*c*) $-27(4+3x)^9$ (*d*) $-27(4-3x)^8$

57. $y = \left(1 - \frac{x}{7}\right)^{-7}$

(*a*) $\left(1-\frac{x}{7}\right)^{8}$ (*b*) $\left(1-\frac{x}{7}\right)^{-8}$ (*c*) $\left(1-\frac{x}{7}\right)^{-5}$ (*d*) $\left(1-\frac{x}{7}\right)^{-4}$

58. $y=\sin(x)+\ln(x^2)+e^{2x}$

(*a*) $\cos(x)+\dfrac{2}{x}-2e^{2x}$ (*b*) $\cos(x)+\dfrac{2}{x}+2e^{2x}$

(*c*) $\sin(x)+\dfrac{2}{x}+2e^{2x}$ (*d*) $\sin(x)+\dfrac{2}{x}-2e^{2x}$

59. $y=2\sin(\omega x+\phi)$ where ω and ϕ constants

(*a*) $2\omega\cos(\omega x+\phi)$ (*b*) $2\omega\cos(\omega x-\phi)$

(*c*) $\omega\cos(\omega x+\phi)$ (*d*) $2\omega\operatorname{cosec}(\omega x+\phi)$

DIFFERENTIATION AS A RATE MEASUREMENT

60. Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. equation that relates dA/dt to dr/dt is.

(*a*) $\dfrac{dA}{dt}=\pi r\dfrac{dr}{dt}$ (*b*) $\dfrac{dA}{dt}=\pi r^2\dfrac{dr}{dt}$

(*c*) $\dfrac{dA}{dt}=2\pi r^2\dfrac{dr}{dt}$ (*d*) $\dfrac{dA}{dt}=2\pi r\dfrac{dr}{dt}$

61. $y=2u^3$, $u=8x-1$. Find $\dfrac{dy}{dx}$

(*a*) $48(8x-1)^2$ (*b*) $48(8x+1)^2$

(*c*) $48(8x-1)$ (*d*) $48(8x+1)$

62. $y=\sin u$, $u=3x+1$. Find $\dfrac{dy}{dx}$

(*a*) $3\cos(3x-1)$ (*b*) $3\cos(3x+1)$

(*c*) $3\sin(3x-1)$ (*d*) $3\sin(3x+1)$

63. $y=\sin u$, $u=\cos x$. Find $\dfrac{dy}{dx}$.

(*a*) $-\cos u.\sin x$ (*b*) $-\cos u.\sqrt{1-u^2}$

(*c*) $-\sin x.\sqrt{1-y^2}$ (*d*) all of above

64. $y=3t^2-1$, $x=t^2$. Find $\dfrac{dy}{dx}$.

(*a*) 3 (*b*) 2 (*c*) 1/3 (*d*) 1/2

MAXIMA & MINIMA

65. Maximum and minimum values of function $2x^3 - 15x^2 + 36x + 11$ is

(*a*) 39, 38 (*b*) 93, 83 (*c*) 45, 42 (*d*) 59, 58

66. Find out minimum/maximum value of y = $1 - x^2$ also find out those points where value is minimum/maximum.

(*a*) max 2, $x = -1$ (*b*) max 1, $x = 0$

(*c*) min 1, $x = -1$ (*d*) min 2, $x = 0$

67. For $y = (x-2)^2$, what is the maximum/minimum value and the point at which y is maximum/minimum?

(*a*) max 2, $x = 0$ (*b*) max 0, $x = 0$

(*c*) min 1, $x = -1$ (*d*) min 0, $x = 2$

68. Particle's position as a function of time is given by $x = -t^2 + 4t + 4$ find the maximum value of position co-ordinate of particle.

(*a*) 2 (*b*) 4 (*c*) –8 (*d*) 8

69. Find out minimum/maximum value of $y = 2x^3 - 15x^2 + 36x + 11$ also find out those points where value is minimum/maximum.

(*a*) max = 39 at $x = 2$, min = 39 at $x = -2$

(*b*) max = 39 at $x = 3$, min = 38 at $x = 2$

(*c*) max = 39 at $x = 2$, min = 38 at $x = 3$

(*d*) max = 39 at $x = 2$, min = 38 at $x = -2$

70. Determine the position where potential energy will be minimum if $U(x) = 100 - 50x + 1000x^2$ J.

(*a*) 0.25×10^{-2} (*b*) 2.5×10^{-2}

(*c*) 2.5×10^{-1} (*d*) 250×10^{-2}

71. Find out minimum/maximum value of $y = 4x^2 - 2x + 3$ also find out those points where value is minimum/maximum.

(*a*) min $=\dfrac{11}{4}, x=\dfrac{1}{2}$ (*b*) max $=\dfrac{11}{4}, x=\dfrac{1}{4}$

(*c*) min $=\dfrac{11}{4}, x=\dfrac{1}{4}$ (*d*) max $=\dfrac{11}{4}, x=\dfrac{1}{2}$

72. $x + y = 10$ then find the maximum value of $f = xy$?

(*a*) 25 (*b*) 30 (*c*) 15 (*d*) 35

73. If $\ell + r = 12$ here ℓ is length of cylinder and r is radius of cylinder then find maximum value of volume of cylinder

(*a*) 156 π (*b*) 350 π (*c*) 256 π (*d*) 250 π

74. If $\ell \times b = 32$ here ℓ is length of rectangle and b is width of rectangle then find minimum perphery of rectangle

(*a*) $8\sqrt{2}$ (*b*) $2\sqrt{16}$ (*c*) $32\sqrt{2}$ (*d*) $16\sqrt{2}$

INTEGRATION

75. $\int(2x)\,dx$ will be

(*a*) $x^2 + C$ (*b*) $2x + C$ (*c*) $2x^2 + C$ (*d*) $-x^2 + C$

76. $\int(x^2)\,dx$ will be

(*a*) $x + C$ (*b*) $2x + C$ (*c*) $\dfrac{x^3}{3}+C$ (*d*) $\dfrac{x^2}{2}+C$

77. $\int(x^2 - 2x + 1)\,dx$ will be

(*a*) $\dfrac{x^3}{3}-x^2-x+C$ (*b*) $\dfrac{x^3}{3}-x^2+x+C$

(*c*) $\dfrac{x^3}{3}+x^2-x+C$ (*d*) $\dfrac{x^3}{3}+x^2+x+C$

78. $\int(-3x^{-4})dx$ will be

(*a*) $x^{-3} + C$ (*b*) $x^3 + C$

(*c*) $-3x^{-3} + C$ (*d*) $3x^{-3} + C$

79. $\int(x^{-4})dx$ will be

(*a*) $\dfrac{1}{3}x^{-3}+C$ (*b*) $\dfrac{1}{3}x^{-5}+C$

(*c*) $\dfrac{1}{2}x^{-2}+C$ (*d*) $-\dfrac{1}{3}x^{-3}+C$

80. $\int \left(\frac{5}{x^2}\right) dx$ will be

(*a*) $-\frac{5}{x} + C$ (*b*) $\frac{5}{x} + C$

(*c*) $\frac{x}{5} + C$ (*d*) $-\frac{x}{5} + C$

81. $\int \left(\frac{3}{2}\sqrt{x}\right) dx$ will be

(*a*) $\sqrt{x^2} + C$ (*b*) $3\sqrt{x^4} + C$

(*c*) $2\sqrt{x^3} + C$ (*d*) $\sqrt{x^3} + C$

82. $\int \left(\frac{3}{2\sqrt{x}}\right) dx$ will be

(*a*) $2\sqrt{x^3} + C$ (*b*) $3\sqrt{x} + C$

(*c*) $\sqrt{x^3} + C$ (*d*) $\sqrt{x^4} + C$

83. $\int \left(\frac{4}{3}\sqrt[3]{x}\right) dx$ will be

(*a*) $x^{4/3} + C$ (*b*) $x^{3/4} + C$

(*c*) $x^{2/3} + C$ (*d*) $x^{1/3} + C$

84. $\int \left(\frac{1}{3\sqrt[3]{x}}\right) dx$ will be

(*a*) $\frac{x^{\frac{3}{4}}}{2} + C$ (*b*) $\frac{x^{\frac{2}{3}}}{3} + C$

(*c*) $x^{\frac{2}{3}} + C$ (*d*) $\frac{x^{\frac{2}{3}}}{2} + C$

85. $\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right) dx$ will be

(*a*) $\frac{3x^{\frac{4}{2}}}{4} + \frac{3x^{\frac{2}{4}}}{2} + C$ (*b*) $\frac{3x^{\frac{4}{2}}}{4} + \frac{3x^{\frac{2}{1}}}{2} + C$

(*c*) $\frac{3x^{\frac{3}{4}}}{4} + \frac{3x^{\frac{3}{2}}}{2} + C$ (*d*) $\frac{3x^{\frac{4}{3}}}{4} + \frac{3x^{\frac{2}{3}}}{2} + C$

86. $\int \left(\frac{1}{2}x^{-1/2}\right) dx$ will be

(*a*) $x^{2/3} + C$ (*b*) $x^{1/2} + C$

(*c*) $x^{3/1} + C$ (*d*) $x^{2/1} + C$

87. $\int \left(-\frac{1}{2}x^{-3/2}\right) dx$ will be

(*a*) $x^{-1/2} + C$ (*b*) $x^{+1/2} + C$

(*c*) $x^2 + C$ (*d*) $x^{-2/1} + C$

88. $\int (3 \sin x)\, dx$ will be

(*a*) $+3 \cos x + C$ (*b*) $+4 \cos x + C$

(*c*) $-3 \cos x + C$ (*d*) $-4 \cos x + C$

89. $\int \left(\frac{1}{3x}\right) dx$ will be

(*a*) $\frac{1}{3}\ell nx + C$ (*b*) $\frac{3}{1}\ell nx + C$

(*c*) $\frac{2}{3}\ell nx + C$ (*d*) $\frac{1}{2}\ell nx + C$

90. $\int \sin 3x\, dx$, will be (use, $u = 3x$)

(*a*) $-\cos 3x + C$ (*b*) $\frac{1}{3}\cos 3x - C$

(*c*) $-\frac{1}{2}\cos 3x - C$ (*d*) $-\frac{1}{3}\cos 3x + C$

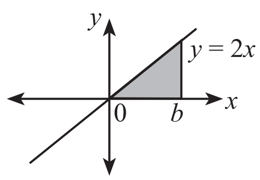
91. $\int_{-2}^{1} 5\, dx$ will be

(*a*) 15 (*b*) 16 (*c*) 17 (*d*) 18

92. $\int_{-4}^{-1} \frac{\pi}{2} d\theta$ will be

(*a*) $\frac{3\pi}{2}$ (*b*) $\frac{2\pi}{2}$ (*c*) 3π (*d*) 2π

93. Use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, b]$ $y = 2x$



(*a*) b^2 (*b*) b^3 (*c*) $2b^2$ (*d*) $\frac{b^3}{3}$

94. $\int_{-2}^{+2} \left(t^2 - 1\right) dt.$

(*a*) $\frac{3}{4}$ (*b*) $\frac{4}{3}$ (*c*) $\frac{1}{4}$ (*d*) $\frac{2}{3}$

95. A particle moves along x-axis with acceleration $a = 6(t - 1)$, where t is in seconds. If the particle is initially at the origin and moves along positive x-axis with $v_0 = 2$ ms^{-1}, find average acceleration of particle from $t = 0$ to $t = 10$ sec.

(*a*) 30 (*b*) 24 (*c*) 15 (*d*) 20

96. A body is moving along x axis as $v = 2t + 3t^2 + 2$ here v is velocity and t is time in second then find average velocity when particle moves from $t = 0$ to $t = 5$ second.

(*a*) 25 (*b*) 40 (*c*) 32 (*d*) 30

97. Current is flowing in conductor as $i = 6t + 9t^2$ here t is time and i is current then find average current in conductor $t = 0$ to $t = 10$ sec.

(*a*) 50 A (*b*) 330 A (*c*) 200 A (*d*) 420 A

98. Find the area of the curve y = sin$x$ between 0 and π.

(*a*) 2 sq. unit (*b*) 5 sq. unit

(*c*) 4 sq. unit (*d*) 10 sq. unit

99. Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

(*a*) 1/3 sq. units (*b*) 1/5 sq. units

(*c*) 1/2 sq. units (*d*) 1/4 sq. units

PW CHALLENGERS

INTEGER TYPE QUESTIONS

1. A person in a boat 'a' miles from the nearest point of the beach, wishes to reach as quickly as possible a point 'b' miles from that point along the shore. The ratio of his rate of walking to his rate of rowing is sec α. Then he should land at a distance of x mile from his destination to reach the destination in minimum time. Take $b = 10$ miles, $a = \dfrac{7}{\sqrt{3}}$ miles and $\alpha = 30°$. Find x.

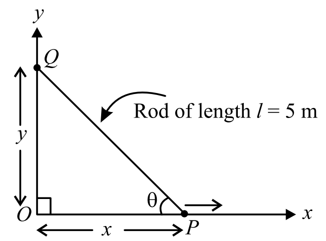
2. A factory D is to be connected by a road with straight railway line on which a town A is situated. The distance DB of the factory to the railway line is $5\sqrt{3}$ km and length AB of railway line is 20 km. Freight charges on the road are twice the charges on railway. At what point P from point B (in km) ($AP < AB$) on the railway line should the road DP be connected so as to ensure minimum freight charges from the factory to the town?

3. Find $\left|\dfrac{dy}{dx}\right|$ at (0,3) for given equation $y^2e^{2x} = 3y + x^2$

SUBJECTIVE TYPE QUESTIONS

4. Acceleration of a particle is defined as $a = \left(\dfrac{dv}{dt}\right)$. If $v^2 = \dfrac{2gl\cot\theta + 4gl(1-\text{cosec}\theta)}{(1+2\cos^2\theta)}$ where, g and l are constants and $\dfrac{d\theta}{dt} = -\left(\dfrac{v}{l}\sin^2\theta\right)$ then find the magnitude of acceleration of particle at $\theta = 45°$.

5. A rod of length $l = 5$ m is slipping on two perpendicular surfaces as shown in figure.



Point P on the rod is moving along x axis with speed 5 m/s when point P is at a distance of 3 m from 'O'. Find the speed of point Q at given instant in m/s (given speed along x axis is $\dfrac{dx}{dt}$ and speed along y axis is $\dfrac{dy}{dt}$).

6. If $x + y + z = \pi$

Prove that

$$\cot\left(\frac{x}{2}\right)+\cot\left(\frac{y}{2}\right)+\cot\left(\frac{z}{2}\right)=\cot\left(\frac{x}{2}\right)\cdot\cot\left(\frac{y}{2}\right)\cdot\cot\left(\frac{z}{2}\right)$$

7. Find $\dfrac{dz}{dx}$ at $(x, y, z) = (1, 0, 1)$ for given equation $x^2\cos(y) = \sin(y^3 + 4z)$.

8. Evaluate $\displaystyle\int_0^{\pi/6}[2\cos(2x) - 3\sin(3x)]dx$

9. Evaluate $\displaystyle\int \tan^7\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)dx$

10. Evaluate $\displaystyle\int_0^{\infty}\frac{xdx}{(1+x)(1+x^2)}$

Answer Key

CONCEPT APPLICATION

1. $\dfrac{\pi}{12}$ **2.** [150°[**3.** (*i*) $-\dfrac{1}{2}$ (*ii*) $\dfrac{-1}{2}$ (*iii*) $-\dfrac{1}{2}$ (*iv*) $-\sqrt{3}$ **4.** (*i*) $\dfrac{5}{13}$ (*ii*) $\dfrac{12}{13}$ **5.** (*i*) 14, 2 (*ii*) 5, –5 (*iii*) 17, –9

6. (*i*) 0.33 (*ii*) 1 (*iii*) 0.158 **7.** (*i*) $\dfrac{7}{25}$ (*ii*) $\dfrac{24}{25}$ **8.** (*i*) $1, -\dfrac{5}{3}$ (*ii*) $\dfrac{-3\pm\sqrt{57}}{4}$ **9.** (*i*) $\dfrac{2}{5}, \dfrac{9}{5}$ (*ii*) $\dfrac{3}{4}, 2$

10. [676] **11.** $\dfrac{2}{3}$ **12.** [77] **13.** (*i*) 7.21 (*ii*) 5.067 (*iii*) 1.02 **14.** 9.61m/s^2 **15.** [13]

16. (*i*) $\sqrt{10}$ (*ii*) $\sqrt{277}$ **17.** [2 or –6] **18.** (A)-R, (B)-T, (C)-P, (D)-Q

19. (*i*) 1.204 (*ii*) 1.678 (*iii*) 1.806 (*iv*) 0.903 **20.** [6 m/s^2] **21.** $\frac{1}{2\sqrt{x}} - 6x$ **22.** $-2\left[\frac{1}{x^3}+1\right]$

23. $\frac{-3}{x^2} + \frac{10}{x^3}$ **24.** $\pm\frac{1}{6\sqrt{x}}$ **25.** $\frac{7}{2}x^{5/2}$ **26.** $-3x^{-4}$ **27.** [1] **28.** $5x^4 + 3x^2 + 2x^{-1/2}$ **29.** $20x^3 + 9x^{1/2} + 9$

30. $2ax + b$ **31.** $15x^4 - 3 + 1/x^2$ **32.** $2t + 5$ **33.** $u + at$ **34.** 30 cm^2 **35.** $2\pi r$ **36.** $4x + 3$

37. $216x^7 - 144x^5 + 105x^4 + 135x^2 - 40$ **38.** [6] **39.** [6] **40.** [2] **41.** [61] **42.** $\frac{-2}{(2x+1)^2}$

43. $\frac{1}{(4x+5)^2}$ **44.** $\frac{2x - x^4}{(x^3+1)^2}$ **45.** (*i*) 0 (*ii*) $\frac{4}{5}$ **46.** (*c*) **47.** $x - \frac{3}{4}x^4 + C$ **48.** $6x - \frac{4}{x} + C$

49. $\sin x - \cos x + C$ **50.** $-\left(\frac{1}{2x^2} + \sin x\right) + C$ **51.** $\frac{\sin 3x}{3} + C$ **52.** $-\left[\frac{1}{x} + \frac{3}{2}\cos 2x\right] + C$ **53.** $\frac{2a}{3} + 2b$

54. $\frac{1}{4}\log_e 17$ **55.** $-\frac{1}{3}\left[\cos\left(\frac{3\pi}{2} - 1\right) - \cos(1)\right]$ **56.** $\frac{1}{2}\left[\sin(2B - 30) - \sin(2A - 30)\right]$

57. $\frac{GMm}{R}$ **58.** $kq_1 q_2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$ **59.** $\frac{1}{2}M(v^2 - u^2)$ **60.** [∞] **61.** [1] **62.** [1] **63.** [2]

64. $\frac{32}{3}$ units **65.** $\frac{1}{6}$ **66.** $\frac{4}{3}$ units

EXERCISE

1. (*a*) $\frac{8\pi}{9}$, (*b*) $\frac{3\pi}{4}$, (*c*) $\frac{5\pi}{12}$, (*d*) $\frac{13\pi}{36}$, (*e*) $\frac{5\pi}{4}$, (*f*) $\frac{25\pi}{18}$, (*g*) $\frac{31\pi}{18}$ **2.** (*a*) 45°, (*b*) 630°, (*c*) 108°, (*d*) 120°, (*e*) 135°

3. (*a*) $\frac{\sqrt{3}+1}{2\sqrt{2}}$, (*b*) $\frac{(\sqrt{3}-1)^2}{2}$, (*c*) $\frac{4}{5}$, (*d*) $\frac{3}{5}$, (*e*) $\frac{3}{4}$, (*f*) $\frac{4}{3}$, (*g*) 0 **4.** (*a*) $-\sqrt{2}$, (*b*) $\frac{\sqrt{6}}{\sqrt{3}+1}$, (*c*) $\frac{\sqrt{3}+1}{2\sqrt{2}}$, (*d*) $\frac{-\sqrt{3}}{2}$, (*e*) $\frac{-1}{2}$, (*f*) 1, (*g*) $\frac{-1}{\sqrt{2}}$, (*h*) $\frac{-1}{\sqrt{2}}$, (*i*) 0 **5.** (*a*) $\frac{1}{2}$, (*b*) $\frac{1}{2\sqrt{2}}$, (*c*) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$, (*d*) $\frac{\sqrt{2}-1}{2\sqrt{2}}$ **6.** (*a*) $-\frac{1}{\sqrt{2}}$, (*b*) $\frac{\sqrt{3}}{2}$, (*c*) $\frac{1}{\sqrt{2}}$, (*d*) $\frac{1}{\sqrt{3}}$

7. (*c*)	**8.** (*b*)	**9.** (*b*)	**10.** (*c*)	**11.** (*a*)	**12.** (*a*)	**13.** (*b*)	**14.** (*a*)	**15.** (*c*)	**16.** (*a*)
17. (*c*)	**18.** (*a*)	**19.** (*c*)	**20.** (*d*)	**21.** (*a*)	**22.** (*c*)	**23.** (*c*)	**24.** (*d*)	**25.** (*b*)	**26.** (*c*)
27. (*d*)	**28.** (*a*)	**29.** (*b*)	**30.** (*b*)	**31.** (*a*)	**32.** (*b*)	**33.** (*a*)	**34.** (*c*)	**35.** (*a*)	**36.** (*a*)
37. (*c*)	**38.** (*a*)	**39.** (*d*)	**40.** (*b*)	**41.** (*b*)	**42.** (*c*)	**43.** (*d*)	**44.** (*b*)	**45.** (*a*)	**46.** (*b*)
47. (*d*)	**48.** (*c*)	**49.** (*a*)	**50.** (*d*)	**51.** (*b*)	**52.** (*a*)	**53.** (*a*)	**54.** (*b*)	**55.** (*b*)	**56.** (*d*)
57. (*b*)	**58.** (*b*)	**59.** (*a*)	**60.** (*d*)	**61.** (*a*)	**62.** (*b*)	**63.** (*a*)	**64.** (*a*)	**65.** (*a*)	**66.** (*b*)
67. (*d*)	**68.** (*d*)	**69.** (*c*)	**70.** (*b*)	**71.** (*c*)	**72.** (*a*)	**73.** (*c*)	**74.** (*d*)	**75.** (*a*)	**76.** (*c*)
77. (*b*)	**78.** (*a*)	**79.** (*d*)	**80.** (*a*)	**81.** (*d*)	**82.** (*b*)	**83.** (*a*)	**84.** (*d*)	**85.** (*d*)	**86.** (*b*)
87. (*a*)	**88.** (*c*)	**89.** (*a*)	**90.** (*d*)	**91.** (*a*)	**92.** (*a*)	**93.** (*a*)	**94.** (*b*)	**95.** (*b*)	**96.** (*c*)
97. (*b*)	**98.** (*a*)	**99.** (*a*)							

PW CHALLENGERS

1. [3] **2.** [5] **3.** [6] **4.** $\frac{g}{4}$ **5.** $\frac{15}{4}$ **7.** $\frac{1}{2\cos(4)}$ **8.** $\left(\frac{\sqrt{3}}{2} - 1\right)$

9. $\frac{1}{2\cos^4\left(\frac{x}{2}\right)} + \cos^2\left(\frac{x}{2}\right) - \frac{3}{\cos^2\left(\frac{x}{2}\right)} - 6\ln\left(\cos\left(\frac{x}{2}\right)\right) + C$ **10.** $\frac{\pi}{4}$