PW

ARJUNA

for

JEE
MAIN & ADVANCED

MATHEMATICS

FULL COURSE STUDY MATERIAL

Class XI

- Introduction to Three Dimensional Geometry
- Limits and Derivatives
- Mathematical Reasoning
- Statistics
- Probability
- Heights and Distances
- Solutions of Triangles

Module-4

CONTENTS

Mathematics Module-4 1-130

CHAPTER

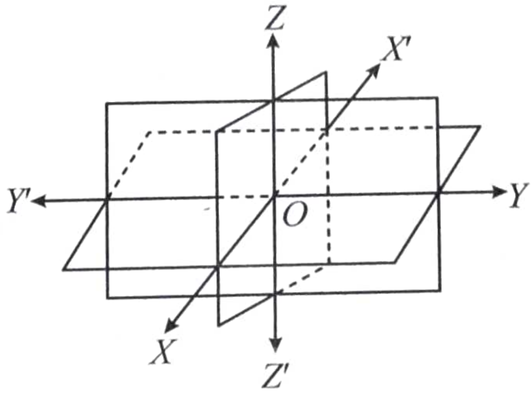
19 Introduction to Three Dimensional Geometry

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

You may recall that to locate the position of a point in a plane, we need two intersecting mutually perpendicular lines in the plane. These lines are called the coordinate axes and the two numbers are called the coordinates of the point with respect to the axes. In actual life, we do not have to deal with points lying in a plane only. For example, if we were to locate the position of the lowest tip of an electric bulb hanging from the ceiling of a room or the position of the central tip of the ceiling fan in a room, we will not only require the perpendicular distances of the point to be located from two perpendicular walls of the room but also the height of the point from the floor of the room. Therefore, we need not only two but three numbers representing the perpendicular distances of the point from three mutually perpendicular planes, namely the floor of the room and two adjacent walls of the room. The three numbers representing the three distances are called the coordinates of the point with reference to the three coordinate planes. So, a point in space has three coordinates.

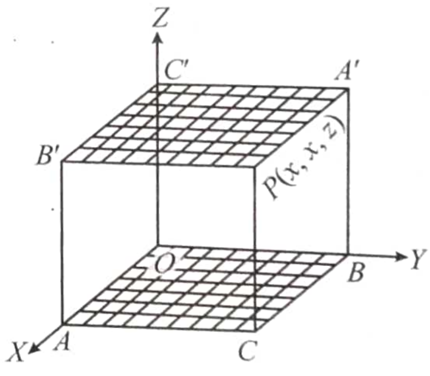
Coordinate Axes and Coordinate Planes in Three Dimensional Space

Consider three planes intersecting at a point O such that these three planes are mutually perpendicular to each other. These three planes intersect along the lines $X'OX$, $Y'OY$ and $Z'OZ$, called the x, y and z-axes, respectively. We may note that these lines are mutually perpendicular to each other. These lines constitute the rectangular coordinate system. The planes XOY, YOZ and ZOX, called, respectively the XY-plane, YZ-plane and the ZX-plane, are known as the three coordinate planes. We take the XOY plane as the plane of the paper and the line $Z'OZ$ as perpendicular to the plane XOY. If the plane of the paper is considered as horizontal, then the line $Z'OZ$ will be vertical. The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ' are taken as negative. Similarly, the distance measured to the right of ZX-plane along OY are taken as positive, to the left of ZX-plane and along OY' as negative, in front of the YZ-plane along OX as positive and to the back of it along OX' as negative. The point O is called the origin of the coordinate system.



Coordinates of a Point in Space

P be a point in space. Through P draw three planes parallel to the coordinate axes to meet the axes in A, B and C respectively. Let $OA = x$, $OB = y$ and $OC = z$. These three real numbers taken this order determined by the point P are called the coordinates of the point P, written as (x, y, z), x, y, z are positive or negative according as they are measured along positive and negative directions of the coordinate axes.

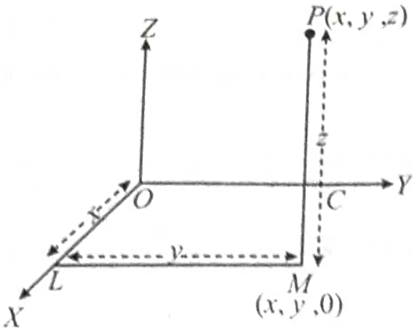


$x = OA = CB' = PA'$ = perpendicular distance from P on the YOZ plane

$y = OB = A'C = PB'$ = Perpendicular distance from P on the ZOX plane

$z = OC = A'B = PC'$ = Perpendicular distance from P on the XOY plane.

Thus, the coordinates of the point P are the perpendicular distance from P on the three mutually rectangular coordinate planes YOZ, ZOX and XOY respectively.



Alternatively, to find the coordinates of a point P in space, we first draw Perpendicualr PM on the xy-plane with M as the foot of this perpendicular as shown in figure. Now, from the point M, we draw perpendicular ML on x-axis with L as the foot of this perpendicular. If $OL = a$, $LM = b$ and $PM = c$, then we say that a, b and c are x, y, and z coordinates, respectively, of the point P in space. In such a case, we say that the point P has coordinates (a, b, c).

Note: The coordinates of the origin O are $(0, 0, 0)$. The coordinates of any point on the x-axis will be as $(x, 0, 0)$ and the coordinates of any point in the YZ-plane will be as $(0, y, z)$.

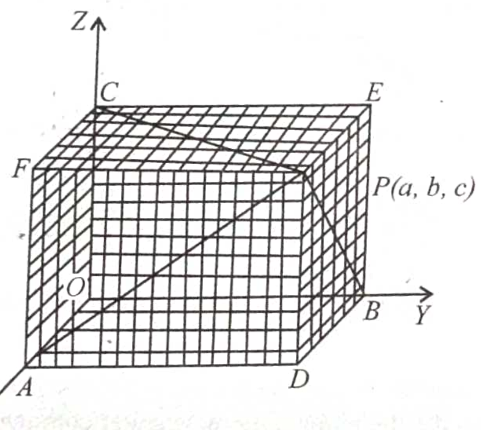
Sign of the Coordinates of a Point

The sign of the coordinates of a point determine the octant in which the point lies. The following table shows the signs of the coordinates in eight octants.

Octants / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

Train Your Brain

Example 1: In figure, if the coordinates of point P are (a, b, c), then



(*i*) write the coordinates of points A, B, C, D, E and F.

(*ii*) write the coordinates of the feet of the perpendiculars from the point P to coordinate axes.

(*iii*) write the coordinates of the feet of the perpendicular from the point P on the coordinate planes XY, YZ and ZX.

(*iv*) Find the perpendicular distances of point P from XY, YZ and ZX-planes.

(*v*) Find the perpendicular distances of the point P from the coordinate axes,

(*vi*) Find the coordinates of the reflection of P in XY, YZ and ZX-planes.

Sol. (*i*) Since the coordinates of P are (a, b, c). Therefore, $OA = a$, $OB = b$ and $OC = c$.

Now, A lies on OX such that $OA = a$. So, the coordinates of A are $(a, 0, 0)$.

Similarly, coordinates of B and C are $(0, b, 0)$ and $(0, 0, c)$ respectively.

Since D lies in XY-plane such that $OA = a$ and $AD = OB = b$. So, the coordinates of D are $(a, b, 0)$. Point E lies in YZ-plane such that $OB = b$ and $BE = OC = c$. So, the coordinates of E are $(0, b, c)$. Similarly, coordinates of F are $(a, 0, c)$ as it lies in XZ-plane.

(*ii*) PA, PB and PC are perpendiculars from P on OX, OY and OZ respectively. So, A, B and C are the feet of perpendiculars from P on OX, OY and OZ respectively. Their coordinates are $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ as discussed in (*i*).

(*iii*) Clearly, PD, PE and PF are the perpendiculars from P on XY, YZ and ZX-planes respectively, So, D, E and F are the feet of the perpendiculars from P on XY, YZ and ZX-planes. The coordinates of D, E and F are $D(a, b, 0)$, $E(0, b, c)$ and $F(a, 0, c)$ respectively as discussed in (*i*).

(*iv*) PD, PE and PF are the perpendicular distances of P from XY, YZ and ZX-planes respectively.

$\therefore$ $PD = OC = c$, $PE = OA = a$ and $PF = OB = b$.

Hence, the perpendicular distances of $P(a, b, c)$ from XY, YZ and ZX planes are c, a and b respectively.

(*v*) PA, PB and PC are the perpendicular distances of point P from OX, OY and OZ respectively.

In right-angled triangle ADP, we have

$AP^2 = AD^2 + DP^2$

$\Rightarrow PA = \sqrt{AD^2 + DP^2} = \sqrt{b^2 + c^2}$

$[\because AD = OB = b$ and $PD = OC = c]$

In right-angled triangle PDB right angled at D, we have

$PB^2 = BD^2 + PD^2$

$\Rightarrow PB = \sqrt{BD^2 + PD^2} = \sqrt{a^2 + c^2}$

$[\because BD = OA = a \text{ and } PD = OC = c]$

In right-angled triangle PCF right angled at F, we have

$PC^2 = PF^2 + CF^2$

$\Rightarrow PC = \sqrt{PF^2 + CF^2}$

$\Rightarrow PC = \sqrt{b^2 + a^2}$

$[\because PF = AD = OB = b \text{ and } CF = OA = a]$

$\Rightarrow PC = \sqrt{a^2 + b^2}$

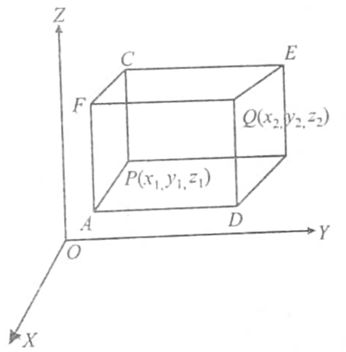
(*vi*) The reflection or image of $P(a, b, c)$ in xy-plane will be as much below the xy-plane as point P is above it, that is, if P' is the reflection of P in xy-plane, then $P'D = PD = c$ and $P'D$ is parallel to OZ'. So, the coordinate of P' are $(a, b, -c)$.

The image of $P(a, b, c)$ in yz-plane will be as much on the back side of yz-plane as the point P is on its front side. Thus, of P'' is the image of P in yz-plane, then P'' lies on PE such that $PE = EP''$. But, $PE = OA = a$. So, the coordinates of P'' are $(-a, b, c)$.

The image of $P(a, b, c)$ in zx-plane will be as much as on the left side of xz-plane as the point P is on its right side. Thus, if P'' is the image of P in zx-plane P'' lies on PF produced such that $PF = FP''$. But, $PF = OB = b$. So, the coordinates of P'' are $(a, -b, c)$.

Example 2: Planes are drawn parallel to the coordinate planes through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Find the length of the edges of the parallelopiped so formed.

Sol. Clearly, PA, PB and PC are the lengths of the edges of the parallelopiped shown in figure.



Clearly, $PBEC$, $QDAF$ are planes parallel to yz-plane such that their distances from yz-plane are x_1 and x_2 respectively. So, $PA =$ (Distance between the planes $PBEC$ and $QDAF$) $= x_2 - x_1$.

PB is the distance between the planes $PAFC$ and $BDQE$ which are parallel to zx-plane and are at distances y_1 and y_2, respectively, from zx-plane.

$\therefore \quad PB = y_2 - y_1$

Similarly, PC is the distance between the parallel planes $PBDA$ and $CEQF$ which are at distance z_1 and z_2, respectively, from xy-plane.

$\therefore \quad PC = z_2 - z_1$

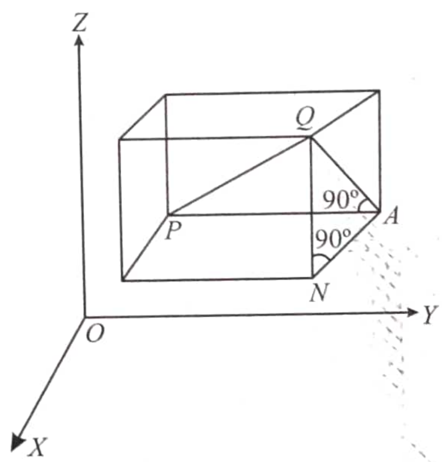
Concept Application

1. Name the octants in which the following points lie : (1, 2, 3), (4, –2, 3), (4, –2, –5), (4, 2, –5), (–4, 2, –5), (–4, 2, 5), (–3, –1, 6) (–2, –4, –7).
2. Find the image of :
   (*i*) (–2, 3, 4) in the yz-plane.
   (*ii*) (–5, 4, – 3) in the xz-plane.
   (*iii*) (5, 2, – 7) in the xy-plane.
3. A cube of side 5 has one vertex at the point (1, 0, –1), and the three edges from this vertex are, respectively, parallel to the negative x and y axes and positive z-axis. Find the coordinates of the other vertices of the cube.
4. Planes are drawn through the points (5, 0, 2) and (3, –2, 5) parallel to the coordinate planes. Find the lengths of the edges of the rectangular parallelepiped so formed.

Distance Between Two Points

We have studied about the distance between two points in two-dimensional coordinate system. Let us now extend this study to three-dimensional system.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points referred to a system of rectangular axes OX, OY and OZ. Through the points P and Q draw planes parallel to the coordinate planes so as to form a rectangular parallelopiped with one diagonal PQ.



Now, since $\angle PAQ$ is a right angle it follows that, in triangle PAQ,

$PQ^2 = PA^2 + AQ^2$ ... (i)

Also, triangle ANQ is right triangle with $\angle ANQ$ a right angle.

Therefore $AQ^2 = AN^2 + NQ^2$ ... (ii)

From (i) and (ii), we have

$PQ^2 = PA^2 + AN^2 + NQ^2$

Now $PA = y_2 - y_1$, $AN = x_2 - x_1$ and $NQ = z_2 - z_1$

Hence $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

Therefore $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

This gives us the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

In particular, if $x_1 = y_1 = z_1 = 0$, i.e., point P is origin O, then $OQ = \sqrt{x_2^2 + y_2^2 + z_2^2}$,

Which gives the distance between the origin O and any point Q (x_2, y_2, z_2).

Train Your Brain

Example 3: Find the distance between the points $P(1, -3, 4)$ and $Q(-4, 1, 2)$.

Sol. The distance PQ between the points $P(1, -3, 4)$ and $Q(-4, 1, 2)$ is

$$PQ = \sqrt{(-4-1)^2 + (1+3)^2 + (2-4)^2}$$

$$= \sqrt{25+16+4} = \sqrt{45} = 3\sqrt{5} \text{ units}$$

Example 4: Show that the points $P(-2, 3, 5)$, $Q(1, 2, 3)$ and $R(7, 0, -1)$ are collinear.

Sol. We know that points are said to be collinear if they lie on a line.

$$\text{Now, PQ} = \sqrt{(1+2)^2 + (2-3)^2 + (3-5)^2} = \sqrt{9+1+4} = \sqrt{14}$$

$$\text{QR} = \sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2}$$

$$= \sqrt{36+4+16} = \sqrt{56} = 2\sqrt{14}$$

$$\text{and PR} = \sqrt{(7+2)^2 + (0-3)^2 + (-1-5)^2}$$

$$= \sqrt{81+9+36} = \sqrt{126} = 3\sqrt{14}$$

Thus, $PQ + QR = PR$. Hence, P, Q and R are collinear.

Example 5: Determine the point in XY-plane which is equidistant from three points $A(2, 0, 3)$, $B(0, 3, 2)$ and $C(0, 0, 1)$.

Sol. we know that z-coordinate of every point on xy-plane zero. So, let $P(x, y, 0)$ be a point on xy-plane such that $PA = PB = PC$.

Now, $PA = PB$

$\Rightarrow PA^2 = PB^2$

$\Rightarrow (x-2)^2 + (y-0)^2 + (0-3)^2 = (x-0)^2 + (y-3)^2 + (0-2)^2$

$\Rightarrow 4x - 6y = 0 \Rightarrow 2x - 3y = 0$ ...(*i*)

$PB = PC$

$\Rightarrow PB^2 = PC^2$

$\Rightarrow (x-0)^2 + (y-3)^2 + (0-2)^2 = (x-0)^2 + (y-0)^2 + (0-1)^2$

$\Rightarrow -6y + 12 = 0 \Rightarrow y = 2$ ...(*ii*)

Putting $y = 2$ in (*i*), we obtain $x = 3$. Hence, the required point has the coordinates $(3, 2, 0)$.

Example 6: Find the locus of the point which is equidistant from the points $A(0, 2, 3)$ and $2, -2, 1)$.

Sol. Let $P(x, y, z)$ be any point which is equidistant from $A(0, 2, 3)$ and $B(2, -2, 1)$.

Then,

$PA = PB \Rightarrow PA^2 = PB^2$

$\Rightarrow (x-0)^2 + (y-2)^2 + (z-3)^2 = (x-2)^2 + (y+2)^2 + (z-1)^2$

$\Rightarrow 4x - 8y - 4z + 4 = 0 \Rightarrow x - 2y - z + 1 = 0$

Hence, the required locus is $x - 2y - z + 1 = 0$.

Concept Application

5. Verify the following :
   (*i*) $(0, 7, -10)$, $(1, 6, -6)$ and $(4, 9, -6)$ are the vertices of an isosceles triangle.
   (*ii*) $(0, 7, 10)$, $(-1, 6, 6)$ and $(-4, 9, 6)$ are the vertices of a right angled triangle.
   (*iii*) $(-1, 2, 1)$, $(1, -2, 5)$, $(4, -7, 8)$ and $(2, -3, 4)$ are the vertices of a parallelogram.
6. Find the equation of the set of points which are equidistant from the points $(1, 2, 3)$ and $(3, 2, -1)$.
7. Are the points $A(3, 6, 9)$, $B(10, 20, 30)$ and $C(25, -41, 5)$, the vertices of a right angled triangle?
8. Find the locus of P if $PA^2 + PB^2 = 2k^2$, where A and B are the points $(3, 4, 5)$ and $(-1, 3, -7)$, respectively.

SECTION FORMULA

Let the two given points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Let the point $R(x, y, z)$ divide PQ in the given ratio $m : n$ internally, then coordinate of point R will be

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

If the point R divides PQ externally in the ratio $m : n$, then is coordinates are obtained by replacing n by $-n$ so that coordinates of point R will be

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

Case 1: Coordinates of the mid-point : In case R is the mid-point of PQ, then $m : n = 1 : 1$ so that $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$ and $z = \frac{z_1 + z_2}{2}$.

These are the coordinates of the mid point of the segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Case 2: The coordinates of the point R which divides PQ in the ratio $k : 1$ are obtained by taking $k = \frac{m}{n}$ which are as given below:

$$\left(\frac{kx_2 + x_1}{1+k}, \frac{ky_2 + y_1}{1+k}, \frac{kz_2 + z_1}{1+k}\right)$$

Generally, this result is used in solving problems involving a general point on the line passing through two given points.

Train Your Brain

Example 7: Find the coordinates of the point which divides the line segment joining the points (1, – 2, 3) and (3, 4, –5) in the ratio 2 : 3 (*i*) internally, and (*ii*) externally.

Sol. (*i*) Let $P(x, y, z)$ be the point which divides line segment joining $A(1, -2, 3)$ and $B(3, 4, -5)$ internally in the ratio 2 : 3. Therefore

$$x = \frac{2(3)+3(1)}{2+3} = \frac{9}{5},\ y = \frac{2(4)+3(-2)}{2+3} = \frac{2}{5},$$

$$z = \frac{2(-5)+3(3)}{2+3} = \frac{-1}{5}$$

Thus, the required point is $\left(\frac{9}{5}, \frac{2}{5}, \frac{-1}{5}\right)$

(*ii*) Let $P(x, y, z)$ be the point which divides segment joining $A(1, -2, 3)$ and $B(3, 4, -5)$ externally in the ratio 2 : 3. Then

$$x = \frac{2(3)+(-3)(1)}{2+(-3)} = -3,\ y = \frac{2(4)+(-3)(-2)}{2+(-3)} = -14,$$

$$z = \frac{2(-5)+(-3)(3)}{2+(-3)} = 19,$$

Therefore, the required point is (–3, –14, 19).

Example 8: Using section formula, Prove that the three points (– 4, 6, 10), (2, 4, 6) and (14, 0, –2) are collinear.

Sol. Let $A(-4, 6, 10)$, $B(2, 4, 6)$ and $C(14, 0, -2)$ be the given point. Let the point P divides AB in the ratio $k : 1$.

Then coordinates of the point P are

$$\left(\frac{2k-4}{k+1}, \frac{4k+6}{k+1}, \frac{6k+10}{k+1}\right)$$

Let us examine whether for same value of k, the point P coincides with point C. on putting $\frac{2k-4}{k+1} = 14$, we get

$$k = -\frac{3}{2}$$

When $k = -\frac{3}{2}$, then $\frac{4k+6}{k+1} = \frac{4(-\frac{3}{2})+6}{-\frac{3}{2}+1} = 0$ and

$$\frac{6k+10}{k+1} = \frac{4(-\frac{3}{2})+10}{-\frac{3}{2}+1} = -2$$

Therefore, $C(14, 0, -2)$ is a point which divides AB externally in the ratio 3 : 2 and is same as P. Hence A, B, C are collinear.

Example 9: Find the coordinates of the centroid of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Sol. Let ABC be the triangle. Let the coordinates of the vertices A, B, C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (z_3, y_3, z_3), respectively. Let D be the mid-point of BC. Hence coordinates of D are

$$\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2}\right)$$

Let G be the centroid of the triangle. Therefore, it divides the median AD in the ratio 2 : 1.

Hence, the coordinates of G are

$$\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+x_1}{2+1}, \frac{2\left(\frac{y_2+y_3}{2}\right)+y_1}{2+1}, \frac{2\left(\frac{z_2+z_3}{2}\right)+z_1}{2+1}\right)$$

or $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

Example 10: Find the ratio in which the line segment joining the points (4, 8, 10) and (6, 10, – 8) is divided by the YZ-plane.

Sol. Let YZ-plane divides the line segment joining $A(4, 8, 10)$ and $B(6, 10, -8)$ at $P(x, y, z)$ in the ratio $k : 1$. The coordinates of P are

$$\left(\frac{4+6k}{k+1}, \frac{8+10k}{k+1}, \frac{10-8k}{k+1}\right)$$

Since P lies on the YZ-plane, its x-coordinate is zero, i.e., $\frac{4+6k}{k+1} = 0$ or $k = -\frac{2}{3}$

Therefore, YZ-plane divides AB externally in the ratio 2 : 3.

Concept Application

9. Find the coordinates of the point which divides the line segment joining the points (– 2, 3, 5) and (1, –4, 6) in the ratio (*i*) 2 : 3 internally, (*ii*) 2 : 3 externally.

10. Given that $P(3, 2, -4)$, $Q(5, 4, -6)$ and $R(9, 8, -10)$ are collinear. Find the ratio in which Q divides PR.

11. Using section formula, show that the points $A(2, -3, 4)$, $B(-1, 2, 1)$ and $C\left(0, \frac{1}{3}, 2\right)$ are collinear.

12. Find the coordinates of the points which trisect the line segment joining the points $P(4, 2, -6)$ and $Q(10, -16, 6)$.

Short Notes

1. In three dimensions, the coordinate axes of a rectangular Cartesian coordinate system are three mutually perpendicular lines. The axes are called the x, y and z-axes.

2. The three planes determined by the pair of axes are the coordinate planes, called XY, YZ and ZX-planes. The three coordinate planes divide the space into eight parts known as octants.

3. The coordinates of a point P in three dimensional geometry is always written in the form of triplet like (x, y, z). Here x, y and z are the distances from the YZ, ZX and XY-planes.

4. (*i*) Any point on x-axis is of the form $(x, 0, 0)$
   (*ii*) Any point on y-axis is of the form $(0, y, 0)$
   (*iii*) Any point on z-axis is of the form $(0, 0, z)$.

5. Distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ given by $PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

6. The coordinates of the point R which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally and externally in the ratio $m : n$ are given by
$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right) \text{ and}$$
$$\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right), \text{ respectively.}$$

7. The coordinates of the mid-point of the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

Solved Examples

1. Show that the points $A(1, 2, 3)$, $B(-1, -2, -1)$, $C(2, 3, 2)$ and $D(4, 7, 6)$ are the vertices of a parallelogram $ABCD$, but it is not rectangle.

Sol. To show $ABCD$ is a parallelogram we need to show opposite side are equal.

$$AB = \sqrt{(-1-1)^2+(-2-2)^2+(-1-3)^2} = \sqrt{4+16+16} = 6$$

$$BC = \sqrt{(2+1)^2+(3+2)^2+(2+1)^2} = \sqrt{9+25+9} = \sqrt{43}$$

$$CD = \sqrt{(4-2)^2+(7-3)^2+(6-2)^2} = \sqrt{4+16+16} = 6$$

$$DA = \sqrt{(1-4)^2+(2-7)^2+(3-6)^2} = \sqrt{9+25+9} = \sqrt{43}$$

Since $AB = CD$ and $BC = AD$, $ABCD$ is a parallelogram.

Now, it is required to prove that $ABCD$ is not a rectangle. For this, we show that diagonals AC and BD are unequal. We have

$$AC = \sqrt{(2-1)^2+(3-2)^2+(2-3)^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$BD = \sqrt{(4+1)^2+(7+2)^2+(6+1)^2} = \sqrt{25+81+49} = \sqrt{155}$$

Since $AC \neq BD$, $ABCD$ is not a rectangle.

Note : We can also show that $ABCD$ is a parallelogram, using the property that diagonals AC and BD bisect each other.

2. Find the equation of the set of the points P such that its distances from the points $A(3, 4, -5)$ and $B(-2, 1, 4)$ are equal.

Sol. If $P(x, y, z)$ be any point such that $PA = PB$.

Now

$$\sqrt{(x-3)^2+(y-4)^2+(z+5)^2} = \sqrt{(x+2)^2+(y-1)^2+(z-4)^2}$$

or $(x-3)^2 + (y-4)^2 + (z+5)^2 = (x+2)^2 + (y-1)^2 + (z-4)^2$

or $10x + 6y - 18z - 29 = 0$.

3. The centroid of a triangle ABC is at the point (1, 1, 1). If the coordinates of A and B are (3, –5, 7) and (–1, 7, – 6), respectively, find the coordinates of the point C.

Sol. Let the coordinates of C be (x, y, z) and the coordinates of the centroid G be (1, 1, 1). Then

$$\frac{x+3-1}{3}=1, \text{ i.e.,} x=1; \frac{y-5+7}{3}=1, \text{ i.e.,} y=1; \frac{z+7-6}{3}=1,$$

i.e., $z = 2$.

Hence, coordinates of C are (1, 1, 2).

4. The mid-points of the sides of a triangle are (1, 5, –1), (0, 4, – 2) and (2, 3, 4). Find its vertices.

Sol. Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of the given triangle, and let $D(1, 5, -1)$, $E(0, 4, -2)$ and $F(2, 3, 4)$ be the mid-points of the sides BC, CA and AB respectively.

D is the mid-point of BC

$$\therefore \quad \frac{x_2+x_3}{2}=1, \frac{y_2+y_3}{2}=5, \frac{z_2+z_3}{2}=-1$$

$\Rightarrow x_2+x_3=2, y_2+y_3=10, z_2+z_3=-2$ ...(*i*)

E is the mid-point of CA

$$\therefore \quad \frac{x_1+x_3}{2}=0, \frac{y_1+y_3}{2}=4, \frac{z_1+z_3}{2}=-2$$

$\Rightarrow x_1+x_3=0, y_1+y_3=8, z_1+z_3=-4$ ...(*ii*)

F is the mid-point of AB

$$\therefore \quad \frac{x_1+x_2}{2}=2, \frac{y_1+y_2}{2}=3, \frac{z_1+z_2}{2}=4$$

$\Rightarrow x_1+x_2=4, y_1+y_2=6, z_1+z_2=8$ ...(*iii*)

Adding first three equations in (*i*), (*ii*) and (*iii*), we obtain $2(x_1+x_2+x_3) = 2+0+4 \Rightarrow x_1+x_2+x_3 = 3$, we obtain: $x_1 = 1, x_2 = 3, x_3 = -1$.

Adding second equation in (*i*), (*ii*) and (*iii*), we obtain $2(y_1+y_2+y_3) = 10+8+6 \Rightarrow y_1+y_2+y_3 = 12$, we obtain: $y_1 = 2, y_2 = 4, y_3 = 6$.

Solving last equations in (*i*), (*ii*) and (*iii*) with $z_1+z_2+z_3 = 1$, we obtain

$z_1 = 3, z_2 = 5, z_3 = -7$.

Thus the vertices of the triangle are $A(1, 2, 3)$, $B(3, 4, 5)$ and $C(-1, 6, -7)$.

5. Given that $P(3, 2, -4)$, $Q(5, 4, -6)$ and $R(9, 8, -10)$ are collinear. Find the ratio in which Q divide PR.

Sol. Suppose Q divides PR in the ratio $\lambda : 1$. Then coordinates of Q are $\left(\frac{9\lambda+3}{\lambda+1}, \frac{8\lambda+2}{\lambda+1}, \frac{-10\lambda-4}{\lambda+1}\right)$

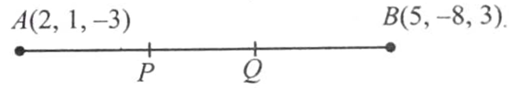
But. Coordinates of Q are (5, 4, –6). Therefore,

$$\frac{9\lambda+3}{\lambda+1}=5, \frac{8\lambda+2}{\lambda+1}=4, \frac{-10\lambda-4}{\lambda+1}=-6$$

These three equations give $\lambda = \frac{1}{2}$. So, Q divides PR in the ratio $\frac{1}{2} : 1$ or $1 : 2$.

6. Find the coordinates of the points which trisect the line segment AB, given that $A(2, 1, -3)$ and $B(5, -8, 3)$.

Sol. Let P and Q be the points which trisect AB. Then, $AP = PQ = QB$. Therefore, P divicles AB in the ratio 1 : 2 and Q divides it in the ratio 2 : 1.



As P divides AB in the ratio 1 : 2, so coordinates of P are

$$\left(\frac{1\times5+2\times2}{1+2}, \frac{1\times-8+2\times1}{1+2}, \frac{1\times3+2\times-3}{1+2}\right) = (3,-2,-1)$$

Since Q divides AB in the ratio 2 : 1. So coordinates of Q are

$$\left(\frac{2\times5+1\times2}{2+1}, \frac{2\times-8+1\times1}{2+1}, \frac{2\times3+1\times-3}{1+2}\right) = (4,-5,1)$$

7. Three vertices of a parallelogram $ABCD$ are $A(3, -1, 2)$, $B(1, 2, -4)$ and $C(-1, 1, 2)$. Find the coordinates of the fourth vertex.

Sol. Let the coordinates of fourth vertex D be (x, y, z). Since diagonals of a parallelogram bisect each other. Therefore, mid-point of AC and BD coincide.

$$\therefore \left(\frac{3-1}{2}, \frac{-1+1}{2}, \frac{2+2}{2}\right) = \left(\frac{1+x}{2}, \frac{2+y}{2}, \frac{-4+z}{2}\right)$$

$$\Rightarrow \quad (1,0,2) = \left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z-4}{2}\right)$$

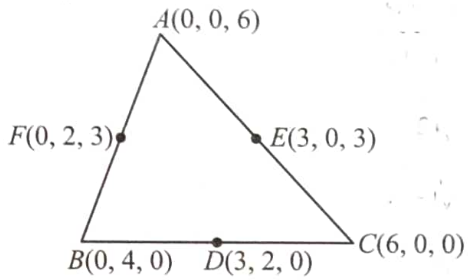
$$\Rightarrow \quad \frac{x+1}{2}=1, \frac{y+2}{2}=0, \frac{z-4}{2}=2$$

$\Rightarrow x = 1, y = -2$, and $z = 8$

Hence, the coordinates of the fourth vertex are (1, –2, 8).

8. Find the lengths of the medians of the triangle with vertices $A(0, 0, 6)$, $B(0, 4, 0)$ and $C(6, 0, 0)$.

Sol. Let D, E and F be the mid-points of sides BC, CA and AB respectively.



The coordinates of D, E and F are

$$\left(\frac{0+6}{2},\frac{4+0}{2},\frac{0+0}{2}\right)=(3,2,0), E\left(\frac{6+0}{2},\frac{0+0}{2},\frac{0+6}{2}\right)=(3,0,3)$$

and $F\left(\frac{0+0}{2},\frac{0+4}{2},\frac{6+0}{2}\right)=(0,2,3)$ respectively.

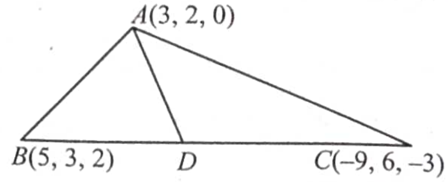
$$\therefore\ AD=\sqrt{(0-3)^2+(0-2)^2+(6-0)^2}=\sqrt{9+4+36}=7$$

$$BE=\sqrt{(0-3)^2+(4-0)^2+(0-3)^2}=\sqrt{9+16+9}=\sqrt{34}$$

and, $CF=\sqrt{(6-0)^2+(0-2)^2+(0-3)^2}=\sqrt{36+4+9}=7$

9. Let $A(3, 2, 0)$, $B(5, 3, 2)$, $C(-9, 6, -3)$ be three points forming a tringle. The bisector AD of $\angle BAC$ meets side BC in D. Find the coordinates of D.

Sol. The bisector AD of $\angle BAC$ divides BC in the ratio $AB : AC$.



Now, $AB=\sqrt{(3-5)^2+(2-3)^2+(0-2)^2}=3$

and, $AC=\sqrt{(3+9)^2+(2-6)^2+(0+3)^2}=13$

Thus, D divides BC in the ratio $AB : AC$ i.e. $3 : 13$. Hence, the coordinates of D are

$$\left(\frac{3\times-9+13\times5}{3+13},\frac{3\times6+3\times13}{3+13},\frac{3\times-3+13\times2}{3+13}\right)=\left(\frac{19}{8},\frac{57}{16},\frac{17}{16}\right)$$

Exercise-1 (Topicwise)

1. A point P lies on the x-axis, then what is possible coordinate of point P.
   (*a*) (2, 0, 0) (*b*) (1, 0, 1)
   (*c*) (0, 3, 0) (*d*) (0, 0, 2)

2. A point $P(x, y, z)$ lies in the XZ-plane, then
   (*a*) $x = 0$ (*b*) $y = 0$
   (*c*) $z = 0$ (*d*) $x = 0$ and $y = 0$

3. Find the octant in which the points (–3, 1, 2) and (–3, 1, –2) lie.
   (*a*) II and V (*b*) II and VI
   (*c*) IV and VI (*d*) V and VIII

4. The X-axis and Y-axis taken together determine a plane known as
   (*a*) XY-plane (*b*) YZ-plane
   (*c*) XZ-plane (*d*) None of these

5. The coordinates of points in the XY-plane are of the form
   (*a*) $(x, 0, z)$ (*b*) $(x, y, 0)$
   (*c*) $(x, 0, 0)$ (*d*) $(0, y, z)$

6. Coordinate planes divide the space into ______ octants.
   (*a*) Eight
   (*b*) Four
   (*c*) Two
   (*d*) None of these

7. Find the distances of the point $P(-4, 3, 5)$ from the coordinate axes.
   (*a*) 10 (*b*) 7
   (*c*) 8 (*d*) 5

8. Find the distance between the points P and Q having coordinates (–2, 3, 1) and (2, 1, 2).
   (*a*) $\sqrt{23}$ (*b*) 4
   (*c*) $\sqrt{21}$ (*d*) $\sqrt{19}$

9. Find the coordinates of the point which divides the join of $P(2, -1, 4)$ and $Q(4, 3, 2)$ internally in the ratio 2 : 3
   (*a*) $\left(\frac{12}{5}, \frac{3}{5}, \frac{16}{5}\right)$ (*b*) $\left(\frac{14}{5}, \frac{3}{5}, \frac{16}{5}\right)$
   (*c*) $\left(-\frac{14}{5}, \frac{3}{5}, \frac{13}{5}\right)$ (*d*) $\left(\frac{12}{5}, -\frac{3}{5}, \frac{13}{5}\right)$

10. Find the ratio in which the line joining the points (1, 2, 3) and (–3, 4, –5) is divided by the XY-plane.
    (*a*) 2 : 3, externally
    (*b*) 5 : 3, internally
    (*c*) 3 : 5, internally
    (*d*) 1 : 2, externally

11. What is the locus of a point for which $y = 0$, $z = 0$?
    (*a*) x-axis (*b*) y-axis
    (*c*) z-axis (*d*) yz-plane

12. The coordinates of the foot of the perpendicular drawn from the point $P(3, 4, 5)$ on the yz-plane are
    (*a*) (3, 4, 0) (*b*) (0, 4, 5)
    (*c*) (3, 0, 5) (*d*) (3, 0, 0)

13. The coordinates of the foot of the perpendicular from a point $P(6, 7, 8)$ on x-axis are
    (*a*) (6, 0, 0) (*b*) (0, 7, 0)
    (*c*) (0, 0, 8) (*d*) (0, 7, 8)

14. The perpendicular distance of the point $P(6, 7, 8)$ from xy-plane is
    (*a*) 8 (*b*) 7
    (*c*) 6 (*d*) 10

15. The length of the perpendicular drawn from the point $P(a, b, c)$ from y-axis is
    (*a*) $\sqrt{a^2 + b^2}$ (*b*) $\sqrt{b^2 + c^2}$
    (*c*) $\sqrt{a^2 + c^2}$ (*d*) $\sqrt{a^2 + b^2 + c^2}$

16. Find the image of the point (–5, 0, 3) in the XZ-plane.
    (*a*) (–5, 0, –3)
    (*b*) (–5, 0, 3)
    (*c*) (5, 0, 3)
    (*d*) (–5, 0, –3)

17. Find the equation of the set of points P, the sum of whose distances from $A(4, 0, 0)$ and $B(-4, 0, 0)$ is equal to 10.
    (*a*) $15x^2 - y^2 - z^2 + 192x + 609 = 0$
    (*b*) $15x^2 + y^2 - z^2 + 192x + 609 = 0$
    (*c*) $15x^2 - y^2 - z^2 + 192x - 609 = 0$
    (*d*) None of these

18. Planes are drawn parallel to the coordinate planes through the points (3, 0, –1) and (–2, 5, 4). Find the lengths of the edges of the parallelepiped so formed.
    (*a*) (5, 5, 5)
    (*b*) (5, 3, 0)
    (*c*) (5, 4, 3)
    (*d*) (5, 4, 4)

19. Determine the point in XY-Plane which is equidistant from three points $A(2, 0, 3)$, $B(0, 3, 2)$ and $C(0, 0, 3)$.
    (*a*) (1, 2, 0) (*b*) $\left(1, \frac{2}{3}, 0\right)$
    (*c*) $\left(2, \frac{3}{2}, 0\right)$ (*d*) (1, 3, 0)

20. Find the coordinates of a point equidistant from the four points $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

(a) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ (b) $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

(c) (a, b, c) (d) None of these

21. Find the ratio in which the join the $A(2, 1, 5)$ and $B(3, 4, 3)$ is divided by the plane $2x + 2y - 2z = 1$.

(a) 5 : 7, internally

(b) 7 : 5, internally

(c) 5 : 7, externally

(d) 3 : 2, internally

22. Determine the point on z-axis which is equidistant from the points (1, 5, 7) and (5, 1, –4).

(a) $\left(0, 0, \frac{1}{2}\right)$ (b) $\left(0, 0, \frac{3}{2}\right)$

(c) $\left(0, \frac{3}{2}, 0\right)$ (d) $\left(0, 0, \frac{5}{2}\right)$

23. Find the point on y-axis which is equidistant from the points (3, 1, 2) and (5, 5, 2).

(a) (0, 5, 0)

(b) (0, 4, 0)

(c) (2, 0, 0)

(d) (0, 1, 0)

24. Find the points on z-axis which are at a distance $\sqrt{21}$ from the point (1, 2, 3).

(a) (0, 0, 1) (b) (0, 0, 3)

(c) (0, 0, 7) (d) (0, 0, –1)

25. Let P and Q be any two points. Find the coordinates of the point R which divides PQ externally in the ratio 2 : 1

(a) $(2x_2 - x_1, 2y_2 - y_1, 2z_2 - z_1)$

(b) $(x_2 - 2x_1, y_2 - 2y_1, z_2 - 2z_1)$

(c) $(2x_2 + x_1, 2y_2 + y_1, 2z_2 + z_1)$

(d) None of these

26. Find the ratio in which the line segment joining the points (2, –1, 3) and (–1, 2, 1) is divided by the plane $x + y + z = 5$.

(a) 1 : 3, externally

(b) 5 : 1, internally

(c) 1 : 5, internally

(d) 3 : 2, internally

27. $A(1, 2, 3)$, $B(0, 4, 1)$, $C(-1, -1, -3)$ are the vertices of a triangle ABC. Find the point in which the bisector of the angle $\angle BAC$ meets BC.

(a) $\left(-\frac{3}{10}, \frac{5}{2}, -\frac{1}{5}\right)$ (b) $\left(\frac{3}{10}, \frac{5}{2}, -\frac{2}{5}\right)$

(c) $\left(\frac{7}{10}, -\frac{5}{2}, -\frac{2}{5}\right)$ (d) $\left(\frac{7}{10}, \frac{7}{2}, -\frac{1}{5}\right)$

28. Find the centroid of a triangle, mid-points of whose sides are (1, 2, –3), (3, 0, 1) and (–1, 1, –4).

(a) (1, 1, 3) (b) (–1, –1, –2)

(c) (1, 1, –2) (d) (–1, 0, 3)

29. The centroid of a triangle ABC is at the point (1, 1, 1). If the coordinates of A and B are (3, –5, 7) and (–1, 7, –6) respectively, find the coordinates of the point C.

(a) (1, –1, 3) (b) (1, 1, 2)

(c) (–1, –1, –3) (d) (2, –1, 3)

30. Find the image of the point (–4, 0, 1) in the XY-plane.

(a) (4, 0, 1) (b) (–4, 0, –1)

(c) (4, 0, –1) (d) (–4, 1, 1)

31. A point C with Z-coordinate 8 lies on the line segment joining the point $A(2, -3, 4)$ and $B(8, 0, 10)$ find its coordinate.

(a) (6, –1, 8) (b) (–6, 1, 8)

(c) (–6, –1, 8) (d) (3, –1, 8)

32. If $P(0, 1, 2)$, $Q(4, -2, 1)$ and $O(0, 0, 0)$ are three points, then $\angle POQ$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

33. If the extremities of the diagonal of a square are (1, –2, 3) and (2, –3, 5), then the length of the side is

(a) $\sqrt{6}$ (b) $\sqrt{5}$

(c) $\sqrt{3}$ (d) $\sqrt{7}$

34. The points (5, –4, 2), (4, –3, 1), (7, 6, 4) and (8, –7, 5) are the vertices of

(a) A rectangle (b) A square

(c) A parallelogram (d) None of these

35. In a three dimensional space the equation $x^2 - 5x + 6 = 0$ represents

(a) Points (b) Planes

(c) Curves (d) Pair of straight lines

36. Let (3, 4, –1) and (–1, 2, 3) be the end points of a diameter of a sphere. Then, the radius of the sphere is equal to

(a) 2 (b) 3

(c) 6 (d) 7

Exercise-2 (Learning Plus)

1. The coordinates of a point are $(3, -2, 5)$. Write down the coordinates of seven points such that the absolute values of their coordinates are the same as those of the coordinates of the given point.
2. Find the distance between the following pairs of points:
   (i) (2,3,5) and (4,3,1)
   (ii) $(-3, 7, 2)$ and $(2, 4, -1)$
   (iii) $(-1, 3, -4)$ and $(1, -3, 4)$
   (iv) $(2, -1, 3)$ and $(-2, 1, 3)$.
3. Show that the points $(-2, 3, 5)$, $(1, 2, 3)$ and $(7, 0, -1)$ are collinear.
4. Find the distance between the following pairs of points :
   (i) $P(1, -1, 0)$ and $Q(2, 1, 2)$
   (ii) $A(3, 2, -1)$ and $B(-1, -1, -1)$
5. Using distance formula prove that the following points are collinear:
   (i) $A(4, -3, -1)$, $B(5, -7, 6)$ and $C(3, 1, -8)$
   (ii) $P(0, 7, -7)$, $Q(1, 4, -5)$ and $R(-1, 10, -9)$
   (iii) $A(3, -5, 1)$, $B(-1, 0, 8)$ and $C(7, -10, -6)$
6. Find the ratio in which the line joining (2, 4, 5) and (3, 5, 4) is divided by the yz-plane.
7. Using section formula, Prove that the three points $A(-2, 3, 5)$, $B(1, 2, 3)$ and $C(7, 0, -1)$ are collinear.
8. Determine the points in (i) xy-plane (ii) yz-plane and (iii) zx-plane which are equidistant from the points $A(1, -1, 0)$, $B(2, 1, 2)$ and $C(3, 2, -1)$.
9. Prove that the triangle formed by joining the three points whose coordinates are (1, 2, 3), (2, 3, 1) and (3, 1, 2) is an equilateral triangle.
10. Show that the points (0, 7, 10), $(-1, 6, 6)$ and $(-4, 9, 6)$ are the vertices of right-angled triangle.
11. Show that the points $A(3, 3, 3)$, $B(0, 6, 3)$, $C(1, 7, 7)$ and $D(4, 4, 7)$ are the vertices of a square.
12. Prove that the point $A(1, 3, 0)$, $B(-5, 5, 2)$, $C(-9, -1, 2)$ and $D(-3, -3, 0)$ taken is order are the vertices of a parallelogram. Also, show that $ABCD$ is not a rectangle.
13. Show that the points $A(1, 3, 4)$, $B(-1, 6, 10)$, $C(-7, 4, 7)$ and $D(-5, 1, 1)$ are the vertices of a rhombus.
14. Prove that the tetrahedron with vertices at the points $O(0,0,0)$, $A(0, 1, 1)$, $B(1, 0, 1)$ and $C(1, 1, 0)$ is a regular one.
15. If the points $A(3, 2, -4)$, $B(9, 8, -10)$ and $C(5, 4, -6)$ are collinear, find the ratio in which C divides AB.
16. Show that the points (3, 2, 2), $(-1, 4, 2)$, (0, 5, 6), (2, 1, 2) lie on a sphere whose is (1, 3, 4). Find also its radius.
17. If $A(-2, 2, 3)$ and $B(13, -3, 13)$ are two points. Find the locus of a point P which moves in such a way that $3PA = 2PB$.
18. Show that the three points $A(2, 3, 4)$, $B(-1, 2 - 3)$ and $C(-4, 1, -10)$ are collinear and find the ratio in which C divides AB.
19. The mid-points of the sides of a triangle ABC are given by $(-2, 3, 5)$, $(4, -1, 7)$ and $(6, 5, 3)$. Find the coordinates of A, B and C.
20. Find the ratio in which the sphere $x^2 + y^2 + z^2 = 504$ divides the line joining the points $(12, -4, 8)$ and $(27, -9, 18)$.
21. Show that the plane $ax + by + cz + d = 0$ divides the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $-\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$.
22. Find the ratio in which the line segment joining the points (4, 8, 10) and $(6, 10, -8)$ is divided by the YZ-plane.
23. Find the coordinates of the points which tisect the line segment joining the points $P(4, 2, -6)$ and $Q(10, -16, 6)$.
24. Using section formula, show that the points $A(2, -3, 4)$, $B(-1, 2, 1)$ and $C(0, 1/3, 2)$ are collinear.
25. Verify the following
    $(5, -1, 1)$, $(7, -4, 7)$, $(1, -6, 10)$ and $(-1, -3, 4)$ are the vertices of a rhombus

ANSWER KEY

CONCEPT APPLICATION

1. (i)-I, (ii)-IV, (iii)-VIII, (iv)-V, (v)-VI, (vi)-VI, (vii)-II, (viii)-III, (ix)-VII
2. (i)-(2, 3, 4), (ii)-(–5, –4, –3), (iii)-(5, 2, 7)
3. B: (1, 5, –1), C: (1, 0, 4), E: (–4, 0, –1), G: (–4, 0, 4), F: (–4, –5,–1), G: (–4, 0, 4), H: (–9, –5, 4)
4. $a = 2$, $b = 2$, $c = 3$
6. $x = 2z$
7. No
8. $2x^2 + 2y^2 - 2z^2 - 4x - 14y + 4z + 109 - 2k^2 = 0$
9. (i) $\left(\frac{-4}{5}, \frac{1}{5}, \frac{27}{5}\right)$, (ii) (–8, 17, 3)
10. 1 : 2
12. (6, –4, –2) and (8, –10, 2)

EXERCISE-1 (TOPICWISE)

1. (*a*)	2. (*b*)	3. (*b*)	4. (*a*)	5. (*b*)	6. (*a*)	7. (*d*)	8. (*c*)	9. (*b*)	10. (*c*)
11. (*a*)	12. (*b*)	13. (*a*)	14. (*a*)	15. (*c*)	16. (*b*)	17. (*d*)	18. (*a*)	19. (*b*)	20. (*a*)
21. (*a*)	22. (*b*)	23. (*a*)	24. (*c, d*)	25. (*a*)	26. (*a*)	27. (*a*)	28. (*c*)	29. (*b*)	30. (*b*)
31. (*a*)	32. (*d*)	33. (*c*)	34. (*d*)	35. (*b*)	36. (*b*)				

EXERCISE-2 (LEARNING PLUS)

1. (3, 2, 5), (3, 2, –5), (3, –2, –5), (–3, 2, 5), (–3, –2, – 5) and (–3, –2, 5)
2. (*i*) $2\sqrt{5}$, (*ii*) $2\sqrt{26}$, (*iii*) $\sqrt{43}$, (*iv*) $2\sqrt{5}$
4. (*i*) 3, (*ii*) 5
6. 2 : 3, externally
8. (*i*) $\left(\frac{3}{2}, 1, 0\right)$, (*ii*) $\left(0, \frac{31}{16}, -\frac{3}{16}\right)$, (*iii*) $\left(\frac{31}{10}, 0, \frac{1}{5}\right)$
15. 1 : 2, internally
17. $5(x^2 + y^2 + z^2) + 140x - 60y + 50z - 1235 = 0$
18. 2 : 1, externally
19. A(12, 1, 5), B(0, 9, 1), C(–4, –3, 9)
20. 2 : 3, –2 : 3
22. 2 : 3, externally
23. (6, 4, –2) (8, –10, 2)

CHAPTER

20 Limits and Derivatives

LIMIT

Definition

The concept of limit is used to discuss the behaviour of a function close to a certain point.

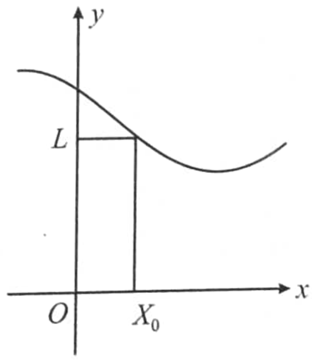
Meaning of $x \to x_0$: Let x is variable and x_0 is a constant, when x takes the values very close to x_0 we say that x approaches to x_0, but x is not equal to x_0. We denotes it by $x \to x_0$

If x approach x_0 from right hand side then we denote it by $x \to x_0^+$ and if x approach x_0 from left hand side then we denote it by $x \to x_0^-$

For e.g.: $x \to 5^- \Rightarrow x$ can take values like 4.999, 4.9999, 4.999999 ... similarly when $x \to 5^+ \Rightarrow x$ can take values like 5.01, 5.0001, 5.0000001 ...

Informal Definition of Limits

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that function approaches the limit L as x approaches x_0, and we write $\lim\limits_{x \to x_0} f(x) = L$.



This definition is "informal" because phrases like arbitrarily close and sufficiently close are imprecise, their meaning depends on the context.

The definition is clear enough and enables us to recognize and evaluate limits of specific function.

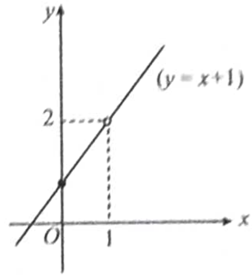
e.g., $f(x) = \dfrac{x^2 - 1}{x - 1}$

Clearly the function is not defined at $x = 1$, but for values close to $x = 1$ the function can be written as

$f(x) = x + 1$

As x approaches 1 (written as $x \to 1$), $f(x)$ approaches the value 2 (i.e., $f(x) \to 2$) we write this as

$\lim\limits_{x \to 1} f(x) = 2$



Left Hand Limit and Right Hand Limit

LHL: It is the value to which function $f(x)$ as x approaches to x_0 from left hand side and we write it as $\lim\limits_{x \to x_0^-} f(x)$.

RHL: It is the value to which function $f(x)$ as x approaches to x_0 from right hand side and we write it as $\lim\limits_{x \to x_0^+} f(x)$.

Existence of limit: $\lim\limits_{x \to x_0} f(x)$ is said to be exist and equal to L and if LHL = RHL = L (a finite quantity).

i.e. $\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x) = L$ (a finite quantity).

e.g. $f(x) = [x]$ (greatest integer function)

For any integer n.

$\lim\limits_{x \to n^-} f(x) = n - 1, \quad \lim\limits_{x \to n^+} f(x) = n$

In such cases we say that $\lim\limits_{x \to n} f(x)$ does not exist.

Train Your Brain

Example 1: Consider the function $f(x) = x^3$. Let us try to find the limit of this function at $x = 1$?

Sol. We tabulate the value of f(x) at x near 1. This is given in the Table.

x	$f(x)$
0.9	0.729
0.99	0.970299
0.999	0.997002999
1.001	1.003003001
1.01	1.030301
1.1	1.331

From this table, we deduce that value of $f(x)$ at $x = 1$ should be greater than 0.997002999 and less than 1.003003001 assuming nothing dramatic happens between

$x = 0.999$ and 1.001. It is reasonable to assume that the value of the $f(x)$ at $x = 1$ as dictated by the numbers to the left of 1 is 1, i.e.,

$$\lim_{x\to 1^-} f(x) = 1.$$

Similarly, when x approaches 1 from the right, $f(x)$ should be taking value 1, i.e.,

$$\lim_{x\to 1^+} f(x) = 1.$$

Hence, it is likely that the left hand limit of $f(x)$ and the right hand limit of f(x) are both equal to 1. Thus,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 1.$$

We note that as x approaches 1 from either right or left, the graph of the function $f(x) = x^3$ approaches the point (1,1).

We observe, again, that the value of the function at $x = 1$ also happens to be equal to 1.

Example 2: We want to find $\lim_{x\to 0} f(x)$, where

$$f(x) = \begin{cases} x-2, & x<0 \\ 0, & x=0 \\ x+2, & x>0 \end{cases}$$

Sol. We make a table of x near 0 with $f(x)$. Observe that for negative values of x we need to evaluate $x - 2$ and for positive values, we need to evaluate $x + 2$.

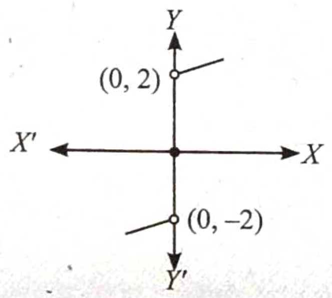
x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	–2.1	–2.01	–2.001	2.001	2.01	2.1

From the first three entries of the Table, we deduce that the value of the function is decreasing to –2 and hence.

$$\lim_{x\to 0^-} f(x) = -2$$

From the last three entires of the table we deduce that the value of the function is increasing from 2 and hence

$$\lim_{x\to 0^+} f(x) = 2$$



Since the left and right hand limits at 0 do not coincide, we say that the limit of the function at 0 does not exist. Graph of this function is given in the figure. Here, we remark that the value of the function at $x = 0$ is well defined and is, indeed, equal to 0, but the limit of the function at $x = 0$ is not even defined.

Example 3: Evaluate the following limits:

$$\lim_{x\to 2}(x+2)$$

Sol. $x + 2$ being a polynomial in x, its limit as $x \to 2$ is given by $\lim_{x\to 2} (x+2) = 2+2 = 4$

Concept Application

1. As a final illustration, we find $\lim_{x\to 1} f(x)$, where

$$f(x) = \begin{cases} x+2 & x \neq 1 \\ 0 & x = 1 \end{cases}$$

FUNDAMENTAL THEOREMS ON LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits. $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$ (where L, K are finite)

1. **Sum Rule:** $\lim_{x\to c} [f(x) + g(x)] = L + K$
2. **Difference Rule:** $\lim_{x\to c} [f(x) - g(x)] = L - K$
3. **Product Rule:** $\lim_{x\to c} [f(x)\, g(x)] = LK$
4. **Quotient Rule:** $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$
5. **Constant Multiplication Rule:** $\lim_{x\to c} [b\, f(x)] = bL$
6. **Composition Rule:** $\text{Limit}_{x\to c} f(g(x)) = f\left(\text{Limit}_{x\to c} g(x)\right) = f(k)$; provided f is continuous at $x = m$.

   For example $\text{Limit}_{x\to c} \ln(f(x)) = \ln\left(\text{Limit}_{x\to a} f(x)\right) = \ln l \,(l > 0)$.

Indeterminate Forms

If on putting $x = a$ in $f(x)$, any one of $\frac{0}{0}, \frac{\infty}{\infty}$, $\mathbf{0 \times \infty, \infty - \infty, \infty^0, 0^0, 1^\infty}$ form is obtained, then the limit has an indeterminate form. All the above forms are interchangeable, i.e. we can change one form to other by suitable substitutions etc.

In such cases $\lim_{x\to a} f(x)$ may exist.

Consider $f(x) = \dfrac{x^2-4}{x-2}$.

Here $\lim_{x\to 2} x^2 - 4 = 0$ and $\lim_{x\to 2} x - 2 = 0$

$\therefore \lim\limits_{x\to 2} f(x)$ has an indeterminate form of the type $\frac{0}{0}$.

$\lim\limits_{x\to\infty} \frac{\ln x}{x}$ has an indeterminate form of type $\frac{\infty}{\infty}$.

$\lim\limits_{x\to 0} (1+x)^{1/x}$ is an indeterminate form of the type 1^{∞}

LIMITS OF POLYNOMIALS AND RATIONAL FUNCTIONS

A function f is said to be a polynomial function of degree n when $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, where a_i's are real numbers such that $a_n \neq 0$ for some natural number n.

We know that $\lim\limits_{x\to a} x = a$. Hence

$$\lim_{x\to a} x^2 = \lim_{x\to a}(x.x) = \lim_{x\to a} x \cdot \lim_{x\to a} x = a.a = a^2$$

An easy exercise in induction on tells us that

$$\lim_{x\to a} x^n = a^n$$

Now, let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ be a polynomial function. Thinking of each of $a_0, a_1 x, a_2 x^2, \ldots, a_n x^n$ as a function, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a}[a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n]$$

$$= \lim_{x\to a} a_0 + \lim_{x\to a} a_1 x + \lim_{x\to a} a_2 x^2 + \ldots + \lim_{x\to a} a_n x^n$$

$$= a_0 + a_1 \lim_{x\to a} x + a_2 \lim_{x\to a} x^2 + \ldots + a_n \lim_{x\to a} x^n$$

$$= a_0 + a_1 a + a^2 a^2 + \ldots + a_n a^n$$

$$= f(a)$$

RATIONALISATION METHOD

We can rationalise the irrational expression in numerator or denominator or in both to remove the indeterminancy.

Train Your Brain

Example 4: Which of the following limits are in indeterminant forms. Also indicate the form

(i) $\lim\limits_{x\to 0} \frac{1}{x}$ (ii) $\lim\limits_{x\to 1} \frac{1-x}{1-x^2}$

(iii) $\lim\limits_{x\to 0} x \ln x$ (iv) $\lim\limits_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2}\right)$

(v) $\lim\limits_{x\to 0} (\sin x)^x$ (vi) $\lim\limits_{x\to 0} (\ln x)^x$

(vii) $\lim\limits_{x\to 0} (1+\sin x)^{\frac{1}{x}}$ (viii) $\lim\limits_{x\to 0} (1)^{\frac{1}{x}}$

Sol. (i) No (ii) Yes $\frac{0}{0}$ form

(iii) Yes $0 \times \infty$ form (iv) Yes $(\infty - \infty)$ form

(v) Yes, 0^0 form (vi) Yes ∞^0 form

(vii) Yes 1^{∞} form (viii) No

Example 5: Evaluate

(i) $\lim\limits_{x\to 3} \frac{x^2 - 2x - 3}{x^2 - 4x + 3}$

(ii) $\lim\limits_{x\to 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x}\right]$

Sol. (i) $\lim\limits_{x\to 3} \frac{x^2 - 2x - 3}{x^2 - 4x + 3} = \lim\limits_{x\to 3} \frac{(x-3)(x+1)}{(x-3)(x-1)} = 2$

(ii) $\lim\limits_{x\to 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x}\right]$

$$= \lim_{x\to 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \left[\frac{x(x-1) - 2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \left[\frac{x^2 - 5x + 6}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right] = \lim_{x\to 2} \left[\frac{x-3}{x(x-1)}\right] = -\frac{1}{2}$$

Example 6: Evaluate:

(i) $\lim\limits_{x\to 1} \frac{4 - \sqrt{15x+1}}{2 - \sqrt{3x+1}}$

(ii) $\lim\limits_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

Sol. (i) $\lim\limits_{x\to 1} \frac{4 - \sqrt{15x+1}}{2 - \sqrt{3x+1}}$

$$= \lim_{x\to 1} \frac{(4 - \sqrt{15x+1})(2 + \sqrt{3x+1})(4 + \sqrt{15x+1})}{(2 - \sqrt{3x+1})(4 + \sqrt{15x+1})(2 + \sqrt{3x+1})}$$

$$= \lim_{x\to 1} \frac{(15 - 15x)}{(3 - 3x)} \times \frac{2 + \sqrt{3x+1}}{4 + \sqrt{15x+1}} = \frac{5}{2}$$

(ii) The form of the given limit is $\frac{0}{0}$ when $x \to 0$.

Rationalising the numerator, we get

$$\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$$

$$= \lim_{x\to 0} \left[\frac{\sqrt{1+x} - \sqrt{1-x}}{x} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right]$$

$$= \lim_{x\to 0}\left[\frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})}\right]$$

$$= \lim_{x\to 0}\left[\frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}\right]$$

$$= \lim_{x\to 0}\left[\frac{2}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{2}{2} = 1$$

Example 7: Evaluate $\lim_{x\to 2}\frac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}}$

Sol. $L = \lim_{x\to 2}\frac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}}\cdot\frac{(\sqrt{x^2-4}-\sqrt{x-2})}{(\sqrt{x^2-4}-\sqrt{x-2})}$

$$= \lim_{x\to 2}\frac{(x-2)(\sqrt{x^2-4}-\sqrt{x-2})}{(x^2-4)-(x-2)}$$

$$= \lim_{x\to 2}\frac{(x-2)(\sqrt{x^2-4}-\sqrt{x-2})}{x^2-x-2}$$

$$= \lim_{x\to 2}\frac{(x-2)(\sqrt{x^2-4}-\sqrt{x-2})}{(x-2)(x+1)}$$

$$= \lim_{x\to 2}\frac{\sqrt{x^2-4}-\sqrt{x-2}}{x+1} = 0.$$

Concept Application

2. $\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x} =$

(a) -1 (b) 1
(c) 2 (d) -2

3. $\lim_{x\to 3}\left\{\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}\right\} =$

(a) 1 (b) 2
(c) -1 (d) -2

4. The value of the limit of $\frac{x^3-8}{x^2-4}$ as x tends to 2 is

(a) 3 (b) $\frac{3}{2}$
(c) 1 (d) 0

5. The value of the limit of $\frac{x^3-x^2-18}{x-3}$ as x tends to 3 is

(a) 3 (b) 9
(c) 18 (d) 21

THEOREM

For any positive integer n,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}.$$

Remark: The expression in the above theorem for the limit is true even if n is any rational number and a is positive.

Proof Dividing $(x^n - a^n)$ by $(x - a)$, we see that

$x^n - a^n = (x-a)\,(x^{n-1}) + x^{n-2}a + x^{n-3}a^2 + ... + x\,a^{n-2} + a^{n-1})$

Thus, $\lim_{x\to a}\frac{x^n-a^n}{x-a} = \lim_{x\to a}(x^{n-1}+x^{n-2}a+x^{n-3}a^2+...+x\,a^{n-2}+a^{n-1})$

$= a^{n-1} + a\,a^{n-2} + ... + a^{n-2}(a) + a^{n-1}$

$= a^{n-1} + a^{n-1} + ... + a^{n-1} + a^{n-1}$ (n terms)

$= na^{n-1}$

Standard Limits

(a) (i) $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = 1$

[Where x is measured in radians]

(ii) $\lim_{x\to 0}\frac{\tan^{-1}x}{x} = \lim_{x\to 0}\frac{\sin^{-1}x}{x} = 1$

(iii) $\lim_{x\to 0}\frac{e^x-1}{x} = 1;\ \lim_{x\to 0}\frac{a^x-1}{x}$

$= \log_e a = \ln a,\ a > 0$

(iv) $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$

(v) $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$

(b) If $f(x) \to 0$, when $x \to a$, then

(i) $\lim_{x\to a}\frac{\sin f(x)}{f(x)} = 1$

(ii) $\lim_{x\to a}\cos f(x) = 1$

(iii) $\lim_{x\to a}\frac{\tan f(x)}{f(x)} = 1$

(iv) $\lim_{x\to a}\frac{e^{f(x)}-1}{f(x)} = 1$

(v) $\lim_{x\to a}\frac{b^{f(x)}-1}{f(x)} = \ln b,\ (b>0)$

(vi) $\lim_{x\to a}\frac{\ln(1+f(x))}{f(x)} = 1$

(c) $\lim_{x\to a} f(x) = A > 0$ and $\lim_{x\to a}\phi(x) = B$(a finite quantity),

then $\lim_{x\to a}[f(x)]^{\phi(x)} = A^B$.

Remark:

(*i*) '0' doesn't means exact zero but represent a value approaching towards zero similarly to '1' and infinity.

(*ii*) $\infty + \infty = \infty$

(*iii*) $\infty \times \infty = \infty$

(*iv*) $(a/\infty) = 0$ if a is finite

(*v*) $\frac{a}{0}$ is not defined for any $a \in R$.

(*vi*) $a\,b = 0$, if & only if $a = 0$ or $b = 0$ and a & b are finite.

(*vii*) $\lim\limits_{x\to 0} \frac{x}{x}$ is an indeterminate form whereas $\lim\limits_{x\to 0} \frac{[x^2]}{x^2}$ is not an indeterminate form (where [.] represents greatest integer function)

Students may remember these forms along with the prefix 'tending to'

i.e. $\frac{\text{tending to zero}}{\text{tending to zero}}$ is an indeterminate form whereas

$\frac{\text{exactly zero}}{\text{tending to zero}}$ is not an indeterminate form, its value is zero.

Similarly $(\text{tending to one})^{\text{tending to }\infty}$ is indeterminate form whereas $(\text{exactly one})^{\text{tending to }\infty}$ is not an indeterminate form, its value is one.

To evaluate a limit, we must always put the value where 'x' is approaching to in the function. If we get a determinate form, then that value becomes the limit otherwise if an indeterminate form comes, we have to remove the indeterminancy, once the indeterminancy is removed the limit can be evaluated by putting the value of x, where it is approaching.

Train Your Brain

Example 8: Evaluate:

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$$

Sol. $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0} \dfrac{\tan x(1-\cos x)}{x^3}$

$$= \lim_{x\to 0} \frac{\tan x \,.\, 2\sin^2 \frac{x}{2}}{x^3}$$

$$= \lim_{x\to 0} \frac{1}{2}\cdot\frac{\tan x}{x}\cdot\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}.$$

Example 9: Evaluate: $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$

Sol. $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 3x} = \lim\limits_{x\to 0}\left[\dfrac{\sin 2x}{2x}\cdot\dfrac{2x}{3x}\cdot\dfrac{3x}{\sin 3x}\right]$

$$= \left[\lim_{2x\to 0} \frac{\sin 2x}{2x}\right]\cdot\frac{2}{3}\cdot\left[\lim_{3x\to 0}\frac{3x}{\sin 3x}\right]$$

$$= 1\cdot\frac{2}{3}\times\left[\lim_{3x\to 0}\frac{\sin 3x}{3x}\right] = \frac{2}{3}\times 1 = \frac{2}{3}$$

Example 10: Show that $\lim\limits_{x\to 0} \dfrac{\sin x(1-\cos x)}{x^3} = \dfrac{1}{2}$

Sol. $\lim\limits_{x\to 0} \dfrac{\sin x(1-\cos x)}{x^3} = \lim\limits_{x\to 0} \dfrac{\sin x\cdot 2\sin^2\frac{x}{2}}{x^3}$

$$= \lim_{x\to 0} 4\cos\frac{x}{2}\cdot\frac{\sin^3\frac{x}{2}}{x^3} = \lim_{x\to 0} 4(1)\left(\frac{1}{2}\right)^3 = \frac{1}{2}$$

Concept Application

6. If $\lim\limits_{x\to 2} \dfrac{x^n - 2^n}{x-2} = 80$, where n is a positive integer, then $n =$
   (*a*) 3 (*b*) 5
   (*c*) 2 (*d*) 1

7. $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{x} =$
   (*a*) 0 (*b*) 1
   (*c*) 2 (*d*) 4

8. $\lim\limits_{x\to 0} \dfrac{x^3 \cot x}{1-\cos x} =$
   (*a*) 0 (*b*) 1
   (*c*) 2 (*d*) –2

DERIVATIVES

Introduction

The rate of change of one quantity with respect to some another quantity has a great importance. For example, the rate of change of displacement of a particle with respect to time is called its velocity and the rate of change of velocity w.r.t time is called its acceleration.

The rate of change of a quantity 'y' with respect to another quantity 'x' is called the derivative or differential coefficient of y with respect to x.

Derivative at a Point

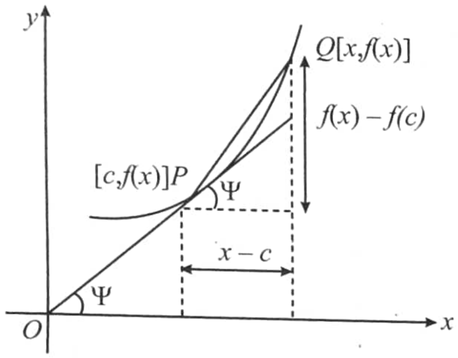
The derivative of a function at a point $x = a$ is defined by $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ (provided the limit exists and is finite)

The above definition of derivative is also called derivative by first principle.

1. **Geometrical meaning of derivatives at a point:** Consider the curve $y = f(x)$. Let $f(x)$ be differentiable at $x = c$. Let $P(c, f(c))$ be a point on the curve and $Q(x, f(x))$ be a neighbouring point on the curve. Then,

   Slope of the chord $PQ = \frac{f(x)-f(c)}{x-c}$. Taking limit as $Q \to P$, *i.e.*, $x \to c$,

   we get $\lim_{Q\to P}$ (slope of the chord PQ) $= \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ ...(*i*)



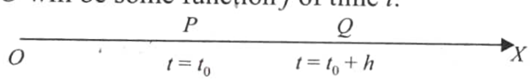
   As $Q \to P$, chord PQ becomes tangent at P.

   Therefore from (*i*), we have

   Slope of the tangent at $P = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \left(\frac{df(x)}{dx}\right)_{x=c}$.

   Note: Thus, the derivatives of a function at a point $x = c$ is the slope of the tangent to curve, $y = f(x)$ at point $(c, f(c))$

2. **Physical interpretation at a point:** Let a particle moves in a straight line OX starting from O towards X. Clearly, the position of the particle at any instant would depend upon the time elapsed. In other words, the distance of the particle from O will be some function f of time t.



   Let at any time $t = t_0$, the particle be at P and after a further time h, it is at Q so that $OP = f(t_0)$ and $OQ = f(t_0 + h)$. Hence, the average speed of the particle during the journey from P to Q is $\frac{PQ}{h}$, *i.e.*, $\frac{f(t_0+h)-f(t_0)}{h} = f(t_0, h)$. Taking the limit of $f(t_0, h)$ as $h \to 0$, we get its instantaneous speed to be $\lim_{h\to 0} \frac{f(t_0+h)-f(t_0)}{h}$, which is simply $f'(t_0)$. Thus, if $f(t)$ gives the distance of a moving particle at time t, then the derivative of f at $t = t_0$ represents the instantaneous speed of the particle at the point P, *i.e.*, at time $t = t_0$.

Important Tips

$\frac{dy}{dx}$ is $\frac{d}{dx}(y)$ in which $\frac{d}{dx}$ is simply a symbol of operation and not '*d*' divided by *dx*.

Finding derivatives from first principles

Consider $y = f(x)$ be a defined function of x. Let us change x by a small quantity, say h.

Then corresponding change in y can be obtained from $f(x + h) - f(x)$.

From the definition of dy/dx, given in last section, we get:

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\text{change in } y}{\text{change in } x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{x+h-x}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

So finding derivative by first principles means using the following formula:

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Illustrating the Concepts

(*i*) Evaluate the derivative of $f(x) = x^n$ w.r.t x from definition of derivative. Hence find the derivative of $\sqrt{x}, 1/x, 1/\sqrt{x}, 1/x^p$ *wrt x*.

Using definition of derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{(x+h)-x}$$

$$= \lim_{t\to x} \frac{t^n - x^n}{t-x} \quad [\text{putting } t = x+h]$$

$$= nx^{n-1}\left[\text{using } \lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

Taking $n = \frac{1}{2}, \frac{d}{dx}\sqrt{x} = \frac{-1}{2\sqrt{x}}$

taking $n = -1, \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

taking $n = \frac{-1}{2}, \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2x\sqrt{x}}$

taking $n = -p, \frac{d}{dx}\left(\frac{1}{x^p}\right) = \frac{-p}{x^{p+1}}$

(*ii*) Find the derivative of $\sin x$ w.r.t x from first principles.

Let $f(x) = \sin x$

Using the definition of derivative,

$$f'(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{2\frac{h}{2}}$$

Theorems for Differentiation

Let $f(x)$, $g(x)$ and $u(x)$ be differentiable functions

1. If at all points of a certain interval. $f'(x)=0$, then the function $f(x)$ has a constant value within this interval.
2. **Chain rule**
   (*i*) **Case I:** If y is a function of u and u is a function of x, then derivative of y with respect to x is $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$ or $y=f(u)\Rightarrow\frac{dy}{dx}=f'(u)\frac{du}{dx}$

   (*ii*) **Case II:** If y and x both are expressed in terms of t, y and x both are differentiable with respect to t then $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$.
3. **Sum and difference rule:** Using linear property $\frac{d}{dx}(f(x)\pm g(x))=\frac{d}{dx}(f(x))\pm\frac{d}{dx}(g(x))$
4. **Product rule:**
   (*i*) $\frac{d}{dx}(f(x)g(x))=f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)$

   (*ii*) $\frac{d}{dx}(u.v.w)=u.v.\frac{dw}{dx}+v.w.\frac{du}{dx}+u.w.\frac{dv}{dx}$
5. **Scalar multiple rule:** $\frac{d}{dx}(k\,f(x))=k\frac{d}{dx}f(x)$
6. **Quotient rule:**

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{d}{dx}(f(x))-f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$, provided $g(x)\neq 0$

Some Standard Differentiation

1. **Differentiation of algebraic functions**
   (*i*) $\frac{d}{dx}x^n=nx^{n-1}, x\in R, n\in R, x>0$

   (*ii*) $\frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}$

   (*iii*) $\frac{d}{dx}\left(\frac{1}{x^n}\right)=-\frac{n}{x^{n+1}}$
2. **Differentiation of trigonometric functions:** The following formulae can be applied directly while differentiating trigonometric functions

   (*i*) $\frac{d}{dx}\sin x=\cos x$

   (*ii*) $\frac{d}{dx}\cos x=-\sin x$

   (*iii*) $\frac{d}{dx}\tan x=\sec^2 x$

   (*iv*) $\frac{d}{dx}\sec x=\sec x\tan x$

   (*v*) $\frac{d}{dx}\operatorname{cosec} x=-\operatorname{cosec} x\cot x$

   (*vi*) $\frac{d}{dx}\cot x=-\operatorname{cosec}^2 x$

Train Your Brain

Example 11: Differentiate $y = \sin(x^2)$

Sol. If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\frac{dy}{dx}=\underbrace{\cos}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}}\underbrace{2x}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}=2x\cos x^2$$

Example 12: Differentiate $g(x) = \cos x^2 + 5\left(\frac{3}{x}+4\right)^6$

Sol. $\frac{dg}{dx}=\frac{d}{dx}\cos x^2+5\frac{d}{dx}(3x^{-1}+4)^6$

$$=-\sin x^2\frac{d}{dx}(x^2)+5\left[6(3x^{-1}+4)^5\frac{d}{dx}(3x^{-1}+4)\right]$$

$$=(-\sin x^2)(2x)+30(3x^{-1}+4)^5(-3x^{-2})$$

$$=-2x\sin x^2-90x^{-2}(3x^{-1}+4)^5$$

Example 13: Differentiate $g(x)=\sqrt[4]{\frac{x}{1-3x}}$.

Sol. Write $g(x)=\left(\frac{x}{1-3x}\right)^{1/4}=u^{1/4}$ where $u=\frac{x}{1-3x}$ is the inner function and $u^{1/4}$ is the outer function.

Then, $g'(x)=(u^{1/4})'\,u'(x)=\frac{1}{4}u^{-3/4}u'(x)$ and we have

$$g'(x)=\frac{1}{4}\left(\frac{x}{1-3x}\right)^{-3/4}\left(\frac{x}{1-3x}\right)'$$

$$=\frac{1}{4}\left(\frac{x}{1-3x}\right)^{-3/4}\left[\frac{(1-3x)(1)-x(-3)}{(1-3x)^2}\right]$$

$$=\frac{1}{4}\left(\frac{x}{1-3x}\right)^{-3/4}\left[\frac{1}{(1-3x)^2}\right]=\frac{1}{4x^{3/4}(1-3x)^{5/4}}.$$

Example 14: If $y = x^2 \cos(\log x)$, find $\frac{dy}{dx}$

Sol. $\frac{dy}{dx} = \frac{d}{dx}\{x^2 \cos(\log x)\}$

$$= x^2 \frac{d\{\cos(\log x)\}}{dx} + \cos(\log x)\frac{d(x^2)}{dx}$$

$$= x^2 \cdot \frac{d\cos(\log x)}{d\log x} \cdot \frac{d(\log x)}{dx} + \cos(\log x)\cdot 2x$$

$$= x^2\left(-\sin(\log x)\cdot\frac{1}{x} + 2x\cos(\log x)\right)$$

$$= -x\sin(\log x) + 2x\cos(\log x)$$

Concept Application

9. Differentiate this $(\sec x - 1)(\sec x + 1)$

10. Differentiate this $\frac{3x+4}{5x^2-7x+9}$

11. Differentiate this $\frac{a+b\sin x}{c+d\cos x}$

12. Differentiate this $x^2 \sin x + \cos 2x$

Short Notes

Limit

- The expected value of the function as dictated by the points to the left of a point defines the left hand limit of the function at that point. Similarly the right hand limit.
- Limit of a function at a point is the common value of the left and right hand limits, if they coincide.
- For a function f and a real number a, $\lim\limits_{x\to a} f(x)$ and $f(a)$ may not be same (In fact, one may be defined and not the other one).

Fundamental Theorems on Limits

- For functions f and g the following holds:

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$

Theorem

- Following are some of the standard limits

$$\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

Derivatives

- The derivative of a function f at a is defined by

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

- Derivative of a function f at any point x is defined by

$$f'(x) = \frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

- For functions u and v the following holds:

$(u \pm v)' = u' \pm v'$

$(uv)' = u'v + uv'$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ provided all are defined.

- Following are some of the standard derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Solved Examples

1. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to

Sol. $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$

$$=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}\frac{(3+\cos x)}{\frac{\sin 4x}{4x}}\frac{\cos 4x}{4}=2$$

2. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x-\cos x}{(\pi-2x)^3}$ equals

Sol. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x-\cos x}{-8\left(x-\frac{\pi}{2}\right)^3}$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\tan\left(\frac{\pi}{2}-x\right)}{8\left(\frac{\pi}{2}-x\right)}\frac{\left(1-\cos\left(\frac{\pi}{2}-x\right)\right)}{\left(\frac{\pi}{2}-x\right)^2}$$

$$=\frac{1}{8}\cdot 1\cdot\frac{1}{2}=\frac{1}{16}$$

3. $\lim\limits_{x\to 0}\dfrac{x\cot(4x)}{\sin^2 x\cot^2(2x)}$ is equal to:

Sol. $\lim\limits_{x\to 0}\dfrac{x\cos 4x\sin^2 2x}{\sin^2 x.\cos^2 2x.\sin 4x}$

$$=\lim_{x\to 0}\frac{x}{\tan 4x}\cdot\frac{1}{\sin^2 x}\frac{\tan^2 2x}{1}$$

$$=\lim_{x\to 0}\frac{1}{4}\frac{4x}{\tan 4x}\frac{x^2}{\sin^2 x}\cdot\frac{\tan^2 2x}{x^2}$$

$$=\lim_{x\to 0}\frac{1}{4}\frac{4x}{\tan 4x}\left(\frac{x}{\sin x}\right)^2\cdot\left(\frac{\tan 2x}{2x}\right)^2\frac{4}{1}$$

$$=\frac{1}{4}\cdot 1\cdot 1\cdot 1\cdot\frac{4}{1}=1$$

4. If $\lim\limits_{x\to 1}\dfrac{x^4-1}{x-1}=\lim\limits_{x\to k}\dfrac{x^3-k^3}{x^2-k^2}$, then k is:

Sol. $\lim\limits_{x\to 1}\dfrac{x^4-1}{x-1}=\lim\limits_{x\to k}\dfrac{x^3-k^3}{x^2-k^2}$

$$\Rightarrow\lim_{x\to 1}(x+1)(x^2+1)=\frac{k^2+k^2+k^2}{2k}$$

$\Rightarrow k=8/3$

5. $\lim\limits_{x\to 0}\dfrac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin^2 x-x+1}}$ is:

Sol. Rationalize

$$\lim_{x\to 0}\frac{(x+2\sin x)\left(\sqrt{x^2+2\sin x+1}+\sqrt{\sin^2 x-x+1}\right)}{x^2+2\sin x+1-\sin^2 x+x-1}$$

$$\lim_{x\to 0}\frac{(x+2\sin x)(2)}{x^2+2\sin x-\sin^2 x+x}$$

$\frac{0}{0}$ form using L' hospital

$$\Rightarrow\lim_{x\to 0}\frac{(1+2\cos x)\times 2}{2x+2\cos x-2\sin x\cos x+1}=\frac{6}{3}\Rightarrow 2$$

6. The value of the limit $\lim\limits_{\theta\to 0}\dfrac{\tan(\pi\cos^2\theta)}{\sin(2\pi\sin^2\theta)}$

Sol. $\lim_{\theta\to 0}\dfrac{\tan(\pi(1-\sin^2\theta))}{\sin(2\pi\sin^2\theta)}$

$$=\lim_{\theta\to 0}\frac{-\tan(\pi\sin^2\theta)}{\sin(2\pi\sin^2\theta)}$$

$$=\lim_{\theta\to 0}-\left(\frac{\tan(\pi\sin^2\theta)}{\pi\sin^2\theta}\right)\left(\frac{2\pi\sin^2\theta}{\sin(2\pi\sin^2\theta)}\right)\times\frac{1}{2}$$

$$=\frac{-1}{2}$$

7. If $y=\sqrt{x}+\dfrac{1}{\sqrt{x}}$, then $\dfrac{dy}{dx}$ at $x=1$ is equal to

Sol. Given, $y=\sqrt{x}+\dfrac{1}{\sqrt{x}}$

Now, $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x}}-\dfrac{1}{2x^{3/2}}$

$$\therefore\left(\frac{dy}{dx}\right)_{x=1}=\frac{1}{2}-\frac{1}{2}=0$$

8. If $f(x)=\dfrac{x-4}{2\sqrt{x}}$, then f '(*l*) is equal to

Sol. Given,

Now,

$$f(x)=\frac{x-4}{2\sqrt{x}}$$

$$f'(x)=\frac{2\sqrt{x}-(x-4)\cdot 2\cdot\frac{1}{2\sqrt{x}}}{4x}$$

$$= \frac{2x-(x-4)}{4x^{3/2}} = \frac{2x-x+4}{4x^{3/2}}$$

$$= \frac{x+4}{4x^{3/2}}$$

$$\therefore f'(1) = \frac{1+4}{4\times(1)^{3/2}} = \frac{5}{4}$$

9. If $y = \dfrac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$, then $\dfrac{dy}{dx}$ is equal to

Sol. Given, $y = \dfrac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}} \Rightarrow y = \dfrac{x^2+1}{x^2-1}$

$$\therefore \frac{dy}{dx} = \frac{(x^2-1)2x-(x^2+1)(2x)}{(x^2-1)^2} \text{ [by quotient rule]}$$

$$\frac{dy}{dx} = \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}$$

$$= \frac{2x(-2)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

10. If $y = \dfrac{\sin x+\cos x}{\sin x-\cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is equal to

Sol. Given, $y = \dfrac{\sin x+\cos x}{\sin x-\cos x}$

$$\therefore \frac{dy}{dx}$$

$$= \frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

[by quotient rule]

$$= \frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$= \frac{-[(\sin x-\cos x)^2+(\sin x+\cos x)^2]}{(\sin x-\cos x)^2}$$

$$= \frac{-[\sin^2 x+\cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x]}{(\sin x-\cos x)^2}$$

$$= \frac{-2}{(\sin x-\cos x)^2}$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=0} = -2$$

11. If $y = \dfrac{\sin(x+9)}{\cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is equal to

Sol. Given,

$$\therefore \frac{dy}{dx} = \frac{\cos x\cos(x+9)-\sin(x+9)(-\sin x)}{(\cos x)^2}$$

[by quotient rule]

$$= \frac{\cos x\cos(x+9)+\sin x\sin(x+9)}{\cos^2 x}$$

$$\therefore \left(\frac{dy}{dx}\right)_{x-0} = \frac{\cos 9}{1}$$

$= \cos 9$

12. Find $\tan x^2$ derivative from first principle.

Sol. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\tan(x+h)^2-\tan x^2}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)^2\cos x^2-\sin x^2\cos(x+h)^2}{h\cdot\cos x^2\cdot\cos(x+h)^2}$$

$$= \lim_{h\to 0} \frac{\sin\left((x+h)^2-x^2\right)}{h\cdot\cos x^2\cdot\cos(x+h)^2}$$

$$= \lim_{h\to 0} \frac{\sin\left(h^2+2hx\right)}{h\cdot\left(\cos x^2\right)^2}$$

$$= \frac{2x}{\cos^2(x^2)}$$

$= 2x \sec^2(x^2)$.

Exercise-1 (Topicwise)

EXISTENCE OF LIMIT

1. If $f(x)=\begin{cases} x, \text{ when } x>1 \\ x^2, \text{ when } x<1 \end{cases}$, then $\lim_{x\to 1} f(x) =$

(a) x^2 (b) x

(c) -1 (d) 1

2. $\lim_{x\to 2}\frac{|x-2|}{x-2}=$

(a) 1

(b) -1

(c) Does not exist

(d) None of these

3. If, $f(x)=\begin{cases} \frac{2}{5-x}, & \text{when } x<3 \\ 5-x, & \text{when } x>3 \end{cases}$, then

(a) $\lim_{x\to 3^+} f(x)=0$ (b) $\lim_{x\to 3^-} f(x)=0$

(c) $\lim_{x\to 3^+} f(x)\neq \lim_{x\to 3^-} f(x)$ (d) None of these

4. Let the function f be defined by the equation

$f(x)=\begin{cases} 3x, \text{ if } 0\le x\le 1 \\ 5-3x, \text{ if } 1<x\le 2 \end{cases}$, then

(a) $\lim_{x\to 1} f(x)-f(1)$ (b) $\lim_{x\to 1} f(x)-3$

(c) $\lim_{x\to 1} f(x)=2$ (d) $\lim_{x\to 1} f(x)$ does not exist

5. $\lim_{x\to 0}\frac{|x|}{x}=$

(a) 1 (b) -1

(c) 0 (d) Does not exist

METHODS OF EVALUATION OF LIMITS

6. $\lim_{x\to 1}(3x^2+4x+5)=$

(a) 12

(b) -1

(c) Does not exist

(d) None of these

7. The value of $\lim_{x\to 2}\frac{3^{x/2}-3}{3^x-9}$ is

(a) 0 (b) $\frac{1}{3}$

(c) $\frac{1}{6}$ (d) ln 3

8. $\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ equals

(a) 1

(b) $\frac{3}{2}$

(c) $\frac{1}{4}$

(d) None of these

9. The value of the limit of $\frac{x^3-x^2-18}{x-3}$ as x tends to 3 is

(a) 3 (b) 9

(c) 18 (d) 21

10. The value of the limit of $\frac{x^3-8}{(x^2-4)}$ as x tends to 2 is

(a) 3 (b) $\frac{3}{2}$

(c) 1 (d) 0

11. $\lim_{x\to 0}\frac{x}{\sqrt{1+x}-\sqrt{1-x}}$ is equal to

(a) $\frac{1}{2}$ (b) 2

(c) 1 (d) 0

12. $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ equals

(a) $\frac{2a}{3\sqrt{3}}$ (b) $\frac{2}{3\sqrt{3}}$

(c) 0 (d) None of these

13. If $f(x)=\frac{2}{x-3}$, $g(x)=\frac{x-3}{x+4}$ and $h(x)=-\frac{2(2x+1)}{x^2+x-12}$ then $\lim_{x\to 3}[f(x)+g(x)+h(x)]$ is

(a) -2 (b) -1

(c) $-\frac{2}{7}$ (d) 0

14. $\lim_{x\to a}\frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}=$

(a) $\frac{5}{3}(a+2)^{2/3}$ (b) $\frac{5}{3}(a+2)^{5/3}$

(c) $\frac{5}{3}a^{2/3}$ (d) $\frac{5}{3}a^{5/3}$

TRIGONOMETRIC LIMITS

15. $\lim_{x\to 0}\frac{x^3}{\sin x^2}=$

(a) 0 (b) $\frac{1}{3}$

(c) 3 (d) $\frac{1}{2}$

16. $\lim_{x\to 0}\frac{\sin 3x+\sin x}{x}=$

(a) $\frac{1}{3}$ (b) 3

(c) 4 (d) $\frac{1}{4}$

17. $\lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right)=$

(a) $\frac{\pi}{2}$ (b) π

(c) $\frac{2}{\pi}$ (d) 0

18. $\lim_{x\to 1}\frac{\sqrt{1-\cos 2(x-1)}}{x-1}$

(a) Exists and it equal $\sqrt{2}$

(b) Exists and it equals $\sqrt{2}$

(c) Does not exist because $x-1\to 0$

(d) Does not exist because left hand limit is not equal to right hand limit

19. $\lim_{x\to 0}\frac{(1-\cos 2x)\sin 5x}{x^2\sin 3x}=$

(a) $\frac{10}{3}$ (b) $\frac{3}{10}$

(c) $\frac{6}{5}$ (d) $\frac{5}{6}$

20. $\lim_{x\to\pi/2}(\sec\theta-\tan\theta)=$

(a) 0 (b) 1/2

(c) 2 (d) ∞

21. $\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}=$

(a) 1

(b) -1

(c) 0

(d) limit does not exist.

22. $\lim_{\theta\to\frac{\pi}{2}}\frac{\frac{\pi}{2}-\theta}{\cot\theta}=$

(a) 0 (b) -1

(c) 1 (d) ∞

DERIVATIVES

23. If $y=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}$ then $\frac{dy}{dx}$ is equal to

(a) y (b) $y+\frac{x^n}{n!}$

(c) $y-\frac{x^n}{n!}$ (d) $y-1-\frac{x^n}{n!}$

24. If $f(x)=\sqrt{1+\cos^2(x^2)}$, then $f'\left(\frac{\sqrt{\pi}}{2}\right)$

(a) $\sqrt{\pi}/6$ (b) $-\sqrt{\pi}/6$

(c) $1/\sqrt{6}$ (d) $\pi/\sqrt{6}$

25. If $y=a\sin x+b\cos x$, then $y^2+\left(\frac{dy}{dx}\right)^2$ is a

(a) function of x (b) function of y

(c) function of x and y (d) constant

26. If $y=\sqrt{\frac{1-x}{1+x}}$, then $(1-x^2)\frac{dy}{dx}$ is equal to

(a) y^2 (b) $1/y$

(c) $-y$ (d) $-y/x$

27. If $y=|\cos x|+|\sin x|$, then $\frac{dy}{dx}$ at $x=\frac{2\pi}{3}$ is

(a) $\frac{1-\sqrt{3}}{2}$ (b) 0

(c) $\frac{1}{2}(\sqrt{3}-1)$ (d) None of these

28. If $f(x)=1+x+\frac{x^2}{2}+\ldots+\frac{x^{100}}{100}$, then $f'(1)$ is equal to

(a) $\frac{1}{100}$

(b) 100

(c) 0

(d) Does not exist

29. If $f(x)=\frac{x^n-a^n}{x-a}$ for some constant a, then $f'(a)$ is equa to

(a) 1

(b) 0

(c) $\frac{1}{2}$

(d) Does not exist

30. If $f(x)=x^{100}+x^{99}+\ldots+x+1$, then $f'(1)$ is equal to

(a) 5050 (b) 5049

(c) 5051 (d) 50051

Exercise-2 (Learning Plus)

1. Evaluate $\lim_{x \to 1/2} \frac{4x^2-1}{2x-1}$.

2. Evaluate $\lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$.

3. Evaluate $\lim_{x \to a} \frac{(2+x)^{5/2}-(a+2)^{5/2}}{x-a}$.

4. Evaluate $\lim_{x \to 1} \frac{x^4-\sqrt{x}}{\sqrt{x}-1}$.

5. Evaluate $\lim_{x \to 0} \frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}$

6. Evaluate $\lim_{x \to -3} \frac{x^3+27}{x^5+243} = \lim_{x \to -3} \frac{\frac{x^3+27}{x+3}}{\frac{x^5+243}{x+3}}$

7. Evaluate $\lim_{x \to 0} \frac{1-\cos mx}{1-\cos nx}$.

8. Evaluate $\lim_{x \to \pi/6} \frac{\sqrt{3}\sin x-\cos x}{x-\frac{\pi}{6}}$.

9. If $\lim_{x \to 1} \frac{x^4-1}{x-1} = \lim_{x \to k} \frac{x^3-k^3}{x^2-k^2}$, then find the value of

10. $\lim_{x \to \pi} \frac{1-\sin\frac{x}{2}}{\cos\frac{x}{2}\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)}$

11. Show that $\lim_{x \to \pi/4} \frac{|x-4|}{x-4}$ does not exist,

12. If $f(x) = \frac{\tan x}{x-\pi'}$ then $\lim_{x \to \pi} f(x) =$

13. $\lim_{x \to 0}\left(\sin mx \cot\frac{x}{\sqrt{3}}\right) = 2$, then $m =$ .....

14. Differentiate this function w.r.t x $\frac{x^4+x^3+x^2+1}{x}$

15. Differentiate this function $\left(x+\frac{1}{x}\right)^3$

16. Differentiate this function $(3x+5)(1+\tan x)$

17. Differentiate this function $\frac{x^5-\cos x}{\sin x}$

18. Differentiate this function $\frac{x^2\cos\frac{\pi}{4}}{\sin x}$

19. Differentiate this function $(ax^2+\cot x)(p+q\cos x)$

20. Differentiate this function $(\sin x+\cos x)^2$

21. Differentiate this function $(2x-7)^2(3x+5)^3$

22. Differentiate this function $\sin^3 x \cos^3 x$

23. Differentiate this function $\frac{1}{ax^2+bx+c}$

24. If $y = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$, then $\frac{dy}{dx} =$ .....

25. Find the derivative of $\sin x$ from first principle.

ANSWER KEY

CONCEPT APPLICATION

1. [3] **2.** (*b*) **3.** (*a*) **4.** (*a*) **5.** (*d*) **6.** (*b*) **7.** (*a*) **8.** (*c*) **9.** $\frac{dy}{dx} = 2\tan x.\sec^2 x$

10. $\frac{dy}{dx} = \frac{55-15x^2-40x}{(5x^2-7x+9)^2}$ **11.** $\frac{dy}{dx} = \frac{bc\cos x + ad\sin x + bd}{(c+d\cos x)^2}$ **12.** $\frac{dy}{dx} = x^2\cos x + 2x\sin x - 2\sin 2x$

EXERCISE-1 (TOPICWISE)

1. (*d*) **2.** (*c*) **3.** (*c*) **4.** (*d*) **5.** (*d*) **6.** (*a*) **7.** (*c*) **8.** (*d*) **9.** (*d*) **10.** (*a*)
11. (*c*) **12.** (*b*) **13.** (*c*) **14.** (*a*) **15.** (*a*) **16.** (*c*) **17.** (*c*) **18.** (*d*) **19.** (*a*) **20.** (*a*)
21. (*d*) **22.** (*c*) **23.** (*c*) **24.** (*b*) **25.** (*d*) **26.** (*c*) **27.** (*c*) **28.** (*b*) **29.** (*d*) **30.** (*a*)

EXERCISE-2 (LEARNING PLUS)

1. [2] **2.** $\frac{1}{2\sqrt{x}}$ **3.** $=\frac{5}{2}(a+2)^{\frac{3}{2}}$ **4.** [7] **5.** [0] **6.** $\frac{1}{15}$ **7.** $\frac{m^2}{n^2}$ **8.** [2] **9.** $\frac{8}{3}$

10. $\frac{1}{\sqrt{2}}$ **11.** LHL $\neq$ RHL **12.** [1] **13.** $m = \frac{2\sqrt{3}}{3}$ **14.** $\frac{3x^4+2x^3+x^2-1}{x^2}$ **15.** $3x^2 - \frac{3}{x^2} - \frac{3}{x^4} + 3$

16. $3x\sec^2 x + 5\sec^2 x + 3\tan x + 3$ **17.** $\frac{5x^4\sin x - x^5\cos x + 1}{\sin^2 x}$ **18.** $\frac{x}{\sqrt{2}}\operatorname{cosec}x[2 - x\cot x]$

19. $-q\sin x\,(ax^2 + \cot x) + (p + q\cos x)(2ax - \operatorname{cosec}^2 x)$ **20.** $2\cos 2x$ **21.** $(2x-7)(3x+5)^2(30x-43)$

22. $\frac{3}{4}\sin^2 2x\cos 2x$ **23.** $\frac{-(2ax+b)}{(ax^2+bx+c)^2}$ **24.** $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ **25.** $(\cos x)$

CHAPTER

21 Mathematical Reasoning

STATEMENTS

In reasoning we communicate our ideas or thoughts with the help of sentences in a particular language "A sentence is called a mathematically acceptable statement if it is either true or false but not both". A statement is assumed to be either true or false. A true statement is known as a valid statement and a false statement is known as an invalid statement.

Note:

- A statement can not be both true and false at the same time.
- In general, statements are denoted by lower case letters p, q, r, s, t, etc.

Example 1: Consider the following sentences:

(*i*) Three plus two equals five.

(*ii*) The sum of two negative numbers is negative.

(*iii*) Every square is a rectangle.

Each of these sentences is a true sentence, therefore they are statements.

Example 2: Consider the following sentences:

(*i*) Three plus four equals six.

(*ii*) All prime numbers are odd.

(*iii*) Every relation is a function.

Each of these sentences is a false sentence, therefore they are statements.

Example 3: Consider the following sentences:

(*i*) The sum of x and y is greater than 0

(*ii*) The square of a number is even.

Here, we are not in a position to determine whether it is true or false unless we know what the numbers are. Therefore these sentences are not a statement.

Note:

(*i*) Imperative sentences (expresses a request or command), exclamatory sentences (expresses some strong feeling), Interrogative sentences (asks some questions) are not considered as a statement in mathematical language.

(*ii*) Sentences involving variable time such as "today", "tomorrow" or "yesterday" are not statements.

Example 4: Consider the following sentences:

(*i*) Give me a glass of water.

(*ii*) Is every set finite?

(*iii*) How beautiful?

(*iv*) Tomorrow is Monday.

All these sentences are not a statement.

TYPE OF STATEMENTS

Simple Statement

A statement is known as simple if it cannot be broken down into two or more statements.

Example:

- 11 is an odd number.
- The Sun is a star.
- $\sqrt{2}$ is a rational number.

Compound Statements

A compound statement is a statement which is made up of two or more simple statements. In this case, each statement is called a component statement.

Example:

- All rational numbers are real and all real numbers are complex numbers.

  The component statements are:
- p : all rational numbers are real.
- q : all real numbers are complex numbers.

Train Your Brain

Example 1: Write three mathematical acceptable statements.

Sol. (*i*) The sum of interior angles of a triangle is 180°.

(*ii*) The product of two odd integers is always even.

(*iii*) There are 8 days in a week.

Example 2: Give three examples of sentences which are not statements.

Sol. (*i*) x^2 is always greater than 1.

(*ii*) He is an engineer.

(*iii*) Mathematics is easy.

Example 3: Write the component statement of statement p, where p: Airplane flies in the air and ships sails on the water.

Sol. It's Component statements are

q : Airplane flies in the air.

r : Ships sails on the water.

Concept Application

1. Write the component statements of the compound statement p and check whether they are true or not.
   p : $\sqrt{12}$ is a rational, an irrational or complex number.
2. Explain why following sentences are not statements.
   (*a*) p : Hisar is far from here.
   (*b*) q : Yesterday was Saturday.
   (*c*) r : Too large!
3. Which of the following is not a statement?
   (*a*) Brush your teeth.
   (*b*) 11 is a prime number.
   (*c*) $\sqrt{p}$ is an irrational number, if p is prime.
   (*d*) 15 is composite number.

TRUTH TABLE

Truth table is that which gives truth values of compound statements.

It has a number of rows and columns. The number of rows depend upon the number of simple statements.

Note that for n statements, there are 2^n rows.

(*i*) Truth table for single statement p:

Number of rows = $2^1 = 2$

p
T
F

(*ii*) Truth table for two statements p and q:

Number of rows = $2^2 = 4$

p	q
T	T
T	F
F	T
F	F

(*iii*) Truth table for three statements p, q and r.

Number of rows = $2^3 = 8$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

NEGATION (OR DENIAL)

The denial of a statement is called the negative of the statement and denoted as ~.

Example: p : All integers are rational numbers".

$\sim p$: At least one integer is not a rational number.

While forming the negation of a statement, phrases like, "it is not the case" or "It is false that" are also used

Example: p : Everyone in Germany speaks German.

$\sim p$: It is false that everyone in Germany speaks German.

If p is true then $\sim p$ must be false and if p is false then $\sim p$ must be true

Truth Table ($\sim p$)		
p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F
Rule ~ is true only when p is false		

It may be noticed that $\sim(\sim p) = p$. Also p and $\sim p$ are contrary.

Important: It may be observed that negation is not a binary operation, it is a unary operation i.e. a modifier.

1. $\sim p$ is true if p is false.
2. $\sim p$ is false if p is true.

Train Your Brain

Example 4: Write the negation of following statement and check whether they are true or not.

(*i*) p : 0 is a natural number.

(*ii*) p : 70 is a multiple of 20.

(*iii*) p : Longest chord of a circle is diameter.

Sol. (*i*) $\sim p$: 0 is not a natural number. (True)

(*ii*) $\sim p$: 70 is not a multiple of 20. (True)

(*iii*) $\sim p$: Diameter is not the longest chord of circle. (False)

Example 5: Are the following pairs of statements negation of each other? Give reason

p : We can insert infinitely many rational numbers between two rational numbers.

q : We cannot insert infinitely many rational numbers between two rational number.

Sol. Yes, they are negation of each other. Since negation of a statement p is also a statement, and we can form negation of statement using phrases like "It is not the case" or "It is false that" before p or, if possible by inserting in p the word "not".

Concept Application

4. Negation of the statement "There does not exist a parallelogram whose diagonals are of equal length" is
   (*a*) It is not the case that there does not exist a parallelogram whose diagonals are of equal length.
   (*b*) It is false that there does not exist a parallelogram whose diagonals are of equal length.
   (*c*) There exists a parallelogram whose diagonals are of equal length.
   (*d*) All of these

5. Negation of the given statement p is
   p : There exists a rational number y such that $y^3 \neq 108$.
   (*a*) There exists an irrational number y such that $y^3 = 108$.
   (*b*) There exists an irrational number y such that $y^3 \neq 108$.
   (*c*) There does not exist a rational number such that $y^3 = 108$.
   (*d*) There does not exist a rational number such that $y^3 \neq 108$.

BASIC LOGIC CONNECTIVES

Conjunction

Simple statements when combined by the word "and" ($\wedge$), the resulting compound statement is called a conjunction denoted as $p \wedge q$.

Example:

A point occupies a position and its location can be determined.

The component statements are

- p : A point occupies a position
- q : Its location can be determined
- Both statements are true.

Important: Do not think that a statement with "And" is always a compound statement.

Example: A mixture of alcohol and water can be separated by chemical methods.

(Here "And" refers to two things).

Note:

(*i*) The compound statement with 'And' is true if all its component statements are true.

(*ii*) The component statement with 'And' is false if any of its component statements is false (this includes the case that some of its component statements are false or all of its component statements are false).

The following truth table shows the truth values of $p \wedge q$ (p and q) and $q \wedge p$ (q and p).

Truth Table ($p \wedge q$, $q \wedge p$)			
p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F
Rule: $p \wedge q$ is true only when p and q are true			

Remark: The above truth table shows that $p \wedge q = q \wedge p$.

Disjunction or Alternation

Simple statements p and q are combined by the connective '*OR*' ($\vee$), then the compound statements denoted as $p \vee q$ (p or q) so formed is called a disjunction.

Example: Two lines in a plane either intersect at one point or they are parallel.

Sometimes we use the connective 'either....or....' to obtain $p \vee q$ and read $p \vee q$ as 'either p or q'.

Note:

(*i*) A compound statement with an 'Or' is true when one component statement is true or both the component statement are true.

(*ii*) A compound statement with an 'Or' is false when both the component statements are false.

Truth Table ($p \vee q$, $q \vee p$)			
p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F
Rule: $p \vee q$ is false only when p and q are false			

Important: A student who has taken Biology or Chemistry can apply for M.Sc. microbiology program.

This means that student who have taken both Biology and Chemistry or only Biology or only Chemistry can apply for the microbiology program. This is example of **inclusive "Or"**. In this case truth table is same as $p \vee q$.

Important: Student can take French or Sanskrit as their third language.

This means that student have to choose only one subject from French and Sanskrit. It exclude the case when one student can choose both subject. This is case of **exclusive "Or"**. This is represented as $p \veebar q$ or $p \otimes q$.

Truth table for exclusive or is as follows.

Truth Table ($p \veebar q$)		
p	q	$p \veebar q$ or $p \otimes q$
T	T	F
T	F	T
F	T	T
F	F	F
Rule: $p \vee q$ is true only when one of p and q is true and the other is false		

Quantifiers

Quantifiers are phrases like "There exists" and "for all".

Negation of Quantifiers

(*i*) P = There exist a number which is equal to its square.

$\sim P$ = There does not exist a number which is not equal to its square.

(*ii*) P = For every real number x, x is less than $x + 1$.

$\sim P$ = There exist a number for which x is not less than $x + 1$.

Implication

There are three types of implications :

(*i*) "If ...... then"

(*ii*) "Only if"

(*iii*) "If and only if"

Conditional Statement

"If ...... then" type of compound statement is called **conditional statement**.

The statement 'if p then q' is denoted by $p \to q$ (to be read as 'p implies q') or by $p \Rightarrow q$. Note that $p \to q$ also means

(*i*) p is sufficient for q (*ii*) q is necessary for p

(*iii*) p lead to q (*iv*) q if p

(*v*) q when p (*vi*) if p, then q

Example: p : *a* number is a multiple of 9

q : *a* number is a multiple of 3.

Then $p \to q$ is false only when p is true and q is false. Truth table for $p \to q$ is as follows.

Truth Table ($p \to q, q \leftarrow p$)			
p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T
Rule: $p \to q$ is false only when p is true and q are false			

Biconditional or Equivalence or 'Double Implication'

"If and only if" type of compound statement is called **Biconditional or equivalence** or **'double implication'**. Symbolically 'p iff q' is represented by $p \leftrightarrow q$ or by $p \Leftrightarrow q$.

(*i*) p is a necessary and sufficient condition for q.

(*ii*) q is necessary and sufficient condition for p.

(*iii*) If p then q and if q then p

(*iv*) q if and only if p.

Example:

p : If the sum of digits of a number is divisible by 3, then the number is divisible by 3.

q : If a numbers is divisible by 3, then the sum of its digits is divisible by 3.

A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

The following are other illustrations which actually do not appear to be so but they in fact are biconditional.

(*i*) If you work hard only then you can succeed.

(*ii*) You can go on leave only if your boss permits.

The truth table for biconditional is as follows:

Truth Table ($p \leftrightarrow q$)				
p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	F	F	F
F	F	T	T	T
Rule: $p \leftrightarrow q$ is true when both are true or false				

Contrapositive, Converse and Inverse

Contrapositive and converse are certain other statements which can be formed from a given statement with "if ...... then".

Contrapositive of $p \to q$ is $\sim q \to \sim p$

Example: If a number is multiple of 6 then it is multiple of 2.

Contrapositive: If a number is not multiple of 2 then it is not multiple of 6.

Converse of $p \to q$ is $q \to p$

Inverse of $p \to q$ is $\sim p \to \sim q$

Example: If the angles of a triangle are equal then it is equilateral triangle.

Converse is if triangle is equilateral then angles of triangle are equal.

Truth Table ($p \to q$)				
p	q	$p \to q$	Contrapositive ($\sim q \to \sim p$)	Converse ($q \to p$)
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Note: Truth table for $p \to q$ is same as its contrapositive.

Negation of Implication

If p and q are two statements, then

$\sim(p \Rightarrow q) = p \wedge \sim q \quad [\because p \Rightarrow q \equiv \sim p \vee q]$

Proof:

p	q	$p \Rightarrow q$	$\sim(p \Rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Negation of Biconditional Statement or Equivalence Theorem

If p and q are two statement, then

$\sim(p \Leftrightarrow q) = (p \wedge \sim q) \vee (q \wedge \sim p)$

Proof:

$p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$

$\therefore \sim(p \Leftrightarrow q) = \sim\{(p \Rightarrow q) \wedge (q \Rightarrow p)\}$

$= [\sim(p \Rightarrow q)] \vee [\sim(q \Rightarrow p)]$

$= (p \wedge \sim q) \vee (q \wedge \sim p)$

Tautologies and Fallacies

The compound statements (or propositions) which are true for any truth value of their components are called '**Tautologies**'. 'eg $(p \vee \sim p)$ is a tautology, p being logical statement. This is illustrated by the truth table given below which shows only T in the last column.

Truth Table ($p \vee \sim p$)		
p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

The negation of tautology is called a **fallacy** or a **contradiction** i.e. a proposition which is false for any truth value of their components is called a fallacy. For example '$p \wedge \sim p$' is a fallacy, p being any logical statement. This is illustrated by the truth table given above which shows only F in the last column.

Truth Table ($p \wedge \sim p$)		
p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Note:

(*i*) $p \vee q$ is true if at least one of p and q is true.

(*ii*) $p \veebar q$ is true if exactly one of p and q is true and the other is false.

(*iii*) $p \wedge q$ is true if both p and q are true

(*iv*) A tautology is always true.

(*v*) A fallacy is always false.

Duality

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ *by* $\vee$ and $\vee$ *by* $\wedge$.

Train Your Brain

Example 6: Write the following statements in the form "if then"

(*a*) p : Three common tangent can be drawn to two circles if they touch each other.

(*b*) q : Chord of a circle which subtends an angle 90° in alternate segment implies it is diameter.

Sol. (*a*) p : If two circle touch other then we can draw three common tangents to both circle.

Example 7: Given below are two pair of statements combine these two statements using "if and only if".

p : If a convex polygon is pentagon then it has 5 diagonals.

q : If a convex polygon has 5 diagonals then it is pentagon.

Sol. A convex polygon is pentagon if and only if it has 5 diagonals.

Example 8: Rewrite the following statement with "If then" in five different ways conveying the same weaning.

p : If two circles are apart from each other then we can draw four common tangents to them.

Sol. Following are the five different ways of writing p which convey the same meaning.

(*i*) Two circles are apart from each other implies we can draw four common tangents to them.

(*ii*) Knowing that two circles are apart from each other is sufficient to conclude that we can draw four common tangents to them.

(*iii*) Two circles are apart from each other only if we can draw four common tangents to them.

(*iv*) For two circles to be apart from each other it is necessary condition that we can draw four common tangents to them.

(*v*) If we cannot draw four common tangents to two circles then the two circles are not apart from each other.

Example 9: The statement $p \Rightarrow q$ is logically equivalent to

(*a*) $(\sim q) \Leftrightarrow p$ (*b*) $\sim q \Rightarrow \sim p$

(*c*) $p \Leftrightarrow (\sim q)$ (*d*) $(\sim p) \Leftrightarrow (\sim q)$

Sol. (*b*)

p	q	$p \Rightarrow q$	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$	$(\sim p) \Leftrightarrow (\sim q)$
T	T	T	F	F	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Example 10: If $\sim q \Rightarrow p$ is false, then

(*a*) p is true and q is true (*b*) p is true and q is false

(*c*) p is false and q is true (*d*) p is false and q is false

Sol. (*d*) $\sim q$ is T

p is F

Hence p and q both are false.

Concept Application

6. Write the contra positive of following statements.

(*i*) p : If two chords bisect each other then they are diameter.

(*ii*) q : If two triangles are similar then their corresponding angles are equal.

7. Write the converse of following statements

(*i*) p : If two circles are congruent then their radius are equal

(*ii*) p : If two polygons are congruent then they will overlap each other.

8. If $q \Rightarrow p$ is false, then

(*a*) p is true and q is true (*b*) p is true and q is false

(*c*) p is false and q is true (*d*) p is false and q is false

9. Check whether following statements are true or not, with reason

(*i*) p : 16 is a multiple of 2, 3 and 4.

(*ii*) q : 16 is a multiple of 2, 3 or 4.

10. Identify the type of "or" used in the following statements.

(*i*) To open bank account a person need a voter I-card, driving licence or Passport.

(*ii*) $\sqrt[3]{64}$ is a rational number or an irrational number.

11. Which of the following option can be used as basic connectives?

(*a*) "Far", "near" (*b*) "And", "Or"

(*c*) It is false (*d*) Is

12. A compound statement with "OR" is true when

(*a*) At least one component statement is true.

(*b*) All the component statements are true.

(*c*) All the component statements are false.

(*d*) Both (*a*) and (*b*)

13. A compound statement with "OR" is false when

(*a*) At least one component statement is false.

(*b*) At most one component statement is false.

(*c*) All the component statements are false.

(*d*) Only one component statement is true other are false.

14. Identify the quantifier in the following statements and also write their negation.

(*i*) p : For every natural number x, $7x$ is greater than 7.

(*ii*) q : There exists a tangent which is chord to the circle.

ALGEBRA OF STATEMENTS

Statements satisfy many laws some of which are given below:

1. **Idempotent Laws:** If p is any statement then

   (*i*) $p \vee p = p$

   (*ii*) $p \wedge p = p$

2. **Associative Laws:** If p, q, r are any three statements, then

   (*i*) $p \vee (q \vee r) = (p \vee q) \vee r$

   (*ii*) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

3. **Commutative Laws:** If p, q are any two statements, then

   (*i*) $p \vee q = q \vee p$

   (*ii*) $p \wedge q = q \wedge p$

4. **Distributive Laws:** If p, q, r are any three statements, then

   (*i*) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

   (*ii*) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

5. **Identity Laws:** If p is any statement, t is tautology and c is a contradiction, then

   (*i*) $p \vee t = t$

   (*ii*) $p \wedge t = p$

   (*iii*) $p \vee c = p$

   (*iv*) $p \wedge c = c$

6. **Complement Laws:** It t is a tautology, c is a contradiction and p is any statement, then

   (*i*) $p \vee (\sim p) = t$

   (*ii*) $p \wedge (\sim p) = c$

   (*iii*) $\sim t = c$

   (*iv*) $\sim c = t$

7. **Involution Law:** If p is any statement, then $\sim(\sim p) = p$.

8. **De-Morgan's Laws:** If p and q are two statements, then

   (*i*) $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$

   (*ii*) $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$

(*i*) **Proof:**

$\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

(*ii*) **Proof:**

$\sim(p \wedge q) \equiv (\sim p) \vee (\sim p)$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

9. **Symmetric Difference:** If p and q are two statements, then

$$p\Delta q = (p \wedge (\sim q)) \vee ((\sim p) \wedge q)$$
$$= (p \vee q) \wedge (\sim(p \wedge q))$$

Train Your Brain

Example 11: $\sim(p \Rightarrow q) \Leftrightarrow \sim p \vee \sim q$ is

(*a*) A tautology

(*b*) A contradiction

(*c*) Neither a tautology nor a contradiction

(*d*) Cannot come to any concusion

Sol. (*c*)

p	q	$p \Rightarrow q$	$\sim(p \Rightarrow q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \Rightarrow q) \Leftrightarrow \sim p \vee \sim q$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	T	F	T	F
F	F	T	F	T	T	. T	F

Last column shows that result is neither a tautology nor a contradiction.

Example 12: In the truth table for the statement $(p \rightarrow q) \leftrightarrow (\sim P \vee q)$, the last column has the truth value in the following order is

(*a*) *TTFF* (*b*) *FFFF*

(*c*) *TTTT* (*d*) *FTFT*

Sol. (*c*)

p	q	$p \Rightarrow q$	$\sim p$	$\sim p \vee q$	$(p \Rightarrow q) \Leftrightarrow \sim(p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Concept Application

15. Let $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. Then, this law is known as

(*a*) Conunutative law (*b*) Associative law

(*c*) De-Morgan's law (*d*) Distributive law

16. If p, q are true and r is false statement, then which of the following is true statement?

(*a*) $(p \wedge q) \vee r$ is F

(*b*) $(p \wedge q) \rightarrow r$ is T

(*c*) $(p \vee q) \wedge (p \vee r)$ is T

(*d*) $(p \rightarrow q) \leftrightarrow (p \rightarrow r)$ is T

17. The negation of $(\sim p \wedge q) \wedge (p \wedge \sim q)$ is

(*a*) $(p \vee \sim q) \vee (\sim p \vee q)$

(*b*) $(p \vee \sim q) \wedge (\sim p \vee q)$

(*c*) $(p \wedge \sim q) \wedge (\sim p \vee q)$

(*d*) $(p \wedge \sim q) \wedge (p \vee \sim q)$

18. $\sim(p \rightarrow q) \rightarrow [(\sim p) \vee (\sim q)]$ is

(*a*) A tautology

(*b*) A contradiction

(*c*) Neither a tautology nor contradiction

(*d*) Cannot come any conclusion.

Short Notes

Mathematical Statements

The basic unit involved in mathematical reasoning is a mathematical statement. A sentence is called a mathematically acceptable statement if it is either true or false but not both. The statements are generally denoted by small letters p, q, r,...

E.g., "Two plus two equals four".

It is true. Therefore it is a statement.

"All prime numbers are odd numbers"

It is not true. Therefore it is not a statement.

Whenever we mention a statement, it is a "Mathematically acceptable" statement.

Negation of a Statement

The denial of a statement is called the negation of the statement.

If p is a statement, then the negation of p is also a statement and is denoted by $\sim p$ and read as 'not p'.

E.g., Consider a statement:

p : sky is blue

Its negation is:

$\sim p$: sky is not blue"

Compound Statement

A compound statement is a statement which is made up of two or more statements. The individual statements are called component statements.

E.g., "All rational numbers are real and all real numbers are complex". Is a compound statement.

The two component statements are:

p : All rational numbers are real.

q : All real numbers are complex.

Connectives

The word 'and'

Any two component statements can be connected by the word 'and' to form a compound statement.

The rules for the compound statement with 'and' are:

Rule 1: The compound statement with 'and' is true if all its component statements are true.

Rule 2: The compound statement with 'and' is false if some or all of its component statements are false.

The word 'or'

Any two component statements can be connected by the word "or" to form a compound statement.

The rules for the compound statement with 'or' are:

Rule 1: A compound statement with an 'or' is true if one component statement is true or both the component statements are true.

Rule 2: A compound statement with an 'or' is false if both the component statements are false.

Quantifiers

In mathematics, sometimes we come across many mathematical statements containing phrases **"There exists"** and **"For every"**. These two phrases are called **quantifiers**. Phrase **"for every (or for all)"** is called the **universal quantifier** and the phrase **"There exists"** is known as the **existential quantifier**.

E.g., p : For every real number x, x is less than $x + 1$.

q : There exists a capital for every country in the world.

Implications

The statements of the form "If p then q", "p only if q", and "if and only if" are called implications.

p implies q is denoted by $p \Rightarrow q$

if and only if is denoted by the symbol $\Leftrightarrow$

General Rules for Validating Statements

Rule 1: If p and q are mathematical statements, then in order to show that the statement "p and q" is true, the following steps are followed.

Case I: Show that the statement p is true.

Case II: Show that the statement q is true.

Rule 2: In order to show that the statement "p or q" is true, one must consider the following.

Case I: By assuming that p is false, show that q must be true.

Case II: By assuming that q is false, show that p must be true.

Rule 3: Statements with 'If-then'

In order to prove the statement "If p then q" we need to show that any of the following cases is true.

Case I: By assuming that p is true, show that q must be true.

Case II: By assuming that q is false, show that p must be false.

Rule 4: Statements with 'If and only if'

In order to prove the statement "p if and only if q", we need to show.

(i) If p is true, then q is true and

(ii) If q is true, then p is true.

Method of Contradiction

To check whether a statement p is true, we assume that p is not true *i.e.* $\sim p$ is true. Then, we arrive at some result which is contradictory to our assumption. Therefore, we conclude that p is true.

This method involves giving an example of a situation where the statement is not valid. Such an example is called a counter example.

Important /Critical Points to Remember

Truth Table

A table that shows the relationship between the truth value of compound statement. $S(p, q, r,...)$ and the truth values of its sub-statements $p, q, r,...$, etc., is called the truth table of statement S. If p and q are two simple statements then truth table for basic logical connectives of:

Conjunction		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation	
p	$\sim p$
T	F
F	T

Conditional		
p	p	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$ or $(p \Rightarrow q) \wedge (q \Leftrightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Logical Equivalence

Two compound statements $S_1(p_1, q_1, r_1 ...)$ and $S_2(p_2\ q_2\ r_2 ...)$ are said to be logically equivalent if they have the same truth values for all logical possibilities.

In other words, two statements S_1 and S_2 are equivalent if they have identical truth table *i.e.* the entries in the last column of the truth tables are same.

If statements S_1 and S_2 are equivalent then we write $S_1 \equiv S_2$

Tautology: A statement is said to be a tautology if it is true for all logical possibilities.

Contradiction: A statement is a contradiction if it is false for all logical possibilities *i.e.* its truth value is always *F*.

Duality

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ *by* $\vee$ and $\vee$ *by* $\wedge$.

Converse: The converse of the conditional statement $p \to q$ is defined as $q \to p$.

Inverse: The inverse of the conditional statement $p \to q$ is defined as $\sim p \to \sim q$

Contrapositive: The contrapositive of the conditional statement $p \to q$ is defined as $\sim q \to \sim p$

Negation of Compound Statements

(*a*) **Negation of conjunction:** If p and q are two statements then

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

(*b*) **Negation of disconjunction:** If p and q are two statements then $\sim (p \vee g) \equiv \sim p \wedge \sim q$

(*c*) **Negation of implication:**

If p and q are two statements, then $\sim (p \Rightarrow q) \equiv p \wedge \sim q$

(*d*) **Negation of Biconditional:**

If p and q are two statements, then

$\sim (p \Leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Algebra of Statements

If p, q, r are any three statements then the some law of algebra of statements are as follow:

(*a*) **Idempotent Laws**

(*i*) $p \vee p \equiv p$

(*ii*) $p \wedge p \equiv p$

(*b*) **Commutative Laws**

(*i*) $p \vee q \equiv q \vee p$

(*ii*) $p \wedge q \equiv q \wedge p$

(c) **Associative Law**

(*i*) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(*ii*) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(*d*) **Distributive Laws**

(*i*) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(*ii*) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(*e*) **De'Morgan's Law**

(*i*) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

(*ii*) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

(*f*) **Contrapositive Laws**

For any statement p, we have

$p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

(g) **Involution Laws (Double Negation Laws)**

$\sim(\sim p) \equiv p$

Solved Examples

1. The logical equivalent proposition of $p \Leftrightarrow q$ is

(*a*) $(p \wedge q) \vee (p \wedge q)$

(*b*) $(p \Rightarrow q) \wedge (p \Rightarrow q)$

(*c*) $(p \wedge q) \vee (q \Rightarrow p)$

(*d*) $(p \wedge q) \Rightarrow (q \vee p)$

Sol. (*b*) $(p \Rightarrow q) \wedge (p \Rightarrow q)$

$\because$ $(p \Rightarrow q) \wedge (q \Rightarrow p)$ means $p \Leftrightarrow q$.

Hence the logically equivalent proposition of $p \Leftrightarrow q$ is $(p \Rightarrow q) \wedge (q \Rightarrow p)$

2. The contrapositive of the statement "If two triangles are identical, then are similar" is

(*a*) If two triangles are not similar, then they are not identical.

(*b*) If two triangles are not identical, then they are not similar.

(*c*) If two triangles are not identical, then they are similar.

(*d*) If two triangles are not similar, then they are identical.

Sol. (*a*) If two triangles are not similar, then they are not identical

Let, p : Two triangles are identical

q : Two triangles are similar

Clearly, the given statement in symbolic form is $p \to q$.

Contrapositive of $p \to q$ is $\sim q \to \sim p$

i.e., If two triangles are not similar, then they are not identical.

3. If $p \Rightarrow (\sim p \vee q)$ is false, then the truth values of p and q are respectively

(a) F, T (b) F, F

(c) T, T (d) T, F

Sol. (d) T, F

$p \Rightarrow (\sim p \vee q)$ is false means p is true and $\sim p \vee q$ is false.

$\Rightarrow p$ is true and both $\sim p$ and q are false.

$\Rightarrow p$ is true and q is false.

4. Consider three statements

P : Suyash will come today evening

q : Shalini will be at his home

r : They will go to party

Then the statement $(\sim q \wedge \sim r) \Rightarrow p$ is logically equivalent to

(a) If Suyash will not come today then Shalini will go to party.

(b) If Suyash will not come today Shalini will not go to party.

(c) Suyash will come today or Shalini will be at his home or they will go to party.

(d) Suyash will come today or Shalini will go to party.

Sol. (c) Suyash will come today or Shalini will be at his home or they will go to party.

$(\sim q \wedge \sim r) \Rightarrow p \equiv \sim (\sim q \wedge \sim r) \vee p$

$\equiv (q \vee r) \vee p \equiv p \vee q \vee r$

5. The contrapositive of the statement: "If the weather is fine then my friends will come and we go for a picnic." is

(a) The weather is fine but my friends will not come of we do not go for a picnic.

(b) If my friends do not come or we not go for picnic then weather will not be fine.

(c) If the weather is not fine then my friends will not come or we do not go for a picnic.

(d) The weather is not fine but my friends will come and we go for a picnic.

Sol. (b) If my friends do not come or we do not go for picnic then weather will not be fine.

$P \rightarrow (q \wedge r)$

Contrapositive $(\sim (q \wedge r)) \rightarrow \sim p$

Contrapositive $(\sim (q \wedge r)) \rightarrow \sim p$

$\therefore (\sim q \vee \sim r) \rightarrow \sim p$

If my friends do not come or we do not go for picnic then weather will not be fine.

6. Let p, q and r be three statements. Consider two compound statements

$S_1 : (p \Rightarrow q) \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$

$S_2 : (p \Leftrightarrow q) \Leftrightarrow r \equiv p \Leftrightarrow (q \Leftrightarrow r)$

State in order, whether S_1, S_2 are true or false.

(where, T represents true and F represents false)

(a) TT (b) TF

(c) FT (d) FF

Sol. (c)

p	q	r	$(p \Rightarrow q) \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$	$(p \Leftrightarrow q) \Leftrightarrow r$	$p \Leftrightarrow (q \Leftrightarrow r)$
T	T	T	T	T	T	T
T	F	T	T	T	F	F
F	T	T	T	T	F	F
F	F	T	T	T	F	F
T	T	F	F	F	F	F
T	F	F	T	T	T	T
F	T	F	F	T	T	T
F	F	F	F	T	F	F

7. Let $\oplus$ and $\otimes$ are two mathematical operations. If $p \oplus (q \otimes r)$ is equivalent to $((p \wedge q) \Rightarrow r)$, then $\oplus$ and $\otimes$

(a) Can be $\vee$ and $\wedge$ respectively

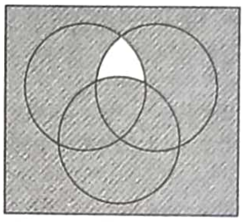
(b) Can be $\wedge$ and $\vee$ respectively

(c) Can both be $\Rightarrow$

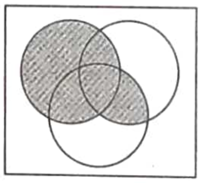
(d) Can be $\Rightarrow$ and $\Leftrightarrow$ respectively

Sol. (c)

$(p \wedge q) \Rightarrow r$

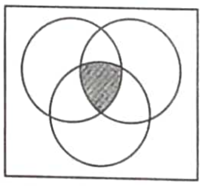


(a) $p \vee (q \wedge r)$



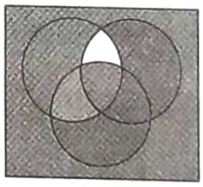
(A) Is incorrect option

(b) $p \wedge (q \wedge r)$



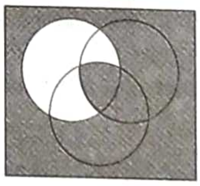
(B) Is incorrect option

(c) $p \Rightarrow (q \Rightarrow r)$



(C) Is correct option

(d) $p \Rightarrow (q \Leftrightarrow r)$



(D) Is incorrect option

8. Verify by method of contradiction p : For every real number $x \in \left[0, \frac{\pi}{2}\right]$, we have $\sin x + \cos x \geq 1$.

Sol. Suppose p is not true. Then there exist an $x \in \left[0, \frac{\pi}{2}\right]$ for which $\sin x + \cos x < 1$.

Since $x \in \left[0, \frac{\pi}{2}\right]$ neither $\sin x$ nor $\cos x$ is negative, So,

$0 \leq \sin x + \cos x < 1$

$\Rightarrow 0^2 \le (\sin x + \cos x)^2 < 1^2$

$\Rightarrow 0 \le \sin^2 x + \cos^2 x + 2\sin x.\cos x < 1$

$\Rightarrow 0 \le 1 + 2\sin x.\cos x < 1$

So, $1 + 2\sin x \cos x < 1$

Subtracting 1 from both sides gives $2\sin x.\cos x < 0$, but This contradicts the fact that neither $\sin x$ or $\cos x$ is negative. So p is true

9. The statement $(p \vee \sim q) \vee (\sim p \vee q)$ is

(*a*) A tautology

(*b*) A fallacy

(*c*) Equivalent to $p \vee (\sim q)$

(*d*) Equivalent to $(\sim p) \wedge q$

Sol. (*a*)

p	q	$\sim q$	$p \vee \sim q$	$\sim p \vee q$	$\sim p \wedge q$	$(p \vee \sim q) \vee (\sim p \vee q)$
T	*T*	*F*	*T*	*T*	*F*	*T*
T	*F*	*T*	*T*	*F*	*F*	*T*
F	*T*	*F*	*F*	*T*	*T*	*T*
F	*F*	*T*	*T*	*T*	*F*	*T*

As all truth values are true, it is a tautology.

10. The boolean expression $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is

(*a*) Equivalent to $(p \Leftrightarrow \sim q)$

(*b*) Equivalent to $(p \Leftrightarrow q)$

(*c*) A fallacy

(*d*) A tautology

Sol. (*d*)

p	q	$p \Rightarrow q$	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$	$p \Leftrightarrow \sim q$	$\sim p \Leftrightarrow \sim q$
T	*T*	*T*	*T*	*T*	*F*	*T*
T	*F*	*F*	*F*	*T*	*T*	*F*
F	*T*	*T*	*T*	*T*	*T*	*F*
F	*F*	*T*	*T*	*T*	*F*	*T*

Hence tautology.

11. Consider the following statements

I. The negation of the statement "The number 2 is greater than 7" is "The number 2 is not greater than 7".

II. The negation of the statement "Every natural number is an integer" is "every natural number is not an integer".

Choose the correct option.

(*a*) Only I is true (*b*) Only II is true

(*c*) Both are true (*d*) Both are false

Sol. (*c*) I. The given statement is "The number 2 is greater than 7". Its negation is "The number 2 is not greater than 7".

II. The given statement is "Every natural number is an integer". Its negation is "Every natural number is not an integer".

12. If $S^*(p, q, r)$ is the dual of the compound statement $S(p, q, r)$ and $S(p, q, r) = \sim p \wedge [\sim (q \vee r)]$ then $S^*(\sim p, \sim q, \sim r)$ is equivalent to

(*a*) $S(p, q, r)$

(*b*) $\sim S(\sim p, \sim q, \sim r)$

(*c*) $\sim S(p, q, r)$

(*d*) $S^*(p, q, r)$

Sol. (*c*) $S^*(p, q, r) = \sim p \vee [\sim (q \wedge r)]$

$S^*(\sim p, \sim q, \sim r) = \sim (\sim p) \vee [\sim (\sim q \wedge \sim r)] = p \vee [q \vee r]$

$\sim S(p, q, r) = \sim [\sim p \wedge [\sim (q \wedge r))] = \sim (\sim p) \vee \sim (\sim (q \vee r))$

$= p \vee (q \vee r) = S^*(\sim p, \sim q, \sim r)$

13. Let f be a function from a set X to a set Y. Consider the following statements:

P: For each $x \in X$, there exists unique $y \in Y$ such that $f(x) = y$

Q: For each $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

R: There exist $x_1, x_2 \in X$ such that $x_1 \ne x_2$ and $f(x_1) = f(x_2)$.

The negation of the statement "f is one-to-one and onto" is

(*a*) P or not R (*b*) R or not P

(*c*) R or not Q (*d*) P and not R

Sol. (*c*) Negation of 'f is one to one and onto' is R or not Q.

14. Match the followings:

	Column-I		Column-II
A.	Dual of statement $[(p \vee q) \wedge (\sim q)] \vee (\sim p)$	p.	$[p \wedge \sim q] \vee (\sim p)$
B.	Logically equivalent of $[(p \vee q) \wedge (\sim q)] \vee (\sim p)$	q.	$[(\sim p \wedge \sim q) \vee q] \wedge p$
C.	Negation of $[(p \vee q) \wedge (\sim q)] \vee (\sim p)$	r.	$[(\sim p \wedge \sim q) \vee q] \vee (\sim p)$
D.	Contrapositive of $[(p \vee q) \wedge (\sim q)] \rightarrow (\sim p)$	s.	$[(p \wedge q) \vee \sim q] \wedge (\sim p)$

Sol. A-(s); B-(p); C-(q); D-(r)

A. Dual of statement $[(p \vee q) \wedge (\sim q)] \vee (\sim p)$ is $[(p \wedge q) \vee (\sim q)] \wedge (\sim p)$

B. Logically equivalent of $[(p \vee q) \wedge (\sim q)] \vee \sim p$ is $[(p \wedge \sim q) \vee (q \wedge \sim q)] \vee \sim p$ or $[p \wedge \sim q] \vee \sim p$

C. Negation of $[(p \vee q) \wedge (\sim q)] \vee (\sim p)$ is $\sim [(p \vee q) \wedge (\sim q)] \wedge \sim (\sim p)$ or $[(\sim p \wedge \sim q) \vee q)] \wedge p$

D. Contrapositive of $[(p \vee q) \wedge (\sim q)] \rightarrow (\sim p)$ is $\sim (\sim p) \rightarrow \sim [(p \vee q) \wedge (\sim q)]$ or $p \rightarrow [\sim (p \vee q) \vee q]$ or $(\sim p) \vee [(\sim p \wedge \sim q) \vee q]$

Exercise-1 (Topicwise)

STATEMENTS

1. Which of the following is not a statement?
   (a) Every set is a finite set.
   (b) 8 is less than 6.
   (c) Where are you going?
   (d) The sum of interior angles of a triangles is 180 degrees.
2. Which of the following is a statement?
   (a) May you live long !
   (b) May God bless you !
   (c) The sun is a star.
   (d) Hurrah ! We have won the match.
3. Which is a statement?
   (a) $x+1=6$ (b) $5 \in N$
   (c) $x+y<12$ (d) $x+y>12$

NEGATIONS

4. Negation of "$2+3=5$ and $8<10$" is
   (a) $2+3 \neq 5$ or $8<10$ (b) $2+3=5$ or $8 \not< 10$
   (c) $2+3 \neq 5$ or $8 \not< 10$ (d) $2+3=5$ or $8<10$
5. If p : Ram is smart.
   q : Ram is intelligent.
   Then, the symbolic form of Ram is smart and intelligent, is
   (a) $(p \wedge q)$ (b) $(p \vee q)$
   (c) $(p \wedge \sim q)$ (d) $(p \vee \sim q)$
6. Negation of "Manu is in class X or Anu is in class XII" is
   (a) Manu is not in class X but Anu is in class XII.
   (b) Manu is not is class X but Anu is not in class XII.
   (c) Manu is not in class X and Anu is not in class XII.
   (d) Manu is in class X and Anu is in class XII.
7. Truth value of the statement "if p then q" is false when
   (a) p is true, q is true (b) p is true, q is false
   (c) p is false, q is true (d) p is false, q is false

ALGEBRA OF STATEMENTS

8. Let p and q be two statements.
   Then, $(\sim p \vee q) \wedge (\sim p \wedge \sim q)$ is a
   (a) Tautology
   (b) Contradiction
   (c) Neither tautology nor contradiction
   (d) Either tautology or contradiction
9. Let $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. Then, this law is known as
   (a) Commutative law (b) Associative law
   (c) De-Morgan's law (d) Distributive law
10. The false statement in the following is
    (a) $p \wedge (\sim p)$ is a contradiction.
    (b) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a contradiction.
    (c) $\sim(\sim p) \Leftrightarrow p$ is a tautology.
    (d) $p \vee (\sim p)$ is a tautology.
11. A compound sentence formed by two simple statements p and q using connective 'or' is called
    (a) Conjunction (b) Disjunction
    (c) Implication (d) Division
12. The contrapositive of $p \Rightarrow \sim q$ is
    (a) $\sim p \Rightarrow q$ (b) $\sim q \Rightarrow p$
    (c) $q \Rightarrow \sim p$ (d) $\sim q \Rightarrow \sim p$
13. If $p = \Delta ABC$ is equilateral and q = each angle is $60°$. Then, symbolic form of statement
    (a) $p \vee p$ (b) $p \wedge q$ (c) $p \Rightarrow q$ (d) $p \Leftrightarrow q$
14. If p = He is intelligent
    q = He is strong
    Then, symbolic form of statement.
    "It is wrong that he is intelligent or strong," is
    (a) $\sim p \vee \sim p$ (b) $\sim(p \wedge q)$
    (c) $\sim p \wedge \sim q$ (d) $p \vee \sim q$
15. If p and q are two statements, then statement $p \Rightarrow q \wedge \sim q$ is
    (a) Tautology
    (b) Contradiction
    (c) Neither tautology nor contradiction
    (d) Either tautology or contradiction
16. If p and q are two statements, then
    $\sim(p \wedge q) \vee \sim(q \Leftrightarrow p)$
    (a) Tautology
    (b) Contradiction
    (c) Neither tautology nor contradiction
    (d) Either tautology or contradiction
17. If p : A man is happy
    q : A man is rich
    Then, the statement, "If a man is not happy, then he is not rich" is written is
    (a) $\sim p \to \sim q$ (b) $\sim q \to p$
    (c) $\sim q \to \sim p$ (d) $q \to \sim p$
18. $\sim(p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to
    (a) $\sim p$ (b) p (c) q (d) $\sim q$
19. The compound statement $p \to (\sim p \vee q)$ is false, then the truth values of p and q are respectively
    (a) T, T (b) T, F (c) F, T (d) F, F
20. $\sim[\sim p \wedge (p \Leftrightarrow q)] \equiv$
    (a) $p \vee q$ (b) $q \wedge p$ (c) T (d) F

Exercise-2 (Learning Plus)

1. If p, q and r are 3 statements, then the truth value of $((\sim P \vee q) \wedge \sim r) \Rightarrow p$ is
   (a) True if truth values of p, q, r are T, F, T respectively.
   (b) False if truth values of p, q, r are T, F, T respectively.
   (c) False if truth values of p, q, r are T, F, F respectively.
   (d) False if truth values of p, q, r are T, T, T respectively.

2. The statement $p \Leftrightarrow q$ is not equivalent to
   (a) $(p \vee q) \Rightarrow (p \wedge q)$
   (b) $(p \wedge q) \Rightarrow (p \vee q)$
   (c) $(p \vee q) \Leftrightarrow (p \Rightarrow q)$
   (d) $\sim (p \vee q) \vee (p \wedge q)$

3. Which of the following statements is false when p is true and q is false?
   (a) $(p \Rightarrow q) \Leftrightarrow r$ (b) $(p \Leftrightarrow q) \Rightarrow r$
   (c) $(q \Rightarrow r) \Rightarrow p$ (d) $(r \Rightarrow p) \Rightarrow q$

4. The negation of $(\sim p \wedge q) \vee (p \wedge \sim q)$ is
   (a) $(p \vee \sim q) \vee (\sim p \vee q)$
   (b) $(p \vee \sim q) \wedge (\sim p \vee q)$
   (c) $(p \wedge \sim q) \wedge (\sim p \vee q)$
   (d) $(p \wedge \sim q) \wedge (p \vee \sim q)$

5. The statement $(\sim(p \Leftrightarrow q)) \wedge p$ is equivalent to
   (a) $p \wedge q$ (b) $q \Leftrightarrow p$
   (c) $p \wedge \sim q$ (d) $\sim p \wedge q$

6. Which of the following is true?
   (a) $p \Rightarrow q = \sim p \Rightarrow \sim q$
   (b) $\sim (p \Rightarrow \sim q) \equiv \sim p \wedge q$
   (c) $\sim (\sim p \Rightarrow \sim q) \equiv \sim p \wedge q$
   (d) $\sim(\sim p \Leftrightarrow q) = [\sim (p \Rightarrow q) \wedge \sim (q \Rightarrow p)]$

7. The inverse of the statement $(p \wedge \sim q) \rightarrow r$ is
   (a) $\sim (p \vee \sim q) \rightarrow \sim r$ (b) $(\sim p \wedge q) \rightarrow \sim r$
   (c) $(\sim p \vee q) \rightarrow \sim r$ (d) None of these

8. Let p, q and r be any three logical statements. Which of the following is true?
   (a) $\sim [p \wedge (\sim q)] \equiv (\sim p) \wedge q$
   (b) $\sim [(p \vee q) \wedge (\sim r) \equiv (\sim p) \vee (\sim q) \vee (\sim r)$
   (c) $\sim [p \vee (\sim q)] \equiv (\sim p) \wedge q$
   (d) $\sim[p \vee (\sim q)] \equiv (\sim p) \wedge \sim q$

9. The inverse of the statement, 'If x is zero then we cannot divide by x' is
   (a) If we cannot divide by x, then x is zero.
   (b) If we cannot divide by x, then x is not zero.
   (c) If x is not zero then we divide by x.
   (d) None of these.

10. If p is any statement, t is tautology and c is a contradiction, then which of the following is not correct?
    (a) $p \vee (\sim p) = c$ (b) $p \vee t = t$
    (c) $p \wedge t = p$ (d) $p \wedge c \equiv c$

11. $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is
    (a) A tautology.
    (b) A contradiction.
    (c) Both a tautology and a contradiction.
    (d) Neither a tautology nor a contradiction.

12. The contrapositive of $(p \vee q) \Rightarrow r$ is
    (a) $r \Rightarrow (p \vee q)$ (b) $\sim r \Rightarrow (p \vee q)$
    (c) $\sim r \Rightarrow \sim p \wedge \sim q$ (d) $p \Rightarrow (q \vee r)$

13. $\sim((\sim p) \wedge q)$ is equal to
    (a) $p \vee (\sim q)$ (b) $p \vee q$
    (c) $\sim q \Rightarrow \sim p$ (d) $\sim p \Rightarrow \sim q$

14. If $p \Rightarrow (q \vee r)$ is false, then the truth values of p, q, r are respectively
    (a) T, F, F (b) F, F, F
    (c) F, T, T (d) T, T, F

15. The propositions $(p \Rightarrow \sim p) \wedge (\sim p \Rightarrow p)$ is a
    (a) Tautology and contradiction.
    (b) Neither tautology nor contradiction.
    (c) Contradiction.
    (d) Tautology.

16. If p : 4 is an even prime number, q : 6 is a divisor of 12 and r : the HCF of 4 and 6 is 2, then which one of the following is true?
    (a) $(p \wedge q)$ (b) $(p \vee q) \wedge \sim r$
    (c) $\sim(q \wedge r) \wedge p$ (d) $\sim p \vee (q \wedge r)$

17. Consider the following statement :
    A : Rishi is a judge.
    B : Rishi is honest.
    C : Rishi is not arrogant.
    The negation of the statement "if Rishi is a judge and he is not arrogant, then he is honest" is
    (a) $B \rightarrow (A \vee C)$ (b) $(\sim B) \wedge (A \wedge C)$
    (c) $B \rightarrow ((\sim A) \vee (\sim C))$ (d) $B \rightarrow (A \wedge C)$

18. the negation of the Boolean expression $((\sim q) \wedge p) \Rightarrow ((\sim p) \vee q)$ is logically equivalent to
    (a) $p \Rightarrow q$ (b) $q \Rightarrow p$
    (c) $\sim(p \Rightarrow q)$ (d) $\sim(q \Rightarrow p)$

19. The Boolean expression $(\sim(p \wedge q)) \vee q$ is equivalent to :
    (a) $q \rightarrow (p \wedge q)$ (b) $p \rightarrow q$
    (c) $p \rightarrow (p \rightarrow q)$ (d) $p \rightarrow (p \vee q)$

20. Let p, q, r be three logical statement. Consider the compound statements

$S_1 : ((\sim p) \vee q) \vee ((\sim p) \vee r)$ and

$S_2 : p \rightarrow (q \vee r)$

Then, which of the following is **NOT** true?

(*a*) If S_2 is True, then S_1 is True.

(*b*) If S_2 is False, then S_1 is False.

(*c*) If S_2 is False, then S_1 is true.

(*d*) If S_1 is False, then S_2 is False.

21. The number of choices of $\Delta \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$, Such that $(p \,\Delta\, q) \Rightarrow ((p \Delta \sim q) \vee ((\sim p) \,\Delta\, q))$ is tautology, is

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4

22. Consider the following statement :

A : Rishi is judge.

B : Rishi is honest.

C : Rishi is not arrogant.

The negation of the statement "if Rishi is a judge and he is not arrogant, then he is honest" is

(*a*) $B \rightarrow (A \vee C)$ (*b*) $(\sim B) \wedge (A \wedge C)$

(*c*) $B \rightarrow ((\sim A) \vee (\sim C))$ (*d*) $B \rightarrow (A \wedge C)$

23. Consider the following two proposition:

$P_1 : \sim (p \rightarrow \sim q)$

$P_2 : (p \wedge \sim q) \wedge ((\sim p) \vee q)$

If the proposition $p \rightarrow ((\sim p) \vee q)$ is evaluated as FALSE, then :

(*a*) P_1 is TRUE and P_2 is FALSE.

(*b*) P_1 is FALSE and P_2 is TRUE.

(*c*) Both P_1 and P_2 are FALSE.

(*d*) Both P_1 and P_2 are TRUE.

24. The negation of the Boolean expression $((\sim q) \wedge p) \Rightarrow ((\sim p) \vee q)$ is logically equivalent to

(*a*) $p \Rightarrow q$ (*b*) $q \Rightarrow p$

(*c*) $\sim (p \Rightarrow q)$ (*d*) $\sim (q \Rightarrow p)$

25. Let $\Delta, \nabla \in \{\wedge, \vee\}$ be such that $p \nabla q \Rightarrow ((p \nabla q) \nabla r)$ is a tautology. Then $(p \nabla q)$ Dr is logically equivalent to :

(*a*) $(p \,\Delta\, r) \vee q$ (*b*) $(p \,\Delta\, r) \wedge q$

(*c*) $(p \wedge r) \,\Delta\, q$ (*d*) $(p \nabla r) \wedge q$

26. Let $r \in \{p, q, \sim p, \sim q\}$ be such that the logical statement $r \vee (\sim p) \Rightarrow (p \wedge q) \vee r$ is a tautology. Then 'r' is equal to :

(*a*) p (*b*) q

(*c*) $\sim p$ (*d*) $\sim q$

27. The Boolean expression $(\sim(p \wedge q)) \vee q$ is equivalent to :

(*a*) $q \rightarrow (p \wedge q)$ (*b*) $p \rightarrow q$

(*c*) $p \rightarrow (p \rightarrow q)$ (*d*) $p \rightarrow (p \vee q)$

28. Which of the following statement is a tautology?

(*a*) $((\sim q) \wedge p) \wedge q$

(*b*) $((\sim q) \wedge p) \wedge (p \wedge (\sim p))$

(*c*) $((\sim q) \wedge p) \vee (p \vee (\sim p))$

(*d*) $(p \wedge q) \wedge (p \wedge q))$

29. Let p, q, r be three logical statements. Consider the compoun statements

$S_1 : ((\sim p) \vee q) \vee ((\sim p) \vee r)$ and

$S_2 : p \rightarrow (q \vee r)$

Then which of the following is **NOT** true?

(*a*) If S_2 is True, then S_1 is True.

(*b*) If S_2 is False, then S_1 is False.

(*c*) If S_2 is False, then S_1 is True.

(*d*) If S_1 is False, then S_2 is False.

30. Suppose a class has 7 students. The average marks of th students in the mathematics examination is 62, and the variance is 20. A student fails in the examination if he/sh gets less than 50 marks, then in worst case, the number o students can fail is

31. The maximum number of compounds propositions. Out o

$p \vee r \vee s, p \vee r \vee \sim s, p \vee \sim q \vee s,$

$\sim p \vee \sim r \vee s, \sim p \vee \sim r \vee \sim s, \sim p \vee \sim q \vee \sim s,$

$q \vee r \vee \sim s, q \vee \sim r \vee \sim s, \sim p \vee \sim q \vee \sim s$

That can be made simultaneously true by an assignment o the truth values to p, q, r and s, is equal to

32. Negation of the Boolean statement $(p \vee q) \Rightarrow ((\sim r) \vee p)$ equivalent to :

(*a*) $p \wedge (\sim q) \wedge r$ (*b*) $(\sim p) \wedge (\sim q) \wedge r$

(*c*) $(\sim p) \wedge q \wedge r$ (*d*) $p \wedge q \wedge (\sim r)$

33. Which of the following statements is a tautology ?

(*a*) $((\sim p) \vee q) \Rightarrow p$ (*b*) $p \Rightarrow ((\sim p) \vee q)$

(*c*) $((\sim p) \vee q) \Rightarrow q$ (*d*) $q \Rightarrow ((\sim p) \vee q)$

34. Consider the following statement :

P : Ramu is intelligent.

Q : Ramu is rich.

R : Ramu is not honest.

The negation of the statement "Ramu is intelligent and hone if and only if Ramu is not rich" can be expressed as :

(*a*) $((P \wedge (\sim R)) \wedge Q) \wedge ((\sim Q) \wedge ((\sim P) \vee R))$

(*b*) $((P \wedge R) \wedge Q) \vee ((\sim Q) \wedge ((\sim P) \vee (\sim R)))$

(*c*) $((P \wedge R) \wedge Q) \wedge ((\sim Q) \wedge ((\sim P) \vee (\sim R)))$

(*d*) $((P \wedge (\sim R)) \wedge Q) \vee ((\sim Q) \wedge ((\sim P) \vee R))$

35. Negation of the Boolean expression $p \Leftrightarrow (q \Rightarrow p)$ is

(*a*) $(\sim p) \wedge q$ (*b*) $p \wedge (\sim q)$

(*c*) $(\sim p) \vee (\sim q)$ (*d*) $(\sim p) \wedge (\sim q)$

36. The statement $(p \wedge q) \Rightarrow (p \wedge r)$ is equivalent to :

(*a*) $q \Rightarrow (p \wedge r)$

(*b*) $p \Rightarrow (p \wedge r)$

(*c*) $(p \wedge r) \Rightarrow (p \wedge q)$

(*d*) $(p \wedge q) \Rightarrow r$

37. The statement $(p \Rightarrow q) \vee (p \wedge r)$ is **NOT** equivalent to:

(*a*) $(p \wedge (\sim r)) \Rightarrow q$

(*b*) $(\sim q) \Rightarrow ((\sim r) \vee p)$

(*c*) $p \Rightarrow (q \vee r)$

(*d*) $(\sim p \wedge (\sim q)) \Rightarrow r$

Exercise-3 (Past Year Questions)

JEE MAIN

1. The Boolean expression $(p \wedge \sim q) \vee q \vee (\sim p \wedge q)$ is equivalent to: **(2016)**
   (a) $p \wedge q$ (b) $p \vee q$
   (c) $p \vee \sim q$ (d) $\sim p$

2. The following statement $(p \to q) \to [(\sim p \to q) \to q]$ is: **(2017)**
   (a) A tauology (b) Equivalent to $\sim p \to q$
   (c) Equivalent to $p \to \sim q$ (d) A fallacy

3. The Boolean expression $\sim (p \vee q) \vee (\sim p \wedge q)$ is equivalent to: **(2018)**
   (a) p (b) q
   (c) $\sim q$ (d) $\sim p$

4. If the Boolean expression $(p \oplus q) \wedge (\sim p \ominus q)$ is equivalent to $p \wedge q$, where $\oplus, \ominus \in \{\wedge, \vee\}$, then the ordered pair $(\oplus, \ominus)$ is: **(2019)**
   (a) $(\wedge, \vee)$ (b) $(\vee, \vee)$
   (c) $(\vee, \wedge)$ (d) $(\wedge, \wedge)$

5. The logical statement
   $[\sim(\sim p \vee q) \vee (p \wedge r) \wedge (\sim q \wedge r)]$ is equivalent to: **(2019)**
   (a) $(p \wedge r) \wedge \sim q$ (b) $(\sim p \wedge \sim q) \wedge r$
   (c) $\sim p \vee r$ (d) $(p \wedge \sim q) \vee r$

6. Consider the following three statements:
   P : 5 is a prime number.
   Q : 7 is a factor of 192
   R : L.C.M. of 5 and 7 is 35.
   Then the truth value of which one of the following statements is true? **(2019)**
   (a) $(\sim P) \vee (Q \wedge R)$ (b) $(P \wedge Q) \vee (\sim R)$
   (c) $(\sim P) \wedge (\sim Q \wedge R)$ (d) $P \vee (\sim Q \wedge R)$

7. Contrapositive of the statement
   "If two numbers are not equal, then their squares are not equals" is: **(2019)**
   (a) If the squares of two numbers are not equal, then the numbers are equal
   (b) If the squares of two numbers are equal, then the numbers are not equal
   (c) If the squares of two numbers are equal, then numbers are equal
   (d) If the squares of two numbers are not equal, then the numbers are not equal

8. If q is false and $p \wedge q \leftrightarrow r$ is true, then which one of the following statements is a tautology? **(2019)**
   (a) $(p \vee r) \to (p \wedge r)$ (b) $(p \wedge r) \to (p \vee r)$
   (c) $p \wedge r$ (d) $p \vee r$

9. The Boolean expression $((p \wedge q) \vee (p \vee \sim q)) \wedge (\sim p \wedge \sim q)$ is equivalent to: **(2019)**
   (a) $p \wedge q$ (b) $p \wedge (\sim q)$
   (c) $(\sim p) \wedge (\sim q)$ (d) $p \vee (\sim q)$

10. The expression $\sim (\sim p \to q)$ is logically equivalent to: **(2019)**
    (a) $\sim p \wedge \sim q$ (b) $p \wedge \sim q$
    (c) $\sim p \wedge q$ (d) $p \wedge q$

11. The contrapositive of the statement "If you are born in India, they you are a citizen of India", is: **(2019)**
    (a) If you are born in India, then you are not a citizen of India.
    (b) If you are not a citizen of India, then you are not born in India.
    (c) If you are a citizen of India, then you are born in India.
    (d) If you are not born in India, then you are not a citizen of India.

12. Which one of the following statements is not a tautology? **(2019)**
    (a) $(p \wedge q) \to p$ (b) $(p \wedge q) \to (\sim p) \vee q$
    (c) $p \to (p \vee q)$ (d) $(p \vee q) \to (p \wedge (\sim q))$

13. For any two statements p and q, the negation of the expression $p \vee (\sim p \wedge q)$ is **(2019)**
    (a) $p \wedge q$ (b) $p \leftrightarrow q$
    (c) $\sim p \vee \sim q$ (d) $\sim p \wedge \sim q$

14. If $P \Rightarrow (q \vee r)$ is false, then the truth values of p, q, r are respectively: **(2019)**
    (a) F, T, T (b) T, F, F
    (c) T, T, F (d) F, F, F

15. Which one of the following Boolean expressions is a tautology? **(2019)**
    (a) $(p \vee q) \wedge (\sim p \vee \sim q)$
    (b) $(p \wedge q) \vee (p \vee \sim q)$
    (c) $(p \vee q) \wedge (p \vee \sim q)$
    (d) $(P \vee q) \vee (p \vee \sim q)$

16. The negation of the boolean expression $\sim s \vee (\sim r \wedge s)$ is equivalent to: **(2019)**
    (a) r (b) $s \wedge r$
    (c) $s \vee r$ (d) $\sim s \wedge \sim r$

17. If the truth value of the statement $p \to (\sim q \vee r)$ is false (F), then the truth values of the statements p, q, r are respectively: **(2019)**
    (a) F, T, T (b) T, F, F
    (c) T, T, F (d) T, F, T

18. The Boolean expression $\sim(p \Rightarrow (\sim q))$ is **(2019)**

(a) $(\sim p) \Rightarrow q$ (b) $p \vee q$
(c) $q \Rightarrow \sim p$ (d) $p \wedge q$

19. The logical statement $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to **(2020)**

(a) p (b) q
(c) $\sim p$ (d) $\sim q$

20. Let A, B, C and D be four non-empty sets. The contrapositive statement of " If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq C$ " is: **(2020)**

(a) If $A \not\subseteq C$, then $A \subseteq B$ and $B \subseteq D$
(b) If $A \not\subseteq C$, then $A \not\subseteq B$, and $B \subseteq D$
(c) If $A \subseteq C$, then $B \subset A$ or $D \subset B$
(d) If $A \not\subseteq C$, then $A \not\subseteq B$ or $B \not\subseteq D$

21. Which one of the following is a tautology? **(2020)**

(a) $(P \wedge (P \to Q)) \to Q$ (b) $(P \wedge (P \vee Q))$
(c) $Q \to (P \wedge (P \to Q))$ (d) $(P \vee (P \wedge Q))$

22. Which of the following statement is a tautology? **(2020)**

(a) $\sim(p \vee \sim q) \to p \wedge q$ (b) $\sim(p \vee \sim q) \to p \vee q$
(c) $\sim(p \wedge \sim q) \to p \vee q$ (d) $p \vee (\sim q) \to p \wedge q$

23. Negation of the statement: **(2020)**

$\sqrt{5}$ is an integer or 5 is irrational is:

(a) $\sqrt{5}$ is not an integer or 5 is not irrational
(b) $\sqrt{5}$ is an integer and 5 is irrational
(c) $\sqrt{5}$ is irrational or 5 is an integer.
(d) $\sqrt{5}$ is not an integer and 5 is not irrational

24. If $p \to (p \wedge \sim q)$ is false, then the truth values of p and q are respectively: **(2020)**

(a) F, F (b) F, T
(c) T, F (d) T, T

25. Which of the following is a tautology? **(2020)**

(a) $(\sim p) \wedge (p \vee q) \to q$ (b) $(\sim q) \vee (p \wedge q) \to q$
(c) $(p \to q) \wedge (q \to p)$ (d) $(q \to p) \vee \sim(p \to q)$

26. The contrapositive of the statement "If I reach the station in time, then I will catch the train" is: **(2020)**

(a) If I will catch the train, then I reach the station in time
(b) If I do not reach the station in time, then I will catch the train
(c) If I do not reach the station in time, then I will not catch the train
(d) If I will not catch the train, then I do not reach the station in time

27. Let p, q, r be three statements such that the truth value of $(p \wedge q) \to (\sim q \vee r)$ is F. Then the truth values of p, q, r are respectively: **(2020)**

(a) T, F, T (b) F, T, F
(c) T, T, T (d) T, T, F

28. The proposition $p \to \sim(p \wedge \sim q)$ is equivalent to: **(2020)**

(a) q (b) $(\sim p) \wedge q$
(c) $(\sim p) \vee (\sim q)$ (d) $(\sim p) \vee q$

29. Given the following two statements: **(2020)**

(S_1) : $(q \vee p) \to (p \leftrightarrow \sim q)$ is a tautology:
(S_2) : $\sim q \wedge (\sim p \leftrightarrow q)$ is a fallacy. Then:

(a) Only (S_1) is correct.
(b) Both (S_1) and (S_2) are correct.
(c) Only (S_2) is correct.
(d) Both (S_1) and (S_2) are not correct.

30. The statement $(p \to (q \to p)) \to (p \to (p \vee q))$ is **(2020)**

(a) A tautology
(b) Equivalent to $(p \vee q) \wedge (\sim p)$
(c) A contradiction
(d) Equivalent to $(p \wedge q) \vee (\sim q)$

31. The negation of the Boolean expression $x \leftrightarrow \sim y$ is equivalent to: **(2020)**

(a) $(x \wedge \sim y) \wedge (\sim x \wedge \sim y)$ (b) $(x \wedge y) \vee (\sim x \wedge \sim y)$
(c) $(x \wedge y) \wedge (\sim x \vee \sim y)$ (d) $(\sim x \wedge y) \vee (\sim x \wedge \sim y)$

32. Consider the statement: "For an integer n, if $n^3 - 1$ is even, then n is odd." The contrapositive statement of this statement is: **(2020)**

(a) For an integer n, if n is odd, then $n^3 - 1$ is even
(b) For an integer n, if n is even, then $n^3 - 1$ is even.
(c) For an integer n, if n is even, then $n^3 - 1$ is odd.
(d) For an integer n, if $n^3 - 1$ is not even, then n is not odd

33. The negation of the Boolean expression $p \vee (\sim p \wedge q)$ is equivalent to: **(2020)**

(a) $\sim p \vee q$ (b) $p \wedge \sim q$
(c) $\sim p \vee \sim q$ (d) $\sim p \wedge \sim q$

34. The statement among the following that is a tautology is **(2021)**

(a) $A \wedge (A \vee B)$ (b) $B \to [A \wedge (A \to B)]$
(c) $A \vee (A \wedge B)$ (d) $[A \wedge (A \to B)] \to B$

35. The negation of the statement $\sim p \wedge (p \vee q)$ is: **(2021)**

(a) $\sim p \wedge q$ (b) $p \wedge \sim q$
(c) $\sim p \vee q$ (d) $p \vee \sim q$

36. For the statement p and q, consider the following compound statements: **(2021)**

A. $(\sim q \wedge (p \to q)) \to \sim p$
B. $((p \vee q)) \wedge \sim p) \to q$

Then which of the following statements is correct?

(a) A is a tautology but not B
(b) A and B both are not tautologies
(c) A and B both are tautologies
(d) B is a tautology but not A

37. The statement $A \to (B \to A)$ is equivalent to: **(2021)**

(a) $A \to (A \wedge B)$ (b) $A \to (A \vee B)$
(c) $A \to (A \to B)$ (d) $A \to (A \leftrightarrow B)$

38. The contrapositive of the statement " If you will work, you will earn money" is: **(2021)**

(a) If you will not earn money, you will not work

(b) You will earn money, if you will not work

(c) If you will earn money, you will work

(d) To earn money, you need to work

39. Let $F_1(A, B, C) = (A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$ and $F_2(A, B) = (A \vee B) \vee (B \to \sim A)$ be two logical expressions Then: **(2021)**

(a) F_1 is not a tautology but F_2 is a tautology

(b) F_1 is a tautology but F_2 is not a tautology

(c) F_1 and F_2 both are a tautologies

(d) Both F_1 and F_2 are not tautologies

40. Which of the following Boolean expression is a tautology? **(2021)**

(a) $(p \wedge q) \vee (p \vee q)$ (b) $(p \wedge q) \vee (p \to q)$

(c) $(p \wedge q) \wedge (p \to q)$ (d) $(p \wedge q) \to (p \to q)$

41. If the Boolean expressions $(p \Rightarrow q) \equiv (q * (\sim p))$ is a tautology, then the Boolean expressions $p * (\sim q)$ is equivalent to **(2021)**

(a) $q \Rightarrow p$ (b) $\sim q \Rightarrow p$

(c) $p \Rightarrow \sim q$ (d) $p \Rightarrow q$

42. If the Boolean expressions $(p \wedge q) * (p \otimes q)$ is a tautology then $*$ and $\otimes$ are respectively given by **(2021)**

(a) $\to, \to$ (b) $\wedge, \vee$

(c) $\vee, \to$ (d) $\wedge, \to$

43. If P and Q are two statement, then which of the following compound statement is a tautology? **(2021)**

(a) $(P \Rightarrow Q) \wedge \sim Q) \Rightarrow Q$

(b) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow \sim P$

(c) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow P$

(d) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow (P \wedge Q)$

44. The number of choices of $\Delta \in \{\wedge, \vee, \Rightarrow, \Leftarrow\}$, such that $(p \Delta q) \Rightarrow ((p \Delta \sim q) \vee ((\sim p) \Delta q))$ is a tautology, is **(2022)**

(a) 1 (b) 2 (c) 3 (d) 4

45. Consider the following two propositions:

$P_1 : \sim (p \to \sim q)$

$P_2 : (p \wedge \sim q) \wedge ((\sim p) \vee q)$

If the proposition $p \to ((\sim p) \vee q)$ is evaluated as FALSE, then: **(2022)**

(a) P_1 is TRUE and P_2 is FALSE

(b) P_1 is FALSE and P_2 is TRUE

(c) Both P_1 and P_2 are FALSE

(d) Both P_1 and P_2 are TRUE

46. Let $\Delta, \nabla \in \{\wedge, \vee\}$ be such that $p \nabla q \Rightarrow ((p \Delta q) \nabla r)$ is a tautology. Then $(p \nabla q) \Delta r$ is logically equivalent to: **(2022)**

(a) $(p \Delta r) \vee q$ (b) $(p \Delta r) \wedge q$

(c) $(p \wedge r) \Delta q$ (d) $(p \nabla r) \wedge q$

47. Let $r \in \{p, q, \sim p, \sim q\}$ be such that the logical statement $r \vee (\sim p) \Rightarrow (p \wedge q) \vee r$ is a tautology. Then 'r' is equal to: **(2022)**

(a) p (b) q

(c) $\sim p$ (d) $\sim q$

48. Which of the following statement is a tautology? **(2022)**

(a) $((\sim q) \wedge p) \wedge q$

(b) $((\sim q) \wedge p) \wedge (p \wedge (\sim p))$

(c) $((\sim q) \wedge p) \vee (p \vee (\sim p))$

(d) $(p \wedge q) \wedge (\sim (p \wedge q))$

49. Suppose a class has 7 students. The average marks of these students in the mathematics examination is 62, and their variance is 20. A student fails in the examination if he/she gets less than 50 marks, then in worst case, the number of students can fail is **(2022)**

50. The maximum number of compound propositions, out of $p \vee r \vee s, p \vee r \vee \sim s, p \vee \sim q \vee s, \sim p \vee \sim r \vee s, \sim p \vee \sim r \vee \sim s, \sim p \vee q \vee \sim s, q \vee r \vee \sim s, q \vee \sim r \vee \sim s, \sim p \vee \sim q \vee \sim s$ that can be made simultaneously true by an assignment of the truth values to p, q, r and s, is equal to **(2022)**

51. Let $\Delta \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$ be such that $(p \wedge q) \Delta ((p \vee q) \Rightarrow q$ is a tautology. Then Δ is equal to **(2022)**

(a) $\wedge$ (b) $\vee$ (c) $\Rightarrow$ (d) $\Leftrightarrow$

52. Negation of the Boolean statement $(p \vee q) \Rightarrow ((\sim r) \vee p)$ is equivalent to: **(2022)**

(a) $p \wedge (\sim q) \wedge r$ (b) $(\sim p) \wedge (\sim q) \wedge r$

(c) $(\sim p) \wedge q \wedge r$ (d) $p \wedge q \wedge (\sim r)$

53. The statement $(\sim(p \Leftrightarrow \sim q)) \wedge q$ is: **(2022)**

(a) A tautology

(b) A contradiction

(c) Equivalent to $(p \Rightarrow q) \wedge q$

(d) Equivalent to $(p \Rightarrow q) \wedge p$

54. $(p \wedge r) \Leftrightarrow (p \wedge (\sim q))$ is equivalent to $(\sim p)$ when r is **(2022)**

(a) p (b) $\sim p$ (c) q (d) $\sim q$

55. Let the operations $*$, $\odot \in \{\wedge, \vee\}$. If $(p * q) \odot (p \odot \sim q)$ is a tautology, then the ordered pair $(*, \odot)$ is **(2022)**

(a) $(\wedge, \vee)$ (b) $(\vee, \vee)$

(c) $(\wedge, \wedge)$ (d) $(\wedge, \vee)$

56. p : Ramesh listens to music.

q : Ramesh is out of his village

r : It is Sunday

s : It is Saturday

Then the statement "Ramesh listens to music only if he is in his village and it is Sunday or Saturday" can be expressed as **(2022)**

(a) $((\sim q) \wedge (r \vee s)) \Rightarrow p$

(b) $(q \wedge (r \vee s)) \Rightarrow p$

(c) $p \wedge (q \wedge (r \vee s))$

(d) $p \Rightarrow ((\sim q) \wedge (r \vee s))$

ANSWER KEY

CONCEPT APPLICATION

1. (i) $\sqrt{12}$ is a rational number. -False

(ii) $\sqrt{12}$ is an irrational number. -True

(iii) $\sqrt{12}$ is a complex number. -True

2. (*b*) **3.** (*a*) **4.** (*d*) **5.** (*c*)

6. (i) If two chords of circle are not diameter then they do not bisect each other.

(ii) If corresponding angles of triangle are not equal then triangle are not similar.

7. (i) If radius of two circles are equal then circles are congruent.

(ii) If two polygons overlap each other, then they are congruent.

8. (*c*)

9. (i) False

(ii) True

10. (i) Inclusive

(ii) Exclusive

11. (*b*) **12.** (*d*) **13.** (*c*)

14. (i) For every

(ii) There exist

15. (*d*) **16.** (*c*) **17.** (*b*) **18.** (*c*)

EXERCISE-1 (TOPICWISE)

1. (*c*)	**2.** (*c*)	**3.** (*b*)	**4.** (*c*)	**5.** (*a*)	**6.** (*c*)	**7.** (*b*)	**8.** (*c*)	**9.** (*d*)	**10.** (*b*)
11. (*b*)	**12.** (*c*)	**13.** (*d*)	**14.** (*c*)	**15.** (*c*)	**16.** (*c*)	**17.** (*a*)	**18.** (*a*)	**19.** (*b*)	**20.** (*a*)

EXERCISE-2 (LEARNING PLUS)

1. (*a*)	**2.** (*b*)	**3.** (*d*)	**4.** (*b*)	**5.** (*c*)	**6.** (*c*)	**7.** (*c*)	**8.** (c)	**9.** (c)	**10.** (*a*)
11. (*b*)	**12.** (*c*)	**13.** (*a*)	**14.** (*a*)	**15.** (*c*)	**16.** (*d*)	**17.** (*b*)	**18.** (*c*)	**19.** (*d*)	**20.** (*c*)
21. (*b*)	**22.** (*b*)	**23.** (*c*)	**24.** (*c*)	**25.** (*a*)	**26.** (*c*)	**27.** (*d*)	**28.** (*c*)	**29.** (*c*)	**30.** [0]
31. [9]	**32.** (*c*)	**33.** (*d*)	**34.** (*d*)	**35.** (*d*)	**36.** (*d*)	**37.** (*b*)			

EXERCISE-3 (PAST YEAR QUESTIONS)

JEE Main

1. (*b*)	**2.** (*a*)	**3.** (*d*)	**4.** (*a*)	**5.** (*a*)	**6.** (*d*)	**7.** (*c*)	**8.** (*b*)	**9.** (*c*)	**10.** (*a*)
11. (*b*)	**12.** (*d*)	**13.** (*d*)	**14.** (*b*)	**15.** (*d*)	**16.** (*b*)	**17.** (*c*)	**18.** (*d*)	**19.** (*c*)	**20.** (*d*)
21. (*a*)	**22.** (*b*)	**23.** (*d*)	**24.** (*d*)	**25.** (*a*)	**26.** (*d*)	**27.** (*d*)	**28.** (*d*)	**29.** (*d*)	**30.** (*a*)
31. (*b*)	**32.** (*c*)	**33.** (*d*)	**34.** (*d*)	**35.** (*d*)	**36.** (*c*)	**37.** (*b*)	**38.** (*a*)	**39.** (*a*)	**40.** (*d*)
41. (*a*)	**42.** (*a*)	**43.** (*b*)	**44.** (*b*)	**45.** (*c*)	**46.** (*a*)	**47.** (*c*)	**48.** (*c*)	**49.** [0]	**50.** [9]
51. (*c*)	**52.** (*c*)	**53.** (*d*)	**54.** (*c*)	**55.** (*b*)	**56.** (*d*)				

CHAPTER

22 Statistics

DATA

Facts, observations and information that come from investigations. Generally three types of data are used.

(*i*) Ungrouped data, raw data or individual series.

(*ii*) Discrete frequency or ungrouped data.

Definition: Data consist of n distinct values $x_1, x_2, \dots x_n$ occuring with frequency $f_1, f_2, \dots, f_n$ respectively. This data in tabular form is called discrete frequency distribution.

(*iii*) Continuous frequency or grouped data:

Definition: A continuous frequency distribution is a series in which the data are classified into different class intervals without gaps along with their respective frequencies.

MEASURES OF CENTRAL TENDENCY

Averages are generally, the central part of the distribution and therefore, they are also called the measures of **Central Tendency.**

It can be divided into two groups:

(*a*) Mathematical Average: (Mean)

(*i*) Arithmetic mean or mean

(*ii*) Geometric mean

(*iii*) Harmonic mean

(*b*) Positional Average:

(*i*) Median

(*ii*) Mode or positional average

ARITHMETIC MEAN

1. **Individual observation or unclassified data:**

If $x_1, x_2, \dots x_n$ be n observations, then their arithmetic mean is given by

$$\bar{x} = \frac{x_1 + x_2 + \dots\dots + x_n}{n} \quad \text{or} \quad \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

2. **Arithmetic mean of discrete frequency distribution**

Let $x_1, x_2, \dots\dots, x_n$ be n observation and let $f_1, f_2, \dots\dots, f_n$ be their corresponding frequencies, then their mean is

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \dots\dots + f_nx_n}{f_1 + f_2 + \dots\dots + f_n} \quad \text{or} \quad \bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

Short cut method: If the values of x or (and) f are large the calculation of arithmetic mean by the previous formula used, is quite tedious and time consuming. In such case we take the deviation from an arbitrarypoint A.

$$\bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$$

where A = Assumed mean

$d_i = x_i - A$ = deviation for each term

Step deviation method: Sometimes during the application of shortcut method of finding the mean, the deviation d_i are divisible by a common number h (say). In such case the calculation is reduced to a great extent by taking

$$u_i = \frac{x_i - A}{h}, \ i = 1, 2, \dots n$$

$$\therefore \ \text{mean } \bar{x} = A + h\left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right)$$

Arithmetic mean of continuous frequency distribution

3. **Continuous frequency distribution**

For calculating mean of continuous frequence distribution the procedure is same as for discrete frequency distribution. Here we have to obtain the mid point of various classes and find their mean.

4. **Weighted arithmetic mean:** If $w_1, w_2, w_3, \dots\dots, w_n$ are the weight assigned to the values $x_1, x_2, x_3, \dots\dots, x_n$ respectively, then the weighted average is defined as -

$$\text{Weighted } A.M. = \frac{w_1x_1 + w_2x_2 + \dots\dots + w_nx_n}{w_1 + w_2 + \dots\dots + w_n}$$

$$\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

5. **Combined mean:** If $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k$ are the mean of k series of sizes $n_1, n_2, \dots, n_k$ respectively then the mean of the composite series is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \dots\dots n_k\bar{x}_k}{n_1 + n_2 + \dots\dots + n_k}$$

6. Properties of Arithmetic Mean:

(*i*) In a statistical data, the sum of the deviation of items from *A.M.* is always zero.

i.e. $\sum_{i=1}^{n}(x_i - \overline{x}) = 0$ or $\sum_{i=1}^{n} x_i - n\overline{x}$

or $n\overline{x} - n\overline{x} = 0$ $\left(\because \overline{x} = \frac{\sum x_i}{n}\right)$

(*ii*) In a statistical data, the sum of squares of the deviation of items from *A.M.* is least i.e. $\sum_{i=1}^{n}(x_i - \overline{x})^2$ is least.

(*iii*) If each of the n given observation be doubled, then their mean is doubled.

(*iv*) If $\overline{x}$ is the mean of $x_1, x_2, ..., x_n$. The mean of $ax_1, ax_2, ..., ax_n$ is $a\overline{x}$ where a is any number different from zero.

(*v*) Arithmetic mean is independent of origin i.e. it is not effected by any change in origin.

Train Your Brain

Example 1: In a class of 40 boys and 30 girls, the average age is 16 years. If the mean age of boys is 1 year more than the mean age of girls, then find the mean age of girls.

Sol. Let mean age of 30 girls is x years so that of 40 boys is $(x + 1)$ years.

$$\therefore \quad 16 = \frac{40(x+1)+30x}{40+30} = \frac{4x+4+3x}{7}$$

$$\therefore \quad 7x + 4 = 112$$

$$7x = 108$$

$$x = 15\frac{3}{7} \text{ years}$$

Example 2: If n persons donate each rupees 1, 2, 4, 8 ...2^{n-1} respectively then the mean donation per person is

Sol. $\overline{x} = \dfrac{1+2+2^2+2^3+......+2^{n-1}}{n}$ this is a *G.P.* with common ratio as 2,

$$= \frac{(2^n - 1)}{n.(2-1)} = \frac{2^n - 1}{n} \text{ rupees}$$

Example 3: Find the mean of first n natural numbers whose frequencies are equal to the corresponding numbers.

Sol. Here $x_1 = 1, x_2 = 2, x_3 = 3 \ldots x_n = n$

also $f_1 = 1, f_2 = 2, f_3 = 3 \ldots f_n = n$

so $\overline{X} = \dfrac{x_1 f_1 + x_2 f_2 + x_3 f_3 + ... + x_n f_n}{f_1 + f_2 + ... + f_n}$

$$= \frac{1.1 + 2.2 + 3.3 + ... + n.n}{1 + 2 + 3 ... + n}$$

$$= \frac{n(n+1)(2n+1) \cdot 2}{6 \cdot n(n+1)} = \frac{2n+1}{3}$$

Example 4: The mean heights of team *A* and team *B* are 6′2″ and 5′10″ respectively. One member from team *A* was transferred to *B* so that average height of team *B* rise by 0.5″. Then the shortest-member from team *B* was transferred to *A* so that now both teams have same average height. Find the heights of members transferred respectively. Each team consists of 6 members.

Sol. Total height of 6 members of team $A = 6'2'' \times 6 = 37'$

Total height of 6 members of team $B = 5'10'' \times 6 = 35'$

Let one member of height h' is transferred from team *A* to *B*

Now total height of 5 members of team *A* remaining $= 37' - h'$

Total height of 7 members of team *B* now $= 35' + h'$

Mean height of team $B = \dfrac{35' + h'}{7} = 5'10.5''$

$$\therefore \quad h' = 7(5'\,10.5'') - 35' = 73.5''$$

$$= 6' \text{ and } \frac{1}{8}'$$

Let one member transferred from *B* to *A* of height H'

So mean height of 6 members of team $B = \dfrac{35 + h - H}{6}$ feet

So mean height of 6 members of team $A = \dfrac{37 - h + H}{6}$ feet

$$\frac{35 + h - H}{6} = \frac{37 - h + H}{6}$$

$$2h - 2H = 2$$

$$H = h - 1 = 5' \text{ and } \frac{1}{8}'$$

So the members transferred were of height 6 and $\frac{1}{8}$ feet, 5 and $\frac{1}{8}$ feet respectively.

Example 5: Mean of 25 observations was found to be 78.4. But later on it was found that 96 was misread as 69. The correct mean is

(*a*) 79.48 (*b*) 76.54

(*c*) 81.32 (*d*) 78.4

Sol. We know that the mean is given by

$$\overline{x} = \frac{\Sigma x}{n} \text{ or } \Sigma x = n\overline{x}$$

Here $\overline{x} = 78.4, n = 25$

$$\therefore \Sigma x = 25 \times 78.4 = 1960$$

But this Σx is incorrect as 96 was misread as 69

correct $\Sigma x = 1960 - 69 + 96 = 1987$

$\therefore$ correct mean $= \frac{1987}{25} = 79.48$

Example 6: If the frequencies of first four numbers out of 1, 2, 4, 6, 8 are 2, 3, 3, 2 respectively, then the frequency of 8 if their AM is 5, is

(*a*) 4 (*b*) 5

(*c*) 6 (*d*) None of these

Sol. Here mean $A = 5$

Let the frequency of 8 be x. Then by the formula

$$A = \frac{\Sigma xf}{\Sigma f}$$

$$5 = \frac{1.2+2.3+4.3+6.2+8x}{2+3+3+2+x} = \frac{32+8x}{10+x}$$

or $18 = 3x$;

$\therefore x = 6$.

Example 7: In a family, there are 8 men, 7 women and 5 children whose mean ages separately are respectively 24, 20 and 6 years. The mean age of the family is

(*a*) 17.1 years (*b*) 18.1 years

(*c*) 19.1 years (*d*) None of these

Sol. Here we have three collections for which $A_1 = 24, n_1 = 8$, $A_2 = 20$, $n_2 = 7$ and $A_3 = 6$, $n_3 = 5$. Their combined mean is the required mean.

By the formula $A = \frac{n_1A_1 + n_2A_2 + n_3A_3}{n_1 + n_2 + n_3}$

$$\therefore A = \frac{8 \times 24 + 7 \times 20 + 5 \times 6}{8+7+5}$$

$$= \frac{192+140+30}{20} = \frac{362}{20} = 18.1$$

Concept Application

1. If the mean of a set of observations $x_1, x_2, ... x_{10}$ is 20 then the mean of $x_1 + 4, x_2 + 8, x_3 + 12, ... x_{10} + 40$ is

(*a*) 34 (*b*) 42 (*c*) 38 (*d*) 40

2. 1, 2, 4, 6, 8, 9 transformed into another series whose mean is 12. Two middle terms of the transformed series are

(*a*) 7, 8 (*b*) 10, 14 (*c*) 9, 15 (*d*) 11, 13

3. The mean of 20 observations is 15. On checking it was found that the two observations were wrongly copied as 3 and 6. The correct values are 8 and 4, then correct mean will be given by

(*a*) 15.15 (*b*) 14.69 (*c*) 14.74 (*d*) 15.25

GEOMETRIC MEAN

1. **Individual data :** If $x_1, x_2, x_3, ..., x_n$ are n values of a variate x, none of them being zero, then the geometric mean G is defined as-

$G = (x_1 x_2 x_3 \dots x_n)^{1/n}$

or $G = \text{antilog}\left(\frac{\log x_1 + \log x_2 + ... + \log x_n}{n}\right)$

or $G = \text{antilog}\left(\frac{1}{n}\sum_{i=1}^{n} \log x_i\right)$

2. **Geometric Mean of grouped data :**

Let x_1, x_2, ...x_n be n observation and let f_1, f_2 ,......f_n be their corresponding frequency then their Geometric Mean is

$$G = \left(x_1^{f_1} x_2^{f_2} .... x_n^{f_n}\right)^{1/N} \text{ where } N = \sum_{i=1}^{n} f_i$$

$$\therefore G = \text{antilog}\left(\frac{\sum_{i=1}^{n} f_i \log x_i}{\sum_{i=1}^{n} f_i}\right)$$

Train Your Brain

Example 8: The geometric mean of the numbers 7, 7^2, 7^3, ..., 7^n is

Sol. $GM = \sqrt[n]{7.7^2, 7^3 \dots 7^n}$

$= \left[7^{1+2+3+...+n}\right]^{1/n} = \left[7^{n(n+1)/2}\right]^{1/n} = 7^{n+1/2}$

Concept Application

4. Let G_1, G_2 be the geometric means of two series $x_1, x_2, ... x_n; y_1, y_2 ... y_n$. If G is the geometric means of x_i/y_i, i = 1, 2.. n, then G is equal to

(*a*) $G_1 - G_2$ (*b*) Log G_1/log G_2

(*c*) Log (G_1/G_2) (*d*) None of these

5. The geometric mean G of the product of n series of data with geometric means, $G_1, G_2, ..., G_n$ respectively, then

(*a*) $\sqrt{G} = G_1, G_2, ..., G_n$ (*b*) $G = G_1 \cdot G_2 \cdot ... G_n$

(*c*) $\sqrt{G} = -G_1, G_2, ..., G_n$ (*d*) None of the above

6. The geometric mean of the numbers 3, 3^2, 3^3,.....,3^n is

(*a*) $3^{2/n}$ (*b*) $3^{(n-1)/2}$

(*c*) $3^{n/2}$ (*d*) $3^{(n+1)/2}$

HARMONIC MEAN

Harmonic mean is reciprocal of mean of reciprocal.

1. **Individual observation:** The $H.M.$ of $x_1, x_2, \ldots, x_n$ of n observation is given by

$$H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots\ldots + \frac{1}{x_n}} \quad \text{i.e. } H = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

2. **H.M. of grouped data:** Let $x_1, x_2, \ldots, x_n$ be n observation and let $f_1, f_2, \ldots, f_n$ be their corresponding frequency then $H.M.$ is

$$H = \frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} \left(\frac{f_i}{x_i}\right)}$$

Relation between A.M. , G.M., and H.M.

$A.M. \geq G.M. \geq H.M.$

Equality sign holds only when all the observations in the series are same.

MEDIAN

Median is the middle most or the central value of the variate in a set of observations, when the observations are arranged either in ascending or in descending order of their magnitudes. It divides the arranged series in two equal parts.

1. **Median of an individual series :**

Let n be the number of observations-

(*i*) Arrange the data in ascending or descending order.

(*ii*) **(*a*) If n is odd then-**

Median (M) = Value of $\left(\frac{n+1}{2}\right)^{th}$ observation

(*b*) If n is even then

Median (M) = mean of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observation

$$\text{i.e. } M = \frac{\left(\frac{n}{2}\right)^{\text{th}} \text{observation} + \left(\frac{n}{2}+1\right)^{\text{th}} \text{observation}}{2}$$

2. **Median of the discrete frequency distribution:**

Arrange data in ascending order

Algorithm to find the median :

Step-I: Find the cumulative frequency ($C.F.$)

Step-II: Find $\frac{N}{2}$, where $N = \sum_{i=1}^{n} f_i$

Step-III: See the cumulative frequency ($C.F.$) just greater than $\frac{N}{2}$ and determine the corresponding value of the variable.

Step-IV: The value obtained in step III is the median.

3. **Median of grouped data or continuous series:**

Let the number of observations be n

(*i*) Prepare the cumulative frequency table

(*ii*) Find the median class *i.e.* the class in which the $\left(\frac{N}{2}\right)^{th}$ observation lies

(*iii*) The median value is given by the formulae

$$\text{Median } (M) = \ell + \left[\frac{\left(\frac{N}{2}\right) - F}{f}\right] \times h$$

N = total frequency = Σf_i

ℓ = lower limit of median class

f = frequency of the median class

F = cumulative frequency of the class preceding the median class

h = class interval (width) of the median class

4. **Properties of Median:**

(*i*) The sum of the absolute value of deviations of the item from median is minimum

(*ii*) It is a positional average and it is not influenced by the position of the items.

Train Your Brain

Example 9: The median of the items 6, 10, 4, 3, 9, 11, 22, 18, is

(*a*) 9 (*b*) 10 (*c*) 9.5 (*d*) 11

Sol. Lets arrange the items in ascending order 3, 4, 6, 9, 10, 11, 18, 22.

In this data the number of items is $n = 8$, which is even.

$\therefore$ Median $= M =$ average of $\left(\frac{n}{2}\right)$ th and $\left(\frac{n}{2}+1\right)$ th terms.

= Average of $\left(\frac{8}{2}\right)$ th and $\left(\frac{8}{2}+1\right)$ th term

= Average of 4$^{\text{th}}$ and 5$^{\text{th}}$ terms

$= \frac{9+10}{2} = \frac{19}{2} = 9.5$

Example 10: The marks obtained by 10 students in a particular subject are 96, 88, 99, 87, 92, 86, 93, 94, 96, 97. Find the median mark.

Sol. We first arrange the data in ascending order as follows 86, 87, 88, 92, 93, 94, 96, 96, 97, 99

So there are altogether 10 observations. Hence the number of observations is even.

Then the median is the average of $\frac{10}{2}$ th and $\left(\frac{10}{2}+1\right)^{\text{th}}$ i.e. 5th and 6th observations.

That is Average of 93 and 94 i.e. $\frac{93+94}{2} = 93.5$

Example 11: In a class of 20 students, the marks obtained by them in a test are grouped as follows

Marks	No. of Students
0	1
1	3
4	2
5	6
7	2
9	3
11	1
15	1
16	1
	20

Find the median marks.

Here we have $n = 60$, $\frac{n}{2} = 30$

We see that 30 lies in the class 50 – 60 with cumulative frequency 31.

Thus the median class 50 – 60

$\therefore$ l = lower limit of the median class i.e. the class in which $\frac{n}{2}$ th or $\frac{n+1}{2}$ th observation lies = 50

f = frequency of the median class = 05

F_1 = 26 (cumulative frequency of the class preceding the median class)

h = class size = 60 – 50 = 10

$$\text{Median} = l + \frac{\left(\frac{n}{2} - F\right)}{f} \times h$$

$$= 50 + \frac{30 - 26}{5} \times 10 = 50 + 8 = 58$$

Sol.

x	f	CF
0	1	1
1	3	4
4	2	6
5	6	12
7	2	14
9	3	17
11	1	18
15	1	19
16	1	20

Here $N = 20$,

so $\frac{N}{2} = 10$

variable corresponding to CF just greater then $\frac{N}{2}$ is 5.

Hence median is 5.

Example 12: The marks obtained by 60 students in a certain examination are given below

Marks	No. of Students
10 – 20	04
20 – 30	05
30 – 40	11
40 – 50	06
50 – 60	05
60 – 70	08
70 – 80	09
80 – 90	07
90 – 100	05

Calculate the median mark.

Sol. We calculate the cumulative frequency from the following table:

Marks Obtained	Frequency (No. of students) (f)	(Cumulative Frequency) (F)
10 – 20	04	04
20 – 30	05	09
30 – 40	11	20
40 – 50	06	26
50 – 60	05	31
60 – 70	08	39
70 – 80	09	48
80 – 90	07	55
90 – 100	05	60
	n = 60	

Concept Application

7. If the median of 21 observations is 40 and if the observations greater than the median are increased by 6 then the median of the new data will be

(*a*) 40 (*b*) 46

(*c*) 46 + 40/21 (*d*) 46 – 40/21

8. The value of median of median for the data

Income (in ₹)	1000	1100	1200	1300	1400	1500
Number of persons	1	1	1	1	1	1

is

(*a*) 1300 (*b*) 1200 (*c*) 1250 (*d*) 1150

9. From the data given, the median of the average deposit balance of saving for the branch during March 1982 is

Average deposit balance (in ₹)	Number of deposit
Less than 100	26
100-200	68
200-300	145
300-400	242
400-500	188
500-600	65
600-700	16

(*a*) 356 (*b*) 300
(*c*) 56.2 (*d*) 356.2

MODE

Mode is that value in a series which occurs most frequently. In a frequency distribution, mode is that variate which has the maximum frequency.

1. **Computation of Mode:**
   Mode for individual series: In the case of individual series, the value which is repeated maximum number of times is the mode of the series.
2. **Mode for grouped data (discrete frequency distribution series):** In the case of discrete frequency distribution, mode is the value of the variate corresponding to the maximum frequency.
3. **Mode for continuous frequency distribution:**
   (*i*) First find the model class i.e. the class which has maximum frequency. The model class can be determined either by inspecting or with the help of grouping data.
   (*ii*) The mode is given by the formula

$$\text{Mode} = \ell + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times h$$

where
$\ell \rightarrow$ lower limit of the model class
$h \rightarrow$ width of the model class
$f_m \rightarrow$ frequency of the model class
$f_{m-1} \rightarrow$ frequency of the class preceding model class
$f_{m+1} \rightarrow$ frequency of the class succeeding model class
4. **Properties of Mode:** It is not effected by presence of extremely large or small items.

Relation between Mean, Median and Mode

(*a*) **Symmetric Distribution** A distribution is a symmetric distribution if values of mean, mode and median coincide.
Mean = Model = Median
(*b*) **Skewed Distribution** A distribution which is not symmetric is called a skewed distribution. In a moderately asymmetric the interval between the mean and median is approximately one-third of the interval between the mean and the mode. We have the following **empirical relation** between them.
Mean - Mode = 3(Mean - Median)
$\Rightarrow$ Mode = 3Median - 2Mean (Approximately)

Train Your Brain

Example 13: Calculate the mode from the following data:

Marks	No. of Students
0 - 10	04
10 - 20	12
20 - 30	24
30 - 40	36
40 - 50	20
50 - 60	16
60 - 70	08
70 - 80	05

Sol.

Value (Class)	Frequency
0 - 10	04
10 - 20	12
20 - 30	24
30 - 40	36
40 - 50	20
50 - 60	16
60 - 70	08
70 - 80	05
	$n = 125$

We choose the highest frequency class as the modal class. So here 30 - 40 is the modal class.

Now
l = lower limit of modal class = 30
h = width of the modal class = 10
f_1 = frequency of the class preceding the modal class = 24
f_2 = frequency of the class following the modal class = 20
f = frequency of the modal class = 36

$$\text{So mode} = l + \frac{(f - f_1)}{(2f - f_1 - f_2)} \times h$$
$$= 30 + \frac{36-24}{72-24-20} \times 10$$
$$= 30 + \frac{12}{28} \times 10$$
$$= 30 + \frac{30}{7} = 34.28$$

Example 14: If the value of the mode and mean is 60 and 66 respectively then the value of median is
(*a*) 60 (*b*) 64
(*c*) 68 (*d*) None of these

Sol. $\text{Median} = \frac{1}{3}[\text{Mode} + 2\text{ Mean}]$
$$= \frac{1}{3}[60 + 2 \times 66] = 64$$

Concept Application

10. The mode of the following distribution is

Class interval	Frequency
0-10	5
10-20	8
20-30	7
30-40	12
40-50	28
50-60	20
60-70	10
70-80	10

(*a*) 46 (*b*) 6.66
(*c*) 46.67 (*d*) None of these

11. Mode of data 3, 2, 5, 2, 3, 5, 6, 6, 5, 3, 5, 2, 5 is
(*a*) 6 (*b*) 4 (*c*) 5 (*d*) 3

12. For the data 2, 2, 3, 3, 3, 4, 4, 5, 6, 8, 8, 8, 8, the mean, median and mode are
(*a*) 64/13, 5, 3 (*b*) 64/13, 14, 3
(*c*) 5, 4, 3 (*d*) 64/13, 4, 8

MEASURES OF DISPERSION

Dispersion is the measure of the variations. The degree to which numerical data tend to spread about an average value is called the dispersion of the data.

The measures of dispersion are commonly used by:
(*i*) Range (*ii*) Mean Deviation
(*iii*) Standard Deviation (*iv*) Variance

(*i*) **Range:** The range is the difference between the greatest and the least observations of the distribution. Thus if A and B are the greatest and the smallest observations respectively in a distribution, then its range = $A - B$.

The co-efficient of range (or scatter) = $\frac{A-B}{A+B}$

(*ii*) **Mean Deviation:** Mean deviation is defined as the arithmetic mean of the absolute deviations of all the values taken about any central value.

(*a*) **Mean deviation of individual observations:**
If $x_1, x_2, ..., x_n$ are n values of a variable x, then the mean deviation from a number A is given by

$$M.D. = \frac{1}{n}\sum_{i=1}^{n}|x_i - A|$$

$$= \frac{1}{n}\sum|d_i|, \text{ where } d_i = x_i - A$$

(*b*) **Mean deviation of a discrete frequency distribution:**
If $x_1, x_2, ..., x_n$ are n observation with frequencies $f_1, f_2, ..., f_n$, then mean deviation from a number A is given by -

$$\text{Mean Deviation} = \frac{1}{N}\sum f_i|x_i - A|$$

where $N = \sum_{i=1}^{n} f_i$

(*c*) **Mean deviation of a grouped or continuous frequency distribution:**
For calculating mean deviation of a continuous frequency distribution the procedure is same as for a discrete frequency distribution. The only difference is that here we have to obtain the mid-point of the various classes and take the deviations of these mid points from the given central value (median or mean)

Train Your Brain

Example 15: For an *A.P.* 2, 5, 8, ..., (21 terms) are given. Find the mean deviation of *A.P.* from its mean.

Sol. $\Sigma x_i = 2 + 5 + 8 \text{ .... } + (21 \text{ terms})$

$$= \frac{21}{2}[4 + 20 \times 3] = 21 \times 32 = 672$$

$$\therefore \bar{x} = \frac{672}{21} = 32$$

$$\Sigma|x_i - \bar{x}| = 2 \times 3[1 + 2 + 3 + ... + 10]$$

$$= 2 \times 3\left[\frac{10 \times 11}{2}\right] = 330$$

$$\therefore \text{M.D.} = \frac{330}{21} = \frac{110}{7}$$

Concept Application

13. If the mean deviation of the numbers 1, 1 + d,... 1+ 100d from their mean is 255, then d is equal to
(*a*) 10.1 (*b*) 20.2
(*c*) 10.0 (*d*) 20.0

14. If the mean deviation about the median of the numbers $a, 2a, ..., 50a$ is 50 then $|a|$ equal
(*a*) 5 (*b*) 2
(*c*) 3 (*d*) 4

15. The mean deviation from mean of the observations $a, a + d, a + 2d, ..., a + 2nd$ is
(*a*) $\frac{n(n+1)d^2}{3}$ (*b*) $\frac{n(n+1)}{2}d^2$
(*c*) $a + \frac{n(n+1)d^2}{2}$ (*d*) None of these

VARIANCE AND STANDARD DEVIATION

The variance of a variate x is the arithmetic mean of the squares of all deviations of x from the arithmetic mean of the observations and is denoted by var (x) or σ^2

The positive square root of the variance of a variate x is known as standard deviation i.e. standard deviation $= +\sqrt{\text{var}(x)}$

(i) Variance of Individual observations :

If $x_1, x_2, \ldots, x_n$ are n values of a variable x, then by definition

$$\text{var}(x) = \frac{1}{n}\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right] = \sigma^2 \quad \ldots (i)$$

$$\text{or var}(x) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 \quad \ldots (ii)$$

If the values of variable x are large, the calculation of variance from the above formulae is quite tedious and time consuming. In that case, we take deviation from an arbitrary point A (say) then

$$\text{var}(x) = \frac{1}{n}\sum_{i=1}^{n} d_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} d_i\right)^2 \quad \ldots (iii)$$

(ii) Variance of a discrete frequency distribution:

If $x_1, x_2, \ldots\ldots, x_n$ are n observations with frequencies $f_1, f_2, \ldots, f_n$ then $\text{var}(x) = \frac{1}{N}\left\{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2\right\}$ ... (i)

$$\text{or var}(x) = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \bar{x}^2 \quad \ldots (ii)$$

If the values of x or f are large, we take the deviations of the values of variable x from an arbitrary point A. (say)

$\therefore d_i = x_i - A;\ i = 1, 2, \ldots. n$

$$\therefore \text{Var}(x) = \frac{1}{N}\left(\sum_{i=1}^{n} f_i d_i^2\right) - \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)^2 \quad \ldots (iii)$$

where $N = \sum_{i=1}^{n} f_i$

Sometime $d_i = x_i - A$ are divisible by a common number h (say)

then $u_i = \frac{x_i - A}{h} = \frac{d_i}{h},\ i = 1, 2, \ldots, n$

$$\text{then var}(x) = h^2\left[\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2\right] \quad \ldots (iv)$$

(iii) Variance of a grouped or continuous frequency distribution: In a grouped or continuous frequency distribution any of the formulae discussed in discrete frequency distribution can be used.

PROPERTIES OF VARIANCE

1. If $x_1, x_2, x_3, \ldots, x_n$ be n values of a variable X. If these values are changed to $x_1 + a$, $x_2 + a$, ... $x_n + a$, where $a \in R$, then the variance remains unchanged.
2. If $x_1, x_2, \ldots, x_n$ values of a variable X and let 'a' be a non-ze real number. Then, the variance of the observation ax_1, ax_2 , ax_n is a^2 Var(X)
3. S.D is $|a|$ S.D(x)

COMBINED STANDARD DEVIATION

If there are two sets of observations containing n_1 and n_2 items w respective mean $\bar{x}_1$ & $\bar{x}_2$ and standard deviation s_1 and s_2, tl the mean $\bar{x}$ and the standard deviations of $n_1 + n_2$ observatio taken together, are

$$\bar{x} = \frac{n_1\bar{x}_1 + n_1\bar{x}_2}{n_1 + n_2}$$

$$\sigma^2 = \frac{1}{n_1 + n_2}[n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)]$$

where $d_1 = \bar{x} - \bar{x}_1,\ d_2 = \bar{x} - \bar{x}_2$

or

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2} \times (\bar{x}_1 - \bar{x}_2)^2$$

Train Your Brain

Example 16: Find the variance of first n natural numbers.

Sol. We have

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

$$= \frac{1}{n}(1^2 + 2^2 + \ldots + n^2) - \left(\frac{1}{n}\cdot(1 + 2 + \ldots + n)\right)^2$$

$$= \frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n^2 - 1}{12}$$

Example 17: A student obtained the mean and standard deviation of 100 observations as 40 and 5.1 respectively. It was later found that he had wrongly copied down an observation as 50 instead of 40. The correct mean and standard deviation are

(a) 39.9, 6 (b) 36.4, 5
(c) 39.9, 5 (d) None of these

Sol. Let the variable be x. Here $n = 100$, incorrect $A = 40$, incorrect $s = 5.1$.

From formula $A = \frac{\Sigma x}{n}$, $\Sigma x = nA$.

$\therefore$ incorrect $\Sigma x = 100 \times 40 = 4000$

$\therefore$ correct Σx = incorrect $\Sigma x - 50 + 40$

$= 4000 - 50 + 40 = 3990$

Again, from $\sigma^2 + A^2 = \frac{\Sigma x^2}{n}$,

we get $\Sigma x^2 = n(s^2 + A^2)$

$\therefore$ incorrect $\Sigma x^2 = 100\{(5.1)^2 + (40)^2\}$

$= 100(26.01 + 1600) = 162601$

$\therefore$ correct Σx^2 = incorrect $\Sigma x^2 - 50^2 + 40^2$

$= 162601 - 2500 + 1600 = 161701$

$\therefore$ correct mean $A = \frac{\Sigma x}{n} = \frac{3900}{100} = 39.9$

Correct $(\sigma^2 + A^2) = \frac{\Sigma x^2}{n} = \frac{161701}{100} = 1617.01$

$\therefore$ correct $\sigma^2 = 1617.01 -$ (correct A^2)

$= 1617.01 - (39.9)^2 = 1617.01 - 1592.01 = 25$

$\therefore \sigma = 5$

Concept Application

16. If $\sum_{i=1}^{18}(x_i - 8) = 9$ and $\sum_{i=1}^{18}(x_i - 8)^2 = 45$ then the standard deviation of $x_1, x_2 \ldots x_{18}$ is

(a) 9/4 (b) 9/8

(c) 1/6 (d) 3/2

17. Assuming the variance of four numbers a, b, c and d as 9, the variance of 5a, 5b, 5c and 5d is

(a) 45 (b) 5/9

(c) 9/5 (d) 225

18. A scientist is weighing each of 30 fishes. Their mean weight worked out is 30 gm and a standard deviation weight of 2 gm. Later, it was found that the measuring scale was misaligned and always under reported every fish weight by 2 gm. The correct mean and standard deviation (in gm) of fishes are respectively:

(a) 32, 2 (b) 32, 4

(c) 28, 2 (d) 28, 4

ANALYSIS OF FREQUENCY DISTRIBUTIONS

Measures of dispersion are unable to compare two or more series which are measured in different units even if they have the same mean. Thus, we require those measures which are independent of the units. The measure of variability which is independent of units is called Coefficient of Variation (C.V.). The coefficient of variation is defined as

$$\text{C.V.} = \frac{\sigma}{\overline{X}} \times 100$$

where σ and $\overline{X}$ are the standard deviation and mean of the data.

For comparing the variability of two series, we calculate the coefficient of variation for each series. the series having greater C.V. is said to be more variable or conversely less consistent, less uniform less stable or less homogeneous than the other and the series having lesser C.V. is said to be more consistent (or homogeneous) than the other.

MEAN AND VARIANCE OF BINOMIAL DISTRIBUTION

If the frequencies of the values 0, 1, 2,...., n of a variate are represented by the following coefficients of a binomial :

$$q^n, {}^nC_1\, q^{n-1}p, {}^nC_2 q^{n-2} p^2, \ldots, p^n$$

where p is the probability of the success of the experiment (variate), q is the probability of its failure and $p + q = 1$ i.e. distribution is a binomial distribution then:

$$P(x = r) = {}^nC_r\, q^{n-r} p^r$$

$$\text{mean } \overline{x} = \sum p_i x_i = np$$

$$\text{Variance } \sigma^2 = npq = \overline{x}\, q$$

Train Your Brain

Example 18. Find coefficient of variation for first n even natural numbers.

Sol. $\because$ 2, 4, 6, ..., $2n$ are first n even natural no.

$$\sigma^2 = \frac{1}{n}\left(\sum_{k=1}^{n}(2k)^2\right) - \left(\frac{1}{n}\sum_{k=1}^{n}(2k)\right)^2$$

$$= \frac{1}{n}\cdot 4\cdot\left(\sum_{k=1}^{n}k^2\right) - 4\left(\frac{1}{n}\sum_{k=1}^{n}k\right)^2$$

$$= 4\left(\frac{1}{n}\sum_{k=1}^{n}n^2 - \left(\frac{1}{n}\sum_{k=1}^{n}k\right)^2\right)$$

$$= 4 \times \frac{n^2 - 1}{12} = \frac{n^2 - 1}{3}$$

$$\therefore \sigma = \sqrt{\frac{n^2 - 1}{3}}$$

$$\overline{x} = \frac{2 + 4 + \ldots.. + 2n}{n} = \frac{2}{n} \times \frac{n(n+1)}{2} = (n+1)$$

Hence

$$C \cdot V \cdot = \frac{\sigma}{x} \times 100 = \frac{\sqrt{\frac{n^2-1}{3}}}{n+1} \times 100$$

Example 19. Find C.V. of a binomial distribution with n = 18 an p = 0.4 ?

Sol. Given n = 18

P = 0.4

$\because p + q = 1 \Rightarrow q = 1 - p$

$\therefore q = 1 - 0.4 = 0.6$

Thus, mean $(\bar{x}) = np = 18 \times 0.4$

$= 7.2$

Variance $(\sigma^2) = npq = 18 \times 0.4 \times 0.6$

$= 4.32$

$S.D. (\sigma) = 2.08$

Hence, $C.V. = \frac{\sigma}{\bar{x}} \times 100$

$= \frac{2.08}{7.2} \times 100$

$= 28.87$

Concept Application

19. Find C.V. of a binomial distribution in which a fai coin is tossed16 times for getting head

(a) 4

(b) 100

(c) 25

(d) 200

20. In the following table mean and variance of earning of male and female is given

	Mean	Variance
Male	162.6 ₹	127.69 ₹
Female	52.36 ₹	23.1361 ₹

Then

(a) Female are more homogeneous than male

(b) Male are more homogeneous than male

(c) S.D. of male is less than the S.D. of female.

(d) None of these

Short Notes

ARITHMETIC MEAN

(i) **Arithmetic Mean for Unclassified (Ungrouped or Raw) Data:** If there are n observations, $x_1, x_2, x_3, \ldots, x_n$, then their arithmetic mean

$$A \text{ or } \bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

(ii) **Arithmetic Mean for Discrete Frequency Distribution or Ungrouped Frequency Distribution:** Let $f_1, f_2, \ldots, f_n$ be corresponding frequencies of $x_1, x_2, \ldots, x_n$. Then, arithmetic mean

$$A = \frac{x_1 f_1 + x_2 f_2 + \ldots + x_n f_n}{f_1 + f_2 + \ldots + f_n} = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i}$$

(iii) **Arithmetic Mean for Classified (Grouped) Data or Grouped Frequency Distribution:** For a classified data, we take the class marks $x_1, x_2, \ldots, x_n$, of the classes, then arithmetic mean by

$$A = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i}$$

Combined Mean: If $A_1, A_2, \ldots, A_r$ are means of $n_1, n_2, \ldots, n_r$ observations respectively, then arithmetic mean of the combined group is called the combined mean of the observation.

$$A = \frac{n_1 A_1 + n_2 A_2 + \ldots + n_r A_r}{n_1 + n_2 + \ldots + n_r} = \frac{\sum_{i=1}^{r} n_i A_i}{\sum_{i=1}^{r} n_i}$$

MEDIAN

Median for Simple Distribution or Raw Data

Firstly, arrange the data in ascending or descending order and then find the number of observations n.

(a) If n is odd, then $\left(\frac{n+1}{2}\right)$ th term is the median.

(b) If n is even, then there are two middle terms namely $\left(\frac{n}{2}\right)$ th and $\left(\frac{n}{2}+1\right)$ th terms, median is mean of these terms.

Median for Classified (Grouped) Data or Grouped Frequency Distribution

For a continuous distribution, median

$$M_d = l + \frac{\frac{N}{2} - C}{f} \times h$$

where, l = lower limit of the median class

f = frequency of the median class

N = total frequency = $\sum_{i=1}^{n} f_i$

C = cumulative frequency of the class just before the median class

h = length of the median class

Mode for Classified (Grouped) Distribution or Grouped Frequency Distribution

$$M_O = l + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h$$

where, l = lower limit of the modal class

f_0 = frequency of the modal class

f = frequency of the pre-modal class

f = frequency of the post-modal class

h = length of the class interval

Relation Between Mean, Median and Mode

(i) Mean – Mode = 3 (Mean – Median)

(ii) Mode = 3 Median – 2 Mean

MEAN DEVIATION (MD)

(i) For simple (raw) distribution, $\delta = \frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$

where, n = number of terms, $\bar{x} = A$ or M_d or M_O

(ii) For unclassified frequency distribution, $\delta = \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i}$

(iii) For classified distribution, $\delta = \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i}$

where, x_i is the class mark of the interval.

STANDARD DEVIATION AND VARIANCE

(i) For simple distribution

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}} = \frac{1}{n}\sqrt{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

where, n is a number of observations and $\bar{x}$ is mean.

(ii) For discrete frequency distribution

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} f(x_i - \bar{x})^2}{N}} = \frac{1}{N}\sqrt{N\sum_{i=1}^{n} f_i\, x_i^2 - \left(\sum_{i=1}^{n} f_i x_i\right)^2}$$

(iii) For continuous frequency distribution

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{N}}$$

where, x_i is class mark of the interval.

Standard Deviation of the Combined Series

If n_1, n_2 are the sizes, $\bar{X}_1, \bar{X}_2$ are the means and σ_1, σ_2, are the standard deviation of the series, then the standard deviation of the combined series is

$$\sigma = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2\left(\sigma_2^2 + d_2^2\right)}{n_1 + n_2}}$$

where, $d_1 = \bar{X}_1 - \bar{X}, d_2 = \bar{X}_2 - \bar{X}$

and $\bar{X} = \frac{n_1\bar{X}_1 + n_2\bar{X}_2}{n_1 + n_2}$

IMPORTANT POINTS TO BE REMEMBERED

(i) The ratio of SD (σ) and the AM ($\bar{x}$) is called the coefficient of standard deviation $\left(\frac{\sigma}{\bar{x}}\right)$

(ii) The percentage form of coefficient of SD i.e. $\left(\frac{\sigma}{\bar{x}}\right) \times 100$ is called coefficient of variation.

(iii) The distribution for which the coefficient of variation is less is more consistent.

(iv) Standard deviation of first n natural numbers is $\sqrt{\frac{n^2 - 1}{12}}$.

(v) Standard deviation is independent of change of origin, but it depends on change of scale.

Solved Examples

1. Let $x_1, x_2, \ldots\ldots, x_n$ be n observations, and let $\overline{x}$ be their arithmetic mean and σ^2 be the variance

Statement-1: Variance of $2x_1, 2x_2, \ldots, 2x_n$ is $4\sigma^2$.

Statement-2: Arithmetic mean $2x_1, 2x_2, .., 2x_n$ is $4\,\overline{x}$.

(1) Statement-1 is false, Statement-2 is true.

(2) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for statement-1.

(3) Statement-1 is true, statement-2 is true; statement-2 is true; statement-2 is **not** a correct explanation for statement-1.

(4) Statement-1 is true, statement-2 is false.

Sol. $\because\ \overline{x} = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$

$$\overline{x}' = \frac{2x_1 + 2x_2 + \ldots + 2x_n}{n} = \frac{2(x_1 + x_2 + \ldots + x_n)}{n}$$

$\therefore$

$\overline{x}' = 2\overline{x}$

Since each variate is multiply by 2

$\therefore$ new variance $= 2^2 \times$ old variance

$= 4\sigma^2$

2. Find the variance and standard deviation of the following frequency distribution :

Variable (x_i)	2	4	6	8	10	12	14	16
Frequency (f_i)	4	4	5	15	8	5	4	5

If each variate is multiplied by 2 then new variance and standard deviation.

Sol. Calculation of variance and standard deviation

Variable x_i	Frequency f_i	f_ix_i	$x_i = \overline{x} = x_i - 9$	$(x_i - \overline{x})^2$	$f_i(x_i - \overline{x})^2$
2	4	8	−7	49	196
4	4	16	−5	25	100
6	5	30	−3	9	45
8	15	120	−1	1	15
10	8	80	1	1	8
12	5	60	3	9	45
14	4	56	5	25	100
16	5	80	7	49	245
	$N = \sum f_i = 50$	$\sum f_ix_i = 450$			$\sum f_i(x_i - \overline{x})^2 = 754$

Here $N = 50,\ \sum f_ix_i = 450$

$$\therefore\ \overline{x} = \frac{1}{N}(\Sigma f_ix_i) = \frac{450}{50} = 9$$

We have $\Sigma f_i(x_i - \overline{x})^2 = 754$

$$\therefore\ \text{var}(X) = \frac{1}{N}[\Sigma f_i(x_i - \overline{x})^2] = \frac{754}{50} = 15.08$$

$$\text{S.D.} = \sqrt{Var(x)} = \sqrt{15.08} = 3.88$$

$\because\ \sigma^2_{\text{new}} = a^2\,\sigma_{\text{old}},\ \sigma_{\text{new}} = |a|\,\sigma_{\text{old}}$

$\therefore\ \sigma^2_{\text{new}} = 60.32,\ \sigma_{\text{new}} = 7.76$

3. For two data sets, each of size 5, the variance are give be 4 and 5 and the corresponding means are given to b and 4, respectively. The variance of the combined data is

(*a*) $\dfrac{11}{2}$ (*b*) 6

(*c*) $\dfrac{13}{2}$ (*d*) $\dfrac{5}{2}$

Sol. (*a*) Given $n_1 = 5,\ n_2 = 5$

$\sigma_1^2 = 4,\ \sigma_2^2 = 5$

$\overline{x}_1 = 2,\ \overline{x}_2 = 4$

Thus

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1n_2}{(n_1 + n_2)^2}(\overline{x}_1 - \overline{x}_2)^2$$

$$= \frac{5\cdot4 + 5\cdot5}{5+5} + \frac{5\cdot5}{(5+5)^2}(2-4)^2$$

$$= \frac{45}{10} + \frac{1}{4} \times 4 = \frac{11}{2}$$

4. If $\overline{x}_1 = 50\ \sigma_1^2 = 25,\ \overline{x}_2 = 100\ \sigma_2^2 = 50$ are mean & varia of two systems. Then which system is more homogene

Sol. $\text{C.V.}_{\text{I}} = \dfrac{\sigma_1}{x_1} \times 100 = \dfrac{\sqrt{25}}{50} \times 100 = 5x_2 = 10$

$\text{C.V.}_{\text{II}} = \dfrac{\sigma_2}{x_2} \times 100 = \dfrac{\sqrt{50}}{100} \times 100 = 9\sqrt{2} = 7.07$

$\because\ \text{C.V.}_{\text{I}} > \text{C.V.}_{\text{II}}$

$\therefore$ System-II is more homogeneous

5. Calculate the mean deviation about the median for the lowing distribution.

C.I.	10-20	20-30	30-40	40-50	50-60	60-70	70
Frequency	2	3	8	14	8	3	

Sol. Let us calculate the median and mean deviation for the given data.

C.I.	Fi	CF	Mid paint (x)	(xi–x)	$Fi\|xi–m\|$
10–20	2	2	15	30	60
20–30	3	5	25	20	60
30–40	8	13	35	10	80
40–50	14	27	45	0	0
50–60	8	35	55	10	80
60–70	3	38	65	20	60
70–80	2	40	75	30	60
Total	$\Sigma fi = 40$				$\Sigma fi \| xi - m \| = 400$

$N = \Sigma f_i = 40$

$N/2 = 40/2 = 20$

The cumulative frequency greater than or equal to $N/2$ is 27 for which the corresponding class interval is 40 – 50.

Median class = 40 – 50

Thus,

I = 40

N = 40

C = 13

f = 14

h = 10

Median = I + [(N/2 – C)/f] × h

= 40 + [(20 – 13)/14] × 10

= 40 + (7/14) × 10

= 40 + 5

= 45

Therefore, median (M) = 45

From the above table,

$\Sigma f_i|x_i - M| = 400$

$N = \Sigma f_i = 40$

Hence, the required mean deviation = 400/40 = 10

Exercise-1 (Topicwise)

ARITHMETIC MEAN

1. If the mean of 3, 4, x, 7, 10 is 6, then the value of x is:
   (a) 4 (b) 5 (c) 6 (d) 7
2. The $A.M.$ of the series 1, 2, 4, 8, 16, ..., 2^n is:
   (a) $\frac{2^n-1}{n}$ (b) $\frac{2^{n+1}-1}{n+1}$
   (c) $\frac{2^n+1}{n}$ (d) $\frac{2^n-1}{n+1}$
3. Mean of the first n terms of the $A.P.$ $a + (a + d) + (a + 2d) + ...$ is
   (a) $a+\frac{nd}{2}$ (b) $a+\frac{(n-1)d}{2}$
   (c) $a+(n-1)d$ (d) $a+nd$
4. If the mean of n observations $1^2, 2^2, 3^2, ..., n^2$ is $\frac{46n}{11}$, then n is equal to
   (a) 11 (b) 12 (c) 23 (d) 22
5. Following table shows the weight of 12 students:

Weight (in kgs.)	67	70	72	73	75
No. of students	4	3	2	2	1

   Find their mean weight
   (a) 70.25 kg. (b) 70.50 kg
   (c) 70.75 kg. (d) None of these
6. A factory employs 100 workers of whom 60 work in the first shift and 40 work in the second shift. The average wage of all the 100 workers is Rs.38. If the average wage of 60 workers of the first shift is Rs.40, then the average wage of the remaining 40 workers of the second shift is:
   (a) 35 (b) 40
   (c) 45 (d) None of these
7. If $\overline{x}$ is the mean of a set of n observations $x_1, x_2, x_3, ..., x_n$ then $\sum_{i=1}^{n}(x_i-\overline{x})$ is equal to
   (a) Mean deviation about mean
   (b) Standard deviation
   (c) Zero
   (d) None of these
8. The mean of a set of observations is $\overline{x}$. If each observation is divided by α, $\alpha \neq 0$, and then is increased by 10 then the mean of the new set is-
   (a) $\frac{\overline{x}}{\alpha}$ (b) $\frac{\overline{x}+10}{\alpha}$
   (c) $\frac{\overline{x}+10\alpha}{\alpha}$ (d) $a\overline{x}+10$
9. If the mean of first n natural numbers is equal to $\frac{n+7}{3}$, t n is equal to -
   (a) 10 (b) 11
   (c) 12 (d) None of these
10. The mean of first three terms is 14 and mean of next terms is 18. The mean of all the five terms is:
    (a) 14.5 (b) 15.0
    (c) 15.2 (d) 15.6
11. The weighted arithmetic mean of first n natural numbers w weight are equal to the corresponding numbers is equal to
    (a) $2n+1$ (b) $\frac{1}{2}(2n+1)$
    (c) $\frac{1}{3}(2n+1)$ (d) $\frac{(2n+1)}{6}$
12. The mean of distribution, in which the values of X are ..., n the frequency of each being unity is
    (a) $\frac{n(n+1)}{2}$ (b) $\frac{n}{2}$
    (c) $\frac{n+1}{2}$ (d) $\frac{n(n-1)}{2}$

GEOMETRIC MEAN & HARMONIC MEAN

13. Geometric mean of the numbers $2, 2^2, 2^3, ..., 2^n$ is
    (a) $2^{2/n}$ (b) $2^{n/2}$
    (c) $2^{\frac{n-1}{2}}$ (d) $2^{\frac{n+1}{2}}$
14. The harmonic mean of 4, 8, 16 is:
    (a) 6.4 (b) 6.7 (c) 6.85 (d) 7.8

MEDIAN

15. The median of 10, 14, 11, 9, 8, 12, 6 is
    (a) 10 (b) 12
    (c) 14 (d) 11
16. If a variable takes the discrete values $\alpha + 4$, $\alpha-\frac{7}{2}, \alpha-\frac{5}{2}, \alpha-3, \alpha-2, \alpha+\frac{1}{2}, \alpha-\frac{1}{2}, \alpha+5 (\alpha > 0$ then the median is:
    (a) $\alpha-\frac{5}{4}$ (b) $\alpha-\frac{1}{2}$
    (c) $\alpha-2$ (d) $\alpha+\frac{5}{4}$
17. The median of the items 6, 10, 4, 3, 9, 11, 22, 18 is:
    (a) 9 (b) 10
    (c) 9.5 (d) 11

18. Find the median from the following distribution

Marks	0-10	10-20	20-30	30-40	40-50	50-60
No. of students	10	20	30	50	40	30

is

(*a*) 30 (*b*) 45 (*c*) 36 (*d*) 40

MODE

19. Mode of the distribution

Marks	4	5	6	7	8
No. of students	3	5	10	6	1

is

(*a*) 6 (*b*) 10
(*c*) 8 (*d*) None of these

20. The mode of the following distribution

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of students	2	18	30	45	35	20	6	3

is

(*a*) 36 (*b*) 37.3 (*c*) 38 (*d*) 39

21. If median = (mode + 2 mean)M, then M is equal to:

(*a*) 3 (*b*) 1/3
(*c*) 2 (*d*) None of these

DISPERSION

22. The standard deviation of a variate x is σ. The standard deviation of the variable $\frac{ax+b}{c}$; a, b, c are constants, is:

(*a*) $\left(\frac{a}{c}\right)\sigma$ (*b*) $\left|\frac{a}{c}\right|\sigma$
(*c*) $\left(\frac{a^2}{c^2}\right)\sigma$ (*d*) None of these

23. The scores of a batsman in ten innings are: 38, 70, 48, 34, 42, 55, 63, 46, 54, 44, then mean deviation about the median is:

(*a*) 8.4 (*b*) 8.5 (*c*) 8.6 (*d*) 8.8

24. Mean deviation from the mean for the observations –1, 0, 4 is

(*a*) $\sqrt{\frac{14}{3}}$ (*b*) $\frac{2}{3}$
(*c*) 2 (*d*) None of these

25. The Standard deviation of 7 scores 1, 2, 3, 4, 5, 6, 7 is

(*a*) 4 (*b*) 2
(*c*) $\sqrt{7}$ (*d*) None of these

26. Let σ be the standard deviation of n observations. Each of the n observations is multiplied by a constant c. Then the standard deviation of the resulting numbers is-

(*a*) σ (*b*) $|c| . \sigma$
(*c*) $\sigma\sqrt{c}$ (*d*) None of these

27. The mean of 100 items is 50 and their Standard deviation is 4, the sum of all the items and also the sum of the squares of the items is

(*a*) 5000, 251600 (*b*) 4000, 251600
(*c*) 5000, 26100 (*d*) 3000, 26100

28. The mean and Standard deviation of distributions of 100 and 150 items are 50, 5 and 40, 6 respectively. The mean and Standard deviation of all the 250 items taken together are

(*a*) 44, 6.46 (*b*) 42, 7.46
(*c*) 44, 7.46 (*d*) 42, 6.46

29. The variance of first n natural numbers is 10 then n is

(*a*) 12 (*b*) 11 (*c*) 10 (*d*) 10 – 11

30. Mean deviationof the series $a, a+d, a+2d, ..., a+2nd$ from its mean is

(*a*) $\frac{(n+1)d}{2n+1}$ (*b*) $\frac{nd}{2n+1}$
(*c*) $\frac{n(n+1)d}{2n+1}$ (*d*) $\frac{(2n+1)d}{n(n+1)}$

Exercise-2 (Learning Plus)

1. The mean weight of 9 items is 15. If one more item is added to the series, the mean becomes 16. The value of 10th item is

(*a*) 35 (*b*) 30
(*c*) 25 (*d*) 20

2. In the frequency distribution of the discrete data given below, the frequency k against value 0 is missing.

Variable x :	0	1	2	3	4	5
Frequency f :	k	20	40	40	20	4

If the mean is 2.5, then the missing frequency k will be

(*a*) 0 (*b*) 1 (*c*) 3 (*d*) 4

3. If mean of n item is $\overline{x}$. If each r^{th} item is increased by $2r$. Then new mean will be

(*a*) $\overline{x}$ (*b*) $\overline{x}+\frac{n}{2}$
(*c*) $\overline{x}+\frac{n+2}{2}$ (*d*) $\overline{x}+n+1$

4. The *SD* of 15 items is 6 and if each item decreases by 1, then standard deviation will be

(*a*) 5 (*b*) 7
(*c*) $\frac{91}{15}$ (*d*) 6

5. The sum of squares of deviations for 10 observations taken from mean 50 is 250. The coefficient of variation is
   (*a*) 10%
   (*b*) 40%
   (*c*) 50%
   (*d*) None of these

6. Mean of n items is $\bar{x}$. If these n items are increased by $1^2, 2^2, 3^2, \ldots, n^2$ successively, then mean gets increased by
   (*a*) $\frac{(n+1)(2n+1)}{6}$
   (*b*) $\frac{n(n+1)(2n+1)}{6}$
   (*c*) $\frac{n^2}{2}$
   (*d*) Remains same

7. The mean value of the median and mean of the odd divisors of 360 is
   (*a*) 13 (*b*) 7
   (*c*) 6 (*d*) 10

8. A sample of 35 observations has the mean 80 and Standard deviation as 4. A second sample of 65 observation from the same population has mean 60 and Standard deviation 3. The Standard deviation of the combined sample is
   (*a*) 5.85 (*b*) 5.58
   (*c*) 10.12 (*d*) None of these

9. The mean and Standard deviation of the marks of 200 candidates were found to be 40 and 15 respectively. Later, it was discovered that a score of 40 was wrongly read as 50. The correct mean and Standard deviation respectively are
   (*a*) 14.98, 39.95
   (*b*) 39.95, 16.12
   (*c*) 39.95, 224.5
   (*d*) None of these

10. The average of n numbers $x_1, x_2, x_3, \ldots, x_n$ is M. If x_1 is replaced by x, then the new average is
   (*a*) $M - x_1 + x$
   (*b*) $\frac{(n-1)M - x_1 + x}{n}$
   (*c*) $\frac{(n+1)M - x_1 + x}{n}$
   (*d*) $\frac{nM - x_1 + x}{n}$

11. The variance of α, β and γ is 9, then variance of 5α, 5β and 5γ is
   (*a*) 45 (*b*) 9/5
   (*c*) 5/9 (*d*) 225

12. The mean of n items is $\bar{x}$. If each item is successively increased by $3, 3^2, 3^3, \ldots, 3^n$, then new mean equals
   (*a*) $\bar{x} + \frac{3^{n+1}}{n}$
   (*b*) $\bar{x} + 3\frac{(3^n - 1)}{2n}$
   (*c*) $\bar{x} + \frac{3^n}{n}$
   (*d*) $\bar{x} + 3\frac{(3^n - 1)}{n}$

13. In a class of 100 students there are 70 boys whose average marks in a subject are 75. If the average marks of the complete class is 72, then what is the average of the girls?
   (*a*) 73 (*b*) 65
   (*c*) 68 (*d*) 74

14. The median of a set of 9 distinct observation is 20.5. If each of the largest 4 observations of the set is increased by 2, then the median of the new set
   (*a*) Is increased by 2
   (*b*) Is decreased by 2
   (*c*) Is two times the original median
   (*d*) Remains the same as that of the original set

15. In an experiment with 15 observations on x, the following results were available
   $\sum x^2 = 2830, \sum x = 170$
   One observation that was 20, we found to be wrong and was replaced by the correct value 30. Then, the corrected variance is
   (*a*) 78.00
   (*b*) 188.66
   (*c*) 177.33
   (*d*) 8.33

16. In a series of $2n$ observations, half of them equal a and remaining half equal $-a$. If the standard deviation of the observations is 2, then $|a|$ equals
   (*a*) $\frac{1}{n}$ (*b*) $\sqrt{2}$
   (*c*) 2 (*d*) $\frac{\sqrt{2}}{n}$

17. If the standard deviation of the observations $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$ is $\sqrt{10}$. The standard deviation of observations 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 will be
   (*a*) $\sqrt{10} + 20$
   (*b*) $\sqrt{10} + 10$
   (*c*) $\sqrt{10}$
   (*d*) None of these

18. Median of ${}^{2n}C_0, {}^{2n}C_1, {}^{2n}C_2, {}^{2n}C_3, \ldots, {}^{2n}C_n$ (where n is even) is

(a) ${}^{2n}C_{\frac{n}{2}}$

(b) ${}^{2n}C_{\frac{n+1}{2}}$

(c) ${}^{2n}C_{\frac{n-1}{2}}$

(d) None of these

NUMERICAL VALUE BASED

19. If the mean deviation of number $1, 1 + d, 1 + 2d, \ldots, 1 + 100d$ from their mean is 255, then the d is equal to

20. The means and variance of n observations $x_1, x_2, x_3, \ldots x_n$ are 5 and 0 respectively. If $\sum_{i=1}^{n} x_i^2 = 400$, then the value of n is equal to

21. The mean of 30 given numbers, when it is given that the mean of 10 of them is 12 and the mean of the remaining 20 is 9, is equal to

22. The algebraic sum of the deviation of 20 observations measured from 30 is 2. Then, mean of observations is

23. If the average of the numbers 148, 146, 144, 142, ... in AP, be 125, then the total numbers in the series will be

24. A group of 10 items has arithmetic mean 6. If the arithmetic mean of 4 of these items is 7.5, then the mean of the remaining items is

25. Mean of 100 observation is 45. If it was later found that two observations 19 and 31 were incorrectly recorded as 91 and 13. The correct mean is

26. In a class of 50 students, 10 have failed and their average marks are 28. The total marks obtained by the entire class are 2800. The average marks of those who have passed, are

Exercise-3 (Past Year Questions)

JEE MAIN

1. If the standard deviation of the number 2, 3, a and 11 is 3.5, then which of the following is true? **(2016)**

(a) $3a^2 - 26a + 55 = 0$

(b) $3a^2 - 32a + 84 = 0$

(c) $3a^2 - 34a + 91 = 0$

(d) $3a^2 - 23a + 44 = 0$

2. If $\sum_{i=1}^{9}(x_i - 5) = 9$ and $\sum_{i=1}^{9}(x_i - 5)^2 = 45$, then the standard deviation of the 9 items $x_1, x_2 \ldots x_9$ is- **(2018)**

(a) 4 (b) 2

(c) 3 (d) 9

3. 5 students of a class have an average height 150 cm and variance 18 cm^2. A new student, whose height is 156 cm, joined them. The variance (in cm^2) of the height of these six students is: **(2019)**

(a) 16 (b) 22

(c) 20 (d) 18

4. A data consists of n observations : $x_1, x_2, \ldots, x_n$. If $\sum_{i=1}^{n}(x_i + 1)^2 = 9n$ and $\sum_{i=1}^{n}(x_i - 1)^2 = 5n$, then the standard deviation of this data is: **(2019)**

(a) 5 (b) $\sqrt{5}$

(c) $\sqrt{7}$ (d) 2

5. The mean of five observations is 5 and their variance is 9.20. If three of the given five observations are 1, 3 and 8, then a ratio of other two observations is: **(2019)**

(a) 10 : 3 (b) 4 : 9

(c) 5 : 8 (d) 6 : 7

6. If mean and standard deviation of 5 observations x_1, x_2, x_3, x_4, x_5 are 10 and 3, respectively, then the variance of 6 observations $x_1, x_2, \ldots, x_5$ and -50 is equal to **(2019)**

(a) 509. 5 (b) 586. 5

(c) 582.5 (d) 507. 5

7. A bag contains 30 white balls and 10 red balls. 16 balls are drawn one by one randomly from the bag with replacement. If X be the number of white balls drawn, then $\left(\dfrac{\text{mean of } X}{\text{standard deviation of } X}\right)$ is equal to: **(2019)**

(a) 4 (b) $4\sqrt{3}$

(c) $3\sqrt{2}$ (d) $\dfrac{4\sqrt{3}}{3}$

8. If the sum of the deviations of 50 observations from 30 is 50, then the mean of these observations is: **(2019)**

(a) 30 (b) 51

(c) 50 (d) 31

9. The mean and the variance of five observations are 4 and 5.20, respectively. If three of the observations are 3, 4 and 4; then the absolute value of the difference of the other two observations, is: **(2019)**

(a) 7 (b) 5

(c) 1 (d) 3

10. The mean and variance of seven observations are 8 and 16, respectively. If 5 of the observations are 2, 4, 10, 12, 14, then the product of the remaining two observations is: **(2019)**

(*a*) 40 (*b*) 49
(*c*) 48 (*d*) 45

11. A student scores the following marks in five tests : 45, 54, 41, 57, 43. His score is not known for the sixth test. If the mean score is 48 in the six tests, then the standard deviation of the marks in six tests is **(2019)**

(*a*) $\frac{10}{\sqrt{3}}$ (*b*) $\frac{100}{\sqrt{3}}$
(*c*) $\frac{100}{3}$ (*d*) $\frac{10}{3}$

12. If the standard deviation of the numbers –1, 0, 1, k is $\sqrt{5}$ where $k > 0$, then k is equal to **(2019)**

(*a*) $2\sqrt{\frac{10}{3}}$ (*b*) $2\sqrt{6}$
(*c*) $4\sqrt{\frac{5}{3}}$ (*d*) $\sqrt{6}$

13. The mean and the median of the following ten numbers in increasing order 10, 22, 26, 29, 34, x, 42, 67, 70, y are 42 and 35 respectively, then $\frac{y}{x}$ is equal to: **(2019)**

(*a*) $\frac{7}{3}$ (*b*) $\frac{9}{4}$
(*c*) $\frac{7}{2}$ (*d*) $\frac{8}{3}$

14. If for some $x \in R$, the frequency distribution of the marks obtained by 20 students in a test is: **(2019)**

Marks	2	3	5	7
Frequency	$(x+1)^2$	$2x-5$	x^2-3x	x

then the mean of the marks is:

(*a*) 2.8 (*b*) 3.2
(*c*) 3.0 (*d*) 2.5

15. If both the mean and the standard deviation of 50 observations $x_1, x_2 ... x_{50}$ are equal to 16, then the mean of $(x_1 - 4)^2, (x_2 - 4)^2 ... (x_{50} - 4)^2$ is: **(2019)**

(*a*) 525 (*b*) 380
(*c*) 480 (*d*) 400

16. If the data $x_1, x_2, ..., x_{10}$ is such that the mean of first four of these is 11, the mean of the remaining six is 16 and the sum of squares of all these is 2,000; then the standard deviation of this data is: **(2019)**

(*a*) 4 (*b*) 2
(*c*) $\sqrt{2}$ (*d*) $2\sqrt{2}$

17. If the variance of the first n natural numbers is 10 and the variance of the first m even natural numbers is 16, then $(m+n)$ is equal to______. **(2020)**

18. If the mean and variance of eight numbers 3, 7, 9, 12, 13, 20, x and y be 10 and 25 respectively, then $x.y$ is equal to ______. **(2020)**

19. The mean and the standard deviation (s.d.) of 10 observations are 20 and 2 respectively. Each of these 10 observations is multiplied by p and then reduced by q, where $p \neq 0$ and $q \neq 0$. If the new mean and new $s.d.$ become half of their original values, then q is equal to **(2020)**

(*a*) – 20 (*b*) 10
(*c*) – 10 (*d*) – 5

20. The mean and variance of 20 observations are found to be 10 and 4, respectively. On rechecking, it was found that an observation 9 was incorrect and the correct observation was 11. Then the correct variance is: **(2020)**

(*a*) 3.99 (*b*) 4.02
(*c*) 4.01 (*d*) 3.98

21. Let the observations $x_i (1 \le i \le 10)$ satisfy the equations, $\sum_{i=1}^{10}(x_i - 5) = 10$ and $\sum_{i=1}^{10}(x_i - 5)^2 = 40$. If μ and λ are the mean and the variance of the observations, $x_1 - 3, x_2 - 3, ..., x_{10} - 3$, then the ordered pair (μ, λ) is equal to: **(2020)**

(*a*) (3, 3) (*b*) (3, 6)
(*c*) (6, 6) (*d*) (6, 3)

22. If the variance of the terms in an increasing $A.P.$, $b_1, b_2, b_3, ..., b_{11}$ is 90, then the common difference of this $A.P.$ is ______. **(2020)**

23. Let $X = \{x \in N : 1 \le x \le 17\}$ and $Y\{ax + b : x \in X\}$ and $a, b \in R, a > 0\}$. If mean and variance of elements of Y are 17 and 216 respectively then $a + b$ is equal to : **(2020)**

(*a*) 7 (*b*) –27
(*c*) 9 (*d*) –7

24. Let x_i $(1 \le i \le 10)$ be ten observations of a random variable X. If $\sum_{i=1}^{10}(x_i - p) = 3$ and $\sum_{i=1}^{10}(x_i - p)^2 = 9$ where $p \neq 0$, $p \in R$, then the standard deviation of these observations is **(2020)**

(*a*) $\frac{7}{10}$ (*b*) $\frac{9}{10}$
(*c*) $\sqrt{\frac{3}{5}}$ (*d*) $\frac{4}{5}$

25. For the frequency distribution : **(2020)**

Variate (x) : $x_1\, x_2\, x_3 ... x_{15}$

Frequency (f) : $f_1 f_2 f_3 ... f_{15}$

Where $0 < x_1 < x_2 < x_3 < ... < x_{15} = 10$ and $\sum_{i=1}^{15} f_i > 0$, then stan.dard deviation cannot be:

(*a*) 1 (*b*) 6
(*c*) 2 (*d*) 4

26. If the variance of the following frequency distribution: (2020)

Class:	10 – 20	20 – 30	30 – 40
Frequency:	2	x	2

is 50, then x is equal to ______

27. The mean and variance of 8 observations are 10 and 13.5, respectively. If 6 of these observations are 5, 7, 10, 12, 14, 15, then the absolute difference of the remaining two observations is: (2020)

(a) 9 (b) 3
(c) 7 (d) 5

28. If the mean and the standard deviation of the data 3, 5, 7, a, b are 5 and 2 respectively, then a and b are the roots of the equation (2020)

(a) $x^2 - 20x + 18 = 0$ (b) $2x^2 - 20x + 19 = 0$
(c) $x^2 - 10x + 18 = 0$ (d) $x^2 - 10x + 19 = 0$

29. The mean and variance of 7 observations are 8 and 16, respectively. If five observations are 2, 4, 10, 12, 14, then the absolute difference of the remaining two observations is: (2020)

(a) 2 (b) 4
(c) 3 (d) 1

30. Consider the data on x taking the values 0, 2, 4, 8, ..., 2^n with frequencies ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ respectively. If the mean of this data is $n\dfrac{728}{2^n}$ then n is equal to ______. (2020)

31. If $\sum_{i=1}^{n}(x_i - a) = n$ and $\sum_{i=1}^{n}(x_i - a)^2 = na$, $(n, a > 1)$
Then the standard deviation of n observations $x_1, x_2, ... x_n$ is: (2020)

(a) $a - 1$ (b) $n\sqrt{a-1}$
(c) $\sqrt{n(a-1)}$ (d) $\sqrt{a-1}$

32. Consider three observations a, b and c such that $b = a + c$. If the standard deviation of $a + 2b + 2, c + 2$ is d, then which of the following is true? (2021)

(a) $b^2 = 3(a^2 + c^2) + 9d^2$
(b) $b^2 = a^2 + c^2 + 3d^2$
(c) $b^2 = 3(a^2 + c^2 + d^2)$
(d) $b^2 = 3(a^2 + c^2) - 9d^2$

33. The mean and variance of the data 4, 5, 6, 6, 7, 8, x, y, where $x < y$, are 6 and $\dfrac{9}{4}$ respectively. Then $x^4 + y^2$ is equal to (2022)

(a) 162 (b) 320
(c) 674 (d) 420

34. Let the mean and the variance of 5 observations x_1, x_2, x_3, x_4, x_5 be $\dfrac{24}{5}$ and $\dfrac{194}{25}$ respectively. If the mean and variance of the first 4 observation are $\dfrac{7}{2}$ and a respectively, then $(4a + x_5)$ is equal to: (2022)

(a) 13 (b) 15 (c) 17 (d) 18

35. The number of value of a $a \in N$ such that the variance of 3, 7, 12, a, 43 – a is a natural number is: (2022)

(a) 0 (b) 2
(c) 5 (d) Infinite

36. If the mean deviation about median for the numbers 3, 5, 7, 2k, 12, 16, 21, 24, arranged in the ascending order, is 6 then the median is (2022)

(a) 11.5 (b) 10.5
(c) 12 (d) 11

37. Let x be a random variable having binomial distribution $B(7,1)$. If $P(X = 3) = 5\,P(x = 4)$, then the sum of the mean and the variance is X is (2022)

(a) $\dfrac{105}{16}$ (b) $\dfrac{7}{16}$
(c) $\dfrac{77}{36}$ (d) $\dfrac{49}{16}$

38. The mean and standard deviation of 50 observations are 15 and 2 respectively. It was found that one incorrect observation was taken such that the sum of correct and incorrect observations is 70. If the correct mean is 16, then the correct variance is equal to (2022)

(a) 10 (b) 36
(c) 43 (d) 60

39. The mean of the numbers a, b, 8, 5, 10 is 6 and their variance is 6.8. If M is the mean deviation of the numbers about the mean, then 25 M is equal to (2022)

(a) 60 (b) 55
(c) 50 (d) 45

ANSWER KEY

CONCEPT APPLICATION

1. (b)	2. (d)	3. (a)	4. (d)	5. (b)	6. (d)	7. (a)	8. (c)	9. (d)	10. (c)
11. (c)	12. (d)	13. (a)	14. (d)	15. (d)	16. (d)	17. (d)	18. (a)	19. (c)	20. (b)

EXERCISE-1 (TOPICWISE)

1. (c)	2. (b)	3. (b)	4. (a)	5. (a)	6. (a)	7. (c)	8. (c)	9. (b)	10. (d)
11. (c)	12. (c)	13. (d)	14. (c)	15. (a)	16. (a)	17. (c)	18. (c)	19. (a)	20. (a)
21. (b)	22. (b)	23. (c)	24. (c)	25. (b)	26. (b)	27. (a)	28. (c)	29. (b)	30. (c)

EXERCISE-2 (LEARNING PLUS)

1. (c)	2. (d)	3. (d)	4. (d)	5. (a)	6. (a)	7. (d)	8. (c)	9. (d)	10. (d)
11. (d)	12. (b)	13. (b)	14. (d)	15. (a)	16. (c)	17. (c)	18. (a)	19. [10.1]	20. [16]
21. [10]	22. [30.1]	23. [24]	24. [5]	25. [44.46]	26. [63]				

EXERCISE-3 (PAST YEAR QUESTIONS)

JEE Main

1. (b)	2. (b)	3. (c)	4. (b)	5. (b)	6. (d)	7. (b)	8. (d)	9. (a)	10. (c)
11. (a)	12. (b)	13. (a)	14. (a)	15. (d)	16. (b)	17. [18]	18. [54]	19. (a)	20. (a)
21. (a)	22. [3]	23. (d)	24. (b)	25. (b)	26. [4]	27. (c)	28. (d)	29. (a)	30. [6]
31. (d)	32. (d)	33. (b)	34. (b)	35. (a)	36. (d)	37. (c)	38. (c)	39. (a)	

CHAPTER

23 Probability

INTRODUCTION

We studied about the concept of probability as a measure of uncertainty of various phenomenon. We have obtained the probability of getting an even number in throwing a die as $\frac{3}{6}$ i.e., $\frac{1}{2}$. Here the total possible outcomes are 1,2,3,4,5 and 6 (six in number).

The outcomes in favour of the event of 'getting an even number' are 2,4,6 (i.e., three in number). In general, to obtain the probability of an event, we find the ratio of the number of outcomes favourable to the event, to the total number of equally likely outcomes. This theory of probability is known as classical theory of probability.

DEFINITIONS

(*a*) **Experiment:** An action or operation resulting in two or more outcome, which are unpredictable in advance.

e.g.

(*i*) Tossing of a coin

(*ii*) Throwing a dice

(*iii*) Drawing a card

(*b*) **Sample space :** A set S that consists of all possible outcomes of a random experiment is called a sample space and each outcome is called a sample point often there will be more than one sample space that can describes outcomes of an experiment, but there is usually only one that will provide the most information. If a sample space has a finite number of points it is called finite sample space and infinite sample space if it has infinite number of points.

e.g.

(*i*) In toss of a coin, S = {H, T} where H and T are sample points representing a head and a tail respectively.

(ii) In throw of a die, S = {1, 2, 3, 4, 5, 6} where the numbers are the sample points representing the six faces.

(c) **Trial:** When an experiment is repeated under similar conditions and it does not give the same result each time but may result in any one of the several possible outcomes, the experiment is called a trial and the outcomes are called cases. The number of times the experiment is repeated is called the number of trials.

Train Your Brain

Example 1: Write the sample space of the following experiment

(*i*) 'Three coins are tossed'.

(*ii*) 'Selection of two children from a group of 3 boys and 2 girls without replacement'.

Sol. (*i*) {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

(*ii*) {B_1 B_2, B_1 B_3, B_1 G_1, B_1 G_2, B_2 B_3, B_2 G_1, B_2 G_2, B_3 G_1, B_3 G_2, G_1 G_2}

Example 2: Consider the experiment in which a coin is tossed repeatedly until a head comes up. Describe the sample space.

Sol. In the experiment head may come up on the first toss, or the 2nd toss, or the 3rd toss and so on till head is obtained. Hence, the desired sample space is S = {H, TH, TTH, TTTH, TTTTH,........}

Concept Application

1. A bag contains one white and one red ball. A ball is drawn from the bag. If the ball drawn in white it is replaced in the bag and again a ball is drawn. Otherwise, a die is tossed. Write the sample space for this experiment.

2. A box contains 1 white and 3 identical black balls. Two balls rare drawn at random in succession without replacement. Writhe the sample space for this experiment.

For example:

- One toss of coin is a trial when coin is tossed 5 times.
- One throw of a die is a trial when the die is thrown 4 times.

(*d*) **Event:** A subset of sample space, i.e. a set of some of possible outcomes of a random experiment is called as event.

e.g. Scoring a six on the throw of a dice

(*e*) **Simple event:** Each sample point in the sample space is called an elementary event or simple event. For example occurrence of head in throw of a coin is simple event.

(*f*) **Sure event:** The set containing all sample points is a sure event as in the throw of a die the occurrence of natural number less than 7, is a sure event.

(g) **Null event:** The set which does not contain any sample point.

(*h*) **Mixed/compound event:** A subset of sample space S containing more than one element is called a mixed event or a compound event.

(*i*) **Compliment of an event:** Let S be the sample space and E be an event then E^c or $\overline{E}$ represents complement of event E which is a subset containing all sample points in S which are not in E. It refers to the non occurrence of event E.

In every case it is set of some or all possible outcomes of the experiment. Therefore event (A) is subset of sample space (S). If outcome of an experiment is an element of A we say that event A has occurred.

- An event consisting of a single point of S is called a simple or elementary event.
- If an event has more than one simple point it is called compound event.
- ϕ is called impossible event and S (sample space) is called sure event.

(*j*) **Mutually Exclusive Events:** Two events are said to be Mutually Exclusive (or disjoint or incompatible) if the occurrence of one precludes (rules out) the simultaneous occurrence of the other. If A and B are two mutually exclusive events then $P(A \cap B) = 0$.

Consider, for example, choosing numbers at random from the set {3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

If, Event A is the selection of a prime number,

Event B is the selection of an odd number,

Event C is the selection of an even number,

Then A and C are mutually exclusive as none of the numbers in this set is both prime and even. But A and B are not mutually exclusive as some numbers are both prime and odd (viz. 3, 5, 7, 11).

(*k*) **Equally Likely Events:** Events are said to be Equally Likely when each event is as likely to occur as any other event.

(*l*) **Exhaustive Events:** Events A, B, C..........L are said to be Exhaustive Event if no event outside this set can result as an outcome of an experiment. For example, if A and B are two events defined on a sample space S, then A and B are exhaustive $\Rightarrow A \cup B = S \Rightarrow P(A \cup B) = 1$.

Comparative study of Equally likely, Mutually Exclusive and Exhaustive events :

Experiment	Events	E/L	M/E	Exhaustive
1. Throwing of a die	A : throwing an odd face {1, 3, 5} B : throwing a composite {4, 6}	No	Yes	No
2. A ball is drawn from an urn containing 2W, 3R and 4G balls	E_1 : getting a W ball E_2 : getting a R ball E_3 : getting a G ball	No	Yes	Yes
3. Throwing a pair of dice	A : Throwing a doublet {11, 22, 33, 44, 55, 66} B : throwing a total of 10 or more {46, 64, 55, 56, 65, 66}	Yes	No	No
4. From a well shuffled pack of cards, a card is drawn	E_1 : getting a heart E_2 : getting a spade E_3 : getting a diamond E_4 : getting a club	Yes	Yes	Yes
5. From a well shuffled pack of a cards, a card is drawn	A = getting a heart B = getting a face card	No	No	No

Train Your Brain

Example 3: In drawing of a card from a well shuffled ordinary deck of playing cards the events 'card drawn is spade' and 'card drawn is an ace' are

(a) Mutually exclusive
(b) Equally likely
(c) Forming an exhaustive system
(d) None of these

Sol. A = getting spade

B = getting an ace

$$P(A) = \frac{13}{52} \Rightarrow P(A) = \frac{1}{4};$$

$$P(B) = \frac{4}{52} \Rightarrow P(B) = \frac{1}{13}$$

They are independent event As $P(A \cap B) = P(A).P(B) = 1/52$

Example 4: In throwing a pair of dice, find whether the two events

(i) E_1 : 'coming up of an odd number on first dice' and E_2 : 'coming up of a total of 8'.

(ii) E_1 : 'coming up of 4 on first dice' and E_2 : 'coming up of 5 on the second dice'. are mutually exclusive or not

Sol. (i) E_1 : {(1, 1) (1, 2)..(1, 6), (3, 1)…(3, 6), (5,1) (5,6)}

E_2 : {: (2, 6) (3, 5) (4, 4) (5, 3) (6, 2)}

(ii) E_1 : {(4,1) (4,2) ... (4,6)}

E_2 : {(1,5) (2,5) … (6,5)}

Example 5: If the letters of INTERMEDIATE are arranged, then the probability no two E's occur together is

Sol. $I \to 2, N \to 1, 0 \to 2, E \to 3, R \to 1, M \to 1, D \to 1, A \to 1$ (3E's Rest 9)

First arrange rest of letters $= \frac{9!}{2!2!}$

Now 3E's can be placed into place in ${}^{10}C_3$ ways so

favourable cases $= \frac{9!}{2!2!} \times {}^{10}C_3 = 3 \times 10!$

Total cases $= \frac{12!}{2!2!3!}$

Probability $= \frac{3 \times 10! \times 2! \times 2! \times 3!}{12!} = \frac{6}{11}$

Example 6: From a group of 10 persons consisting of 5 lawyers, 3 doctors and 2 engineers, four persons are selected at random. The probability that the selection contains at least one of each category is

Sol. $n(S) = {}^{10}C_4 = 210;$

$n(E) = {}^5C_2 \times {}^3C_1 \times {}^2C_1 + {}^5C_1 \times {}^3C_2 \times {}^2C_1 + {}^5C_1 \times {}^3C_1 \times {}^2C_2 = 105.$

$\therefore\ P(E) = \frac{105}{210} = \frac{1}{2}.$

Concept Application

3. The number 1, 2, 3 and 4 are written separately on four slips of paper. The slips are then put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the following events:

A = The number on the first slip is larger than the one on the second slip.

B = The number on the second slip is greater than 2

C = The sum of the numbers on the two slips is 6 or 7

D = The number on the second slips is twice that on the first slip.

Which pair(s) of events is (are) mutually exclusive?

4. A card is picked up from a deck of 52 playing cards.
   (i) What is the sample space of the experiment ?
   (ii) What is the event that the chosen card is black faced card ?

PROBABILITY OF AN EVENT

If there are 'n' elementary events associated with a random experiment and 'm' of then are favorable to an event A, then the probability of happening or occurrence of A is denoted by $P(A)$ and is defined as the ratio $\frac{m}{n}$.

Thus, $P(A) = \frac{m}{n}$

Clearly, $0 \le m \le n$. Therefore,

$$0 \le \frac{m}{n} \le 1 \Rightarrow 0 \le P(A) \le 1$$

If $P(A) = 1$, then A is called certain event and A is called an impossible event, if $P(A) = 0$.

The number of elementary events which will ensure the non-occurrence of A i.e. which ensure the occurrence of $\overline{A}$ is (n –m). Therefore,

$$P(\overline{A}) = \frac{n-m}{n}$$

$$\Rightarrow P(\overline{A}) = I - \frac{m}{n}$$

$$\Rightarrow P(\overline{A}) = 1 - P(A)$$

$$\Rightarrow P(A) + P(\overline{A}) = 1$$

The odds in favour of occurrence of the event A are defined by m: $(n - m)$ i.e. $P(A)$: $P(\overline{A})$ and the odds against the occurrence of A are defined by $n - m : m$ i.e. $P(\overline{A}) : P(A)$.

Train Your Brain

Example 7: IF the letters of the word 'ALGORITHM' are arranged at random in a row what is the probability the letters 'GOR' must remain together as a unit ?

Sol. Number o letters in the word 'ALGORITHM' = 9

If 'GOR' remain together, then considered it as 1 group.

$\therefore$ Number of letters = 6 + 1 = 7

Number of word, if 'GOR' remain together = 7!

Total number of words from the letters of the word 'ALGORITHM' = 9!

$\therefore$ Required probability $=\dfrac{7!}{9!}=\dfrac{1}{72}$

Example 8: Four candidates A, B, C and D have applied for the assignment to coach a school cricket team. If A is twice as likely to be selected as B and B and C are given about the same chance of being selected, while C is twice as likely to be selected as D, then what are the probabilities that

(*i*) C will be selected ?

(*ii*) A will not be selected?

Sol. It is given that A is twice as likely to be selected as D.

$P(A) = 2P(B)$

$\Rightarrow \dfrac{P(A)}{2} = P(B)$

While C is twice as likely to be selected as D.

$P(C) = 2P(D) \Rightarrow P(B) = 2P(D)$

$\Rightarrow \dfrac{P(A)}{2} = 2P(D) \Rightarrow P(D) = \dfrac{P(A)}{4}$

B and C are given about the same change of being selected.

$P(B) = P(C)$

Now, sum of probability = 1

$P(A) + P(B) + P(C) + P(D) = 1$

$P(A)+\dfrac{P(A)}{2}+\dfrac{P(A)}{2}+\dfrac{P(A)}{4}=1$

$\Rightarrow \dfrac{4P(A)+2P(A)+2P(A)+P(A)}{4}=1$

$\Rightarrow 9P(A) = 4 \Rightarrow P(A) = \dfrac{4}{9}$

(*i*) P(C will be selected) = $P(C) = P(B) = \dfrac{P(A)}{2}=\dfrac{4}{9\times 2}$

$\left[\because P(A)=\dfrac{4}{9}\right]$

(*ii*) P(A will not be selected) $= P(A') = 1 - P(A) = 1-\dfrac{4}{9}=\dfrac{5}{9}$.

Example 9: If a card is drawn from a deck of 52 cards, then find the probability of getting a king or a heart or a red card.

Sol. $\because$ Number of possible event = 52

And favorable events = 4 king + 13 heart + 26 red − 13 − 2 = 28

$\therefore$ Required probability $=\dfrac{28}{52}=\dfrac{7}{13}$

Example 10: Words are formed with the letters of the word *PEACE*. Find the probability that 2*E*'s come together

Sol. Total number of words which can be formed with the letters $P, E, A, C, E = \dfrac{\lfloor 5}{\lfloor 2} = 60$.

Number of words in which 2 *E*'s come together

$= \lfloor 4 = 24$ $\therefore$ required probability $=\dfrac{24}{60}=\dfrac{2}{5}$.

Example 11: A bag contains 5 red and 4 green balls. Four balls are drawn at random then find the probability that two balls are of red and two balls are of green colour.

Sol. $n(s)$ = the total number of ways of drawing 4 balls out of total 9 balls : ${}^9C_4$

A : Drawing 2 red and 2 green balls $n(A) = {}^5C_2 \times {}^4C_2$

$$\therefore \quad P(A)=\frac{n(A)}{n(s)}=\frac{{}^5C_2\times{}^4C_2}{{}^9C_4}=\frac{\dfrac{5\times4\times4\times3}{2\times2}}{\dfrac{9\times8\times7\times6}{4\times3\times2}}=\frac{10}{21}$$

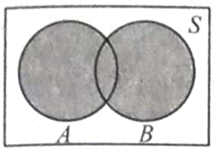
Concept Application

5. In shuffling a pack of 52 playing cards, four are accidently dropped; find the chance that the missing cards should be one from each suit.
6. From a deck of 52 cards, four cards are drawn simultaneously, find the chance that they will be the four honours of the same suit.
7. A committee of two persons is selected from two men and two women. What is the probability the the committee will have
   (*i*) No man ?
   (*ii*) One man?
   (*iii*) Tow men ?
8. In a hand at Whist, what is the probability that four kings are held by a specified player ?
9. In a lottery, a person choses six different natural numbers at random form 1 to 20, and if these six numbers match with the six number already fixed by the lottery committee, he wins the prize. What is the probability of winning the prize in the game ?

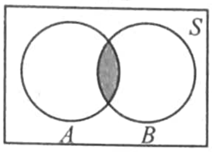
VENN DIAGRAMS

A diagram used to illustrate relationships between sets. Commonly, a rectangle represents the universal set and a circle within it represents a given set (all members of the given set are represented by points within the circle). A subset is represented by a circle within a circle and union and intersection are indicated by overlapping circles. Let S is the sample space of an experiment and A, B, C are three events corresponding to it

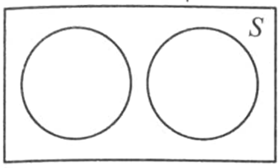
$A \cup B$



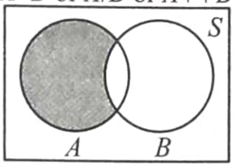
$A \cap B$



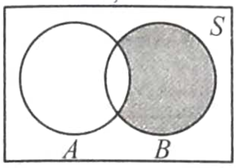
$A \cap B = \phi$



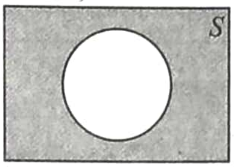
$A–B$ or A/B or $A \cap \overline{B}$



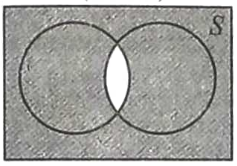
B-A, $B \backslash A$



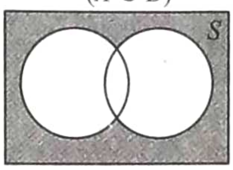
A', A^c or $\overline{A}$



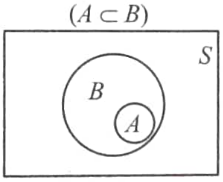
$(A \cap B)'$

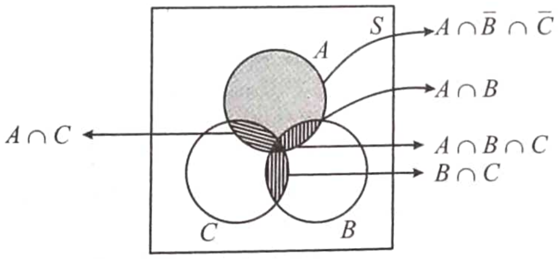


$(A \cup B)'$



$(A \subset B)$





ADDITION THEOREM

$A \cup B = A + B = A$ or B denotes occurrence of at least A or B.

For 2 events A and B:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note: (*a*) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(This is known as generalized addition theorem)

$$\left.\begin{array}{l} P(A+B) \\ P(A)+P(B\cap\overline{A}) \\ P(A\,or\,B) \\ P(B)+P(A\cap\overline{B}) \\ P \text{ (occurrence of atleast} \\ \text{one } A \text{ or } B) \end{array}\right] \Rightarrow \begin{array}{c} P(A\cap\overline{B})+P(A\cap B)+P(B\cap\overline{A}) \\ 1-P(\overline{A}\cap\overline{B}) \\ 1-P(\overline{A\cup B}) \end{array}$$

(*b*) $P(A \cup B) = P(A\,.\,\overline{B}) + P(\overline{A}\,.\,B) + P(A\,.\,B) = 1 - P(\overline{A}\,.\,\overline{B})$

(*c*) Opposite of "atleast A or B" is neither A nor B

i.e. $\overline{A+B} = 1 - (A \text{ or } B) = \overline{A} \cap \overline{B}$

Note that $P(A+B) + P(\overline{A} \cap \overline{B}) = 1$

(*d*) If A and B are mutually exclusive then

$P(A \cup B) = P(A) + P(B)$

(*e*) For any two events A and B, P (exactly one of A, B occurs)

$= P(A \cap \overline{B}) + P(B \cap \overline{A}) = P(A) + P(B) - 2P(A \cap B)$

$= P(A \cup B) - P(A \cap B) = P(\overline{A} \cup \overline{B}) - P(\overline{A} \cap \overline{B})$

(*f*) **De Morgan's Law:** If A and B are two subsets of a universal set U, then

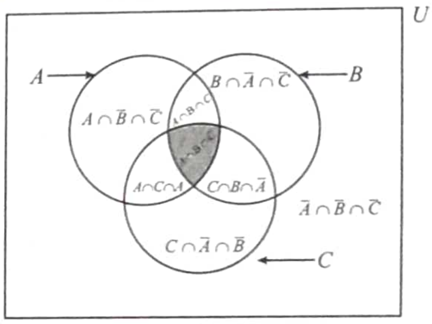
(*i*) $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

(*ii*) $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

(*iii*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and

(*iv*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

PROBABILITY OF THREE EVENTS



For any three events A, B and C we have

(*a*) $P(A$ or B or $C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(*b*) P (at least two of A, B, C occur) $= P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(*c*) P (exactly two of A, B, C occur) $= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(*d*) P (exactly one of A, B, C occurs)

$= P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

Note: If three events A, B and C are pair wise mutually exclusive then they must be mutually exclusive. i.e. $P(A \cap B) = P(B \cap C) = P(C \cap A) = 0 \Rightarrow P(A \cap B \cap C) = 0$. However the converse of this is not true.

Train Your Brain

Example 12: If A and B are mutually exclusive events, $P(A) = 0.35$ and $P(B) = 0.45$, then find

(i) $P(A')$ (ii) $P(B')$

(iii) $P(A \cup B)$ (iv) $P(A \cap B)$

(v) $P(A \cap B')$ (vi) $P(A' \cap B')$

Sol. Since, it is given that, A and B are mutually exclusive events.

$\therefore \quad P(A \cap B) = 0 \quad [\because A \cap B = f]$

and $P(A) = 0.35$, $P(B) = 0.45$

(i) $P(A') = 1 - P(A) = 1 - 0.35 = 0.65$

(ii) $P(B') = 1 - P(B) = 1 - 0.45 = 0.55$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.35 + 0.45 - 0 = 0.80$

(iv) $P(A \cap B) = 0$

(v) $P(A \cap B') = P(A) - P(A \cap B) = 0.35 - 0 = 0.35$

(vi) $P(A' \cap B') = P(A) - P(A \cap B)' = 1 - P(A \cup B)$
$= 1 - 0.8 = 0.2$

(vii) $P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$
$= 1 - 0.8 = 0.2$

Example 13: The accompanying Venn diagram shows three events, A, B and C and also the probabilities of the various intersections [for instance, $P(A \cap B) = 0.7$]. Determine

(i) $P(A)$

(ii) $P(B \cap \overline{C})$

(iii) $P(A \cup C)$

(iv) Probability of exactly one of the three occurs.

Sol. Same the above Venn diagram,

(i) $P(A) = 0.13 + 0.07 = 0.20$

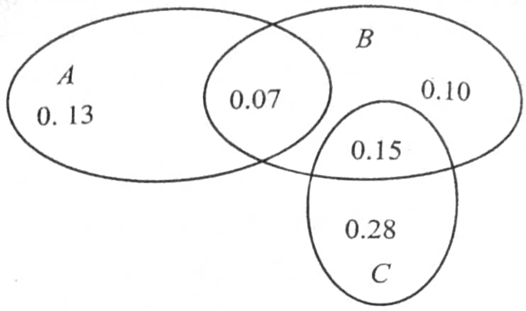
(ii) $P(B \cap \overline{C}) = P(B) - P(B \cap C)$
$= 0.07 + 0.10 + 0.15 - 0.15$
$= 0.07 + 0.10 = 0.17$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.13 + 0.07 + 0.07 + 0.10 + 0.15 - 0.07$
$= 0.13 + 0.07 + 0.10 + 0.15 = 0.45$

(iv) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$
$= 0.13 + 0.07 = 0.13$

(v) $P(B \cap C)\ 0.15$

(vi) P(exactly one of the three occurs)
$= 0.13 + 0.10 + 0.28 = 0.51$



Example 14: A bag contains 6 white, 5 black and 4 red balls. Find the probability of getting either a white or a black ball in a single draw.

Sol. Let A = event that we get a white ball, B = event that we get a black ball

So, the events are mutually exclusive

$$P(A) = \frac{{}^6C_1}{{}^{15}C_1},\ P(B) = \frac{{}^5C_1}{{}^{15}C_1}$$

$$\text{So } P(A+B) = P(A) + P(B) = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}$$

Concept Application

10. If A, B and C are three arbitrary events. Find the expression for the events noted below, in the context of A, B and C.

(i) Only A occurs

(ii) Both A and B, but not C occur

(iii) All the three events occur

(iv) At least one occurs

(v) At least two occur

(vi) One and no more occurs

(vii) Two and no more occur

(viii) None occurs

(ix) Not more than two occur

11. For the three events A, B and C, P (exactly one of the events A or B occurs) $= P$ (exactly one of the events B or C occurs) $= P$ (exactly one of the events C and A occurs) $= p$ and P (all the three events occur simultaneously) $= p^2$, where $0 < p < 1/2$. Then, find the probability of occurrence of at least one of the three events A, B, and C.

Short Notes

Mutually Exclusive Events

A set of events is said to be mutually exclusive if occurrence of one of them precludes the occurrence of any of the remaining events.

Thus, $E_1, E_2, \ldots, E_n$ are mutually exclusive if and only if $E_i \cap E_j = \phi$ for $i \neq j$.

Independent Events

Two events are said to be independent, if the occurrence of one does not depend on the occurrence of the other.

For example, when a coin is tossed twice, the event of occurrence of head in the first throw and the event of occurrence of head in the second throw are independent events.

Complement of An Event

The complement of an event E, denoted by $\overline{E}$ or E' or E^c, is the set of all sample points of the space other than the sample points in E.

For example, when a die is thrown, sample space

$S = \{1, 2, 3, 4, 5, 6\}$.

If $E = \{1, 2, 3, 4\}$, then $\overline{E} = \{5, 6\}$.

Note that $E \cup \overline{E} = S$.

Mutually Exclusive and Exhaustive Events

A set of events $E_1, E_2, \ldots, E_n$ of a sample space S form a mutually exclusive and exhaustive system of events, if

(*i*) $E_i \cap E_j = \phi$ for $i \neq j$ and

(*ii*) $E_1 \cup E_2 \cup \ldots \cup E_n = S$

Notes:

(*i*) $O \leq P(E) \leq 1$, i.e. the probability of occurrence of an event is a number lying between 0 and 1.

(*ii*) $P(\phi) = 0$, i.e. probability of occurrence of an impossible event is 0.

(*iii*) $P(S) = 1$, i.e. probability of occurrence of a sure event is 1.

ODDs in Favour of An Event and ODDs Against An Event

If the number of ways in which an event can occur be m and the number of ways in which it does does not occur be n, then

(*i*) Odds in favour of the event $= \frac{m}{n}$ and

(*ii*) Odds against the event $= \frac{n}{m}$.

Some Important Results on Probability

1. $P(\overline{A}) = 1 - P(A)$.
2. If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cup B)$.
3. If A and B are mutually exclusive events, then $A \cap B = \phi$ and hence $P(A \cap B) = 0$.
   $\therefore P(A \cup B) = P(A) + P(B)$.
4. If A, B, C are any three events, then $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$.
5. If A, B, C are mutually exclusive events, then $A \cap B = \phi, B \cap C = \phi, C \cap A = \phi, A \cap B \cap C = \phi$ and hence $P(A \cap B) = 0, P(B \cap C) = 0, P(C \cap A) = 0, P(A \cap B \cap C) = 0$.
   $\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C)$.
6. $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$.
7. $P(\overline{A} \cup \overline{B}) = 1 - P(A \cap B)$.
8. $P(A) = P(A \cap B) + P(A \cap \overline{B})$.
9. $P(B) = P(B \cap A) + P(B \cap \overline{A})$.
10. If $A_1, A_2, \ldots, A_n$ are independent events, then $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2), \ldots, P(A_n)$.
11. If $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$.
12. If $A_1, A_2, \ldots, A_n$ are exhaustive events, then $P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1$.
13. If $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive events, then
    $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n) = 1$.
14. If $A_1, A_2, \ldots, A_n$ are n events, then
    (i) $P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq P(A_1) + P(A_2) + \ldots + P(A_n)$.
    (ii) $P(A_1 \cap A_2 \cap \ldots \cap A_n) \geq 1 - P(\overline{A}_1) - P(\overline{A}_2) \ldots - P(\overline{A}_n)$.

Solved Examples

1. One card is drawn from a pack of 52 card, each of the 52 cards being equally likely to be drawn. Find the probability that the card drawn is :

(*i*) An ace (*ii*) Red

(*iii*) Either red or king (*iv*) Red and king

Sol. Out of 52 cards, one card can be drawn in $^{52}C_1$ ways.

$\therefore$ Total number of elementary events = $^{52}C_1 = 52$.

(*i*) There are four aces in a pack of 52 cards, out of which one ace can be drawn in $^4C_1$ ways. So, required probability $= \frac{4}{52} = \frac{1}{13}$.

(*ii*) There are 26 red cards, out of which one red card can be drawn in $^{26}C_1 = 26$.

$\therefore$ Favourable number of elementary events $= {}^{26}C_1 = 26$.

So, required probability $= \frac{26}{52} = \frac{1}{2}$.

(*iii*) There are 26 red cards including 2 red kings and there are 2 more kings. Therefore, there are 28 cards which are either red or king, out of 28 cards, one can be drawn in $^{28}C_1$ ways.

$\therefore$ Favourable number of elementary events $= {}^{28}C_1 = 28$.

So, required probability $= \frac{28}{52} = \frac{7}{13}$.

(*iv*) There are 2 cards which are red and king i.e., red kings.

$\therefore$ Favourable number of elementary events $= {}^{2}C_1 = 2$

So, required probability $= \frac{2}{52} = \frac{7}{26}$.

2. Four cards are drawn at random from a pack of 52 playing cards. Find the probability of getting

(*i*) All the four cards of the same suit

(*ii*) All the four cards of the same number

(*iii*) One card from each suit

(*iv*) Two red cards and two black cards

(*v*) All cards of the same colour

(*vi*) All face cards

Sol. Four cards can be drawn from a pack of 52 cards in $^{52}C_4$ ways.

So, total number cards of elementary events $= {}^{52}C_4$

(*i*) There are four suits viz. club, spade, heart and diamond, each of 13 cards. All the four cards are of the same suit means that either four cards drawn are club cards or spade cards or heart cards diamond cards. So, the total number of ways of getting all the four cards of the same suit is $^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4({}^{13}C_4)$

So, required probability $= \frac{4({}^{13}C_4)}{{}^{52}C_4} = \frac{198}{20825}$

(*ii*) Four cards drawn can be of the same number in any one of the following ways:

(1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3),..., (13, 13, 13, 13)

$\therefore$ Favourable number of elementary events = 13.

So, required probability $= \frac{13}{{}^{52}C_4} = \frac{13}{270725}$

(*iii*) There are four suits each of 13 cards. One card from each suit means that there is one diamond card, one club card, one spade card and one heart card. There are 13 diamond cards, out of which one can be selected in $^{13}C_1$ ways, Similarly, one spade and one heart, each can be selected in $^{13}C_1$ ways.

$\therefore$ The number of ways of selecting 4 cards, one from each suit $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$

So, required probability $= \frac{({}^{13}C_1)^4}{{}^{52}C_4} = \frac{2197}{20825}$

(*iv*) There are 26 red cards and 26 black cards. Out of 26 red cards, 2 cards can be drawn in $^{26}C_2$ ways. Similarly, 2black cards can be drawn in $^{26}C_2$ ways. Therefore, 2 red and 2 black cards can be drawn in $^{26}C_2 \times {}^{23}C_2$ ways.

So, required probability $= \frac{{}^{26}C_2 \times {}^{26}C_2}{{}^{52}C_4}$

(*v*) There are two colours viz. red and black. Out of 26 red colur cards can be drawn in $^{26}C_4$ ways. 4 black cards can be drawn in $^{26}C_4$ ways. Therefore, 4 red or 4 black cards can be drawn in $^{26}C_4 + {}^{26}C_4 = 2({}^{26}C_4)$ ways.

So, required probability $= \frac{2({}^{26}C_4)}{{}^{52}C_4}$

(*vi*) There are 12 face cards (4 king, 4 queens and 4 jacks). Out of these 12 face cards, 4 cards can be selected in $^{12}C_4$ ways,

$\therefore$ Favourable number of elementary events $= {}^{12}C_4$

So, required probability $= \frac{{}^{12}C_4}{{}^{52}C_4}$

3. Five marbles are drawn from a bag which contains 7 blue marbles and 4 black marbles. What is the probability that:

(*i*) All will be blue ?

(*ii*) 3 will be blue and 2 black ?

Sol. There are $7+4=11$ marbles in the bag out of which 5 marbles can be drawn in $^{11}C_5$ ways.

$\therefore$ Total number of elementary events $= {}^{11}C_5$.

(*i*) There are 7 blue marbles out of which 5 blue marbles can be drawn in $^{7}C_5$ ways.

$\therefore$ Favourable number of elementary events = ${}^7C_5$

Hence, required probability

$$= \frac{{}^7C_5}{{}^{11}C_5} = \frac{7!}{2!5!} \times \frac{5!6!}{11!} = \frac{1}{22}$$

(*ii*) Three blue out of 7 blue balls and 2 black out of 4 black balls can be drawn in ${}^7C_3 \times {}^4C_2$ ways.

$\therefore$ Favourable number of elementary events = ${}^7C_3 \times {}^4C_2$

Hence, required probability

$$= \frac{{}^7C_3 \times {}^4C_2}{{}^{11}C_5} = \frac{7!}{3!4!} \times \frac{4!}{2!2!} \times \frac{5! \times 6!}{11!} = \frac{5}{11}$$

4. A card is drawn from an ordinary pack of 52 cards and a gambler bets that, it is a spade or cot ace, What are the odds against his winning this bet?

Sol. Let A be the event of getting a spade or an ace from a pack of 52 cards. Then, Total number of elementary events = ${}^{52}C_1$ = 52

Since there are 13 spade cards including an ace of spade and three aces other than an ace of

So, $P(A) = \frac{16}{52} = \frac{4}{13}$.

Hence, odds against A are $P(\overline{A}) : P(A) = \frac{9}{13} : \frac{4}{13} = 9 : 4$

5. Six new employees, two of whom are married to each other, are to be assigned six desks that are lined up in a row. If the assignment of employees to desks is made randomly, what is the probability that the married couple will have non-adjacent desks ?

Sol. Let the couple occupied adjacent desks consider those two as 1.

There are (4 + 1) *i.e.*, 5 persons to be assigned.

$\therefore$ Number of ways of assigning these five person = 5! × 2!

Total number of ways of assigning 6 persons = 6!

$\therefore$ Probability that the couple has adjacent desk

$$= \frac{5! \times 2!}{6!} = \frac{2}{6} = \frac{1}{3}$$

Probability that the married couple will have non-adjacent desks $= 1 - \frac{1}{3} = \frac{2}{3}$

6. A term of medical students doing their internship have to assist during surgeries at a city hospital. The probabilities of surgeries rates as very complex, complex, routine, simple or very simple are respectively, 0.15, 0.20, 0.31, 0.26 and 0.08. Find the probabilities that a particular surgery will be rated

(*a*) Complex or very comples

(*b*) Neither very complex nor very simple

(*c*) Routine or complex

(*d*) Routine or simple

Sol. Let E_1, E_2, E_3, E_4 and E_5 be the event that surgeries are rated as very complex, complex, routine, simple or very simple, respectively.

$\therefore$ P(complex or very Complex) = P(E_1 or E_2)

= P(Complex or very Complex) = P(E_1 or E_2)

$= P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= 0.15 + 0.20 - 0$ $[P(E_1 \cap E_2) = 0]$

[Because all events are independent] = 0.35

(*ii*) P(neither very complex nor very simple),

$(P(E_1' \cap E_5') = P(E_1 \cup E_5)' = 1 - P(E_1 \cup E_5)$

$= 1 - [P(E_1) + P(E_5)]$

$= 1 - 0.23 = 0.77$

(*iii*) P(routine or complex)

$P(E_3 \cup E_2) = P(E_3) + P(E_2)$

$= 0.31 + 0.20 = 0.51$

(*iv*) P(routine or simple) = $P(E_3 \cup E_4)$

$= P(E_3) + P(E_4)$

$= 0.31 + 0.26 = 0.57$

7. Determine the probability p, for the each of the following events.

(*i*) An odd number appears in a single toss of a fair die.

(*ii*) Atleast one head appears in two tosses of a fair coin.

(*iii*) A king, 9 of hearts or 3 of spades appears in drawing a single card from a well shuffled ordinary deck of 52 cards.

(*iv*) The sum of 6 appears in a single toss of a pair of fair dice.

Sol. (*i*) When a die is throw the possible outcomes are

$S = \{1, 2, 3, 4, 5, 6\}$ out of the which 1, 3, 5 are odd,

$\therefore$ Required probability $= \frac{3}{6} = \frac{1}{2}$

(*ii*) When a fair coinistossed two times the sample space is S = {HH, HT, TH, TT}

In at least one head favorable enonts are

HH, HT, TH

$\therefore$ Required probability $= \frac{3}{4}$

(*iii*) Total cards = 52

Favorable = 4 king + 2 of heart + 3 spade

= 4 + 1 + 1 = 6

$\therefore$ Required probability $= \frac{6}{52} = \frac{3}{26}$

(*iv*) When a pair of dice is rolled total sample parts are 36. Out of which (1, 5), (5, 1), (2, 4), (4, 2) and (3, 3).

$\therefore$ Required probability $= \frac{5}{36}$

8. There are three events A, B, C, one of which must, and only one can, happen; the odds are 8 to 3 against A, 5 to 2 against B : find the odds against C.

Sol. $P(A) = \frac{3}{11}$; $P(B) = \frac{2}{7}$; P(C) = ? ; P(A) + P(B) + P(C) = 1

$\Rightarrow P(C) = \frac{34}{77}$.

9. Three persons A, B and C speak at a function along with 5 other persons. If the persons speak at random, find the probability that A speaks before B and B speaks before C

Sol. A ~ B ~ C; $P = \dfrac{\frac{8!}{3!}}{8!}$

10. Tickets are numbered from 1 to 100. One ticket is picked up at random. Then find the probability that the ticket picked up has a number which is divisible by 5 or 8.

Sol. Let A (B) be the event that the number on the ticket is divisible by 5(8). Then

$A = \{5, 10, 15, 20, \ldots, 95, 100\}$; $B = \{8, 16, 24, 32, \ldots, 88, 96\}$

$\Rightarrow A \cap B = \{40, 80\}$; $n(A) = \dfrac{100}{5} = 20$,

$n(B) = 12$, $n(A \cap B) = 2$

The reqd. prob. $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \dfrac{20}{100} + \dfrac{12}{100} - \dfrac{2}{100} = \dfrac{3}{10}$

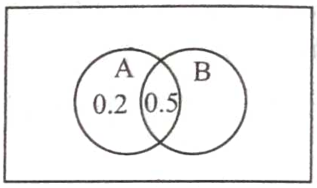
11. If $P(A) = 0.7$ and $P(A \cap B) = 0.5$ find

(*i*) $P(A - \bar{B})$

(*ii*) $P(\bar{A} \cup B)$

Sol. (*i*) $P(A - \bar{B}) = P(A \cap B) = 0.5$

(*ii*) $P(\bar{A} \cup B) = 1 - P(\overline{(\bar{A} \cup B)}) = 1 - \{P(A \cap \bar{B})\} = 1 - \{P(A) - P(A \cap B)\} = 1 - P(A \cap B) = 1 - 0.7 + 0.5 = 0.3 + 0.5 = 0.8$



12 From a group of 2 boys and 3 girls, two children are selected at random. Describe the events.

(*i*) A both selected children are girls.

(*ii*) B = the selected group consists of one boy and one girl.

(*iii*) C = at least one boy is selected.

Which pair (s) of events is (are) mutually exclusive?

Sol. Let B1, B2 be two boys and G1, G2, G3 be three girls. Then, the sample space associated with the random experiment is

$S = [B_1 B_2, B_1 G_1, B_1 G_2, B_1 G_3, B_2G_1, B_2G_2, B_2G_3, G_1G_2, G_1G_3, G_2G_3]$

(*i*) We have,

B = The selected children are girls = $[G_1 G_2, G_1 G_3, G_2 G_3]$

(*ii*) We have,

B = The selected group consists of one boy and one girl.

$\Rightarrow B = \{B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3\}$

(*iii*) We have,

C = At least one boy is selected = $[B_1B_2, B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3]$

$\Rightarrow A \cap B = \phi$ and $A \cap C = \phi$. So, A and B, A and C are two pairs of mutually exclusive.

Exercise-1 (Topicwise)

1. If a dice is thrown twice, then the probability of getting 1 in the first throw only is

   (a) $\frac{1}{36}$ (b) $\frac{3}{36}$
   (c) $\frac{5}{36}$ (d) $\frac{1}{6}$

2. The probability of getting a total of 5 or 6 in a single throw of 2 dice is

   (a) $\frac{1}{2}$ (b) $\frac{1}{4}$
   (c) $\frac{1}{3}$ (d) $\frac{1}{6}$

3. In a non-leap year, the probability of having 53 Tuesday or 53 Wednesday is

   (a) $\frac{1}{7}$ (b) $\frac{2}{7}$
   (c) $\frac{3}{7}$ (d) None of these

4. Three numbers are chosen from 1 to 20. Find the probability that they are not consecutive

   (a) $\frac{186}{190}$ (b) $\frac{187}{190}$
   (c) $\frac{188}{190}$ (d) $\frac{18}{{}^{20}C_3}$

5. Two dice are thrown simultaneously. What is the probability of obtaining a multiple of 2 on one of them and a multiple of 3 on the other

   (a) $\frac{5}{36}$ (b) $\frac{11}{36}$
   (c) $\frac{1}{6}$ (d) $\frac{1}{3}$

6. A coin is tossed 4 times. The probability that at least one head turns up is

   (a) $\frac{1}{16}$ (b) $\frac{2}{16}$
   (c) $\frac{14}{16}$ (d) $\frac{15}{16}$

7. While shuffling a pack of 52 playing cards, 2 are accidentally dropped. Find the probability that the missing cards to be of different colours.

   (a) $\frac{29}{52}$ (b) $\frac{1}{2}$
   (c) $\frac{26}{51}$ (d) $\frac{27}{51}$

8. If a coin be tossed n times then probability that the head comes odd times is

   (a) $\frac{1}{2}$ (b) $\frac{1}{2^n}$
   (c) $\frac{1}{2^{n-1}}$ (d) 2^{n-1}

9. A bag contains 3 red, 7 white and 4 black balls. If three balls are drawn from the bag, then the probability that all of them are of the same colour is

   (a) $\frac{6}{71}$ (b) $\frac{7}{81}$
   (c) $\frac{10}{91}$ (d) $\frac{7}{91}$

10. If seven persons are to be seated in a row. Then, the probability that two particular persons sit next to each other is

    (a) $\frac{1}{3}$ (b) $\frac{1}{6}$
    (c) $\frac{2}{7}$ (d) $\frac{1}{2}$

11. If with out repetition of the numbers, four-digit numbers are formed with the numbers 0, 2, 3 and 5 then the probability of such a number divisible by 5 is

    (a) $\frac{1}{5}$ (b) $\frac{4}{5}$
    (c) $\frac{1}{30}$ (d) $\frac{5}{9}$

12. The letter of the word 'ASSASSIN' are written down at random in a row. The probability that no two S occur together is

    (a) $\frac{1}{35}$ (b) $\frac{1}{14}$
    (c) $\frac{4}{13}$ (d) $\frac{2}{35}$

13. A bag contains 4 white and 3 red balls. Two draws of one ball each are made without replacement. Then the probability that both the balls are red is

    (a) $\frac{1}{7}$ (b) $\frac{2}{7}$
    (c) $\frac{3}{7}$ (d) $\frac{4}{7}$

14. If A and B are mutually exclusive events, then

    (a) $P(A) \le P(\overline{B})$ (b) $P(A) \ge P(\overline{B})$
    (c) $P(A) < P(\overline{B})$ (d) None of these

15. If a committee of 3 is to be chosen from a group of 38 people of which you are a member. What is the probability that you will be on the committee

(a) $\binom{38}{3}$ (b) $\binom{37}{2}$

(c) $\binom{37}{2}\Big/\binom{38}{3}$ (d) $\frac{666}{8436}$

16. If $P(A \cup B) = P(A \cap B)$ for any two events A and B, then

(a) $P(A) = P(B)$ (b) $P(A) > P(B)$

(c) $P(A) < P(B)$ (d) None of these

17. If A and B are two events such that $P(A) = 0.4$, $P(A + B) = 0.7$ and $P(AB) = 0.2$, then $P(B) =$

(a) 0.1 (b) 0.3

(c) 0.5 (d) 0.4

18. A card is drawn at random from a pack of cards. The probability of this card being a red or a queen is

(a) $\frac{1}{13}$ (b) $\frac{1}{26}$

(c) $\frac{1}{2}$ (d) $\frac{7}{13}$

19. If A and B are two events such that $P(A \cup B) + P(A \cap B) = \frac{7}{8}$ and $P(A) = 2P(B)$, then $P(A) =$

(a) $\frac{7}{12}$ (b) $\frac{7}{24}$

(c) $\frac{5}{12}$ (d) $\frac{17}{24}$

20. The probability that at least one of A and B occurs is 0.9. If A and B occur simultaneously with probability 0.3, then $P(A') + P(B') =$

(a) 0.9 (b) 0.2

(c) 0.8 (d) 1.2

21. Let A and B are two events and $P(A') = 0.3$ $P(B') = 0.4$, $P(A \cap B') = 0.5$, then $P(A \cup B')$ is

(a) 0.5 (b) 0.8

(c) 1 (d) 0.1

22. If 6 boys and 6 girls sit in a row at random, then the probability that all the girls sit together is

(a) $\frac{12}{132}$ (b) $\frac{12}{431}$

(c) $\frac{1}{432}$ (d) None of these

23. If a single letter is selected at random from the wo 'PROBABILITY', then the probability that it is a vowel

(a) $\frac{1}{3}$ (b) $\frac{4}{11}$

(c) $\frac{2}{11}$ (d) $\frac{3}{11}$

24. If the probability for A to fair in an examination is 0.2 and t for B is 0.3, then the probability that either A or B fails is

(a) > 0.5 (b) 0.5

(c) ≤ 0.5 (d) 0

25. The probability that at least one of the events A and B occu is 0.6. If A and B occur simultaneously with probability 0. then $P(\overline{A}) + P(\overline{B})$ is equal to

(a) 0.4 (b) 0.8

(c) 1.2 (d) 1.6

26. If M and N are any two events, the probability that at lea one of them occurs is

(a) $P(M) + P(N) - 2P(M \cap N)$

(b) $P(M) + P(N) - P(M \cap N)$

(c) $P(M) + P(N) + P(M \cap N)$

(d) $P(M) + P(N) + 2P(M \cap N)$

27. A 9 digit number using the digits 1, 2, 3, 4, 5, 6, 7, 8 and is written randomly without repetetion. The probability tha the number will be divisible by 9 is:

(a) 1/9 (b) 1/2

(c) 1 (d) $9!/9^9$

28. Two dies are rolled simultaneously. The probability that the sum of the two numbers on the top faces will be atleast 10 is:

(a) 1/6 (b) 1/12

(c) 1/18 (d) 1/16

29. Three numbers are chosen at random without replacement from {1, 2, 3 ...., 10}. The probability that the minimum of the chosen numbers is 3 or their maximum is 7 is

(a) 1/2 (b) 1/3

(c) 1/4 (d) 11/40

30. If atleast one child in a family with 3 children is a boy then the probability that 2 of the children are boys, is

(a) 3/7 (b) 4/7

(c) 1/3 (d) 3/8

Exercise-2 (Learning Plus)

1. Two dice are thrown simultaneously. Find probability of getting :
   (i) An even number as the sum
   (ii) the sum as a prime number
   (iii) A total of at least 10
   (iv) A doublet of even number
   (v) A multiple of 2 on one dice and a multiple of 3 on the other dice
   (vi) Same number of both dice
   (vii) A multiple of 3 as the sum
2. Three dice are thrown together. Find the probability of getting a total of at least 6.
3. Tickets numbered from 1 to 20 are mixed up together and then a ticket is draw at random. What is the probability that the tricket has a number which is a multiple of 3 or 7?
4. An urn contains 9 red, 7 white and 4 black balls, if two balls are drawn at random, find the probability that:
   (i) Both the balls are red.
   (ii) One ball is white
   (iii) The balls are of the same colour
   (iv) One is white and other red.
5. Four persons are to be chosen at random from a group of 3 men, 2 women and 4 children. Find the probability of electing:
   (i) 1 man, 1 woman and 2 children
   (ii) Exactly 2 children
   (iii) 2 women
6. In a single throw of three dice, determine the probability of getting
   (i) A total of 5
   (ii) A total of al most 5
   (iii) A total of at least 5.
7. The odds in favour of an event are 3 : 5. Find the probability of occurrence of the event.
8. A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the tan digits. If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol ?
9. If an integer from 1 through 1000 is chosen at random, then find the probability that the integer is a multiple of 2 or a multiple of 9.
10. An experiment consists of rolling a die until a 2 appears.
    (i) How many elements of the sample space correspond to the events that the 2 appears on the kth roll of the die?
    (ii) How many elements of the sample space correspond to the events the 2 appears not tater than the kth roll of the die ?
11. A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find $P(G)$, where G is that a number greater than 3 occurs on a single roll of the die.
12. In a large metropolitan area , the probabilities are 0.87, 0.36, 0.30 that a family (randomly chosen for a sample survey) owns a colour television set, a black an white television set or both kinds of sets. What is the probability that a family owns either anyone or both kinds of sets ?
13. One of the four persons john, Rita, Aslam or Gurpreet will be promoted next month. Consequently, the simple space of four elementary outcomes S = {John promoted, Rita, promoted, Aslam promoted, Gurpreet promoted}. You are told that the chances of john's promotion is same as that of Gurpreet Rita's Chances of promotion are twice as likely as Johns. Aslam's chances are four times that of John.
    (i) Determine
    P(John promoted), P(Rita promoted),
    P(Aslam promoted), P(Gurpreet promoted).
    (ii) If A = {John promoted or Gurpreet promoted}, Find P(A)
14. One urn contains two black balls (Labelled B_1 and B_2) and two white ball. A second urn contains one black ball and two white balls (labelled W_1 and W_2). Suppose the following experiment is performed. One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then, a second ball is chosen at random from the same urn without replacing the first ball.
    (i) Write the sample space showing all possible outcomes.
    (ii) What is the probability that two black balls are chosen?
    (iii) What is the probability that two balls of opposite colour are chosen?
15. A bag contains 8 red and 5 white balls. Three balls are drawn at random. Find the probability that
    (i) All the three balls are white
    (ii) All the three balls are red.
    (iii) One ball is red and two balls are white.
16. If the letters of the word 'ASSASSINATION' are arranged at random. Find the probability that
    (i) Four S's come consecutively in the word.
    (ii) Two I's and two N's come together.
    (iii) All A's are not coming together.
    (iv) No. two A's are coming together.

17. A sample space consists of 9 elementary outcomes E_1, E_2, ... E_9 whose probabilities are

$P(E_1) = P(E_2) = 0.08$, $P(E_3) = P(E_4) = P(E_5) = 0.1$

$P(E_6) = P(E_7) = 0.2$, $P(E_8) = P(E_9) = 0.07$

Suppose $A = \{E_1, E_5, E_8\}$, $B = \{E_2, E_5, E_8, E_9\}$

(*i*) Calculate P(A), P(B) and $P(A \cap B)$.

(*ii*) Using the addition law od probability, calculate $P(A \cup B)$.

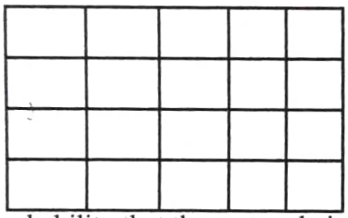
(*iii*) List the composition of the event $A \cup B$ and calculate $P(A \cup B)$ by adding the probabilities of the elementary outcomes.

(*iv*) Calculate $P(\overline{B})$ from $P(\overline{B})$, also calculate $P(\overline{B})$ directly from the elementary outcomes of $\overline{B}$.

18. If the letters of the word BANANA are arranged randomly, then find the probability that the word thus formed does not contain the pattern BAN.

19. Nine cards are labelled 0, 1, 2, 3, 4, 5, 6, 7, 8. Two cards are drawn at random and put on a table in a sucessive order, and then the resulting number is read say 07 (seven), 42 (fourty two) and so on. Find the probability that the number is even.

20. (*i*) A rectangle is randomly selected from the grid of equally spaced squares as shown.



Find the probability that the rectangle is a square.

(*ii*) Three of the sex vertices of a regular hexagon are chosen at random. Then the probability that the triangle with three vertices is equilateral is 'p' then 100p equals

21. In throwing of a pair of dice, find the probability of the event: total is 'not 8' and 'not 11'.

22. Prove that

$$P(A - B) = P(A) - P(A \cap B) = P(A \cup B) - P(B)$$
$$= P(A \cap \overline{B}) = 1 - P(\overline{A} \cup B)$$

23. If $P(A) = 0.4$, $P(B) = 0.48$ and $P(A \cap B) = 0.16$, then find the value of each of the following :

(*i*) $P(A \cup B)$

(*ii*) $P(A' \cap B)$

(*iii*) $P(A' \cap B')$

(*iv*) $P((A \cap B') \cup (A' \cap B))$

24. There are three clubs A, B, C in a town with 40, 50, 60 members respectively 10 people are members of all the three clubs, 70 members belong to only one club. A member is randomly selected. Find the probability that he has membership of exactly two clubs

25. Two dice are thrown and the sum of the numbers which come up on the dice is noted. Let us consider the following events.

A = The sum is even, B = The sum is multiple of 3, C = The sum is less than 4,

D = The sum is greater than 11

Which pairs of these events are mutually exclusive?

ANSWER KEY

CONCEPT APPLICATION

1. S = {(W, W), (W, R), (R, 1), (R, 2) (R, 3), (R, 4), (R, 5), (R, 6)}
2. S = {WB, BW, BB}
3. $A = [(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)]$, $B = [(1, 3), (2, 3), (1, 4), (2, 4), (3, 4), (4, 3)]$, $C = [(2, 4), (3, 4), (4, 2), (4, 3)]$, $D = [(1, 2), (2, 4)]$ A and D form a pair of mutually exclusive events.
4. (i) The sample space is the set of 52 cards; (ii) Required event is the set of jack, king and queen of spades and clubs.
5. $\frac{2197}{20825}$, **6.** $\frac{4}{270725}$, **7.** (i) $\frac{1}{6}$, (ii) $\frac{2}{3}$, (iii) $\frac{1}{6}$, **8.** $\frac{11}{4165}$, **9.** $=\frac{1}{38760}$,
10. (i) $A\cap\bar{B}\cap\bar{C}$, (ii) $A\cap B\cap\bar{C}$, (iii) $A\cap B\cap C$, (iv) $A\cup B\cup C$, (v) $(A\cup B)\cup(B\cap C)\cup(A\cap C)\cup(A\cap B\cap C)$, (vi) $(A\cap\bar{B}\cap\bar{C})\cup(\bar{A}\cap B\cap\bar{C})\cup(\bar{A}\cap\bar{B}\cap C)$, (vii) $(A\cap B\cap\bar{C})\cup(\bar{A}\cap B\cap C)\cup(A\cap\bar{B}\cap C)$ (viii) $\bar{A}\cap\bar{B}\cap\bar{C}=\overline{A\cup B\cup C}$, (ix) $(A\cap\bar{B}\cap\bar{C})\cup(\bar{A}\cap B\cap\bar{C})\cup(\bar{A}\cap\bar{B}\cap C)$, $\cup(A\cap B\cap\bar{C})\cup(\bar{A}\cap B\cap C)\cup(A\cap\bar{B}\cap C)$.
11. $\frac{3p+2p^2}{2}$

EXERCISE-1 (TOPICWISE)

1. (*c*)	**2.** (*b*)	**3.** (*a*)	**4.** (*b*)	**5.** (*b*)	**6.** (*d*)	**7.** (*c*)	**8.** (*a*)	**9.** (*c*)	**10.** (*c*)
11. (*d*)	**12.** (*b*)	**13.** (*a*)	**14.** (*a*)	**15.** (*c*)	**16.** (*a*)	**17.** (*c*)	**18.** (*d*)	**19.** (*a*)	**20.** (*c*)
21. (*b*)	**22.** (*c*)	**23.** (*b*)	**24.** (*c*)	**25.** (*c*)	**26.** (*b*)	**27.** (*c*)	**28.** (*a*)	**29.** (*d*)	**30.** (*a*)

EXERCISE-2 (LEARNING PLUS)

1. (i) $\frac{1}{2}$. (ii) $\frac{5}{12}$. (iii) $\frac{1}{6}$. (iv) $\frac{1}{12}$. (v) $\frac{11}{36}$. (vi) $\frac{1}{6}$. (vii) $\frac{1}{3}$. **2.** $\frac{103}{108}$. **3.** 2/5

4. (i) $\frac{18}{95}$. (ii) $\frac{91}{190}$. (iii) $\frac{63}{190}$. (iv) $\frac{63}{190}$. **5.** (i) $\frac{2}{7}$ (ii) $\frac{10}{21}$ (iii) $\frac{1}{6}$ **6.** (i) $\frac{1}{36}$ (ii) $\frac{5}{108}$ (iii) $\frac{53}{54}$

7. $\frac{3}{8}$ **8.** [0.1583] **9.** [0.556] **10.** (i) 5^{k-1} (ii) $\frac{5^k-1}{4}$ **11.** $\frac{4}{9}$ **12.** [0.93] **13.** (i) $=\frac{1}{8}$ (ii) $\frac{1}{4}$

14. (i) $\{B_1B_2, B_1W, B_2W_1, B_2W, WB_1, WB_2, BW_1, BW_2, W_1B, W_1W_2, W_2B, W_2W_1\}$ (ii) $\frac{1}{6}$ (iii) $\frac{2}{3}$

15. (i) $\frac{5}{143}$ (ii) $\frac{28}{143}$ (iii) $\frac{40}{143}$ **16.** (i) $\frac{2}{143}$ (ii) $\frac{2}{143}$ (iii) $\frac{25}{26}$ (iv) $\frac{15}{26}$

17. (i) P(A) = 0.25 (ii) $P(A\cup B) = 0.40$ (iii) 0.40 (iv) 0.68 **18.** $\frac{4}{5}$ **19.** $\frac{5}{9}$ **20.** (i) $\frac{4}{15}$ (ii) $\frac{1}{10}$.

21. $\frac{29}{36}$. **23.** (i) 0.72 (ii) 0.32 (iii) 0.28 (iv) 0.56 **24.** $\frac{5}{21}$

25. A and B are not mutually exclusive events.
C and D are mutually exclusive events.

CHAPTER

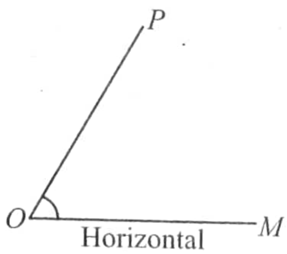
24 Heights and Distance

INTRODUCTION

This chapter deals with the applications of trigonometry in practical situations concerning measurement of heights and distances, which are otherwise not directly measurable. We need to first define certain terms and state some properties before applying the principles of trigonometry.

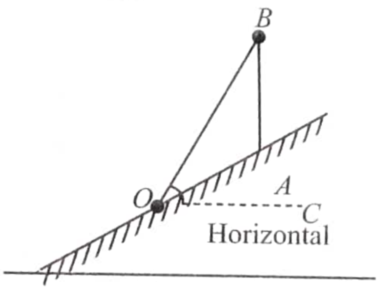
ANGLE OF ELEVATION

Consider a point P (at higher level) being observed from a point O (usually called observer) at a lower horizontal level. Draw a horizontal line OM through O, OP is called the line of sight and $\angle POM$ is called the angle of elevation of point P as seen from O.



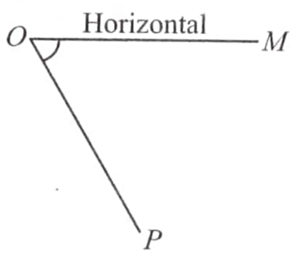
Unless otherwise stated, the height of the observer is neglected because mostly the heights and distances to be measured are very large compared to the height of the observer.

Suppose, there is a flagstaff AB mounted on a sloping level and its top B is observed from O. Then the angle $\angle AOB$ is not the angle of elevation of B because it is not measured w.r.t the horizontal at O. The correct angle of elevation is $\angle COB$.



ANGLE OF DEPRESSION

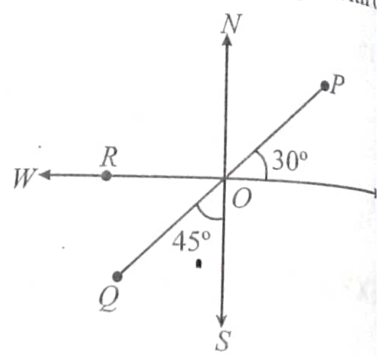
Consider a point P being observed from a point O at a higher horizontal level. $\angle POM$ here is called the angle of depression of the point P as seen from O. This is also measured w.r.t the horizontal at O.



DIRECTIONS OF A POINT

When the observer (O) and the object (P) are at the same horizontal level, to specify the direction or location of P w.r.t O, we take the help of cardinal directions North, East, West and South. Bear of a point is defined as the angle made by the line of sight with of the principal directions North, East, West or South.

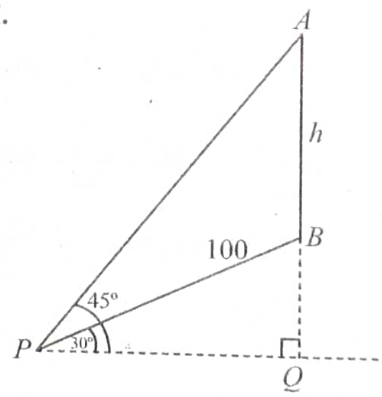
e.g., Bearing of point P can be stated as $30°$ north of east or $60°$ east of north, which can be symbolically written as $E30°N$ or $N60°E$. Bearing of Q is $W45°S$ or $S45°W$ or simply WS or SW and that of R is W.



Train Your Brain

Example 1: The angle of elevation of the top of a temple from a point on a hill of slope $\frac{1}{\sqrt{3}}$ is $45°$. If the point is at a distance of 100m from the temple along the hill, find the height of the temple.

Sol.



Given $\tan\angle BPQ = \frac{1}{\sqrt{3}} \Rightarrow \angle BPQ = 30°$

$\angle APQ = 45°$ and $PB = 100$

Now, $BQ = 100 \sin 30° = 50\text{m}$

and $PQ = 100 \cos 30° = 50\sqrt{3}$ m

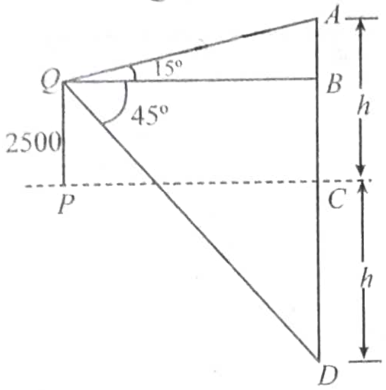
So, $\Delta APQ \Rightarrow \tan 45° = \frac{AQ}{PQ}$

$\Rightarrow$ $h + 50 = 50\sqrt{3} \Rightarrow h = 50(\sqrt{3}-1) = 36.6\text{m}.$

Example 2: The angle of elevation of a stationary cloud from a point 2500 metres above a lake is $15°$ and the angle of depression of its reflection in the lake is 45°. What is the height of the cloud above the lake level?

Sol. $\tan 15° = \dfrac{AB}{BQ}$

$\Rightarrow \tan(45° - 30°) = \dfrac{h - 2500}{BQ}$ ... (i)



$\Rightarrow \dfrac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \dfrac{h - 2500}{BQ}$ $\tan 45° = \dfrac{BD}{BQ}$

$\Rightarrow 1 = \dfrac{h + 2500}{BQ}$ ... (ii)

(i) ÷ (ii) $\Rightarrow \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = \dfrac{h-2500}{h+2500} \Rightarrow h = 2500\sqrt{3}$ m

Example 3: An observer on the top of a cliff 200 m above the sea level observes the angle of depression of two objects on the opposite sides of the cliff. to be 45° and 30°. Find the distance between the objects.

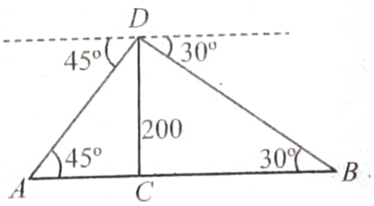
Sol.

$AC = 200 \cot 45°$

$= 200\ m$

$BC = 200 \cot 30°$

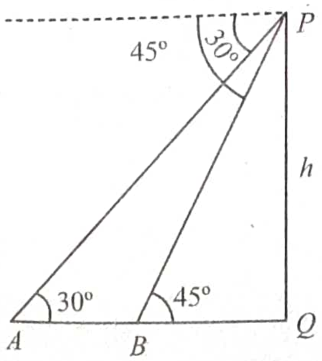
$= 200\sqrt{3}\ m$



$\Rightarrow AB = 200(\sqrt{3} + 1) = 546.4\ m.$

Example 4: The shadow of a tower standing on a level plane is found to be 60 m longer when the sun's elevation is 30° then when it is 45°. Find the height of tower.

Sol.



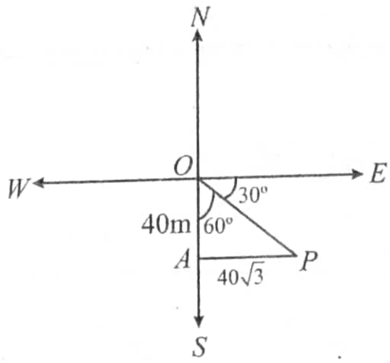
$AB = 60\ m = AQ - BQ$

$= h \cot 30° - h \cot 45°$

$\Rightarrow h = \dfrac{60}{\sqrt{3}-1} m = \dfrac{60(\sqrt{3}+1)}{3-1} = 81.96 m$

Example 5: Find the distance and bearing of a point situated 40 m south and $40\sqrt{3}$ m east of the reference point.

Sol.



$\tan\angle POA = \dfrac{40\sqrt{3}}{40} = \sqrt{3}$

$\Rightarrow \angle POA = 60°$

$\therefore$ Bearing of P can be written as $E30°S$ or $S60°E$ or 30° south of east 'or' 60^0 east of south.

Distance $OP = \sqrt{(40) + (40\sqrt{3})^2} = 80m$

Concept Application

1. Two men are on the opposite side of a tower. They measure the angles of elevation of the top of the tower 45° and 30° respectively. If the height of the tower is 40 *m*, find the distance between the men

2. A ladder 5 *metre* long leans against a vertical wall. The bottom of the ladder is 3 *metre* from the wall. If the bottom of the ladder is pulled 1 *metre* farther from the wall, how much does the top of the ladder slide down the wall

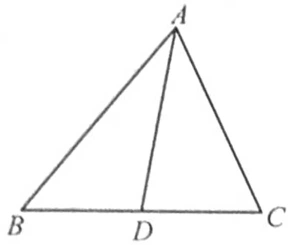
3. The angle of elevation of the top of a pillar at any point *A* on the ground is 15°. On walking 40 *metres* towards the pillar, the angle become 30°. The height of the pillar is

4. The shadow of a tower standing on a level ground is found to be 60 *m* longer when the sun's altitude is 30° than when it is 45°. The height of the tower is

APOLLONIUS THEOREM

If AD is the median of a ΔABC, then

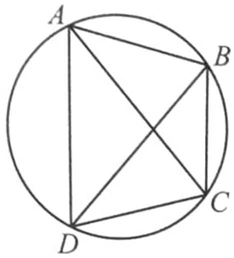
$$AB^2 + AC^2 = 2(AD^2 + BD^2) \text{ or } 2(AD^2 + DC^2)$$



PTOLEMY'S THEOREM

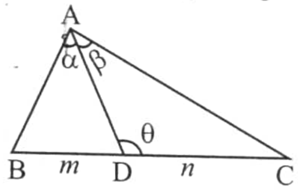
In a cyclic quadrilateral $ABCD$

$$AC.BD = AB.CD + AD.BC$$



m-n THEOREM

If D be the point on the side BC of a ΔABC which divides the side BC in the ratio $m : n$, then according the figure



we have,

(i) $(m + n) \cot \theta = m \cot \alpha - n \cot \beta$

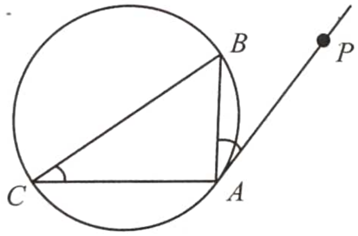
(ii) $(m + n) \cot \theta = n \cot B - m \cot C$

PROPERTIES OF CIRCLES

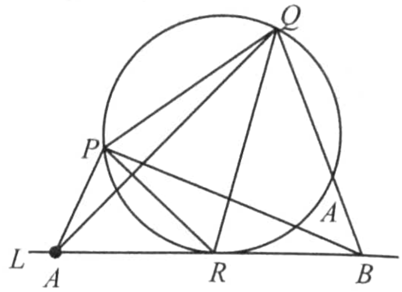
(i) If AB subtends equal angles at two points P and Q, the points A, B, P and Q are concyclic. ($\because$ angles on the same segment of a circle are equal)

(ii) Angle subtended by a chord at the center is twice the angle subtended at any point on the circumference.

(iii) Let AP be the tangent at a point A on the circumference of a circle passing through A, B and C. Then $\angle BAP = \angle ACB$.



(iv) A line segment PQ subtends various angles at different points on another line L. Obviously, at some point (say R) PQ will subtend the greatest angle also. Then, the circle passing through P, Q, R touches the line L at R.

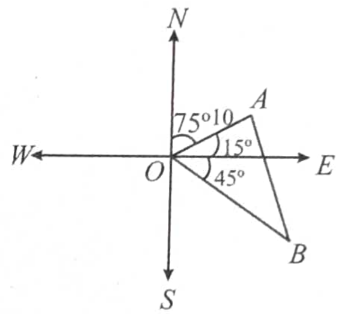


Three dimensional problems in heights and distances are generally solved by decomposing the picture into two 2-dimensional views [top (or plane) view and front view] and then principles of plane trigonometry are used.

Train Your Brain

Example 6: Two cars leave a place at the same time. One travels 10 km in the direction $N75°E$ and the other 20 km in the south-east direction. What is the final distance between the cars?

Sol.



$$\cos\angle AOB = \frac{OA^2 + OB^2 - AB^2}{2(OA)(OB)}$$

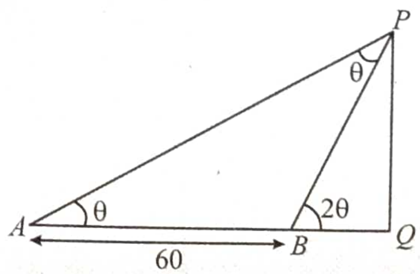
$\Rightarrow$ $AB^2 = (10)^2 + (20)^2 - 2(10)(20) \cos 60°$

$= 100 + 400 - 200 = 300$

$\Rightarrow$ $AB = 10\sqrt{3}$ km

Example 7: A man observes the angle of elevation of the top of a tower to be $\theta°$. He moves a distance 60 m towards the tower and finds that the elevation has doubled. Find the value of $\sin 2\theta$. If height of the tower is 50 m.

Sol.



In ΔPAB, $\angle ABP = \pi - 2\theta$ and $\angle APB = \theta$ (exterior angle is the sum of two opposite interior angles)

Applying sine rule,

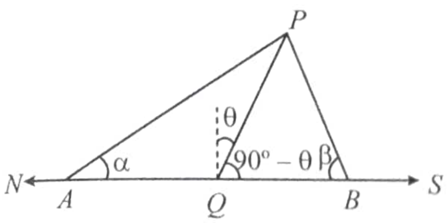
$$\frac{\sin\theta}{AB} = \frac{\sin(\pi - 2\theta)}{AP}$$

$$\Rightarrow \frac{\sin\theta}{60} = \frac{\sin 2\theta}{PQ/\sin\theta} = \frac{\sin 2\theta.\sin\theta}{50}$$

$$\Rightarrow \sin 2\theta = \frac{5}{6}$$

Example 8: A chimney leans towards the south. At equal distances due north and south of it the angles of elevation of the top of chimney are α and β respectively. Find the inclination of the chimney to the vertical.

Sol.



We require the value of θ.

Applying $m - n$ theorem to ΔPAB, We have

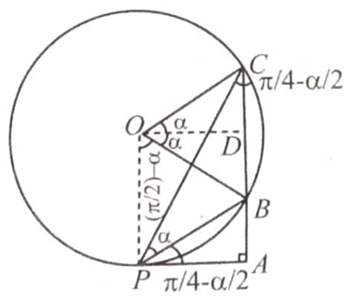
$(1 + 1)\cot(90° - \theta) = 1.\cot\alpha - 1.\cot\beta$

$$\Rightarrow \tan\theta = \frac{\cot\alpha - \cot\beta}{2}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\cot\alpha - \cot\beta}{2}\right)$$

Example 9: A man is moving towards a building on which a flagstaff is mounted. The flagstaff subtends its maximum angle α at the man's eye when he is at a distance 'c' from the base of building. Obtain the height of building and flagstaff in terms of c and α.

Sol.



Let AB be the building and BC be the flagstaff.

Since angle subtended by BC is maximum at P, a circle drawn through P, B and C touches PA at P. The perpendicular bisector of chord BC and a line perpendicular to PA at P intersect at the center O of the circle. Consider chord BC.

$\because$ $\angle BPC = \alpha \Rightarrow \angle BOC = 2\alpha$

$\Rightarrow$ $\angle COD = \alpha$

$\therefore$ Height of flagstaff $= BC = 2CD$

$= 2OD \tan\angle COD$

$= 2PA \tan\alpha = 2c \tan\alpha$

Now consider chord PB.

$\because$ $\angle POB = \pi/2 - \alpha$

$\Rightarrow$ $\angle PCB = \pi/4 - \alpha/2$

$\Rightarrow$ $\angle BPA = \pi/4 - \alpha/2$ (alternate segment theorem)

$\therefore$ Height of building $= AB = AP\tan\angle BPA = C\tan(\pi/4 - \alpha/2)$

Concept Application

5. A vertical lamp post stands at the mid-point of the edge BC of a park in the shape of a triangle. The angle of elevation of the top of the lamp post from A is 30°. If the sides AB, BC and CA respectively are 40m, $20\sqrt{6}$ m and 20m, obtain the height of lamp post.

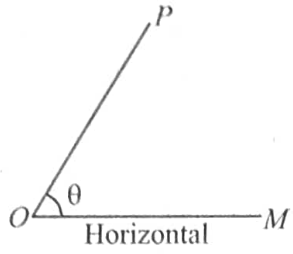
6. A rocket of height h metres is fired vertically upwards. Its velocity at time t seconds is $(2t + 3)$ metres/second. If the angle of elevation of the top of the rocket from a point on the ground after 1 second of firing is $\pi/6$ and after 3 seconds it is $\pi/3$ then the distance of the point from the rocket is

7. A piece of paper in the shape of a sector of a circle of radius 10 cm and of angle 216° just covers the lateral surface of a right circular cone of vertical angle 2θ. Then $\sin\theta$ is

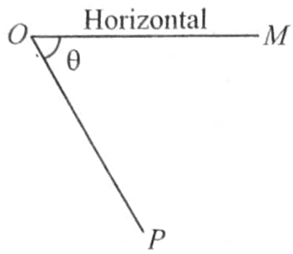
8. A man standing between two vertical posts finds that the angle subtended at his eyes by the tops of the posts is a right angle. If the heights of the two posts are two times and four times the height of the man, and the distance between them is equal to the length of the longer post, then the ratio of the distances of the man from the shorter and the longer post is

Short Notes

Angle of Elevation

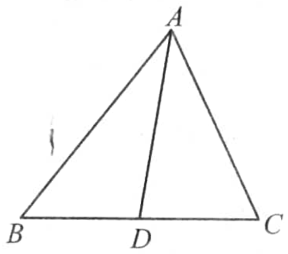


Angle of Depression



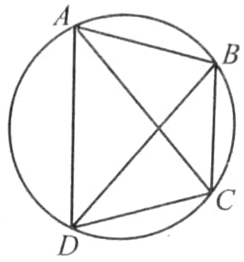
Apollonius Theorem

$AB^2 + AC^2 = 2(AD^2 + BD^2)$ or $2(AD^2 + DC^2)$

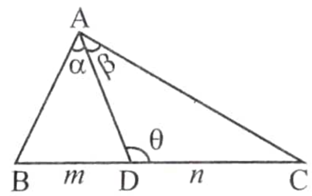


Ptolemy's Theorem

$AC.BD = AB.CD + AD.BC$



m-n Theorem



(*i*) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(*ii*) $(m+n)\cot\theta = n\cot B - m\cot C$

Properties of Circles

- If AB subtends equal angles at two points P and Q, the point A, B, P and Q are concyclic. ($\because$ Angles on the same segmen of a circle are equal)
- Angle subtended by a chord at the center is twice the angl subtended at any point on the circumference.
- Let AP be the tangent at a point A on the circumference o a circle passing through A, B and C. Then $\angle BAP = \angle ACB$.

Solved Examples

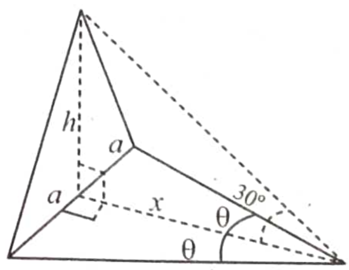
1. An isosceles triangle of wood is placed in a vertical plane, vertex upwards and faces the sun. If $2a$ be the base of the triangle, h its height and 30° the altitude of the sun, then the tangent of the angle at the apex of the shadow is

(*a*) $\dfrac{2ah\sqrt{3}}{3h^2-a^2}$ (*b*) $\dfrac{2ah\sqrt{3}}{3h^2+a^2}$

(*c*) $\dfrac{ah\sqrt{3}}{h^2-a^2}$ (*d*) None of these

Sol. (*a*) $x = h\cot 30° = h\sqrt{3}$

$$\tan\theta = \frac{a}{x} = \frac{a}{h\sqrt{3}}$$



$$\Rightarrow \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2a}{h\sqrt{3}}\cdot\frac{3h^2}{3h^2-a^2} = \frac{2ah\sqrt{3}}{3h^2-a^2}$$

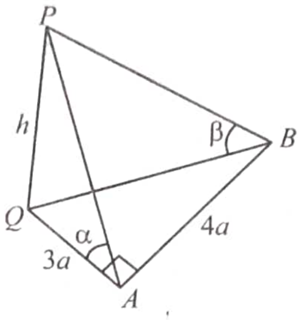
2. A person standing at the foot of a tower walks a distance 3a away from the tower and observes that the angle of elevation of the top of the tower is α. He then walks a distance 4a

perpendicular to the previous direction and observes the angle of elevation to be β. The height of the tower is

(a) $3a \tan \alpha$ (b) $5a \tan \beta$

(c) $4a \tan \beta$ (d) $7a \tan \beta$

Sol. (a,b)



$QB^2 = QA^2 + AB^2$

$\Rightarrow (h \cot\beta)^2 = (3a)^2 + (4a)^2$

$\Rightarrow h^2 = 25a^2\tan^2\beta$

$\Rightarrow h = 5a \tan\beta$

Also, $3a = h \cot\alpha$

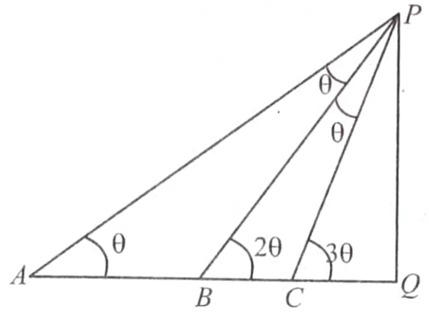
$\Rightarrow h = 3a \tan\alpha$

3. A tower subtends angle θ, 2θ and 3θ at 3 points A, B, C respectively, lying on a horizontal line through the foot of the tower then the ratio AB/BC equals to

(a) $\dfrac{\sin 3\theta}{\sin\theta}$ (b) $\dfrac{\sin\theta}{\sin 3\theta}$

(c) $\dfrac{\cos 3\theta}{\cos\theta}$ (d) $\dfrac{\tan\theta}{\tan 3\theta}$

Sol. (a)



$$\frac{AB}{BC} = \frac{BP}{BC} = \frac{\sin(180° - 3\theta)}{\sin\theta}$$

(From isosceles triangle ABP and sine rule in ΔBCP)

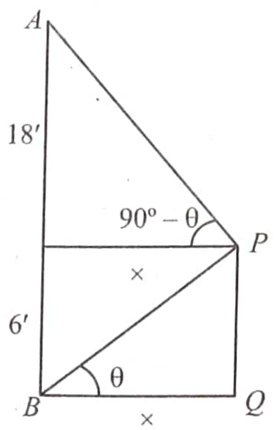
$$\Rightarrow \frac{AB}{BC} = \frac{\sin 3\theta}{\sin\theta}$$

4. A 6-ft-tall man finds that the angle of elevation of the top of a 24-ft-high pillar and the angle of depression of its base are complementary angles. The distance of the man from the pillar is

(a) $2\sqrt{3}$ ft (b) $8\sqrt{3}$ ft

(c) $6\sqrt{3}$ ft (d) None of these

Sol. (c)



Here, $\tan\theta = \dfrac{6}{x}$, $\tan(90° - \theta) = \dfrac{18}{x}$

$\therefore \dfrac{6}{x} \cdot \dfrac{18}{x} = 1$

or $x^2 = 6 \times 18$

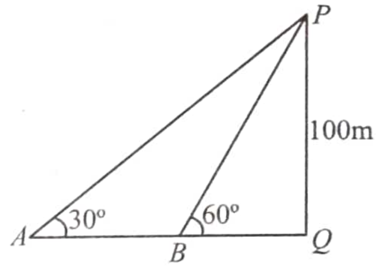
or $x = 6\sqrt{3}$

5. A man from the top of a 100 metres high tower sees a car moving towards the tower at an angle of depression of 30°. After some time, the angle of depression becomes 60°. The distance (in metres) travelled by the car during this time is

(a) $100\sqrt{3}$ (b) $200\sqrt{3/3}$

(c) $100\sqrt{3/3}$ (d) $200\sqrt{3}$

Sol. (b)



$AB = 100\cot 30° - 100\cot 60°$

$$100\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = \frac{200}{\sqrt{3}} = \frac{200\sqrt{3}}{3}$$

6. The angle of elevation of the top of a tower observed from each of the three points A, B, C on the ground, forming an equilateral triangle of side length a, is the same angle α. The height of the tower is

(a) $a \sin\alpha$ (b) $\dfrac{a}{\sqrt{3}}\sin\alpha$

(c) $a \tan\alpha$ (d) $\dfrac{a}{\sqrt{3}}\tan\alpha$

Sol. (d) Obviously foot of the tower is the circumcentre of the triangle ABC. The circum radius of the circumcircle of $\Delta ABC = \dfrac{a}{2\sin 60°} = \dfrac{a}{\sqrt{3}}$

Hence height of the tower $= \dfrac{a}{\sqrt{3}}\tan\alpha$

7. Three vertical poles of heights h_1, h_2 and h_3 at the vertices A, B and C of a ΔABC subtend angles α, β and γ respectively at the circumcentre of the triangle. If $\cot\alpha$, $\cot\beta$ and $\cot\gamma$ are in AP then h_1, h_2, h_3 are in

(a) AP (b) GP

(c) HP (d) None of these

Sol. (c) $\dfrac{R}{h_1} = \cot\alpha$, etc., where R is the circumradius

$\cot\alpha$, $\cot\beta$, $\cot\gamma$ are in $A.P.$

$\Rightarrow \dfrac{1}{h_1}, \dfrac{1}{h_2}, \dfrac{1}{h_3}$ are in $A.P.$

$\Rightarrow h_1, h_2, h_3$ are in $H.P.$

8. The angle of elevation of the top of a hill from each of the vertices A, B, C of a horizontal triangle is α. The height of the hill is

(*a*) $b \tan\alpha . \operatorname{cosec}\beta$ (*b*) $\frac{1}{2}$ a $\tan\alpha \operatorname{cosec} A$

(*c*) $\frac{1}{2}$ $c \tan\alpha . \operatorname{cosec} C$ (*d*) None of these

Sol. (*b*) The distance of the foot from each vertex = $h\cot\alpha$

$\therefore$ the foot is at the circumcentre of the triangle.

So, $R = h\cot\alpha$

$\therefore\ h = R\tan\alpha = \dfrac{a}{2\sin A}\tan\alpha$

9. If the angular elevations of the tops of two spires which appear in a straight line is α and the angular depression of their reflections in a lake, h feet below the point of observation are β and γ, show that the distance between the spires is $2h\cos^2\alpha\sin(\gamma-\beta)\operatorname{cosec}(\beta-\alpha)\operatorname{cosec}(\gamma-\alpha)$ ft where $\gamma > \beta$.

Sol. Let P and Q be the tops of two spires, P' and Q' be their reflections. From question, $OA = h$

Let $BP = BP' = h_1$

$CQ = CQ' = h_2$

Let the distance between the spires be x.

We have to find x.

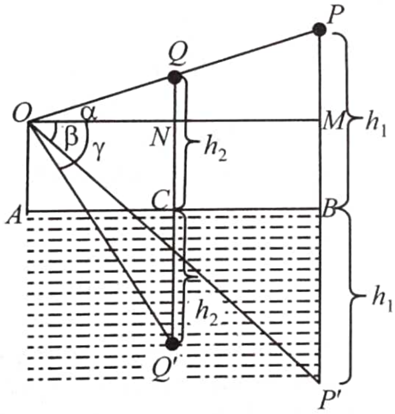
Clearly $x = MN = OM - ON$

From right angled triangle OMP'

$\tan\beta = \dfrac{P'M}{OM} = \dfrac{h_1 + h}{OM}$ or,

$OM\tan\beta = h_1 + h = (h + PM) + h$

$[\because h_1 = BP = BM + PM = h + PM]$



or, $OM\tan\beta = 2h + PM = 2h + OM\tan\alpha$

$\left[\because \text{From triangle PMO, } \tan\alpha = \dfrac{PM}{OM}\right]$

or, $OM(\tan\beta - \tan\alpha) = 2h$

$\Rightarrow\quad OM = \dfrac{2h}{\tan\beta - \tan\alpha}$ ... (i)

Similarly $ON = \dfrac{2h}{\tan\gamma - \tan\alpha}$

Hence $x = OM - ON$

$= \dfrac{2h}{\tan\beta - \tan\alpha} - \dfrac{2h}{\tan\gamma - \tan\alpha}$

$= 2h\left[\dfrac{1}{\tan\beta - \tan\alpha} - \dfrac{1}{\tan\gamma - \tan\alpha}\right]$

$= 2h\left[\dfrac{\tan\gamma - \tan\alpha - \tan\beta + \tan\alpha}{(\tan\beta - \tan\alpha)(\tan\gamma - \tan\alpha)}\right]$

$= 2h\left[\dfrac{\tan\gamma - \tan\beta}{(\tan\beta - \tan\alpha)(\tan\gamma - \tan\alpha)}\right]$

$= 2h\left[\dfrac{\sin(\gamma-\beta)\cos\beta\cos\alpha\cos\gamma\cos\alpha}{\cos\gamma.\cos\beta\sin(\beta-\alpha)\sin(\gamma-\alpha)}\right]$

$= 2h\cos^2\alpha\sin(\gamma-\beta)\operatorname{cosec}(\beta-\alpha)\operatorname{cosec}(\gamma-\alpha)$ ft.

10. A pole stands vertically on the center of a square. Whe is the elevation of the sun its shadow just reaches the sid the square and is at a distance x and y from the ends of side. Show that the height of the pole is $\sqrt{\dfrac{x^2+y^2}{2}}$ tan

Sol. Let O be the center of the square, OP the pole. Shadow the pole OP is OQ. From question $BQ = y$ and $CQ = x$.

Then, $BC = x + y$

Let $OR \perp BC$

$\therefore OR = \dfrac{x+y}{2}$ and $BR = \dfrac{x+y}{2}$

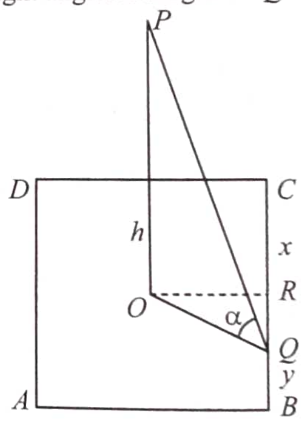
$RQ = \dfrac{x+y}{2} - y = \dfrac{x-y}{2}$

Let h be the height of the pole.

From right angled triangle POQ,

$\tan\alpha = \dfrac{h}{OQ}\quad \therefore OQ = h\cot\alpha$

Now, from right angled triangle ORQ.



$OQ^2 = OR^2 + RQ^2$

or, $h^2\cot^2\alpha = \left(\dfrac{x+y}{2}\right)^2 + \left(\dfrac{x-y}{2}\right)^2$

or, $h^2\cot^2\alpha = \dfrac{2(x^2+y^2)}{4}$

$\therefore\ h = \sqrt{\dfrac{x^2+y^2}{2}}.\tan\alpha$

11. A man observes a tower AB of height h from a point P on the ground. He moves a distance d towards the foot of the tower and finds that the angle of elevation has doubled. He further moves a distance $\frac{3}{4}d$ in the same direction and finds that the angle of elevation is three times that at P. Prove that $36h^2 = 35d^2$.

Sol. Let $AB = h$, $PQ = d$ and $QR = \frac{3}{4}d$.

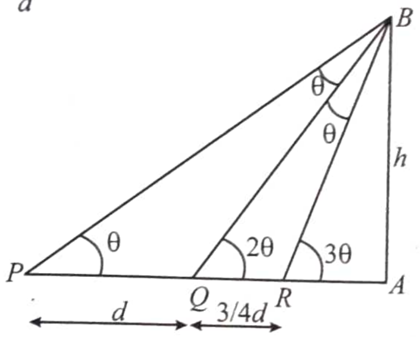
$\because \angle BPQ = \angle PBQ = \theta$

$\therefore PQ = QB = d$

[Here angles are given, hence we select right angled triangles].

From right angled triangle BAQ.

$\sin 2\theta = \frac{h}{d}$ $\therefore h = d\sin 2\theta$



or, $h = 2d\sin\theta\cos\theta$ ...(i)

Now, applying sine rule in triangle BQR,

$$\frac{\frac{3}{4}d}{\sin\theta} = \frac{d}{\sin(180° - 3\theta)}$$

or, $$\frac{3}{4\sin\theta} = \frac{1}{3\sin\theta - 4\sin^3\theta}$$

or, $9 - 12\sin^2\theta = 4$ $[\because \sin\theta \neq 0]$

or, $\sin^2\theta = \frac{5}{12}$ $\therefore \cos^2\theta = \frac{7}{12}$

From (i), we have $h^2 = 4d^2\sin^2\theta\cos^2\theta$

or, $h^2 = 4d^2 \cdot \frac{5}{12} \cdot \frac{7}{12}$ or, $36h^2 = 35d^2$

12. A man observes that when he moves up a distance c meters on a slope, the angle of depression of a point on the horizontal plane from the base of the slope is 30°; and when moves up further a distance c metres, then angle of depression of the point is 45°. Obtain the angle of inclination of the slope with the horizontal.

Sol. Let the slope be OB, making an angle θ with the horizontal OQ, i.e. $\angle BOQ = \theta$.

Let P be the point on the horizontal plane through the base of the slope.

From question $OA = c$ and $AB = c$.

Also $\angle XAP = 30° = \angle APO$

and $\angle YBP = 45° = \angle BPO$.

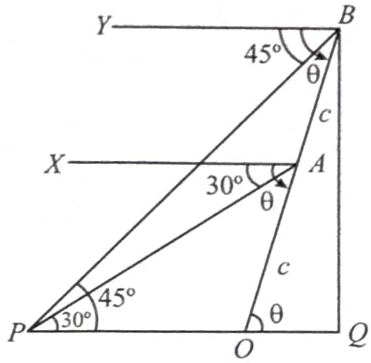
Clearly, $\angle XAO = \angle AOQ = \theta$

$\therefore \angle PAO = \theta - 30°$

and $\angle YBO = \angle BOQ = \theta$ $\therefore \angle PBO = \theta - 45°$

Suppose $OP = x$. We have to find θ.

Note: since given angles are not contained in right angled triangles we will use sine rule.



Now, applying sine rule in triangle POA.

$$\frac{x}{\sin(\theta - 30°)} = \frac{c}{\sin 30°}$$

$$\therefore x = \frac{c\sin(\theta - 30°)}{\sin 30°} \quad \text{... (i)}$$

Applying sine rule in triangle POB,

$$\frac{x}{\sin(\theta - 45°)} = \frac{2c}{\sin 45°}$$

$$\therefore x = \frac{2c\sin(\theta - 45°)}{\sin 45°} \quad \text{... (ii)}$$

From (i) and (ii), we get

$$\frac{c\sin(\theta - 30°)}{\sin 30°} = \frac{2c\sin(\theta - 45°)}{\sin 45°}$$

or, $$2\left(\sin\theta \cdot \frac{\sqrt{3}}{2} - \cos\theta \cdot \frac{1}{2}\right)$$

$$= \sqrt{2}.2\left(\sin\theta \cdot \frac{1}{\sqrt{2}} - \cos\theta \frac{1}{\sqrt{2}}\right)$$

$$\tan\theta = \frac{1}{2 - \sqrt{3}} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2 - \sqrt{3}}\right)$$

13. A tower is observed from two stations A and B, where B is east of A at a distance 100 metres. The tower is due north of and due north west of B. The angle of elevation of the tower from A and B are complementary. Find the height of the tower.

Sol. Let OP be the tower of height h. A is a point due south of the tower and the angle of elevation of the tower at B will be $(90° - \theta)$ (complementary of θ).

i.e. $\angle PAO = \theta$; $\angle PBO = 90° - \theta$

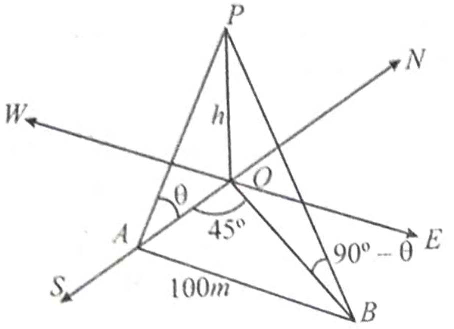
From right angled triangle POA.

$$\tan\theta = \frac{h}{OA} \quad \text{... (i)}$$

From right angle triangle POB,

$$\tan(90° - \theta) = \frac{h}{OB} \text{ or, } \tan\theta = \frac{OB}{h} \quad \text{... (ii)}$$

From (i) and (ii), we have



$$\frac{h}{OA} = \frac{OB}{h} \quad \therefore h = \sqrt{OA.OB} \quad \text{... (iii)}$$

Since BO is north-west, OB will be south-east

$\therefore \angle AOB = 45° = \angle OBA$ $\quad [\because \angle OAB = 90°]$

$\Rightarrow OA = AB = 100$ m

Now, From right angled triangle OAB,

$OB^2 = OA^2 + AB^2 = 100^2 + 100^2$

$\therefore OB = 100\sqrt{2}$ m $\quad$ From (iii)

$$h = \sqrt{100.100\sqrt{2}} = 100(2)^{1/4} \text{ metres}$$

14. A tower stands on the edge of the circular lake $ABCD$. The foot of the tower is at D and the angle of elevation of the top at A, B, C are respectively α, β, γ. If $\angle ACB = \theta = \angle BAC$. Show that $2\cos\theta \cot\beta = \cot\alpha + \cot\gamma$

Sol. Let A, B, C be there points on the circular lake and DP be the tower of height h standing at D.

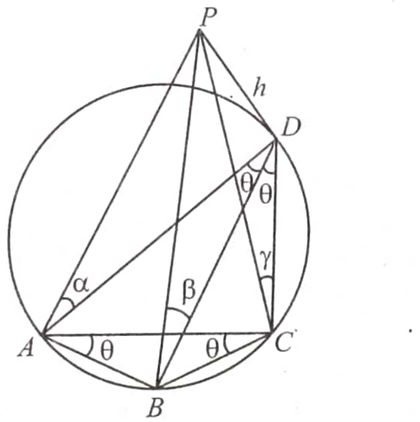
$\angle PAD = \alpha$, $\angle PBD = \beta$

and $\angle PCD = \gamma$

Also $\angle BAC = \theta = \angle ACB$

Since, angles on the same segment of a circle are equal.

$\therefore \angle ADB = \angle ACB = \theta$ $\quad$ [angles on the segment AB]



From right angled ΔPDA,

$$\tan\alpha = \frac{h}{AD} \quad \therefore AD = h\cot\alpha \quad \text{... (i)}$$

Similarly, $BD = h\cot\beta$ $\quad$ ... (ii)

and $CD = h\cot\gamma$ $\quad$ ... (iii)

In triangle ABC,

$\angle BAC = \angle ACB = \theta$

$\therefore AB = BC$

or, $AB^2 = BC^2$ $\quad$ ... (iv)

Applying cos in e rule in triangle ADB.

$$\cos\theta = \frac{AD^2 + BD^2 - AB^2}{2AD.BD}$$

or, $AB^2 = AD^2 + BD^2 - 2AD.BD\cos\theta$

Similarly from triangle BDC,

$BC^2 = BD^2 + CD^2 - 2BD.CD\cos\theta$

Hence from (iv), we have

$AD^2 + BD^2 - 2AD.BD\cos\theta = BD^2 + CD^2 - 2BD.CD\cos\theta$

or, $2BD\cos\theta\,[CD - AD] = CD^2 - AD^2$

or, $2.h\cot\beta\cos\theta\,(h\cot\gamma - h\cot\alpha] = h^2\cot^2\gamma - h^2\cot^2\alpha$

or, $2\cot\beta\cos\theta = \cot\gamma + \cot\alpha$

15. A spherical ball of diameter d subtends an angle α at a man's eye when the elevation of its centre is β. Prove that the height of the center of the ball is $1/2\, d \sin\beta$ cosec $(\alpha/2)$.

Sol. Let O be the center of the ball and E, the man's eye. EA and EB are the tangents covering the ball and hence

$\angle AEB = \alpha$

Draw $OO' \perp EO'$

$\angle OEO' = \beta$

(From question)

Let $OO' = h$

From right angled triangle $OO'E$,

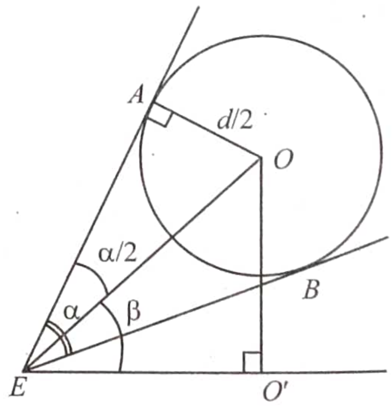
$$\sin\beta = \frac{h}{EO}$$

$\therefore h = EO\sin\beta$ $\quad$ ... (i)

From geometry, it is clear that

$$\angle AEO = \angle BEO = \frac{\alpha}{2} \quad [\angle AEB = \alpha]$$

Now, From right angled triangle OAE,



$$\sin\frac{\alpha}{2} = \frac{d/2}{OE} \text{ or, } OE = \frac{1}{2}d\,\text{cosec}\frac{\alpha}{2} \quad \text{... (ii)}$$

From (i) and (ii), $h = \frac{1}{2}d\sin\beta\,\text{cosec}\,\frac{\alpha}{2}$

16. A right circular cylindrical tower of height h and radius r stands on a horizontal plane. Let A be a point in the horizontal plane and PQR be the semi-circular edge of the top of the tower such that Q is the point in it nearest to A. The angles of elevation of the points P and Q are 45° and 60° respectively.

Show that $\frac{h}{r} = \frac{\sqrt{3}(1+\sqrt{5})}{2}$.

Sol. Let C be the center of the top of the circular tower.

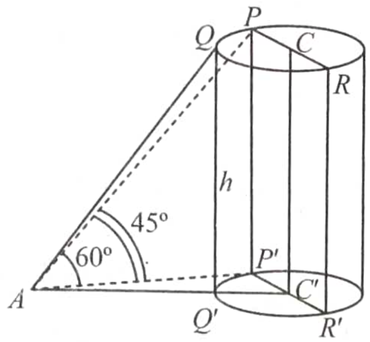
Draw QQ', PP' CC' and RR' perpendicular to the horizontal plane.

Since A is the point on the horizontal plane nearest to Q, hence A will be on the line $Q'A$, where $Q'A \perp QQ'$.

From question $QQ' = h$, $C'Q' = r$, $\angle QAQ' = 60°$ and $\angle PAP' = 45°$

From right angled triangle $QQ'A$

$$\tan 60° = \frac{h}{AQ'}, \quad \therefore AQ' = \frac{h}{\sqrt{3}} \quad \text{... (i)}$$



From right angle triangle $PP'A$,

$$\tan 45° = \frac{h}{AP'}, \quad \therefore AP' = h \quad \text{... (ii)}$$

Now, $AC' = AQ' + Q'C' = \frac{h}{\sqrt{3}} + r$

$C'P' = r$ ... (iii)

and $\angle AC'P' = 90°$ ... (iv)

From right angled triangle $AC'P'$, $AP'^2 = AC'^2 + C'P'^2$

or, $h^2 = \left(\frac{h}{\sqrt{3}} + r\right)^2 + r^2$

or, $h^2 = \frac{h^2}{3} + 2\frac{h}{\sqrt{3}}.r + r^2 + r^2$

or, $2h^2 - 2\sqrt{3}\ hr - 6r^2 = 0$ or, $h^2 - \sqrt{3}\ rh - 3r^2 = 0$

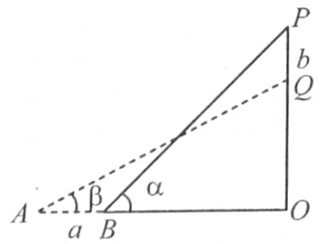
$$\therefore h = \frac{\sqrt{3}r \pm \sqrt{3r^2 - 4.1(-3r^2)}}{2}$$

or, $h = \frac{\sqrt{3}r(1 \pm \sqrt{5})}{2}$

As $h > 0 \Rightarrow \frac{h}{r} = \frac{\sqrt{3}(1+\sqrt{5})}{2}$

17. A ladder rests against a wall at an angle α to the horizontal. Its foot is pulled away from the wall through a distance a, so that it slides a distance b down the wall making an angle β with the horizontal. Show that $a = b \tan \frac{1}{2}(\alpha + \beta)$

Sol.



Let l be the length of the ladder.

$a = OA - OB = l \cos \beta - l \cos \alpha$

$b = OP - OQ = l \sin \alpha - l \sin \beta$

$$\Rightarrow \frac{a}{b} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \frac{2\sin\frac{\alpha+\beta}{2} \cdot \sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}}$$

$$\Rightarrow \quad a = b \tan\frac{\alpha+\beta}{2}$$

Exercise-1 (Topicwise)

1. The angle of elevation of the top of a tower at point on the ground is 30°. If on walking 20 *metres* toward the tower, the angle of elevation become 60°, then the height of the tower is

(a) 10 meter (b) $\frac{10}{\sqrt{3}}$ meter

(c) $10\sqrt{3}$ meter (d) $100\sqrt{3}$ meter

2. The angle of elevation of the top of the tower observed from each of the three points *A*, *B*, *C* on the ground, forming a triangle is the same angle α. If *R* is the circum-radius of the triangle *ABC*, then the height of the tower is

(a) R sin α (b) R cos α

(c) R cot α (d) R tan α

3. A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°. When he retires 40 meters from the bank, he finds the angle to be 30°. The breadth of the river is

(a) 20 m (b) 40 m

(c) 30 m (d) 60 m

4. A vertical pole consists of two parts, the lower part being one third of the whole. At a point in the horizontal plane through the base of the pole and distance 20 *meters* from it, the upper part of the pole subtends an angle whose tangent is $\frac{1}{2}$. The possible heights of the pole are

(a) 20 m and $20\sqrt{3}$ m (b) 20 m and 60 m

(c) 16 m and 48 m (d) 20 m and 16 m

5. From a 60 meter high tower angles of depression of the top and bottom of a house are α and β respectively. If the height of the house is $\frac{60\sin(\beta-\alpha)}{x}$, then $x =$

(a) sin α sin β (b) cos α cos β

(c) sin α cos β (d) cos α sin β

6. An observer on the top of a tree, finds the angle of depression of a car moving towards the tree to be 30°. After 3 minutes this angle becomes 60°. After how much more time, the car will reach the tree

(a) 4 min. (b) 4.5 min.

(c) 1.5 min. (d) 2 min.

7. A house of height 100 *metres* subtends a right angle at the window of an opposite house. If the height of the window be 64 *metres*, then the distance between the two houses is

(a) 48 m (b) 36 m

(c) 54 m (d) 72 m

8. The length of the shadow of a pole inclined at 10° vertical towards the sun is 2.05 *metres*, when the el of the sun is 38°. The length of the pole is

(a) $\frac{2.05\sin 38^o}{\sin 42^o}$ (b) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)30\text{m}$

(c) $\frac{2.05\cos 38^o}{\cos 42^o}$ (d) $\frac{2.05\cos 42^o}{\sin 38^o}$

9. From the top of a light house 60 *meters* high with its the sea level, the angle of depression of a boat is 15° distance of the boat from the foot of light house is

(a) $\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)60$ m (b) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)60$ m

(c) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$ m (d) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)30\text{m}$

10. An observer in a boat finds that the angle of elevat a tower standing on the top of a cliff is 60° and that bottom of cliff is 30°. If the height of the tower be 60 m then the height of the cliff is

(a) 30 m (b) $40\sqrt{3}$ m

(c) $20\sqrt{3}$ m (d) $40\sqrt{3}$ m

11. From a point *a metre* above a lake the angle of elevat a cloud is α and the angle of depression of its reflect β. The height of the cloud is

(a) $\frac{a\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$ metre (b) $\frac{a\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$ metre

(c) $\frac{a\sin(\beta-\alpha)}{\sin(\alpha+\beta)}$ metre (d) $\frac{2a\sin(\beta-\alpha)}{\sin(\alpha+\beta)}$

12. The shadow of a tower is found to be 60 *metre* shorter the sun's altitude changes from 30° to 60°. The height tower from the ground is approximately equal to

(a) 62 m (b) 301 m

(c) 101 m (d) 52 m

13. If the angles of elevation of two towers from the middle of the line joining their feet be 60° and 30° respectively, the ratio of their heights is

(a) 2 : 1 (b) $1:\sqrt{2}$

(c) 3 : 1 (d) $1:\sqrt{3}$

14. Some portion of a 20 meters long tree is broken by the and the top struck the ground at an angle of 30°. The he of the point where the tree is broken is

(a) 10 m (b) $(2\sqrt{3}-3)20$ m

(c) $\frac{20}{3}$m (d) $10\sqrt{3}$ m

15. The base of a cliff is circular. From the extremities of a diameter of the base the angles of elevation of the top of the cliff are 30° and 60°. If the height of the cliff be 500 *metres*, then the diameter of the base of the cliff is

(a) $1000\sqrt{3}$ m (b) $2000/\sqrt{3}$ m

(c) $1000/\sqrt{3}$ m (d) $2000\sqrt{2}$ m

16. The angle of elevation of the top of a tower from the top of a house is 60° and the angle of depression of its base is 30°. If the horizontal distance between the house and the tower be 12 *m*, then the height of the tower is

(a) $48\sqrt{3}$ m (b) $16\sqrt{3}$ m

(c) $24\sqrt{3}$ m (d) $16/\sqrt{3}$ m

17. A man whose eye level is 1.5 metres above the ground observes the angle of elevation of a tower to be 60°. If the distance of the man from the tower be 10 meters, the height of the tower is

(a) $(1.5+10\sqrt{3})$ m (b) $10\sqrt{3}$ m

(c) $\left(1.5+\dfrac{10}{\sqrt{3}}\right)$m (d) $100\sqrt{3}$m

18. A tower subtends an angle of 30° at a point distant d from the foot of the tower and on the same level as the foot of the tower. At a second point h vertically above the first; the depression of the foot of the tower is 60°. The height of the tower is

(a) $h/3$ (b) $h/3d$

(c) $3h$ (d) $\dfrac{3h}{d}$

19. A tree is broken by wind, its upper part touches the ground at a point 10 *metres* from the foot of the tree and makes an angle of 45° with the ground. The total length of tree is

(a) 15 metres (b) 20 metres

(c) $10(1+\sqrt{2})$ metres (d) $10\left(1+\dfrac{\sqrt{3}}{2}\right)$ metres

20. The angle of elevation of a stationary cloud from a point 2500 *m* above a lake is 15° and the angle of depression of its reflection In the lake is 45°. The height of cloud above the lake level is

(a) $2500\sqrt{3}$ metres (b) 2500 metres

(c) $500\sqrt{3}$ metres (d) 5000 metres

21. From an aeroplane vertically over a straight horizontally road, the angles of depression of two consecutive mile stones on opposite sides of the aeroplane are observed to be α and β, then the height in miles of aeroplane above the road is

(a) $\dfrac{\tan\alpha.\tan\beta}{\cot\alpha+\cot\beta}$ (b) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha.\tan\beta}$

(c) $\dfrac{\cot\alpha+\cot\beta}{\tan\alpha.\tan\beta}$ (d) $\dfrac{\tan\alpha.\tan\beta}{\tan\alpha+\tan\beta}$

22. A flag-post 20 m high standing on the top of a house subtends an angle whose tangent is $\dfrac{1}{6}$ at a distance 70 m from the foot of the house. The height of the house is

(a) 30 m (b) 60 m

(c) 50 m (d) 20 m

23. A balloon is coming down at the rate of 4 *m/min.* and its angle of elevation is 45° from a point on the ground which has been reduced to 30° after 10 *minutes*. Balloon will be on the ground at a distance of how many meters from the observer

(a) $20\sqrt{3}$ m (b) $20(3+\sqrt{3})$ m

(c) $10(3+\sqrt{3})$ m (d) $10(3-\sqrt{3})$m

24. *AB* is a vertical pole resting at the end *A* on the level ground. *P* is a point on the level ground such that $AP = 3\,AB$. If *C* is the mid- point of *AB* and *CB* subtends an angle β at *P*, the value of tan β is

(a) $\dfrac{18}{19}$ (b) $\dfrac{3}{19}$

(c) $\dfrac{1}{6}$ (d) $\dfrac{1}{64}$

25. Two straight roads intersect at an angle of 60°. A bus on one road is 2 km away from the intersection and a car on the other road is 3 km away from the intersection. Then the direct distance between the two vehicles is

(a) 1 km (b) $\sqrt{2}$ km

(c) 4 km (d) $\sqrt{7}$ km

26. The angle of elevation of a cliff at a point *A* on the ground and a point *B*, 100 *m* vertically at *A* are α and β respectively. The height of the cliff is

(a) $\dfrac{100\cot\alpha}{\cot\alpha-\cot\beta}$ (b) $\dfrac{100\cot\beta}{\cot\alpha-\cot\beta}$

(c) $\dfrac{100\cot\beta}{\cot\beta-\cot\alpha}$ (d) $\dfrac{100\cot\beta}{\cot\beta+\cot\alpha}$

27. The angular elevation of a tower *CD* at a point *A* due south of it is 60° and at a point *B* due west of *A*, the elevation is 30°. If $AB = 3$ *km*, the height of the tower is

(a) $2\sqrt{3}$ km (b) $2\sqrt{6}$ km

(c) $\dfrac{3\sqrt{3}}{2}$ km (d) $\dfrac{3\sqrt{6}}{4}$ km

28. The angles of elevation of the top of a tower (*a*) from the top (*b*) and bottom (*d*) at a building of height *a* are 30° and 45° respectively. If the tower and the building stand at the same level, then the height of the tower is

(a) $a\sqrt{3}$ (b) $\dfrac{a\sqrt{3}}{\sqrt{3}-1}$

(c) $\dfrac{a(3+\sqrt{3})}{2}$ (d) $a(\sqrt{3}-1)$

29. The top of a hill observed from the top and bottom of a building of height h is at the angle of elevation p and q respectively. The height of the hills is

(a) $\dfrac{h\cot q}{\cot q-\cot p}$ (b) $\dfrac{h\cot p}{\cot p-\cot q}$

(c) $\dfrac{h\tan p}{\tan p-\tan q}$ (d) $\dfrac{h\cot p}{\cot q+\cot p}$

30. For a man, the angle of elevation of the highest point of the temple situated east of him is 60°. On walking 240 metres to north, the angle of elevation is reduced to 30°, then the height of the temple is

(a) $60\sqrt{6}$ m (b) 60 m

(c) $50\sqrt{3}$ m (d) $30\sqrt{6}$ m

31. A tower subtends angles α, 2α, 3α respectively at points A, B and C, all lying on a horizontal line through the foot of the tower. Then $AB/BC =$

(a) $\dfrac{\sin 3\alpha}{\sin 2\alpha}$ (b) $1+2\cos 2\alpha$

(c) $2+\cos 3\alpha$ (d) $\dfrac{\sin 2\alpha}{\sin \alpha}$

32. Two pillars of equal height stand on either side of a roadway which is 60 metres wide. At a point in the roadway between the pillars, the elevation of the top of pillars are 60° and 30°. The height of the pillars is

(a) $15\sqrt{3}$ m (b) $\dfrac{15}{\sqrt{3}}$ m

(c) 15 m (d) 20 m

33. A ladder rests against a wall making an angle α with the horizontal. The foot of the ladder is pulled away from the wall through a distance x, so that it slides a distance y down the wall making an angle β with the horizontal. The correct relation is

(a) $x=y\tan\dfrac{\alpha+\beta}{2}$ (b) $y=x\tan\dfrac{\alpha+\beta}{2}$

(c) $x=y\tan(\alpha+\beta)$ (d) $y=x\tan(\alpha+\beta)$

Exercise-2 (Learning Plus)

1. A pole 25m long stands on the top of a tower 225m high. If θ is the angle subtended by the pole at a point on the ground which is at a distance of 2.25 km from the foot of the tower, then $\tan\theta$ is equal to

(a) 1/90 (b) 1/91

(c) 1/10 (d) 1/9

2. The angle of elevation of the top of a vertical pole when observed from each vertex of a regular hexagon is $\pi/3$. If the area of the circle circumscribing the hexagon be A metre^2 then the area of the hexagon is

(a) $\dfrac{3\sqrt{3}}{8}A$ metre^2 (b) $\dfrac{\sqrt{3}}{\pi}A$ metre^2

(c) $\dfrac{3\sqrt{3}}{4\pi}A$ metre^2 (d) $\dfrac{3\sqrt{3}}{2\pi}A$ metre^2

3. A vertical pole PO is standing at the center O of a square $ABCD$. If AC subtends an angle 90° at the top P of the pole then the angle subtended by a side of the square at P is

(a) 45° (b) 30°

(c) 60° (d) 90°

4. The angles of elevation of the top of a tower standing on a horizontal plane, from two points on a line passing through its foot at distances a and b respectively, are complementary angles. If the line joining the two points subtends an angle θ at the top of the tower, then

(a) $\sin\theta=\dfrac{a-b}{a+b}$ (b) $\tan\theta=\dfrac{2\sqrt{ab}}{a-b}$

(c) $\sin\theta=\dfrac{a+b}{a-b}$ (d) $\cos\theta=\dfrac{2\sqrt{ab}}{a-b}$

5. A vertical lamp-post, 6 m high, stands at a distance of 2 m from a wall, 4 m high. A 1.5-m-tall man starts to walk away from the wall on the other side of the wall, in line with the lamp-post. The maximum distance to which the man can walk remaining in the shadow is

(a) $\dfrac{5}{2}$ m (b) $\dfrac{3}{2}$ m

(c) 4 m (d) $\dfrac{4}{3}$ m

6. A circular ring of radius 3 cm is suspended horizontally from a point 4 cm vertically above the center by 4 strings attached at equal intervals to its circumference. If the angle between two consecutive strings be θ then $\cos\theta$ is

(a) 4/5 (b) 4/25

(c) 16/25 (d) 5/26

7. A flagstaff stands vertically on a pillar, the height of the flagstaff being double the height of the pillar. A man on the ground at a distance finds that both the pillar and the flagstaff subtend equal angles at his eyes. The ratio of the height of the pillar and the distance of the man from the pillar, is

(a) $\sqrt{3}:1$ (b) $1:3$

(c) $1:\sqrt{3}$ (d) $\sqrt{3}:2$

8. As seen from A, due west of a hill HL itself leaning east, the angle of elevation of top H of the hill is $60°$; and after walking a distance of one kilometer along an incline of $30°$ to a point B, it was seen that the hill LH was printed at right angles to AB, the height LH of the hill is

(a) $\frac{1}{\sqrt{3}}$ km (b) $\sqrt{3}$

(c) $2\sqrt{3}$ km (d) $\frac{2}{\sqrt{3}}$ km

9. ABC is a triangular park with $AB = AC = 100$ metres. A clock tower is situated at the mid point of BC. The angles of elevation of the top of the tower at A and B are $\cot^{-1} 3.2$ and $\operatorname{cosec}^{-1} 2.6$ respectively. The height of the tower is

(a) 16 m (b) 25 m

(c) 50 m (d) 75 m

10. The angles of elevation of the top of a tower from the top and bottom of a building of height 'a' are $30°$ and $45°$, respectively. If the tower and the building stand at the same level, the height of the tower is

(a) $a\sqrt{3}$ (b) $\frac{3\sqrt{3}a}{2}$

(c) $\frac{\sqrt{3}a}{\sqrt{3}-1}$ (d) $a(\sqrt{3}+1)$

11. A flag staff of 5 mt high stands on a building of 25 mt high. At an observer at a height of 30 mt. the flag staff and the building subtend equal angles. The distance of the observer from the top of the flag staff is

(a) $\frac{5\sqrt{3}}{2}$ (b) $5\sqrt{\frac{3}{2}}$

(c) $5\sqrt{\frac{2}{3}}$ (d) $5\sqrt{\frac{2}{5}}$

12. If a flag-staff of 6 metres high placed on the top of a tower throws a shadow of $2\sqrt{3}$ metres along the ground then the angle (in degrees) that the sun makes with the ground is

(a) $60°$ (b) $30°$

(c) $45°$ (d) $90°$

13. The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 100 mt. from its base is $45°$. If the angle of elevation of the top of the complete pillar at the same point is to be $60°$, then the height of the incomplete pillar is to be increased by

(a) $50\sqrt{2}$ mt (b) 100 mt

(c) $100(\sqrt{3}-1)$ mt (d) $100(\sqrt{3}+1)$ mt

14. The angles of elevation of a cliff at a point A on the ground and a point B, 100 meters vertically at A are α and β respectively. The height of the cliff is

(a) $\frac{100\cot\alpha}{\cot\alpha-\cot\beta}$ (b) $\frac{100\cot\beta}{\cot\alpha-\cot\beta}$

(c) $\frac{100\cot\beta}{\cot\beta-\cot\alpha}$ (d) $\frac{100\cot\beta}{\cot\beta+\cot\alpha}$

15. The angle of elevation of a cloud from a point h mt. above is θ and the angle of depression of its reflection in the lake is ϕ. Then the height is

(a) $\frac{h\sin(\phi-\theta)}{\sin(\phi+\theta)}$ (b) $\frac{h\sin(\phi+\theta)}{\sin(\phi-\theta)}$

(c) $\frac{h\sin(\theta+\phi)}{\sin(\theta-\phi)}$ (d) $\frac{h\sin(\phi-\theta)}{\sin(\theta-\phi)}$

16. On the level ground the angle of elevation of the top of a tower is $30°$. On moving 20 m. nearer the tower, the angle of elevation is found to be $60°$. The height of the tower is

(a) 10 m (b) 20 m

(c) $10\sqrt{3}$ m (d) $100\sqrt{3}$ m

17. Each side of a square subtends an angle of $60°$ at the top of a tower h metres high standing in the centre of the square. If a is the length of each side of the square, then

(a) $2a^2 = h^2$ (b) $2h^2 = a^2$

(c) $3a^2 = 2h^2$ (d) $3h^2 = 2a^2$.

18. From the top of light house 60 metres high with its base at the sea level, the angle of depression of a boat is $15°$. The distance of the boat from the foot of the light house is

(a) $\frac{\sqrt{3}-1}{\sqrt{3}+1}.60$ metres (b) $\frac{\sqrt{3}+1}{\sqrt{3}-1}.60$ metres

(c) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ metres (d) $\sqrt{3}+1$ metres

19. A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is $60°$, when he retires 40 metres from the bank he finds the angle to be $30°$. Then the breadth of the river is

(a) 40 m (b) 60 m

(c) 20 m (d) 30 m

20. AB is a vertical pole. The end A is on the level ground. C is the middle point of AB. P is a point on the level ground. The portion BC subtends an angle β at P. If $AP = n\,AB$, then $\tan\beta =$

(a) $\frac{n}{2n^2+1}$ (b) $\frac{n}{n^2-1}$

(c) $\frac{n}{n^2+1}$ (d) $\frac{n^2}{n+1}$

21. A tree is broken by wind, its upper part touches the ground at a point 10 metres from the foot of the tree and makes an angle of $45°$ with the ground. The entire length of the tree is

(a) 15 metres (b) 20 metres

(c) $10(1+\sqrt{2})$ metres (d) $\left(1+\frac{\sqrt{3}}{2}\right)$ metres

22. An aeroplane flying at a height of 300 metres above the ground passes vertically above another plane at an instant when the angles of elevation of the two planes from the same

point on the ground are 60° and 45° respectively. The height of the lower plane from the ground (in metres) is

(*a*) $100\sqrt{3}$ (*b*) $\frac{100}{\sqrt{3}}$

(*c*) 50 (*d*) $150(\sqrt{3}+1)$

23. A tower subtends an angle α at a point A in the plane of its base and the angle of depression of the foot of the tower at a point b ft. just above A is β. Then height of the tower is

(*a*) $b \tan\alpha \cot\beta$ (*b*) $b \cot\alpha \tan\beta$

(*c*) $b \tan\alpha \tan\beta$ (*d*) $b \cot\alpha \cot\beta$

Exercise-3 (Past Year Questions)

JEE MAIN

1. A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 min from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then, the time taken (in min) by him, from B to reach the pillar, is: **(2016)**

(*a*) 6 (*b*) 10

(*c*) 20 (*d*) 5

2. Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that $AP = 2AB$. If $\angle BPC = \beta$, then $\tan\beta$ is equal to: **(2017)**

(*a*) $\frac{4}{9}$ (*b*) $\frac{6}{7}$

(*c*) $\frac{1}{4}$ (*d*) $\frac{2}{9}$

3. PQR is a triangular park with $PQ = PR = 200$m. A T.V. tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P, Q and R are respectively 45°, 30° and 30°, then the height of the tower (in m) is: **(2018)**

(*a*) 50 (*b*) $100\sqrt{3}$

(*c*) $50\sqrt{2}$ (*d*) 100

4. Consider a triangular plot ABC with sides AB = 7 m, BC = 5m and CA = 6m. A vertical lamp-post at the mid point D of AC subtendes an angle 30° at B. The height (in m) of the lamp-post is: (2019)

(*a*) $\frac{3}{2}\sqrt{21}$ (*b*) $\frac{2}{3}\sqrt{21}$

(*c*) $2\sqrt{21}$ (*d*) $7\sqrt{21}$

5. If the angle of elevation of a cloud from apoint P which is 25 m above a lake be 30° and the angle of depression of reflection of the cloud in the lake from P be 60°, then the height of the cloud (in meters) from the surface of the lake is (2019)

(*a*) 60 (*b*) 50

(*c*) 45 (*d*) 42

6. Two vertical poles of heights, 20m and 80m stand a part on a horizontal plane. The height (in meters) of the points of intersection of the lines joining the top of each pole to the foot of the other, from this horizontal plane is: (2019)

(*a*) 12 (*b*) 15

(*c*) 16 (*d*) 18

7. Two poles standing on a horizontal ground are of heights 5m and 10 m respectively. The line joining their tops makes an angle of 15° with ground. Then the distance (**in m**) between the poles, is: (2019)

(*a*) $\frac{5}{2}(2+\sqrt{3})$ (*b*) $5(\sqrt{3}+1)$

(*c*) $5(2+\sqrt{3})$ (*d*) $10(\sqrt{3}-1)$

8. ABC is a triangular park with AB = AC = 100 metres. A vertical tower is situated at the mid point of BC. If the angles of elevation of the top of the tower at A and B are $\cot^{-1}(3\sqrt{2})$ and $\text{cosec}^{-1}(2\sqrt{2})$ respectively, then the height of the tower (in metres) is: (2019)

(*a*) $10\sqrt{5}$ (*b*) $\frac{100}{3\sqrt{3}}$

(*c*) 20 (*d*) 25

9. The angle of elevation of the top of vertical tower standing on a horizontal plane is observed to be 45° from a point A on the plane. Let B be the point 30 m vertically above the point . If the angle of elevation of the top of the tower from B be 30° then the distance (in m) of the foot of the tower from the point A is: (2019)

(*a*) $15(3-\sqrt{3})$ (*b*) $15(3+\sqrt{3})$

(*c*) $15(1+\sqrt{3})$ (*d*) $15(5-\sqrt{3})$

10. The angle of elevation of a cloud C from a point P 200 m above a still lake is 30°. If the angle of depression of the image of C in the lake from the point P is 60°, then PC (in m) is equal to: (2020)

(*a*) 400 (*b*) $400\sqrt{3}$

(*c*) 100 (*d*) $200\sqrt{3}$

11. Two vertical poles $AB = 15$ m and $CD = 10$ m are standing apart on a horizontal ground with points A and C on the ground. If P is the point of intersection of BC and AD, then the height of P (in m) above the line AC is: **(2020)**

(a) 6 (b) 20/3
(c) 10/3 (d) 5

12. The angle of elevation of the summit of a mountain from a point on the ground is 45°. After climbing up one km towards the summit at an inclination of 30° from the ground, the angle of elevation of the summit is found to be 60°. Then the height (in km) of the summit from the ground is: **(2020)**

(a) $\frac{1}{\sqrt{3}+1}$ (b) $\frac{1}{\sqrt{3}-1}$
(c) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

13. The angle of elevation of the top of a hill from a point on the horizontal plane passing through the foot of the hill is found to be 45º. After walking a distance of 80 meters towards the top, up a slope inclined at an angle of 30º to the horizontal plane, the angle of elevation of the top of the hill becomes 75º. Then the height of the hill (in meters) is: **(2020)**

(a) 20 (b) 80
(c) 40 (d) 70

14. A vertical pole fixed to horizontal ground is divided in the ratio of 3 : 7 by a mark on it with lower part shorter than upper part. If the two parts subtend equal angles at a point on the ground 18m away from the base of other pole, then the height of the pole (in m) is: (2021)

(a) $12\sqrt{15}$ (b) $12\sqrt{10}$
(c) $8\sqrt{10}$ (d) $6\sqrt{10}$

15. Two poles, AB of length a metres and CD of length $a + b$ metres $(a \neq b)$ are exected at same horizontal level with bases at B and D. If $BD = x$ and tan $\angle ACB = \frac{1}{2}$, then, **(2021)**

(a) $x^2 + 2(a + 2b)x - b(a + b) = 0$
(b) $x^2 + 2(a + 2b)x + a(a + b) = 0$
(c) $x^2 - 2ax + b(a + b) = 0$
(d) $x^2 - 2ax + a(a + b) = 0$

16. The angle of elevation of jet plane from a point A on the ground is 60° After a flight of 20 seconds at the speed of 432 km/hr, the angle of elevation changes to 30°. If the jet plane is flying at a constant height, then it's height is: **(2021)**

(a) $3600\sqrt{3}$ m (b) $1200\sqrt{3}$ m
(c) $1800\sqrt{3}$ m (d) $2400\sqrt{3}$ m

17. Two vertical poles are 150m apart and height of one is three times that of other. If from the middle point of the line joining their feet, an observer finds the angle of elevation of their tops to be complementary, then height of shorter pole (in m) is: **(2021)**

(a) 30 (b) 25 (c) $20\sqrt{3}$ (d) $25\sqrt{3}$

18. A man is observing from the top of tower a boat speeding towards the lower from a certain point A, with uniform speed. At that point, angle of depression of boat with man's eye is 30° (ignore man's height). After Sailing for 20 seconds, towards the base of tower (same level as of water), the boat has reached a point B, where angle of depression is 45°. Then time taken by boat (in seconds) from B to reach base of tower is: **(2021)**

(a) $10\left(\sqrt{3}+1\right)$ (b) $10\left(\sqrt{3}-1\right)$
(c) 10 (d) $10(\sqrt{3})$

19. Let AB and PQ be two vertical poles, 160 m apart from each other. Let C be the middle point of B and Q. which are feet of these two poles. Let and θ be the angles of elevation from C to P and A, respectively. If the height of pole PQ is twice the height of pole AB. then $\tan^2\theta$ is equal to **(2021)**

(a) $\frac{3-2\sqrt{2}}{2}$ (b) $\frac{3+\sqrt{2}}{2}$
(c) $\frac{3-2\sqrt{2}}{4}$ (d) $\frac{3-\sqrt{2}}{4}$

20. From the base of a pole of height 20 meter, the angle of elevation of the top of a tower is 60°. The pole subtends an angle 30° at the top of the tower. Then the height of the tower is: **(2022)**

(a) $15\sqrt{3}$ (b) $20\sqrt{3}$
(c) $20+10\sqrt{3}$ (d) 30

21. A tower PQ stands on a horizontal ground with base Q on the ground. The point R divides the tower in two parts such that $QR = 15$ m. If from a point A on the ground the angle of elevation of R is 60° and the part PR of the tower subtends an angle of 15° at A, then the height of the lower is: **(2022)**

(a) $5\left(2\sqrt{3}+3\right)$m (b) $5\left(\sqrt{3}+3\right)$m
(c) $10\left(\sqrt{3}+1\right)$m (d) $10\left(2\sqrt{3}+1\right)$m

22. Let a vertical tower AB of height $2h$ stands on a horizontal ground. Let from a point P on the ground a man can see upto height h of the tower with an angle of elevation 2α. When from P, he moves a distance d in the direction of AP, he can see the top B of the tower with an angle of elevation α. If $d = \sqrt{7}h$. then $\tan\alpha$ is equal to **(2022)**

(a) $\sqrt{5}-2$ (b) $\sqrt{3}-1$
(c) $\sqrt{7}-2$ (d) $\sqrt{7}-\sqrt{3}$

23. The angle of elevation of the top P of a vertical tower PQ of height 10 from a point A on the horizontal ground is 45°. Let R be a point on AQ and from a point B, vertically above R, the angle of elevation of P is 60°. If $\angle BAQ = 30°$, AB = d and the area of the trapezium $PQRB$ is α, then the ordered pair (d, α) is: **(2022)**

(a) $\left(10\left(\sqrt{3}-1\right), 25\right)$ (b) $\left(10\left(\sqrt{3}-1\right), \frac{25}{2}\right)$

(c) $\left(10\left(\sqrt{3}+1\right), 25\right)$ (d) $\left(10\left(\sqrt{3}+1\right), \frac{25}{2}\right)$

24. A horizontal park is in the shape of a triangle OAB with $AB = 16$. A vertical lamp post OP is erected at the point O such that $\angle PAO = \angle PBO = 15°$ and $\angle PCO = 45°$, where C is the midpoint of AB. Then $(OP)^2$ is equal to **(2022)**

(a) $\frac{32}{\sqrt{3}}\left(\sqrt{3}-1\right)$ (b) $\frac{32}{\sqrt{3}}\left(2-\sqrt{3}\right)$

(c) $\frac{16}{\sqrt{3}}\left(\sqrt{3}-1\right)$ (d) $\frac{16}{\sqrt{3}}\left(2-\sqrt{3}\right)$

25. The angle of elevation of the top of a tower from a point A due north of it is a and from a point B at a distance of 9 units due west of A is $\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)$. If the distance of the point B from the tower is 15 units, then cot a is equal to: **(2022)**

(a) $\frac{6}{5}$ (b) $\frac{9}{5}$

(c) $\frac{4}{3}$ (d) $\frac{7}{3}$

ANSWER KEY

CONCEPT APPLICATION

1. 109.28 m
2. 1 m
3. 20 m
4. $30(\sqrt{3}+1)$ m
5. $h = \frac{20}{\sqrt{3}}$ m.
6. $AP = 7\sqrt{3}$
7. $\sin\theta = \frac{3}{5}$
8. 3 : 1 and 1 : 3

EXERCISE-1 (TOPICWISE)

1. (c)	2. (d)	3. (a)	4. (b)	5. (d)	6. (c)	7. (a)	8. (a)	9. (b)	10. (a)
11. (b)	12. (d)	13. (c)	14. (c)	15. (b)	16. (b)	17. (a)	18. (a)	19. (c)	20. (a)
21. (d)	22. (c)	23. (b)	24. (b)	25. (d)	26. (c)	27. (d)	28. (c)	29. (b)	30. (a)
31. (b)	32. (a)	33. (a)							

EXERCISE-2 (LEARNING PLUS)

1. (b)	2. (d)	3. (c)	4. (a)	5. (a)	6. (c)	7. (c)	8. (a)	9. (b)	10. (c)
11. (b)	12. (a)	13. (c)	14. (c)	15. (b)	16. (c)	17. (b)	18. (b)	19. (c)	20. (a)
21. (c)	22. (a)	23. (a)							

EXERCISE-3 (PAST YEAR QUESTIONS)

JEE Main

1. (d)	2. (d)	3. (d)	4. (b)	5. (b)	6. (c)	7. (c)	8. (c)	9. (b)	10. (a)
11. (a)	12. (b)	13. (b)	14. (b)	15. (c)	16. (b)	17. (d)	18. (a)	19. (c)	20. (d)
21. (a)	22. (c)	23. (a)	24. (b)	25. (a)					

CHAPTER

25 Solutions of Triangles

INTRODUCTION

In any Δ, the three sides and the three angles are generally called the elements of the triangle.

A triangle which does not contain a right angle is called an oblique triangle.

In any ΔABC, the measures of the angles $\angle BAC$, $\angle CBA$ and $\angle ACB$ are denoted by the letters.

A, B and C respectively, and the sides BC, AC and AB opposite to the angles A, B and C are respectively denoted by a, b and c. These six elements of a triangle are not independent and are connected by the relations.

(i) $A + B + C = \pi$

(ii) $a + b > c;\ b + c > a;\ c + a > b$

(iii) $|a - b| < c$, $|a - c| < b$, $|b - c| < a$

SINE FORMULA

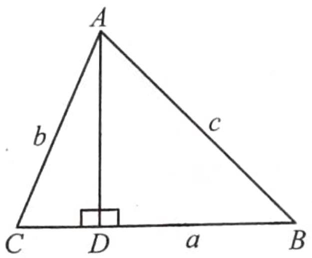
In any triangle the sides are proportional to the sines of the opposite angles i.e.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

(1) Let the triangle ABC be acute-angled.

From A draw AD perpendicular to the opposite side; then $AD = AB \sin(\angle ABD) = c \sin B$ and $AD = AC \sin(\angle ACD) = b \sin C$

$\therefore\ b \sin C = c \sin B$ i.e. $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

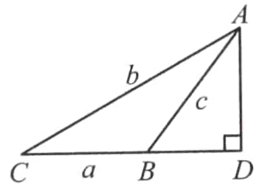


(2) Let the triangle ABC have an obtuse angle at B Draw AD perpendicular to CB produced; then $AD = AC \sin \angle ACD = b \sin C$ and

$AD = AB \sin \angle ABD$

$= c \sin(180^\circ - B) = c \sin B;$

$\therefore$ b sin C = c sin B i.e. $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$



In a similar manner it may be proved that either of these ratios is equal to $\dfrac{a}{\sin A}$

Thus $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Train Your Brain

Example 1: If the angles of a ΔABC are $\frac{\pi}{7}, \frac{2\pi}{7}$ and $\frac{4\pi}{7}$ and R is the radius of the circumcircle then $a^2 + b^2 + c^2$ has the value equal to

Sol. $a^2 + b^2 + c^2 = 4R^2(\sin^2A + \sin^2B + \sin^2C) = 2R^2$ $\left[1 - \cos\frac{2\pi}{7} + 1 - \cos\frac{4\pi}{7} + 1 - \cos\frac{8\pi}{7}\right]$

$= 2R^2[3 - (\cos\theta + \cos 2\theta + \cos 4\theta)]$ where $\theta = 2\pi/7$

now let $S = \cos\theta + \cos 2\theta + \cos 3\theta$ ($\cos 4\theta = \cos 3\theta$)

$2 \sin \frac{\theta}{2} S = \sin\frac{3\theta}{2} - \sin\frac{\theta}{2} + \sin\frac{5\theta}{2} - \sin\frac{3\theta}{2} + \sin\frac{7\theta}{2} - \sin\frac{5\theta}{2} = \sin\frac{7\theta}{2} - \sin\frac{\theta}{2}$

$= \sin\pi - \sin\frac{\theta}{2}$

$= -\sin\frac{\theta}{2}$, $S = -\frac{1}{2}$

$\Rightarrow a^2 + b^2 + c^2 = 2R^2(3 + 1/2) = 7R^2$

Example 2: In a triangle ABC, $a\cos A + b\cos B + c\cos C = s$. Prove that the triangle is equilateral.

Sol. The given result can be written as $2a\cos A + 2b\cos B + 2c\cos C = a + b + c$

Using sine rule we get $2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C = \sin A + \sin B + \sin C$

$\Rightarrow \sin 2A + \sin 2B + \sin 2C = \sin A + \sin B + \sin C$

$\Rightarrow 4\sin A\sin B\sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$\Rightarrow 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1$

$\Rightarrow 4\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right]\sin\frac{C}{2} = 1$

$\Rightarrow 4\sin^2\frac{C}{2} - 4\cos\frac{A-B}{2}\sin\frac{C}{2} + 1 = 0$. This is a quadratic equation in $\sin\frac{C}{2}$ which must have real roots.

Hence $16\cos^2\frac{A-B}{2} \le 1$

$\Rightarrow \cos^2\frac{A-B}{2} \le 1$. But $\cos^2\frac{A-B}{2} \le 1$

$\Rightarrow \cos^2\frac{A-B}{2} = 1 \Rightarrow A = B$

Similarly it can be prove that $B = C \Rightarrow A = B = C$

Example 3: In any ΔABC, prove that $\dfrac{a+b}{c} = \dfrac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$.

Sol. Since $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (let)

$\Rightarrow a = k\sin A,\ b = k\sin B$ and $c = k\sin C$

$\therefore$ L.H.S. $= \dfrac{a+b}{c}$

$= \dfrac{k(\sin A + \sin B)}{k\sin C}$

$= \dfrac{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}\cos\frac{C}{2}} = \dfrac{\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}\cos\frac{C}{2}}$

$= \dfrac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}} =$ R.H.S.

Hence L.H.S. = R.H.S. Proved

Concept Application

1. If A, B, C are in AP and $b : c = \sqrt{3} : \sqrt{2}$, then what is the value of $\sin C$?
   (a) 1 (b) $\frac{1}{\sqrt{3}}$
   (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{2}}$

2. In a ΔABC, $a + b = 3(1+\sqrt{3})$ cm and $a - b = 3(1-\sqrt{3})$ cm. If $\angle A$ is 30°, then what is the measure$\angle$B of?
   (a) 120° (b) 90°
   (c) 75° (d) 60°

3. How many triangle can be constructed with the data: $a = 5, b = 7\ \sin A = 3/4$
   (a) 0 (b) 1
   (c) 2 (d) None of these

COSINE FORMULA

To find an expression for one side (c) of a triangle in terms of other to sides and the included angle (C).

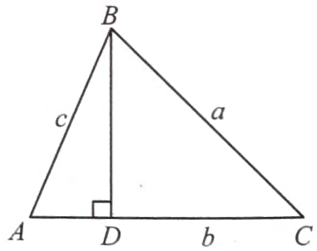
(1) Let C be an acute angle.

Draw BD perpendicular to AC ;

$AB^2 = BC^2 + CA^2 - 2AC\,.\,CD$;

$\therefore c^2 = a^2 + b^2 - 2ba\cos C$

$= a^2 + b^2 - 2ab\cos C$.



(2) Let C be an obtuse angle.

Draw BD perpendicular to AC produced ;

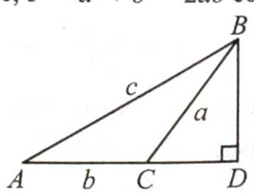
$AB^2 = BC^2 + CA^2 + 2AC\,.\,CD$;

$\therefore c^2 = a^2 + b^2 + 2ba\cos BCD$

$= a^2 + b^2 + 2ab\cos(180° - C)$

$= a^2 + b^2 - 2ab\cos C$

Hence in each case, $c^2 = a^2 + b^2 - 2ab\cos C$



Similarly it may be shown that

$a^2 = b^2 + c^2 - 2bc \cos A$ and $b^2 = c^2 + a^2 - 2ac \cos B$

From the above result we obtain

$$\cos A = \frac{b^2+c^2-a^2}{2bc};$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca};$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

These results enable us to find the cosines of the angles when the numerical values of the sides are given.

Train Your Brain

Example 4: In a ΔABC, prove that $a(b\cos C - c\cos B) = b^2 - c^2$

Sol. $\because$ We have to prove $a(b \cos C - c \cos B) = b^2 - c^2$

$\because$ from cosine rule we know that

$$\cos C = \frac{a^2+b^2-c^2}{2ab} \ \& \cos B = \frac{a^2+c^2-b^2}{2ac}$$

$\therefore$ L.H.S.

$$= a\left\{b\left(\frac{a^2+b^2-c^2}{2ab}\right) - c\left(\frac{a^2+c^2-b^2}{2ac}\right)\right\}$$

$$= \frac{a^2+b^2-c^2}{2} - \frac{(a^2+c^2-b^2)}{2}$$

$= (b^2 - c^2) =$ R.H.S.

Example 5: If in ΔABC, $\angle A = 60°$ then find the value of $\left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right)$.

Sol. $\because \angle A = 60°$

$$\because \left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right) = \left(\frac{c+a+b}{c}\right)\left(\frac{b+c-a}{b}\right)$$

$$= \frac{(b+c)^2-a^2}{bc} = \frac{(b^2+c^2-a^2)+2bc}{bc}$$

$$= \frac{b^2+c^2-a^2}{bc} + 2 = 2\left(\frac{b^2+c^2-a^2}{2bc}\right) + 2$$

$= 2 \cos A + 2 = 3$

$\left(\because \angle A = 60° \Rightarrow \cos A = \frac{1}{2}\right)$

$$\therefore \left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right) = 3$$

Example 6: If the sides a, b, c of a ABC satisfy the relation, $a^4 + b^4 + c^4 = 2c^2 (a^2 + b^2)$, find the possible values of the angle C.

Sol. Solving as a quadratic equation in c we get, $c^2 = a^2 + b^2 \pm \sqrt{2}ab$ or $a^2 + b^2 - c^2 = \pm\sqrt{2}ab = \frac{a^2+b^2+c^2}{2ab}$

$$= \pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow C = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

Concept Application

4. In a triangle ABC, $a = 4$, $b = 3$, $\angle BAC = 60°$, then the equation for which c is the root, is
   (a) $c^2 + 3c + 7 = 0$ (b) $c^2 + 3c - 7 = 0$
   (c) $c^2 - 3c + 7 = 0$ (d) $c^2 - 3c - 7 = 0$
5. In a ΔABC, $(c + a + b)(a + b - c) = ab$. The measure of the angle C is:
   (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
   (c) $\frac{2\pi}{3}$ (d) None of these
6. The angle between the two sides of the triangle having length 6 cm and 8 cm is 60°, find the third side of the triangle.
   (a) $\sqrt{26}$ cm (b) 12 cm
   (c) $\sqrt{52}$ cm (d) 13 cm

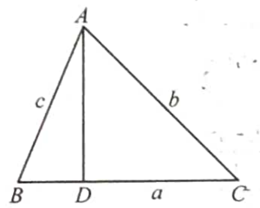
PROJECTION FORMULA

To express one side of a triangle in terms of the adjacent angles and the other two sides.

(1) Let ABC be an acute-angled triangle

Draw AD perpendicular to BC;

then BC = BD + CD = AB cos $\angle ABD$ + AC cos $\angle ACD$;



i.e. $a = c \cos B + b \cos C$

(2) Let the triangle ABC have an obtuse angle C.

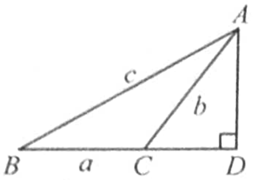
Draw AD perpendicular to BC produced; then

BC = BD – CD = AB cos

$\angle ABD - AC \cos \angle ACD$;

$\therefore a = \cos B - b\cos(180° - C)$

$= c\cos B + b\cos C$



Thus in each case $a = b\cos C + c\cos B$.

Similarly it may be shown that

$b = c\cos A + a\cos C$, and $c = a\cos B + b\cos A$

Train Your Brain

Example 7: In a ΔABC prove that $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a + b + c$

Sol. $\because$ L.H.S. $= (b+c)\cos A + (c+a)\cos B + (a+b)\cos C$

$= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C$

$= (b\cos A + a\cos B) + (c\cos A + a\cos C)$

$+ (c\cos B + b\cos C) = a + b + c =$ R.H.S.

Example 8: In any triangle ABC, find.

$(a + b)^2 \sin^2 \dfrac{C}{2} + (a - b)^2 \cos^2 \dfrac{C}{2}$

Sol. $a^2\left(\sin^2\dfrac{C}{2}+\cos^2\dfrac{C}{2}\right)+b^2\left(\sin^2\dfrac{C}{2}+\cos^2\dfrac{C}{2}\right)$

$-2ab\left(\cos^2\dfrac{C}{2}-\sin^2\dfrac{C}{2}\right)$

$= a^2 + b^2 - 2ab\cos C = c^2$

Concept Application

7. Prove that $a(b\cos C - c\cos B) = b^2 - c^2$

8. If in a triangle a $a\cos^2\dfrac{C}{2}+c\cos^2\dfrac{A}{2}, = \dfrac{3b}{2}$ then find the relation between the sides of the triangle.

NAPIER'S ANALOGY - TANGENT RULE

(*i*) $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

(*ii*) $\tan\dfrac{C-A}{2}=\dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

(*iii*) $\tan\dfrac{A-B}{2}=\dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

Train Your Brain

Example 9: Find the unknown elements of the ΔABC in which $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$, $C = 60°$.

Sol. $\because a = \sqrt{3} + 1,\ b = \sqrt{3} - 1,\ C = 60°$

$\because\ A + B + C = 180°$

$\therefore\ A + B = 120°$ ... (i)

$\because$ From law of tangent, we know that $\tan\left(\dfrac{A-B}{2}\right)$

$=\dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

$=\dfrac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\cot 30°$

$=\dfrac{2}{2\sqrt{3}}\cot 30°$

$\Rightarrow \tan\left(\dfrac{A-B}{2}\right)=1$

$\therefore\ \dfrac{A-B}{2}=\dfrac{\pi}{4}=45° \Rightarrow A - B = 90°$ ... (ii)

From equation (i) and (ii), we get

$A = 105°$ and $B = 15°$

Now,

$\because$ From **sine-rule,**

we know that $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

$\therefore\ c=\dfrac{a\sin C}{\sin A}=\dfrac{(\sqrt{3}+1)\sin 60°}{\sin 105°}=\dfrac{(\sqrt{3}+1)\dfrac{\sqrt{3}}{2}}{\dfrac{\sqrt{3}+1}{2\sqrt{2}}}$

$\because\ \sin 105°=\dfrac{\sqrt{3}+1}{2\sqrt{2}} \Rightarrow c=\sqrt{6}$

$\therefore\ c=\sqrt{6},\ A=105°,\ B=15°$

Example 10: Prove that $\tan\left(\dfrac{B-C}{2}\right)=\dfrac{b-c}{b+a}\cot\dfrac{A}{2}$

Sol. In any tnangle ABC we have

$\Rightarrow \dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

$\Rightarrow \dfrac{b}{c}=\dfrac{\sin B}{\sin C}$

$\Rightarrow \left(\dfrac{b-c}{b+c}\right)=\dfrac{\sin B-\sin C}{\sin B+\sin C}$

[Applying Dividendo and Componendo]

$$\Rightarrow \left(\frac{b-c}{b+c}\right) = \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}$$

[Since, $A + B + C = \pi \Rightarrow \frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$]

$$\left(\frac{b-c}{b+c}\right) = \tan\frac{A}{2}\tan\left(\frac{B-C}{2}\right)$$

$$\left(\frac{b-c}{b+c}\right) = \frac{\tan\frac{B-c}{2}}{\cot\frac{A}{2}}$$

Therefore, $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$.

Concept Application

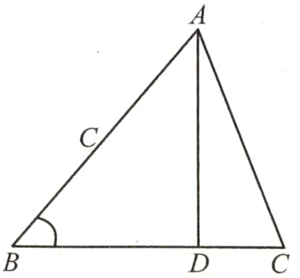
9. In a triangle ABC, $\angle A = 60°$ and $b : c = \sqrt{3}+1 : 2$, then find the value of $(\angle B - \angle C)$.
10. If one side of a triangle is double the other, and the angles on opposite sides differ by 60°, then the triangle is
    (a) Equilateral
    (b) Obtuse angled
    (c) Right angled
    (d) Acute angled

AREA OF A TRIANGLE

To find the area of a triangle. Let Δ denote the area of the triangle ABC. Draw AD perpendicular to BC.

$$\therefore \Delta = \frac{1}{2}(\text{base} \times \text{altitude})$$

$$= \frac{1}{2}\text{BC . AD} = \frac{1}{2}\text{ BC . AB sin B} = \frac{1}{2}\text{ ca sin B}$$



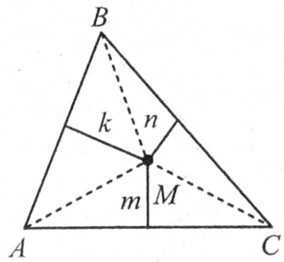
Train Your Brain

Example 11: Find the area of a triangle with angles α, β and γ knowing that the distances from an arbitrary point M taken inside the triangle to its sides are equal to m, n and k. (fig.)

Sol. The area S of the triangle ABC can be found by the formula $S = \frac{1}{2}$ AC. BC. sin γ, by sin rule

$$\frac{AC}{\sin\beta} = \frac{BC}{\sin\alpha} = \frac{AB}{\sin\gamma},$$

whence we find that $AC = \frac{a\sin\beta}{\sin\alpha}$ and $AB = \frac{a\sin\gamma}{\sin\alpha}$.



We have:

$$S = \frac{1}{2}\text{ AC . BC . }\sin\gamma = \frac{1}{2}\frac{a\sin\beta}{\sin\alpha}.\, a\sin\gamma$$

$$= \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha}.$$ On the other hand,

$$S = S_{AMB} + S_{BMC} + S_{AMC}$$

$$= \frac{1}{2}AB\cdot k + \frac{1}{2}BC\cdot n + \frac{1}{2}\cdot AC\cdot m$$

$$= \frac{1}{2}\cdot\frac{a\sin\gamma}{\sin\alpha}\cdot k + \frac{1}{2}an + \frac{1}{2}\cdot\frac{a\sin\beta}{\sin\alpha}\cdot m$$

$$= \frac{a(k\sin\gamma + n\sin\alpha + m\sin\beta)}{2\sin\alpha}$$

Hence, $\frac{a^2\sin\beta\sin\gamma}{2\sin\alpha} = \frac{a(k\sin\gamma + n\sin\alpha + m\sin\beta)}{2\sin\alpha}$,

we get: $x = \frac{k\sin\gamma + n\sin\alpha + m\sin\beta}{\sin\beta\sin\gamma}$.

Substituting this value of a into the first of the above formulas for the area of the triangle ABC, we obtain:

$$S \Rightarrow \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha} = \frac{(k\sin\gamma + n\sin\alpha + m\sin\beta)^2}{2\sin\alpha\sin\beta\sin\gamma}.$$

Example 12: The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides.

Sol. Let $ABCD$ be the cyclic quadrilateral in which $AB = 2$ and $BC = 5, \angle ADC$ $180^\circ - 60^\circ$

Area of cyclic quadrilateral $ABCD$ = Area of ΔABC + Area of ΔACD

$$\Rightarrow 4\sqrt{3} = \frac{1}{2}\ AB\ BC \sin 60^\circ + \frac{1}{2} CD\ DA \sin 120^\circ$$

$$= \frac{1}{2} 2 \times 5 \times (\sqrt{3}/2) + \frac{1}{2} xy(\sqrt{3}/2)$$

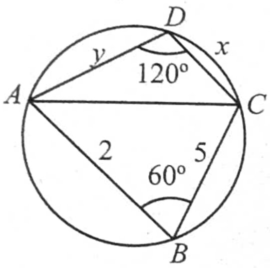
where $CD = x$, $AD = y$

$\therefore xy = 6$ ...(i)

From ΔABC, we get

$AC^2 = AB^2 + BC^2 - 2AB\ BC$

$\cos 60^\circ = 4 + 25 - 20(1/2) = 19$ ...(ii)



Also from ΔACD,

$AC^2 = CD^2 + DA^2 - 2CD\ DA \cos 120^\circ$

$= x^2 + y^2 + xy = ax^2 + y^2 + 6$ [using Eq. (i)] ...(iii)

Now from Eqs. (ii) and (iii),

we have $x^2 + y^2 + 6 = 19$ or $x^2 + y^2 = 13$ ...(iv)

Solving Eqs. (ii) and (iv), we get

$x^4 - 13x^2 + 36 = 0 \Rightarrow x^2 = 4, 9 \Rightarrow x = 2, 3 \Rightarrow y = 3, 2$

Hence, the other two sides of the cyclic quadrilateral are 3 and 2.

Concept Application

11. A cyclic quadrilateral $ABCD$ of area $3\sqrt{3}/4$ is inscribed in a unit circle. If one of its sides $AB = 1$ and $\angle A$ is acute and the diagonal $BD = \sqrt{3}$, find the lengths of the other sides.

12. Let P be a point inside the triangle ABC such that $\angle APB = \angle BPC = \angle CPA$. Prove that $PA + PB + PC = \sqrt{\frac{a^2 + b^2 + c^2}{2} + 2\sqrt{3}\Delta}$, where a, b, c, Δ are the sides and the area of triangle ABC.

TRIGONOMETRIC FUNCTIONS OF HALF ANGLES

(*i*) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$;

$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$;

$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(*ii*) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$;

$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$;

$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(*iii*) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$

where $s = \frac{a+b+c}{2}$ & Δ = area of triangle

(*iv*) Area of triangle = $\sqrt{s(s-a)(s-b)(s-c)}$

Train Your Brain

Example 13: If a, b, c are in A.P., then the numerical value of $\tan\frac{A}{2}\tan\frac{C}{2}$ is

Sol. Given $2b = a + c$

$\Rightarrow 3b = 2s = a + b + c$

$$\tan\frac{A}{2}\tan\frac{C}{2} = \frac{\Delta}{s(s-a)} \cdot \frac{\Delta}{s(s-c)} \cdot \frac{s-b}{s-b}$$

$$= \frac{2s-2b}{2s} = \frac{b}{3b} = \frac{1}{3}$$

Example 14: With usual notions, prove that in a triangle ABC, $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s}{r}$.

Sol. Using $\cot\frac{A}{2} = \frac{s(s-a)}{\Delta}$ etc.

$$\text{LHS} = \frac{s(s-a) + s(s-b) + s(s-c)}{\Delta}$$

$$= \frac{3s^2 - s(a+b+c)}{\Delta} = \frac{3s^2 - 2s^2}{\Delta} = \frac{s^2}{\Delta} = \frac{s}{r}$$

$= a^2 + b^2 - 2ab\cos C = c^2$

Concept Application

13. In any ΔABC, if $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in *A. P.*, then a, b, c are in

(*a*) *A.P.* (*b*) *G.P.*

(*c*) *H.P.* (*d*) None of these

14. In ΔABC, if $\sin^2\frac{A}{2}, \sin^2\frac{B}{2}$ and $\sin^2\frac{C}{2}$ are in HP., then a, b and c will be in

(*a*) A.P. (*b*) GP.

(*c*) H.P. (*d*) None of these

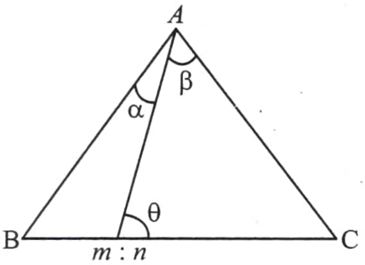
15. In a triangle ΔABC, 1 - tan (A/2) tan (B/2) is equal to

(*a*) $\frac{2a}{b+c-a}$ (*b*) $\frac{2a}{c+a-b}$

(*c*) $\frac{2c}{a+b-c}$ (*d*) $\frac{2c}{a+b+c}$

m-n RULE

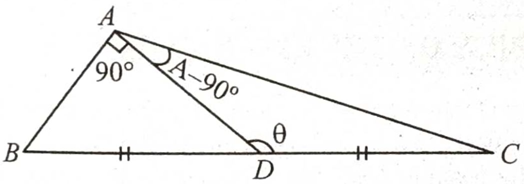
In any triangle, $(m+n)\cot\theta = m\cot\alpha - n\cot\beta = n\cot B - m\cot C$



Train Your Brain

Example 15: If the median AD of a triangle ABC is perpendicular to AB, prove that tan A + 2tan B = 0.

Sol. From the figure, we see that θ = 90° + B (as θ is external angle of ΔABD)



Now if we apply **m-n rule** in ΔABC, we get

(1 + 1) cot (90° + B) = 1. cot 90° – 1.cot (A – 90°)

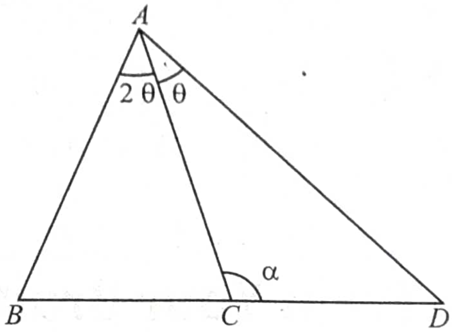
⇒ –2 tan B = cot (90° – A)

⇒ – 2 tan B = tan A

⇒ tan A + 2 tan B = 0 **Hence proved.**

Example 16: In ΔABC, the median AD divides ∠BAC such that ∠BAD:∠CAD = 2:1. Then cos (*A*/3) is equal to

Sol.



Let $\frac{A}{3} = \angle CAD = \theta$

Now, by m-n theorem,

$(1 + 1)\cot\alpha = 1\cot 2\theta - 1\cot\theta$

$\Rightarrow 2\cot(B + 2\theta) = \cot 2\theta - \cot\theta$

$\Rightarrow \cot(B + 2) + \cot\theta - \cot 2\theta - \cot(B + 2\theta)$

$\Rightarrow \frac{\sin(B+3\theta)}{\sin(B+2\theta)\sin\theta} = \frac{\sin B}{\sin(B+2\theta)\sin 2\theta}$

$\Rightarrow \frac{\sin(B+A)}{\sin\theta} = \frac{\sin B}{\sin 2\theta}$

$\Rightarrow \sin C = \frac{\sin B}{2\cos\theta} \quad \Rightarrow \cos\frac{A}{3} = \frac{\sin B}{2\sin C}$

Concept Application

16. Prove that the median through A divides angle A into two parts whose cotangents are, 2 cot *A* + cot *C* and 2 cot *A* + cot *B* makes on angle with the side *BC* whose cotangent is $\frac{1}{2}$ (cot *B* – cot *C*).

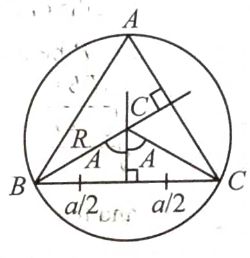
17. Find the angles of a triangle in which the altitude and a median drawn from the same vertex divide the angle at that vertex into 3 equal parts.

RADIUS OF CIRCUMCIRCLE

Circumcircle is the circle passing thorugh vertices of triangle ABC. Its radius is known as circumradius (R) & centre is circumcentre (C).

Circumcentre is the point of intersection of perpendicular bisectors of sides.

In ΔBCD



$\frac{a/2}{R} = \sin A$

$\Rightarrow a = 2R\sin A$

or $R = \frac{a}{2\sin A}$

Area of triangle and circumradius (R)

we know that

$\Delta = \frac{1}{2} a.b. \sin c$ $\left(\because \sin c = \frac{c}{2R}\right)$

$\Rightarrow \quad \Delta = \frac{1}{2} a.b. \frac{c}{2R}$

or $\quad R = \frac{abc}{4\Delta}$

Train Your Brain

Example 17: In a ΔABC, if $a = 18$ cm., $b = 24$ cm. and $c = 30$ cm., then find its circum-radius

Sol. Clearly, the triangle is right angled.

$(\because 18^2 + 24^2 = 30^2)$

Thus, the area of the triangle

$= \frac{1}{2} \times 24 \times 18 = 12 \times 18$

Therefore, the circum-radius = R

$= \frac{abc}{4\Delta} = \frac{18 \times 24 \times 30}{4 \times 12 \times 18} = 15$ cm

Example 18: In an euqilateral triangle of side $2\sqrt{3}$ cm., then find the circum - radius

Sol. As we know that,

$\frac{a}{\sin A} = 2R$

$\Rightarrow 2R = \frac{2\sqrt{3}}{\sin(60^\circ)}$

$R = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}} = 2$

Hence, the circum-radius is 2cm.

Example 19: If the length of the sides of a triangle are 3, 4 and 5 units, then find its circum-radius R.

Sol. Let a = 3, b = 4 and c = 5. Clearly, it is a right angled triangle

Thus, $\Delta = \frac{1}{2} \times 4 \times 3 = 6$ sq. u

Hence, the circum radius R

$= \frac{abc}{4\Delta}$

$= \frac{3 \times 4 \times 5}{6} = 10$ unit

Concept Application

18. Find the radius of the circumcircle of a triangle, whose sides are given as $a = 8$, $b = 10$ and $c = 6$ in respectively.

19. In a triangle ABC $\angle A = 60^\circ$, $\angle B = 40^\circ$ and $\angle C = 80^\circ$ If P is the centre of the circumcircle of triangle ABC with radius unity, then the radius of the circumcircle of triangle BPC is

20. If in a triangle ABC, $AB = 5$ units, $\angle B = \cos^{-1}\left(\frac{3}{5}\right)$ and radius of circumcircle of ΔABC is 5 units, then the area (in sq. units) of ΔABC is

RADIUS OF THE INCIRCLE

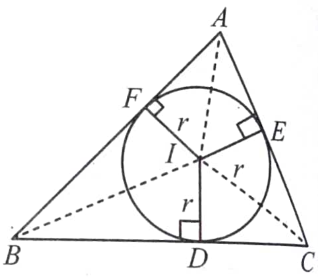
To find the radius of the circle inscribed in a triangle. Let ℓ be the circle inscribed in the triangle ABC, and D, E, F the points of contact; then ID, IE, IF are perpendicular to the sides.

Now Δ = sum of the areas of the triangles BIC, CIA, AIB

$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$

$= \frac{1}{2}(a+b+c)r = sr$

$\Rightarrow r = \frac{\Delta}{s}$



(a) $r = \frac{\Delta}{s}$ where $s = \frac{a+b+c}{2}$

(b) $r = (s-a)\tan\frac{A}{2}$

$= (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(c) $r = \frac{a \sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ & so on

(d) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

RADIUS OF THE EX-CIRCLES

A circle which touches one side of a triangle and the produced of other two sides is said to be an escribed circle of the triangle. Thus the triangle ABC has three escribed circles, one touching BC, and produced AB and AC, a second touching CA, and produced BC and AB; a third touching AB, and produced CA, CB.

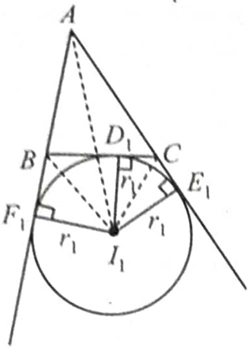
To find the radius of an escribed circle of a triangle. Let I_1 be the centre of the circle touching the side BC and the two sides AB and AC produced. Let D_1, E_1, F_1 be the points of contact; then the lines joining I_1 to these points are perependicular to the sides.

Let r_1 be the radius; then

Δ = area ABC

= area ABI_1C – area BI_1C

= area BI_1A + area CI_1A – area BI_1C



$$= \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1$$

$$= \frac{1}{2}(c+b+a)\, r_1 = (s-a)\, r_1$$

$$\therefore \quad r_1 = \frac{\Delta}{s-a}$$

Similarly, if r_2, r_3 be the radii of the escribed circles opposite to the angles

B and C respectively

$$r_2 = \frac{\Delta}{s-b},\ r_3 = \frac{\Delta}{s-c}.$$

Many important relations connecting a triangle and its circles may be established by elementary geometry.

With the notation of previous articles, since tangents to a circle from the same point are equal.

we have, AF = AE, BD = BF, CD = CE;

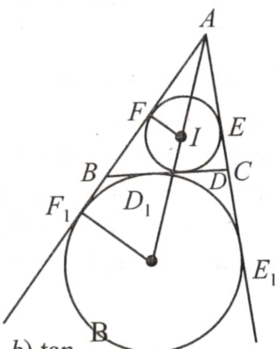
$\therefore$ AF + (BD + CD) = half the sum of the sides;

$\therefore$ AF + a = s

$\therefore$ AF = $s - a$ = AE

Similarly, BD = BF = $s - b$, CD = CE = $s - c$. Also

$$r_1 = AF \tan\frac{A}{2} = (s-a)\tan\frac{A}{2}$$



Similarly, $r_2 = (s-b)\tan\frac{B}{2}$,

$r_3 = (s-c)\tan\frac{C}{2}$.

Again, $AF_1 = AE_1$, $BF_1 = BD_1$, $CE_1 = CD_1$

$\therefore \quad 2AF_1 = AF_1 + AE_1 = (AB + BD_1) + (AC + CD_1)$

= sum of the sides

$\therefore \quad AF_1 = s = AE_1$

$\therefore \quad BD_1 = BF_1 = s - c,\ CD_1 = CE_1 = s - b$

Also $r_1 = AF_1 \tan\frac{A}{2} = s\tan\frac{A}{2}$

Similarly, $r_2 = s\tan\frac{B}{2}$, $r_3 = s\tan\frac{C}{2}$.

Note: $r_1 = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

$= 4R\sin\frac{A}{2}\,.\cos\frac{B}{2}\,.\cos\frac{C}{2}$ & so on

Train Your Brain

Example 20: With usual notation in a triangle ABC, prove that $r^2 + s^2 + 4Rr = ab + bc + ca$.

Sol. $r^2 + s^2 + 4Rr$

$$= \frac{\Delta^2}{s^2} + s^2 + \frac{abc}{\Delta}\cdot\frac{\Delta}{s} \qquad \left(r = \frac{\Delta}{s}, R = \frac{abc}{4\Delta}\right)$$

$$= \frac{s(s-a)(s-b)(s-c)}{s.s} + s^2 + \frac{abc}{s}$$

$$= \frac{(s-a)(s-b)(s-c) + s^2 + abc}{s}$$

$$= \frac{2s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc + abc}{s}$$

$= ab + bc + ca$ (proved)

Example 21: In a ΔABC, prove that sinA + sin B + sin C = $\frac{s}{R}$

Sol. In a ΔABC, we know that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\therefore \quad \sin A = \frac{a}{2R},$$

$$\sin B = \frac{b}{2R} \text{ and } \sin C = \frac{c}{2R}.$$

$$\therefore \quad \sin A + \sin B + \sin C = \frac{a+b+c}{2R} = \frac{2s}{2R}$$

$$\because \quad a + b + c = 2s$$

$$\Rightarrow \quad \sin A + \sin B + \sin C = \frac{s}{R}.$$

Example 22: In a ΔABC if $a = 13$ cm, $b = 14$ cm and $c = 15$ cm, then find its circumradius.

Sol. $\because R = \dfrac{abc}{4\Delta}$ ... (i)

$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\because s = \dfrac{a+b+c}{2} = 21$ cm

$\therefore \Delta = \sqrt{21\times8\times7\times6} = \sqrt{7^2\times4^2\times3^2}$

$\Rightarrow \Delta = 84\ \text{cm}^2$

$\therefore R = \dfrac{13\times14\times15}{4\times84} = \dfrac{65}{8}$ cm $\therefore R = \dfrac{65}{8}$ cm.

Example 23: In a ΔABC, prove that $s = 4R\cos\dfrac{A}{2}.\cos\dfrac{B}{2}.\cos\dfrac{C}{2}$.

Sol. In a ΔABC,

$\because \cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}, \cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ca}}$

and $\cos\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$ and $R = \dfrac{abc}{4\Delta}$

$\because$ R.H.S. $= 4R\cos\dfrac{A}{2}.\cos\dfrac{B}{2}.\cos\dfrac{C}{2}$.

$= \dfrac{abc}{\Delta}\cdot s\sqrt{\dfrac{s(s-a)(s-b)(s-c)}{(abc)^2}} = s$

$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)} =$ L.H.S.

Hence R.H.S = L.H.S. proved.

Example 24: The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides.

Sol. $\dfrac{s-a}{\Delta}, \dfrac{s-b}{\Delta}, \dfrac{s-c}{\Delta}$ are in A.P.

$\Rightarrow a, b, c$ are in A.P.

$\Rightarrow 2b = a + c$

$\Rightarrow 2s = 24 \Rightarrow s = 12$

$\Rightarrow \sqrt{12(12-a)4(12-16+a)} = 24$

$\Rightarrow 12 \times 4(12-a)(a-4) = 24 \times 24$

$\Rightarrow -a^2 + 16a - 48 = 12$

$\Rightarrow a^2 - 16a + 60 = 0$

$\Rightarrow (a-10)(a-6) = 0$

$\Rightarrow a = 10$ or $a = 6$

Hence, the lengths of side are 6, 8, 10 cm.

Concept Application

21. Find the radius of the incircle of the triangle whose sides measure 10.24 and 26 m?

22. In an isosceles triangle, if one angle is 120° and radius of its incircle is $\sqrt{3}$, then the area of the triangle in square units is

23. If A, A_1, A_2, A_3 are the areas of incircle and the ex-circles of a triangle, then $\dfrac{1}{\sqrt{A_1}} + \dfrac{1}{\sqrt{A_2}} + \dfrac{1}{\sqrt{A_3}} =$

(a) $\dfrac{2}{\sqrt{A}}$ (b) $\dfrac{1}{\sqrt{A}}$ (c) $\dfrac{1}{2\sqrt{A}}$ (d) $\dfrac{3}{\sqrt{A}}$

24. If the diameter of any excircle of a triangle is equal to its perimeter, then tire triangle is

(a) Equilateral (b) Isosceles

(c) Right angled (d) Scalene

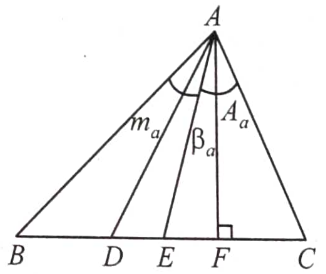
LENGTH OF ANGLE BISECTORS, MEDIANS & ALTITUDE

(i) Length of an angle bisector from the angle $A = \beta_a$

$$= \frac{2bc\cos\frac{A}{2}}{b+c}.$$

(ii) Length of median from the angle $A = m_a$

$$= \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$



(iii) Length of altitude from the angle $A = A_a = \dfrac{2\Delta}{a}$.

Note: $m_a^2 + m_b^2 + m_c^2 = 3/4\,(a^2 + b^2 + c^2)$

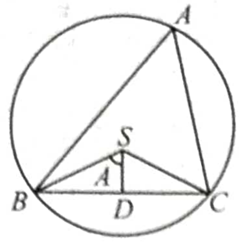
To find the radius of the circle circumscribing a triangle. Let S be the centre of the circle circumscribing the triangle ABC, and R its radius. Bisect $\angle BSC$ by SD, which will also bisect BC at right angles.

$\angle BSC = 2A$

and $\dfrac{a}{2} = BD = BS\sin BSD = R\sin A$

$\therefore R = \dfrac{a}{2\sin A}$

Thus $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$ or $a = 2R \sin A$, $b=2R\sin B$, $c=2R\sin C$



The circum-radius may be expressed in a form not involving the angles, as

$$R=\frac{a}{2\sin A}=\frac{abc}{2bc\sin A}=\frac{abc}{4\Delta}$$

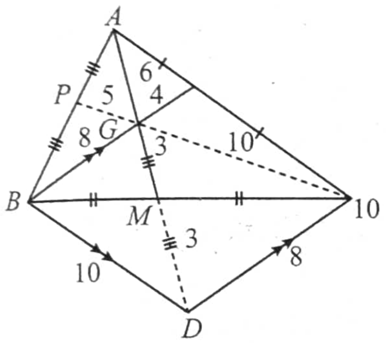
Train Your Brain

Example 25: Show that $2R^2 \sin A \sin B \sin C = \Delta$.

Sol. L.H.S. $=\frac{1}{2}.2R\sin A.\,2R\sin B.\sin C=\frac{1}{2}ab\sin C=\Delta$

Example 26: The medians of a triangle ABC are 9 cm, 12 cm and 15 cm respectively. Then the area of the triangle is

Sol.



Produce the median AM to D such that $GM = MD$. Join D to B and C.

Now $GBDC$ is a parallelogram. Note that the sides of the $AGDC$ are 6, 8, 10

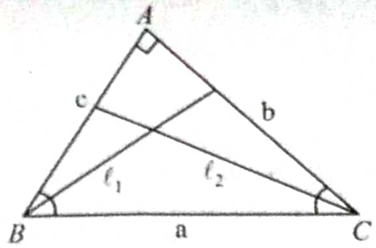
$\Rightarrow \angle GDC = 90°$

$$\left.\begin{aligned}&\text{Area of a } \Delta ADC=\frac{12.8}{2}=48\\&\text{Area of a } \Delta MDC=\frac{3.8}{2}=12\end{aligned}\right]$$

$\Rightarrow$ Area of $\Delta AMC = 36$

$\Rightarrow$ Area of $\Delta ABC = 72\ \text{cm}^2$

Example 27: The ratios of the lengths of the sides BC & AC of a triangle ABC to the radius of a circumscribed circle are equal to 2 & 3/2. respectively. Show that the ratio of the lengths of the bisectors of the interior angles B & C is, $\frac{7\left(\sqrt{7}-1\right)}{9\sqrt{2}}$



Sol. $\frac{a}{R}=2;\frac{b}{R}=\frac{3}{2}\therefore\frac{2R\sin A}{R}=2\ \sin B=\frac{3}{4}$

$\Rightarrow \sin A = 1;\ C^2 = 4R^2-\frac{9R^2}{4}\Rightarrow A=90°;\ C=\frac{\sqrt{7}}{2}R$

Now, $I_1=\frac{2ac}{a+c}\cos\frac{B}{2}=\frac{2ac}{a+c}\sqrt{\frac{1+\cos B}{2}}$ and

$I_2=\frac{2ab}{a+b}\cos\frac{c}{2}=\frac{2ab}{a+b}\sqrt{\frac{1+\cos C}{2}}$

$\therefore \frac{I_1}{I_2}=\frac{a+b}{a+c}\cdot\frac{c}{b}\sqrt{\frac{1+\cos B}{1+\cos C}}=\frac{c(a+b)}{b(a+c)}\sqrt{\frac{1+\frac{c}{a}}{1+\frac{b}{a}}}$

$=\frac{c}{b}\sqrt{\frac{a+b}{a+c}}$

Substituting $a=2R$; $b=\frac{3}{2}R$ & $c=\frac{\sqrt{7}}{2}R$, we get the desired result.

Concept Application

25. AD is a median of the ΔABC. If AE and AF are medians of the triangles ABD and ADC respectively, and $AD = m_1$, $AE = m_2$, then prove that $m_2{}^2+m_2{}^2=\frac{a^2}{B}$

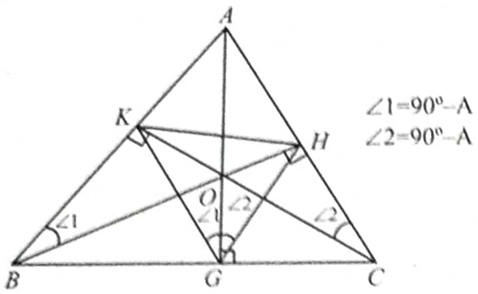
26. In ΔABC, in the usual notation, the area is y be sq. units AD is the median to BC. Prove that $\angle ABC=\frac{1}{2}\angle ADC$. A

27. In a ΔABC, the bisector of the angle A meets the side BC in D and the circumscribed circle in E. Show that,

$$DE=\frac{a^2\sec\frac{A}{2}}{2(b+c)}$$

ORTHOCENTRE AND PEDAL TRIANGLE

Let G, H, K be the feet of the perpendiculars from the angular points on the opposite sides of the triangle ABC, then GHK is called the Pedal triangle of ABC. The three perpendiculars AG, BH, CK meet in a point O which is called the orthocentre of the triangle ABC.



To find the sides and angles of the pedal triangle.

In the figure, the points K, O, G, B are concyclic:

$\therefore \quad \angle OGK = OBK = 90^\circ - A$

Also the points H, O, G, C are concyclic:

$\therefore \quad \angle OGH = \angle OCH = 90^\circ - A$

$\therefore \quad \angle KGH = 180^\circ - 2A$

Thus the angles of the pedal triangle are

$180^\circ - 2A$, $180^\circ - 2B$, $180^\circ - 2C$

Again, the triangles AKH, ABC are similar:

$$\therefore \quad \frac{HK}{BC} = \frac{AK}{AC} = \cos A$$

$\therefore \quad HK = a \cos A$

Thus the sides of the pedal triangle are $a \cos A$, $b \cos B$, $c \cos C$.

In terms of R, the equivalent forms become R sin 2A, R sin 2B, R sin 2C.

If the angle ACB of the given triangle is obtuse, the expression $180^\circ - 2C$, and c cos C are both negative, and the values we have obtained required some modification. In this case the angles are 2A, 2B, $2C - 180^\circ$, and the sides $a \cos A$, $b \cos B$, $-c \cos C$.

Remarks:

(*i*) The distances of the orthocentre from the angular points of the ΔABC are 2 R cos A, 2R cos B and 2R cos C.

(*ii*) The distances of orthocentre from sides are 2R cos B cos C, 2R cos C cos A & 2R cos B cos C.

(*iii*) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

To find the area and circum-radius of the pedal triangle.

$$\text{Area} = \frac{1}{2} \text{ (product of two sides)} \times \text{(sine of included angle)}$$

$$= \frac{1}{2} R \sin 2B \,.\, R \sin 2C \,.\, \sin(180^\circ - 2A)$$

$$= \frac{1}{2} R^2 \sin 2A \sin 2B \sin 2C. \text{ The circum-radius}$$

$$= \frac{HK}{2\sin HGK} = \frac{2\sin 2A}{2\sin(180^\circ - 2A)} = \frac{R}{2}.$$

EXCENTRAL TRIANGLE

Let ABC be a triangle and I_1, I_2, I_3 its ex-centres; then triangle $I_1I_2I_3$ is called the Ex-central triangle of ABC. Let I be the in-centre; then from the construction for finding the positions of the in-centre and ex-centres, it follows that:

(*i*) The points I, I_1 lie on the line bisecting the angle BAC; the points I, I_2 lie on the line bisecting the angle ABC; the points I, I_3 lie on the line bisecting the angle ACB.

(*ii*) The points I_2, I_3 lie on the line bisecting the angle BAC externally; the points I_3, I_1 lie on the line bisecting the angle ABC externally; the points I_1, I_2 lie on the line bisecting the angle ABC externally.

(*iii*) The line A_1 is perpendicular to I_2I_3; the line BI_2 is perpendicular to I_3I_1; the line CI_3 is perpendicular to I_1I_2. Thus the triangle ABC is the Pedal triangle of its ex-central triangle $I_1I_2I_3$.

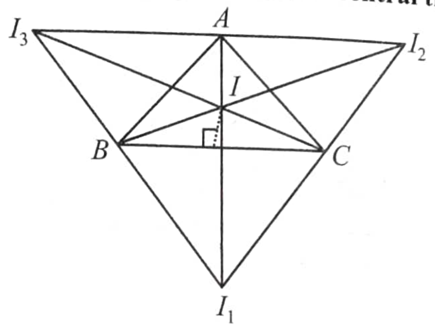
(*iv*) The angles IBI_1 and ICI_1 are right angles; hence the points B, I, C, I_1 are concyclic . Similarly, the points C, I, A, I_2, and the points A, I, B, I_3 are concyclic.

(*v*) The lines AI_1, BI_2, CI_3 meet at the in-centre I, which is therefore the orthocentre of the ex-central triangle $I_1I_2I_3$.

(*vi*) The lines AI_1, BI_2, CI_3 meet at the in-centre I, which is therefore the orthocentre of the ex-central triangle $I_1I_2I_3$.

(*vii*) Each of the four points I, I_1, I_2, I_3 is the orthocentre of the triangle formed by joining the other three points.

To find the sides and angles of the ex-central triangle.



With the figure of the-last article.

$\angle BI_1C = \angle BI_1I + \angle CI_1I$

$= \angle BCI + \angle CBI \ \dfrac{C}{2} + \dfrac{B}{2}$

$= 90^\circ - \dfrac{A}{2}$. Thus the angles are

$90^\circ - \dfrac{A}{2}$ $90^\circ - \dfrac{B}{2}$, $90^\circ - \dfrac{C}{2}$.

Again, the points B, I_3, I_2, C are concyclic.

$\therefore \angle I_1I_2I_3 = I_3BC = \angle I_1BC$

$\therefore$ the triangles $I_1I_2I_3$

= supplement of $\angle I_1BC$ are similar

$$\therefore \quad \frac{I_2I_3}{BC} = \frac{I_3I_1}{I_1C} = \sec\left(90^\circ - \frac{A}{2}\right) = \operatorname{cosec}\frac{A}{2}$$

$$\therefore \ I_2I_3 = a \operatorname{cosec}\frac{A}{2} = 4R\cos\frac{A}{2}$$

Thus the sides are $4R\cos\dfrac{A}{2}$, $4R\cos\dfrac{B}{2}$, $4R\cos\dfrac{C}{2}$

To find the area and circum-radius of the ex-central triangle.

AREA:

$$= \frac{1}{2} \text{ (product of two sides)} \times \text{(sine of included angle)}$$

$$= \frac{1}{2} \times 4R \cos \frac{B}{2} \times 4R \cos \frac{C}{2} \times \sin \left(90^\circ - \frac{A}{2}\right)$$

$$= 8R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

THE CIRCUM-RADIUS

$$= \frac{I_2I_3}{2\sin I_2I_1I_3} = \frac{4R\cos\frac{A}{2}}{2\sin\left(90^\circ - \frac{A}{2}\right)} = 2R$$

To find the distance between the in-centre and ex-centres.

The angles $\angle IBI_1$, $\angle ICI_1$ are right angles $\therefore$ II_1 is the diameter of the circum-circle of the triangle BCI_1

$$\therefore II_1 = \frac{BC}{\sin BI_1C} = \frac{a}{\cos\frac{A}{2}} = 4R \sin \frac{A}{2}.$$

Thus the distances are

$$4R \sin \frac{A}{2},\ 4R \sin \frac{B}{2},\ 4R \sin \frac{C}{2}$$

We have proved that OG, OH, OK bisect the angles HGK, KHG, GHK respectively, so that O is the in-centre of the triangle GHK. Thus the orthocentre of a triangle is the in-centre of the pedal triangle.

Again, the line CGB which is at right angles to OG bisect $\angle$HGK externally. Similarly the lines AHC and BKA bisect $\angle$KHG and $\angle$GKH externally, so that ABC is the ex-central triangle of its pedal triangle GHK.

N. DISTANCE OF SPECIAL POINTS FROM VERTICES AND SIDES OF A TRIANGLE

(*i*) Circumcentre (O): $OA = R$ & $O_a = R\cos A$

(*ii*) Incentre (I): $IA = r \operatorname{cosec} \frac{A}{2}$ & $I_a = r$

(*iii*) Excentre (I_1): $I_1A = r_1 \operatorname{cosec} \frac{A}{2}$ & $I_{1a} = r_1$

(*iv*) Orthocentre (H): HA $= 2R\cos A$ & $H_a = 2R \cos B \cos C$

(*v*) Centrod (G): GA $= \frac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}$ & $G_a = \frac{2\Delta}{3a}$

DISTANCE BETWEEN SPECIAL POINTS

(*a*) The distance between circumcentre and orthocentre is $= R\cdot\sqrt{1 - 8\cos A \cos B \cos C}$

(*b*) The distance between circumcentre and incentre is

$$= \sqrt{R^2 - 2Rr} = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

(*c*) Distance between circumcentre and centroid

$$OG = \sqrt{R^2 - \frac{1}{9}(a^2 + b^2 + c^2)}$$

(*d*) The distance between incentre and orthocentre is $\sqrt{2r^2 - 4R^2 \cos A \cos B \cos C}$

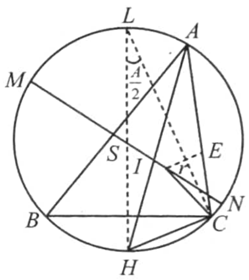
To find the distance between the in-centre and circum-centre.

Let S be the circum-centre and I the in-centre.Produce AI to meet the circum-circle in H; join CH and CI.

Draw IE perpendicular to AC. Produce HS to meet the circumference in L, and join CL. Then

$$\angle HIC = \angle IAC + \angle ICA = \frac{A}{2} + \frac{C}{2};$$

$$\angle HCI = \angle ICB + \angle BCH = \frac{C}{2} + \angle BAH = \frac{C}{2} + \frac{A}{2};$$



$\therefore\ \angle HCI = HIC;$

$\therefore\ HI = HC = 2R \sin \frac{A}{2}.$

Also $AI = IE \operatorname{cosec} \frac{A}{2} = r \operatorname{cosec} \frac{A}{2}$

$\therefore\ AI \,.\, IH = 2Rr$

Produce SI to meet the circumference in M and N. Since MIN, AIH are chords of the circle.

$AI.IH = MI.\ IN = (R + SI)(R - SI);$

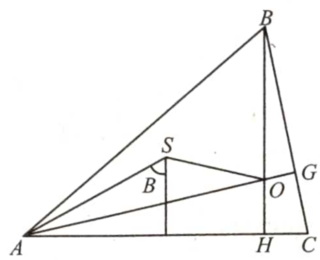
$\therefore 2Rr = R^2 - SI^2;\ SI^2 = R^2 - 2Rr$

To find the distance of the orthocentre from the circum-centre.

With the usual notation, we have

$SO^2 = SA^2 + AO^2 - 2SA\,.\,AO \cos SAO.$

Now $AS = R$; $AO = AH \operatorname{cosec} C = c \cos A \operatorname{cosec} C$



$= 2R \sin C \cos A \operatorname{cosec} C = 2R \cos A$; $\angle SAO$

$= \angle SAC - \angle OAC = (90° - B) - (90° - C) = C - B$

$\therefore SO^2 = R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos (C - B)$

$= R^2 - 4R^2 \cos^2 A \{\cos (B + C) + \cos (C - B)\}$

$= R^2 - 8R^2 \cos A \cos B \cos C.$

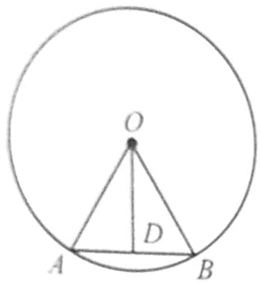
INSCRIBED AND CIRCUMSCRIBED POLYGONS

To find the perimeter and area of a regular polygon of n sides inscribed a circle.

Let r be the radius of the circle, and AB a side of the polygon.

Join OA, OB, and draw OD bisecting $\angle AOB$; then AB is bisected at right angles in D.

And $\angle AOB = \frac{1}{n}$ (four right angles) $= \frac{2\pi}{n}$



Perimeter of polygon

$= nAB = 2nAD = 2nAO \sin \Delta AOD = 2nr \sin \frac{\pi}{n}$

Area of polygon = n (area of triangle AOB)

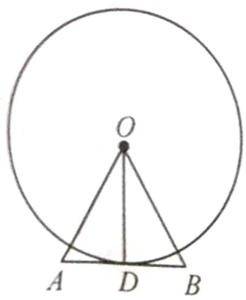
$= \frac{1}{2} nr^2 \sin \frac{2\pi}{n}.$

To find the perimeter and area of a regular polygon of n sides circumscribed about a given circle.

Let r be the radius of the circle, and AB a side of the polygon. Let AB touch the circle at D. Join OA, OB, OD; then OD bisects AB at right angles, and also bisects $\angle AOB$.

Perimeter $= nAB = 2nAD = 2nOD \tan \angle AOD = 2nr \tan \frac{\pi}{n}$.

Area of polygon $= n$ (area of triangle AOB)



$= nOD \,.\, AD = nr^2 \tan \frac{\pi}{n}$

Train Your Brain

Example 28: If f, g, h denote sides, the pedal triangle of a ΔABC, then show that

$$f \cdot \frac{(b^2 - c^2)}{a^2} + g \cdot \frac{(c^2 - a^2)}{b^2} + h \cdot \frac{(a^2 - g^2)}{c^2} = 0$$

Sol. Sides are a cos A, b cos B, c cos C. Hence LHS

$$\left(\frac{b^2 - c^2}{a}\right)\cos A + \left(\frac{c^2 - a^2}{b}\right)\cos B + \left(\frac{a^2 - b^2}{c}\right)\cos C$$

Put the values of cos A etc. get the result.

Example 29: Vertex A of a variable triangle ABC, inscribed in a circle of radius R, is a fixed point. If the angles subtended by the side BC at orthocentre (H), circumcentre (O) and incentre (I) are equal than identify the locus of orthocentre of triangle ABC.

Sol. The angles subtended by the side BC at points H, O and I are B + C, 2A and $\pi - \left(\frac{B + C}{2}\right)$ respectively.

$\Rightarrow B + C = 2A = 180 - \left(\frac{B + C}{2}\right)$

$\Rightarrow A = \frac{\pi}{3}$ and $B + C = \frac{2\pi}{3}$.

Also in triangle ABC, HA = 2R cos A = R

$\Rightarrow$ HA is contant

$\Rightarrow$ locus of orthocentre is a circle having centre at the vertex A.

Example 30: If I is the incentre and I_1, I_2, I_3 are the centre of escribed circles of the ΔABC, Prove that

(i) $II_1.II_2.II_3 = 16R^2r$

(ii) $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = I_1I_2^2 + II_3^2$

Sol. (i) $\because$ We know that

$II_1 = a \sec \frac{A}{2}$, $II_2 = b \sec \frac{B}{2}$ and $II_3 = c \sec \frac{C}{2}$

$\because I_1I_2 = c, \operatorname{cosec} \frac{C}{2}$,

$I_2I_3 = a \operatorname{cosec} \frac{A}{2}$ and $I_3I_1 = b \operatorname{cosec} \frac{B}{2}$

$\because II_1.II_2.II_3 = abc \sec \frac{A}{2} \sec \frac{B}{2} \sec \frac{C}{2}$ ... (i)

$\because a = 2R \sin A$, $b = 2R \sin B$ and $c = 2R \sin C$

$\therefore$ equation (i) becomes

$\because II_1.II_2.II_3 = (2R \sin A)(2R \sin B)$

$(2R \sin C) \sec \frac{A}{2} \sec \frac{B}{2} \sec \frac{C}{2}$

$$= 8R^3 \cdot \frac{\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)}{\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}$$

$$= 64R^3 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\because r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\therefore II_1.II_2.II_3 = 16R^2r$$

(*ii*) $II_1^2 + I_2I_3^2 = II_2^2\, I_3I_1^2 = II_3^2 + I_1I_2^2$

$$\because II_1^2 + I_2I_3^2$$

$$= a^2\sec^2\frac{A}{2} + a^2.\operatorname{cosec}^2\frac{A}{2} = \frac{a^2}{\sin^2\frac{A}{2}\cos^2\frac{A}{2}}$$

$$\because a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$$

$$\therefore II_1^2 + I_2I_3^2 = \frac{16R^2\sin^2\frac{A}{2}.\cos^2\frac{A}{2}}{\sin^2\frac{A}{2}.\cos^2\frac{A}{2}} = 16R^2$$

Hence $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$

Example 31: If r and R are radii of the incircle and circumcircle of ΔABC, prove that

$$8rR\left(\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}\right)$$

$$= 2\,bc + 2\,ca + 2\,ab - a^2 - b^2 - c^2$$

Sol. $\text{LHS} = \frac{8\Delta\, abc}{s\ \ 4\Delta}\Sigma\cos^2\frac{A}{2} = \frac{abc}{s}\left[\Sigma 2\cos^2\frac{A}{2}\right]$

$$= \frac{abc}{s}\left[(1 + \cos A) + (1 + \cos B) + (1 + \cos C)\right]$$

$$= \frac{abc}{s}\left[\frac{(b+c)^2 - a^2}{2bc} + \frac{(c+a)^2 - b^2}{2ca} + \frac{(a+b)^2 - c^2}{2ab}\right]$$

$$= \frac{abc}{s}\frac{(2s)}{2abc}\left[a(b+c-a) + b(c+a-b) + c(a+b-c)\right]$$

Example 32: If a, b, c denote the sides of a ΔABC, show that the value of the expression,

$a^3(p - q)(p - r) + b^2(q - r) + b^2(q - r)(q - p) + c^2(r - p)(r - q)$ cannot be negative where $p, q, r \in R$.

Sol. Let $p > q > r$ and $\left.\begin{matrix} p - q = y > 0 \\ q - r = z > 0 \end{matrix}\right\} \Rightarrow p - r = y + z > 0$

Consider $E = a^2y(y + z) - b^2zy + c^2z(y + z)$

$= a^2y^2 + c^2z^2 + yz(a^2 + c^2 - b^2)$

Now $b < a + c$

$\Rightarrow\ b_{max} = a + c$

$E_{least} = a^2y^2 + c^2z^2 + yz[a^2 + c^2 - (a + c)^2]$

$= a^2y^2 + c^2z^2 - 2acyz = (ay - cz)^2 \geq 0$

Concept Application

28. If in ΔABC, the distances of the vertices from the orthocenter are x, y, and z, then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$

29. In ΔABC, let L, M, N be the feet of the altitudes. Then prove that $\sin(\angle MLN) + \sin(\angle LMN) + \sin(\angle LMN)$ $= 4\sin A \sin B \sin C$

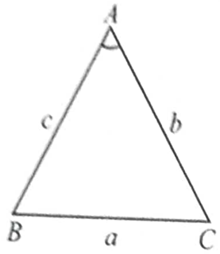
30. Let ABC be an acute triangle whose orthocenter is at H. If altitude from A is produced to meet the circumcircle of triangle ABC at D, then find HD

31. In a triangle ABC if $2\cos\frac{B}{2}\cos\frac{C}{2} = \frac{1}{2} + \left(\frac{b+c}{a}\right)$ $\sin\frac{A}{2}$ then find the measure of angle A.

Short Notes

1. Sine Rule:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$



2. Cosine Formula:

(i) $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$

(ii) $\cos B=\dfrac{c^2+a^2-b^2}{2ca}$

(iii) $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$

3. Projection Formula

(i) $a=b\cos C+c\cos B$

(ii) $b=c\cos A+a\cos C$

(iii) $c=a\cos B+b\cos A$

4. Napier's Analogy - Tangent Rule

(i) $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

(ii) $\tan\dfrac{C-A}{2}=\dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

(iii) $\tan\dfrac{A-B}{2}=\dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

5. Trigonometric Functions of Half Angles

(i) $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$; $\sin\dfrac{B}{2}=\sqrt{\dfrac{(s-c)(s-a)}{ca}}$;

$\sin\dfrac{C}{2}=\sqrt{\dfrac{(s-a)(s-b)}{ab}}$

(ii) $\cos\dfrac{A}{2}=\sqrt{\dfrac{s(s-a)}{bc}}$; $\cos\dfrac{B}{2}=\sqrt{\dfrac{s(s-b)}{ca}}$; $\cos\dfrac{C}{2}=\sqrt{\dfrac{s(s-c)}{ab}}$

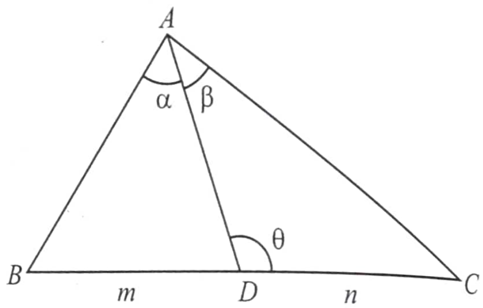
(iii) $\tan\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}=\dfrac{\Delta}{s(s-a)}$ where $s=\dfrac{a+b+c}{2}$ is semi perimetre of triangle.

(iv) $\sin A=\dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\dfrac{2\Delta}{bc}$

6. Area of Triangle (Δ)

$$\Delta=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B$$
$$=\sqrt{s(s-a)(s-b)(s-c)}.$$

7. m-n Rule



If $BD:DC=m:n$, then

$(m+n)\cot\theta=m\cot\alpha-n\cot\beta$

$=n\cot B-m\cot C$

8. Radius of Circumcircle

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}$$

9. Radius of The Incircle

(i) $r=\dfrac{\Delta}{s}$

(ii) $r=(s-a)\tan\dfrac{A}{2}=(s-b)\tan\dfrac{B}{2}=(s-c)\tan\dfrac{C}{2}$

(iii) $r=\dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ and so on

(iv) $r=4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

10. Radius of The Ex-Circles

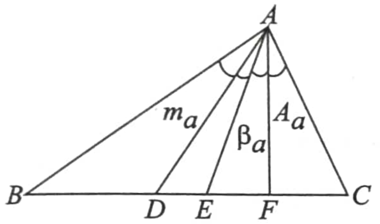
(i) $r_1 = \frac{\Delta}{s-a}$; $r_2 = \frac{\Delta}{s-b}$; $r_3 = \frac{\Delta}{s-c}$

(ii) $r_1 = s\tan\frac{A}{2}$; $r_2 = s\tan\frac{B}{2}$; $r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ and so on.

(iv) $r_1 = 4R\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$

11. Length of Angle Bisectors, Medians and Altitudes



(i) Length of an angle bisector from the angle $A = \beta_a = \frac{2\,bc\cos\frac{A}{2}}{b+c}$.

(ii) Length of median from angle $A = m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$.

(iii) Length of altitude from the angle $A = A_a = \frac{2\Delta}{a}$.

12. The Distances of the special Points from Vertices and Sides of Triangle

(i) Circumcentre (O) : $OA = R$ and $O_a = R\cos A$

(ii) Incentre (I) : $IA = r\,\text{cosec}\,\frac{A}{2}$ and $I_a = r$

(iii) Excentre (I_1) : $I_1A = r_1\,\text{cosec}\,\frac{A}{2}$

(iv) Orthocentre : $HA = 2R\cos A$ & $H_a = 2R\cos B\cos C$

(v) Centroid (G) : $GA = \frac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}$ and $G_a = \frac{2\Delta}{3a}$

13. Orthocentre and Pedal Triangle

The triangle KLM which is formed by joining the feet of the altitudes is called the Pedal Triangle.

(i) Its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$.

(ii) Its sides are $a\cos A = R\sin 2A$,

$b\cos B = R\sin 2B$ and

$c\cos C = R\sin 2C$

(iii) Circumradii of the triangles PBC, PCA, PAB and ABC are equal. Where P is orthocenter of ΔABC.

14. Excentral Triangle

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

(i) ΔABC is the pedal triangle of the $\Delta I_1I_2I_3$.

(ii) Its angles are $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$.

(iii) Its sides are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$.

(iv) $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

(v) Incentre I or ΔABC is the orthocentre of the excentral $\Delta I_1I_2I_3$.

15. Distance Between Special Points

(i) Distance between circumcentre and orthocentre

$OH^2 = R^2(1 - 8\cos A\cos B\cos C)$

(ii) Distance between circumcentre and incentre

$OI^2 = R^2\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = R^2 - 2Rr$

(iii) Distance between circumcentre and centroid

$OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$

Solved Examples

1. If in a triangle ABC $\sin A = \sin^2 B$ and $2\cos^2 A = 3\cos^2 B$, then the ΔABC is -

(a) right angled (b) obtuse angled

(c) isosceles (d) equilateral

Sol. $\sin A = \sin^2 B$ ... (i)

$2\cos^2 A = 3\cos^2 B$ ... (ii)

from (i) and (ii)

$2(1 - \sin^2 A) = 3(1 - \sin A)$

$\Rightarrow 2\sin^2 A - 3\sin A + 1 = 0$

$\Rightarrow (2\sin A - 1)(\sin A - 1) = 0$

$\Rightarrow \sin A = \frac{1}{2}$ or $\sin A = 1$

But $\sin A \neq 1$

$\Rightarrow \angle A = 30^\circ$

from (i) $\sin^2 B = \frac{1}{2}$

$\angle B = 45^\circ$

and $\angle C = 180^\circ - 75^\circ = 105^\circ$

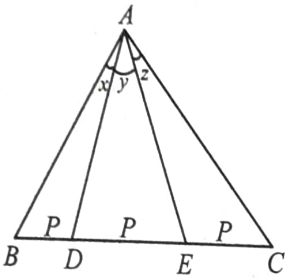
ΔABC is obtuse angled.

2. Points D, E are taken on the side BC of a triangle ABC such that $BD = DE = EC$. If $\angle BAD = x$, $\angle DAE = y$, $\angle EAC = z$, then the value of $\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z} =$

(*a*) 1 (*b*) 2

(*c*) 4 (*d*) none

Sol.



$\because \frac{AD}{BD} = \frac{\sin B}{\sin x}$ and $\frac{AD}{DC} = \frac{\sin C}{\sin(y+z)}$ ... (i)

and $\frac{AE}{BE} = \frac{\sin B}{\sin(x+y)}$ and $\frac{AE}{EC} = \frac{\sin C}{\sin z}$ ... (ii)

From (i) $\frac{DC}{BD} = \frac{\sin B}{\sin C} \cdot \frac{\sin(y+z)}{\sin x}$ ... (iii)

From (ii) $\frac{BE}{EC} = \frac{\sin C}{\sin B}, \frac{\sin(x+y)}{\sin z}$ ... (iv)

$\because \frac{DC}{BD} = 2$ and $\frac{BE}{EC} = 2$

Then from (iii) and (iv), we get

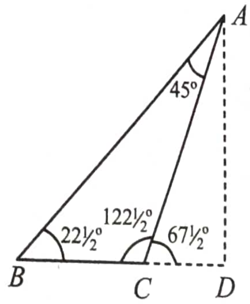
$\frac{\sin(x+y)\sin(y+z)}{\sin x \sin z} = 4$

3. If the base angles of a triangle are $22\frac{1}{2}^\circ$ and $112\frac{1}{2}^\circ$, the height of the triangle is equal to -

(*a*) half the base (*b*) the base

(*c*) twice the base (*d*) four times the base

Sol.



From sine rule, we have

$\frac{BC}{\sin 45^\circ} = \frac{AC}{\sin 22^{1/2^\circ}}$ ...(i)

Also $\frac{AD}{AC} = \sin 671/2^\circ$

$AB = AC \cos 22\frac{1}{2}^\circ$

$\Rightarrow AD = \frac{BC}{\sin 45^\circ} \sin 22\frac{1}{2}^\circ \cos 22\frac{1}{2}^\circ$ from (i)

$\Rightarrow AD = \frac{BC}{2} \cdot \frac{\sin 45^\circ}{\sin 45^\circ}$

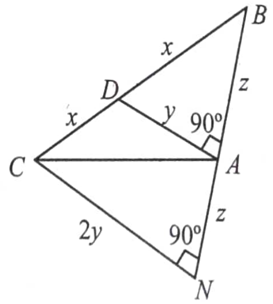
$\Rightarrow AD = \frac{1}{2} BC = \frac{1}{2}$ of base

4. In a ΔABC if median AD is perpendicular to AB, then $\tan A + 2\tan B$ is equal to -

(*a*) 1 (*b*) 3

(*c*) 0 (*d*) $\frac{1}{2}$

Sol.



Given that $BD = DC$; $\angle DAB = 90^\circ$

Draw a perpendicular CN at extend BA

In ΔBCN; $DA = \frac{1}{2} CN = y$ and $AB = AN = z$

$\therefore \tan A = \tan(\pi - \angle CAN) = -\tan(\angle CAN)$

$= -\frac{CN}{AN} = -\frac{2y}{z}$

$\Rightarrow \tan A = -\frac{2AD}{AB}$

$\Rightarrow \tan A = -2\tan B$

$\Rightarrow \tan A + 2\tan B = 0$

5. In a triangle, $a^2 + b^2 + c^2 = ca + ab\sqrt{3}$, then the triangle is -

(*a*) equilateral

(*b*) right-angled and isosceles

(*c*) right-angled with

(*d*) None of these

Sol. Write $a^2 = \frac{a^2}{4} + \frac{3a^2}{4}$

then, $\left(\frac{a^2}{4} + c^2 - ca\right) + \left(\frac{3a^2}{4} - ab\sqrt{3} + b^2\right) = 0$

$\Rightarrow \left(\frac{a}{2} - c\right)^2 + \left(\frac{\sqrt{3}a}{2} - b\right)^2 = 0$

$\Rightarrow c = \frac{a}{2}$ and $b = \frac{\sqrt{3}a}{2}$

then $b^2+c^2=\frac{3a^2}{4}+\frac{a^2}{4}=a^2$

triangle is right triangled, $A=90^\circ$

$\cos B=\frac{C}{1}=\frac{1}{2}; B=60^\circ$ and $C=30^\circ$

6. If in a ΔABC, $a^2\cos^2 A=b^2+c^2$, then-

(a) $A<\frac{\pi}{2}$ (b) $\frac{\pi}{4}<A<\frac{\pi}{2}$

(c) $A>\frac{\pi}{2}$ (d) $A=\frac{\pi}{2}$

Sol. Given $a^2\cos^2 A=b^2+c^2$

we know that $\cos A=\frac{b^2+c^2-a^2}{2bc}$

$\Rightarrow \cos A=\frac{a^2\cos^2 A-a^2}{2bc}$

or $\cos A=-\frac{a^2\sin^2 A}{2bc}=-ve$

$A>\frac{\pi}{2}$

7. In a ΔABC, if $a^4+b^4+c^4=2c^2(a^2+b^2)$, then $\angle C$ is equal to-

(a) 60° (b) 135°

(c) 90° (d) 75°

Sol. $a^4+b^4+c^4-2a^2c^2-2b^2c^2=0$

$a^4+b^4+c^4-2a^2c^2-2b^2c^2+2a^2b^2=2b^2c^2$

$\Rightarrow (a^2+b^2-c^2)^2=2a^2b^2$

$\Rightarrow \left(\frac{a^2+b^2-c^2}{2ab}\right)^2=\frac{1}{2}$

$\Rightarrow \cos^2 C=\frac{1}{2}$

$\Rightarrow \cos C=\pm\frac{1}{\sqrt{2}}$

$\Rightarrow C=45^\circ, 135^\circ$

8. In ΔABC, $a=5$, $b=4$ and $\tan\frac{C}{2}=\sqrt{\frac{7}{9}}$. The side is -

(a) 6 (b) 3

(c) 2 (d) None

Sol. $\because \cos C=\frac{1-\tan^2\frac{C}{2}}{1+\tan^2\frac{C}{2}}$

$\because \tan\frac{C}{2}=\sqrt{\frac{7}{9}}$

$\Rightarrow \cos C=\frac{1-\frac{7}{9}}{1+\frac{7}{9}}$

$\Rightarrow \cos C=\frac{1}{8}$

$\because \cos C=\frac{a^2+b^2-c^2}{2ab}$

$\because a=5, b=4$

$\Rightarrow \frac{1}{8}=\frac{25+16-c^2}{40}$

$\Rightarrow c^2=36$

$\Rightarrow c=6$

9. In a ΔABC, $A=\frac{2\pi}{3}, b-c=3\sqrt{3}\,cm$ and ar (ΔABC) $=\frac{9\sqrt{3}}{2}cm^2$. Then a is -

(a) $6\sqrt{3}\,cm$ (b) 9 cm

(c) 18 cm (d) None of these

Sol. Given $A=\frac{2\pi}{3}, b-c=3\sqrt{3}$ & ar$(\Delta ABC)=\frac{2\sqrt{3}}{2}$

$\because \Delta=\frac{1}{2}bc\sin A$

$\Rightarrow \frac{1}{2}bc\sin\frac{2\pi}{3}=\frac{9\sqrt{3}}{2}$

$\Rightarrow bc=18$ and $b-c=3\sqrt{3}$

$\Rightarrow b^2+c^2=27+2bc=27+36$

$\Rightarrow b^2+c^2=36$

$\because \cos A=\frac{b^2+c^2-a^2}{2bc}$

$\Rightarrow \cos\frac{2\pi}{3}=\frac{63-a^2}{2\times 18}$

$\Rightarrow -\frac{1}{2}=\frac{63-a^2}{36}$

$\Rightarrow a^2=63+18=81$

$\Rightarrow a=9$

10. In a ΔABC, $\cos B.\cos C+\sin B.\sin C\sin^2 A=1$. Then the triangle is-

(a) right-angle isosceles

(b) isosceles whose equal angles are greater than

(c) equilateral

(d) none of these

Sol. $\sin^2 A=\frac{1-\cos B\cos C}{\sin B\sin C}\le 1$

$\Rightarrow 1\le \cos B\cos C+\sin B\sin C$

$\Rightarrow 1\le \cos(B-C)$

$\Rightarrow 1=\cos(B-C)$

$\Rightarrow B-C=0$

$\Rightarrow$ B = c $\quad(\therefore \cos\theta\ge 1)$

then from (i)

$$\sin^2 A = \frac{1-\cos^2 B}{\sin^2 B} = 1$$

$\Rightarrow \sin A = 1$

$\Rightarrow A = 90°$

$B = C = 45°$

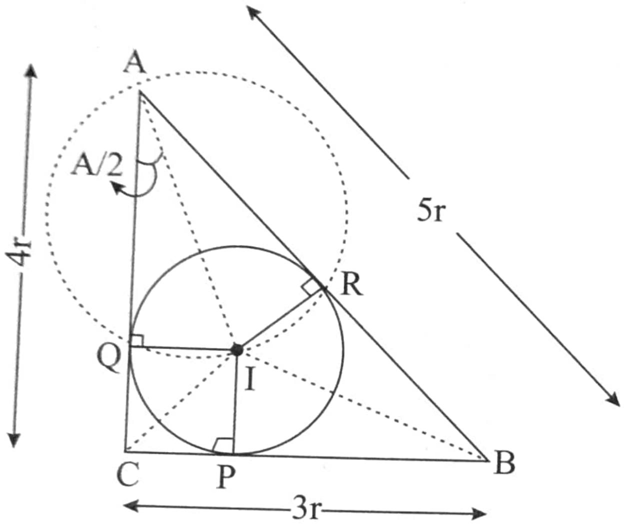
triangle is right angled isosceles.

11. Tangents at P, Q, R on a circle of radius r form a triangle whose sides are $3r, 4r, 5r$ and $PR^2+RQ^2+QP^2$ equal to $\frac{ar^2}{b}$ then a + b is equal to:

Sol. In $\Delta ARQ = \frac{RQ}{\sin A} = 2AI = \frac{2r}{\sin\left(\frac{A}{2}\right)}; RQ = 4r\cos\left(\frac{A}{2}\right)$

Similarly, $RP = 4r\cos\left(\frac{B}{2}\right), PQ = 4r\cos\left(\frac{C}{2}\right)$

$\therefore PR^2 + RQ^2 + QP^2$



$$= 16r^2\left[\cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) + \cos^2\left(\frac{C}{2}\right)\right]$$

$$= 16r^2\left[\frac{1+\cos A}{2} + \frac{1+\cos B}{2} + \frac{1}{2}\right]$$

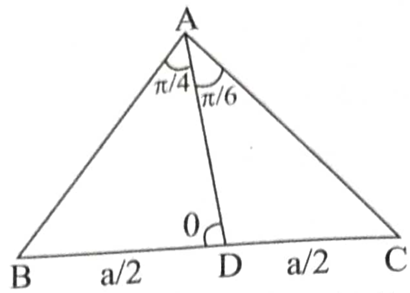
$$= 8r^2\left[3 + \frac{3}{5} + \frac{4}{5}\right] = 8r^2\left[\frac{15+7}{5}\right] = \frac{176r^2}{5}$$

12. In a ΔABC, the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides angle A into the angles of measure $\frac{\pi}{6}$ and $\frac{\pi}{4}$.

Then the length of side BC is.

Sol. By m–n theorem, $2\cot\theta = \cot\frac{\pi}{6} - \cot\frac{\pi}{4}$

$\Rightarrow 2\cot\theta = \sqrt{3}-1 \;\Rightarrow \cot\theta = \frac{\sqrt{3}-1}{2}$ is acute



Again applying sine rule to ΔABD, we have

$$\frac{AD}{\sin B} = \frac{AB}{\sin\theta} = \frac{BD}{\sin\pi/4}$$

$$\Rightarrow \frac{AD}{\sin\left(\pi - \frac{\pi}{4} - \theta\right)} = \frac{c}{\sin\theta} = \frac{a/2}{1/\sqrt{2}}$$

$$\Rightarrow \frac{AD}{\sin\left(\frac{\pi}{4}+\theta\right)} = \frac{c}{\sin\theta} = \frac{a}{\sqrt{2}}$$

$$\Rightarrow AD = \frac{a}{\sqrt{2}}\sin\left(\frac{\pi}{4}+\theta\right)$$

$$\Rightarrow AD = \frac{a}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right]$$

$$\Rightarrow AD = \frac{a}{2}(\sin\theta + \cos\theta)$$

$$\Rightarrow AD = \frac{a}{2}\left[\frac{2}{4-\sqrt{3}} + \sqrt{\frac{4-2\sqrt{3}}{8-2\sqrt{3}}}\right]$$

$$\Rightarrow AD = \frac{a}{2}\left[\frac{2}{\sqrt{8-\sqrt{3}}} + \frac{\sqrt{4-2\sqrt{3}}}{\sqrt{8-2\sqrt{3}}}\right]$$

$$\Rightarrow AD = \frac{a}{2}\left[\frac{2+\sqrt{4-2\sqrt{3}}}{\sqrt{8-2\sqrt{3}}}\right]$$

$$\Rightarrow AD = \frac{a}{2}\left[\frac{\sqrt{3}+1}{\sqrt{8-2\sqrt{3}}}\right]$$

$$\Rightarrow BC = a = \frac{2AD\left(\sqrt{8-2\sqrt{3}}\right)}{\left(\sqrt{3}+1\right)}$$

$$= 2\times\frac{1}{\sqrt{11-6\sqrt{3}}}\times\frac{\sqrt{8-2\sqrt{3}}}{\left(\sqrt{3}+1\right)}$$

$$= \frac{2\sqrt{8-2\sqrt{3}}}{\sqrt{\left(11-6\sqrt{3}\right)\left(4+2\sqrt{3}\right)}}$$

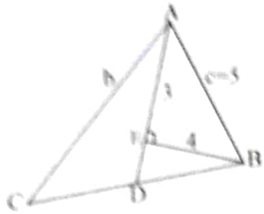
$$= 2\sqrt{\frac{8-2\sqrt{3}}{8-2\sqrt{3}}} = 2 \text{ Thus } BC = 2$$

13. In ΔABC, $AC > AB$, the internal angle bisector of angle A meets BC at D and E is the foot of the perpendicular from

B onto AD. Suppose AB = 5 and BE = 4, find the value of the expression $\left(\frac{AC+AB}{AC-AB}\right)(ED)$.

Sol. $\cos\frac{A}{2} = \frac{3}{5}(AE = 3)$

$$x = \frac{2bc}{b+c}\cos\frac{A}{2} \Rightarrow x = \frac{2b\cdot 5}{b+5}\cdot\frac{3}{5}$$



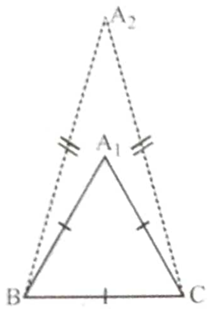
$$x = \frac{6b}{b+5} = AD; \qquad ED = AD - AE$$

$$ED = \frac{6b}{b+5} - 3 = \frac{6b-3b-15}{b+5} =$$

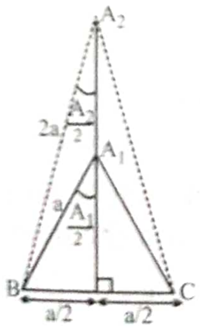
$$\text{Expression} = \frac{AC+AB}{AC-AB}\cdot \text{ED}$$

$$= \left(\frac{b+5}{b-5}\right)\cdot 3\left(\frac{b-5}{b+5}\right) = 3$$

14. The length of two sides of an equilateral triangle are doubled, creating an isosceles triangle, as shown in the diagram. The two longer sides are doubled again creating a third isosceles triangle (all three triangles have the same base BC). This process is continued indefinitely. If the measure of the vertex angle of each triangle is represented by A1, A2, A3, ..., and $\sum_{n=1}^{\infty}(1-\cos A_n) = \frac{p}{q}$ $(p, q \Rightarrow N)$, then find the minimum value of $(p+q)$.



Sol.



$$\sin\frac{A_1}{2} = \frac{1}{2}(A_1 = 60^\circ)$$

$$\sin\frac{A_2}{2} = \frac{a}{2\times 2a} = \frac{1}{4}$$

$$\sin\frac{A_3}{2} = \frac{a}{2\times 4a} = \frac{1}{8}$$

$$\vdots$$

$$\sin\frac{A_n}{2} = \frac{a}{2\times 2^{n-1}a} = \frac{1}{2^n}$$

$$\Sigma_{n=1}^{\infty}(1-\cos A_n) = 2\Sigma_{n=1}^{\infty}\left(\sin^2\frac{A_n}{2}\right)$$

$$= \text{Lim}_{n\to\infty} 2\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{8^2}+........++\frac{1}{2^{2n}}\right) = \frac{2\times\frac{1}{2^2}}{1-\frac{1}{4}} = \frac{2}{3}$$

15. In a triangle ABC, if α, β, γ are the altitudes and r is the inradius then minimum value of $\frac{\alpha+r}{\alpha-r}+\frac{\beta+r}{\beta-r}+\frac{\gamma+r}{\gamma-r}$ is

Sol.

$$AD = \alpha, BE = \beta, CF = \gamma.$$

$$\Delta = \frac{1}{2}a\alpha \Rightarrow \alpha = \frac{2\Delta}{a}$$

Similarly $\beta = \frac{2\Delta}{b}$ and $\gamma = \frac{2\Delta}{c}$

$$\frac{\alpha+r}{\alpha-r}+\frac{\beta+r}{\beta-r}+\frac{\gamma+r}{\gamma-r}$$

$$\text{Now} = \frac{\frac{2\Delta}{a}+\frac{\Delta}{s}}{\frac{2\Delta}{a}-\frac{\Delta}{s}}+\frac{\frac{2\Delta}{b}+\frac{\Delta}{s}}{\frac{2\Delta}{b}-\frac{\Delta}{s}}+\frac{\frac{2\Delta}{c}+\frac{\Delta}{s}}{\frac{2\Delta}{c}-\frac{\Delta}{s}}$$

$$= \frac{2s+a}{2s-a}+\frac{2s+b}{2s-b}+\frac{2s+c}{2s-c}$$

$$= \left(\frac{2s+a}{2s-a}+1\right)+\left(\frac{2s+b}{2s-b}+1\right)+\left(\frac{2s+c}{2s-c}+1\right)-3$$

$$= \frac{4s}{2s-a}+\frac{4s}{2s-b}+\frac{4s}{2s-c}-3$$

Now $AM \geq HM$

$$\frac{1}{3}\left(\frac{4s}{2s-a}+\frac{4s}{2s-b}+\frac{4s}{2s-c}\right) \geq \frac{3}{\left(\frac{2s-a}{4s}+\frac{2s-b}{4s}+\frac{2s-c}{4s}\right)}$$

$$\Rightarrow \frac{4s}{2s-a}+\frac{4s}{2s-b}+\frac{4s}{2s-c} \geq 9$$

$$= \left\{\frac{\alpha+r}{\alpha-r}+\frac{\beta+r}{\beta-r}+\frac{\gamma+r}{\gamma-r}\right\} \geq 6$$

$$\Rightarrow \frac{\alpha+r}{a-r}+\frac{\beta+r}{\beta-r}+\frac{\gamma+r}{\gamma-r} \geq 6$$

Exercise-1 (Topicwise)

1. If the angles of a triangle ABC be in A.P., then
   (*a*) $c^2 = a^2 + b^2 - ab$ (*b*) $b^2 = a^2 + c^2 - ac$
   (*c*) $a^2 = b^2 + c^2 - ac$ (*d*) $b^2 = a^2 + c^2$

2. In ΔABC, $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) =$
   (*a*) 0 (*b*) $a + b + c$
   (*c*) $a^2 + b^2 + c^2$ (*d*) $2(a^2 + b^2 + c^2)$

3. In ΔABC, $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C =$
   (*a*) 0 (*b*) $a^2 + b^2 + c^2$
   (*c*) $2(a^2 + b^2 + c^2)$ (*d*) $\frac{1}{2abc}$

4. If the sides of a triangle are p, q and $\sqrt{p^2 + pq + q^2}$, then the biggest angle is
   (*a*) $\pi/2$ (*b*) $2\pi/3$
   (*c*) $5\pi/4$ (*d*) $7\pi/4$

5. In a triangle ABC, if B = 3C, then the values of $\sqrt{\left(\frac{b+c}{4c}\right)}$ and $\left(\frac{b-c}{2c}\right)$ are
   (*a*) $\sin C, \sin\frac{A}{2}$ (*b*) $\cos C, \sin\frac{A}{2}$
   (*c*) $\sin C, \cos\frac{A}{2}$ (*d*) $\cos C, \cos\frac{A}{2}$

6. In a ΔABC, if $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then cos C =
   (*a*) $\frac{7}{5}$ (*b*) $\frac{5}{7}$
   (*c*) $\frac{17}{36}$ (*d*) $\frac{16}{17}$

7. In a triangle ABC, $b = \sqrt{3}$, $c = 1$ and $\angle A = 30^o$, then the largest angle of the triangle is
   (*a*) 135° (*b*) 90°
   (*c*) 60° (*d*) 120°

8. In a ΔABC, $2ac \sin\left(\frac{A-B+C}{2}\right)$ is equal to
   (*a*) $a^2 + b^2 - c^2$ (*b*) $c^2 + a^2 - b^2$
   (*c*) $b^2 - c^2 - a^2$ (*d*) $c^2 - a^2 - b^2$

9. If in triangle ABC, $\cos A = \frac{\sin B}{2 \sin C}$, then the triangle is
   (*a*) Equilateral (*b*) Isosceles
   (*c*) Right angled (*d*) Scalene

10. If a, b and c are the sides of a triangle such that $a^4 + b^4 + c^4 = 2c^2(a^2 + b^2)$ then the angles opposite to the side c is
    (*a*) 45° or 135° (*b*) 30° or 100°
    (*c*) 50° or 100° (*d*) 60° or 120°

11. In a triangle ABC, $(a + b + c)(b + c - a) = \lambda bc$ if -
    (*a*) $\lambda < 0$ (*b*) $\lambda > 0$
    (*c*) $0 < \lambda < 4$ (*d*) $\lambda > 4$

12. Let ABC be a triangle such that $\angle A = 45°$, $\angle B = 75°$ then $a + c\sqrt{2}$ equal to -
    (*a*) 0 (*b*) b
    (*c*) $2b$ (*d*) $-b$

13. Angles A, B and C of a triangle ABC are in A.P. If $\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$, then angle A is equal to -
    (*a*) $\frac{\pi}{6}$ (*b*) $\frac{\pi}{4}$
    (*c*) $\frac{5\pi}{12}$ (*d*) $\frac{\pi}{2}$

14. If in a triangle ABC, $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A-B)}{\sin(A+B)}$ then the triangle is -
    (*a*) Right angled or isosceles
    (*b*) Right angled and isosceles
    (*c*) Equilateral
    (*d*) None of these

15. In any triangle ABC,
    $\frac{a^2 \sin(B-C)}{\sin B + \sin C} + \frac{b^2 \sin(C-A)}{\sin C + \sin A} + \frac{c^2 \sin(A-B)}{\sin A + \sin B}$ is equal to
    (*a*) $a + b + c$ (*b*) $a + b - c$
    (*c*) $a - b + c$ (*d*) 0

16. The expression $\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$ is equal to
    (*a*) $\cos^2 A$ (*b*) $\sin^2 A$
    (*c*) $\cos A \cos B \cos C$ (*d*) None of these

Projection Formula, Napier's Analogy & Area of Triangle Half-Angled Formulae m-n theorem

17. If in a triangle ABC, $b = \sqrt{3}$, c = 1 and B – C = 90° then $\angle A$ is
    (*a*) 30° (*b*) 45°
    (*c*) 75° (*d*) 15°

18. If a^2, b^2, c^2 are in A. P. then which of the following are also in A.P.

(a) sin A, sin B, sin C
(b) tan A, tan B, tan C
(c) cot A, cot B, cot C
(d) cosec A, cosec B, cosec C

19. If in a triangle ABC, cos A cos B + sin A sin B sin C = 1, then the sides are proportional to

(a) $1:1:\sqrt{2}$ (b) $1:\sqrt{2}:1$
(c) $\sqrt{2}:1:1$ (d) $1:1:1$

20. If in the ΔABC, AB = 2BC, then $\tan\frac{B}{2}:\cot\left(\frac{C-A}{2}\right)$

(a) 3:1 (b) 2:1
(c) 1:2 (d) 1:3

21. If in a triangle ABC, $a = 5$, $b = 4$, $A = \frac{\pi}{2}+B$, then $C =$

(a) $\tan^{-1}\left(\frac{1}{9}\right)$ (b) $\tan^{-1}\frac{1}{40}$
(c) Cannot be evaluated (d) $2\tan^{-1}\left(\frac{1}{9}\right)$

22. If Δ is the area and 2s the sum of 3 sides of triangle, then

(a) $\Delta \le \frac{s^2}{3\sqrt{3}}$ (b) $\Delta \le \frac{s^2}{2}$
(c) $\Delta > \frac{s^2}{\sqrt{3}}$ (d) $\Delta \ge \frac{s^2}{3\sqrt{3}}$

23. If in a triangle ABC right angled at B, $s-a=3$, $s-c=2$, then the values of a and c are respectively

(a) 2, 3 (b) 3, 4
(c) 4, 3 (d) 6, 8

24. In a ΔABC a, c, A are given and b_1, b_2 are two values of the third side b such that $b_2 = 2b_1$. Then sin A =

(a) $\sqrt{\frac{9a^2-c^2}{8a^2}}$ (b) $\sqrt{\frac{9a^2-c^2}{8c^2}}$
(c) $\sqrt{\frac{9a^2+c^2}{8a^2}}$ (d) $\sqrt{\frac{9a^2+c^2}{8c^2}}$

25. In a ΔABC, $a^2\sin 2C + c^2\sin 2A =$

(a) Δ (b) 2Δ
(c) 3Δ (d) 4Δ

26. The product of the arithmetic mean of the lengths of the sides of a triangle and harmonic mean of the lengths of the altitudes of the triangle is equal to

(a) Δ (b) 2Δ
(c) 3Δ (d) 4Δ

27. In a ΔABC, if AB = 5 cm, BC = 13 cm and CA = 12 cm, then the distance of vertex A from the side BC is (in cm)

(a) $\frac{25}{13}$ (b) $\frac{60}{13}$
(c) $\frac{65}{12}$ (d) $\frac{144}{13}$

28. In a ΔABC, if $b = 2$, $c = \sqrt{3}$ and $\angle A = \frac{\pi}{6}$, then value of R is equal to

(a) $\frac{1}{2}$ (b) 1
(d) 2 (d) $\frac{1}{4}$

29. In a triangle ABC, a2cos2B + b2cos2A + 2abcos(A − B) =

(a) a^2 (b) c^2
(c) b^2 (d) a^2+b^2

30. If in a triangle ABC, $b=\sqrt{3}$, $c = 1$ and $B-C=90°$ then $\angle A$ is

(a) 30° (b) 45° (c) 75° (d) 15°

31. If i a triangle ABC, $(s-a)(s-b) = s(s-c)$, then angle C is equal to

(a) 90° (b) 45°
(c) 30° (d) 60°

32. In ΔABC, if $\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}$ be in H. P. then a, b, c will be in

(a) A. P. (b) G. P.
(c) H. P. (d) A.G.P.

33. In any triangle ABC, $\frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}} =$

(a) $\frac{a-b}{a+b}$ (b) $\frac{a-b}{c}$
(c) $\frac{a-b}{a+b+c}$ (d) $\frac{c}{a+b}$

34. In any triangle ABC, the value of $a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2)\cos C$ is

(a) $3abc^2$ (b) $3a^2bc$
(c) $3abc$ (d) $3ab^2c$

35. In a triangle ABC, AD is altitude from A. Given b > c, $\angle C = 23°$ and $AD = \frac{abc}{b^2-c^2}$, then $\angle B =$

(a) 67° (b) 44°
(c) 113° (d) 157°

36. In a ΔABC if the sides are $a = 3$, $b = 5$ and $c = 4$, then $\sin\frac{B}{2}+\cos\frac{B}{2}$ is equal to

(a) $\sqrt{2}$ (b) $\frac{\sqrt{3}+1}{2}$
(c) $\frac{\sqrt{3}-1}{2}$ (d) 1

37. Which of the following is true in a triangle ABC

(a) $(b+c)\sin\frac{B-C}{2}=2a\cos\frac{A}{2}$

(b) $(b+c)\cos\frac{A}{2}=2a\sin\frac{B-C}{2}$

(c) $(b-c)\cos\frac{A}{2}=a\sin\frac{B-C}{2}$

(d) $(b-c)\sin\frac{B-C}{2}=2a\cos\frac{A}{2}$

38. In a ΔABC, if $3a=b+c$, then the value of $\cot\frac{B}{2}\cot\frac{C}{2}$ is

(a) 1 (b) 2

(c) $\sqrt{3}$ (d) $\sqrt{2}$

39. In ΔABC, $2R^2\sin A\sin B\sin C=$

(a) s^2 (b) $ab+bc+ca$

(c) Δ (d) Zero

40. In an equilateral triangle the inradius and the circum-radius are connected by

(a) $r=4R$ (b) $r=R/2$

(c) $r=R/3$ (d) $r=R/4$

41. If the sides of the triangle are 5K, 6K, 5K and radius of incircle is 6 then value of K is equal to

(a) 4 (b) 5

(c) 6 (d) 7

42. The circum-radius of the triangle whose sides are 13, 12 and 5 is

(a) 15 (b) 13/2

(c) 15/2 (d) 6

43. In a ΔABC, if $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$, and the side $a=2$, then area of the triangle is

(a) 1 (b) 2

(c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{3}$

44. We are given and such that is acute and . Then -

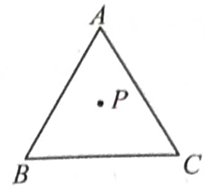
(a) No triangle is possible

(b) One triangle is possible

(c) Two triangles are possible

(d) A right angled triangle is possible

45. In the adjacent figure 'P' is any interior point of the equilateral triangle ABC of side length 2 unit



If x_a, x_b and represent the distance of P from the sides BC, CA and AB respectively then $x_a+x_b+x_c$ is equal to -

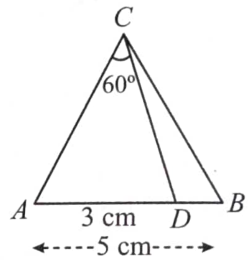
(a) 6 (b) $\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) $2\sqrt{3}$

46. If two sides a, b and the angle A be such that two triangles are formed, then the sum of the two values of the third side is

(a) b^2-a^2 (b) $2b\cos A$

(c) $2b\sin A$ (d) $\frac{b-c}{b+c}$

47. In the figure, ABC is triangle in which $C=90°$ and $AB=5$ cm. D is a point on ABsh that $AD=3$ cm and $\angle ACD=60°$. Then th length of AC is



(a) $5\sqrt{\frac{3}{7}}\,cm$ (b) $\sqrt{\frac{3}{7}}\,cm$

(c) $\frac{3}{\sqrt{7}}\,cm$ (d) None of these

Exercise-2 (Learning Plus)

1. In a ΔABC, A: B: C = 3: 5: 4. Then $a+b+c\sqrt{2}$ is equal to

(a) $2b$ (b) $2c$

(c) $3b$ (d) $3a$

2. If in a ΔABC, $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$, then the triangle is:

(a) Right angled (b) Isosceles

(c) Equilateral (d) Obtuse angled

3. In a ΔABC $\frac{bc\sin^2 A}{\cos A+\cos B\cos C}$ is equal to

(a) b^2+c^2 (b) bc

(c) a^2 (d) a^2+bc

4. In a triangle ABC, $(a+b+c)(b+c-a)=kbc$, if

(a) $k<0$ (b) $k>6$

(c) $0<k<4$ (d) $k>4$

5. In a ΔABC, $A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ cm and area (ΔABC) = $\frac{9\sqrt{3}}{2}$ cm^2. Then 'a' is

(a) $6\sqrt{3}$ cm (b) 9 cm
(c) 18 cm (d) None of these

6. If AD, BE and CF are the medians of ΔABC, then $(AD^2 + BE^2 + CF^2) : (BC^2 + CA^2 + AB^2)$ is equal to

(a) 4: 3 (b) 3: 2
(c) 3: 4 (d) 2: 3

7. The distance between the middle point of BC and the foot of the perpendicular form A is

(a) $\frac{-a^2+b^2+c^2}{2a}$ (b) $\frac{b^2-c^2}{2a}$

(c) $\frac{b^2+c^2}{\sqrt{bc}}$ (d) None of these

8. In a acute angled triangle ABC, AP is the altitude. Circle drawn with AP as its diameter cuts the sides AB and AC at D and E respectively, then length DE is equal to

(a) $\frac{\Delta}{2R}$ (b) $\frac{\Delta}{3R}$

(c) $\frac{\Delta}{4R}$ (d) $\frac{\Delta}{R}$

9. If in a ΔABC, $\Delta = a^2 - (b-c)^2$, then tan A =

(a) 15/16 (b) 8/15
(c) 8/17 (d) 1/2

10. Let f, g, h be the lengths of the perpendiculars from the circumcentre of the ΔABC on the sides a, b and c respectively. If $\frac{a}{f}+\frac{b}{g}+\frac{c}{h}=\lambda\frac{abc}{fgh}$ then the value of λ is

(a) 1/4 (b) 1/2
(c) 1 (d) 2

11. If in a triangle ABC, right angle at B, $s - a = 3$ and $s - c = 2$, then

(a) $a = 2, c = 3$ (b) $a = 3, c = 4$
(c) $a = 4, c = 3$ (d) $a = 6, c = 8$

12. If in a triangle ABC, $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3}{2}c$, then a, c, b are:

(a) in A.P. (b) in G.P.
(c) in H.P. (d) None of these

13. In a ΔABC if $b + c = 3a$, then $\cot\frac{B}{2}\cdot\cot\frac{C}{2}$ has the value equal to:

(a) 4 (b) 3
(c) 2 (d) 1

14. If in a ΔABC, $\angle A = \frac{\pi}{2}$, then $\tan\frac{C}{2}$ is equal to

(a) $\frac{a-c}{2b}$ (b) $\frac{a-b}{2c}$

(c) $\frac{a-c}{b}$ (d) $\frac{a-b}{c}$

15. If R denotes circumradius, then in ΔABC, $\frac{b^2-c^2}{2aR}$ is equal to

(a) cos (B – C) (b) sin (B – C)
(c) cos B – cos C (d) None of these

16. In a ΔABC, the value of $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is equal to:

(a) $\frac{r}{R}$ (b) $\frac{R}{2r}$

(c) $\frac{R}{r}$ (d) $\frac{2r}{R}$

17. If the sides of a triangle are 3: 7: 8, then R: r is equal to

(a) 2: 7 (b) 7: 2
(c) 3: 7 (d) 7: 3

18. In a right angled triangle R is equal to

(a) $\frac{s+r}{2}$ (b) $\frac{s-r}{2}$

(c) $s-r$ (d) $\frac{s+r}{a}$

19. If the area of triangle is 100 sq. cm, r_1 = 10 cm and r_2 = 50 cm, then the value of $(b - a)$ is equal to

(a) 20 (b) 16
(c) 8 (d) 4

20. In a ΔABC, the inradius and three exradii are r, r_1, r_2 and r_3 respectively. In usual notations the value of $r \cdot r_1 \cdot r_2 \cdot r_3$ is equal to

(a) 2Δ (b) Δ^2

(c) $\frac{abc}{4R}$ (d) None of these

21. In a ΔABC if $r_1 > r_2 > r_3$, then

(a) $a > b > c$ (b) $a < b < c$
(c) $a > b$ and $b < c$ (d) $a < b$ and $b > c$

22. In a triangle ABC, right angled at B, the inradius is

(a) $\frac{AB+BC-AC}{2}$ (b) $\frac{AB+AC-BC}{2}$

(c) $\frac{AB+BC+AC}{2}$ (d) None of these

23. If H is the orthocentre of a triangle ABC, then the radii of the circle circumscribing the triangles BHC, CHA and AHB are respectively equal to

(*a*) R, R, R (*b*) $\sqrt{2}$R, $\sqrt{2}$R, $\sqrt{2}$R

(*c*) 2R, 2R, 2R (*d*) $\frac{2}{R}, \frac{2}{R}, \frac{2}{R}$

24. In a triangle ABC, if $\frac{a-b}{b-c} = \frac{s-a}{s-c}$, then r_1, r_2, r_3 are in

(*a*) A.P. (*b*) G.P.

(*c*) H.P. (*d*) None of these

25. A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of length 3, 4 and 5 units. Then area of the triangle is equal to

(*a*) $\frac{9\sqrt{3}(1+\sqrt{3})}{\pi^2}$ (*b*) $\frac{9\sqrt{3}(\sqrt{3}-1)}{\pi^2}$

(*c*) $\frac{9\sqrt{3}(1+\sqrt{3})}{2\pi^2}$ (*d*) $\frac{9\sqrt{3}(\sqrt{3}-1)}{2\pi^2}$

26. If in a ΔABC, $\frac{r}{r_1} = \frac{1}{2}$, then the value of $\tan\frac{A}{2}\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)$ is equal to

(*a*) 2 (*b*) $\frac{1}{2}$

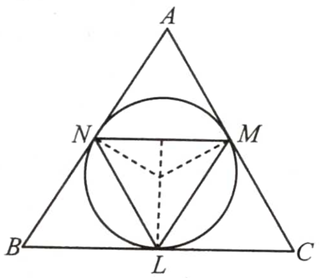
(*c*) 1 (*d*) None of these

27. If in a triangle ABC, the line joining the circumcentre and incentre is parallel to BC, then cos B + cos C is equal to

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) None of these

28. If the incircle of the Δ ABC touches its sides respectively at L, M and N and if x, y, z be the circumradii of the triangles MIN, NIL and LIM where I is the incentre then the product xyz is equal to



(*a*) R r^2 (*b*) r R^2

(*c*) $\frac{1}{2}$ R r^2 (*d*) $\frac{1}{2}$ r R^2

29. The perimeter of a triangle *ABC* is 6 times the arithmetic mean of the sines of its angles. If the side a is 1, then the angle *A* is

(*a*) $\pi/6$ (*b*) $\pi/3$

(*c*) $\pi/2$ (*d*) π

30. If a^2, b^2, c^2 are in *A*.P , then cot*A*, cot*B*, cot*C* are in

(*a*) A.P (*b*) G.P

(*c*) H.P (*d*) None of these

31. The area of the circle and the regular polygon of n sides and of equal perimeter are in the ratio of

(*a*) $\tan(\pi/n)$: π/n (*b*) $\cos(\pi/n)$: π/n

(*c*) $\sin(\pi/n)$: π/n (*d*) $\cot(\pi/n)$: π/n

32. In a triangle *ABC*, $(a+b+c)(b+c-a) = \lambda bc$ if

(*a*) $\lambda < 0$ (*b*) $\lambda > 0$

(*c*) $0 < \lambda < 4$ (*d*) $\lambda > 4$

33. In a triangle *ABC*, *AD* is the altitude from *A*. Given $b > c$, C $= 23°$ and $AD = \frac{abc}{b^2 - c^2}$ then $\angle B$ is equal to

(*a*) 23° (*b*) 113° (*c*) 67° (*d*) 90°

34. In any triangle *ABC*, $a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B)$ is equal to

(*a*) 6*abc* (*b*) 9*abc*

(*c*) 3*abc* (*d*) None of these

35. In a triangle *ABC*, $\sqrt{a} + \sqrt{b} - \sqrt{c}$ is

(*a*) Always positive

(*b*) Always negative

(*c*) Positive only when *c* is smallest

(*d*) Mone of these .

36. In a triangle with sides *a*,*b*, and *c*, a semicircle touching the sides *AC* and *CB* is inscribed whose diameter lies on *AB*. Then, the radius of the semicircle is

(*a*) *a*/2

(*b*) *A*/*s*

(*c*) $\frac{2\Delta}{a+b}$

(*d*) $\frac{2abc}{(s)(a+b)}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

37. A triangle is inscribed in a circle. The vertices of the triangle divide the circle in to three arcs of length 3, 4 and 5 units, then area of the triangle is equal to,

(*a*) $\frac{9\sqrt{3}(1+\sqrt{3})}{\pi^2}$ (*b*) $\frac{9\sqrt{3}(\sqrt{3}-1)}{\pi^2}$

(*c*) $\frac{9\sqrt{3}(1+\sqrt{3})}{2\pi^2}$ (*d*) $\frac{9\sqrt{3}(\sqrt{3}-1)}{2\pi^2}$

38. If in ΔABC, sec*A* , sec*B*, sec*C* are in Harmonic progression, then

(*a*) *a*, *b*, *c*, are in harmonic progression.

(*b*) $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in harmonic progression

(*c*) r_1, r_2, r_3 are in arithmetic progression

(*d*) $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in arithmetic progression.

39. If $\sin A$ and $\sin B$ of a triangle ABC satisfy $c^2x^2 - c(a + b)x + ab = 0$, then the triangle is

(a) Equilateral (b) Isosceles

(c) Right angled (d) Acute angled

40. The sides of a triangle are a, b and $\sqrt{a^2+ab+b^2}$, then the greatest angle is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) $\frac{2\pi}{3}$ (d) None of these

41. In a triangle ABC, R = circumradius and r = inradius. The value of $a\cos A + b\cos B + c\cos C$ $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is equal to

(a) $\frac{R}{r}$ (b) $\frac{R}{2r}$ (c) $\frac{r}{R}$ (d) $\frac{2r}{R}$

42. The distance of the circumcentre of the acute angled ΔABC from the sides BC, CA and AB are in the ratio

(a) $a \sin A : b \sin B : c \sin C$

(b) $\cos A : \cos B : \cos C$

(c) $a \cot A : b \cot B : c \cot C$

(d) None of these

43. If p_1, p_2 and p_3 are respectively the lengths of perpendiculars from the vertices of a triangle ABC to the opposite sides, then the value of $p_1, p_2\, p_3$ is

(a) $\frac{a^2b^2c^2}{8R^2}$ (b) $\frac{a^2b^2c^2}{8R^3}$

(c) $\frac{a^2b^2c^2}{8R^4}$ (d) $\frac{a^2b^2c^2}{4R^2}$

44. Suppose the angles of a triangle ABC are in A.P. and sides b and c satisfy $b : c = \sqrt{3} : \sqrt{2}$ then the angle A equals

(a) 45° (b) 60°

(c) 75° (d) 90°

45. If a^2, b^2, c^2 are the roots of the equation $x^3 - Px^2 + Qx - R = 0$ where a, b, c be the sides of a triangle ABC then the value of $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$ equals $a\,b\,c$

(a) $\frac{P}{\sqrt{R}}$ (b) $\frac{P}{2\sqrt{R}}$

(c) $\frac{P}{4\sqrt{R}}$ (d) None of these

Exercise-3 (JEE Advanced Level)

1. If A is the area and $2s$ the sum of the sides of a triangle, then-

(a) $A \le \frac{s^2}{3\sqrt{3}}$ (b) $A \ge \frac{s^2}{3\sqrt{3}}$

(c) $A > \frac{s^2}{\sqrt{3}}$ (d) None of these

2. If the angles of a triangle are in the ratio 4 : 1 : 1, then the ratio of the longest side to the perimeter is

(a) $\sqrt{3}:(2+\sqrt{3})$ (b) 1 : 6

(c) $1:2+\sqrt{3}$ (d) 2 : 3

3. In any ΔABC having sides a, b, c opposite to angles A, B, C respectively.

(a) $a\sin\left(\frac{B-C}{2}\right)=(b-c)\cos\frac{A}{2}$

(b) $a\cos\frac{A}{2}=(b-c)\sin\frac{B-C}{2}$

(c) $a\cos\frac{A}{2}=(b+c)\sin\frac{B+C}{2}$

(d) $a\sin\frac{B+C}{2}=(b+c)\cos\frac{A}{2}$

4. Let PQR be a triangle of area Δ with $a=2$, $b=\frac{7}{2}$ and $c=\frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P-\sin 2P}{2\sin P+\sin 2P}$ equals

(a) $\frac{3}{4\Delta}$ (b) $\frac{45}{4\Delta}$ (c) $\left(\frac{3}{4\Delta}\right)^2$ (d) $\left(\frac{45}{4\Delta}\right)^2$

5. If a,b,c, are the sides of a triangle, then the minimum value of $\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}$ is equal to

(a) 3 (b) 6

(c) 9 (d) 12

6. Triangle ABC is right angled at A. The points P and Q are on the hypotenuse BC such that $BP = PQ = QC$. If $AP = 3$ and $AQ = 4$ then the length BC is equal to

(a) $\sqrt{27}$ (b) $\sqrt{36}$

(c) $\sqrt{45}$ (d) $\sqrt{54}$

7. A sector $OABO$ of central angle θ is constructed in a circle with centre O and of radius 6. The radius of the circle that is circumscribed about the triangle OAB, is

(a) $6\cos\frac{\theta}{2}$ (b) $6\sec\frac{\theta}{2}$

(c) $3\left(\cos\frac{\theta}{2}+2\right)$ (d) $3\sec\frac{\theta}{2}$

8. If the incircle of the Δ ABC touches its sides respectively at L, M and N and if x, y, z be the circumradii of the triangles MIN, NIL and LIM where I is the incentre then the product xyz is equal to :

(a) Rr^2 (b) rR^2

(c) $\frac{1}{2}Rr^2$ (d) $\frac{1}{2}rR^2$

9. In a ΔABC, $a = a_1 = 2$, $b = a_2$, $c = a_3$ such that $a_{p+1} = \frac{5^p}{3^{2-p}}a_p\left(2^{2-p} - \frac{4p-2}{5^p}a_p\right)$

where $p = 1,2$ then

(a) $r_1 = r_2$ (b) $r_3 = 2r_1$

(c) $r_2 = 2r_1$ (d) $r_2 = 3r_1$

10. If 'O' is the circumcentre of the Δ ABC and R_1, R_2 and R_3 are the radii of the circumcircles of triangles OBC, OCA and OAB respectively then $\frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3}$ has the value equal to:

(a) $\frac{abc}{2R^3}$ (b) $\frac{R^3}{abc}$

(c) $\frac{4\Delta}{R^2}$ (d) $\frac{\Delta}{4R^2}$

11. Let ABC be a triangle with $\angle BAC =$ and $AB = x$ such that $(AB)(AC) = 1$. If x varies then the longest possible length of the angle bisector AD equals

(a) 1/3 (b) 1/2

(c) 2/3 (d) 3/2

12. Let L and M be the respective intersections of the internal and external angle bisectors of the triangle ABC at C and the side AB produced. If $CL = CM$, then the value of $(a^2 + b^2)$ is (where a and b have their usual meanings)

(a) $2R^2$ (b) $2\sqrt{2}R^2$

(c) $4R^2$ (d) $4\sqrt{2}R^2$

13. If r_1, r_2, r_3 are radii of the escribed circles of a triangle ABC and r is the radius of its incircle, then the roots of equation $x^2 - r(r_1r_2 + r_2r_3 + r_3r_1)x + (r_1r_2r_3) - 1 = 0$ is (are)

[Note : All symbols used have usual meaning in triangle ABC.]

(a) r_1 (b) $r_2 + r_3$

(c) 1 (d) $(r_1 r_2 r_3) - 1$

14. Which of the following holds good for any triangle ABC?

(a) $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$

(b) $\frac{\sin A}{a}+\frac{\sin B}{b}+\frac{\sin C}{c}=\frac{3}{2R}$

(c) $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$

(d) $\frac{\sin 2A}{a^2}=\frac{\sin 2B}{b^2}=\frac{\sin 2C}{c^2}$

15. If in a ΔABC, $a = 5$, $b = 4$ and $\cos(A - B) = \frac{31}{32}$, then

(a) $c = 6$ (b) $\sin A = \left(\frac{5\sqrt{7}}{16}\right)$

(c) area of $\Delta ABC = \frac{15\sqrt{7}}{4}$ (d) None of these

16. In a triangle ABC, with usual notations the length of the bisector of internal angle A is:

(a) $\frac{2bc\cos\frac{A}{2}}{b+c}$ (b) $\frac{2bc\sin\frac{A}{2}}{(b+c)}$

(c) $\frac{abc\operatorname{cosec}\frac{A}{2}}{2R(b+c)}$ (d) $\frac{2\Delta}{b+c}\operatorname{cosec}\frac{A}{2}$

17. If $r_1 = 2r_2 = 3r_3$, then

(a) $\frac{a}{b}=\frac{4}{5}$ (b) $\frac{a}{b}=\frac{5}{4}$

(c) $\frac{a}{c}=\frac{3}{5}$ (d) $\frac{a}{c}=\frac{5}{3}$

18. In a ΔABC, following relations holds good. In which case(s) the triangle is a right angled triangle?

(a) $r_2 + r_3 = r_1 - r$ (b) $a^2 + b^2 + c^2 = 8R^2$

(c) $r_1 = s$ (d) $2R = r_1 - r$

19. If in a triangle ABC, $\cos A\cos B + \sin A\sin B\sin C = 1$, then the triangle is

(a) Isosceles (b) Right angled

(c) Equilateral (d) None of these

20. AD, BE and CF are the perpendiculars from the angular points of an acute angled Δ ABC upon the opposite sides, then:

(a) $\frac{\text{Perimeter of }\Delta DEF}{\text{Perimeter of }\Delta ABC}=\frac{r}{R}$

(b) Area of $\Delta DEF = 2\Delta\cos A\cos B\cos C$

(c) Area of $\Delta AEF = \Delta\cos^2 A$

(d) Circum-radius of $\Delta DEF = \frac{R}{2}$

21. The product of the distances of the incentre from the angular points of a Δ ABC is:

(a) $4\,R^2 r$ (b) $4\,Rr^2$

(c) $\frac{(a\,b\,c)R}{s}$ (d) $\frac{(a\,b\,c)\,r}{s}$

22. In a triangle ABC, points D and E are taken on side BC such that BD = DE = EC. If angle ADE = angle AED = θ, then:

(a) $\tan\theta = 3\tan B$ (b) $3\tan\theta = \tan C$

(c) $\frac{6\tan\theta}{\tan^2\theta - 9} = \tan A$ (d) angle B = angle C

23. With usual notations, in a Δ ABC the value of $\Pi\,(r_1 - r)$ can be simplified as:

(a) $abc\,\Pi\tan\frac{A}{2}$ (b) $4\,r\,R^2$

(c) $\frac{(abc)^2}{R(a+b+c)^2}$ (d) $4\,R\,r^2$

24. In a ΔABC, tan A and tan B satisfy the in-equation $\sqrt{3}x^2 - 4x + \sqrt{3} < 0$. Then-

(a) $a^2 + b^2 + ab > c^2$ (b) $a^2 + b^2 - ab > c^2$

(c) $a^2 + b^2 > c^2$ (d) None of these

25. In a ΔABC, $A = \frac{\pi}{3}$ and $b : c = 2 : 3$.

If $\tan\alpha = \frac{\sqrt{3}}{5}, 0 < \alpha < \frac{\pi}{2}$ then-

(a) $B = 60° + \alpha$ (b) $C = 60° + \alpha$

(c) $B = 60° - \alpha$ (d) $C = 60° - \alpha$

26. If in a ΔABC, $a = 6$, $b = 3$ and $\cos(A - B) = \frac{4}{5}$ then-

(a) $C = \frac{\pi}{4}$ (b) $A = \sin^{-1}\frac{2}{\sqrt{5}}$

(c) ar $(\Delta ABC) = 9$ (d) None of these

27. In ΔABC, let AD, BE and CF be internal bisectors of angles A, B and C respectively. If incentre I divide AD, BE and CF in the ratio 2 : 1, 7 : 5 and 3 : 1 respectively and semiperimeter of the triangle is 12 then (points D, E and F lie on sides BC, CA and AB respectively)

[Note: Symbols used have their usual meaning in triangle ABC.]

(a) $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{6}$

(b) $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = 1$

(c) $r +$ area of ΔABC $= 26$

(d) $R + r = 11$

COMPREHENSION BASED QUESTIONS

Comprehension–1 (Q. 28 to 30): Consider a triangle ABC, where x, y, z are the length of perpendicular drawn from the vertices of the triangle to the opposite sides a, b, c respectively and let the letters R, r, S, Δ denote the circumradius, inradius semi-perimeter and area of the triangle respectively.

28. If $\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = \frac{a^2 + b^2 + c^2}{k}$, then the value of k is:

(a) R (b) S

(c) 2R (d) $\frac{3}{2}$R

29. If $\cot A + \cot B + \cot C = k\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$, then the value of k is

(a) R^2 (b) rR

(c) Δ (d) $a^2 + b^2 + c^2$

30. The value of $\frac{c\sin B + b\sin C}{x} + \frac{a\sin C + c\sin A}{y} + \frac{b\sin A + a\sin B}{z}$ is equal to

(a) $\frac{R}{r}$ (b) $\frac{S}{R}$

(c) 2 (d) 6

Comprehension–2 (Q. 31to 33): G is the centroid of triangle ABC. Perpendiculars from vertices A, B, C meet the sides BC, CA, AB at D, E, F respectively. P, Q, R are feet of the perpendiculars from G on sides BC, CA, AB respectively. L, M, N are the mid points of sides BC, CA, AB respectively, then

31. Length of the side PG is

(a) $\frac{1}{2}\,b\sin C$ (b) $\frac{1}{2}\,c\sin C$

(c) $\frac{2}{3}\,b\sin C$ (d) $\frac{1}{3}\,c\sin B$

32. (Area of ΔGPL) to (Area of ΔALD) is equal to

(a) $\frac{1}{3}$ (b) $\frac{1}{9}$

(c) $\frac{2}{3}$ (d) $\frac{4}{9}$

33. Area of ΔPQR is

(a) $\frac{1}{9}(a^2 + b^2 + c^2)\sin A\sin B\sin C$

(b) $\frac{1}{18}(a^2 + b^2 + c^2)\sin A\sin B\sin C$

(c) $\frac{2}{9}(a^2 + b^2 + c^2)\sin A\sin B\sin C$

(d) $\frac{1}{3}(a^2 + b^2 + c^2)\sin A\sin B\sin C$

Comprehension–3 (Q. 34 to 36): In a ΔABC two sides a, be and $\angle A$ are given. Then the numbers of triangles that are formed in the following cases are

On the basis of above passage, answer the following questions :

34. If $a < b \sin A$

(a) 1 (b) 2 (c) 0 (d) Infinite

35. If $a = b\sin A$ and $\angle A$ is acute

(a) 1 (b) 2 (c) 0 (d) nfinite

36. If $a = b\sin A$ and $\angle A$ is obtuse

(a) 1 (b) 2 (c) 0 (d) Infinite

Comprehension–4 (Q. 37 to 39): The internal bisectors of the angle of ΔABC meet the sides BC, CA and AB at D, E and F respectively. Then

On the basis of above passage, answer the following questions :

37. The length BE is given by

(a) $\dfrac{2bc}{b+c}\cos\left(\dfrac{A}{2}\right)$ (b) $\dfrac{bc}{b+c}\cos\left(\dfrac{A}{2}\right)$

(c) $\dfrac{2ac}{a+c}\cos\left(\dfrac{B}{2}\right)$ (d) $\dfrac{ca}{c+a}\cos\left(\dfrac{B}{2}\right)$

38. $\dfrac{1}{AD}\cos\dfrac{A}{2}+\dfrac{1}{BE}\cos\dfrac{B}{2}+\dfrac{1}{CF}\cos\dfrac{C}{2}=$

(a) $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$ (b) abc

(c) $a+b+c$ (d) $ab+bc+ca$

39. Area of ΔDBF is given by

(a) $\dfrac{ac\sin B}{2(a+b)}$ (b) $\dfrac{\Delta ac}{(a+b)(b+c)}$

(c) $\dfrac{ac\sin B}{(a+b)(b+c)}$ (d) $\dfrac{ab\sin C}{(a+b)}$

Comprehension–5 (Q. 40 to 42): Consider a triangle ABC with $b = 3$. Altitude from the vertex B meets the opposite side in D, which divides AC internally in the ratio 1 : 2. A circle of radius 2 passes through the point A and D and touches the circumcircle of the triangle BCD at D.

40. If E is the centre of the circle with radius 2 then angle EDA equals

(a) $\sin^{-1}\left(\dfrac{\sqrt{15}}{4}\right)$ (b) $\sin^{-1}\left(\dfrac{3}{4}\right)$

(c) $\sin^{-1}\left(\dfrac{1}{4}\right)$ (d) $\sin^{-1}\left(\dfrac{15}{16}\right)$

41. If F is the circumcentre of the triangle BDC then which one of the following does not hold good?

(a) $\angle FCD = \sin^{-1}\left(\dfrac{\sqrt{15}}{4}\right)$

(b) $\angle FDC = \cos^{-1}\left(\dfrac{1}{4}\right)$

(c) Triangle DFC is an isosceles triangle

(d) Area of ΔADE = (1/4)th of the area of Δ DBC

42. If R is the circumradius of the ΔABC, then R equal

(a) 4 (b) 6

(c) $2\left(\sqrt{\dfrac{61}{15}}\right)$ (d) $4\left(\sqrt{\dfrac{61}{15}}\right)$

MATCH THE COLUMN TYPE QUESTIONS

43. Match the following:

	Column-I		Column-II
A.	In a ΔABC, $2B = A + C$ and $b^2 = ac$. Then the value o $\dfrac{a^2(a+b+c)}{3abc}$ is equal to	p.	8
B.	In any right angled triangle ABC, the value of $\dfrac{a^2+b^2+c^2}{R^2}$ is always equal to (where R is the circumradius of ΔABC)	q.	1
C.	In a ΔABC if $a = 2$, $bc = 9$, then the value of $2R\Delta$ is equal to	r.	5
D.	In a ΔABC, $a = 5$, $b = 3$ and $c = 7$, then the value of 3 cos C + 7 cos B is equal to	s.	9

(a) (A) → (s), (B) → (p), (C) → (r), (D) → (q)

(b) (A) → (q), (B) → (p), (C) → (s), (D) → (r)

(c) (A) → (r), (B) → (p), (C) → (s), (D) → (q)

(d) (A) → (q), (B) → (s), (C) → (q), (D) → (p)

44. Match the following:

	Column-I		Column-II
A.	In a ΔABC, $a = 4$, $b = 3$ and the medians AA_1 and BB_1 are mutually perpendicular, then square of area of the ΔABC is equal to	p.	3
B.	If in an acute angled ΔABC, line joining the circumcentre and orthocentre is parallel to side AC, then value of tan A.tan C is equal to	q.	7
C.	In a ΔABC, $a = 5$, $b = 4$ and $\tan\dfrac{C}{2} = \sqrt{\dfrac{7}{9}}$, then side '$c$' is equal to	r.	6
D.	In a ΔABC, $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then value of (8 cos B) is equal to	s.	11

(a) (A) → (s), (B) → (p), (C) → (r), (D) → (q)

(b) (A) → (q), (B) → (p), (C) → (s), (D) → (r)

(c) (A) → (r), (B) → (p), (C) → (s), (D) → (q)

(d) (A) → (q), (B) → (s), (C) → (q), (D) → (p)

45. In a triangle ABC, AD is perpendicular to BC and DE is perpendicular to AB.

	Column-I		Column-II
A.	Area of ΔADB	p.	$(b^2/4)\sin 2C$
B.	Area of ΔADC	q.	$(c^2/4)\cos^2 B \sin 2B$
C.	Area of ΔADE	r.	$(c^2/4)\sin 2B$
D.	Area of ΔBDE	s.	$(c^2/4)\sin^2 B \sin 2B$

(a) (A) → (s), (B) → (p), (C) → (r), (D) → (q)
(b) (A) → (q), (B) → (p), (C) → (s), (D) → (r)
(c) (A) → (r), (B) → (p), (C) → (s), (D) → (q)
(d) (A) → (q), (B) → (s), (C) → (q), (D) → (p)

46. Match the following:

	Column-I		Column-II
A.	In a triangle ABC, if a is the arithmetic mean and b, c(b ≠ c) are the two geometric means between any two positive real numbers then $\frac{\sin^3 B+\sin^3 C}{\sin A\sin B\sin C}$ is equal to	p.	9
B.	In a triangle ABC; $\frac{(a^2-b^2-c^2)\tan A+(a^2+c^2-b^2)\tan B}{2}$ is equal to	q.	2
C.	In a triangle ABC, if $B=30°$ and $C=\sqrt{3}b$, then $\frac{A}{45°}$ can be equal to	r.	1
D.	In a ΔABC, $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$, then $\Sigma\, 4\sin A\sin B$	s.	0

(a) (A) → (s), (B) → (p), (C) → (r), (D) → (q)
(b) (A) → (q), (B) → (p), (C) → (s), (D) → (r)
(c) (A) → (r), (B) → (p), (C) → (s), (D) → (q)
(d) (A) → (q), (B) → (s), (C) → (q), (D) → (p)

NUMERICAL BASED QUESTIONS

47. The value of
$a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=$

48. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose $a=2$, $b=3$, $c=4$ and H be the orthocentre. Find $\frac{\sqrt{15}}{2}(\text{HA})$.

49. A triangle has base 6 cm and an area of 12 sq. cm. The difference of the base angles is 60°. The opposite angle is given by the equation 8 sinA – 6cosA =

50. The ratio of the circum-radius and the in-radius of ΔABC, whose sides are in the ratio 4: 5: 6, is expressed as $\left(\frac{m}{n}\right)$; $m, n \in \mathbb{N}$, L.C.M. $(m, n)=1$. Then find $(m-n)$.

51. Let ABC and ABC' be two non-congruent triangles with sides AB = 4, AC = AC' = $2\sqrt{2}$ and angle B = 30°. Find the absolute value of the difference between the areas of these triangles.

52. The sides of a ΔABC are in A.P. as well as in G.P., then the value of $\left(\frac{r_1}{r_2}-\frac{r_2}{r_3}\right)$ is (where r_1, r_2, r_3 are radii of excircles)

53. In a triangle ABC, if $\frac{\sin A}{4}=\frac{\sin B}{5}=\frac{\sin C}{6}$, then the value of

$\cos A+\cos B+\cos C=\frac{K}{M}$ then find $\left(\frac{K}{M}\right)=$ ——
(where $(\cdot)$ denotes least integer function)

54. If two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them 60^0. If the third side is 3, the remaining fourth side is

55. In triangle ABC, E is the midpoint of side BC and D is on side AC. Let the length of side AC is unity and $\angle BAC=60°$, $\angle ABC=100°$, $\angle ACB=20°$, $\angle DEC=80°$. If the area of the triangle ABC plus twice the area of triangle CDE equals $\sqrt{m/n}$ where m and n are coprime then find the value of $(m+n)$.

56. Two circles are passing through vertex A of triangle ABC and one of the circle touches the side BC at B and other circle touches the side BC at C. If $a=5$ and $\angle A=30°$ then find the product of radii of two circles.

[Note: All symbols used have their usual meaning in the triangle.]

57. If the quadratic equation $ax^2+bx+c=0$ has equal roots where a, b and c denote the lengths of the sides opposite to vertices A, B and C of a triangle ABC respectively, then find the sum of integers in the range of $\left(\frac{\sin A}{\sin C}+\frac{\sin C}{\sin A}\right)$.

58. In a triangle ABC, if the sides a, b, c are the roots of x3 – 11x2 + 38x – 40 = 0. If the value of $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$ can be expressed as a rational number as $\frac{m}{n}$ in the lowest form then find the value of $(m+n)$

Exercise-4 (Past Year Questions)

JEE MAIN

1. If 5, 5r, $5r^2$ are the lengths of the sides of a triangle, then r cannot be equal to: **(2019)**

(a) $\frac{3}{4}$ (b) $\frac{5}{4}$

(c) $\frac{7}{4}$ (d) $\frac{3}{2}$

2. With the usual notation, in ΔABC, If $\angle A + \angle B = 120°$, $a = \sqrt{3} + 1$ and $b = \sqrt{3} - 1$, then the ratio ∠A: ∠B, is: **(2019)**

(a) 7: 1 (b) 5: 3

(c) 9: 7 (d) 3: 1

3. Given $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ for ΔABC with usual notation.

If $\frac{\cos A}{\alpha} = \frac{\cos B}{\beta} = \frac{\cos C}{\gamma}$, then the ordered tried (α, β, γ) has a value: **(2019)**

(a) (7, 19, 25) (b) (3, 4, 5)

(c) (5, 12 ,13) (d) (19, 7, 25)

4. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y. If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is: **(2019)**

(a) $\frac{3}{2}y$ (b) $\frac{c}{\sqrt{3}}$

(c) $\frac{c}{3}$ (d) $\frac{y}{\sqrt{3}}$

5. If the lengths of the sides of a triangle are in A.P. and the greatest angle is double the smallest, then a ratio of lengths of the sides of this triangle is: **(2019)**

(a) 5: 9: 13 (b) 5: 6: 7

(c) 4: 5: 6 (d) 3: 4: 5

6. The angles A, B and C of a triangle ABC are in A.P. and $a : b = 1 : \sqrt{3}$. If $c = 4$cm, then the area (in sq. cm) of this triangle is: **(2019)**

(a) $4\sqrt{3}$ (b) $\frac{2}{\sqrt{3}}$

(c) $2\sqrt{3}$ (d) $\frac{4}{\sqrt{3}}$

7. The triangle of maximum area that can be inscribed in a given circle of radius 'r' is: **(2019)**

(a) An equilateral triangle having each of its side of length $\sqrt{3}$.

(b) An equilateral triangle of height $\frac{2r}{3}$.

(c) A right angle triangle having two of its sides of length $2r$ and r.

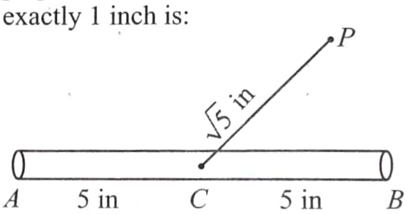
(d) An isosceles triangle with base equal to $2r$.

8. If in a triangle ABC, AB = 5 units, $\angle B = \cos^{-1}\left(\frac{3}{5}\right)$ and radius of circumcircle of ΔABC is 5 units, then the area (in sq. units) of ΔABC is: **(2019)**

(a) $10+6\sqrt{2}$ (b) $8+2\sqrt{2}$

(c) $6+8\sqrt{3}$ (d) $4+2\sqrt{3}$

9. 10 inches long pencil AB with mid point C and a small eraser P are placed on the horizontal top of a table such that $PC = \sqrt{5}$ inches and $\angle PCB = \tan^{-1}(2)$. The acute angle through which the pencil must be rotated about C so that the perpendicular distance between eraser and pencil becomes exactly 1 inch is: **(2019)**



(a) $\tan^{-1}\left(\frac{3}{4}\right)$ (b) $6\tan^{-1}(1)$

(c) $\tan^{-1}\left(\frac{1}{3}\right)$ (d) $\tan^{-1}\left(\frac{1}{2}\right)$

10. Let $\frac{\sin A}{\sin B} = \frac{\sin(A-C)}{\sin(C-B)}$, where A, B, C are angles of triangle ABC. If the lengths of the sides opposite these angles are a, b, c respectively, then: **(2019)**

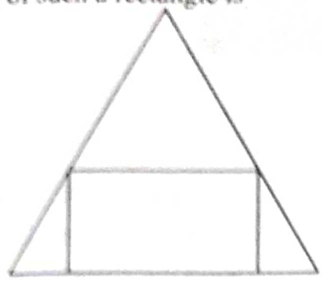
(a) $b^2 - a^2 = a^2 + c^2$ (b) b^2, c^2, a^2 Are in A.P.

(c) c^2, a^2, b^2 Are in A.P. (d) a^2, b^2, c^2 Are in A.P.

11. Let A B C D be a square of side of unit length. Let a circle C_1 cantered at A with unit radius is drawn. Another circle C_2 which touches C_1 and the lines AD and AB are tangent to it, is also drawn. Let a tangent line from the point C to the circle C_2 meet the side AB at E. If the length of EB is $\alpha + \sqrt{3}\beta$, where α, β are integers, then α + β is equal to **(2019)**

12. In ΔABC, the lengths of sides AC and AB are 12 cm and 5 cm, respectively. If the area of ΔABC is 30 cm^2 and R and r are respectively the radii of circumcircle and incircle of ΔABC, then the value of 2R + r (in cm) is equal to **(2019)**

13. If a rectangle is inscribed in an equilateral triangle of side length $2\sqrt{2}$ as shown in the figure, then the square of the largest area of such a rectangle is **(2019)**



14. Let a, b and c be the length of sides of a triangle ABC such the $\frac{a+b}{7} = \frac{b+c}{8} = \frac{c+a}{9}$. If r and R are the radius of incircle and radius of circumcircle of thetriangle ABC, respectively then the value of $\frac{R}{r}$ is equal to **(2022)**

(a) $\frac{5}{2}$ (b) 2 (c) $\frac{3}{2}$ (d) 1

JEE ADVANCED

15. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals **(2019)**

(a) $\frac{3}{4\Delta}$ (b) $\frac{45}{4\Delta}$

(c) $\left(\frac{3}{4\Delta}\right)^2$ (d) $\left(\frac{45}{4\Delta}\right)^2$

16. In a triangle *PQR*, *P* is the largest angle and $\cos P = \frac{1}{3}$.

Further the incircle of the triangle touches the sides *PQ*, *QR* and *RP* at *N*, *L* and *M* respectively, such that the lengths of *PN*, *QL* and *RM* are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are) **(2019)**

(a) 16 (b) 18

(c) 24 (d) 22

17. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is **(2019)**

(a) $\frac{3y}{2x(x+c)}$ (b) $\frac{3y}{2c(x+c)}$

(c) $\frac{3y}{4x(x+c)}$ (d) $\frac{3y}{4c(x+c)}$

Paragraph (Q. 4 to 5): Let *O* be the origin, and $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}$, $\overrightarrow{RP}$, $\overrightarrow{PQ}$ respectively, of a triangle PQR.

18. If the triangle PQR varies, then the minimum value of cos (P + Q) + (Q + R) + cos(R + P) is **(2019)**

(a) $-\frac{3}{2}$ (b) $\frac{3}{2}$

(c) $\frac{5}{3}$ (d) $-\frac{5}{3}$

19. $\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| =$ **(2019)**

(a) sin (P + Q) (b) sin (P + R)

(c) sin (Q + R) (d) $\sin^2 R$

20. In a triangle PQR, let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is(are) TRUE ? **(2019)**

(a) $\angle QPR = 45^\circ$

(b) The area of the traingel PQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$

(c) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$

(d) The area of the circumcircle of the triangle PQR is 100π.

21. In a non-right-angled triangle ΔPQR, let *p*, *q*, *r* denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct? **(2019)**

(a) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$

(b) Radius of incircle of $\Delta PQR \frac{\sqrt{3}}{2}(2-\sqrt{3})$

(c) Length of RS $= \frac{\sqrt{7}}{2}$

(d) Length of OE $= \frac{1}{6}$

22. Let *x*, *y* and *z* be positive real numbers. Suppose *x*, *y* and *z* are lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$$\tan\frac{X}{2} + \tan\frac{Z}{2} = \frac{2Y}{x+y+z},$$

then which of the following statements is/are TRUE? **(2019)**

(a) $2Y = X + Z$ (b) $Y = X + Z$

(c) $\tan\frac{X}{2} = \frac{x}{y+z}$ (d) $x^2 + z^2 - y^2 = xz$

ANSWER KEY

CONCEPT APPLICATION

1. (d) 2. (d) 3. (a) 4. (d) 5. (c) 6. (c) 7. 8. A.P. 9. [30°] 10. (c)
11. $AD=2, BC=CD=1$ 12. 13. (a) 14. (c) 15. (d) 16. 17. [30°] 18. [5] 19. [1]
20. $\left[6+8\sqrt{3}\right]$ 21. 4m 22. $\left[12+7\sqrt{3}\right]$ 23. (a) 24. (c) 25. 26. 27.
28. 29. 30. HD=4R cos B cos C 31. [60°]

EXERCISE-1 (TOPICWISE)

1. (b) 2. (a) 3. (a) 4. (b) 5. (b) 6. (b) 7. (d) 8. (b) 9. (b) 10. (a)
11. (c) 12. (c) 13. (c) 14. (a) 15. (d) 16. (b) 17. (a) 18. (c) 19. (a) 20. (b)
21. (d) 22. (a) 23. (b) 24. (b) 25. (d) 26. (b) 27. (b) 28. (b) 29. (b) 30. (a)
31. (a) 32. (c) 33. (b) 34. (c) 35. (c) 36. (a) 37. (c) 38. (b) 39. (c) 40. (b)
41. (a) 42. (b) 43. (d) 44. (a) 45. (b) 46. (b) 47. (a)

EXERCISE-2 (LEARNING PLUS)

1. (c) 2. (c) 3. (c) 4. (c) 5. (b) 6. (c) 7. (b) 8. (d) 9. (b) 10. (a)
11. (b) 12. (a) 13. (c) 14. (d) 15. (b) 16. (a) 17. (b) 18. (b) 19. (c) 20. (b)
21. (a) 22. (a) 23. (a) 24. (a) 25. (a) 26. (b) 27. (b) 28. (c) 29. (a) 30. (a)
31. (a) 32. (c) 33. (b) 34. (c) 35. (a) 36. (c) 37. (a) 38. (c) 39. (c) 40. (c)
41. (c) 42. (b) 43. (b) 44. (c) 45. (b)

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (a) 2. (a) 3. (a) 4. (c) 5. (a) 6. (c) 7. (d) 8. (c) 9. (d) 10. (c)
11. (b) 12. (c) 13. (c) 14. (a,b) 15. (a,b,c) 16. (a,c,d) 17. (b,d) 18. (a,b,c,d) 19. (a,b)
20. (a,b,c,d) 21. (b,d) 22. (a,c,d) 23. (a,c,d) 24. (a,b) 25. (b,c) 26. (b,c) 27. (a,b) 28. (c) 29. (c)
30. (d) 31. (d) 32. (b) 33. (b) 34. (c) 35. (a) 36. (c) 37. (c) 38. (a) 39. (b)
40. (a) 41. (d) 42. (c) 43. (b) 44. (a) 45. (c) 46. (d) 47. [0] 48. [7] 49. [3]
50. [9] 51. [4] 52. [0] 53. [2] 54. [2] 55. [67] 56. [25] 57. [12] 58. [25]

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (c) 2. (a) 3. (a) 4. (b) 5. (c) 6. (c) 7. (a) 8. (c) 9. (a) 10. (b)
11. [1] 12. [15] 13. [3] 14. (a)

JEE Advanced

15. (c) 16. (b, d) 17. (b) 18. (a) 19. (a) 20. (b,c,d) 21. (b,c,d) 22. (b,c)