ARJUNA

for

JEE

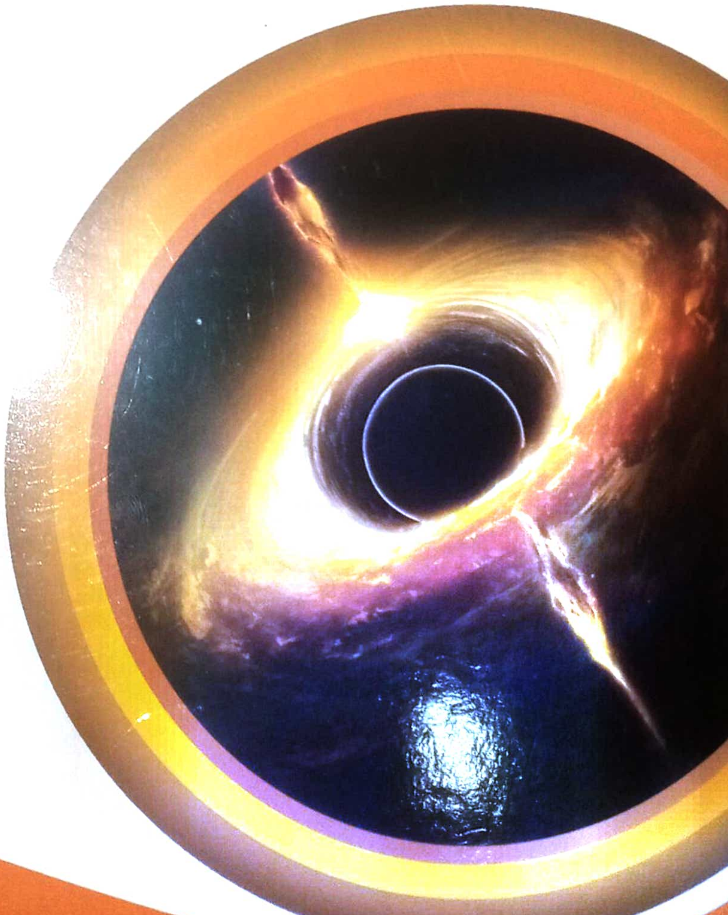
MAIN & ADVANCED

PHYSICS

FULL COURSE STUDY MATERIAL

Class XI

- Rotational Motion
- Gravitation
- Mechanical Properties of Solids
- Mechanical Properties of Fluids



Module-3

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CHAPTER

11 Rotational Motion

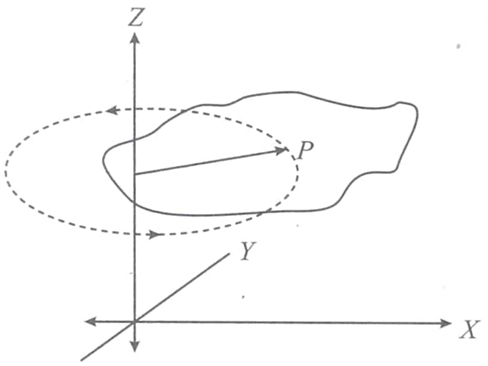
RIGID BODY

A system of particles is said to be a rigid body if distance between any two particles of the body remains constant no matter what force is applied on it. Remember, rigid body is an idealized concept. No body is perfectly rigid but many body come close to it. An iron rod, steel ball, stone etc, are almost rigid bodies.

Up until now, we dealt with translation motion of particles or of rigid bodies. A rigid body, however, can also rotate.

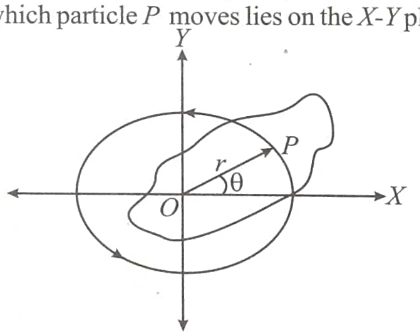
Kinematics of rotation about a fixed axis

We shall now consider rotational motion of a rigid body about an axis fixed in an inertial frame. (see Fig.)



Let P represent an arbitrary particle in the rigid body described by the position vector $\vec{r}$. A rigid body moves in pure rotation, if every particle of the body moves in a circle, the centers of which are on a straight line called the axis of rotation (the z-axis in this figure). If we draw a perpendicular from any point in the body to the axis, each such line will sweep through the same angle in any given time interval as another such line. Therefore, we can describe the pure rotation of rigid body by considering the motion of any one of the particles.

Let us pass *a* plane through P at right angles to the axis of rotation. The circle in which particle P moves lies on the X-Y plane (see Fig.)



From the discussion in the preceding paragraph it is clear that location of entire body is known if we know the location of single particle such as P. Thus for kinematical purposes we need only consider the circular motion of a single particle.

The angular displacement point of particle is θ and the angular velocity ω is given by $\omega = \dfrac{d\theta}{dt}$.

If the angular velocity is not constant then the particle will have angular acceleration denoted by α and is equal to $\alpha = \dfrac{d\omega}{dt}$.

The dimension of angular velocity ω and angular acceleration α are rad/sec and rad/sec^2 respectively.

The rotation of a rigid body about a fixed axis has a direct correspondence to the translation motion of a particle along a fixed direction. The kinematics variable in the translation case are x, v and a while in case of rotation they are θ, ω and α. The correspondence is

$x \to \theta$

$v \to \omega$

$a \to \alpha$

If the rigid body has constant angular acceleration, we can derive the following formulae in direct analogy to translation motion.

$$\omega_f = \omega_i + \alpha t \quad ...(i)$$

$$\theta_f - \theta_i = \omega_i t + 1/2\alpha t^2 \quad ...(ii)$$

$$\omega_f^2 - \omega_i^2 = 2\alpha\left(\theta_f - \theta_i\right) \quad ...(iii)$$

where the subscript i & f correspond to initial and final values.

The linear velocity of the particle P is ωr and its tangential and centripetal acceleration are αr and $\omega 2r$ respectively.

The angular velocity and angular acceleration are actually vector quantities. We are able to treat them as scalar quantities because we consider rotation about a fixed axis of rotation. But if the axis of rotation is changing direction then we will have to take their vector nature into consideration.

The magnitude of angular velocity $\vec{\omega}$ is $d\theta/dt$' but its direction is along the axis of rotation. By convention, if the fingers of right hand curl around the axis of rotation in the direction of rotation of the body , the extended thumb points out in the direction $\vec{\omega}$. In Fig. (1) the direction of $\vec{\omega}$ is in the +ve Z-direction. If the body was rotating in the clock wise direction then its direction will be along the+ve *Z axis*.

Note : Nothing moves in the direction of $\vec{\omega}$ as opposed to the linear velocity $\vec{v}$ in translatory motion which points in the direction of motion.

The direction of angular acceleration $\vec{\alpha}$ can now be determined from $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$. In vector form, the following relations hold good for the motion of a particle in the body.

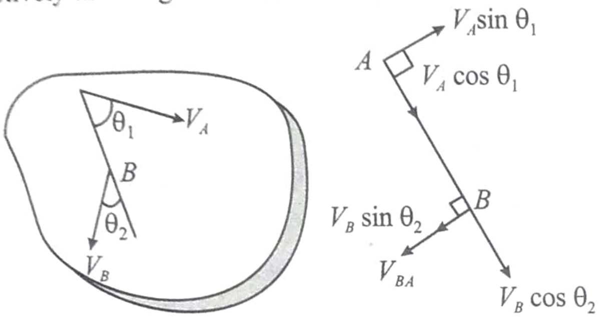
$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \vec{a}_T + \vec{a}_R$$

In which $\vec{a}_T = \vec{\alpha} \times \vec{r}$ and $\vec{a}_R = \vec{\omega} \times \vec{v}$

For every pair of particles in a rigid body, there is no velocity of separation or approach between the particles i.e. with respect to any particle A of a rigid body the motion of any other particle B of that rigid body is circular motion.

Let velocities of A and B with respect ground be $\vec{V}_A$ and $\vec{V}_B$ respectively in the figure below.



If the above body is rigid, $V_A \cos\theta_1 = V_B \cos\theta_2$

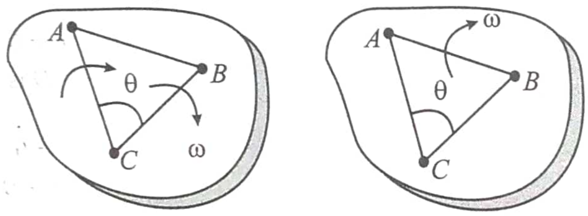
V_{BA} = relative velocity of B with respect to A.

$V_{BA} = V_A \sin\theta_1 + V_B \sin\theta_2$

B will appear to move in a circle to an observer fixed at A.

With respect to any point of the rigid body the angular velocity of all other points of the that rigid body is same.

Suppose A, B, C is a rigid system hence during any motion sides AB, BC and CA must rotate through the same angle. Hence all the sides rotate by the same rate.



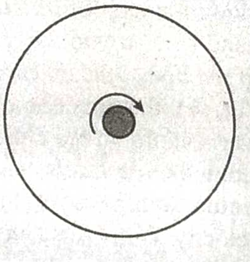
(*i*) (*ii*)

From figure (*i*) angular velocity of A and B w.r.t. C is ω,

From figure (*ii*) angular velocity of A and C w.r.t. B is ω,

Train Your Brain

Example 1: A wheel rotates at a rate of 100 rev/min in the clock wise direction (see Fig.). What is the magnitude and direction of the constant angular acceleration so that it stops in 10 seconds? How many revolution will it cover before stopping ?



Sol. Let z-axis point out of the plane of the paper.

$$\omega_i = \frac{2\pi \times 100}{60} = \frac{10\pi}{3} \text{ rad / sec}$$

Curving our right hand fingers in the direction of motion of the body we find that thumb points in the –ve z-direction . Thus

$\omega_i = -\frac{10\pi}{3}\vec{k}$ where $\vec{k}$ is the unit vector in the z-direction.

Here, $\omega_f = 0 \Rightarrow \alpha = \frac{\omega_f - \omega_i}{10} = \frac{0 - \left(-\frac{10\pi}{3}\right)\hat{k}}{10}$

$$= \frac{\pi}{3}(\text{rad / s}^2)\vec{k}$$

So, α points out of the plane of the paper and its magnitude is $\frac{10}{3}$ rad / s^2

Angular displacement θ is given by the eqn.

$$\omega_f^2 - \omega_i^2 = 2\alpha\theta \Rightarrow \theta = \frac{0 - \left(\frac{10\pi}{3}\right)^2}{2\left(\frac{\pi}{3}\right)} = \frac{50\pi}{3} \text{rad}$$

Number of revolutions $= \frac{50\pi}{3 \times 2\pi} = \frac{50}{6}$, clock wise.

Example 2: A turn table of radius 50 cm starts from rest and acceleration with constant angular acceleration of 1.0 rad/s^2. Compute the tangential, radial and resultant acceleration of a point on the rim after $t = 2$ sec.

Sol. We have $\alpha = 1.0$ rad/s^2 , $\omega_i = 0$, r = 0.5

After $t = 2$sec, $\omega = \omega_i + 2\alpha = 2$rad

Tangential acc $= a_t = \alpha r = (1.0)(0.5) = 0.5$ m/s^2

Radial acc $a_r = \omega^2 r = (4.0)(0.5) = 2.0$ m/s^2

Since, a_t and a_r are perpendicular to each other, the resultant acceleration is

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{2^2 + 0.5^2} = \frac{\sqrt{17}}{2} \text{ m/s}^2$$

Concept Application

1. A disc of radius 10 cm is rotating about its axis at an angular speed of 20 rad/s. Find the linear speed (m/s) of
   (*a*) a point on the rim
   (*b*) the middle point of radius
2. A disc rotates about its axis with a constant angular acceleration of 4 rad/s^2. Find the radial (cm/s^2) and tangential accelerations (cm/s^2) of a particle at a distance of 1 cm from the axis at the end of the first second after the disc starts rotating.

3. A wheel starting from rest is uniformly accelerated at 4 rad/s² for 10 seconds. It is allowed to rotate uniformly for the next 10 seconds and is finally brought to rest in the next 10 seconds. Find the total angle (rad) rotated by the wheel.

MOMENT OF INERTIA

Like the centre of mass, the moment of inertia is a property of an object that is related to its mass distribution. The moment of inertia (denoted by I) is an important quantity in the study of system of particles that are rotating. The role of the moment of inertia in the study of rotational motion is analogous to that of mass in the study of linear motion. Moment of inertia gives a measurement of the resistance of a body to a change in its rotational motion.

If a body is at rest, the larger the moment of inertia of a body the more difficult it is to put that body into rotational motion. Similarly, the larger the moment of inertia of a body, the more difficult to stop its rotational motion. The moment of inertia is calculated about some axis (usually the rotational axis).

Moment of inertia depends on:

(*i*) density of the material of body

(*ii*) shape & size of body

(*iii*) axis of rotation

In totality we can say that it depends upon distribution of mass relative to axis of rotation.

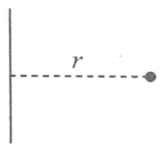
Note: Moment of inertia does not change if the mass :

(*i*) is shifted parallel to the axis of the rotation

(*ii*) is rotated with constant radius about axis of rotation

MOMENT OF INERTIA OF A SINGLE PARTICLE

For a very simple case the moment of inertia of a single particle about an axis is given by,



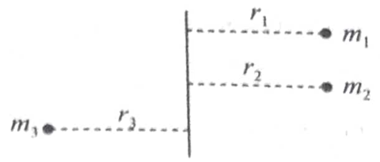
$I = mr^2$ ...(*i*)

Here, m is the mass of the particle and r its distance from the axis under consideration.

MOMENT OF INERTIA OF A SYSTEM OF PARTICLES

The moment of inertia of a system of particles about an axis is given by,

$I = \sum_i m_i r_i^2$ ...(*ii*)



where r_i is the perpendicular distance of the i^{th} particle from the axis, which has a mass m_i.

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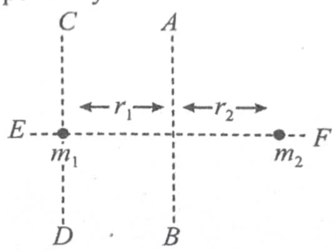
Example 3: Two heavy particles having masses m_1 & m_2 are situated in a plane perpendicular to line AB at a distance of r_1 and r_2 respectively.



(*i*) What is the moment of inertia of the system about axis AB?

(*ii*) What is the moment of inertia of the system about an axis passing through m_1 and perpendicular to the line joining m_1 and m_2?

(*iii*) What is the moment of inertia of the system about an axis passing through m_1 and m_2?

Sol. (*i*) Moment of inertia of particle on left is $I_1 = m_1 r_1^2$.

Moment of Inertia of particle on right is $I_2 = m_2 r_2^2$.

Moment of Inertia of the system about AB is

$I = I_1 + I_2 = m_1 r_1^2 + m_2 r_2^2$

(*ii*) Moment of inertia of particle on left is $I_1 = 0$

Moment of Inertia of the system about CD is

$I = I_1 + I_2 = 0 + m_2(r_1 + r_2)^2$

(*iii*) Moment of inertia of particle on left is $I_1 = 0$

Moment of inertia of particle on right is $I_2 = 0$

Moment of Inertia of the system about EF is

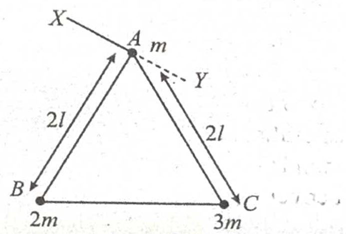
$I = I_1 + I_2 = 0 + 0$

Example 4: Three light rods, each of length 2ℓ, are joined together to form a triangle. Three particles A, B, C of masses m, 2 m, 3 m are fixed to the vertices of the triangle. Find the moment of inertia of the resulting body about

(*a*) an axis through A perpendicular to the plane ABC,

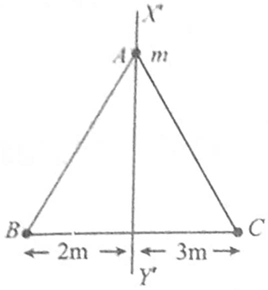
(*b*) an axis passing through A and the midpoint of BC.

Sol.



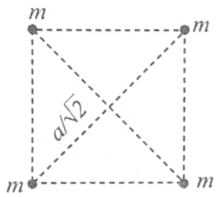
(a) B is at a distant 2ℓ from the axis XY so the moment of inertia of B (I_B) about XY is $2\ m\ (2\ell)^2 + 3m\ (2\ell)^2 = 20\ m\ell^2$

(b) The moment of inertia of the body about $X'\ Y'$ is $m(0)^2 + 2m(\ell)^2 + 3m(\ell)^2 = 5\ m\ell^2$



Example 5: Four particles each of mass m are kept at the four corners of a square of edge a. Find the moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.

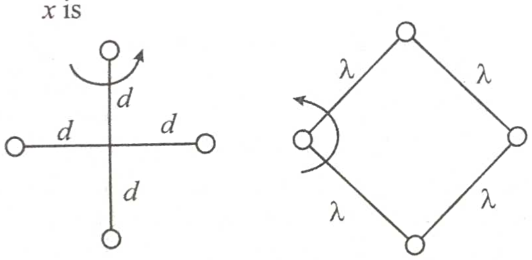
Sol. The perpendicular distance of every particle from the given line is $a/\sqrt{2}$. The moment of inertia of one particle is, therefore, $m(a/\sqrt{2})^2 = \frac{1}{2}ma^2$. The moment of inertia of the system is, therefore,



$4 \times \frac{1}{2}ma^2 = 2\ ma^2$.

Concept Application

4. Four particles with equal masses m are connected by rods of negligible mass. Find the moment of inertia, about the indicated axis, of
   (a) the cross in figure (a) is xmd^2 then the value of x is
   (b) The square in figure (b) is xml^2 then the value of x is



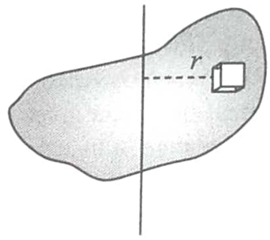
(a) (b)

5. Three particles of masses 1gm, 2gm & 3gm are at 1cm, 2cm, & 3cm from the axis of rotation respectively then the moment of inertia of the system is ______ gm cm^2
   (a) 63 (b) 60
   (c) 36 (d) 36

MOMENT OF INERTIA OF RIGID BODIES

For a continuous mass distribution such as found in a rigid body, we replace the summation of $I = \sum_i m_i r_i^2$ by an integral. If the system is divided into infinitesimal element of mass dm and if r is the distance of a mass element from the axis of rotation, then moment of inertia is,

$$I = \int r^2\, dm$$

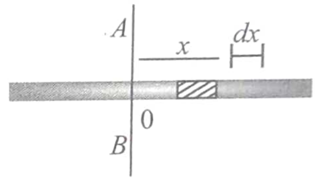


(A) Uniform rod about a perpendicular bisector

Consider a uniform rod of mass M and length l figure and suppose the moment of inertia is to be calculated about the bisector AB. Take the origin at the middle point O of the rod. Consider the element of the rod between a distance x and $x + dx$ from the origin. As the rod is uniform,

Mass per unit length of the rod = M/l

so that the mass of the element = $(M/l)dx$.



The perpendicular distance of the element from the line AB is x. The moment of inertia of this element about AB is

$$dI = \frac{M}{l}dx\ x^2.$$

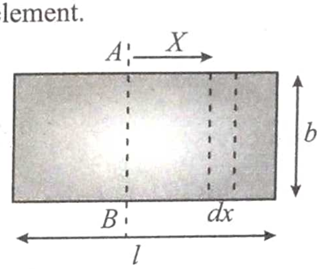
When $x = -l/2$, the element is at the left end of the rod. As x is changed from $-l/2$ to $l/2$, the elements cover the whole rod.

Thus, the moment of inertia of the entire rod about AB is

$$I = \int_{-l/2}^{l/2} \frac{M}{l}x^2 dx = \left[\frac{M}{l}\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{Ml^2}{12}$$

(B) Moment of inertia of a rectangular plate about a line parallel to an edge and passing through the centre

The situation is shown in figure. Draw a line parallel to AB at a distance x from it and another at a distance $x + dx$. We can take the strip enclosed between the two lines as the small element.



It is "small" because the perpendiculars from different points of the strip to AB differ by not more than dx. As the plate is uniform,

its mass per unit area $= \dfrac{M}{bl}$

Mass of the strip $= \dfrac{M}{bl} b\,dx = \dfrac{M}{l} dx$.

The perpendicular distance of the strip from $AB = x$.

The moment of inertia of the strip about

$AB = dI = \dfrac{M}{l} dx\,x^2$. The moment of inertia of the given plate is, therefore,

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2 dx = \frac{M\,l^2}{12}$$

The moment of inertia of the plate about the line parallel to the other edge and passing through the centre may be obtained from the above formula by replacing l by b and thus,

$$I = \frac{Mb^2}{12}.$$

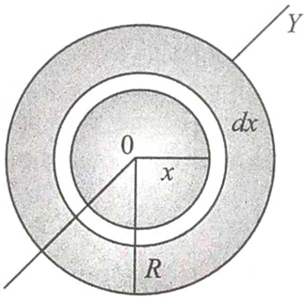
(*C*) **Moment of inertia of a circular ring about its axis (the line perpendicular to the plane of the ring through its centre)**

Suppose the radius of the ring is R and its mass is M. As all the elements of the ring are at the same perpendicular distance R from the axis, the moment of inertia of the ring is

$$I = \int r^2\,dm = \int R^2\,dm = R^2 \int dm = MR^2 .$$

(*D*) **Moment of inertia of a uniform circular plate about its axis**

Let the mass of the plate be M and its radius R. The centre is at O and the axis OY is perpendicular to the plane of the plate.



Draw two concentric circles of radii x and $x + dx$, both centred at O and consider the area of the plate in between the two circles.

This part of the plate may be considered to be a circular ring of radius x. As the periphery of the ring is $2\pi\,x$ and its width is dx, the area of this elementary ring is $2\pi x dx$. The area of the plate is $\pi\,R^2$. As the plate is uniform,

Its mass per unit area $= \dfrac{M}{\pi\,R^2}$

Mass of the ring $= \dfrac{M}{\pi\,R^2} 2\pi x\,dx = \dfrac{2M\,x\,dx}{R^2}$

Using the result obtained above for a circular ring, the moment of inertia of the elementary ring about OY is

$$dI = \left[\frac{2Mx\,dx}{R^2}\right] x^2 .$$

The moment of inertia of the plate about OY is

$$I = \int_0^R \frac{2M}{R^2} x^3\,dx = \frac{MR^2}{2} .$$

(*E*) **Moment of inertia of a hollow cylinder about its axis**

Suppose the radius of the cylinder is R and its mass is M. As every element of this cylinder is at the same perpendicular distance R from the axis, the moment of inertia of the hollow cylinder about its axis is

$$I = \int r^2\,dm = R^2 \int dm = MR^2$$

(*F*) **Moment of inertia of a uniform hollow sphere about a diameter**

Let M and R be the mass and the radius of the sphere, O its centre and OX the given axis (figure). The mass is spread over the surface of the sphere and the inside is hollow.

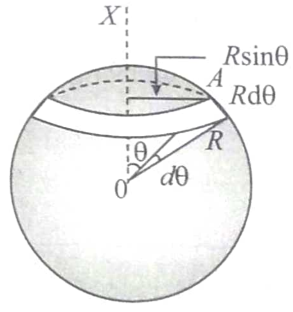
Let us consider a radius OA of the sphere at an angle θ with the axis OX and rotate this radius about OX. The point A traces a circle on the sphere. Now change θ to $\theta + d\theta$ and get another circle of somewhat larger radius on the sphere. The part of the sphere between these two circles, shown in the figure, forms a ring of radius $R \sin\theta$. The width of this ring is $Rd\theta$ and its periphery is $2\pi R \sin\theta$.

Hence, the area of the ring $= (2\pi R \sin\theta)\,(Rd\theta)$.

Mass per unit area of the sphere $= \dfrac{M}{4\pi R^2}$.

The mass of the ring

$$= \frac{M}{4\pi R^2}(2\pi R \sin\theta)(Rd\theta) = \frac{M}{2} \sin\theta\,d\theta.$$



The moment of inertia of this elemental ring about OX is

$$dI = \left(\frac{M}{2} \sin\theta\,d\theta.\right)(R\sin\theta)^2 = \frac{M}{2} R^2 \sin^3\theta\,d\theta$$

As θ increases from 0 to π, the elemental rings cover the whole spherical surface. The moment of inertia of the hollow sphere is, therefore,

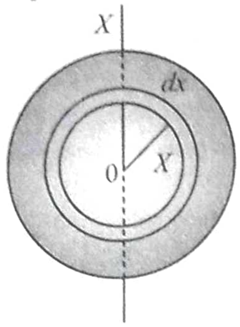
$$I = \int_0^\pi \frac{M}{2} R^2 \sin^3\theta\,d\theta = \frac{MR^2}{2}\left[\int_0^\pi (1-\cos^2\theta)\sin\theta\,d\theta\right]$$

$$= \frac{MR^2}{2}\left[\int_{\theta=0}^{\pi} -(1-\cos^2\theta)\,d\,(\cos\theta)\right]$$

$$= \frac{-MR^2}{2}\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_0^\pi = \frac{2}{3} MR^2$$

(*G*) **Moment of inertia of a uniform solid sphere about a diameter**

Let M and R be the mass and radius of the given solid sphere. Let O be centre and OX the given axis. Draw two spheres of radii x and $x + dx$ concentric with the given solid sphere. The thin spherical shell trapped between these spheres may be treated as a hollow sphere of radius x.



The mass per unit volume of the solid sphere

$$= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

The thin hollow sphere considered above has a surface area $4\pi x^2$ and thickness dx. Its volume is $4\pi x^2 dx$ and hence its mass is

$$= \left(\frac{3M}{4\pi R^3}\right)(4\pi x^2 dx) = \frac{3M}{R^3} x^2 dx$$

Its moment of inertia about the diameter OX is, therefore,

$$dI = \frac{2}{3}\left[\frac{3M}{R^3} x^2 dx\right] x^2 = \frac{2M}{R^3} x^4 dx$$

If $x = 0$, the shell is formed at the centre of the solid sphere. As x increases from 0 to R, the shells cover the whole solid sphere.

The moment of inertia of the solid sphere about OX is, therefore,

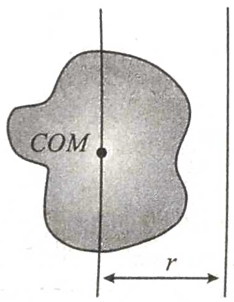
$$I = \int_0^R \frac{2M}{R^3} x^4 dx = \frac{2}{5} MR^2.$$

THEOREMS OF MOMENT OF INERTIA

There are two important theorems on moment of inertia, which, in some cases enable the moment of inertia of a body to be determined about an axis, if its moment of inertia about some other axis is known. Let us now discuss both of them.

Theorem of Parallel axes

A very useful theorem, called the parallel axes theorem relates the moment of inertia of a rigid body about two parallel axes, one of which passes through the centre of mass.



Two such axes are shown in figure for a body of mass M. If r is the distance between the axes and I_{COM} and I are the respective moments of inertia about them, these moments are related by,

$$I = I_{COM} + Mr^2$$

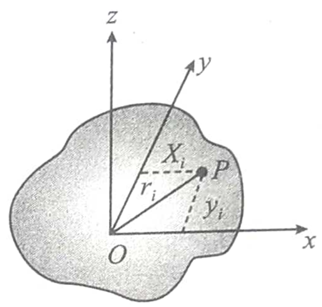
- Theorem of parallel axis is applicable for any type of rigid body whether it is a two dimensional or three dimensional

THEOREM OF PERPENDICULAR AXIS

The theorem states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about two axes perpendicular to each other, in its own plane and intersecting each other, at the point where the perpendicular axis passes through it.

Let x and y axes be chosen in the plane of the body and z-axis perpendicular, to this plane, three axes being mutually perpendicular, then the theorem states that.

$$I_z = I_x + I_y$$



Important point in perpendicular axis theorem

(i) This theorem is applicable only for the plane bodies (two dimensional).

(ii) In theorem of perpendicular axes, all the three axes (x, y and z) intersect each other and this point may be any point on the plane of the body (it may even lie outside the body).

(iii) Intersection point may or may not be the centre of mass of the body.

RADIUS OF GYRATION

If a particle of same mass is placed at distance equal to radius of gyration from the axis of rotation, then it will bear same moment of inertia as the rigid body has.

Where

$I = MK^2$

I = Moment of Inertia of a body

M = Mass of a body

K = Radius of gyration

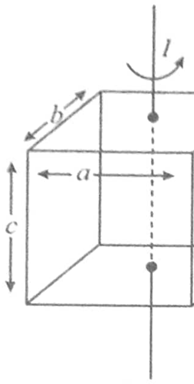
$$K = \sqrt{\frac{I}{M}}$$

Length K is the geometrical property of the body and axis of rotation.

S.I. Unit of K is meter.

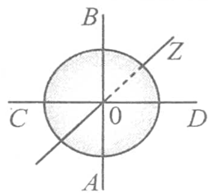
Train Your Brain

Example 6: Find the moment of Inertia of a cuboid along the axis as shown in the figure.



Sol. After compressing the cuboid parallel to the axis $I = \dfrac{M(a^2+b^2)}{12}$

Example 7: Find the moment of inertia of uniform ring of mass M and radius R about a diameter.

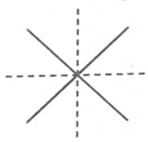


Sol. Let AB and CD be two mutually perpendicular diameters of the ring. Take them as X and Y-axis and the line perpendicular to the plane of the ring through the centre as the Z-axis. The moment of inertia of the ring about the Z-axis is $I = MR^2$. As the ring is uniform, all of its diameter equivalent and so $I_x = I_y$. From perpendicular axes theorem,

$I_z = I_x + I_y$ Hence $I_x = \dfrac{I_z}{2} = \dfrac{MR^2}{2}$

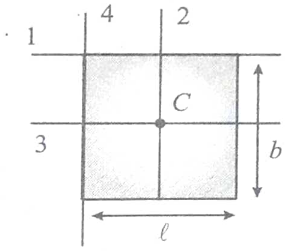
Similarly, the moment of inertia of a uniform disc about a diameter is $MR^2/4$.

Example 8: Two uniform identical rods each of mass M and length ℓ are joined perpendicularly to form a cross as shown in figure. Find the moment of inertia of the cross about a bisector as shown dotted in the figure.



Sol. Consider the line perpendicular to the plane of the figure through the centre of the cross. The moment of inertia of each rod about this line is $\dfrac{M\ell^2}{12}$ and hence the moment of inertia of the cross is $\dfrac{M\ell^2}{6}$. The moment of inertia of the cross about the two bisector are equal by symmetry and according to the theorem of perpendicular axes, the moment of inertia of the cross about the bisector is $\dfrac{M\ell^2}{12}$.

Example 9: In the figure shown find moment of inertia of a plate having mass M, length ℓ and width b about axis 1, 2, 3 and 4. Assume that C is centre and mass is uniformly distributed



Sol. Moment of inertia of the plate about axis 1 (by taking rods perpendicular to axis 1)

$$I_1 = \frac{mb^2}{12} + m\left(\frac{b}{2}\right)^2 = \frac{mb^2}{3}$$

Moment of inertia of the plate about axis 2 (by taking rods perpendicular to axis 2)

$I_2 = M\ell^2/12$

Moment of inertia of the plate about axis 3 (by taking rods perpendicular to axis 3)

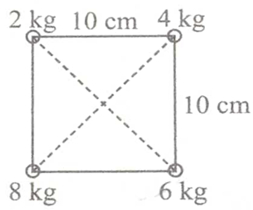
$$I_3 = \frac{Mb^2}{12}$$

Moment of inertia of the plate about axis 4 (by taking rods perpendicular to axis 4)

$$I_4 = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3}$$

Concept Application

6. Four point masses 2 kg, 4 kg, 6 kg and 8 kg are placed at four corners of a square of side 10 cm. The radius of gyration of system about an axis passing through centre O and perpendicular to square plane is



(a) 10 cm (b) $10\sqrt{2}$ cm

(c) $\frac{10}{\sqrt{2}}$ cm (d) $20\sqrt{2}$ cm

7. Moment of inertia of a thin rod about an axis passing through centre and inclined at an angle θ with the axis.

(a) $\frac{Ml^2}{12}\sin^2\theta$ (b) $\frac{Ml^2}{3}\sin^2\theta$

(c) $\frac{Ml^2}{12}\cos^2\theta$ (d) $\frac{Ml^2}{3}\cos^2\theta$

8. I_1, I_2 are moments of inertia of two solid spheres of same mass about axes passing through their centres If first is made of wood and the second is made of steel, then

(a) $I_2 = I_2$

(b) $I_1 < I_2$

(c) $I_1 > I_2$

(d) $I_1 \le I_2$

9. The radius of gyration of a rotating metallic disc is independent of the following physical quantity.

(a) Position of axis of rotation

(b) Mass of disc

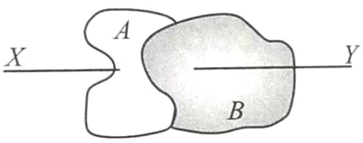
(c) Radius of disc

(d) Temperature of disc

MOMENT OF INERTIA OF COMPOUND BODIES

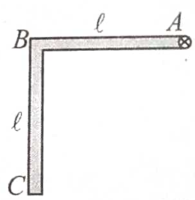
Consider two bodies A and B, rigidly joined together. The moment of the compound body, about an axis XY, is required. If I_A is the moment of inertia of body A about XY. I_B is the moment of inertia of body B about XY.Then, moment of Inertia of compound body $I = I_A + I_B$

Extending this argument to cover any number of bodies rigidly joined together, we see that the moment of inertia of the compound body, about a specified axis, is the sum of the moments of inertia of the separate parts of the body about the same axis.



Train Your Brain

Example 10: Two rods each having length l and mass m joined together at point B as shown in figure. Then find out moment of inertia about axis passing through A and perpendicular to the plane of page as shown in figure.



Sol. We find the resultant moment of inertia I by dividing in two parts such as

I = M.I of rod AB about A + M.I of rod BC about A

$I = I_1 + I_2$ ...(i)

first calculate I_1:

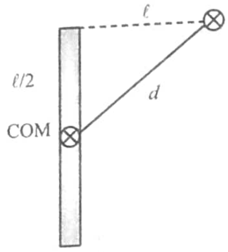
$$I_1 = \frac{m\ell^2}{3} \quad ...(ii)$$

Calculation of I_2:

Use parallel axis theorem

$I_2 = I_{CM} + md^2$

$$\frac{m\ell^2}{12} + m\left(\frac{\ell^2}{4} + \ell^2\right)$$



$$= \frac{m\ell^2}{12} + \frac{5\ell^2}{4}m \quad ...(iii)$$

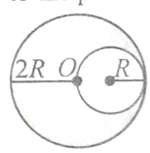
Put value from eq. (2) & (3) into (1), we get

$$\Rightarrow I = \frac{m\ell^2}{3} + \frac{m\ell^2}{12} + \frac{5\ell^2 m}{4}$$

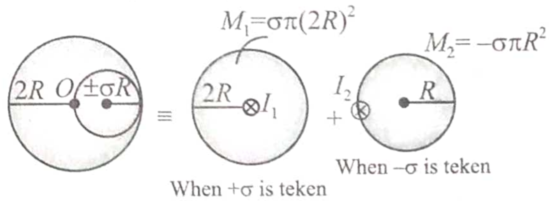
$$I = \frac{m\ell^2}{12}(4+1+15)$$

$$\Rightarrow I = \frac{5m\ell^2}{3}$$

Example 11: A uniform disc having radius $2R$ and mass density σ as shown in figure. If a small disc of radius R is cut from the disc as shown. Then find out the moment of inertia of remaining disc around the axis that passes through O and is perpendicular to the plane of the page.



Sol. We assume that in remaning part a disc of radius R and mass density ± σ is placed. Then



Total Moment of Inertia $I = I_1 + I_2$

$$I_1 = \frac{M_1(2R)^2}{2}$$

$$I_1 = \frac{\sigma\pi 4R^2 . 4R^2}{2} = 8\,\pi\,\sigma\,R^4$$

To calculate I_2 we use parallel axis theorem.

$$I_2 = I_{CM} + M_2R^2$$

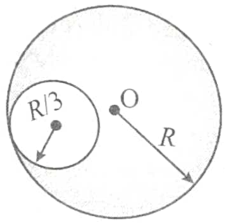
$$I_2 = \frac{M_2R^2}{2} + M_2R^2$$

$$I_2 = \frac{3}{2}M_2R^2 = \frac{3}{2}(-\sigma\pi R^2)R^2 \;;\; I_2 = -\frac{3}{2}\sigma\pi R^4$$

Now $I = I_1 + I_2$

$$I = 8\pi\sigma R^4 - \frac{3}{2}\sigma\pi R^4 \;;\; I = \frac{13}{2}\sigma\pi R^4$$

Example 12: A uniform disc of radius R has a round disc of radius $R/3$ cut as shown in Fig. The mass of the remaining (shaded) portion of the disc equals M. Find the moment of inertia of such a disc relative to the axis passing through geometrical centre of original disc and perpendicular to the plane of the disc.



Sol. Let the mass per unit area of the material of disc be σ. Now the empty space can be considered as having density $-\sigma$ and σ.

Now $I_0 = I\sigma + I\_\sigma$

$I\sigma = (\sigma\pi R^2)R^2/2 = M.I$ of σ about O

$$I\_\sigma = \frac{-\sigma\pi(R/3)^2(R/3)^2}{2} + [-\sigma\pi(R/3)^2](2R/3)^2$$

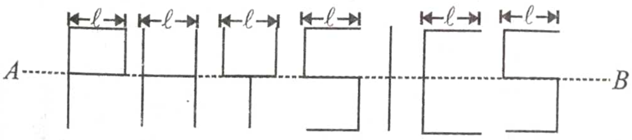
$= M.I$ of $-\sigma$ about 0

$$\therefore\; I_0 = \frac{4}{9}\sigma\pi R^4$$

where $\sigma = \dfrac{M}{\pi R^2 - \pi\left(\dfrac{R^2}{9}\right)}$; $\sigma = \dfrac{9M}{8\pi R^2}$

Concept Application

10. Find out the moment of inertia of the following structure (written as Physics) about axis AB made of thin uniform rods of mass per unit length λ.

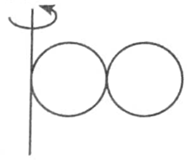


(a) $5\,\lambda l^3$ (b) $13\,\lambda l^3$
(c) $10\,\lambda l^3$ (d) $12\,\lambda l^3$

11. Two identical rings each of mass m with their planes mutually perpendicular, radius R are welded at their point of contact O. If the system is free to rotate about an axis passing through the point P perpendicular to the plane of the paper, then the moment of inertia of the system about this axis is equal to

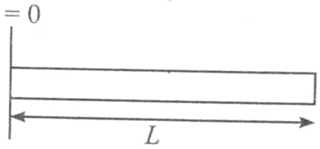
(a) 6.5 mR^2 (b) 12 mR^2
(c) 6 mR^2 (d) 11.5 mR^2

12. Two identical hollow spheres of mass M and radius R are joined together, and the combination is rotated about an axis tangential to one sphere and perpendicular to the line connecting them. The rotational inertia of the combination is



(a) $10\,MR^2$ (b) $\frac{4}{3}\,MR^2$
(c) $\frac{32}{3}\,MR^2$ (d) $\frac{34}{3}\,MR^2$

13. Mass M is distributed over the rod of length L. If linear mass density (λ) of the rod of linearly increasing with length as $l = Kx$, where 'x' is measured from pne end as shown in figure and K is constant. Then $M.I$ of the rod about the end perpendicular to rod where, linear mass density is zero.



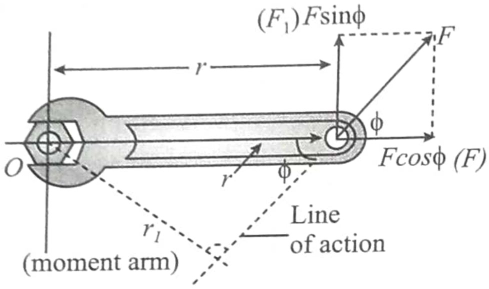
(a) $\frac{ML^2}{3}$ (b) $\frac{KL^2}{12}$
(c) $\frac{2}{3}ML^2$ (d) $\frac{KL^4}{4}$

MOMENT OF FORCE OR TORQUE

Rotational Analogue of Force

Let us take the example of opening or closing of a door. A door is a rigid body which can rotate about a fixed vertical axis passing through the hinges. What makes the door rotate? It is clear that unless a force is applied the door does not rotate. But any force does not do the job. A force applied to the hinge line cannot produce any rotation at all, whereas a force of given magnitude applied at right angles to the door at its outer edge is most effective in producing rotation. It is not the force alone, but how and where the force is applied is important in rotational motion. The quantitative measure of the tendency of a force to cause or change the rotational motion of a body is called torque. Consider an example to understand this.

In the figure below, the wrench is trying to open the nut. Now the ability of wrench to open the nut will depend not only on the applied force, but the distance at which force is applied. This gives birth to a new physical quantity called torque.



If only radial force F_r were present, the nut could not be turned. Thus the force causing the rotation is tangential force F_T only. The magnitude of the torque about an axis due to a force is given by

τ = (Force causing the rotation) × (distance of point of application of force from the axis)

$= (F \sin\phi)\, r$

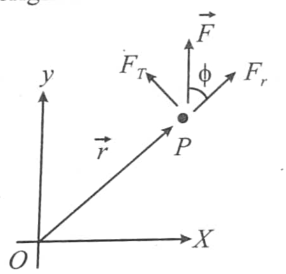
we may also write $\tau = F\,(r\sin\phi) = F\,(r_\perp)$

$\therefore\ \tau = r\,F\sin\phi = (r\sin\phi)F$

$= rF_T = r\perp F$

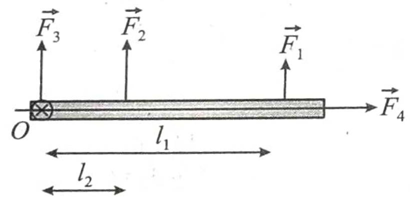
where $r\perp$ is known as moment arm (lever arm)

Thus, if a force acts on a single particle at a point P whose position with respect to the origin O is given by the position vector $\vec{r}$, (see figure), the moment of force (torque) acting on the particle with respect to the origin O is defined as the vector product;

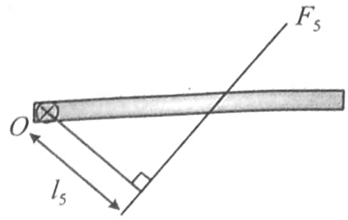


$\vec{\tau} = \vec{r} \times \vec{F}$

Direction of torque is found by sliding the force vector at the axis of rotation and using right hand thumb rule.



Torque about O due to F_1, F_2, F_3, F_4 and F_5 and their moment arm or force arm are shown in the figure.



(Torque) $\tau_1 = F_1\, l_1$

$\tau_2 = F_2\, l_2$

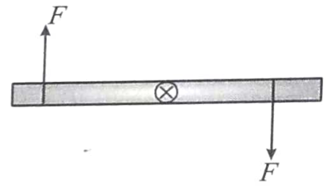
$\tau_3 = 0$

$\tau_4 = 0$

$\tau_5 = F_5\, l_5$

Important Note

- The *SI* unit of torque is newton-metre. This is also the unit of work and energy. But torque is not work or energy. Usually, the unit of work or energy is written as "joule". But the torque should not be expressed in "joule". However, both have same dimensions ML^2T^{-2}.
- Torque is always defined with reference to a given point (or given line). On changing the reference point (or line) the torque may change.
- Torque is a vector quantity, whose direction is perpendicular to the plane of force and position vector and its direction is given by right hand screw rule.
- There can be non-zero torque on body even when net force is zero (couple) and there can be force producing zero torque.



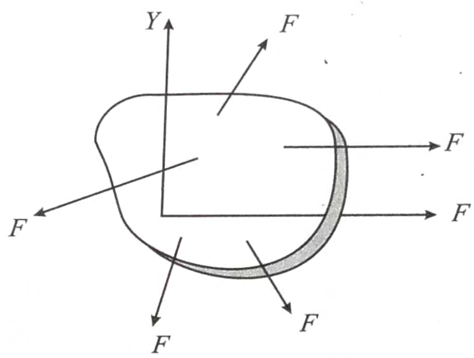
- **Force couple:** A pair of forces each of same magnitude and in opposite direction is called a force couple. Torque due to a couple about any point is same and its magnitude = Magnitude of one force X distance between their lines of action

Rotational Equilibrium

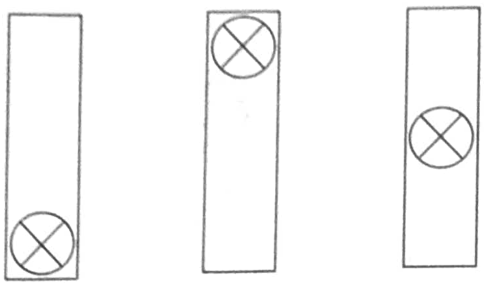
A system is in mechanical equilibrium if it is in translational as well as rotational equilibrium. For this :

$\vec{F}_{\text{net}} = 0$, $\vec{\tau}_{\text{net}} = 0$ (about every point)

Of $\vec{F}_{\text{net}} = 0$ then $\vec{\tau}_{\text{net}}$ is same about every point



Hence necessary and sufficient condition for equilibrium is $\vec{F}_{net} = 0, \vec{\tau}_{net} = 0$ about any one point, which we can choose as per our convenience. ($\vec{\tau}_{net}$ will automatically be zero about every point)



The equilibrium of a body is called stable if the body tries to regain its equilibrium position after being slightly displaced and released. It is called unstable if it gets further displaced after being slightly displaced and released. If it can stay in equilibrium even after being slightly displaced and released, it is said to be in neutral equilibrium.

Train Your Brain

Example 13: Given that, $\vec{r} = 2\hat{i} + 3\hat{j}$ m and $\vec{F} = 2\hat{i} + 6\hat{k}$ N. The magnitude of torque will be-

(a) $\sqrt{405}$ N.m (b) $\sqrt{410}$ N.m

(c) $\sqrt{504}$ N.m (d) $\sqrt{510}$ N.m

Sol. We know that, $\vec{\tau} = \vec{r} \times \vec{F}$

$\Rightarrow \quad \vec{\tau} = (2\hat{i} + 3\hat{j}) \times (2\hat{i} + 6\hat{k})$

$= 12(-\hat{j}) + 6(-\hat{k}) + 18\hat{i} = -12\hat{j} - 6\hat{k} + 18\hat{i}$

[Note : $\hat{i} \times \hat{i} = 0$, $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{i} = -\hat{k}$ etc]

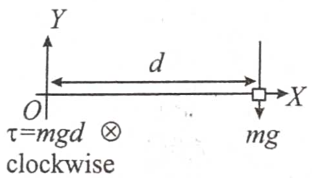
Now, $|\vec{\tau}| = \sqrt{(-12)^2 + (-6)^2 + (18)^2}$

$= \sqrt{144 + 36 + 324} = \sqrt{504}$

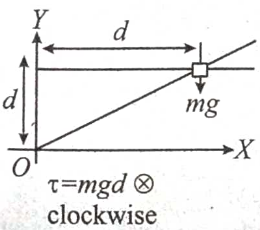
Example 14: A particle is falling freely along line $x = d$. Find torque on this particle due to gravity, about origin when it

(i) Crosses x axis (ii) is at $y = d$

Sol. (i)

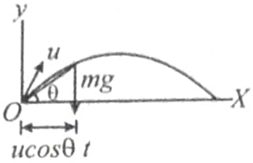


(ii)



Example 15: Find torque due to gravity at any time t about point of projection O, if a body is projected with velocity u at an angle θ.

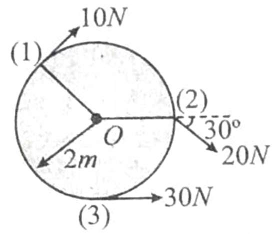
Sol.



At every instant force mg is in downward direction and perpendicular distance will be displacement along x-axis i.e. $x = (u \cos\theta)t$

$\tau_0 = mg\, u \cos\theta\, t$, it is increasing with time

Example 16: Find net torque about axis of rotation passing through O.



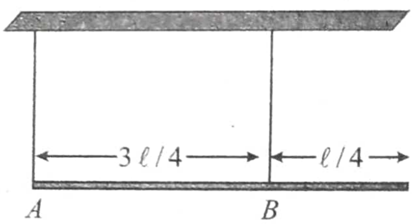
Sol. $\tau_1 = 10 \times 2 \quad \otimes \quad = 20 \otimes$

$\tau_2 = 20 \times 2 \sin 30° \quad \otimes \quad = 20 \otimes$

$\tau_3 = 30 \times 2 \quad \odot \quad = 60 \odot$

$\Sigma\vec{\tau} = 20 \odot$

Example 17: A uniform rod of length l, mass m is hung from two strings of equal length from a ceiling as shown in figure. Determine the tensions in the strings ?



Sol. $T_A = \dfrac{mg}{3}, T_B = \dfrac{2mg}{3}$

$T_A + T_B = mg \quad$ ...(i)

Torque about point A is zero

So, $T_B \times \dfrac{3\ell}{4} = mg\dfrac{\ell}{2} \quad$ ...(ii)

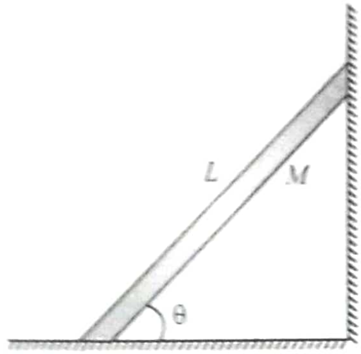
From eq. (i) & (ii),

$T_A = mg/3, T_B = 2mg/3$.

Example 18: A rod of length placed an ragh horizontal surface rests against a smooth wall as shown in the figure.

(a) find the frictional force acting on it.

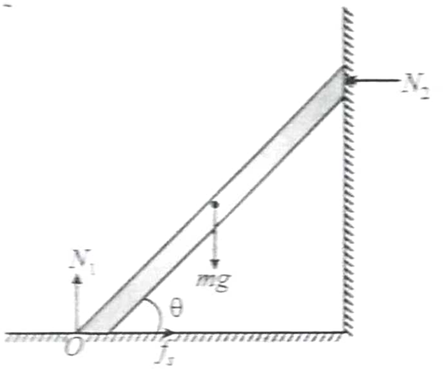
(b) is coefficient of static friction is $\mu_s = \frac{\sqrt{3}}{2}$ find the maximum angle 'θ' that the rod can make with the horizontal.



Sol. Balancing force in horizontal and vertical directions

$N_1 - mg = 0$

$fs - N_2 = 0$



F.B.D of the rod

Balancing torque about 0,

$mgL/2 \cos\theta = N_2 L \sin\theta$

$\Rightarrow N_2 = \frac{Mg}{2}\cot\theta$ or $fs = \frac{mg}{2}\cot\theta$

$\frac{mg}{2}\cot\theta \le \mu_s N_1 = \mu_s mg$

$\cot\theta \le 2\mu_s$ or $\cot\theta \le \sqrt{3}$

$\Rightarrow \theta_{max} = 30°$

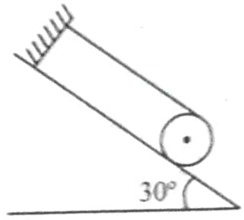
Concept Application

14. The beam and pans of a balance have negligible mass. An object weighs W_1, when placed in one pan and W_2 when placed in the other pan. The weight W of the object is :

(a) $\sqrt{W_1 W_2}$ (b) $\sqrt{(W_1 + W_2)}$

(c) $W_1^{\,2} + W_2^{\,2}$ (d) $\left(W_1^{-1} + W_2^{-1}\right)/2$

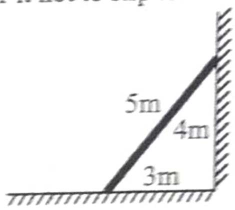
15. A thin hoop of weight 500 N and radius 1 m rests on a rough inclined plane as shown in the figure. The minimum coefficient of friction needed for this configuration is



(a) $\frac{1}{3\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{1}{2}$ (d) $\frac{1}{2\sqrt{3}}$

16. A uniform ladder of length 5 m is placed against the wall as shown in the figure. If coefficient of friction μ is the same for both the wall and the floor then minimum value of μ for it not to slip is



(a) μ = 1/2 (b) μ = 1/4

(c) μ = 1/3 (d) μ = 1/5

17. In case of torque of a couple if the axis is changed by displacing it parallel to itself, torque will

(a) Increase

(b) Decrease

(c) Remain constant

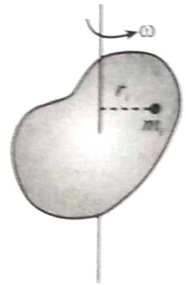
(d) None of these

KINETIC ENERGY OF A RIGID BODY ROTATING ABOUT A FIXED AXIS

Suppose a rigid body is rotating about a fixed axis with angular speed ω.

Then, kinetic energy of the rigid body will be :

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\omega r_i)^2$$



$$= \frac{1}{2}\omega^2 \sum_i m_i r_i^2 = \frac{1}{2} I\omega^2 \ \left(\text{as } \sum_i m_i r_i^2 = I\right)$$

Thus, KE = $\frac{1}{2}I\omega^2$

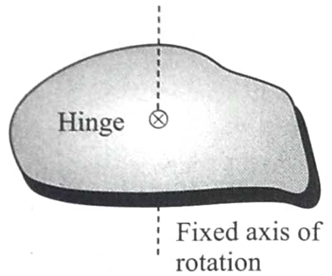
Sometimes it is called the rotational kinetic energy.

DYNAMICS OF ROTATION ABOUT A FIXED AXIS

If I_{Hinge} = moment of inertia about the axis of rotation (since this axis passes through the hinge, hence the name I_{Hinge}).

$\vec{\tau}_{ext}$ = resultant external torque

acting on the body about axis of rotation



α = angular acceleration of the body.

$(\vec{\tau}_{ext})_{Hinge} = I_{Hinge}\,\vec{\alpha}$

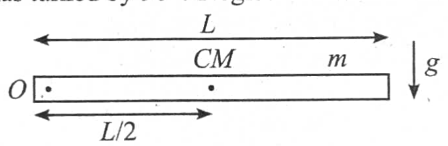
$\vec{P} = M\vec{v}_{CM}$; $\vec{F}_{external} = M\vec{a}_{CM}$

Net external force acting on the body has two component tangential and centripetal.

$$\Rightarrow F_C = ma_C = m\frac{v^2}{r_{CM}} = m\omega^2 r_{CM} \Rightarrow F_t = ma_t = m\alpha r_{CM}$$

Train Your Brain

Example 19: A rod of mass m and length L is hinged at point O is free to rotate about the horizontal axis in the vertical plane. If it is released from rest in the position shown, find its angular velocity w and velocity of its center of mass when it has turned by 90°? Neglect friction dute to hinge.



Sol. Loss in potential energy = $mg\,\Delta h_{cm} = mg\frac{L}{2}$

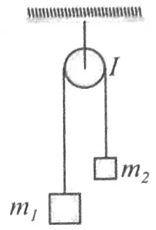
Gain in kinetic energy = $\frac{1}{2}I_o\omega^2 = \frac{1}{2}\frac{ML^2}{3}\omega^2$

Applying energy conservation, we get

$$\frac{1}{2}\frac{ML^2}{3}\omega^2 = mg\frac{L}{2} \Rightarrow \omega = \sqrt{\frac{3g}{L}}$$

$$V_{cm} = \frac{\omega L}{2} = \sqrt{\frac{3g}{L}} = \frac{\sqrt{3gL}}{2}$$

Example 20: The pulley shown in figure has a moment of inertia I about its axis and its radius is r. Calculate the magnitude of the acceleration of the two blocks. Assume that the string is light and does not slip on the pulley.



Sol. Suppose the tension in the left string is T_1 and that in the right string in T_2. Suppose the block of mass m_1 goes down with an acceleration α and the other block moves up with the same acceleration a. This is also the tangential acceleration of the rim of the wheel as the string does not slip over the rim. The angular acceleration of the wheel is, therefore, $\alpha = a/r$. The equations of motion for the mass m_1, the mass m_2 and the pulley are as follows:

$m_1g - T_1 = m_1a$ ...(i)

$T_2 - m_2g = m_2a$ ...(ii)

$T_1r - T_2r = I\alpha = Ia/r$ ...(iii)

Putting T_1 and T_2 from (i) and (ii) into (iii),

$$[(m_1g - a) - m_2(g + a)]\, r = I\frac{a}{r}$$

which gives a = $\frac{(m_1 - m_2)gr^2}{I + (m_1 + m_2)r^2}$.

Alternative Method : (Using energy conservation)

Let the velocity of m_1 and m_2 be v when they have undergone displacement S. Loss in potential energy = $m_1gs - m_2gs$ Gain in kinetic energy

$$= \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + \frac{1}{2}I\left(\frac{v}{r}\right)^2$$

Applying energy conservation :

$$\Rightarrow \frac{1}{2}\left(m_1 + m_2 + \frac{I}{r^2}\right)v^2 = (m_1 - m_2)gs \quad ...(i)$$

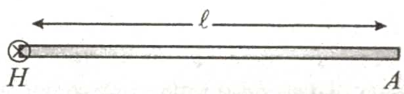
$$\Rightarrow v^2 = 2\frac{(m_1 - m_2)gr^2}{((m_1 + m_2)r^2 + I)}S \quad ...(ii)$$

But $v^2 = 2\,as$

Comparing (i) and (ii) we get

$$a = \frac{(m_1 - m_2)gr^2}{I + (m_1 + m_2)r^2}$$

Example 21: A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H.



(*i*) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest ?

(*ii*) Calculate the acceleration (tangential and radial) of point *A* at this moment.

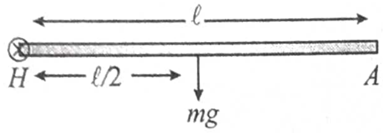
(*iii*) Calculate net hinge force acting at this moment.

(*iv*) Find α and ω when rod becomes vertical.

(*v*) Find hinge force when rod become vertical.

Sol. (*i*) $\tau_H = I_H \alpha$

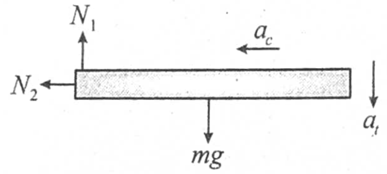
$$mg.\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha \Rightarrow \alpha = \frac{3g}{2\ell}$$



(*ii*) $a_{tA} = \alpha\ell = \frac{3g}{2\ell}.\ell = \frac{3g}{2}$

$a_{CA} = \omega^2 r = 0.\ell = 0 \quad (\because \omega = 0$ just after release)

(*iii*) Suppose hinge exerts normal reaction in component form as shown



In vertical direction

$F_{ext} = ma_{CM}$

$\Rightarrow \quad mg - N_1 = m.\frac{3g}{4}$

(we get the value of a_{CM} from previous example)

$\Rightarrow \quad N_1 = \frac{mg}{4}$

In horizontal direction

$F_{ext} = ma_{CM}$

$\Rightarrow N_2 = 0$

($\therefore a_{CM}$ in horizontal = 0 as ω = 0 just after release).

(*iv*) Torque = 0 when rod becomes vertical.

so α = 0

using energy conservation

$$\frac{mg\ell}{2} = \frac{1}{2}I\omega^2 \left(I = \frac{m\ell^2}{3}\right)$$

$$\omega = \sqrt{\frac{3g}{\ell}}$$

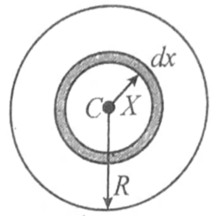
(*v*) When rod becomes vertical

$$\alpha = 0, \omega = \sqrt{\frac{3g}{\ell}}$$

$$F_H - \text{mg} = \frac{m\omega^2\ell}{2} \; ; F_H = \frac{5mg}{2}$$

Example 22: A uniform disc of radius *R* and mass *M* is spinned to an angular speed ω_0 in its own plane about its centre and then placed on a rough horizontal surface such that plane of the disc is parallel to the horizontal plane. If co-efficient of friction between the disc and the surface is μ then how long will it take for the disc to come to stop.

Sol. Consider a differential circular strip of the disc of radius *x* and thickness *dx* Mass of this strip is $dm = 2\rho\pi x dx$.



where $\rho = \frac{M}{\pi R^2}$. Frictional force on this strip is along the tangent and is equal to $dF = mr2\pi xdxg$

Torque on the strip due to frictional force is equal to $d\tau = \mu\rho g 2\pi x^2 dx$ disc is supposed to be the combination of number of such strips hence torque on the disc is given by

$$\tau = \int d\tau = \mu\rho g 2\pi \int_0^R x^2 dx = \mu\rho g 2\pi \frac{R^3}{3}$$

$$\Rightarrow \tau = \mu Mg(2/3)R \Rightarrow \alpha = \frac{2\mu MgR}{3\left(\frac{MR^2}{2}\right)} = \frac{4}{3}\frac{\mu g}{R}$$

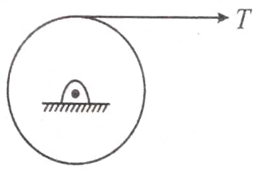
The α is opposite to the ω

$\because \; \omega(t) = \omega_0 + \alpha t$

$$\Rightarrow 0 = \omega_0 - \frac{4\mu g}{3R}t \Rightarrow t = \frac{3\omega_0 R}{4\mu g}.$$

Concept Application

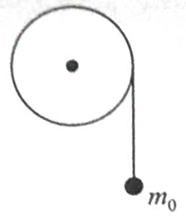
18. A wheel 4 m in diameter rotates about a fixed frictionless horizontal axis, about which its moment of inertia is 10 kg m². A constant tension of 40 N is maintained on a rope wrapped around the rim of the wheel. If the wheel starts from rest at $t = 0$ s, find the length of rope unwounded till $t = 3$s.



(*a*) 36.0 m (*b*) 72.0 m

(*c*) 18.0 m (*d*) 720 m

19. A uniform solid cylinder of mass m can rotate freely about its fixed axis, which is kept horizontal. A particle of mass m_o hangs from the end of a light string which is fightly wound round the cylinder. When system is allowed to move, acceleration with which the particle descends

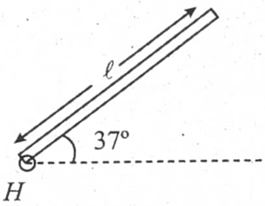


(a) $\dfrac{mg}{m+2m_o}$ (b) $\dfrac{2m_o g}{m+2m_o}$

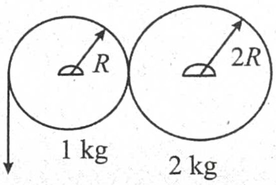
(c) $\dfrac{m_o g}{m+m_o}$ (d) $\dfrac{2mg}{m+2m_o}$

20. A uniform rod of mass m and length l can rotate in vertical plane about a smooth horizontal axis hinged at point H. The angular acceleration α is $\dfrac{xg}{5l}$, of the rod just after it is released from initial position making an angle of 37° with horizontal from rest then the value of x is.

The force exerted by the hinge is $\dfrac{\sqrt{xmg}}{5}$, just after the rod is released from rest then the value of x is.

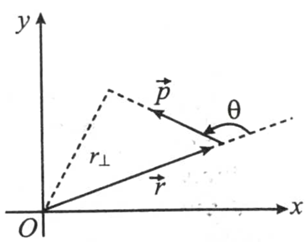


21. Two discs A and B touch each other as in figure. A rope lightly wound on A is pulled down at 2 m/s^2. Find the friction force (N) between A and B if slipping is absent.



ANGULAR MOMENTUM AND ITS CONSERVATION

Consider a particle that has linear momentum $\vec{p}$ and is located at position $\vec{r}$ relative to an origin O, as shown in the figure. Its angular momentum about the origin is defined as



$\vec{L} = \vec{r} \times \vec{p}$

The magnitude of momentum is given by

$L = rp\sin\theta$

The direction of angular momentum is given by the right hand rule.

The SI unit of angular momentum is kg m^2/s.

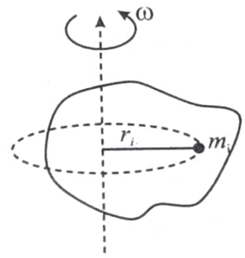
Note: that angular momentum is defined always with respect to a point

The total angular momentum $\vec{L}$ of a system of particles relative to a given origin is the sum of the angular momentum of the particles.

$$\vec{L} = \sum \vec{r}_i \times \vec{p}_i$$

Angular momentum of a body rotating about a fixed axis : Let the rigid body rotate about z-axis as shown in the figure.

The angular momentum of the i^{th} particle about axis of rotation is $L_{iz} = r_i p_i$



$L_z = \Sigma r_i p_i$

Also $p_i = m_i v_i = m_i r_i \omega$

or $L_z = \Sigma m_i r_i^2\, \omega$ or $L_z = I\omega$

where $I = \Sigma m_i r_i^2$

If the mass of the body is distributed symmetrically about axis of rotation then $\vec{L} = I\vec{\omega}$

As force changes the linear momentum of a particle, torque changes the angular momentum of a particle. The rate of change of angular momentum with time is given by

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v}) \text{ or } \frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times m\frac{d\vec{v}}{dt}$$

or $\dfrac{d\vec{L}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}$

or $\dfrac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$ because $\vec{v} \times m\vec{v}$ is zero

Thus, $\vec{\tau} = \dfrac{d\vec{L}}{dt}$

The above equation is the rotational analog of equation $\vec{F} = \dfrac{d\vec{P}}{dt}$

If the net external torque on a system is zero, the total angular momentum is constant in magnitude and direction.

That is, if $\vec{\tau}_{ext} = \vec{0}$ $\dfrac{d\vec{L}}{dt} = \vec{0}$

Thus, $\vec{L}$ = constant

For a rigid body rotating about a fixed axis,

$L_f = L_i$

or $I_f\omega_f = I_i\omega_i$

ANGULAR IMPULSE

In complete analogy with the linear momentum, angular impulse is defined as

$\vec{J} = \int \vec{\tau}_{ext} dt$

Using Newton's second law for rotation motion,

$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$

$\therefore \vec{J} = \Delta\vec{L} = \vec{L}_f - \vec{L}_i$

The net angular impulse acting on a rigid body is equal to the change in angular momentum of the body. This is called the impulse – momentum theorem for rotational dynamics.

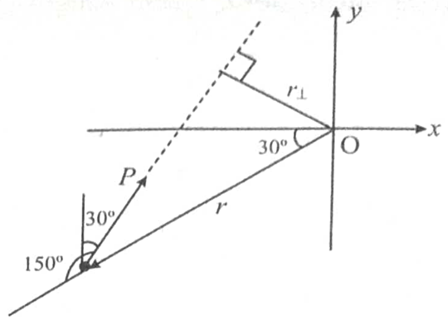
Finally, we present an analogy between rotational dynamics and linear dynamics.

	Quantity	Linear	Rotational
1.	Inertia	m	$\sum m_i r_i^2$ or $\int r^2 dm$
2.	Newton's Second Law	$F_{ext} = ma$ $\vec{F}_{ext} = \frac{d\vec{p}}{dt}$	$\tau_{ext} = I\alpha$ $\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$
3.	Work	$W_{lin} \int \vec{F}.d\vec{s}$	$W_{rot} = \vec{\tau}.d\vec{\theta}$
4.	Kinetic Energy	$K_{lin} = \frac{1}{2}mv^2$	$K_{rot} = \frac{1}{2}I\omega^2$
5.	Work Energy Theorem	$W_{lin} = \Delta K_{lin}$	$W_{rot} = \Delta K_{rot}$
6.	Impulse	$I = \int F_{ext} dt$	$\vec{J} = \int \vec{\tau}_{ext} dt$
7.	Momentum	$p = mv$	$L = I\omega$
8.	Impulse Momentum Theorem	$\vec{I} = \Delta\vec{p}$	$\vec{J} = \Delta\vec{L}$
9.	Power	$P = \vec{F}.\vec{v}$	$P = \vec{\tau}.\vec{\omega}$

Train Your Brain

Example 23: What is the angular momentum of a particle of mass $m = 2$ kg that is located 15 m from the origin in the direction 30° south of west and has a velocity $v = 10$ m/s in the direction 30° east of north?

Sol. In the figure (a), the x – axis points east. We know $r = 15\ m$; $p = mv = 20$ kg m/s. The angle between r and p is $(180° - 30°) = 150°$



Thus,

$L = rp \sin\theta = (15)(20) \sin 150° = 150$ kg m^2/s

In unit vector notation,

$\vec{r} = -15 \cos 30° \hat{i} - 15 \sin 30° \hat{j}$ m

$\vec{p} = 20 \sin 30° \hat{i} + 20 \cos 30° \hat{j}$ kg m/s.

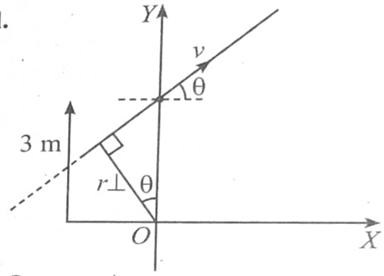
Therefore,

$\vec{L} = \left(-\frac{15\sqrt{3}}{2}\hat{i} - \frac{15}{2}\hat{j}\right) \times (10\hat{i} + 10\sqrt{3}\hat{j})$

$= -150\ \hat{k}$ kg m²/s

Example 24: A particle of mass m moves along the line $y = 2x + 3$ with uniform velocity v. Find it angular momentum about the origin with it crosses y-axis.

Sol.



$L_o = mvr\perp$

$r\perp = 3 \cos\theta$

Slope = $\tan\theta = 2$

$\cos\theta = \frac{1}{\sqrt{5}}$

$L_o = \frac{3mv}{\sqrt{5}}, \otimes$

The angular momentum will remain constant as $r\perp = 3 \cos\theta$ = constant.

Example 25: A disc of moment of inertia 4 kg m² is spinning freely at 3 rad/s. A second disc of moment of inertia 2 kg m² slides down the spindle and they rotate together.

(a) What is the angular velocity of the combination?

(b) What is the change in kinetic energy of the system?

Sol. (a) Since there are no external torques acting, we may apply the conservation of angular momentum.

$I_f\omega_f = I_i\omega_i$

(6 kg m²) ω_f = (4 kg m²)(3 rad/s)

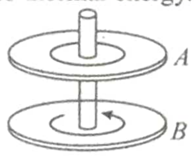
Thus, $\omega_f = 2$ rad/s

(*b*) The kinetic energies before and after the collision are

$$K_i = \frac{1}{2} I_i \omega_i^2 = 18\ J;\ K_f = K_f = \frac{1}{2} I_f \omega_f^2 = 12J$$

The change is $\Delta K = K_f - K_i = -6\ J$.

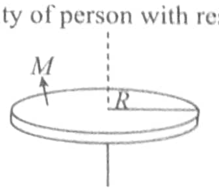
In order for the two discs to spin together at the same rate, there had to be friction between them. The lost kinetic energy is converted to thermal energy.



Example 26: A person of mass m stands at the edge of a circular platform of radius R and moment of inertia I. A platform is at rest initially. But the platform rotate when the person jumps off from the platform tangentially with velocity u with respect to platform. Determine the angular velocity of the platform.

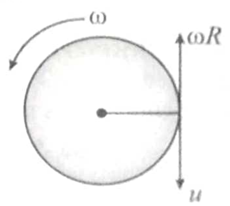
Sol. Let the angular velocity of platform is ω.

Then the velocity of person with respect to ground v.



$v_{mD} = v_{mG} - V_{DG}$; $u = v_m + \omega R$; $v_m = u - \omega R$

Now from angular momentum conservation



$L_i = L_f$

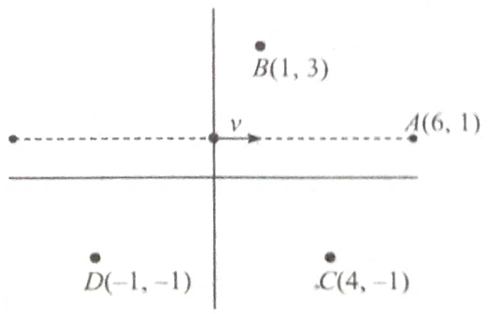
$0 = mv_m R - I\omega$

$\Rightarrow\ I\omega = m(u - \omega R) . R$

$$\Rightarrow\ \omega = \frac{muR}{I + mR^2}$$

Concept Application

22. A particle of mass m is moving with a constant velocity v as shown in the figure. Magnitude of its angular momentum is minimum about :



(*a*) A (*b*) B
(*c*) C (*d*) D

23. A particle is at time $t = 0$ from a point P with a speed v_o at an angle of 45° to the horizontal. Find the direction. The magnitude of the angular momentum of the particle about the point P at time $t = v_o/g$ is $\frac{mv_0^3}{xg}$ then the value of x is.

24. If the radius of earth shrinks to η_R ($\eta < 1$), where R is the radius of earth, then the time period of rotation of earth about its axis will become

(*a*) $24\,\eta^2$ hour (*b*) $\frac{24}{\eta}$ hour

(*c*) $24\,\eta$ hour (*d*) $\frac{24}{\eta^2}$ hour

25. A playground merry-go-round is at rest, pivoted about a frictionless axis. A child of mass m runs along a path tangential to the rim with speed v and jumps on to the merry-go-round. If R is the radius of the merry-go-round and I is its moment of inertia, then the angular velocity of the merry-go-round and the child is –

(*a*) $\frac{mvR}{mR^2 + I}$ (*b*) $\frac{mvR}{I}$

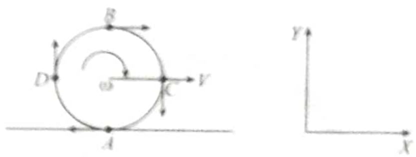
(*c*) $\frac{mR^2 + I}{mvR}$ (*d*) $\frac{I}{mvR}$

THE COMBINED TRANSLATIONAL AND ROTATIONAL MOTION OF A RIGID BODY

So far we have considered only pure rotation of a rigid body. However, a rigid body while rotating about an axis can also be moving translationally. We can separate the general motion of a rigid body into translational motion of its center of mass and rotational motion about its center of mass. The motion of rigid body is thus completely characterized by the velocity V_{cm} of center of mass and angular velocity ω about the axis passing through the center of mass. In general ,axis of rotation can also change direction but that is dealt with in advanced level course.

Once V_{cm} and ω are specified , the velocity of any point on a rigid body can be determined.

Illustration: A uniform disc of radius R is rotating about the cylindrical axis with angular velocity ω and the cylindrical axis is moving with velocity V with respect to ground.(see figure below). Find the velocity of point A, B, C and D with respect to ground.



Let X and Y axis point in the horizontal and vertical directions respectively. The velocity V_{cm} is equal to V. The velocity of any point can be determined from eqns of relative velocities.

$\vec{V}_A = \vec{V}_{A/cm} + \vec{V}_{cm}$

$= -\omega R\hat{i} + V\hat{i} = (V - \omega R)\hat{i}$

$$\vec{V}_B = \vec{V}_{B/cm} + \vec{V}_{cm} = \omega R\hat{i} + V\hat{i} = (V + \omega R)\hat{i}$$

$$\vec{V}_C = \vec{V}_{C/cm} + \vec{V}_{cm} = -\omega R\hat{j} + V\hat{i} = V\hat{i} - \omega R\hat{j}$$

$$\vec{V}_D = \omega R\hat{j} + V\hat{i} = V\hat{i} + \omega R\hat{j}$$

Kinetic energy: The total kinetic of the body in this case is given by the sum of translational and rotational kinetic energy which is given as.

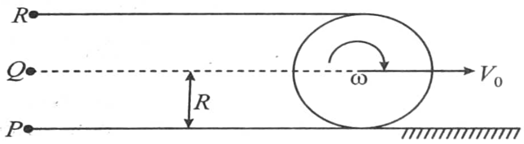
$$KE = \frac{1}{2}MV_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2$$

Angular momentum. The total angular momentum in this general case about a point O is given by

$$\vec{L} = \vec{L}_{\text{c.m.}} + \vec{r}_{\text{cm}} \times M\vec{V}_{\text{cm}}$$
$$= I_{\text{cm}}\vec{\omega} + \vec{r}_{\text{cm}} \times M\vec{V}_{\text{cm}}$$

where $\vec{L}_{\text{cm}}$ is the angular momentum relative to center of mass and $\vec{r}_{\text{cm}}$ is the position vector of center of mass w.r.t O. This relation can be interpreted as regarding the total angular momentum to be sum of angular momentum about the center of mass and angular momentum of the motion of center of mass C.M . with respect to O as if all the mass was concentrated at the center of mass.

Illustration: A cylinder of mass and radius R is moving on a horizontal surface with angular velocity ω and its cylindrical axis is moving with velocity V_0 (see figure). Find the angular momentum of the cylinder w.r.t. to P, Q and R shown in the figure.



Let Z-axis be perpendicular to the plane of the paper and point in to the plane of the paper. Since cylinder is rotating in the clockwise direction, $\vec{\omega}$ point in the +ve z-direction.

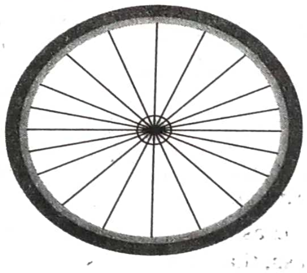
$$\vec{L}_P = (MV_oR + I_{cm}\omega)\hat{k}$$
$$\vec{L}_Q = I_{\text{cm}}\omega\hat{k} + 0 \quad \text{[momentum arm is zero]}$$
$$\vec{L}_R = (I_{\text{cm}}\omega - MVR)\hat{k}$$

where $\hat{k}$ is a unit vector in direction of Z-axis.

ROLLING

Rolling Motion

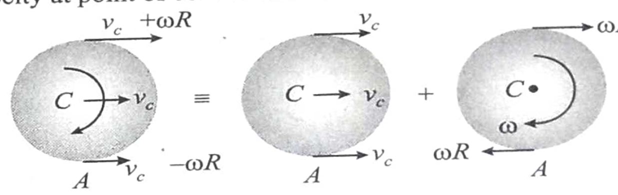
A photograph of a rolling bicycle wheel.



The spokes near the top of the wheel are more blurred than those near the bottom of the wheel because they are moving faster

Pure rolling means no sliding. Now, the motion of any body can be divided into pure translation & pure rotation as discussed. The velocity V_P of point of contact A is $V_C - \omega R$ (see figure).

Sliding refers to the condition under which two bodies in contact have relative velocity. And under pure rolling, the relative velocity at point of contact should be zero.

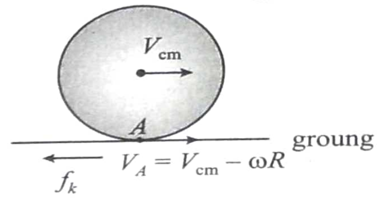


Rolling = Translation + Rotation

Now for the wheel shown, point A is in contact with the ground. The point A on the ground has zero velocity. Thus, the point A on the wheel should also have zero velocity.

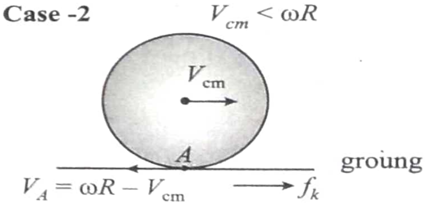
Rolling with slipping

Case-I $V_{cm} > \omega R$



Note: The friction will be kinetic in nature & its magnitude can be determined using $f_k = \mu_k N$. Direction will be backwards (because point of contact A is moving forward w.r.t. ground)

Case -2 $V_{cm} < \omega R$



Point of contact A moves backwards. So kinetic friction acts in forward direction.

Rolling without slipping

$V_{\text{cm}} = \omega R$, $V_A = 0$

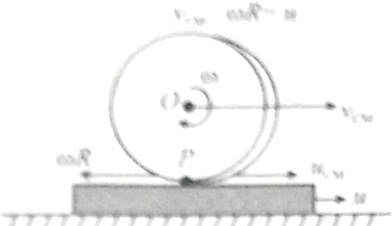
This is known as perfect rolling

In Maximum problems we come across this situation

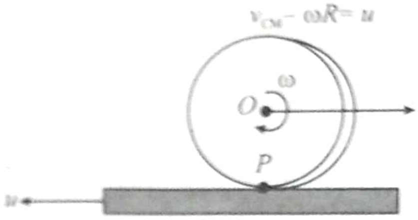
Note:
1. The friction will be be static in nature. Its magnitude is self-adjusting, it can vary from 0 to $\mu_s N$.
2. Static friction may act in forward or backward to prevent slipping.
3. The total work done by static friction is zero because point of application of the frictional force does not move.
4. If there is no applied force, mechanical energy will be conserved.

ROLLING BODIES ON MOVING PLATFORM

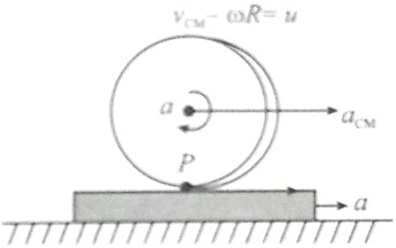
1. If point of contact of the surface is moving with a velocity u with respect to the ground, then



2. For no sliding on the moving platform, $u = \omega R - v_{CM}$



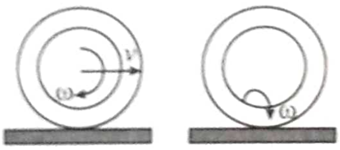
3. For accelerated surface,



$a_{CM} - \alpha R = a$

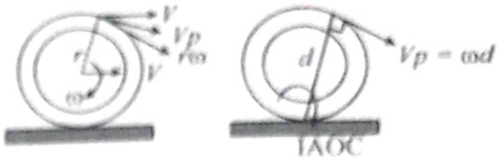
INSTANTANEOUS AXIS OF ROTATION

The combined effects of translation of the centre of mass and rotation an axis through the centre of mass are equivalent to a pure rotation with the same angular speed about an axis passing through a point of zero velocity.



Such an axis is called the instantaneous axis of rotation (IAOR). This axis is always perpendicular to the plane used to represent the motion and the intersection of the axis with this plane defines the location of instantaneous centre of zero velocity (IC).

For example consider a wheel which rolls without slipping. In this case the point of contact with the ground has zero velocity. Hence, this point represents the I.A.R. for the wheel. If it is imagined that the wheel is momentarily pinned at his point, the velocity of any point on the wheel can be found using $v = \omega d$.



Here d is the distance of the point from IC. Similarly, the kinetic energy of the body can be assumed to be pure rotational about IAOR or.

Rotation + Translation ⇒ Pure rotation about IAOR passing through IC]

$$K.E. = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

Where $\frac{1}{2}mv_c^2$ is the translational kinetic energy and

$\frac{1}{2}I_{cm}\omega^2$ is the rotational kinetic energy about the center of mass

In pure rolling motion, $v_c = \omega R$

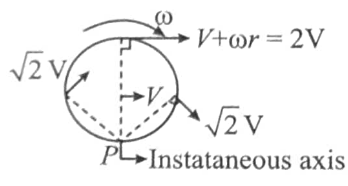
$$\therefore\ K = \frac{1}{2}m(\omega R)^2 + \frac{1}{2}I_{cm}\omega^2$$

or $K = (I_{cm} + mR^2)\,\omega^2$

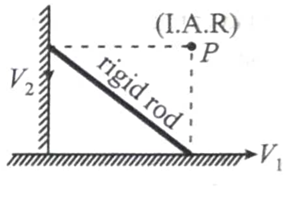
$$\Rightarrow K.E. = \frac{1}{2}I_{IAOR}\omega^2$$

GEOMETRICAL CONSTRUCTION OF INSTANTANEOUS AXIS OF ROTATION

Draw velocity vector at any two points on the rigid body. The I.A.R. is the point of the point of intersection of the perpendicular drawn on them. (See fig (*ii*))



(*i*)



(*ii*)

In case of pure rolling the lower point is instantaneous axis of rotation. (Fig (i))

The motion of body in pure rolling can therefore by analysed as pure rotation about this axis. Consequently, kinetic energy can be written as

$KE = 1/2I_P\omega^2$

Where I_P is moment of inertia about instantaneous axis of rotation passing through P.

Application of Newton's Second Law in Rolling Motion

1. Write $F_{net} = M\,a_{cm}$ for the object as if it were a point-mass, that is, ignoring rotation.
2. Write $\tau = I_{cm}\,\alpha$ as if the object were only rotating about the centre of mass, that is ignoring translation.

3. Use of no-slip condition
4. Solve the resulting equations simultaneously for any unknown.

Caution:

- In general, it is not the case that $f = \mu N$
- Be certain that the sign convention of forces and torques are consistent.

Train Your Brain

Example 27: A spherical ball rolls on a table without slipping. Then the fraction of its total energy associated with rotation is-

(*a*) 2/5 (*b*) 3/5
(*c*) 2/7 (*d*) 3/7

Sol. Total energy,

$$E = (1/2)\, I\omega^2 + (1/2)\, mv^2$$
$$= (1/2)\,(2/5\; mr^2)\,\omega^2 + (1/2)\, mr^2\omega^2$$
$$= (1/5)\, mr^2\omega^2 + (1/2)\, mr^2\omega^2 = (7/10)\, mr^2\omega^2$$

Rotational energy = $(1/5)\, mr^2\omega^2$

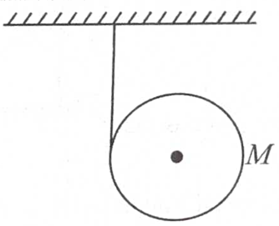
$$\therefore \quad \frac{\text{Rotational energy}}{\text{Total energy}} = \frac{\frac{1}{5}mr^2\omega^2}{\frac{7}{10}mr^2\omega^2} = \frac{2}{7}$$

Example 28: A uniform cylinder of mass M and radius R starts descending at a moment $t = 0$ due to gravity (see fig. below)

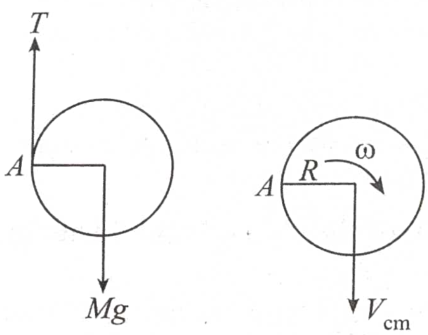
Find (*a*) tension in the thread

(*b*) acceleration of the C.M

(*c*) the angular acceleration.



Sol. The f.b.d of the cylinder is as shown below.



From eqns of motion, we have

$$Mg - T = Ma_{cm} \quad \text{...(1)}$$

If disc does not slip, $f_s \le \mu_s N$

$F/3 \le \mu_s Mg$

$$\Rightarrow \mu_s \ge \frac{F}{3\,mg}\;;\; \mu_{s,\min} = \frac{F}{3\,mg}$$

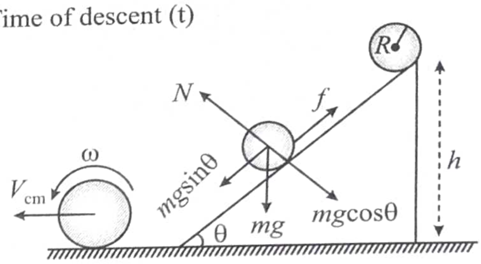
Note : Static friction is a variable force in pure rolling you may take f_s in any which direction (forward or backward). If after solving, f_s is negative then it acts in opposite direction to that assumed.

Example 29: A rigid body of mass M rolls on an rough inclined plane without slipping where k is the radius of gyration of the rigid body about center of mass then find the following parameters.

(*a*) Velocity of center of mass when body reaches the ground

(*b*) Acceleration of center of mass when body reaches the ground

(*c*) Time of descent (t)



Sol: (*a*) Applying energy conservation

$$Mgh = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}MK^2\omega^2$$

(where K is the radius of gyration)

$$= \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}\cdot\frac{K^2}{R^2}\cdot MR^2\omega^2$$

$$= \frac{1}{2}M.\left(1+\frac{K^2}{R^2}\right)v_{cm}^2, \text{ [where } v_{cm} = R\omega \text{ for pure rolling]}$$

$$\text{or, } v_{cm} = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}}$$

[where $\frac{K^2}{R^2}$ is a pure number between 0 and 1 that depends on the shape of the body]

It is important to note that the velocity of a rolling body is independent of its mass (M) and radius (R). All uniform solid spheres have the same speed at the bottom even if their masses and radii are different because they have same $\frac{K^2}{R^2} = \left(\frac{2}{5}\right)$. All solid cylinders have the same speed $\left(\frac{K^2}{R^2} = \frac{1}{2}\right)$, all hollow cylinders have the same speed $\left(\frac{K^2}{R^2} = 1\right)$ and so on. Smaller the value of $\frac{K^2}{R^2}$, faster the body is moving.

(*b*) **Acceleration :** Let the linear acceleration of the rolling body be a_{cm}. Then for the linear motion, the

net force $= mg\sin\theta - f = ma_{cm}$ ...(*i*)

For the rotational motion, the net torque

$$fR = I_{cm}\cdot\alpha = MK^2\cdot\frac{a_{cm}}{R}$$

$$\text{or, } f = \frac{MK^2}{R^2}a_{cm} \quad ...(ii)$$

solving (*i*) and (*ii*) for a_{cm}, we get

$$\boxed{a_{cm} = \frac{g\sin\theta}{1+\frac{K^2}{R^2}}}$$

It is also independent of mass (M) and radius R.

(*c*) **Time of descent (t) :** $\frac{1}{2}a_{cm}.t^2 = \frac{h}{\sin\theta}$

$$\text{or, } t = \sqrt{\frac{2h}{a_{cm}\cdot\sin\theta}}$$

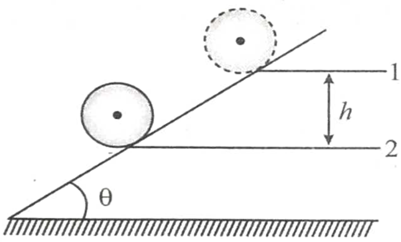
$$\boxed{t = \frac{1}{\sin\theta}\sqrt{\frac{2h\left(1+\frac{K^2}{R^2}\right)}{g}}}$$

It is also independent of mass and radius. Clearly, a body of smaller $\frac{K^2}{R^2}$, takes less time.

Example 30: A solid cylinder of mass m and radius r starts rolling down an inclined plane of inclination θ. Friction is enough to prevent slipping. Find the speed of its centre of mass when, its centre of mass has fallen a height h.

Sol. Consider the two shown positions of the cylinder. As it does not slip, total mechanical energy will be conserved.

Energy at position 1 is $E_1 = mgh$



Energy at position 2 is $E_2 = \frac{1}{2}mv^2_{c.m.} + \frac{1}{2}I_{c.m.}\,\omega^2$

$$\therefore\ \omega = \frac{V_{c.m.}}{r} = \text{and } I_{c.m.} = \frac{mr^2}{2}$$

$$E_2 = \frac{1}{2}mv^2_{cm} + \frac{1}{2}\left(\frac{mr^2}{2}\right)\left(\frac{v_{cm}}{r}\right)^2$$

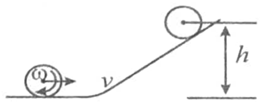
$$\Rightarrow E_2 = \frac{3}{4}mv^2_{c.m.}$$

From conservation of energy, $E_1 = E_2$

$$\Rightarrow V_{c.m.} = \sqrt{\frac{4}{3}gh}$$

Example 31: A ball of radius R and mass m is rolling without slipping on a horizontal surface with velocity of its centre of mass v. It then rolls without slipping up a hill to a height h before momentarily coming to rests. Find h.

Sol. $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$



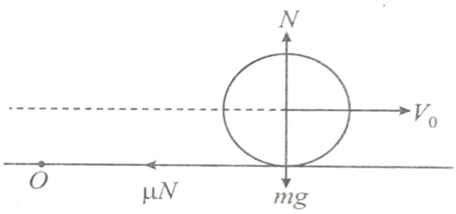
$$v = \omega R\ ,\ I = \frac{2}{5}mR^2$$

$$h = \frac{7v^2_{CM}}{10g}$$

Example 32: A bowling ball is thrown down the alley in such a way that it slides with speed V_0 initially without rolling (*a*) Find the velocity of the ball when it starts rolling on the surface.

(*b*) How much kinetic energy is lost before it starts rolling. The co-efficient of kinetic friction is μ.

Sol.



From the free body diagram it is clear that N = mg.

From eqns. of motion we have

$m\,a_{cm} = -\mu mg$ ...(1)

$I_{cm}\alpha = \mu mg\,R$ ...(2)

R = radius of the ball

$\Rightarrow a_{cm} = -\mu g$

$$\Rightarrow \alpha = \frac{\mu mgR}{I_{cm}} = \frac{\mu mgR}{\frac{2}{5}mR^2} = \frac{5\mu g}{2R}$$

After time 't', the velocity V_f of center of ball and angular velocity ω_f will be

$V_f = V_o - \mu_k gt$ ...(3)

$\omega_f = 0 + \frac{5\mu g}{2R}$ ...(4)

If it starts rolling after time t then

$V_f - \omega_f R = 0$ ...(5)

After solving eqns (3), (4) and (5) we have

(*a*) $V_f = \frac{5}{7}V_o$ and $\omega = \frac{5V_o}{7R}$

(*b*) The change in kinetic energy is

$$\Delta KE = \left(\frac{1}{2}mV_{cm}^2 + \frac{1}{2}\times\frac{2m}{5}\omega^2 R^2\right) - \left(\frac{1}{2}mV_0^2\right)$$

$$= \left(\frac{1}{2}m\left(\frac{5}{7}V_o\right)^2 + \frac{m}{5}\left(\frac{5}{7}\frac{V_o}{R}\right)^2 R^2\right) - \frac{1}{2}mV_0^2 = -\frac{mV_0^2}{7}$$

Alternative method: Angular momentum of the cylinder about point O is conserved (see fig.). Since N = mg the torque due these forces cancels out. The torque due to friction is zero since its moment arm is zero about O. Hence

$$L_i = L_f$$

$$mV_o R = mV_f R + I_{cm}\omega_f = \frac{7}{5}mRV_f$$

$$\Rightarrow V_f = \frac{5}{7}V_o$$

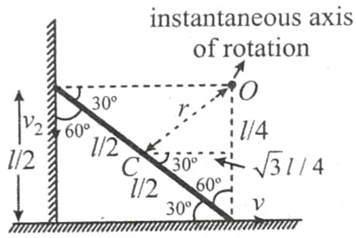
Example 33: A ladder of length L is slipping with its ends against a vertical wall and a horizontal floor. At a certain moment, the speed of the end in contact with the horizontal floor is v and the ladder makes and angle $\alpha = 30°$ with the horizontal. Then the speed of the ladder's center must be

(*a*) $2v/\sqrt{3}$ (*b*) $\frac{v}{2}$

(*c*) v (*d*) None

Sol. (*c*) $\omega = \frac{v}{l/2}$

$$\omega = \frac{2v}{l}$$



Also, $v_c = r\omega = \omega\sqrt{\left(\frac{\sqrt{3}}{4}l\right)^2 + \left(\frac{l}{4}\right)^2}$

$$= \frac{1}{4}l\sqrt{3+1}\,\omega = \frac{l}{2}\omega$$

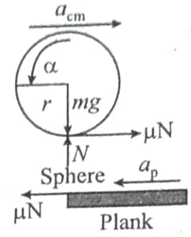
$$= \frac{l}{2}\times\frac{2v}{l} = v$$

Example 34: A plank of mass M rests on a smooth horizontal plane. A sphere of mass m is placed on the rough upper surface of the plank and the plank is suddenly given a velocity v in the direction of its length. Find the time after which the sphere begins pure rolling, if the coefficient of friction between the plank and the sphere is μ and the plank is sufficiently long.

Sol. Let t be the time after which slipping between the sphere and plank ceases.

For the sphere, $N = mg$, $\mu N = ma_{cm}$

$$\Rightarrow a_{cm} = \mu g$$



$$\tau = I\alpha \Rightarrow \mu mgr = 2/5mr^2\alpha$$

$$\Rightarrow \alpha = \frac{5\mu g}{2r}$$

For the plank,

$$\mu N = Ma_p \Rightarrow a_p = \frac{\mu mg}{M}$$

After time t,

velocity of plank, $v_P = v - \frac{\mu mg}{M}t$

velocity of sphere, $v_{cm} = \mu gt$

angular velocity of sphere, $\omega = \frac{5\mu g}{2r}t$

For no slipping, the point of contact of sphere should have same speed as that of plank,

$$\Rightarrow v_{cm} + \omega r = v_P$$

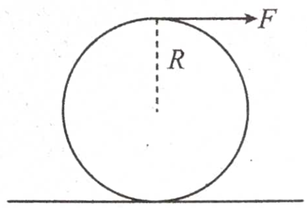
Substituting and solving

$$\mu gt + \left(\frac{5\mu g}{2r}t\right)r = v - \frac{\mu mg}{M}t$$

$$\Rightarrow t = \frac{2v}{g(2\mu + 5\mu + 2\mu m/M)} = \frac{2v}{\mu g[7 + (2m/M)]}$$

Concept Application

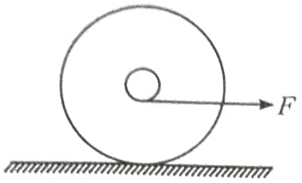
26. A tangential force F acts at the top of a thin spherical shell of radius R. The acceleration of the shell if it rolls without slipping is:



(*a*) $\frac{6F}{m}$ (*b*) $\frac{5F}{6m}$

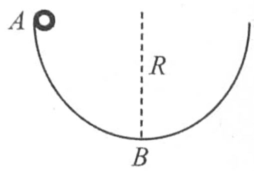
(*c*) $\frac{F}{m}$ (*d*) $\frac{6F}{5m}$

27. A yo-yo arranged as shown, rests on a frictionless surface. When a force F is applied to the string, the yo-yo



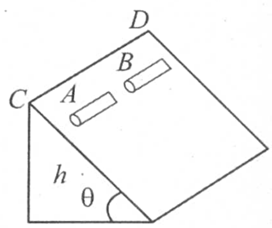
(a) moves to the left and rotates counterclockwise
(b) moves to the right and rotates counterclockwise
(c) moves to the left and rotates clockwise
(d) moves to the right and rotates clockwise

28. A small sphere of mass m and radius r rolls without slipping inside a large fixed hemispherical bowl of radius $R(>> r)$ as shown in figure. If the sphere starts from rest at the top point of the hemisphere find the normal force exerted by the small sphere on the hemisphere when it is at the bottom B of the hemisphere.



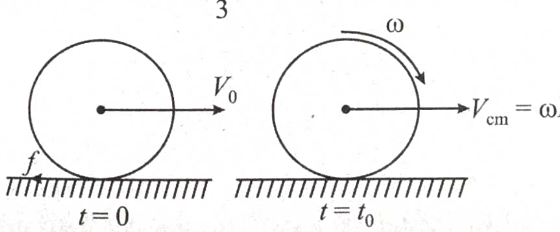
(a) $\frac{10}{7}$mg (b) $\frac{17}{7}$mg
(c) $\frac{5}{7}$mg (d) $\frac{7}{5}$mg

29. Two cylinders A and B are released on a fixed rough inclined plane of angle θ with the horizontal from height 'h'. A is a hollow cylinder and 'B' is a solid cylinder. Both perform pure rolling. Find the distance between the two cylinders when one of them reaches the bottom. Assume that the axis of both cylinders is parallel to the edge CD always.



(a) $\frac{h}{2\sin\theta}$ (b) $\frac{h}{4\sin\theta}$
(c) $\frac{h}{\sin\theta}$ (d) $\frac{h}{8\sin\theta}$

30. A uniform disc of mass m and radius R is projected horizontally with velocity v_0 on a rough horizontal floor so that it starts off with a purely sliding motion at $t = 0$. After t_0 seconds, it acquires a purely rolling motion as figure. The velocity of the centre of mass of the disc at t_0 is $\frac{xv_0}{3}$, then the value x is.



IMPULSIVE FORCE AND IMPULSIVE TORQUE

When two bodies collide, impulsive force act between them. Associated with an impulsive force is an impulsive torque. Hence if an impulsive force acts on a rigid body it will not only change the velocity of V_{cm} but will also change the angular velocity ω.

From eqns of motion , we have

$$\vec{F}_{ext} = M\,a_{cm}$$

$$\tau_m = r_\perp F = I_{cm}\alpha$$

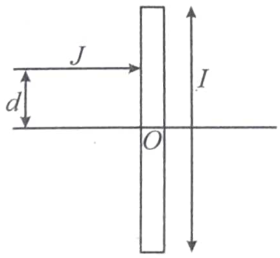
Multiplying both eqn by Δt yield

$$\vec{J} = \left(\vec{F}_{ext}\cdot\Delta t\right) = M\vec{a}_{cm}\ \Delta t = M\left(\vec{V}_f - \vec{V}_i\right)$$

$$Jr_\perp = I_{cm}\alpha\Delta t = I_{cm}(\omega_f - \omega_i)$$

where J is the impulse. and $Jr\perp$ is the impulsive torque.

Illustration 3: An impulse J is applied on a rod of mass M and length L lying on a horizontal surface (see fig.) . Find the velocity of the c.m. of the rod and its angular velocity ω after the application of impulse.



Since $V_i = 0$,. $J = MV_{cm}$

$\Rightarrow V_{cm} = J/M$

$$Jd = \frac{ML^2}{12}\omega \qquad \Rightarrow \omega = \frac{12Jd}{ML^2}$$

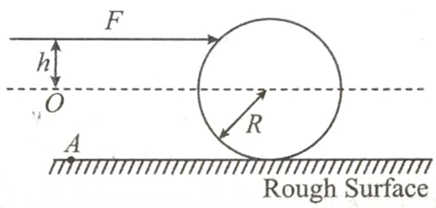
Note. If $d = 0$, then rod will only have translatory motion as external impulsive torque about center of mass vanishes.

Train Your Brain

Example 35: A billiards ball, initially are rest is given a sharp impulse by a cue. The cue is held a distance h above center line (see fig.) .The ball leaves the cue with a speed v_0 and because of its forward english finally acquires a final speed of $9/7\ v_0$.

Show that $h = 4/5R$

where R is the radius of ball



Sol. F acts for time Δt. Then

$$V_{cm} = v_o = \frac{F\Delta t}{M} \ \& \ \omega = \frac{F\Delta th}{2/5MR^2}$$

$$\Rightarrow \omega = \frac{5v_o h}{2R^2}$$

After the application of impulse,angular momentum about A will be conserved.

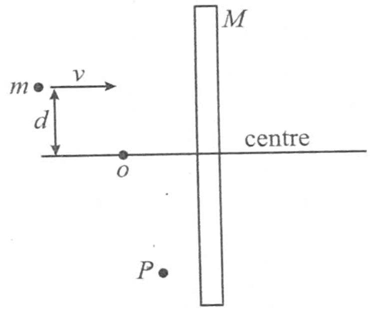
$$\Rightarrow L_i = L_f$$

$$\Rightarrow Mv_o R + I_{cm} w = MV_f R + I_{cm}\frac{V_f}{R}$$

$$\Rightarrow Mv_o R + \frac{2}{5}MR^2 \times \frac{5v_o h}{2R^2} = \left(MR + \frac{2}{5}\frac{MR^2}{R}\right)\frac{9}{7}v_o$$

$$\Rightarrow MR + Mh = \frac{9}{5}MR \Rightarrow h = \frac{4}{5}R$$

Example 36: A rod of length L lies on a smooth table. It has a mass M and is free to move in any way on the table. A small ball of mass m moving with speed v as shown collides elastically with stick. (a) What quantities are conserved during collision. (b) What must be the mass of the ball so that it remains at rest immediately after collision ?



Sol. During collision, impulsive force act on the ball and the rod. The balls velocity will reduce while rod's center of mass acquires certain velocity V_{cm} and it will start rotating with certain angular velocity ω. Let the reduced velocity of the ball be v'.

If we consider the system consisting of ball and the rod, there is no external impulsive force acting on it. Consequently, momentum and angular momentum of the system should be conserved. Since the collision is elastic , mechanical energy will also be conserved.

(b) Equating momentum, kinetic energy and angular momentum about O, before and after the collision we have

$$mv = mv' + MV_{cm} \quad ...(1)$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}MV_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \quad ...(2)$$

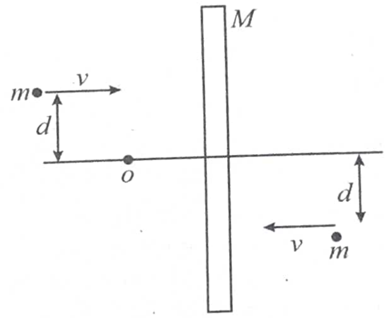
$$mvd = mv'd + I_{cm}\omega \quad ..(3)$$

Setting $v' = 0$ in the eqns and solving them, we obtain

$$m = \frac{ML^2}{12d^2 + L^2}$$

Note: Since angular momentum is conserved about any point, you can choose any point about which you write the third eqn. Choose any other point P (see fig.) and write a eqn for angular momentum. Though you shall get a different eqn it has no effect on the final answer.

Example 37: A stick of mass M and length L lie on a horizontal table. It is hit by two bullets fired with velocity v in the opposite direction. The bullets stick to the rod. If the mass of each bullet is m and they simultaneous hit the rod (see fig.), find the angular velocity of the stick after the collision.



Sol. Let the velocity of the center of mass of the stick be V_{cm} and its angular velocity be ω. Since the bullets stick to the plate, mechanical energy will be lost.

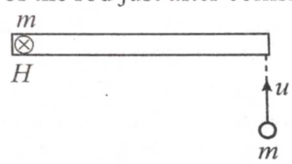
If we consider the system consisting of stick and two bullets, no external force acts . From conservation laws, we obtain

$$mv - mv = (M + 2m)V_{cm} \quad ...(1)$$

$$mvd + mvd = \left(\frac{ML^2}{12} + 2md^2\right)\omega \quad ...(2)$$

$$\Rightarrow V_{cm} = 0, \quad \omega = \frac{2mvd}{\frac{ML^2}{12} + 2md^2}$$

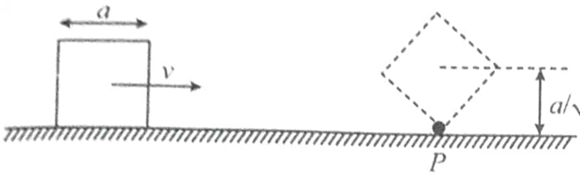
Example 38: A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having same mass m coming with an initial speed u perpendicular to the rod, strikes the rod in-elastically at its free end. Find out the angular velocity of the rod just after collision?



Sol. Angular momentum is conserved about H because external impulsive force due to the hinge act at H and it produces no torque about H

$$mu\ell = \left(\frac{m\ell^2}{3} + m\ell^2\right)\omega \Rightarrow \omega = \frac{3u}{4\ell}$$

Example 39: A cube of side a moves on smooth horizontal surface (see fig.). At point P it collides inelastically with a small obstacle. As a result it starts rotating about it. Find the minimum velocity v that should be imparted to the block so that it topples over.



Sol. During collision angular momentum about P would be conserved (Why?). Energy will not be conserved as collision is not elastic. Let ω_0 be the angular velocity of cube just after collision.

Conserving angular momentum about P, we obtain

$$mv\frac{a}{2} = I_p\omega_o$$

Where $I_p = I_{cm} + md^2$

$$= \left(m\left(\frac{a^2}{12}+\frac{a^2}{12}\right)\right) + ma^2/2 = \frac{2}{3}ma^2$$

$$\Rightarrow v = \frac{4}{3}a\omega$$

If the cube manages to rotate by $\pi/4$, then it will topple over. Applying energy conservation after the collision, we have

$$\left[0-\frac{1}{2}I_p\omega^2\right] + mg\left[\frac{a}{\sqrt{2}}-\frac{a}{2}\right] = 0$$

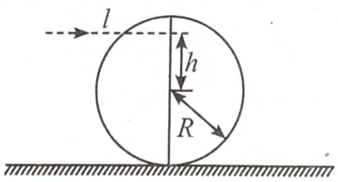
$$\Rightarrow \omega^2 = \frac{mga}{I_p\sqrt{2}}\left[1-\frac{1}{\sqrt{2}}\right] \Rightarrow \omega^2 = \frac{3}{4}g/a\left(\sqrt{2}-1\right)$$

Substituting it in $v = 4/3a\,\omega$, gives

$$v = \sqrt{\frac{4}{3}ga\left(\sqrt{2}-1\right)}$$

Concept Application

31. An impulsive force F acts horizontally on a solid sphere of radius R placed on a horizontal surface. The line of action of the impulsive force is at a height h above the centre of the sphere. If the rotational and translational kinetic energies of the sphere just after the impulse are equal, then the value of h will be-



(a) $\frac{2}{5}$R (b) $\sqrt{\frac{2}{3}}$R

(c) $\sqrt{\frac{2}{5}}$R (d) None of these

32. A uniform rod of mass M and length a lies on a smooth horizontal plane. A particle of mass m moving at a speed v perpendicular to the length of the rod strikes it a distance $a/4$ from the centre and stop after the collision. (a) The velocity of the centre of the rod is $\frac{xm}{M}v$ then the value of x is (b) The angular velocity of the rod about its centre just after the collision is $\frac{xm}{Ma}v$ then value of x is

33. A thin uniform rod of mass m and length l is free to rotate about its upper end. When it is at rest, it receives an impulse J at its lowest point, normal to its length. Immediately after impact

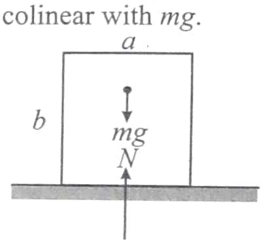
(a) The angular velocity of the rod is $\frac{xJ}{ml}$ then the value of x is

(b) The linear velocity of the midpoint of the rod is $3J/xm$ then the value of x is

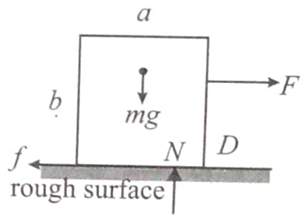
TOPPLING

In many situations an external force is applied to a body to cause it to slide along a surface. In certain cases, the body may tip over before sliding ensues. This is known as topping.

(1) There is a no horizontal force so pressure at bottom is uniform and normal is colinear with mg.

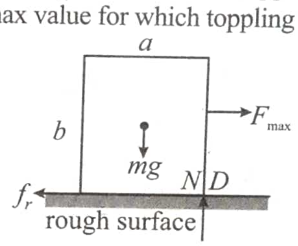


(2) If a force is applied at COM, pressure is not uniform Normal shifts right so that torque of N can counter balance torque of friction.



(3) If F is continuously increased N keeps shifting towards right until it reaches the right most point D.

Here we have assumed that the surface is sufficiently rough so that there is no sliding as F is increased to F_{max}. If force is increased any further, then torque of N can not counter balance torque of friction f_r and body will topple. The value of force now is the max value for which toppling will not occure.



$F_{max} = f_r$

$N = mg$

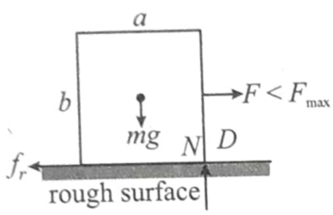
$f_r \cdot b/2 = N \cdot a/2 \Rightarrow f_r = Na/b = mga/b, F_{max} = mga/b$

(4) If surface is not sufficiently rough and the body slides before F is increased to F_{max} ($= mg\, a/b$) then body will slide before toppling.

This is the case if

$$F_{max} > f_{limit} \Rightarrow mga/b > \mu mg$$

or $\mu < a/b$



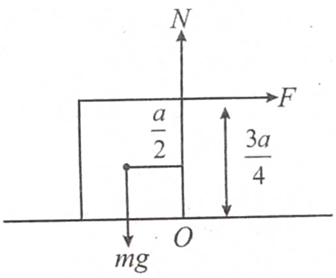
Train Your Brain

Example 40: A uniform cube of side a and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly above the centre of the face, at a height $\frac{3a}{4}$ above the base. What is the minimum value of F for which the cube begins to topple about an edge?

Sol. In the limiting case normal reaction will pass through O. The cube will topple about O if torque of F exceeds the torque of mg.

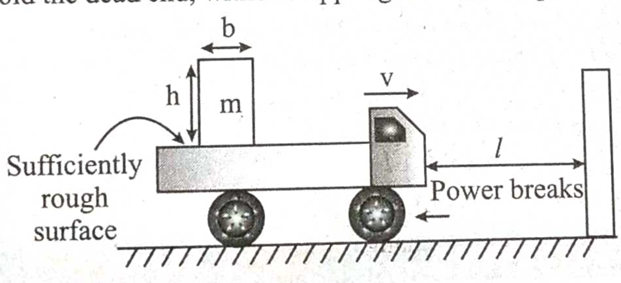
$$\Rightarrow F\left(\frac{3a}{4}\right) > mg\left(\frac{a}{2}\right)$$

$$\Rightarrow F > \frac{2}{3}mg$$



So, the minimum value of F is $\frac{2}{3}mg$

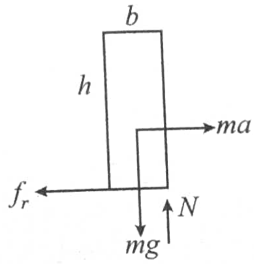
Example 41: Find minimum value of l so that truck can avoid the dead end, without toppling the block kept on it.



Sol. If the truck decelerates at a rate a then in the frame of the truck, pseudo force will act on the block in the forward direction at its center of mass.

$$ma\frac{h}{2} \le mg\frac{b}{2}$$

$$a \le \frac{b}{h}g$$



Final velocity of truck is zero. So that

$$0 = v^2 - 2\left(\frac{b}{h}g\right)l$$

$$l = \frac{h}{2b}\frac{v^2}{g}$$

Example 42: A uniform cylinder of height h and radius is placed with its circular face on a rough inclined plane and the inclination of the plane to. the horizontal is gradually increased. If is the coefficient of friction is μ, then under what conditions the cylinder will

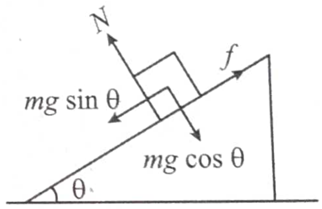
(a) side before toppling

(b) topple before sliding.

Sol. (a) The cylinder will slide if

$mg \sin\theta > \mu\, mg \cos\theta$

$\tan\theta > \mu$ (i)



The cylinder will topple if

$$(mg\sin\theta)\frac{h}{2} > (mg\cos\theta)r$$

$$\Rightarrow \tan\theta > \frac{2r}{h}$$

Thus, the condition of sliding is tan θ > μ and condition of toppling is $\tan\theta > \frac{2r}{h}$. Hence, the cylinder will slide before toppling if

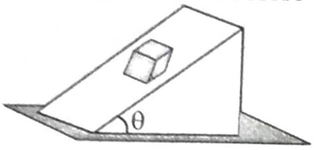
$$\mu < \frac{2r}{h}$$

(b) The cylinder will topple before sliding if

$$\mu > \frac{2r}{h}$$

Concept Application

34. A cube is placed on a rough inclined plane of inclination θ as shown in figure. The coefficient of friction between the cube and the plane is μ. If the angle θ is gradually. Increased, the cube slides before toppling when



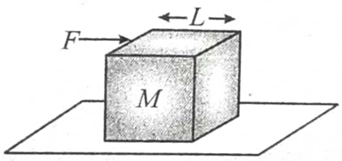
(a) $\mu < 1$

(b) $\mu < \frac{1}{2}$

(c) $\mu > 1$

(d) $\mu > \frac{1}{2}$

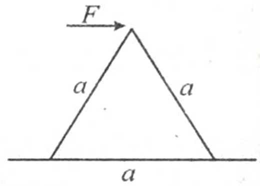
35. A cubical block of side L rests on a rough horizontal surface with coefficient of friction μ. A horizontal force F is applied on the block as shown. If the coefficient of friction is sufficiently high so that the block does not slide before toppling, the minimum force required to topple the block is



(a) Infinitesimal (b) $\frac{mg}{4}$

(c) $\frac{mg}{2}$ (d) $mg(1-\mu)$

36. An equilateral prism of mass 'm' rests on a rough horizontal surface with coefficient of friction μ. A horizontal force 'F' is applied on the prism as shown in Fig. If the coefficient of friction is sufficiently high so that the prism does not slide before toppling, the minimum force required to topple the prism is



(a) $\frac{mg}{\sqrt{3}}$

(b) $\frac{mg}{4}$

(c) $\frac{\mu mg}{\sqrt{3}}$

(d) $\frac{\mu mg}{4}$

37. A solid homogeneous cylinder of height h and base radius r is kept vertically on a conveyor belt moving horizontally with an increasing velocity $v = a + bt^2$. If the cylinder is not allowed to slip find the time when the cylinder is about to topple.

(a) $2gr/bh$

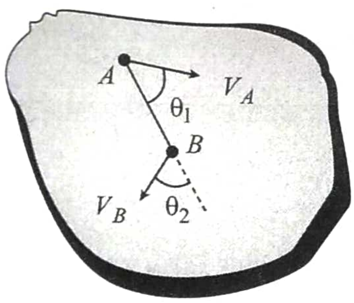
(b) $4gr/bh$

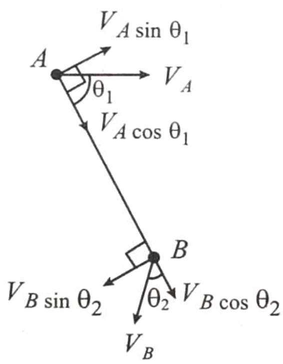
(c) gr/bh

(d) $gr/2bh$

Short Notes

Rigid body





If the above body is rigid

$V_A \cos\theta_1 = V_B \cos\theta_2$.

V_{BA} = relative velocity of point B with respect to point A.

Types of Motion of rigid body

- Pure Translational Motion
- Pure Rotational Motion
- Combined Translational and Rotational Motion

Moment of inertia (i)

Definition: Moment of Inertia is defined as the capability of system to oppose the change produced in the rotational motion of a body.

Moment of Inertia is a scalar (positive quantity).

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots$$
$$= I_1 + I_2 + I_3 + \ldots$$

SI unit of Moment of Inertia is Kgm^2.

Moment of Inertia of

A single particle

$$I = mr^2$$

where m = mass of the particle

r = perpendicular distance of the particle from the axis about which moment of Inertia is to be calculated

For a continuous object

$$I = \int dI = r^2 \int dm$$

where, dI = moment of inertia of a small element

dm = mass of a small element

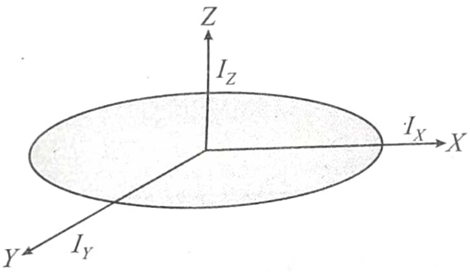
r = perpendicular distance of the particle from the axis

Two Important theorems on moment of inertia

Perpendicular Axis Theorem

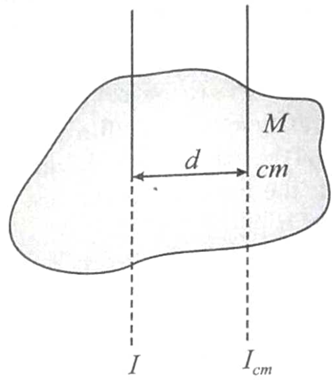
Only applicable to plane lamina (that means for 2-D objects only)

$I_Z = I_X + I_Y$ (when object is in $x-y$ plane).



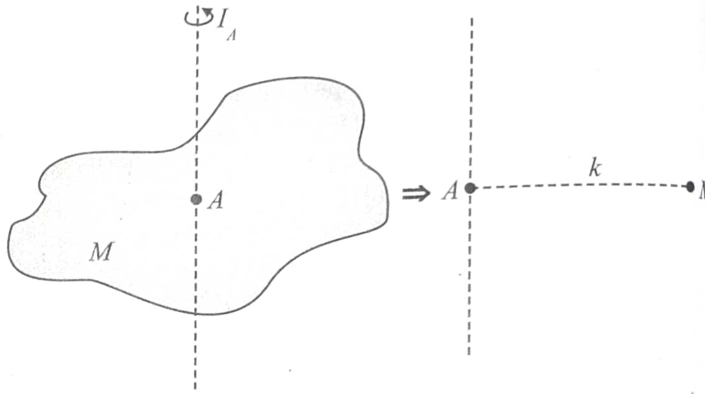
Parallel Axis Theorem

(Applicable to any type of object) :



$$I = I_{cm} + Md^2$$

Radius of gyration (k)



$$I_A = Mk^2$$

$$k = \sqrt{\frac{I_A}{M}}$$

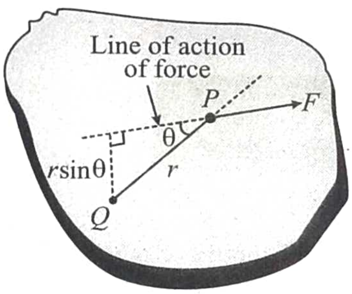
Table: Moment of Inertia of Some Regular Rigid Bodies

Rigid Body	Axis of Rotation	Moment of Inertia (I)	Radius of Gyration (K)
1. Circular ring of mass M and radius R.	1. $\perp^r$ to the plane of ring and passing through its centre	MR^2	R
	2. In the plane of the ring and passing through its centre (or) passing through any diameter of ring	$MR^2/2$	$R/\sqrt{2}$
2. Thin circular plate of mass M and radius R	1. $\perp^r$ to the plane of plate and passing through its centre	$MR^2/2$	$R/2$
	2. In the plane of plate and passing through its centre (or) passing through any diameter of plate	$MR^2/4$	$R/2$
3. Thin hollow sphere of mass M and radius R	1. Passing through its centre or any diameter	$2MR^2/3$	$\sqrt{2}R/\sqrt{3}$
4. Solid sphere of mass M and radius R	1. Passing through its centre or any diameter	$2MR^2/5$	$\sqrt{2}R/\sqrt{5}$

Rigid Body	Axis of Rotation	Moment of Inertia (I)	Radius of Gyration (K)
5. Thin uniform rod of mass M and L	1. $\perp^r$ to the length of rod and passing through its centre	$ML^2/12$	$L/2\sqrt{3}$
	2. $\perp^r$ to the length of rod and passing through its end	$ML^2/3$	$L/\sqrt{3}$
6. Thin uniform rectangular plate of mass M Length L and Breadth B.	1. $\perp^r$ to the plane and passing through its centre	$\frac{M}{12}(L^2+B^2)$	$\frac{\sqrt{L^2+B^2}}{2\sqrt{3}}$
7. Thin square plate of mass M and side length L.	1. $\perp^r$ to the plane of plate and passing through its centre	$ML^2/6$	$L/\sqrt{6}$
	2. In the plane of plate parallel to any side and passing through centre of plate	$ML^2/12$	$L/2\sqrt{3}$
8. Thin hollow cylinder of mass M radius R and Length L	1. About geometrical or natural axis	MR^2	R
	2. $\perp^r$ to the axis of cylinder and passing through its centre	$M\left(\frac{L^2}{12}+\frac{R^2}{2}\right)$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{2}}$
9. Solid cylinder of Mass M radius R and length L.	1. About geometrical or natural axis	$MR^2/2$	$R/\sqrt{2}$
	2. $\perp^r$ to the axis of cylinder and passing through its centre	$M\left(\frac{L^2}{12}+\frac{R^2}{4}\right)$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{4}}$

Torque

$\vec{\tau} = \vec{r} \times \vec{F}$



$\vec{P} = M\vec{v}_{CM}$ $\quad\quad$ $\vec{F}_{external} = M\vec{a}_{CM}$

net external force acting on the body has two perts tangential and centripetal.

$$F_C = ma_C = m\frac{v^2}{r_{CM}} = m\omega^2 r_{CM}$$

$$F_t = ma_t = m\alpha r_{CM}$$

Rotational equilibrium

For transitional equilibrium.

$$\Sigma F_x = 0$$

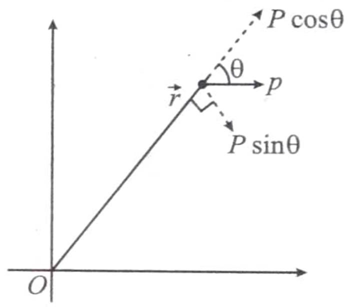
and $\Sigma F_y = 0$

The condition of rotational equilibrium is

$$\Sigma \vec{\tau} = 0$$

Angular momentum ($\vec{L}$)

Angular Momentum of a Particle about a Point



$$\vec{L} = \vec{r} \times \vec{P}$$

$$L = rP\sin\theta$$

$$|L| = r_\perp \times P$$

$$|L| = P_\perp \times r$$

Angular momentum of a rigid body rotating about fixed axis

$L_H = I_H\omega$

L_H = angular momentum of object about axis H.

I_H = Moment ofi Inertia of rigid object about axis H.

ω = angular velocity of the object.

Conservation of Angular Momentum

Angular momentum of a particle or a system remains constant if $\tau_{ext} = 0$ about that point or axis of rotation.

$$L_i = L_f \Rightarrow I_i\omega_i = I_f\omega_f$$

Relation between Torque and Angular Momentum

$$\tau = \frac{dL}{dt}$$

Torque is change in angular momentum

Impulse of Torque

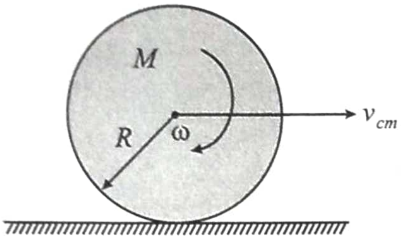
$$\int \tau dt = \Delta J$$

ΔJ = Change in angular momentum.

Rolling Motion

Total kinetic energy $= \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{cm}\omega^2$

Total angular momentum $= Mv_{CM}R + I_{cm}\omega$



Pure Rolling (or Rolling without Slipping) on Stationary Surface

Condition : $v_{cm} = R\omega$

In accelerated motion $a_{cm} = R\alpha$

If $v_{cm} > R\omega$ then rolling with forward slipping.

If $v_{cm} < R\omega$ then rolling with backward slipping.

Total kinetic energy in pure rolling

$$K_{total} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}(Mk^2)\left(\frac{v_{cm}^2}{R^2}\right) = \frac{1}{2}Mv_{cm}^2\left(1+\frac{k^2}{R^2}\right)$$

Dynamics :

$$\vec{\tau}_{cm} = I_{cm}\vec{\alpha},\ \vec{F}_{ext} = M\vec{a}_{cm},\ \vec{P}_{system} = M\vec{v}_{cm}$$

Total K.E. $= \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$

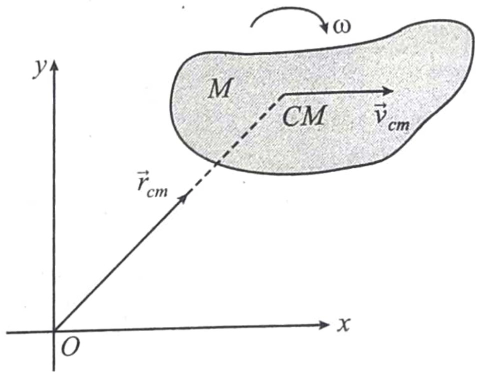
Pure rolling motion on an inclined plane

Acceleration $a = \dfrac{g\sin\theta}{1+k^2/R^2}$

Minimum frictional coefficient $\mu_{min} = \dfrac{\tan\theta}{1+R^2/k^2}$

Angular momentum about axis O = $\vec{L}$ about C.M. + $\vec{L}$ of C.M. about O

$$\vec{L}_O = I_{CM}\vec{\omega} + \vec{r}_{CM} \times M\vec{v}_{CM}$$



Solved Examples

1. A spherical ball rolls on a table without slipping. Then the fraction of its total energy associated with rotation is -

(a) 2/5 (b) 2/7
(c) 3/5 (d) 3/7

Sol. (b) Total energy is, $K = K_R + K_T = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$

$$= \frac{1}{2}\left(\frac{2}{5}mr^2\right)\omega^2 + \frac{1}{2}mr^2\omega^2$$

$$= \frac{1}{5}mr^2\omega^2 + \frac{1}{2}mr^2\omega^2 = \frac{7}{10}mr^2\omega^2$$

Now, rotational kinetic energy

$$K_R = \frac{1}{2}I\omega^2 = \frac{1}{5}mr^2\omega^2$$

$$\therefore \frac{K_R}{K} = \frac{\frac{1}{5}mr^2\omega^2}{\frac{7}{10}mr^2\omega^2} = \frac{2}{7}$$

2. The moment of inertia of a body about a given axis is $1.2\ kg \times m^2$. Initially, the body is at rest. In order to produce a rotational KE of 1500 joule, an angular acceleration of 25 rad/sec^2 must be applied about that axis for a duration of -

(a) 4 s (b) 2 s
(c) 8 s (d) 10 s

Sol. $K_R = \frac{1}{2}I\omega^2 = \frac{1}{2}I(\alpha t)^2 = \frac{1}{2}I\alpha^2t^2$

$$1500 = \frac{1}{2} \times 1.2 \times (25)^2t^2$$

or $t^2 = 4$ or $t = 2s$

3. A body of radius R and mass m is rolling horizontally without slipping with speed v. It then rolls up a hill to a maximum height $h = \dfrac{3v^2}{4g}$. The body might be

(a) solid sphere (b) hollow sphere
(c) disc (d) ring

Sol. (*c*) Let I be the moment of inertia of the body. Then total

$KE = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

or $KE = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{R^2} \quad \left(\omega = \frac{v}{R}\right)$

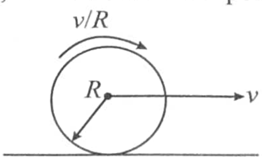
According to energy conservation loss in KE = gain in PE.

or $\frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2 = mgh = mg\left(\frac{3v^2}{4g}\right)$

Solving this, we get $I = \frac{1}{2}mR^2$

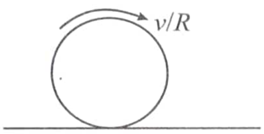
i.e., the solid body is a disc

4. A disc is performing pure rolling on a smooth stationary surface with constant angular velocity as shown in figure. At any instant, for the lower most point of the disc.



(*a*) Velocity is v, acceleration is zero
(*b*) Velocity is zero, acceleration is zero
(*c*) Velocity is v, acceleration is a $\frac{v^2}{R}$
(*d*) Velocity is zero, acceleration is nonzero

Sol. (*d*) From figure,



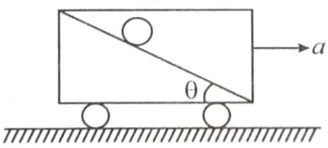
$v = R\omega$

v_{net} (for lowest point) $= v - R\omega = v - v = 0$

and Acceleration $= \frac{v^2}{R} + 0 = \frac{v^2}{R}$

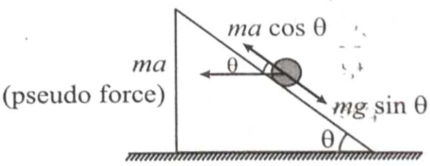
(Since linear speed is constant)

5. Figure shows a smooth inclined plane of inclination θ fixed in a car. A sphere is set in pure rolling on the incline. For what value of 'a' (the acceleration of car in horizontal direction) the sphere will continue pure rolling ?



(*a*) $g\cos\theta$ (*b*) $g\sin\theta$
(*c*) $g\cot\theta$ (*d*) $g\tan\theta$

Sol. (*d*)



The sphere will continue pure rolling if

$ma\cos\theta = mg\sin\theta$

or $a = g\tan\theta$

6. The speed of wave traveling on the uniform circular ring, which is rotating about an axis passing through its center and perpendicular to its plane with tangential speed v in gravity free space is -

(*a*) v (*b*) $\frac{v}{2}$

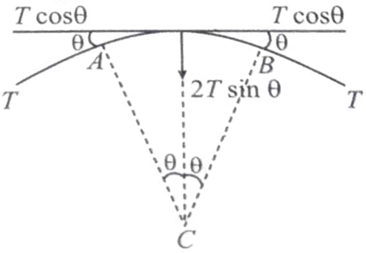
(*c*) $\frac{v}{\sqrt{2}}$ (*d*) $\sqrt{2}v$

Sol. Let T be the tension in the string. Consider a small circular element AB of the string of length,

$\Delta l = R(2\theta)$ (R = radius of hoop)

The components of tension $T\cos\theta$ are equal and opposite and thus cancel out. The components towards centre C (i.e. $2T\sin\theta$) provides the necessary force to element AB

$\therefore\ 2T\sin\theta = \frac{mv^2}{R}$ ...(*i*)



Here, $m = \mu\Delta l = 2\mu R\theta \quad \left(\mu = \frac{\text{mass}}{\text{length}}\right)$

As θ is small, $\sin\theta \approx \theta$

Substituting in Eq. (*i*), we get

$2T\theta = \frac{2\mu R\theta v^2}{R}$ or $\frac{T}{\mu} = v^2$

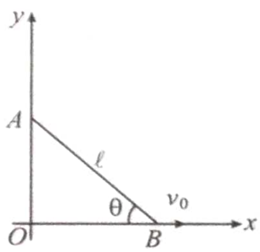
or $\sqrt{\frac{T}{\mu}} = v$ ...(*ii*)

Speed of wave traveling on this string,

$V = \sqrt{\frac{T}{\mu}} = v$ [From eq. (*ii*)]

i.e. the velocity of the transverse wave along the hoop of string is the same as the velocity of rotation of the hoop, viz. v.

7. In the figure given below, the end B of the rod AB which makes angle θ with the floor is pulled with a constant velocity v_0 as shown. The length of rod is ℓ. At an instant when $\theta = 37°$



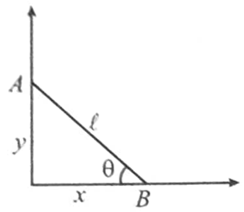
(a) Velocity of end A is $\frac{4v_0}{3}$

(b) angular velocity of rod is $\frac{5v_0}{6\ell}$

(c) angular velocity of rod is constant

(d) velocity of end A is constant

Sol. (a)



$x^2 + y^2 = \ell^2$

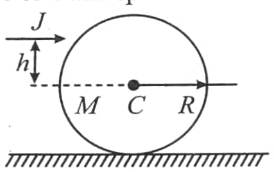
$\Rightarrow \frac{dy}{dt} = -\left(\frac{x}{y}\right)\frac{dx}{dt} = -\frac{l\cos 37}{l\sin 37}\frac{dx}{dt}$

$\therefore v_A = -\frac{4}{3}v_0$

Now, $5 = \ell\cos\theta$

$\frac{dx}{dt} = -\ell\sin\theta\frac{d\theta}{dt} \Rightarrow \omega = -\frac{5}{3}\left(\frac{v_0}{\ell}\right)$

8. A solid sphere of mass M and radius R is placed on a smooth horizontal surface. It is given a horizontal impulse J at a height h above the centre of mass and sphere starts rolling then, the value of h and speed of centre of mass are –



(a) $h = \frac{2}{5}R$ and $v = \frac{J}{M}$

(b) $h = \frac{2}{5}R$ and $v = \frac{2}{5}\frac{J}{M}$

(c) $h = \frac{7}{5}R$ and $v = \frac{7}{5}\frac{J}{M}$

(d) $h = \frac{7}{5}R$ and $v = \frac{J}{M}$

Sol. (a) Let the force producing impulse J is F then

$F \times h = \frac{2}{5}mR^2 \times \alpha$

$Fh = \frac{2}{5}MR^2 \times \frac{a}{R}$

$Fh = \frac{2}{5}MRa$ and $F = ma$ (where $a = R\alpha$)

$MaH = \frac{2}{5}MRa$

$h = \frac{2R}{5}$

Also impulse = change in momentum

or $J = Mv$

9. What must be the relation between length 'L' and radius 'R' of the cylinder if its moment of inertia about its axis is equal to that about the equatorial axis ?

(a) $L = R$ (b) $L = 2R$

(c) $L = 3R$ (d) $L = \sqrt{3}R$

Sol. (d) $\frac{mR^2}{2} = M\left(\frac{L^2}{12} + \frac{R^2}{4}\right)$

or $\frac{R^2}{2} = \frac{L^2}{12} + \frac{R^2}{4}$

or $L = \sqrt{3}R$

10. A particle performs uniform circular motion with angular momentum 'L'. If the frequency of particles motion is halved and its KE is doubled then the angular momentum becomes –

(a) $\frac{L}{4}$ (b) $4L$

(c) $2L$ (d) $L/2$

Sol (b) K.E. $= \frac{1}{2}I\omega^2 = \frac{1}{2}(I\omega)(\omega)$

or K.E. $= \frac{1}{2}L\omega$

or $L = \frac{2K.E.}{\omega}$

Now $L' = \frac{2(2K.E.)}{(\omega/2)} = 4L$

11. The angular speed of rotating rigid body is increased from 4ω to 5ω. The percentage increase in its K.E. is –

(a) 20 % (b) 25 %

(c) 125 % (d) 56 %

Sol. (d) K.E. $= \frac{1}{2}I\omega^2 \Rightarrow$ K.E. $\alpha\ \omega^2$

% increase K.E. $= \frac{KE_f - KE_i}{KE_i} \times 100$

$= \frac{5^2 - 4^2}{4^2} \times 100 = \frac{9}{16} \times 100 = 56\%$

12. Two loops P and Q are made from a uniform wire. The radii of P and Q are r_1 and r_2 respectively, and their moments of inertia are I_1 and I_2 respectively. If $I_2 = 4I_1$, then $\frac{r_2}{r_1}$ equals –

(a) $4^{2/3}$ (b) $4^{1/3}$

(c) $4^{-2/3}$ (d) $4^{-1/3}$

Sol. (b) $I = MR^2 = (2\pi R\lambda)R^2$

or $I\ \alpha\ R^3$ or $R\ \alpha\ I^{1/3}$

or $\frac{R_2}{R_1} = \left(\frac{I_2}{I_1}\right)^{1/3} = \left(\frac{4}{1}\right)^{1/3}$

13. A loop of radius 3 meter and weighs 150 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 15 cm/sec. How much work has to be done to stop it –

(a) 3.375 J

(b) 7.375 J

(c) 5.375 J

(d) 9.375 J

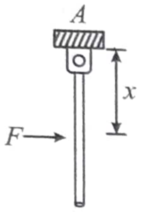
Sol. (a) Required work = Total K.E.

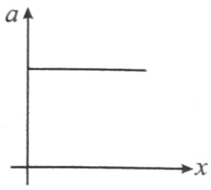
$$= \frac{1}{2} mv^2 \left(1 + \frac{k^2}{R^2}\right)$$

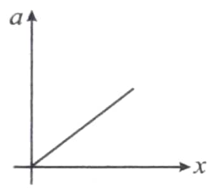
$$= \frac{1}{2} Mv^2 \left[1 + \frac{k^2}{R^2}\right]$$

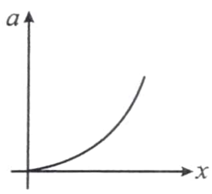
$$= \frac{1}{2} \times 150 \times (0.15)^2 (1 + 1) = 3.375 \text{ J}$$

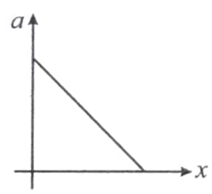
14. A rod of mass m and length l is hinged at one of its end A as shown in figure. A force F is applied at a distance x from A. The acceleration of centre of mass (a) varies with x as –



(a) 

(b) 

(c) 

(d) 

Sol. (b) The rod will rotate about point A with angular acceleration:

$$\alpha = \frac{\tau}{I} = \frac{Fx}{\frac{ml^2}{3}} = \frac{3Fx}{ml^2}$$

$$\therefore a = \frac{l}{2} \alpha = \frac{3}{2} \frac{Fx}{ml}$$

or $a \propto x$

i.e., a-x graph is a straight line passing through origin.

15. Two points of a rod move with velocities $3v$ and v perpendicular to the rod and in the same direction, separated by a distance 'r'. Then the angular velocity of the rod is -

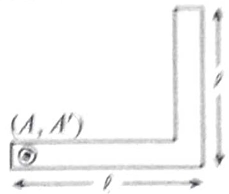
(a) $\frac{3v}{r}$ (b) $\frac{4v}{r}$

(c) $\frac{5v}{r}$ (d) $\frac{2v}{r}$

Sol. (d) $\omega_{rod} = \omega_{point} = \left[\frac{v_{rel.}}{r}\right]$; v_{rel} being the velocity of one point w.r.t other $= \frac{3v - v}{r}$ and 'r' being the distance between them

$$= \frac{2v}{r}$$

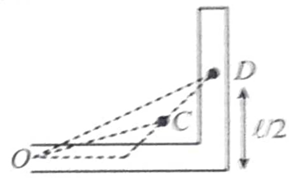
16. A L shaped rod of mass M is free to rotate in a vertical plane about axis AA' as shown in figure. Maximum angular acceleration of rod is-



(a) $\frac{3g}{\sqrt{10\ell}}$ (b) $\frac{9g\ell}{10}$

(c) $\frac{9g\ell}{5}$ (d) $\frac{6g}{5\sqrt{2\ell}}$

Sol. (a) Moment of inertia about axis AA' is



$$I_{AA'} = \left\{\left(\frac{M}{2}\right)\frac{\ell^2}{12} + \left(\frac{M}{2}\right)\cdot\frac{5\ell^2}{4} + \left(\frac{M}{2}\right)\frac{\ell^2}{3}\right\}$$

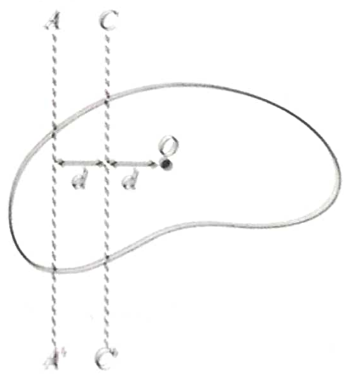
$$\therefore OD = \frac{\sqrt{5}\ell}{2}$$

$$OC = \frac{\sqrt{10}\ell}{4}$$

Torque of Mg will maximum when 'OC' is horizontal

$$\therefore \alpha = \frac{Mg.\sqrt{10}\ell/4}{\frac{5M\ell^2}{6}} = \frac{3g}{\sqrt{10\ell}}$$

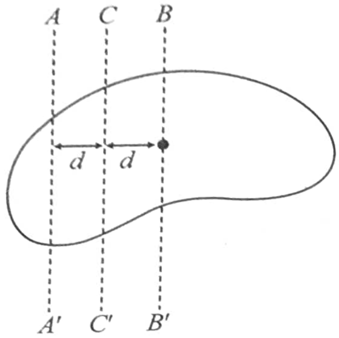
17. Figure shows a body of arbitrary shape 'O' is the centre of mass of the body and mass of the body is M. If $I_{CC}' = I_o$ then I_{AA}' will be equal to –



(a) $I_{CC}' + Md^2$ (b) $I_{CC}' - Md^2$

(c) $I_{CC}' + 3Md^2$ (d) $I_{CC}' + 4Md^2$

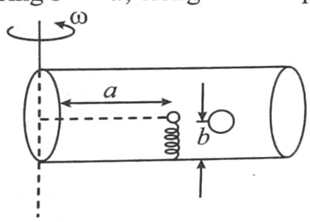
Sol. (*c*) Let BB' be an axis passing through centre of mass and parallel to CC'.



$\therefore I_{AA}' = I_{BB}' + 4\,Md^2 = I_{CC}' - Md^2 + 4\,Md^2$

$I_{AA}' = I_{CC}' + 3\,Md^2$

18. A plastic ball is suspended inside liquid filled in a closed cylindrical container shown in figure. The elongation in spring is 1 mm. The cylinder is now rotated about vertical axis shown with angular velocity $\omega = \sqrt{\dfrac{g}{a}}$. Assuming that length of spring $b << a$, elongation in spring will be –



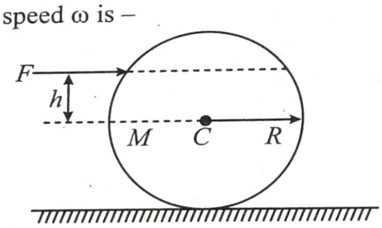
(*a*) 1.7 mm (*b*) 2 mm
(*c*) 1 mm (*d*) 1.4 mm

Sol. (*d*) $g_{eff} = \sqrt{g^2 + (\omega^2 a)^2} = \sqrt{2}g$

$\therefore \dfrac{x'}{x} = \dfrac{g_{eff}}{g} \Rightarrow x' = \sqrt{2}x$

(x' = Final elongation) = 1.4 mm

19. A solid sphere of radius R and M is placed on a smooth horizontal floor. If it given a horizontal impulse F at a height h above centre of mass and the sphere starts rolling, then its angular speed ω is –



(*a*) $\omega = \dfrac{2F.h}{5R^2M}$ (*b*) $\omega = \dfrac{5}{2} \times \dfrac{Fh}{MR^2}$

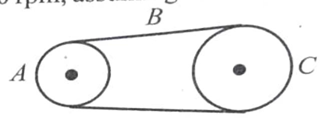
(*c*) $\omega = \dfrac{F}{hM}$ (*d*) $\omega = \dfrac{7Fh}{5MR^2}$

Sol. (*b*) Angular impulse = Change in angular momentum

$\therefore F \times h = \dfrac{2}{5} MR^2 \times \omega$

$\therefore \omega = \dfrac{5}{2} \times \dfrac{Fh}{MR^2}$

20. As shown in figure, wheel A of radius $r_A = 10$ cm is coupled by belt B to wheel C of radius $r_C = 25$ cm. The angular speed of wheel A is increased from rest at a constant rate of 1.6 rad/s². Time after which wheel C reaches a rotational speed of 100 rpm, assuming the belt does not slip, is nearly.



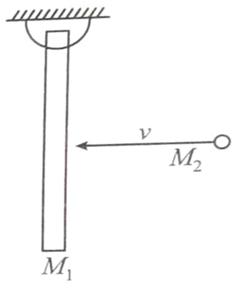
(*a*) 4 sec (*b*) 8 sec
(*c*) 12 sec (*d*) 16 sec

Sol. (*d*) $a_t = \alpha_A r_A = \alpha_C r_C$

$\alpha_C = \alpha_A \left(\dfrac{r_A}{r_C}\right) = 1.6 \times \dfrac{10}{25} = 0.64 \text{ rad/s}^2$

$t = \dfrac{\omega_C}{\alpha_C} = \dfrac{\dfrac{100 \times 2\pi}{60}}{0.64} = 16.35 \text{ sec.}$

21. A uniform rod of mass M_1 is hinged at its upper end. A particle of mass M_2 moving horizontally strikes the rod at its mid point elastically. If the particle comes to rest after collision, the value of $\dfrac{M_1}{M_2}$ is –



(*a*) $\dfrac{3}{4}$ (*b*) $\dfrac{4}{3}$

(*c*) $\dfrac{2}{3}$ (*d*) $\dfrac{3}{2}$

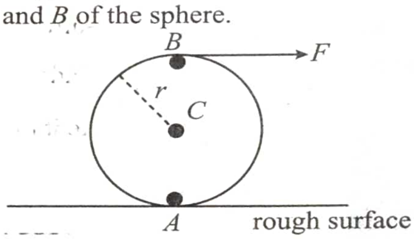
Sol. (*a*) $L_H = M_2 v \dfrac{L}{2} = \dfrac{M_1 L^2}{3}\,\omega$

$\omega = \dfrac{3M_2 v}{2M_1 L}$ ...(i)

for $e = 1$, $v_{CM} = v = \dfrac{\omega L}{2}$ ...(ii)

$\Rightarrow \dfrac{2v}{L} = \dfrac{3M_2 v}{2M_1 L} \Rightarrow \dfrac{M_1}{M_2} = \dfrac{3}{4}$

22. A force F acts tangentially at the highest point of a sphere of mass m kept on a rough horizontal plane. If the sphere rolls without slipping, find the acceleration of the centre (C) and point A and B of the sphere.



Sol. The situation is shown in figure. As the force F rotates the sphere, the point of contact has a tendency to slip towards left so that the static friction on the sphere will act towards right. Let r be the radius of the sphere and a be the linear acceleration of the centre of the sphere. The angular acceleration about the centre of the sphere is $\alpha = a/r$, as there is no slipping.

For the linear motion of the centre, $F + f = ma$

and for the rotational motion about the centre,

$$Fr - fr = I\alpha = \left(\frac{2}{5}mr^2\right)\left(\frac{a}{r}\right) \text{ or } F - f = \frac{2}{5}ma,$$

From (*i*) and (*ii*), $2F = \frac{7}{5}ma$ or $a = \frac{10F}{7m}$.

Acceleration of point A is zero.

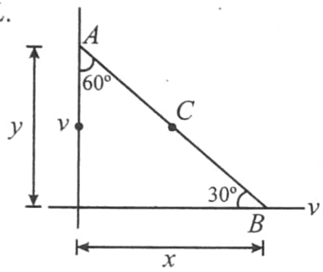
Acceleration of point B is $2a = \left(\frac{20F}{7m}\right)$

23. A ladder of length L is slipping with its ends against a vertical wall and a horizontal floor at a certain moment, the speed of the end in contact with the horizontal floor is v and the l angle $\alpha = 30°$ with the horizontal. Then the speed of the ladder's center of mass must be

(*a*) $\frac{2v}{\sqrt{3}}$ (*b*) $\frac{v}{2}$

(*c*) v (*d*) None

Sol. Let the length of the ladder be L. Since the ladder is constrained to move along the x and y-axis, let the position co-ordinate of the ends of the ladder be $(x, 0)$, $(y, 0)$.



$AB = \sqrt{x^2 + y^2} = L$; where L is constant.

Differentiating both sides.

$$\left(\frac{1}{2}(x^2 + y^2)^{\frac{-1}{2}}\right)2\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = 0$$

$$\Rightarrow \left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = 0 \qquad ...(i)$$

Let the velocity of the upper end along the y-axis is v_y.

The velocity of the lower end along the x-axis is $v_x = v$ as given.

Component of v_y along AB = Component of v along AB.

$v_y \cos 60 = v \sin 60°$

$$\Rightarrow v_y \times \frac{1}{2} = v \times \frac{\sqrt{3}}{2} \Rightarrow v_y = \sqrt{3}v$$

The end points of the ladder have co-ordinates $(x, 0)$,$(0, y)$.

Position co-ordinate of the com of the ladder $= \frac{1}{2}(x\hat{i} + y)\hat{j}$

Since v_y is along the –ve y-axis, we take it to be negative.

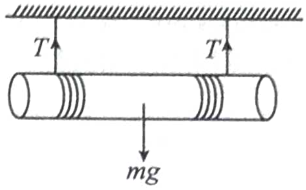
$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{2}\left(\hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt}\right)$$

$$\Rightarrow \vec{v}_{cm} = \frac{1}{2}(\hat{i}v_x + \hat{j}v_y) \Rightarrow \vec{v}_{cm} = \frac{1}{2}(\hat{i}v + \hat{j}\sqrt{3}v)$$

$\therefore$ Magnitude of the velocity of com: $|v_{cm}|$

$$= \frac{1}{2}\sqrt{((v^2 + (\sqrt{3}v)^2)} = v$$

24. A cylinder of mass m is suspended through two strings wrapped around it as shown in figure. Find (*a*) the tension T in the string and (*b*) the speed of the cylinder as it falls through a distance h.



Sol. The portion of the strings between the ceiling and the cylinder is at rest. Hence the points of the cylinder where the strings leave it are at rest. The cylinder is thus rolling without slipping on the strings. Suppose the centre of the cylinder falls with an acceleration a. The angular acceleration of the cylinder about its axis is $\alpha = a/R$, as the cylinder does not slip over the strings.

The equation of motion for the centre of mass of the cylinder is

$$mg - 2T = ma \qquad ...(i)$$

and for the motion about the centre of mass, it is

$$2Tr = I\alpha$$

$$2Tr = \frac{1}{2}mr^2\frac{a}{r} \text{ or } 2T = \frac{ma}{2} \qquad ...(ii)$$

From (*i*) and (*ii*), $a = \frac{2}{3}g$ and $T = \frac{mg}{6}$.

As the centre of the cylinder starts moving from rest, the velocity after it has fallen through a distance h is given by

$$v^2 = 2\left(\frac{2}{3}g\right)h \text{ or } v = \sqrt{\frac{4gh}{3}}.$$

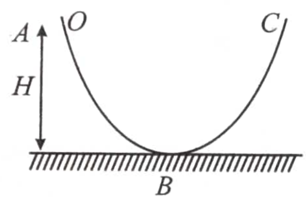
25. A sphere of mass m and radius r is released from rest at point A on a track in vertical plane. The track is rough enough to support rolling between A and B and from B onwards it is smooth. The maximum height attained by sphere from ground on its journey from B onwards is

(*a*) H

(*b*) $\frac{5}{7}$H

(*c*) $\frac{2}{5}$H

(*d*) $\frac{2}{7}$H



Sol. At B

$$mgH = \frac{1}{2}mV_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}m\omega^2r^2 + \frac{1}{2}\times\frac{2}{5}mr^2\omega^2 = \frac{7}{10}mr^2\omega^2$$

$$\Rightarrow \omega^2 = \frac{10gH}{7r^2}$$

At highest point along curve BC, $mgH = mgh + \frac{1}{2}I_{cm}\omega^2$

$$\Rightarrow mgH = mgh + \frac{1}{2}\times\frac{2}{5}mr^2\times\frac{10gH}{7r^2} \Rightarrow h = H\left(1 - \frac{2}{7}\right) = \frac{5H}{7}.$$

Exercise-1 (Topicwise)

ANGULAR DISPLACEMENT, VELOCITY AND ACCELERATION

1. The shaft of a motor rotates at a constant angular velocity of 3000 *rpm*. The radians it has turned in 1 sec are

(*a*) 1000π (*b*) 100π

(*c*) π (*d*) 10π

2. The angular speed of secondhand of a clock is

(*a*) (1/60) rad/s (*b*) (π/60) rad/s

(*c*) (2π/60) rad/s (*d*) (360/60) rad/s

3. Two bodies of mass 10 kg and 5 kg moving in concentric orbits of radius r_1 and $r_2$such that their periods are same. The ratio of centripetal accelerations is

(*a*) r_1/r_2 (*b*) r_2/r_1

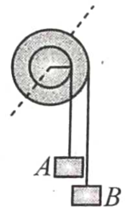
(*c*) $(r_1/r_2)3$ (*d*) $(r_2/r_1)^2$

4. A wheel starts rotating from rest and attains an angular velocity of 60 rad/sec in 5 seconds. The total angular displacement in radians will be-

(*a*) 60 (*b*) 80

(*c*) 100 (*d*) 150

5. Figure shows a small wheel fixed coaxially on a bigger one of double the radius. The system rotates uniformly about the common axis. The strings supporting *A* and *B* do not slip on the wheels. If *x* and *y* be the distances travelled by *A* and *B* in the same time interval, then-



(*a*) $x = 2y$ (*b*) $x = y$

(*c*) $y = 2x$ (*d*) None of these

6. The linear and angular acceleration of a particle are 10 m /sec^2 and 5 rad/sec^2 respectively It will be at a distance from the axis of rotation.

(*a*) 50 m (*b*) $\frac{1}{2}$ m

(*c*) 1 m (*d*) 2 m

7. A particle is moving with a constant angular velocity about an exterior axis. Its linear velocity will depend upon -

(*a*) perpendicular distance of the particle form the axis

(*b*) the mass of particle

(*c*) angular acceleration of the particle

(*d*) the linear acceleration of particle

8. A body is in pure rotation. The linear speed *v* of a particle, the distance *r* of the particle from the axis and the angular velocity ω of the body are related as $\omega = \frac{v}{r}$. Thus-

(*a*) $\omega \propto \frac{1}{r}$

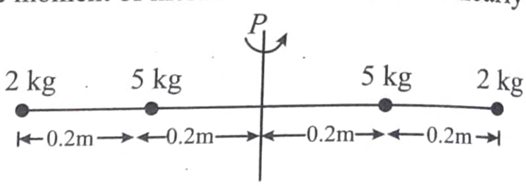
(*b*) $\omega \propto r$

(*c*) $\omega = 0$

(*d*) ω is independent of *r*.

MOMENT OF INERTIA

9. Four masses are fixed on a massless rod as shown in fig. The moment of inertia about the axis *P* is nearly



(*a*) 2 kg m^2

(*b*) 1 kg m^2

(*c*) 0.5 kg m^2

(*d*) 0.3 kg m^2

10. By the theorem of perpendicular axes, if a body be in *X*-*Z*-plane then :-

(*a*) $I_x - I_y = I_z$

(*b*) $I_x + I_z = I_y$

(*c*) $I_x + I_y = I_z$

(*d*) $I_y + I_z = I_x$

11. The axis *X* and *Z* in the plane of a disc are mutually perpendicular and *Y*-axis is perpendicular to the plane of the disc. If the moment of inertia of the body about *X* and *Y* axes is respectively 30 kg m^2 and 40 kg m^2 then *M.I.* about *Z*-axis in kg m^2 will be :

(*a*) 70

(*b*) 50

(*c*) 10

(*d*) Zero

12. Two rods each of mass *m* and length ℓ are joined at the centre to form a cross. The moment of inertia of this cross about an axis passing through the common centre of the rods and perpendicular to the plane formed by them, is :

(*a*) $\frac{m\ell^2}{12}$ (*b*) $\frac{m\ell^2}{6}$

(*c*) $\frac{m\ell^2}{3}$ (*d*) $\frac{m\ell^2}{2}$

13. For the same total mass which of the following will have the largest moment of inertia about an axis passing through the centre of mass and perpendicular to the plane of the body

(a) A disc of radius a

(b) A ring of radius a

(c) A square lamina of side $2a$

(d) Four rods forming a square of side $2a$

14. The moment of inertia of a thin uniform circular disc about one of the diameters is I. Its moment of inertia about an axis perpendicular to the plane of disc and passing through its centre is

(a) $(\sqrt{2})I$ (b) $2I$

(c) $I/2$ (d) $I/\sqrt{2}$

15. The moment of inertia of a uniform semicircular wire of mass M and radius r about a line perpendicular to the plane of the wire through the centre is

(a) Mr^2

(b) $\frac{1}{2}Mr^2$

(c) $\frac{1}{4}Mr^2$

(d) $\frac{2}{5}Mr^2$

16. The density of a rod AB increases linearly from A to B. Its midpoint is O and its centre of mass is at C. Four axes pass through A, B, O and C, all perpendicular to the length of the rod. The moments of inertia of the rod about these axes are I_A, I_B, I_O and I_C respectively.

(a) $I_A > I_B$ (b) $I_A < I_B$

(c) $I_O = I_C$ (d) $I_O < I_C$

17. A stone of mass 4 kg is whirled in a horizontal circle of radius 1m and makes 2 rev/sec. The moment of inertia of the stone about the axis of rotation is-

(a) 64 kg × m^2

(b) 4 kg × m^2

(c) 16 kg × m^2

(d) 1 kg × m^2

18. In an arrangement four particles, each of mass 2 gram are situated at the coordinate points $(3, 2, 0)$, $(1, -1, 0)$, $(0, 0, 0)$ and $(-1, 1, 0)$. The moment of inertia of this arrangement about the Z- axis will be-

(a) 8 units (b) 16 units

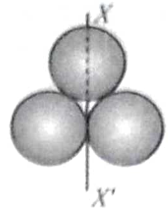
(c) 43 units (d) 34 units

19. A solid sphere and a hollow sphere of the same mass have the same $M.I.$ about their respective diameters. The ratio of their radii will be :

(a) 1 : 2 (b) $\sqrt{3}:\sqrt{5}$

(c) $\sqrt{5}:\sqrt{3}$ (d) 5 : 4

20. Three solid spheres of mass M and radius R are shown in the figure. The moment of inertia of the system about XX' axis wil be:



(a) $\frac{7}{2}MR^2$ (b) $\frac{14}{5}MR^2$

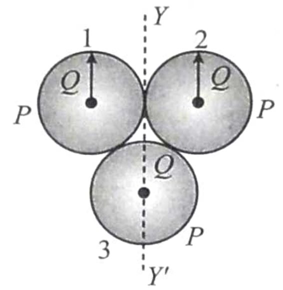
(c) $\frac{16}{5}MR^2$ (d) $\frac{21}{5}MR^2$

21. The moment of inertia of a square lamina about the perpendicular axis through its centre of mass is 20 kg-m^2. Then the moment of inertia about an axis touching its side and in the plane of the lamina will be :-

(a) 10 kg-m^2 (b) 30 kg-m^2

(c) 40 kg-m^2 (d) 25 kg-m^2

22. Three rings, each of mass P and radius Q are arranged as shown in the figure. The moment of inertia of the arrangement about YY' axis will be-



(a) $\frac{7}{2}PQ^2$

(b) $\frac{2}{7}PQ^2$

(c) $\frac{2}{5}PQ^2$

(d) $\frac{5}{2}PQ^2$

23. The moment of inertia of a rod of mass M and length L about an axis passing through one edge and perpendicular to its length will be:

(a) $\frac{ML^2}{12}$ (b) $\frac{ML^2}{6}$

(c) $\frac{ML^2}{3}$ (d) ML^2

24. Three thin uniform rods each of mass M and length L and placed along the three axis of a Cartesian coordinate system with one end of each rod at the origin. The $M.I.$ of the system about z-axis is-

(a) $\frac{ML^2}{3}$ (b) $\frac{2ML^2}{3}$

(c) $\frac{ML^2}{6}$ (d) ML^2

25. Four particles each of mass m are placed at the corners of a square of side length ℓ. The radius of gyration of he system about an axis perpendicular to the square and passing through centre is:

(a) $\frac{\ell}{\sqrt{2}}$ (b) $\frac{\ell}{2}$

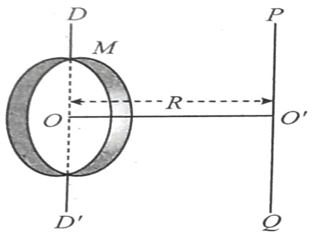
(c) ℓ (d) $\ell\sqrt{2}$

26. The $M.I.$ of a thin rod of length ℓ about the perpendicular axis through its centre is I. The $M.I.$ of the square structure made by four such rods about a perpendicular axis to the plane and through the centre will be:

(a) 4 I (b) 8 I

(c) 12 I (d) 16 I

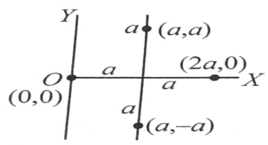
27. The moment of inertia of a ring of mass M and radius R about PQ axis will be :



(a) MR^2 (b) $\frac{MR^2}{2}$

(c) $3/2MR^2$ (d) $2MR^2$

28. Four point masses (each of mass m) are arranged in the $X-Y$ plane the moment of inertia of this array of masses about Y-axis is



(a) ma^2 (b) $2ma^2$

(c) $4ma^2$ (d) $6ma^2$

29. An equilateral triangular wire frame is made from 3 rods of equal mass and length ℓ each. The frame is rotated about an axis perpendicular to the plane of the frame and passing through its end. What is the radius of gyration of the frame?

(a) $\frac{\ell}{2}$

(b) ℓ

(c) $\frac{\ell}{\sqrt{2}}$

(d) $\frac{\ell}{2\sqrt{3}}$

FIXED AXIS ROTATION + TOPPLING

30. For a system to be in equilibrium, the torques acting on i must balance. This is true only if the torques are taken abou

(a) the centre of the system

(b) the centre of mass of the system

(c) any point on the system

(d) any point on the system or outside it

31. A rectangular block has a square base measuring $a \times a$, and its height is h. It moves on a horizontal surface in a directio perpendicular to one of the edges h being vertical. Th coefficient of friction is μ. It will topple if

(a) $\mu > \frac{h}{a}$ (b) $\mu > \frac{a}{h}$

(c) $\mu > \frac{2a}{h}$ (d) $\mu > \frac{a}{2h}$

32. A force of $\left(2\hat{i} - 4\hat{j} + 2\hat{k}\right)$ Newton acts at a point $\left(3\hat{i} + 2\hat{j} - 4\hat{k}\right)$ metre from the origin. The magnitude of torque is -

(a) zero (b) 24.4 N-m

(c) 0.244 N-m (d) 2.444 N-m

33. Rate of change of angular momentum with respect to tim is proportional to :

(a) angular velocity (b) angular acceleration

(c) moment of inertia (d) torque

34. When constant torque is acting on a body then :

(a) body maintain its state or moves in straight line with same velocity

(b) acquire linear acceleration

(c) acquire angular acceleration

(d) rotates with a constant angular velocity

35. If $I = 50$ kg-m^2, then how much torque will be applied to stop it in 10 sec. Its initial angular speed is 20 rad/sec.:

(a) 100 N-m (b) 150 N-m

(c) 200 N-m (d) 250 N-m

36. A particle is at a distance r from the axis of rotation. A given torque τ produces some angular acceleration in it. If the mass of the particle is doubled and its distance from the axis is halved, the value of torque to produce the same angular acceleration is -

(a) $\tau/2$ (b) τ

(c) 2τ (d) 4τ

ENERGY ANALYSIS

37. A ring of radius r and mass m rotates about an axis passing trough its centre and perpendicular to its plane with angular velocity ω. Its kinetic energy is

(a) $mr\omega$ (b) $\frac{1}{2}mr\omega^2$

(c) $mr^2\omega^2$ (d) $\frac{1}{2}mr^2\omega^2$

38. A rod of length L is hinged at one end. It is brought to a horizontal position and released. The angular velocity of the rod when it is in vertical position is

(a) $\sqrt{2g/L}$ (b) $\sqrt{3g/L}$

(c) $\sqrt{g/2L}$ (d) $\sqrt{g/L}$

39. Two bodies A and B having same angular momentum and $I_A > I_B$, then the relation between $(K.E.)_A$ and $(K.E.)_B$ will be:

(a) $(K.E.)_A > (K.E.)_B$ (b) $(K.E.)_A = (K.E.)_B$

(c) $(K.E.)_A < (K.E.)_B$ (d) $(K.E.)_A \neq (K.E.)_B$

ANGULAR MOMENTUM

40. A thin circular ring of mass M and radius r is rotating about its axis with a constant angular velocity ω. Two objects, each of mass m are attached gently to the opposite ends of a diameter of the ring. The wheel now rotates with an angular velocity

(a) $\frac{\omega M}{(M+m)}$ (b) $\frac{\omega(M-2m)}{(M+2m)}$

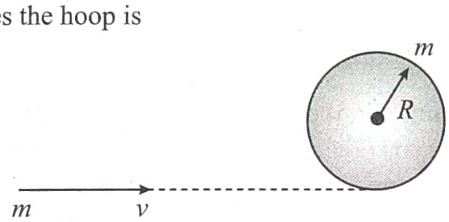
(c) $\frac{\omega M}{(M+2m)}$ (d) $\frac{\omega(M+2m)}{M}$

41. A rotating table completes one rotation is 10 sec. and its moment of inertia is 100 kg-m^2. A person of 50 kg mass stands at the centre of the rotating table. If the person moves 2m. from the centre, the angular velocity of the rotating table in rad/sec. will be:

(a) $\frac{2\pi}{30}$ (b) $\frac{20\pi}{30}$

(c) $\frac{2\pi}{3}$ (d) 2π

42. A circular hoop of mass m, and radius R rests flat on a horizontal frictionless surface. A bullet, also of mass m, and moving with a velocity v, strikes the hoop and gets embedded in it. The thickness of the hoop is much smaller than R. The angular velocity with which the system rotates after the bullet strikes the hoop is



(a) $\frac{v}{4R}$ (b) $\frac{v}{3R}$

(c) $\frac{2v}{3R}$ (d) $\frac{3v}{4R}$

43. A uniform heavy disc is rotating at constant angular velocity (ω) about a vertical axis through its centre O. Some wax is dropped gently on the disc. The angular velocity of the disc-

(a) does not change

(b) increases

(c) decreases

(d) becomes zero

44. A girl sits near the edge of a rotating circular platform. If the girl moves from circumference towards the centre of the platform then the angular velocity of the platform will-

(a) decrease

(b) increase

(c) remain same

(d) becomes zero

45. Two wheels P and Q are mounted on the same axle. The moment of inertia of P is 6 kg-m^2 and it is rotating at 600 rotations per minute and Q is at rest. If the two are joined by means of a clutch then they combined and rotate at 400 rotations per minute. The moment of inertia of Q will be -

(a) 3 kg-m^2 (b) 4 kg-m^2

(c) 5 kg-m^2 (d) 8 kg-m^2

COMBINED ROTATION AND TRANSLATION

46. If a spherical ball rolls on a table without slipping, the fraction of its total kinetic energy associated with rotation is

(a) 3/5 (b) 2/7

(c) 2/5 (d) 3/7

47. The speed of a homogeneous solid sphere after rolling down an inclined plane of vertical height h, from rest without sliding is

(a) $\sqrt{gh}$ (b) $\sqrt{(g/5)gh}$

(c) $\sqrt{(4/3)gh}$ (d) $\sqrt{(10/7)gh}$

48. The rotational kinetic energy of a body is E. In the absence of external torque, if mass of the body is halved and radius of gyration doubled, then its rotational kinetic energy will be:

(a) 0.5 E (b) 0.25E

(c) E (d) 2E

49. A ring is rolling without slipping. Its energy of translation is E. Its total kinetic energy will be :

(a) E (b) 2E

(c) 3E (d) 4E

50. One hollow and one solid cylinder of the same outer radius rolls down on a rough inclined plane. The foot of the inclined plane is reached by

(a) solid cylinder earlier

(b) hollow cylinder earlier

(c) simultaneously

(d) the heavier earlier irrespective of being solid or hollow

51. If a solid sphere, disc and cylinder are allowed to roll down an inclined plane from the same height

(a) cylinder will reach the bottom first

(b) disc will reach the bottom first

(c) sphere will reach the bottom first

(d) all will reach the bottom at the same time

52. A solid sphere rolls down without slipping two different inclined planes of the same height but of different inclinations

(*a*) in both cases the speeds and time of descend will be same

(*b*) the speeds will be same but time of descend will be different

(*c*) the speeds will be different but time of descend will be same

(*d*) speeds and time of descend both will be different

53. A solid homogeneous sphere is moving on a rough horizontal surface, partly rolling and partly sliding. During this kind of motion of the sphere

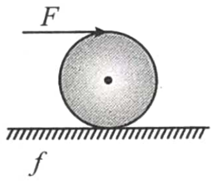
(*a*) total kinetic energy is conserved

(*b*) angular momentum of the sphere about the point of contact with the plane is conserved

(*c*) only the rotational kinetic energy about the centre of mass is conserved

(*d*) angular momentum about centre of mass is conversed

54. A ring of mass M is kept on a horizontal rough surface. A force F is applied tangentially at its rim as shown. The coefficient of friction between the ring and the surface is μ. Then



(*a*) friction will act in the forward direction

(*b*) friction will act in the backward direction

(*c*) frictional force will not act

(*d*) frictional force will be μ Mg.

55. A disc is rolling on an inclined plane without slipping then what fraction of its total energy will be in form of rotational kinetic energy:

(*a*) 1 : 3

(*b*) 1 : 2

(*c*) 2 : 7

(*d*) 2 : 5

56. A ring takes time t_1 and t_2 for sliding down and rolling down an inclined plane of length L respectively for reaching the bottom. The ratio of t_1 and t_2 is :-

(*a*) $\sqrt{2} : 1$

(*b*) $1 : \sqrt{2}$

(*c*) 1 : 2

(*d*) 2 : 1

57. A ladder rests against a frictionless vertical wall, with its upper end 6 m above the ground and the lower end 4 m away from the wall. The weight of the ladder is 500 N ad its C.G. at 1/3rd distance from the lower end. Wall's reaction will be, (in Newton)

(*a*) 111

(*b*) 333

(*c*) 222

(*d*) 129

Exercise-2 (Learning Plus)

1. A fan is running at 3000 rpm. It is switched off. It comes to rest by uniformly decreasing its angular speed in 10 seconds. The total number of revolutions in this period.

(*a*) 150 (*b*) 250

(*c*) 350 (*d*) 300

2. The moment of inertia of a non-uniform semicircular wire having mass m and radius r about a line perpendicular to the plane of the wire through the centre is

(*a*) mr^2 (*b*) $\frac{1}{2}mr^2$

(*c*) $\frac{1}{4}mr^2$ (*d*) $\frac{2}{5}mr^2$

3. Let I_A and I_B be the moments of inertia of two solid cylinders of identical geometrical shape and size about their axes, the first made of aluminium and the second of iron.

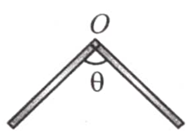
(*a*) $I_A < I_B$

(*b*) $I_A = I_B$

(*c*) $I_A > I_B$

(*d*) relation between I_A and I_B depends on the actual shapes of the bodies.

4. A uniform thin rod of length L and mass M is bent at the middle point O as shown in figure. Consider an axis passing through its middle point O and perpendicular to the plane of the bent rod. Then moment of inertia about this axis is :



(*a*) $\frac{2}{3}mL^2$

(*b*) $\frac{1}{3}mL^2$

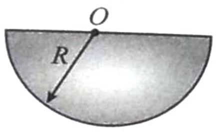
(*c*) $\frac{1}{12}mL^2$

(*d*) dependent on θ

5. Two spheres of same mass and radius are in contact with each other. If the moment of inertia of a sphere about its diameter is I, then the moment of inertia of both the spheres about the tangent at their common point would be -
   (a) $3I$ (b) $7I$
   (c) $4I$ (d) $5I$

6. The $M.I.$ of a disc about its diameter is 2 units. Its $M.I.$ about axis through a point on its rim and in the plane of the disc is
   (a) 4 units. (b) 6 units
   (c) 8 units (d) 10 units

7. Moment of inertia of a thin semicircular disc (mass = M & radius = R) about an axis through point O and perpendicular to plane of disc, is given by:



   (a) $\frac{1}{4}MR^2$ (b) $\frac{1}{2}MR^2$
   (c) $\frac{1}{8}MR^2$ (d) MR^2

8. A rigid body can be hinged about any point on the x-axis. When it is hinged such that the hinge is at x, the moment of inertia is given by $I = 2x^2 - 12x + 27$ The x-coordinate of centre of mass is
   (a) $x = 2$
   (b) $x = 0$
   (c) $x = 1$
   (d) $x = 3$

9. On applying a constant torque on a body-
   (a) linear velocity may be increases
   (b) angular velocity may be increases
   (c) it will rotate with constant angular velocity
   (d) it will move with constant velocity

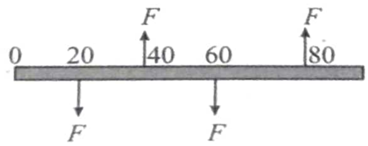
10. A wheel starting with angular velocity of 10 radian/sec acquires angular velocity of 100 radian/sec in 15 seconds. If moment of inertia is 10kg-m^2, then applied torque (in N-m) is
   (a) 900 (b) 100
   (c) 90 (d) 60

11. A torque of 2 N-m produces an angular acceleration of 2 rad/sec^2 a body. If its radius of gyration is 2m, its mass will be:
   (a) 2kg
   (b) 4 kg
   (c) 1/2 kg
   (d) 1/4 kg

12. A uniform circular disc A of radius r is made from a metal plate of thickness t and another uniform circular disc B of radius $4r$ is made from the same metal plate of thickness $t/4$. If equal torques act on the discs A and B, initially both being at rest. At a later instant, the angular speeds of a point on the rim of A and another point on the rim of B are ω_A and ω_B respectively. We have
   (a) $\omega_A > \omega_B$
   (b) $\omega_A = \omega_B$
   (c) $\omega_A < \omega_B$
   (d) the relation depends on the actual magnitude of the torques.

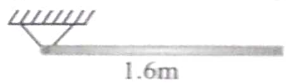
13. A force $\vec{F} = 4\hat{i} - 10\hat{j} =$ acts on a body at a point having position vector $-5\hat{i} - 3\hat{j}$ relative to origin of co-ordinates on the axis of rotation . The torque acting on the body about the origin is :
   (a) $38\,\hat{k}$ (b) $-25\,\hat{k}$
   (c) $62\,\hat{k}$ (d) none of these

14. Four equal and parallel forces are acting on a rod (as shown in figure) at distances of 20 cm, 40 cm, 60 cm and 80 cm respectively from one end of the rod. Under the influence of these forces the rod :



   (a) is at rest
   (b) experiences a torque
   (c) experiences a linear motion
   (d) experiences a torque and also a linear motion

15. The uniform rod of mass 20 kg and length 1.6 m is pivoted at its end and swings freely in the vertical plane. Angular acceleration of rod just after the rod is released from rest in the horizontal position as shown in figure is



   (a) $\frac{15g}{16}$ (b) $\frac{17g}{16}$
   (c) $\frac{16g}{15}$ (d) $\frac{g}{15}$

16. Two men support a uniform horizontal rod at its two ends. If one of them suddenly lets go, the force exerted by the rod on the other man will:
   (a) remain unaffected
   (b) increase
   (c) decrease
   (d) become unequal to the force exerted by him on the beam.

17. A cubical block of mass M and edge a slides down a rough inclined plane of inclination θ with a uniform velocity. The torque of the normal force on the block about its centre has a magnitude.
   (a) zero
   (b) Mga
   (c) $Mga \sin\theta$
   (d) $\frac{1}{2}Mga\sin\theta$

18. A uniform cube of side a and mass m rests on a rough horizontal table. A horizontal force 'F' is applied normal to one of the faces at a point that is directly above the centre of the face, at a height $\frac{3a}{4}$ above the base. The minimum value of 'F' for which the cube begins to tilt about the edge is (assume that the cube does not slide).

(*a*) $\frac{2}{3}mg$ (*b*) $\frac{4}{3}mg$

(*c*) $\frac{5}{4}mg$ (*d*) $\frac{1}{2}mg$

19. The moment of inertia and rotational kinetic energy of a fly wheel are 20kg-m^2 and 1000 joule respectively. Its angular frequency per minute would be -

(*a*) $\frac{600}{\pi}$ (*b*) $\frac{25}{\pi^2}$

(*c*) $\frac{5}{\pi}$ (*d*) $\frac{300}{\pi}$

20. A circular ring of wire of mass M and radius R is making n revolutions/sec about an axis passing through a point on its rim and perpendicular to its plane. The kinetic energy of rotation of the ring is given by-

(*a*) $4\pi^2MR^2n^2$ (*b*) $2\pi^2MR^2n^2$

(*c*) $\pi^2MR^2n^2$ (*d*) $8\pi^2MR^2n^2$

21. Rotational kinetic energy of a disc of constant moment of inertia is -

(*a*) directly proportional to angular velocity

(*b*) inversely proportional to angular velocity

(*c*) inversely proportional to square of angular velocity

(*d*) directly proportional to square of angular velocity

22. A constant torque acting on a uniform circular wheel changes its angular momentum from A_0 to $4A_0$ in 4 sec. the magnitude of this torque is :

(*a*) $4A_0$ (*b*) A_0

(*c*) $3A_0/4$ (*d*) $12A_0$

23. A particle moves with a constant velocity parallel to the Y-axis. Its angular momentum about the origin.

(*a*) is zero

(*b*) remains constant

(*c*) goes on increasing

(*d*) goes on decreasing

24. A boy sitting firmly over a rotating stool has his arms folded. If he stretches his arms, his angular momentum about the axis of rotation

(*a*) increases (*b*) decreases

(*c*) remains unchanged (*d*) doubles

25. The rotational kinetic energy of a rigid body of moment of inertia 5 kg-m^2 is 10 joules. The angular momentum about the axis of rotation would be -

(*a*) 100 joule-sec (*b*) 50 joule-sec

(*c*) 10 joule-sec (*d*) 2 joule -sec

26. The angular velocity of a body changes from one revolution per 9 second to 1 revolution per second without applying any torque. The ratio of its radius of gyration in the two cases is

(*a*) 1 : 9 (*b*) 3 : 1

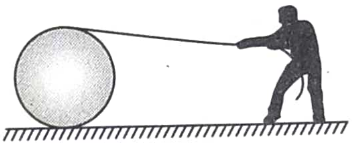
(*c*) 9 : 1 (*d*) 1 : 3

27. A disc rolls on a table. The ratio of its K.E. of rotation to the total K.E. is -

(*a*) 2/5 (*b*) 1/3

(*c*) 5/6 (*d*) 2/3

28. A thin string is wrapped several times around a cylinder kep on a rough horizontal surface. A boy standing at a distance from the cylinder draws the string towards him as shown i figure. The cylinder rolls without slipping. The length of th string passed through the hand of the boy while the cylinde reaches his hand is



(*a*) ℓ (*b*) 2ℓ

(*c*) 3ℓ (*d*) 4ℓ

29. A solid sphere, a hollow sphere and a ring, all having equ mass and radius, are placed at the top of an incline ar released. The friction coefficients between the objects and t incline are equal and but not sufficient to allow pure rollin The greatest kinetic energy at the bottom of the incline w be achieved by

(*a*) the solid sphere

(*b*) the hollow sphere

(*c*) the ring

(*d*) all will achieve same kinetic energy.

30. A body is given translational velocity and kept on a surfa that has sufficient friction. Then:

(*a*) body will move forward before pure rolling

(*b*) body will move backward before pure rolling

(c) body will start pure rolling immediately

(*d*) none of these

31. A disc and a ring of the same mass are rolling to have same kinetic energy. What is ratio of their velocities of cen of mass

(*a*) $(4:3)^{1/2}$

(*b*) $(3:4)^{1/2}$

(*c*) $(2:3)^{1/2}$

(*d*) $(3:2)^{1/2}$

32. A body kept on a smooth horizontal surface is pulled b constant horizontal force applied at the top point of the bo If the body rolls purely on the surface, its shape can be

(*a*) thin pipe

(*b*) uniform cylinder

(*c*) uniform sphere

(*d*) thin spherical shell

33. A solid sphere with a velocity (of centre of mass) v and angular velocity ω is gently placed on a rough horizontal surface. The frictional force on the sphere:

(*a*) must be forward (in direction of v)

(*b*) must be backward (opposite to v)

(*c*) cannot be zero

(*d*) none of the above

34. A cylinder is pure rolling up an incline plane. It stops momentarily and then rolls back. The force of friction.

(*a*) on the cylinder is zero throughout the journey

(*b*) is directed opposite to the velocity of the centre of mass throughout the journey

(*c*) is directed up the plane throughout the journey

(*d*) is directed down the plane throughout the journey

35. A sphere is released on a smooth inclined plane from the top. When it moves down its angular momentum is:

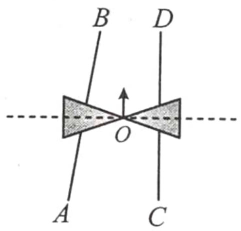
(*a*) Conserved about every point

(*b*) Conserved about the point of contact only

(*c*) Conserved about the centre of the sphere only

(*d*) Conserved about any point on a fixed line parallel to the inclined plane and passing through the centre of the ball.

36. A roller is made by joining together two cones at their vertices O. It is kept on two ails AB and CD which are placed asymmetrically (see figure), with its axis perpendicular to CD and its centre O at the centre of line joining AB and CD (see figure). It is given a light push so that it starts rolling with its centre O moving parallel to CD in the direction shown. As it moves, the roller will tend to :



(*a*) turn left and right alternately

(*b*) turn left.

(*c*) turn right.

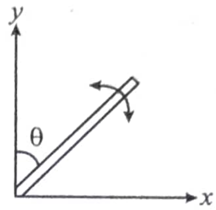
(*d*) go straight.

37. The moment of inertia of a uniform cylinder of length ℓ and radius R about its perpendicular bisector is I. What is the ratio ℓ/R such that the moment of inertia is minimum?

(*a*) 1 (*b*) $\frac{3}{\sqrt{2}}$

(c) $\sqrt{\frac{3}{2}}$ (*d*) $\frac{\sqrt{3}}{2}$

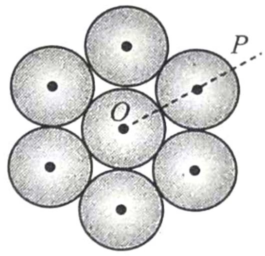
38. A slender uniform rod of mass M and length ℓ is pivoted at one end so that it can rotate in a vertical plane (see figure). There is negligible friction at the pivot. The free end is held vertically above the pivot and then released. The angular acceleration of the rod when it makes an angle θ with the vertical is:



(*a*) $\frac{3g}{2\ell}\cos\theta$ (*b*) $\frac{2g}{3\ell}\cos\theta$

(*c*) $\frac{3g}{2\ell}\sin\theta$ (*d*) $\frac{2g}{3\ell}\sin\theta$

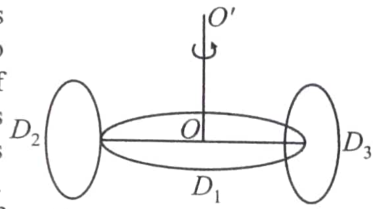
39. Seven identical circular planar disks, each of mass M and radius R are welded symmetrically as shown. The moment of inertia of the arrangement about the axis normal to the plane and passing through the point P is:



(*a*) $\frac{55}{2}MR^2$ (*b*) $\frac{73}{2}MR^2$

(*c*) $\frac{181}{2}MR^2$ (*d*) $\frac{19}{2}MR^2$

40. A circular disc D_1 of mass M and radius R has two identical discs D_2 and D_3 of the same mass M and radius R attached rigidly as its opposite ends (see figure). The moment of inertia of the system about the axis OO', passing through the centre of D_1 as shown in the figure, will:



(*a*) MR^2 (*b*) $3MR^2$

(*c*) $\frac{4}{5}MR^2$ (*d*) $\frac{2}{3}MR^2$

41. The magnitude of torque on a particle of mass 1 kg is 2.5 Nm about the origin. If the force acting on it is 1N, and the distance of the particle from the origin is 5m, the angle between the force and the position vector is (in radians):

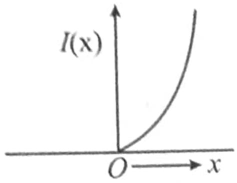
(*a*) $\frac{\pi}{6}$ (*b*) $\frac{\pi}{3}$ (*c*) $\frac{\pi}{8}$ (*d*) $\frac{\pi}{4}$

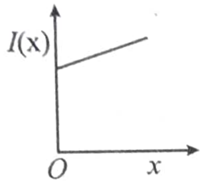
42. A simple pendulum, made of a string of length ℓ and a bob of mass m, is released from a small angle θ_0. It strikes a block of mass M, kept on a horizontal surface at its lowest point of oscillations, elastically. It bounces back and goes up to an angle θ_1. Then M is given by:

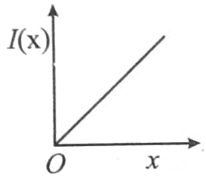
(*a*) $\frac{m}{2}\left(\frac{\theta_0+\theta_1}{\theta_0-\theta_1}\right)$ (*b*) $m\left(\frac{\theta_0-\theta_1}{\theta_0+\theta_1}\right)$

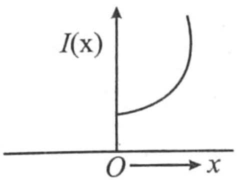
(*c*) $m\left(\frac{\theta_0+\theta_1}{\theta_0-\theta_1}\right)$ (*d*) $\frac{m}{2}\left(\frac{\theta_0-\theta_1}{\theta_0+\theta_1}\right)$

43. The moment of inertia of a solid sphere, about an axis parallel to its diameter and at a distance of x from it, is '$I(x)$'. Which one of the graphs represents the variation of $I(x)$ with x correctly?

(a)


(b)


(c)


(d)


44. A stationary horizontal disc is free to rotate about its axis. When a torque is applied on it, its kinetic energy as a function of θ, where θ is the angle by which it has rotated, is given as $k\theta^2$. If its moment of inertia is I then the angular acceleration of the disc is:

(a) $\frac{k}{2I}\theta$

(b) $\frac{k}{I}\theta$

(c) $\frac{k}{4I}\theta$

(d) $\frac{2k}{I}\theta$

45. The radius of gyration of a uniform rod of length ℓ, about an axis passing through a point $l/4$ away from the centre of the rod, and perpendicular to it, is

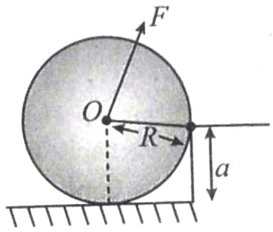
(a) $\sqrt{\frac{7}{48}}\ell$

(b) $\frac{1}{8}\ell$

(c) $\sqrt{\frac{3}{8}}\ell$

(d) $\frac{1}{4}\ell$

46. A uniform cylinder of mass M and radius R is to be pulled over a step of height a ($a < R$) by applying a force F at its centre 'O' perpendicular to the plane through the axes of the cylinder on the edge of the step (see figure). The minimum value of F required is



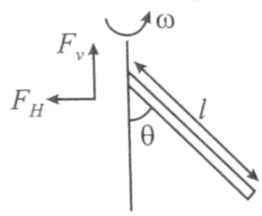
(a) $Mg\sqrt{1-\left(\frac{R-a}{R}\right)^2}$

(b) $Mg\sqrt{\left(\frac{R}{R-a}\right)^2-1}$

(c) $Mg\sqrt{1-\frac{a^2}{R^2}}$

(d) $Mg\frac{a}{R}$

47. A uniform rod of length 'l' is pivoted at one of its ends o[n a] vertical shaft of negligible radius. When the shaft rotate[s at] angular speed ω the rod makes an angle θ with it (see figu[re]). To find θ equate the rate of change of angular moment[um] (direction going into the paper) about the centre of m[ass] (CM) to the torque provided by the horizontal and vert[ical] forces F_H and F_V about the CM. The value of θ is then s[uch] that



(a) $\cos\theta = \frac{g}{l\omega^2}$

(b) $\cos\theta = \frac{2g}{3l\omega^2}$

(c) $\cos\theta = \frac{g}{2l\omega^2}$

(d) $\cos\theta = \frac{3g}{2l\omega^2}$

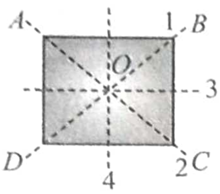
48. A person of 80 kg mass is standing on the rim [of a] circular platform of mass 200 kg rotating about its [axis] at 5 revolutions per minute (rpm). The person now s[tarts] moving towards the centre of the platform. What will b[e the] rotational speed (in rpm) of the platform when the pe[rson] reaches its centre ________.

49. A circular disc of mass M and radius R is rotating abo[ut its] axis with angular speed ω_1. If another stationary disc ha[ving] radius $\frac{R}{2}$ and same mass M is dropped co-axially on t[o the] rotating disc. Gradually both discs attain constant an[gular] speed ω_2. The energy lost in the process is $p\%$ of the i[nitial] energy. Value of p is _____.

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. The moment of inertia of a thin uniform square plate $ABCD$ of uniform thickness about an axis passing through the centre O and perpendicular to the plate is



where I_1, I_2, I_3, and I_4 are respectively the moments of inertia about axes 1, 2, 3, and 4 which are in the plane of the plate.

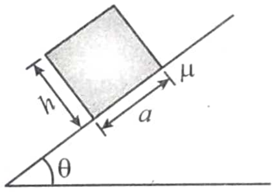
(a) $I_1 + I_2$

(b) $I_3 + I_4$

(c) $I_1 + I_3$

(d) $I_1 + I_2 + I_3 + I_4$

2. A block with a square base measuring $a \times a$ and height h, is placed on an inclined plane. The coefficient of friction is μ. The angle of inclination (θ) of the plane is gradually increased. The block will:



(a) Topple before sliding if $\mu > a/h$

(b) Topple before sliding if $\mu < a/h$

(c) Slide before toppling if $\mu > a/h$

(d) Slide before toppling if $\mu < a/h$

3. A rod of weight w is supported by two parallel knife edges A and B and is in equilibrium in a horizontal position. The knives are at a distance d from each other. The centre of mass of the rod is at a distance x from A.

(a) The normal reaction at A is $\frac{wx}{d}$

(b) The normal reaction at A is $\frac{w(d-x)}{d}$

(c) The normal reaction at B is $\frac{wx}{d}$

(d) The normal reaction at B is $\frac{w(d-x)}{d}$

4. A body is in equilibrium under the influence of a number of forces. Each force has a different line of action. The minimum number of forces required is

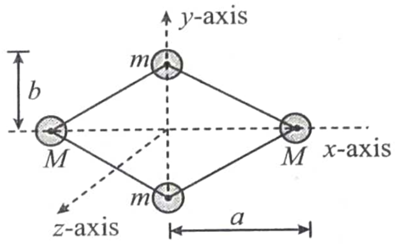
(a) 2, if their lines of action pass through the centre of mass of the body

(b) 3, if their lines of action are not parallel

(c) 3, if their lines of action are parallel

(d) 4, if their lines of action are parallel and all the forces have the same magnitude

5. Four point masses are fastened to the corners of a frame of negligible mass lying in the xy plane. Let ω be the angular speed of rotation. Then



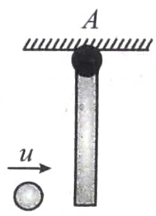
(a) Rotational kinetic energy associated with a given angular speed depends on the axis of rotation.

(b) Rotational kinetic energy about y-axis is independent of m and its value is $Ma^2\omega^2$

(c) Rotational kinetic energy about z-axis depends on m and its value is $(Ma^2 + mb^2)\omega^2$

(d) Rotational kinetic energy about z-axis is independent of m and its value is $Mb^2\omega^2$

6. In the given figure a ball strikes a rod elastically and rod is hinged at point A. Then which of the statement(s) is/are correct for the collision?



(a) Linear momentum of system (ball + rod) is conserved

(b) Angular momentum of system about hinged point A is conserved

(c) Initial KE of the system is equal to final KE of the system

(d) Linear momentum of ball is conserved.

7. A particle falls freely near the surface of the earth. Consider a fixed point O (not vertically below the particle) on the ground.

(a) Angular momentum of the particle about O is increasing

(b) Torque of the gravitational force on the particle about O is decreasing

(c) The moment of inertia of the particle about O is decreasing

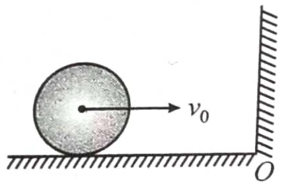
(d) The angular velocity of the particle about O is increasing

8. A man spinning in free space changes the shape of his body, eg. by spreading his arms or curling up. By doing this, he can change his
(a) Moment of inertia
(b) Angular momentum
(c) Angular velocity
(d) Rotational kinetic energy

9. When a bicycle is in motion (accelerating) on a rough horizontal plane, the force of friction exerted by the plane on the two wheels is such that it acts :
(a) In the backward direction on the front wheel and in the forward direction on the rear wheel
(b) In the forward direction on the front wheel and in the backward direction on the rear wheel
(c) In the backward direction on both front and the rear wheels
(d) In the forward direction on both the front and the rear wheels

10. A ring rolls without slipping on the ground. Its centre C moves with a constant speed u. P is any point on the ring. The speed of P with respect to the ground is v.
(a) $0 \le v \le 2u$
(b) $v = u$, if CP is horizontal
(c) $v = u$, if CP makes an angle of 30° with the horizontal and P is below the horizontal level of C
(d) $v = \sqrt{2}u$, if CP is horizontal

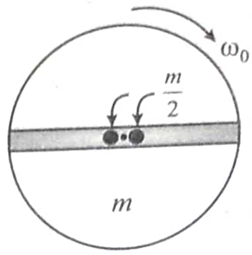
11. Consider a sphere of mass 'm' radius 'R' doing pure rolling motion on a rough surface having velocity $\vec{v}_0$ as shown in the figure. It makes an elastic impact with the smooth wall and moves back and starts pure rolling after some time again.



(a) Change in angular momentum about 'O' in the entire motion equals $2mv_0R$ in magnitude.
(b) Moment of impulse provided by wall during impact about O equals $2mv_0R$ in magnitude
(c) Final velocity of ball will be $\frac{3}{7}\vec{v}_0$
(d) Final velocity of ball will be $-\frac{3}{7}\vec{v}_0$

COMPREHENSION BASED QUESTIONS

Comprehension (Q. 12 to 14): A uniform disc of mass 'm' and radius R is free to rotate in horizontal plane about a vertical smooth fixed axis passing through its centre. There is a smooth groove along the diameter of the disc and two small balls of mass $\frac{m}{2}$ each are placed in it on either side of the centre of the disc as shown in figure. The disc is given initial angular velocity ω_0 and released.



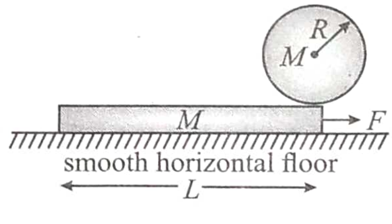
12. The angular speed of the disc when the balls reach the end of the disc is :
(a) $\frac{\omega_0}{2}$
(b) $\frac{\omega_0}{3}$
(c) $\frac{2\omega_0}{3}$
(d) $\frac{\omega_0}{4}$

13. The speed of each ball relative to ground just after they leave the disc is :
(a) $\frac{R\omega_0}{\sqrt{3}}$
(b) $\frac{R\omega_0}{\sqrt{2}}$
(c) $\frac{2R\omega_0}{3}$
(d) $\frac{R\omega_0}{3}$

14. The net work done by forces exerted by disc on one of the ball for the duration ball remains on the disc is
(a) $\frac{2mR^2\omega_0^2}{9}$
(b) $\frac{mR^2\omega_0^2}{18}$
(c) $\frac{mR^2\omega_0^2}{6}$
(d) $\frac{mR^2\omega_0^2}{9}$

Comprehension (Q. 15 to 17): A uniform disc of mass M and radius R initially stands vertically on the right end of a horizontal plank of mass M and length L, as shown in the figure.

The plank rests on smooth horizontal floor and friction between disc and plank is sufficiently high such that disc rolls on plank without slipping. The plank is pulled to right with a constant horizontal force of magnitude F.



15. Th magnitude of acceleration of plank is-
(a) $\frac{F}{8M}$
(b) $\frac{F}{4M}$
(c) $\frac{3F}{2M}$
(d) $\frac{3F}{4M}$

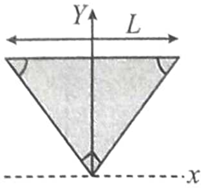
16. The magnitude of angular acceleration of the disc is -
(a) $\frac{F}{4mR}$
(b) $\frac{F}{8mR}$
(c) $\frac{F}{2mR}$
(d) $\frac{3F}{2mR}$

17. The distance travelled by centre of disc from its initial position till the left end of plank comes vertically below the centre of disc is

(a) $\frac{L}{2}$ (b) $\frac{L}{4}$

(c) $\frac{L}{8}$ (d) L

Comprehension (Q. 18 to 21): The figure shows an isosceles triangular plate of mass M and base L. The angle at the apex is $90°$. The apex lies at the origin and the base is parallel to x-axis.



18. The moment of inertia of the plate about the z-axis is

(a) $\frac{ML^2}{12}$ (b) $\frac{ML^2}{24}$

(c) $\frac{ML^2}{6}$ (d) none of these

19. The moment of inertia of the plate about the x-axis is

(a) $\frac{ML^2}{8}$ (b) $\frac{ML^2}{32}$

(c) $\frac{ML^2}{24}$ (d) $\frac{ML^2}{6}$

20. The moment of inertia of the plate about its base parallel to the x-axis is

(a) $\frac{ML^2}{18}$ (b) $\frac{ML^2}{36}$

(c) $\frac{ML^2}{24}$ (d) none of these

21. The moment of inertia of the plate about the y-axis is

(a) $\frac{ML^2}{6}$

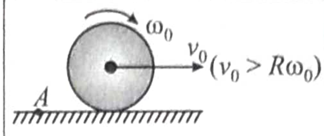
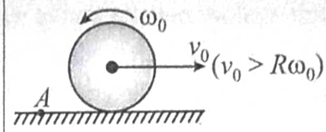
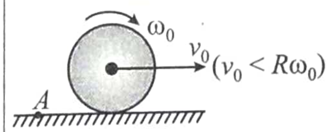
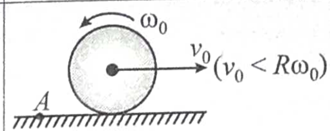
(b) $\frac{ML^2}{8}$

(c) $\frac{ML^2}{24}$

(d) none of these

MATCH THE COLUMN TYPE QUESTIONS

22. In each situation of column-I, a uniform disc of mass m and radius R rolls on a rough fixed horizontal surface as shown in the figure. At $t = 0$ (initially) the angular velocity of disc is ω_o and velocity of centre of mass of disc is v_o (in horizontal direction). The relation between v_o and ω_o for each situation and also initial sense of rotation is given for each situation in column-I. Then match the statements in column-I with the corresponding results in column-II.

	Column-I		Column-II
A.		p.	The angular momentum of disc about point A (as shown in figure) remains conserved.
B.		q.	The kinetic energy of disc after it starts rolling without slipping is less than its initial kinetic energy.
C.		r.	In the duration disc rolls with slipping, the friction acts on disc towards left.
D.		s.	In the duration disc rolls with slipping, the friction acts on disc for some time towards right and for some time towards left.

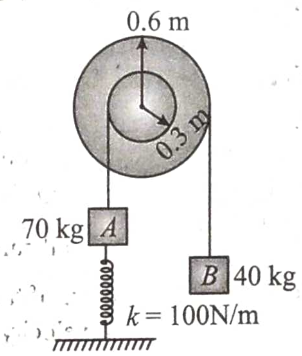
(a) A → (p,r); B → (p,r); C → (p); D → (q,r)
(b) A → (p,q,r); B → (p,q,r); C → (p,q); D → (p,q,r)
(c) A → (q,r); B → (q,r); C → (q); D → (q,r)
(d) A → (p,q); B → (p,q); C → (q); D → (p,q,r)

NUMERICAL TYPE QUESTIONS

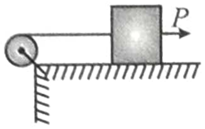
23. A merry-go-round is a common piece of playground equipment. A 4 m diameter merry-go-round with a moment of inertia of 500 kg-m^2 is spinning at 5.6 rad/s. John runs tangent to the merry-go-round at 5 m/s, in the same direction that it is turning, and jumps onto the outer edge. John's mass is 30 kg. What is the merry-go-round's angular velocity (in rad/s) after John jumps on ?

24. A solid hemisphere rests in equilibrium on a rough ground and against a smooth wall. The curved surface touches the wall and the ground. The angle of inclination of the circular base to the horizontal is 30°, Find minimum coefficient of friction required between ground and hemisphere. Quote 32μ in Answer.

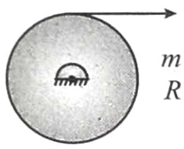
25. The figure shown is in equilibrium. Find out the extension in the spring (in cm).



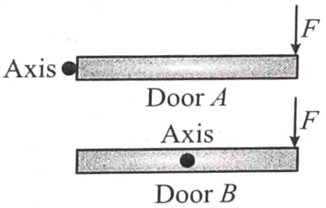
26. A wheel of radius 0.20 m is mounted on a frictionless horizontal axis. The rotational inertia of the wheel about the axis is 0.07 kg m^2. A massless cord wrapped around the wheel is attached to a 2.0 kg block that slides on a horizontal frictionless surface. If a horizontal force of magnitude $P = 3.0$ N is applied to the block as shown in figure, what is the magnitude of the angular acceleration (in rad/s^2) of the wheel? Assume the string does not slip on the wheel.



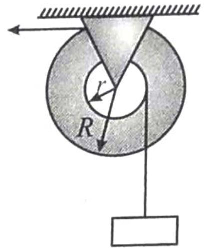
27. A force of 100 N is applied on a disc along the tangent for 10 sec. During this time, the disc attains an angular velocity of 10 rad/s. There is constant frictional torque at the axis which opposes the rotation of the disc. Now the force is removed. After what time (in sec) from the removal of force will the disc come to rest?



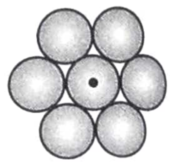
28. The drawing shows the top view of two doors. The doors are uniform and identical. Door A rotates about an axis through its left edge, while door B rotates about an axis through the center. The same force F is applied perpendicular to each door at its right edge, and the force remains perpendicular as the door turns. Starting from rest, door A rotates through a certain angle in 3 s. How long does it take door B to rotate through the same angle? Round off to nearest integer.



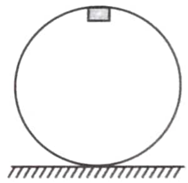
29. A yo-yo-shaped device mounted on a horizontal frictionles axis is used to lift a 30 kg box as shown in figure. The oute radius R of the device is 0.50 m, and the radius r of the hub i 0.20 m. When a constant horizontal force of magnitude 152 N is applied in the left direction to a rope wrapped around the outside of the device, the box, which is suspended from a rope wrapped around the hub, has an upward acceleration of magnitude 0.80 m/s^2. What is the rotational inertia I (in kg-m^2) of the device about its axis of rotation?



30. Seven pennies are arranged in a hexagonal, planar pattern so as to touch each neighbor, as shown in the figure below. Each penny is a uniform disk of mass $m = 2$ kg and radius $r = 1$ m. What is the moment of inertia of the system (in kg-m^2) of seven pennies about an axis that passes through the center of the central penny and is normal to the plane of the pennies ?



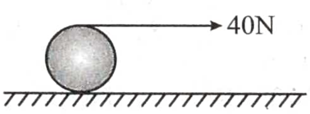
31. A ring of mass $m = 1$ kg and radius $R = 1.25$ m is kept on a rough horizontal ground. A small body of mass m is struck to the top of the ring. When it was given a slight push forward, the ring started rolling purely on the ground. What is the maximum speed of the centre of the ring (in m/s)?



Exercise-4 (Past Year Questions)

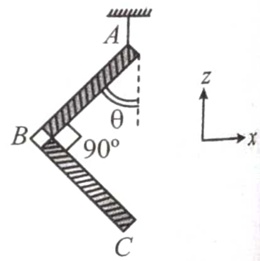
JEE MAIN

1. A string is wound around a hollow cylinder of mass 5 kg and radius 0.5m. If the string is now pulled with a horizontal force of 40 N. and the cylinder is rolling without slipping on a horizontal surface (see figure), then the angular acceleration of the cylinder will be Neglect th mass an thickness of the string) **(2019)**



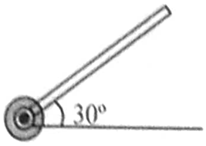
(a) 20 rad/s^2 (b) 16 rad/s^2
(c) 12 rad/s^2 (d) 10 rad/s^2

2. An L-shaped object, made of thin rods of uniform mass density, is suspended with a string as shown in figure. If $AB = BC$, and the angle made by AB with downward vertical is θ, then: **(2019)**



(a) $\tan\theta = \dfrac{1}{2\sqrt{3}}$ (b) $\tan\theta = \dfrac{1}{2}$
(c) $\tan\theta = \dfrac{2}{\sqrt{3}}$ (d) $\tan\theta = \dfrac{1}{3}$

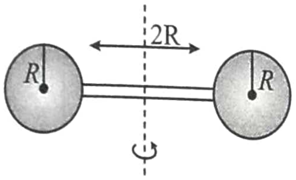
3. A rod of length 50 cm is pivoted at one end. It is raised such that if makes an angle of 30° for the horizontal as shown and released from rest. Its angular speed when it passes through the horizontal (in rad s^{-1}) will be ($g = 10\ ms^{-2}$). **(2019)**



(*a*) $\sqrt{\frac{30}{2}}$ (*b*) $\sqrt{30}$

(*c*) $\sqrt{\frac{20}{2}}$ (*d*) $\sqrt{\frac{30}{2}}$

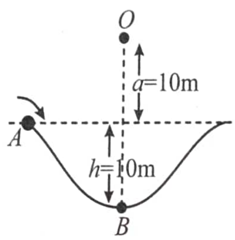
4. Two identical spherical balls of mass M and radius R each are stuck on two ends of a rod of length $2R$ and mass M (see figure). The moment of inertia of the system about the axis passing perpendicularly through the centre of the rod is: **(2019)**



(*a*) $\frac{137}{15}MR^2$ (*b*) $\frac{17}{15}MR^2$

(*c*) $\frac{209}{15}MR^2$ (*d*) $\frac{152}{15}MR^2$

5. A particle of mass 20 g is released with an initial velocity 5 m/s along the curve from the point A, as shown in the figure. The point A is at height h from point B. The particle slides along the frictionless surface. When the particle reaches point B, its angular momentum about O will be : [Take $g = 10\ m/s^2$] **(2019)**



(*a*) 2 kg-m²/s (*b*) 8 kg-m²/s

(*c*) 6 kg-m²/s (*d*) 3 kg-m²/s

6. A homogeneous solid cylindrical roller of radius R and mass M is pulled on a cricket pitch by a horizontal force. Assuming rolling without slipping, angular acceleration of the cylinder is : **(2019)**

(*a*) $\frac{3F}{2mR}$ (*b*) $\frac{F}{3mR}$

(*c*) $\frac{F}{2mR}$ (*d*) $\frac{2F}{3mR}$

7. To mop–clean a floor, a cleaning machine presses a circular mop of radius R vertically down with a total force F and rotates it with a constant angular speed about its axis. If the force F is distributed uniformly over the mop and if the coefficient of friction between the map and the floor is μ, the torque applied by the machine on the mop is: **(2019)**

(*a*) $\mu FR/3$ (*b*) $\mu FR/6$

(*c*) $\mu FR/2$ (*d*) $\frac{2}{3}\mu FR$

8. A thin smooth rod of length L and mass M is rotating freely with angular speed ω_0 about an axis perpendicular to the rod and passing through its center. Two beads of mass m and negligible size are at the center of the rod initially. The beads are free to slide along the rod. The angular speed of the system, when the beads reach the opposite ends of the rod, will be :- **(2019)**

(*a*) $\frac{M\omega_0}{M+3m}$ (*b*) $\frac{M\omega_0}{M+m}$

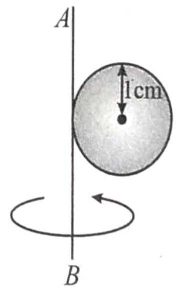
(*c*) $\frac{M\omega_0}{M+2m}$ (*d*) $\frac{M\omega_0}{M+6m}$

9. A thin circular plate of mass M and radius R has its density varying as $\rho(r) = \rho_0 r$ with ρ_0 as constant and r is the distance from its centre. The moment of inertia of the circular plate about an axis perpendicular to the plate and passing through its edge is $I = aMR^2$. The value of the coefficient a is : **(2019)**

(*a*) $\frac{3}{2}$ (*b*) $\frac{1}{2}$

(*c*) $\frac{3}{5}$ (*d*) $\frac{8}{5}$

10. A metal coin of mass 5 g and radius 1 cm is fixed to a thin stick AB of negligible mass as shown in the figure. The system is initially at rest. The constant torque, that will make the system rotate about AB at 25 rotations per second in 5 s, is close to: **(2019)**



(*a*) 4.0×10^{-6}Nm

(*b*) 2.0×10^{-5}Nm

(*c*) 1.6×10^{-5}N

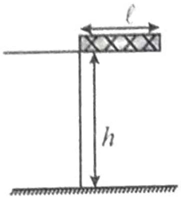
(*d*) 7.9×10^{-6}Nm

11. A solid sphere of mass M and radius R is divided into two unequal parts. The first part has a mass of $\frac{7M}{8}$ and is converted into a uniform disc of radius 2R. The second part is converted into a uniform solid sphere. Let I_1 be the moment of inertia of the disc about its axis and I_2 be the moment of inertia of the new sphere about its axis. The ratio I_1/I_2 is given by : **(2019)**

(*a*) 185 (*b*) 65

(*c*) 285 (*d*) 140

12. A rectangular solid box of length 0.3 mis held horizontally, with one of its sides on the edge of a platform of height 5m. When released, it slips off the table in a very short time $\tau = 0.01s$, remaining essentially horizontal. The angle by which it would rotate when it hits the ground will be (in radians) close to: **(2019)**

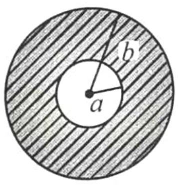


(a) 0.02 (b) 0.28
(c) 0.5 (d) 0.3

13. The following bodies are made to roll up (without slipping) the same inclined plane from a horizontal plane : (i) a ring of radius R, (ii) a solid cylinder of radius $\frac{R}{2}$ and (iii) a solid sphere of radius $\frac{R}{4}$. If in each case, the speed of the centre of mass at the bottom of the incline is same, the ratio of the maximum height they climb is : **(2019)**

(a) 4 : 3 : 2 (b) 20 : 15 : 14
(c) 10 : 15 : 7 (d) 2 : 3 : 4

14. A circular disc of radius b has a hole of radius a at its centre (see figure). If the mass per unit area of the disc varies as $\rho = \rho_0/r$ then the radius of gyration of the disc about its axis passing through the centre is : **(2019)**



(a) $\frac{a+b}{2}$

(b) $\frac{a+b}{3}$

(c) $\sqrt{\frac{a^2+b^2+ab}{2}}$

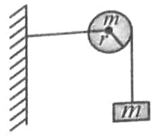
(d) $\sqrt{\frac{a^2+b^2+ab}{3}}$

15. Two coaxial discs, having moments of inertia I_1 and $\frac{I_1}{2}$, are rotating with respective angular velocities ω_1 and $\frac{\omega_1}{2}$, about their common axis. They are brought in contact with each other and thereafter they rotate with a common angular velocity. If E_f and E_i are the final and initial total energies, then $(E_f - E_i)$ is: **(2019)**

(a) $\frac{I_1\omega_1^2}{12}$ (b) $\frac{3}{8}I_1\omega_1^2$

(c) $\frac{I_1\omega_1^2}{6}$ (d) $\frac{I_1\omega_1^2}{24}$

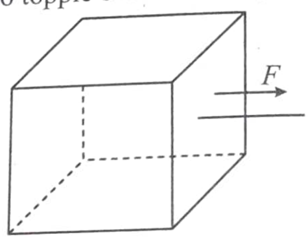
16. As shown in the figure, a bob a mass m is tied by a massless string whose other end portion is wound on a fly wheel (disc) of radius r and mass m. When released from rest, the bob starts falling vertically. When it has covered a distance of h, the angular speed of the wheel will be : **(2020)**



(a) $r\sqrt{\frac{3}{4gh}}$ (b) $\frac{1}{r}\sqrt{\frac{2gh}{3}}$

(c) $r\sqrt{\frac{3}{2gh}}$ (d) $\frac{1}{r}\sqrt{\frac{4gh}{3}}$

17. Consider uniform cubical box of side a on a rough floor that is to be moved by applying minimum possible force F at a point b above its centre of mass (see figure). If the coefficient of friction is $\mu = 0.4$, the maximum possible value of $100 \times \frac{b}{a}$ for box not to topple before moving is **(2020)**



18. Mass per unit area of a circular disc of radius a depends on the distance r from its centre as $\sigma(r) = A + Br$. The moment of inertia of the disc about the axis, perpendicular to the plane passing through its centre is: **(2020)**

(a) $2\pi a^4\left(\frac{aA}{4}+\frac{B}{5}\right)$

(b) $2\pi a^4\left(\frac{A}{4}+\frac{aB}{5}\right)$

(c) $\pi a^4\left(\frac{A}{4}+\frac{aB}{5}\right)$

(d) $2\pi a^4\left(\frac{A}{4}+\frac{B}{5}\right)$

19. Consider a uniform rod of mass $M = 4m$ and length ℓ pivoted about its centre. A mass m moving with velocity v making angle $\theta = \frac{\pi}{4}$ to the rod's long axis collides with one end of the rod and sticks to it. The angular speed of the rod-mass system just after the collision is: **(2020)**

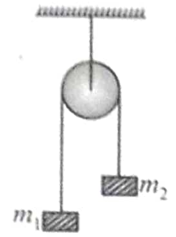
(a) $\frac{4}{7}\frac{v}{\ell}$

(b) $\frac{3\sqrt{2}}{7}\frac{v}{\ell}$

(c) $\frac{3}{7}\frac{v}{\ell}$

(d) $\frac{3}{7\sqrt{2}}\frac{v}{\ell}$

20. A uniformly thick wheel with moment of inertial I and radius R is free to rotate about its centre of mass (see figure). A massless string is wrapped over its rim and two blocks of masses m_1 and m_2 $(m_1 > m_2)$ are attached to the ends of the string. The system is released from rest. The angular speed of the wheel when m_1 descents by a distance h is: **(2020)**



(a) $\left[\dfrac{m_1+m_2}{(m_1+m_2)R^2+I}\right]^{\frac{1}{2}} gh$

(b) $\left[\dfrac{2(m_1-m_2)gh}{(m_1+m_2)R^2+I}\right]^{\frac{1}{2}}$

(c) $\left[\dfrac{2(m_1+m_2)gh}{(m_1+m_2)R^2+I}\right]^{\frac{1}{2}}$

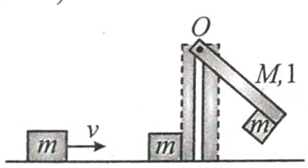
(d) $\left[\dfrac{(m_1+m_2)}{(m_1+m_2)R^2+I}\right]^{\frac{1}{2}} gh$

21. Two uniform circular discs are rotating independently in the same direction around their common axis passing through their centres. The moment of inertia and angular velocity of the first disc are 0.1 kg-m^2 and 10 rad s^{-1} respectively while those for the second one are 0.2 kg-m^2 and 5 rad s^{-1} respectively. At some instant they get stuck together and start rotating as a single system about their common axis with some angular speed. The kinetic energy of the combined system is **(2020)**

(a) $\frac{20}{3}$J (b) $\frac{5}{3}$J

(c) $\frac{10}{3}$J (d) $\frac{2}{3}$J

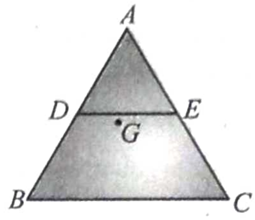
22. A block of mass $m = 1$ kg slides with velocity $v = 6$ m/s on a frictionless horizontal surface and collides with a uniform vertical rod and sticks to it as shown. The rod is pivoted about O and swings as a result of the collision making angle θ before momentarily coming to rest. If the rod has mass $M = 2$ kg and length $l = 1$ m, the value of θ is approximately (take $g = 10$ m/s^2) **(2020)**



(a) 49° (b) 55°

(c) 63° (d) 69°

23. ABC is a plane lamina of the shape of an equilateral triangle. D, E are mid points of AB, AC and G is the centroid of the lamina. Moment of inertia of the lamina about an axis passing through G and perpendicular to the plane ABC is I_0. If part ADE is removed, the moment of inertia of the remaining part about the same axis is $\frac{NI_0}{16}$ where N is an integer. Value of N is ______. **(2020)**

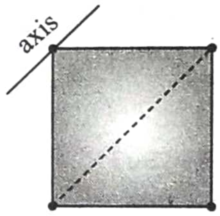


24. A wheel is rotating freely with an angular speed ω on a shaft. The moment of inertia of the wheel is I and the moment of inertia of the shaft is negligible. Another wheel of moment of inertia $3I$ initially at rest is suddenly coupled to the same shaft. The resultant fractional loss in the kinetic energy of the system is **(2020)**

(a) $\frac{5}{6}$ (b) $\frac{1}{4}$

(c) 0 (d) $\frac{3}{4}$

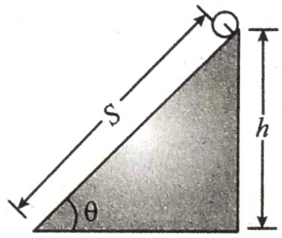
25. Four point masses, each of mass m, are fixed at the corners of a square of side ℓ. The square is rotating with angular frequency ω, about an axis passing through one of the corners of the square and parallel to its diagonal, as shown in the figure. The angular momentum of the square about this axis is **(2020)**



(a) $3\,m\ell^2\omega$ (b) $4\,m\ell^2\omega$

(c) $m\ell^2\omega$ (d) $2\,m\ell^2\omega$

26. The following bodies (1) Ring, (2) disc, (3) solid sphere, (4) Solid cylinder of same mass 'm' and radius 'R' are allowed to roll down without slipping simultaneously from the top of the inclined plane. The body which will reach first at the bottom of the inclined plane is [Mark the body as per their respective numbering given in the question] **(2021)**

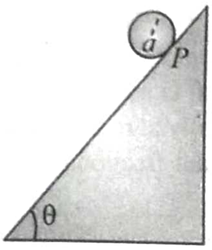


(a) a ring (b) a disc

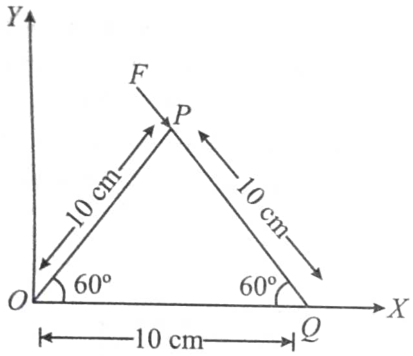
(c) a solid cylinder (d) a solid sphere,

27. The angular speed of truck wheel is increased from 900 rpm to 2460 rpm in 26 seconds. The number of revolutions by the truck engine during this time is (Assuming the acceleration to be uniform). **(2021)**

28. A solid disc of radius 'a' and mass 'm' rolls down without slipping on an inclined plane making an angle θ with the horizontal. The acceleration of the disc will be $\frac{2}{b} g \sin\theta$ where b is (Round off to the Nearest Integer) (g = acceleration due to gravity, θ = angle as shown in figure) **(2021)**



29. A triangular plate is shown. A force $\vec{F} = 4\hat{i} - 3\hat{j}$ is applied at point P. The torque at point P with respect to point 'O' and 'Q' are :



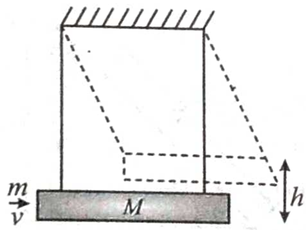
(a) $-15+20\sqrt{3}, 15+20\sqrt{3}$

(b) $15-20\sqrt{3}, 15+20\sqrt{3}$

(c) $15+20\sqrt{3}, 15-20\sqrt{3}$

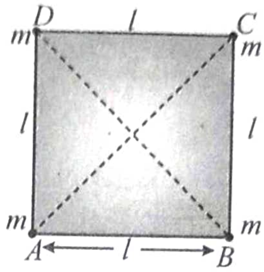
(d) $-15-20\sqrt{3}, 15-20\sqrt{3}$

30. A large block of wood of mass $M = 5.99$ kg is hanging from two long massless cords. A bullet of mass $m = 10$ g is fired into the block and gets embedded in it. The (block + bullet) then swing upwards, their centre of mass rising a vertical distance $h = 9.8$ cm before the (block + bullet) pendulum comes momentarily to rest at the end of its arc. The speed of the bullet just before collision is: (take $g = 9.8\ \text{ms}^{-2}$) **(2021)**



(a) 831.5 m/s
(b) 811.5 m/s
(c) 841.5 m/s
(d) 821.5 m/s

31. Four equal masses, m each are placed at the corners of a square of length (l) as shown in the figure. The moment of inertia of the system about an axis passing through A and parallel to DB would be: **(2021)**



(a) $\sqrt{3}ml^2$
(b) $2ml^2$
(c) ml^2
(d) $3ml^2$

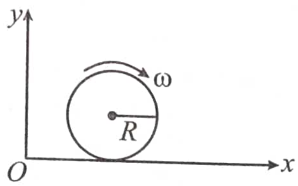
32. A metre scale is balanced on a knife edge at its centre. When two coins, each of mass 10 g are put one on the top of the other at the 10.0 cm mark the scale is found to be balanced at 40.0 cm mark. The mass of the metre scale is found to be $x \times 10^{-2}$ kg. The value of x is **(2022)**

33. Solid spherical ball is rolling on a frictionless horizontal plane surface about its axis of symmetry. The ratio of rotational kinetic energy of the ball to its total kinetic energy is :

(a) 2/5
(b) 2/7
(c) 1/5
(d) 7/10

34. The moment of inertia of a uniform thin rod about a perpendicular axis passing through one end is I_1. The same rod is bent into a ring and its moment of inertia about a diameter is I_2. If $\frac{I_1}{I_2}$ is $\frac{x\pi^2}{3}$, then the value of x will be ________. **(2022)**

35. A spherical shell of 1 kg mass and radius R is rolling with angular speed ω on horizontal plane (as shown in figure). The magnitude of angular momentum of the shell about the origin O is $\frac{a}{3}R^2\omega$. The value of a will be : **(2022)**



(a) 2
(b) 3
(c) 5
(d) 4

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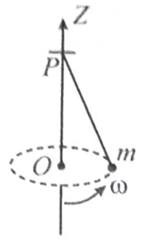
36. A small mass m is attached to a massless string whose other end is fixed at P as shown in the figure. The mass is undergoing circular motion in the x-y plane with centre at O and constant angular speed ω. If the angular momentum of the system, calculated about O and P are denoted by $\vec{L}_O$ and $\vec{L}_P$ respectively, then **(2012)**



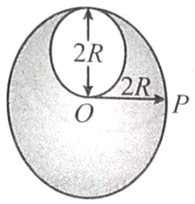
(*a*) $\vec{L}_0$ and $\vec{L}_P$ do not vary with time

(*b*) $\vec{L}_0$ varies with time while $\vec{L}_P$ remains constant

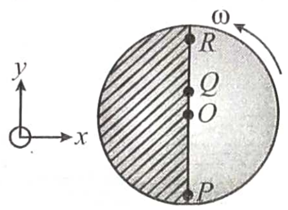
(*c*) $\vec{L}_0$ remains constant while $\vec{L}_P$ varies with time

(*d*) $\vec{L}_0$ and $\vec{L}_P$ both vary with time.

37. A lamina is made by removing a small disc of diameter $2R$ from a bigger disc of uniform mass density and radius $2R$, as shown in the figure. The moment of inertia of this lamina about axes passing through O and P is I_o and I_P, respectively. Both these axes are perpendicular to the plane of the lamina. The ratio $\frac{I_P}{I_o}$ to the nearest integer is **(2012)**



38. Consider a disc rotating in the horizontal plane with a constant angular speed ω about its centre O. The disc has a shaded region on one side of the diameter and an unshaded region on the other side as shown in the figure. When the disc is in the orientation as shown, two pebbles P and Q are simultaneously projected at an angle towards R. The velocity of projection is in the y-z plane and is same for both pebbles with respect to the disc. Assume that (*i*) they land back on the disc before the disc has completed 1/8 rotation, (*ii*) their range is less than half the disc radius, and (*iii*) ω remains constant throughout. Then **(2012)**



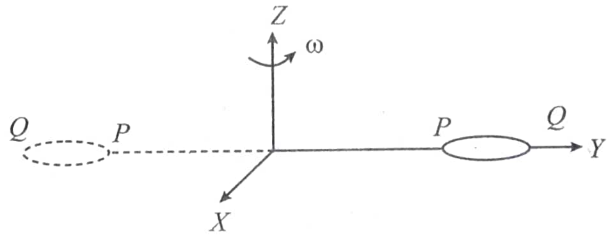
(*a*) P lands in the shaded region and Q in the unshaded region

(*b*) P lands in the unshaded region and Q in the shaded region

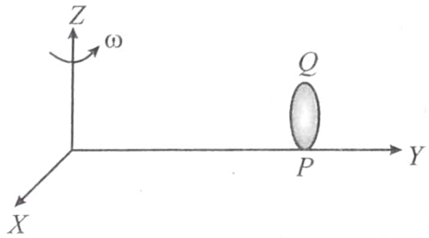
(*c*) Both P and Q land in the unshaded region

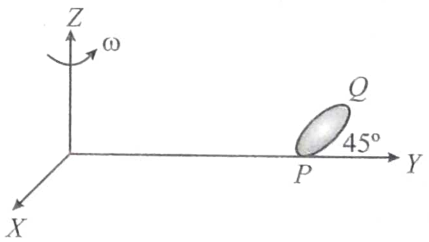
(*d*) Both P and Q land in the shaded region

Comprehension (Q. 39 to 40): The general motion of a rigid body can be considered to be a combination of (*i*) a motion of its centre of mass about an axis, and (*ii*) its motion about an instantaneous axis passing through center of mass. These axes need not be stationary. Consider, for example, a thin uniform welded (rigidly fixed) horizontally at its rim to a massless stick, as shown in the figure. Where disc-stick system is rotated about the origin on a horizontal frictionless plane with angular speed ω, the motion at any instant can be taken as a combination of (*i*) a rotation of the centre of mass the disc about the z-axis, and (*ii*) a rotation of the disc through an instantaneous vertical axis passing through its centre of mass (as is seen from the changed orientation of points P and Q). Both the motions have the same angular speed ω in the case.



Now consider two similar systems as shown in the figure: case (*a*) the disc with its face vertical and parallel to x-z plane; Case (*b*) the disc with its face making an angle of 45° with x-y plane its horizontal diameter parallel to x-axis. In both the cases, the disc is welded at point P, and systems are rotated with constant angular speed ω about the z-axis.





39. Which of the following statements about the instantaneous axis (passing through the centre of mass) is correct ? **(2012)**

(*a*) It is vertical for both the cases (*a*) and (*b*).

(*b*) It is vertical for case (*a*); and is at 45° to the x-z plane and lies in the plane of the disc for case (*b*).

(*c*) It is horizontal for case (*a*); and is at 45° to the x-z plane and is normal to the plane of the disc for case (*b*).

(*d*) It is vertical for case (*a*); and is at 45° to the x-z plane and is normal to the plane of the disc for case (*b*).

40. Which of the following statements regarding the angular speed about the instantaneous axis (passing through the centre of mass) is correct **(2012)**

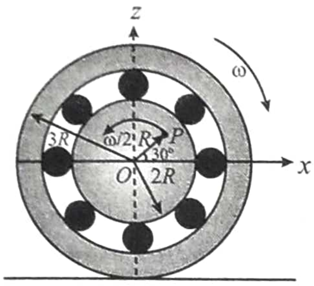
(*a*) It is $\sqrt{2}\omega$ for both the cases.

(*b*) It is ω for case (*a*); and $\omega/\sqrt{2}$ for case (*b*).

(*c*) It is ω for case (*a*); and $\sqrt{2}\omega$ for case (*b*).

(*d*) It is ω for both the cases.

41. The figure shows a system consisting of (i) a ring of outer radius $3R$ rolling clockwise without slipping on a horizontal surface with angular speed ω and (ii) an inner disc of radius $2R$ rotating anti-clockwise with angular speed $\omega/2$. The ring and disc are separated by frictionless ball bearings. The system is in the 8 x-z plane. The point P on the inner disc is at a distance R from the origin, where OP makes an angle of 30° with the horizontal. Then with respect to the horizontal surface. **(2012)**



(a) the point O has a linear velocity $3R\omega\hat{i}$.

(b) the point P has a linear velocity $\frac{11}{4}R\omega\hat{i}+\frac{\sqrt{3}}{4}R\omega\hat{k}$

(c) the point P has a linear velocity $\frac{13}{4}R\omega\hat{i}-\frac{\sqrt{3}}{4}R\omega\hat{k}$

(d) the point P has a linear velocity

$$\left(3-\frac{\sqrt{3}}{4}\right)R\omega\hat{i}+\frac{1}{4}R\omega\hat{k}$$

42. Two solid cylinders P and Q of same mass and same radius start rolling down a fixed inclined plane from the same height at the same time. Cylinder P has most of its mass concentrated near its surface, while Q has most of its mass concentrated near the axis. Which statement (s) is (are) correct? **(2012)**

(a) Both cylinders P and Q reachthe ground at the same time.

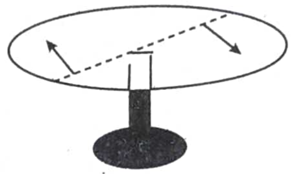
(b) Cylinder P has larger linear acceleration than cylinder

(c) Both cylinders reach the ground with same translational kinetic energy.

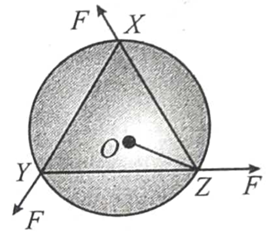
(d) Cylinder Q reaches the ground with larger angular speed.

43. A uniform circular disc of mass 50 kg and radius 0.4 m is rotating with an angular velocity of 10 rad s^{-1} about its own axis, which is vertical. Two uniform circular rings, each of mass 6.25 kg and radius 0.2 m, are gently placed symmetrically on the disc in such a manner that they are touching each other along the axis of the disc and are horizontal. Assume that the friction is large enough such that the rings are at rest relative to the disc and the system rotates about the original axis. The new angular velocity (in rad s^{-1}) of the system is: **(2013)**

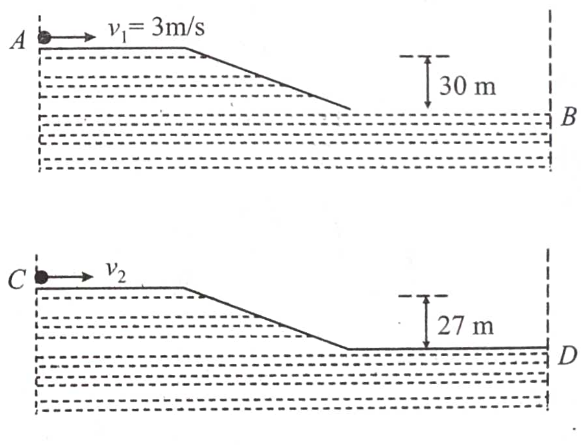
44. A horizontal circular platform of radius 0.5 m and mass 0.45 kg is free to rotate about its axis. Two massless spring toy-guns, each carrying a steel ball of mass 0.05 kg are attached to the platform at a distance 0.25 m from the centre on its either sides along is diameter (see figure). Each gun simultaneously fires the balls horizontally and perpendicular to the diameter in opposite directions. After leaving the platform the balls have horizontal speed of 9ms^{-1} with respect to the ground. The rotational speed of the platform in rad s^{-1} after the balls leave the platform is **(2014)**



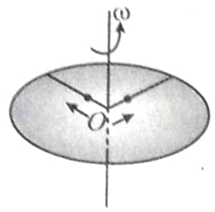
45. A uniform circular disc of mass 1.5 kg and radius 0.5 m is ini8tially at rest on a horizontal friction less surface. Three forces of equal magnitude $F = 0.5$ N are applied simultaneously along the three sides of an equilateral triangle XYZ with its vertices on the perimeter of the disc (see figure). One second after applying the forces, the angular speed of the disc in rad/s is **(2014)**



46. Two identical uniform discs roll without slipping on two different surfaces AB and CD (see figure) starting at A and C with linear speeds v_1 and v_2, respectively, and always remain in contact with the surfaces. If they reach B and D with the same linear speed and $v_1 = 3$ m/s, then v_2 in m/s is ($g = 10$ m/s^2) **(2015)**



47. A ring of mass M and radius R is rotating with angular speed ω about a fixed vertical axis passing through its centre O with two point masses each of mass $\frac{M}{8}$ at rest at O. These masses can move readily outwards along two massless rods fixed on the ring as shown in the figure. At some instant angular velocity is $\frac{8}{9}\omega$ and one of the masses is at a distance of $\frac{3}{5}$ R from O. At this instant the distance of the other mass from O is **(2015)**



(a) $\frac{2R}{3}$ (b) $\frac{R}{3}$

(c) $\frac{3R}{5}$ (d) $\frac{4R}{5}$

48. The densities of two solid sheres A and B of the same radii R vary with radial distance r as $\rho_A(r) = k\left(\frac{r}{R}\right)$ and $\rho_B(r) = k\left(\frac{r}{R}\right)^5$, respectively, where k, is a constant. The moments of inertia of the individual spheres about axes passing through their centres are I_A and I_B, respectively. If $\frac{I_B}{I_A} = \frac{n}{10}$, the value of n is: **(2015)**

49. A uniform wooden stick of mass 1.6 kg and length ℓ rests in an inclined manner on a smooth, vertical wall of height h ($<\ell$) such that a small portion of the stick extends beyond the wall. The reaction force of the wall on the stick is perpendicular to the stick. The stick makes an angle of 30° with the wall and the bottom of the stick is on a rough floor. The reaction of the wall on the stick is equal in magnitude to the reaction of the floor on the stick. The ratio h/ℓ and the frictional force f at the bottom of the stick are: ($g = 10\ ms^{-2}$) **(2016)**

(a) $\frac{h}{\ell} = \frac{\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3}N$

(b) $\frac{h}{\ell} = \frac{3}{16}, f = \frac{16\sqrt{3}}{3}N$

(c) $\frac{h}{\ell} = \frac{3\sqrt{3}}{16}, f = \frac{8\sqrt{3}}{3}N$

(d) $\frac{h}{\ell} = \frac{3\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3}N$

50. The position vector $\vec{r}$ of a particle of mass m is given by the following equation $\vec{r}(t) = \alpha t^3\hat{i} + \beta t^2\hat{j}$,

where $\alpha = \frac{10}{3} ms^{-3}$, $\beta = 5\ ms^{-2}$ and $m = 0.1$ kg. At $t = 1$ s, which of the following statement(s) is (are) true about the particle? **(2016)**

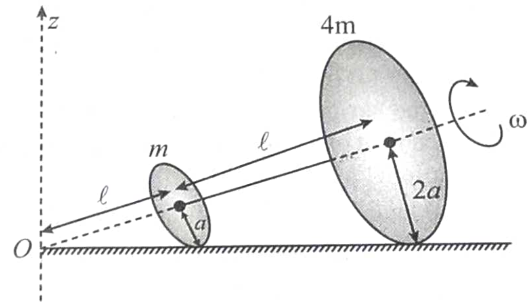
(a) The velocity $\vec{v}$ is given by $\vec{v} = (10\hat{i} + 10\hat{j})ms^{-1}$

(b) The angular momentum $\vec{L}$ with respect to the origin is given by $\vec{L} = -\left(\frac{5}{3}\right)\hat{k}Nms$

(c) The force $\vec{F}$ is given by $\vec{F} = (\hat{i} + 2\hat{j})N$

(d) The torque $\vec{\tau}$ with respect to the origin is given by $\vec{\tau} = -\left(\frac{20}{3}\right)kNm$

51. Two thin circular discs of mass m and $4m$, having radii of a and $2a$, respectively, are rigidly fixed by a massless, right rod of length $\ell = \sqrt{24}a$ through their center. This assembly is laid on a firm and flat surface, and set rolling without slipping on the surface so that the angular speed about the axis of the rod is ω. The angular momentum of the entire assembly about the point 'O' is $\vec{L}$ (see the figure). Which of the following statement(s) is(are) true ? **(2016)**



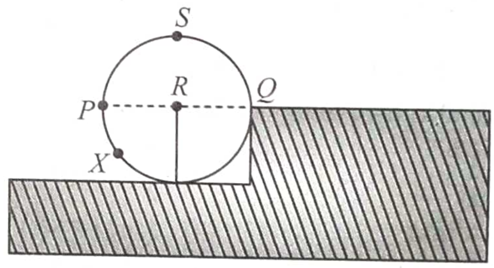
(a) Te magnitude of angular momentum of the assembly about its center of mass is $17ma^2\omega/2$

(b) The magnitude of the z-component of $\vec{L}$ is $55\ ma^2\omega$

(c) The magnitude of angular momentum of center of mass of the assembly about the point O is $81\ ma^2\omega$

(d) The center of mass of the assembly rotates about the z-axis with an angular speed of $\omega/5$

52. A wheel of radius R and mass M is placed at the bottom of a fixed step of height R as shown in the figure. A constant force is continuously applied on the surface of the wheel so that it just climbs the step without slipping. Consider the torque τ about an axis normal to the plane of the paper passing through the point Q. Which of the following options is/are correct? **(2017)**



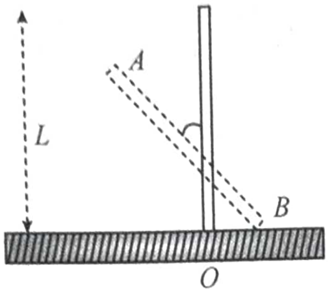
(a) If the force is applied normal to the circumference at point P then τ is zero

(b) If the force is applied tangentially at point S then $\tau \neq 0$ but the wheel never climbs the step

(c) If the force is applied at point P tangentially then τ decreases continuously as the wheel climbs

(d) If the force is applied normal t the circumference at point X then τ is constant

53. A rigid uniform bar AB of length L is slipping from it vertical position on a frictionless floor (as shown in the figure). At some instant of tie, th angle made by the bar with the vertical is θ. Which of the following statements about its motion is/are correct? **(2017)**



(a) The trajectory of the point A is a parabola

(b) Instantaneous torque about the point in contact with the floor is proportional to sin θ.

(c) When the bar makes an angle θ with the vertical, the displacement of its midpoint from the initial position is proportional to $(1 - \cos\theta)$

(d) The midpoint of the bar will fall vertically downward.

Comprehension (Q. 54 to 55): One twirls a circular ring (mass M and radius R) near the tip of one's finger as shown in Figure-1. In the process the finger never loses contact with the inner rim of the ring. The finger traces out the surface of a cone, shown by the dotted line. The radius of the path traced out by the point where the ring and the finger is in contact is r. The finger rotates with an angular velocity ω_0. The rotating ring rolls without slipping on the outside of a smaller circle described by the point where the ring and the finger is in contact (Figure-2). The coefficient of friction between the ring and the finger is μ and the acceleration due to gravity is g. **(2017)**

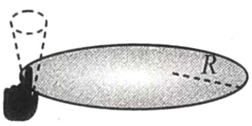


Figure-1

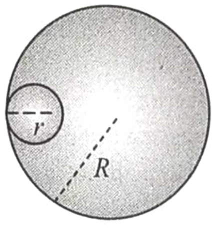


Figure-2

54. The total kinetic energy of the ring is :

(a) $\frac{1}{2}M\omega_0^2(R-r)^2$ (b) $\frac{3}{2}M\omega_0^2(R-r)^2$

(c) $M\omega_0^2R^2$ (d) $M\omega_0^2(R-r)^2$

55. The minimum value of ω_0 below which the ring will drop down is :

(a) $\sqrt{\frac{g}{\mu(R-r)}}$ (b) $\sqrt{\frac{g}{2\mu(R-r)}}$

(c) $\sqrt{\frac{3g}{2\mu(R-r)}}$ (d) $\sqrt{\frac{2g}{\mu(R-r)}}$

56. The potential energy of a particle of mass m at a distance r from a fixed point O is given by $V(r) = kr^2/2$, where k is a positive constant of appropriate dimensions. This particle is moving in a circular orbit of radius R about the point O. If v is the speed of the particle and L is the magnitude of its angular momentum about O, what of the following statements is(are) true? **(2018)**

(a) $v = \sqrt{\frac{k}{2m}}R$

(b) $v = \sqrt{\frac{k}{m}}R$

(c) $L = \sqrt{mk}R^2$

(d) $L = \sqrt{\frac{mk}{2}}R^2$

57. Consider a body of mass 1.0 kg at rest at the origin at tim $t = 0$. A force $\vec{F} = (\alpha t\hat{i} + \beta\hat{j})$ is applied on the body, wher $\alpha = 1.0$ Ns^{-1} and $\beta = 1.0$ N. The torque acting on the bod about the origin at time $t = 1.0$ s is $\vec{\tau}$. Which of the followin statements is(are) true? **(201**

(a) $|\vec{\tau}| = \frac{1}{3}Nm$

(b) The torque $\vec{\tau}$ is in the direction of the unit vector $+\hat{k}$

(c) The velocity of the body at $t = 1$ s is $\vec{v} = \frac{1}{2}(\hat{i} + \hat{j})$ ms^{-1}

(d) The magnitude of displacement of the body at $t = 1$ s i $\frac{1}{6}m$.

58. A ring and a disc are initially at rest, side by side, at th top of an inclined plane which makes an angle 60° with th horizontal. They start to roll without slipping at the sam instant of time along the shortest path. If the time differenc between their reaching the ground is $(2-\sqrt{3})/\sqrt{10}$ s, the the height of the top of the inclined plane, in meters, is ____ Take $g = 10$ ms^2 **(201**

59. In the List-I below, four different paths of a particle ar given a functions of time. In these functions α and β ar positive constants of appropriate dimensions and $\alpha \neq$ In each case, the force acting on the particle is either zer or conservative. In List-II, five physical quantities of th particle are mentioned; $\vec{P}$ is the linear momentum $\vec{L}$ is th angular momentum about the origin, K is the kinetic energ U is the potential energy and E is the total energy. Matc each path in List-I with those quantities in List-II which ar conserved for that path **(201**

	List - I		List - II
P.	$\vec{r}(t) = \alpha t\hat{i} + \beta t\hat{j}$	1.	$\vec{P}$
Q.	$\vec{r}(t) = \alpha\cos\omega t\hat{i} + \beta\sin\omega t\hat{j}$	2.	$\vec{L}$
R.	$\vec{r}(t) = \alpha(\cos\omega t\hat{i} + \sin\omega t\hat{j})$	3.	K
S.	$\vec{r}(t) = \alpha t\hat{i} + \frac{\beta}{2}t^2\hat{j}$	4.	U
		5.	E

(a) $P \to 1, 2, 3, 4, 5$; $Q \to 2, 5$; $R \to 2, 3, 4, 5$; $S \to 5$

(b) $P \to 1, 2, 3, 4, 5$; $Q \to 3, 5$; $R \to 2, 3, 4, 5$; $S \to 2, 5$

(c) $P \to 2, 3, 4$; $Q \to 5$; $R \to 1, 2, 4$; $S \to 2, 5$

(d) $P \to 1, 2, 3, 5$; $Q \to 2, 5$; $R \to 2, 3, 4, 5$; $S \to 2, 5$

60. A thin and uniform rod of mass M and length L is held vertical on a floor with large friction. The rod is released from rest so that it falls by rotating about its contact-point with the floor without slipping. Which of the following statement(s) is/are correct, when the rod makes an angle 60° with vertical ? [g is the acceleration due to gravity] **(2018)**

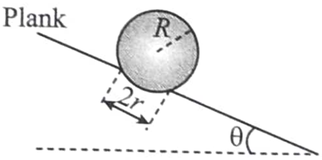
(*a*) The radial acceleration of the rod's center of mass will be $\frac{3g}{4}$

(*b*) The angular acceleration of the rod will be $\frac{2g}{L}$

(*c*) The angular speed of the rod will be $\sqrt{\frac{3g}{2L}}$

(*d*) The normal reaction force from the floor on the will be $\frac{Mg}{16}$

61. A football of radius R is kept on a hole of radius r ($r < R$) made on a plank kept horizontally. One end of the plank is now lifted so that it gets tilted making an angle θ from the horizontal as shown in the figure below. The maximum value of θ so that the football does not start rolling down the plank satisfies (figure is schematic and not drawn to scale) **(2020)**

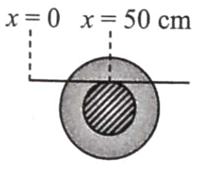


(*a*) $\sin\theta = \frac{r}{R}$ (*b*) $\tan\theta = \frac{r}{R}$

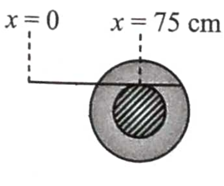
(c) $\sin\theta = \frac{r}{2R}$ (*d*) $\cos\theta = \frac{r}{2R}$

62. A small roller of diameter 20 cm has an axle of diameter 10 cm (see figure below on the left). It is on a horizontal floor and a meter scale is positioned horizontally on its axle with one edge of the scale on top of the axle (see figure on the right). The scale is now pushed slowly on the axle so that it moves without slipping on the axle, and the roller starts rolling without slipping. After the roller has moved 50 cm, the position of the scale will look like (figures are schematic and not drawn to scale)- **(2020)**

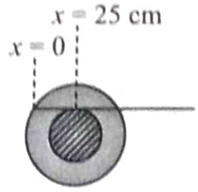
(*a*)



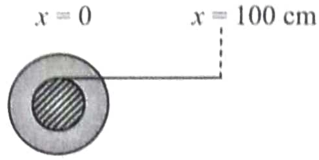
(*b*)



(*c*)

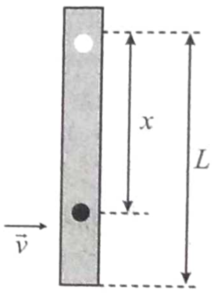


(*d*)



63. Put a uniform meter scale horizontally on your extended index fingers with the left one at 0.00 cm and the right one at 90.00 cm. When you attempt to move both the fingers slowly towards the center, initially only the left finger slips with respect to the scale and the right finger does not. After some distance the left finger stops and the right one starts slipping. Then the right finger stops at a distance x_R from the center (50.00 cm) of the scale and the let one starts slipping again. This happens because of the difference in the frictional forces on he two fingers. If the coefficients of static and dynamic friction between the fingers and the scale are 0.40 and 0.32, respectively, the value of x_R (in cm) is ______ **(2020)**

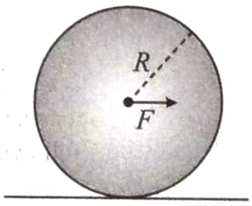
64. A rod of mass m and length L, pivoted at one of its ends, is hanging vertically. A bullet of the same mass moving at speed v strikes the rod horizontally at a distance x from its pivoted end and gets embedded in it. The combined system now rotates with angular speed ω about the pivot. The maximum angular speed ω_m is achieved for $x = x_M$. Then **(2020)**



(*a*) $\omega = \frac{3vx}{L^2+3x^2}$ (*b*) $\omega = \frac{12vx}{L^2+12x^2}$

(c) $x_M = \frac{L}{\sqrt{3}}$ (*d*) $\omega_M = \frac{v}{2L}\sqrt{3}$

65. A horizontal force F is applied at the centre of mass of a cylindrical object of mass m and radius, perpendicular to its axis as shown in the figure. The coefficient of friction between the object and the ground is μ. The centre of mass of the object has an acceleration a. The acceleration due to gravity is g. Given that the object rolls without slipping, which of the following statement(s) is(are) correct? **(2021)**



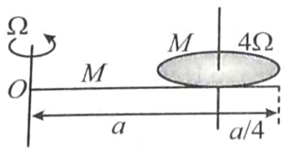
(a) For the same F, the value of a does not depend on whether the cylinder is solid or hollow

(b) For a solid cylinder, the maximum possible value of a is $2\mu g$

(c) The magnitude of the frictional force on the object due to the ground is always μmg

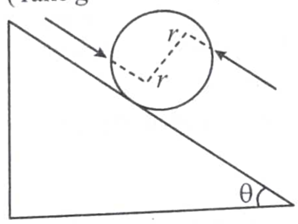
(d) For a thin-walled hollow cylinder, $a = \frac{F}{2m}$

66. A thin rod of mass M and length a is free to rotate in horizontal plane about a fixed vertical axis passing through point O. A thin circular disc of mass M and of radius $a/4$ is pivoted on this rod with its center at a distance $a/4$ from the free end so that it can rotate freely about its vertical axis, as shown in the figure. Assume that both the rod and the disc have uniform density and they remain horizontal during the motion. An outside stationary observer finds the rod rotating with an angular velocity Ω and the disc rotating about its vertical axis with angular velocity 4Ω. The total angular momentum of the system about the point O is. The value of $\left(\frac{Ma^2\Omega}{48}\right)n$ is **(2021)**



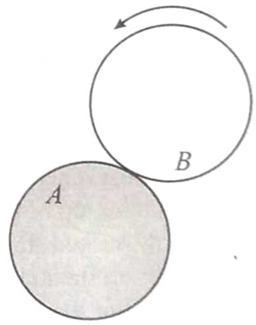
67. At time $t = 0$, a disk of radius 1 m starts to roll without slipping on a horizontal plane with an angular acceleration of $\alpha = \frac{2}{3}$ rad s^{-2}. A small stone is stuck to the disk. At $t = 0$, it is at the contact point of the disk and the plane. Later, at time $t = \sqrt{\pi}s$, the stone detaches itself and flies off tangentially from the disk. The maximum height (in m) reached by the stone measured from the plane is $\frac{1}{2} + \frac{x}{10}$. The value of x is [Take $g = 10$ ms^{-2}]. **(2022)**

68. A solid sphere of mass 1 kg and radius 1 m rolls without slipping on a fixed inclined plane with an angle of inclination $\theta = 30°$ from the horizontal. Two forces of magnitude 1 N each, parallel to the incline, act on the sphere, both at distance $r = 0.5$ m from the center of the sphere, as shown in the figure. The acceleration of the sphere down the plane is_____ ms^{-2}. (Take $g = 10$ ms^{-2}.) **(2022)**



69. A particle of mass 1 kg is subjected to a force which depends on the position as with $\vec{F} = -k(x\hat{i} + y\hat{j})$ kg ms^{-2}. At time $t = 0$, the particle's position $\vec{r} = \left(\frac{1}{\sqrt{2}}\hat{i} + \sqrt{2}\hat{j}\right)$m and its velocity $\vec{v} = \left(-\sqrt{2}\hat{i} + \sqrt{2}\hat{j} + \frac{2}{\pi}\hat{k}\right)$ ms^{-1}. Let v_x and v_y denote the x and the y components of the particle's velocity, respectively. Ignore gravity. When $z = 0.5$ m, the value of $(xv_y - yv_x)$ is _________ m^2 s^{-1}. **(2022)**

70. A flat surface of a thin uniform disk A of radius R is glued to a horizontal table. Another thin uniform disk B of mass M and with the same radius R rolls without slipping on the circumference of A, as shown in the figure. A flat surface of B also lies on the plane of the table. The center of mass of B has fixed angular speed ω about the vertical axis passing through the center of A. The angular momentum of B is $nM\omega R^2$ with respect to the center of A. Which of the following is the value of n? **(2022)**



(a) 2
(b) 5
(c) 7/2
(d) 9/2

ANSWER KEY

CONCEPT APPLICATION

1. (a) [2], (b) [1] 2. [16,4] 3. [800] 4. (a) [8], (b) [4] 5. (d) 6. (c) 7. (a) 8. (c) 9. (b) 10. (b) 11. (d) 12. (d) 13. (d) 14. (a) 15. (d) 16. (c) 17. (c) 18. (b) 19. (b) 20. [6,10] 21. [2] 22. (a) 23. [$\hat{k}, 2\sqrt{2}$] 24. (a) 25. (a) 26. (d) 27. (b) 28. (b) 29. (b) 30. [2] 31. (c) 32. [1,3] 33. [3,2] 34. (a) 35. (c) 36. (a) 37. (c)

EXERCISE-1 (TOPICWISE)

1. (b) 2. (c) 3. (a) 4. (d) 5. (c) 6. (d) 7. (a) 8. (d) 9. (b) 10. (b) 11. (c) 12. (b) 13. (d) 14. (b) 15. (a) 16. (a) 17. (b) 18. (d) 19. (c) 20. (c) 21. (c) 22. (a) 23. (c) 24. (b) 25. (a) 26. (d) 27. (c) 28. (d) 29. (c) 30. (d) 31. (b) 32. (b) 33. (d) 34. (c) 35. (a) 36. (a) 37. (d) 38. (b) 39. (c) 40. (c) 41. (a) 42. (b) 43. (c) 44. (b) 45. (a) 46. (b) 47. (d) 48. (a) 49. (b) 50. (a) 51. (c) 52. (b) 53. (b) 54. (c) 55. (a) 56. (b) 57. (a)

EXERCISE-2 (LEARNING PLUS)

1. (b) 2. (a) 3. (a) 4. (c) 5. (b) 6. (d) 7. (b) 8. (d) 9. (b) 10. (d) 11. (d) 12. (a) 13. (c) 14. (b) 15. (a) 16. (c) 17. (d) 18. (a) 19. (d) 20. (a) 21. (d) 22. (c) 23. (b) 24. (c) 25. (c) 26. (b) 27. (b) 28. (b) 29. (a) 30. (a) 31. (a) 32. (a) 33. (d) 34. (c) 35. (d) 36. (b) 37. (c) 38. (c) 39. (c) 40. (b) 41. (a) 42. (c) 43. (d) 44. (d) 45. (a) 46. (a) 47. (d) 48. [9] 49. [20]

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (a,b,c) 2. (a,d) 3. (b,c) 4. (b,c,d) 5. (a,b,c) 6. (b,c) 7. (a,c,d) 8. (a,c,d) 9. (a,c) 10. (a,c,d) 11. (a,b,d) 12. (b) 13. (c) 14. (d) 15. (d) 16. (c) 17. (a) 18. (c) 19. (a) 20. (c) 21. (c) 22. (b) 23. [5 rad/s] 24. [0006] 25. [0100] 26. [0004] 27. [0010] 28. [0002] 29. [28] 30. [0055] 31. [0005]

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (b) 2. (d) 3. (b) 4. (a) 5. (c) 6. (d) 7. (d) 8. (d) 9. (d) 10. (b) 11. (d) 12. (c) 13. (b) 14. (d) 15. (d) 16. (d) 17. [50.00] 18. (b) 19. (b) 20. (b) 21. (a) 22. (c) 23. [11] 24. (d) 25. (a) 26. (d) 27. [728] 28. [3] 29. (d) 30. (a) 31. (d) 32. [6] 33. (b) 34. [8] 35. (c)

JEE Advanced

36. (c) 37. [3] 38. (c) 39. (a) 40. (d) 41. (a,b) 42. (d) 43. [8] 44. [4] 45. [2] 46. [7] 47. (d) 48. [6] 49. (d) 50. (a,b,d) 51. (a,d) 52. (a) 53. (b,c,d) 54. (Bonus) 55. (a) 56. (b,c) 57. (a,c) 58. [0.75] 59. (a) 60. (a,c,d) 61. (a) 62. (b) 63. [25.60] 64. (a,c,d) 65. (b, d) 66. [49] 67. [0.52] 68. [2.86] 69. [3] 70. (b)

CHAPTER 12

Gravitation

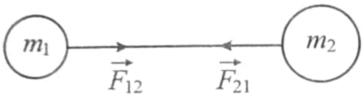
NEWTON'S LAW OF GRAVITATION

Newton's law of gravitation states that every point mass in the universe attracts every other point mass with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the particles.

Therefore from Newton's law of gravitation

$$\vec{F} = \frac{Gm_1m_2}{r^2}\hat{r} \qquad \dots \text{(i)}$$

where G is called the gravitational constant and $\hat{r}$ is the unit vector along the line joining the two point mass particles.

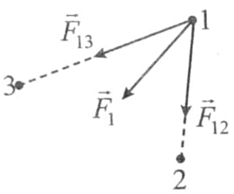


$\vec{F}_{12} + \vec{F}_{21} = 0$

The gravitational force between two particles form an action reaction pair.

Superposition Principle

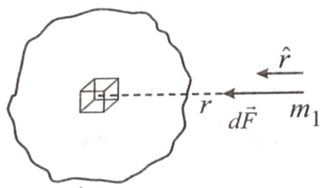
If there are more than two point objects then the net force on any particular body is equal to the vector sum of forces due to all other objects.



Net force on 1 is $\vec{F}_1 = \vec{F}_{13} + \vec{F}_{12}$

Force Because of a Continuous Object

Suppose we have a continuous object as shown. We need to find force of gravitation on a point mass m_1 as shown.



Force due to dm will be $\overrightarrow{dF} = \frac{Gdmm_1}{r^2}\hat{r}$

Net force will be given by integration of this vector.

$$\vec{F} = \int \overrightarrow{dF} = \int \frac{Gdmm_1}{r^2}\hat{r}$$

Train Your Brain

Example 1: A mass M is split into two parts m and $(M - m)$, which are then separated by a certain distance. What ratio (M/m) maximizes the gravitational force between the parts?

Sol. If 'r' is the distance between m and $(M - m)$, the gravitational force will be

$$F = G\frac{(M-m)m}{r^2} = \frac{G}{r^2}[Mm - m^2]$$

For F to be maximum, $\frac{dF}{dm} = 0$,

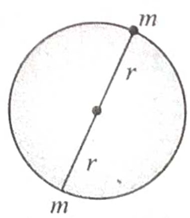
i.e., $\frac{d}{dm}\left[\frac{G}{r^2}(Mm - m^2)\right] = 0$

or, $M - 2m = 0$ $\qquad [\because G/r^2 \neq 0]$

or, $M/m = 2$, i.e., the force will be maximum when two parts are equal.

Example 2: Two particles of equal mass $m = 10$ kg are moving in a circle of radius $r = 1.65$ m under the action of their mutual gravitational attraction. Find the speed (in μm/s) of each particle. ($G = 6.6 \times 10^{-11}$ Nm²/kg²)

Sol. The particles will always remain diametrically opposite so that the force on each particle will be directed along the radius.



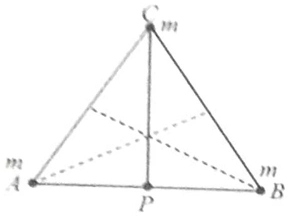
Considering the circular motion of one particle, we have, $\frac{mv^2}{r} = \frac{Gm.m}{(2r)^2}$

$\because \; v = \sqrt{\frac{Gm}{4r}} = 10$ μm/s

Example 3: Three identical particles each of mass $m = 10$ kg are placed at the three corners of an equilateral triangle of side $a = 1$ mm. Find the force exerted (in CGS units) by this system on another particle of mass m placed at (i) The mid-point of a side (ii) At the center of the triangle.

$(G = 6.6 \times 10^{-11}\ \text{Nm}^2/\text{kg}^2)$

Sol. As gravitational force is a two body interaction, the principle of superposition is valid, i.e., resultant force on the particle of mass m at P is $\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C$



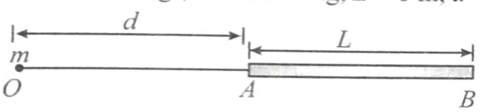
(i) As shown in the above figure, when P is at the mid-point of a side, $\vec{F}_A$ and $\vec{F}_B$ will be equal in magnitude but opposite in direction. So they will cancel each-other. So the point mass m at P will experience a force due to C only, i.e.,

$$F_C = \frac{Gmm}{(CP)^2} = \frac{Gm^2}{(a\sin 60°)^2} = \frac{4Gm^2}{3a^2}$$

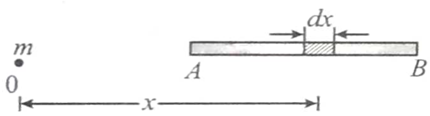
$F_C = 880$ dynes along PC

(ii) From symmetry, the net force on the particle at the center of triangle = 0

Example 4: Find the gravitational force in micro newton of attraction on the point mass 'm' placed at O by a thin rod of mass M and length L as shown in figure. [Take $m = 10$ kg, $G = 6.6 \times 10^{-11}$ Nm²/kg², $M = 1000$ kg, $L = 1$ m, $d = 10$ cm]



Sol. First we need to find the force due to an element of length dx.



The mass of the element is $dm = \left(\frac{M}{L}\right)dx$.

So, $dF = G\frac{Mm}{L}\frac{dx}{x^2}$

$\therefore$ The net gravitational force is

$$F = \frac{GMm}{L}\int_d^{d+L}\frac{dx}{x^2} = \frac{GMm}{L}\left[\frac{1}{d} - \frac{1}{L+d}\right] = \frac{GMm}{d(L+d)}$$

$= 66\ \mu\text{N}$

Notice that when $d >> L$, we find $F = \frac{GMm}{d^2}$, the result for two point masses.

Concept Application

1. The magnitude of the force of gravity between two identical objects is given by F_0. If the mass of each object is doubled but the distance between them is halved, then the new force of gravity between the objects will be
   (a) $16F_0$ (b) $4F_0$
   (c) F_0 (d) $F_0/2$
2. A 3 kg mass and a 4 kg mass are placed on x and y axes at a distance of 1 metre from the origin and a 1 kg mass is placed at the origin. Then the resultant gravitational force on 1 kg mass is
   (a) $7G$ (b) G
   (c) $5G$ (d) $3G$
3. Three particles of identical masses 'm' are kept at the vertices of an equilateral triangle of each side length 'a'. The gravitational force of attraction on any one of the particles is
   (a) $\sqrt{2}\frac{Gm^2}{a^2}$ (b) $\sqrt{3}\frac{Gm^2}{a^2}$
   (c) $\frac{3Gm^2}{a^2}$ (d) $\frac{2Gm^2}{a^2}$
4. Two masses M and $4M$ are kept at a distance of 12 m apart. A small particle of mass m is to be placed so that the net gravitational force on it is zero. What will be its distance (in mm) from mass M?
5. The distance of the centres of moon and earth is D. The mass of the earth is 81 times the mass of the moon. At what distance from the centre of the earth, the gravitation force will be zero?
   (a) $\frac{D}{2}$ (b) $\frac{2D}{3}$ (c) $\frac{4D}{3}$ (d) $\frac{9D}{10}$

GRAVITATIONAL FIELD OR GRAVITATIONAL FIELD STRENGTH

All the bodies on or above earth's surface experience gravitational force known as the weight of the bodies. Therefore the space surrounding the earth, where the gravitational force (weight) is experienced is known as the gravitational field of the earth. Similarly the space surrounding each and every material particle is known as gravitational field of that particle. Gravitational field is a vector field.

Gravitational field strength at any point is defined as gravitational force exerted on a unit point mass. It is numerically equal to acceleration due to gravity. Its unit is N/kg and dimension [M^0LT^{-2}].

Now if we want to measure the strength of the gravitational field at any point we will have to calculate the force acting on a point mass placed at that point. We see that , different masses experience different forces. The larger the mass, the larger the force it will experience. When we take the ratio of the gravitational force $\vec{F}_g$ and the point mass m we obtain a constant value for that point. This constant is known as the strength of the gravitational field. If the field exerts a large force on the point mass, we say that the strength of the gravitational field is stronger at that point and vice-versa.

The strength of the gravitational field $\vec{E}_g = (\vec{F}_g / m)$

Gravitational field strength is defined as gravitational force per unit mass.

In earth's gravitational field $\vec{E}_g = \dfrac{\text{weight of the particle}}{\text{mass of the particle}} = \dfrac{\vec{W}}{m}$

The above expression is equal to the acceleration due to gravity '$\vec{a}_g$'

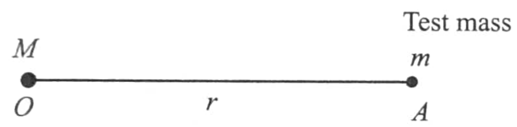
Gravitational Field Intensity Due to a Point Mass

Consider a point mass M at O and let us calculate gravitational intensity at A due to this point mass. Suppose a test mass is placed at A.

By Newton's law of gravitation, force on test mass

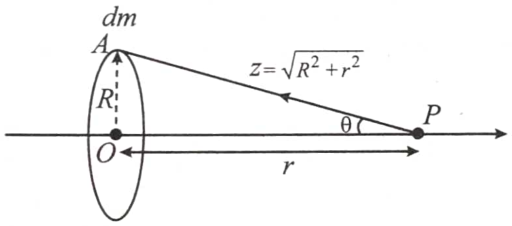
$$F = \frac{GMm}{r^2} \text{ along } AO$$

$$E = \frac{F}{m} = -\frac{GM}{r^2}\hat{e}_r$$



Gravitational Field Intensity Due to a Uniform Circular Ring at a Point on its Axis

Figure shows a ring of mass M and radius R. Let P is the point at a distance r from the centre of the ring. By symmetry the field must be towards the centre that is along PO.



Let us assume that a particle of mass dm on the ring say, at point A. Now the distance AP is $\sqrt{R^2 + r^2}$. Again the gravitational field at P due to dm is along $\overline{PA}$ and its magnitude is

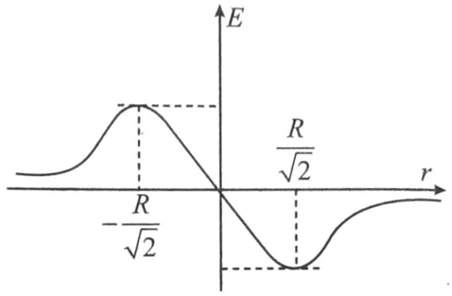
$$dE = \frac{Gdm}{z^2}$$

$$\therefore\ dE\cos\theta = \frac{Gdm}{z^2}\cos\theta$$

Net gravitational field $E = \dfrac{G\cos\theta}{z^2}\displaystyle\int dm = \dfrac{GM}{z^2}\dfrac{r}{z}$

$$= \frac{GMr}{(r^2+R^2)^{3/2}} \text{ along } PO$$

Variation of gravitational field due to a ring as a function of its axial distance.



Important points:

- If $r >> R$, $r^2 + R^2 \simeq r^2$

  $\therefore\quad E = -\dfrac{GMr}{r^3} = -\dfrac{GM}{r^2}$

  [where negative sign is because of attraction]

  Thus, for a distant point, a ring behaves as a point mass placed at the center of the ring.

- If $r << R$, $r^2 + R^2 \simeq R^2$

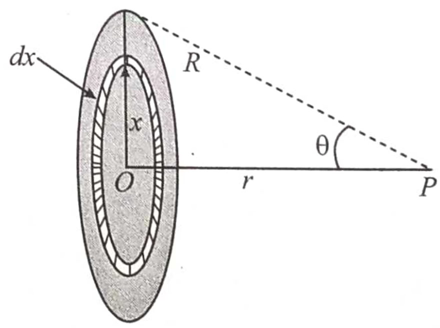
  $\therefore\quad E = -\dfrac{GMr}{R^3}$

  i.e., $E \propto r$

Gravitational Field Intensity Due to a Uniform Disc at a Point on its Axis

Let the mass of disc be M and its radius is R and P is the point on its axis where gravitational field is to be calculated.

Let us draw a circle of radius x and centre at O. We draw another concentric circle of radius $x + dx$. The part of disc enclosed between two circle can be treated as a uniform ring of radius x. The area of this ring is $2\pi x dx$



Therefore mass dm of the ring $= \dfrac{M}{\pi R^2}2\pi x\,dx = \dfrac{2Mxdx}{R^2}$

Gravitational field at P due to the ring is,

$$dE = \frac{G\left(\dfrac{2Mx\,dx}{R^2}\right)r}{(r^2+x^2)^{3/2}}$$

$$E = \int dE = \frac{2GM\,r}{R^2}\int_0^R \frac{x\,dx}{(r^2+x^2)^{3/2}}$$

$$= \frac{2GM\,r}{R^2}\left[-\frac{1}{\sqrt{r^2+x^2}}\right]_0^R = \frac{2GM\,r}{R^2}\left[\frac{1}{r}-\frac{1}{\sqrt{r^2+R^2}}\right]$$

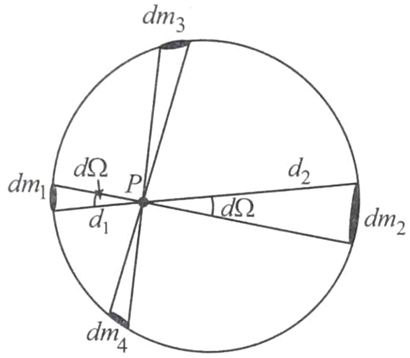
In terms of θ, E can be expressed as

$$E = \frac{2GM}{R^2}(1-\cos\theta)$$

Gravitational Field Due to a Spherical Shell

(i) **Gravitational field inside the shell of uniform density.** Consider an arbitrary point 'P' inside the shell of mass density σ.

We consider two opposite infinitesimal mass elements dm_1 and dm_2 subtending solid angle $d\Omega$.



$$dm_1 = \sigma dA_1 = \sigma d_1^2 d\Omega \qquad \left(\because d\Omega = \frac{dA}{r^2}\right)$$

$$dm_2 = \sigma dA_2 = \sigma d_2^2\, d\Omega$$

$$\Rightarrow dE_1 = \frac{G\,dm_1}{d_1^2} = G\sigma\, d\Omega, \leftarrow$$

$$\text{and } dE_2 = \frac{G\,dm_2}{d_2^2} = G\sigma\, d\Omega, \rightarrow$$

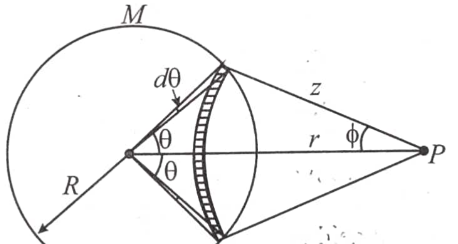
Thus, $dE = dE_2 - dE_1 = 0$

Similarly dE_3 cancels dE_4 and so on.

Therefore gravitation field inside the shell is zero,

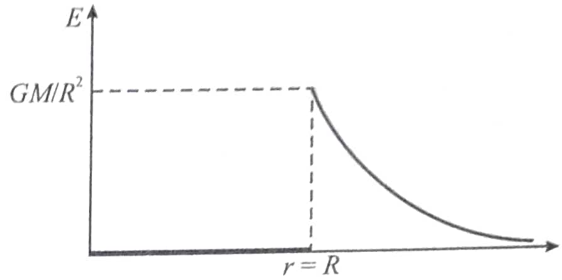
$E(r) = 0\ (r < R)$

(ii) Gravitational field outside the spherical shell: Gravitation field outside shell can be found by integrating gravitational field due to elemental rings as shown by varying θ from 0 to π.



The integration is a bit tedious and cumbersome, so we directly state the result $E(r) = \frac{GM}{r^2}\ (r > R)$

The gravitational field at the outside points is same as that of a point mass whose mass is equal to shell. The figure below graphically represents 'E' as function of distance.



Gravitational fixed due to a spherical shell.

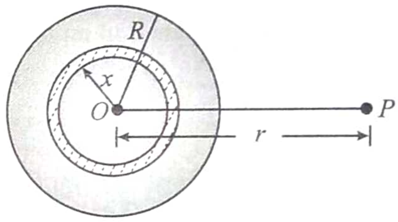
Shell Theorem:

- A uniform spherical shell attracts a point mass lying outside it in such a way as if the entire of mass of the shell were concentrated at its centre.
- A uniform spherical shell exerts no gravitational force on a point mass lying inside it.

Gravitational Field Due to a Uniform Solid Sphere

(i) **Field at an external point:** Let the mass of sphere is M and its radius is R we have to calculate the gravitational field at P.

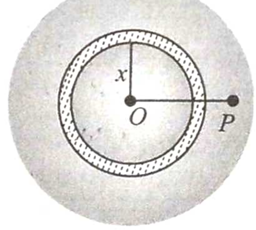
Consider spherical shell of radius x as shown. The field due to this is $dE = \frac{Gdm}{r^2}$ (independent of x)



$$\int dE = \int \frac{Gdm}{r^2} = \frac{G}{r^2}\int dm = \frac{Gm}{r^2}$$

Thus, a uniform sphere may be treated as a single particle of equal mass placed at its centre for calculating the gravitational field at an external point.

(ii) **Field at an internal point:** Suppose the point P is inside the solid sphere, in this case $r < R$ the sphere may be divided into thin spherical shells all centered at O. Only those shells will contribute whose radius is less than r. Suppose the mass of such a shell is dm. Then



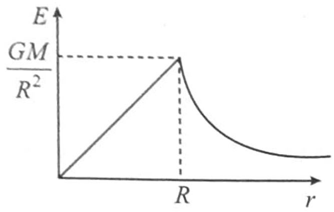
$$dE = \frac{Gdm}{r^2} \text{ along } PO$$

$$E=\frac{G}{r^2}\int dm \text{ where } \int dm=\frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3=\frac{Mr^3}{R^3}$$

$$\therefore\ E=\frac{GM}{R^3}r$$

Therefore gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere. At the centre $r = 0$ the field is zero. At the surface of the sphere $r = R$

$$E=\frac{GM}{R^2}$$



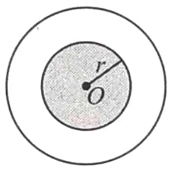
Gravitational field due to solid sphere is continuous but it is not differentiable function.

Gravitational field due to thin spherical shell is both discontinuous and non-differentiable function.

Gravitational Field in Spherical Symmetry

For spherical symmetric mass distribution, in which density is either constant or only depends on distance r from the centre of symmetry, gravitational field is given by $E(r)=\frac{GM_{enc}}{r^2}$

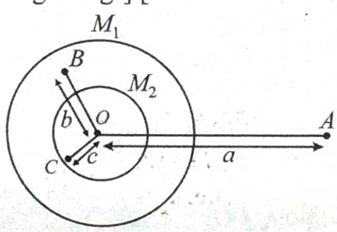
where M_{enc} is the mass contained in an imaginary spherical shell of radius r.



If density depends on r then $M_{enc}=\int_0^r \rho(r)4\pi r^2 dr$

Train Your Brain

Example 5: Two concentric shells of masses M_1 and M_2 are situated as shown in figure. Find the force on a particle of mass m when the particle is located at (i) $r = a$ (ii) $r = b$ (iii) $r \le c$. The distance r is measured from the center of the shell. [M_1 = 500 kg, M_2 = 1500 kg, $a=\sqrt{6.6}\,m$, $b=\sqrt{3.3}$, $G = 6.6 \times 10^{-11}$ kgm²/kg²] [Take 10^{-8} N as unit]



Sol. We know that attraction at an external point due to spherical shell of mass M is $\frac{GM}{r^2}$ while at an internal point is zero. So

(i) For $r = a$, the point is external for both the shell; so

$$E_A=\frac{G(M_1+M_2)}{a^2}$$

$$\therefore\ F_A=mE_A=\frac{G[M_1+M_2]m}{a^2}=2\text{ units}$$

(ii) For $r = b$, the point is external to the shell of mass M_2 and internal to the shell of mass M_1; so

$$E_B=\frac{GM_2}{b^2}+O$$

$$\therefore\ F_B=mE_B=\frac{GM_2m}{b^2}=3\text{ units}$$

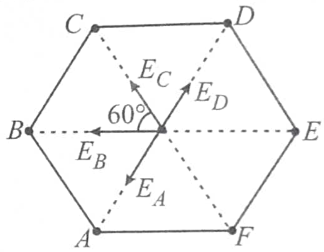
(iii) For $r = c$, the point is internal to both the shells, so

$E_C = 0 + 0 = 0$

$\therefore\ F_C = m\,E_C = 0$

Example 6: Four point masses each of mass M are placed at four vertices A, B, C and D of a regular hexagon of side L. Find the field strength at the centre of the hexagon.

Sol. Gravitation field strength is a vector quantity. So it is a vector sum of four vectors of equal magnitudes.

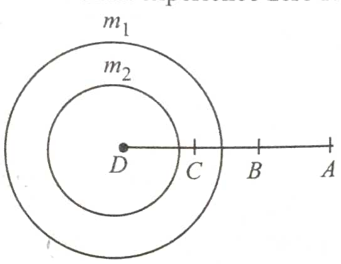


E_A and E_D are cancel each other. So, net field strength is a vector sum of E_B and E_C at angle 60°.

$$E_{net}=\sqrt{E^2+E^2+2(E)(E)\cos 60^\circ}=\sqrt{3}E=\frac{\sqrt{3}GM}{L^2}$$

Concept Application

6. The adjacent figure shows two shells of masses m_1 and m_2. The shells are concentric. At which point, a particle of mass m shall experience zero force?



(a) A (b) B (c) C (d) D

7. The gravitational field in a region is $10(\hat{i}-\hat{j})$N / kg. The work done by gravitational force to shift slowly a particle of mass 1 kg from point (1 m, 1 m) to a point (2 m, – 2 m) is

(*a*) 10 J (*b*) –10 J

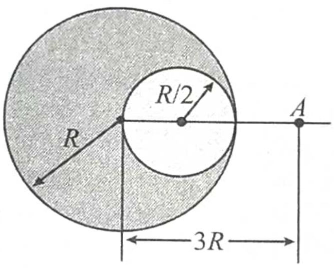
(*c*) –40 J (*d*) + 40 J

8. The gravitational field at a distance $R/2$ from the centre of a uniform solid sphere of mass M and radius R is

(*a*) zero (*b*) $\frac{GM}{R^2}$

(*c*) $\frac{GM}{2R^2}$ (*d*) $\frac{GM}{4R^2}$

9. A solid sphere of uniform density and radius R applies a gravitational force of attraction equal to F_1 on a particle placed at a distance $3R$ from the centre of the sphere. A spherical cavity of radius $R/2$ is now made in the sphere as shown in the figure. The sphere with cavity now applies a gravitational force F_2 on the same particle. The ratio F_2/F_1 is



(*a*) 9/50 (*b*) 41/0

(*c*) 3/25 (*d*) 22/25

ACCELERATION DUE TO GRAVITY

Earth can be approximately treated as a sphere of uniform density. So, earth's gravitational field and acceleration due to gravity is

$$a_g = E = \frac{GM_e}{r^2}(r \geq R_e)$$

$$= \frac{GM_e}{R_e^3}r(r < R_e)$$

At the surface of earth $r \simeq R_e$. Therefore,

$$a_g = g = \frac{GM_e}{R_e^2}$$

where $G = 6.67 \times 10^{-11} \frac{N-m^2}{Kg^2}$

(Universal gravitational constant)

$M_e = 5.983 \times 10^{24}$ Kg (mass of earth)

$R_e = 6.378 \times 10^6$ m (radius of earth)

Also, if the particle is very close to the earth's surface then

$r = R_e + h \simeq R_e$

Putting all the values we obtain $g = 9.8$ m/sec^2

This value near the surface of the earth can be determined by various experiments.

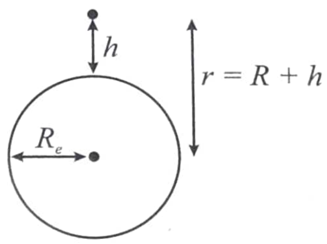
From $g = \frac{GM_e}{R_e^2}$, we get $M_e = \frac{gR_e^2}{G}$.

Using this formula, Cavendish computed the mass M of the Earth when he measured $G = 6.67 \times 10^{-11}$ N.m^2/kg^2. The mass of the Earth comes out to be 5.98×10^{24} kg

Variation in the Value of Acceleration Due to Gravity

(i) **Variation with altitude or height from the surface of the Earth:** If the object is placed at a distance h above the surface of the earth, acceleration due to gravity at that height is

$$a_g = E = \frac{GM_e}{r^2} = \frac{GM_e}{(R_e + h)^2} \quad (\text{as } r = R_e + h)$$



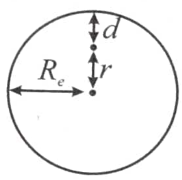
We see that the value of g decreases as one goes up. We can write,

$$a_g = \frac{GM_e}{R_e^2\left(1+\frac{h}{R_e}\right)^2} = \frac{g}{\left(1+\frac{h}{R_e}\right)^2}$$

where $g = \frac{GM_e}{R_e^2}$ is the value of a_g at the surface of the earth. If $h << R_e$,

$$a_g = g\left(1+\frac{h}{R_e}\right)^{-2} \approx g\left(1-\frac{2h}{R_e}\right).$$

(ii) **Variation with depth:**



For point inside earth $a_g = E = \frac{GM_e}{R_e^3}r$

Thus, inside the Earth, $a_g \propto r$.

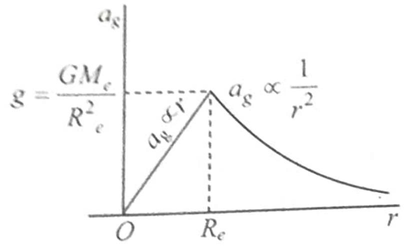
Hence the value of a_g decreases as we go inside or outside the Earth's surface, i.e. a_g is maximum at the surface of the Earth.

We can also write,

$$a_g = \frac{GM_e}{R_e^3}r = \frac{GM_e}{R_e^3}(R_e - d) = \frac{GM_e}{R_e^2}\left(1 - \frac{d}{R_e}\right)$$

or, $\boxed{a_g = g\left(1 - \frac{d}{R_e}\right)}$.

[acceleration due to gravity at a depth d]

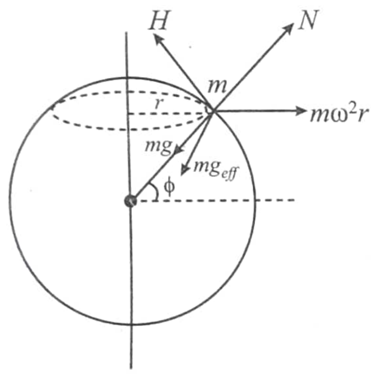


Figure, shows the graph of the value "a_g" at a distance 'r' from the centre

Here $g = \frac{GM_e}{R_e^2}$ is the value of g at the surface of the Earth.

At the centre of the Earth, $d = R_e$, so $a_g = 0$ at the centre

(iii) **Variation with latitude (Due to rotation of earth)**



In figure m is a mass placed on a weighing machine situated at a latitude of ϕ the real forces acting on it are:

(*i*) The gravitational force mg towards the center of earth.

(*ii*) The normal reaction N of the weighing machine directed away from the center of earth,

(*iii*) The horizontal reaction H of the weighing machine directed along the tangent as shown.

In the reference frame fixed to the earth's surface the body would also be acted upon by the pseudo force (centrifugal force) $m\omega^2 r$ directed as shown,

where r is the distance of P from earth's axis, $r = R_e \cos\phi$, where R_e is radius of earth.

Now in this reference frame m is at rest. Resolving forces along the radial and normal direction we get,

$mg = m\omega^2 r\cos\phi + N$

$\quad = m\omega^2 R_e \cos^2\phi + N$

$\Rightarrow N = mg - m\omega^2 R_e \cos^2\phi$

and $H = m\omega^2 R_e \cos\phi \sin\phi$

Now the effective weight of the body is the net force experienced by the weighing machine which is equal and opposite to the force exerted by the weighing machine on the body.

$$\therefore\ m\vec{g}_{eff} = -(\vec{N} + \vec{H}) \Rightarrow g_{eff} = |\vec{g}_{eff}| = \frac{1}{m}\sqrt{N^2 + H^2}$$

$$= \sqrt{(g - \omega^2 R_e \cos^2\phi)^2 + (\omega^2 R_e \cos\phi \sin\phi)^2}$$

Now $\omega^2 R_e \simeq 0.0337$, hence $(\omega^2 R_e \cos\phi \sin\phi)^2$ may be neglected and thus $\boxed{g_{eff} \simeq g - \omega^2 R_e \cos^2\phi}$

where g_{eff} = apparent acceleration due to gravity at a latitude of ϕ

- At poles, $\phi = \pi/2$ and hence $g_{eff} = g$
- At the equator, $\phi = 0$ and hence $g_{eff} = g - \omega^2 R_e$

Important Points:

- **Nonsphericity of the Earth:** All formulae and equations have been derived by assuming that the earth is a uniform solid sphere. the shape of the earth slightly deviates from the perfect sphere. It is an ellipsoid, flattened at the poles and bulging at the equator. The radius in the equatorial plane is about 21 km larger than the radius along the poles. Due to this the force of gravity is more at the poles and less at the equator. The value of g is accordingly larger at the poles and less at the equator. Note that due to rotation of earth also, the value of g is smaller at the equator than that at the poles.
- **Nonuniformity of the Earth:** The earth is not a uniformly dense object. There are a variety of minerals, metals, water, oil, etc., inside the earth. Then at the surface there are mountains, seas, etc. Due to these nonuniformities in the mass distribution, the value of g is locally affected.

Train Your Brain

Example 7: If the radius of the earth were to shrink by one percent, its mass remaining the same, what would happen to the acceleration due to gravity on the earth's surface.

Sol. Consider the case of a body of mass m placed on the earth's surface (mass of the earth is M and radius is R).

$$g = \frac{GM}{R^2} \quad ...(i)$$

Now, when the radius reduced by 1%, i.e. become 0.99 R, let acceleration due to gravity be g', then

$$g' = \frac{GM}{(0.99R)^2} \quad ...(ii)$$

From equation (i) and (ii), we get

$$\frac{g'}{g} = \frac{R^2}{(0.99)R^2} = \frac{1}{(0.99)^2} \text{ or } g' = g \times \left(\frac{1}{(0.99)}\right)^2$$

or $g' = 1.02g$

Thus the value of g is increased by 2%

Example 8: At what rate should the earth rotate so that the apparent g at the equator becomes zero? What will be the length of the day in this situation?

Sol. At earth's equator effective value of gravity is

$$g_{eff} = g_s - \omega^2 R_e$$

If g_{eff} at equator to be zero, we have $g_s - \omega^2 R_e = 0$

$$\text{or } \omega = \sqrt{\frac{g_s}{R_e}}$$

Thus length of the day will be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_e}{g_s}}$$

$$= 2 \times 3.14\sqrt{\frac{6.4\times10^6}{9.8}} = 5075 \text{ s} \simeq 84.5 \text{ min.}$$

Example 9: Calculate the value of acceleration due to gravity at a point

(i) 5.0 km above the earth's surface and

(ii) 5.0 km below the earth's surface.

Radius of earth = 6400 km and the value of g at the surface of the earth is 9.80 m s^{-2}.

Sol. (i) The value of g at a height h is (for $h << R$)

$$g_h = g_0\left(1-\frac{2h}{R}\right)$$

$$= (9.80 \text{ m s}^{-2})\left(1-\frac{2\times5.0 \text{ km}}{6400 \text{ km}}\right) = 9.78 \text{ ms}^{-2}$$

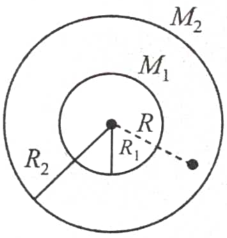
(ii) The value at a depth h is

$$g = g_0\left(1-\frac{h}{R}\right)$$

$$= (9.8 \text{ m s}^{-2})\left(1-\frac{5.0 \text{ km}}{6400 \text{ km}}\right) = 9.79 \text{ m s}^{-2}$$

Example 10: Two concentric spherical shells have masses M_1, M_2 and radii R_1, $R_2 (R_1 < R_2)$. What is the force exerted by this system on a particle of mass m if it is placed at a distance $\frac{(R_1+R_2)}{2}$ from the centre?

Sol.



$$R = \frac{R_1+R_2}{2}$$

As the particle 'm' is inside shell M_2 no force is exerted due to it and M_1 will exert force as if whole of the mass is concentrated at its centre.

$$\therefore\ F \text{ on } 'm' = \frac{GM_1m}{R^2} = \frac{4GM_1m}{(R_1+R_2)^2}$$

Concept Application

10. If R is the radius of earth, ω is its angular velocity and g_p is the value of acceleration due to gravity at the poles, then effective value of acceleration due to gravity at the latitude $\lambda = 60°$ will be equal to

(*a*) $g_p - \frac{1}{4}R\omega^2$ (*b*) $g_p - \frac{3}{4}R\omega^2$

(*c*) $g_p - R\omega^2$ (*d*) $g_p + \frac{1}{4}R\omega^2$

11. Find the ratio of acceleration due to gravity g at depth d and height h, where $d = 2h$.

(*a*) 1 : 1 (*b*) 1 : 2

(*c*) 2 : 1 (*d*) 1 : 4

12. If g is the acceleration due to gravity on the surface of earth, its value at a height equal to double the radius of earth is

(*a*) g (*b*) $\frac{g}{2}$

(*c*) $\frac{g}{3}$ (*d*) $\frac{g}{9}$

13. A planet A has a mass M and radius R. Another planet B with the same density has a double mass but in one fourth radius. The ratio of acceleration due to gravity of the two planets is

(*a*) 1 (*b*) 8 (*c*) 16 (*d*) 1/16

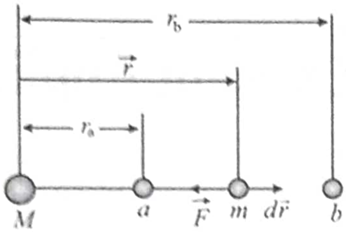
GRAVITATIONAL POTENTIAL ENERGY

In analyzing the motion of planets and satellites, it is often easier and more informative to use energy rather than force. In this section we shall evaluate the potential energy of a system consisting of two bodies that interact through the gravitational force. We obtained the potential energy change due to gravity for a body that moves through a height y near the Earth's surface : $\Delta U = mgy$. However, this applies only near the Earth's surface, (for changes in height that are small compared with the distance from the center of the Earth) where we can regard the gravitational force as approximately constant. Our goal here is to find a general expression that applies at all location, such as at the altitude of an orbiting satellite.

The potential energy difference can be found from equation: $\Delta U = U_b - U_a = -W_{ab}$, where W_{ab} is the work done by gravitational force to change configuration a to configuration b. However, this equation applies only if the force is conservative.

Calculating the Potential Energy

The gravitational force is conservative so we can calculate the potential energy. Figure shows a particle of mass m moving from a to b along a radial path. A particle of mass M, which we assume to be at rest at the origin, exerts a gravitational force on m. The vector $\vec{r}$ locates the position of m relative to M.



The work done by the gravitational force is

$$W_{ab} = \int_a^b \vec{F}.d\vec{r} = -\int_a^b Fdr$$

($\because$ $\vec{F}$ and $d\vec{r}$ are in opposite direction)

$$= -\int_{r_a}^{r_b} \frac{GMm}{r^2} dr = -GMm \int_{r_a}^{r_b} \frac{dr}{r^2}$$

$$= -GMm\left(-\frac{1}{r}\right)\Bigg|_{r_a}^{r_b} = GMm\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

Applying equation ($\Delta U = -W_{ab}$), we can find the change in the potential energy of the system as m moves between points a and b

$$\Delta U = U_b - U_a = -W_{ab} = \text{GMm}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

= Negative of work done by gravitational force

Instead of differences in potential energy, we can consider the value of the potential energy at a single point if we define a reference point. We choose our reference point to be an infinite separation of the particles, and the we define the potential energy to be zero in that point. Let us evaluate equation for $r_b = \infty$ and $U_b = 0$. If a represents any arbitrary point, where the separation between the particles is r, then the above equation becomes

$$U(\infty) - U(r) = GMm\left(\frac{1}{r} - 0\right)$$

$$\text{or } U(r) = -\frac{GMm}{r}$$

The potential energy given by this equation is a property of the system of two particles rather than of either particle alone. There is no way to divide this energy and say that so much belongs to one particle and so much to the other. Nevertheless, if $M >> m$, as is true for Earth and a block, we often speak of "the potential energy of the block". We can get away with this because, when a baseball moves in the vicinity of Earth, changes in the potential energy of the block - Earth system appear almost entirely as changes in the kinetic energy of the block, since changes in the kinetic energy of Earth are too small to be measured.

In the situation discussed by us, a particle of mass M exerts a gravitational force $\vec{F}$ on a particle of mass m that moves from a to b. If m moves outward from a to b, the change in potential energy is positive ($U_b > U_a$). That is, if the particle passes through point a with a certain kinetic energy K_a, as it travels to b its gravitational potential energy increases as its kinetic energy decreases ($K_b < K_a$). Conversely, if the particle is moving inward, its potential energy decreases as its kinetic energy increases.

Important Points:

- We can reverse the previous calculation and derive the gravitational force from the potential energy. For spherically symmetric potential energy functions, the relation $F = -dU/dr$ gives the radial component of the force; see equation.

$$F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) = -\frac{GMm}{r^2}$$

The minus sign here shows that the force is attractive, directed inward along a radius.

- We can show that the potential energy defined according to equation $U_{(r)} = -\frac{GMm}{r}$ leads to the familiar 'mgy' for a small difference in elevation y near the surface of the Earth. Let us evaluate equation for the difference in potential energy between the location at a height y above the surface (that is, $r = R_E + y$, where R_E is the radius of the Earth) and the surface ($r_a = R_E$)

$$\Delta U = U(R_E + y) - U(R_E)$$

$$= GM_E m\left(\frac{1}{R_E} - \frac{1}{R_E + y}\right)$$

$$= \frac{GM_E m}{R_E}\left(1 - \frac{1}{1 + y/R_E}\right)$$

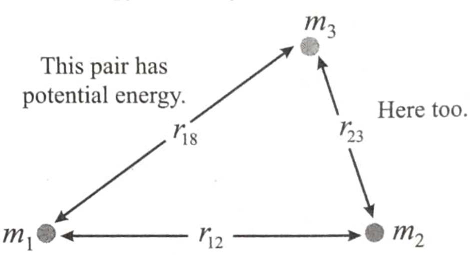
When $y << R_E$, which would be the case for small displacements of bodies near the Earth's surface, we can use the binomial expansion to approximate the last term as $(1 + x)^{-1} = 1 - x + \ldots \approx 1 - x$, which gives

$$\Delta U \approx \frac{GM_E m}{R_E}\left[1 - \left(1 - \frac{y}{R_E}\right)\right]$$

$$= \frac{GM_E my}{R_E^2} = mgy$$

using equation to replace GM_E/R_E^2 with g.

- If our system contains more than two particles, we consider each pair of particles in turn, calculate the gravitational potential energy of that pair with equation as if the other particles were not there, and then algebraically sum the results. Each of the three pairs of figure, for example, gives the potential energy of the system as



$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right)$$

Train Your Brain

Example 11: An object is dropped from height $h = 2R_e$ on the surface of earth. Find the speed with which it will collide with ground [Where R_e is radius of earth, and take mass of earth as M]

Sol. The initial potential energy (U_i) of object is

$$U_i = -\frac{GMm}{3R_e} \quad r = R_e + 2R_e = 3R_e$$

Final potential energy $U_f = -\dfrac{GMm}{R_e}$

By law of conservation of mechanical energy

$\Delta KE = -\Delta PE$

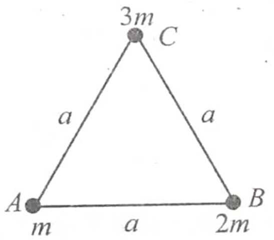
$$\Rightarrow \frac{1}{2}mv^2 = -[U_f - U_i] = U_i - U_f$$

$$\Rightarrow \frac{1}{2}mv^2 = -\frac{GMm}{3R_e} + \frac{GMm}{R_e}$$

$$\Rightarrow \frac{1}{2}v^2 = \frac{2GM}{3R_e}$$

$$\Rightarrow v = \sqrt{\frac{4GM}{3R_e}} = 2\sqrt{\frac{GM}{3R_e}}$$

Example 12: Three particles of masses m, $2m$ and $3m$ are placed at the corners of an equilateral triangle of side a. Calculate (i) The potential energy of the system (ii) The work done on the system if the side of the triangle is changed from a to $2a$. Assume the potential energy to be zero when the separation is infinity.



Sol. (i) The potential energy of the system = Sum of the potential energies of all the three possible distinct pairs.

i.e., $U = U_{AB} + U_{BC} + U_{CA}$

$$= \frac{-G(m)(2m)}{a} + \frac{-G(2m)(3m)}{a} + \frac{-G(3m)(m)}{a}$$

$$= \frac{-11Gm^2}{a}$$

(ii) When the side a is changed to $2a$, the potential energy $U' = \dfrac{-11Gm^2}{2a}$

The work done on the system $= U' - U = \dfrac{11Gm^2}{2a}$

Note:

- The work done by the system = The work done by the gravitational force $= -(U' - U) = \dfrac{11Gm^2}{2a}$
- The work done on the system = The work done against the gravitational force = The work done by the external force $= U' - U = \dfrac{-11Gm^2}{2a}$

Concept Application

14. What is the weight of a body at a distance $2r$ from the centre of the earth if the gravitational potential energy of the body at a distance r from the centre of the earth is U?

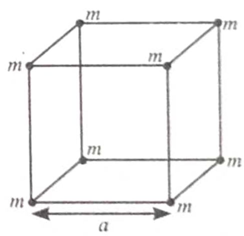
(a) $\dfrac{U}{2r}$ (b) $\dfrac{U}{3r}$ (c) $\dfrac{U}{4r}$ (d) Ur

15. An objective of mass m taken from the earth's surface to the altitude equal to the twice the earth radius R of the earth. The change in potential energy of the body will be

(a) $2mgR$ (b) $\dfrac{2}{3}mgR$

(c) $3mgR$ (d) $\dfrac{mgR}{3}$

16. Eight point masses, each of mass m are placed on the corners of a cube of side a. Find the gravitational potential energy of system of point masses.



GRAVITATIONAL POTENTIAL

"The gravitational potential at a point is defined as the work done per unit mass by an external agent (or against the gravitational force) in bringing a particle slowly from the reference point to the given point".

This definition is equivalent to the definition given below:

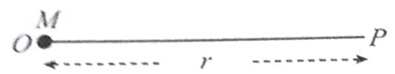
The gravitational potential at a point is equal to the change in potential energy per unit mass, as the mass is brought from the reference point to the given point.

The SI unit of gravitational potential is $\dfrac{J}{kg}$.

Potential Due to a Point Mass

Suppose a particle of mass M lies at O. We want to know the gravitational potential at a point P distant r from O

If we take the reference point of zero potential at infinity, then on placing another particle of mass m from infinity to P, the change in potential energy.



$$U - U_\infty = \frac{-GMm}{r} \qquad (\because U_\infty = 0)$$

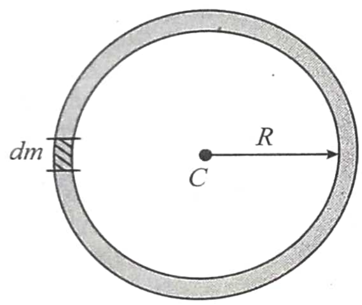
$\therefore$ The potential at P is $V = \frac{U - U_\infty}{m} = \frac{-GM}{r}$

Remember that potential is a scalar quantity.

Potential Due to Uniform Ring

(i) **At its centre:** To find potential at the centre C of the ring, we first find potential dV at centre due to an elemental mass dm of ring which is given as

$$dV = -\frac{Gdm}{R}$$



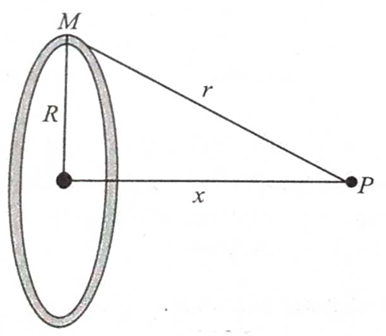
Total potential at C is

$$V = \int_0^V dV = \int_0^M -\frac{Gdm}{R} = -\frac{GM}{R}$$

As all dm's of the ring are situated at same distance R from the ring centre C, simply the potential due to all is added as being a scalar quantity.

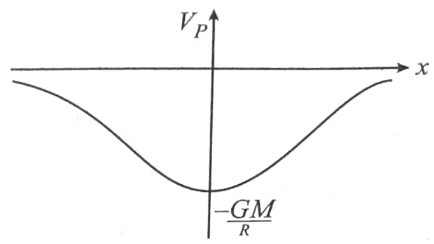
Note: Even if mass M is non-uniformly distributed on ring, the gravitational potential at C will remain same.

(ii) **At a point on axis of ring**: Gravitational potential at a point P on the axis of ring as shown, we can directly state the result as here also all points of ring are at same distance $r = \sqrt{x^2 + R^2}$ from the point P.



Potential at P can be given as; $V_P = -\frac{GM}{r} = \frac{GM}{\sqrt{R^2 + x^2}}$

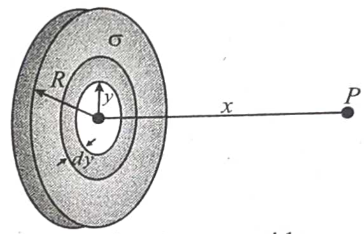
Graphs of V_p vs x



Potential due to a Uniform Disc

Figure shows a uniform disc of radius R with surface mass density

$$\sigma = \frac{M}{\pi R^2}$$



To find potential at point P, we consider an elemental ring of radius y and width dy. Mass on this elemental ring is

$$dm = \sigma\, 2\pi y dy$$

Due to this elemental ring, the potential at point P can be given as

$$dV = -\frac{Gdm}{\sqrt{x^2+y^2}} = -\frac{G\sigma 2\pi y\, dy}{\sqrt{x^2+y^2}}$$

Net potential at point P due to entire disc can be given as

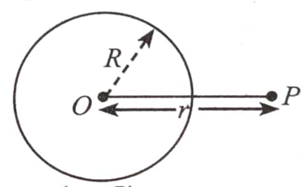
$$V = \int dV = -\int_0^R \sigma 2\pi G \frac{ydy}{\sqrt{x^2+y^2}} = -\sigma 2\pi G \left[\sqrt{x^2+y^2}\right]_0^R$$

$$V_P = -\sigma 2\pi G[\sqrt{x^2+R^2} - x] = -\frac{2GM}{R^2}\left[\sqrt{x^2+R^2} - x\right]$$

Potential Due to a Uniform Thin Spherical Shell

Let us take a thin uniform spherical shell of mass M and radius R with centre at O.

(i) **Outside the sphere**



$(r > R)$

According to definition of gravitational potential, at point P

$$V = -\int_\infty^r \vec{E}.\vec{dr} = \int_\infty^r \frac{GM}{r^2} dr \qquad \left[\because E_{\text{out}} = -\frac{GM}{r^2}\right]$$

$$V = GM\int_\infty^r \frac{1}{r^2} dr = -GM\left[\frac{1}{r}\right]_\infty^r = -\frac{GM}{r}$$

(ii) **On the surface**

$$V = -\int_{\infty}^{R} \vec{E}\cdot d\vec{r} = -\int_{\infty}^{R}\left(-\frac{GM}{r^2}\right)dr \qquad \left[\because E_{out} = -\frac{GM}{r^2}\right]$$

$$V = GM\int_{\infty}^{R}\left(\frac{1}{r^2}\right)dr = -GM\left[\frac{1}{r}\right]_{\infty}^{R} \Rightarrow V = -\frac{GM}{R}$$

(iii) **Inside the surface**

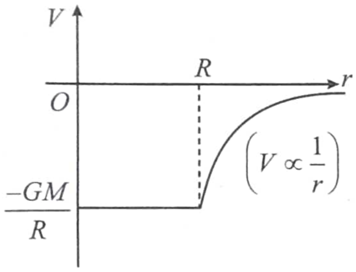
Inside the surface $E = 0$,

$\frac{dV}{dr} = 0$ or $V = \text{constant}$

$\left[\because E = -\frac{dV}{dr}\right]$, so $V = -\frac{GM}{R}$ (Same as that on the surface)

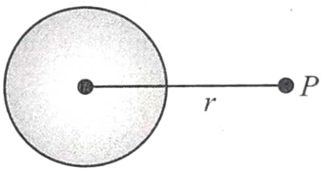
This is independent of the location of P inside the spherical shell.

Variation of V with r:



Potential Due to a Uniform Solid Sphere

Let us consider a uniform solid of mass M and radius R.



(i) **When P is outside the sphere**

$$V = -\frac{GM}{r}$$

(As if whole mass is concentrated at its centre, same as shell.)

(ii) **When P is inside the sphere**

$$V = -\int_{\infty}^{r} \vec{E}\cdot\vec{dr}$$

$$\Rightarrow V = -\left[\int_{\infty}^{R} E_1 dr + \int_{R}^{r} E_2 dr\right] = -\left[\int_{\infty}^{R}\left(-\frac{GM}{r^2}\right)dr + \int_{R}^{r}\left(-\frac{GMr}{R^3}\right)dr\right]$$

$$V = \left[GM\left(-\frac{1}{r}\right)_{\infty}^{R} + \frac{GM}{R^3}\left(\frac{r^2}{2}\right)_{R}^{r}\right]$$

$$\Rightarrow V = GM\left[-\frac{1}{R} + \frac{r^2}{2R^3} - \frac{R^2}{2R^3}\right]$$

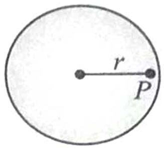
$$\Rightarrow V = -\frac{GM}{2R^3}[3R^2 - r^2]$$

Special cases:

(i) At the centre of sphere; $r = 0$

$$V_{centre} = -\frac{3GM}{2R}$$

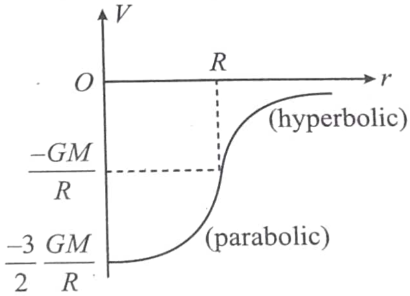
(ii) At the surface of sphere



$$V_{surface} = \frac{-GM}{R}$$

$$\therefore\ V_{\text{centre}} = \frac{3}{2}V_{\text{surface}}$$

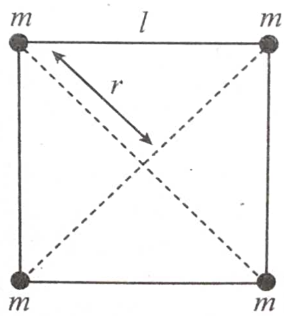
Variation of V with r:



Train Your Brain

Example 13: Find the potential energy of a system of four particles placed at the vertices of a square of side ℓ. Also obtain the potential at the centre of the square.

Sol. Consider four masses each of mass m at the corners of a square of side ℓ. We have four mass pairs at distance ℓ and two diagonal pairs at distance $\sqrt{2}\ \ell$.

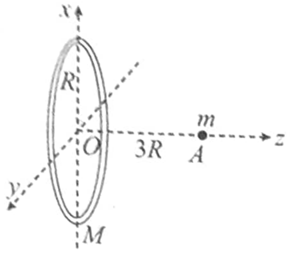


Hence, $U = -4\frac{Gm^2}{\ell} - 2\frac{Gm^2}{\sqrt{2}\ell} = -\frac{2Gm^2}{\ell}\left(2 + \frac{1}{\sqrt{2}}\right)$

The gravitational potential at the centre of the square $(r = \sqrt{2}\ell/2)$ is

$$V = -4\sqrt{2}\frac{Gm}{\ell}$$

Example 14: A circular ring of mass M and radius R is placed in xy plane with centre at origin. A particle of mass m is released from rest at a point $z = 3R$. Find the speed with which it will pass the centre of ring.



Sol. As shown in figure, first we find potential at A due to the ring.

$$V_A = -\frac{GM}{\sqrt{R^2+(3R)^2}} = -\frac{GM}{\sqrt{10}R}$$

Now potential at centre of ring due to the ring is

$$V_0 = -\frac{GM}{R}$$

When the mass moves from A to O, work done on it due to gravitation forces is

$$W = m(V_A - V_0) = m\left(-\frac{GM}{\sqrt{10}R}+\frac{GM}{R}\right)$$

$$= \frac{GMm}{R}\left(\frac{\sqrt{10}-1}{\sqrt{10}}\right)$$

This work done by gravitational force on mass must be equal to the increase in kinetic energy of the mass. Thus,

$$\frac{1}{2}mv^2 = \frac{\sqrt{10}-1}{\sqrt{10}}\frac{GMm}{R} \text{ or } v = \left[\frac{2\left(\sqrt{10}-1\right)GM}{\sqrt{10}\,R}\right]^{1/2}$$

Concept Application

17. A hollow spherical shell is compressed to half its radius. The gravitational potential at the centre

(a) Increases

(b) Decreases

(c) Remains same

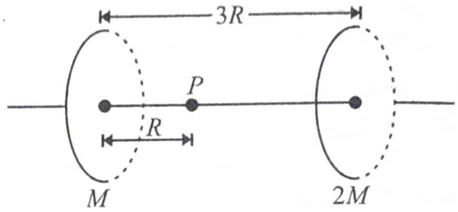
(d) During the compression increases then returns at the previous value.

18. Two small and heavy spheres, each of mass M, are placed a distance r apart on a horizontal surface. The gravitational potential at the mid-point on the line joining the centre of the spheres is

(a) Zero (b) $-\frac{GM}{r}$

(c) $-\frac{2GM}{r}$ (d) $-\frac{4GM}{r}$

19. Two rings having masses M and $2M$, respectively, having same radius are placed coaxially as shown in figure. If the mass distribution on both the rings is non - uniform,then gravitational potential at point P is



(a) $-\frac{GM}{R}\left[\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{5}}\right]$ (b) $-\frac{GM}{R}\left[1+\frac{2}{2}\right]$

(c) Zero (d) None of these

20. A particle of mass m is placed at a distance l from the centre of a hollow sphere of mass m and radius r ($l > r$).

(a) At the centre of the shell gravitational potential and field both will be zero.

(b) At the centre of the shell gravitational potential is zero only.

(c) At the centre of the shell gravitational field is zero only.

(d) Gravitational field is zero only at a point outside the shell.

ESCAPE SPEED

It is the minimum speed with which a body must be projected from the surface of a planet (usually the earth) so that it permanently overcomes and escapes from the gravitational field of the planet (the earth). We can also say that a body projected with escape speed will be able to go to a point which is at infinite distance from the earth.

If a body of mass m is projected with speed v from the surface of a planet of mass M and radius R, then

$$K.E. = \frac{1}{2}mv^2 \text{ ; } G.P.E. = -\frac{GMm}{R}$$

Total mechanical energy (T.M.E.) of the body $= \frac{1}{2}mv^2 - \frac{GMm}{R}$

If the v' is the speed of the body at infinity, then

$$\text{T.M.E. at infinity} = 0+\frac{1}{2}mv'^2 = \frac{1}{2}mv'^2$$

Applying the principle of conservation of mechanical energy, we have

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{1}{2}mv'^2 \text{ or, } v^2 = \frac{2GM}{R}+v'^2$$

v will be minimum when $v' \to 0$, i.e.,

$$v_e = v_{min} = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} \qquad \left[\because g = \frac{GM}{R^2}\right]$$

Important Points

- Escape speed is independent of the mass and direction of projection of the body.
- For earth as g = 9.8 m/s^2 and R = 6400 km

$$v_e = \sqrt{2\times 9.8\times 6.4\times 10^6} = 11.2\,\text{km/s}$$

Escape speed depends on:

(i) Mass (M_e) and size (R) of the planet

(ii) Position from where the particle is projected.

Escape speed does not depend on:

(i) Mass of the body which is projected (m_0)

(ii) **Angle of projection:** If a body is thrown from Earth's surface with escape speed, it goes out of earth's gravitational field and never returns to the earth's surface. But it starts revolving around the sun.

PLANETS AND SATELLITES

(*a*) **Planets:** Planets move round the sun due to the gravitational attraction of the sun. The path of these planets are elliptical with the sun at a focus. However, the difference in major and minor axis is not large. The orbits can be treated as nearly circular for not too advanced calculations. The motion of planets is largely governed by Kepler's Laws.

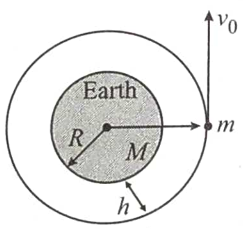
(*b*) **Satellite:** Satellites are launched from the earth so as to move around it. A number of rockets are fired from the satellite at proper time to establish the satellite in the desired orbit. Once the satellite is placed in the desired orbit with the correct speed for that orbit, it will continue to move in that orbit under gravitational attraction of the earth.

We make two assumptions that simplify the analysis:

(i) We consider the gravitational force only between the orbiting body and the central body, ignoring the perturbing effect of the gravitational force of other bodies (such as other planets)

(ii) We assume that the central body is so much more massive than the orbiting body that we can ignore its motion under their mutual interaction.

Let the speed of an artificial earth satellite in its orbits of radius r be v_0. The satellite is accelerating towards the centre of earth by earth's gravitational pull $\dfrac{GMm}{r^2}$



$$\Rightarrow F_{CP} = \frac{GMm}{r^2}$$

$$\frac{mv_0^2}{r} = \frac{GMm}{r^2} \Rightarrow v_0 = \sqrt{\frac{GM}{r}} \qquad \text{...(i)}$$

Putting $\dfrac{GM}{r^2} = a_g$ (acceleration due to gravity at the orbit), we obtain,

$$\Rightarrow v_0 = \sqrt{a_g r} \qquad \text{...(ii)}$$

When its orbit at an altitude h, putting $r = (R + h)$

$$\Rightarrow v_0 = \sqrt{\frac{GM}{R+h}} = \sqrt{\frac{GM}{R(1+h/R)}}$$

$$= \sqrt{\frac{gR}{(1+h/R)}}$$

Angular Speed

The angular speed

$$\omega = \frac{v_0}{r}$$

Putting $v_0 = \sqrt{\dfrac{GM}{r}}$, we obtain

$$\omega = \sqrt{\frac{GM}{r^3}} = \sqrt{\frac{GM}{(R+h)^3}}$$

Angular Momentum

The angular momentum of an earth satellite or a planet is given as

$$L = mv_0 r = m\sqrt{\frac{GM}{r}}\times r$$

$$= m\sqrt{GMr}$$

Time Period of Revolution

The period of revolution, $T = \dfrac{2\pi}{\omega}$

Putting, $\omega = \sqrt{\dfrac{GM}{r^3}}$, $T = 2\pi\sqrt{\dfrac{r^3}{GM}}$

ENERGY CONSIDERATION IN PLANETARY AND SATELLITE MOTION

$$U(r) = -\frac{GMm}{r}$$

Where r is the radius of the circular orbit.

The kinetic energy of the system is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 r^2$$

the Earth being at rest $\omega^2 r^2 = \dfrac{GM}{r}$

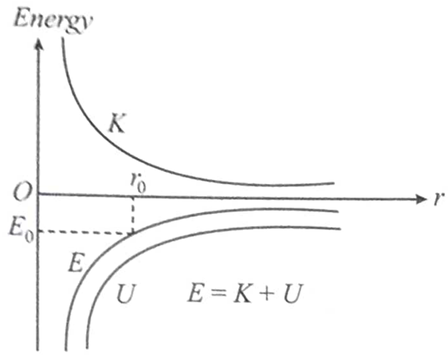
so that (with $v = \omega r$)

$$K = \frac{GMm}{2r}$$

The total mechanical energy is

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

This energy is constant and negative. The kinetic energy can never be negative.



Important Points

- Kinetic energy K, potential energy U, and total energy $E = K + U$ of a body in circular planetary motion A satellite with total energy $E < 0$ will remain in a orbit with radius r_0. The greater the distance from the planet, the greater (that is, less negative) its total energy E. To escape from the center of force and still have kinetic energy at infinity, the planet would need positive total energy.
- **Gravitational binding energy:** We have seen that if a particle of mass m placed on the earth is given an energy $\frac{1}{2}mv^2 = \frac{GMm}{R}$ or more, it finally escapes from earth. The minimum energy needed to take the particle infinitely away from the earth is called the binding energy of the earth-particle system. Thus, the binding energy of the earth-particle system is $\frac{GMm}{R}$.

GEOSTATIONARY SATELLITE

A satellite which appears to be stationary when seen from earth is called a Geostationary satellite.

For a satellite to be geostationary.

(i) Its orbit must be circular

(ii) It must rotate about the same axis as earth, i.e. it must move in the equatorial plane.

(iii) It must revolve from west to east.

(iv) Its time period must be 24 hours.

$$\Rightarrow m\omega^2(R+h) = \frac{GMm}{(R+h)^2}$$

$$\left(\frac{2\pi}{T}\right)^2 = \frac{GM}{(R+h)^3}$$

$$(R+h) = \left\{GM\left(\frac{T}{2\pi}\right)^2\right\}^{1/3}$$

where $T = 24 \times 3600$ sec.

Solving $h = 35800$ km and $r = R + h = 42200$ km

(v) The orbital speed of geostationary satellite is

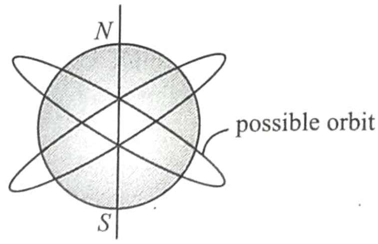
$$v_0 = \frac{2\pi r}{T} = 3.07\ km/s$$

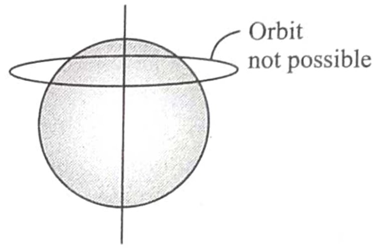
(vi) The angular orbital speed of geostationary satellite is

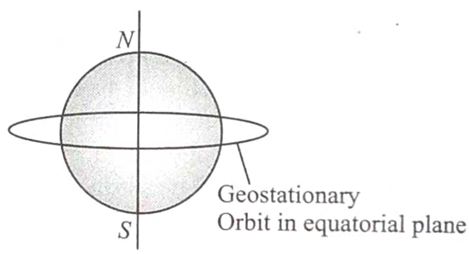
$$\omega = \frac{2\pi}{T} = 7.3 \times 10^{-5}\ rad/s$$

(vii) The INSAT group of satellites sent up by India are one such group of Geostationary satellites widely used for telecommunications in India.

Any satellite must rotate about the center of earth.







Train Your Brain

Example 15: Estimate the mass of the sun, assuming the orbit of the earth round the sun to be a circle. The distance between the sun and earth is 1.49×10^{11} m and $G = 6.66 \times 10^{-11}$ Nm²/kg².

Sol. Here the revolving speed of earth can be given as

$$v = \sqrt{\frac{GM}{r}} \qquad \text{[Orbital speed]}$$

Where M is the mass of sun and r is the orbit radius of earth.

We know time period of earth around sun is $T = 365$ days, thus we have

$$T = \frac{2\pi r}{v} \text{ or } T = 2\pi r\sqrt{\frac{r}{GM}}$$

$$\text{or } M = \frac{4\pi^2 r^3}{GT^2} = \frac{4\times(3.14)^2\times(1.49\times10^{11})^3}{(365\times24\times3600)^2\times(6.66\times10^{-11})}$$

$$= 1.972 \times 10^{22} \text{ kg}$$

Example 16: If escape speed on the surface of earth is v, then find the escape speed on the surface of another planet having radius α times and density β times the density of earth.

Sol. $v_e = \sqrt{\dfrac{2GM}{R}}$ *and* $M = \rho\dfrac{4}{3}\pi R^3$

$$\therefore\ v_e = \sqrt{\frac{2G}{R}\cdot\rho\frac{4}{3}\pi R^3} \Rightarrow v_e = 2R\sqrt{\frac{2G\rho\pi}{3}}$$

$$\text{Now, } \frac{v_{planet}}{v_{earth}} = \frac{R_p\sqrt{\rho_p}}{R_e\sqrt{\rho_e}} = \frac{\alpha R_e\sqrt{\beta\rho_e}}{R_e\sqrt{\rho_e}} \Rightarrow \frac{v_p}{v_e} = \alpha\sqrt{\beta}$$

$\therefore$ Escape speed on planet = $(\alpha\sqrt{\beta})v$

Example 17: Two satellites s_1 and s_2 of equal masses revolve in the same sense around a heavy planet in coplanar circular orbit of radii R and $4R$

(*a*) The ratio of period of revolution of s_1 & s_2 is 1 : 8

(*b*) Their velocities are in the ratio 2 : 1

(*c*) Their angular momentum about the planet are in the ratio 2 : 1

(*d*) The ratio of angular velocities of s_2 w.r.t. s_1 when all three are in the same line is 9 : 5

Sol. (*a,b,d*)

$$v_1 = \sqrt{\frac{GM}{R}} \quad v_2 = \sqrt{\frac{GM}{4R}} \Rightarrow v_1 = 2v_2$$

$$T_1 = \frac{2\pi R}{v_1}, \quad T_2 = \frac{2\pi(4R)}{v_2} \Rightarrow \frac{T_1}{T_2} = \frac{1}{8}$$

$$L_1 = mv_1R, \quad L_2 = mv_2 4R \Rightarrow \frac{L_1}{L_2} = \frac{1}{2}$$

$$\omega_1 = \sqrt{\frac{GM}{R^3}}, \quad \omega_2 = \sqrt{\frac{GM}{(4R)^3}}$$

Example 18: Find the energy required to launch a satellite of mass m from earth's surface in a circular orbit at an altitude 3 R. (where R radius of earth)

Sol. Circular orbit's radius = $4R$

$\therefore$ Total mechanical energy of satellite is $-\dfrac{GM_gm}{2(4R)}$

Thus; Energy required

$$-\frac{GM_gm}{R} = -\frac{GM_gm}{8R}$$

$$\Rightarrow \text{Energy required} = \frac{GM_gm}{R} - \frac{GM_gm}{8R} = \frac{7}{8}\frac{GM_gm}{R}$$

$$= \frac{7}{8}mgR$$

(in the form of Kinetic Energy)

Example 19: A satellite revolve around an imaginary planet, on the surface of which gravitational force (F) varies with distance (r) from its centre as $F \propto r^{-3}$. Find

(i) The proportionality relation between its orbital speed and radius of its orbit.

(ii) The proportionality relation between its time period and radius of its orbit.

Sol. $F \propto r^{-3}$, but this force is responsible for providing centripetal acceleration

$$\therefore\ \frac{v_0^2}{r} \propto r^{-3}$$

$$\Rightarrow v_0^2 \propto r^{-2} \Rightarrow v_0 \propto r^{-1}$$

$$\text{and } T = \frac{2\pi r}{v_0} \propto \frac{r}{r^{-1}} = r^2 \Rightarrow T \propto r^2$$

Concept Application

21. Two satellites A and B go round a planet P in circular orbits having radii $4R$ and R respectively. If the speed of the satellite A is $3v$, the speed of the satellite B will be

(*a*) $12v$ (*b*) $6v$ (*c*) $\dfrac{4v}{3}$ (*d*) $\dfrac{3v}{2}$

22. The escape velocity on the surface of the earth is 11.2 km/s. What would be the escape velocity on the surface of another planet of the same mass but 1/4 times the radius of the earth?

(*a*) 44.8 km/s (*b*) 22.4 km/s

(*c*) 5.6 km/s (*d*) 11.2 km/s

23. The period of a satellite in a circular orbit around a planet is independent of

(*a*) The mass of the planet

(*b*) The radius of the orbit

(*c*) The mass of the satellite

(*d*) All of three parameters given in options *a*, *b* and *c*

24. When a satellite going round the earth in a circular orbit at a distance from a proton with kinetic energy E. To escape to infinity, the energy which must be supplied to the electron is

(*a*) E (*b*) $2E$

(*c*) $0.5E$ (*d*) $\sqrt{2}E$

25. A satellite of mass m is orbiting around the earth at a height h above the surface of the earth. Mass of the earth is M and its radius is R. The angular momentum of the satellite is independent of

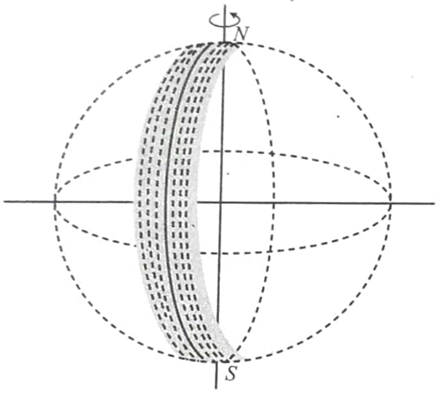
(*a*) m (*b*) M

(*c*) h (*d*) None of these

POLAR SATELLITE

These are low altitude ($h \approx 500$ to 800 km) satellites, but they go around the poles of the earth in a north-south direction whereas the earth rotates around its axis in an east-west direction. Since its time period is around 100 minutes it crosses any altitude many times a day. However, since its height h above the earth is about 500-800 km, a camera fixed on it can view only small strips of the earth in one orbit. Adjacent strips are viewed in the next orbit, so that in effect the whole earth can be viewed strip by strip during the entire day. These satellites can view polar and equatorial regions at close distances with good resolution.

Information gathered from such satellites is extremely useful for remote sensing, meterology as well as for environmental studies of the earth.

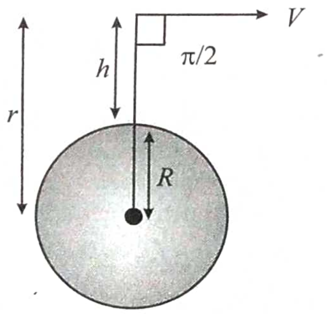


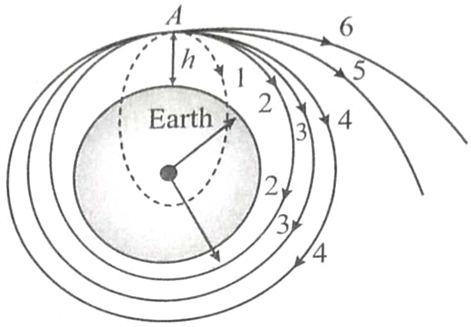
A Polar satellite. A strip on earth's surface (shown shaded) is visible from the satellite during one cycle. For the next revolution of the satellite, the earth has rotated a little on its axis so that an adjacent strip becomes visible.

The PSLV (polar satellite Launch vehicle] is an expandable launch system developed and operated by ISRO (Indian space research organisation). PSLV is designed and developed at VSSC [Vikram Sarabhai Space Centre]. Thiruvananthapuram, Kerala. The PSLV was first time successfully launched in 1996.

CONDITIONS FOR DIFFERENT TRAJECTORIES

For a body being projected tangentially from above earth's surface, say at a distance r from earth's center, the trajectory would depend on the velocity of projection v.





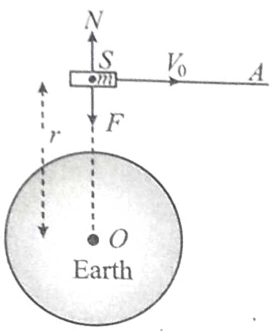
	Velocity	Orbit
1.	Velocity, $v << \sqrt{\frac{GM}{r}}$	Body returns to earth following elliptical path.
2.	$v < \sqrt{\frac{GM}{r}}$	Body acquires an elliptical orbit with earth as the far-focus w.r.t. the point of projection.
3.	Velocity is equal to the critical velocity of the orbit, $v = \sqrt{\frac{GM}{r}}$	Circular orbit with radius r
4.	Velocity is between the critical and escape velocity of the orbit $\sqrt{\frac{2GM}{r}} > v > \sqrt{\frac{GM}{r}}$	Body acquires an elliptical orbit with earth as the near focus w.r.t. the point of projection.
5.	$v = v_{esc} = \sqrt{\frac{2GM}{r}}$	Body just escapes earth's gravity, along a parabolic path.
6.	$v > v_{esc} = \sqrt{\frac{2GM}{r}}$	Body escape earth's gravity along a hyperbolic path.

Important Points

- Closed orbits are always elliptical and total mechanical energy is always negative.
- Open orbits are either parabolic or hyperbolic and total mechanical energy are always non-negative.
- When the mechanical energy of a satellite is zero or positive, it escapes away from the gravitational field of the central planet.
- When a satellite is orbiting then no energy is required to keep it moving in its orbit.
- When the speed of satellite in circular orbit is increased, then its energy increases. It starts moving in an elliptical path of greater semi-major axis.
- When the height of a satellite is increased, its potential energy increases and kinetic energy decreases.
- The numerical value of negative potential energy of a satellite orbiting in circular orbit is always more than its kinetic energy.

WEIGHTLESSNESS

Life of astronauts inside an orbiting satellite is very different. Absence of gravity has severe physiological effects. They develop puffy faces, their noses get blocked by fluids, their height may grow by few centimetre, the cardiovascular system does less work and causes headaches. Bones grow weak; muscles face deleterious effects in the absence of gravity which may cripple their day to day activity once they come back on the Earth.



Now, consider a body of mass m in a satellite S orbiting the Earth in a circular orbit of radius r with orbital speed v_0.

You know $v_0 = \sqrt{\dfrac{GM}{r}}$, where M is the mass of the Earth.

The body of mass m is also orbiting with the satellite with speed v_0. Suppose the normal reaction of the floor of the satellite on the body is N. The gravitational force F on the body is $\dfrac{GMm}{r^2}$.

The net force towards the centre will be equal to the centripetal force on the body.

$$\therefore F - N = \frac{mv_0^2}{r}$$

$$\text{or, } \frac{GMm}{r^2} - N = \frac{mv_0^2}{r}$$

$$\text{or, } N = \frac{GMm}{r^2} - \frac{m}{r}.\frac{GM}{r} = 0$$

Thus, the normal reaction becomes zero, Hence apparent weight of the body is zero. Thus **"everybody inside the satellite is in a sate of weightlessness"**.

If we suspend a body by a spring balance in the satellite, the balance will read zero.

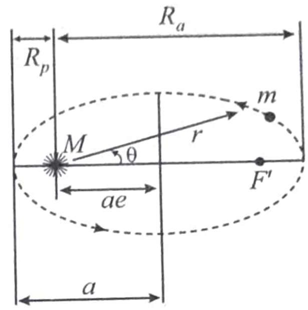
Figure shows that the force of the Earth's gravitation causes the satellite S to deviate from its natural straight line path SA towards the Earth. Hence a satellite may be called to **"fall towards the Earth freely because its acceleration is g"**.

KEPLER'S LAWS

The empirical basis for understanding the motion of the planets is three laws deduced by Kepler (1571-1630, well before Newton) from studies of the motion of the planet Mars.

Law of Orbits

All planets move in elliptical orbits having the Sun at one focus. Newton was the first to realize that there is a direct mathematical relationship between inverse-square ($1/r^2$) force and elliptical orbits. Figure shows a typical elliptical orbit.



A planet of mass m moving in a elliptical orbit around the Sun. The Sun, of mass M, is at one focus of the ellipse. F' marks the other or "empty" focus. The semimajor axis a of the ellipse, the perihelion distance "R_p", and the aphelion distance "R_a" are also shown. The distance "ae" locates the focal points, e being the eccentricity of the orbit.

For other planets in the solar system, the eccentricities are small and the orbits are nearly circular.

The maximum distance R_a of the orbiting body from the central body is indicated by the prefix (ap-), as in aphelion (the maximum distance from the Sun) or (apo-) apogee.

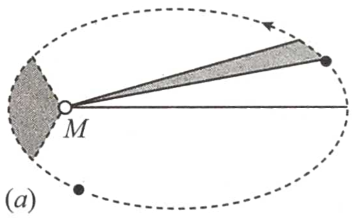
Similarly, the closest distance R_p is indicated by the prefix (peri-), as in perihelion or perigee. As you can see from figure $R_a = a(1 + e)$ and $R_p = a(1 - e)$. For circular orbits: $R_a = R_p = a$

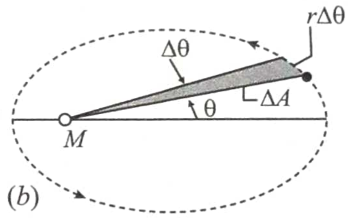
Law of Areas

A line joining any planet to the Sun sweeps out equal areas in equal times. Figure illustrates this law: in effect it says that the orbiting body moves more rapidly when it is close to the central body than it does when it is far away. We now show that the law of areas is identical with the law of conservation of angular momentum.

Consider the small area increment ΔA covered in a time interval Δt, as shown in figure. The area of this approximately triangular wedge is one-half its base, $r\,\Delta\theta$, times its height r. The rate at which this area is swept out is $\Delta A/\Delta t = \dfrac{1}{2}(r\Delta\theta)(r)/\Delta t$. In the instantaneous limit this becomes

$$\frac{dA}{dt} = \lim_{\Delta \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta \to 0} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} = \frac{1}{2} r^2 \omega$$





(*a*) The equal shaded areas are covered in equal times by a line connecting the planet to the Sun, demonstrating the law of areas.

(*b*) The area ΔA is covered in a time Δt, during which the line sweeps through an angle $\Delta\theta$.

Assuming we can regard the more massive body M as at rest, the angular momentum of the orbiting body m relative to the origin at the central body is, according to equation $L_z = I\,\omega = mr^2\omega$ (choosing the z-axis perpendicular to the plane of the orbit). Thus

$$\frac{dA}{dt} = \frac{L_z}{2m}$$

If the system of M and m is isolated, meaing that there is no net external torque on the system, then L_z is a constant; therefore dA/dt is also constant. That is in every interval dt in the orbit, the line connecting m and M sweeps out equal areas dA, which verifies kepler's second law.

The speeding up of a comet as it passes close to the Sun is an example of this effect and is thus a direct consequence of the law of conservation of angular momentum.

Law of Periods

The square of the period of any planet about the Sun is proportional to the cube of semi-major axis of the elliptical orbit.

If 'T' is the period of revolution and 'a' be the semi-major axis of the path of planet then according to Kepler's III Law, we have $T^2 \propto a^3$

Let us prove this result for circular orbits. The gravitational force provides the necessary centripetal acceleration for circular motion.

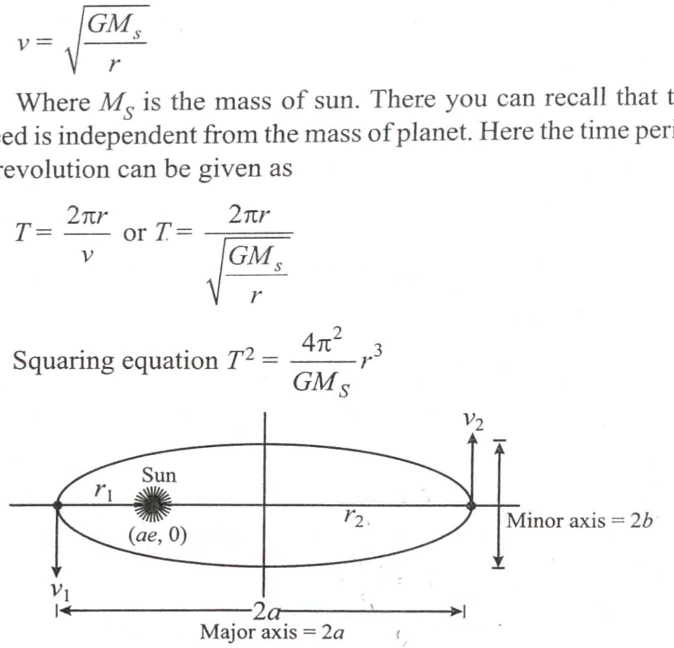
For circular orbits, it is a special case of ellipse when its major and minor axis are equal. If a planet is in a circular orbit of radius r around the sun then its revolution speed must be given as

$$v = \sqrt{\frac{GM_s}{r}}$$

Where M_S is the mass of sun. There you can recall that this speed is independent from the mass of planet. Here the time period of revolution can be given as

$$T = \frac{2\pi r}{v} \text{ or } T = \frac{2\pi r}{\sqrt{\frac{GM_s}{r}}}$$

Squaring equation $T^2 = \frac{4\pi^2}{GM_S}r^3$



Here r_1 and r_2 are the shortest and farthest distance of planet from sun during its motion, which are given as

$$r_1 = a\,(1 - e) \text{ and } r_2 = a\,(1 + e)$$

From geometry we know that the relation in semi major axis 'a' and semi minor axis 'b' is given as $b = a\sqrt{1-e^2}$

If v_1 and v_2 are the planet speeds at perihelion and aphelion points then from conservation of momentum we have

$$L_z = mv_1r_1 = m\,v_2r_2$$

Train Your Brain

Example 20: If the earth be one-half of its present distance from the sun, how many days will be there in one year?

Sol. If orbit of earth is r, time period is given as

$$T = 2\pi r\sqrt{\frac{r}{GM}} = \frac{2\pi}{\sqrt{GM}}r'^{3/2} \quad \text{...(i)}$$

If radius changes to $r' = \frac{r}{2}$, new time period become

$$T' = \frac{2\pi}{\sqrt{GM}}r'^{3/2} \quad \text{...(ii)}$$

From equations (i) and (ii),

$$\frac{T}{T'} = \left(\frac{r}{r'}\right)^{3/2}$$

$$\text{or } T' = T\left(\frac{r'}{r}\right)^{3/2}$$

$$= 365\left(\frac{1}{2}\right)^{3/2} = \left(\frac{365}{2\sqrt{2}}\right)\text{days} \simeq 129\text{ days}$$

Example 21: Two satellites S_1 and S_2 are revolving round a planet in coplanar and concentric circular orbits of radii R_1 and R_2 in the same sense respectively. Their respective periods of revolution are 1 h and 8 h. The radius of the orbit of satellite S_1 is equal to 10^4 km. Find the relative speed in km/h when they are closest.

Sol. By Kepler's 3rd law,

$$\frac{T^2}{R^3} = \text{constant} \;\therefore\; \frac{T_1^2}{R_1^3} = \frac{T_2^2}{R_2^3} \text{ or } \frac{1}{(10^4)^3} = \frac{64}{R_3^3}$$

or $R_2 = 4 \times 10^4$ km

Distance travelled in one revolution,

$S_1 = 2\pi R_1 = 2\pi \times 10^4$ and $S_2 = 2\pi R_2 = 2\pi \times 4 \times 10^4$

$$\text{Now, } v_1 = \frac{S_1}{t_1} = \frac{2\pi\times10^4}{1} = 2\pi\times10^4 \text{ km/h}$$

$$\text{and } v_2 = \frac{S_2}{t_2} = \frac{2\pi\times4\times10^4}{8} = \pi\times10^4 \text{ km/h}$$

$\therefore$ Relative velocity $= v_1 - v_2 = 2\pi \times 10^4 - \pi \times 10^4$

$= \pi \times 10^4$ km/h

Example 22: A planet revolves about the sun in elliptical orbit. The areal velocity $\left(\frac{dA}{dt}\right)$ of the planet is 4.0×10^{16} m^2/s. The least distance between planet and the sun is 2×10^{12} m. Then the maximum speed of the planet in km/s is

(*a*) 10 (*b*) 20

(*c*) 40 (*d*) None of these

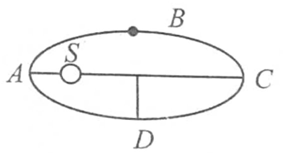
Sol. (*c*) $\frac{dA}{dt}=\frac{L}{2m}=\frac{mvr}{2m}$

$\frac{vr}{2}=4\times10^{16}$

$v=\frac{4\times10^{16}\times2}{2\times10^{12}}=4\times10^4$ m/s $=40$ km/s

Concept Application

26. A planet revolves in elliptical orbit around the sun shown in the figure. The linear speed of the planet will be maximum at



(*a*) *A* (*b*) *B* (*c*) *C* (*d*) *D*

27. A planet of mass *M* is revolving round the sun in an elliptical orbit. If its angular momentum is *J* then the area swept per second by the line joining planet to sun will be

(*a*) $\frac{JM}{2}$ (*b*) $\frac{J}{M}$ (*c*) $\frac{J}{2M}$ (*d*) *JM*

28. A planet of mass *m* is in an elliptical orbit about the sun ($m << M_{sun}$) with an orbital period *T*. If *A* be the area of orbit, then its angular momentum would be:

(*a*) $\frac{2mA}{T}$ (*b*) mAT

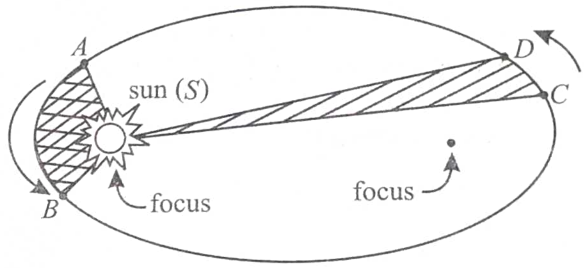
(*c*) $\frac{mA}{2T}$ (*d*) $2mAT$

29. A satellite is moving on a circular path of radius *r* around the earth has a time period *T*. If its radius slightly increased by Δr, the change in its time period is:

(*a*) $\frac{3}{2}\left(\frac{T}{r}\right)\Delta r$ (*b*) $\left(\frac{T}{r}\right)\Delta r$

(*c*) $\frac{3}{2}\left(\frac{T}{r}\right)\Delta T$ (*d*) None of these

30. Figure shows elliptical orbit of a planet *P* about the sun *S*. The shaded area *SCD* is twice the shaded area *SAB*. If t_1 is the time for the planet to move from *C* to *D* and t_2 is the time to move from *A* to *B*, then



(*a*) $t_1=t_2$

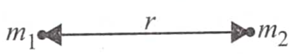
(*b*) $t_1=2t_2$

(*c*) $t_1=4t_2$

(*d*) $t_1<t_2$

Short Notes

Newton's Universal law of gravitation

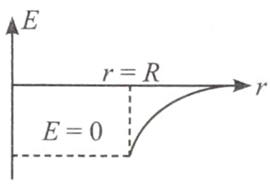


- Force of attraction between two point masses is $F=\frac{Gm_1m_2}{r^2}$, where $G=6.67\times10^{-11}$ Nm^2/kg^2
- Direction is along the line joining the point masses.

Gravitational Force

- It is a conservative force ⇒ mechanical energy will be conserved.
- It is a central force ⇒ angular momentum will be conserved.

Gravitational Field due to Spherical Shell



Outside region ($r>R$): $E=\frac{GM}{r^2}$

On the surface ($r=R$): $E=\frac{GM}{R^2}$

Inside region ($r<R$): $E=0$

Gravitational Field due to Solid Sphere

Outside region ($r > R$): $E = \frac{GM}{r^2}$

On the surface ($r = R$): $E = \frac{GM}{R^2}$

Inside region ($r < R$): $E = \frac{GMr}{R^3}$

Acceleration due to gravity

On the surface of earth: $g_s = \frac{GM}{R^2}$

At height h: $g_h = \frac{GM}{(R+h)^2}$

if $h << R$: $g_h \approx g_s\left(1 - \frac{2h}{R}\right)$

At depth d: $g_d = \frac{GM(R-d)}{R^3} = g_s\left(1 - \frac{d}{R}\right)$

Effect of rotation:

$g' = g - \omega^2 R\cos^2\lambda$

where λ is angle of latitude.

Gravitational Potential

- Due to a point mass at a distance r is

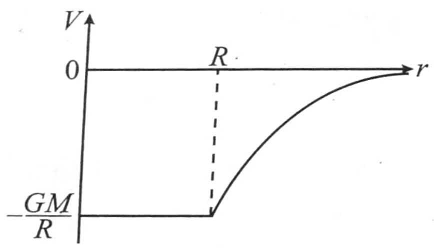
  $V = -\frac{GM}{r}$

- Gravitational potential due to spherical shell

  Outside the shell: $V = -\frac{GM}{r}, r > R$

  Inside/on the surface of the shell:

  $V = -\frac{GM}{R}, r < R$

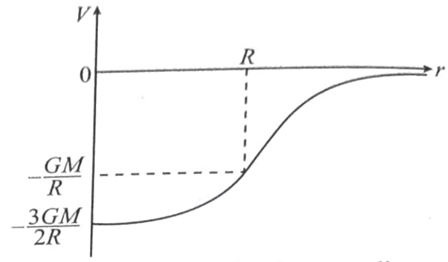


- Potential Due to a Solid Sphere

  Outside Region: $V = -\frac{GM}{r}, r > R$

  On the surface: $V = -\frac{GM}{R}, r = R$

  Inside Region: $V = -\frac{GM(3R^2 - r^2)}{2R^3}, r < R$



- Potential on the axis of a thin ring at a distance x from the centre $V = -\frac{GM}{\sqrt{R^2 + x^2}}$

Escape velocity

Escape velocity from a planet of mass M and radius R:

$v_e = \sqrt{\frac{2GM}{R}}$

Orbital velocity

Orbital velocity of satellite (orbital radius r)

$v_o = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{(R+h)}}$

For nearby satellite:

$v_o = \sqrt{\frac{GM}{R}} = \frac{v_e}{\sqrt{2}}$

where v_e = escape velocity on earth surface.

Time Period of Satellite

$T = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{\sqrt{GM}}$

Energies of a Satellite

Potential energy: $U = -\frac{GMm}{r}$

Kinetic energy: $K = \frac{1}{2}mv^2 = \frac{GMm}{2r}$

Mechanical energy: $E = U + K = -\frac{GMm}{2r}$

Binding energy: $\text{BE} = -E = \frac{GMm}{2r}$

Kepler's laws

- **Law of orbits:** Path of a planet is elliptical with the sun at one of the focus.
- **Law of areas:** Areal velocity

  $\frac{d\vec{A}}{dt} = \text{constant} = \frac{\vec{L}}{2m}$

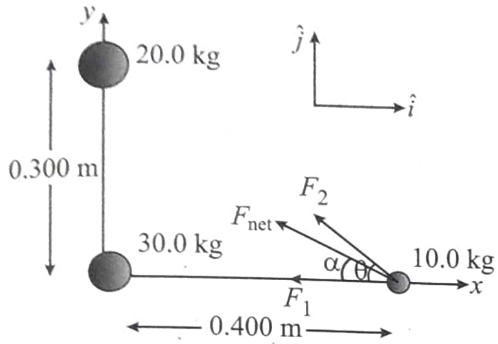
- **Law of periods:** $T^2 \propto a^3$

  or $T^2 \propto \left(\frac{r_{max} + r_{min}}{2}\right)^3 \propto (\text{mean radius})^3$

  For circular orbits, $T^2 \propto R^3$

Solved Examples

1. Figure shows three masses assuming them to be point masses, find the magnitude and angle of the total gravitational force on 10.0 kg mass.



Sol. The magnitude of the gravitational force of 30. 0 kg mass on 10.0 kg mass,

$$F_1 = \frac{GMm}{r^2} = \frac{\left(6.67\times10^{-11}\right)(30.0)(10.0)}{(0.400)^2} = 1.25 \times 10^{-7}\text{ N}$$

$$\vec{F}_1 = \left(-1.25\times10^{-7}\right)\hat{i}$$

Distance between 20.0 kg and 10.0 kg masses can be determined by the Pythagorean theorem:

$[(0.30)^2 + (0.40)^2]^{1/2} = 0.50$ m

The magnitude of the gravitational force of the 20.0 kg mass on the 10.0 kg mass is

$$F_2 = \frac{\left(6.67\times10^{-11}\right)(20.0)(10.0)}{(0.50)^2} = 5.34 \times 10^{-8}\text{ N}$$

$$\vec{F}_2 = \left(5.34\times10^{-8}\right)\left[(\cos\theta)\left(-\hat{i}\right)+(\sin\theta)\hat{j}\right]$$

where cos θ and sin θ from geometry of the arrangement are

$\cos\theta = 0.80$, $\sin\theta = 0.60$

Thus $\vec{F}_2 = \left(5.34\times10^{-8}\right)\left(-0.80\hat{i}+0.60\hat{j}\right)$

$$= -\left(4.27\times10^{-8}\right)\hat{i}+\left(3.20\times10^{-8}\right)\hat{j}$$

According to the principle of superposition,

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$$

$$= \left(-1.25\times10^{-7}\right)\hat{i}+\left[\left(-4.27\times10^{-8}\right)\hat{i}+\left(3.20\times10^{-8}\right)\hat{j}\right]$$

$$= -\left(1.68\times10^{-7}\right)\hat{i}+\left(3.20\times10^{-8}\hat{j}\right)$$

The magnitude of total gravitational force

$$F_{net} = \left[\left(-1.68\times10^{-7}\right)^2+\left(3.20\times10^{-8}\right)^2\right]^{1/2}$$

$$= 1.71 \times 10^{-7}\text{ N}$$

The angle α in figure is found from the components of $\vec{F}_{net}$

$$\tan\alpha = \frac{\left|\vec{F}_{net}\right|_y}{\left|\vec{F}_{net}\right|_x} = \frac{3.20\times10^{-8}}{1.68\times10^{-7}} = 0.190$$

So, $\alpha = 10.8°$

The angle of the total force with positive x-axis is, therefore,

$180° - \alpha = 180° - 10.8° = 169.2°$

2. Suppose the earth increases its speed of rotation. At what new time period will the weight of a body on the equator becomes zero [$g = 10$ m/s^2 and $R = 6400$ km]

Sol. The weight will become zero when

$g' = 0 \Rightarrow g - R\omega^2 = 0$

$$\Rightarrow \omega = \sqrt{\frac{g}{R}} \text{ or } \frac{2\pi}{T} = \sqrt{\frac{g}{R}} \Rightarrow T = 2\pi\sqrt{\frac{R}{g}}$$

$$= \frac{2\pi\sqrt{\dfrac{6400\times10^3}{10}}}{3600}\text{ hr}$$

$T = 1.4$ hr

3. A thin rod of mass M and length L lies along the x-axis, with centre at the origin. Find the field strength due to rod at a point x_0 on the x-axis, where $x_0 > L/2$.

Sol. We choose a mass element dm at a distance x, as shown in figure. All such elements produce a gravitational field at point P, that points towards negative x-axis. The resultant field can be calculated by integrating.

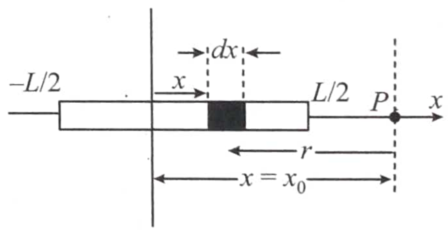


Figure the magnitude of the field produced by dm from $x = -\frac{L}{2}$ to $x = +\frac{L}{2}$

As $dg = -\frac{Gdm}{r^2}$

where $dm = \frac{M}{L}dx$ and $r = x_0 - x$

Substituting these results in the expression for dg, we get

$$dg = -\frac{Gdm}{r^2} = \frac{G(M/L)dx}{(x_0 - x)^2}$$

$$g = \int dg = -\frac{GM}{L}\int_{-L/2}^{L/2}\frac{dx}{(x_0-x)^2}$$

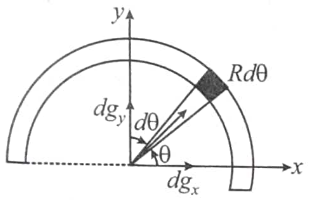
$$= -\frac{GM}{L}\left[\frac{1}{x_0-x}\right]_{-L/2}^{L/2}$$

$$= -\frac{GM}{L}\left(\frac{1}{x_0-L/2}-\frac{1}{x_0+L/2}\right)$$

$$= -\frac{GM}{x_0^2-L^2/4}$$

where, negative sign shows gravitational field points towards negative 'x' direction.

4. Find the field strength at the centre of a thin semicircular ring of radius R and mass M as shown in figure. Take the linear mass density as λ kg/m.



Sol: We choose an element that is an arc of length $ds = Rd\theta$, its mass is $dm = \lambda R d\theta$. For every element at $+x$ there is an equivalent element at $-x$ whose contribution is symmetrically opposite. From this symmetry we can see that the field strength has no x-component at the centre.

The y-component of the field is

$$dg_y = dg\sin\theta = \frac{Gdm\sin\theta}{R^2} = \frac{G\lambda R\,d\theta\sin\theta}{R^2} = \frac{G\lambda}{R}\sin\theta\,d\theta$$

The resultant field strength is

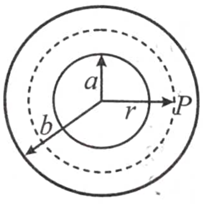
$$g_{\text{net}} = \int dg_y = \frac{G\lambda}{R}\int_0^{\pi}\sin\theta d\theta = \frac{2G\lambda}{R}\text{ (along }+y\text{ direction)}$$

5. A uniform hollow sphere has internal radius a and external radius b. Taking the potential at infinity to be zero, show that the ratio of the gravitational potential at a point on the outer surface to that on the inner surface is $\dfrac{2(b^3-a^3)}{3b(b^2-a^2)}$.

Sol. Let the density of the hollow sphere be ρ. The mass M of the hollow sphere is $M = \frac{4}{3}\pi\rho(b^3-a^3)$

The gravitational potential on the outer surface is $-\frac{GM}{b}$;

so we have $V(b) = -\dfrac{G\cdot\frac{4}{3}\pi\rho(b^3-a^3)}{b}$



Now we consider a point P at a distance r from the centre of the sphere such that $a < r < b$. The gravitational field at this point is due to the mass contained within the radius r, this mass is $\frac{4}{3}\pi\rho(r^3-a^3)$

The gravitational field strength at P is

$$g = -\frac{G\cdot\frac{4}{3}\pi\rho(r^3-a^3)}{r^2}$$

From the relation between the field intensity g and the gravitational potential, we have

$$V(a)-V(b) = -\int_b^a g\,dr$$

$$= \frac{4\pi\rho G}{3}\int_b^a\frac{r^3-a^3}{r^2}dr$$

$$= \frac{4\pi\rho G}{3}\left[\frac{r^2}{2}+\frac{a^3}{r}\right]_b^a$$

$$= \frac{4\pi\rho G}{3}\left[\frac{a^2}{2}-\frac{b^2}{2}+\frac{a^3}{a}-\frac{a^3}{b}\right]$$

$$V(a)-V(b) = \frac{4\pi\rho G}{3}\left[\frac{3a^2}{2}-\frac{b^2}{2}-\frac{a^3}{b}\right]$$

Now, using the expression of $V(b)$, we obtain

$$V(a) = \frac{4\pi\rho G}{3}\left(-b^2+\frac{a^3}{b}+\frac{3a^2}{2}-\frac{b^2}{2}-\frac{a^3}{b}\right)$$

$$= \frac{4\pi\rho G}{3}\left(-\frac{3b^2}{2}+\frac{3a^2}{2}\right)$$

$$\frac{V(b)}{V(a)} = \frac{-4\pi\rho G}{3}\left(\frac{b^3-a^3}{b}\right)\Bigg/\frac{4\pi\rho G}{3}\left(\frac{3a^2-3b^2}{2}\right)$$

$$\frac{V(b)}{V(a)} = \frac{2(b^3-a^3)}{3b(b^2-a^2)}$$

6. Two particle m_1 and m_2 are initially at rest at infinite distance. Find their relative velocity of approach due to gravitational attraction when their separation is d.

Sol. In the absence of any external force, the momentum of system is conserved. Due to work done by gravitational field, the kinetic energy of point masses increases. Initially when the separation was large i.e., at infinity there was no gravitational potential energy and when they gets closer system's gravitational energy decreases and the kinetic energy increases.

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2-\frac{Gm_1m_2}{d} = 0 \quad ...(i)$$

We have from momentum conservation

$$m_1v_1 = m_2v_2 \quad ...(ii)$$

From equation (i) and (ii)

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}\frac{m_1^2}{m_2}v_1^2 = \frac{Gm_1m_2}{d}$$

$$v_1 = \sqrt{\frac{2Gm_2^2}{d(m_1+m_2)}} = \left(\sqrt{\frac{2G}{d(m_1+m_2)}}\right)m_2$$

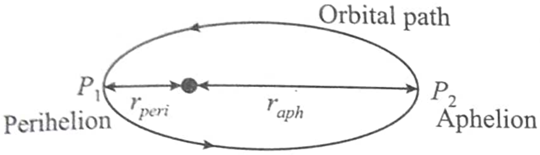
And from equation (2)

$$v_2 = \left(\sqrt{\frac{2G}{d(m_1+m_2)}}\right)m_1$$

Thus approach velocity is given as

$$v_{app} = v_1 + v_2 = \sqrt{\frac{2G(m_1+m_2)}{d}}$$

7. When a planet is at its closest distance to the sun, the planet is at perihelion, position P_1 in figure. When the planet is at its greatest distance from the sun, the planet is at aphelion, position P_2 in figure.



(*a*) What is the angle θ in the polar equation for an ellipse, when the planet is at aphelion?

(*b*) Show that the aphelion distance is related to the semi-major axis and the eccentricity of the ellipse by $r_{aph} = a(1+e)$.

(*c*) What is the angle θ when the planet is at perihelion?

(*d*) Show that the perihelion distance is related to the semi-major axis '*a*' and the eccentricity by $r_{peri} = a(1-e)$

Sol. (*a*) When the planet is at aphelion point P_2, the angle θ is equal to 0°.

(*b*) Polar equation of ellipse is $r = \dfrac{a(1-e^2)}{1-e\cos\theta}$

Substituting θ = 0°,

$$r_{aph} = \frac{a(1-e^2)}{1-e\cos 0^\circ} = \frac{a(1-e)(1+e)}{(1-e)} = a(1+e)$$

(*c*) When the planet is at perihelion the angle θ = 180°.

(*d*) Substitute θ = 180° into the equation for an ellipse.

$$r_{peri.} = \frac{a(1-e^2)}{1-e\cos 180^\circ} = \frac{a(1+e)(1-e)}{(1+e)} = a(1-e)$$

Note that the ratio of the aphelion and perihelion distances depends only on the eccentricity:

$$\frac{r_{aph.}}{r_{peri.}} = \frac{a(1+e)}{a(1-e)} = \frac{1+e}{1-e}$$

8. A projectile is fired straight up from the surface of the earth with an initial speed v_i = 9 km/s Neglecting air resistance, find the maximum height to which the projectile rises.

Sol: We will determine the maximum height by applying energy conservation law. We take earth's surface as initial point and maximum height as final point.

$$U_i + KE_i = U_f + KE_f$$

$$-\frac{GM_Em}{R_E} + \frac{1}{2}mv_i^2 = -\frac{GM_Em}{r} + 0$$

$$\frac{1}{2}v_i^2 = \frac{GM_E}{R_E}\left(1-\frac{R_E}{r}\right) = gR_E\left(1-\frac{R_E}{r}\right), \quad \text{...(i)}$$

$$\because \frac{GM_E}{R_E^2} = g$$

Equation (i) becomes

$$1-\frac{R_E}{r} = \frac{v_i^2}{2gR_E}$$

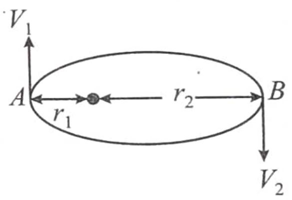
$$\Rightarrow r = \frac{R_E}{1-v_i^2/2gR_E}$$

$$\text{So, } \frac{v_i^2}{2gR_E} = \frac{(9000)^2}{2(9.81)(6.40\times10^6)} = 0.645$$

$$r = \frac{R_E}{1-0.645} = 2.81\ R_E$$

$$\therefore h = r - R_E = 1.81\ R_E$$

9. A satellite moves in an elliptical orbit around the earth. It was put in the orbit at a point with a speed 1.2*v*, where *v* is the speed for a circular orbit at that point. Find the ratio of maximum to minimum distance of the satellite from the earth.



Sol: The satellite moves in an elliptical orbit with r_1 and r_2 the minimum and maximum distances, then

$$v = \sqrt{\frac{GM_E}{r_1}} \text{ i.e. } GM_E = v^2 r_1$$

From conservation of energy between points *A* and *B*, we have

$$\frac{1}{2}m(1.2v)^2 - \frac{GM_Em}{r_1} = \frac{1}{2}mv_2^2 - \frac{GM_Em}{r_2}$$

$$\frac{1.44v^2}{2} - \frac{GM_E}{r_1} = \frac{v_2^2}{2} - \frac{GM_E}{r_2} \quad \text{...(i)}$$

From conservation of angular momentum,

$1.2\, vr_1 = v_2 r_2$ ...(ii)

$\Rightarrow v_2 = \dfrac{1.2vr_1}{r_2} = 1.2vk$

Let $k = r_1/r_2$.

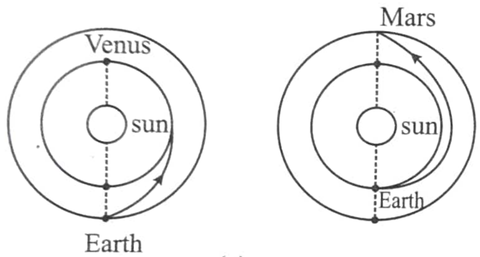
Now substituting v_2 and GM_E in equation (i), we obtain

$1 - k = 0.721\,(1 - k^2)$

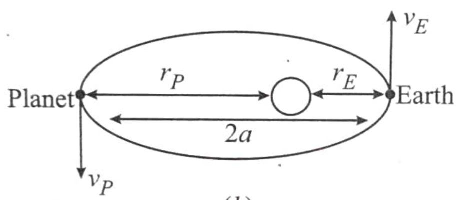
which on solving for k gives $k = 2.57$

Thus $\dfrac{r_2}{r_1} = 2.57$

10. Several planets of the solar system have been explored using unmanned vehicles or probes. First the vehicle must acquire the earth's escape velocity, then it must be placed in a solar orbit under the sun's gravitational attraction which carries the vehicle to a rendervous with the planet. The path of the vehicle has the earth at launch and the planet at arrival on a line that passes through the sun. These paths are called Hohmann orbits and are ellipses with sun at one focus. What is the velocity v_E of the probe from the earth, relative to sun and the velocity of the probe, v_p, relative to the sun as it approaches the planet.



(*a*)



(*b*)

Sol. The total energy of a probe of mass m in a Hohmann orbit is $E = -\dfrac{GMm}{2a}$, where M is the solar mass.

In terms of its kinetic energy and potential energy at launch point, earth,

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a} \quad \text{...(i)}$$

Solving for velocity we have $v^2 = GM\left(\dfrac{2a - r}{ar}\right)$ ...(ii)

As for an ellipse $2a = r_E + r_P$ ...(iii)

when the probe is launched from the earth, $r = r_E$

From equation (ii) we have,

$$v_E^2 = GM\left(\frac{2a - r_E}{ar_E}\right)$$

From equation (iii), $2a - r_E = r_p$

$$v_E = \left(\frac{GMr_p}{ar_E}\right)^{1/2}$$

This is the insertion velocity relative to the sun.

The velocity v_p, also relative to sun, of the probe as it approaches the planet is found by taking $r = r_p$ in eqn. (ii).

$$v_p^2 = GM\left(\frac{2a - r_p}{ar_p}\right)$$

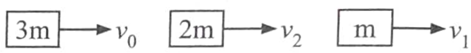
From equation (3),

$2a - r_p = r_E$

$$v_p = \left(\frac{GMr_E}{ar_p}\right)^{1/2}$$

11. A launching pad with a spaceship is moving along a circular orbit of the moon, whose radius R is triple that of moon's radius R_m. The space ship leaves the launching pad with a relative velocity equal to the launching pad's initial orbital velocity $\vec{v}_0$ and the launching pad then falls to the moon. Determine the angle θ with the horizontal at which the launching pad crashes into the surface of the moon if its mass is twice that of the spaceship.

Sol. Let's us assume launch pad leaves the ship with a velocity $\vec{v}_2$



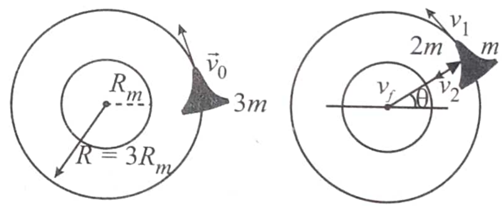
$\vec{v}_1 = \vec{v}_2 + \vec{v}_0$ $\quad \left\{v_0 = \sqrt{\dfrac{GM}{3R}}\right\}$ ...(i)

From conservation of linear momentum

$3m\vec{v}_0 = 2m\vec{v}_2 + m\vec{v}_1$ ...(ii)

From equation (i) and (ii)

$\vec{v}_2 = \dfrac{2}{3}\vec{v}_0 \Rightarrow v_2^2 = \dfrac{4}{9}\cdot\dfrac{GM}{3R}$



Assuming just before crash launchpad makes and angle θ with horizontal and lands with velocity v_f

Using angular momentum conservation

$2m \cdot v_2 \cdot 3R = 2m\, v_f \cos\theta \cdot R$

$$v_f = \frac{3v_2}{\cos\theta} \Rightarrow v_f^2 = \frac{9v_2^2}{\cos^2\theta} = \frac{9}{\cos^2\theta}\cdot\frac{4}{9}\cdot\frac{GM}{3R}$$

$$v_f^2 = \frac{4}{3\cos^2\theta}\cdot\frac{GM}{3R}$$

Using energy conservation:

$$\frac{-GmM}{3R} + \frac{1}{2}mv_2^2 = \frac{-GmM}{R} + \frac{1}{2}mv_f^2$$

$$\frac{-GmM}{3R}+\frac{1}{2}m\cdot\frac{4}{9}\cdot\frac{GM}{3R}=\frac{-GmM}{R}+\frac{1}{2}m\frac{4}{3\cos^2\theta}\cdot\frac{GM}{R}$$

$$\frac{-1}{3}+\frac{2}{27}=-1+\frac{2}{3\cos^2\theta}$$

$$\frac{2}{3\cos^2\theta}=1-\frac{1}{3}+\frac{2}{27}=\frac{27-9+2}{27}=\frac{20}{27}$$

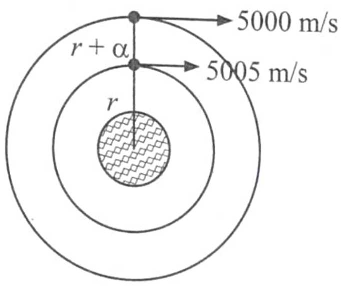
$$\cos^2\theta=\frac{2\times 27}{3\times 20}=\frac{9}{10}$$

$$\cos\theta=\frac{3}{\sqrt{10}}$$

12. An astronaut in a circular orbit around earth observes a celestial body moving in a lower circular orbit around earth in same plane as his orbit and in the same sense. He observes that the body moves at a speed of 5 m/s relative to himself when it is closest to him. The minimum distance between him and the body if he is moving at a speed of 5000 m/s is α km. Fill the value of $\frac{\alpha}{4}$ in OMR sheet.

(Mass of earth = 6×10^{24} kg & round off the value of α to the nearest integer).

Sol. $V=\sqrt{\frac{GM}{r}} \Rightarrow r=\frac{GM}{V^2}$



$r=\frac{GM}{(5005)^2}$ ...(i), $r+\alpha=\frac{GM}{(5000)^2}$ ...(ii)

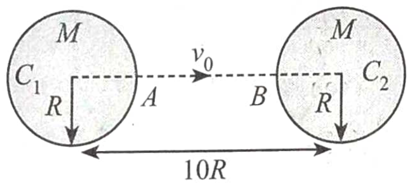
From equation (i) and (ii)

$$\alpha=\frac{GM}{(5000)^2}-\frac{GM}{(5005)^2}$$

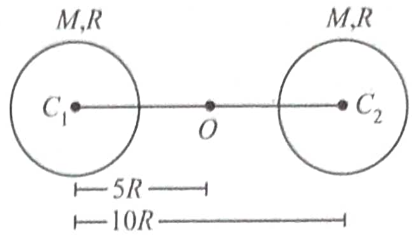
Put $G=6.67\times 10^{-11}\frac{\text{N}-\text{m}^2}{\text{Kg}^2}$ & $M = 6 \times 10^{24}$ kg

We have $\frac{\alpha}{4}=8$ km

13. Two large spherical object of mass M each (uniformly distributed) are fixed as shown in figure. A small point mass m is projected from point A heading towards center C_2 of second sphere. The minimum velocity of point mass so that it can reach upto second object at point B is $\frac{n}{3}\sqrt{\frac{GM}{3R}}$ then calculate n. [Neglect other gravitational forces]



Sol.



O is the point where gravitational field is zero due to both sphere. So, small particle have to cross point O only.

Using Energy Conservation

$$-\frac{GMm}{R}-\frac{GMm}{9R}+\frac{1}{2}mv^2=-\frac{GMm}{5R}\times 2+0$$

$$\frac{1}{2}mv^2=\frac{10GMm}{9R}-\frac{2GMm}{5R}$$

$$v^2=\frac{20GM}{9R}-\frac{4GM}{5R}$$

$$v^2=\frac{4GM}{R}\left(\frac{5}{9}-\frac{1}{5}\right)$$

$$v^2=\frac{4GM}{R}\left(\frac{16}{45}\right)$$

$$v=\frac{8}{3}\sqrt{\frac{GM}{5R}}$$

14. A satellite S is moving in an elliptical orbit around the earth. The mass of the satellite is very small compared to the mass of the earth

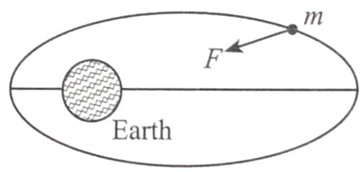
(*a*) The acceleration of S is always directed towards the centre of the earth

(*b*) The angular momentum of S about the centre of the earth changes in direction, but its magnitude remains constant

(*c*) The total mechanical energy of S varies periodically with time

(*d*) The linear momentum of S remains constant in magnitude

Sol. (*a*)



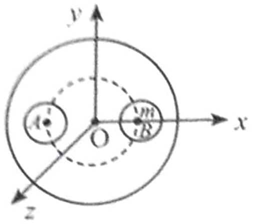
A. Since force on S will always act towards earth,
So, direction of acceleration of S is always directed towards centre of earth.

B. Since Force acting on S always pass through earth, So, angular momentum of S is constant about earth.

C. Total mechanical energy remains constant.

D. Linear momentum varies in magnitude as distance of S from earth changes with time.
Hence speed changes with time.

15. A solid sphere of uniform density and radius 4 units is located with its centre at the origin O of coordinates. Two spheres of equal radii 1 unit, with their centres at $A(-2, 0, 0)$ and $B(2, 0, 0)$ respectively, are taken out of the solid leaving behind spherical cavities as shown in figure. Then



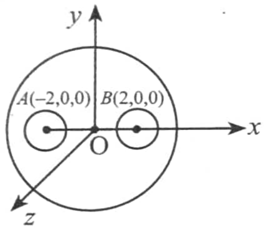
(*a*) The gravitational field due to this object at the origin is zero

(*b*) The gravitational field at the point B (2, 0, 0) is zero

(*c*) The gravitational potential is the same at all points of circle $y^2 + z^2 = 36$

(*d*) The gravitational potential is the same at all points on the circle $y^2 + z^2 = 4$

Sol. (*a,c,d*)



(i) Let $\vec{F}_A$ = gravitational field due to sphere *A* (if it were present at that position)

$\vec{F}_B$ = gravitational field due to sphere *B* (if it were present at that position)

$\vec{F}_R$ = gravitational field due to remaining portion after the cavities are made.

Then from superposition principle, we can see that $\vec{F}_A + \vec{F}_B + \vec{F}_R = 0$, as the force due to entire, sphere is zero at centre.

Now since $\vec{F}_A + \vec{F}_B = 0$ due to symmetry.

Hence $\vec{F}_R = 0$, hence option (i) is correct

(ii) Now at point *B*, Gravitation field inside a solid sphere of mass *M* at distance *r* from centre

$$\vec{F} = \frac{GM}{R^3}r(\hat{i}) = \frac{GM}{64}2(\hat{i}) = \frac{GM}{32}(\hat{i})$$

whereas gravitational field due to sphere *A* at point *B*

$$\vec{F}_A = \frac{Gm}{4^2}(\hat{i}) = \frac{GM}{16\times 64}(\hat{i}) = \frac{GM}{1024}(\hat{i})$$

(m = mass of sphere $A = M/64$) and

Gravitational field due to sphere *B* at its centre, $\vec{F}_B = 0$

Now, from superposition principle, we have

$$\vec{F}_A + \vec{F}_B + \vec{F}_R = \vec{F}$$

$$\Rightarrow \vec{F}_R = \vec{F} - \vec{F}_A = \frac{GM}{32}\hat{i} - \frac{GM}{1024}\hat{i} \neq 0$$

Hence option (ii) is not correct.

Regarding potential at a point on $y^2 + z^2 = 36$, we can see that the radius of the circle is 6 units, however, all the points on it are symmetrically located from remaining sphere. Hence potential must be same at every point on this circle. Same logic holds for the circle $y^2 + z^2 = 4$ also, though this circle lies inside the remaining sphere. Hence options (iii) and (iv) are also correct.

Hence options (i), (iii) and (iv) are correct.

Note, we can use superposition principle to calculate the potential at these points.

In option (iii) it will be equal to $-\frac{GM}{2}\left(\frac{1}{3} - \frac{1}{32\sqrt{10}}\right)$

In option (iv) it will be equal to $-\frac{GM}{2}\left(\frac{1}{3} - \frac{1}{32\sqrt{10}}\right)$

16. A double star is a system of two stars of masses m and 2m, rotating about their centre of mass only under their mutual gravitational attraction. If r is the separation between these two stars then their time period of rotation about their centre of mass will be proportional to :

(*a*) $r^{3/2}$ (*b*) r

(*c*) $m^{1/2}$ (*d*) $m^{-1/2}$

Sol.(*a,d*)

$r_1 = \frac{mr}{3m} = \frac{r}{3}$

$$\frac{Gm.2m}{r^2} = 2m\omega^2\left(\frac{r}{3}\right)$$

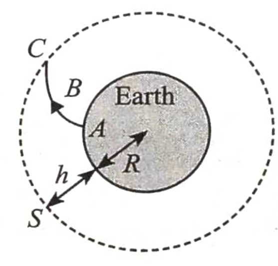
$$\omega^2 \propto \frac{m}{r^3} \Rightarrow \omega \propto \frac{m^{1/2}}{r^{3/2}}$$

$$T \propto \frac{1}{\omega} \Rightarrow T \propto r^{3/2}\ ;\ T \propto m^{-1/2}$$

Comprehension (Q. 17 to 18): An Earth satellite is a body placed in a stable orbit about the Earth. These satellites are used for communications, meteorology, Earth measurements (gravitation and magnetic fields), resource evaluation (water, minerals), transmission of radio and TV signals, and as reference points for navigation.

Although most satellites are launched from ground based stations, more recently some have been placed in orbit from one of NASA's space shuttles.

There are two requirements needed to place a satellite in a stable orbit at an insertion point *C*. (Figure). It is first necessary to bring the satellite to that altitude and then the satellite must be given the necessary orbiting velocity. The orbiting velocity for a circular orbit, also called the insertion velocity.



17. What is insertion velocity for very small h (<<R)

(a) $\left(\frac{GM}{R}\right)^{1/2}\left[1-\frac{h}{2R}\right]$ (b) $\left(\frac{2GM}{R}\right)^{1/2}\left[1-\frac{h}{2R}\right]$

(c) $\left(\frac{GM}{R}\right)^{1/2}\left[1-\frac{2h}{2R}\right]$ (d) $\left(\frac{GM}{2R}\right)^{1/2}\left[1-\frac{h}{2R}\right]$

Sol. (a) $\frac{-GmM}{(R+h)}+\frac{1}{2}mv^2=\frac{-1}{2}m\left(\sqrt{\frac{GM}{R+h}}\right)^2$

$$\frac{1}{2}mv^2=\frac{1}{2}\frac{GmM}{R+h}$$

$$V=\sqrt{\frac{GM}{R+h}}=\sqrt{\frac{GM}{R\left(1+\frac{h}{R}\right)}}$$

$$V=\sqrt{\frac{GM}{R}}\left(1+\frac{h}{R}\right)^{-1/2}$$

Using binomial approximation

$$V=\sqrt{\frac{GM}{R}}\left(1-\frac{h}{2R}\right)$$

18. If velocity at insertion point is k times escape velocity of the satellite at the insertion height h, then mark incorrect statement.

(a) For $k=1$ satellite moves in a circular orbit

(b) For $k\geq 1$ satellite moves in an unbound orbit

(c) For $k<1$ satellite may move in an elliptical orbit

(d) For $k<1$ satellite can move in circular orbit

Sol. (a) For escape

$$\frac{-GmM}{2(R+h)}+\frac{1}{2}mv^2=0$$

$$V_e=\sqrt{\frac{GM}{R+h}}\Rightarrow K=1$$

19. A particle is projected from surface of Earth with velocity v, such that it's total energy at height h becomes same as the total energy of a satellite of same mass, moving in circular orbit of same height. Value of v is

(a) $\sqrt{\frac{2GM}{R}}\left(1+\frac{h}{R+h}\right)^{1/2}$ (b) $V=\sqrt{\frac{GM}{R}}\left(1+\frac{h}{R+h}\right)^{1/2}$

(c) $\sqrt{\frac{2GM}{R}}\left(1-\frac{h}{R+h}\right)^{1/2}$ (d) $V=\sqrt{\frac{GM}{R}}\left(1-\frac{h}{R+h}\right)^{1/2}$

Sol. (b) $\frac{-GMm}{R}+\frac{1}{2}mv^2=\frac{-GmM}{2(R+h)}$

$$\frac{1}{2}mv^2=\frac{GMm}{R}-\frac{GMm}{2(R+h)}$$

$$V^2=\frac{2GM}{R}-\frac{GM}{R+h}$$

$$V^2=GM\left(\frac{2(R+h)-R}{R(R+h)}\right)$$

$$V^2=GM\cdot\frac{(R+2h)}{R.(R+h)}$$

$$V^2=\frac{GM}{R}\cdot\left(\frac{R+h}{R+h}+\frac{h}{R+h}\right)$$

$$V=\sqrt{\frac{GM}{R}}\left(1+\frac{h}{R+h}\right)^{1/2}$$

20. In elliptical orbit of a planet, as the planet moves from aphelion position to perihelion position,

	Column-I		Column-II
A.	Speed of planet	p.	Remains same
B.	Distance of planet from centre of Sun	q.	Decreases
C.	Potential energy	r.	Increases
D.	Angular momentum about centre of Sun	s.	Cannot say

(a) A-(r); B-(q); C-(q); D-(p)

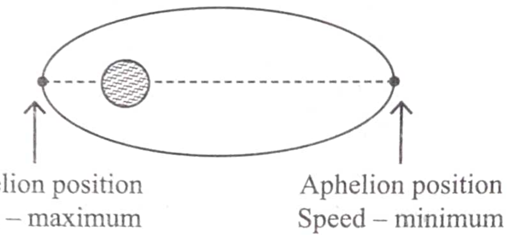
(b) A-(s); B-(q); C-(p); D-(p)

(c) A-(q); B-(r); C-(p); D-(s)

(d) A-(p); B-(r); C-(q); D-(p)

Sol. (a) Aphelion position → Fastest point

Perihelion position → Nearest point.



Potential Energy $=\frac{-GmM}{r}$

(Minimum at fastest point. So potential Energy decreases.)

Exercise-1 (Topicwise)

FORCE/FIELD

1. If the distance between two masses is doubled, the gravitational attraction between them
   (a) Is doubled
   (b) Becomes four times
   (c) Is reduced to half
   (d) Is reduced to a quarter

2. Two particles of equal mass go round a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is
   (a) $v = \frac{1}{2R}\sqrt{\frac{1}{Gm}}$ (b) $v = \sqrt{\frac{Gm}{2R}}$
   (c) $v = \frac{1}{2}\sqrt{\frac{Gm}{R}}$ (d) $v = \sqrt{\frac{4Gm}{R}}$

3. Gravitational mass is proportional to gravitational
   (a) Field (b) Force
   (c) Intensity (d) All of these

4. Earth binds the atmosphere because of
   (a) Gravity
   (b) Oxygen between earth and atmosphere
   (c) Both (a) and (b)
   (d) None of these

5. Which of the following statements about the gravitational constant is true
   (a) It is a force
   (b) It has no unit
   (c) It depends on the value of the masses
   (d) It does not depend on the nature of the medium in which the bodies are kept.

6. Two identical solid copper spheres of radius R placed in contact with each other. The gravitational attraction between them is proportional to
   (a) R^2 (b) R^{-2}
   (c) R^4 (d) R^{-4}

VARIATION OF g

7. The time period of a simple pendulum on a freely moving artificial satellite is
   (a) Zero (b) 2 sec
   (c) 3 sec (d) Infinite

8. A solid metallic sphere is placed in vacuum. It is given a small charge. Due to the sphere, the gravitational acceleration at a distance x from the centre (inside the sphere) is
   (a) Zero
   (b) Proportional to x
   (c) Inversely proportional to x^2
   (d) Inversely proportional to x

9. When a body is taken from the equator to the poles, its weight
   (a) Remains constant
   (b) Increases
   (c) Decreases
   (d) Increases at N-pole and decreases at S-pole

10. A body weighs 700 gm wt on the surface of the earth. How much will it weigh on the surface of a planet whose mass is $\frac{1}{7}$ and radius is half that of the earth
    (a) 200 gm wt (b) 400 gm wt
    (c) 50 gm wt (d) 300 gm wt

11. The radius of the earth is 6400 km and $g = 10$ m/sec^2. In order that a body of 5 kg weighs zero at the equator, the angular speed of the earth is
    (a) 1/80 radian/sec
    (b) 1/400 radian/sec
    (c) 1/800 radian/sec
    (d) 1/1600 radian/sec

12. Select the wrong statement
    The acceleration due to gravity 'g' decreases if
    (a) We go down from the surface of the earth towards its centre.
    (b) We go up from the surface of the earth.
    (c) We go from the equator towards the poles on the surface of the earth.
    (d) The rotational velocity of the earth is increased.

13. If the earth suddenly shrinks (without changing mass) to half of its present radius, the acceleration due to gravity will be
    (a) $g/2$ (b) $4g$
    (c) $g/4$ (d) $2g$

14. If the angular speed of the earth is doubled, the value of acceleration due to gravity (g) at the north pole
    (a) Doubles (b) Becomes half
    (c) Remains same (d) Becomes zero

15. Assume that the acceleration due to gravity on the surface of the moon is 0.2 times the acceleration due to gravity on the surface of the earth. If R_e is the maximum range of a projectile on the earth's surface, what is the maximum range on the surface of the moon for the same velocity of projection

(a) $0.2\ R_e$ (b) $2\ R_e$

(c) $0.5\ R_e$ (d) $5\ R_e$

16. What will be the acceleration due to gravity at height h if $h >> R$. Where R is radius of earth and g is acceleration due to gravity on the surface of earth

(a) $\dfrac{g}{\left(1+\dfrac{h}{R}\right)^2}$ (b) $g\left(1-\dfrac{2h}{R}\right)$

(c) $\dfrac{g}{\left(1-\dfrac{h}{R}\right)^2}$ (d) $g\left(1-\dfrac{h}{R}\right)$

17. Acceleration due to gravity on moon is 1/6 of the acceleration due to gravity on earth. If the ratio of densities of earth (ρ_e) and moon (ρ_m) is $\left(\dfrac{\rho_e}{\rho_m}\right)=\dfrac{5}{3}$ then radius of moon R_m in terms of R_e will be

(a) $\dfrac{5}{18}R_e$ (b) $\dfrac{1}{6}R_e$

(c) $\dfrac{3}{18}R_e$ (d) $\dfrac{1}{2\sqrt{3}}R_e$

GRAVITATION POTENTIAL, POTENTIAL ENERGY AND ESCAPE VELOCITY

18. A body of mass m rises to height $h = R/5$ from the earth's surface, where R is earth's radius. If g is acceleration due to gravity at earth's surface, the increase in potential energy is

(a) mgh (b) $\dfrac{4}{5}mgh$

(c) $\dfrac{5}{6}mgh$ (d) $\dfrac{6}{7}mgh$

19. In a gravitational field, at a point where the gravitational potential is zero

(a) The gravitational field is necessarily zero

(b) The gravitational field is not necessarily zero

(c) Nothing can be said definitely about the gravitational field

(d) None of these

20. The mass of the earth is 6.00×10^{24} kg and that of the moon is 7.40×10^{22} kg. The constant of gravitation $G = 6.67 \times 10^{-11}\ N\text{–}m^2/kg^2$. The potential energy of the system is -7.79×10^{28} joules. The mean distance between the earth and moon is

(a) 3.80×10^8 metres (b) 3.37×10^6 metres

(c) 7.60×10^4 metres (d) 1.90×10^2 metres

21. There are two bodies of masses 100 kg and 10000 kg separated by a distance 1 m. At what distance from the smaller body, the intensity of gravitational field will be zero

(a) $\dfrac{1}{9}m$ (b) $\dfrac{1}{10}m$

(c) $\dfrac{1}{11}m$ (d) $\dfrac{10}{11}m$

22. The gravitational potential energy of a body of mass 'm' at the earth's surface $-mgR_e$. Its gravitational potential energy at a height R_e from the earth's surface will be (Here R_e is the radius of the earth)

(a) $-2mgR_e$ (b) $2mgR_e$

(c) $\dfrac{1}{2}mgR_e$ (d) $-\dfrac{1}{2}mgR_e$

23. A body is projected vertically upwards from the surface of a planet of radius R with a velocity equal to half the escape velocity for that planet. The maximum height attained by the body is

(a) $R/3$ (b) $R/2$

(c) $R/4$ (d) $R/5$

24. Energy required to move a body of mass m from an orbit of radius 2R to 3R is

(a) $GMm/12R^2$ (b) $GMm/3R^2$

(c) $GMm/8R$ (d) $GMm/6R$

25. The escape velocity from the earth is about 11 km/second. The escape velocity from a planet having twice the radius and the same mean density as the earth, is

(a) 22 km/sec (b) 11 km/sec

(c) 5.5 km/sec (d) 15.5 km/sec

26. A missile is launched with a velocity less than the escape velocity. The sum of its kinetic and potential energy is

(a) Positive

(b) Negative

(c) Zero

(d) May be positive or negative depending upon its initial velocity

27. Four particles each of mass M, are located at the vertices of a square with side L. The gravitational potential due to this at the centre of the square is

(a) $-\sqrt{32}\dfrac{GM}{L}$ (b) $-\sqrt{64}\dfrac{GM}{L^2}$

(c) Zero (d) $\sqrt{32}\dfrac{GM}{L}$

SATELLITE MOTION

28. The escape velocity of a projectile from the earth is approximately

(a) 11.2 m/sec (b) 112 km/sec

(c) 11.2 km/sec (d) 11200 km/sec

29. If the radius of a planet is R and its density is ρ, the escape velocity from its surface will be

(a) $v_e \propto \rho R$
(b) $v_e \propto R\sqrt{\rho}$
(c) $v_e \propto \frac{\sqrt{\rho}}{R}$
(d) $v_e \propto \frac{1}{\sqrt{\rho R}}$

30. Which is constant for a satellite in orbit

(a) Velocity
(b) Angular momentum
(c) Potential energy
(d) Acceleration

31. Select the correct statement from the following

(a) The orbital velocity of a satellite increases with the radius of the orbit
(b) Escape velocity of a particle from the surface of the earth depends on the speed with which it is fired
(c) The time period of a satellite does not depend on the radius of the orbit
(d) The orbital velocity is inversely proportional to the square root of the radius of the orbit

32. The ratio of the K.E. required to be given to the satellite to escape earth's gravitational field to the K.E. required to be given so that the satellite moves in a circular orbit just above earth atmosphere is

(a) One
(b) Two
(c) Half
(d) Infinity

33. For a satellite escape velocity is 11 km/s. If the satellite is launched at an angle of 60° with the vertical, then escape velocity will be

(a) 11 km/s
(b) $11\sqrt{3}$ km/s
(c) $\frac{11}{\sqrt{3}}$ km/s
(d) 33 km/s

34. Which of the following statements is correct in respect of a geostationary satellite

(a) It moves in a plane containing the Greenwich meridian
(b) It moves in a plane perpendicular to the celestial equatorial plane
(c) Its height above the earth's surface is about the same as the radius of the earth
(d) Its height above the earth's surface is about six times the radius of the earth

35. An earth satellite S has an orbit radius which is 4 times that of a communication satellite C. The period of revolution of S is

(a) 4 days
(b) 8 days
(c) 16 days
(d) 32 days

36. If v_e and v_0 represent the escape velocity and orbital velocity of a satellite corresponding to a circular orbit of radius R, then

(a) $v_e = v_0$
(b) $\sqrt{2}v_0 = v_e$
(c) $v_e = v_0/\sqrt{2}$
(d) v_e and v_0 are not related

KEPLER'S LAWS

37. A satellite which is geostationary in a particular orbit is taken to another orbit. Its distance from the centre of earth in new orbit is 2 times that of the earlier orbit. The time period in the second orbit is

(a) 4.8 hours
(b) $48\sqrt{2}$ hours
(c) 24 hours
(d) $24\sqrt{2}$ hours

38. Kepler discovered

(a) Laws of motion
(b) Laws of rotational motion
(c) Laws of planetary motion
(d) Laws of curvilinear motion

39. The period of a satellite in a circular orbit of radius R is T, the period of another satellite in a circular orbit of radius $4R$ is:

(a) $4\,T$
(b) T/4
(c) $8\,T$
(d) T/8

40. Orbit of a planet around a star is

(a) A circle
(b) An ellipse
(c) A parabola
(d) A straight line

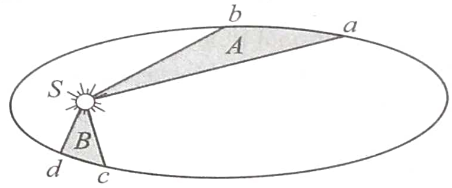
41. Two planets move around the sun. The periodic times and the mean radii of the orbits are T_1, T_2 and r_1, r_2 respectively. The ratio T_1/T_2 is equal to

(a) $(r_1/r_2)^{1/2}$
(b) r_1/r_2
(c) $(r_1/r_2)^2$
(d) $(r_1/r_2)^{3/2}$

42. A satellite of mass m is circulating around the earth with constant angular velocity. If radius of the orbit is R_0 and mass of the earth M, the angular momentum about the centre of the earth is

(a) $m\sqrt{GMR_0}$
(b) $M\sqrt{GMR_0}$
(c) $m\sqrt{\frac{GM}{R_0}}$
(d) $M\sqrt{\frac{GM}{R_0}}$

43. The figure shows the motion of a planet around the sun in an elliptical orbit with sun at the focus. The shaded areas A and B are also shown in the figure which can be assumed to be equal. If t_1 and t_2 represent the time for the planet to move from a to b and d to c respectively, then



(a) $t_1 < t_2$
(b) $t_1 > t_2$
(c) $t_1 = t_2$
(d) $t_1 \leq t_2$

44. The distance of a planet from the sun is 5 times the distance between the earth and the sun. The time period of the planet is

(a) $5^{3/2}$ years
(b) $5^{2/3}$ years
(c) $5^{1/3}$ years
(d) $5^{1/2}$ years

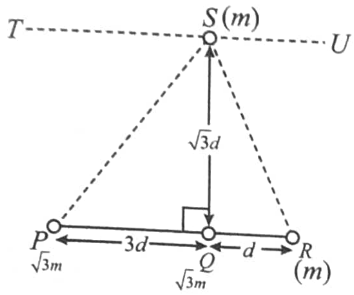
Exercise-2 (Learning Plus)

1. If the distance between sun and earth is made 3 times of the present value then gravitational force between them will become

(a) 9 times (b) $\frac{1}{9}$ times

(c) $\frac{1}{3}$ times (d) 3 times

2. Three particles P, Q and R are placed as per given figure. Masses of P, Q and R are $\sqrt{3}$ m, $\sqrt{3}$ m and m respectively. The gravitational force on a fourth particle 'S' of mass m is equal to



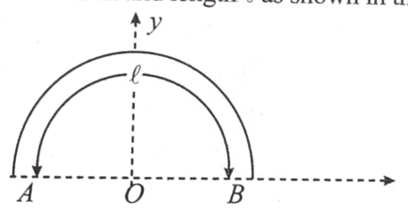
(a) $\frac{\sqrt{3}\,GM^2}{2d^2}$ in ST direction only

(b) $\frac{\sqrt{3}\,Gm^2}{2d^2}$ in SQ direction and $\frac{\sqrt{3}\,Gm^2}{2d^2}$ in SU direction

(c) $\frac{\sqrt{3}\,Gm^2}{2d^2}$ in SQ direction only

(d) $\frac{\sqrt{3}\,Gm^2}{2d^2}$ in SQ direction and $\frac{\sqrt{3}\,Gm^2}{2d^2}$ in ST direction

3. Gravitational field at the centre of a semicircle formed by a thin wire AB of mass m and length ℓ as shown in the figure is:



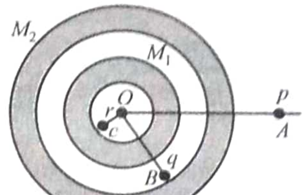
(a) $\frac{Gm}{\ell^2}$ along $+x$ axis

(b) $\frac{Gm}{\pi\ell^2}$ along $+y$ axis

(c) $\frac{2\pi Gm}{\ell^2}$ along $+x$ axis

(d) $\frac{2\pi Gm}{\ell^2}$ along $+y$ axis

4. Two concentric shells of uniform density of mass M_1 and M_2 are situated as shown in the figure. The forces experienced by a particle of mass m when placed at positions A, B and C respectively are (given $OA = p$, $OB = q$ and $OC = r$).



(a) 0, $G\frac{M_1m}{q^2}$ and $G\frac{(M_1+m_2)m}{p^2}$

(b) $G\frac{(M_1+M_2)m}{p^2}$, $G\frac{(M_1+M_2)m}{q^2}$ and $G\frac{M_1m}{r^2}$

(c) $G\frac{M_1m}{q^2}$, $\frac{G(M_1+M_2)m}{p^2}$, $G\frac{M_1m}{q^2}$ and 0

(d) $\frac{G(M_1+M_2)m}{p^2}$, $G\frac{M_1m}{q^2}$ and 0

5. If R is the radius of the earth and g the acceleration due to gravity on the earth's surface, the mean density of the earth is

(a) $4\pi G/3gR$ (b) $3\pi R/4gG$

(c) $3g/4\pi RG$ (d) $\pi Rg/12G$

6. The height above surface of earth where the value of gravitational acceleration is one fourth of that at surface, will be

(a) $R_e/4$ (b) $R_e/2$ (c) $3R_e/4$ (d) R_e

7. The decrease in the value of g on going to a height $R/2$ above the earth's surface will be

(a) $g/2$ (b) $\frac{5g}{4}$ (c) $\frac{4g}{9}$ (d) $\frac{g}{3}$

8. If the rotational motion of earth increases, then the weight of the body

(a) will remain same (b) will increase

(c) will decrease (d) none of these

9. If the acceleration due to gravity inside the earth is to be kept constant, then the relation between the density d and the distance r from the centre of earth will be

(a) $d \propto r$ (b) $d \propto r^{1/2}$

(c) $d \propto 1/r$ (d) $d \propto \frac{1}{r^2}$

10. If the radius of the earth be increased by a factor of 5, by what factor its density be changed to keep the value of g the same?

(a) 1/25 (b) 1/5

(c) $1/\sqrt{5}$ (d) 5

11. The mass and diameter of a planet are twice those of earth. What will be the period of oscillation of a pendulum on this planet if it is two seconds pendulum on earth?

(a) $\sqrt{2}$ second (b) $2\sqrt{2}$ seconds

(c) $\frac{1}{\sqrt{2}}$ second (d) $\frac{1}{2\sqrt{2}}$ second

12. If a body is carried from surface of earth to moon, then

(a) the weight of a body will continuously increase

(b) the mass of a body will continuously increase

(c) the weight of a body will decrease first, become zero and then increase,

(d) the mass of a body will decrease first, become zero and then increase.

13. Let gravitation field in a space be given as $E = -(k/r)$. If the reference point is at distance d_i where potential is V_i then relation for potential is

(a) $V = k\,\ell n\frac{1}{V_i} + 0$ (b) $V = k\,\ell n\frac{r}{d_i} + V_i$

(c) $V = \ell n\frac{r}{d_i} + kV_i$ (d) $V = \ell n\frac{r}{d_i} + \frac{V_i}{k}$

14. A very large number of particles of same mass m are kept at horizontal distances of 1m, 2m, 4m, 8m and so on from (0, 0) point. The total gravitational potential at this point is

(a) – 8G m (b) – 3G m

(c) – 4G m (d) – 2G m

15. A body starts from rest at a point, distance R_0 from the centre of the earth of mass M, radius R. The velocity acquired by the body when it reaches the surface of the earth will be

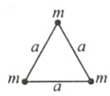
(a) $GM\left(\frac{1}{R}-\frac{1}{R_0}\right)$ (b) $2\,GM\left(\frac{1}{R}-\frac{1}{R_0}\right)$

(c) $\sqrt{2GM\left(\frac{1}{R}-\frac{1}{R_0}\right)}$ (d) $2GM\sqrt{\left(\frac{1}{R}-\frac{1}{R_0}\right)}$

16. Three equal masses each of mass 'm' are placed at the three-corners of an equilateral triangle of side 'a'. If a fourth particle of equal mass is placed at the centre of triangle, then net force acting on it, is equal to

(a) $\frac{Gm^2}{a^2}$ (b) $\frac{4Gm^2}{3a^2}$ (c) $\frac{3Gm^2}{a^2}$ (d) zero

17. If three particles system of equilateral triangle side a is to be changed to side of 2a, then work done on the system is equal to



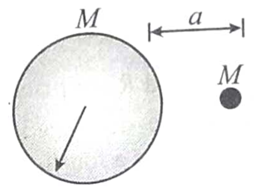
(a) $\frac{3Gm^2}{a}$ (b) $\frac{3Gm^2}{2a}$ (c) $\frac{4Gm^2}{3a}$ (d) $\frac{Gm^2}{a}$

18. In the above given three particle system, if two particles are kept fixed and third particle is released. Then speed of the particle when it reaches to the mid-point of the side connecting other two masses

(a) $\sqrt{\frac{2Gm}{a}}$ (b) $2\sqrt{\frac{Gm}{a}}$

(c) $\sqrt{\frac{Gm}{a}}$ (d) $\sqrt{\frac{Gm}{2a}}$

19. A particle of mass M is at a distance a from surface of a thin spherical shell of equal mass and having radius a.



(a) Gravitational field and potential both are zero at centre of the shell

(b) Gravitational field is zero not only inside the shell but at a point outside the shell also

(c) Inside the shell, gravitational field alone is zero

(d) Neither gravitational field nor gravitational potential is zero inside the shell

20. If the kinetic energy of a satellite orbiting around the earth is doubled then

(a) the satellite will escape into the space.

(b) the satellite will fall down on the earth

(c) radius of its orbit will be doubled

(d) radius of its orbit will become half.

21. Two planets A and B have the same material density. If the radius of A is twice that of B, then the ratio of the escape velocity $\frac{v_A}{v_B}$ is

(a) 2 (b) $\sqrt{2}$

(c) $1/\sqrt{2}$ (d) 1/2

22. A projectile is fired from the surface of earth of radius R with a speed kv_e in radially outward direction (where v_e is the escape velocity and $k < 1$). Neglecting air resistance, the maximum distance of rise from centre of earth is

(a) $\frac{R}{k^2+1}$ (b) k^2R (c) $\frac{R}{1-k^2}$ (d) kR

23. Two different masses are dropped from same heights, then just before these strike the ground, the following is same

(a) kinetic energy (b) potential energy

(c) linear momentum (d) Acceleration

24. Work done in taking a body of mass m to a height nR above surface of earth will be: (R = radius of earth)

(a) $mgnR$ (b) $mgR\ (n/n+1)$

(c) $mgR\frac{(n+1)}{n}$ (d) $\frac{mgR}{n(n+1)}$

25. The potential energy of a body of mass 3kg on the surface of a planet is 54 joule. The escape velocity will be
(a) 18m/s (b) 162 m/s
(c) 36 m/s (d) 6 m/s

26. A body of mass m is situated at a distance $4R_e$ above the earth's surface, where R_e is the radius of earth. How much minimum energy be given to the body so that it may escape
(a) mgR_e (b) $2mgR_e$
(c) $\frac{mgR_e}{5}$ (d) $\frac{mgR_e}{16}$

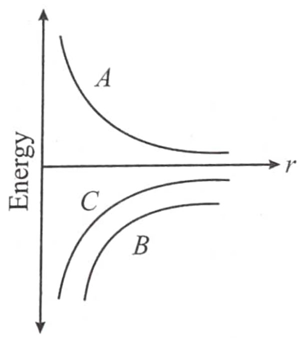
27. Periodic-time of satellite revolving around the earth is (ρ is density of earth)
(a) Proportional to $\frac{1}{\rho}$ (b) Proportional to $\frac{1}{\sqrt{\rho}}$
(c) Proportional ρ (d) does not depend on ρ.

28. A satellite of the earth is revolving in circular orbit with a uniform velocity V. If the gravitational force suddenly disappears, the statellite will
(a) continue to move with the same velocity in the same orbit
(b) move tangentially to the original orbit with velocity V
(c) fall down with increasing velocity
(d) come to a stop somewhere in its original orbit

29. A satellite revolves in the geostationary orbit but in a direction east to west. The time interval between its successive passing about a point on the equator is
(a) 48 hrs (b) 24 hrs (c) 12 hrs (d) never

30. Two point masses of mass 4m and m respectively separated by d distance are revolving under mutual force of attraction. Ratio of their kinetic energies will be
(a) 1 : 4 (b) 1 : 5 (c) 1 : 1 (d) 1 : 2

31. The figure shows the variation of energy with the orbit radius of a body in circular planetary motion. Find the correct statement about the curves A, B and C



(a) A shows the kinetic energy, B the total energy and C the potential energy of the system
(b) C shows the total energy, B the kinetic energy and A the potential energy of the system
(c) C and A are kinetic and potential energies respectively and B is the total energy of the system
(d) A and B are kinetic and potential energies and C is the total energy of the system

32. A planet of mass m revolves around the sun of mass M in an elliptical orbit. The minimum and maximum distance of the planet from the sun are r_1 & r_2 respectively. If the minimum velocity of the planet is $\sqrt{\frac{2GMr_1}{(r_1+r_2)r_2}}$ then it's maximum velocity will be
(a) $\sqrt{\frac{2GMr_2}{(r_1+r_2)r_1}}$ (b) $g\sqrt{\frac{2GMr_1}{(r_1+r_2)r_2}}$
(c) $\sqrt{\frac{2Gmr_2}{(r_1+r_2)r_1}}$ (d) $\sqrt{\frac{2GM}{r_1+r_2}}$

33. A planet has mass 1/10 of that of earth, while radius is 1/3 that of earth. If a person can throw a stone on earth surface to a height of 90m, then he will be able to throw the stone on that planet to a height
(a) 90m (b) 40 m
(c) 100 m (d) 45 m

34. The ratio of distances of satellites A and B from the centre of the earth is 1.4 : 1, then the ratio of energies of satellites B and A will be
(a) 1.4 : 1 (b) 2 : 1
(c) 1 : 3 (d) 4 : 1

35. A satellite can be in a geostationary orbit around earth at a distance r from the centre. If the angular velocity of earth about its axis doubles, a satellite can now be in a geostationary orbit around earth if its distance from the center is
(a) $\frac{r}{2}$ (b) $\frac{r}{2\sqrt{2}}$
(c) $\frac{r}{(4)^{1/3}}$ (d) $\frac{r}{(2)^{1/3}}$

36. A body is dropped by a satellite in its geo-stationary orbit
(a) it will burn on entering into the atmosphere.
(b) it will remain in the same place with respect to the earth.
(c) it will reach the earth in 24 hours
(d) it will perform uncertain motion

37. The orbital velocity of an artificial satellite in a circular orbit just above the earth's surface is v. For a satellite orbiting at an altitude of half of the earth's radius, the orbital velocity is
(a) $\frac{3}{2}v$ (b) $\sqrt{\frac{3}{2}}v$
(c) $\sqrt{\frac{2}{3}}v$ (d) $\frac{2}{3}v$

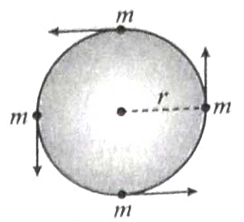
38. When a satellite moves around the earth in a certain orbit, the quantity which remains constant is
(a) Angular velocity
(b) Kinetic energy
(c) Areal velocity
(d) Potential energy

39. A planet of mass m is in an elliptical orbit about the sun ($m << M_{sun}$) with an orbital period T. If A be the area of orbit, then its angular momentum would be

(a) $\frac{2mA}{T}$ (b) mAT

(c) $\frac{mA}{2T}$ (d) $2mAT$

40. Four similar particles of mass m are orbiting in a circle of radius r in the same direction and same speed because of their mutual gravitational attractive force as shown in the figure. Velocity of a particle is given by



(a) $\left[\frac{Gm}{r}\left(\frac{1+2\sqrt{2}}{4}\right)\right]^{\frac{1}{2}}$ (b) $\sqrt[3]{\frac{Gm}{r}}$

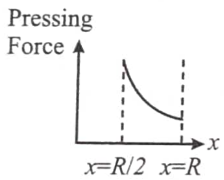
(c) $\sqrt{\frac{Gm}{r}\left(1+2\sqrt{2}\right)}$ (d) zero

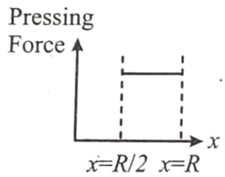
Exercise-3 (JEE Advanced Level)

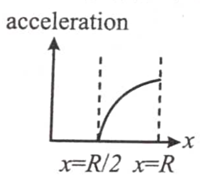
MULTIPLE CORRECT TYPE QUESTIONS

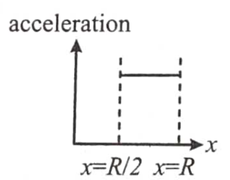
1. Assuming the earth to be a sphere of uniform density the acceleration due to gravity
   (a) At a point outside the earth is inversely proportional to the square of its distance from the center
   (b) At a point outside the earth is inversely proportional to its distance from the centre
   (c) At a point inside is zero
   (d) At a point inside is proportional to its distance from the centre

2. A tunnel is dug along a chord of the earth at a perpendicular distance $R/2$ from the earth's centre. The wall of the tunnel may be assumed to be frictionless. A particle is released from one end of the tunnel. The pressing force by the particle on the wall and the acceleration of the particle varies with x (distance of the particle from the centre) according to:

(a)


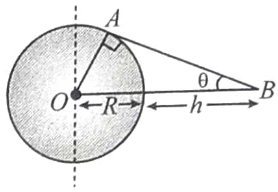
(b)


(c)


(d)


3. In side a hollow spherical shell
   (a) Everywhere gravitational potential is zero
   (b) Everywhere gravitational field is zero
   (c) Everywhere gravitational potential is same
   (d) Everywhere gravitational field is same

4. A geostationary satellite is at a height h above the surfac of earth. If earth radius is R



   (a) The minimum colatitude on earth upto which the satellit can be used for communication is $\sin^{-1}\{R/(R+h)\}$
   (b) The maximum colatitudes on earth upto which th satellite can be used for communication is $\sin^{-1}\{R/(R+h)\}$
   (c) The area on earth escaped from this satellite is given a $2\pi R^2(1+\sin\theta)$
   (d) The area on earth escaped from this satellite is given a $2\pi R^2(1+\cos\theta)$

5. Inside an isolated uniform spherical shell:
   (a) The gravitation potential is not zero
   (b) The gravitational field is not zero
   (c) The gravitational potential is same everywhere
   (d) The gravitational field is same everywhere.

6. A satellite close to the earth is in orbit above the equator wit a period of revolution of 1.5 hours. If it is above a point P on the equator at some time, it will be above P again afte time
   (a) 1.5 hours
   (b) 1.6 hours if it is rotating from west to east
   (c) 24/17 hours if it is rotating from east to west
   (d) 24/17 hours if it is rotating from west to east

7. In case of earth
   (a) Field is zero, both at centre and infinity
   (b) Potential is zero, both at centre and infinity
   (c) Potential is same, both at centre and infinity but not zero
   (d) Potential is minimum at the centre

8. A communications Earth satellite
   (a) Goes round the earth from east to west
   (b) Can be in the equatorial plane only
   (c) Can be vertically above any place on the earth
   (d) Goes round the earth from west to east

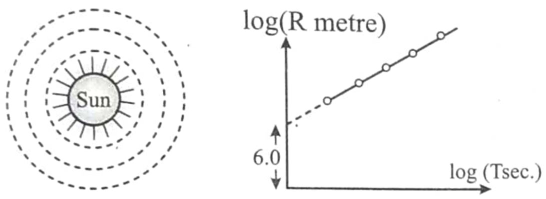
9. For a satellite to orbit around the earth, which of the following must be true?
   (a) It must be above the equator at some time
   (b) It cannot pass over the poles at any time
   (c) Its height above the surface cannot exceed 36,000 km
   (d) Its period of rotation must be $> 2\pi\sqrt{R/g}$ where R is radius of earth

COMPREHENSION BASED QUESTIONS

Comprehension (Q. 10 to 12): Many planets are revolving around the fixed sun, in circular orbits of different radius (R) and different time period (T). To estimate the mass of the sun, the orbital radius (R) and time period (T) of planets were noted. Then $\log_{10} T$ v/s $\log_{10} R$ curve was plotted.

The curve was found to be approximately straight line (as shown in figure) having y intercept = 6.0. (Neglect the gravitational interaction among the planets.)

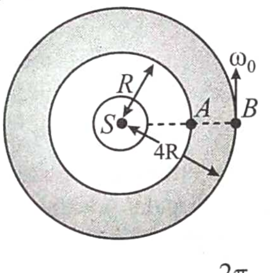
[Take $G = \frac{20}{3} \times 10^{-11}$ Nm^2/kg^2, $\pi^2 = 10$]



10. The slope of the line should be
    (a) 1 (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $\frac{19}{4}$

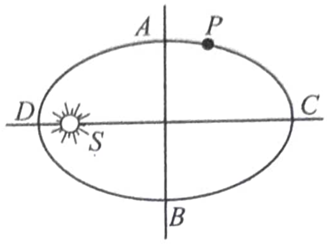
11. Estimate the mass of the sun
    (a) 6×10^{29} kg (b) 5×10^{20} kg
    (c) 8×10^{25} kg (d) 3×10^{35} kg

12. Two planets A and B, having orbital radius R and $4R$ are initially at the closest position and rotating in the same direction. If angular velocity of planet B is ω_0, then after how much time will both the planets be again in the closest position? (Neglect the interaction between planets)



    (a) $\frac{2\pi}{7\omega_0}$ (b) $\frac{2\pi}{9\omega_0}$
    (c) $\frac{2\pi}{\omega_0}$ (d) $\frac{2\pi}{5\omega_0}$

Comprehension (Q. 13 to 14): Figure shows the orbit of a planet P round the sun S. AB and CD are the minor and major axes of the ellipse.



13. If t_1 is the time taken by the planet to travel along ACB and t_2 the time along BDA, then
    (a) $t_1 = t_2$
    (b) $t_1 > t_2$
    (c) $t_1 < t_2$
    (d) nothing can be concluded

14. If U is the potential energy and K kinetic energy then $|U| > |K|$ at
    (a) Only D (b) Only C
    (c) Both D & C (d) neither D nor C

MATCH THE COLUMN TYPE QUESTIONS

15. A particle is taken to a distance r (> R) from centre of the earth. R is radius of the earth. It is given velocity V which is perpendicular to $\vec{r}$. With the given values of V in column I you have to match the values of total energy of particle in column II and the resultant path of particle in column III. Here 'G' is the universal gravitational constant and 'M' is the mass of the earth.

Column I (Velocity)		Column II (Total energy)		Column III (Path)	
A.	$V = \sqrt{GM/r}$	p.	Negative	t.	Elliptical
B.	$V = \sqrt{2GM/r}$	q.	Positive	u.	Parabolic
C.	$V > \sqrt{2GM/r}$	r.	Zero	v.	Hyperbolic
D.	$\sqrt{GM/r} < V < \sqrt{2GM/r}$	s.	Infinite	w.	Circular

    (a) A-(p,w); B-(r,u); C-(q,v); D-(p,t)
    (b) A-(p,t); B-(p,w); C-(p,r); D-(q,u)
    (c) A-(p,u); B-(q,w); C-(t,r); D-(q,v)
    (d) A-(r,w); B-(q,t); C-(q,r); D-(p,t)

16. Let V and E denote the gravitational potential and gravitational field respectively at a point due to certain uniform mass distribution described in four different situations of column-I. Assume the gravitational potential at infinity to be zero. The value of E and V are given in column-II. Match the statement in column-I with results in column-II.

	Column-I		Column-II
A.	At centre of thin spherical shell	p.	$E = 0$
B.	At centre of solid sphere	q.	$E \neq 0$
C.	A solid sphere has a non-concentric spherical cavity At the centre of the spherical cavity	r.	$V \neq 0$
D.	At centre of line joining two point masses of equal magnitude	s.	$V = 0$

(*a*) A-(r); B-(p); C-(q); D-(s)

(*b*) A-(q); B-(p); C-(r); D-(s)

(*c*) A-(p); B-(q); C-(q); D-(r)

(*d*) A-(r); B-(q); C-(r); D-(q)

NUMERICAL TYPE QUESTIONS

17. A satellite of earth is in near earth circular orbit. What is its speed (in km/s). Round off to nearest integer.

18. Gravitational potential difference between a point on surface of planet and another point 10 m above is 4J /kg. Considering gravitational field to be uniform , how much work is done by an external agent in moving a mass of 2.0 kg from the surface to a point 5.0 m above the surface slowly.

19. You are at a distance of $R = 1.5 \times 10^6$ m from the centre of an unknown planet. You notice that if you throw a ball horizontally it goes completely around the planet hitting you in the back 90,000 seconds later with exactly the same speed that you originally threw it. If the length of semi major axis of ball is $2R$, what is the mass of the planet. Express in form $a \times 10^b$ kg and fill $a + b$ in *OMR* sheet.

[Take: $G = \frac{20}{3} \times 10^{-11}\ Nm^2/kg^2$, $\pi^2 = 10$]

20. Imagine a frictionless tunnel along a chord of earth having length equal to radius of earth R. Particle is thrown from surface of earth inside it along its length at a velocity $\sqrt{\frac{3gR}{4}}$ where g is acceleration due to gravity on surface of earth. Find the time (in sec) taken by particle to cross the tunnel. **Take** $g = 10$ m/s^2, $\pi = 3.14$ and $R = 6400$ km.

21. A spherical planet has no atmosphere and consists of pure gold. Find the minimum orbital period T (in sec.) for a satellite circling the planet.

Take density of gold as $5\pi \times 10^3$ kg/m^3 and

$G = \frac{20}{3} \times 10^{-11}\ Nm^2/kg^2$

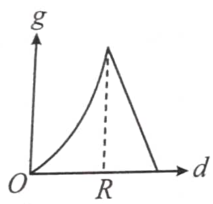
22. Two particles of mass 'm' and $3m$ are initially at rest an infinite distance apart. Both the particles start moving due to gravitational attraction. At any instant their relative velocity of approach is $\sqrt{\frac{\eta Gm}{d}}$, where 'd' is their separation at that instant. Find η.

Exercise-4 (Past Year Questions)

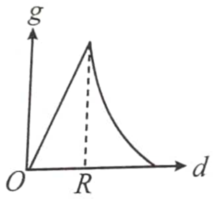
JEE MAIN

1. The variation of acceleration due to gravity g with distance d from centre of the earth is best represented by (R = Earth's radius): **(2017)**

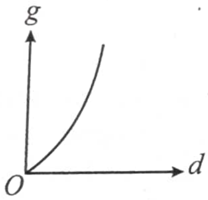
(*a*)



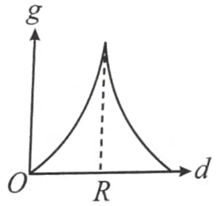
(*b*)



(*c*)



(*d*)



2. A satellite is revolving in a circular orbit at a height 'h' from the earth's surface (radius of earth R ; $h << R$). The minimum increase in its orbital velocity required, so that the satellite could escape from the earth's gravitational field, is close to: (Neglect the effect of atmosphere.) **(2019)**

(*a*) $\sqrt{2gr}$ (*b*) $\sqrt{gr}$

(*c*) $\sqrt{gr/2}$ (*d*) $\sqrt{gr}(\sqrt{2}-1)$

3. A satellite is moving with a constant speed υ in circular orbit around the earth. An object of mass 'm' is ejected from the satellite such that it just escapes from the gravitational pull of the earth. At the time of ejection, the kinetic energy of the object is **(2019)**

(*a*) $2m\upsilon^2$ (*b*) $m\upsilon^2$

(*c*) $\frac{1}{2}m\upsilon^2$ (*d*) $\frac{3}{2}m\upsilon^2$

4. If the angular momentum of a planet of mass m, moving a round the Sun in a circular orbit is L, about the center of the Sun, its areal velocity is **(2019)**

(*a*) $\frac{L}{m}$ (*b*) $\frac{4L}{m}$

(*c*) $\frac{L}{2m}$ (*d*) $\frac{2L}{m}$

5. Two satellites, A and B, have masses m and 2m respectively. A is in a circular orbit of radius R, and B is in a circular orbit of radius $2R$ around the earth. The ratio of their kinetic energies, T_A/T_B is **(2019)**

(a) $\frac{1}{2}$ (b) 1

(c) 2 (d) $\sqrt{\frac{1}{2}}$

6. A satellite of mass M is in a circular orbit of radius R about the centre of the earth. A meteorite of the same mass, falling towards the earth collides with the satellite completely inelastically. The speeds of the satellite and the meteorite are the same, just before the collision. The subsequent motion of the combined body will be **(2019)**

(a) such that it escapes to infinity
(b) in an elliptical orbit
(c) in the same circular orbit of radius R
(d) in a circular orbit of a different radius

7. A straight rod of length L extends from $x = a$ to $x = L + a$. The gravitational force it exerts on a point mass 'm' at $x = 0$, if the mass per unit length of the rod is $A + Bx^2$, is given by **(2019)**

(a) $Gm\left[A\left(\frac{1}{a+L}-\frac{1}{a}\right)-BL\right]$

(b) $Gm\left[A\left(\frac{1}{a}-\frac{1}{a+L}\right)-BL\right]$

(c) $Gm\left[A\left(\frac{1}{a+L}-\frac{1}{a}\right)+BL\right]$

(d) $Gm\left[A\left(\frac{1}{a}-\frac{1}{a+L}\right)+BL\right]$

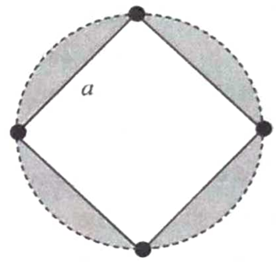
8. Two stars of masses 3×10^{31} kg each, and at distance 2×10^{11} m rotate in a plane about their common centre of mass O. A meteorite passes through O moving perpendicular to the star's rotation plane. In order ot escape from the gravitational field of this double star, the minimum speed that meteorite should have at O is (Take Gravitational constant $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}) **(2019)**

(a) 2.4×10^4 m/s
(b) 1.4×10^5 m/s
(c) 3.8×10^4 m/s
(d) 2.8×10^5 m/s

9. A test particle is moving in a circular orbit in the gravitational field produced by a mass density $\rho(r) = \frac{K}{r^2}$. Identify the correct relation between the radius R of the particle's orbit and its period T **(2019)**

(a) T/R^2 is a constant
(b) TR is a constant
(c) T^2/R^3 is a constant
(d) $T\backslash R$ is a constant

10. Four identical particles of mass M are located at the corners of a square of side 'a'. What should be their speed if each of them revolves under the influence of other's gravitational field in a circular orbit circumscribing the square? **(2019)**



(a) $1.21\sqrt{\frac{GM}{a}}$ (b) $1.41\sqrt{\frac{GM}{a}}$

(c) $1.16\sqrt{\frac{GM}{a}}$ (d) $1.35\sqrt{\frac{GM}{a}}$

11. A spaceship orbits around a planet at a height of 20 km from its surface. Assuming that only gravitational field of the planet acts on the spaceship, what will be the number of complete revolutions made by the spaceship in 24 hours around the planet?
[**Given:** Mass of planet = 8×10^{22} kg;
Radius of planet = 2×10^6 m,
Gravitational constant $G = 6.67 \times 10^{-11}$ Nm^2/ kg^2] **(2019)**

(a) 9 (b) 11
(c) 13 (d) 17

12. A rocket has to be launched from earth in such a way that it never returns. If E is the minimum energy delivered by the rocket launcher, what should be the minimum energy that the launcher should be the minimum energy that the launcher should have if the same rocket is to be launched from the surface of the moon? Assume that the density of the earth and the moon are equal and that the earth's volume is 64 times the volume of the moon **(2019)**

(a) $\frac{E}{4}$ (b) $\frac{E}{16}$

(c) $\frac{E}{32}$ (d) $\frac{E}{64}$

13. A solid sphere of mass 'M' and radius 'a' is surrounded by a uniform concentric spherical shell of thickness $2a$ and mass $2M$. The gravitational field at distance '$3a$' from the centre will be: **(2019)**

(a) $\frac{2GM}{9a^2}$ (b) $\frac{GM}{3a^2}$ (c) $\frac{GM}{9a^2}$ (d) $\frac{2GM}{3a^2}$

14. The ratio of the weights of a body on the Earth's surface to that on the surface of a planet is 9 : 4. The mass of the planet is 1/9 th of that of the Earth. If 'R' is the radius of the Earth, what is the radius of the planet? (Take the planets to have the same mass density) **(2019)**

(a) $\frac{R}{3}$ (b) $\frac{R}{2}$ (c) $\frac{R}{4}$ (d) $\frac{R}{9}$

15. The value of acceleration due to gravity at Earth's surface is $9.8\ ms^{-2}$. The altitude above its surface at which the acceleration due to gravity decreases to $4.9\ ms^{-2}$, is close to: (Radius of earth $= 6.4 \times 10^6$ m) **(2019)**

(*a*) 1.6×10^6 m (*b*) 6.4×10^6 m

(*c*) 9.0×10^6 m (*d*) 2.6×10^6 m

16. A satellite of mass m is launched vertically upwards with an initial speed u from the surface of the earth. After it reaches height R(R = radius of the earth), it ejects a rocket of mass $\frac{m}{10}$ so that subsequently the satellite moves in a circular orbit. The kinetic energy of the rocket is (G is the gravitational constant; M is the mass of the earth) **(2020)**

(*a*) $\frac{m}{20}\left(u^2+\frac{113}{200}\frac{GM}{R}\right)$ (*b*) $5m\left(u^2-\frac{119}{200}\frac{GM}{R}\right)$

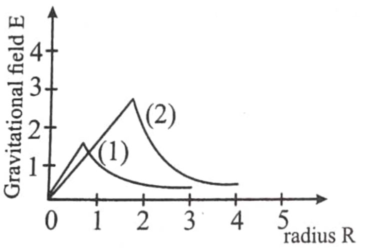
(*c*) $\frac{3m}{8}\left(u+\sqrt{\frac{5GM}{6R}}\right)^2$ (*d*) $\frac{m}{2}\left(u-\sqrt{\frac{2GM}{3R}}\right)^2$

17. A box weights 196 N on a spring balance at the north pole. Its weight recorded on the same balance if it is shifted to the equator is close to (Take $g = 10\ ms^{-2}$ at the north pole and the radius of the earth = 6400 km) **(2020)**

(*a*) 194.66 N (*b*) 195.66 N

(*c*) 195.32 N (*d*) 194.32 N

18. Consider two solid spheres of radii $R_1 = 1$m, $R_2 = 2$m and masses M_1 and M_2, respectively. The gravitational field due to sphere (*a*) and (*b*) are shown. The value of $\frac{M_1}{M_2}$ is **(2020)**



(*a*) $\frac{2}{3}$ (*b*) $\frac{1}{3}$

(*c*) $\frac{1}{2}$ (*d*) $\frac{1}{6}$

19. An asteroid is moving directly towards the centre of the earth. When at a distance of $10R$ (R is the radius of the earth) from the earths centre, it has a speed of 12 km/s. Neglecting the effect of earths atmosphere, what will be the speed of the asteroid when it hits the surface of the earth (escape velocity from the earth is 11.2 km/s^{-1})? Give your answer to the nearest integer in kilometer/s____________. **(2020)**

20. Planet A has mass M and radius R. Planet B has half the mass and half the radius of Planet A. If the escape velocities from the Planets A and B are V_A and V_B, respectively, then $V_A/V_B = n/4$. The value of n is **(2020)**

(*a*) 3 (*b*) 2

(*c*) 1 (*d*) 4

21. The height 'h' at which the weight of a body will be the same as that at the same depth 'h' from the surface of the earth is (Radius of the earth is R and effect of the rotation of the earth is neglected) **(2020)**

(*a*) $\frac{\sqrt{5}R-R}{2}$ (*b*) $\frac{\sqrt{3}R-R}{2}$

(*c*) $\frac{R}{2}$ (*d*) $\frac{\sqrt{5}}{2}R-R$

22. The mass density of a spherical galaxy varies as $\frac{K}{r}$ over a large distance 'r' from it s centre. In that region, a small star is in a circular orbit of radius R. Then the period of revolution, T depends on R as **(2020)**

(*a*) $T^2 \propto \frac{1}{R^3}$ (*b*) $T^2 \propto R$

(*c*) $T \propto R$ (*d*) $T^2 \propto R^3$

23. The mass density of a planet of radius R varies with the distance r from its centre as $\rho(r) = \rho_0\left(1-\frac{r^2}{R^2}\right)$. Then the gravitational field is maximum at **(2020)**

(*a*) $r = R\frac{1}{\sqrt{3}}$ (*b*) $r = \sqrt{\frac{3}{4}}R$

(*c*) $r = \sqrt{\frac{5}{9}}R$ (*c*) $r = R$

24. A satellite is moving in a low nearly circular orbit around the earth. Its radius is roughly equal to that of the earth's radius R_e. By firing rockets attached to it, its speed is instantaneously increased in the direction of its motion so that it become $\sqrt{\frac{3}{2}}$ times larger. Due to this the farthest distance from the centre of the earth that the satellite reaches is R. Value of R is **(2020)**

(*a*) $3R_e$ (*b*) $4R_e$

(*c*) $2.5R_e$ (*d*) $2R_e$

25. A body is moving in a low circular orbit about a planet of mass M and radius R. The radius of the orbit can be taken to be R itself. Then the ratio of the speed of this body in the orbit to the escape velocity from the planet is **(2020)**

(*a*) 1 (*b*) $\sqrt{2}$

(*c*) 2 (*d*) $\frac{1}{\sqrt{2}}$

26. On the x-axis and at a distance x from the origin, the gravitational field due to a mass distribution is given by $\frac{Ax}{(x^2+a^2)^{3/2}}$ in the x-direction. The magnitude of gravitational potential on the x-axis at a distance x, taking its value to be zero at infinity, is **(2020)**

(*a*) $A(x^2+a^2)^{3/2}$ (*b*) $\frac{A}{(x^2+a^2)^{3/2}}$

(*c*) $\frac{A}{(x^2+a^2)^{1/2}}$ (*d*) $A(x^2+a^2)^{1/2}$

27. The acceleration due to gravity on the earth's surface at the poles is g and angular velocity of the earth about the axis passing through the pole is ω. An object is weighed at the equator and at a height h above the poles by using a spring balance. If the weights are found to be same, then h is ($h << R$, where R is the radius of the earth) **(2020)**

(a) $\frac{R^2\omega^2}{2g}$ (b) $\frac{R^2\omega^2}{g}$

(c) $\frac{R^2\omega^2}{8g}$ (d) $\frac{R^2\omega^2}{4g}$

28. The value of the acceleration due to gravity is g_1 at a height $h = \frac{R}{2}$ (R = radius of the earth) from the surface of the earth. It is again equal to g_1 at a depth d below the surface of the earth. The ratio $\left(\frac{d}{R}\right)$ equals **(2020)**

(a) $\frac{5}{9}$ (b) $\frac{1}{3}$

(c) $\frac{7}{9}$ (d) $\frac{4}{9}$

29. Two planets have masses M and $16M$ and their radii are a and $2a$, respectively. The separation between the centres of the planets is $10a$. A body of mass m is fired from the surface of the larger planet towards the smaller planet along the line joining their centres. For the body to be able to reach at the surface of smaller planet, the minimum firing speed needed is **(2020)**

(a) $\frac{3}{2}\sqrt{\frac{5GM}{a}}$ (b) $\sqrt{\frac{GM^2}{ma}}$

(c) $2\sqrt{\frac{GM}{ma}}$ (d) $4\sqrt{\frac{GM}{a}}$

30. A satellite is in an elliptical orbit around a planet P. It is observed that the velocity of the satellite when it is farthest from the planet is $\frac{1}{6}$ times of that when it is closest to the planet. The ratio of distances between the satellite and the planet at closest and farthest points is **(2020)**

(a) 1 : 6 (b) 1 : 2

(c) 3 : 4 (d) 1 : 3

31. The radius in kilometre to which the present radius of earth (R = 6400 km) to be compressed so that the escape velocity is increased 10 times is **(2021)**

32. If one wants to remove all the mass of the earth to infinity in order to break it up completely. **(2021)**
The amount of energy that needs to be supplied will be $\frac{x}{5}\frac{GM^2}{R}$ where x is (Round off to the Nearest Integer)
(M is the mass of earth, R is the radius of earth, G is the gravitational constant)

33. The approximate height from the surface of earth at which the weight of the body becomes 1/3 of its weight on the surface of earth is [Radius of earth R = 6400 km and $\sqrt{3}$ = 1.732] **(2022)**

(a) 3840 km
(b) 4685 km
(c) 2133 km
(d) 4267 km

34. The distance between Sun and Earth is R. The duration of year if the distance between Sun and Earth becomes $3R$ will be **(2022)**

(a) $\sqrt{3}$ years
(b) 3 years
(c) 9 years
(d) $3\sqrt{3}$ years

35. The height of any point P above the surface of earth is equal to diameter of earth. Hie value of acceleration due to gravity at point P will be (Given g = acceleration due to gravity at the surface of earth) **(2022)**

(a) $g/2$ (b) $g/4$

(c) $g/3$ (d) $g/9$

36. Two satellites S_1 and S_2 are revolving in circular orbits around a planet with radius R_1 = 3200 km and R_2 = 800 km respectively. The ratio of speed of satellite S_1 to the speed of satellite S_2 in their respective orbits would be $1/x$ where x = **(2022)**

37. Two planets A and B of equal mass are having their period of revolutions T_A and T_B such that $T_A = 2T_B$. These planets are revolving in the circular orbits of radii r_A and r_B respectively. Which out of the following would be the correct relationship of their orbits? **(2022)**

(a) $2r_A^2 = r_B^2$
(b) $r_A^3 = 2r_B^3$
(c) $r_A^3 = 4r_B^3$
(d) $T_A^2 - T_B^2 = \frac{\pi^2}{GM}(r_B^3 - 4r_A^3)$

JEE ADVANCED

38. Two spherical planets P and Q have the same unfirom density ρ, masses M_P and M_Q, with surface areas A and $4A$, respectively. A spherical planet R also has unfirom density ρ and its mass is $(M_P + M_Q)$. The escape velocities from the planets P, Q and R, are V_P, V_Q and V respectively. Then **(2012)**

(a) $V_Q > V_R > V_P$
(b) $V_R > V_Q > V_P$
(c) $V_R/V_P = 3$
(d) $V_P/V_Q = \frac{1}{2}$

39. Two bodies, each of mass M, are kept fixed with a separation $2L$. A particle of mass m is projected from the midpoint of the line joining their centres, perpendicular to the line. The gravitational constant is G. The correct statement(s) is (are) **(2013)**

(a) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $4\sqrt{\frac{GM}{L}}$.

(b) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $2\sqrt{\frac{GM}{L}}$.

(c) The minimum initial velocity of the mass m to escape the gravitational field of the two bodies is $\sqrt{\frac{2GM}{L}}$.

(d) The energy of the mass m remains constant.

40. A planet of radius $R = \frac{1}{10} \times$ (radius of Earth) has the same mass density as Earth. Scientists dig a well of depth $\frac{R}{5}$ on it and lower a wire of the same length and of linear mass density 10^{-3} kgm^{-1} into it. If the wire is not touching anywhere, the force applied at the top of the wire by a person holding it in place is (take the radius of Earth $= 6 \times 10^6$ m and the acceleration due to gravity on Earth is 10 ms^{-2}) **(2014)**

(a) 96 N (b) 108 N

(c) 120 N (d) 150 N

41. A rocket is launched normal to the surface of the Earth, away from the Sun, along the line joining the Sun and the Earth. The Sun is 3×10^5 times heavier than the Earth and is at a distance 2.5×10^4 times larger than the radius of the Earth. The escape velocity from Earth's gravitational field is $v_e = 11.2$ km s^{-1}. The minimum initial velocity (v_s) required for the rocket to be able to leave the Sun-Earth system is closest to:

(Ignore the rotation and revolution of the Earth and the presence of any other planet) **(2017)**

(a) $v_s = 72$ km s^{-1} (b) $v_s = 22$ km s^{-1}

(c) $v_s = 42$ km s^{-1} (d) $v_s = 62$ km s^{-1}

42. A planet of mass M, has two natural satellites with masses m_1 and m_2. The radii of their circular orbits are R_1 and R_2 respectively. Ignore the gravitational force between the satellites. Define v_1, L_1, K_1 and T_1 to be, respectively, the orbital speed, angular momentum, kinetic energy and time period of revolution of satellite 1; and v_2, L_2, K_2 and T_2 to be the corresponding quantities of satellite 2. Given $m_1/m_2 = 2$ and $R_1/R_2 = 1/4$, match the ratios in column-I to the numbers in column-II. **(2018)**

	Column-I		Column-II
A.	$\frac{v_1}{v_2}$	p.	$\frac{1}{8}$
B.	$\frac{L_1}{L_2}$	q.	1
C.	$\frac{K_1}{K_2}$	r.	2
D.	$\frac{T_1}{T_2}$	s.	8

(a) A-(s); B-(q); C-(p); D-(r)

(b) A-(r); B-(q); C-(s); D-(p)

(c) A-(p); B-(r); C-(p); D-(s)

(d) A-(p); B-(r); C-(s); D-(p)

43. Consider a spherical gaseous cloud of mass density $\rho(r)$ in free space where r is the radial distance from its center. The gaseous cloud is made of particles of equal mass m moving in circular orbits about the common center with the same kinetic energy K. The force acting on the particles is their mutual gravitational force.

If $\rho(r)$ is constant in time, the particle number density $n(r) = \rho(r)/m$ is

[G is universal gravitational constant] **(2019)**

(a) $\frac{K}{\pi r^2 m^2 G}$ (b) $\frac{K}{6\pi r^2 m^2 G}$

(c) $\frac{3K}{\pi r^2 m^2 G}$ (d) $\frac{K}{2\pi r^2 m^2 G}$

44. The distance between two stars of masses $3M_s$ and $6M_s$ is $9R$. Here R is the mean distance between the centers of the Earth and the Sun, and M_s is the mass of the Sun. The two stars orbit around their common center of mass in circular orbits with period $n\,T$, where T is the period of Earth's revolution around the Sun. The value of n is **(2021)**

45. Two spherical stars A and B have densities r_A and r_B, respectively. A and B have the same radius, and their masses M_A and M_B are related by $M_B = 2M_A$. Due to an interaction process, star A loses some of its mass, so that its radius is halved, while its spherical shape is retained, and its density remains ρ_A. If v_A and v_B are the escape velocities from A and B after interaction process, the ratio $\frac{v_B}{v_A} = \sqrt{\frac{10n}{15^{1/3}}}$. **(2022)**

ANSWER KEY

CONCEPT APPLICATION

1. (a)	2. (c)	3. (c)	4. [3]	5. (d)	6. (d)	7. (d)	8. (c)	9. (b)	10. (b)
11. (a)	12. (d)	13. (c)	14. (c)	15. (b)	16. $\left(-12-\frac{12}{\sqrt{2}}-\frac{4}{\sqrt{3}}\right)\frac{Gm^2}{a}$		17. (b)	18. (c)	
19. (a)	20. (d)	21. (b)	22. (b)	23. (c)	24. (a)	25. (d)	26. (a)	27. (c)	28. (a)
29. (a)	30. (b)								

EXERCISE-1 (TOPICWISE)

1. (d)	2. (c)	3. (d)	4. (a)	5. (d)	6. (c)	7. (d)	8. (b)	9. (b)	10. (b)
11. (c)	12. (c)	13. (b)	14. (c)	15. (d)	16. (a)	17. (a)	18. (c)	19. (a)	20. (a)
21. (c)	22. (d)	23. (a)	24. (d)	25. (a)	26. (b)	27. (a)	28. (c)	29. (b)	30. (b)
31. (d)	32. (b)	33. (a)	34. (d)	35. (b)	36. (b)	37. (b)	38. (c)	39. (c)	40. (b)
41. (d)	42. (a)	43. (c)	44. (a)						

EXERCISE-2 (LEARNING PLUS)

1. (b)	2. (c)	3. (d)	4. (d)	5. (c)	6. (d)	7. (b)	8. (c)	9. (c)	10. (b)
11. (b)	12. (c)	13. (b)	14. (d)	15. (c)	16. (d)	17. (b)	18. (b)	19. (d)	20. (a)
21. (a)	22. (c)	23. (d)	24. (b)	25. (d)	26. (c)	27. (b)	28. (b)	29. (c)	30. (a)
31. (d)	32. (a)	33. (c)	34. (a)	35. (c)	36. (b)	37. (c)	38. (c)	39. (a)	40. (a)

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (a,d)	2. (b,c)	3. (b,c,d)	4. (a,c)	5. (a,c,d)	6. (b,c)	7. (a,d)	8. (b,d)	9. (a,d)	10. (c)
11. (a)	12. (a)	13. (b)	14. (c)	15. (a)	16. (a)	17. [0008]	18. [0004]	19. [0023]	20. [837.33]
21. [3000]	22. [8]								

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (b)	2. (d)	3. (b)	4. (c)	5. (b)	6. (b)	7. (d)	8. (d)	9. (d)	10. (c)
11. (b)	12. (b)	13. (b)	14. (b)	15. (d)	16. (b)	17. (c)	18. (d)	19. [16]	20. (d)
21. (a)	22. (b)	23. (c)	24. (a)	25. (d)	26. (c)	27. (a)	28. (a)	29. (a)	30. (a)
31. [64]	32. [3]	33. (b)	34. (d)	35. (d)	36. [2]	37. (c)			

JEE Advanced

38. (b,d)	39. (b,d)	40. (b)	41. (c)	42. (b)	43. (d)	44. [9]	45. [2.30]

CHAPTER

13 Mechanical Properties of Solids

RIGID BODY

When the external forces do not produce any deformation in the body, the body is called a rigid body eg. Diamond.

ELASTICITY

The solid is said to possess elasticity, if the Deformed shape is retained by the solid as long as the forces act, and the original shape is regained when the force cease to act.

If the body regains its shape completely it is known as perfectly elastic body example quartz tiber.

PLASTICITY

If the modified shape is retained after the deforming forces cease to act, the body is said to be perfectly plastic.

If the body partially recovers its original shape and size, it is known as plastic body.

Cause of Elasticity

We know that in a solid, each atom or molecule is surrounded by neighboring atoms or molecules. These are bonded together by inter-atomic or intermolecular forces and stay in a stable equilibrium position. When a solid is deformed, the atoms or molecules are displaced from their equilibrium positions causing a change in the inter-atomic (or intermolecular) distances. When the deforming force is removed, the inter-atomic forces tend to drive them back to their original positions. Thus, the body regains its original shape and size.

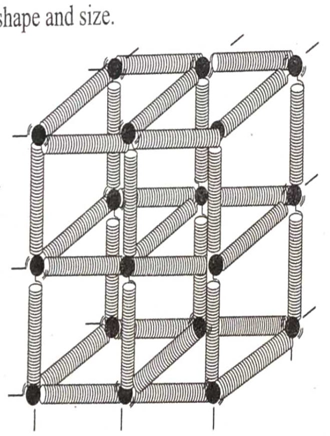


Figure: Spring-ball model for the illustration of elastic behavior of solids.

Effect of temperature on Elasticity

A rise of temperature generally decreases the elastic properties of the material.

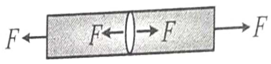
LONGITUDINAL STRESS

Restoring force per unit area set up inside the body against deformation is called stress and is measured by the magnitude of force acting on unit area of the body in equilibrium.

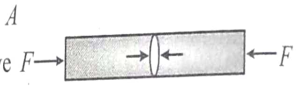
If F is the force applied and A is the area of cross section of the body,

$$\text{stress } (\sigma) = F/A.$$

The SI unit of stress is Nm^{-2} or pascal (Pa) and its dimensional formula is $[ML^{-1}T^{-2}]$.

Tensile 

$$\text{Tensile stress} = \frac{F}{A}$$

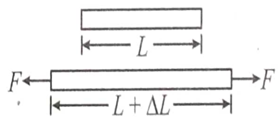
Compressive 

$$\text{Compressive stress} = \frac{F}{A}$$

Strain (Longitudinal)

The fractional change in length is called the strain. It is a **dimensionless** measure of the degree of deformation.

$$\text{Strain } (\in) = \frac{\Delta L}{L}$$



HOOKE'S LAW

As per Hooke's Law

Longitudinal stress $\propto$ Longitudinal strain

$$\frac{F}{A} \propto \frac{\Delta L}{L}$$

$$\frac{F}{A} = Y\frac{\Delta L}{L}$$

Y is called the **elastic modulus or Young's modulus**, Y has the same units as those of stress (Pa or N/m^2). Since, strain is dimensionless.

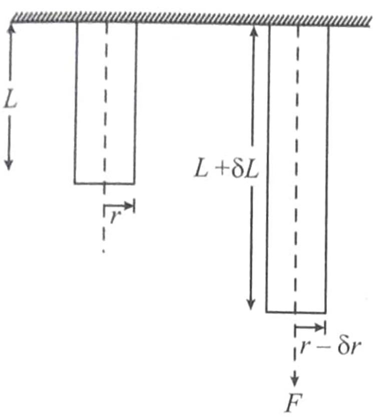
$$\text{Young's modulus } (Y) = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

$$Y = \frac{F/A}{l/L} = \frac{FL}{lA}$$

Young's modulus can be thought of as the **inherent stiffness** of a material. For example, rubber has a low Young's modulus and steel has a high Young's modulus

POISSONS RATIO

When a long bar is stretched by a force along its length then its length increases, its radius decreases as shown in figure



$$\text{Lateral strain} = \frac{-\delta r}{r}; \quad \text{Longitudinal strain} = \frac{\delta L}{L}$$

$$\mu = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{-\delta r/r}{\delta L/L}$$

$-1 \le \mu \le 0.5$ (Theoretical limit)

$\mu \approx \le 0.2 - 0.4$ (experimental limit)

STRESS-STRAIN CURVE

The relation between the stress and the strain for a given material under tensile stress can be found experimentally. As we can see that in the region between O to A, the curve is linear. In this region, Hook's law is obeyed. The body regains its original dimensions when the applied force is removed. In this region, the solid behaves as an elastic body.

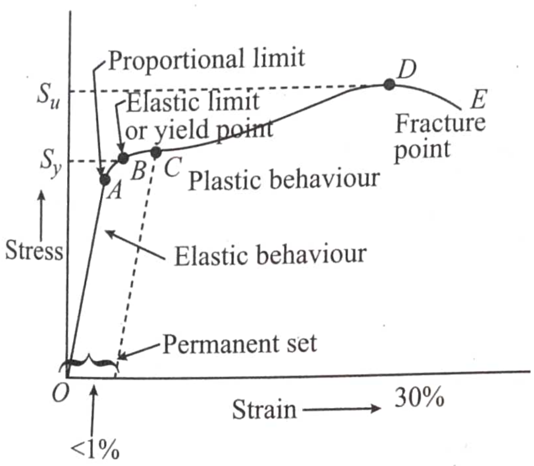


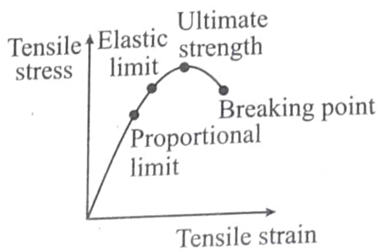
Figure: Stress-strain curve for a metal.

In the region from A to B, stress and strain are not proportional. Nevertheless, the body still returns to its original dimension when the load is removed. The point B in the curve is known as **yield point** (also known as **elastic limit**) and the corresponding stress is known as **yield strength** (S_y) of the material.

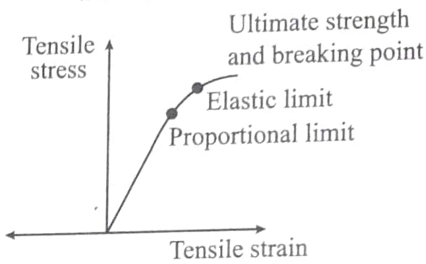
If the tensile or compressive stress exceeds the proportional limit, the strain is no longer proportional to the stress (see the figure).

If the stress exceeds the elastic limit, the material is permanently deformed. For still larger stresses, the solid fractures when the stress reaches the breaking point. The maximum stress that can be withstood without breaking is called the **ultimate strength**. A ductile material continues to stretch beyond its ultimate tensile strength without breaking. Examples of ductile solids are relatively soft metals, such as gold, silver, copper, and lead. These metals can be pulled like taffy, becoming thinner and thinner until finally reaching the breaking point.

Ductile



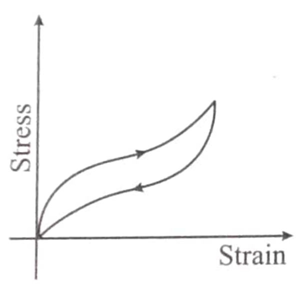
Brittle



ELASTIC HYSTERESIS

The Lagging of strain behind the stress is defined as elastic hysteresis while increasing and decreasing the load. That is a only reason why the values of strain for same stress are different.

The area of the stress-strain curve is called hysteresis loop and it is numerically equal to the work done in loading the material and then unloading it.



Train Your Brain

Example 1: A light wire of length 4m is suspended to the ceiling by one of its ends. If its cross-sectional area is 19.6 mm^2, what is its extension under a load of 10kg. Young's modulus of steel = 2×10^{11} Pa.

Sol. Given ; original length $L = 4$m;

force $F = 10 \times 9.8 = 98$ N;

and $Y = 2 \times 10^{11}$ Nm^{-2}; $l = ?$

Using the relation,

We have

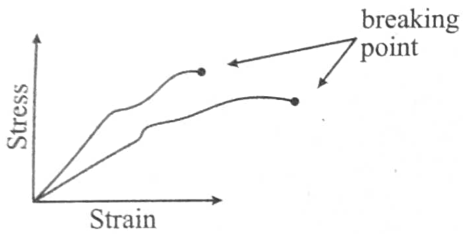
$$\Rightarrow l = \frac{FL}{YA}$$

$$\therefore \quad l = \frac{98 \times 4}{2 \times 10^{11} \times 19.6 \times 10^{-6}}$$

$= 1 \times 10^{-4}$ m

$= 0.1$ mm

Example 2: Select the correct statement on the basis of the given graph:



(*a*) Young's modulus of A is greater but it is less ductile

(*b*) Young's modulus of A is greater and it is more ductile

(*c*) Young's modulus of A is less and it is less ductile

(*d*) Young's modulus of A is less but it is more ductile

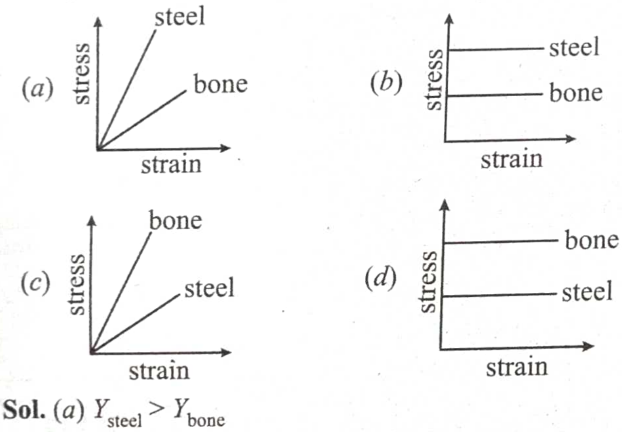
Sol. (*a*) $Y = \frac{\text{stress}}{\text{strain}}$ for same strain (stress)A > (stress)B

$YA > YB$

Breaking point of B > Breaking point of A

So, A is less ductile.

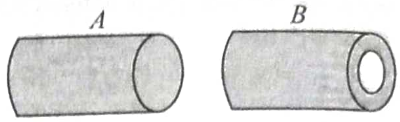
Example 3: Which of the following graphs of stress versus strain is consistent with the information presented in Table? (Given : Young's modulus of steel $= 2 \times 10^{11}$ Pa ; Young's modulus of bone $= 1.50 \times 10^{10}$ Pa)



Sol. (*a*) $Y_{\text{steel}} > Y_{\text{bone}}$

$$\left(\frac{\text{stress}}{\text{strain}}\right)_{\text{steel}} > \left(\frac{\text{stress}}{\text{strain}}\right)_{\text{bone}}$$

Example 4: The drawing shows two cylinders. They are identical in all respects, except one is hollow. In a setup like that in figure, identical forces are applied to the right end of each cylinder while the left end is fixed.



(*a*) The elongation of A is more than that of B

(*b*) The elongation of B is more than that of A

(*c*) The energy stored in B is more than that in A

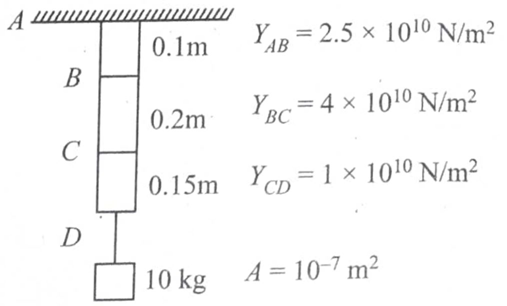
(*d*) The energy stored in A is more than that in B

Sol. (*b*, *c*) $\frac{F}{A} = Y\frac{\Delta\ell}{\ell} \quad \Rightarrow \Delta\ell = \left(\frac{\ell F}{Y}\right)\frac{1}{A}$

So, more elongation corresponds to less area

$$\Rightarrow \quad U = \frac{1}{2}\left(\frac{AY}{\ell}\right)\Delta\ell^2 = \frac{1}{2}\frac{AY}{\ell}\left(\frac{\ell F}{r}\right)^2\frac{1}{A^2} = U \propto \frac{1}{A}$$

Example 5: Find out the shift in point B, C and D (Assume rod to be massless)



Sol. $\Delta L_B = \Delta L_{AB} = \frac{FL}{AY} = \frac{MgL}{AY} = \frac{10 \times 10 \times 0.1}{10^{-7} \times 2.5 \times 10^{10}}$

$= 4 \times 10^{-3}$ m $= 4$ mm

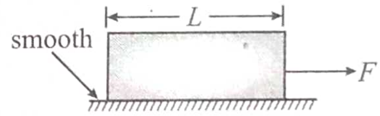
$\Delta L_C = \Delta L_B + \Delta L_{BC} = 4 \times 10^{-3} + \frac{100 \times 0.2}{10^{-7} \times 4 \times 10^{10}}$

$= 4 \times 10^{-3} + 5 \times 10^{-3} = 9$ mm

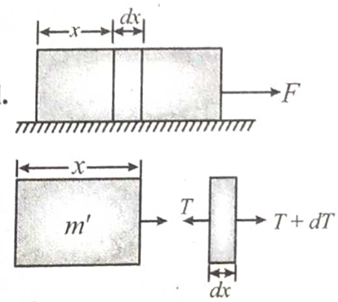
$L_D = \Delta L_C + \Delta L_{CD} = 9 \times 10^{-3} + \frac{100 \times 0.15}{10^{-7} \times 1 \times 10^{10}}$

$= 9 \times 10^{-3} + 15 \times 10^{-3} = 24$ mm

Example 6: Find out the elongation in block. If mass, area of cross-section and young modulus of block are m, A and Y respectively.



Sol.



Acceleration, $a = \frac{F}{m}$ then $T = m'a$

Where $\Rightarrow m' = \frac{m}{\ell}x$

$T = \frac{m}{\ell}x\frac{F}{m} = \frac{F\,x}{\ell}$

Elongation in element

$'d\ell' = \frac{Tdx}{AY}$

Total elongation,

$$\delta = \int_0^\ell \frac{Tdx}{AY}\, d = \int_0^\ell \frac{Fxdx}{A\ell Y} = \frac{F\ell}{2AY}$$

Example 7: A uniform bar of length L and cross sectional area A is subjected to a tensile load F. If Y be the Young's modulus of the material of the bar and σ be its Poisson's ratio, then determine the volumetric strain.

Sol. Longitudinal stress $= \frac{F}{A}$.

Longitudinal strain $= \frac{F}{AY} = \varepsilon_1$ (say) ...(*i*)

Now, by definition of Poisson's ratio,

$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{-\delta r/r}{\delta L/L}$$

or $\delta r/r = -\sigma\delta L/L$

$\Rightarrow -\frac{\sigma F}{AY}$ [From eqn. (*i*)]

Since

Volumetric strain = Strain in length + Twice strain in radius.

$\therefore$ Volumetric strain $= \frac{\delta L}{L} + \frac{2\delta r}{r}$

$$= \frac{F}{AY} + 2\left(-\frac{\sigma F}{AY}\right) = \frac{F}{AY}(1-2\sigma).$$

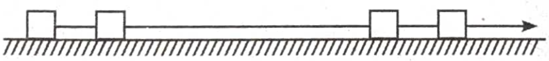
Concept Application

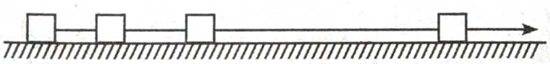
1. Two wires of the same material and length are stretched by the same force. Their masses are in the ratio 3 : 2. Their elongations are in the ratio

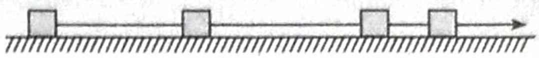
   (*a*) 3 : 2 (*b*) 9 : 4

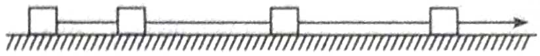
   (*c*) 2 : 3 (*d*) 4 : 0

2. Each of the pictures shows four objects tied together with rubber bands being pulled to the right across a horizontal frictionless surface by a horizontal force F. All the objects have the same mass; all the rubber bands obey Hook's law and have the same equilibrium length and the same force constant. Which of these pictures is drawn most correctly?

   (*a*) 

   (*b*) 

   (*c*) 

   (*d*) 

3. The ratio of diameters of two wires of same material is n : 1. The length of each wire is 4 m. On applying the same load, the increase in the length of the thin wire will be ($n > 1$)

   (*a*) n^2 times (*b*) n times

   (*c*) $2n$ times (*d*) $(2n + 1)$ times

4. A copper wire and a steel wire of the same cross-sectional area and length are joined end-to-end (at one end). Equal and opposite longitudinal forces are applied to the free ends giving a total elongation of l. Then the two wires will have

   (*a*) Same stress and same strain

   (*b*) Same stress and different strains

   (*c*) Different stresses and same strain

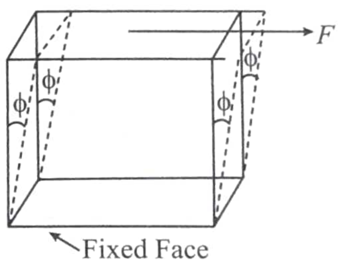
   (*d*) Dsifferent stresses and different strains

SHEARING STRESS (TANGENTIAL STRESS)

Let us consider a cube whose lower face is fixed and a tangential force F acts on the upper face whose area is A.

$\therefore$ Tangential stress $= F/A$.

Let the vertical sides of the cube shifts through an angle θ, called shear strain



Modulus of Rigidity: (Shear Modulus)

It is defined as the ratio of the tangential stress to the shear strain.

$\therefore$ Modulus of rigidity is given by

$$\eta = \frac{\text{Tangential stress}}{\text{Shear strain}} \text{ or } \eta = \frac{F/A}{\phi} = \frac{F_{\text{tangential}}}{A\phi}$$

Note: ϕ is measured in radian.

VOLUMETRIC STRESS

Since the fluid presses inward on all sides of the object (figure), the solid is compressed and its volume is reduced. The fluid pressure (P) is the force per unit surface area and can be through of as the volume stress on the solid object.

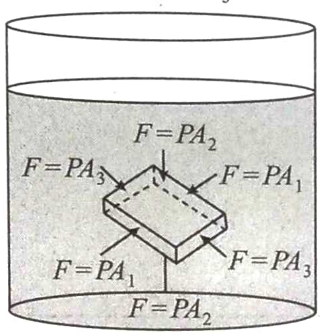


Figure All Forces on an object when submerged in a fluid

volume stress = pressure = $\frac{F}{A} = P$

Volume strain = $\frac{\text{change in volume}}{\text{original volume}} = \frac{\Delta V}{V}$

BULK MODULUS (B)

It is defined as the ratio of the normal stress to the volume strain

i.e. $B = \frac{\text{Pressure}}{\text{Volume strain}}$

The stress being the normal force applied per unit area and is equal to the pressure applied (P).

$$B = \frac{P}{\frac{-\Delta V}{V}} = -\frac{PV}{\Delta V}$$

Negative sign shows that increase in pressure (P) causes decrease in volume (ΔV).

COMPRESSIBILITY (K)

The reciprocal of bulk modulus of elasticity is called compressibility.

Unit of compressibility in SI is $N^{-1}\ m^2$ or pascal^{-1} (Pa^{-1}). $K = \frac{1}{B}$

- $B_{\text{solids}} > B_{\text{liquids}} > B_{\text{gases}}$
- Isothermal bulk modulus of elasticity of gas
  $B = P$ (pressure of gas)
- Adiabatic bulk modulus of elasticity of gas
  $B = \gamma \times P$ where $\gamma = \frac{C_p}{C_v}$

Relation Among Modulus

(1) $Y = 3B(1 - 2\mu)$ (2) $Y = 2\eta(1 + \mu)$

(3) $Y = \frac{9B\eta}{3B + \eta}$ (4) $\mu = \frac{3B - 2\eta}{6B + 2\eta}$

Where: Y = young's modulus
B = Bulk modulus
η = Modulus of rigidity
μ = Poisson's Ratio.

Train Your Brain

Example 8: If B is the bulk modulus of a metal and a pressure P is applied uniformly on all sides of the metal with density D, then the fractional increase in density is given by:

(a) $\frac{B}{P}$ (b) $\frac{P}{B}$

(c) $\frac{PD}{B}$ (d) $\frac{BD}{P}$

Sol. $D = \frac{M}{V}$, $D' = \frac{M}{V - \Delta V}$,

$$\therefore \frac{D'}{D} = \frac{V}{V - \Delta V} = \left(1 - \frac{\Delta V}{V}\right)^{-1} = 1 + \frac{\Delta V}{V}$$

$$\frac{D' - D}{D} = \frac{D'}{D} - 1 = \frac{\Delta V}{V}$$

$$\frac{\Delta V}{V} = \frac{P}{B} \quad \therefore \frac{D' - D}{D} = \frac{P}{B}$$

or fractional increase in density = P/B.

Example 9. The bulk modulus of rubber is 9.1×10^8 N/m^2. To what depth (approximately) a rubber ball be taken in a lake so that its volume is decreased by 0.1%?

(a) 25 m (b) 100 m

(c) 200 m (d) 500 m

Sol. $B = \frac{P}{\Delta V / V}$ or $P = B \cdot \frac{\Delta V}{V}$

$$\therefore P = 9.1 \times 10^8 \times \left(\frac{0.1}{100}\right) \text{ or } h\rho g = 9.1 \times 10^8 \times \left(\frac{0.1}{100}\right)$$

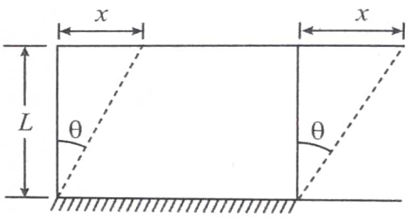
$$\therefore h = \frac{9.1 \times 10^8}{1000 \times 9.8} \times \frac{0.1}{100} \text{ m}$$

$= 92.85 \text{ m} \simeq 100$ m.

Example 10: A rubber cube of side 5 cm has one side fixed while a tangential force equal to 1800 N is applied to opposite face find the shearing strain and the lateral displacement of the strained face. Modulus of rigidity for rubber is 2.4×10^6 N/m^2.

Sol. $L = 5 \times 10^{-2}$ m

$$\Rightarrow \frac{F}{A} = \eta\ \frac{x}{L}$$



$$\text{strain} = \tan\theta = \frac{F}{A\eta} = \frac{1800}{25 \times 10^{-4} \times 2.4 \times 10^6}$$

$$= \frac{180}{25 \times 24} = \frac{3}{10} = 0.3 \text{ radian}$$

$$\frac{x}{L} = 0.3$$

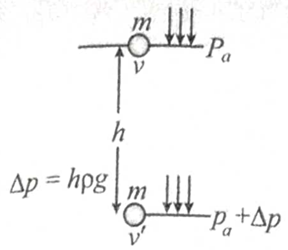
$\Rightarrow x = 0.3 \times 5 \times 10^{-2}$

$= 1.5 \times 10^{-2}$ m = 1.5 cm

If we increase the load gradually on a vertical suspended metal wire,

Example 11: Find the depth of lake at which density of water is 1% greater than at the surface. Given compressibility $K = 50 \times 10^{-6}$ /atm.

Sol.



m = constant

ρV = constant

$d\rho\, V + dV \,.\, \rho = 0 \Rightarrow \frac{d\rho}{\rho} = -\frac{dV}{V}$

i.e. $\frac{\Delta\rho}{\rho} = \frac{\Delta p}{B} \Rightarrow \frac{\Delta\rho}{\rho} = \frac{1}{100}$

$\frac{1}{100} = \frac{h\rho g}{B}$ [assuming ρ = constant]

$h\rho g = \frac{B}{100} = \frac{1}{100K}$

$\Rightarrow h\rho g = \frac{1 \times 10^5}{100 \times 50 \times 10^{-6}}$

$h = \frac{10^5}{5000 \times 10^{-6} \times 1000 \times 10}$

$= \frac{100 \times 10^3}{50} = 2$ km

Concept Application

5. A cube is shifted to a depth of 100 m is a lake. The change in volume is 0.1%. The bulk modulus of the material is nearly

(*a*) 10 Pa (*b*) 10^4 Pa

(*c*) 10^7 Pa (*d*) 10^9 Pa

6. The Bulk modulus of a perfectly rigid body is equal to:

(*a*) zero

(*b*) unity

(*c*) infinity

(*d*) may have any finite non-zero value

7. The shear modulus of a liquid is:

(*a*) zero

(*b*) unity

(*c*) infinity

(*d*) may have any finite non-zero value

8. A uniform pressure P is exerted on all sides of a solid cube. It s heated through $\Delta t°$ in order to bring its volume back to the value it had before the application of pressure. Then:

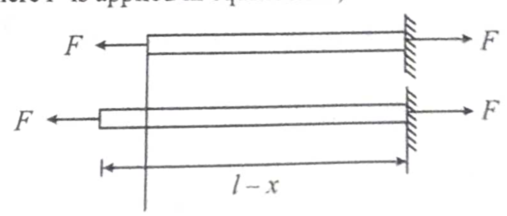
(*a*) $\Delta t = \frac{P}{B\gamma}$ (*b*) $\Delta t = \frac{B}{P\gamma}$

(*c*) $\Delta t = B\gamma P$ (*d*) $\Delta t = \frac{B\gamma}{P}$

(where γ is the cubical coefficient of expansion.)

ELASTIC POTENTIAL ENERGY

(where F is applied in equilibrium)



$dw = Fdx$ $\left(F = \frac{AY}{L}x\right)$

$= \frac{AY}{L} x\, dx$

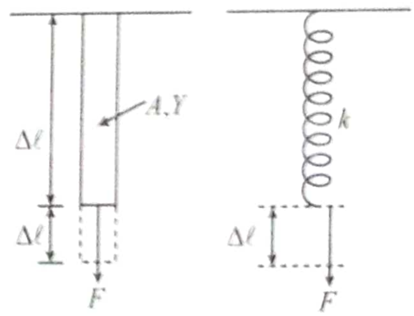
$w = \frac{AY}{L}\int_0^l x\, dx$

$U = \frac{AYl^2}{2L} = \frac{1}{2}\left(\frac{Yl}{L}\right)\left(\frac{l}{L}\right)(AL)$

$U = \frac{1}{2}$ (stress) (strain) (volume)

ANALOGY OF ROD AS A SPRING

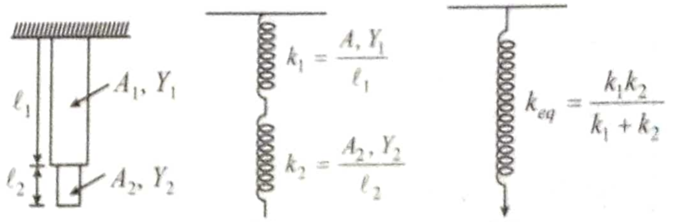
$Y = \frac{\text{stress}}{\text{strain}} \Rightarrow Y = \frac{F\ell}{A\Delta\ell}$



or $F = \frac{AY}{\ell}\Delta\ell$

$\frac{AY}{\ell}$ = constant, depends on type of material and geometry of rod. $F = k\Delta\ell$

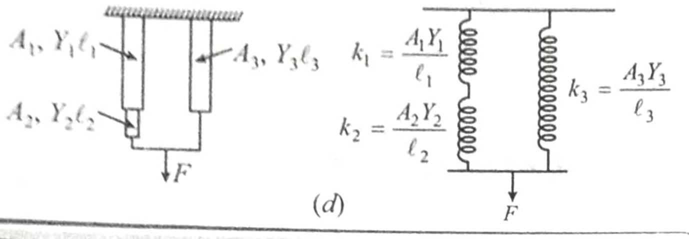
Where = equivalent spring constant. $k = \frac{AY}{\ell}$



(*a*) (*b*) (*c*)

for the system of rods shown in figure (*a*), the replaced spring system is shown in figure (*b*) two spring in series. Figure (*c*) represents equivalent spring system.

Figure (*d*) represents another combination of rods and their replaced spring system.



(*d*)

THERMAL STRESS AND THERMAL STRAIN

If a rod is fixed between two support, due to change in temperature its length will changes and so it will exert a normal stress. Stress will be tensile if temperature decreases and compressive if temperature Increases.

Change in length (ΔL) due to change in temperature (ΔT) = $\alpha L \Delta T$

Thermal strain $(E_T) = \dfrac{\Delta L}{L}$

$$\boxed{E_T = \alpha \Delta T}$$

Thermal stress (σT) = Youngs Modulus × Thermal strain

$$\boxed{\sigma_T = Y.\alpha \Delta T}$$

TORSIONAL STRESS (τ)

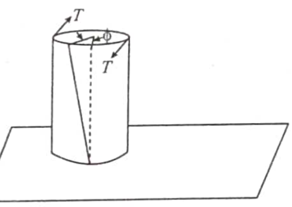
If we hold one end and twist the other end as shown in figure, the developed stress is known as torsional or twisting stress

$$\boxed{\text{Torsional stress } (\tau_{max}) = r \times \frac{T}{J}}$$

for solid bar

$$\boxed{J = \frac{\pi D^4}{32}}$$

r = Radius
J = Polar moment of inertia of the cross section
T = Twisting moment



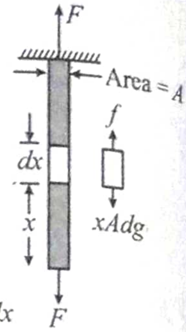
Train Your Brain

Example 12: A wire having a length l = 2m, and cross sectional area A = 5mm^2 is suspended at one of its ends from a ceiling. What will be its strain energy due to its own weight, if the density and Young's modulus of the material of the wire be $d = 9\text{g/cm}^3$ and $Y = 1.5 \times 10^{11}$ Nm^{-2}?

Sol. Consider an elemental length of the wire of length dx, at a distance x from the lower end.

Clearly, this length is acted upon by the external force equal to the weight of the portion of wire below it d is the density of wire = $xAdg$. In equilibrium, the restoring force $f = xAdg$.

$\therefore$ Stress $= \dfrac{f}{A} = xdg.$



Now, elastic potential energy stored in the elemental length will be

$$dU = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}$$

$$= \frac{1}{2} \times \frac{(\text{stress})^2}{Y} \times \text{volume}$$

$$= \frac{1}{2} \times \frac{(xdg)^2}{Y} \times Adx = \frac{1}{2}\frac{d^2g^2A}{Y}x^2dx$$

$\therefore$ Total elastic potential energy $U = \int dU$

$$= \int_0^l \frac{1}{2}\frac{d^2g^2A}{Y}x^2\,dx = \frac{1}{6}d^2g^2\frac{Al^3}{Y}$$

Substituting the values,

$$U = \frac{1}{6} \times \frac{(9 \times 10^3)(9{\cdot}8)^2 \times 5 \times 10^{-6} \times 2^3}{1.5 \times 10^{11}}$$

$= 3.841 \times 10^{-5}\,J$

Example 13: Two rods of different materials having coefficients of linear expansion α_1, α_2 and Young's moduli Y_1 and Y_2 respectively are fixed between two rigid massive walls. The rods are heated such that they undergo the same increase in temperature. There is no bending of rods. If $\alpha_1 : \alpha_2 = 2 : 3$, the thermal stresses developed in the two rods are equal provided $Y_1 : Y_2$ is equal to:

(*a*) 2 : 3
(*b*) 1 : 1
(*c*) 3 : 2
(*d*) 4 : 9

Sol. $\sigma_1 = \sigma_2$

$Y_1\alpha_1\Delta T = Y_2\alpha_2\Delta T$

$$\frac{Y_1}{Y_2} = \frac{\alpha_2}{\alpha_1} = \frac{3}{2}$$

Example 14: A steel wire of length 20 cm and uniform cross-section 1 mm^2 is tied rigidly at both the ends. The temperature of the wire is altered from 40°C to 20°C. Coefficient of linear expansion for steel $\alpha = 1.1 \times 10^{-5}$/°C and Y for steel is 2.0×10^{11}N/m^2. The change in tension of the wire is:

(*a*) 2.2×10^6 newton
(*b*) 16 newton
(*c*) 8 newton
(*d*) 44 newton

Sol. l = 20 cm

$A = 1$ mm^2 ; $\Delta T = 20$°C

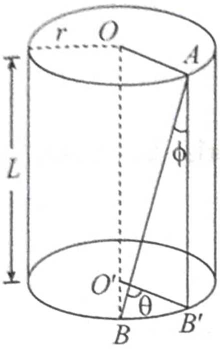
$\alpha = 1.1 \times 10^{-5}$/°C ; $Y = 2 \times 10^{11}$ N/m^2

$T = 6 \times A = Y\alpha\Delta T \times A$
$= 2 \times 10^{11} \times 1.1 \times 10^{-5} \times 20 \times 10^{-6}$
$= 44$ Newton

Example 15: The upper end of wire 1m long and 4mm radians is clamped. The lower end is twisted by an angle of 30°. The angle of shear is-

(*a*) 12°
(*b*) 1.2°
(*c*) 0.12°
(*d*) 0.012°

Sol.



$BB' = r\theta = L\phi$
ϕ = angle of shear
θ = angle of twist

$$\phi = \frac{r\theta}{L} = \frac{0.4 \times 30°}{100} = 0.12°.$$

Concept Application

9. Two wires of the same material and length but diameters in the ratio 1 : 2 are stretched by the same force. The potential energy per unit volume for the two wires when stretched will be in the ratio

(*a*) 16 : 1 (*b*) 4 : 1
(*c*) 2 : 1 (*d*) 1 : 1

10. A steel rod of length l, area of cross-section A, Young's modulus E and linear coefficient of expansion α is heated through t°C. The work that can be performed by the rod when heated is:

(*a*) $(EA\alpha t)\,(l\alpha t)$
(*b*) $(1/2)(EA\alpha t)(l\alpha t)$
(*c*) $(1/2(EA\alpha t) \times (1/2)(l\alpha t)$
(*d*) $2(EA\alpha t)\,(l\alpha t)$

11. An iron bar of length L and area of cross-section A is heated from 20°C to 80°C. The bar is so held between supports that it is neither allowed to extend nor allowed to bend. If the stress developed in the bar be S, then:

(*a*) $S \propto L$ (*b*) $S \propto (1/L)$
(*c*) $S \propto A$ (*d*) $S \propto (1/A)$

Short Notes

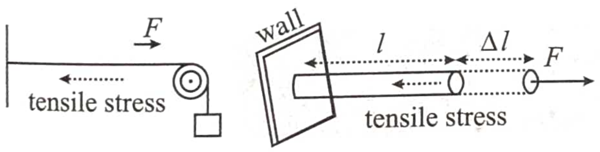
STRESS

$$\text{Stress} = \frac{\text{Internal restoring force}}{\text{Area of cross} - \text{section}} = \frac{F_{Res}}{A}.$$

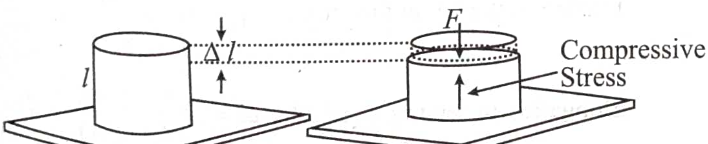
There are three types of stress

Longitudinal Stress (2 Types)

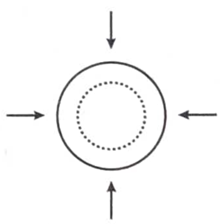
(*a*) **Tensile Stress:**



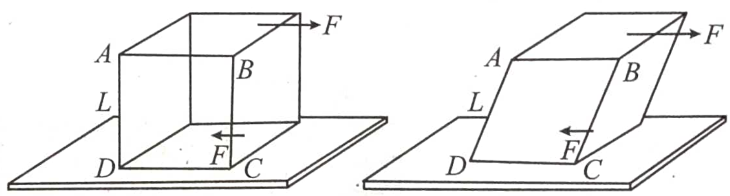
(*b*) **Compressive Stress:**



Volume Stress



Tangential Stress or Shear Stress



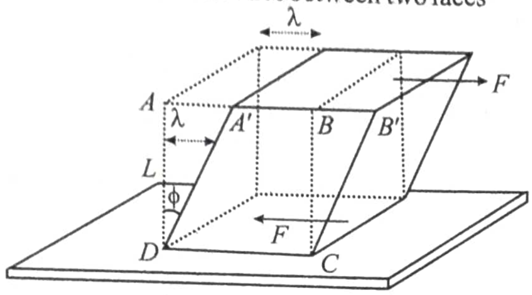
$$\textbf{Strain} = \frac{\text{Change in Size of the body}}{\text{Original size of the body}} \text{ (3 types)}$$

1. $$\textbf{Longitudinal strain} = \frac{\text{Change in length of the body}}{\text{initial length of the body}} = \frac{\Delta L}{L}$$

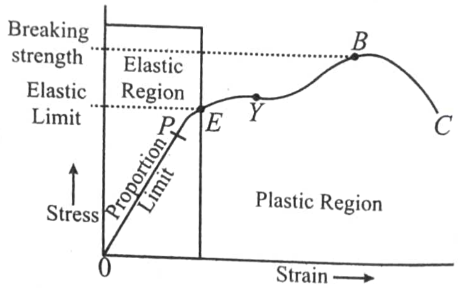
2. $$\textbf{Volume strain} = \frac{\text{Change in volume of the body}}{\text{original volume of the body}} = \frac{\Delta V}{V}$$

3. Shear starin:

$$\tan\phi = \frac{\ell}{L} \text{ or } \phi = \frac{\ell}{L} = \frac{\text{displacement of upper face}}{\text{distance between two faces}}$$



Stress – Strain Curve



Hooke's Law

$\boxed{\text{Stress } \alpha \text{ strain}}$ (within limit of elasticity)

Young's modulus of elasticity $Y = \frac{\text{Longitudinal stress}}{\text{Longitudinal strain}} = \frac{FL}{A\Delta\ell}$

If L is the length of wire, r is radius and ℓ is the increase in length of the wire by suspending a weight Mg at its one end, then Young's modulus of elasticity of the material of wire

$$Y = \frac{(Mg/\pi r^2)}{(\ell/L)} = \frac{MgL}{\pi r^2 \ell}$$

Increment in Length due to Own Weight

Bulk Modulus of Elasticity

$$\Delta\ell = \frac{MgL}{2AY} = \frac{\rho g L^2}{2Y}$$

$$K = \frac{\text{Volumetric stress}}{\text{Volumetric strain}} = \frac{F/A}{\left(\frac{-\Delta V}{V}\right)} = \frac{P}{\left(\frac{-\Delta V}{V}\right)}$$

$$k = \frac{\text{Volume stress}}{\text{Volume strain}} = \frac{\frac{F}{A}}{-\frac{\Delta V}{V}} = \frac{P}{-\frac{\Delta V}{V}}$$

- Compressibility $C = \frac{1}{\text{Bulk modulus}} = \frac{1}{K}$
- Modulus of rigidity $\eta = \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{(F_{\text{tangential}})/A}{\phi}$
- Poisson's ratio $(\mu) = \frac{\text{lateral strain}}{\text{Longitudinal strain}}$
- Work done in stretching wire :

$$W = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}$$

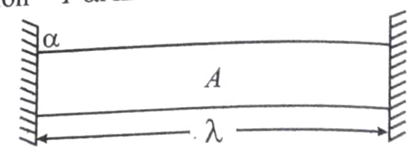
$$W = \frac{1}{2} \times \frac{F}{A} \times \frac{\Delta\ell}{\ell} \times A \times \ell = \frac{1}{2} F \times \Delta\ell$$

Rod is Rigidly Fixed at the ends, between Walls

Thermal strain = $\alpha\Delta\theta$

Thermal stress = $Y\alpha\Delta\theta$

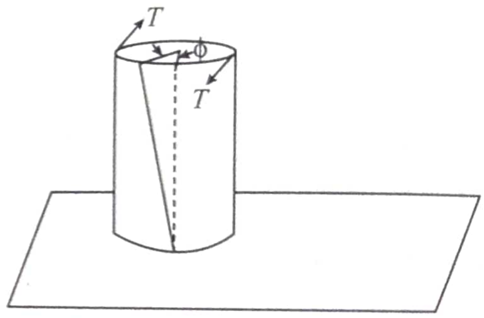
Thermal tension = $Y\alpha A\Delta\theta$



Effect of Temperature on Elasticity

When temperature is increased then due to weakness of inter molecular force the elastic properties in general decrease i.e. elastic constant decreases. Plastic properties increase with temperature.

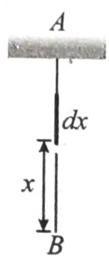
Torsional Stress (τ)



$\boxed{\text{Torsional stress } (\tau) = r \times \frac{T}{J}}$

Solved Examples

1. A bar of mass m and length l is hanging from point A as shown in figure. Find the increase in its length due to its own weight. The Young's modulus of elasticity of the wire is Y and area of cross-section of the wire is A.



Sol. Consider a small section of the bar at a distance x from B.

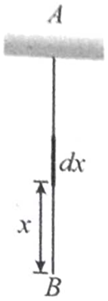
Tension in the bar at this point is $T_x = W_x = \left(\frac{mg}{l}\right)x$

Elongation in section dx will be $dl = \frac{T_x dx}{AY} = \left(\frac{mg}{lAY}\right)x\,dx$

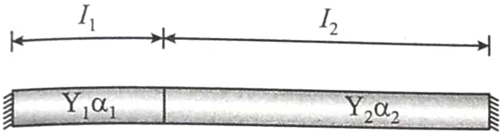
Total elongation in the bar can be obtained by integrating this expression from $x = 0$ to $x = l$.

$$\therefore\ \Delta l = \int_{x=0}^{x=l} dl = \left(\frac{mg}{lAY}\right)\int_0^l x\,dx$$

or $\Delta l = \dfrac{mgl}{2AY}$



2. Two rods of different metals having the same area of cross-section A are placed between the two massive walls as shown in figure. The first rod has a length l_1, coefficient of linear expansion a_1 and Young's modulus Y_1. The corresponding quantities for second rod are l_2, a_2 and Y_2. The temperature of both the rods is now raised by t°C.



(a) find the force with which the rods act on each other at higher temperature.

(b) the lengths of the rods at higher temperature.

Sol. (a) Let Δl be the displacement of the joint towards right.

Strain on first rod $= \dfrac{l_1\alpha_1 t - \Delta l}{l_1}$

or force exerted by the first rod on the joint is

$$F_1 = Y_1 A\left(\frac{l_1\alpha_1 t - \Delta l}{l_1}\right) \qquad ......(i)$$

Strain on second rod $= \dfrac{l_2\alpha_2 t + \Delta l}{l_2}$

or force exerted by second rod on the joint is

$$F_2 = Y_2 A\left(\frac{l_2\alpha_2 t + \Delta l}{l_2}\right) \qquad .....(ii)$$

In equilibrium $F_1 = F_2$

Solving this, we get

$$\Delta l = \frac{l_1 l_2 t(Y_1\alpha_1 - Y_2\alpha_2)}{(Y_1 l_2 + Y_2 l_1)} \qquad ....(iii)$$

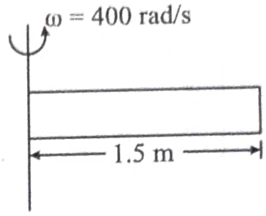
Solving (i), (ii) and (iii), we get

$$F_1 = F_2 = \frac{At(l_1\alpha_1 + l_2\alpha_2)}{\left(\dfrac{l_1}{Y_1} + \dfrac{l_2}{Y_2}\right)}$$

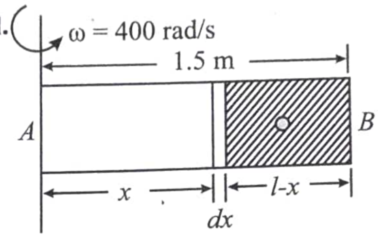
(b) $l'_1 = l_1 + \Delta l = l_1 + \dfrac{l_1 l_2 t(Y_1\alpha_1 - Y_2\alpha_2)}{(Y_1 l_2 + Y_2 l_1)}$

and $l'_2 = l_2 - \Delta l = l_2 + \dfrac{l_1 l_2 t(Y_1\alpha_1 - Y_2\alpha_2)}{Y_1 l_2 + Y_2 l_1}$

3. A rod having uniform mass distribution is rotated with constant angular velocity ω (400 rad/s) as shown in fig. Find out elongation in rod.
[Given $Y = 2 \times 10^{11}$ N/m^2; density of rod $= 10^4$ kg/m^3]



Sol.



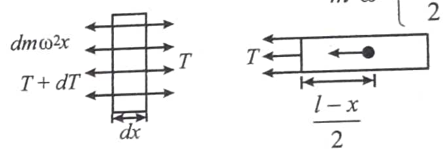
mass of shaded portion

$m' = \dfrac{m}{\ell}(\ell - x)$ [where m = total mass = $\rho A l$]

$$T = m'\omega^2\left[\frac{\ell - x}{2} + x\right]$$

$$\Rightarrow T = \frac{m}{\ell}(l - x)\,\omega^2\left(\frac{\ell + x}{2}\right);\ T = \frac{m\omega^2}{2\ell}(\ell^2 - x^2)$$

$$m'\omega^2\left\{\frac{l-x}{2} + x\right\}$$



This tension will be maximum at A $\left(\dfrac{m\omega^2\ell}{2}\right)$ and minimum at 'B' (zero), elongation in element of width 'dy' $= \dfrac{Tdx}{AY}$

Total elongation

$$y = \int \frac{Tdx}{AY} = \int_0^\ell \frac{m\omega^2(\ell^2 - x^2)}{2\ell\ AY}dx$$

$$y = \frac{m\omega^2}{2\ell\ AY}\left[\ell^2 x - \frac{x^3}{3}\right]_0^\ell$$

$$= \frac{m\omega^2 \times 2\ell^3}{2\ell AY \times 3} = \frac{m\omega^2\ell^2}{3AY} = \frac{\rho A\ell\omega^2\ell^2}{3AY}$$

$$y = \frac{\rho\omega^2\ell^3}{3Y} = \frac{10^4 \times (400) \times (1.5)^3}{3 \times 2 \times 10^{11}} = 9 \times 10^{-3}\text{ m} = 9\text{mm}$$

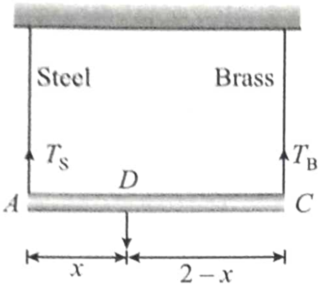
4. A light rod of length 2.00 m is suspended from the ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross-section 10^{-3} m^2 and the other is of brass of cross-section 2×10^{-3} m^2. Find out the position along the rod at which a weight may be hung to produce,

(a) equal stresses in both wires

(b) equal strains on both wires.

Young's modulus for steel is 2×10^{11} N/m^2 and for brass is 10^{11} N/m^2.

Sol. (*a*) Given, stress in steel = stress in brass



$$\therefore \frac{T_S}{A_S} = \frac{T_B}{A_B}$$

$$\therefore \frac{T_S}{A_B} = \frac{T_S}{A_B}$$

$$= \frac{10^{-3}}{2\times10^{-3}} = \frac{1}{2} \qquad \text{....}(i)$$

As the system is in equilibrium, taking moments about D, we have

$$T_S{\cdot}x = T_B(2-x)$$

$$\therefore \frac{T_S}{T_B} = \frac{2-x}{x} \qquad \text{......}(ii)$$

From Eqs. (*i*) and (*ii*), we get

$x = 1.33\ m$

(*b*) Strain $= \dfrac{\text{Stress}}{Y}$

Given, strain in steel strain in brass

$$\therefore \frac{T_S / A_S}{Y_S} = \frac{T_B / A_B}{Y_B}$$

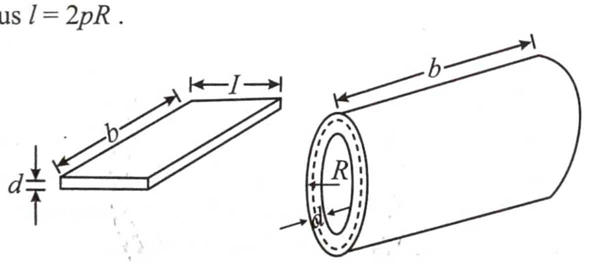
$$\therefore \frac{T_S}{Y_B} = \frac{A_S Y_S}{A_B Y_B} = \frac{(1\times10^{-3})(2\times10^{11})}{(2\times10^{-3})(10^{11})} = 1 \qquad \text{.....}(iii)$$

From Eqs. (*ii*) and (*iii*), we have

$x = 1.0\ m$

5. Show that work performed to make a hoop out of a steel band (Young's modulus Y) of length l, width b and thickness d is equal to $\dfrac{1}{6}\left(\dfrac{\pi^2 Ybd^3}{l}\right)$.

Sol. While making a hoop out of a steel band, the central layer remains unstretched. The layers above the central layer are extended in length while the lower layers are compressed in length. Let R be the radius of the central unstretched layer. Thus $l = 2pR$.



Consider a coaxial layer of radius x. The strain produced in this layer will be

$$\frac{2\pi x - 2\pi R}{2\pi R} = \left(\frac{x}{R} - 1\right)$$

Hence, stress at this layer $= Y \times \left(\dfrac{x}{R} - 1\right)$

Now the energy density (u) at this layer is given by

$$u = \frac{1}{2} \times \text{stress} \times \text{strain}$$

$$= \frac{1}{2}\left\{Y \times \left(\frac{x}{R} - 1\right)\right\}\left\{\frac{x}{R} - 1\right\} = \frac{1}{2}Y\left(\frac{x}{R} - 1\right)^2$$

The energy stored in an elementary volume at x,

$$dU = u \times \text{volume} = \frac{1}{2}Y\left(\frac{x}{R} - 1\right)^2 \{(2\pi x dx)b\}$$

$$\therefore U = \int_{R-\frac{d}{2}}^{R+\frac{d}{2}} \frac{1}{2}.Y.\left(\frac{x}{R} - 1\right)^2 .2\pi x dx b$$

$$= \frac{\pi Yb}{R^2} \int_{R-\frac{d}{2}}^{R+\frac{d}{2}} (x-R)^2 x\, dx$$

$$= \frac{\pi Yb}{R^2}\left[\int_{R-\frac{d}{2}}^{R+\frac{d}{2}} (x^3 - 2Rx^2 + R^2 x)\, dx\right]$$

$$= \frac{\pi Ybd^3}{12R} \qquad (\because d << R)$$

$$= \frac{\pi Ybd^3}{12l/(2\pi)} \qquad (\because l = 2pR)$$

$$= \frac{1}{6}\left(\frac{\pi^2 Ybd^3}{l}\right)$$

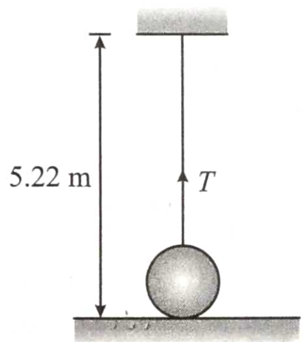
6. A sphere of radius 0.1 m and mass 8π kg is attached to the lower end of a steel wire of length 5.0 m and diameter 10^{-3}. The wire is suspended from 5.22 m high ceiling of a room. When the sphere is made to swing as a simple pendulum, it just grazes the floor at its lowest point. Calculate the velocity of the sphere at the lowest position. Young's modulus of steel is 1.994×10^{11} N/m^2.

Sol. Let Δl be the extension of wire when the sphere is at mean position. Then, we have

$l + \Delta l + 2r = 5.22$

or $\Delta l = 5.22 - l - 2r = 5.22 - 5 - (2 \times 0.1) = 0.02$ m

Let T be the tension in the wire at mean position during oscillations, then



$$Y = \frac{T/A}{\Delta l / l}$$

$$\therefore T = \frac{YA\Delta l}{l} = \frac{Y\pi r^2 \Delta l}{l}$$

Substituting the values, we have

$$T = \frac{(1.994\times 10^{11})\times\pi\times(0.5\times 10^{-3})^2\times 0.02}{5} = 626.43\ N$$

The equation of motion at mean position is,

$$T - mg = \frac{mv^2}{R} \qquad .....(i)$$

Here, $R = 5.22 - r = 5.22 - 0.01 = 5.12$ m

and $m = 8\pi$ kg $= 25.13$ kg

Substituting the proper values in Eq. (i), we have

$$(626.43) - (25.13\times 9.4) = \frac{(25.13)v^2}{5.12}$$

Solving this equation, we get

$v = 8.8$ m/s

7. Find the volume density of the elastic deformation energy in fresh water at a depth of $h = 1$ km. Take density of water $= 10^3$ kg / m^3, $g = 98$ m/s^2 and bulk modulus of elasticity of water $B = 2\times 10^9$ N/m^2. 224. Bulk modulus is given by

$$B = -\frac{dP}{(dV/V)} \quad \text{or} \quad \frac{dV}{V} = -\frac{dP}{B}$$

$$\text{or} \quad \frac{P\cdot dV}{V} = -\frac{PdP}{B}$$

$\frac{PdV}{V}$ is the work done per unit volume. The negative sign implies that a decrease in pressure gives rise to increase in volume and vice-versa.

Hence volume density of elastic potential energy

$$u = \int_0^{h\rho g}\frac{P\cdot dP}{B} = \frac{1}{2}\frac{(h\rho g)^2}{B}$$

Substituting the values, we obtain

$$u = \frac{1}{2}\frac{\left(10^3\right)^2\left(10^3\right)^2(9.8)^2}{2\times 10^9} = 2.4\times 10^4\ \text{J/m}^3$$

8. A copper rod of length l is suspended from the ceiling by one of its ends. Find (a) the elongation Δl of the rod due to its own weight; (b) the fractional decrement of its volume, $-\frac{\Delta V}{V}$; (c) the elastic potential energy stored in the rod due to its own weight.

Sol. (a) Consider a differential element dx at a distance x from top. The increment in length dx due to the weight of $(l-x)$ length of the rod below it,

$$\frac{\Delta x}{dx} = \frac{F/A}{Y}$$

Let dx be increment in length of differential element.

$$\text{or } \Delta x = \frac{m(l-x)}{AY}gdx \qquad ...(i)$$

Net elongation in length of rod can be obtained by integrating eqn. (i) in limits $x = 0$ to $x = l$.

$$\Delta l = \int\Delta x = \frac{mg}{AY}\int_0^l (l-x)\,dx$$

$$= \frac{mg}{AY}\left(l^2 - \frac{l^2}{2}\right) = \frac{mg}{AY}\frac{l^2}{2} = \frac{\rho g l^2}{2Y}$$

where $\rho = \frac{m}{A\times 1}$ is the density of the rod.

(b) Initial volume of the $rod = \frac{\pi D^2}{4}l = V$

$$\text{Poisson's ratio } \sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = +\frac{\Delta D/D}{\Delta l/l}$$

$$\text{Final volume, } V' = \frac{\pi D^2 l}{4}\left\{\left(1-\frac{\Delta D}{D}\right)^2\left(1+\frac{\Delta l}{l}\right)\right\}$$

$$\text{Fractional change in volume} = -\frac{\Delta V}{V} = \text{volumetric strain}$$

$$= \frac{V'-V}{V} = \left(1-\frac{\Delta D}{D}\right)^2\left(1+\frac{\Delta l}{l}\right) - 1 \qquad \left[\text{as } V = \frac{\pi D^2}{4}l\right]$$

$$= \left(1-\frac{2\Delta D}{D}\right)\left(1+\frac{\Delta l}{l}\right) - 1 \qquad \left[\text{as } \frac{\Delta D}{D} \ll 1 \text{ and } \frac{\Delta l}{l} \ll 1\right]$$

$$\equiv 1 + \frac{\Delta l}{l} - \frac{2\Delta D}{D} - 1$$

$$= \frac{\Delta l}{l}(1-2\sigma) = \frac{\rho g l}{2Y}(1-2\sigma)$$

(c) Energy density at $x = \frac{1}{2}\times$ stress $\times$ strain

$$= \frac{1}{2}\times\frac{(l-x)mg}{\pi r^2 Y}\times\frac{(l-x)mg}{\pi r^2}$$

$$= \left\{\frac{1}{2}\times\frac{(l-x)^2}{Y}\times\rho^2\right\}\times\left(\pi r^2 dx\right)$$

Where r = radius of the rod

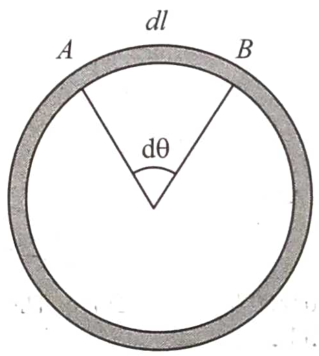
Total potential energy stored,

$$U = \frac{1}{2}\frac{\pi r^2\rho^2}{Y}\int_0^1 (l-x)^2\,dx = \frac{1}{6}\frac{\pi r^2\rho^2 l^3}{Y}$$

9. A thin ring of radius R is made of a material of density ρ and Young's modulus Y. If the ring is rotated about its centre and its own plane with angular velocity ω. Find the small increase in its radius.

Sol. Consider an element of length dl and mass dm of the ring.

Let S be the cross-sectional area of wire and the tension.



$dl = R \cdot d\theta$

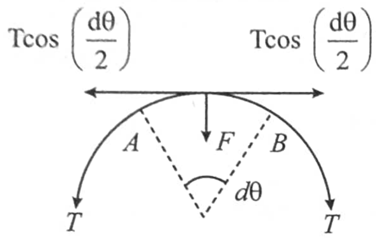
$F = 2T \sin\left(\frac{d\theta}{2}\right) = T(d\theta)$ $\left(\sin\frac{d\theta}{2} \approx \frac{d\theta}{2}\right)$ $\because$ As θ is small

This F provides the necessary centripetal force.

Hence, $T.d\theta = (dm) R\omega^2$

$= (R.d\theta \cdot S.\rho) R\omega^2$ or $\frac{T}{S} = \rho\omega^2 R^2$

Let ΔR be the increase in radius. Then



$$\text{longitudinal strain} = \frac{\Delta L}{L} = \frac{\Delta(2\pi R)}{2\pi R} = \frac{\Delta R}{R}$$

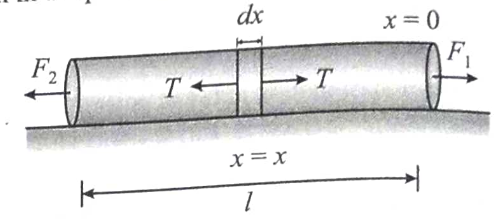
Now $Y = \frac{\text{stress}}{\text{strain}} = \frac{T/S}{\Delta R/R}$

$$Y = \frac{\rho\omega^2 R^3}{\Delta R}$$

Hence $\Delta R = \frac{\rho\omega^2 R^3}{Y}$

10. Two opposite forces F_1 and F_2 $(< F_1)$ act on an elastic plank of modulus of elasticity Y and lengt l placed over a smooth horizontal surface. The cross-sectional area of the plank is S. Find the change in length of the plank in the direction of the force.

Sol. Tension in the plank varies linearly from F_1 to F_2. Therefore, tension in the plank at a distance x from the front edge is :



change in length in element dx is :

$$dl = \frac{T \cdot dx}{SY}\left(\Delta l = \frac{Fl}{AY}\right)$$

$$dl = \frac{\left\{F_1 - (F_1 - F_2)\frac{x}{l}\right\}}{SY} \cdot dx$$

$\therefore$ total change in length will be

$$\Delta l = \int_{x=0}^{x=l} dl$$

or $\Delta l = \int_0^l \left\{\frac{F_1 - (F_1 - F_2)\frac{x}{l}}{SY}\right\} dx$

$$\Delta l = \frac{(F_1 + F_2)l}{2\,SY}$$

Exercise-1 (Topicwise)

ELASTIC BEHAVIOUR, LONGITUDINAL STRESS, YOUNG MODULUS

1. An elastic rod will not change its length when it is:
(a) Pulled along the rough surface
(b) Pulled along the smooth surface
(c) Is hanging under gravity
(d) Freely falls vertically under gravity

2. The length of an iron wire is L and area of cross-section is A. The increase in length is l on applying the force F on its two ends. Which of the statement is correct
(a) Increase in length is inversely proportional to its length L
(b) Increase in length is proportional to area of cross-section A
(c) Increase in length is inversely proportional to A
(d) Increase in length is proportional to Young's modulus

3. Two wires have the same diameter and length. One is made of copper and the other brass. They are connected together at one end. When free ends are pulled in opposite direction by same force.
(a) the wires will have same strain
(b) the wires will have same stress
(c) Both the wires will break at the same force
(d) Both wires will have same elongation

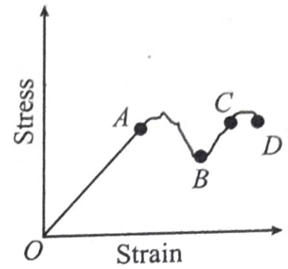
4. The ratio of the lengths of two wires A and B of same material is 1 : 2 and the ratio of their diameter is 2 : 1. They are stretched by the same force, then the ratio of increase in length will be
(a) 2 : 1 (b) 1 : 4
(c) 1 : 8 (d) 8 : 1

5. The area of cross-section of a wire of length 1.1 metre is 1 mm^2. It is loaded with 1 kg. If Young's modulus of copper is 1.1×10^{11} N/m^2, then the increase in length will be (If $g = 10$ m/s^2)
(a) 0.01 mm (b) 0.075 mm
(c) 0.1 mm (d) 0. 15 mm

6. If the temperature increases, the modulus of elasticity
(a) Decreases (b) Increases
(c) Remains constant (d) Becomes zero

7. A force F is needed to break a copper wire having radius R. The force needed to break a copper wire of radius $2R$ will be
(a) $F/2$ (b) $2F$
(c) $4F$ (d) $F/4$

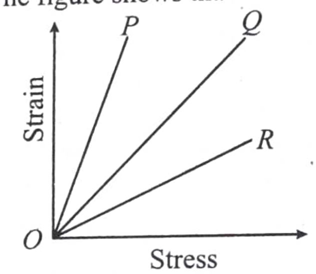
8. A graph is shown between stress and strain for a metal. The part in which Hook's law holds good is



(a) OA (b) AB
(c) BC (d) CD

9. A force of 400 kg. weight can break a wire. The force required to break a wire of double the area of cross-section will be
(a) 800 kg.wt (b) 200 kg.wt
(c) 1600 kg.wt (d) 100 kg.wt

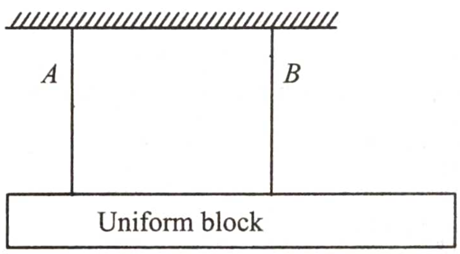
10. The strain-stress curves of three wires of different materials are shown in the figure. P, Q and R are the elastic limits of the wires. The figure shows that



(a) Elasticity of wire P is maximum
(b) Elasticity of wire Q is maximum
(c) Tensile strength of R is maximum
(d) None of the above is true

11. A metal wire of length L area of cross-section A and Young modulus Y behaves as a spring of spring constant k, where k is equal to
(a) $\frac{YL}{A}$ (b) $\frac{YA}{L}$ (c) $\frac{2YA}{L}$ (d) $\frac{YA}{2L}$

12. The figure shows a horizontal block that is suspended by two wires A and B are identical except their original length. Which wire was originally shorter?



(a) A

(b) B

(c) Both had equal length

(d) Any one of them could be shorter

13. An iron wire of negligible mass, length L and cross-section are A has one end fixed. A ball of mass m is attached to the other end of wire. The wire and ball are rotating with an angular velocity w in a horizontal plane. If ΔL is extension produced in wire, then the Young's modulus of wire is

(a) $\dfrac{mL^2\omega}{A\Delta L}$ (b) $\dfrac{m\omega^2 L^2}{A\Delta L}$

(c) $\dfrac{m\omega^2 L}{A\Delta L}$ (d) $\dfrac{m\omega L}{A\Delta L}$

14. The Young's modulus of wire of length L and radius r is Y. If the length and radius are reduced to $\dfrac{L}{3}$ and $\dfrac{r}{2}$, then its Young's modulus will be

(a) Y (b) $\dfrac{3}{4}Y$

(c) $3Y$ (d) $4Y$

15. On increasing the length by 0.5 mm of a steel wire of length 4m and area of cross section 5 mm^2, the force required is ($Y = 11.90 \times 10^{12}$ N/m^2)

(a) 4×10^3 N (b) 2×10^3 N

(c) 7.4×10^3 N (d) 10^3 N

16. A block of mass 500 kg is suspended by wire of length 70 cm. The area of cross-section of wire is 10 mm^2. When the load is removed, the wire contracts by 0.5 cm. The Young's modulus of the material of wire will be

(a) 10×10^{14} N/m^2 (b) 4×10^{14} N/m^2

(c) 8×10^{11} N/m^2 (d) 7×10^{10} N/m^2

17. Two wires A and B are of same materials. Their length are in the ratio 3 : 4 and diameters are in the ratio 5 : 1 when stretched by force FA and FB respectively they get equal increase in their lengths. The ratio of FB/FA is.

(a) 0.01 (b) 0.03

(c) 0.04 (d) 0.05

18. Four wires made of same material are stretched by the same load. Their dimensions are given below. The one which elongates more is

(a) Wire of length 1 m and diameter 1 mm

(b) Length 2 m, diameter 2 mm

(c) Length 3 m, diameter 3 mm

(d) Length 0.5 m, diameter 0.5 mm

19. A uniform bar of length 'L' and cross sectional area 'A' is subjected to a tensile load 'F'. 'Y' be the Young's modulus and 'σ' be the Poisson's ratio then volumetric strain is

(a) $\dfrac{F}{AY}(1-\sigma)$ (b) $\dfrac{F}{AY}(2-\sigma)$

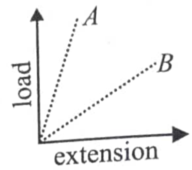
(c) $\dfrac{F}{AY}(1-2\sigma)$ (d) $\dfrac{F}{AY}.\sigma$

20. The mean distance between the atoms of iron is 3×10^{-10} m and interatomic force constant for iron is 7 N/m. The Young's modulus of elasticity for iron is

(a) 2.33×10^5 N/m^2 (b) 23.3×10^{10} N/m^2

(c) 233×10^{10} N/m^2 (d) 2.33×10^{10} N/m^2

21. The dimensions of two wires A and B are the same. But their materials are different. Their load-extension graphs are shown. If Y_A and Y_B are the values of Young's modulus of elasticity of A and B respectively then



(a) $Y_A > Y_B$ (b) $Y_A < Y_B$

(c) $Y_A = Y_B$ (d) $Y_B = 2Y_A$

22. A rubber cord of length L is suspended vertically. Density of rubber is D and Young's modulus is Y. If the cord extends by a length l under its own weight then

(a) $l = \dfrac{L^2 Dg}{Y}$ (b) $l = \dfrac{L^2 Dg}{2Y}$

(c) $l = \dfrac{L^2 Dg}{4Y}$ (d) $l = \dfrac{L^2 Dg}{8Y}$

23. The metal ring of initial radius r and cross-sectional area A is fitted onto a wooden disc of radius $R > r$. If young's modulus of the metal is Y, then the tension in the ring is

(a) $\dfrac{AYR}{r}$ (b) $\dfrac{AY(R-r)}{r}$

(c) $\dfrac{Y}{A}\left(\dfrac{R-r}{r}\right)$ (d) $\dfrac{Yr}{AR}$

24. The force required to stretch a steel wire 100 m^2 in cross section to triple its length is ($Y = 2 \times 10^{11}$ Nm^{-2})

(a) 4×10^{10} N (b) 4×10^{12} N

(c) 4×10^{11} N (d) 4×10^{13} N

TANGENTIAL STRESS AND STRAIN, SHEAR MODULUS

25. Modulus of rigidity of diamond is

(a) Too less (b) Greater than all matters

(c) Less than all matters (d) Zero

26. Modulus of rigidity of a liquid

(a) Non zero constant (b) Infinite

(c) Zero (d) Can not be predicted

27. For a given material, the Young's modulus is 2.4 times tha of rigidity modulus. Its Poisson's ratio is

(a) 2.4 (b) 1.2 (c) 0.4 (d) 0.2

28. Shearing stress causes change in

(a) Length (b) Breadth

(c) Shape (d) Volume

29. A 3 cm long copper wire is stretched to increase its length by 0.3 cm. Find the lateral strain in the wire, if the Poisson's ratio for it is 0.26.
(*a*) 0.013 (*b*) 0.018
(*c*) 0.026 (*d*) 0.016

30. Y, k, n represent respectively the young's modulus,bulk modulus and rigidity modulus of a body. If rigidity modulus is twice the bulk modulus, then
(*a*) $Y = 5k/18$ (*b*) $Y = 5n/9$
(*c*) $Y = 9k/5$ (*d*) $Y = 18k/5$

31. For a material $Y = 6.6 \times 10^{10}$ N/m^2 and bulk modulus $K = 11 \times 10^{10}$ N/m^2, then its Poissons's ratio is
(*a*) 0.8 (*b*) 0.35
(*c*) 0.7 (*d*) 0.4

32. If the poisson's ratio of a solid is 2/5, then the ratio of its young's modulus to the rigidity modulus is
(*a*) 5/4 (*b*) 7/15
(*c*) 14/9 (*d*) 14/5

33. A material has Poisson's ratio 0.30. If a uniform rod suffers a longitudinal strain of 3×10^{-4}, then the percentage change in volume.
(*a*) 0.05% (*b*) 0.02%
(*c*) 0.01% (*d*) 0.08%

PRESSURE AND VOLUMETRIC STRAIN, BULK MODULUS OF ELASTICITY

34. The isothermal elasticity of a gas is equal to
(*a*) Density (*b*) Volume
(*c*) Pressure (*d*) Specific heat

35. The adiabatic elasticity of a gas is equal to
(*a*) $\gamma \times$ density (*b*) $\gamma \times$ volume
(*c*) $\gamma \times$ pressure (*d*) $\gamma \times$ specific heat

36. The only elastic modulus that applies to fluids is
(*a*) Young's modulus (*b*) Shear modulus
(*c*) Modulus of rigidity (*d*) Bulk modulus

37. A ball falling in a lake of depth 200 m shows 0.1% decrease in its volume at the bottom. What is the bulk modulus of the material of the ball
(*a*) 19.6×10^8 N/m^2 (*b*) 19.6×10^{-10} N/m^2
(*c*) 19.6×10^{10} N/m^2 (*d*) 19.6×10^{-8} N/m^2

38. What is the density of water at a depth where the pressure is 501 atm, given its density at the surface is 1.0×10^3 kg/m^3? Given : Bulk modulus of water $K = 2.0 \times 10^9$ Pa. 1 atm = 1.01×10^5 Pa
(*a*) 1×10^3 kg/m^3 (*b*) 3.51×10^3 kg/m^3
(*c*) 2.12×10^3 kg/m^3 (*d*) 0.52×10^3 kg/m^3

39. To what depth below the surface of sea should a rubber ball be taken so as to decrease its volume by 0.4% [density of sea water = 1000 kg m^{-3}, Bulk modulus of rubber = 9×10^8 Nm^{-2}, g = 10 m/s^2]
(*a*) 5×10^2 m (*b*) 30×10^6 m
(*c*) 40×10^6 m (*d*) 36×10^1 m

40. The fractional change in volume per unit increase in pressure is called
(*a*) Pressure coefficient (*b*) Volume coefficient
(*c*) Bulk modulus (*d*) Compressibility

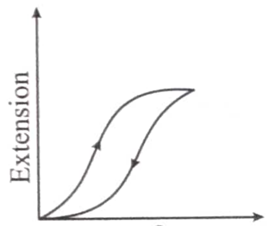
41. An increase in pressure required to decrease the 200 litres volume of a liquid by 0.004% in container is : (Bulk modulus of the liquid = 2100 MPa)
(*a*) 188 kPa (*b*) 8.4 kPa
(*c*) 18.8 kPa (*d*) 84 kPa

42. A metal wire elongates by hanging load on it, then fractional change in volume $\frac{\Delta V}{V}$ is proportional to
(*a*) $l\Delta l$ (*b*) $\frac{\Delta l}{l}$
(*c*) $\frac{2\Delta l}{l}$ (*d*) $\sqrt{\frac{\Delta l}{l}}$

ELASTIC POTENTIAL ENERGY

43. An iron bar of length 'L' cross section 'A' and Young's modulus 'Y' is pulled by a force 'F', from both the ends so as to produce an elongation 'e'. The elastic energy stored in the deformed rod will be:
(*a*) $\frac{F^2L}{AY}$ (*b*) $\frac{F^2L}{2AY}$
(*c*) $\frac{F^2}{2A^2Y}$ (*d*) None of these

44. The diagram shows a force-extension graph for a rubber band. Consider the following statements



I. It will be easier to compress this rubber than expand it
II. Rubber does not return to its original length after it is stretched
III. The rubber band will get heated if it is stretched and released

Which of these can be deduced from the graph
(*a*) III only (*b*) II and III
(*c*) I and III (*d*) I only

45. A wire suspended vertically from one of its ends is stretched by attaching a weight of 200 N to the lower end. The weight stretches the wire by 1 mm. Then the elastic energy stored in the wire is
(*a*) 20 J (*b*) 0.1 J
(*c*) 0.2 J (*d*) 10 J

46. A 5 metre long wire is fixed to the ceiling. A weight of 10 kg is hung at the lower end and is 1 metre above the floor. The wire was elongated by 1 mm. The energy stored in the wire due to stretching is

(a) Zero (b) 0.05 joule

(c) 100 joule (d) 500 joule

47. If the tension on a wire is removed at once, then

(a) It will break

(b) Its temperature will reduce

(c) There will be no change in its temperature

(d) Its temperature will increases

48. A wire is suspended by one end. At the other end, a weight equivalent to 20 N force is applied. If the increase in length is 1.0 mm, the increase in energy of the wire will be

(a) 0.01 J (b) 0.02 J

(c) 0.04 J (d) 1.00 J

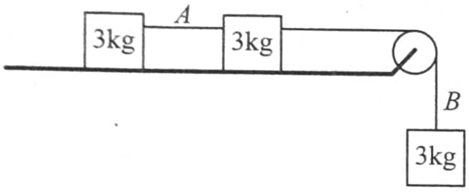
49. The elastic potential energy of a wire of length 'L' and radius 'R' twisted by an angle α, modulus of rigidity of wire is 'η':

(a) $\frac{\pi}{4}\frac{\eta R^4\alpha^2}{L}$ (b) $\frac{\pi R^4\alpha^2}{L}$

(c) $\frac{\pi^2 R^4\alpha^2}{2L}$ (d) $\frac{\pi^2 R^2\alpha^2}{4L}$

Exercise-2 (Learning Plus)

1. Three equal masses of 3 kg are connected by two massless strings of cross sectional area 0.005 cm^2 and Young modulus is 2×10^{11} N/m^2 each. The longitudinal strain in the wires



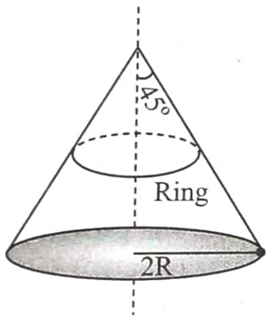
(a) are equal

(b) cannot be different

(c) must be different

(d) may or may not be different

2. A ring of mass m, radius R, cross-sectional area a and Young's modulus Y is kept on a smooth cone of radius $2R$ and semi-vertical angle 45°, as shown in the figure. Assume that the extension in the ring is small. Then choose the wrong statement:



(a) The tension in the ring will be same throughout

(b) The tension in the ring will be independent of the radius of ring if radius of the ring is less than $2R$

(c) The extension in the ring will be $\frac{mgR}{aY}$

(d) Elastic potential energy stored in the ring will be $\frac{m^2g^2R}{8\pi Ya}$

3. The diameter of a brass rod is 4 mm and Young's modulus of brass is 9×10^{10} N/m^2. The force required to stretch it by 0.1% of its length is:

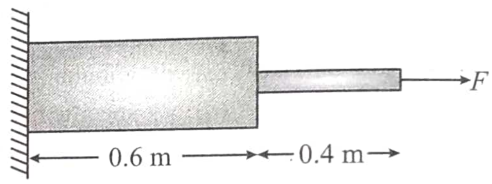
(a) 360 π N

(b) 36 N

(c) 144 $\pi \times 10^3$ N

(d) 36 $\pi \times 10^5$ N

4. Two bars of steel ($Y = 2 \times 10^{11}$ N/m^2) are joined together as shown. The area of cross section of the left bar is 15 cm^2 and the area of right bar is unknown. The extension in both bars is the same.



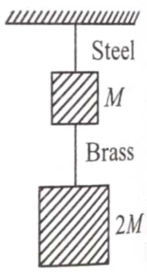
(a) The area of right bar is 10 cm^2

(b) The stresses in left and right bar are in ratio 3 : 2

(c) The decrease in thickness of bar is more for the left

(d) The decrease in thickness of bar is less for right bar

5. If the ratio of lengths, radii and Young's moduli of steel and brass wires in the figure are a, b and c respectively, then the corresponding ratio of increases in their lengths is:



(a) $\frac{2a^2c}{b}$

(b) $\frac{3a}{2b^2c}$

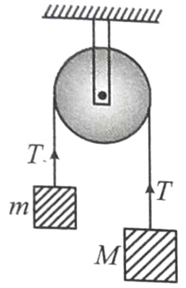
(c) $\frac{2ac}{b^2}$

(d) $\frac{3ac}{2ab^2}$

6. The dimensions of four wires of the same material are given below. In which wire the increase in length will be maximum when the same tension is applied

(a) Length 100 cm, Diameter 1 mm
(b) Length 200 cm, Diameter 2 mm
(c) Length 300 cm, Diameter 3 mm
(d) Length 50 cm, Diameter 0.5 mm

7. Two blocks of masses m and M are connected by means of a metal wire of cross-sectional area A passing over a frictionless fixed pulley as shown in the figure. The system is then released. If $M = 2$ m, then the stress produced in the wire is:

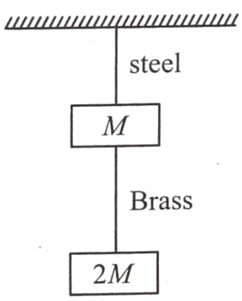


(a) $\dfrac{2mg}{3A}$ (b) $\dfrac{3mg}{4A}$
(c) $\dfrac{mg}{A}$ (d) $\dfrac{4mg}{3A}$

8. Two bodies of masses 2 kg and 3 kg are connected by a metal wire of cross-section 0.04 mm^2 and are placed on a frictionless horizontal surface. Breaking stress of metal wire is 2.5 GPa. The maximum force F that can be applied to 3 kg block so that wire does not break is

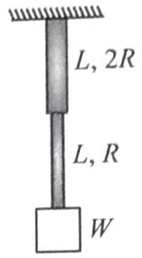
(a) 100 N (b) 150 N
(c) 200 N (d) 250 N

9. If the ratio of lengths, radii and young's modulus of steel and brass wires in figure are 2 : 1, 2 : 1, 3 : 1 respectively. Then corresponding ratio of increase in their length would be. (Neglect the mass of the wires and take $g = 10$ m/s^2)



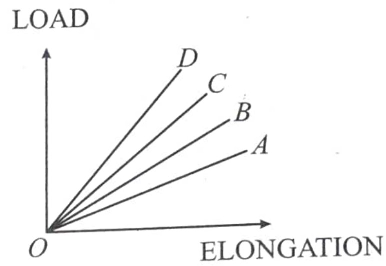
(a) 1 : 4 (b) 8 : 1
(c) 1 : 3 (d) 2 : 4

10. Two light wires of the same material (Young's modulus Y) and same length L but different radii R and $2R$, as shown in the figure, are joined end to end and suspended from a fixed support. A weight W is suspended from the combination. The elastic potential energy in the system is



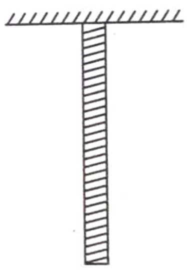
(a) $\dfrac{3W^2L}{4\pi R^2Y}$ (b) $\dfrac{3W^2L}{8\pi R^2Y}$
(c) $\dfrac{5W^2L}{8\pi R^2Y}$ (d) $\dfrac{W^2L}{\pi R^2Y}$

11. The load versus elongation graph for four wires of the same materials is shown in the figure. The thinnest wire is represented by the line:



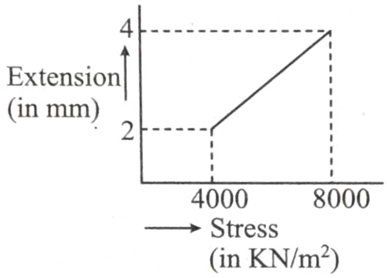
(a) OC (b) OD
(c) OA (d) OB

12. The extension in a uniform rod of length ℓ, mass m, cross-section radius r and young's modulus Y when it is suspended at one of its end is: (consider area of cross-section remains same)



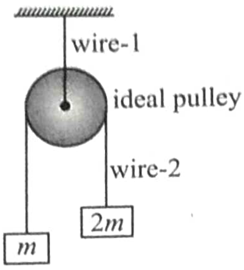
(a) $\dfrac{mg\ell}{\pi r^2Y}$ (b) $\dfrac{mg\ell}{2\pi r^2Y}$
(c) $\dfrac{2mg\ell}{\pi r^2Y}$ (d) $\dfrac{mg\ell}{4\pi r^2Y}$

13. In determination of young modulus of elasticity of wire, a force is applied and extension is recorded. Initial length of wire is 1 m. The curve between extension and stress is depicted then young modulus of wire will be:



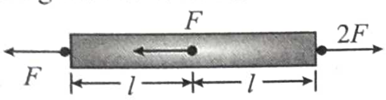
(a) 2×10^9 N/m^2 (b) 1×10^9 N/m^2
(c) 2×10^{10} N/m^2 (d) 1×10^{10} N/m^2

14. Find the ratio of stress present in wire-1 to that in wire-2 shown in the diagram while both wires have same material & thickness.



(a) 1 (b) 2
(c) 3 (d) 4

15. Forces acting on a uniform rod having length $2l$, area of cross-section A and Young's modulus Y are shown in the figure. Elongation of the rod is



(a) $\frac{Fl}{AY}$ (b) $\frac{3Fl}{2AY}$
(c) $\frac{2Fl}{AY}$ (d) $\frac{3Fl}{AY}$

16. One end of a cylindrical solid rod of length L and radius r is clamped in a fixed position. The other end is turned by an external torque τ resulting in a twist θ. The shear modulus is given by η. The twist angle θ is proportional to
(a) $\tau r^1 \eta/\pi L^2$
(b) $\tau\, r^4/\pi\, \eta L$
(c) $\tau\, \eta\, L/\pi\, r^4$
(d) $\tau\, L/\pi\, \eta\, r^4$

17. The upper end of a wire of radius 4 mm and length 100 cm is clamped and its other end is twisted through an angle of 30°. Then angle of shear is
(a) 12° (b) 0.12°
(c) 1.2° (d) 0.012°

18. A uniform wire (Young's modulus 2×10^{11} Nm^{-2}) is subjected to longitudinal tensile stress of 5×10^{7} Nm^{-2}. If the overall volume change in the wire is 0.02%, the fractional decrease in the radius of the wire is close to:
(a) 0.25×10^{-4} (b) 1.0×10^{-4}
(c) 5×10^{-4} (d) 1.5×10^{-4}

19. A metal block is experiencing an atmospheric pressure of 1×10^5 N/m^2, when the same block is placed in a vacuum chamber, the fractional change in its volume is (the bulk modulus of metal is 1.25×10^{11} N/m^2)
(a) 4×10^{-7} (b) 2×10^{-7}
(c) 8×10^{-7} (d) 1×10^{-7}

20. Expansion during heating –
(a) occurs only in a solid
(b) increases the density of the material
(c) decreases the density of the material
(d) occurs at the same rate for all liquids and solids

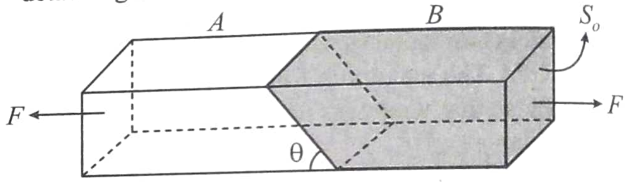
21. For a constant hydraulic stress on an object, the fractional change in the object's volume $\left(\frac{\Delta V}{V}\right)$ and its bulk modulus (B) are related as
(a) $\frac{\Delta V}{V} \propto B$ (b) $\frac{\Delta V}{V} \propto \frac{1}{B}$
(c) $\frac{\Delta V}{V} \propto B^2$ (d) $\frac{\Delta V}{V} \propto B^{-2}$

22. If work done in stretching a wire by 1mm is 2J, the work necessary for stretching another wire of same material, but with double the radius and half the length by 1mm in joule is -
(a) 1/4 (b) 4
(c) 8 (d) 16

23. If the potential energy of a spring is V on stretching it by 2 cm, then its potential energy when it is stretched by 10 cm will be
(a) $V/25$ (b) $5V$
(c) $V/5$ (d) $25V$

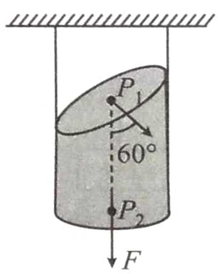
24. Two wires of same diameter of the same material having the length l and $2l$. If the force F is applied on each, the ratio of the work done in the two wires will be
(a) 1 : 2 (b) 1 : 4
(c) 2 : 1 (d) 1 : 1

25. Two bars A and B are stuck using an adhesive. The contact surface of the bars make an angle θ with the length. Area of cross section of each bar is S_0. It is known that the adhesive yields if normal stress at the contact surface exceeds σ_0. Find the maximum pulling force F that can be applied without detaching the bars.



(a) $\frac{\sigma_o S_o}{\sin^2\theta}$ (b) $\frac{\sigma_o}{S_0 \sin^2\theta}$
(c) $\frac{\sigma_o S_0}{2\sin^2\theta}$ (d) $\frac{\sigma_0}{2S_0 \sin^2\theta}$

26. A massless uniform rod is subjected to force F at its free end as shown in figure. The ratio of tensile stress at plane P_1 to stress at P_2 is



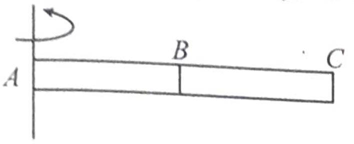
(a) 1 : 2 (b) $\sqrt{2}:1$
(c) 1 : 4 (d) 3 : 2

27. A rod of length L kept on a smooth horizontal surface is pulled along its length by a force F. The area of cross-section is A and Young's modulus is Y. The extension in the rod is

(a) $\frac{FL}{AY}$ (b) $\frac{2FL}{AY}$

(c) $\frac{FL}{2AY}$ (d) Zero

28. A rigid rod of mass m and lengths l. Is being rotated in horizontal plane about a vertical axis, passing through one end A. If T_A, T_B and T_C are the tensions in rod at point A, mid point B and point C of rod respectively, then



(a) $TC = 0$ (b) $T_B = \frac{3}{4}T_A$

(c) $T_B = \frac{T_A}{2}$ (d) $TA = m\omega^2 l$

29. A mild steel wire of length 1.0 m and cross-sectional area 0.50×10^{-2} cm^2 is stretched well within its elastic limit horizontally between two pillars. A mass of 100 g is suspended from the midpoint of the wire. Calculate the depression at the midpoint. ($Y_{steel} = 200$ GPa)

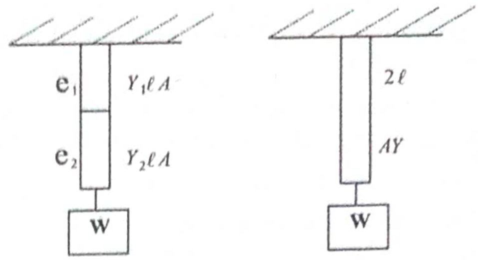
(a) 1.074 m (b) 0.712 m

(c) 0.85 m (d) 1.516 m

30. Wires A and B are made from the same material. A has twice the diameter and three times the length to that of B. If the elastic limits are not reached, when each wire is stretched by the same tension, Ratio of energy stored in wire A and B is:

(a) 2 : 3 (b) 3:4 (c) 3 : 2 (d) 6:1

31. Two wires of same length and radius are joined end to end and loaded. The Young's modulii of the materials of the two wires are Y_1 and Y_2. If the combination behaves as a single wire then its Young's modulus is



(a) $\frac{Y_1Y_2}{Y_1+Y_2}$ (b) $\frac{Y_1+Y_2}{2}$

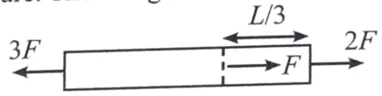
(c) $\frac{2Y_1Y_2}{Y_1+Y_2}$ (d) $\frac{Y_1+Y_2}{Y_1Y_2}$

32. An aluminum wire and a steel wire of the same length and cross-section are joined end to end. The composite wire is hung from a rigid support and a load is suspended from the free end. If the increase in the length of the composite wire is 2.7 mm, then the increase in the length of each wire is (in mm). ($Y_{Al} = 2 \times 10^{11}$Nm^{-2}, $Y_{steel} = 7 \times 10^{10}$ Nm^{-2})

(a) 1.7,1 (b) 1.3,1.4

(c) 1.5,1.2 (d) 2,0.7

33. A uniform slender rod of length L, cross- sectional area A and Young's modulus Y is acted upon by the forces shown in the figure. The elongation of the rod is:



(a) $\frac{3FL}{5AY}$ (b) $\frac{2FL}{5AY}$ (c) $\frac{3FL}{8AY}$ (d) $\frac{8FL}{3AY}$

34. To what depth below the surface of sea should a rubber ball be taken as to decrease the volume by 0.1%?
(take, density of sea water = 1000 kgm^{-3}, Bulk modulus of rubber = 9×10^8 Nm^{-2}, acceleration due to gravity = 10 ms^{-2})

(a) 9 m (b) 18 m (c) 180 m (d) 90 m

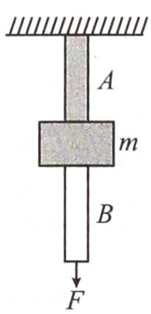
35. Two wires A and B are of same material. Their lengths are in the ratio 1 : 2 and diameters are in the ratio 2 : 1. When stretched by forces F_A and F_B respectively, they get equal increase in their lengths. Then, the ratio $F_A : F_B$ should be

(a) 1:2 (b) 1:1 (c) 2:1 (d) 8:1

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. The wires A and B shown in the figure, are made of the same material and have radii r_A and r_B. A block of mass m kg is tied between them: If the force F is mg/3, one of the wires breaks. Then,



(a) A will break before B if $r_A < 2r_B$

(b) A will break before B if $r_A = r_B$

(c) Either A or B will break if $r_A = 2r_B$

(d) The lengths of A and B must be known to decide which wire will break

2. A body of mass M is attached to the lower end of a metal wire, whose upper end is fixed. The elongation of the wire is l.

(a) Loss in gravitational potential energy of M is Mgl

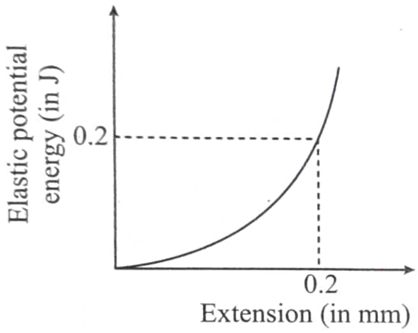
(b) The elastic potential energy stored in the wire is Mgl

(c) The elastic potential energy stored in the wire is 1/2 Mgl

(d) Heat produced is 1/2 Mgl

3. Four rods A, B, C, D of same length and material but of different radii r, $r\sqrt{2}$, $r\sqrt{3}$ and $2r$ respectively are held between two rigid walls. The temperature of all rods is increased by same amount. If the rods do not bend, then
   (*a*) the stress in the rods are in the ratio 1 : 2 : 3 : 4
   (*b*) the force on the rod exerted by the wall are in the ratio 1 : 2 : 3 : 4
   (*c*) the energy stored in the rods due to elasticity are in the ratio 1 : 2 : 3 : 4
   (*d*) the strains produced in the rods are in the ratio 1 : 2 : 3 : 4

4. Figure shows the graph of elastic potential energy (U) stored versus extension, for a steel wire ($Y = 2 \times 10^{11}$ Pa) of volume 200 cc. If area of cross-section A and original length L, then



(*a*) $A = 10^{-4}$ m^2 (*b*) $A = 10^{-3}$ m^2
(*c*) $L = 1.5$ m (*d*) $L = 2$ m

COMPREHENSION BASED QUESTIONS

Comprehension (Q. 5 to 9): When a tensile or compressive load 'P' is applied to rod or cable, its length changes. The change in length x for an elastic material is proportional to the force (Hook's law).

$$P \propto x \text{ or } P = kx$$

The above equation is similar to the equation of spring. For a rod of length L, area A and young modulus Y, the extension x can be expressed as-

$$x = \frac{PL}{AY} \text{ or } P = \frac{AY}{L}x, \text{ hence } K = \frac{AY}{L}$$

Thus rods or cables attached to lift can be treated as springs.

The energy stored in rod is called strain energy & equal to $\frac{1}{2}Px$.

The loads placed or dropped on the floor of lift cause stresses in the cables and can be evaluated by spring analogy. If the cable of lift is previously stressed and load is placed or dropped, then maximum extension in cable can be calculated by energy conservation.

5. If rod of length 4 m, area 4 cm^2 and young modules 2×10^{10} N/m^2 is attached with mass 200 kg, then angular frequency of SHM (rad/sec.) of mass is equal to
   (*a*) 1000 (*b*) 10
   (*c*) 100 (*d*) 10π

6. In above problem if mass of 10 kg falls on the massless collar attached to rod from the height of 99 cm then maximum extension in the rod is equal ($g = 10$ m/sec^2)
   (*a*) 9.9 cm (*b*) 10 cm
   (*c*) 0.99 cm (*d*) 1 cm

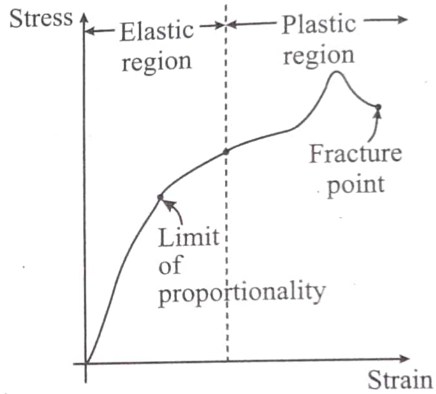
7. In the above problem, the maximum stress developed in the rod is equal to - (N/m^2)
   (*a*) 5×10^7 (*b*) 5×10^8
   (*c*) 4×10^7 (*d*) 4×10^8

8. If two rods of same length (4m) and cross section areas 2 cm^2 and 4 cm^2 with same young modulus 2×10^{10} N/m^2 are attached one after the other with mass 600 kg then angular frequency is
   (*a*) $\frac{1000}{3}$ (*b*) $\frac{10}{3}$
   (*c*) $\frac{100}{3}$ (*d*) $\frac{10\pi}{3}$

9. Four identical rods of geometry as described in problem (*b*) are attached with lift. If weight of the lift cage is 1000 N, and elastic limit of each rod is taken as 9×10^6 N/m^2 then the number of persons it can carry safely is equal to. ($g = 10$ m/sec^2, assume average mass of a person as 50 kg and lift moves with constant speed)
   (*a*) 7 (*b*) 26 (*c*) 24 (*d*) 25

Comprehension (Q. 10 to 12): On gradual loading, stress-strain relationship for a metal wire is as follows.

Within proportionality limit, stress $\propto$ strain or, $\frac{\text{Stress}}{\text{Strain}}$ = a constant for the material of wire.



10. Two wires of same material have length and radius (L, r) and $\left(2L, \frac{r}{2}\right)$. The ratio of their Young's modulii is
    (*a*) 1 : 2 (*b*) 2 : 3
    (*c*) 2 : 1 (*d*) 1 : 1

11. Just on crossing the yield region, the material will have
    (*a*) increased and breaking stress
    (*b*) reduced and breaking stress
    (*c*) constant stress
    (*d*) None of these

12. If $\frac{\text{Stress}}{\text{Strain}}$ is x in elastic region and y in the region of yield, then
    (*a*) $x > y$ (*b*) $x = y$
    (*c*) $x < y$ (*d*) $x = 2y$

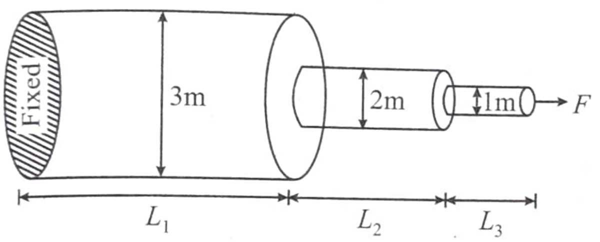
MATCH THE COLUMN TYPE QUESTIONS

13. A metal wire of length L is suspended vertically from a rigid support. When a bob of mass M is attached to the lower end of wire, the elongation of the wire is ℓ:

	Column-I		Column-II
A.	The loss in gravitational potential energy of mass M is equal to	p.	$Mg\ell$
B.	The elastic potential energy stored in the wire is equal to	q.	$\frac{1}{2}Mg\ell$
C.	The elastic constant of the wire is equal to	r.	Mg/ℓ
D.	Heat produced during extension is equal to	s.	$\frac{1}{4}Mg\ell$

(*a*) A-(p); B-(q); C-(r); D-(q)
(*b*) A-(p); B-(q); C-(r); D-(s)
(*c*) A-(s); B-(r); C-(p); D-(q)
(*d*) A-(q); B-(p); C-(r); D-(s)

14. Three wires of lengths L_1, L_2, L_3 and Young's moduii Y_1, Y_2 and Y_3 respectively are pulled by a force F shown in fig. The extensions produced in wires are ΔL_1, ΔL_2, ΔL_3

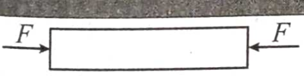
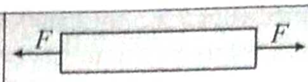
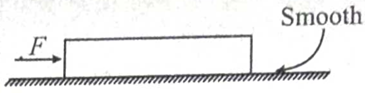


Match the Column-I with Column-II

	Column-I		Column-II
A.	If $9L_2 = 4L_1$ and $\Delta L_1 = \Delta L_2$, then	p.	$Y_2 = Y_1$
B.	If $L_2 < 4L_3$ and $\Delta L_2 = \Delta L_3$, then	q.	$Y_2 > Y_1$
C.	If $\Delta L_1 = \Delta L_2$ and $L_1 = L_2$, then	r.	$Y_2 < Y_3$
D.	If $L_2 = L_3$ and $\Delta L_2 = \Delta L_3$, then	s.	$4Y_2 = Y_3$

(*a*) A-(q); B-(p); C-(r); D-(s)
(*b*) A-(p); B-(r); C-(q); D-(r,s)
(*c*) A-(s); B-(r); C-(p); D-(q)
(*d*) A-(p); B-(q); C-(r); D-(s)

15. In Column I, a unsiform bar of uniform cross-sectional area under the application of forces is shown in the figure and in Column II, some effects/phenomena are given. Match the two columns.

	Column-I		Column-II
A.		p.	Uniform stresses are developed in the rod.
B.		q.	Non-uniform stresses are developed in the rod.
C.		r.	Compressive stresses are developed in the rod.
D.		s.	Tensile stresses are developed in the rod.

(*a*) A-(r); B-(p,q); C-(s); D-(p,q,r,s)
(*b*) A-(r,s); B-(r,q); C-(p,q,r,s); D-(s)
(*c*) A-(p,r); B-(p,s); C-(q,s); D-(q,r)
(*d*) A-(r,s); B-(p,r); C-(p,q,r,s); D-(p,q)

NUMERICAL TYPE QUESTIONS

16. Find the greatest length (in km) of a steel wire of uniform cross-section that can hang vertically without breaking. Breaking stress of steel = 8.0×10^8 N/m^2. Density of steel = 8.0×10^3 kg/m^3.

17. The elastic limit of a steel cable is 2.40×10^8 Pa and the cross-section area is 4.00 cm^2. Find the maximum upward acceleration (in m/s^2) that can be given to a 800 kg elevator supported by the cable if the stress should not exceed one third of the elastic limit.

18. Two steel wires of same length but radii r and $2r$ are connected together end to end and tied to a wall as shown.

The force stretches the combination by 10 mm. How far does the midpoint A move. (in mm)

19. What is the maximum height (in m) of a brick column of uniform cross section for which column deformation due to its own weight is within the elastic limit?

$P_{atmospheric} = 100$ kPa,

$\rho = 1.8 \times 10^3$ kg/m^3.

Elastic limit, $\sigma = 3.7$ M Pa.

20. A substance breaks down under a stress of 10^5 Pa. If the density of the wire is 2×10^3 kg/m^3, find the minimum length of the wire which will break under its own weight ($g = 10$ m/s^2)

21. A steel wire, 3.2 m long, has a diameter of 1.2 mm. The wire stretches by 1.6 mm when it bears a load. Young's modulus for steel is 2.0×10^{11} Pa. Find the mass (in kg) of the load closest?

22. A horizontally oriented copper rod of length l is rotated about a vertical axis passing through its middle. If the rotated frequency at which the rod ruptures is $\frac{1}{2\pi}\sqrt{\frac{N\sigma}{\rho l_2}}$ then N is ________. (Given : Breaking or rupture strength of copper of σ and density of copper is ρ.)

23. A sphere of radius 10 cm and mass 25 kg is attached to the lower end of a steel wire of length 5 m and diameter 4 mm which is suspended from the ceiling of a room. The point of support is 521 cm above the floor. When the sphere is set swinging as a simple pendulum, its lowest point just grazes the floor. What is the velocity of the ball at its lowest position in m/s. ($Y_{steel} = 2 \times 10^{11}$ N/m^2).

Exercise-4 (Past Year Questions)

JEE MAIN

1. A man grows into a giant such that his linear dimensions increase by a factor of 9. Assuming that his density remains same, the stress in the leg will change by a factor of: **(2017)**

(a) 81 (b) 1/81
(c) 9 (d) 1/9

2. A solid sphere of radius r made of a soft material of bulk modulus K is surrounded by a liquid in a cylindrical container. A massless piston of area a floats on the surface of the liquid, covering entire cross section of cylindrical container. When a mass m is placed on the surface of the piston to compress the liquid, the fractional decrement in the radius of the sphere, $\left(\frac{dr}{r}\right)$, is: **(2018)**

(a) $\frac{Ka}{3mg}$ (b) $\frac{mg}{3Ka}$
(c) $\frac{mg}{Ka}$ (d) $\frac{Ka}{mg}$

3. A load of mass M kg is suspended from a steel wire of length 2 m and radius 1.0 mm in Searle's apparatus experiment. The increase in length produced in the wire is 4.0 mm. Now the load is fully immersed in a liquid of relative density 2. The relative density of the material of load is 8. The new value of increase in length of the steel wire is **(2019)**

(a) 3.0 mm (b) 4.0 mm
(c) 5.0 mm (d) zero

4. A boy's catapult is made of rubber cord which is 42 cm long, with 6 mm diameter of cross-section and of negligible mass. The boy keeps a stone weighing 0.02 kg on it and stretches the cord by 20 cm by applying a constant force. When released, the stone flies off with a velocity of 20 ms^{-1}. Neglect the change in the area of cross-section of the cord while stretched. The Young's modulus of rubber is closest to: **(2019)**

(a) 10^4 Nm^{-2} (b) 10^8 Nm^{-2}
(c) 10^6 Nm^{-2} (d) 10^3 Nm^{-2}

5. A steel wire having a radius of 2.0 mm, carrying a load of 4 kg, is hanging from a ceiling. Given that $g = 3.1\pi$ ms^{-2}, what will be the tensile stress that would be developed in the wire? **(2019)**

(a) 4.8×10^6 Nm^{-2} (b) 5.2×10^6 Nm^{-2}
(c) 6.2×10^6 Nm^{-2} (d) 3.1×10^6 Nm^{-2}

6. The elastic limit of brass is 379 MPa. What should be the minimum diameter of a brass rod if it is to support a 400 N load without exceeding its elastic limit? **(2019)**

(a) 1.16 mm (b) 0.90 mm
(c) 1.36 mm (d) 1.00 mm

7. In an experiment, brass and steel wires of length 1 m each with areas of cross section 1 mm^2 are used; the wires are connected in series and one end of the combined wire is connected to a rigid support and other end is subjected to elongation. The stress required to produce a net elongation of 0.2 mm is: **(2019)**
(Given, the Young's Modulus for steel and brass are respectively, 120×10^9 N/m^2 and 60×10^9 N/m^2)

(a) 0.2×10^6 N/m^2 (b) 4.0×10^6 N/m^2
(c) 1.8×10^6 N/m^2 (d) 1.2×10^6 N/m^2

8. Young's moduli of two wires A and B in the ratio 7 : 4. Wire A is 2 m long and has radius R. Wire B is 1.5 m long and has radius 2 mm. If the two wires stretch by the same length for a given load, then the value of R is close to **(2019)**

(a) 1.9 mm (b) 1.7 mm
(c) 1.5 mm (d) 1.3 mm

9. A non-isotropic solid metal cube has coefficients of linear expansion as: 5×10^{-5} /°C along the x-axis and 5×10^{-6} /°C along the y and the z-axis. If the coefficient of volume expansion of the solid is $C \times 10^{-6}$ /°C then the value of C is **(2020)**

10. A leak proof cylinder of length 1m, made of a metal which has very low coefficient of expansion is floating vertically in water at 0°C such that its height above the water surface is 20 cm. When the temperature of water is increased to 4°C, the height of the cylinder above the water surface becomes 21 cm. The density of water at $T = 4$°C, relative to the density at $T = 0$°C is close to: **(2020)**

(a) 1.0 (b) 1.04
(c) 1.26 (d) 1.01

11. A body of mass m = 10 kg is attached to one end of a wire of length 0.3 m. The maximum angular speed (in rad s^{-1}) with which it can be rotated about its other end in space station is (Breaking stress of wire = 4.8×10^7 Nm^{-2} and area of cross section of the wire = 10^{-2} cm^2) is: **(2020)**

12. Two steel wires having same length are suspended from a ceiling under the same load. If the ratio of their energy stored per unit volume is 1 : 4, the ratio of their diameters is **(2020)**

(*a*) 1 : 2
(*b*) $1:\sqrt{2}$
(*c*) 2 : 1
(*d*) $\sqrt{2}:1$

13. An object of mass m is suspended at the end of a massless wire of length L and area of cross-section A. Young modulus of the material of the wire is Y. If the mass is pulled down slightly its frequency of oscillation along the vertical direction is **(2020)**

(*a*) $f = \frac{1}{2\pi}\sqrt{\frac{mL}{YA}}$

(*b*) $f = \frac{1}{2\pi}\sqrt{\frac{YA}{mL}}$

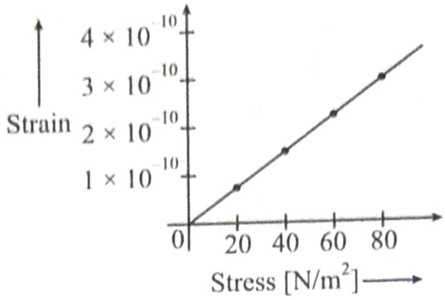
(*c*) $f = \frac{1}{2\pi}\sqrt{\frac{YL}{mA}}$

(*d*) $f = \frac{1}{2\pi}\sqrt{\frac{mA}{YL}}$

14. A stone of mass 20 g is projected from a rubber catapult of length 0.1 m and area of cross section 10^{-6} m^2 stretched by an amount 0.04 m. The velocity of the projected stone is m/s. (Young's modulus of rubber = 0.5×10^9 N/m^2) **(2021)**

15. The area of cross-section of a railway track is 0.01 m^2. The temperature variation is 10°C. Coefficient of liner expansion of material of track is 10^{-5}/°C. The energy stored per meter in the track is J/m. (Young's modulus of material of track is 10^{11} Nm^{-2}) **(2021)**

16. The elastic behaviour of material for linear stress and linear strain, is shown in the figure. The energy density for a linear strain of 5×10^{-4} is __________ kJ/m^3. Assume that material is elastic upto the linear strain of 5×10^{-4}. **(2022)**



17. A wire of length L and radius r is clamped rigidly at one end. When the other end of the wire is pulled by a force F, its length increases by 5 cm. Another wire of the same material of length $4L$ and radius $4r$ is pulled by a force $4F$ under same conditions. The increase in length of this wire is __________ cm. **(2022)**

18. The area of cross section of the rope used to lift a load by a crane is 2.5×10^{-4} m^2. The maximum lifting capacity of the crane is 10 metric tons. To increase the lifting capacity of the crane to 25 metric tons, the required area of cross section of the rope should be : **(2022)**
(take $g = 10$ ms^{-2})

(*a*) 6.25×10^{-4} m^2 (*b*) 10×10^{-4} m^2
(*c*) 1×10^{-4} m^2 (*d*) 1.67×10^{-4} m^2

19. A uniform heavy rod of mass 20 kg. Cross sectional area 0.4 m^2 and length 20 m is hanging from a fixed support. Neglecting the lateral contraction, the elongation in the rod due to its own weight is $x \times 10^{-9}$ m. The value of x is __________ : (Given. Young's modulus $Y = 2 \times 10^{11}$ Nm^{-2} and $g = 10$ ms^{-2}) **(2022)**

JEE ADVANCED

20. One end of a horizontal thick copper wire of length 2L and radius $2R$ is welded to an end of another horizontal thin copper wire of length L and radius R. When the arrangement is stretched by a applying forces at two ends, the ratio of the elongation in the thin wire to that in the thick wire is: **(2013)**

(*a*) 0.25 (*b*) 0.50
(*c*) 2.00 (*d*) 4.00

21. A cubical solid aluminium (bulk modulus $= -V\frac{dP}{dV}$ = 70 GPa) block has an edge length of 1 m on the surface of the earth. It is kept on the floor of a 5 km deep ocean. Taking the average density of water and the acceleration due to gravity to be 10^3 kg m^{-3} and 10 ms^{-2}, respectively, the change in the edge length of the block in mm is **(2020)**

ANSWER KEY

CONCEPT APPLICATION

1. (c) **2.** (b) **3.** (a) **4.** (b) **5.** (d) **6.** (c) **7.** (d) **8.** (d) **9.** (a) **10.** (d)
11. (a)

EXERCISE-1 (TOPICWISE)

1. (d) **2.** (c) **3.** (b) **4.** (c) **5.** (c) **6.** (a) **7.** (c) **8.** (a) **9.** (c) **10.** (c)
11. (b) **12.** (b) **13.** (b) **14.** (a) **15.** (c) **16.** (d) **17.** (b) **18.** (d) **19.** (c) **20.** (d)
21. (a) **22.** (b) **23.** (b) **24.** (d) **25.** (b) **26.** (c) **27.** (d) **28.** (c) **29.** (c) **30.** (d)
31. (d) **32.** (d) **33.** (c) **34.** (c) **35.** (c) **36.** (d) **37.** (a) **38.** (a) **39.** (d) **40.** (d)
41. (d) **42.** (b) **43.** (b) **44.** (c) **45.** (b) **46.** (b) **47.** (d) **48.** (a) **49.** (a)

EXERCISE-2 (LEARNING PLUS)

1. (c) **2.** (d) **3.** (a) **4.** (a) **5.** (b) **6.** (d) **7.** (d) **8.** (d) **9.** (a) **10.** (c)
11. (c) **12.** (b) **13.** (a) **14.** (b) **15.** (d) **16.** (d) **17.** (b) **18.** (a) **19.** (c) **20.** (c)
21. (b) **22.** (d) **23.** (d) **24.** (a) **25.** (a) **26.** (c) **27.** (c) **28.** (b) **29.** (a) **30.** (b)
31. (c) **32.** (d) **33.** (d) **34.** (d) **35.** (d)

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (a,b,c) **2.** (a,c,d) **3.** (b,c) **4.** (a,d) **5.** (c) **6.** (d) **7.** (a) **8.** (c) **9.** (b) **10.** (d)
11. (b) **12.** (a) **13.** (a) **14.** (b) **15.** (c) **16.** [0010] **17.** [0030] **18.** [8] **19.** [200] **20.** [5]
21. [0012] **22.** [0008] **23.** [0031]

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (c) **2.** (b) **3.** (a) **4.** (c) **5.** (d) **6.** (a) **7.** (c) **8.** (b) **9.** [60] **10.** (d)
11. [4] **12.** (d) **13.** (b) **14.** [20] **15.** [05] **16.** [25] **17.** [05] **18.** (a) **19.** [25]

JEE Advanced

20. (c) **21.** [0.24]

CHAPTER

14 Mechanical Properties of Fluids

DEFINITION OF FLUID

The term fluid refers to a substance that can flow and does not have a shape of its own. For example, liquid and gases.

Properties of fluids :

(*i*) Density (*ii*) Viscosity (*iii*) Bulk modulus of elasticity (*iv*) Pressure (*v*) Specific gravity

Density

The density ρ of a substance is defined as the mass per unit volume of a sample of the substance.

$$\rho = \frac{m}{V}$$

The SI unit of density is kg m^{-3} and dimension is $[ML^{-3}]$

Relative Density/Specific Gravity

The **specific gravity** of a substance is the ratio of its density to that of water at 4°C, which is 1000 kg/m^3. For example, the specific gravity of mercury is 13.6, and the specific gravity of water at 100°C is 0.998.

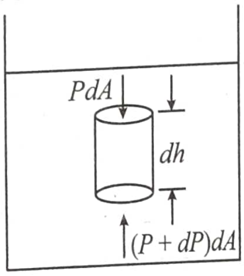
THRUST AND PRESSURE

A **perfect fluid** resists forces normal to its surface and offers no resistance to forces acting tangential to its surface.

Force exerted perpendicular to a surface is called thrust and thrust per unit area is called pressure.

Variation of Pressure with Depth in Liquid

In all fluids at rest, the pressure is a function of height or depth. To determine this, consider the forces acting on a vertical column of fluid of cross sectional area dA as shown in figure. The positive direction of vertical measurement h is taken downward. The pressure on the upper side is P, and that on the lower face is $P + dP$. The weight of the element is $\rho g dh dA$.



$\because$ The element of fluid will be at rest

$\therefore$ $F_{net} = 0$

$\Rightarrow$ $PdA + \rho g dA dh - (P + dP)\, dA = 0$

$\Rightarrow$ $dP = \rho g dh$

$$\int_{P_0}^{P} dP = \int_0^h \rho g dh$$

- This differential relation shows that the pressure in a fluid increases with depth or decreases with increased elevation.
- Liquids are generally treated as incompressible so their density ρ is constant. With ρ as constant, equation may be integrated and the result is

$$P = P_0 + \rho g h$$

$\Rightarrow$ $\Delta P = P - P_0 = \rho g h$ (Hydrostatics pressure equality)

The pressure P_0 is the pressure at the surface of the liquid where $h = 0$.

Variation of Pressure with Height

For gases, the constant density assumed in the compressible model is often not adequate. However, an alternative simplifying assumption can be made that the density is proportional to the pressure, i.e., $\rho = kp$

Let ρ_o be the density of air at the earth's surface where the pressure is atmospheric p_o, then $\rho_o = kp_o$

After eliminating k, we get $\rho = \frac{\rho_o}{p_o} p$

Substituting the value of ρ in equation,

$$dp = -\rho g dy \quad \text{or} \quad dp = -\left(\frac{\rho_o}{p_o} p\right) g dy$$

On rearranging, we get $\int_{p_o}^{p} \frac{dp}{p} = -\frac{\rho_o}{p_o} g \int_0^h dy$

where p is the pressure at a height $y = h$ above the earth's surface.

After integrating, we get

$$\ln\left|\frac{p}{p_o}\right| = -\frac{\rho_o}{p_o} gh \quad \text{or} \quad p = p_0 e^{\frac{-\rho_0}{p_0} gh}$$

Note: Instead of a linear decrease in pressure with increasing height as in the case of an incompressible fluid, in this case pressure decreases exponentially.

Characteristics of Fluid Pressure

(i) Pressure at a point acts equally in all directions.

(ii) Liquids at rest exerts lateral pressure, which increases with depth.

(iii) Pressure acts normally on any area in whatever orientation the area may be held.

(iv) Free surface of a liquid at rest remains horizontal.

(v) Pressure at every point in the same horizontal line is the same inside a liquid at rest.

(vi) Liquid at rest stands at the same height in communicating vessels.

ABSOLUTE PRESSURE AND GAUGE PRESSURE

- Absolute pressure is the total pressure at a point while gauge pressure is relative to the local atmospheric pressure.

$p_{gauge} = p_{absolute} - p_{atm}$

- Gauge pressure may be positive or negative depending upon the fact whether the pressure is more or less than the atmospheric pressure.

PASCAL'S LAW

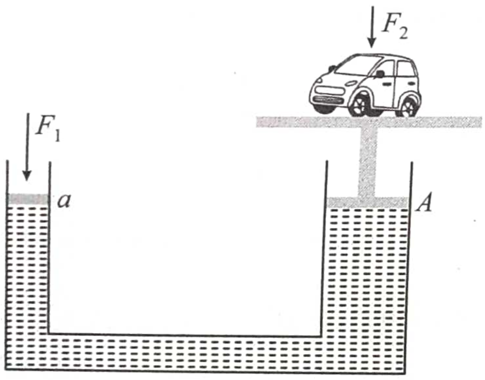
- A pressure applied to a confined fluid at rest is transmitted equally undiminished to every part of the fluid and the walls of the container.
- This principle is used in a hydraulic jack or lift, as shown in the figure.

Hydraulic Jack (lift)

- The pressure due to a small force F_1 applied to a piston of area a is transmitted to the larger piston of area A. The pressure at the two pistons is the same because they are at the same level.

$$p = \frac{F_1}{a} = \frac{F_2}{A} \quad \text{or} \quad F_2 = \left(\frac{A}{a}\right)F_1$$

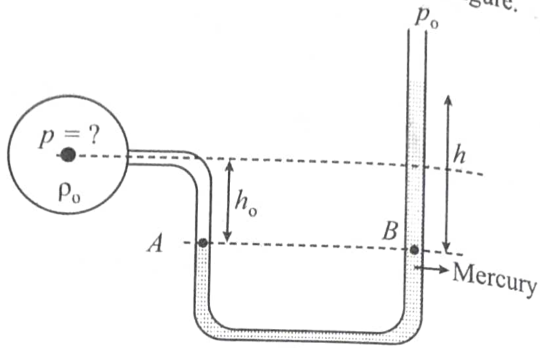
- Consequently, the force on the larger piston is large Thus, a small force F_1 acting on a small area a results in a larger force F_2 acting on a larger area A.



Pressure Measuring Devices

(1) Manometer

A **manometer** is a tube open at both the ends and bent into the shape of a "U" and partially filled with mercury. When one end of the tube is subjected to an unknown pressure p, the mercury level drops on that side of the tube and rises on the other so that the difference in mercury level is h as shown in the figure.



Since, pressure at every point in the same horizontal line is the same inside a liquid at rest.

$\therefore \quad P_A = P_B \Rightarrow p + \rho_0 g h_0 - \rho_m g h = 0$

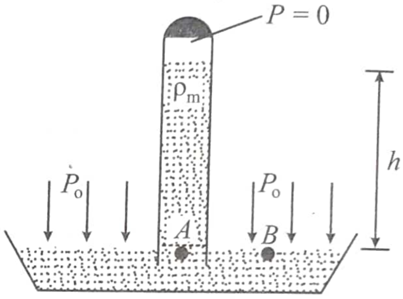
(2) Mercury Barometer

It is a straight glass tube (closed at one end) completely filled with mercury and inserted into a dish which is also filled with mercury as shown in the figure.

Pressure at points A and B are equal and thus

$\Rightarrow \quad P_B = P_A \Rightarrow p_0 = 0 + \rho g h$

Hence, $p_0 = (13.6 \times 10^3)(9.81)(0.76) = 1.01 \times 10^5\ \text{Nm}^{-2}$ (or Pa)



(3) Water Barometer

Let us suppose water is used in the barometer instead of mercury

$$h\rho g = 1.013 \times 10^5 \text{ or } h = \frac{1.013 \times 10^5}{\rho g} = 10.3 \text{ m}$$

The height of the water column in the tube will be 10.3 m. Such a long tube cannot be managed easily, thus water barometer is not feasible.

Train Your Brain

Example 1: A liquid is filled upto the top in a rectangular tank of square cross section. The sides of cross section is $a = 1$m and height of the tank is $H = 3$m. If density of liquid is $\rho = 100$ kgm^3 find force on the bottom of the tank and on one of its wall.

(a) 5000 N (b) 4000 N

(c) 4500 N (d) 6000 N

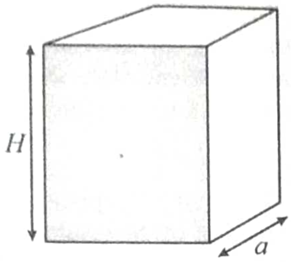
Sol. (c) Force on the bottom of tank: pressure at the bottom of tank due to liquid is uniform of magnitude $\rho g h$.

Area of the bottom of the tank = a^2

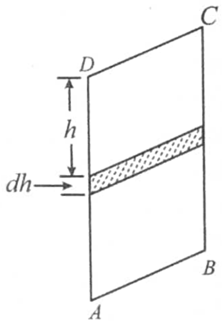
$\therefore$ force on the bottom = pressure × area

$= \rho g H a^2 = 3000$ N

Force on the wall and its points of application:



Force on the wall of the tank can't be calculated using pressure × area as pressure is not uniform over the surface. The problem can be solved by finding force dF on a thin strip of thickness dh at a depth h below the free surface and then taking its integral.



Pressure of liquid at depth h,

$P = \rho g h$

Area of thin strip = adh

$\therefore$ Force of the strip,

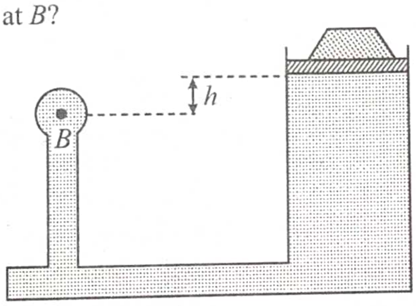
$dF = P\,adh = \rho g h a dh$

$\therefore$ Total force on the wall

$= \int_0^H \rho g a h dh$

$F = \rho g a \dfrac{H^2}{2} = 4500$ N

Example 2: A weighted piston confines a fluid of density ρ in a closed container, as shown in the figure. The combined weight of piston and weight is $W = 200$ N, and the cross-sectional area of the piston is $A = 8$ cm^2. Find the total pressure at point B if the fluid is mercury and $h = 25$ cm ($\rho m = 13600$ kg/m^3). What would an ordinary pressure gauge read at B?



Sol. Notice what Pascal's principle tells us about the pressure applied to the fluid by the piston and atmosphere. This added pressure is applied at all points within the fluid. Therefore, the total pressure at B is composed of three parts:

Pressure of atmosphere = 1.0×10^5 Pa

Pressure due to piston and weight $= \dfrac{W}{A} = \dfrac{200N}{8\times10^{-4}m^2}$

$= 2.5 \times 10^5$ Pa

Pressure due to height h of fluid = $h\rho g = 0.33 \times 10^5$ Pa

In this case, the pressure of the fluid itself is relatively small. We have

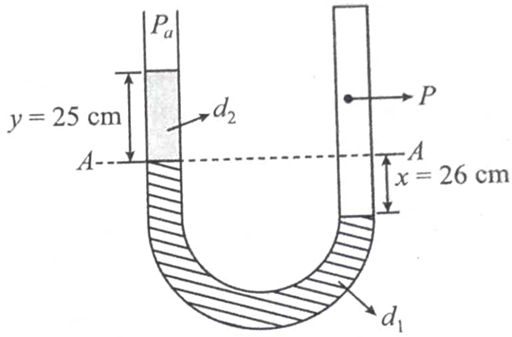
Total pressure at $B \approx 3.8 \times 10^5$ $Pa = 380$ kPa

The gauge pressure does not include atmospheric pressure. Therefore,

Gauge pressure at $B = 280$ kPa

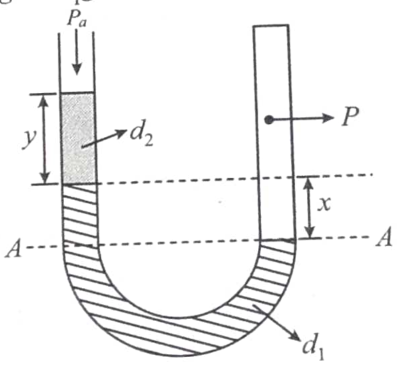
Example 3: In a given U-tube (open at one-end) find out relation between P and Pa.

Given $d_2 = 2 \times 13.6$ gm/cm^3 $d_1 = 13.6$ gm/cm^3



Sol. Pressure in a liquid at same level is same i.e. at $A - A$,

$P_a + d_2 y g + x d_1 g = P$



In C.G.S.

$P_a + 13.6 \times 2 \times 25 \times g + 13.6 \times 26 \times g = P$

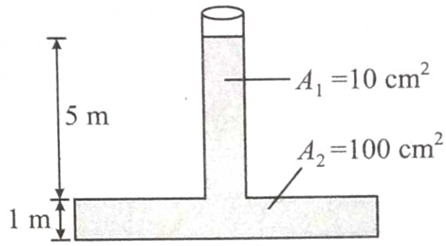
$P_a + 13.6 \times g\,[50 + 26] = P$

$2P_a = P$ $\quad [P_a = 13.6 \times g \times 76]$

Example 4: In the figure shown, find

(*a*) the total force on the bottom of the tank due to the water pressure.

(*b*) the total weight of water.



Sol. (*a*) Pressure at the base due to water is

$p = \rho_w g\,[5 + 1] = (10^3)(10)(5 + 1) = 6 \times 10^4$ N/m^2

$\therefore$ Force $= pA_2 = (6 \times 10^4)(100 \times 10^{-4}) = 600$ N

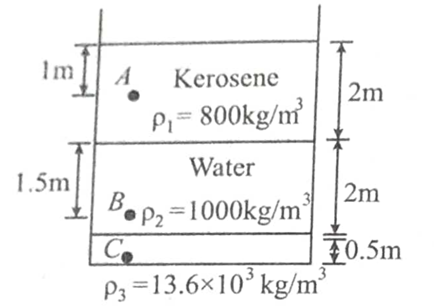
(*b*) Weight of water $= \rho_w\, g\,[5A_1 + A_2]$

$= 10^4\,[5 \times 10 \times 10^{-4} + 100 \times 10^{-4}] = 150$ N

Concept Application

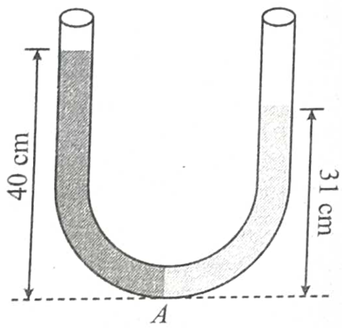
1. Find the density and specific gravity of gasoline if 51 g occupies 75 cm^3?
   (a) 680 kg/m^3 and 0.8
   (b) 680 kg/m^3 and 0.68
   (c) 720 kg/m^3 and 0.68
   (d) 720 kg/m^3 and 0.8

2. Find the absolute pressure and gauge pressure at points A, B and C as shown in the figure. (1 atm = 10^5 Pa)



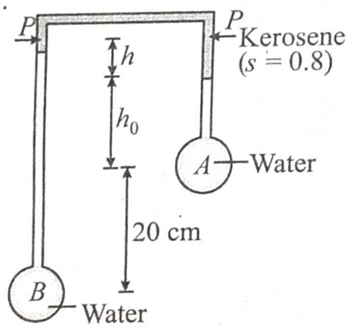
   (a) $P_A = 108$ kPa, $P_B = 131$ kPa, $P_C = 204$ kPa
   (b) $P_A = 100$ kPa, $P_B = 120$ kPa, $P_C = 204$ kPa
   (c) $P_A = 108$ kPa, $P_B = 121$ kPa, $P_C = 204$ kPa
   (d) $P_A = 108$ kPa, $P_B = 120$ kPa, $P_C = 131$ kPa

3. As shown in the figure, a column of water 40 cm high supports a 31 cm of an unknown fluid. What is the density of the unknown fluid?



   (a) 1090 kg/m^3
   (b) 1290 kg/m^3
   (c) 1410 kg/m^3
   (d) 1190 kg/m^3

4. For the arrangement shown in the figure, determine h if the pressure difference between the vessels A and B is 3 kN/m^2.



   (a) 10 cm
   (b) 15 cm
   (c) 12 cm
   (d) 20 cm

HYDROSTATICS

Hydro statics is the branch of mechanics, which deals with the forces on fluids (liquids and gases) at rest with respect to container.

Force due to Fluid on a Plane Submerged Surface

A surface submerged in liquid such as bottom of a tank or wall of tank or gate valve in a dam, is subjected to pressure acting normal to its surface and distributed over its area.

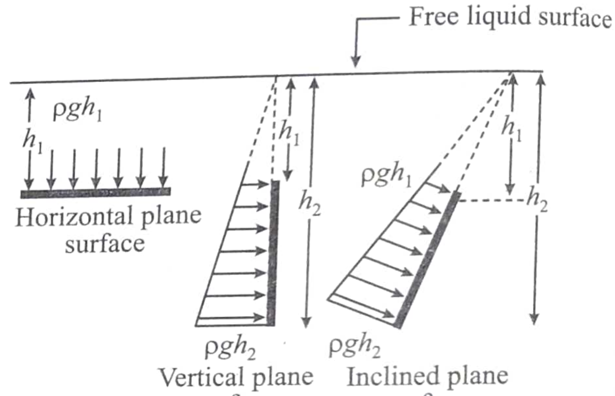
In general pressure at different point on the submerged surface varies so to calculate resultant force, we divide the surface into number of elementary areas and we calculate force on it first by treating pressure as constant then we integrate it to get net force.

i.e., $F_R = \int PdA$

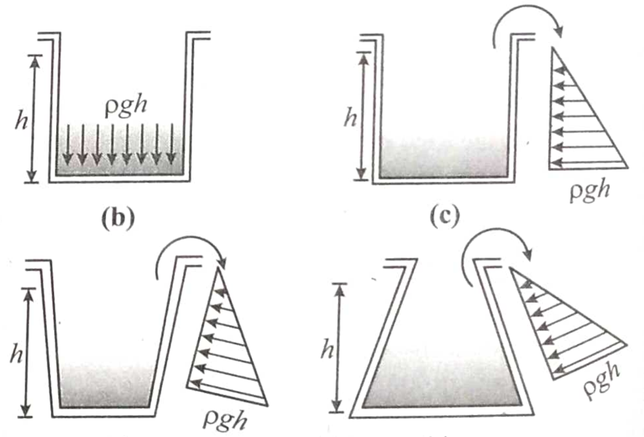
The point of application of resultant force must be such that the moment of the resultant force **about any axis** is equal **to the moment of the distributed force about the axis.**

PRESSURE DIAGRAMS AND FORCE ON BOUNDARIES

A pressure diagram is a graphical representation of the variation of the pressure intensity over a surface. Such a diagram may be prepared by plotting to some convenient scale the pressure intensities at various points on the surface. Net force (total pressure) as well as point of application of force (centre of pressure) for a plane surface wholly submerged in a static liquid of either vertically or inclined, may also be determined by drawing a pressure diagram which is shown below.



(a)

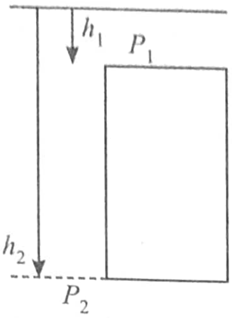


(b) (c) (d) (e)

Since force at any point acts in the direction normal to the surface, the pressure intensities at various points on the surface are plotted normal to the surface.

FORCE ON SURFACE IMMERSED IN A LIQUID AND CENTRE OF FORCE

Consider a rectangular plane surface of area 'A' held vertically in a static liquid of density ρ as shown in figure. Let top and bottom edges of the plane surface be at vertical depths of h_1 and h_2 respectively, below the free surface of the liquid. The pressure intensity at the top edge is $P_1 = \rho g h_1$. The pressure intensity at the bottom edge is $P_2 = \rho g h_2$. As the pressure intensity increases linearly from $\rho g h_1$ to $\rho g h_2$, so the average pressure intensity over the entire surface is



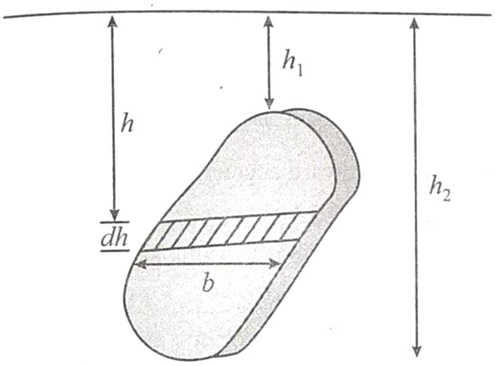
$$P_{av} = \frac{P_1 + P_2}{2} = \rho g\left(\frac{h_1 + h_2}{2}\right)$$

Thus, the net force on the vertical surface is

$$F = P_{avg} \times \text{Area of the surface}$$

or $$F = \rho g\left(\frac{h_1 + h_2}{2}\right) \times A$$

For any general surface



This result can be generated for a surface of any shape. Let us analyse the same case by another approach. It is easy to see that the thrust due to the liquid on any surface can be obtained as

Thrust = (Area) × (Hydrostatic pressure at the centroid of the area)

Consider a plane surface of any arbitrary shape, immersed in a liquid of density ρ. Let us subdivide the area into infinitesimally thin horizontal strips. Consider one such strip of width 'b' and thickness dh, at a depth h below the free liquid surface. Normal thrust on this strip is dF = Pressure × Area = $\rho g h\,(bdh)$. Total thrust exerted by the liquid on the area is

$$F = \Sigma dF = \Sigma \rho g h b dh = \rho g \sum_{h=h_1}^{h_2} bhdh$$

Now, if h_o be the depth of the centroid of the area, below the free surface and A is the total area, then

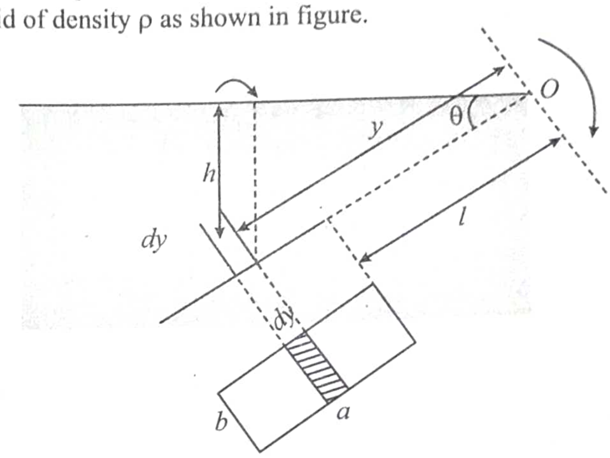
$$h_o = \sum_{h=h_1}^{h_2} \frac{(bdh)h}{A} \Rightarrow \sum_{h=h_1}^{h_2} (bdh)h = Ah_o$$

$$\therefore \; F = (\rho g h_o)A \quad \text{...(i)}$$

Hence, force on a vertical immersed surface is the product of the pressure on the centroid of the surface and the area of the surface.

HYDROSTATIC FORCES ON INCLINED SURFACES IMMERSED IN LIQUID

Consider a plane surface of length a and width b which is inside a liquid of density ρ as shown in figure.



Choose an element of length (dy) at a distance y from O. The depth of the element $h = y \sin\theta$. The intensity of pressure at the position of the element is $P = \rho g h = \rho g\,(y \sin\theta)$.

The force on the element is $dF = P(bdy)$

$$= \rho g(y\sin\theta)\, bdy = \rho g b \sin\theta\, y\, dy$$

The force on the entire surface is

$$F = \int_l^{(l+a)} dF = \rho g b \sin\theta \int_l^{(l+a)} y dy$$

$$= \rho g \frac{b\sin\theta}{2} \left| y^2 \right|_l^{(l+a)} = \rho g b \frac{\sin\theta}{2}[(l+a)^2 - l^2]$$

$$= \frac{\rho g b \sin\theta}{2}[a^2 + 2al] = \frac{\rho g a b}{2}[a + 2l]\sin\theta$$

Alternate Method :

Force on inclined surface is given by

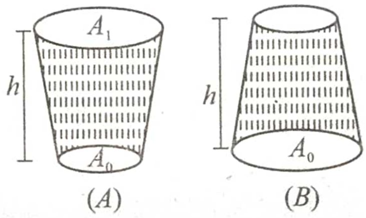
$$F = P_{avg} \times \text{Area of surface}$$

$$= \frac{1}{2}[\rho g l \sin\theta + \rho g(l+a)\sin\theta] \times ab$$

$$= \frac{1}{2}[\rho g\, ab \sin\theta\, [a + 2l]$$

Train Your Brain

Example 5: Force on the bottom of given container A and B will be related as

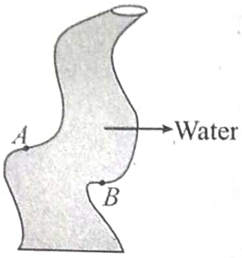


(a) $F_A > F_B$
(b) $F_A < F_B$
(c) $F_A = F_B$
(d) $F_A = 2F_B$

Sol. (c) Pressure on the bottom will be same and equal to ρgh

So, force $(F) = \rho g h A_0$ = same for both

Example 6: Force at point A & B on the container by water will be

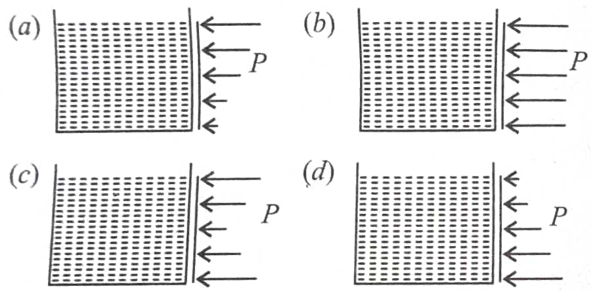


(a) Upward at A and upward at B
(b) Upward at A and downward at B
(c) Downward at A and upward at B
(d) Downward at A and downward at B

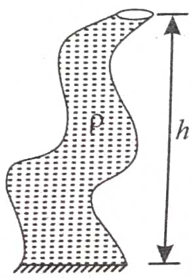
Sol. (b)

Concept Application

5. Which of the following is correct representation of variation of pressure on the surface of container field with water



6. Pressure at the bottom of following container is



(a) Less than ρgh
(b) More than ρgh
(c) Equal to ρgh
(d) None of these

LIQUID IN ACCELERATED VESSEL

A Liquid in accelerated vessel can be considered as in the rigid body motion i.e. motion without deformation as though it were a solid body.

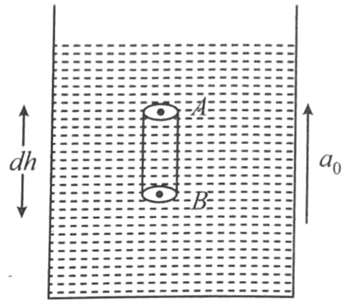
Vertically accelerated

Consider a liquid of density ρ kept in a vessel moving with acceleration a_0 in upward direction.

Let A and B are two points separated vertically by a distance dh. The forces acting an a vertical liquid column of cross sectional area dA are shown in the figure. For the vertical motion of this liquid column,

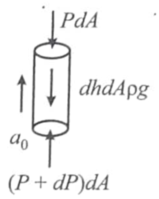
$$(P + dP)\,dA - PdA - (dh)\,dA\rho g = (dh)\,dA\rho a_0$$

$$\Rightarrow dP = \rho(g + a_0)dh$$



If pressure at the free surface of liquid is P_0 then pressure P at a depth h from the free surface is given by $\int_{P_0}^{P} dp = \int_{P_0}^{P} \rho(g + a_0)dh$

$$\Rightarrow \quad P = P_0 + \rho\,(g + a_0)h$$



In case of liquid in downward accelerated vessel can be written as

$$P = P_0 + \rho\,(g - a_0)\,h$$

We can generalize the above results as

$$P = P_0 + \rho g_{eff} h$$

Where, $g_{eff} = g + a_0$ in case of upward acceleration and

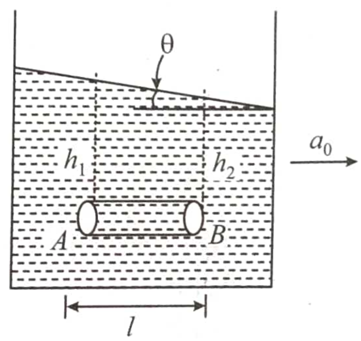
$g_{eff} = g - a_0$ in case of downward acceleration

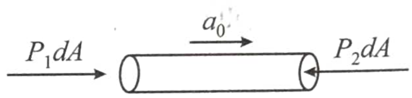
Horizontally accelerated

When a vessel filled with liquid accelerates horizontally. We observe its free surface inclined at some angle with horizontal.

$$P_1 dA - P_2\,dA = \rho(dA)\,l(a_0)$$

$$(h_1 - h_2)g = la_0$$





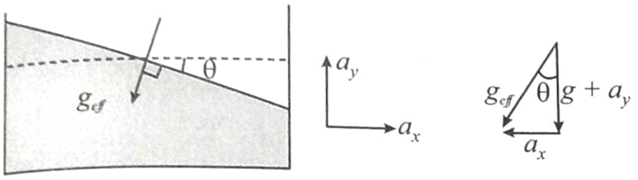
$$\Rightarrow \frac{h_1 - h_2}{l} = \frac{a_0}{g}$$

$$\Rightarrow \boxed{\tan\theta = \frac{a_0}{g} \Rightarrow \theta = \tan^{-1}\frac{a_0}{g}}$$

Combined Horizontal and Vertical Acceleration

In general, when the fluid has both horizontal and vertical acceleration, the inclination of the free surface with the horizontal may be obtained as

$$\tan\theta = \frac{dy}{dx} = \frac{dp/dx}{dp/dy} = \frac{a_x}{g + a_y}$$



Conclusion:

(1) In a non-accelerating fluid, pressure increases with depth and decreases with height:

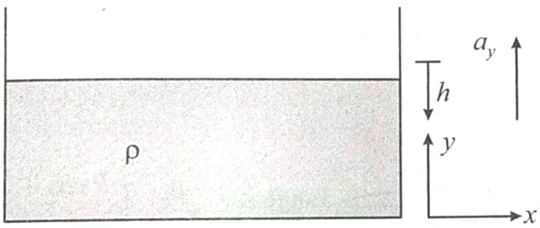
$$\frac{dp}{dh} = \rho g \text{ and } \frac{dp}{dy} = -\rho g$$

(2) When a fluid is subjected to a vertical acceleration, the above two equations may be modified as

Variation with depth: $\frac{dp}{dh} = \rho(g + a_y)$

Variation with height: $\frac{dp}{dy} = -\rho(g + a_y)$

Here, a_y is positive for upward acceleration and negative for downward acceleration.



A container accelerated in the vertical direction

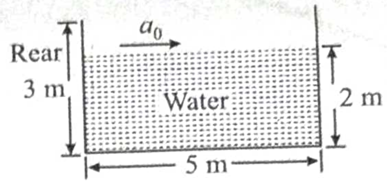
Train Your Brain

Example 7: An open rectangular tank 5 m × 4 m × 3 m high containing water upto a height of 2 m is accelerated horizontally along the longer side.

(a) Determine the maximum acceleration that can be given without spilling the water.

(b) Calculate the percentage of water split over, if this acceleration is increased by 20%

(c) If initially, the tank is closed at the top and is accelerated horizontally by 9 m/s², find the gauge pressure at the bottom of the front wall of the tank. (Take $g = 10$ m/s²).



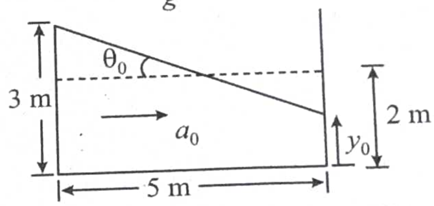
Sol. (a) Volume of water inside the tank remains constant

$$\frac{3 + y_0}{2} \times 5 \times 4 = 5 \times 2 \times 4$$

or $y_0 = 1$ m

$$\therefore \tan\theta_0 = \frac{3-1}{5} = 0.4$$

since, $\tan\theta_0 = \frac{a_0}{g}$, therefore $a_0 = 0.4\text{ g} = 4\text{ m/s}^2$



(b) When acceleration increased by 20%

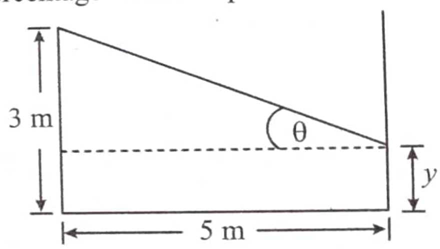
$a = 1.2\, a_0 = 0.48$ g

$\therefore \tan\theta = \frac{a}{g}\ a/g = 0.48$

Now, $y = 3 - 5\tan\theta = 3 - 5\,(0.48) = 0.6$ m

$$= \frac{4\times 2\times 5 - \frac{(3+0.6)}{2}\times 5\times 4}{2\times 5\times 4} = 0.1$$

percentage of water split over = 10%



(c) $a' = 0.9$ g

$$\tan\theta' = \frac{a'}{g} = 0.9$$

volume of air remains constant

$$4 \times \frac{1}{2}yx = (5)(1)\times 4$$

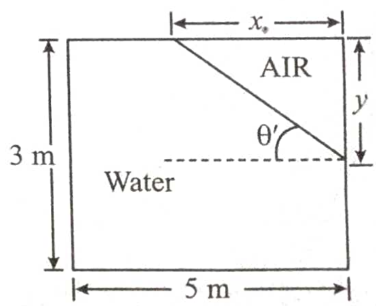
since $y = x\tan\theta'$

$$\therefore \frac{1}{2}x^2\tan\theta' = 5$$

or $x = 3.33$ m;

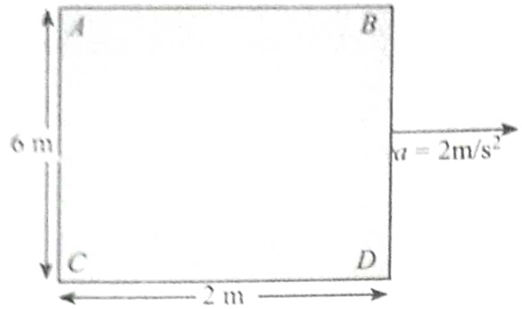
$y = 3.0$ m

(so front wall contact with air)



Gauge pressure at the bottom of the Front wall p_f = zero

Example 8: A closed container shown in figure is filled with water ($\rho = 10^3$ kg/m^3)



This is accelerated in horizontal direction with an acceleration, $a = 2$ m/s^2. Find (*a*) $p_C - p_D$ and (*b*) $p_A - p_D$

Sol. (*a*) In horizontal direction, pressure decreases in the direction of acceleration.

Thus, $p_C > p_D$

or $p_C - p_D = \rho ax$

Substituting the values, we have

$$p_C - p_D = (10^3)(2)(2)$$

$$p_C - p_D = 4.0 \times 10^3 \text{ N/m}^2$$

(*b*) In vertical direction, pressure increases with depth.

$\therefore p_C > p_A$

or $p_A - p_C = -\rho gh$

$= -(10^3)(10)(6) = -60 \times 10^3$ N/m^2

Now, $p_A - p_D = (p_A - p_C) + (p_C - p_D)$

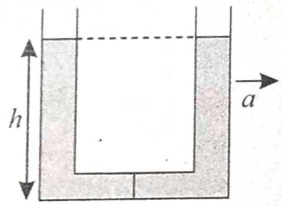
$= (-60 \times 10^3) + (4.0 \times 10^3)$

$= -56 \times 10^3$ N/m^2

$= -5.6 \times 10^4$ N/m^2

Concept Application

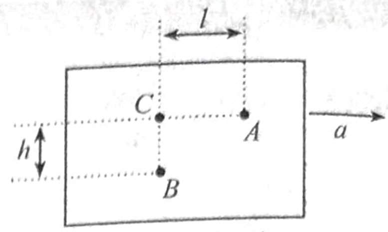
7. A *U*-tube of base length '*l*' filled with the same volume of two liquids of densities ρ and 2ρ is moving with an acceleration '*a*' on the horizontal plane. If the height difference between the two surfaces (open to atmosphere) becomes zero, then the height *h* is given by



(*a*) $\frac{a}{2g}l$ (*b*) $\frac{3a}{2g}l$

(*c*) $\frac{a}{g}l$ (*d*) $\frac{2a}{3g}l$

8. A sealed tank containing a liquid of density ρ moves with horizontal acceleration *a* as shown in the figure. The difference in pressure between two points *A* and *B* will be



(*a*) $h\rho g$ (*b*) $l\rho g$

(*c*) $h\rho g - l\rho a$ (*d*) $h\rho g + l\rho a$

9. When at rest, a liquid stands at the same level in the tubes as shown in the figure. But as indicated, a height difference *h* occurs when the system is given an acceleration *a* towards the right. Then *h* is equal to

(*a*) $\frac{aL}{2g}$ (*b*) $\frac{gL}{2a}$

(*c*) $\frac{gL}{a}$ (*d*) $\frac{aL}{g}$

LIQUID IN A VESSEL ROTATING WITH CONSTANT ANGULAR VELOCITY

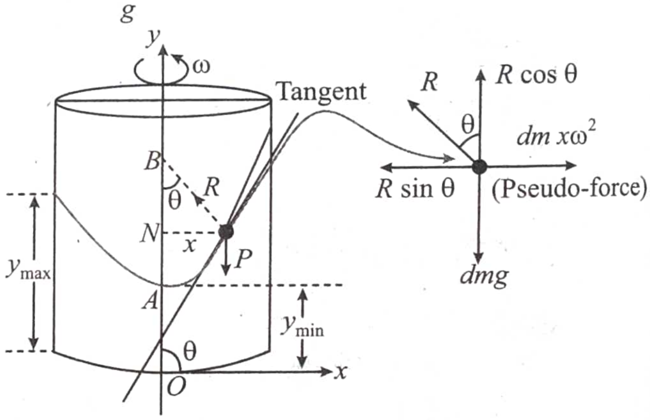
When the vessel is rotated there is no relative motion between fluid elements. The liquid surface orients itself in permanent position relative to the vessel. We will analyse a differential element *dm* in the reference frame of vessel. The forces acting on it are shown in Fig. (*a*) and (*b*). *R* is the force exerted on the element due to neighbouring elements of fluid.

$$R\cos\theta = dmg \quad ...(i)$$

$$R\sin\theta = dmx\omega^2 \quad ...(ii)$$

Dividing Eq. (*ii*) by Eq. (*i*), we have

$$\tan\theta = \frac{\omega^2 x}{g} \quad ...(iii)$$



$$\tan\theta = \frac{dy}{dx} = \frac{\omega^2 x}{g} \Rightarrow dy = \frac{\omega^2 x}{g}dx$$

After integration, $y = \frac{\omega^2 x^2}{2g} + c$

It represents a parabola i.e., liquid surface is a paraboloid.

At $x = R$, $y = y_{max} = \frac{R^2\omega^2}{2g} + c$

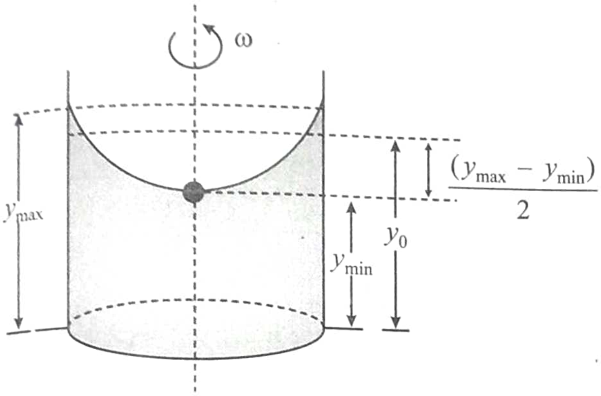
At $x = 0, y = y_{min} = c$

Therefore, $y_{max} = \dfrac{R^2\omega^2}{2g} + y_{min}$

and the general equation can be written as

$$y = \frac{x^2\omega^2}{2g} + y_{min} \qquad ...(iv)$$

Note: Volume of a paraboloid of revolution is half that of the circumscribing cylinder.



Hence, volume of a paraboloid $= \dfrac{(y_{max} - y_{min})}{2} A$

If originally liquid filled up to height y_0, then

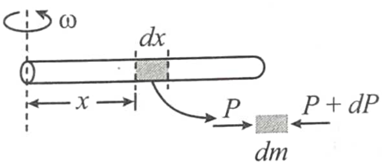
$$y_0 = y_{min} + \frac{y_{max} - y_{min}}{2} = \frac{y_{max} + y_{min}}{2}$$

$$y_{max} = y_0 + \frac{\omega^2 R^2}{4g} \qquad \text{(using Eq. } (iv))$$

PRESSURE DIFFERENCE IN ROTATING FLUIDS

In case of accelerating fluids, the pressure increases in the direction opposite to the direction of the acceleration. The rotating fluid is also the case of accelerating fluid. So in this case the pressure increases in moving away from the rotational axis.

Suppose that liquid of density ρ kept inside a tube of area of cross-section A is rotating with an angular velocity 'ω' as shown.



Let us take a small element of length 'dx' at a distance x from the axis of rotation. Mass of this element is,

dm = (density) (volume) or $dm = (\rho\, Adx)$

This element is rotating in a circle of radius 'x'. So, this is accelerated towards centre with a centripetal acceleration

$$a = x\omega^2 \qquad (\text{as } a = R\omega^2)$$

To provide this acceleration, pressure on right hand side of the element should be more.

$$\therefore\ (p + dp)A - (p)A = (dm)\,a = (\rho Adx)(x\omega^2) \text{ or } dp = (\rho x\omega^2)\,dx$$

$$\therefore\quad \Delta p = \int_0^x (\rho\omega^2)\,xdx \text{ or } \Delta p = \frac{\rho\omega^2 x^2}{2}$$

It means at a distance 'x' from the rotational axis, increases in pressure is $\dfrac{\rho\omega^2 x^2}{2}$

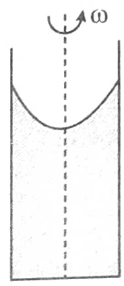
Important Points:

Take $\Delta p = +\dfrac{\rho\omega^2 x^2}{2}$ in moving away from the rotational axis, as pressure increases in this direction and take $\Delta p = -\dfrac{\rho\omega^2 x^2}{2}$ in moving towards the rotational axis.

Train Your Brain

Example 9: A liquid of density ρ is in a bucket that spins with angular velocity ω as shown in figure. Show that the pressure at a radial distance r from the axis is

$$P = P_0 + \frac{\rho\omega^2 r^2}{2}$$



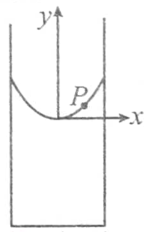
where P_0 is the atmospheric pressure.

Sol. Consider a fluid particle P of mass m at coordinates (x, y). From a non-inertial rotating frame of reference two forces are acting on it,

(*i*) pseudo force ($mx\omega^2$)

(*ii*) weight (mg)

in the directions shown in figure.

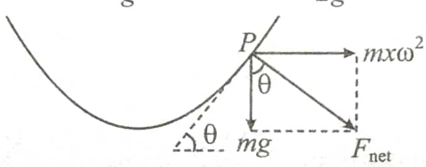


$P \equiv (x, y)$

Net force on it should be perpendicular to the free surface (in equilibrium). Hence.

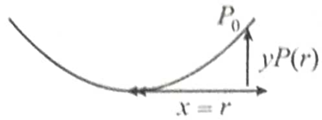
$$\tan\theta = \frac{mx\omega^2}{mg} = \frac{x\omega^2}{g} \text{ or } \frac{dy}{dx} = \frac{x\omega^2}{g}$$

$$\therefore\quad \int_0^y dy = \int_0^x \frac{x\omega^2}{g}.dx \quad \therefore\quad y = \frac{x^2\omega^2}{2g}$$



This is the equation of the free surface of the liquid, which is a parabola.

At $x = r$, $y = \dfrac{r^2\omega^2}{2g}$



$\therefore \quad P(r) = P_0 + \rho g y$

or $\quad P(r) = P_0 + \dfrac{\rho\omega^2 r^2}{2}$

Example 10: A cylindrical vessel of diameter 0.3 m, height 0.6 m, is filled two thirds with liquid of specific gravity 0.8. The vessel is rotated about its axis. Determine the speed of rotation:

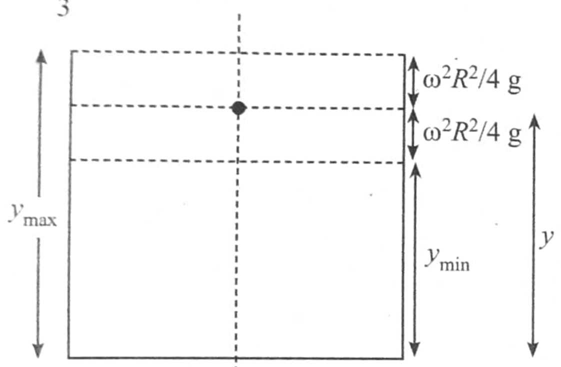
(*a*) When the liquid just starts spilling,

(*b*) Wshen the base is just visible.

(*c*) What is the percentage of liquid left in the vessel?

Sol. Initial height of liquid in the vessel = 0.6 m

and $\dfrac{2}{3}$ of 0.6 = 0.4 m



Height of container = 0.6 m

$$y_{max} - y_{min} = \frac{\omega^2 R^2}{2g} \qquad ...(i)$$

$$y_{max} = y + \frac{\omega^2 R^2}{4g} \qquad ...(ii)$$

$$y_{min} = y - \frac{\omega^2 R^2}{4g} \qquad ...(iii)$$

(*a*) $y_{max.} = 0.6$ m

From equation (*ii*)

$$0.6 = 0.4 + \frac{\omega^2 \times \left(\frac{0.3}{2}\right)^2}{4 \times 9.81}$$

$\omega = 18.68$ rad/sec

(*b*) $y_{max} = 0.6, y_{min} = 0$

$$y_{max} = \frac{R^2\omega^2}{2g} \qquad \text{[From } (i)\text{]}$$

$$0.6 = \frac{(0.3)^2 \times \omega^2}{2 \times 9.81}$$

$\omega = 11.43$ rad/sec

(*c*) Volume of paraboloid of revolution is half the volume of enclosing cylinder.

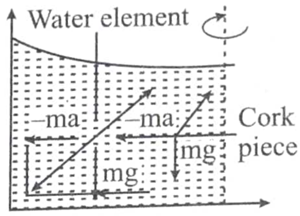
Volume of liquid left in the vessel = 0.3 A m^3

where A is cross-sectional area.

Initial volume was 0.4A m^3

$$\text{Percentage of liquid left} = \frac{0.3A}{0.4A} \times 100 = 75\%$$

Example 11: A piece of cork is fixed to the bottom of a cylindrical vessel filled with water and is rotating about the vertical axis with a constant angular velocity ω. At some instant the cork gets free and comes to the surface. What is the trajectory along which the cork moves to the surface: does it approach the wall or the axis or does it move vertically upward?



Sol. We consider the reference frame of vessel; an element of water that has the same volume as that of the piece of cork is in the state of equilibrium due to three forces:

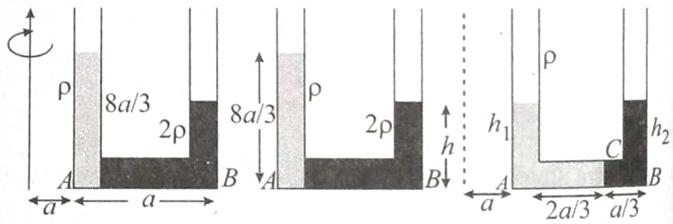
(*i*) Force of gravity,

(*ii*) Pseudo force $|m\bar{a}| = mr\omega^2$,

(*iii*) Force due to pressure of surrounding water.

Same forces act on the cork piece. The force of pressure of the surrounding water is same but the pseudo force and gravitational forces are lesser. Hence there is a net force, the difference between the force of pressure and the force of gravity and pseudo force. This force makes the cork piece to move towards the surface and at the same time towards the axis of the vessel. Similarly an object with a density greater than the density of water, in a rotating vessel filled with water, will sink and move towards the wall of the vessel.

Example 12: The interface of two liquids of densities ρ and 2ρ respectively lies at the point A in a light U-tube at rest. The height of liquid column above A is $8/3a$, where $AB = a$. The cross-sectional area of the tube is S. With what angular velocity must it be whirled about a vertical axis at a distance 'a' such that the interface of the liquids shifts towards B by $2/3a$?



Sol. When the tube is at rest using pascal's law we have $P_A = P_B$

Let P_0 be atmospheric pressure.

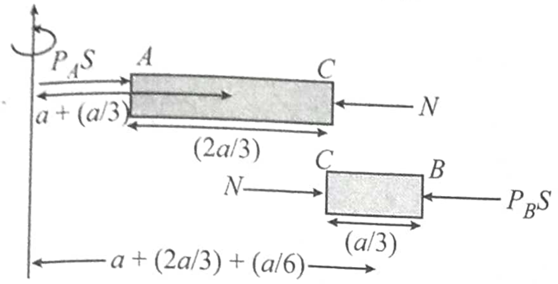
$$P_0 + \rho g\left(\frac{8}{3}a\right) = P_0 + (2\rho)gh$$

When the tube is whirled about vertical axis such that the interface shifts to C as shown in figure the height of liquid columns becomes

$h_1 = \left(\frac{8}{3}a\right) - \left(\frac{2}{3}a\right) = 2a$; $h_2 = \left(\frac{4}{3}a\right) + \left(\frac{2}{3}a\right) = 2a$

$P_A = \rho g h_1 = 2\rho g a$; $P_B = (2\rho) g h_2 = 2\rho g(2a) = 4\rho g a$

Now from force diagram of liquids figure in horizontal portion of U-tube for liquid density ρ,



or $N - P_A S = m_1 \omega^2 r_1$

$$N - 2\rho g a S = \left(\rho S \frac{2}{3}a\right)\omega^2\left(a + \frac{a}{3}\right)$$

$$N - 2\rho g a S = \frac{8}{9}\rho S \omega^2 a^2 \qquad \text{...(i)}$$

For liquid of density 2ρ,

or $P_B S - N = m_2 \omega^2 r_2$

$$4\rho g a S - N = \left(2\rho S \frac{a}{3}\right)\omega^2\left(a + \frac{2}{3}a + \frac{a}{6}\right)$$

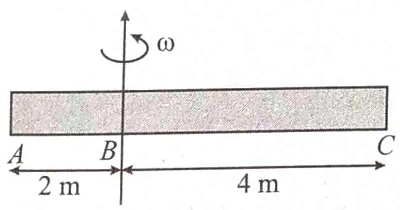
$$4\rho g a S - N = \frac{11}{9}\rho S \omega^2 a^2 \qquad \text{...(ii)}$$

From equation (*i*) and (*ii*), we have, $2\rho g a S = \frac{19}{9}\rho S \omega^2 a^2$

$$\omega = \sqrt{\frac{18g}{19a}}$$

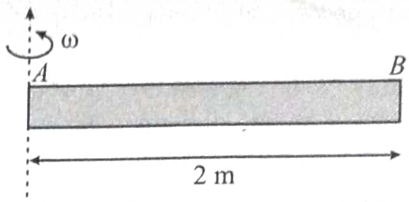
Concept Application

10. A closed tube is filled with water ($\rho = 10^3$ kg/m³). It is rotating about an axis shown in figure with an angular velocity $\omega = 2$ rad/s. What is $p_A - p_C$.
($AB = 2$ m, $BC = 4$ m)



(*a*) -2.4×10^3 N/m² (*b*) 2.4×10^4 N/m²
(*c*) -2.1×10^3 N/m² (*d*) 2.1×10^4 N/m²

11. Water $(\rho = 10^3 \text{ kg/m}^3)$ is filled in tube AB as shown in figure $\omega = 10$ rad/s. Tube is open at end A. Atmospheric pressure is $p_0 = 10^5$ N/m². Find absolute pressure at end B.



(*a*) 2×10^5 N/m² (*b*) 3×10^5 N/m²
(*c*) 4×10^5 N/m² (*d*) 6×10^5 N/m²

12. A liquid is kept in a cylindrical vessel, which is rotated along its axis. The liquid rises at the side. If the radius of the vessel is 5 cm and the frequency of rotation is 4 rev/s, then the difference in the height of the liquid at the centre of the vessel and its sides is:
(*a*) 8 cm (*b*) 2 cm
(*c*) 40 cm (*d*) 4 cm

BUOYANCY

If a body is partially or wholly immersed in a fluid, it experiences an upward force due to the fluid surrounding it.

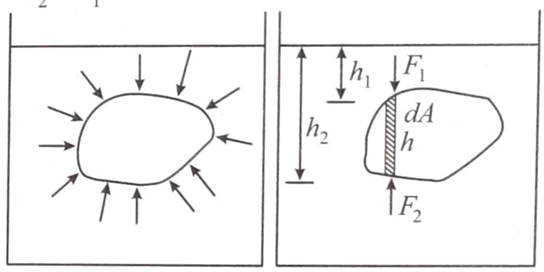
This phenomenon of force exerted by fluid on the body is called buoyancy and the force is called buoyant force.

Archimedes Principle

A body immersed in a fluid experiences an upward buoyant force equivalent to the weight of the fluid displaced by it.

The force acting on the upper surface of the element is F_1 (downward) and that on the lower surface is F_2 (upward). Since $F_2 > F_1$, therefore, the net upward force acting on the element is

$dF = F_2 - F_1$



It can be easily seen from the figure that

$F_1 = (\rho g h_1) dA$ and $F_2 = (\rho g h_2) dA$

$\therefore$ $dF = \rho g(h_2) dA - (\rho g h_1) dA$

Also, $h_2 - h_1 = h$ and $h(dA) = dV$

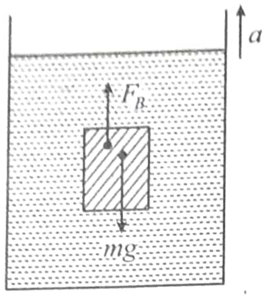
$\therefore$ The net upward force is

$$F = \int \rho g dV = \rho V g \Rightarrow F_{\text{buoyant}} = \rho V g$$

- Hence, for the entire body, the buoyant force is the weight of the volume of the fluid displaced.
- The buoyant force arises because the pressure in the fluid is not uniform; it increases with depth.

LAW OF FLOATATION

An object floats on water if it can displace a volume of water whose weight is greater than that of the object. If the density of the material is less than that of the liquid, it will float even if the material is a uniform solid, such as a block of wood floats on water surface.



If the density of the material is greater than that of water, such as iron, the object can be made to float provided it is not a uniform solid. An iron hulled ship is an example to this case.

Buyant force in accelerated Fluid

$$F_B - mg = ma$$

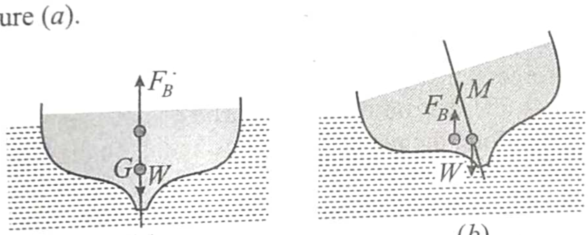
$$F_B = m(g + a)$$

$$\boxed{F_B = \rho V(g + a)}$$

Here, m = mass of displaced liquid $= \rho V$

STABILITY OF A FLOATING BODY

The stability of a floating body depends on the effective point of application of the buoyant force. The weight of the body acts at its centre of gravity. The buoyant force acts at the centre of gravity of the displaced liquid. This is called the centre of buoyancy. Under equilibrium condition the centre of gravity G and the centre of buoyancy B lies along the vertical axis of the body as shown in the figure (a).



(a) (b)

When the body tilts to one side, the centre of buoyancy shifts relative to the centre of gravity as shown in the figure (b). The two forces act along different vertical lines. As a result the buoyant force exerts a torque about the centre of gravity. The line of action of the buoyant force crosses the axis of the body at the point M, called the metacentre.

If G is below M, the torque will tend to restore the body to its equilibrium position.

If G is above M, the torque will tend to rotate the body away from its equilibrium position and the body will be unstable.

At equilibrium :

$$W = F_B$$

$$V\rho_s g = V_l \rho_l g$$

$$\frac{V_l}{V} = \frac{\rho_s}{\rho_l}$$

where, V = Volume of object

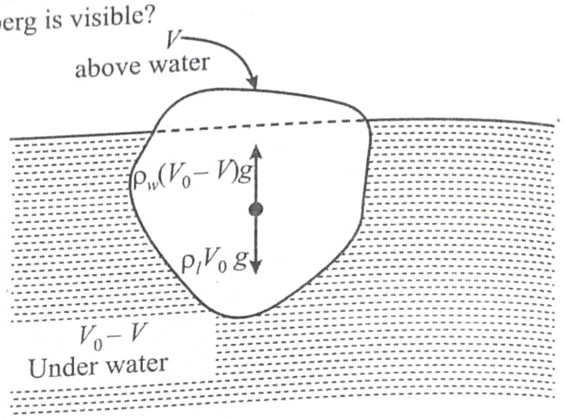
V_l = Volume of displaced liquid

ρ_s = Density of object

ρ_l = Density of liquid.

Train Your Brain

Example 13: An iceberg with a density of 920 kg m^{-3} floats on an ocean of density 1025 kg m^{-3}. What fraction of the iceberg is visible?



Sol. Let V be the volume of the iceberg above the water surface, then the volume under water will be $V_o - V$. Under floating conditions, the weight ($\rho_i V_o g$) of the iceberg is balanced by the buoyant force $\rho_w(V_o - V)g$.

Thus,

$$\rho_i V_o g = \rho_w (V_o - V)g$$

or $\rho_w V = (\rho_w - \rho_i)V_o$

or $\frac{V}{V_o} = \left(\frac{\rho_w - \rho_i}{\rho_w}\right)$

Since ρ_w = 1025 kg m^{-3} and ρ_i = 920 kg m^{-3}, therefore,

$$\frac{V}{V_o} = \frac{1025 - 920}{1025} = 0.10$$

Hence 10 % of the total volume is visible.

Example 14: When a 2.5 kg crown is immersed in water, it has an apparent weight of 22 N. What is the density of the crown?

Sol. Let W = actual weight of the crown

W' = apparent weight of the crown

ρ = density of crown

ρ_o = density of water

The buoyant force is given by

$F_B = W - W'$ or $\rho_o Vg = W - W'$

Since $W = \rho Vg$, therefore, $V = \frac{W}{\rho g}$

Eliminating V from the above two equations, we get

$$\rho = \frac{\rho_o W}{W - W'}$$

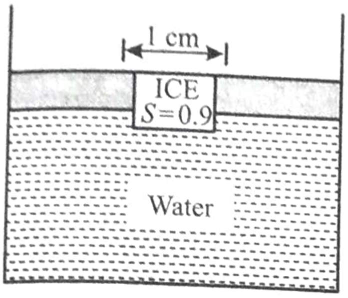
Here W = 25 N; W' = 22 N; ρ_o = 10^3 kg m^{-3}

$$\therefore \quad \rho = \frac{(10^3)(25)}{25 - 22} = 8.3 \times 10^3 \text{ kg m}^{-3}$$

Example 15: An ice cube of side 1cm is floating at the interface of kerosene and water in a beaker of base area 10 cm^2. The level of kerosene is just covering the top surface of the ice cube.

(*a*) Find the depth of submergence in the kerosene and that in the water.

(*b*) Find the change in the total level of the liquid when the whole ice melts into water.



Sol. (*a*) Condition of floating

$0.8\,\rho_w g h_k + \rho_w g h_w = 0.9\,\rho_w g h$

or $0.8h_k + h_w = 0.9h$ ...(*i*)

where h_k and h_w be the submerged depth of the ice in the kerosene and water, respectively.

Also, $h_k + h_w = h$ ...(*ii*)

Solving equations (*i*) and (*ii*), we get

$h_k = 0.5$ cm,

$h_w = 0.5$ cm

(*b*) $\underset{(\text{Ice})}{1\,\text{cm}^3} \xrightarrow{\text{melts}} \underset{(\text{Water})}{0.9\,\text{cm}^3}$

Fall in the level of kerosene $\Delta h_k = \dfrac{0.5}{A}$

Rise in the level of water $\Delta h_w = \dfrac{0.9-0.5}{A} = \dfrac{0.4}{A}$

Net fall in the overall level

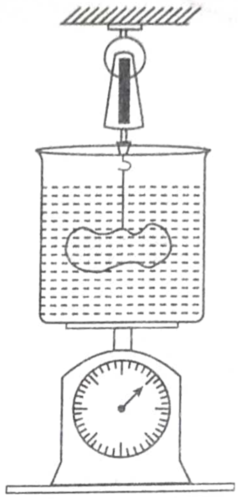
$$\Delta h = \frac{0.1}{A} = \frac{0.1}{10} = 0.01\,\text{cm} = 0.1\,\text{mm}$$

Example 16: A tank containing water is placed on spring balanced. A stone of weight w is hung and lowered into the water without touching the sides and the bottom of the tank. Explain how the reading will change.

Sol. The situation is shown in figure. Make free-body diagrams of the bodies separately and consider their equilibrium. Like all other forces, buoyancy is also exerted equally on the two bodies in contact. Hence it the water exerts a buoyant force, say, B on the stone upward, the stone exerts the same force on the water downward. The forces acting on the 'water + container' system are : W, weight of the system downward, B, buoyant force of the stone downward, and the force R of the spring in the upward direction. For equilibrium

$R = W + B$

Thus the reading of the spring scale will increase by an amount equal to the weight of the liquid displaced, that is, by an amount



Equal to the buoyant force.

Example 17: A cube of wood supporting 200 gm mass just floats in water ($\rho = 1$ g/cc). When the mass is removed, the cube rises by 2 cm. What is the size of the cube?

Sol. If, l = side of cube, h = height of cube above water and ρ = density of wood. Mass of the cube = $l^3\rho$

Volume of cube in water = $l^2(l-h)$. Volume of the displaced water = $l^2(l-h)$. As the tube is floating weight of cube + weight of wood = weight of liquid displaced

or $l^3\rho_{\text{wood}} + 200\,g = l^3(\rho_w g)$ ...(*i*)

After the removal of mass, the cube rises 2 cm.

$= l^3\rho_{\text{wood}}\,g = l^2(1-2)g$ ...(*ii*)

wt.of block = upthrust

Now from eqn. (*i*) & (*ii*)

$l^3\rho_w - 200 = l^2(l-2)$

We know, $\rho_w = 1$

So, $l^3 - 200 = l^3 - 2l^2$

$\Rightarrow\ l^2 = 100$

Example 18: A hemisphere of radius R is just submerged is just sinking in water of density ρ. Find the

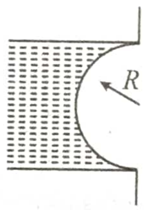
(*a*) Horizontal thrust.

(*b*) Vertical thrust.

(*c*) Total hydrostatic force.

(*d*) Angle of orientation of total hydrostatic force acting on the hemisphere

Do not count atmospheric pressure

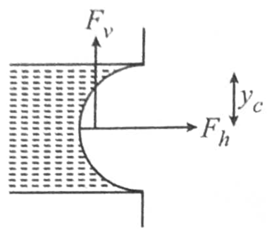


Sol. (*a*) Let the horizontal and vertical thrusts on the hemisphere be F_h and F_v respectively

We know that

where $F_h = \rho g y_c A_v$ and $y_c = R$

$A_v = \pi R^2$ then $F_h = \rho g \pi R^3$



(*b*) V = volume of the hemisphere $= \frac{2}{3}\pi R^3$

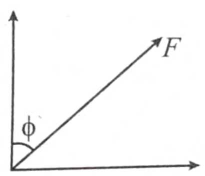
we have $F_v = \frac{2}{3}\rho g \pi R^3$ (up)

(*c*) Hence the net hydrostatic force on the hemisphere is

$$F = \sqrt{F_h^2 + F_v^2}$$

$$F = \sqrt{\left(\rho g \pi R^3\right)^2 + \left(\frac{2}{3}\rho g \pi R^3\right)^2}$$

$$= \frac{\sqrt{13}}{3}\rho g \pi R^3$$

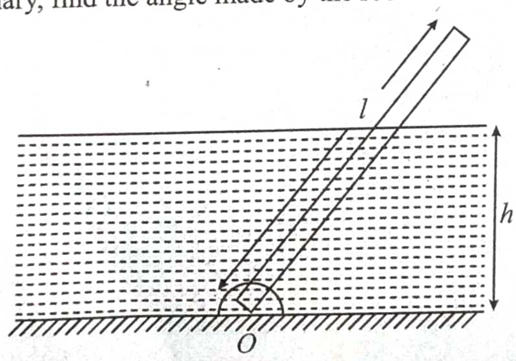


(*d*) The angle of orientation of the force F is

$$\phi = \tan^{-1}\frac{F_h}{F_v}$$

$$= \tan^{-1}\frac{\rho g \pi R^3}{\frac{2}{3}\rho g \pi R^3} = \tan^{-1}\frac{3}{2}$$

Example 19: A uniform rod of length l, density σ is pivoted smoothly at the point O below the water level at a depth h, as shown in Fig. If the density of water is ρ and the rod remains stationary, find the angle made by the rod with horizontal.



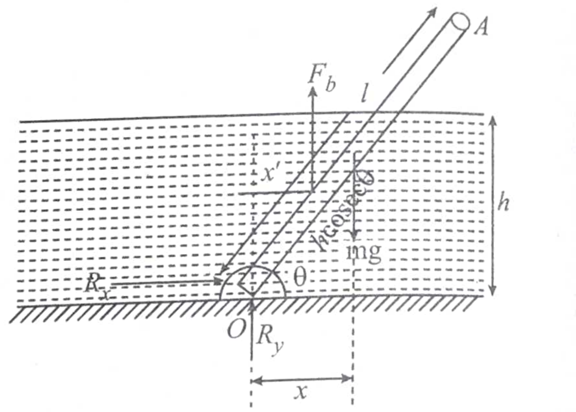
Sol. The forces acting on the rod are:

(*i*) Weight mg (downward)

(*ii*) Buoyant force F_b (upward)

(*iii*) Assumed reaction forces R_x (right) and R_y (up) at the pivot.

The net torque acting on the rod is zero because it is in rotational equilibrium.



Taking the torques of all force about the pivot O, we

$\tau_0 = F_b x' - mgx = 0$ where $x = \frac{l}{2}\cos\theta$ and

$x' = \frac{h}{2}\text{cosec}\theta\cos\theta$

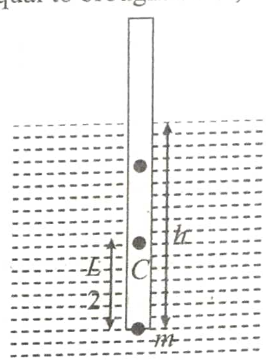
This gives $F_b h\,\text{cosec}\theta\cos\theta = mgl\cos\theta$

Substituting $F_b = V\rho g = A(h\,\text{cosec}\theta)\rho g, m = (A/\sigma)$,

we have $\theta = \sin^{-1}\frac{h}{l}\sqrt{\frac{\rho}{\sigma}}$ A wooden stick of length L, radius R and density ρ has a small metal piece of mass (of negligible volume) attached to its one end. Find the minimum value for the mass m (in terms of given parameters) that would make the stick float vertically in equilibrium in a liquid of density σ.

The stick will be vertical, that is, in rotational equilibrium if centre of gravity lies below the centre of buoyancy. For minimum m, the two points will coincide.

Let h be the length of immersed portion. For translatory equilibrium, combined weight of stick and mass attached is equal to brought force,



$(M+m)g = \pi R^2 h\sigma g$ ...(1)

where $M = \pi R^2 L\rho$

The height of centre of mass from bottom

$$= \frac{(M)L/2 + m \times 0}{m+M}$$

$$= \frac{ML}{2(m+M)}$$

For rotatory equilibrium and for minimum m, this should be equal to $h/2$.

$$\therefore \frac{h}{2} = \frac{ML}{2(m+M)} \quad \therefore h = \frac{ML}{(m+M)}$$

Substituting in eqn. (1), we get

$$(M+m)g = \pi R^2 \sigma g \cdot \frac{ML}{(m+M)}$$

$$(M+m) = \sqrt{M\pi R^2 \sigma L} = \sqrt{\pi R^2 L\rho \cdot \pi R^2 \sigma L}$$

$$m = \pi R^2 L\sqrt{\sigma\rho} - \pi R^2 L\rho = \pi R^2 L\rho\left[\sqrt{\frac{\sigma}{\rho}} - 1\right]$$

Example 20: A piece of an alloy of mass 96 gm is composed of two metals whose specific gravities are 11.4 and 7.4. If the weight of the alloy is 86 gm in water, find the mass of each metal in the alloy.

Sol. Suppose the mass of the metal of specific gravity 11.4 be m and the mass of the second metal of specific gravity will be $(96 - m)$

Volume of first metal $= \frac{m}{11.4}$ cm^3

Volume of second metal $= \frac{96-m}{7.4}$ cm^3

Total volume $= \frac{m}{11.4} + \frac{96-m}{7.4}$

Buoyancy force in water $= \left(\frac{m}{11.4} + \frac{96-m}{7.4}\right)$ gm weight

Apparent wt. in water $= 96 - \left[\left(\frac{m}{11.4}\right) + \frac{(96-m)}{7.4}\right]$

According to the given problem,

$$96 - \left[\left(\frac{m}{11.4}\right) + \frac{(96-m)}{7.4}\right] = 86$$

Solving we get, $m = 62.7$ gm

Thus mass of second metal is $= 96 - 62.7 = 33.3$ gm

Concept Application

13. An iceberg of density 700 kg/m^3 is floating in water of density 1000 kg/m^3. The percentage of volume of ice cube outside the water is

(a) 30% (b) 40%
(c) 50% (d) 60%

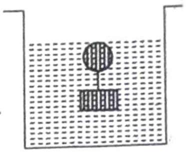
14. A block floats in water with two - third of its volume above the surface of water. If it is placed in liquid, it floats with half of its volume above the surface of the liquid. The specific gravity of the liquid is ,

(a) 4 : 1 (b) 3 : 2
(c) 1 : 4 (d) 2 : 3

15. A motor boat having a length of 10m and breadth 5 m is floating on a lake. The boat sinks by 5cm when load gets on it. What is mass of load?

(a) 3000 kg (b) 2500 kg
(c) 4010 kg (d) None of these

16. A body floats in a liquid contained in a beaker. The whole system is falling under gravity. The upthrust on the body due to liquid is



(a) Zero
(b) Equal to weight of liquid displaced
(c) Equal to weight of the body in air
(d) Equal to weight of the immersed body

17. Two solid uniform spheres each of radius 5 cm are connected by a light string and totally immersed in a tank of water. If the specific gravities of the sphere are 0.5 and 2, the tension in the string and the contact force between the bottom of tank and the heavier sphere.

(a) 2.6N & 2.26N
(b) 2.6N & 1.3N
(c) 1.6N & 2.6N
(d) 2.6N & 5.2N

FLUID DYNAMICS

In order to describe the motion of a fluid, in principle one might apply Newton's laws to a particle (a small volume element of fluid) and follow its progress in time. This is a difficult approach. Instead, we consider the properties of the fluid, such as velocity and pressure, at fixed points in space.

In order to simplify the discussion we make several assumptions:

(i) The fluid is non viscous. Which means that there is no loss of energy due to internal friction between adjacent layer in the fluid.

(ii) The flow is steady. Which means that the velocity and pressure at each point are constant in time.

(iii) The flow is irrotational which means that a tiny paddle wheel placed in the liquid will not rotate.
In rotational flow, for example, in eddies, the fluid has net angular momentum about a given point.

TYPES OF FLOW

Steady and Unsteady Flow

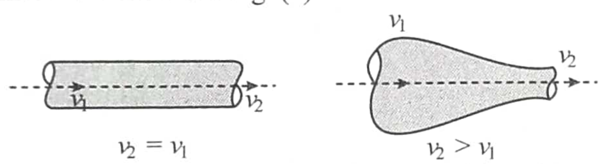
A flow is said to be steady if the velocity, pressure and density at any point in the flow do not change with time, *i.e.*,

$$\frac{dv}{dt} = 0, \frac{dp}{dt} = 0 \text{ and } \frac{d\rho}{dt} = 0$$

In an unsteady flow, the velocity at a point in the flow varies with time, i.e., $\frac{dv}{dt} \neq 0$.

Uniform and Non-uniform Flow

A flow is said to be **uniform** at any instant of time if the velocity (both in magnitude and in direction) does not vary along the direction of flow. In a non-uniform flow, velocity varies in the direction of flow. For example, if a water tap is kept fully open and left untouched, then after some time the velocity of the flow at any point in the pipe will remain unchanged, provided the other conditions remain the same. Hence, the flow is steady. However, during the period of opening and closing the water tap, the flow is unsteady. If the diameter of a pipe remains constant and also the quantity of liquid flowing through it remains the same, then the flow will be uniform as shown in Fig. (*a*). However, if the diameter of the pipe varies along its length and the quantity of liquid flowing through it remains the same, then the flow will be non-uniform as shown in Fig. (*b*).



(a) Uniform flow **(b) Non-uniform flow**

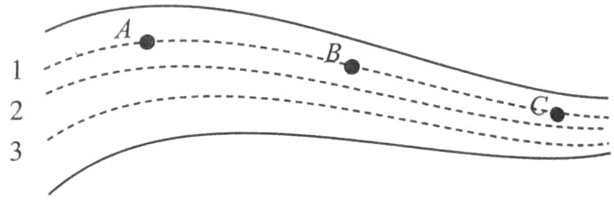
VISUALIZATION OF FLOW

The fluid flow pattern may be visualized in terms of pathlines and streamlines.

1. **Pathline:** It is a line drawn such that it describes the trajectory of a given fluid particle as it moves with the passage of time. The concept of the pathline in fluids is identical to the trajectory of a solid body. Tangent drawn at a point on a pathline gives the direction of velocity at that point at the time when the particle passes that point.
   Important properties of a pathline:
   - A pathline can intersect itself at different times.
   - Pathlines are the history lines of individual fluid particles over a period of time.
2. **Streamline:** It is a line drawn such that a tangent at every point of it is in the direction of velocity at that instant.
   A collection of streamlines drawn in this manner represents the complete flow pattern at an instant. Thus, for an unsteady flow, the streamline pattern may or may not remain the same at the next instant. Note that if the unsteadiness is due to a change in the magnitude of the velocity, then the streamline pattern still remains the same at different instants of time. On the other hand, if the unsteadiness is due to change in the direction of the velocity, the pattern will change from time to time.
   Unlike pathlines, a streamline cannot intersect itself. This is because instantaneously fluid can have a unique velocity at a point.

Streamline Flow (Laminar Flow)

If every point of a steadily flowing liquid follows exactly the same path that has been followed by the particles preceding it, the flow is said to be streamlined. The path is known as 'streamline'. In figure, the paths 1, 2, 3 are streamlines. If a liquid follows the path *ABC*, particles following it move along the same path.

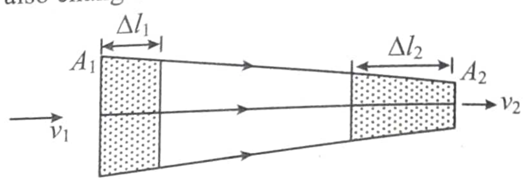


Turbulent Flow

When the velocity exceeds a certain critical value, the nature of flow becomes complicated, random, irregular, local currents (called vortices) develop throughout the fluids. The resistance to the flow increases tremendously. This type of flow is called 'turbulent flow'.

EQUATION OF CONTINUITY

In general, the velocity of a particle will not be constant along a streamline. The density and the cross-sectional area of a tube of flow will also change.



Since no mass is lost or gained.

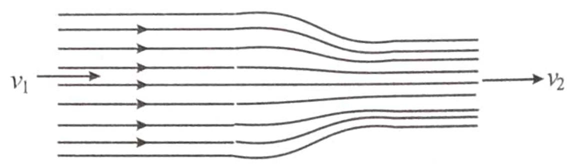
$\Delta m_1 = \Delta m_2$, and $\rho_1 A_1 v_1 = \rho_2 A_2 v_2$

This is called the **equation of continuity**. It is a statement of the conservation of mass.

If the fluid is incompressible, i.e. $\rho_1 = \rho_2$

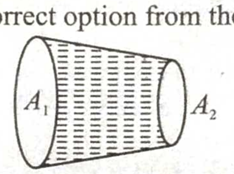
$A_1 v_1 = A_1 v_2$

The product Av is the volume rate of flow (m^3/s). From equation we conclude that the speed of a fluid is greatest where the cross-sectional area is the least. Streamlines are close together where the speed is higher.



Train Your Brain

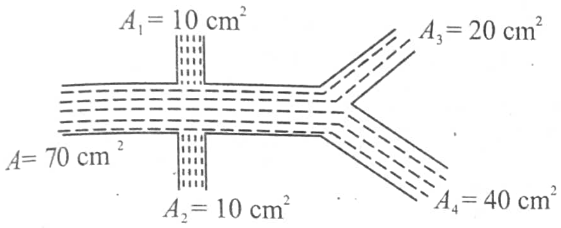
Example 21: A liquid is allowed to flow into a cone shape pipe. Choose the correct option from the following.



(a) The velocity is high at the wider end and low at narrow end.
(b) The velocity low at the wider end and high at the narrow end
(c) The velocity is same at both the ends
(d) The liquid / fluid flows with uniform velocity in pipe

Sol. (b) By equation of continuity $A_1v_1 = A_2v_2 =$ constant
If area decreases then velocity increases.

Example 22: A water connection is made in a colony from main water supply plant. Water is flowing through a pipe line of area of cross - section 70 cm² after entering into colony it is divided into five pipe line network of area A_1, A_2, A_3, A_4 in which water flows with velocities 5 m/s, 5 m/s, 2 m/s and 1 m/s respectively. What is the velocity of water in main pipe line? (in m/s)

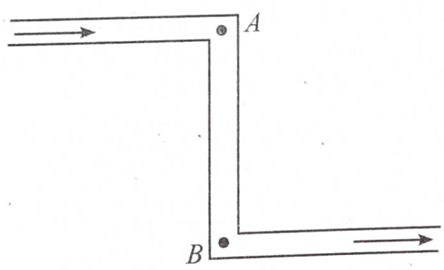


(a) 2.57 (b) 3.4
(c) 5.1 (d) 6.2

Sol. (a) According to the equation of continuity

$$AV = A_1v_1 + A_2v_2 + A_3v_3 + A_4v_4$$
$$\Rightarrow 70 \times 10^{-4} \times v = 10 \times 10^{-4} \times 5 + 10 \times 10^{-4} \times 5 + 20 \times 10^{-4} \times 2 + 40 \times 10^{-4} \times 1$$
$$\Rightarrow 70 \times 10^{-4}\, v = 10^{-4}\,(50 + 50 + 40 + 40)$$
$$\Rightarrow 70v = 180 \Rightarrow v = \frac{180}{70} = 2.57 \text{ m/s}$$

Example 23: An ideal liquid flows through the pipe, which is of uniform cross - section area. The speed v_A and v_B, and pressure P_A and P_B at points A and B respectively are



(a) $v_A = v_B$
(b) $v_B > v_A$
(c) $P_A = P_B$
(d) $P_B > P_A$

Sol. (d) Pressure at B is due to the water column AB and pressure at A which is applied by flowing liquid but $v_A > v_B$.

Example 24: Water is flowing in a pipe of diameter 4 cm with a velocity of 20 m/s. The water then enters into a tube of diameter 2 cm. The velocity of water in the other pipe is:

(a) 10 m/s (b) 40 m/s
(c) 80 m/s (d) 50 m/s

Sol. (c) $A_1v_1 = A_2v_1$

$$\pi\left(\frac{D_1}{2}\right)^2 v_1 = \pi\left(\frac{D_2}{2}\right)^2 v_2 \;;\; \left(\frac{D_1}{D_2}\right)^2 v_1 = v_2$$

$$v_2 = \left(\frac{4}{2}\right)^2 \times 20 = 4 \times 20 = 80 \text{ m/s}$$

Concept Application

18. Streamline flow is more likely for liquids with
(a) High density and high viscosity
(b) Low density and low viscosity
(c) High density and low viscosity
(d) Low density and high viscosity

19. Which of the following is correct?
(a) Two stream lines of fluid cannot cut each other
(b) Two stream line can cut each other
(c) Path and velocity of each particle is same
(d) Both (a) & (c)

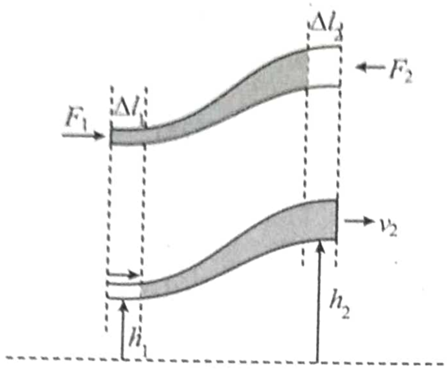
20. An incompressible fluid flows steadily through a cylindrical pipe which has radius $4R$ at point A and radius at point B farther from the point A in the direction of flow of water. If the velocity at point A is 5 m/s then velocity at B is
(a) 5 m/s (b) 20 m/s
(c) 40 m/s (d) 80 m/s

21. An ideal flow of any liquid or fluid must satisfy
(a) Pascal law
(b) Stoke's law
(c) Continuity equation
(d) Bernoulli's theorem

22. Water is flowing through a tube of non-uniform cross-section. If the radii of the tube at the entrance and the exit points are in the ratio of , then the ratio of the velocities of flow of water at the entrance and exit is.
(a) 25 : 16 (b) 16 : 25
(c) 12 : 15 (d) 10 : 13

BERNOULLI'S EQUATION

Let us focus our attention on the motion of the shaded region. This is our "system". the lower cylindrical element of fluid of length Δl_1 and area A_1 is at height y_1, and moves at speed v_1. After some time, the leading section of our system fills the upper cylinder of fluid of length Δl_2 and area A_2 at height y_2, and is then moving at speed v_2.



A pressure force F_1 acts on the lower cylinder due to fluid to its left , and a pressure force F_1 acts on the upper cylinder in the opposite direction. The net work done on the system by F_1 and F_2 is

$$W = F_1\Delta l_1 - F_2\Delta l_2 = P_1A_1\Delta l_1 - P_2A_2\Delta l_2 = (P_1 - P_2)\Delta V$$

The changes in the potential and kinetic energies are

$$\Delta U = \Delta m\, g(h_2 - h_1)\ ;\ \Delta K = \frac{1}{2}\Delta m\,(v_2^2 - v_1^2)$$

These changes are brought about by the net work done on the system, $W = \Delta U + \Delta K$

$$(P_1 - P_2)\Delta V = \Delta mg(h_2 - h_1) + \frac{1}{2}\Delta m\,(v_2^2 - v_1^2)$$

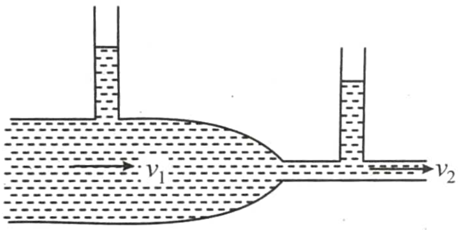
Since the density is $\rho = \Delta m/\Delta V$, we have

$$p_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2$$

Since the points 1 and 2 can be chosen arbitrarily, we can express this result as Bernoulli's Equation

$$p + \rho gh + \frac{1}{2}\rho v^2 = \text{constant}$$

It is applied to all points along a streamline in a nonviscous, incompressible fluid.



OTHER FORMS OF BERNOULLI'S EQUATION

Other Forms of Bernoulli's Equations

If the kinetic, potential and pressure energies are expressed in terms of per unit mass or per unit weight, then the Bernoulli's equation may be written as

$$\frac{p}{\rho} + \frac{v^2}{2} + gh = \text{constant} \qquad \text{(per unit mass)}$$

$$\frac{p}{\rho g} + \frac{v^2}{2g} + h = \text{constant} \qquad \text{(per unit weight)}$$

The term $v^2/2g$ is known as the velocity head, $p/\rho g$ is known as the pressure head and h is known as the potential head.

Assumptions and Limitations

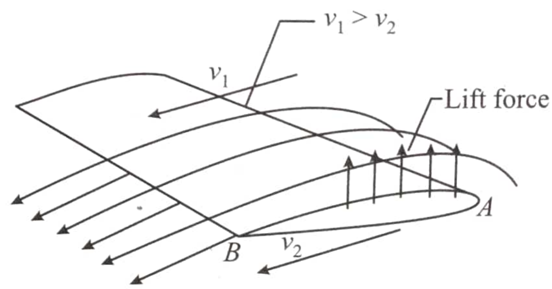
- The flow is assumed to be steady, i.e., there is no change in pressure, velocity and density of the fluid at any point with respect to time.
- However, in the problems of unsteady flow with gradually changing conditions, Bernoulli's equation can be applied without appreciable error. For example, a problem of emptying a large tank can be solved by applying Bernoulli's equation.
- The fluid is assumed to be incompressible. Since liquids are incompressible, Bernoulli's equation can be applied to all liquids. However, it can be applied to the problems of gas flow when there is little variation in pressure, velocity and temperature so that density of gas can be assumed to be constant.
- The fluid is assumed to be irrotational. The irrotationality means the net angular momentum at any point in the fluid flow is zero.
- The fluid is assumed to be ideal, i.e., the energy loss due to friction is ass umed to be absent.

APPLICATIONS OF BERNOULLI'S THEOREM

The fluid dynamic lift or simply dynamic lift is a force that acts on a body because of its motion through a fluid. It is a consequence of Bernoulli's theorem according to which pressure is low where speed is high and vice-versa. Further, dynamic lift is different from static lift, which is the buoyant force that acts on a body according to Archimedes' principle. The examples of dynamic lift are to be found in an airplane wing, a spinning ball and many others. Let us consider the first two in some detail.

The Physics of Airplane Wings

In figure, AB represents an aerofoil of an aeroplane. Its shape is such that it is slightly convex upwards and concave downwards. On account of this shape, when the aerofoil moves through the air, the molecules of air which separate from each other at A simultaneously, will meet at B at the same time because the motion is streamlined. The molecules of air above the airfoil have to travel a longer distance than the molecules which are below the aerofoil in the same time.



Air above the wing is moving with a larger velocity than that below it. The pressure at top of wing is less than pressure at bottom of wing.

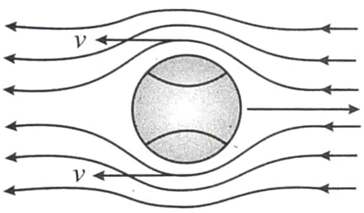
This means that he air above the wing is moving with a larger velocity than that below it. Since air is moving more rapidly (with velocity v_1) above the wing than below it (with velocity v_2), according to Bernoulli's theorem, the pressure above the wing is less than the pressure below it. It is this difference in pressure which provides a net upward lift.

Dynamic Lift on a Spinning Ball

Let us first consider a non-spinning ball (baseball, cricket ball or a tennis ball) as it moves through the air from left to right. If we look at the motion of the ball through the air with reference to the coordinate system fixed on the ball, the flow of air is steady and the streamlines are symmetrical as shown in Fig. (*a*). As the streamlines approach the ball, half bend upward and half downward. Thus, the velocity of the air flow rightward the ball is equal to the velocity leftward the ball. According to Bernoulli's principle, the pressure upward of the ball is equal to the pressure downward it. Hence, there is no net upward or downward force. Let us now consider a ball that has no translatory motion but is spinning about an axis perpendicular to the plane of the paper. Due to friction, air is dragged with the ball and the resulting streamlines, as seen from the spinning ball, are as shown in Fig. (*b*). Note that the tangential velocity at the top is to the right, i.e., vt, while at the bottom it is to the left, i.e., $(-vt)$.

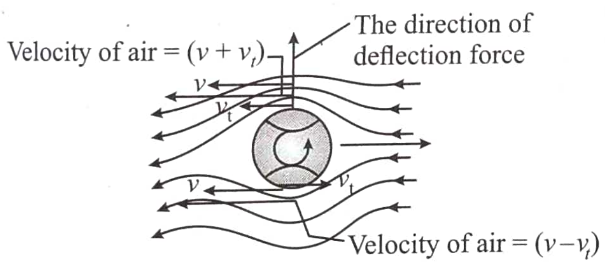
Let us now consider the case in which the ball has both translatory and spinning motions. It is clear that the streamlines are much closer above the ball than below the ball because the velocity of air above the ball is $(v + vt)$ and below the ball is $(v - vt)$, both to the left. Thus, according to Bernoulli's principle, the pressure above the ball is less than the pressure below the ball.

This results in a net force F acting upward as shown in Fig. (*c*). Thus, as the spinning ball moves through the air, there is an upward lift which is called the dynamic lift.

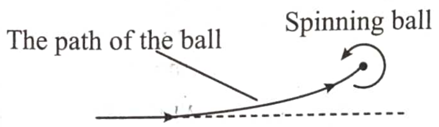


The path of the ball

(*a*) The ball moving without spin



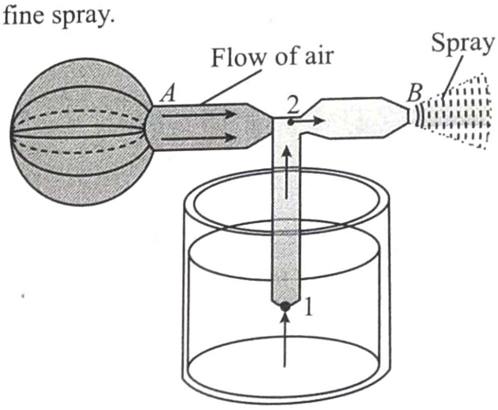
(*b*) The bal moving with spin



(*c*) The trajectory with ball

The dynamic lift due to spinning is called Magnus effect. It is on account of dynamic lift that a spinning ball deviates from its parabolic trajectory as it moves through air. For example, a cricket ball bowled with an initial spin or a tennis ball 'cut' (i.e., given a rapid spin along an axis at right angles to its direction of flight by striking at the centre) has a curved flight.

Atomizer or sprayer: An atomiser is usually used in perfume and deodorant bottles. When the rubber balloon is pressed, the air passes with a large velocity over the tube dipping in the liquid to be sprayed. Due to this, pressure over the tube dipping in the liquid decreases. It makes the liquid rise up in the tube. Due to the applied pressure, the air rushing out with a large velocity from the balloon blows away the liquid coming out of the nozzle in the form of a fine spray.

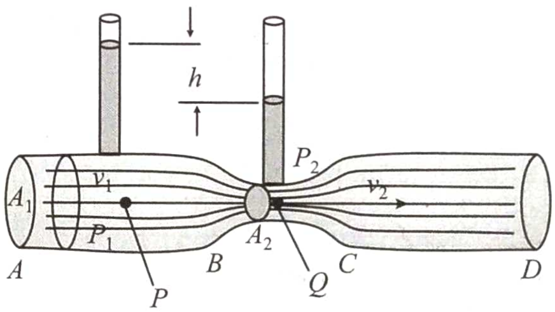


Blowing off of the roofs of houses during a storm: In a storm, cyclone or hurricane, sometimes the light roofs of thatched houses are blown off. This is because, due to the high velocity wind blowing over the roof, the pressure above the roof decreases. As the atmospheric pressure below the roof is greater than the pressure above the roof, the roof gets lifted and is blown away by the wind.

VENTURIMETER

Venturimeter is a device based on Bernoulli's theorem. It is used to measure the rate of flow of a liquid.

As shown in figure, a venturimeter consists of a wide tube having a contraction in the middle. The liquid enters through the broad end *AB*, called convergent cone. After passing through the narrow short horizontal part *BC*, called throat, the liquid, then leaves through the other broad end *CD*, called divergent cone. Two vertical tubes are connected to the venturimeter, one to the convergent cone at the point *P* and the other to the throat at the point *Q*.



When liquid flows through the venturimeter, its velocity (or its kinetic energy) increases at the throat (contraction) and hence pressure energy (or pressure) decreases. This decrease in pressure is measured by measuring the difference in levels (h) of the liquid in the two vertical tubes.

Let A_1 and A_2 be the areas of cross-section of the venturimeter at the points P and Q respectively and v_1 and v_2 be the velocities of the liquid, while crossing at these points. According to the equation of continuity,

$$A_1 v_1 = A_2 v_2$$

$$\text{or} \quad v_2 = \frac{A_1 v_1}{A_2} \qquad \text{...}(i)$$

Since flow of the liquid through the venturimeter is horizontal, the potential energy of the liquid at point Q remains the same as that at point P. If ρ is density of liquid, then according to Bernoulli's theorem

$$\frac{P_1}{\rho} + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + \frac{1}{2}\rho v_2^2$$

$$\text{or} \quad P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 \qquad \text{...}(ii)$$

Here, P_1 and P_2 are the values of the pressure of the liquid at points P and Q respectively. Equation (ii) gives

$$P_1 - P_2 = \frac{1}{2}\rho\,(v_2^2 - v_1^2)$$

From the Eq. (i), substituting for v_2 in the above equation, we have

$$P_1 - P_2 = \frac{1}{2}\rho\left(\frac{A_1^2 v_1^2}{A_2^2} - v_1^2\right) = \frac{1}{2}\rho v_1^2\left(\frac{A_1^2}{A_2^2} - 1\right)$$

$$\text{or} \quad P_1 - P_2 = \frac{1}{2}\rho\, v_1^2\left(\frac{A_1^2 - A_2^2}{A_2^2}\right) \qquad \text{...}(iii)$$

$$\text{or} \quad v_1 = A_2\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}} \qquad \text{...}(iv)$$

If difference of levels of the liquid in the two vertical tubes is h, then

$$P_1 - P_2 = h\rho g \qquad \text{...}(v)$$

Therefore, from the Eqs. (iv) and (v), we have

$$\text{or} \quad v_1 = A_2\sqrt{\frac{2gh}{A_1^2 - A_2^2}} \qquad \text{...}(vi)$$

It gives the velocity of flow of the liquid. The volume of liquid flowing per second i.e. rate of flow of the liquid is given by

$$V = A_1 v_1$$

$$\text{or} \quad V = A_1 A_2\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}} = A_1 A_2\sqrt{\frac{2gh}{A_1^2 - A_2^2}} \qquad \text{...}(vii)$$

If instead of liquid gas is flowing through the pipe, we connect a differential manometer to measure pressure difference. In this case,

$$P_1 - P_2 = hdg \qquad \text{...}(viii)$$

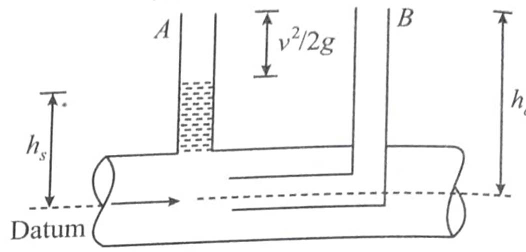
here d is the density of the manometer liquid. From (vii) and ($viii$), we get

$$V = A_1 A_2\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}} = A_1 A_2\sqrt{\frac{2hdg}{\rho\,(A_1^2 - A_2^2)}}$$

STATIC PRESSURE AND DYNAMIC PRESSURE

The height hs to which the liquid rises in it with respect to the reference level (datum) is a measure of the static pressure in the fluid at a point on the axis. That is,

$$p_s = \rho g h_s \quad \text{or} \quad \frac{p_s}{\rho g} = h_s$$



The height h_s is also called the static pressure head. When a tube bent at right angle is inserted in the fluid flow such that the open end faces against the direction of fluid flow as shown by tube B in figure. At the open end of the tube the velocity of the fluid suddenly reduces to zero and the kinetic energy gets converted into pressure energy. The liquid in tube B rises to a higher level than that in tube A. The difference in the levels of liquid is a measure of velocity of the fluid. Tube B measures the dynamic pressure and the height h_d of the liquid in the tube is called the dynamic head, which is equal to the sum total of the static pressure head and velocity head.

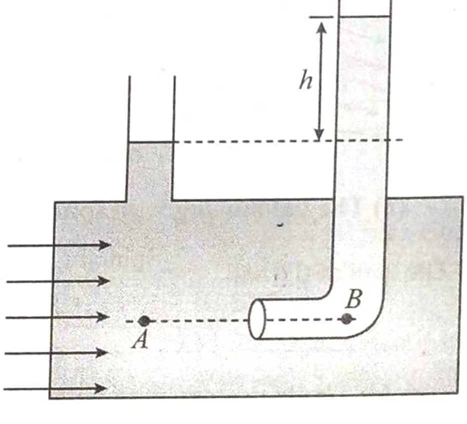
That is, $h_d = h_s + \dfrac{v^2}{2g}$.

Dynamic Pressure, $p_d = \dfrac{1}{2}\rho v^2$

PITOT TUBE

Pitot tube is used to measure the speed of flow of a fluid. The cross section of tube at B is perpendicular to the direction of flow and tube is at rest, so that at B, velocity of fluid becomes zero, i.e., $v_B = 0$. Let v be the velocity of fluid at A and P_A and P_B are pressures at A and B. Then, $P_B - P_A = h\rho g$, where ρ is density of liquid in vertical tubes.

For liquids: If points A and B are in the same horizontal line and area of is cross section same at A and B, gravitational head is the same and the pressure difference is due to difference in velocity. Applying Bernoulli's theorem at A and B, we have



$$P_A + \frac{1}{2}\rho v^2 = P_B + 0$$

$$\Rightarrow \quad \frac{2(P_B - P_A)}{\rho} = v^2 \Rightarrow v = \sqrt{\frac{2(P_B - P_A)}{\rho}} \Rightarrow v = \sqrt{2gh}$$

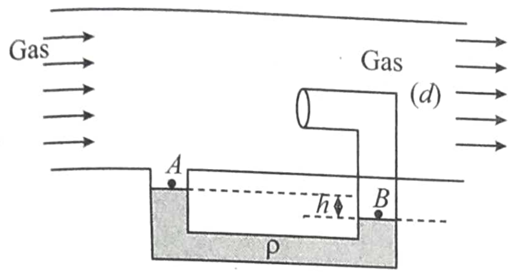
For gases: Let ρ be the density of the liquid in the vertical tube.

$$P_B - P_A = h\rho g$$

Applying Bernoulli's theorem at A and B, we have

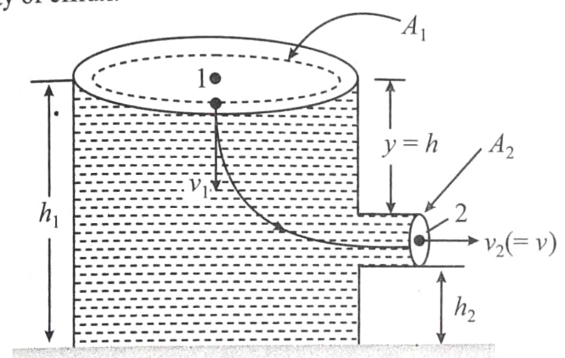
$$P_A + \frac{1}{2}dv^2 = P_B \Rightarrow v^2 = \frac{2(P_B - P_A)}{d}$$

$$\Rightarrow \quad v^2 = \frac{2h\rho g}{d} \quad \Rightarrow v = \sqrt{\frac{2h\rho g}{d}}$$



VELOCITY OF EFFLUX: TORRICELLI'S THEOREM

The liquid comes out of a hole made at a depth y. The areas of the base and hole are A_1 and A_2, respectively. Let us calculate the velocity of efflux.



Choose two points 1 and 2 at the free surface of liquid and at just outside the hole, respectively. Since the points 1 and 2 are exposed to atmosphere, $P_1 = P_2 = P_{atm}$. We have done this to get a mathematical advantage. However, you can take any other point inside the tube instead of taking at the free surface of the liquid. Equation of continuity gives

$$A_1 v_1 = A_2 v_2 \quad \text{...}(i)$$

$$P_{atm} + \frac{\rho v_1^2}{2} + \rho g h_1 = P_{atm} + \frac{\rho v_2^2}{2} + \rho g h_2$$

Substituting $h_1 - h_2 = h$, we have

$$v_2 = \sqrt{v_1^2 + 2gh} \quad \text{...}(ii)$$

Substituting $v_1 = (A_2/A_1)\, v_2$ from Eq. (i) in Eq. (ii), we have

$$v_2 = \sqrt{\frac{2gh}{1 - (A_2/A_1)^2}}$$

Note:

If area of opening is much lesser than area of cross section of tank ($A_2 << A_1$), then velocity of efflux

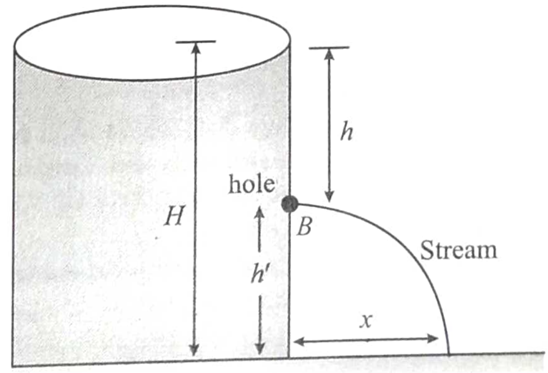
$$v = \sqrt{2gh}$$

which is the same speed that an object would acquire in falling from rest through a distance h under gravity.

The velocity of effiux is the velocity of escaping liquid relative to the container (but not necessarily relative to ground when the container moves).

HORIZONTAL RANGE OF THE ESCAPING LIQUID

Water stands at a depth H in a tank whose sides are vertical. A hole is made in one of the walls at a depth h below the hole water surface. Let container remains at rest.



When the liquid stream emerges out of the hole, it goes along a parabolic path. So time taken by water to fall through a height of $H - h$ is given as

$$H - h = 0 \times t + 1/2\, gt^2 \Rightarrow t = \sqrt{\frac{2(H-h)}{g}}$$

Horizontal range is given by

$$x = v \times t$$

$$= \sqrt{2gh}\sqrt{\frac{2(H-h)}{g}} = \sqrt{4h(H-h)} = 2\sqrt{hh'}$$

or $\quad x = \sqrt{4h(H-h)} = \sqrt{4hH - 4h^2 + H^2 - H^2}$

$$= \sqrt{H^2 - (H^2 - 4hH + 4h^2)} = \sqrt{H^2 - (2h - H)^2}$$

For the range to be maximum, $(2h - H)^2 = 0$

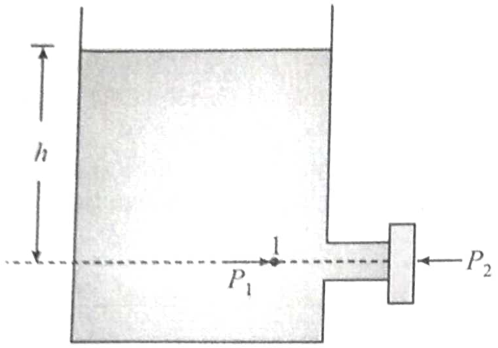
$\Rightarrow \quad 2h - H = 0 \Rightarrow h = H/2$

So, the maximum range $x_{max} = 2\sqrt{hh'}$.

The formula $= 2\sqrt{hh'}$ tells us if we make two holes at equal vertical distances from top and bottom, both liquids jets will strike the same spot (but not simultaneously).

REACTION FORCE DUE TO EJECTION OF LIQUID

A cylindrical vessel has an opening of cross-sectional area 'a' near the bottom. A disc is held against the opening to prevent liquid of density ρ from coming out. If the height of the liquid above opening is h. Let us analyse the force on the disc in this situation.



The disc experiences a hydrostatic pressure from the liquid inside the vessel. Pressure at the level of the disc is

$$P_1 = P_{atm} + \rho gh$$

The air pressure on the outside of disc is

$$P_2 = P_{atm}$$

The net outward force = $(P_2 - P_1)a = \rho gha$.

Now, the disc is moved a short distance away in horizontal direction. The liquid comes out, strikes the disc inelastically and drops vertically downward. The water in this case will impart impulsive (impact) force on the disc.

When the disc is moved away, the liquid moves out with speed $v = \sqrt{2gh}$. The (mass per second), i.e., the rate of mass coming out of the opening is given by $dm/dt = \rho av = \rho a\sqrt{2gh}$.

Momentum per second imparted by water rightward is given by

$$\frac{dm}{dt}v = (\rho av)v = \rho av^2 = 2\rho gha$$

The change in momentum per second after striking the disc is

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

$$\Rightarrow \quad 0 - (\rho av^2) = -\rho av^2 = -2\rho gha$$

Taking the rightward direction as positive, the force on the liquid is towards the left and its reaction with disc is towards the right. The force obtained is twice the hydrostatic force. The force due to atmospheric pressure is cancelled out on the both sides.

Train Your Brain

Example 25: A siphon tube is discharging a liquid of specific gravity 0.9 from a reservoir as shown in the figure.

(*a*) Find the velocity of the liquid through the siphon

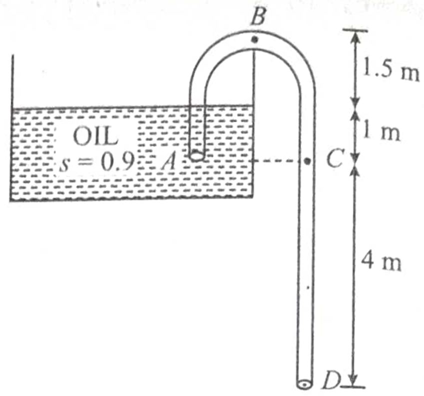
(*b*) Find the pressure at the highest point B.

(*c*) Find the pressure at the points A (out side the tube) and C. State and explain the following:

(*d*) Would the rate of flow be more, less or the same if the liquid were water ?

(*e*) Is there a limit on the maximum height of B above the liquid level in the reservoir?

(*f*) Is there a limit on the vertical depth of the right limb of the siphon?



Sol. Assume datum at the free surface of the liquid.

(*a*) Applying Bernoulli's equation on point 1 and 2, as shown in the figure.

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + y_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + y_2$$

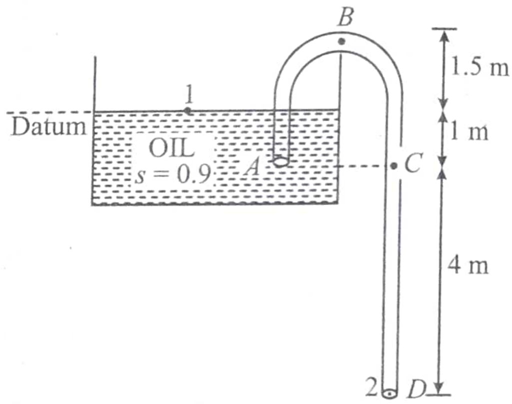
Here $p_1 = p_2 = p_0 = 10^5$ N/m^2; $y_1 = 0, y_2 = -5$ m

Since area of the tube is very small as compared to that of the reservoir, therefore,

$v_1 << v_2$. Thus $\frac{v_1^2}{2g} \approx 0$

$$\therefore \quad v_2 = \sqrt{2g(y_1 - y_2)} = \sqrt{2(10)(5)} = 10 \text{ m/s}$$

(*b*) Applying Bernoulli's equation at D and B.



$$\frac{p_B}{\rho g} + \frac{v_B^2}{2g} + y_B = \frac{p_1}{\rho g} + \frac{v_1^2}{2g} + y_1$$

Here, $p_1 = 10^5$ N/m^2; $\frac{v_1^2}{2g} \approx 0$;

$v_1 = 0$, $v_D = v_2 = 10$m/s, $y_B = 1.5$ m

$$\therefore \quad p_B = p_1 - \frac{1}{2}\rho v_2^2 - \rho g y_B$$

or $p_B = 10^5 - 1/2(900)(10)^2 - (900)(10)(1.5)$

$= 41.5$ kN/m^2

(*c*) Applying Bernoulli's equation at 1 and A

$p_A = p_1 + \rho g\,(y_1 - y_A)$

or $p_A = 10^5 + (900)(10)(1)$

$= 109$ kN/m^2.

Applying Bernoulli's equation at 1 and C,

$$p_C = p_1 - \frac{1}{2}\rho v_C^2 - \rho g y_C$$

$$= 10^5 - \frac{1}{2}(900)(10)^2 - (900)(10)(-1)$$

$$= 10^5 - 45000 + 9000 = 64 \text{ kN/m}^2.$$

(*d*) The velocity of flow is independent of the density of the liquid, therefore, the discharge would remain the same.

(*e*) Since the pressure at B is less than atmospheric, the liquid, therefore, has a tendency to get vapourised if the pressure becomes equal to the vapour pressure of it. Thus, $p_B > p_{\text{vapour}}$.

(*f*) The velocity of flow depends on the depth of the point D, below the free surface

$$\frac{v^2}{2g} = y_1 - y_2 = H \text{ and}$$

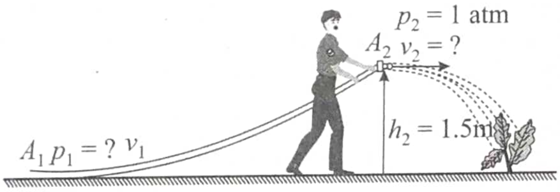
$$p_B = p_1 - \frac{1}{2}\rho v^2 - \rho g y_B = p_1 - \rho g H - \rho g y_B$$

For working of siphon, $H \neq 0$, and H should not be high enough so that p_B may not reduce to vapour pressure.

Example 26: A garden hose has an inside cross-sectional area of 3.60 cm^2, and the opening in the nozzle is 0.250 cm^2. If the water velocity is 50 m/s in a segment of the hose that lies on the ground

(*a*) with what velocity does the water come from the nozzle when it is held 1.50 m above the ground and

(*b*) What is the water pressure in the hose on the ground?



Sol. (*a*) We first apply the Equation of Continuity, to find the velocity of the fluid at the nozzle.

$$v_2 = \frac{A_1}{A_2}v_1 = \left(\frac{3.6\,\text{cm}^2}{0.25\,\text{cm}^2}\right)(50\,\text{cms}^{-1})$$

$= 720$ cms^{-1}

$= 7.2$ ms^{-1}

(*b*) We next apply Bernoulli's Equation to find the pressure p_1. We know that $h_1 = 0$ and $h_2 = 1.5$ m. The pressure at the nozzle is atmospheric pressure $p_2 = 1.01 \times 10^5 Pa$.

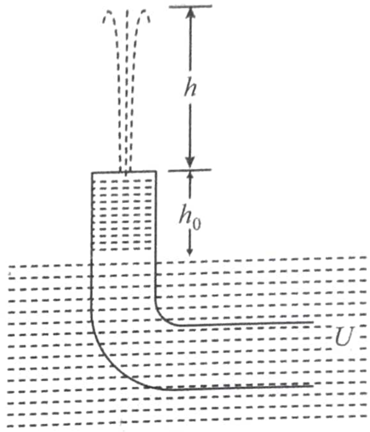
Solving for p_1 and using the density of water $\rho = 1 \times 10^3$ kg/m^3, we have

$$p_1 = p_2 + 1/2\rho\left(v_2^2 - v_1^2\right) + \rho g(h_2 - h_1)$$

$$= (1.01 \times 10^5) + 1/2(1.5 \times 10^3)[(7.2)^2 - (0.50)^2] + (1 \times 10^3)(9.8)(1.5 - 0)$$

$$= 1.41 \times 10^5\ Pa$$

Example 27: A bent tube is lowered into a water stream as shown in Fig, The velocity of the stream relative to the tube is equal to $v = 2.5$ m/s. The closed upper end of the tube located at the height $h_0 = 12$ cm has a small office. To what height h will the water jet spurt?



Sol. Let the velocity of the water jet, near the orifice be v', then applying Bernoullis theorem,

$$\frac{1}{2}\rho v^2 = h_0\rho g + \frac{1}{2}\rho v^2\ ;\ v' = \sqrt{v^2 - 2gh_0} \quad ...(i)$$

Here the pressure term on both sides is the same and equal to atmospheric pressure.

Now, if it rises upto a height then at this height, whole of its kinetic energy will be converted into potential energy. So,

$$\frac{1}{2}\rho v'^2 = \rho g h \text{ or } h = \frac{v'^2}{2g}$$

$$= \frac{v^2}{2g} - h_0 = 20 \text{ cm} \quad \text{[Using equation } (i)]$$

Example 28: A horizontal pipe has a cross section of 10 cm^2 in one region and of 5 cm^2 in another. The water velocity at the first is 5 ms^{-1} and the pressure in the second is 2×10^5 Nm^{-2}. Find the pressure of water in 1st region.

Sol. $A_1v_1 = A_2v_2$

$10 \times 5 = 5 \times v_2$

$v_2 = 10$ m/s

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g}$$

$$\frac{p_1}{10^4} + \frac{25}{20} = \frac{2\times10^5}{10^4} + \frac{100}{20}$$

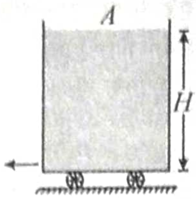
$$\frac{p_1}{10^4} = 25 - 1.25 = 23.75$$

$$p_1 = 2375 \times 10^2\ Pa$$

Example 29: An open tank of cross sectional area A contains water up to height H. It is kept on a smooth horizontal surface. A small orifice of area is punched at the bottom of the wall of the tank. Water begins to drain out. Mass of the empty tank may be neglected.

(i) Prove that the tank will move with a constant acceleration till it is emptied. Find this acceleration.

(ii) Find the find speed acquired by the tank when it is completely empty.



Sol. (i) let v = Instantaneous velocity of the tank

u = Instantaneous speed of efflux relative to the lank $= \sqrt{2gh}$.

Where; h = Instantaneous height of water in the tank.

Rate of mass coming out of the tank is $\frac{dm}{dt} = (uA_0)\rho$

We can use the equation for variable mass and write

$$m\frac{dv}{dt} = u\left(\frac{dm}{dt}\right)$$

$m = Ah\,\rho$ = instantaneous mass of tank

$$\therefore Ah\rho\frac{dv}{dt} = u^2 A_0\rho \;;\; Ah\frac{dv}{dt} = A_0(2gh)$$

$$\therefore \frac{dv}{dt} = \frac{2A_0 g}{A} = a \qquad \text{constant ...(i)}$$

(ii) if h fall by dh in a small time interval dt then.

$$-\rho A dh = \rho A_0 u dt$$

$$\Rightarrow \frac{dh}{dt} = -\frac{A_0 u}{A} = -\frac{A_0}{A}\sqrt{2gh} \text{ and}$$

$$\frac{dv}{dh}\frac{dh}{dt} = \frac{2A_0 g}{A} \text{ From (i)}$$

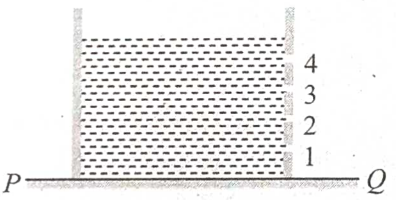
$$\frac{dv}{dh}\left(-\frac{A_0}{A}\sqrt{2gh}\right) = \frac{2A_0 g}{A}$$

$$\therefore \frac{dv}{dh} = -\sqrt{2g}\frac{1}{\sqrt{h}} \quad \therefore \int_0^v dv = -\sqrt{2g}\int_H^0 \frac{dh}{\sqrt{h}}$$

This give $v = 2\sqrt{2gh}$

Concept Application

23. A cylindrical vessel of 90 cm height is kept filled upto the brim. It has four holes 1, 2, 3, 4 which are respectively at heights of 20 cm, 30 cm, 45 cm and 50 cm from the horizontal floor P The water falling at the maximum horizontal distance from the vessel comes from



(a) Hole number 4 (b) Hole number 3
(c) Hole number 2 (d) Hole number 1

24. A streamlined body falls through air from a height h on the surface of a liquid. If d and $D(D > d)$ represents the densities of the material of the body and liquid respectively, then the time after which the body will be instantaneously at rest, is

(a) $\sqrt{\frac{2h}{g}}$ (b) $\sqrt{\frac{2h}{g}\cdot\frac{D}{d}}$

(c) $\sqrt{\frac{2h}{g}\cdot\frac{d}{D}}$ (d) $\sqrt{\frac{2h}{g}}\left(\frac{d}{D-d}\right)$

25. Which of the following is not an assumption for an ideal fluid flow for which Bernoulli's principle is valid

(a) Steady flow (b) Incompressible
(c) Viscous (d) Irrotational

26. The rate of flowing of water from the orifice in a wall of a tank will be more if the orifice is:

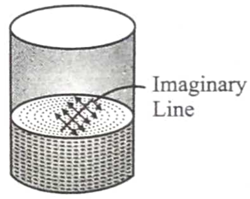
(a) Near the bottom
(b) Near the upper end
(c) Exactly in the middle
(d) Does not depend upon the position of orifice

SURFACE TENSION

Surface tension is a property of liquid at rest by virtue of which a liquid surface gets contracted to a minimum area and behaves like a stretched membrane.

Surface tension of a liquid is measured by force per unit length on either side of any imaginary line drawn tangentially over the liquid surface, force being normal to the imaginary line as shown in figure i.e. surface tension.

$$(T) = \frac{\text{Total force on either of the imaginary line } (F)}{\text{Length of the line } (l)}$$



Unit of Surface Tension

In C.G.S. system the unit of surface tension is dyne/cm (dyne cm^{-1}) and SI system its units in Nm^{-1} and Dimensions is $[M^1 T^{-2}]$.

APPLICATION OF SURFACE TENSION

(i) The wetting property is made use in detergents and water proofing. When the detergent materials are added to liquids, the angle of contact decreases and hence the wett ability increases. On the other hand, when water proofing material is added to a fabric, it increases the angle of contact, making the fabric water-repellent.

(ii) The low value of surface tension prevents the formation of drops that may otherwise block the entrance to skin or a wound. Due to low surface tension the antiseptics spreads properly over the wound.

(iii) Surface tension of all lubricating oils and paints is kept low so that they spread over a large area.

(iv) Oils spreads over the surface of water because the surface tension of oil is less than the surface tension of cold water.

(v) A rough sea can be calmed by pouring oil on its surface.

EFFECT OF TEMPERATURE AND IMPURITIES ON SURFACE TENSION

The surface tension of a liquid decreases with the rise in temperature and vice versa.

According to Ferguson,

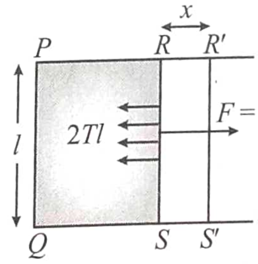
$T = T_0\left(1-\dfrac{\theta}{\theta_C}\right)^n$, where T_0 is surface tension at $0°C$, θ is absolute temperature of the liquid, θ_C is the critical temperature and n is a constant varies slightly from liquid and has means value 1.21. This formula shows that the surface tension becomes zero at the critical temperature, when the machinery parts get jammed in winter.

The surface tension of a liquid change appreciably with addition of impurities. For example, surface tension of water increases with addition of highly soluble substance like NaCl, $ZnSO_4$ etc. On the other hand surface tension of water gets reduced with addition of sparingly soluble substances like phenol, soap etc.

SURFACE ENERGY

The potential energy of surface molecules per unit area of the surface is called surface energy.

Unit of surface energy is erg cm^{-2} in C.G. S. system and Jm^{-2} in SI system. Dimensional formula of surface energy is $[ML^0T^{-2}]$ surface energy depends on number of surfaces e.g. a liquid drop is having one liquid air surface while bubble is having two liquid air surface.



Consider a rectangular frame $PQRS$ of wire, whose arm RS can slide on the arms PR and QS. If this frame is dipped in a soap solution, then a soap film is produced in the frame $PQRS$ in figure due to the surface tension (T), the film exerts a force on the frame (towards the interior of the film). Let l be the length of the arm RS, then the force acting on the arm RS towards the film is $F = T \times 2l$ [Since soap film has two surfaces upper and down, that is way the length is taken twice.]

$\therefore$ Work done $(W) = F\,x = 2Tlx$

Increase in potential energy of the soap film.

$EA = E(2lx)$ = work done in increasing the area (ΔW)

Where E = surface energy of the soap film per unit area.

According the law of conversation of energy, the work done must be equal to the increase in the potential energy.

$\therefore \quad 2Tlx = E(2lx)$

or $\quad T = E = \dfrac{\Delta W}{A}$

Thus, surface tension is numerically equal to surface energy or work done per unit increase surface area. So, if area is increased surface energy is increased so this is not a natural process.

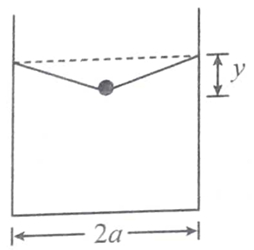
Total surface energy

$U = TA$

where, T = Surface tension, A = Exposed Area (Area which is in contact with Air)

Train Your Brain

Example 30: A container of width $2a$ is filled with a liquid. A thin wire of weight per unit length λ is gently placed over the liquid surface in the middle of the surface as shown in the figure. As a result the liquid surface is depressed by a distance $y(y << a)$. Determine the surface tension of the liquid.



Sol. By force balance on the wire,

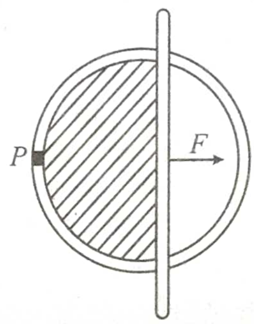
$$2T\cos\theta\,\ell = \lambda\ell g$$

$$\frac{2Ty}{\sqrt{a^2+y^2}}\,\ell = \lambda\ell g$$

Taking approximation

$$\left[T = \frac{\lambda a g}{2y}\right]$$

Example 31: A circular wire, 10 cm in diameter, with a slider wire on it, is in a horizontal plane. A liquid film is formed, bounded by the wires, on the left side of the slider, as shown. The surface tension of the liquid is 0.1 N/m. An applied force 16 mN, perpendicular to the slider, maintains the film in equilibrium. Ignore the sag in the film. What can be the distance between point P and slider.



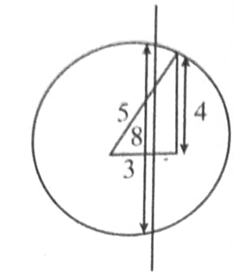
(*a*) 8 cm

(*b*) 2 cm

(*c*) 5cm

(*d*) slider cannot be in equilibrium

Sol. $16 \times 10^{-3} = 2T \times l$



$$l = \frac{16 \times 10^{-3}}{2 \times 0.1} = 8 \text{ cm}$$

$\Rightarrow$ Distance from centre = 3 cm

$\Rightarrow$ 5 + 3 = 8cm

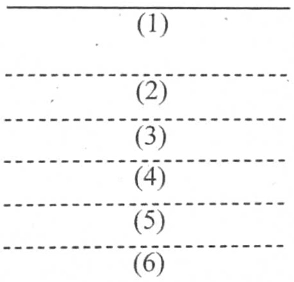
or 5 – 3 = 2cm

Example 32: The average force that acts on any molecule from the side of all the others, however, is always equal to zero if the liquid is in equilibrium. This is why the work done to move the liquid from a depth to the surface should also be zero. What is the origin, in this case, of the surface energy?

Sol. The forces of attraction acting on a molecule in the surface layer from all the other molecules produce a resultant directed downward. The closest neighbours, however, exert a force of repulsion on the molecule which is therefore in equilibrium.

Owing to the forces of attraction and repulsion, the density of the liquid is smaller in the surface layer than inside. Indeed, molecule 1 (figure) is acted upon by the force of repulsion from molecule 2 and the forces of attraction from all the other molecules (3, 4, ...). Molecule 2 is acted upon by the forces of repulsion from 3 and 1 and the forces of attraction from the molecules in the deep layers.

As a result, distance 1-2 should be greater than 2-3 etc.



This course of reasoning is quite approximate (thermal motion, etc is disregarded), but nevertheless it gives a qualitatively correct energy.

An increase in the surface of the liquid causes new sections of the rarefied surface layer to appear. Here work should be performed against the forces of attraction between the molecules. It is this work that constitutes the surface energy.

Example 33: A film of water is formed between two straight parallel wires each 10 cm long and at a separation 0.5 cm. Calculate the work required to increase 1 mm distance between them. Surface tension of water 72×10^{-3} N/m.

Sol. Since this a water film, it has two surface, therefore increase in area

$\Delta S = 2 \times 10 \times 0.1 = 2\text{cm}^2$.

$\therefore$ Work required to be done

$W = \Delta S \times T = 2 \times 10^{-4} \times 72 \times 10^{-3}$

$= 144 \times 10^{-7}$ joule

$= 1.44 \times 10^{-5}$ joule.

Example 34: Find the radius of the drop that is formed due to combining of 8 drop of radius *R*. If surface tension is *T*. Find the energy released in the process.

Sol. $8 \times \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R'^3$

$\Rightarrow 8R^3 = R'^3$

$\Rightarrow R' = 2R$

$A_1 = 8 \times 4\pi R^2 = 32\pi R^2$.

$A_2 = 4\pi(2R)^2 = 16\pi R^2$

$\Delta A = 16\pi R^2$

So, work done $= T \times 16\pi R^2$.

- Surface area of final drop is less than initial drop.
- Energy is released in this process. So, combining of drop is a natural process.
- If you want to break a drop you have to increase the surface area. So you have to do work so breaking of drop is not a natural process.

Example 35: In the above question if energy released goes in increasing the temperature of the liquid find temperature rise. Given that density of liquid is ρ, specific heat is *S*.

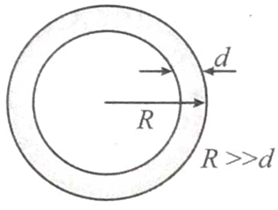
Sol. Heat released $= T \times 16\pi R^2$

$Q = mS\Delta\theta$

$T \times 16\pi R^2 = 8 \times 4/3\pi R^3 \rho \times S\Delta\theta$

$$\Delta\theta = \frac{T \times 16 \times 3 \times R^2}{8 \times 4R^3 \times \rho \times S} = \frac{3T}{2R\rho S}$$

Example 36: A soap bubble has radius *R* and thickness $d(<< R)$ as shown. It collapse into a spherical drop. Then to energy released, and pressure in that process is



Sol. $2 \times 4\pi R^2 \times d = \frac{4}{3}\pi r^3$

$6R^2 d = r^3 \Rightarrow r = (6R^2 d)^{\frac{1}{3}}$

$S' = 4\pi r^2 = (6R^2 d)^{\frac{2}{3}}$

$\Delta U = T \times (2 \times 4\pi R^2 - 4\pi r^2)$ $\quad [\therefore DU = T \times DA]$

$= 4T\pi\, [2R^2 - (6R^2 d)^{\frac{2}{3}}$.

Example 37: The work done in increasing the size of a soap film from 11 cm × 7 cm to 20 cm × 8 cm is 8×10^{-2} J. The surface tension of the film is

(*a*) 4.8 N/m (*b*) 5 N/m
(*c*) 6.2 N/m (*d*) None of these

Sol. (*b*) (*a*) As we know that $W = T \times \Delta A \Rightarrow T = \frac{W}{\Delta A}$

$$T = \frac{8\times10^{-2}}{2\times[(20\times8)10^{-4}-(11\times7)10^{-4}]}$$

$$= \frac{8\times10^{-2}}{2(160\times10^{-4}-77\times10^{-4})}$$

$$T = \frac{8\times10^{-2}}{2(83\times10^{-4})} = \frac{4\times10^{-2}}{83\times10^{-4}} = 0.048\times10^{2} = 4.8 \text{ N/m}$$

Concept Application

27. A drop of oil is placed on the surface of water. Which of the following statement is correct?
(*a*) It will remain on it as a sphere
(*b*) It will spread as a thin layer
(*c*) It will be partly as spherical droplets and party as thin film
(*d*) It will flat as a distorted drop on the water surface

28. Due to which property of water, tiny particles of camphor dance on the surface of water?
(*a*) Viscosity (*b*) Surface tension.
(*c*) Weight (*d*) Floating force

29. The surface tension of a liquid is 10N/m. If a thin film of the area 0.05 m^2 is formed on a loop. Then the surface energy will be
(*a*) 5J (*b*) 3J (*c*) 2J (*d*) 1J

30. If two soap bubbles of equal radii r coalesce then the radius of curvature of interface between two bubbles will be
(*a*) 2 (*b*) 1 (*c*) 0 (*d*) ∞

31. 27 mercury drops coalesce to form one mercury drop, the energy changes by a factor of
(*a*) 2 (*b*) 4 (*c*) 9 (*d*) 3

32. A soap bubble in vacuum has a radius 8 cm and another soap bubble in vacuum has a radius of 6cm. If the two bubbles coalesce under isothermal conditions then the radius of new bubble will be
(*a*) 10 cm (*b*) 12 cm
(*c*) 16 cm (*d*) None of theses

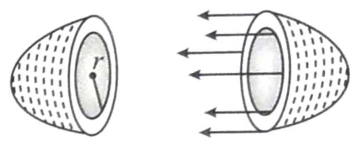
33. The surface tension does not depend upon:
(*a*) Presence of impurities
(*b*) Temperature
(*c*) Atmospheric pressure
(*d*) Nature of the liquid

34. The surface tension of soap solution is 0.05 N/m. The work done in blowing to form a soap bubble of surface area 80 cm^2 is
(*a*) 2×10^{-4} J
(*b*) 8×10^{-4} J
(*c*) 7×10^{-3} J
(*d*) 3×10^{-4} J

35. At critical temperature the surface tension of a liquid becomes:
(*a*) Zero
(*b*) Infinity
(*c*) Same as that at any other temperature
(*d*) Cannot be determined 65. (b) Density of oil is lower than that of water that's why oil spreads on water surface.

EXCESS PRESSURE INSIDE A LIQUID DROP AND A BUBBLE

1. **Inside a bubble:** Consider a soap bubble of radius r. Let p be the pressure inside the bubble and p_a outside. The excess pressure $= p - p_a$. Imagine the bubble broken into two halves and consider one half of it as shown in figure. Since there are two surface, inner and outer, so the force due to surface tension is



F = Surface tension × Length = $T \times 2$ (circumference of the bubble) $= T \times 2\,(2\pi r)$ ...(*i*)

$\Rightarrow F = (p - p_a)\,\pi r^2.$ ...(*ii*)

Maintain the equilibrium,

$(p - p_a)\,\pi r^2 = T \times 2\,(2\pi r)$

or $(p - p_a) = \frac{4T}{r}$

$\Rightarrow p_{excess} = \frac{4T}{r}$

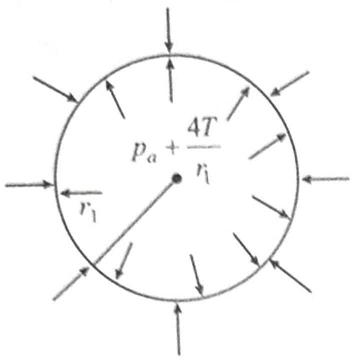
2. **Inside the drop:** In a drop, there is only one surface and hence excess pressure can be written as

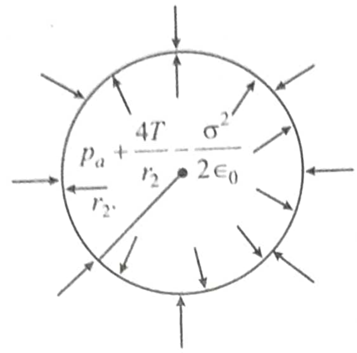
$(p - p_a) = \boxed{\frac{2T}{r} = p_{excess}}$

3. **Inside air bubble in a liquid:**

$(p - p_a) = \frac{2T}{r}$; $p_{excess} = \frac{2T}{r}$

4. **A charged bubble:** If bubble is charged, it's radius increases. Bubble has pressure excess due to charge too. Initially pressure inside the bubble $= p_a + \frac{4T}{r_1}$





For charge bubble, pressure inside $= p_a + \frac{4T}{r_2} - \frac{\sigma^2}{2\epsilon_0}$, where σ is surface charge density.

Taking temperature remains constant, then from Boyle's law

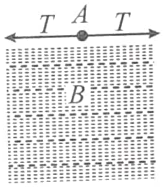
$P_1V_1 = P_2V_2$

$$\left(p_a + \frac{4T}{r_1}\right)\frac{4}{3}\pi r_1^3 = \left[p_a + \frac{4T}{r_2} - \frac{\sigma^2}{2\epsilon_0}\right]\frac{4}{3}\pi r_2^3$$

From above expression the radius of charged drop may be calculated. It can be conclude that radius of charged bubble increases i.e. $r_2 > r_1$.

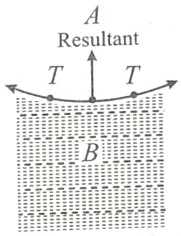
EXCESS PRESSURE INSIDE A CURVED SURFACE

1. **Plane Surface:** If the surface of the liquid is plane [as shown in figure (*a*)], the molecule on the liquid surface is attracted equally in all directions. The resultant force due to surface tension is zero. The pressure, therefore on the liquid surface is normal.



(a) Plane Surface

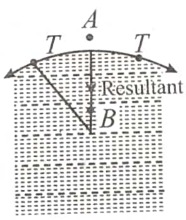
2. **Concave Surface:** If the surface is concave upward [as shown in figure (*b*)], there will be upward resultant force due to surface tension acting on the molecule. Since the molecule on the surface is in equilibrium, there must be an excess of pressure on the concave side in the downward direction to balance the resultant force of surface tension



(b) Concave Surface

$$p_A - p_B = \frac{2T}{r}.$$

3. **Convex Surface:** If the surface is convex [as shown figure (*c*)], the resultant force due to surface tension acts in the downward direction. Since the molecule on the surface are in equilibrium, there must be an excess of pressure on the concave side of the surface acting in the upward direction to balance the downward resultant force of surface tension. Hence, there is always in excess of pressure on concave side of a curved surface over that on the convex side.

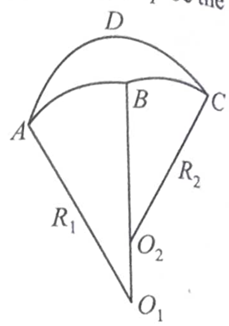


(c) Convex Surface

$$p_B - p_A = \frac{2T}{r}$$

Relation between Surface Tension, Radii of Curvature and Excess Pressure on a Curved Surface

Let us consider a small element *ABCD* (figure) of a curved liquid surface which is convex on the upper side. R_1 and R_2 are the maximum and minimum radii of curvature respectively. They are called the "principal radii of curvature" of the surface. Let p be the pressure on the concave side.



Then, $p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$.

For liquid film

$p = 2T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$,

(because a film has two surfaces.)

FORCE OF COHESION

The force of attraction between the molecules of the same substance is called cohesion.

In case of solids, the force of cohesion is very large and due to this solids have definite shape and size. On the other hand, the force of cohesion in case of liquids is weaker than that of solids. Hence liquids do not have definite shape but have definite volume. The force of cohesion is negligible in case of gases. Because of this fact, gases have neither fixed shape nor volume.

Example:

(*i*) Two drops of a liquid combine into one when brought in mutual contact because of the cohesive force.

(*ii*) It is difficult to separate two sticky plates of glass wetted with water because a large force has to be applied against the cohesive force between the molecules of water.

(*iii*) It is very difficult to break a drop of mercury into small droplets because of large cohesive force between mercury molecules.

FORCE OF ADHESION

The force of attraction between molecules of different substance is called adhesion.

Example:

(*i*) Adhesive force enables us to write on the black board with a chalk.

(*ii*) Adhesive force helps us to write on the paper with ink.

(*iii*) Large force of adhesion between cement and bricks helps us in construction work.

(*iv*) Due to force of adhesive, water wets the glass plate.

(*v*) Fevicol and gum are used in joint two surfaces together because of adhesive force.

SHAPE OF LIQUID MENISCUS

When a capillary tube or a tube is dipped in a liquid, the liquid surface becomes curved near the point of contact. This curved surface is due to the two forces i.e.

(i) Due to the force of cohesion and

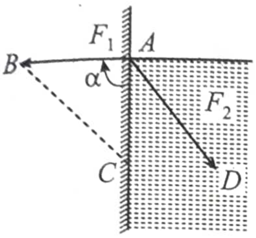
(ii) Due to the force of adhesion. The curved surface of the liquid is called meniscus of the liquid. Various forces acting on molecule A are

(iii) Force F_1 due to force of adhesion which acts outwards at right angle to the wall of the tube. This force is represented by AB.

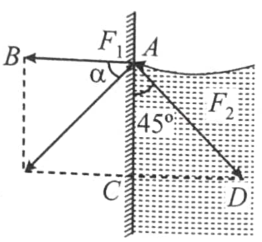
(iv) Force F_2 due to force of cohesion which acts at an angle of 45° to the vertical. This force is represented by AD.

(v) The weight of the molecule A which acts vertically downward along the wall of the tube.

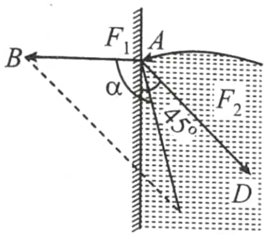
Since the weight of the molecule is negligible as compared to F_1 and F_2 and hence can be neglected. Thus, there are only two forces (F_1 and F_2) acting on the liquid molecules. These forces are inclined at an angle of 135°.



(a)



(b)



(c)

The resultant force represented by AC will depend upon the values of F_1 and F_2. Let the resultant force makes an angle α with F_1.

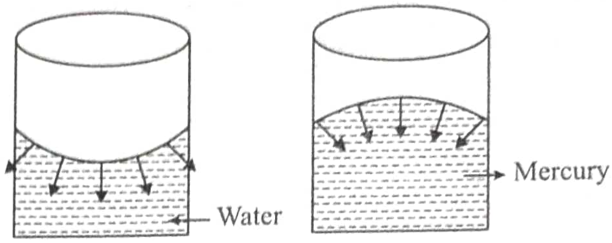
According to parallelogram law of vectors

$$\tan\alpha = \frac{F_2 \sin 135°}{F_1 + F_2 \cos 135°} = \frac{\frac{F_2}{\sqrt{2}}}{F_1 - \frac{F_2}{\sqrt{2}}} = \frac{F_2}{\sqrt{2}\,F_1 - F_2}$$

ANGLE OF CONTACT

The angle which the tangent to the liquid surface at the point of contact makes with the solid surface inside the liquid is called angle of contact.

Those liquids which we the wall of the container (say in case of water and glass) have meniscus concave upwards and their value of angle of contact is less than 90° (also called acute angle). However, those liquids which don't wet the walls of the container (say in case of mercury and glass) have meniscus convex upwards and their value of angle of contact is greater than 90° (also called obtuse angle). The angle of contact of mercury with glass about 140°, whereas the angle of contact of water with glass is about 8°. But, for pure water, the angle of contact θ with glass is taken as 0°.



SPECIAL CASES

(i) If $F_2 = \sqrt{2}\,F_1$, then $\tan\alpha = \infty$. $\therefore$ $\alpha = 90°$.
Then the resultant force will act vertically downward and hence the meniscus will be plane or horizontal shown in figure (a). Example: pure water contained in silver capillary tube.

(ii) If $F_2 < \sqrt{2}\,F_1$, then $\tan\alpha$ is positive. $\therefore$ α is acute angle.
Thus, the resultant will be directed outside the liquid and hence the meniscus will be concave upward shown in figure (b). This is possible is case of liquids which wet the walls of the capillary tube. Example: water in glass capillary tube.

(iii) If $F_2 > \sqrt{2}\,F_1$, then $\tan\alpha$ is negative. $\therefore$ α is obtuse angle.
Thus, the resultant will be directed inside the liquid and hence the meniscus will be convex upward shown in figure (c). This is possible in case of liquid which do not wet the walls of the capillary tube. Example: mercury in glass capillary tube.

Train Your Brain

Example 38: A minute spherical air bubble is rising slowly through a column of mercury contained in a deep jar. If the radius of the bubble at a depth of 100 cm is 0.1 mm, calculate its depth where its radius is 0.126 mm, given that the surface tension of mercury is 567 dyne/cm. Assume that the atmospheric pressure is 76 cm of mercury.

Sol. The total pressure inside the bubble at depth h_1 is

$$= (P + h_1\rho g) + \frac{2T}{r_1} = P_1 \qquad (P \text{ atmospheric pressure})$$

and the total pressure inside the bubble at depth h_2 is

$$= (P + h_2\rho g) + \frac{2T}{r_2} = P_2$$

Now, according to Boyle's Law

$$P_1V_1 = P_2V_2 \text{ where } V_1 = \frac{4\pi}{3}\pi r_1^3 \text{ and } V_2 = \frac{4\pi}{3}\pi r_2^3$$

Hence, we get $\left[(P + h_1\rho g) + \frac{2T}{r_1}\right]\frac{4}{3}\pi r_1^3$

$$= \left[(P + h_2\rho g) + \frac{2T}{r_2}\right]\frac{4}{3}\pi r_2^3$$

or $\left[(P + h_1\rho g) + \frac{2T}{r_1}\right] r_1^3 = \left[(P + h_2\rho g) + \frac{2T}{r_2}\right] r_2^3$

Given that: $h_1 = 100$ cm,

$r_1 = 0.1$ mm $= 0.01$ cm, $r_2 = 0.126$ cm, $T = 567$ dyne/cm,

$P = 76$ cm of mercury

Substituting all the values, we get $h_2 = 9.48$ cm.

Example 39: A small hollow sphere which has a small hole in it is immersed in water to a depth of before any water is penetrated into it. If the surface tension of water is, what is the radius of the hole?

(*a*) 0.07 mm (*b*) 0.04 mm

(*c*) 0.01 mm (*d*) 0.05 mm

Sol. (*a*) $P = \frac{2S}{R}$

$$\Rightarrow R = \frac{2S}{P} = \frac{2\times 0.073}{0.2\times 10^3 \times 9.8} = 0.0744\times 10^{-3}$$

$R = 0.07$ mm

Example 40: The excess pressure inside a spherical drop of water is four times that of another drop. Then their respective mass ratio is

(*a*) 1 : 16 (*b*) 8 : 1

(*c*) 1 : 4 (*d*) 1 : 64

Sol. (*d*) Excess pressure inside spherical drop $P = \frac{2S}{R}$

$$P_1 = \frac{2T}{R_1}, P_2 = \frac{2T}{R_2}$$

$$\therefore \frac{2T}{R_1} = 4\times\frac{2T}{R_2} \Rightarrow R_2 = 4R_1$$

$$\frac{m_1}{m_2} = \frac{V_1\rho}{V_2\rho} = \frac{4/3\pi R_1^3\rho}{4/3\pi R_2^3\rho} = \frac{R_1^3}{64R_1^3} = \frac{1}{64}$$

Example 41: Pressure inside two soap bubbles are and . Ratio between their volumes is

(*a*) 16 : 25 (*b*) 64 : 27

(*c*) 16 : 9 (*d*) 16 : 9

Sol. (*b*) Excess pressure $(\Delta P_1) = P_1 - P_0 = 1.06 - 1 = 0.06$ atm

$(\Delta P_2) = P_1 - P_0 = 1.08 - 1 = 0.08$ atm

$$\Delta P = \frac{4S}{R} \Rightarrow R = \frac{4S}{\Delta P} \Rightarrow \frac{R_1}{R_2} = \frac{\Delta P_2}{\Delta P_1}$$

$$\frac{R_1}{R_2} = \frac{0.08}{0.06} = \frac{8}{6} = \frac{4}{3}$$

$$\therefore \frac{V_1}{V_2} = \left(\frac{R_1}{R_2}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$$

Example 42: When a large bubble rises from the bottom of a lake to the surface, its radius triples. If atmospheric pressure is equal to that of column of water height H, then the depth of the lake (h) is

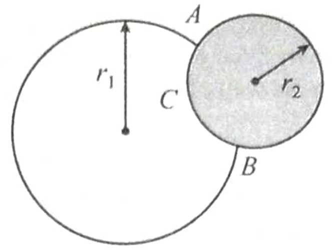
(*a*) 28 H (*b*) 25 H

(*c*) 27 H (*d*) 26 H

Sol. (*d*) According to the relation, $P_1V_1 = P_2V_2$

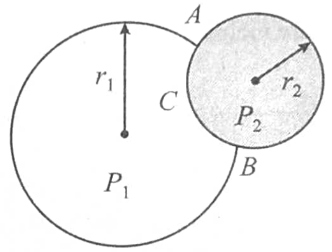
$(H + h)r^3 = 27Hr^3 \therefore h = 26$ H

Example 43: Two soap bubbles of radii and are attached as shown. Find the radius of curvature of the common film



Sol. Let the radius of curvature of surface ACB be R.

$$P_2 - P_1 = \frac{4T}{R}$$



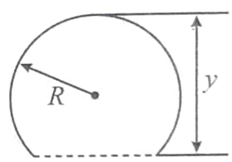
$\left[\frac{4T}{R}\right.$ because there are two surfaces in the wall ACB]

$$\therefore \frac{4T}{r_2} - \frac{4T}{r_1} = \frac{4T}{R} \Rightarrow R = \frac{r_1r_2}{r_1 - r_2}$$

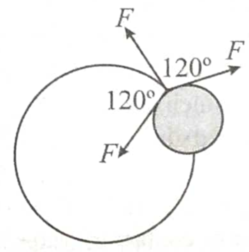
Example 44: (*i*) In the last question find the angle between the tangents drawn to the bubble surfaces at point A.

(*ii*) In the above question assume that $r_1 = r_2 = r$. What is the shape of the common interface ACB ? Find length AB in this case.

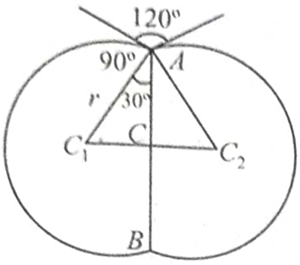
(*iii*) With $r_1 = r_2 = r$ the common wall bursts and the two bubbles form a single bubble find the radius of this new bubble. It is given that volume of a truncated sphere of radius and height y is $\frac{\pi}{3}y^2(3R - y)$ [see figure]



Sol. (*i*) Consider a particle on the common meeting line of the three surface. It experience three forces of same magnitude (surface tension is same for all three films) and is in equilibrium. This is possible only if the three forces are at 120° to each other



(ii) When $r_1 = r_2 = r$ the pressure on two sides of the common wall is same. The common wall remains flat in shape of a disc. AB is diameter of the disc.



$$AC = r\cos30^\circ = \frac{\sqrt{3}r}{2} \Rightarrow AB = 2AC = \sqrt{3}r$$

(iii) In above figure $CC_1 = r\sin30^\circ = \dfrac{r}{2}$

$$\therefore y = r + \frac{r}{2} = \frac{3r}{2}$$

$\therefore$ Volume of each bubble $= \dfrac{\pi}{3}\left(\dfrac{3r}{2}\right)^2\left(3r - \dfrac{3r}{2}\right) = \dfrac{9\pi r^3}{8}$

If radius of new bubble $= R$, then $\dfrac{4}{3}\pi r^3 = 2 \times \dfrac{9\pi r^3}{8}$

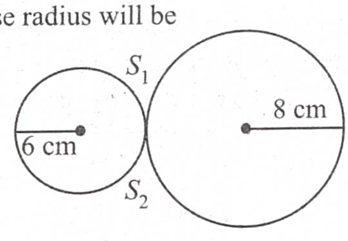
$$R = \frac{3r}{2(2)^{1/3}}$$

Concept Application

36. The pressure at the bottom of a tank containing a liquid does not depends on

(a) Acceleration due to gravity
(b) Height of the liquie column
(c) Area of the bottom surface
(d) Nature of the liquid

37. Two soap bubbles of radii r_1 and r_2 equals to 6 cm and 8 cm are touching each other over a common surface S_1S_2 whose radius will be

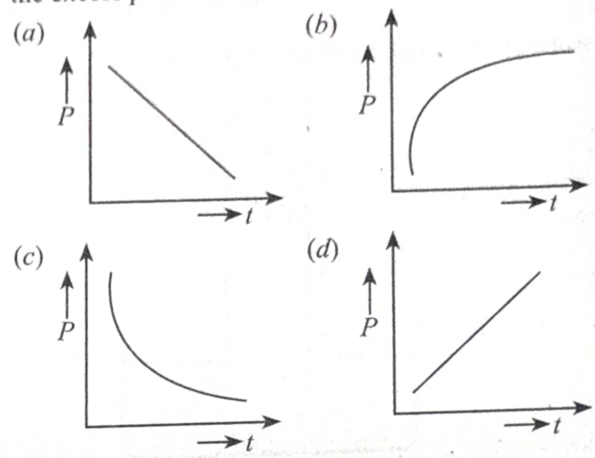


(a) 24 cm (b) 22 cm
(c) 12 cm (d) 10 cm

38. If excess of pressure inside a soap bubble of radius is balanced by that due to column of oil of (specific gravity high. What is the surface tension of soap bubble?

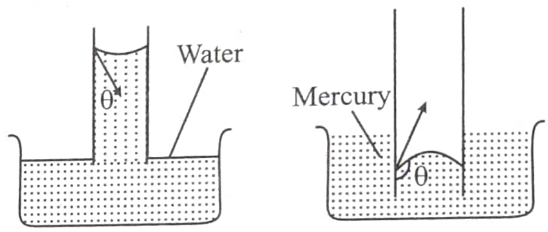
(a) 12×10^2 N/m (b) 6×10^{-2} N/m
(c) 8.12×10^{-2} N/m (d) 7.84×10^{-2} N/m

39. The soap bubble formed at the end of the tube is blown very slowly with time. Which graph is best suited for the excess pressure inside the bubble with time.



CAPILLARITY

A glass tube of very fine bore throughout the length of the tube is called capillary tube. If the capillary tube is dipped in water, the water wets the inner side of the tube and rises in it [shown in figure (a)]. If the same capillary tube is dipped in the mercury, then the mercury is depressed [shown in figure (b)]. The phenomenon of rise or fall of liquids in a capillary tube is called capillarity.



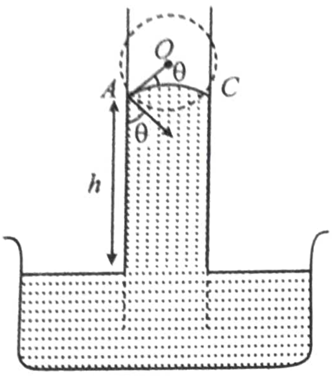
(a) (b)

PRACTICAL APPLICATIONS OF CAPILLARITY

1. The oil in a lamp rises in the wick by capillary action.
2. The tip of nib of a pen is split up, to make a narrow capillary so that the ink rises upto the tin or nib continuously.
3. Sap and water rise upto the top of the leaves of the tree by capillary action.
4. If one end of the towel dips into a bucket of water and the other end hangs over the bucket the towel soon becomes wet throughout due to capillary action.
5. Ink is absorbed by the blotter due to capillary action.
6. Sandy soil is more dry than clay. It is because the capillaries between sand particles are not so fine as to draw the water up by capillaries.
7. The moisture rises in the capillaries of soil to the surface, where it evaporates. To preserve the moisture in the soil, capillaries must be broken up. This is done by ploughing and leveling the fields.
8. Bricks are porous and behave like capillaries.

CAPILLARY RISE (ASCENT FORMULA)

Let p be the pressure on the concave side of the meniscus and p_a be the pressure on the convex side of the meniscus. The excess pressure $(p - p_a)$ is given by $(p - p_a) = \frac{2T}{R}$.



Where R is the radius of the meniscus. Due to this excess pressure, the liquid will rise in the capillary tube till it becomes equal to the hydrostatic pressure $h\rho g$. Thus in equilibrium state.

Excess pressure = Hydrostatic pressure or $\frac{2T}{R} = h\rho g$.

Let θ be the angle of contact and r be the radius of the capillary tube shown in figure.

From $\Delta OAC = \frac{OC}{OA} = \cos\theta$ or $R = \frac{r}{\cos\theta}$

$$\Rightarrow \quad h = \frac{2T\cos\theta}{r\rho g}$$

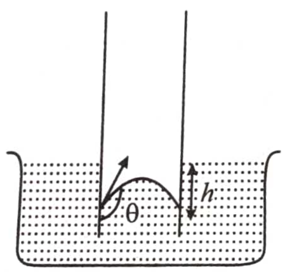
The expression is called ASCENT Formula.

Discussion

(*i*) For liquids which wet the glass tube or capillary tube, angle of contact $\theta < 90°$. Hence $\cos\theta$ = positive $\Rightarrow h$ = positive.

It means that these liquids rise in the capillary tube. Hence, the liquids which wet capillary tube rise in the capillary tube. For example. Water milk, kerosene oil, Petrol etc.

(*ii*) For liquids which do not wet the glass tube or capillary tube, angle of contact $\theta > 90°$. Hence, $\cos\theta$ = negative $\Rightarrow$ h = negative. Hence, the liquids which do not wet capillary tube are depressed in the capillary tube. For example, mercury.

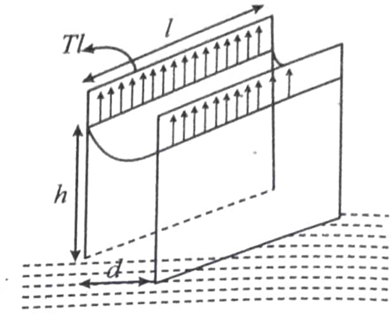


(*iii*) T, θ, ρ and g are constant and hence $h \propto 1/r$. Thus, the liquid rises more in a narrow tube and less in a wider tube. This is called Jurin's Law.

(*iv*) If two parallel plates with the spacing 'd' are placed in water, then height of rise

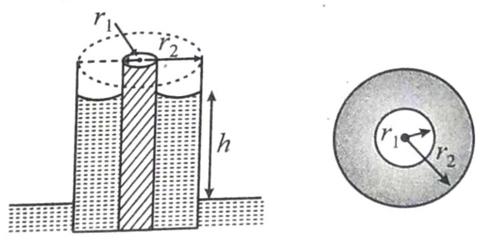
$$\Rightarrow 2Tl = \rho lhdg$$

$$\Rightarrow h = \frac{2T}{\rho dg} \quad (\theta = 0)$$



(*v*) If two concentric tube of radius 'r_1' and 'r_2' (inner one is solid) are placed in water reservoir, then height of rise

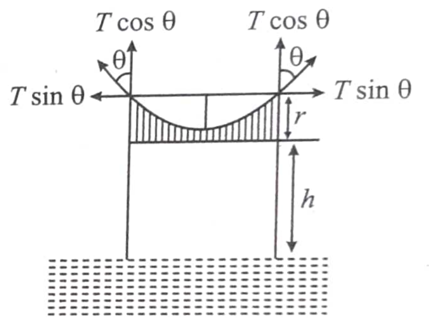
$$\Rightarrow T[2\pi r_1 + 2\pi r_2] = [\pi r_2^2 h - \pi r_1^2 h]\rho g$$



$$\Rightarrow h = \frac{2T}{(r_2 - r_1)\rho g} \quad (\theta = 0)$$

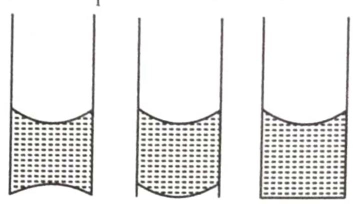
(*vi*) If weight of the liquid in the meniscus is to be consider:

$$T\cos\theta \times 2\pi r = \left[\pi r^2 h + \frac{1}{3}\pi r^2 \times r\right]\rho g$$



$$\left[h + \frac{r}{3}\right] = \frac{2T\cos\theta}{r\rho g}$$

(*vii*) When capillary tube (radius 'r') is in vertical position, the upper meniscus is concave and pressure due to surface tension is directed vertically upward and is given by $p_1 = 2T/R_1$ where R_1 = radius of curvature of upper meniscus.



(*a*) (*b*) (*c*)

The hydrostatic pressure $p_2 = h\rho g$ is always directed downwards.

If $p_1 > p_2$ i.e. resulting pressure is directed upward. For equilibrium, the pressure due to lower meniscus should be downward. This makes lower meniscus concave downward [figure (*a*)]. The radius of lower meniscus R_2 can be given by

$$\frac{2T}{R_2} = (p_1 - p_2).$$

If $p_1 < p_2$ i.e. resulting pressure is directed downward for equilibrium, the pressure due to lower meniscus should be upward. This makes lower meniscus convex upward [Figure (*b*)].

The radius of lower meniscus can be given by

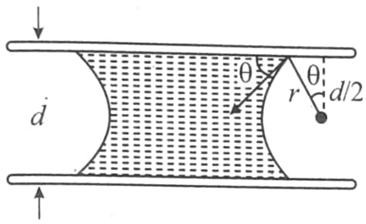
$$\frac{2T}{R_2} = (p_2 - p_1).$$

If $p_1 = p_2$, then is no resulting pressure, then

$$p_1 - p_2 = \frac{2T}{R_2} = 0$$

or $R_2 = \infty$ i.e. lower surface will be flat [Figure (*c*)].

(*viii*) **Liquid between two plates:** When a small drop of water is placed between two glass plates put face to face, it forms a thin film which is concave outward along its boundary. Let 'R' and 'r' be the radii of curvature of the enclosed film in two perpendicular directions.



Hence, the pressure inside the film is less than the atmospheric pressure outside it by an amount p given by $p = T\left(\frac{1}{r} + \frac{1}{r = \infty}\right)$ and we have, $p = \frac{T}{r}$.

If d be the distance between the two plates and θ the angle of contact for water and glass, then from the figure

$$\cos\theta = \frac{\frac{d}{2}}{r} \text{ or } \frac{1}{r} = \frac{2\cos\theta}{d}.$$

Substituting for $\frac{1}{r}$ in, we get $p = \frac{2T}{d}\cos\theta$.

θ can be taken zero water and glass, i.e. $\cos\theta = 1$. Thus, the upper plate is pressure downward by the atmospheric pressure minus $\frac{2T}{d}$. Hence, the resultant downward pressure acting on the upper plate is $\frac{2T}{d}$.

If A be the arc of the plate wetted by the film, the resultant force F pressing the upper plate downward is given by F = resultant pressure × area = $\frac{2TA}{d}$. For very nearly plane surface, d will be very small and hence the pressing force F very large. Therefore, it will be difficult to separate the two plates normally.

CAPILLARY RISE IN A TUBE OF INSUFFICIENT LENGTH

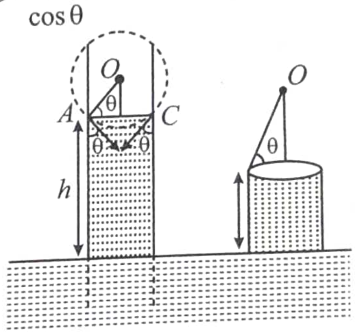
We know, then height through which a liquid rises in the capillary tube of radius r is given by

$$\therefore \quad h = \frac{2T}{R\rho g} \text{ or } h\,R = \frac{2T}{\rho g} = \text{constant}$$

When the capillary tube is cut and its length is less than h (i.e. h') then the liquid rises upto the top of the tube and spreads in such a way that the radius (R') of the liquid meniscus increases and it becomes more flat so that $hR = h'R'$ = constant. Hence the liquid does not overflow.

If $h' < h$ then $R' > R$

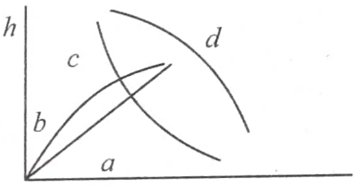
$$\text{or} \quad \frac{r}{\cos\theta'} > \frac{r}{\cos\theta}$$



$$\Rightarrow \quad \cos\theta' < \cos\theta \Rightarrow \theta' > \theta$$

Train Your Brain

Example 45: Which of the following graphs may represent the relation between the capillary rise h and the radius r of the capillary?



(*a*) a (*b*) b (*c*) c (*d*) d

Sol. (*c*) $h = \frac{2T\cos\theta}{r\rho g}$; $h \propto \frac{1}{r}$

Example 46: A 20 cm long capillary tube is dipped in water. The water rises up to 8 cm. If the entire arrangement is put in a freely falling elevator, the length of water column in the capillary tube will be:

(*a*) 8 cm (*b*) 6 cm (*c*) 10 cm (*d*) 20 cm

Sol. (*d*) In the freely falling elevator. The water column in capillary tube will be 20 cm full length of capillary.

Example 47: A barometer contains two uniform capillaries of radii 1.44×10^{-3} m and 7.2×10^{-4} m. If the height of the liquid in the tube is 0.2 m more than that in the wide tube, calculate the true pressure difference. Density of liquid = 10^3 kg/m^3, surface tension = 72×10^{-3} N/m and g = 9.8 m/s^2.

Sol. Let the pressure in the wide and narrow capillaries of radii r_1 and r_2 respectively be P_1 and P_2. then pressure just below the meniscus in the wide and narrow tubes respectively are:

$$\left(P_1 - \frac{2T}{r_1}\right) \text{ and } \left(P_2 - \frac{2T}{r_2}\right) \qquad \left[\text{Excess pressure} = \frac{2T}{r}\right]$$

Difference in these pressure

$$= \left(P_1 - \frac{2T}{r_1}\right) - \left(P_2 - \frac{2T}{r_2}\right) = h\rho g$$

$\therefore$ True pressure difference

$$= P_1 - P_2 = h\rho g + 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= 0.2 \times 10^3 \times 9.8 + 2 \times 72 \times 10^{-3}$$

$$\left[\frac{1}{1.44\times10^{-3}} - \frac{1}{7.2\times10^{-4}}\right]$$

$$= 1.86 \times 10^3 = 1860 \text{ N/m}^2.$$

Example 48: A glass tube of circular cross-section is closed at one end. This end is weighted and the tube floats vertically in water, heavy end down. How far below the water surface is the end of the tube? Give outer radius of the tube 0.14 cm, mass of weighted tube 0.2 gm, surface tension of water 73 dyne/cm and g = 980 cm/sec^2.

Sol. Let l be the length of the tube inside water. The forces acting on the tube are:

$\rho = 10^3 \text{kg/m}^3 = 1 \text{ gm/cm}^3$

(*i*) Upthrust of water acting upward $= \pi r^2 l \times 1 \times 980$

$$= \frac{22}{7} \times (0.14)^2 \times l \times 980 = 60.368\, l \text{ dyne.}$$

(*ii*) Weight of the system acting downward = mg

$= 0.2 \times 980 = 196$ dyne.

(*iii*) Force of surface tension acting downward $= 2\pi r T$

$$= 2 \times \frac{22}{7} \times 0.14 \times 73 = 64.24 \text{ dyne.}$$

Since, the tube is in equilibrium, the upward force is balanced by the downward forces.

i.e. $60.368\, l = 196 + 64.24 = 260.24$

$$\therefore \quad l = \frac{260.24}{60.368} = 4.31 \text{ cm.}$$

Example 49: If a 5 cm long capillary tube with 0.1 mm internal diameter open at both ends is slightly dipped in water having surface tension 75 dyne cm^{-1}, state whether (*i*) water will rise half way in the capillary. (*ii*) water will rise up to the upper end of capillary (*iii*) what will overflow out of the upper end of capillary. Explain your answer.

Sol. Given that surface tension of water, $T = 75$ dyne/cm

Radius $r = \frac{0.1}{2}$ mm $= 0.05$ mm $= 0.005$ cm

density $\rho = 1$ gm/cm^3, angle of contact $\theta = 0°$

Let h be the height to which water rise in the capillary tube. Then

$$h = \frac{2T\cos\theta}{r\rho g} = \frac{2\times75\times\cos0°}{0.005\times1\times981} \text{ cm} = 30.58 \text{ cm}$$

But length of capillary tube $h' = 5$ cm

(*i*) Because $h > \frac{h'}{2}$ therefore the first possibility does not exist.

(*ii*) Because the tube is of insufficient length therefore the water will rise upto the upper end of the tube.

(*iii*) The water will not overflow out of the upper end of the capillary. It will rise only upto the upper end of the capillary.

The liquid meniscus will adjust its radius of curvature R' in such a way that

$R'h' = Rh$

$$\left[\because hR = \frac{2T}{\rho g} = \text{constant}\right]$$

where R is the radius of curvature that the liquid meniscus would possess if the capillary tube were of sufficient length.

$$\therefore\ R' = \frac{Rh}{h'} = \frac{rh}{h'} \quad \left[\because R = \frac{r}{\cos\theta} = \frac{r}{\cos 0°} = r\right]$$

$$= \frac{0.005\times30.58}{5} = 0.0306 \text{ cm.}$$

Example 50: Angle of contact of liquid with a solid depends on

(*a*) Solid only

(*b*) Liquid only

(*c*) Both solid and liquid only

(*d*) Orientation of the solid surface in liquid

Sol. (*c*) Because force of adhesion and cohesion considered.

Concept Application

40. Due to capillary action, a liquid will rise in a tube, if the angle of contact is

(*a*) Acute (*b*) Obtuse (*c*) 90° (*d*) Zero

41. Two capillaries made of same material but of different radii are dipped in a liquid. The rise of liquid in one capillary is 4 cm and that in second tube is 5 cm. The ratio of their radii is

(*a*) 1/2 (*b*) 2/3

(*c*) 1/4 (*d*) 5/4

42. Two capillary tubes of same diameter are put vertically one each in two liquids whose relative densities are 0.8 and 0.6 and surface tensions are 70 and 40 dyne/cm respectively. The ratio of heights of liquids in the tubes h_1/h_2 is

(*a*) 5/4 (*b*) 2/3

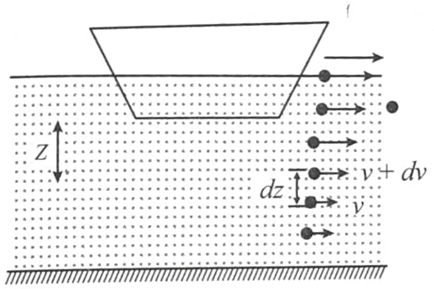
(*c*) 21/16 (*d*) 16/21

43. When a capillary tube is dipped in a liquid, the capillary rise is h_1, when the inner surface is coated with wax, the capillary rise is h_2 then
(a) $h_1 = h_2$ (b) $h_1 < h_2$
(c) $h_1 > h_2$ (d) None of theses

VISCOSITY

When a layer of a fluid slips or tend to slip on another layer in contact, the two layers exert tangential forces on each other. The directions are such that the relative motion between the layers is opposed, this property of a fluid to oppose relative motion between its layers is called viscosity. The forces between the layers opposing relative motion between them are known as the forces of viscosity. Thus, viscosity may be thought of as the internal friction of a fluid in motion.

If a solid surface is kept in contact with a fluid and is moved, forces of viscosity appear between the solid surface and the fluid layer in contact. The fluid in contact is dragged with the solid. If the viscosity is sufficient, the layer moves with the solid and there is no relative slipping. When a boat moves slowly on the water of a calm river, the water in contact with the boat is dragged with it, whereas the water in contact with the bed of the river remains at rest. Velocities of different layers are different. Let v be the velocity of the layer at a distance z from the bed and $(v + dv)$ be the the velocity at a distance $(z + dz)$ [Figure].



Thus, the velocity differs by dv in going through a distance dz perpendicular to it. The quantity $\frac{dv}{dz}$ is called the **velocity gradient**.

FORCE OF VISCOSITY BETWEEN TWO LAYERS

The force of viscosity between two layers of a fluid is proportional to the velocity gradient in the direction perpendicular to the layers. Also the force is proportional to the area of the layer.

Thus, if F is the force exerted by a layer of area A on a layer in contact.

$$F \propto A \text{ and } F \propto \frac{dv}{dz} \text{ or } F = -\eta A \frac{dv}{dz}$$

The negative sign is included as the force is frictional in nature and opposes relative motion. The constant of proportionality η is called the coefficient of viscosity.

The SI unit of viscosity is $N\text{-}s/m^2$. CGS unit is dyne-s/cm^2 is in common use and is called a poise in honour of the French Scientist Poiseuille.

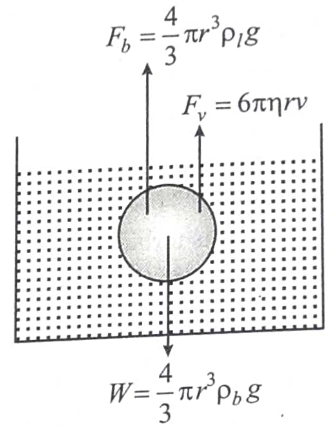
We have, 1 poise = 0.1 N–s/m^2

STOKE'S LAW AND TERMINAL VELOCITY

Whenever a spherical body of radius r moves in a fluid of viscosity η with velocity v, it experience a viscous drag F which is given by

$$\boxed{F = 6\pi\eta r v}$$

The viscous force on a ball moving through a fluid is proportional to its velocity. When a solid is dropped in a fluid, the forces acting on it area



(a) Weight W acting vertically downward,
(b) The viscous force F_V acting vertically upward and
(c) The buoyancy force F_B acting vertically upward.

The weight W and the buoyancy F_B are constant but the force F_V is proportional to the velocity v, initially, the velocity is zero and hence the viscous force F is zero and the solid is accelerated due to the net force $W\text{-}F_B$. Because of the acceleration, the velocity increases. Accordingly, the viscous force also increases. At a certain instant the viscous force becomes equal to $(W\text{-}F_B)$, the net force then becomes zero and the solid falls with constant velocity. The constant velocity is known as the terminal velocity.

Consider a spherical body falling through a liquid. Suppose the density of the body = ρ, density of the liquid = σ, radius of the sphere = r and the terminal velocity = v_T. The viscous force is

$$F = 6\pi\eta r v_T$$

the weight $W = \frac{4}{3}\pi r^3 \rho g$

and the buoyancy force $F_B = \frac{4}{3}\pi r^3 \sigma g$

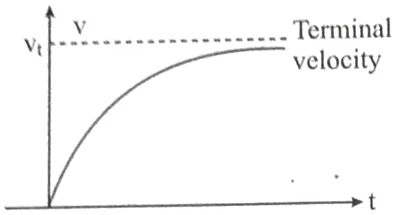
We have $6\pi\eta r v_T = W - F_B = \frac{4}{3}\pi r^3 \rho g - \frac{4}{3}\pi r^3 \sigma g$

or $v_T = \frac{2r^2(\rho - \sigma)g}{9\eta}$.

Effect of temperature on viscosity:

1. For liquid $\eta \propto \frac{1}{\sqrt{T}}$ and $\eta = 0$ at boiling point.

2. For gas $\eta \propto \frac{1}{\sqrt{T}}$



Reynold's Number-

$$R = \rho\frac{vd}{\eta}$$

where, d = diameter of tube

v = velocity of liquid

h = viscosity of fluid

r = density of liquid

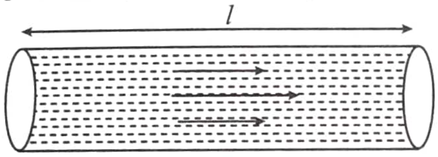
1) $R < 2000$, liquid flow is laminar

2) $R > 3000$, liquid flow is turbulent

3) $2000 < R < 3000$, liquid flow is transient.

POISEUILLE'S FORMULA

Poiseuille studied the stream-line flow of liquid in capillary tubes. He found that if a pressure difference (P) is maintained across the two ends of a capillary tube of length l and radius r, then the volume of liquid coming out of the tube per second is



(*i*) Directly proportional to the pressure difference (P).

(*ii*) Directly proportional to the fourth power of radius (r) of the capillary tube

(*iii*) Inversely proportional to the coefficient of viscosity (η) of the liquid.

(*iv*) Inversely proportional to the length (l) of the capillary tube.

i.e. $V \propto \frac{Pr^4}{\eta l}$ or $V = \frac{KPr^4}{\eta l}$

$\therefore V = \frac{\pi Pr^4}{8\eta l}$

[Where $K = \frac{\pi}{8}$ is the constant of proportionality]

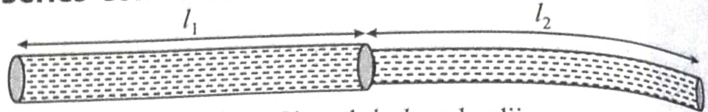
This is known as Poiseuille's equation.

This equation also can be written as.

$V = \frac{P}{R}$ where $V = \frac{\pi P_1 r_1^4}{8\eta l_1}$

R is called as liquid resistance.

Series Combination of Tubes



(*i*) When two tubes of length l_1, l_2 and radii r_1 r_2 are connected in series across a pressure difference P,

Then $P = P_1 + P_2$ ...(*i*)

Where P_1 and P_2 are the pressure difference across the first and second tube respectively

(*ii*) The volume of liquid flowing through both the tubes i.e. rate of flow of liquid is same.

Therefore $V = V_1 = V_2$; $V = \frac{\pi P_1 r_1^4}{8\eta l_1} = \frac{\pi P_2 r_2^4}{8\eta l_2}$ ...(*ii*)

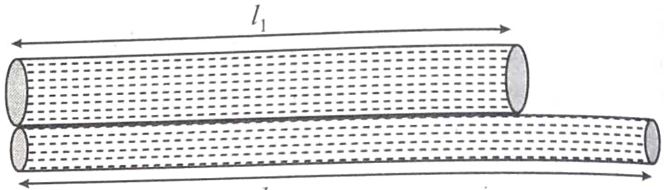
Substituting the value of P_1 and P_2 from equation (*ii*) to equation (*i*) we get, $P = P + P_2 = V\left[\frac{8\eta l_1}{\pi r_1^4} + \frac{8\eta l_2}{\pi r_2^4}\right]$

$$\therefore V = \frac{P}{\left[\frac{8\eta l_1}{\pi r_1^4} + \frac{8\eta l_2}{\pi r_2^4}\right]} = \frac{P}{R_1 + R_2} = \frac{P}{R_{eff}}$$

Where R_1 and R_2 are the liquid resistance in tubes

(*iii*) Effective liquid resistance in series combination $R_{eq} = R_1 + R$

Parallel Combination of Tubes



(*i*) $P = P_1 = P_2$

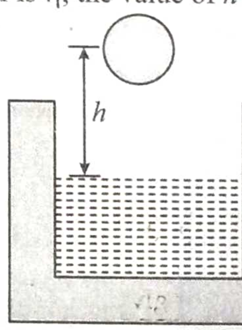
(*ii*) $V = V_1 + V_2 = \frac{P\pi r_1^4}{8\eta l_1} + \frac{P\pi r_2^4}{8\eta l_2}$

(*iii*) Effective liquid resistance in parallel combinatio

$$\therefore V = P\left[\frac{1}{R_1} + \frac{1}{R_2}\right] = \frac{P}{R_{eff}} \text{ or } = P\left[\frac{\pi r_1^4}{8\eta l_1} + \frac{\pi r_2^4}{8\eta l_2}\right]$$

Train Your Brain

Example 51: A ball of radius r and density ρ falls freely under gravity through a distance h before entering water. Velocity of ball does not change even on entering water. If viscosity of water is η, the value of h is given by:



(a) $\frac{2}{9}r^2\left(\frac{1-\rho}{\eta}\right)g$ (b) $\frac{2}{81}r^2\left(\frac{\rho-1}{\eta}\right)g$

(c) $\frac{2}{81}r^4\left(\frac{\rho-1}{\eta}\right)^2 g$ (d) $\frac{2}{9}r^4\left(\frac{\rho-1}{\eta}\right)^2 g$

Sol. (c) Refer to figure given in question

Velocity of ball does not change on entering water

Velocity attained $v = \sqrt{2gh} = VT$.

$\therefore \quad \sqrt{2gh} = \frac{2}{9}\frac{r^2(\rho-1)g}{\eta}$; $\sigma = 1$ for water

$\Rightarrow \quad 2gh = \frac{4}{81}r^4\left[\frac{(\rho-1)}{\eta}\right]g^2$; $h = \frac{2}{81}r^4\left[\frac{(\rho-1)}{\eta}\right]^2 g$.

Example 52: Calculate the Reynolds number for blood flowing at 30 cm/s through an aorta of radius 1.0 cm. Assume that the blood has a viscosity of 4mPas and a density of 1060 kg/m^3. What is the nature of flow?

Sol. Here, $v = 30$ cm/s $= 3 \times 10^{-1}$ m/s, $r = 1060$ kg/m^3,

$d = 2 \times 1.0$ cm $= 20 \times 10^{-2}$ m

$\eta = 4$ mPas $= 4 \times 10^{-3}$ Pas $= 4 \times 10^{-3}$ Ns/m^2

Thus, $R_e = \frac{v\rho d}{\eta}$

or $v = \frac{(3\times10^{-1}\text{m/s})(1060\text{ kg/m}^3)(2\times10^{-3}\text{m})}{(4\times10^{-3}\text{ Ns/m}^3)} = 1590$

Since, R_e is less than 2000, this flow is laminar.

Example 53: An engineer wants to have the same flow rate of water and light machine oil from the pipes of the same length and with the same pressure head. What should be ratio of the radii of the two pipes? Given that viscosity of water = 0.01 poise and that of light machine oil = 11 poise.

Sol. Let L and R be the length and radius of the pipe. Let Δp be the pressure difference between the two ends of the pipe. If η is viscosity of the water, then the rate of flow of the water through the pipe,

$$Q = \frac{\pi\Delta pR^4}{8\eta L} = \frac{\pi\Delta pR^4}{8\times0.01\times L} \quad ...(i)$$

Example 54: Suppose that the radius of the pipe through which light machine oil flows has to be taken as R', so as to keep the rate of flow same in the two cases. If η' is viscosity of the light machine oil, then the rate of flow of the oil through the pipe,

Sol. $Q = \frac{\pi\Delta pR'^4}{8\eta' L}$ or $Q = \frac{\pi\Delta pR'^4}{8\times11\times L}$ ...(ii)

From Eqs. (i) and (ii), we have

$$\frac{\pi\Delta pR'^4}{8\times11\times L} = \frac{\pi\Delta pR^4}{8\times0.01\times L}$$

or $\frac{R'}{R} = \left(\frac{11}{0.01}\right)^{1/4} = (1100)^{1/4} = 5.76$

Concept Application

44. A spherical body falling through a viscous liquid of infinite extent ultimately attains a constant value. when
(a) Upthrust + weight = viscous drag
(b) Weight + viscous drag = upthrust
(c) Viscous drag + upthrust = weights
(d) Viscous drag + upthrust > weight

45. If the terminal speed of a sphere of gold (density $=19.5 \times 10^3$ kg/m^3 is 0.5 *m/s* in a viscous liquid (density = 3 kg/m^3). Find the terminal velocity of a sphere of silver (density = 10.5×10^3 kg/m^3) of the same size in the same liquid
(a) 0.30 m/s (b) 0.20 m/s
(c) 0.51 m/s (d) 0.22 m/s

46. A layer of glycerin of thickness 3 mm is present between a large surface area and a surface area of 0.2 m^2, with what force the small surface is to be pulled, so that it can move with a velocity of 5 m/s? ($\eta = 0.09$ kgm^{-1} s^{-1}).
(a) 10 N (b) 30 N
(c) 40 N (d) 50 N

Short Notes

Properties of Fluid

Density $= \frac{\text{mass}}{\text{volume}}$ (kg m^{-3})

Specific weight $= \frac{\text{weight}}{\text{volume}} = \rho g$ (kg m^{-2} s^{-2})

Relative density $= \frac{\text{density of given liquid}}{\text{density of pure water at 4°C}}$ (Unitless)

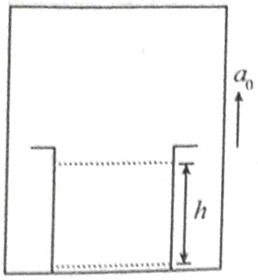
Pressure $= \frac{\text{normal force}}{\text{area}} = \frac{\text{thrust}}{\text{Area}}$ (Nm^{-2})

Variation of pressure with depth *h*

$$\Delta P = h\rho g$$

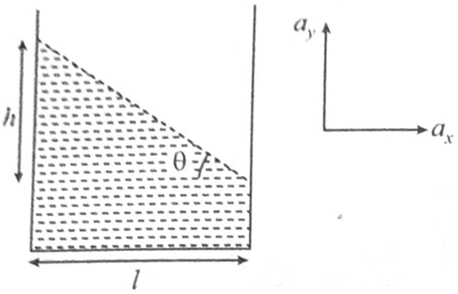
Pressure in case of accelerating fluid

Liquid placed in elevator: When elevator accelerates upward with acceleration a_0 then pressure in the fluid, at depth h may be given by, $P = h\rho[g + a_0]$



Free surface of liquid in accelerating container

Container accelerating in horizontal as well as in vertical direction.



$\tan\theta = \dfrac{h}{l} = \dfrac{a_x}{g+a_y}$ (goint up i.e. $a_y = +$)

$\tan\theta = \dfrac{h}{l} = \dfrac{a_x}{g-a_y}$ (going down i.e. $a_y = -$)

Buoyant force = weight of displaced liquid

$= \rho_l \cdot v_l \cdot g$

ρ_l = density of liquid, v_l = volume of displaced liquid

Steady and Unsteady Flow : The fluid characteristics like velocity, pressure and density at a point do not change with time is steady flow, whereas if any change in fluid characteristics then the flow is unsteady.

Streamline Flow : All the particles passing through a given point follow the same path and hence a unique line of flow. This line or path is called a streamline.

Laminar and Turbulent Flow : In Laminar flow fluid particles move along well-defined streamlines which are straight and parallel, whereas the motion of fluid particles is random.

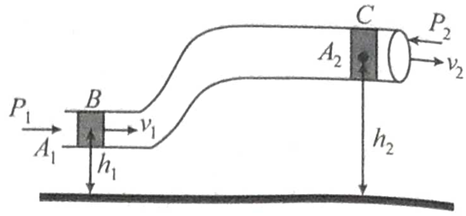
Compressible and Incompressible Flow : In compressible flow the density is not constant for the fluid whereas in incompressible flow the density of the fluid remains constant throughout.

Rotational and Irrotational Flow : The fluid particles while flowing along path-lines also rotate about their own axis, In irrotational flow particles do not rotate about their axis.

Equation of continuity $A_1v_1 = A_2v_2$ (Based on conservation of mass.)

Bernoulli's theorem : $\boxed{P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}}$

Based on the conservation of energy.



$$P_1 + \rho g h_1 + \frac{\rho v_1^2}{2} = P_2 + \rho g h_2 + \frac{\rho v_2^2}{2}$$

Kinetic Energy

Kinetic energy per unit volume

$$= \frac{\text{Kinetic Energy}}{\text{volume}} = \frac{1}{2}\frac{m}{V}v^2 = \frac{1}{2}\rho v^2$$

Potential Energy

Potential energy per unit volume

$$= \frac{\text{Potential Energy}}{\text{volume}} = \frac{m}{V}gh = \rho g h$$

Pressure Energy

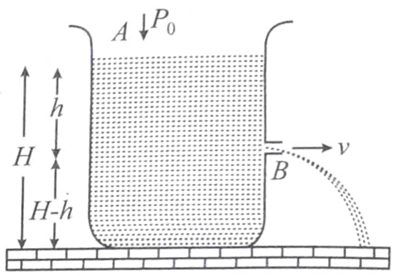
Pressure energy per unit volume $= \dfrac{\text{Pressure energy}}{\text{volume}} = P$

Rate of flow

Volume of water flowing per second $Q = A_1v_1 = A_2v_2$

Velocity of efflux $V = \sqrt{2gh}$

Horizontal range $R = 2\sqrt{h(H-h)}$



Surface Tension

$$\text{Surface Tension }(T) = \frac{\text{Total Force on either of imaginary line }(F)}{\text{Length of Line }(l)}$$

Surface Energy (U) = Surface Tension (T) × Exposed Area (A)

For liquid drop or water bubble $A = 4\pi r^2$

For soap bubble $A = 8\pi r^2$

Splitting of bigger drop into smaller droplets $R = n^{1/3} r$

Where, R = Radius of bigger drop

r = Radius of smaller drops

Work done = Change in surface energy $= 4\pi R^2 T(n^{1/3} - 1)$

Excess pressure $P_{ex} = P_{in} - P_{out}$

In liquid drop $P_{ex} = \dfrac{2T}{R}$

In soap bubble $P_{ex} = \dfrac{4T}{R}$

Angle of Contact (θ_C)

The angle enclosed between the tangent plane at the liquid surface and the tangent plane at the solid surface at the point of contact inside the liquid is defined as the angle of contact.

Angle of contact $\theta < 90^\circ \Rightarrow$ concave shape, Liquid rise up in capillary

Angle of contact $\theta > 90^\circ \Rightarrow$ convex shape, Liquid falls down in capillary

Angle of contact $\theta = 90^\circ \Rightarrow$ plane shape, Liquid neither rise nor falls

Capillary rise $h = \dfrac{2T\cos\theta}{r\rho g}$

When two soap bubbles are in contact then, radius of curvature of the common surface.

$r = \dfrac{r_1 r_2}{r_1 - r_2} (r_1 > r_2)$

When two soap bubbles are combined to form a new bubble then radius of new bubble.

$r = \sqrt{r_1^2 + r_2^2}$

Viscosity

Newton's law of viscosity $F = \eta A \dfrac{\Delta V_x}{\Delta y}$

SI Units of η : $\dfrac{N \times s}{m^2}$

CGS Units : dyne–s/cm^2 or poise (1 decapoise = 10 poise)

Poiseuille's formula $Q = \dfrac{dV}{dt} = \dfrac{\pi p r^4}{8\eta L}$

Viscous force $F_V = 6\pi\eta r v$

Terminal velocity $V_T = \dfrac{2}{9}\dfrac{r^2(\rho-\sigma)g}{\eta} \Rightarrow V_T \,\alpha\, r^2$

Reynolds number $R_e = \dfrac{\rho V d}{\eta}$

$R_e < 1000$, then laminar flow, If $R_e > 2000$, than turbulent flow

Solved Examples

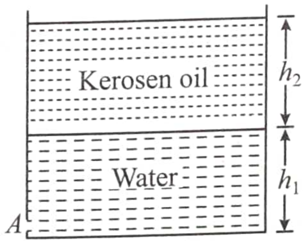
1. A wide vessel with a small hole in the bottom is filled with water and kerosene. Neglecting the viscosity, find the velocity of the water flow, if the thickness of the water layer is equal to $h_1 = 30$ cm and that of the kerosene layer to $h_2 = 20$ cm.

Sol. Since, the density of water is greater than that of kerosene oil, it will collect at the bottom. Now, pressure due to water level equals $h_1\rho_1 g$ and pressure due to kerosene oil level equals $h_2\rho_2 g$. So, net pressure becomes $h_1\rho_1 g + h_2\rho_2 g$.

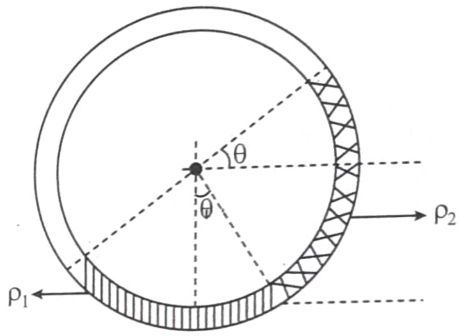
From Bernoulli's theorem, this pressure energy will be converted into kinetic energy while flowing through the hole A.

i.e. $h_1\rho_1 g + h_2\rho_2 g = \dfrac{1}{2}\rho_1 v^2$

Hence $v = \sqrt{2\left(h_1 + h_2 \dfrac{\rho_2}{\rho_1}\right)g} = 3 \text{ m/s}$

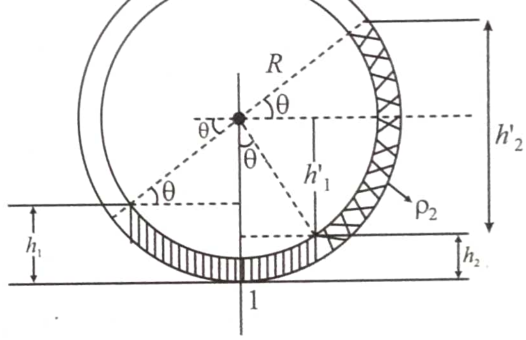


2. A circular tube of uniform cross-section is filled with two liquids of densities ρ_1 and ρ_2 such that half of each liquid occupies a quarter of volume of the tube. If the line joining the free surface of the liquids makes an angle θ with horizontal, find the value of θ.



Sol. Let us find the pressure at the lowest point 1. Since, the liquid of density ρ_2 and height of liquid column h'_2 is there, in right hand side of the point 1, we have.

$P_1 = \rho_1 g h_1$



Since two liquid columns of heights h_1 and h_2 and densities ρ_1 and ρ_2 are situated above the point 1, in left hand side, we have

$P_2 = \rho_1 g h_2 + \rho_2 g h'_2$

Equating P_1 and P_2 from equation (1) and (2), we have

$\rho_1 h_2 + \rho_2 h_2' = \rho_1 h_1$

Substituting $h_2' = R\sin\theta + R\cos\theta$

$h_2 = R(1-\cos\theta)$

and $h_1 = R(1-\sin\theta)$ in eqn. (3), we have

$\rho_1 R(1-\cos\theta) + \rho_2 R(\sin\theta + \cos\theta) = \rho_1 R(1-\sin\theta)$

This gives; $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \dfrac{\rho_1}{\rho_2}$

$\tan\theta = \dfrac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$

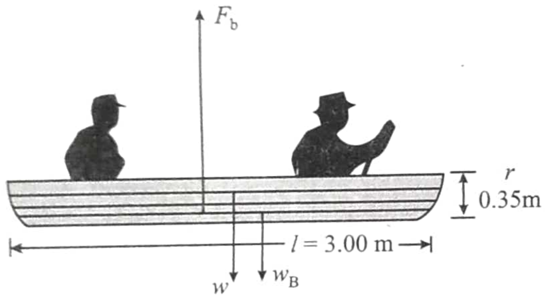
3. A wooden boat is in the shape of a half cylinder of length 3 m and radius 0.350 m. It weighs 1.00×10^3 N. What is the maximum weight it can hold without sinking? Density of water is 1.0×10^3 kg/m^3.

Sol. As the weight in the boat increases, it sinks lower in the water. It will carry maximum weight when its top is at the level of the water. Since the boat is in equilibrium,

$\Sigma F_{y=0}$

$F_b - w_B - w_P = 0$

$w_P = F_b - w_B$



where F_b is buoyant force; w_B is weight of boat and w_P is weight of passengers.

Volume of the displaced water is the volume of the boat

$$V = \frac{1}{2}\pi r^2 l$$

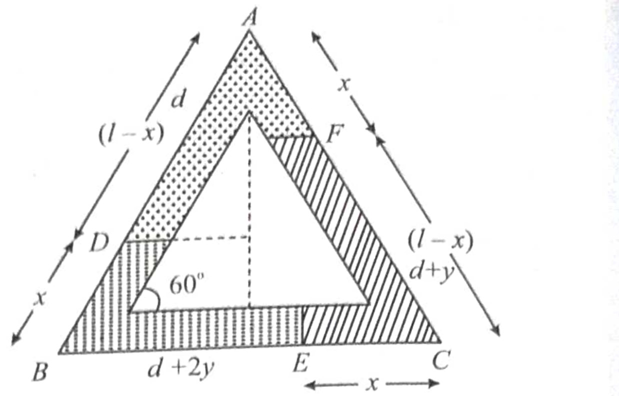
$$F_b = \rho g V = \rho g\left(\frac{1}{2}\pi r^2 l\right)$$

$$= (1.00\times10^3)(9.80)\left(\frac{\pi}{2}\right)(0.350)^2(3)$$

$$= 5.66\times10^3 \text{ N}$$

$$w_P = 5.66\times10^3 - 1.00\times10^3 = 4.66\times10^3$$

4. Figure shows a glass tube in the form of an equilateral triangle of uniform cross-section. It lies in the vertical plane, with base horizontal. The tube is filled with equal volumes of three immiscible liquids whose densities are in arithmetic progression. Determine the length as shown in Fig.



Sol. Let the densities of liquids in DAF, FCE and EBD are d, $d+y$ and $d+2y$ respectively. Pressure at a point inside a fluid depends on the vertical height of the liquid above that point. We will calculate pressure at from left arm and right arm, these values must be the same for equilibrium.

$P_E = dg(l-x)\sin 60^\circ + (d+2y)gx\sin 60^\circ$ [Left arm]

$P_E = dgx\sin 60^\circ + (d+y)g(l-x)\sin 60^\circ$ [Right arm]

Equating the two values of pressures,

$$dg(l-x)\frac{\sqrt{3}}{2} + \frac{x\sqrt{3}}{2}(d+2y)g$$

$$= \frac{x\sqrt{3}}{2}dg + (l-x)\frac{\sqrt{3}}{2}(d+y)g$$

$$(l-x)\sqrt{3}d + x\sqrt{3}(d+2y) = x\sqrt{3}d + (l-x)\sqrt{3}(d+y)$$

$$(l-x)\sqrt{3}d + x\sqrt{3}d + 2yx\sqrt{3}$$

$$= x\sqrt{3}d + (l-x)\sqrt{3}d + y(l-x)\sqrt{3}$$

$$2yx = y(l-x);\ 2x = l-x;\ x = \frac{l}{3}$$

5. The side wall of a wide vertical cylindrical vessel of height h = 75 cm has a narrow vertical slit running all the way down to the bottom of the vessel. The length of the slit is l = 50 cm and the width b = 1.0 mm. With the slit closed, the vessel is filled with water. Find the resultant force of reaction of the water flowing out of the vessel immediately after the slit is opened.

Sol. Consider an element of height dy at a distance y from the top. The velocity of the fluid coming out of the element is

$v = \sqrt{2gy}$

The force of reaction dF due to this is $dF = \rho dAv^2$, as in the previous problem,

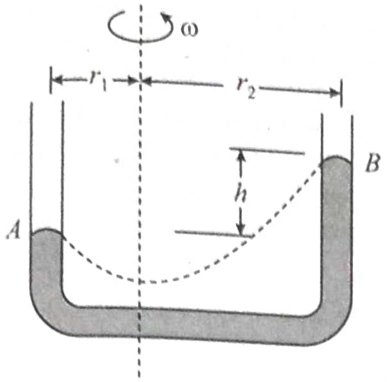
$= \rho(bdy)2gy$

Integrating $F = \rho gb\int_{h-l}^{h} 2y\,dy$

$$= \rho gb\left[h^2 - (h-l)^2\right] = \rho gbl(2h-l)$$

(The slit runs from a depth $h-l$ to a depth h from the top.)

6. A U-tube shown in Figure rotates with angular velocity ω about the vertical axis. What is the difference in fluid level h in terms of ω, the radii r_1 and the fluid density ρ.



Sol. Let y_{min} be the lowest point on the dotted parabola shown

$$y_1 = \frac{r_1^2\omega^2}{2g} + y_{min}$$

$$y_2 = \frac{r_2^2\omega^2}{2g} + y_{min}$$

$$h = y_2 - y_1 = \frac{(r_2^2 - r_1^2)\omega^2}{2g}$$

Alternatively, we may apply Bernoulli's equation between points A and B.

$$P_{atm.} + \frac{1}{2}\rho(r_1\omega)^2 = P_{atm.} + \frac{1}{2}\rho(r_2\omega)^2 + \rho gh$$

$$\text{or } h = \left(\frac{r_2^2 - r_1^2}{2g}\right)\omega^2$$

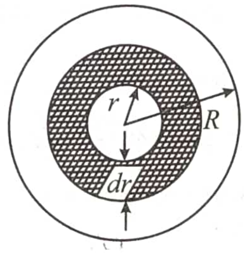
7. A tube of length l and radius R carries a steady flow of fluid whose density is ρ and viscosity η. The fluid flow velocity depends on the distance r from the axis of the tube as $v = v_0(1 - r^2/R^2)$. Find:
(a) The volume of the fluid flowing across the section of the tube per unit time;
(b) The kinetic energy of the fluid within the tube's volume;
(c) The friction force exerted on the tube by the fluid;
(d) The pressure difference at the ends of the tube.

Sol. (a) Let dV be the volume flowing per second through the cylindrical shell of thickness dr then,

$$dV = -(2\pi r dr)v_0\left(1 - \frac{r^2}{R^2}\right) = 2\pi v_0\left(r - \frac{r^3}{R^2}\right)dr$$

and the total volume,

$$V = 2\pi v_0\int_0^R\left(r - \frac{r^3}{R^2}\right)dr = 2\pi v_0\frac{R^2}{4} = \frac{\pi}{2}R^2 v_0$$



(b) Let dE be the kinetic energy, within the above cylindrical shell. Then

$$dE = \frac{1}{2}(dm)v^2 = \frac{1}{2}(2\pi r l dr\rho)v^2$$

$$= \frac{1}{2}(2\pi l\rho)r dr v_0^2\left(1 - \frac{r^2}{R^2}\right)^2 = \pi l\rho v_0^2\left[r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right]dr$$

Hence, total energy of the fluid,

$$E = \pi l\rho v_0^2\int_0^R\left(r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right)dr = \frac{\pi R^2\rho l v_0^2}{6}$$

(c) Here frictional force is the shearing force on the tube, exerted by the fluid, which is $= \eta S\frac{dv}{dt}$
Given,

So, $v = v_0\left(1 - \frac{r^2}{R^2}\right)$; $\frac{dv}{dr} = -2v_0\frac{r}{R^2}$

And at $r = R, \frac{dv}{dr} = -\frac{2v_0}{R}$

Then, viscous force is given by,

$$F = -\eta(2\pi Rl)\left(\frac{dv}{dr}\right)_{r=R}$$

$$= -2\pi R\eta l\left(-\frac{2v_0}{R}\right) = 4\pi\eta v_0 l$$

(d) Taking a cylindrical shell of thickness dr and r radius viscous force,

$$F = -\eta(2\pi rl)\frac{dv}{dr}$$

Let Δp be the pressure difference, then net force on the element $= \Delta p\pi r^2 + 2\pi\eta lr\frac{dv}{dr}$. But, since the flow is steady, $F_{net} = 0$

$$\text{or } \Delta p = \frac{-2\pi l\eta r\frac{dv}{dr}}{\pi r^2} = \frac{-2\pi l\eta r\left(-2v_0\frac{r}{R^2}\right)}{\pi r^2} = 4\eta v_0 l / R^2$$

8. A lead sphere is steadily sinking in glycerin whose viscosity is equal to $\eta = 13.9$P. What is the maximum diameter of the sphere at which the flow around that sphere still remains laminar? It is known that the transition to the turbulent flow corresponds to Reynolds number Re = 0.5. (Here the characteristic length is taken to be the sphere diameter).

Sol. We have $R = \frac{v\rho_0 d}{\eta}$ and is given by

$$6\pi\eta rv = \frac{4\pi}{3}r^2(\rho - \rho_0)g$$

($\rho =$ density of lead, $\rho_0 =$ density of glycerine.)

$$v = \frac{2}{9\eta}(\rho - \rho_0)gr^2 = \frac{1}{18\eta}(\rho - \rho_0)gd^2$$

Thus, $\frac{1}{2} = \frac{1}{18\eta^2}(\rho - \rho_o)g\rho_0 d^3$

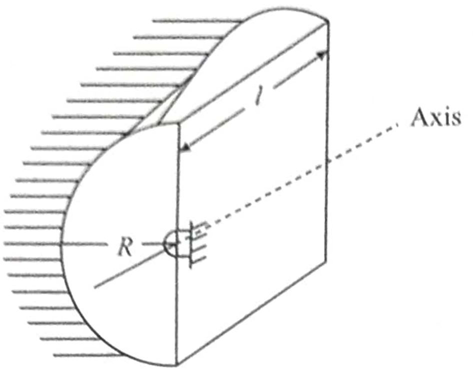
and $d = \left[9\eta^2/\rho_0(\rho - \rho_0)g\right]^{1/3} = 5\cdot 2$ mm on putting the values.

9. A hemi-cylindrical gate of radius R and length l is pivoted about its axis. Find the:

(a) Torque offered by the liquid about the axis.

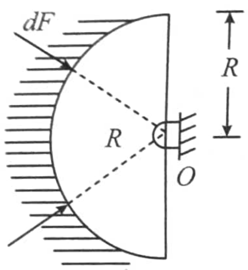
(b) Horizontal thrust.

(c) Vertical thrust of liquid on the gate.



Sol. (a) Since hydrostatic force acting at each element is normal to surface it passes through the centre (pivot O), it cannot produce a torque about O.

(b) The net horizontal thrust is



where $y_c = R$ and $A = (2R)(l)$

This gives $F_x(= F_b) = V\rho g$, where V = volume of liquid displaced

$$= \frac{\pi}{2} R^2 l$$

This gives $F_y = \frac{\pi \rho R^2 g l}{2}$

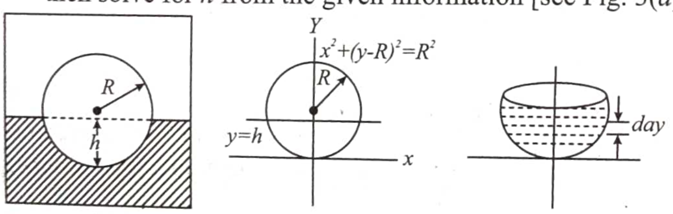
10. How far does a wooden (spherical) ball of specific gravity and radius 2 feet sink in water?

Sol. By "Archimedes' principle" we know that the ball will sink until it displaces a weight of water equal to the entire weight of the ball: (weight of ball) = (weight of displaced water

We know that

(weight of ball) = (volume) (specific gravity) (density of water) where the density of water w is measured in pounds per cubic foot.

We must first calculate the volume of the part of the sphere immersed in water when the sphere sinks to a depth of h, and then solve for h from the given information [see Fig. 3(a)].



(a) (b) (c)

Finding this volume is equivalent to finding the volume generated by rotating the area between the curves $x^2 + (y - R)^2$ and $y = h$, about the y-axis [see Fig. (b)].

We note that the general formula for a circle is $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) are the coordinates of its centre and r is the length of its radius. In this case the coordinates of the centre are $(0, R)$ and $r = R$.

Recall that

$$dV = \pi x^2 dy = \pi\left[R^2 - (y - R)^2\right] dy$$

$$= \pi\left(2Ry - y^2\right) dy$$

where dV is the differential cylindrical volume element, x being its radius and dy its height. Finally, we integrate dV between $y = 0$ and $y = h$.

$$y = \pi \int_0^h \left(2Ry - y^2\right) dy$$

$$= \pi\left[2R\frac{y^2}{2} - \frac{y^3}{3}\right]_0^h = \pi\left(Rh^2 - \frac{h^3}{3}\right) = \pi h^2\left(R - \frac{h}{3}\right)$$

Hence, from the equation for the weight of the ball:

(Weight of ball) = (volume of submerged portion) (density of water)

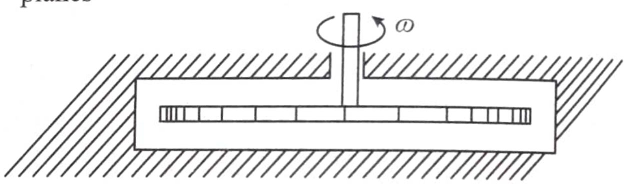
$$\left(\frac{4}{3}\pi R^3\right)(0.4) w = \pi h^2\left(R - \frac{h}{3}\right) w$$

$$\frac{4}{3}(2)^3 (0.4) = h^2\left(2 - \frac{h}{3}\right)$$

$$h^3 - 6h^2 + 12.8 = 0$$

We find by synthetic division that $h \approx 1.75$ feet.

11. A thin horizontal disc of radius $R = 10$ cm is located within a cylindrical cavity filled with oil whose viscosity $\eta = 0.08\,P$. The clearance between the disc and the horizontal planes



of the cavity is equal to $h = 1.0$ mm. Find the power developed by the viscous forces acting on the disc when it rotates with the angular velocity $\omega = 60$ rad/s. The end effects are to be neglected.

Sol. When the disc rotates the fluid in contact with, corotates but the fluid in contact with the walls of the cavity does not rotate. A velocity gradient is then set up leading to viscous forces.

At a distance r from the axis the linear velocity is ωr so there is a velocity gradient $\frac{\omega r}{h}$ both in the upper and lower clearance. The corresponding force on the element whose radial width is dr is

$$F = \eta 2\pi r dr \frac{\omega r}{h} \text{(from the formular } F = \eta A \frac{dv}{dx}\text{)}$$

The torque due to this force is

$$\tau = \eta 2\pi r dr \frac{\omega r}{h} r$$

and the net torque considering both the upper and lower clearance is

$$\tau_{net} = 2\int_0^R \eta 2\pi r^3 dr \frac{\omega}{h} = \pi R^4 \omega\eta / h$$

So power developed is

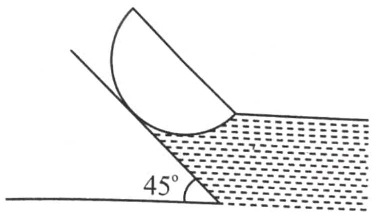
$P = \pi R^4 \omega^2 \eta / h = 9{\cdot}05$ W (on putting the values).

(As instructed end effects i.e. rotation of fluid in the clearance $r > R$ has been neglected.)

12. A solid semicylinder of uniform density ρ rests in equilibrium on a rough inclined plane with a liquid of density ρ on its right as shown in Fig.

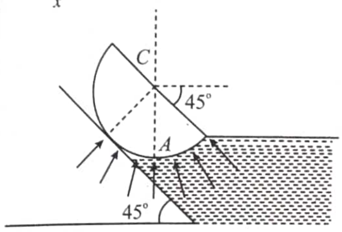
(*a*) Determine the minimum coefficient of friction to ensure equilibrium and

(*b*) The ratio ρ_1/ρ_2. Assume that flat surface of the cylinder is parallel to the inclined plane.

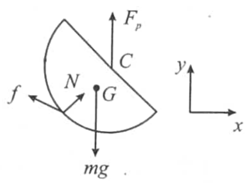


Sol. The hydrostatic pressure force acts radially inwards, towards the axis of the cylinder that passes through and is normal to plane of paper. The torque due to this force about C is zero. Secondly the submerged portion is symmetrical about the line CA. Therefore

$F_x = 0$



(*b*)



(*c*)

From the diagram of pressure force it can be deduced

(*i*) The torque of pressure force about C to zero

(*ii*) The resultant pressure force acts vertically (by symmetry), i.e., $F_x = 0$.

Calculation of y-component of the pressure force,

$F_y = \int PdA\sin\theta$

$$\text{and } = \int_{\pi/4}^{3\pi/4}\left[\rho g R\left(\sin\theta - \sin\frac{\pi}{4}\right)\right](LRd\theta)\sin\theta$$

$$= \rho g R^2 L \int_{\pi/4}^{3\pi/4}\left(\sin^2\theta - \frac{1}{\sqrt{2}}\sin\theta\right)d\theta$$

$$= \rho g R^2 L\left(\frac{\pi}{4} - \frac{1}{2}\right) \ldots \qquad \ldots(i)$$

Balancing force in x-direction.

$$\frac{f}{\sqrt{2}} = \frac{N}{\sqrt{2}}$$

or $f = N$

$$\mu_s \geq \frac{f}{N}$$

Hence the minimum coefficient of static friction.

In y-direction, $F_p + \frac{1}{\sqrt{2}}(f + N) = mg$

$$\left(\frac{\pi}{4} - \frac{1}{2}\right)LR^2\rho g + \sqrt{2}f = mg \text{ [using } (i)]$$

Balancing torque about C,

$$fR = mg\frac{4}{3\pi}R\sin\frac{\pi}{4}$$

$$f = \frac{4mg}{3\sqrt{2}\pi}$$

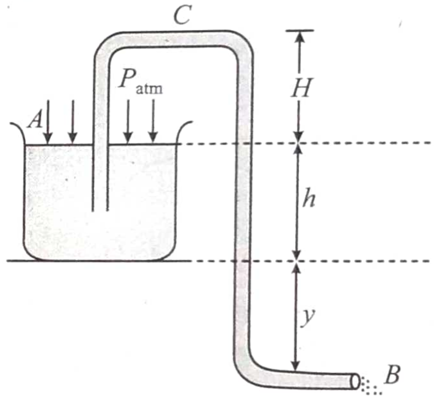
Putting in (2),

$$\text{or } \left(\frac{\pi}{4} - \frac{1}{2}\right)LR^2\rho g + \frac{4mg}{3\pi} = mg$$

$$\frac{m}{LR^2\rho} = \frac{3\pi(\pi - 2)}{4(3\pi - 4)}$$

The required ratio, $\frac{\rho_1}{\rho_2} = \frac{2m}{L\rho\pi R^2} = \frac{3(\pi - 2)}{4(3\pi - 4)}$

13. Figure shows a siphon, which is used to remove fluid from a container. The tube must initially be filled with fluid but then it operates until the fluid level reaches the submerged end of the tube.



(*a*) Derive an expression for the speed with which the water leaves the tube at B.

(b) What is the greatest value of 'h' for which the siphon will work? Assume uniform cross-section of the tube.

Sol. (*a*) From Bernoulli's theorem, energy at points A and B must be equal. Taking B as reference level,

$$E_A = P_{atm} + \rho g(h + y) + 0 \qquad \ldots(i)$$

We have ignored kinetic energy of fluid, assuming cross-section of vessel is large as compared to that of tube.

$$E_B = P_{atm.} + \frac{1}{2}\rho v^2 \qquad \ldots(ii)$$

On equating equation (*i*) and (*ii*), and solving for v, we obtain

$$v=\sqrt{2g(h+y)}$$

(*b*) Now we apply Bernoulli's theorem at the highest point of the tube and at level A.

$$E_A = P_{atm} + \rho g(h+y) \quad ...(iii)$$

$$E_C = P+\rho g(h+y+H)+\frac{1}{2}\rho v^2 \quad ...(iv)$$

where P denotes pressure at C, highest point of tube, and v is velocity of fluid at C. Since the fluid is incompressible, the continuity equation yields

$$v_A = v_B = v_C = \sqrt{2g(h+y)} \quad ...(v)$$

From equations (*iii*), (*iv*) and (*v*),

$$P_{atm}+\rho g(h+y) = P+\rho g(h+y+H)+\frac{1}{2}\rho[\sqrt{2g(h+y)}]^2$$

$$P = P_{atm} - \rho gH - \rho g(h+y)$$

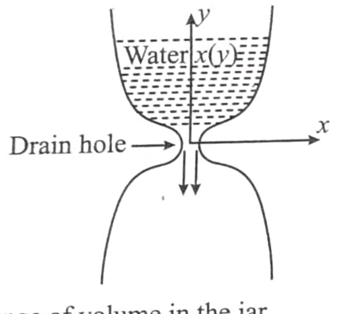
$$P = P_{atm} - \rho g[h+y+H] \quad ...(vi)$$

The pressure at the topmost point C must be non-negative, liquids cannot with stand vacuum.

Putting $P = 0$ in equation (*vi*), we obtain

$$H=\frac{P_{atm}}{\rho g(h+y)}$$

14. The ancient water clock clepsydra shown in the figure has such a shape that the water level descended at a constant rate at all times. If the water level falls by 4 cm every hour, determine the shape of the jar, i.e., specify x as a function of y. The drain hole diameter is 2 mm and can be assumed to be very small compared to x.



Sol. Rate of change of volume in the jar

= Rate of water flowing out through drain hole

$$\pi x^2 \frac{dy}{dt} = (vA)_{\text{drain hole}} \quad ...(i)$$

Now we use Bernoulli's equation to get velocity of water at hole.

or $\rho gy = \frac{1}{2}\rho v^2$ and $v=\sqrt{2gy}$; $A=\pi(1\times10^{-3})^2$

Putting the values in equation (*i*), we get

$$\pi x^2 \frac{4\times10^{-2}}{60\times60} = \sqrt{2g}\sqrt{y}\cdot\pi(1\times10^{-3})^2$$

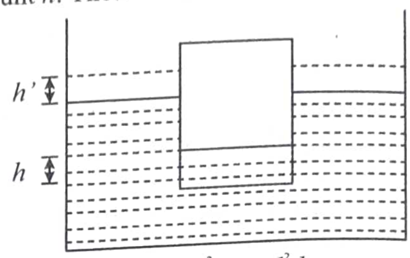
$$x^2 = 0.3984\sqrt{y}$$

$$x = 0.631y^{1/4}$$

15. A cubical block of wood has density $\rho_1 = 500$ kg/m^3 and side $l = 30$ cm. It is floating in a rectangular tank partially filled with water of density $\rho_2 = 1000$ kg/m^3 and having base area, $A = 45$ cm × 60 cm. Calculate work done to press the block slowly so that it is just immersed in water. ($g = 10$ m/s^2)

Sol. $\frac{\rho_1}{\rho_2}=\frac{1}{2}$ i.e., in equilibrium, block is half immersed in water.

Let h' be the increase in level when the block is pressed by an amount h. Then



$$(A-l^2)h' = l^2.h$$

Substituting the values, we get $h' = 0.5$ h. We have to immerse further 1/2.

Hence $\frac{1}{2} = h+h' = 1.5h$ or $h=\frac{1}{3}$

When the block is depressed by h, extra upthrust

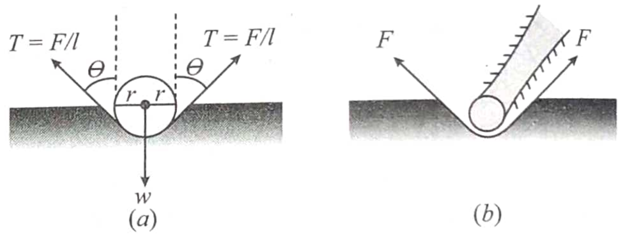
$$F=(h+h')\rho_2 l^2\cdot g = 1.5l^2\rho_2 gh$$

Work has to be done against this upthrust.

Hence $W=\int_0^{1/3} Fdh = \int_0^{1/3}(1.5l^2\rho_2 gh)dh$

Substituting the values, we get $W = 6.75$ J

16. The base of an insect's leg is approximately spherical in shape, with a radius of about 2.0×10^{-5} m. The 0.0030 gm mass of the insect is supported equally by the six legs. Estimate the angle θ for an insect on the surface of water. Assume the water temperature is 20°C. Coefficient of surface tension $T = 0.072$ N/m



Sol. Since the insect is in equilibrium, the upward surface tension force is equal to the effective pull of gravity downward on each leg:

$$2\pi rT\cos\theta \approx mg$$

where mg is one-sixth the weight of the insect (since it has six legs). Then

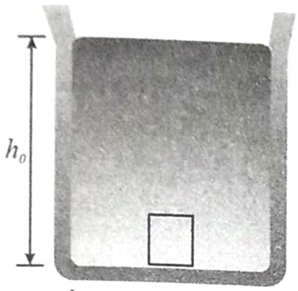
$$(6.28)(2.0\times10^{-5}\text{ m})(0.072\text{ N/m})\cos\theta$$

$$\approx\frac{1}{6}(3.0\times10^{-6}\text{ kg})(9.8\text{ m/s}^2)$$

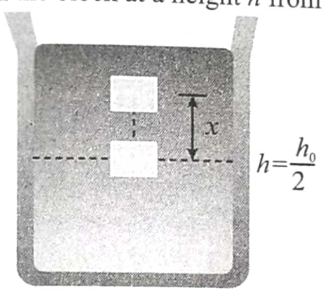
$$\cos\theta\approx\frac{0.49}{0.90}=0.54$$

So $\theta = 57°$, notice that if $\cos\theta$ were greater than 1, this would indicate that the surface tension would not be great enough to support the weight.

17. Figure shows a container having liquid of variable density. The density of liquid varies as $\rho = \rho_0\left(4 - \frac{3h}{h_0}\right)$ Here h0 and ρ_0 are constants and h is measured from bottom of the container. A solid block of small dimensions whose density is $\frac{5}{2}\rho_0$ and mass m is released from bottom of the tank. Prove that the block will execute simple harmonic motion. Find the frequency of oscillation.



Sol. Net force on the block at a height h from the bottom is



F_{net} = upthrust – weight (upwards)

$$= \left(\frac{m}{5/2\rho_0}\right)\rho_0\left(4 - \frac{3h}{h_0}\right)g - mg$$

$F_{net} = 0$ we get $h = \frac{h_0}{2}$ [For mean position]

So, $h = \frac{h_0}{2}$ is the equilibrium position of the block.

For $h > \frac{h_0}{2}$, weight > upthrust

i.e., net force is downwards and line change for $h < \frac{h_0}{2}$

weight < upthrust

i.e., net force is upwards

For upward displacement x from mean position, net downward force is

$$F = -\left[\left(\frac{m}{5/2\rho_0}\right)\rho_0\left\{4 - \frac{3(h+x)}{h_0}\right\}g - mg\right] \qquad \left(h = \frac{h_0}{2}\right)$$

$$F = -\frac{6mg}{5h_0}x \qquad ...(i)$$

(because at $h = \frac{h_0}{2}$ upthrust and weight are equal)

Since $F \propto -x$

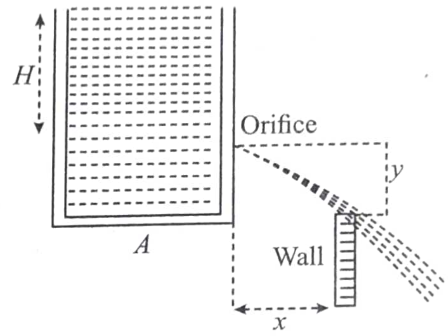
Oscillations are simple harmonic in nature.

Rewriting equation (i)

$$ma = -\frac{6mg}{5h_0}x \qquad (a \text{ acceleration of block})$$

or $a = -\frac{6g}{5h_0}x$; $f = \frac{1}{2\pi}\sqrt{\left|\frac{a}{x}\right|}$; $f = \frac{1}{2\pi}\sqrt{\frac{6g}{5h_0}}$

18. For the arrangement shown in figure, find the time interval after which the water jet ceases to cross the wall. Area of the tank is A and area of orifice is a.



Sol. First of all let us find velocity of efflux at time t

$$A\left(-\frac{dh}{dt}\right) = a\sqrt{2gh}$$

$$\int_H^h \frac{dh}{\sqrt{h}} = \int_0^t \frac{-a\sqrt{2g}}{A}dt$$

Solving this equation, we get

$$\sqrt{h} = \sqrt{H} - \frac{a}{A}\sqrt{\frac{g}{2}}t$$

$\therefore$ Velocity of efflux, $v = \sqrt{2gh}$

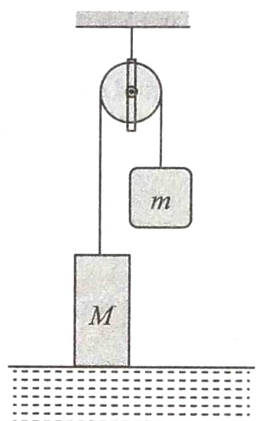
$$= \sqrt{2g}\left[\sqrt{H} - \frac{a}{A}\sqrt{\frac{g}{2}}t\right]$$

Now time to fall a height y is, $t = \sqrt{\frac{2y}{g}}$

Putting, $vt = x$ we get

$$t = \frac{A}{a}\sqrt{\frac{2}{g}}\left[\sqrt{H} - \frac{x}{\sqrt{4y}}\right]$$

19. The pulley mass system is as shown in the figure. Mass M is a uniform cylinder ($M > m$). String is inextensible and there is no friction. Find the acceleration of the cylinder as a function of length of cylinder inside the liquid. Find the velocity of system when cylinder is totally immersed. ρ_s and ρ_l are the densities of cylinder and the liquid respectively. Length of the cylinder is L.



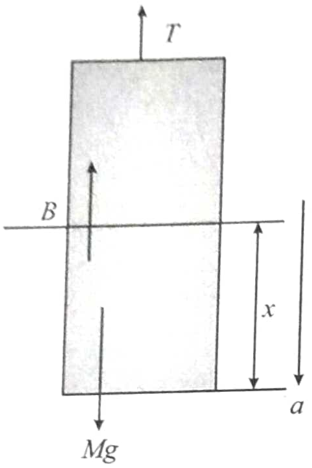
Sol. *FBD* of cylinder

$$B = \text{buoyant force} = \left(\frac{M}{L}\frac{x}{\rho_s}\right)\rho_l g$$

$$Mg - T - \left(\frac{M}{L}\frac{\rho_l}{\rho_s}\right)gx = Ma$$

For the mas 'm'

$$T - mg = ma$$



$$\Rightarrow \left(M - \frac{M}{L}\frac{\rho_l}{\rho_s}x - m\right)g = (m+M)a$$

$$a = \frac{\left(M - \frac{M}{L}\frac{\rho_l}{\rho_s}x - m\right)}{M+m}g$$

$$a = v\frac{dv}{dx}$$

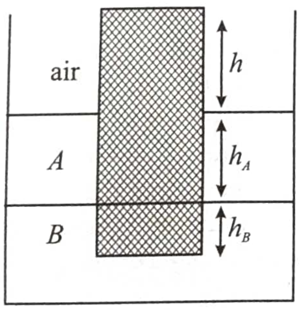
$$vdv = adx = \frac{g}{M+m}\left(M - \frac{M}{L}\frac{\rho_l}{\rho_s}x - m\right)dx$$

$$\text{or } \frac{v^2}{2} = \frac{g}{M+m}\left(Mx - \frac{M}{L}\frac{\rho_l}{\rho_s}\frac{x^2}{2} - mx\right) + c$$

at $x = 0$, $v = 0 \Rightarrow c = 0$

$$v = \sqrt{\frac{2g}{m+M}\left(ML - \frac{M\rho_l L}{2\rho_s} - mL\right)}$$

20. The uniform solid cylinder of density 0.8 g/cm^3 floats in equilibrium in a combination of two non-mixing liquids A and B with its axis vertical. The densities of the liquid A and B are 0.7 g/cm^3 and 1.2 g/cm^3, respectively. The height of liquid A is $h_A = 1.2$ cm. The length of the part of the cylinder immersed in liquid B is $h_B = 0.8$ cm.

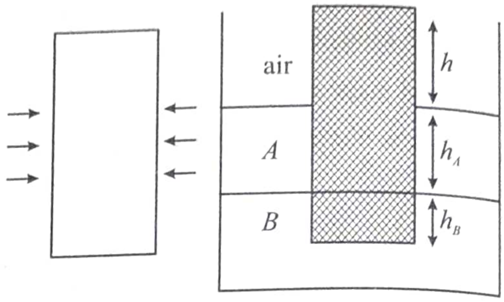


(*a*) Find the total force exerted by liquid on the cylinder.

(*b*) Find h, the length of the part of the cylinder in air.

(*c*) The cylinder is depressed in such a way that its top surface is just below the upper surface of liquid and is then released. Find the acceleration of the cylinder immediately after it is released.

Sol. (*a*)



Liquid is applying the hydrostatic force on cylinder from all the sides. So net force is zero.

(*b*) In equilibrium: Weight of cylinder = Net upthrust on the cylinder

Let s be the area of the cross-section of the cylinder, then

weight $= (s)(h + h_A + h_B)\rho_{\text{cylinder}} g$

and upthrust on the cylinder = upthrust due to liquid

+ upthrust due to liquid $B = sh_A\rho_A g + sh_B\rho_B g$

Equating these two

$$s(h + h_A + h_B)\rho_{\text{cylinder}} g = sh_A\rho_A g + sh_B\rho_B g$$

$$\text{or } (h + h_A + h_B)\rho_{\text{cylinder}} = h_A\rho_A + h_B\rho_B$$

Substituting $h_A = 1.02$ cm, $h_B = 0.8$ cm,

$\rho_A = 0.7\ \text{g/cm}^3$, $\rho_B = 1.02\ \text{g/cm}^3$

and $\rho_{\text{cylinder}} = 0.8\ \text{g/cm}^3$

in the above equation we get $h = 0.25$ cm

(*c*) Net upward force = extra upthrust = $sh\rho_B g$

$$\text{Net acceleration } a = \frac{\text{force}}{\text{mass of cylinder}}$$

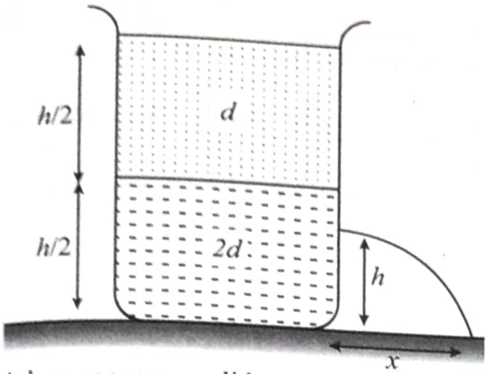
$$\text{or } a = \frac{sh\rho_B g}{s(h + h_A + h_B)\rho_{\text{cylinder}}}$$

$$a = \frac{h\rho_B g}{(h + h_A + h_B)\rho_{\text{cylinder}}}$$

substituting the values of h, h_A, h_B, ρ_B and ρ_{cylinder} we get,

$$a = \frac{g}{6} \text{ (upwards)}$$

21. A container of large uniform cross-sectional area A resting on a horizontal surface, holds two immiscible, non-viscous and incompressible liquids of densities d and $2d$, each of height $L/4$ as shown in figure. The lower density liquid is open to the atmosphere having pressure.



(*a*) A homogeneous solid cylinder of length $L(L < H/2)$, cross-sectional area $A/2$ is immersed such that if floats with its axis vertical at the liquid-liquid interface with length in the denser liquid. Determine:

(*i*) The density d of the solid,

(*ii*) The total pressure at the bottom of the container.

(*b*) The cylinder is removed and the original arrangement is restored. A tiny hole of area $S(S << A)$ is punched on the vertical side of the container at a height h $(h < H/2)$. Determine:

(*i*) The initial speed of efflux of the liquid at the hole,

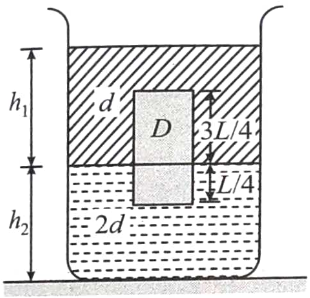
(*ii*) The horizontal distance travelled by the liquid initially, and

(*iii*) The height h_m at which the hole should be punched so that the liquid travels the maximum distance initially. Also calculate x_m.

(Neglect the air resistance in these calculations).

Sol. (*a*) (*i*) Considering vertical equilibrium of cylinder

Weight of cylinder = Upthrust due to upper liquid + upthrust due to lower liquid



Note that h_1 and $h_2 \neq \frac{H}{2}$

$$\therefore (A/5)(L)D \cdot g = (A/5)(3L/4)(d)g + (A/5)(L/4)(2d)(g)$$

$$\therefore D = \left(\frac{3}{4}\right)d + \left(\frac{1}{4}\right)(2d) = \frac{5}{4}d$$

(*ii*) Considering vertical equilibrium of two liquids and the cylinder.

$(P - P_0)A$ = weight of two liquids + weight of cylinder

$$\therefore P = P_0 + \frac{\text{weight of cylinder}}{A} \qquad ...(i)$$

Now Weight of cylinder

$$= (A/5)(L)(D)g = (A/5Lg)(5/4d) = \frac{ALdg}{4}$$

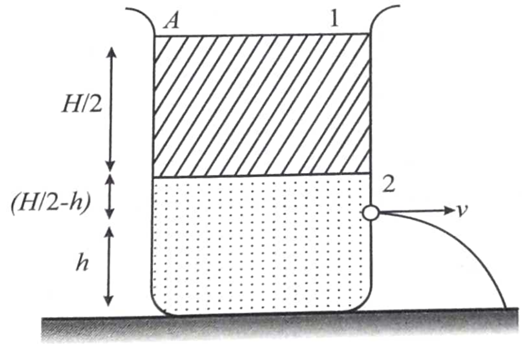
Weight of two liquids

$$\text{Weight of upper liquid} = \left\{\frac{H}{2}Adg\right\}$$

$$\text{and Weight of lower liquid} = \frac{H}{2}A(2d)g = HAgd$$

$$\text{Total weight of two liquids} = \frac{3}{2}HAdg$$

From equation (*i*), pressure at the bottom of the container will be



$$\text{or} \quad P = P_0 + \frac{dg(6H + L)}{4}$$

(*b*) (*i*) Applying Bernoulli's theorem at 1 and 2

$$P_0 + dg(H/2) + 2dg(H/2 - h) = P_0 + \frac{1}{2}(2d)v^2$$

Here v is velocity of efflux at 2 .

Solving this, we get

$$v = \sqrt{(3H - 4h)g/2}$$

(*ii*) Time taken to reach the liquid to the bottom will be

$$t = \sqrt{2h/g}$$

Horizontal distance x travelled by the liquid is

$$x = vt = \left(\sqrt{(3H - 4h)g/2)}\right)\left(\sqrt{\frac{2h}{g}}\right)$$

$$x = \sqrt{h(3H - 4h)}$$

(*iii*) For x to be maximum $-\frac{dx}{dh} = 0$

$$\text{or} \quad \frac{1}{2\sqrt{h(3H - 4h)}}(3H - 8h) = 0 \Rightarrow h = \frac{3H}{8}$$

Therefore, x will be maximum at $h = 3H/8$

The maximum value of x will be

$$x_m = \sqrt{\left(\frac{3H}{8}\right)\left[3H - 4\left(\frac{3H}{8}\right)\right]} \quad \text{or} \quad x_m = \frac{3}{4}H$$

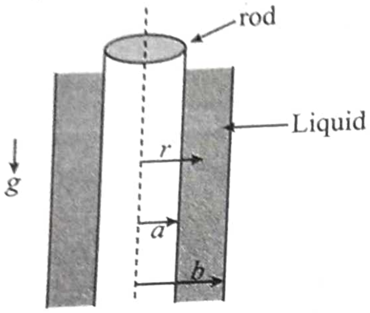
22. A vertical steel rod has radius a. The rod has a coat of a liquid film on it. The liquid slides under gravity. It was found that the speed of liquid layer at radius r is given by

$$v = \frac{\rho g b^2}{2\eta}\ln\left(\frac{r}{a}\right) - \frac{\rho g}{4\eta}\left(r^2 - a^2\right)$$

Where b is the outer radius of liquid film, η is coefficient of viscosity and ρ is density of the liquid.

(*i*) Calculate the force on unit length of the rod due to the viscous liquid?

(*ii*) Set up the integral to calculate the volume flow rate of the liquid down the rod. [you may not evaluate the integral]



Sol. (*i*) Velocity of fluid layer at radius r is

$v = \frac{\rho g b^2}{2\eta}\ln\left(\frac{r}{a}\right) - \frac{\rho g}{4\eta}\left(r^2 - a^2\right)$ Velocity gradient along radial direction is $\frac{dv}{dr} = \frac{\rho g b^2}{2\eta}\frac{1}{r} - \frac{\rho g}{4\eta}2r$.

At $r = a$ (i.e., on the surface of the rod)

$$\frac{dv}{dr} = \frac{\rho g b^2}{2\eta a} - \frac{\rho g a}{2\eta} = \frac{\rho g a}{2\eta}\left[\frac{b^2}{a^2} - 1\right]$$

Area of unit length of the rod's surface

$$A = (2\pi a)(1) = 2\pi a$$

(*ii*) Volume flow rate is

$$\therefore F_v = \eta A\frac{dv}{dr} = \pi\rho g a^2\left[\frac{b^2}{a^2} - 1\right]$$

$$Q = \int_{r=a}^{r=b} v \cdot 2\pi r dr$$

$$= \frac{\pi\rho g b^2}{\eta}\int_a^b r\ln\left(\frac{r}{a}\right)dr - \frac{\pi\rho g}{2\eta}\int_a^b\left(r^2 - a^2\right)r dr$$

You can evaluate the above integral if you want some practice in mathematics. To help you it is being given that

$$\int x\ln x = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

23. A spherical ball of radius r and density d is dropped from rest in a viscous fluid having density ρ and coefficient of viscosity η.

(*a*) Calculate the power (P_1) of gravitational force acting on the ball at a time after it is dropped.

(*b*) Calculate the rate of heat generation (P_2) due to rubbing of fluid molecules with the ball, at time t after it is dropped.

(*c*) How do P_1 and P_2 change if the radius of the ball were doubled?

(*d*) Find P_1 and P_2 when both become equal.

Sol. $m\frac{dv}{dt} = mg - F_B - F_v$

$$\frac{4}{3}\pi r^3 d\frac{dv}{dt} = \frac{4}{3}\pi r^3 d\cdot g - \frac{4}{3}\pi r^3\rho\cdot g - 6\pi\eta r v$$

$$\frac{dv}{dt} = g - \frac{\rho}{d}g - \frac{9}{2}\frac{\eta v}{dr^2}$$

$$= g\left(1 - \frac{\rho}{d}\right) - \frac{9\eta}{2\cdot dr^2}\cdot v = a - bv \qquad \text{[Let]}$$

$$\Rightarrow \int_0^v \frac{dv}{a - bv} = \int_0^t dt \Rightarrow [\ln(a - bv)]_0^v = -bt$$

$$\Rightarrow \ln\left(\frac{a - bv}{a}\right) = -bt \Rightarrow 1 - \frac{b}{a}v = e^{-bt}$$

$$\Rightarrow v = \frac{a}{b}\left(1 - e^{-bt}\right) \Rightarrow v = \frac{2g(d-\rho)r^2}{9\eta}\left(1 - e^{-\frac{9\eta}{2\cdot d\cdot r^2}t}\right)$$

(*a*) $P_1 = m\cdot g\cdot v = \frac{4}{3}\pi r^3 d\cdot g\cdot\frac{2g(d-\rho)r^2}{9\eta}\left[1 - e^{-\frac{9\eta}{2dr^2}t}\right]$

$$= \frac{8\pi}{27}\frac{d(d-\rho)g^2\cdot r^5}{\eta}\left[1 - e^{-\frac{9\eta t}{2dr^2}}\right]$$

(*b*) $P_2 = F_v\cdot v = 6\pi\eta r v\cdot v = 6\pi\eta r v^2$

$$= 6\pi\eta r\cdot\frac{4g^2(d-\rho)^2\cdot r^4}{81\eta^2}\left[1 - e^{-\frac{9\eta t}{2dr^2}}\right]^2$$

$$= \frac{8\pi}{27}\frac{(d-\rho)^2 g^2\cdot r^5}{\eta}\left[1 - e^{-\frac{9\eta t}{2dr^2}}\right]^2$$

(*c*) $P_1 \propto r^5$ and $P_2 \propto r^5$

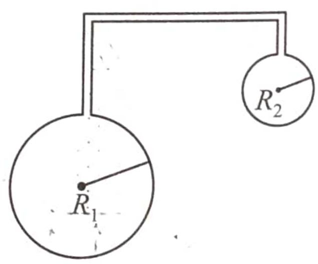
$\therefore$ On doubling the radius, both become 32 times

(*d*) $P_1 = P_2$, when

$e^{\frac{9\eta t}{2dr^2}} \to 0$ [i.e., at $t = \infty$]

$$\therefore P_1 = P_2 = \frac{8\pi}{27}\frac{(d-\rho)^2 g^2 r^5}{\eta}$$

24. Two soap bubbles of radius R_1 and $R_2 (< R_2)$ are joined by a straw. Air flows from one bubble to another and a single bubble of radius R_3 remains.



(a) From which bubble does the air flow out ?

(b) Assuming no temperature change and atmospheric pressure to be P_0, find the surface tension of the soap solution.

Sol. (a) As excess pressure for a soap bubble is $(4T/r)$ and external pressure p_0,

$p_i = p_0 + (4T/r)$

Pressure inside the smaller bubble is higher, Hence, air flows from smaller bubble to the larger one.

(b) The larger bubble grows in size till the entire air of smaller bubble is transferred into it.

$$p_1 = \left[p_0 + \frac{4T}{R_1}\right], p_2 = \left[p_0 + \frac{4T}{R_2}\right] \text{ and}$$

$$p_3 = \left[p_0 + \frac{4T}{R_3}\right] \quad ...(i)$$

$$\text{and } V_1 = \frac{4}{3}\pi R_1^3,\ V_2 = \frac{4}{3}\pi R_2^3 \text{ and } V_3 = \frac{4}{3}\pi R_3^3 \quad ...(ii)$$

Now, as mass is conserved, $n_1 + n_2 = n_3$

i.e., $\frac{P_1V_1}{RT_1} + \frac{P_2V_2}{RT_2} = \frac{P_3V_3}{RT_3}$

$$\left[\because PV = nRT \text{ as, i.e., } n = \frac{PV}{RT}\right]$$

As temperature is constant, i.e., the above expression reduces to

$P_1V_1 + P_2V_2 = P_3V_3$

Which, in the light of Eqn. (i) and (ii) becomes

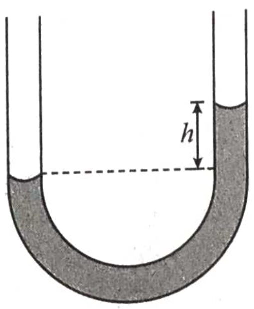
$$\left[P_0 + \frac{4T}{R_1}\right]\left[\frac{4}{3}\pi R_1^3\right] + \left[P_0 + \frac{4T}{R_2}\right]\left[\frac{4}{3}\pi R_2{}^3\right]$$

$$= \left[P_0 + \frac{4T}{R_3}\right]\left[\frac{4}{3}\pi R_3^3\right] \text{b}$$

i.e, $4T\left(R_1{}^2 + R_2{}^2 - R_3{}^2\right) = P_0\left(R_3^3 - R_1^3 - R_2^3\right)$

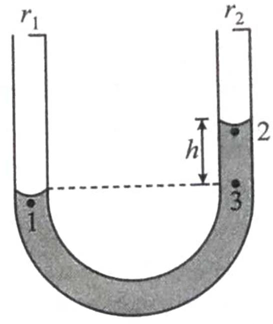
i.e, $T = \dfrac{P_0\left(R_3{}^3 - R_1^3 - R_2{}^3\right)}{4\left(R_1{}^2 + R_2{}^2 - R_3{}^2\right)}$

25. The radii of two columns in a U tube are r_1 and r_2 $(r_1 > r_2)$. A liquid of density ρ is filled in it. The contact angle of the liquid with the tube wall is θ. If the surface tension of the liquid is T then plot the graph of the level difference (h) of the liquid in the two arms versus contact angle. Plot the graph for angle θ changing from 0° to 90°. Assume the curved surface of meniscus to be part of a sphere.



Sol. $P_0 - \dfrac{2T}{R_1} = P_2 + \rho gh$

$$\Rightarrow P_0 - \frac{2T}{R_1} = P_0 - \frac{2T}{R_2} + \rho gh$$

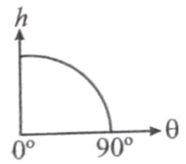


R_1 & R_2 are radii of curvature of the meniscus and it is known that $R = \dfrac{r}{\cos\theta}$

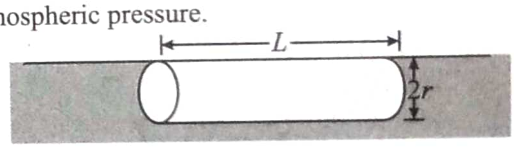
$$\rho gh = 2T\left[\frac{1}{R_2} - \frac{1}{R_1}\right]$$

$$\Rightarrow h = \frac{2T\cos\theta}{\rho g}\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$$

$$h = \frac{2T}{\rho g}\left(\frac{r_1 - r_2}{r_1 r_2}\right)\cos\theta \qquad \therefore h \propto \cos\theta$$

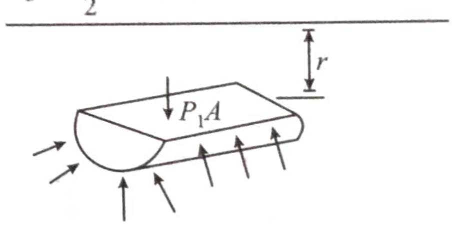


26. A cylindrical wooden log of length L and radius r is floating in water (density = ρ) while remaining completely submerged as shown in figure. Calculate the force of water pressure on the lower half of the cylinder. Exclude contribution due to atmospheric pressure.



Sol. Assume that half the cylinder (cut along its length) is held in lite position shown Buoyancy force on half cylinder

$$F_B = \frac{\pi r^2}{2} L\rho g(\uparrow)$$



Force on the flat surface at depth r is

$F_1 = P_1A\,(\downarrow) = (\rho gr)\,(2rL)\,(\downarrow) = 2\,\rho gLr^2\,(\uparrow)$

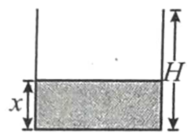
Upward force on curved part due to water pressure is given by

$$F = F_B + F_1 = \frac{\pi r^2 L\rho g}{2} + 2\rho g L r^2 = \rho g L r^2\left(\frac{\pi}{2}+2\right)(\uparrow)$$

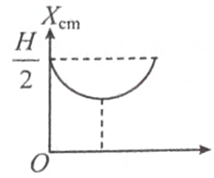
27. A cylindrical container has mass M and height H. The centre of mass of the empty container is at height $\frac{H}{2}$ from the base. A liquid, when completely filled in the container, has mass $\frac{M}{2}$. This liquid is poured in the empty container.

(*i*) How does die centre of mass of the system (container + liquid) move as the height (x) of liquid column changes from zero to H? Explain your answer qualitatively. Draw a graph showing the variation of height of centre of mass of the system (x_{cm}) with x.

(*ii*) Find the height of liquid column x for which the centre of mass is at its lowest position.



Sol. (*i*) COM of empty container is at $x_{cm} = \frac{H}{2}$ When liquid is poured, the mass of the system increases on the lower side and hence, the COM moves down. But the final position of COM when container is completely filled is once again $x_{cm} = \frac{H}{2}$. It means the COM begins to rise after falling through a certain distance.



(*ii*) Mass for unit length of liquid column is $\frac{M}{2H}$

Height of COM when liquid column has length x is

$$x_{cm} = \frac{M\frac{H}{2} + \frac{M}{2H}x\cdot\frac{x}{2}}{M + \frac{Mx}{2H}} = \frac{2H^2 + x^2}{2[2H + x]} \quad ...(i)$$

x_{cm} is minimum when $\frac{dx_{cm}}{dx} = 0$

i.e.. $(2H + x)(2x) - (2H^2 + x^2) = 0$

$x^2 + 4Hx - 2H^2 = 0$

$$\therefore x = \frac{-4H \pm \sqrt{16H^2 + 8H^2}}{2}$$

x cannot be negative $\therefore x_0 = (\sqrt{6} - 2)H$

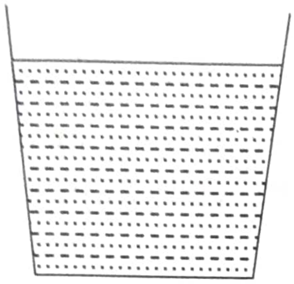
When $x = x_0$, the COM of the system is at lowest position

To find the minimum value of x_{cm} you can put $x = (\sqrt{6} - 2)H$ in equation (*i*)

Exercise-1 (Topicwise)

MEASUREMENT AND CALCULATION OF PRESSURE

1. A liquid of mass 1 kg is filled in a flask as shown in figure. The force exerted by the flask on the liquid is ($g = 10$ m/s^2) [Neglect atmospheric pressure]



(a) 10 N (b) Greater than 10 N
(c) Less than 10 N (d) Zero

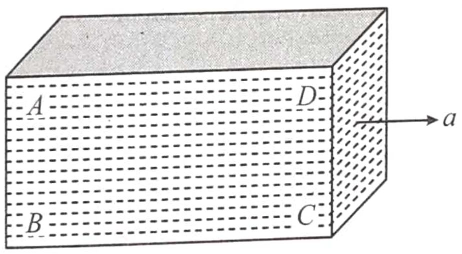
2. Density of ice is ρ water is σ. What will be the decrease in volume when a mass M of ice melts

(a) $\frac{M}{\sigma-\rho}$ (b) $\frac{\sigma-\rho}{M}$
(c) $M\left[\frac{1}{\rho}-\frac{1}{\sigma}\right]$ (d) $\frac{1}{M}\left[\frac{1}{\rho}-\frac{1}{\sigma}\right]$

3. The volume of an air bubble becomes three times as it rises from the bottom of a lake to its surface. Assuming atmospheric pressure to be 75 cm of Hg and the density of water to be 1/10 of the density of mercury, the depth of the lake is

(a) 5 m (b) 10 m
(c) 15 m (d) 20 m

4. A closed rectangular tank is completely filled with water and is accelerated horizontally with an acceleration a towards right. Pressure is (i) maximum at, and (ii) minimum at

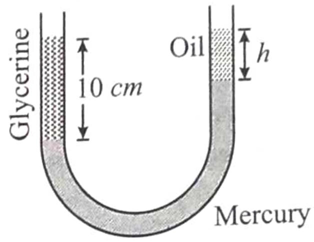


(a) (i) B (ii) D (b) (i) C (ii) D
(c) (i) B (ii) C (d) (i) B (ii) A

5. Why the dam of water reservoir is thick at the bottom
(a) Quantity of water increases with depth
(b) Density of water increases with depth
(c) Pressure of water increases with depth
(d) Temperature of water increases with depth

PASCAL'S LAW

6. A vertical U-tube of uniform inner cross section contains mercury in both sides of its arms. A glycerin (density = 1.3 g/cm^3) column of length 10 cm is introduced into one of its arms. Oil of density 0.8 gm/cm^3 is poured into the other arm until the upper surfaces of the oil and glycerin are in the same horizontal level. Find the length of the oil column, Density of mercury = 13.6 g/cm^3



(a) 10.4 cm (b) 8.2 cm (c) 7.2 cm (d) 9.7 cm

7. Two stretched membranes of areas 2 and 3 m^2 are placed in a liquid at the same depth. The ratio of the pressure on them is:

(a) 1 : 1 (b) 2 : 3
(c) $\sqrt{2} : \sqrt{3}$ (d) $2^2 : 3^2$

8. In a hydraulic lift, used at a service station the radius of the large and small piston are in the ratio of 20 : 1. What weight placed on the small piston will be sufficient to lift a car of mass 1500 kg?

(a) 3.75 kg (b) 37.5 kg
(c) 7.5 kg (d) 75 kg

9. A tank with length 10 m, breadth 8 m and depth 6m is filled with water to the top. If $g = 10$ m s^{-2} and density of water is 1000 kg m^{-3}, then the thrust on the bottom is

(a) $6 \times 1000 \times 10 \times 80\ N$ (b) $3 \times 1000 \times 10 \times 48\ N$
(c) $3 \times 1000 \times 10 \times 60\ N$ (d) $3 \times 1000 \times 10 \times 80\ N$

ARCHEMEDES PRINCIPLE AND FORCE OF BUOYANCY

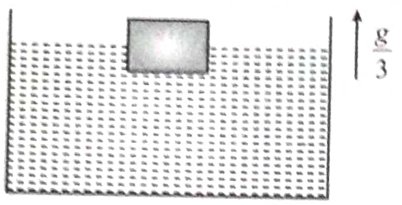
10. A concrete sphere of radius R has a cavity of radius r which is packed with sawdust. The specific gravities of concrete and sawdust are respectively 2.4 and 0.3 for this sphere to float with its entire volume submerged under water. Ratio of mass of concrete to mass of sawdust will be

(a) 8 (b) 4 (c) 3 (d) Zero

11. A metallic block of density 5 gm cm^{-3} and having dimensions 5 cm × 5 cm × 5 cm is weighed in water. Its apparent weight will be

(a) $5 \times 5 \times 5 \times 5\ gf$ (b) $4 \times 4 \times 4 \times 4\ gf$
(c) $5 \times 4 \times 4 \times 4\ gf$ (d) $4 \times 5 \times 5 \times 5\ gf$

12. A cubical block is floating in a liquid with half of its volume immersed in the liquid. When the whole system accelerates upwards with acceleration of $g/3$, the fraction of volume immersed in the liquid will be



(a) $\frac{1}{2}$ (b) $\frac{3}{8}$

(c) $\frac{2}{3}$ (d) $\frac{3}{4}$

13. A boat carrying steel balls is floating on the surface of water in a tank. If the balls are thrown into the tank one by one, how will it affect the level of water

(a) It will remain unchanged

(b) It will rise

(c) It will fall

(d) First it will first rise and then fall

14. A boy carries a fish in one hand and a bucket (not full) of water in the other hand. If he places the fish in the bucket, the weight now carried by him (assume that water does not spill):

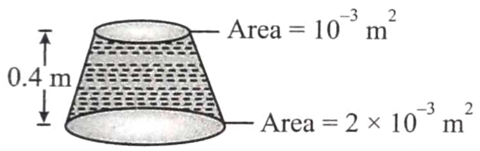
(a) is less than before

(b) is more than before

(c) is the same as before

(d) depends upon his speed

15. A uniformly tapering vessel shown in Fig. is filled with liquid of density 900 kg/m^3. The force that acts on the base of the vessel due to liquid is (Take $g = 10$ m/s^2):



(a) 3.6 N (b) 7.2 N

(c) 9.0 N (d) 12.6 N

16. Ram and Shyam are having the same weight when measured outside the water. When measured under water, it is found that weight of Ram is more than that of Shyam, then we can say that

(a) Ram is having more fat content than Shyam.

(b) Shyam is having more fat content that Ram.

(c) Ram and Shyam both are having the same fat content.

(d) None of these.

17. The reading of a spring balance when a block is suspended from it in air is 60 newton. This reading is changed to 40 newton when the block is submerged in water. The specific gravity of the block must be therefore:

(a) 3 (b) 2

(c) 6 (d) 3/2

18. An ice block floats in a liquid whose density is less than water. A part of block is outside the liquid. When whole of ice has melted, the liquid level will:

(a) rise (b) go down

(c) remain same (d) first rise then go down

19. A piece of steel has a weight W in air, W_1 when completely immersed in water and W_2 when completely immersed in an unknown liquid. The relative density (specific gravity) of liquid is:

(a) $\frac{W - W_1}{W - W_2}$ (b) $\frac{W - W_2}{W - W_1}$

(c) $\frac{W_1 - W_2}{W - W_1}$ (d) $\frac{W_1 - W_2}{W - W_2}$

CONTINUITY EQUATION, BERNOULLI THEOREM AND THEIR APPLICATION

20. In which one of the following cases will the liquid flow in a pipe be most streamlined

(a) Liquid of high viscosity and high density flowing through a pipe of small radius

(b) Liquid of high viscosity and low density flowing through a pipe of small radius

(c) Liquid of low viscosity and low density flowing through a pipe of large radius

(d) Liquid of low viscosity and high density flowing through a pipe of large radius

21. Two water pipes of diameters 2 cm and 4 cm are connected with the main supply line. The velocity of flow of water in the pipe of 2 cm diameter is

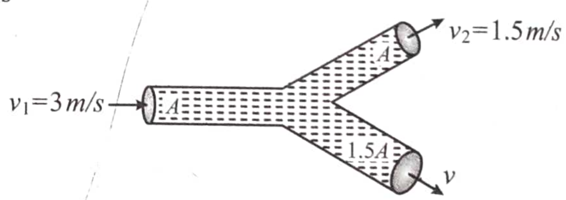
(a) 4 times that in the other pipe

(b) $\frac{1}{4}$ times that in the other pipe

(c) 2 times that in the other pipe

(d) $\frac{1}{2}$ times that in the other pipe

22. An incompressible liquid flows through a horizontal tube as shown in the following fig. Then the velocity v of the fluid is



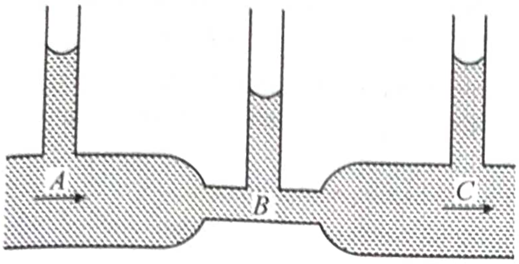
(a) 3.0 m/s

(b) 1.5 m/s

(c) 1.0 m/s

(d) 2.25 m/s

23. In the following fig. is shown the flow of liquid through a horizontal pipe. Three tubes A, B and C are connected to the pipe. The radii of the tubes A, B and C at the junction are respectively 2 cm, 1 cm and 2 cm. It can be said that the



(a) Height of the liquid in the tube A is maximum

(b) Height of the liquid in the tubes A and B is the same

(c) Height of the liquid in all the three tubes is the same

(d) Height of the liquid in the tubes A and C is the same

24. A cylinder of height 20 m is completely filled with water. The velocity of efflux of water (in m/s) through a small hole on the side wall of the cylinder near its bottom is

(a) 10 (b) 20

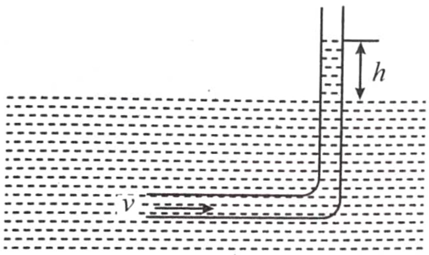
(c) 25.5 (d) 5

25. Fig. represents vertical sections of four wings moving horizontally in air. In which case the force is upwards

(a)  (b) 

(c)  (d) 

26. An L-shaped glass tube is just immersed in flowing water such that its opening is pointing against flowing water. If the speed of water current is v, then



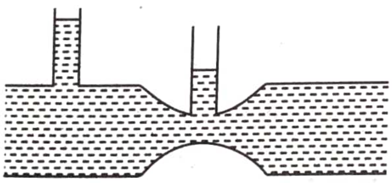
(a) The water in the tube rises to height $\frac{v^2}{2g}$

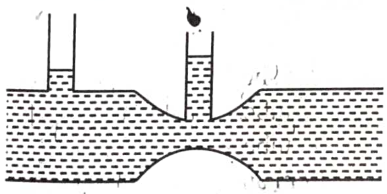
(b) The water in the tube rises to height $\frac{g}{2v^2}$

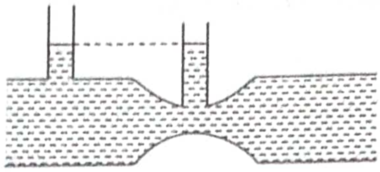
(c) The water in the tube does not rise at all

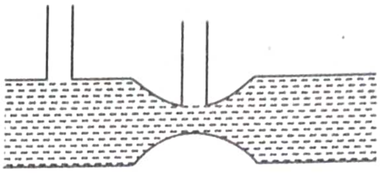
(d) None of these

27. For a fluid which is flowing steadily in the figure shown, the level in the vertical tubes is best represented by

(a) 

(b) 

(c) 

(d) 

28. Water is flowing in a horizontal pipe of non-uniform cross-section. At the most contracted place of the pipe:

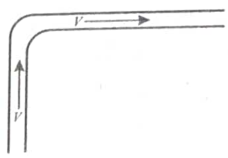
(a) Velocity of water will be maximum and pressure minimum

(b) Pressure of water will be maximum and velocity minimum

(c) Both pressure and velocity of water will be maximum

(d) Both pressure and velocity of water will be minimum

29. A fire hydrant delivers water of density ρ at a volume rate L. The water travels vertically upward through the hydrant and then does 90° turn to emerge horizontally at speed V. The pipe and nozzle have uniform cross-section throughout. The force exerted by the water on the corner of the hydrant is:



(a) $\rho v L$ (b) zero

(c) $2\rho v L$ (d) $\sqrt{2}\,\rho v L$

30. A cyclindrical vessel of cross-sectional area 1000 cm^2, is fitted with a frictionless piston of mass 10 kg, and filled with water completely. A small hole of cross-sectional area 10 mm^2 is opened at a point 50 cm deep from the lower surface of the piston. The velocity of efflux from the hole will be

(a) 10.5 m/s (b) 3.4 m/s

(c) 0.8 m/s (d) 0.2 m/s

SURFACE TENSION, SURFACE ENERGY

31. The value of surface tension of a liquid at critical temperature is

(a) Zero (b) Infinite

(c) Between 0 and ∞ (d) Can not be determined

32. Soap helps in cleaning clothes, because

(a) Chemicals of soap change

(b) It increases the surface tension of the solution

(c) It absorbs the dirt

(d) It lowers the surface tension of the solution

33. A pin or a needle floats on the surface of water, the reason for this is

(a) Surface tension (b) Less weight
(c) Upthrust of liquid (d) None of the above

34. Coatings used on raincoat are waterproof because

(a) Water is absorbed by the coating
(b) Cohesive force becomes greater
(c) Water is not scattered away by the coating
(d) Angle of contact decreases

35. The surface tension of a liquid

(a) Increases with area
(b) Decreases with area
(c) Increase with temperature
(d) Decrease with temperature

36. A thin metal disc of radius r floats on water surface and bends the surface downwards along the perimeter making an angle θ with vertical edge of the disc. If the disc displaces a weight of water W and surface tension of water is T, then the weight of metal disc is

(a) $2\pi rT + W$
(b) $2\pi rT \cos\theta - W$
(c) $2\pi rT \cos\theta + W$
(d) $W - 2\pi rT \cos\theta$

37. A 10 cm long wire is placed horizontally on the surface of water and is gently pulled up with a force of 2×10^{-2} N to keep the wire in equilibrium. The surface tension, in Nm^{-1}, of water is

(a) 0.1 (b) 0.2
(c) 0.001 (d) 0.002

38. Two droplets merge with each other and forms a large droplet. In this process

(a) Energy is liberated
(b) Energy is absorbed
(c) Neither liberated nor absorbed
(d) Some mass is converted into energy

39. Radius of a soap bubble is 'r', surface tension of soap solution is T. Then without increasing the temperature, how much energy will be needed to double its radius

(a) $4\pi r^2 T$ (b) $2\pi r^2 T$
(c) $12\pi r^2 T$ (d) $24\pi r^2 T$

40. If T is the surface tension of soap solution, the amount of work done in blowing a soap bubble from a diameter D to $2D$ is

(a) $2\pi D^2 T$ (b) $4\pi D^2 T$
(c) $6\pi D^2 T$ (d) $8\pi D^2 T$

41. The radius of a soap bubble is increased from $\frac{1}{\sqrt{\pi}}$ cm to $\frac{2}{\sqrt{\pi}}$ cm. If the surface tension of water is 30 dynes per cm, then the work done will be

(a) 180 ergs (b) 360 ergs
(c) 720 ergs (d) 960 ergs

42. A water drop is divided into 8 equal droplets. The pressure difference between the inner and outer side of the big drop will be :

(a) same as for smaller droplet
(b) 1/2 of that for smaller droplet
(c) 1/4 of that for smaller droplet
(d) twice that for smaller droplet

CAPILLARY RISE

43. Water does not wet an oily glass because

(a) Cohesive force of oil >> adhesive force between oil and glass
(b) Cohesive force of oil > cohesive force of water
(c) Oil repels water
(d) Cohesive force for water > adhesive force between water and oil molecules

44. A water drop takes the shape of a sphere in a oil while the oil drop spreads in water, because

(a) $C.F.$ for water > $A.F.$ for water and oil
(b) $C.F.$ for oil > $A.F.$ for water and oil
(c) $C.F.$ for oil < $A.F.$ for water and oil
(d) None of the above

($A.F.$ = adhesive force $C.F.$ = cohesive force)

45. Which of the fact is not due to surface tension

(a) Dancing of a camphor piece over the surface of water
(b) Small mercury drop itself becomes spherical
(c) A liquid surface comes at rest after stirring
(d) Mercury does not wet the glass vessel

46. Mercury does not wet glass, wood or iron because

(a) Cohesive force is less than adhesive force
(b) Cohesive force is greater than adhesive force
(c) Angle of contact is less than 90°
(d) Cohesive force is equal to adhesive force

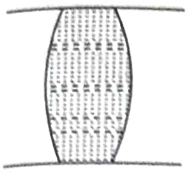
47. A mercury drop does not spread on a glass plate because the angle of contact between glass and mercury is

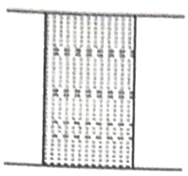
(a) Acute
(b) Obtuse
(c) Zero
(d) 90°

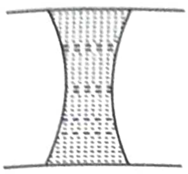
48. A glass plate is partly dipped vertically in the mercury and the angle of contact is measured. If the plate is inclined, then the angle of contact will

(a) Increase
(b) Remain unchanged
(c) Increase or decrease
(d) Decrease

49. If a water drop is kept between two glass plates, then its shape is

(a) 

(b) 

(c) 

(d) None of these

50. Water rises in a capillary tube to a height h. it will rise to a height more than h

(a) on the surface of sun
(b) in a lift moving down with an acceleration
(c) at the poles
(d) in a lift moving up with an acceleration

51. A capillary tube of radius R is immersed in water and water rises in it to a height H. Mass of water in capillary tube is M. if the radius of the tube is doubled, mass of water that will rise in capillary tube will be

(a) $2M$ (b) M
(c) $\frac{M}{2}$ (d) $4M$

52. In a surface tension experiment with a capillary tube water rises upto 0.1 m. If the same experiment is repeated in an artificial satellite, which is revolving around the earth; water will rise in the capillary tube upto a height of

(a) 0.1 m (b) 0.2 m
(c) 0.98 m (d) full length of tube

EXCESS PRESSURE IN DROP AND BUBBLE

53. A soap bubble assumes a spherical surface. Which of the following statement is wrong

(a) The soap film consists of two surface layers of molecules back to back
(b) The bubble encloses air inside it
(c) The pressure of air inside the bubble is less than the atmospheric pressure; that is why the atmospheric pressure has compressed it equally from all sides to give it a spherical shape
(d) Because of the elastic property of the film, it will tend to shrink to as small a surface area as possible for the volume it has enclosed

54. If two soap bubbles of different radii are in connected with each other

(a) Air flows from larger bubble into the smaller one
(b) The size of the bubbles remains the same
(c) Air flows from the smaller bubble into the large one and the larger bubble grows at the expense of the smaller one
(d) The air flows from the larger

55. The excess of pressure inside a soap bubble than that of the outer pressure is

(a) $\frac{2T}{r}$ (b) $\frac{4T}{r}$ (c) $\frac{T}{2r}$ (d) $\frac{T}{r}$

56. A soap bubble in vacuum has a radius of 3 cm and another soap bubble in vacuum has a radius of 4 cm. If the two bubbles coalesce under isothermal condition, then the radius of the new bubble is

(a) 2.3 cm (b) 4.5 cm (c) 5 cm (d) 7 cm

57. Excess pressure of one soap bubble is four times more than the other. Then the ratio of volume of first bubble to another one is

(a) 1 : 64 (b) 1 : 4 (c) 64 : 1 (d) 1 : 2

58. When charge is given to a soap bubble, it shows:

(a) a decrease in size
(b) no change in size
(c) an increase in size
(d) sometimes an increase and sometimes a decreases in size

59. A soap bubble of radius r_1 is placed on another soap bubble of radius $r_2 (r_1 < r_2)$. The radius R of the soapy film separating the two bubbles is:

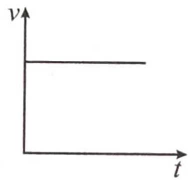
(a) $r_1 + r_2$ (b) $\sqrt{r_1^2 + r_2^2}$
(c) $(r_1^3 + r_2^3)$ (d) $\frac{r_2 r_1}{r_2 - r_1}$

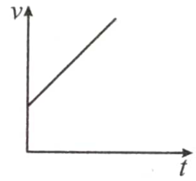
60. Two unequal soap bubbles are formed one on each side of a tube closed in the middle by a tap. What happens when the tap is opened to put the two bubbles in communication?

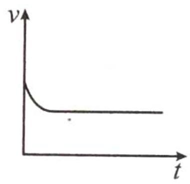
(a) No air passes in any direction as the pressure are the same on two sides of the tap
(b) Larger bubble shrinks and smaller bubble increases in size till they become equal in size
(c) Smaller bubble gradually collapses and the bigger one increases in size
(d) None of the above

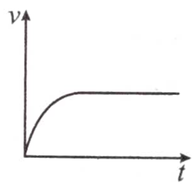
VISCOSITY AND VISCOUS FORCE

61. From amongst the following curves, which one shows the variation of the velocity v with time t for a small sized spherical body falling vertically in a long column of a viscous liquid

(a) 

(b) 

(c) 

(d) 

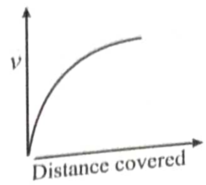
62. Two spherical raindrops of equal size are falling vertically through air with a terminal velocity of 1 m/s. What would be the terminal speed if these two drops were to coalesce to form a large spherical drop?

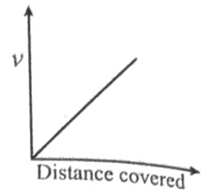
(a) 2.1 m/s (b) 1.58 m/s
(c) 2.21 m/s (d) 2.34 m/s

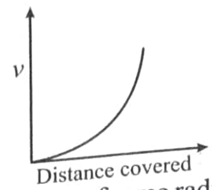
63. What is the velocity v of a metallic ball of radius r falling in a tank of liquid at the instant when its acceleration is one-half that of a freely falling body? (The densities of metal and of liquid are ρ and σ respectively, and the viscosity of the liquid is η).

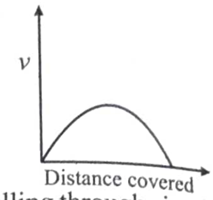
(a) $\frac{r^2g}{9\eta}(\rho-2\sigma)$ (b) $\frac{r^2g}{9\eta}(2\rho-\sigma)$
(c) $\frac{r^2g}{9\eta}(\rho-\sigma)$ (d) $\frac{2r^2g}{9\eta}(\rho-\sigma)$

64. A lead shot of 1 mm diameter falls through a long column of glycerine. The variation of its velocity v. with distance covered is represented by

(a)


(b)


(c)


(d)


65. Two drops of same radius are falling through air with steady velocity of v cm/s. If the two drops coalesce, what would be the terminal velocity?

(a) $4v$
(b) $(4)^{1/3}v$
(c) $2v$
(d) $64v$

Exercise-2 (Learning Plus)

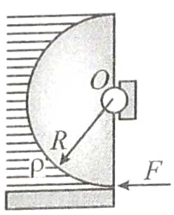
1. The density of ice is x gm/cc and that of water is y gm/cc. What is the change in volume in cc, when m gm of ice melts?

(a) $m(y-x)$ (b) $(y-x)/m$
(c) $mxy(x-y)$ (d) $m(1/y-1/x)$

2. An inverted bell lying at the bottom of a lake 47.6 m deep has 50 cm^3 of air trapped in it. The bell is brought to the surface of the lake. The volume of the trapped air will be (atmospheric pressure = 70 cm of Hg and density of Hg = 13.6 g/cm^3)

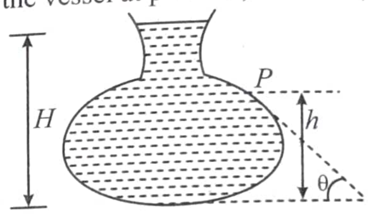
(a) 350 cm^3 (b) 300 cm^3
(c) 250 cm^3 (d) 22 cm^3

3. A light semi cylindrical gate of radius R is pivoted at its mid point O, of the diameter as shown in the figure holding liquid of density ρ. The force F required to prevent the rotation of the gate is equal to



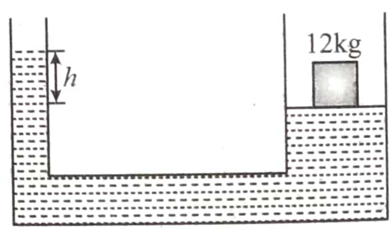
(a) $2\rho R^3 rg$
(b) $2\rho g R^3 l$
(c) $\frac{2R^2 l\rho g}{3}$
(d) None of these

4. Figure here shows the vertical cross-section of a vessel filled with a liquid of density ρ. The normal thrust per unit area on the walls of the vessel at point. P, as shown, will be



(a) $h\rho g$ (b) $H\rho g$
(c) $(H-h)\rho g$ (d) $(H-h)\rho g\cos\theta$

5. The area of cross-section of the wider tube shown in figure is 800 cm^2. If a mass of 12 kg is placed on the massless piston, the difference in heights h in the level of water in the two tubes is:

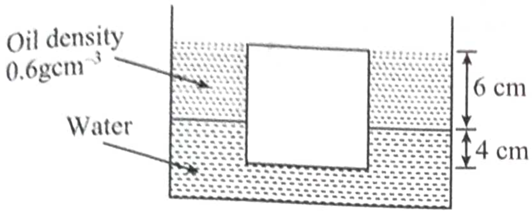


(a) 10 cm (b) 6 cm
(c) 15 cm (d) 2 cm

6. Two bodies are in equilibrium when suspended in water from the arms of a balance. The mass of one body is 36 g and its density is 9 g/cc. If the mass of the other is 48 g, its density in g/cc is :

(a) 4/3 (b) 3/2 (c) 3 (d) 5

7. A cubical block of wood 10 cm on a side, floats at the interface of oil and water as shown in figure. The density of oil is 0.6 g cm^{-3} and density of water is 1 g cm^{-3}. The mass of the block is



(*a*) 706 g (*b*) 607 g
(*c*) 760 g (*d*) 670 g

8. A metal ball of density 7800 kg/m^3 is suspected to have a large number of cavities. It weighs 9.8 kg when weighed directly on a balance and 1.5 kg less when immersed in water. The fraction by volume of the cavities in the metal ball is approximately:

(*a*) 20% (*b*) 30%
(*c*) 16% (*d*) 11%

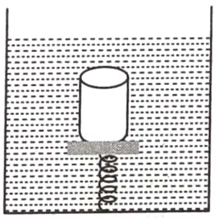
9. A sphere of radius R and made of material of relative density σ has a concentric cavity of radius r. It just floats when placed in a tank full of water. The value of the ratio R/r will be

(*a*) $\left(\frac{\sigma}{\sigma-1}\right)^{1/3}$ (*b*) $\left(\frac{\sigma-1}{\sigma}\right)^{1/3}$
(*c*) $\left(\frac{\sigma+1}{\sigma}\right)^{1/3}$ (*d*) $\left(\frac{\sigma-1}{\sigma+1}\right)^{1/3}$

10. A beaker containing water is placed on the platform of a spring balance. The balance reads 1.5 kg. A stone of mass 0.5 kg and density 500 kg/m^3 is immersed in water without touching the walls of beaker. What will be the balance reading now?

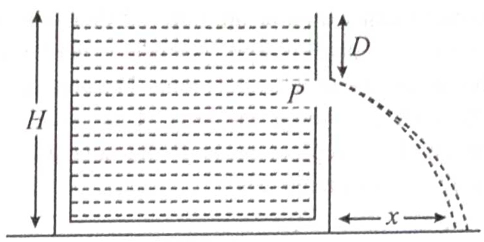
(*a*) 2 kg (*b*) 2.5 kg
(*c*) 1 kg (*d*) 3 kg

11. A cylindrical block of area of cross-section A and of material of density ρ is placed in a liquid of density one-third of density of block. The block compresses a spring and compression in the spring is one-third of the length of the block. If acceleration due to gravity is g, the spring constant of the spring is



(*a*) ρAg (*b*) $2\rho Ag$
(*c*) $2\rho Ag/3$ (*d*) $\rho Ag/3$

12. A tank is filled with water up to height H. Water is allowed to come out of a hole P in one of the walls at a depth D below the surface of water as shown in the figure. Express the horizontal distance x in terms of H and D:

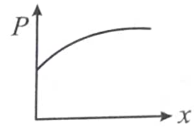


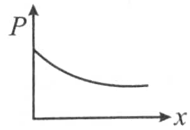
(*a*) $x = \sqrt{D(H-D)}$ (*b*) $x = \sqrt{\frac{D(H-D)}{2}}$
(*c*) $x = 2\sqrt{D(H-D)}$ (*d*) $x = 4\sqrt{D(H-D)}$

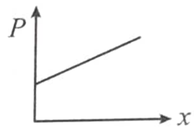
13. A cylindrical tank of height 0.4 m is open at the top and has a diameter 0.16 m. Water is filled in it up to a height of 0.16 m. How long it will take to empty the tank through a hole of radius 5×10^{-3} m in its bottoms?

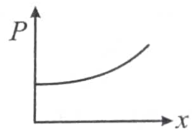
(*a*) 46.26 sec. (*b*) 4.6 sec.
(*c*) 462.6 sec. (*d*) 0.46 sec.

14. The cross sectional area of a horizontal tube increases along its length linearly, as we move in the direction of flow. The variation of pressure, as we move along its length in the direction of flow (x-direction), is best depicted by which of the following graphs

(*a*)


(*b*)


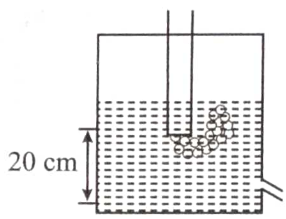
(*c*)


(*d*)


15. Water is flowing steadily through a horizontal tube of non uniform cross-section. If the pressure of water is 4×10^4 N/m^2 at a point where cross-section is 0.02 m^2 and velocity of flow is 2 m/s, what is pressure at a point where cross-section reduces to 0.01 m^2

(*a*) 1.4×10^4 N/m^2 (*b*) 3.4×10^4 N/m^2
(*c*) 2.4×10^{-4} N/m^2 (*d*) none of these

16. A tube is attached as shown in closed vessel containing water. The velocity of water coming out from a small hole is:



(*a*) $\sqrt{2}$ m/s
(*b*) 2 m/s
(*c*) depends on pressure of air inside vessel
(*d*) None of these

17. A cylindrical vessel open at the open at the top is 20 cm high and 10 cm in diameter. A circular hole whose cross-sectional area 1 cm^2 is cut at the centre of the bottom of the vessel. Water flows from a tube above it into the vessel at the rate 100 cm^{3}s^{-1}. The height of water in the vessel under state is (Take g = 1000 cms^{-2})

(*a*) 20 cm (*b*) 15 cm (*c*) 10 cm (*d*) 5 cm

18. A vertical tank, open at the top, is filled with a liquid and rests on a smooth horizontal surface. A small hole is opened at the centre of one side of the tank. The area of cross-section of the tank is N times the area of the hole, where N is a large number. Neglect mass of the tank itself. The initial acceleration of the tank is

(a) $\frac{g}{2N}$ (b) $\frac{g}{\sqrt{2}N}$

(c) $\frac{g}{N}$ (d) $\frac{g}{2\sqrt{N}}$

19. A horizontal right angle pipe bend has cross-sectional area = 10 cm^2 and water flows through it at speed = 20 m/s. The force on the pipe bend due to the turning of water is :

(a) 565.7 N

(b) 400 N

(c) 20 N

(d) 282.8 N

20. Water is pumped from a depth of 10 m and delivered through a pipe of cross section $10^{-2}m^2$. If it is needed to deliver a volume of $10^{-1}m^3$ per second the power required will be:

(a) 10 kW (b) 9.8 kW

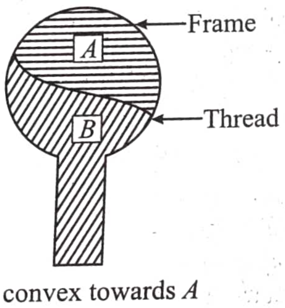
(c) 15 kW (d) 4.9 kW

21. Fountains usually seen in gardens are generated by a wide pipe with an enclosure at one end having many small holes. Consider one such fountain which is produced by a pipe of internal diameter 2 cm in which water flows at a rate $3ms^{-1}$. The enclosure has 100 holes each of diameter 0.05 cm. The velocity of water coming out of the holes is (in ms^{-1}):

(a) 0.48 (b) 96

(c) 24 (d) 48

22. A thread is tied slightly loose to a wire frame as shown in the figure. And the frame is dipped into a soap solution and taken out. The frame is completely covered with the film. When the portion A is punctured with a pin, the thread:



(a) becomes convex towards A

(b) becomes concave towards A

(c) remains in the initial position

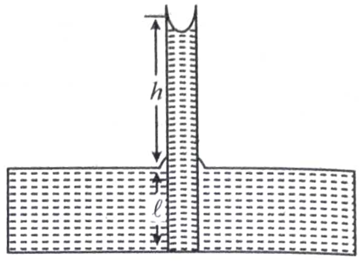
(d) either (a) or (b) depending on size of A w.r.t. B

23. The work done to get n smaller equal size spherical drops from a bigger size spherical drop of water is proportional to:

(a) $\left(\frac{1}{n^{2/3}}\right)-1$ (b) $\left(\frac{1}{n^{1/3}}\right)-1$

(c) $n^{1/3}-1$ (d) $n^{4/3}-1$

24. Water rises to a height h in a capillary tube lowered vertically into water to a depth ℓ as shown in the figure. The lower end of the tube is closed, the tube is then taken out of the water and opened again. The length of the water column remaining in the tube will be:



(a) $2h$ if $\ell \geq h$ and $\ell + h$ if $\ell \leq h$

(b) h if $\ell \geq h$ and $\ell + h$ if $\ell \leq h$

(c) $4h$ if $\ell \geq h$ and $\ell - h$ if $\ell \leq h$

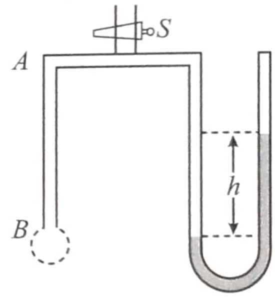
(d) $h/2$ if $\ell \geq h$ and $\ell + h$ if $\ell \leq h$

25. Two parallel glass plates are dipped partly in the liquid of density 'd'. keeping them vertical. If the distance between the plates is 'x', Surface tension for liquid is T & angle of contact is θ then rise of liquid between the plates due to capillary will be:

(a) $\frac{T\cos\theta}{xd}$ (b) $\frac{2T\cos\theta}{xdg}$

(c) $\frac{2T}{xdg\cos\theta}$ (d) $\frac{T\cos\theta}{xdg}$

26. A tube of fine bore AB is connected to a manometer M as shown. the stop cock S controls the flow of air. AB is dipped into a liquid whose surface tension is σ. On opening the stop cock for a while, a bubble is formed at B and the manometer level is recorded, showing a difference h in the levels in the two arms. if ρ be the density of manometer liquid and r the radius of curvature of the bubble, then the surface tension σ of the liquid is given by



(a) ρhrg (b) $2\rho hgr$

(c) $4\rho hrg$ (d) $\frac{rh\rho g}{4}$

27. An air bubble of radius r in water is at a depth h below the water surface at some instant. If P is atmospheric pressure, d and T are density and surface tension of water respectively, the pressure inside the bubble will be:

(a) $P+h\,dg-\frac{4T}{r}$ (b) $P+h\,dg+\frac{2T}{r}$

(c) $P+h\,dg-\frac{2T}{r}$ (d) $P+h\,dg+\frac{4T}{r}$

28. A cylinder with a movable piston contains air under a pressure p_1 and a soap bubble of radius 'r'. The pressure p_2 to which the air should be compressed by slowly pushing the piston into the cylinder for the soap bubble to reduce its size by half will be: (The surface tension is σ, and the temperature T is maintained constant)

(a) $\left[8p_1 + \frac{24\sigma}{r}\right]$ (b) $\left[2p_1 + \frac{24\sigma}{r}\right]$

(c) $\left[2p_1 + \frac{24\sigma}{r}\right]$ (d) $\left[2p_1 + \frac{12\sigma}{r}\right]$

29. A Newtonian fluid fills the clearance between a shaft and a sleeve. When a force of 800*N* is applied to the shaft, parallel to the sleeve, the shaft attains a speed of 1.5 cm/sec. If a force of 2.4 *kN* is applied instead, the shaft would move with a speed of

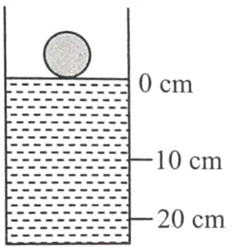
(a) 1.5 cm/sec (b) 13.5 cm/sec

(c) 4.5 cm/sec (d) None of these

30. A solid metallic sphere of radius r is allowed to fall freely through air. If the frictional resistance due to air is proportional to the cross-sectional area and to the square of the velocity, then the terminal velocity of the sphere is proportional to which of the following?

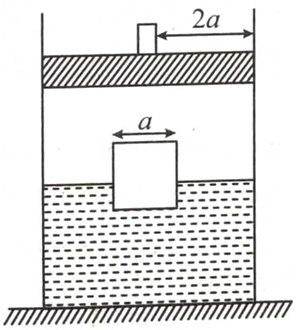
(a) r^2 (b) r (c) $r^{3/2}$ (d) $r^{1/2}$

31. A spherical ball of density ρ and radius 0.003 m is dropped into a tube containing a viscous fluid filled up to the 0 cm mark as shown in the figure. Viscosity of the fluid = 1.260 N.m^{-2} and its density $\rho_L = \rho/2 = 1260$ kg . m^{-3}. Assume the ball reaches a terminal speed by the 10 cm mark. The time taken by the ball to traverse the distance between the 10 cm and 20 cm mark is.
(g = acceleration due to gravity = 10 ms^{-2})



(a) 500 μs (b) 50 ms (c) 0.5 s (d) 5 s

32. A cubical block of side 'a' is floating in a fixed and closed cylindrical container of radius $2a$ kept on the ground. Density of the block is ρ, whereas the density of liquid is 2ρ. Container is made up of conducting wall so that the temperature remains constant. A piston is mounted in the cylinder which can move inside the cylinder without friction. If piston oscillates with large amplitude A :



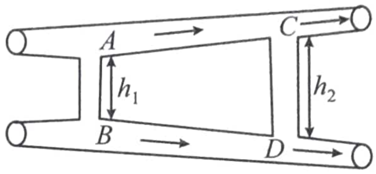
(a) The cube will remain stationary

(b) The cube will oscillate with very small amplitude in same phase with piston

(c) The cube will oscillate with very small amplitude in opposite phase with piston

(d) The cube will oscillate with amplitude A

33. An ideal liquid is flowing in two pipes, one is inclined and second is horizontal. Both the pipes are connected by two vertical tubes of length h_1 and h_2 as shown in the figure. The flow is streamline in both the pipes. If velocity of liquid at A, B and C are 2m/s, 4m/s and 4m/s respectively, the velocity at D will be :



(a) 4 m/s (b) $\sqrt{14}$ m/s

(c) $\sqrt{28}$ m/s (d) Nsone of these

34. A small body with relative density d_1 falls in air from a height 'h' on to the surface of a liquid of relative density d_2 where $d_2 > d_1$. The time elapsed after entering the liquid to the instant when it comes to instantaneous rest inside liquid :

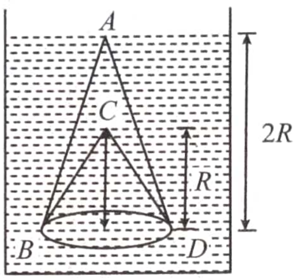
(a) $\sqrt{\frac{2h}{g}\frac{d_2}{d_1}}$

(b) $\sqrt{\frac{2h}{g}\cdot\frac{d_1}{d_2 - d_1}}$

(c) $\sqrt{\frac{2h}{g}\frac{d_1}{d_2}}$

(d) $\sqrt{\frac{2h}{g}\frac{d_2 - d_1}{d_1}}$

35. A solid cone of uniform density and height $2R$ and base radius R has a conical portion scooped out from its base with the same base radius but height R as shown in the figure. The solid cone is floating in a liquid of density ρ with vertex A touching the fluid surface. If atmospheric pressure is P_0, the force acting on the surface BCD due to liquid is :



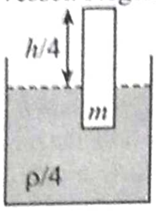
(a) $\frac{\pi r^3 \rho g}{3}\left[5 + \frac{3P_o}{R\rho g}\right]$

(b) $\frac{\pi r^3 \rho g}{3}\left[2 + \frac{P_o}{R\rho g}\right]$

(c) $\frac{\pi R^3 \rho g}{2}\left[5 + \frac{P_0}{R\rho g}\right]$

(d) $2\pi R^3 \rho g\left[5 + \frac{P_0}{R\rho g}\right]$

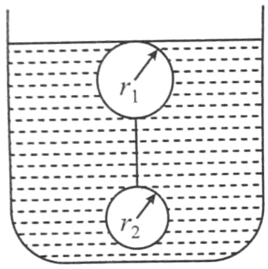
36. A solid cylinder of height h and mass m floats in a liquid of density ρ as shown in figure. Now the cylinder is released inside a liquid of density $\rho/4$, contained in a downward accelerated vessel. Determine the magnitude of acceleration of vessel a, for which cylinder sinks with relative acceleration $a/3$ with respect to vessel. Neglect any dissipative force :



(a) $\frac{2}{3}g$ (b) $\frac{4}{3}g$

(c) $\frac{3}{4}g$ (d) $\frac{1}{3}g$

37. Two spherical balls of radius r_1 and r_2 ($< r_1$) and of density σ are tied up with a long string and released in a viscous liquid column of lesser density ρ with the string just taut as shown. Find the tension in the string when terminal velocity is attained :



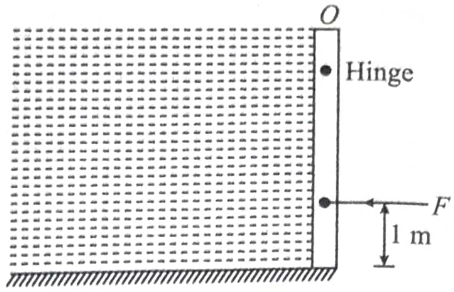
(a) $\frac{3}{4}\pi\left(\frac{r_2^4 - r_1^4}{r_2 - r_1}\right)(\sigma - \rho)g$

(b) $\frac{2}{3}\pi\left(r_2^4 - r_1^4\right)(\sigma - \rho)g$

(c) $\frac{4}{3}\pi\left(r_2^4 - r_1^3\right)(\sigma - \rho)g$

(d) $\frac{4}{3}\pi\left(\frac{r_2^4 - r_1^4}{r_2 + r_1}\right)(\sigma - \rho)g$

38. A square gate of size 4 m × 4 m is hinged at topmost point. A fluid of density ρ fills the space left of it. The force which acting 1 m from lowest point can hold the gate stationary is :



(a) $\frac{256}{3}\rho g$ (b) $\frac{256}{9}\rho g$

(c) $\frac{128}{9}\rho g$ (d) $\frac{128}{3}\rho g$

39. A wide vessel with small hole in the bottom is filled wi water and kerosene. Neglecting viscosity, the velocity o water flow v, if the thickness of water layer is h_1 and that o kerosene layer is h_2, is: (density of water ρ_1 gm/cc and tha of kerosene is ρ_2 gm/cc.)

(a) $v = \sqrt{2g(h_1 + h_2)}$

(b) $v = \sqrt{2g(h_1\rho_1 + h_2\rho_2)}$

(c) $v = \sqrt{2g\left[h_1 + h_2\left(\frac{\rho_2}{\rho_1}\right)\right]}$

(d) $v = \sqrt{2g\left[h_1\left(\frac{\rho_1}{\rho_2}\right) + h_2\right]}$

40. A soap bubble of radius r is placed on another soap bubble of radius R. What is the radius of the film separating the two bubbles ?

(a) $\frac{Rr}{R-r}$ (b) $\frac{Rr}{R+r}$

(c) $\frac{R}{R+r}$ (d) $\frac{r}{R+r}$

41. A drop of water of mass 0.2 g is placed between two glass plates. The distance between them is 0.01 cm. Find the force of attraction between the plates if surface tension of water $= 0.07\ \text{Nm}^{-1}$:

(a) 2.8 N (b) 3.5 N

(c) 0.7 N (d) 1.25 N

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. In drops of a liquid each with surface energy E join to form a single drop. Then :

(a) Some energy will be released in the process

(b) Some energy will be absorbed in the process

(c) The energy released or absorbed will be $E(n - n^{2/3})$

(d) The energy released or absorbed will be $nE(2^{2/3} - 1)$

2. Mercury of density (ρ_{Hg}) is poured into cylindrical communicating vessels of cross-sectional area A_1 and A_2 respectively ($A_1 > A_2$). A solid iron cube of volume V_0 and

density ρ_{iron} is dropped into the broad vessel, and as a result the level of the mercury in it rises. Then liquid of density ρ_{liq} poured into the broader vessel until the mercury reaches the previous level in it. The height of water column h is :

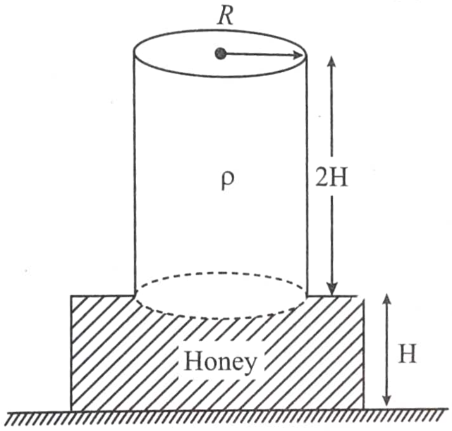
(a) $\dfrac{V_0\rho_{iron}}{\rho_{liq}\left(A_2 + V_0^{2/3}\right)}$ if rhe liquid does not submerge the block

(b) $\dfrac{V_0\rho_{iron}}{\rho_{liq}A_2}$ if the liquid does not submerge the block

(c) $\dfrac{V_0}{A_2}$ if the liquid submerge the block

(d) $\dfrac{V_0}{A_2}\left(\dfrac{\rho_{iron}-\rho_{liq}}{\rho_{Hg}-\rho_{liq}}\right)\dfrac{\rho_{Hg}}{\rho_{liq}}$ if the liquid submerge the block

3. A bottle is kept on the ground as shown in the figure. The bottle can be modelled as having two cylindrical zones. The lower zone of the bottle has a cross-sectional radius of $R\sqrt{2}$ and is filled with honey of density 2ρ. The upper zone of the bottle is filled with the water of density ρ and has a cross-sectional radius R. The height of the lower zone is H while that of the upper zone is $2H$. If now the honey and the water parts are mixed together to form a homogeneous solution : (Assume that total volume does not change)



(a) The pressure inside the bottle at the base will remain unaltered.

(b) The normal reaction on the bottle from the ground will remain unaltered

(c) The pressure inside the bottle at the base will increase by an amount (1/2) ρgH

(d) The pressure inside the bottle at the base will decrease by an amount (1/4) ρgH

4. Pressure gradient in a static fluid is represented by (z–direction is vertically upwards, and x-axis is along horizontal, d is density of fluid):

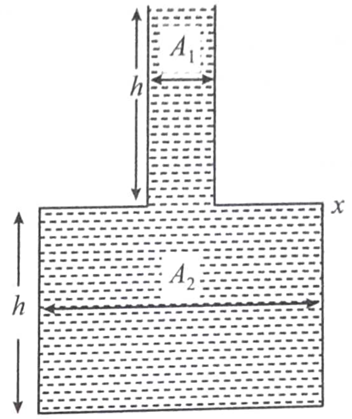
(a) $\dfrac{\partial p}{\partial z} = -dg$

(b) $\dfrac{\partial p}{\partial z} = dg$

(c) $\dfrac{\partial p}{\partial z} = 0$

(d) $\dfrac{\partial p}{\partial z} = 0$

5. The vessel shown in Figure has two sections of area of cross-section A_1 and A_2. A liquid of density r fills both the sections, up to height h in each. Neglecting atomospheric pressure,



(a) the pressure at the base of the vessel is $2\,h\,r\,g$

(b) The weight of the liquid in vessel is equal to $2\,h\,r\,g$

(c) The force exerted by the liquid on the base of vessel is $2\,h\,r\,g\,A_2$

(d) The walls of the vessel at the level x exert a force $h\,r\,g$ $(A_2 - A_1)$ downwards on the liquid.

6. A cubical block of wood of edge 10cm and mass 0.92 kg floats on a tank of water with oil of rel. density 0.6. Thickness of oil is 4cm above water. When the block attains equilibrium with four of its side edges vertical:

(a) 1 cm of it will be above the free surface of oil.

(b) 5 cm of it will be under water.

(c) 2 cm of it will be above the common surface of oil and water.

(d) 8 cm of it will be under water.

7. An air bubble in a water tank rises from the bottom to the top. Which of the following statements are true?

(a) Bubble rises upwards because pressure at the bottom is less than that at the top.

(b) Bubble rises upwards because pressure at the bottom is greater than that at the top.

(c) As the bubble rises, its size increases.

(d) As the bubble rises, its size decreases.

8. If for a liquid in a vessel, force of a cohesion is twice of adhesion:

(a) The meniscus will be convex upwards

(b) The angle of contact will be obtuse

(c) The liquid will descend in the capillary tube

(d) The liquid will wet the solid

9. The rise of liquid in a capillary tube depends on:

(a) The material of tube and nature of liquid

(b) The length of tube

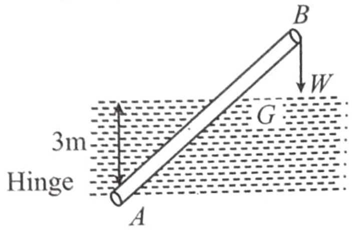
(c) The outer radius

(d) The inner radius of the tube

10. Which of the following statements are true in case when two water drops coalesce and make a bigger drop:
(a) Energy is released
(b) Energy is absorbed
(c) The surface area of the bigger drop is greater than the sum of the surface areas of both the drops
(d) The surface area of the bigger drop is smaller than the sum of the surface areas of both the drops

COMPREHENSION TYPE QUESTIONS

Comprehension (Q. 11 to 12): A rod of length 6 m has a mass 12 kg. It is hinged at one end A at a distance of 3 m below water surface. The specific gravity of the material of rod is 0.5.

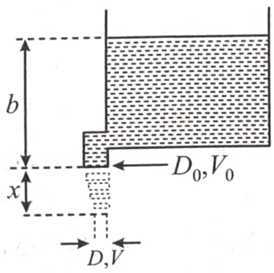


11. What weight must be attached to the other end B so that 5 m of the rod is immersed in water ?
(a) $7kgf$
(b) $\frac{7}{3}kgf$
(c) $\frac{7}{5}kgf$
(d) $\frac{7}{2}kgf$

12. Find the magnitude and direction of the force exerted by the hinge on the rod:
(a) $\frac{17}{3}kgf$ in the downward direction
(b) 8 kgf in the downward direction
(c) 4 kgf in the downward direction
(d) 5 kgf in the downward direction

Comprehension (Q. 13 to 17): The figure shows the commonly observed decrease in diameter of a water stream as it falls from a tap. The tap has internal diameter D_0 and is connected to a large tank of water. The surface of the water is at a height b above the end of the tap.

By considering the dynamics of a thin "cylinder" of water in the stream answer the following: (Ignore any resistance to the flow and any effects of surface tension, given ρ_w = density of water)



13. Equation for the flow rate, i.e. the mass of water flowing through a given point in the stream per unit time, as function of the water speed v will be
(a) $v\rho_w \pi D^2/4$
(b) $v\rho_w (\pi D^2/4 - \pi D_0^2/4)$
(c) $v\rho_w \pi D^2/2$
(d) $v\rho_w \pi D_0^2/4$

14. Which of the following equation expresses the fact that the flow rate at the tap is the same as at the stream point with diameter D and velocity v (i.e. D in terms of D_0, v_0 and v will be):
(a) $D = \frac{D_0 v_0}{v}$
(b) $D = \frac{D_0 v_0^2}{v^2}$
(c) $D = \frac{D_0 v}{v_0}$
(d) $D = D_0\sqrt{\frac{v_0}{v}}$

15. The equation for the water speed v as a function of the distance x below the tap will be:
(a) $v = \sqrt{2gb}$
(b) $v = [2g(b+x)]^{1/2}$
(c) $v = \sqrt{2gx}$
(d) $v = [2g(b-x)]^{1/2}$

16. Equation for the stream diameter D in terms of x and D_0 will be:
(a) $D = D_0\left(\frac{b}{b+x}\right)^{1/4}$
(b) $D = D_0\left(\frac{b}{b+x}\right)^{1/2}$
(c) $D = D_0\left(\frac{b}{b+x}\right)$
(d) $D = D_0\left(\frac{b}{b+x}\right)^{2}$

17. A student observes after setting up this experiment that for a tap with $D_0 = 1$ cm at $x = 0.3$ m the stream diameter $D = 0.9$ cm. The heights b of the water above the tap in this case will be:
(a) 5.7 cm (b) 57 cm
(c) 27 cm (d) 2.7 cm

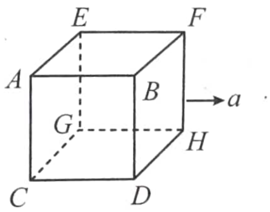
MATCH THE COLUMN TYPE QUESTIONS

18. A wooden block floats in water in a sealed container. Which copper in litre, at rest, 25% of the block is above the water. In Column-I the description of the motion of lift is given, while in Column-II the effect of motion of lift on the block is mentioned. Match the entries of Column-I with the entries of Column-II :

Column-I		Column-II	
A.	When the lift is going up or coming down with constant velocity.	p.	Greater than or less than 25% of the block is above the water.
B.	When the lift is accelerating up.	q.	25% of the block is above die water.
C.	When the lift is accelerating down.	r.	Exact fraction of the block above the water can't be determined from given information.
D.	When the lift is moving up with constant speed and air pressure above the water in the container is increased.	s.	Exact fraction of the block above the water can be determined from given information.

(*a*) A→ (q, s) B → (q, s) C → (q, s) D → (q, s)
(*b*) A → (s); B → (r); C → (p); D → (s)
(*c*) A → (p); B → (r); C → (q); D → (t)
(*d*) A → (p); B → (q); C → (t); D → (s)

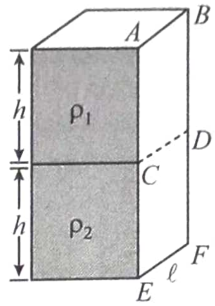
19. A cubical box is completely filled with mass m of a liquid and is given horizontal acceleration a as shown in the figure. Match the force due to fluid pressure on the faces of the cube with their appropriate values (assume zero pressure as minimum pressure)



Column-I		Column-II	
A.	force on face *ABFE*	p.	$\frac{ma}{2}$
B.	force on face *BFHD*	q.	$\frac{mg}{2}$
C.	force on face *ACGE*	r.	$\frac{ma}{2}+\frac{mg}{2}$
D.	force on face *CGHD*	s.	$\frac{mg}{2}+mg$
		t.	$\frac{mg}{2}+ma$

(*a*) A → (p); B → (r); C → (q); D → (t)
(*b*) A → (s); B → (r); C → (p); D → (s)
(*c*) A → (p); B → (r); C → (q); D → (t)
(*d*) A → (p); B → (q); C → (t); D → (s)

20. A cuboid is filled with liquid of density ρ_2 upto height h & with liquid of density ρ_1, also upto height h as shown in the figure

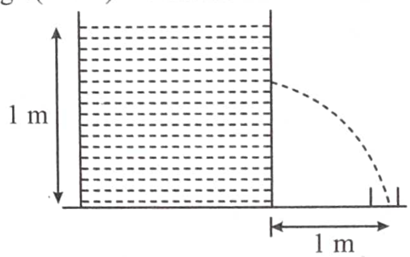


Column-I		Column-II	
A.	Force on face *ABCD* due to liquid of density ρ_1	p.	zero
B.	Force on face *ABCD* due to liquid of density ρ_2	q.	$\frac{\rho_1 gh^2\ell}{2}$
C.	Force on face *CDEF* transferred due to liquid of density ρ_1	r.	$\rho_1 gh^2\ell$
D.	Force on face *CDEF* due to liquid of density ρ_2 only	s.	$\frac{\rho_2 gh^2\ell}{2}$

(*a*) A → (p); B → (r); C → (q); D → (t)
(*b*) A → (s); B → (r); C → (p); D → (s)
(*c*) A → (p); B → (r); C → (q); D → (t)
(*d*) A → (p); B → (q); C → (r); D → (s)

NUMERICAL TYPE QUESTIONS

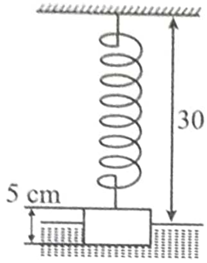
21. A tank is filled with water upto height of 1 m. A hole is to be made in the side wall so that the liquid coming out from the side falls first in a cup kept 1 m away from the right side. How high (in cm) should the hole be made?



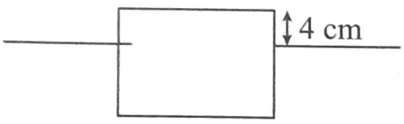
22. The cross-sectional area of the aorta (the major blood vessel emerging from the heart) of a normal resting person is 3 cm^2, and the speed of blood through it is 30 cm/s. A typical capillary has cross-sectional area of 3×10^{-7} cm^2 and a flow speed of 0.05 cm/s. How many capillaries (in multiple of 10^9) does such a person have? (Assume that all the blood that passes through the capillaries must have passed through the aorta).

23. A uniform block of wood placed in water floats with exactly 2/3 of its volume submerged. Assume that the water has a density of 1×10^3 kg/m^3. The exact same block of wood, placed in oil, floats with $\frac{5}{6}$ of its volume submerged. Using this information, what is the density of the oil in kg/m^3?

24. The original length of a spring is 25 cm. It elongates 2 cm if a force of 0.96 N is exerted on it. A container is filled with water and one end of the spring is fixed 30 cm above the surface of the water in the container. A wooden block of mass 32 g and of density 0.4 g/cm^3 is hanged onto the other end of the spring. The height of the block is 5 cm. To what depth (in mm) does the block sink into the water?



25. A wooden block of mass 0.6 kg of size 10 cm × 10 cm × 10 cm is floating over an unknown liquid as shown in the figure

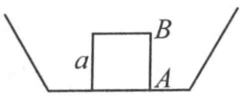


How much minimum mass (in gm) should be kept on the wooden block so that it completely submerges into liquid?

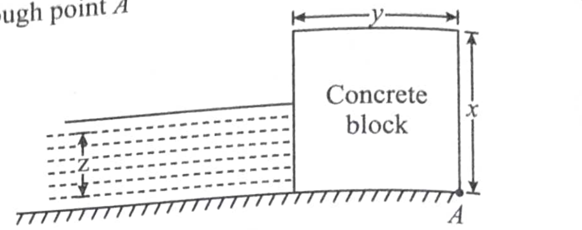
26. A microscope slide measures 6.0 cm × 1.5 cm × 0.20 cm. It is suspended with its face vertical and with its longest side horizontal and is lowered into a liquid until it is half immersed. Its apparent weight is then found to be the same as its weight in air. Calculate the surface tension of liquid in 10^{-3} N/m, assuming the angle of contact to be zero. Density of liquid is 1000 kg/m^3. Round off to nearest integer.

27. Two solid balls have different radii but are made of same material. The balls are linked together with a long thin thread and released from a large height. At the terminal velocity, the thread is under tension. The larger ball has a fixed mass, but we have choice of the smaller ball with different masses. At what ratio of larger and smaller mass will this tension be maximum?

28. A non uniform cube of side length a is kept inside a container as shown in the figure. The average density of the material of the cube is 2 ρ where ρ is the density of water. Water is gradually filled in the container. It is observed that the cube begins to topple, about its edge (into the plane of the figure) passing through point A, when the height of the water in the a container becomes $\frac{a}{2}$. Find N if the distance of the centre of mass of the cube from the face AB of the cube is a/N. Assume that water seeps under the cube.



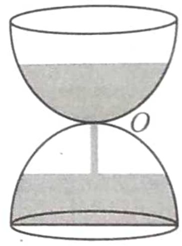
29. A rectangular concrete block (specific gravity = 2.5) is used as a retaining wall in a reservoir of water. The height and width of the block are x and y respectively. The height of water 3 in the reservoir is $z = \frac{3}{4}x$. The concrete block cannot slide on the horizontal base but can rotate about an axis perpendicular to the plane of the figure and passing through point A



(*a*) Calculate k if the minimum value of the ratio $\frac{y}{x}$ for which the block will not begin to overturn about A is $\frac{3}{4\sqrt{k}}$

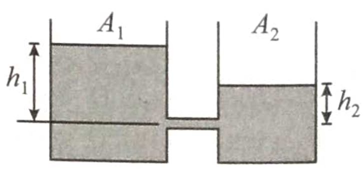
(*b*) Redo the above problem for the case when there is a seepage and a thin film of water is present under the block. Assume that a seal at A prevents the water from flowing out underneath the block.

30. A water clock consist of a vessel which has a small orifice O. The upper container is filled with water which trickles clown into the lower container. The shape of the (upper or lower) container is such that height of waler in I lie upper container changes at a uniform rate. Assume that atmospheric air can enter inside the lower container through a hole in it and that the upper container is open at the top. Vessel is axially symmetric. If the relation between radius (x) of cross-section of water level and the water level hight (z) is $z = \frac{\pi^2 v_o^2}{2gA_o^2} \cdot x^k$. The value of k is



31. There are two tanks next to each other having cross sectional area A_1 and A_2. They are interconnected by a narrow pipe having area of cross section equal to A_0. Initial height of water in the two tanks is h_1 and h_2 measured from the level of the pipe. Assume that the flow is ideal and behaves in a way similar to the discharge in air. Calculate N if the time needed for the water level in two tanks to become same is

$$\frac{A_1 A_2}{A_o(A_1 + A_2)}\sqrt{\frac{N(h_1 - h_2)}{g}}$$



Exercise-4 (Past Year Questions)

JEE MAIN

1. A liquid of density ρ is coming out of a hose pipe of radius a with horizontal speed v and hits a mesh. 50% of the liquid passes through the mesh unaffected. 25% looses all of its momentum and 25% comes back with the same speed. The resultant pressure on the mesh will be: **(2019)**

   (*a*) $\frac{1}{4}\rho v^2$ (*b*) $\frac{3}{4}\rho v^2$

   (*c*) $\frac{1}{2}\rho v^2$ (*d*) ρv^2

2. A cylindrical plastic bottle of negligible mass of filled with 310 ml of water and left floating in a pond with still water. If pressed downward slightly and released, it starts performing simple harmonic motion at angular frequency ω. If the radius of the bottle is 2.5 cm then ω is close to : (density of water $= 10^3$ kg/m^3) **(2019)**

   (*a*) 3.75 rad s^{-1} (*b*) 1.25 rad s^{-1}

   (*c*) 2.50 rad s^{-1} (*d*) 5.00 rad s^{-1}

3. A long cylindrical vessel is half filled with a liquid. When the vessel is rotated about its own vertical axis, the liquid rises up near the wall. If the radius of vessel is 5 cm and its rotational speed is 2 rotations per second. then the difference in the heights between the centre and the sides, in cm, will be: **(2019)**

   (*a*) 2.0 (*b*) 0.1

   (*c*) 0.4 (*d*) 1.2

4. Water flows into a large tank with flat bottom at the rate of 10^{-4} m^{3}s^{-1}. Water is also leaking out of a hole of area 1 cm^2 at its bottom. If the height of the water in the tank remains steady, then this height is: **(2019)**

   (*a*) 5.1 cm (*b*) 1.7 cm

   (*c*) 4 cm (*d*) 2.9 cm

5. The top of a water is open to air and its water level is maintained. It is giving out 0.74 m^3 water per minute through a circular opening of 2 cm radius in its wall. The depth of the centre of the opening from the level of water in the tank is close to: **(2019)**

   (*a*) 6.0 m (*b*) 4.8 m

   (*c*) 9.6 m (*d*) 2.9 m

6. A wooden block floating in a bucket of water has $\frac{4}{5}$ of its volume submerged. When certain amount of an oil is poured into the bucket, it is found that the block is just under the oil surface with half of its volume under water and half in oil. The density of oil relative to that of water is: **(2019)**

   (*a*) 0.5 (*b*) 0.7

   (*c*) 0.6 (*d*) 0.8

7. Water from a pipe is coming at a rate of 100 litres per minute. If the radius of the pipe is 5 cm, the Reynolds number for the flow is of the order of : (density of water = 1000 kg/m^3, coefficient of viscosity of water = 1mPas) **(2019)**

   (*a*) 10^6 (*b*) 10^3

   (*c*) 10^4 (*d*) 10^2

8. A submarine experiences a pressure of 5.05×10^6 *Pa* at a depth of d_1 in a sea. When it goes further to a depth of d_2, it experiences a pressure of 8.08×10^6 *Pa*., Then $d_2 - d_1$ is approximately (density of water = 10^3 kg/m^3 and acceleration due to gravity = 10 ms^{-2}) **(2019)**

   (*a*) 500 m (*b*) 400 m

   (*c*) 300 m (*d*) 600 m

9. Water from a tap emerges vertically downwards with an initial speed of 1.0 ms^{-1}. The cross sectional area of the tap is 10^{-4}m^2. Assume that the pressure is constant throughout the stream of water and that the flow is streamlined. The cross-sectional area of the stream, 0.15 m below the tap would be: **(2019)**

   (Take g = 10 ms^{-2})

   (*a*) 1×10^{-5} m^2 (*b*) 5×10^{-5} m^2

   (*c*) 2×10^{-5} m^2 (*d*) 5×10^{-4} m^2

10. A cubical block of side 0.5 m floats on water with 30% of its volume under water. What is the maximum weight that can be put on the block without fully submerging it under water? (Take density of water = 10^3 kg/m^3) **(2019)**

    (*a*) 65.4 kg (*b*) 87.5 kg

    (*c*) 30.1 kg (*d*) 46.3 kg

11. An ideal fluid flows (laminar flow) through a pipe of non-uniform diameter. the maximum and minimum diameters of the pipes are 6.4 cm and 4.8 cm, respectively. The ratio of the minimum and the maximum velocities of fluid in this pipe is: **(2020)**

    (*a*) $\frac{9}{16}$ (*b*) $\frac{\sqrt{3}}{2}$

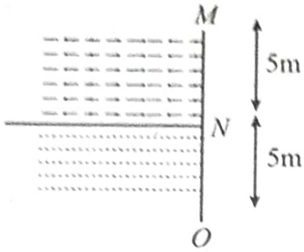
    (*c*) $\frac{81}{256}$ (*d*) $\frac{3}{4}$

12. Consider a solid sphere of radius R and mass density $\rho(r) = \rho_0\left(1-\frac{r^2}{R^2}\right) 0 < r \le R$. The minimum density of a liquid in which it will float is: **(2020)**

    (*a*) $\frac{\rho_0}{3}$ (*b*) $\frac{2\rho_0}{3}$

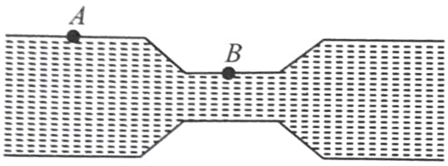
    (*c*) $\frac{\rho_0}{5}$ (*d*) $\frac{2\rho_0}{5}$

13. Two liquids of densities ρ_1 and ρ_2 ($\rho_2 = 2\rho_1$) are filled up in a closed vessel behind a square wall of side 10m as shown in figure. Each liquid has a height of 5m. The ratio of the forces due to these liquids exerted on upper part MN to that at the lower part NO is (Assume that the liquids are not mixing): **(2020)**



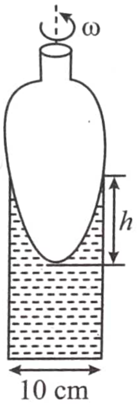
(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{2}{3}$

14. Water flows in a horizontal tube (see figure). The pressure of water changes by 700 Nm^{-2} between A and B where the area of cross section are 40 cm^2 and 20 cm^2, respectively. Find the rate of flow of water thorugh the tube. (density of water = 1000 kgm^{-3}) **(2020)**



(a) 2720 cm^3/s (b) 2420 cm^3/s
(c) 3020 cm^3/s (d) 1810 cm^3/s

15. A cylindrical vessel containing a liquid is rotated about its axis so that the liquid rises at its sides as shown in the figure. The radius of vessel is 5 cm and the angular speed of rotation is ω rad s^{-1}. The difference in the height, h (in cm) of liquid at the centre of vessel and at the side will be **(2020)**



(a) $\frac{2\omega^2}{25g}$ (b) $\frac{5\omega^2}{2g}$

(c) $\frac{2\omega^2}{5g}$ (d) $\frac{25\omega^2}{2g}$

16. Two identical cylindrical vessels are kept on the ground and each contain the same liquid of density d. The area of the base of both vessels is S but the height of liquid in one vessel is x_1 and in the other, x_2. When both cylinders are connected through a pipe of negligible volume very close to the bottom, the liquid flows from one vessel to the other until it comes to equilibrium at a new height. The change in energy of the system in the process is **(2020)**

(a) $gdS(x_2 + x_1)^2$ (b) $\frac{1}{4}gdS(x_2 - x_1)^2$

(c) $\frac{3}{4}gdS(x_2 - x_1)^2$ (d) $gdS(x_2^2 + x_1^2)$

17. A hollow spherical shell at outer radius R floats just submerged under the water surface. The inner radius of the shell is r. If the specific gravity of the shell material is $\frac{27}{8}$ w.r.t water, the value of r is **(2020)**

(a) $\frac{2}{3}R$ (b) $\frac{4}{9}R$

(c) $\frac{1}{3}R$ (d) $\frac{8}{9}R$

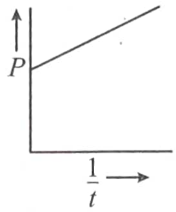
18. A fluid is flowing through a horizontal pipe of varying cross-section, with speed v ms^{-1} at a point where the pressure is P pascal.

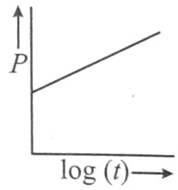
At another point where pressure is $\frac{P}{2}$ pascal its speed is V ms^{-1}. If the density of the fluid is ρ kg m^{-3} and the flow is streamline, then V is equal to: **(2020)**

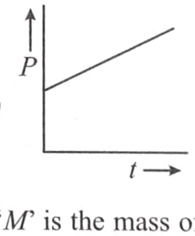
(a) $\sqrt{\frac{P}{\rho}+v^2}$ (b) $\sqrt{\frac{2P}{\rho}+v^2}$

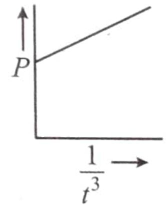
(c) $\sqrt{\frac{P}{\rho}+v}$ (d) $\sqrt{\frac{P}{2\rho}+v^2}$

19. A soap bubble, blown by a mechanical pump at the mouth of a tube, increases in volume, with time, at a constant rate. The graph that correctly depicts the time dependence of pressure inside the bubble is given by: **(2019)**

(a) 

(b) 

(c) 

(d) 

20. If 'M' is the mass of water that rises in a capillary tube of radius 'r'. then mass of water which will rise in a capillary tube of radius '$2r$' is: **(2019)**

(a) $4M$ (b) M

(c) $2M$ (d) $\frac{M}{2}$

21. A solid sphere, of radius R acquires a terminal velocity v_1 when falling (due to gravity) through a viscous fluid having a coefficient of viscosity η. The sphere is broken into 27

identical solid spheres. If each of these spheres acquires a terminal velocity, v_2, when falling through the same fluid, the ration (v_1/v_2) equals. **(2019)**

(a) 1/27 (b) 1/9

(c) 27 (d) 9

22. The ratio of surface tensions of mercury and water is given to be 7.5 while the ratio of thier densities is 13.6. Their contact angles, with glass, are close to 135° and 0°, respectively. It is observed that mercury gets depressed by an amount h in a capillary tube of radius r_1, while water rises by the same amount h in a capillary tube of radius r_2. The ratio, (r_1/r_2), is then close to:

(a) 2/3 (b) 3/5

(c) 2/5 (d) 4/5

23. A small spherical droplet of density d is floating exactly half immerse in a liquid ρ and surface tension T. The radius of the droplet is (take note that the surface tension applies an upward force on the droplet); **(2020)**

(a) $r=\sqrt{\dfrac{T}{(d+\rho)g}}$ (b) $r=\sqrt{\dfrac{3T}{(2d-\rho)g}}$

(c) $r=\sqrt{\dfrac{T}{(d-\rho)g}}$ (d) $r=\sqrt{\dfrac{2T}{3(d-\rho)g}}$

24. A capillary tube made of glass of radius 0.15 mm is dipped vertically in a beaker filled with methylene iodide (surface tension = 0.05 Nm, density = 667 kg m^{-3}) which rises to height h in the tube. It is observed that the two tangents drawn from liquid-glass interfaces (from opp. sides of the capillary) make an angle of 60° with one another. Then h is close to ($g = 10\ ms^{-2}$) **(2020)**

(a) 0.049 m (b) 0.087 m

(c) 0.137 m (d) 0.172 m

25. When a long glass capillary tube of radius 0.015 cm is dipped in a liquid, the liquid rises to a height of 15 cm within it. If the contact angle between the liquid and glass to close to 0°, the surface tension of the liquid, in milli Newton m^{-1}, is **(2020)**

$[\rho_{liquid} = 900\ kgm^{-3}, g = 10\ ms^{-2}]$

(Give answer in closest integer) ________.

26. Pressure inside two soap bubbles are 1.01 atm and 1.02 atm, respectively. The ratio of their volumes is **(2020)**

(a) 2 : 1 (b) 0.8 : 1 (c) 4 : 1 (d) 8 : 1

27. A air bubble of radius 1 cm in water has an upward acceleration 9.8 cm s^{-2}. The density of water is 1 gm cm^{-3} and water offers negligible drag force on the bubble. The mass of the bubble is ($g = 980\ cm/s^2$) **(2020)**

(a) 1.52 gm (b) 4.51 gm

(c) 3.15 gm (d) 4.15 gm

28. In an experiment to verify Stokes law, a small spherical ball of radius r and density ρ falls under gravity through a distance h in air before entering a tank of water. If the terminal velocity of the ball inside water is same as its velocity just before entering the water surface, then the value of h is proportional to

(ignore viscosity of air) **(2020)**

(a) r^4 (b) r^3 (c) r (d) r^2

29. The water is filled up to height of 12 m in a tank having vertical sidewalls. A hole is made in one of the walls at a depth 'h' below the water level. The value of 'h' for which the emerging steam of water strikes the ground at the maximum range is ________ m. **(2021)**

30. Given below are two statements : One is labelled as Assertion (A) and the other is labelled as Reason (R).

Assertion (A) : Clothes containing oil or grease stains cannot be cleaned by water wash.

Reason (R): Because the angle of contact between the oil/ grease and water is obtuse. In the light of the above statements, choose the correct answer from the option given below. **(2022)**

(a) Both (A) and (R) are true and (R) is the correct explanation of (A)

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A)

(c) (A) is true but (R) is false

(d) (A) is true but (R) is true

31. A drop of liquid of density ρ is floating half immersed in a liquid of density σ and surface tension 7.5 × 10^{-1} Ncm^{-1}. The radius of drop in cm will be : (Take : g = 10 m/s^2)

(a) $\dfrac{15}{\sqrt{2\rho-\sigma}}$ (b) $\dfrac{15}{\sqrt{\rho-\sigma}}$

(c) $\dfrac{3}{2\sqrt{\rho-\sigma}}$ (d) $\dfrac{3}{20\sqrt{2\rho-\sigma}}$

32. A water drop of radius 1 μm falls in a situation where the effect of buoyant force is negligible. Coefficient of viscosity of air is 1.8 × 10^{-5} Nsm^{-2} and its density is negligible as compared to that of water $10^6 gm^{-3}$. Terminal velocity of the water drop is: **(2022)**

(Take acceleration due to gravity = 10 ms^{-2})

(a) 145.4 × 10^{-6} ms^{-1} (b) 118.0 × 10^{-6} ms^{-1}

(c) 132.6 × 10^{-6} ms^{-1} (d) 123.4 × 10^{-6} ms^{-1}

33. A water drop of radius 1cm is broken into 729 equal droplets. If surface tension of water is 75 dyne/cm, then the gain in surface energy upto first decimal place will be : [Given π = 3.14] **(2022)**

(a) 8.5 × 10^{-4}J (b) 8.2 × 10^{-4} J

(c) 7.5 × 10^{-4} J (d) 5.3 × 10^{-4}J

34. Two cylindrical vessels of equal cross-sectional area 16 cm^2 contain water upto heights 100 cm and 150 cm respectively. The vessels are interconnected so that the water levels in them become equal. The work done by the force of gravity during the process, is [Take density of water = 103 kg/m^3 and g = 10 m/s^2] **(2022)**

(a) 0.25 J (b) 1 J

(c) 8 J (d) 12 J

35. The velocity of a small ball of mass 'm' and density d_1 when dropped in a container filled with glycerine, becomes constant after some time. If the density of glycerine is d_2, then the viscous force acting on the ball, will be : **(2022)**

(a) $mg\left(1-\frac{d_1}{d_2}\right)$ (b) $mg\left(1-\frac{d_2}{d_1}\right)$

(c) $mg\left(\frac{d_1}{d_2}-1\right)$ (d) $mg\left(\frac{d_2}{d_1}-1\right)$

36. The area of cross-section of a large tank is 0.5 m^2 It has a narrow opening near the bottom having area of cross-section 1 cm^2 A load of 25 kg is applied on the water at the top in the tank. Neglecting the speed of water in the tank, the velocity of the water, coming out of the opening at the time when the height of water level in the tank is 40 cm above the bottom, will be ____________ cms^{-1}. **(2022)**

[Take $g = 10\ ms^{-2}$]

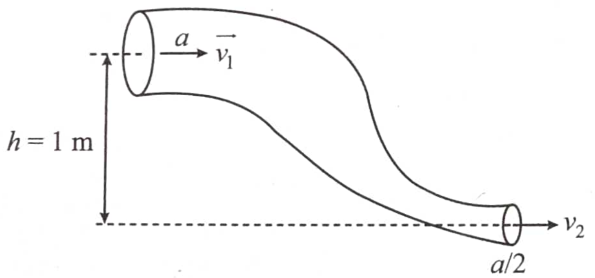
37. A small spherical ball of radius 0.1 mm and density 10^4 kg m^{-3} falls freely under gravity through a a distance h before entering a tank of water. If after entering the water the velocity of ball does not change and it continue to fall with same constant velocity inside water, then the value of h will be ______ m. (Given $g = 10\ ms^{-2}$, viscosity of water = 1.0×10^{-5} N-sm^{-2}). **(2022)**

38. The velocity of upper layer of water in a river is 36 kmh^{-1}. Shearing stress between horizontal layers of water is $10^{-3}\ Nm^{-2}$. Depth of the river is ______ m. (Co-efficiency of viscosity of water is 10^{-2} Pa.s) **(2022)**

39. A liquid of density 750 kgm^{-3} flows smoothly through a horizontal pipe that tapers in cross-sectional area from $A_1 = 1.2 \times 10^{-2}\ m^2$ to $A_2 = \frac{A_1}{2}$. Pressure difference between the wide and narrow sections of the pipe is 4500 Pa. The rate of flow of liquid is $\times\ 10^{-3} m^3 s^{-1}$. (2022)

40. An ideal fluid of density 800 kg/m^3, flows smoothly through a bent pipe (as shown in figure) that tapers in cross-sectional area from a to $\frac{a}{2}$. The pressure difference between the wide and narrow sections of pipe is 4100 Pa. At wider section, the velocity of fluid is $\frac{\sqrt{x}}{6} ms^{-1}$ for $x =$ ....

(Given $g = 10\ m^{-2}$) **(2022)**



JEE ADVANCED

41. A solid sphere of radius R and density ρ is attached to one end of a mass-less spring of force constant k. The other end of the spring is connected to another solid sphere of radius R and density 3ρ. The complete arrangement is placed in a liquid of density 2ρ and is allowed to reach equilibrium. The correct statement(s) is (are) **(2013)**

(a) The net elongation of the spring is $\frac{4\pi R^3 \rho g}{3k}$

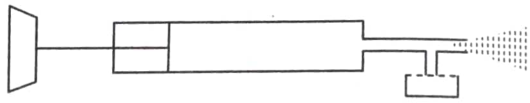
(b) The net elongation of the spring is $\frac{8\pi R^3 \rho g}{3k}$

(c) The light sphere is partially submerged.

(d) The light sphere is completely submerged.

Comprehension–1 (No. 42 to 43)

A spray gun is shown in the figure where a piston pushes air out of a nozzle. A thin tube of uniform cross section is connected to the nozzle. The other end of the tube is in a small liquid container. As the piston pushes air through the nozzle, the liquid from the container rises into the nozzle and is sprayed out. For the spray gun shown, the radii of the piston and the nozzle are 20 mm and 1mm respectively. The upper end of the container is open to the atmosphere.



42. If the piston is pushed at a speed of 5mms^{-1}, the air comes out of the nozzle with a speed of: **(2014)**

(a) 0.1ms^{-1} (b) 1ms^{-1}

(c) 2ms^{-1} (d) 8ms^{-1}

43. If the density of air is ρ_a and that of the liquid ρ_ℓ, then for a given piston speed the rate (volume per unit time) at which the liquid is sprayed will be proportional to **(2014)**

(a) $\sqrt{\frac{\rho_a}{\rho_\ell}}$ (b) $\sqrt{\rho_a \rho_\ell}$

(c) $\sqrt{\frac{\rho_\ell}{\rho_a}}$ (d) ρ_ℓ

44. A person in a lift is holding a water jar, which has a small hole at the lower end of its side. When the lift is at rest, the water jet coming out of the hole hits the floor of the lift at a distance d of 1.2 m from the person. In the following, state of the lift's motion is given in List-I and the distance where the water jet hits the floor of the lift is given in List-II. Match the statements from List-I with those in List-II and select the correct answer using the code given below the lists. **(2014)**

List-I

P. Lift is accelerating vertically up.

Q. Lift is accelerating vertically down with an accelerating less than the gravitational acceleration.

R. List is moving vertically up with constant Speed

S. Lift is falling freely.

List-II

1. $d = 1.2$ m

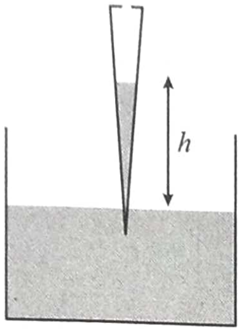
2. $d > 1.2$m

3. $d < 1.2$ m
4. No water leaks out of the jar

Code :

(a) P-2, Q-3, R-2, S-4 (b) P-2, Q-3, R-1, S-4
(c) P-1, Q-1, R-1, S-4 (d) P-2, Q-3, R-1, S-1

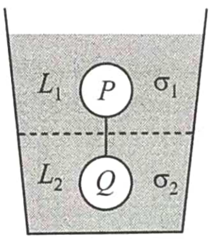
45. A glass capillary tube is of the shape of a truncated cone with an apex angle α so that its two ends have cross sections of different radii. When dipped in water vertically, water rises in it to a height h, where the radius of its cross section is b. If the surface tension of water is S, its density is ρ, and its contact angle with glass is θ, the value of h will be (g is the acceleration due to gravity) **(2014)**



(a) $\frac{2S}{b\rho g}\cos(\theta-\alpha)$ (b) $\frac{2S}{b\rho g}\cos(\theta+\alpha)$

(c) $\frac{2S}{b\rho g}\cos(\theta-\alpha/2)$ (d) $\frac{2S}{b\rho g}\cos(\theta+\alpha/2)$

46. Two spheres P and Q of equal radii have densities ρ_1 and ρ_2, respectively. The spheres are connected by a massless string and placed in liquids L_1 and L_2 of densities σ_1 and σ_2 and viscosities η_1 and η_2, respectively. They float in equilibrium with the sphere P in L_1 and sphere Q in L_2 and the string being taut (see figure). If sphere P alone in L_2 has terminal velocity $\vec{V}_P$ and Q alone in L_1 has terminal velocity $\vec{V}_Q$, then **(2015)**



(a) $\frac{|\vec{V}_P|}{|\vec{V}_Q|}=\frac{\eta_1}{\eta_2}$ (b) $\frac{|\vec{V}_P|}{|\vec{V}_Q|}=\frac{\eta_2}{\eta_1}$

(c) $\vec{V}_P\cdot\vec{V}_Q>0$ (d) $\vec{V}_P\cdot\vec{V}_Q<0$

47. Consider two solid spheres P and Q each of density 8 gm cm^{-3} and diameters 1 cm and 0.5 cm, respectively. Sphere P is dropped into a liquid of density of 0.8 gm cm^{-3} and viscosity $\eta = 3$ poiseuilles. Sphere Q is dropped into a liquid of density 1.6 gm cm^{-3} and viscosity $\eta = 2$ poiseulles. The ratio of the terminal velocities of P and Q is **(2016)**

48. A drop of liquid of radius $R = 10^{-2}$ m having surface tension $S = \frac{0.1}{4\pi}$Nm^{-1} divides itself into K identical drops. In this process the total change in the surface energy $\Delta U = 10^{-3}J$. If $K = 10^{\alpha}$ then the value of α is: **(2017)**

49. A uniform capillary tube of inner radius r is dipped vertically into a beaker filled with water. The water rises to a height h in the capillary tube above the water surface in the beaker. The surface tension of water is σ. The angle of contact between water and the wall of the capillary tube is θ. Ignore the mass of water in the meniscus. Which of the following statements is(are) true? **(2018)**

(a) For a given material of the capillary tube, h decreases with increase in r.

(b) For a given material of the capillary tube, h is independent of σ.

(c) If this experiment is performed in a lift going up with a constant acceleration, then h decreases.

(d) h is proportional to contact angel θ.

50. Consider a thin square plate floating on a viscous liquid in a large tank. The height h of the liquid in the tank is much less than the width of the tank. The floating plate is pulled horizontally with a constant velocity u_0. Which of the following statements is (are) true? **(2018)**

(a) The resistive force of liquid on the plate is inversely proportional to h

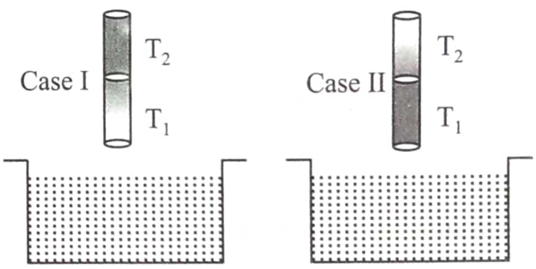
(b) The resistive force of liquid on the plate is independent of the area of the plate

(c) The tangential (shear) stress on the floor of the tank increases with u_0

(d) The tangential (shear) stress on the plate varies linearly with the viscosity η of the liquid.

51. A cylindrical capillary tube of 0.2 mm radius is made by joining two capillaries T_1 and T_2 of different materials having water contact angles of 0° and 60°, respectively. The capillary tube is dipped vertically in water in two different configurations, case I and II as shown in figure. Which of the following option(s) is(are) correct?

(Surface tension of water = 0.075 N/m, density of water = 1000 kg/m^3, take g = 10m/s^2) **(2019)**



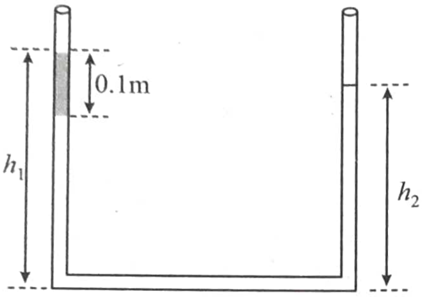
(a) The correction in the height of water column raised in the tube, due to weight of water contained in the meniscus, will be different for both cases.

(b) For case I, if the capillary joint is 5 cm above the water surface, the height of water column raised in the tube will be more than 8.75 cm. (Neglect the weight of the water in the meniscus)

(c) For case I, if the joint is kept at 8 cm above the water surface, the height of water column in the tube will be 7.5 cm. (Neglect the weight of the water in the meniscus)

(d) For case II, if the capillary joint is 5 cm above the water surface, the height of water column raised in the tube will be 3.75 cm. (Neglect the weight of the water in the meniscus)

52. An open-ended U-tube of uniform cross-sectional area contains water (density 10^3 kg m^{-3}). Initially the water level stands at 0.29 m from the bottom in each arm. Kerosene oil (a water-immiscible liquid) of density 800 kg m^{-3} is added to the left arm until its length is 0.1 m, as shown in the schematic figure below. The ratio $\left(\frac{h_1}{h_2}\right)$ of the heights of the liquid in the two arms is:



(2020)

(a) $\frac{15}{14}$ (b) $\frac{35}{33}$

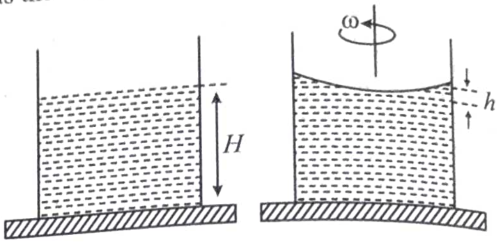
(c) $\frac{7}{6}$ (d) $\frac{5}{4}$

53. A train with cross-sectional area S_t is moving with speed v_t inside a long tunnel of cross-sectional area $S_0(S_0 = 4S_t)$. Assume that almost all the air (density ρ) in front of the train flows back between its sides and the walls of the tunnel. Also, the air flow with respect to the train is steady and laminar. Take the ambient pressure and that inside the train to be p_0. If the pressure in the region between the sides of the train and the tunnel walls is p, then $p_0 - p = \frac{7}{2N}\rho v_t^2$. The value of N is _____ **(2020)**

54. A hot air balloon is carrying some passengers, and a few sandbags of mass 1 kg each so that its total mass is 480 kg. Its effective volume giving the balloon its buoyancy is V. The balloon is floating at an equilibrium height of 100 m. When N number of sandbags are thrown out, the balloon rises to a new equilibrium height close to 150 m with its volume V remaining unchanged. If the variation of the density of air with height h from the ground is $\rho(h) = \rho_0 e^{\frac{h}{h_0}}$, where ρ_0=1.25 kg m^{-3} and h_0= 6000 m, the value of N is _________. **(2020)**

55. A beaker of radius r is filled with water (refractive index $\frac{4}{3}$) up to a height H as shown in the figure on the left. The beaker is kept on a horizontal table rotating with angular speed ω. This makes the water surface curved so that the difference in the height of water level at the center and at the circumference of the beaker is $h(h << H, h << r)$, as shown in the figure on the right. Take this surface to be approximately spherical with a radius of curvature R. Which of the following is/are correct?

(g is the acceleration due to gravity) **(2020)**



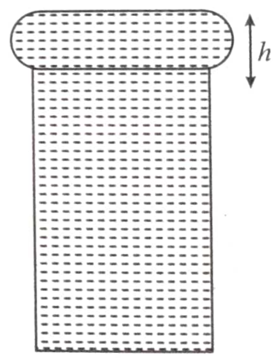
(a) $R = \frac{h^2 + r^2}{2h}$

(b) $R = \frac{3r^2}{2h}$

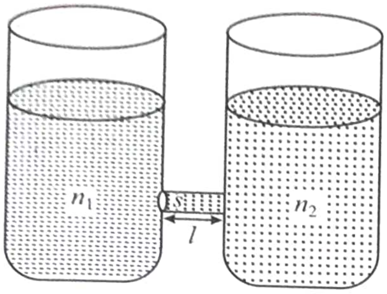
(c) Apparent depth of the bottom of the beaker is close to $\frac{3H}{2}\left(1 + \frac{\omega^2 H}{2g}\right)^{-1}$

(d) Apparent depth of the bottom of the beaker is close to $\frac{3H}{4}\left(1 + \frac{\omega^2 H}{4g}\right)^{-1}$

56. When water is filled carefully in a glass, one can fill it to a height h above the rim of the glass due to the surface tension of water. To calculate h just before water starts flowing, model the shape of the water above the rim as a disc of thickness h having semicircular edges, as shown schematically in the figure. When the pressure of water at the bottom of this disc exceeds what can be withstood due to the surface tension, the water surface breaks near the rim and water starts flowing from there. If the density of water, its surface tension and the acceleration due to gravity are 10^3 kg m^{-3}, 0.07 Nm^{-1} and 10 ms^{-2}, respectively, the value of h (in mm) is ________ **(2020)**



57. As shown schematically in the figure, two vessels contain water solutions (at temperature T) of potassium permanganate ($KMnO_4$) of different concentrations n_1 and n_2 $(n_1 > n_2)$ molecules per unit volume with $\Delta n = (n_1 - n_2)$ $<< n_1$. When they are connected by a tube of small length ℓ and cross-sectional area S, $KMnO_4$ starts to diffuse from the left to the right vessel through the tube. Consider the collection of molecules to behave as dilute ideal gases and the difference in their partial pressure in the two vessels causing the diffusion. The speed v of the molecules is limited by the viscous force $-\beta v$ on each molecule, where β is a constant. Neglecting all terms of the order $(\Delta n)^2$, which of the following is/are correct? (k_B is the Boltzmann constant): **(2020)**



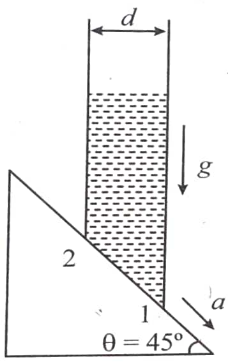
(*a*) The force causing the molecules to move across the tube is $\Delta n k_B T S$

(*b*) Force balance implies $n_1 \beta v \ell = \Delta n k_B T$

(*c*) Total number of molecules going across the tube per sec is $\left(\frac{\Delta n}{\ell}\right)\left(\frac{k_B T}{\beta}\right) S$

(*d*) Rate of molecules getting transferred through the tube does not change with time

58. A cylindrical tube, with its base as shown in the figure, is filled with water. It is moving down with a constant acceleration a along a fixed inclined plane with angle $\theta = 45°$. P_1 and P_2 are pressures at points 1 and 2, respectively, located at the base of the tube. Let $\beta = (P_1 - P_2)/(\rho g d)$, where ρ is density of water, d is the inner diameter of the tube and g is the acceleration due to gravity. Which of the following statement(s) is (are) correct? **(2021)**



(*a*) $\beta = 0$ when $a = g/\sqrt{2}$

(*b*) $\beta > 0$ when $a = g/\sqrt{2}$

(*c*) $\beta = \frac{\sqrt{2}-1}{\sqrt{2}}$ when $a = g/2$

(*d*) $\beta = \frac{1}{\sqrt{2}}$ when

59. A bubble has surface tension S. The ideal gas inside the bubble has ratio of specific heats $\gamma = \frac{5}{3}$. The bubble is exposed to the atmosphere and it always retains its spherical shape. When the atmospheric pressure is P_{a_1}. the radius of the bubble is found to be r_1 and the temperature of the enclosed gas is T_1. When the atmospheric pressure is P_{a_2} radius of the bubble and the temperature of the enclosed gas are r_2 and T_2 respectively. **(2022)**

Which of the following statement(s) is(are) correct?

(*a*) If the surface of the bubble is a perfect heat insulator,

then $\left(\frac{r_1}{r_2}\right)^5 = \frac{P_{a2} + \frac{2S}{r_2}}{P_{a1} + \frac{2S}{r_1}}$.

(*b*) If the surface of the bubble is a perfect heat insulator, then the total internal energy of the bubble including its surface energy does not change with the external atmospheric pressure.

(*c*) If the surface of the bubble is a perfect heat conductor and the change in atmospheric temperature is negligible,

then $\left(\frac{r_1}{r_2}\right)^3 = \frac{P_{a2} + \frac{4S}{r_2}}{P_{a1} + \frac{4S}{r_1}}$

(*d*) If the surface of the bubble is a perfect heat insulator,

then $\left(\frac{T_2}{T_1}\right)^{\frac{5}{2}} = \frac{P_{a2} + \frac{4S}{r_2}}{P_{a1} + \frac{4S}{r_1}}$

ANSWER KEY

CONCEPT APPLICATION

1. (b)	2. (a)	3. (b)	4. (b)	5. (d)	6. (c)	7. (b)	8. (d)	9. (d)	10. (a)
11. (b)	12. (a)	13. (a)	14. (d)	15. (b)	16. (a)	17. (a)	18. (d)	19. (d)	20. (d)
21. (c)	22. (b)	23. (b)	24. (d)	25. (c)	26. (a)	27. (b)	28. (b)	29. (d)	30. (d)
31. (c)	32. (a)	33. (c)	34. (b)	35. (a)	36. (c)	37. (a)	38. (d)	39. (c)	40. (a)
41. (d)	42. (c)	43. (c)	44. (c)	45. (d)	46. (b)				

EXERCISE-1 (TOPICWISE)

1. (a)	2. (c)	3. (c)	4. (a)	5. (c)	6. (d)	7. (a)	8. (a)	9. (a)	10. (b)
11. (d)	12. (a)	13. (c)	14. (c)	15. (b)	16. (b)	17. (a)	18. (b)	19. (b)	20. (b)
21. (a)	22. (c)	23. (d)	24. (b)	25. (a)	26. (a)	27. (a)	28. (a)	29. (d)	30. (b)
31. (a)	32. (d)	33. (a)	34. (b)	35. (d)	36. (c)	37. (a)	38. (a)	39. (d)	40. (c)
41. (c)	42. (b)	43. (d)	44. (a)	45. (c)	46. (b)	47. (b)	48. (b)	49. (c)	50. (b)
51. (a)	52. (d)	53. (c)	54. (c)	55. (b)	56. (c)	57. (a)	58. (c)	59. (d)	60. (c)
61. (d)	62. (b)	63. (a)	64. (a)	65. (b)					

EXERCISE-2 (LEARNING PLUS)

1. (d)	2. (b)	3. (d)	4. (c)	5. (c)	6. (c)	7. (c)	8. (c)	9. (a)	10 (b)
11. (b)	12. (c)	13. (a)	14. (a)	15. (b)	16. (b)	17. (d)	18. (c)	19. (a)	20. (c)
21. (d)	22. (b)	23. (c)	24. (a)	25. (b)	26. (d)	27. (b)	28. (a)	29. (c)	30. (d)
31. (d)	32. (c)	33. (c)	34. (b)	35. (a)	36. (a)	37. (d)	38. (b)	39. (c)	40. (a)
41. (a)									

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (a,c)	2. (a,d)	3. (b,c)	4. (a,c)	5. (a,c,d)	6. (c,d)	7. (b,c)	8. (a,b,c)	9. (a,b,d)	10. (a,d)
11. (b)	12. (a)	13. (a)	14. (d)	15. (b)	16. (a)	17. (b)	18. (a)	19. (d)	20. (d)
21. [50 cm]	22. [6×10^9]		23. [800]	24. [5]	25. [400]	26. [73]	27. [8]	28. [8]	29. [10, 7]
30. [4]	31. [2]								

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (b)	2. (b)	3. (a)	4. (a)	5. (b)	6. (c)	7. (c)	8. (c)	9. (b)	10. (b)
11. (a)	12. (d)	13. (c)	14. (a)	15. (d)	16. (b)	17. (d)	18. (a)	19. (d)	20. (c)
21. (d)	22. (c)	23. (b)	24. (b)	25. [101]	26. (d)	27. (d)	28. (a)	29. [6]	30. (a)
31. (a)	32. (d)	33. (c)	34. (b)	35. (b)	36. [300]	37. [20]	38. [100]	39. [24]	40. [363]

JEE Advanced

41. (a,d)	42. (c)	43. (a)	44. (c)	45. (d)	46. (a,d)	47. [3]	48. [6]	49. (a,c)	50. (a,c,d)
51. (a,c,d)	52. (b)	53. [9]	54. [4]	55. (a,d)	56. [3.74]	57. (a,b,c)	58. (a,c)	59. (c,d)	