PROPERTIES AND SOLUTION OF A TRIANGLE

01 Sine Rule

The sides of a triangle (any type of triangle) are proportional to the sines of the angle opposite to them in triangle ABC, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

Note :- The above rule can also be written as

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

Cosine Formulae

In any $\Delta ABC, \cos A=\frac{b^2+c^2-a^2}{2bc}, \cos B=\frac{c^2+a^2-b^2}{2ac}, \cos C=\frac{a^2+b^2-c^2}{2ab}$

Projection Formulae

In any ΔABC, $a=b\cos C+c\cos B, b=c\cos A+a\cos C, c=a\cos B+b\cos A$

02 Trigonometrical Ratios of Half of The Angles of A Triangle

In any ΔABC, we have

(i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}, \sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ac}}, \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}, \cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}, \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

03 Area Of A Triangle

If Δ be the area of a triangle ABC, then

(i) $\Delta=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$

(ii) $\Delta=\frac{1}{2}\frac{a^2\sin B\sin C}{\sin(B+C)}=\frac{1}{2}\frac{b^2\sin C\sin A}{\sin(C+A)}=\frac{1}{2}\frac{c^2\sin A\sin B}{\sin(A+B)}$

(iii) $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$ (Heron's formula)

Form above results, we obtain following values of sinA, sinB, sinC,

(iv) $\sin A=\frac{2\Delta}{bc}=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

(v) $\sin B=\frac{2\Delta}{ca}=\frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}$

(vi) $\sin C=\frac{2\Delta}{ab}=\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$

04 Napier's Analogy

In any triangle ABC,

$$\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\frac{B}{2}$$

$$\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$$

05 Circumcircle of A Triangle

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}$$

Where Δ is area of triangle and $s=\frac{a+b+c}{2}$

06 Incircle Of A Triangle

(i) $r=\frac{\Delta}{s}$

(ii) $r=(s-a)\tan\left(\frac{A}{2}\right), r=(s-b)\tan\left(\frac{B}{2}\right)$ and $r=(s-c)\tan\left(\frac{C}{2}\right)$

(iii) $r=\frac{a\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}, r=\frac{b\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{B}{2}\right)}$ and $r=\frac{c\sin\left(\frac{B}{2}\right)\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{C}{2}\right)}$

(iv) $r=4R\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right)$

07 Formulae For r_1, r_2, r_3

In any Δ ABC, we have

(i) $r_1=\frac{\Delta}{s-a},\quad r_2=\frac{\Delta}{s-b},\quad r_3=\frac{\Delta}{s-c}$

(ii) $r_1=s\tan\frac{A}{2}, r_2=s\tan\frac{B}{2}, r_3=s\tan\frac{C}{2}$

(iii) $r_1=a\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}, r_2=b\frac{\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}, r_3=c\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(iv) $r_1=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}, r_2=4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}, r_3=4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

08 Pedal Triangle

The point of intersection of perpendiculars drawn from the vertices on the opposite sides of a triangle is called its orthocentre. Let the perpendicular AD, BE and CF from the vertices A, B and C on the opposite sides BC, CA and AB of ABC, respectively, meet at O. Then O is the orthocentre of the Δ ABC.

1. The triangle DEF is called the pedal triangle of the Δ ABC.
2. The distances of the orthocentre from the vertices and the sides - If O is the orthocentre and DEF the pedal triangle of the DABC, where AD, BE, CF are the perpendiculars drawn from A, B, C on the opposite sides BC, CA, AB respectively, then

(i) OA = 2R cos A, OB = 2R cos B and OC = 2R cos C

(ii) OD = 2R cos B cos C, OE = 2R cos C cos A and OF = 2R cos A cos B

(iii) The circumradius of the pedal triangle = $\frac{R}{2}$

(iv) The area of pedal triangle = 2Δ cosA cos B cos C.

09 Some Important Results

(1) $\tan\frac{A}{2}\tan\frac{B}{2}=\frac{s-c}{s}\quad\therefore\quad\cot\frac{A}{2}\cot\frac{B}{2}=\frac{s}{s-c}$

(2) $\tan\frac{A}{2}+\tan\frac{B}{2}=\frac{c}{s}\cot\frac{C}{2}=\frac{c}{\Delta}(s-c)$

(3) $\tan\frac{A}{2}-\tan\frac{B}{2}=\frac{a-b}{\Delta}(s-c)$

(4) $\cot\frac{A}{2}+\cot\frac{B}{2}=\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{c}{s-c}\cot\frac{C}{2}$

10 Height and Distance

Terms Related to Height and Distance

01 **Line of Sight:** It is the straight line that is drawn from the eye of an observer to the point of an object which is to be viewed.

02 **Horizontal Level:** It is the horizontal line drawn from the eye of the viewer.

03 **The angle of elevation:** It is the angle formed between the line of sight and horizontal level if the object is above the horizontal level.

04 **The Angle of Depression:** It is the angle formed between the line of sight and the horizontal level if the object is below the horizontal level.

To calculate the angle of elevation or depression we can use the following formul

$$\sin\theta=\frac{\text{Perpendicular}}{\text{Hypotenuse}},\cos\theta=\frac{\text{Base}}{\text{Hypotenuse}},\tan\theta=\frac{\text{Perpendicular}}{\text{Base}}$$