ARJUNA

for

JEE

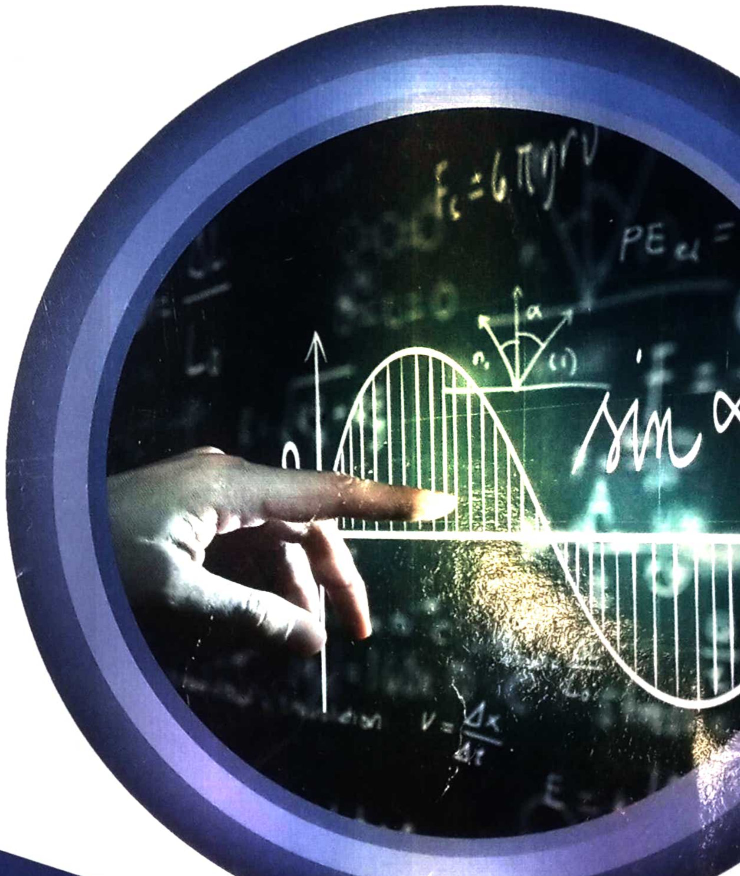
MAIN & ADVANCED

MATHEMATICS

FULL COURSE STUDY MATERIAL

Class XI

- Straight Lines
- Circle
- Parabola
- Ellipse
- Hyperbola
- Complex Number-II



Module-3

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CHAPTER

13 Straight Lines

CO-ORDINATE GEOMETRY

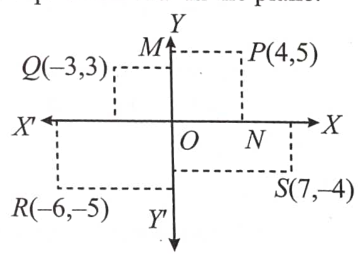
A french mathematician and a greatest philosopher named Rene Descartes, pioneered the use of algebra in Geometry. He suggested methods to study geometry by algebraic methods without making direct reference to the actual figures

This geometry was called co-ordinate geometry or analytical geometry and it is the branch of geometry in which algebraic equations are used to denote points, lines and curves.

Coordinate Geometry is the unification of algebra and geometry in which algebra is used in the study of geometrical relations and geometrical figures are represented by means of equations. The most popular coordinate system is the rectangular Cartesian system. Coordinates of a point are the real variables associated in an order to describe its location in space. Here we consider the space to be two-dimensional. Through a point O, referred to as the origin, we take two mutually perpendicular lines XOX' and YOY' and call them x and y axes respectively. The position of a point is completely determined with reference to these axes by means of an ordered pair of real numbers (x, y) called the coordinates of P where $|x|$ and $|y|$ are the distances of the point P from the y-axis and the x-axis respectively, x is called the x-coordinate or the abscissa of P and y is called the y-coordinate or the ordinate of the point P.

RECTANGULAR CARTESIAN CO-ORDINATE SYSTEMS

Let P be a point in a plane; draw two perpendicular lines $X'OX, Y'OY$ in the plane, and draw PM, PN parallel to OX, OY respectively. $X'OX, Y'OY$ are known as the coordinate axes. MP is the x-coordinate, called the abscissa, and NP is the y-coordinate called the ordinate of P. These two distances MP and NP together fix the position of P in the plane.

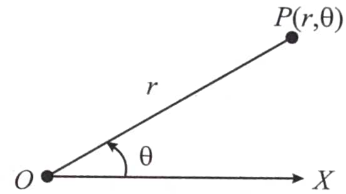


If $MP = 4$, $NP = 5$, then the position of P is denoted by $(4, 5)$ the x-coordinate being written first. Q is the point $(-3, 3)$, for the x-coordinate, is measured to the left of the y-axis, and is negative in sign; R is the point $(-6, -5)$ for both coordinates are negative in sign. Similarly S is the point $(7, -4)$. It should be observed that O, the origin is the point $(0, 0)$ and the y-coordinate of any point on the x-axis is zero while the x-coordinate of any point on the y-axis is zero.

Every point in the plane of the axes of coordinates is represented correspondingly by an ordered pair (x, y) the first one, namely, x, representing the x-coordinate of the point and the second one, namely y, representing the y-coordinate of the point. Conversely to every ordered pair there corresponds only one point. Thus this method of representation of points in a plane by an ordered pair of real number is, what is usually called one-one and onto by which we mean that for every point there is only one ordered pair of real numbers and for every ordered pair of real numbers there is only one point, and neither any point nor any ordered pairs is left out without its associated ordered pair of real numbers or a point as the case may be.

POLAR COORDINATES

Let OX be a given line and P be any point in a given plane. Let θ be the angle through which a line rotates in moving from OX to the position OP. Then, if r is the length of OP, the position of P is known when r and θ are known.



r is called the radius vector, θ the vectorial angle and (r, θ) is the polar coordinates of the point P, which is briefly referred to as the point (r, θ). OX is called the initial line and O the pole.

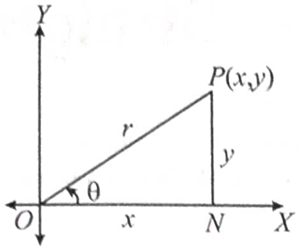
Relation between Cartesian and Polar Coordinates

Let P be the point (x, y) with reference to rectangular axes OX, OY and the point (r, θ) with reference to the pole O and initial line OX. i.e., the X-axis.

$\therefore$ we have $x = r\cos\theta$; $y = r\sin\theta$

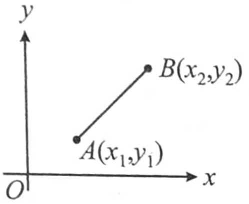
Hence $\tan\theta = \frac{y}{x}$ and $x^2 + y^2 = r^2$

These relations enable us to change from one system of coordinates to the other.



DISTANCE FORMULA

(a) Let A and B be two given points, whose coordinates are given by



$A(x_1, y_1)$ and $B(x_2, y_2)$ respectively.

Then $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

(b) Distance of (x_1, y_1) from origin : $\sqrt{x_1^2 + y_1^2}$

Notes:

(i) If two vertex $A(x_1, y_1)$, $B(x_2, y_2)$ are given then third vertex of equilateral triangle C is

$$\left[\frac{x_1 + x_2 \mp \sqrt{3}(y_2 - y_1)}{2}, \frac{y_1 + y_2 \pm \sqrt{3}(x_2 - x_1)}{2}\right]$$

(ii) If the three points P, Q, R are collinear, then $|PQ| \pm |QR| = |PR|$.

(iii) When 3 points are given then

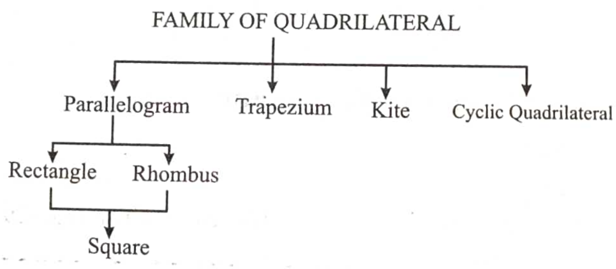
(a) an isosceles triangle, when any two sides are equal.

(b) an equilateral triangle, when all sides are equal

(c) a scalene triangle, when all sides are unequal.

(d) a right-angled triangle, say ΔABC, when $AB^2 + BC^2 = AC^2$

TYPES OF QUADRILATERAL



Parallelogram

Definition: If opposite sides of quadrilateral are parallel and equal, then quadrilateral is called parallelogram.

Four ways to prove that a quadrilateral is parallelogram.

(i) Opposite sides are parallel

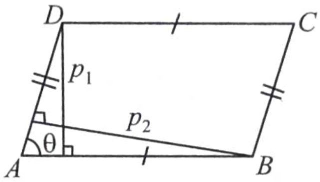
i.e. $AB \parallel DC$ and $AD \parallel BC$

(ii) Opposite sides are equal

i.e. $AB = DC$ and $AD = BC$

(iii) One pair of opposite sides are equal and parallel.

(iv) Diagonals bisect each other

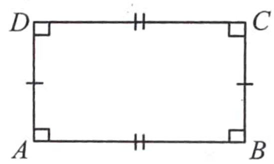


Note: Area of parallelogram $= \frac{P_1P_2}{\sin\theta}$

Rectangle

Definition: If all angles of parallelogram are equal then it is called rectangle.

(i) Diagonal are equal and bisect each other.

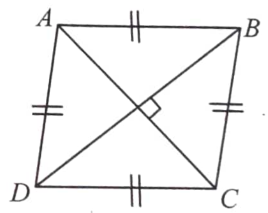


(ii) Each diagonal divides the rectangle into two triangles of equal area.

Rhombus

Definition: If all sides of a parallelogram are equal then it is called Rhombus

(i) Diagonals are perpendicular



(ii) Area $= \frac{1}{2} d_1 d_2$ where d_1 and d_2 are diagonal.

Note: If distance between pair of parallel sides are equal then it is a rhombus.

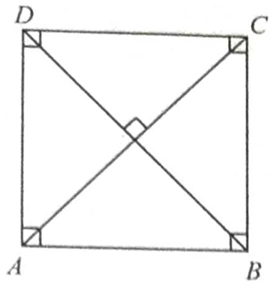
Square

Definition: If all the sides and all the interior angles of a parallelogram are equal then it is called a square.

(i) All sides are equal

$AB = BC = CD = DA$

(ii) Diagonals are equal and bisect each other at 90°.



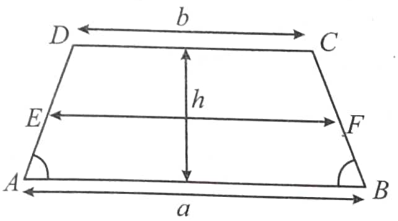
(iii) Area $= \frac{d^2}{2}$ (d = diagonal)

Note: Every square is a rectangle but not the converse.

Trapezium

Definition: Trapezium is a quadrilateral which has exactly one pair of opposite sides parallel.

(i) Area of trapezium $= \frac{1}{2}$(sum of parallel sides) × distance between sides



(ii) $\angle DAB$ and $\angle CBA$ = Base angle

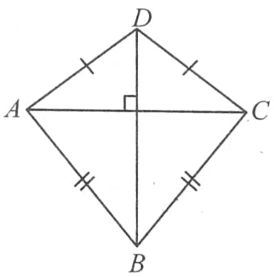
(iii) Median $(EF) = \frac{1}{2}(a+b)$

(iv) For equilateral /isosceles trapezium, non parallel sides are equal i.e. $AD = BC$

Kite

Definition: It is a quadrilateral in which two pairs of adjacent sides are equal.

(i) $AD = DC$ and $AB = BC$



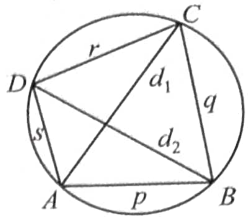
(ii) Diagonals are perpendicular but not bisect

(iii) Only one diagonal divide the figure into two congruent triangles.

Cyclic Quadrilateral

Definition: If all vertices of quadrilateral lies on the circumference of a circle, then it is called cyclic quadrilateral.

(i) Opposite angles are supplementary.



(ii) Sum of product of opposite sides are equal to product of diagonals $pr + qs = d_1d_2$

Note: Let A, B, C & D be the four given points in a plane. Then the quadrilateral will be

(i) Square if $AB = BC = CD = DA$ & $AC = BD$; $AC \perp BD$

(ii) Rhombus if $AB = BC = CD = DA$ and $AC \neq BD$; $AC \perp BD$

(iii) Parallelogram if $AB = DC$, $BC = AD$; $AC \neq BD$; $AC \not\perp BD$

(iv) Rectangle if $AB = CD$, $BC = DA$; $AC = BD$; $AC \not\perp BD$

Train Your Brain

Example 1: Match the Column

Column-I	Column-II
(a) The triangle with vertices $A(7, 10)$, $B(4, 5)$ and $C(10, 15)$ is	(P) Equilateral
(b) The triangle with vertices $P(2, 7)$, $Q(4, -1)$ and $R(-2, 6)$ is	(Q) Isosceles
(c) The triangle with vertices $L(3, 1)$, $M(5, 6)$ and $N(9, 16)$ is	(R) Right angled
(d) The triangle with vertices $R(a, a)$, $S(-a, -a)$ and $T(\sqrt{3}a, -\sqrt{3}a)$ is	(S) Collinear

Sol. (a) In a ΔABC

$$\therefore \quad AB = \sqrt{(4-7)^2 + (5-10)^2} = \sqrt{34}$$

$$BC = \sqrt{(10-4)^2 + (15-5)^2} = \sqrt{136}$$

$$AC = \sqrt{(10-7)^2 + (15-10)^2} = \sqrt{34}$$

Hence $AB = AC$

Hence ΔABC is isosceles triangle.

(b) $\therefore \quad PQ = \sqrt{(2-4)^2 + (7+1)^2} = \sqrt{68}$

$$QR = \sqrt{(4+2)^2 + (-1-6)^2} = \sqrt{85}$$

$$PR = \sqrt{(-2-2)^2 + (6-7)^2} = \sqrt{17}$$

Hence $PQ^2 + PR^2 = QR^2$

Hence ΔABC is right angled.

(c) $\therefore \quad LM = \sqrt{(5-3)^2 + (6-1)^2} = \sqrt{29}$

$$MN = \sqrt{(9-5)^2 + (16-6)^2} = \sqrt{116} = 2\sqrt{29}$$

$LN = \sqrt{(9-3)^2+(16-1)^2} = \sqrt{261} = 3\sqrt{29}$

Hence $LM + MN = LN$

Hence points L, M, N are collinear.

(*d*) $\therefore$ side $RS = \sqrt{(a+a)^2+(a+a)^2} = 2\sqrt{2}\,a$

$$ST = \sqrt{(\sqrt{3}a+a)^2+(-\sqrt{3}a+a)^2} = 2\sqrt{2}\,a$$

$$RT = \sqrt{(\sqrt{3}a-a)^2+(-\sqrt{3}a-a)^2} = 2\sqrt{2}\,a$$

Hence $RS = ST = RT$

Hence ΔRST is equilateral.

Example 2: If $A(0, -1)$, $B(6, 7)$, $C(-2, 3)$ and $D(\lambda, 3)$ forms a rectangle then find the value of λ.

Sol. $AB = CD$ and $AC = BD$.

$AB = \sqrt{6^2+8^2} = 10$

$CD = \sqrt{(\lambda+2)^2} = |\lambda+2|$

$\lambda + 2 = 10$ or $\lambda + 2 = -10$

$\lambda = 8$ or $\lambda = -12$ ... (1)

Now $AC = BD$

$4 + 16 = (6-\lambda)^2 + 4^2$

$4 = 36 + \lambda^2 - 12\lambda$

$\Rightarrow \lambda^2 - 12\lambda + 32 = 0$

$\lambda = 4, 8$

Hence from (1) and (2)

$\lambda = 8$

Example 3: Prove that the four points $A(0, 0)$, $B(2, 2)$, $C(2(\sqrt{2}+1), 2)$ and $D(2\sqrt{2}, 0)$ form a Rhombus but not a rectangle.

Sol. Sides are $AB = 2\sqrt{2}$, $BC = 2\sqrt{2}$, $CD = 2\sqrt{2}$, $DA = 2\sqrt{2}$

Diagonals, $AC = \sqrt{2^2(\sqrt{2}+1)^2+4}$;

$BD = \sqrt{2^2(\sqrt{2}-1)^2+4}$

Since $AC \neq BD$ and all sides are equal hence given points from a Rhombus but not a rectangle.

Concept Application

1. If M is the mid-point of the side BC of the triangle ABC, prove that $AB^2 + AC^2 = 2AM^2 + 2BM^2$.
2. The distance between the point $P(a \cos \alpha, a \sin \alpha)$ and $P(a \cos \beta, a \sin \beta)$ is-

   (*a*) $4a\sin\dfrac{\alpha-\beta}{2}$ (*b*) $2a\sin\dfrac{\alpha+\beta}{2}$

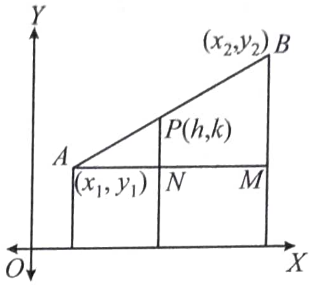
   (*c*) $2a\sin\dfrac{\alpha-\beta}{2}$ (*d*) $2a\cos\dfrac{\alpha-\beta}{2}$

SECTION FORMULA

Let A, B are the points (x_1, y_1), (x_2, y_2), here we will find the coordinates of the point P on AB such that $AP : PB = l : m$.

Let the coordinates of P be (h, k)

In the Figure, triangles ANP and AMB are similar.



$$\therefore \quad \frac{PN}{BM} = \frac{AP}{AB} = \frac{l}{l+m}$$

Since $\dfrac{AP}{PB} = \dfrac{l}{m}$ i.e., $\dfrac{k-y_1}{y_2-y_1} = \dfrac{l}{l+m}$

$\therefore \quad (l+m)k = (l+m)y_1 + l(y_2 - y_1) = ly_2 + my_1$

$$\therefore \quad k = \frac{ly_2+my_1}{l+m}$$

similarly, $h = \dfrac{lx_2+mx_1}{l+m}$

Note: Putting $l = m$, we find that the mid-point of the line joining the points (x_1, y_1), (x_2, y_2).

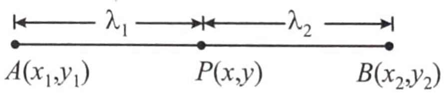
i.e. $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

Note: When P lies outside AB i.e., external to AB such that AP: $BP = l : m$.

We have $h = \dfrac{lx_2-mx_1}{l-m}$, $k = \dfrac{ly_2-my_1}{l-m}$

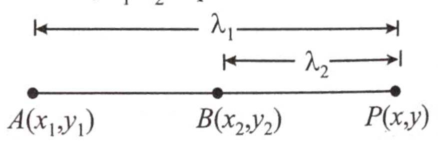
It has been assumed that l and m are positive numbers; if, however, we take $AP : BP$ to be positive in the former case and negative in the latter case, we have in both cases the formula

$$h = \frac{lx_2+mx_1}{l+m}, k = \frac{ly_2+my_1}{l+m}$$



Coordinates of the point P dividing the join of two points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio of $\lambda_1 : \lambda_2$ are $P\left(\dfrac{\lambda_2 x_1 - \lambda_1 x_2}{\lambda_2 - \lambda_1}, \dfrac{\lambda_2 y_1 - \lambda_1 y_2}{\lambda_2 - \lambda_1}\right)$.

In both the cases, λ_1/λ_2 is positive.



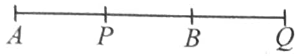
Notes:

(i) If the ratio, in which a given line segment is divided, is to be determined, then sometimes, for convenience (instead of taking the ratio $\lambda_1 : \lambda_2$), we take the ratio $k : 1$. If the value of k turns out to be positive, it is an internal division otherwise it is an external division.

(ii) The coordinates of the mid-point of the line-segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

HARMONIC CONJUGATE

If P is a point that divides AB internally in the ratio $m_1 : m_2$ and Q is another point which divides AB externally in the same ratio $m_1 : m_2$, then the point P and Q are said to be Harmonic conjugate to each other with respect to A and B.



i.e. AP, AB and AQ forms a $HP \Rightarrow \frac{1}{AP} + \frac{1}{AQ} = \frac{2}{AB}$

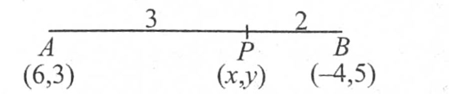
Note: Internal and External angle bisector of an angle divides the base harmonically.

Train Your Brain

Example 4: Find the co-ordinates of the point which divides the line segment joining the points (6, 3) and (– 4, 5) in the ratio 3 : 2 (i) internally and (ii) externally.

Sol. Let $P(x, y)$ be the required point.

(i) For internal division :

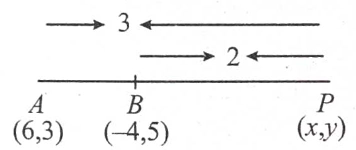


$$x = \frac{3\times(-4)+2\times 6}{3+2} \quad \text{and} \quad y = \frac{3\times 5+2\times 3}{3+2}$$

or $x = 0$ and $y = \frac{21}{5}$

So the co-ordinates of P are $\left(0, \frac{21}{5}\right)$

(ii)



For external division

$$x = \frac{3\times(-4)-2\times 6}{3-2}$$

and $y = \frac{3\times 5-2\times 3}{3-2}$

or $\quad x = -24 \quad$ and $\quad y = 9$

So the co-ordinates of P are (–24, 9)

Concept Application

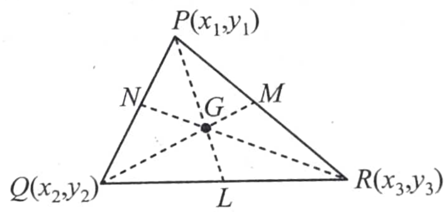
3. Find the coordinates of the point which divides the line connecting the points (8, 9) and (–7, 4).
   (i) Internally in the ratio 2: 3
   (ii) Externally in the ratio 4 : 3
4. Find the ratio in which the point (2, 1) divides the line connecting the points (1, –2) & (4, 7).
5. Find the ratio in which the x-axis divides the line joining the points (2, 5) and (1, 9).

COORDINATES OF SPECIAL POINTS WITH RESPECT TO A TRIANGLE

(a) Centroid (G):

Definition: The point of concurrence of the medians of a triangle is called the centroid of the triangle.

(i) G divides median into 2 : 1.



$$\frac{PG}{GL} = \frac{QG}{GM} = \frac{RG}{GN} = \frac{2}{1}$$

(ii) G always lies inside the triangle.

(iii) Co-ordinates of G is

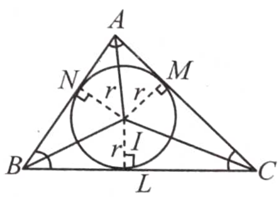
$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) \text{ or } \left(\frac{\Sigma x_1}{3}, \frac{\Sigma y_1}{3}\right)$$

(b) Incentre (I)

Definition: The point of concurrency of the internal bisectors of the angles of a triangle is called the incentre of the triangle.

(i) I always lies inside the triangle.

(ii) Internal angle bisector divides the base in the ratio of adjacent sides.



$c : b$

$$\frac{AB}{AC} = \frac{BL}{CL} \Rightarrow \frac{BL}{CL} = \frac{c}{b}$$

lly: $\frac{CM}{MA} = \frac{a}{c}$ & $\frac{AN}{NB} = \frac{b}{a}$

(iii) Co-ordinates of **I** is

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

where a, b, c are the lengths of the sides of the Δ

(c) Ex-centres (I_1, I_2, I_3)

Definition: The centre of the described circle which is opposite to vertices.

To get I_1 (or I_2 or I_3) replace a by $-a$ (b by $-b$ or c by $-c$) in formula of coordinate of I

$$I_1=\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

$$I_2=\left(\frac{ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right)$$

$$I_3=\left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

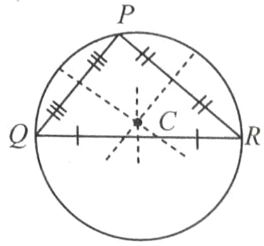
(d) Circumcentre (C)

Definition: The point of concurrency of the perpendicular bisectors of the sides of a triangle is called circumcentre of the triangle.

(i) For acute angle $\Delta \Rightarrow$ lies inside

(ii) For obtuse angle $\Delta \Rightarrow$ lies outside

(iii) For right angle $\Delta \Rightarrow$ Mid point of hypotenuse



(iv) Co-ordinates of circumcentre is

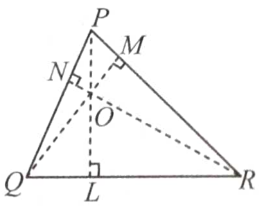
$$\left(\frac{x_1\sin 2A+x_2\sin 2B+x_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}, \frac{y_1\sin 2A+y_2\sin 2B+y_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}\right)$$

(e) Orthocentre (O)

Definition: The point of concurrency of the altitudes of a triangle is called the orthocentre of the triangle.

(i) For acute angle $\Delta \Rightarrow$ lies inside

(ii) For obtuse angle $\Delta \Rightarrow$ lies outside



(iii) For right angle $\Delta \Rightarrow$ vertex having right angle

(iv) Co-ordinates of orthocentre is

$$\left(\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C}, \frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right)$$

Notes:

(i) In any triangle **O, G, C** are collinear.

(ii) In any triangle **G** divides the line joining **O** & **C** in ratio $2:1$.

(iii) In an equilateral triangle **O, G, C, I** are coincident.

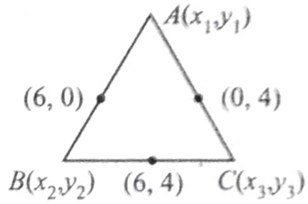
(iv) In an isosceles triangle **O, G, C, I** are collinear.

(v) The orthocentre of ΔABC are (a, b), (b, a) and (a, a) is (a, a).

Train Your Brain

Example 5: If midpoints of the sides of a triangle are (0, 4), (6, 4) and (6, 0), then find the vertices of triangle, centroid and circumcentre of triangle.

Sol. Let points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of ΔABC.



$x_1+x_3=0,\ y_1+y_3=8$

$x_2+x_3=12,\ y_2+y_3=8$

$x_1+x_2=12,\ y_1+y_2=0$

Solving we get $A(0, 0)$, $B(12, 0)$ and $C(0, 8)$.

Hence ΔABC is right angled triangle $\angle A=\pi/2$.

$\therefore$ Circumcentre is midpoint of hypotenuse which is (6, 4) itself and centroid

$$\equiv\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)=\left(4, \frac{8}{3}\right)$$

Example 6: Find the distance between the orthocentre and circumcentre of a triangle whose vertices are $P(3, 0)$, $Q(0, 0)$ and $R\left(\frac{3}{2}, \frac{-3\sqrt{3}}{2}\right)$

Sol. $\therefore$ side $PQ=\sqrt{(3-0)^2+(0-0)^2}=3$

$$QR=\sqrt{\left(\frac{3}{2}-0\right)^2+\left(\frac{-3\sqrt{3}}{2}-0\right)^2}=3$$

$$PR=\sqrt{\left(3-\frac{3}{2}\right)^2+\left(0+\frac{3\sqrt{3}}{2}\right)^2}=3$$

Hence $PQ=QR=PR$

Hence, the triangle is equilateral.

Now, since in an equilateral triangle orthocentre and circumcentre coincides therefore distance between them is zero.

Example 7: Find the co-ordinates of circumcentre of the triangle whose vertices are (8, 6), (8, –2) and (2, –2).

Sol. Let $A(8, 6)$ and $B(8, -2)$ and $C(2, -2)$

$P(h, k)$ be the circumcentre

$PA = PB = PC$

$\Rightarrow PA^2 = PB^2$

$(h-8)^2 + (k-6)^2 = (h-8)^2 + (k+2)^2$

$16k = 32 \quad \Rightarrow \quad k = 2$

$PB^2 = PC^2$

$(h-8)^2 + (k+2)^2 = (h-2)^2 + (k+2)^2$

$12h = 60$

$\Rightarrow h = 5$

Hence the co-ordinate of the circumcentre is $(5, 2)$.

Concept Application

6. Find the coordinates of the incentre of the triangle whose vertices are $(4, -2)$, $(-2, 4)$ and $(5, 5)$.

7. If $\left(\frac{3}{2}, 0\right)$, $\left(\frac{3}{2}, 6\right)$ and $(-1, 6)$ are mid points of sides of triangle, then find the centroid of the triangle.

AREA OF A TRIANGLE

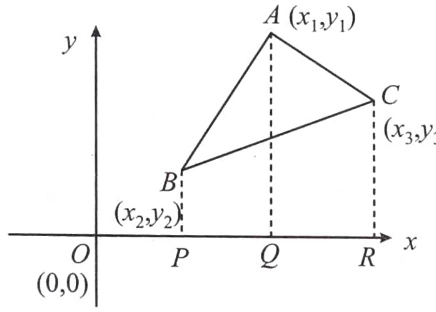
Let the vertices of the triangle be

$A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$

We have considered the vertices of the triangle in the first quadrant for the ease of calculations. Drop perpendiculars form A, B and C on the x-axis.

Area of ΔABC = Area of trapezium $ABPQ$

+ Area of trapezium $AQRC$ – Area of trapezium $BCRP$



$$= \frac{1}{2}(BP + AQ)(PQ) + \frac{1}{2}(AQ + CR)(QR) - \frac{1}{2}(BP + CR)(PR)$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

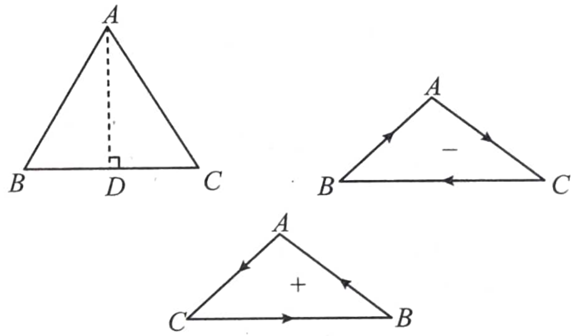
expressed in the determinant form,

$$\Rightarrow \Delta = \text{absolute value of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \quad \text{... (1)}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{... (2)}$$

While using formula (1) or (2). order of the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) has not been taken into account. If we plot the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then the area of the triangle as obtained by using formula (1) or (2) will be positive or negative as the points A, B, C are in anti-clockwise or clockwise directions.



So, while finding the area of triangle ABC, we use the formula:

Area of

$$\Delta ABC = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

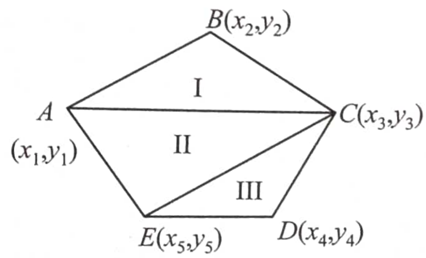
Notes:

(*i*) If three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

(*ii*) Equation of straight line passing through (x_1, y_1) and (x_2, y_2) is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(*iii*) Similarly to calculate the area of pentagon whose vertices are given, we can divide the pentagon into a number of triangles and then sum up the areas. e.q.



Area of pentagon $ABCDE$ = Area of I + Area of II + Area of III

$$=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}+\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_3 & y_3 & 1\\ x_5 & y_5 & 1\end{vmatrix}+\frac{1}{2}\begin{vmatrix}x_3 & y_3 & 1\\ x_4 & y_4 & 1\\ x_5 & y_5 & 1\end{vmatrix}$$

(*iv*) The area of a polygon, with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots (x_n, y_n)$ is

$$\frac{1}{2}\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}+\frac{1}{2}\begin{vmatrix}x_2 & y_2\\ x_3 & y_3\end{vmatrix}+\dots+\frac{1}{2}\begin{vmatrix}x_n & y_n\\ x_1 & y_1\end{vmatrix}$$

Train Your Brain

Example 8: The vertices of quadrilateral in order are $(-1, 4)$, $(5, 6)$, $(2, 9)$ and (x, x^2). The area of the quadrilateral is 15/2 sq. units, then find the point (x, x^2)

Sol. Area of quadrilateral

$$=\frac{1}{2}\left\|\begin{vmatrix}-1 & 4\\ 5 & 6\end{vmatrix}+\begin{vmatrix}5 & 6\\ 2 & 9\end{vmatrix}+\begin{vmatrix}2 & 9\\ x & x^2\end{vmatrix}+\begin{vmatrix}x & x^2\\ -1 & 4\end{vmatrix}\right\|$$

$$=\frac{1}{2}\left|-26+33+2x^2-9x+4x+x^2\right|=\frac{15}{2}$$

$\therefore\quad 3x^2 - 5x + 7 = \pm 15$

$\therefore\quad 3x^2 - 5x - 8 = 0,\ 3x^2 - 5x + 22 = 0$

$\Rightarrow\quad x = 8/3, x = -1$

Hence point is $\left(\frac{8}{3}, \frac{64}{9}\right)$ or $(-1, 1)$. But $(-1, 1)$ will not form a quadrilateral as per given order of the points.

Hence the required point is $\left(\frac{8}{3}, \frac{64}{9}\right)$

Concept Application

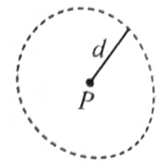
8. Find the area of the triangle whose vertices are (1, 3), (2, 4) and (5, 6).
9. Prove that the points (0, 5), (0, –9) and (3, 6) are non-collinear.

LOCUS

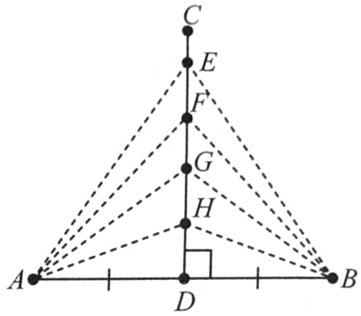
The locus of a point, as it moves in accordance with a given geometrical condition, is the path traced out by the moving point. In geometry we mean, by locus, the curve itself. We say that the locus is a straight line (or) the locus is a circle. On the contrary in coordinate geometry the moving point can be represented by the ordered pair (x, y). The geometrical condition imposed on the moving point gets transformed into an algebraic relation connecting (x, y). This relation, between x and y (the coordinates of the moving point), is called the equation to the curve described by the points.

Note:

1. All those points which satisfy the given geometrical condition will definitely lie on the locus. But converse is not true always.
2. If a point moves in a plane in such a way that its distance from a fixed point is always d the same. Then the locus of the movable point P is called a circle.



3. If a points moves in a plane in such a way that its distances from two fixed points are always the same, the locus of the point is a perpendicular bisector.



EQUATION OF LOCUS

The equation to a locus is the relation which exists between the coordinates of any point on the path, and which holds for no other point except those lying on the path.

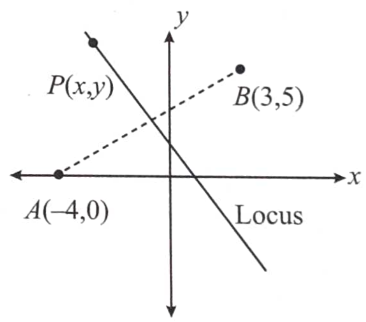
For example, if $P(x, y)$ is any point equidistant from $A(-4, 0)$ and $B(3, 5)$,

We have $PA^2 = PB^2$ and this can be expressed in terms of coordinates of A, P and B as

$(x + 4)^2 + y^2 = (x - 3)^2 + (y - 5)^2$

i.e., $14x + 10y - 18 = 0$

i.e., $7x + 5y - 9 = 0$



Moreover, if $P(x, y)$ is any point whose coordinates satisfy the equation $7x + 5y - 9 = 0$.

We have $PA^2 = PB^2$

i.e., P is equidistant from A and B. Hence the relation $7x + 5y - 9 = 0$, is called the equation of the locus of points equidistant from A and B.

It is said that the equation $7x + 5y - 9 = 0$, represents the perpendicular bisector of AB.

Coordinate Geometry is based on the concept of locus of a point. Large number of problems will involve the idea of the locus of a point.

Procedure to find the Equation of the Locus of a Point

(i) If we are finding the equation of the locus of a point P, assign coordinates (h, k) to P.

(ii) Express the given conditions as equations in terms of the known quantities to facilitate calculations. Sometimes we include some unknown quantities known as parameters.

(iii) Eliminate the parameters, so that the eliminate contains only h, k and known quantities.

(iv) Replace h by x, and k by y, in the eliminate. The resulting would be the equation of the locus of P.

(v) If x and y coordinates of the moving point are obtained in terms of a third variable t (called the parameter), eliminate t to obtain the relation in x and y and simplify this relation. This will give the required equation of locus.

Train Your Brain

Example 9: Find the locus of point P if P is equidistant from points

(i) $A(3, 4)$ and $B(5, -2)$

(ii) $A(a + b, a - b)$ and $B(a - b, a + b)$

Sol. (i) If point $P(h, k)$ is equidistant from $A(3, 4)$ & $B(5, -2)$ then $PA = PB$

$\Rightarrow \sqrt{(h-3)^2+(k-4)^2} = \sqrt{(h-5)^2+(k+2)^2}$

$\Rightarrow h^2 - 6h + 9 + k^2 - 8k + 16 = h^2 - 10h + 25 + k^2 + 4k + 4$

$\Rightarrow 4h - 12k = 4 \Rightarrow h - 3k = 1$

hence locus of P is $x - 3y = 1$

(ii) $PA = PB$

$\Rightarrow [h - (a + b)]^2 + [k - (a - b)]^2 = [h - (a - b)]^2 + [k - (a + b)]^2$

$\Rightarrow -2h(a + b) - 2k(a - b) = -2h(a - b) - 2k(a + b)$

$\Rightarrow 2h(2b) + 2k(2b) = 0 \Rightarrow h + k = 0$

Hence locus of P is $x + y = 0$

Example 10: Find the equation to the locus of a point which moves so that

(i) Its distance from the point $(a, 0)$ is always four times its distance from the axis of y.

(ii) Sum of the squares of its distances from the axes is equal to 3.

(iii) Its distance from x-axis is 3 times of its distance from y-axis.

Sol. (i) Let the point be $P(h, k)$

Distance of P from axis of $y = |h|$

Distance of P from $(a, 0) = \sqrt{(h-a)^2+k^2}$

$\Rightarrow \sqrt{(h-a)^2+k^2} = 4|h|$

$\Rightarrow (h - a)^2 + k^2 = 16h^2 \Rightarrow h^2 - 2ah + a^2 + k^2 = 16h^2$

$\Rightarrow 15h^2 - k^2 + 2ah = a^2$

hence locus of P is $15x^2 - y^2 + 2ax = a^2$

(ii) Let the point be $P(h, k)$

Distance of P from y-axis $= |h|$

Distance of P from x-axis $= |k|$

$h^2 + k^2 = 3$

hence Locus of P is

$x^2 + y^2 = 3$

(iii) Let the point be $P(h, k)$

Distance from x-axis $= |k|$

Distance from y-axis $= |h|$

$|k| = 3|h|$

$3h - k = 0$ or $3h + k = 0$

$3x - y = 0$ or $3x + y = 0$

Example 11: $A(0, 1)$ and $B(0, -1)$ are 2 points if a variable point P moves such that sum of its distance from A and B is 4. Then the locus of P is the equation of the form of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Find the value of $(a^2 + b^2)$.

Sol. Let the point P is (h, k)

Given that $PA + PB = 4$ where $A(0, 1)$ and $B(0, -1)$

$\Rightarrow \sqrt{(h-0)^2+(k-1)^2} + \sqrt{(h-0)^2+(k+1)^2} = 4$

$\Rightarrow \sqrt{h^2+k^2-2k+1} = 4 - \sqrt{h^2+k^2+2k+1}$

squaring both sides, we get

$\Rightarrow h^2 + (k - 1)^2 = 16 + h^2 + (k + 1)^2 - 8\sqrt{h^2+(k+1)^2}$

$\Rightarrow 8\sqrt{h^2+(k+1)^2} = 16 + 4k$

squaring again, we get

$\Rightarrow 4h^2 + 3k^2 = 12 \Rightarrow \frac{h^2}{3}+\frac{k^2}{4}=1$

Hence locus of P is $\frac{x^2}{3}+\frac{y^2}{4}=1$

$\Rightarrow a^2 + b^2 = 3 + 4 = 7$

Concept Application

10. Find the locus of a variable point which is at a distance of 2 units from the y-axis.

11. Find the locus of a variable point whose distance from $A(4, 0)$ is equal to its distance from $B(0, 2)$.

12. Q is a variable point whose locus is $2x + 3y + 4 = 0$; corresponding to a particular position of Q, P is a point on OQ, O being the origin, such that $OP : PQ = 3 : 1$. Find the locus of P.

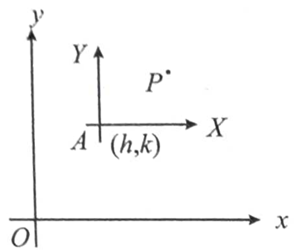
SHIFTING OF AXES

Shifting of Origin

When origin is shifted to a new position $A(h, k)$ without changing the directions of the axes, then coordinates of all points in the plane are obtained by the formulae:

$x = X + h \qquad y = Y + k$

where, $P(x, y)$ referred to ox, oy and $P(X, Y)$ referred to AX, AY



Rotation of Coordinate Axes

Suppose an xy-coordinate system and an $x'y'$-coordinate system have the same origin and θ is the angle from the positive x-axis to the positive x'-axis. If the coordinates of point P are (x, y) in the xy-system and (x', y') in the rotated $x'y'$-system, then

$x = x \cos\theta - y \sin\theta$ ...(i)

$y = x' \sin\theta + y' \cos\theta$ ...(ii)

Note: Amount of Rotation Formula: The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, B \neq 0$$

can be rewritten as an equation in x' and y' without an $x'y'$-term by rotating the axes through angle θ, where $\cot 2\theta = \dfrac{A-C}{B}$.

Proof:

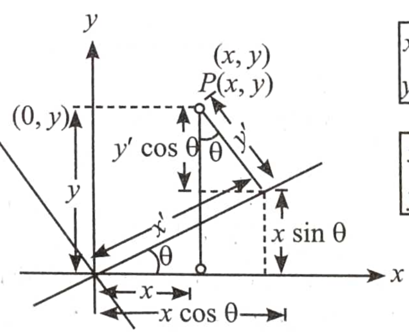
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, B \neq 0.$$

$(A \cos^2\theta + B \sin\theta \cos\theta + C \sin^2\theta)x'^2 + [B(\cos^2\theta - \sin^2\theta) + 2(C - A)(\sin\theta \cos\theta)]x'y' + (A \sin^2\theta - B \sin\theta \cos\theta + C \cos^2\theta)y'2 + (D \cos\theta + E \sin\theta)x' + (-D \sin\theta + E \cos\theta)y' + F = 0.$

{Using (i), (ii)}

$[B(\cos^2\theta - \sin^2\theta) + 2(C - A)(\sin\theta \cos\theta)] = 0$

Where P is (x, y) referred to coordinate system oxy and P is (x', y') referred to new coordinate system $ox'y'$.



$\boxed{\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ y &= x' \cos\theta - y' \cos\theta \end{aligned}}$

$\boxed{\begin{aligned} x' &= x \cos\theta - y \sin\theta \\ y' &= x \sin\theta - y \cos\theta \end{aligned}}$

Train Your Brain

Example 12: Find the equation of the curve $x^2 + 2y^2 + 3x - 4y + 7 = 0$, when the origin is transferred to the point $(1, -2)$ without changing the direction of axes.

Sol. If $P(x, y)$ be any point on the curve and (x_1, y_1) be the coordinates of P. w.r.to new axes then $x_1 = x - \alpha = x - 1$ and $y_1 = y - \beta = y + 2$

$\therefore \quad x = x_1 + 1, y = y_1 - 2$

Hence new equation will be

$(x_1 + 1)^2 + 2(y_1 - 2)^2 + 3(x_1 + 1) - 4(y_1 - 2) + 7 = 0$

or

$x_1^2 + 2x_1 + 1 + 2(y_1^2 - 4y_1 + 4) + 3x_1 + 3 - 4y_1 + 8 + 7 = 0$

or $\quad x_1^2 + 2y_1^2 + 5x_1 - 12y_1 + 27 = 0$

Thus new equation of the curve will be

$x^2 + 2y^2 + 5x - 12y + 27 = 0$

Note: New equation of the curve can be directly obtained by putting $(x + 1)$ in place of x and $(y - 2)$ in place of y in the given equation of the curve.

Concept Application

13. Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain term in y and the constant.

14. If (x, y) and (X, Y) be the co-ordinate of the same point referred to two sets of rectangular axes with same origin and if $ux + vy$, where u and v are independent of x and y, becomes $VX + UY$, show that $u^2 + v^2 = U^2 + V^2$.

15. If the axes are turned through 45°, find the transformed form of the equation $3x^2 + 3y^2 + 2xy - 2 = 0$.

DEFINITION OF STRAIGHT LINE

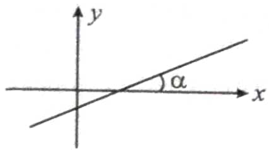
It is a locus of a point $P(h, k)$ which moves in such a way that point $P(h, k)$ is collinear with the two given points.

'or'

A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

Inclination of a Line

Its a measure of the smallest non-negative angle which the line makes with +ve direction of the x-axis [angle being measured in anti-clockwise direction]. $0 \leq \alpha < \pi$



Slope of the Line

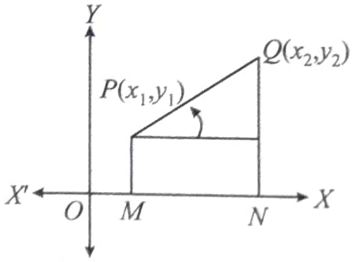
If the inclination of line is θ and $\theta \neq \frac{\pi}{2}$ then its slope is defined as $\tan\theta$ and denoted by 'm'

(*i*) If $\theta = 0$, then $m = 0$ i.e. line parallel to x-axis.

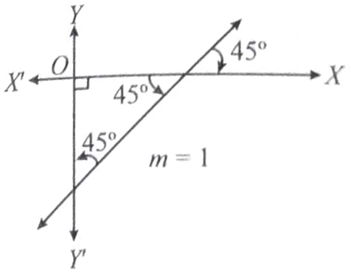
(*ii*) If $\theta = 90°$, then m does not exist i.e. line parallel to y-axis

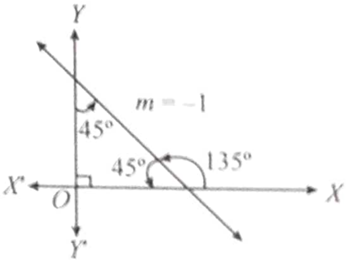
(*iii*) Slope of line joining two points $A(x_1, y_1)$ & $B(x_2, y_2)$ is

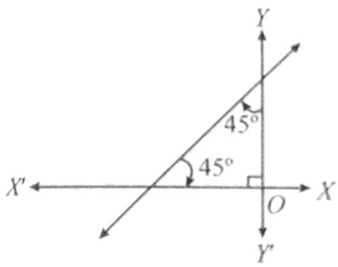
$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

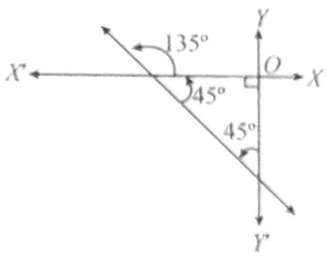


(*iv*) If a line equally inclined with co-ordinate axes then slope is ± 1.

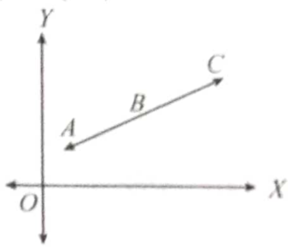








(*v*) If three points A, B, C are collinear, then $m(AB) = m(BC) = m(CA)$.



(*vi*) If two lines, having slopes m_1 and m_2 are parallel, then $m_1 = m_2$

If two lines, having slopes m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$

Intercepts:

The point where a line cuts the x-axis (or y-axis) is called its x-intercept (or y-intercept).

(*i*) Intercepts may be +ve, –ve or zero.

(*ii*) A line making an intercept of $-a$ with y-axis means the line passing through $(0, -a)$

(*iii*) A line makes equal non-zero intercept with both co-ordinate axes then slope is -1.

(*iv*) A line makes non-zero intercept with both co-ordinate axes equal in magnitude then slope is ± 1.

STANDARD EQUATIONS OF STRAIGHT LINES

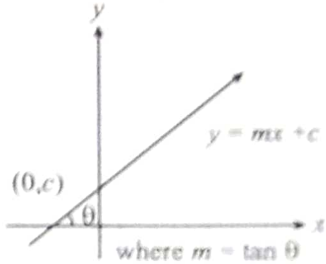
1. General Form: Any first degree equation of the form $Ax + By + C = 0$,

where A, B, C are constant always represents general equation of a straight line (at least one out of A and B is non zero.)

2. Slope - Intercept Form :

$y = mx + c$

where m = slope of the line = $\tan\theta$



where $m = \tan\theta$ $\qquad$ $c = y$ intercept

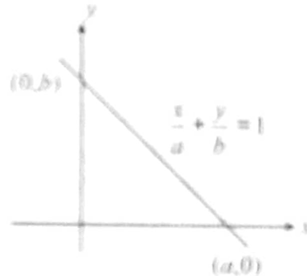
The general equation of any line passing through the origin is $y = mx$.

3. Intercept Form:

$$\frac{x}{a} + \frac{y}{b} = 1$$

x intercept = a

y intercept = b



Notes:

(*i*) The intercepts a and b may be positive or negative.

(*ii*) The intercept cut on negative side of x and y axes are taken as negative.

(*iii*) Area of the triangle formed by the line $\frac{x}{a} + \frac{y}{b} = 1$ and coordinate axes is $= \frac{1}{2}|ab|$ sq. unit

4. Normal Form: If p is the length of the perpendicular from the origin upon a straight line, and that α is the angle the perpendicular makes with the axis of x, equation of the straight line can be obtained as follows:

Let PQ be the straight line, OQ ($= p$) the perpendicular drawn to it from the origin O, and $\angle QOX = \alpha$

Draw the ordinate PN also draw NR perpendicular to OQ, and PM perpendicular to RN

We have $\angle PNM = 90° - \angle RNO = \alpha$

$\therefore p = OQ = OR + RQ = OR + PM = ON \cos\alpha + PN \sin\alpha$

or $p = x\cos\alpha + y\sin\alpha$

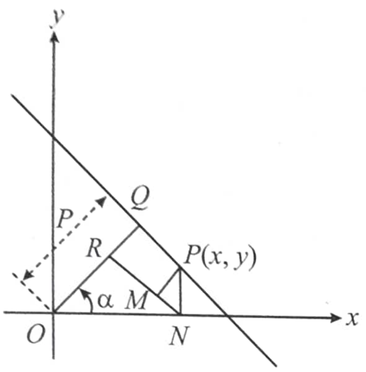
$\therefore x\cos\alpha + y\sin\alpha = p$ is the required equation.

This is called the perpendicular form.

Alternatively,

Suppose $\frac{x}{a}+\frac{y}{b}=1$ is the equation of the straight line

In this case, $a = \frac{p}{\cos\alpha}, b = \frac{p}{\sin\alpha}$



$\therefore$ by substitution, the equation becomes

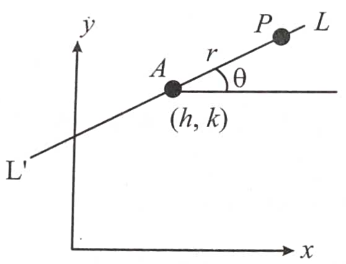
$\frac{x\cos\alpha}{p}+\frac{y\sin\alpha}{p}=1$ or $x\cos\alpha + y\sin\alpha = p$

This is known as the normal form of the equation to a straight line.

5. **Slope Point Form:** Equation : $y - y_1 = m(x - x_1)$, is the equation of line passing through the point (x_1, y_1) and having the slope 'm'

6. **Two points Form:** Equation: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, is the equation of line passing through two points (x_1, y_1) and (x_2, y_2)

7. **Parametric Form:** To find the equation of a straight line which passes through a given point $A(h, k)$ and makes a given angle θ with the positive direction of the x-axis. $P(x, y)$ is any point on the line LAL'. Let $AP = r$, $x - h = r\cos\theta$, $y - k = r\sin\theta$



$\frac{x-h}{\cos\theta}=\frac{y-k}{\sin\theta}=r$ is the equation of the straight line LAL'.

Any point on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k)

Note: If point P is taken relatively upward to A then r is positive otherwise negative. If line is parallel to x-axis then for the point right to A, r is positive and for left to A, r is negative.

Train Your Brain

Example 13: What is the slope of a line whose inclination with the positive direction of x-axis is:

(*i*) 0° (*ii*) 90°
(*iii*) 120° (*iv*) 150°

Sol. (*i*) Here $\theta = 0°$

Slope $= \tan\theta = \tan 0° = 0$.

(*ii*) Here $\theta = 90°$

$\therefore$ The slope of line is not defined.

(*iii*) Here $\theta = 120°$

$\therefore$ Slope $= \tan\theta = \tan 120° = \tan(180° - 60°)$
$= -\tan 60° = -\sqrt{3}$.

(*iv*) Here $\theta = 150°$

$\therefore$ Slope $= \tan\theta = \tan 150° = \tan(180° - 30°)$
$= -\tan 30° = -1/\sqrt{3}$

Example 14: Find the slope of the line passing through the points:

(*i*) (1, 6) and (– 4, 2) (*ii*) (5, 9) and (2, 9)

Sol. (*i*) Let $A = (1, 6)$ and $B = (-4, 2)$

$\therefore$ Slope of $AB = \frac{2-6}{-4-1}=\frac{-4}{-5}=\frac{4}{5}$

$\left(\text{Using slope } \frac{y_2 - y_1}{x_2 - x_1}\right)$

(*ii*) Let $A = (5, 9)$, $B = (2, 9)$

$\therefore$ Slope of $AB = \frac{9-9}{2-5}=\frac{0}{-3}=0$

Concept Application

16. Find the slope of a line whose coordinates are (3,7) and (9,1)?

17. If the slope of a line passing through the points (4, α) and (2, –9) is 3, then what is the value of α?

REDUCTION OF GENERAL EQUATION TO DIFFERENT STANDARD FORMS

1. **Slope Form:** To reduce the equation $Ax + By + C = 0$ to the form $y = mx + c$

Given equation is $Ax + By + C = 0 \Rightarrow m = -A/B, c = -\frac{C}{B}$ $(B \neq 0)$

Note: Slope of the line $Ax + By + c = 0$ is $-\frac{A}{B}$.

i.e $-\left(\frac{\text{coefficient of } x}{\text{coefficient of } y}\right)$. y intercept the line $= -\frac{C}{B}$

2. **Intercept Form:** To reduce the equation $Ax + By + C = 0$ the form $\frac{x}{a}+\frac{y}{b}=1$. This reduction is possible only when $C \neq 0$

Given equation is $Ax + By = -C$

$$\Rightarrow \frac{x}{\frac{-C}{A}}+\frac{y}{\frac{-C}{B}}=1 \text{ which is the form } \frac{x}{a}+\frac{y}{b}=1.$$

where $a = \frac{-C}{A}, b = -\frac{C}{B}$

3. **Normal form:** To reduce the equation $Ax + By + C = 0$ to the form $x \cos\alpha + y \sin\alpha = p$

Given equation is $Ax + By + C = 0$ or, $Ax + By = -C$

Case I: When $-C > 0$, then normal form is $\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = \frac{-C}{\sqrt{A^2+B^2}}$

where $\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \frac{B}{\sqrt{A^2+B^2}}$; $p = \frac{-C}{\sqrt{A^2+B^2}}$

Case II: When $-C < 0$, the write the equation as $-Ax - By = C$

$$\frac{-A}{\sqrt{A^2+B^2}}x+\frac{-B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}}$$

where $\cos\alpha$

$$= \frac{-A}{\sqrt{A^2+B^2}}, \sin\alpha = \frac{-B}{\sqrt{A^2+B^2}}, p = \frac{C}{\sqrt{A^2+B^2}}$$

Note: In the normal form $x \cos\alpha + y \sin\alpha = p$, p is always taken as positive.

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Example 15: Find the equation of a line passing through $(2, -3)$ and inclined at an angle of $135°$ with the positive direction of x-axis.

Sol. Here, m = slope of the line $= \tan 135° = \tan(90° + 45°) = -\cot 45° = -1$, $(x_1, y_1) = (2, -3)$

So, the equation of the line is $y - y_1 = m(x - x_1)$

i.e. $y - (-3) = -1(x - 2)$ or $y + 3 = -x + 2$

or $x + y + 1 = 0$

Example 16: Find the equation of a line with slope -1 and cutting off an intercept of 4 units on negative direction of y-axis.

Sol. Here $m = -1$ and $c = -4$. So, the equation of the line is $y = mx + c$ i.e. $y = -x - 4$ or $x + y + 4 = 0$

Example 17: Find the equation of the line joining the points $(-1, 3)$ and $(4, -2)$

Sol. Here the two points are $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (4, -2)$.

So, the equation of the line in two-point form is

$$y - 3 = \frac{3-(-2)}{-1-4}(x + 1)$$

$$\Rightarrow y - 3 = -x - 1 \Rightarrow x + y - 2 = 0$$

Example 18: Find the equation of the line which is at a distance 3 from the origin and the perpendicular from the origin to the line makes an angle of $30°$ with the positive direction of the x-axis.

Sol. Here $p = 3$, $\alpha = 30°$

$\therefore$ Equation of the line in the normal form is

$x \cos 30° + y \sin 30° = 3$

or $x\frac{\sqrt{3}}{2} + \frac{y}{2} = 3$ or $\sqrt{3}x + y = 6$

Example 19: Reduce the line $2x - 3y + 5 = 0$, in slope intercept, intercept and normal forms.

Sol. Slope - Intercept Form :

$$y = \frac{2x}{3}+\frac{5}{3}, \tan\theta = m = \frac{2}{3}, c = \frac{5}{3}$$

Intercept Form: $\frac{x}{\left(-\frac{5}{2}\right)}+\frac{y}{\left(\frac{5}{3}\right)}=1$, $a = -\frac{5}{2}$, $b = \frac{5}{3}$.

Normal Form: $-\frac{2x}{\sqrt{13}}+\frac{3y}{\sqrt{13}}=\frac{5}{\sqrt{13}}$

$$\sin\alpha = \frac{3}{\sqrt{13}}, \cos\alpha = \frac{-2}{\sqrt{13}}, p = \frac{5}{\sqrt{13}}$$

Concept Application

18. The equation of the lines which pass through the point $(3, 4)$ and the sum of intercepts on the axes is 14 is-
(a) $4x - 3y = 24, x - y = 7$
(b) $4x + 3y = 24, x + y = 7$
(c) $4x + 3y + 24 = 0, x + y + 7 = 0$
(d) $4x - 3y + 24 = 0, x - y + 7 = 0$

19. Find the equation of the straight line on which the perpendicular from origin makes an angle $30°$ with positive x-axis and which forms a triangle of area $\left(\frac{50}{\sqrt{3}}\right)$ sq. units with the co-ordinate axes.

20. The angle between the lines $y - x + 5 = 0$ and $\sqrt{3}x - y + 7 = 0$ is
(a) 15° (b) 60°
(c) 45° (d) 75°

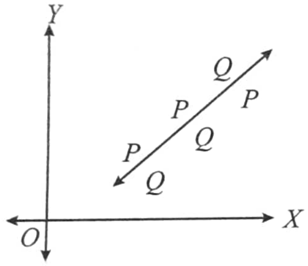
21. The obtuse angle between the line $y = -2$ and $y = x + 2$ is
(a) 120° (b) 135°
(c) 150° (d) 160°

22. If the sum of the distances of a moving point in a plane from the axes is 1, then find the locus of the point.

POSITION OF A POINT WITH RESPECT TO A LINE

$L : Ax + By + C = 0$

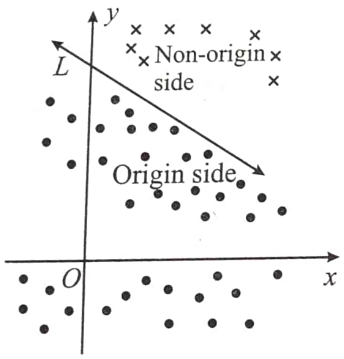
(i) If the points $P(x_1, y_1)$ & $Q(x_2, y_2)$ lies on the same side of the line $Ax + By + C = 0$ then the expressions $Ax_1 + By_1 + C$ & $Ax_2 + By_2 + C$ have same sign otherwise if P and Q lies on opposite side then $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will have opposite sign.



(ii) If only one point is given then position of that point is checked w.r. to origin.

Origin and Non-Origin Sides

Let L be a straight line in the coordinate plane which is not passing through the origin. The side of the region in which the origin lies is called the origin side of the line and the other is called the non-origin side of the line L.



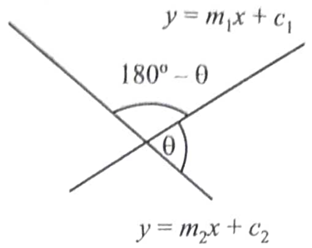
Let $L \equiv ax + by + c = 0$ be a line which is not passing through the origin. Then a point $A(x_1, y_1)$ (not on the line $L = 0$) lies

(i) on the non-origin side of
$L = 0 \Leftrightarrow c$ and L_{11} are of opposite signs

(ii) $A(x_1, y_1)$ lies on the origin side of
$L = 0 \Leftrightarrow c$ and L_{11} have the same sign

ANGLE BETWEEN TWO STRAIGHT LINES

If θ is the acute angle between two lines, then $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$



where m_1 and m_2 are the slopes of the two lines and are finite.

Notes:

(i) If the two lines are perpendicular to each other then $m_1 m_2 = -1$

(ii) Any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$

(iii) If the two lines are parallel or coincident, then $m_1 = m_2$

(iv) Any line parallel to $ax + by + c = 0$ is of the form $ax + by + k = 0$

(v) If any of the two lines is perpendicular to x-axis, then the slope of that line is infinite.

Let $m_1 = \infty$,

$$\text{Then } \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{1 - \frac{m_2}{m_1}}{\frac{1}{m_1} + m_2}\right| = \left|\frac{1}{m_2}\right|$$

or $\theta = |90° - \alpha|$, where $\tan\alpha = m_2$

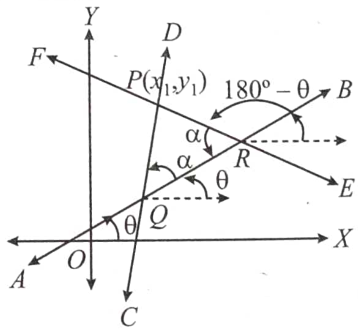
i.e. angle θ is the complimentary to the angle which the oblique line makes with the x-axis.

(vi) If lines are equally inclined to the coordinate axis then $m_1 + m_2 = 0$

+ **Equation of straight lines passing through a given point and making a given angle with a given line.**

The equation of the straight lines which pass through a given point (x_1, y_1) and makes an angle α with the given straight line $y = mx + c$ are $y - y_1 = \tan(\theta \pm \alpha)(x - x_1)$,

Where $m = \tan\theta$.



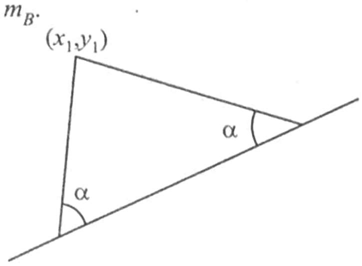
Equation of straight Lines passing through a given point and equally inclined to a given line :

Let the straight passing through the point (x_1, y_1) and make equal angles with the given straight line

$y = mx + c$. If m is the slope of the required line and α is the angle which this line makes with the given line then

$$\tan\alpha = \pm\frac{m_1 - m}{1+m_1 m}$$

The above expression for $\tan\alpha$, given two values of m, say m_A and m_B.



The required equations of the lines through the point (x_1, y_1) and making equal angles α with the given line are $y - y_1 = m_A(x - x_1)$, $y - y_1 = m_B(x - x_1)$

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Example 20: Find the equations of the straight lines passing through the point (2, 3) and inclined at $\pi/4$ radians to the line $2x + 3y = 5$.

Sol. Let the line $2x + 3y = 5$ make an angle θ with positive x-axis.

Then $\tan\theta = -\frac{2}{3}$

Slopes of required lines are $\tan\left(\theta+\frac{\pi}{4}\right)$ and $\tan\left(\theta-\frac{\pi}{4}\right)$

$$\therefore\ \tan\left(\theta+\frac{\pi}{4}\right) = \frac{\tan\theta+\tan\left(\frac{\pi}{4}\right)}{1-\tan\theta\tan\left(\frac{\pi}{4}\right)} = \frac{\left(-\frac{2}{3}\right)+1}{1-\left(-\frac{2}{3}\right)(1)} = \frac{1}{5}$$

$$\text{and } \tan\left(\theta-\frac{\pi}{4}\right) = \frac{\tan\theta-\tan\left(\frac{\pi}{4}\right)}{1+\tan\theta\tan\left(\frac{\pi}{4}\right)} = \frac{\left(-\frac{2}{3}\right)-1}{1+\left(-\frac{2}{3}\right)(1)} = -5$$

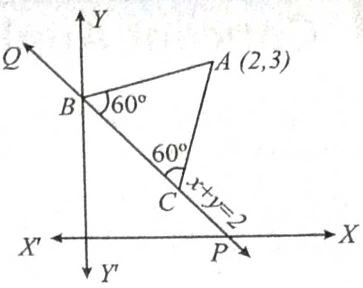
$\therefore$ Equations of required lines are

$$y-3=\frac{1}{5}(x-2) \text{ and } y-3=-5(x-2)$$

i.e., $x - 5y + 13 = 0$ and $5x + y - 13 = 0$

Example 21: A vertex of an equilateral triangle is (2, 3) and the opposite side is $x + y = 2$. Find the equations of the other sides.

Sol. Let $A(2,3)$ be one vertex and $x+y=2$ be the opposite side of an equilateral triangle. Clearly remaining two sides pass through the point $A(2,3)$ and make an angle 60° with $x+y=2$



$\because$ Slope of $x + y = 2$ is -1

Let $\tan\theta = -1$ $\therefore$ $\theta = 135°$

$\therefore$ Equations of the other two sides are

$$y-3 = \tan(135° \pm 60°)\,(x-2)$$

i.e., sides are

$y - 3 = \tan(195°)(x - 2)$ (taking '+' sign)

$\Rightarrow$ $y - 3 = \tan(180° + 15°)(x - 2)$

$\Rightarrow$ $y - 3 = \tan 15°(x - 2) \Rightarrow y - 3 = (2-\sqrt{3})(x-2)$

$\Rightarrow$ $(2-\sqrt{3})x - y = 1-2\sqrt{3}$

and $y - 3 = \tan(75°)(x - 2)$ (taking '–' sign)

$\Rightarrow$ $y - 3 = \cot 15°(x - 2)$

$\Rightarrow$ $y-3 = (2+\sqrt{3})(x-2) \Rightarrow (2+\sqrt{3})x - y = 1+2\sqrt{3}$

Hence equations of other sides are

$(2-\sqrt{3})x - y = 1-2\sqrt{3}$ and $(2+\sqrt{3})x - y = 1+2\sqrt{3}$

Example 22: Find the equation to the sides of an isosceles right-angled triangle, the equation of whose hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point (2, 2).

Sol. The problem can be restarted as :

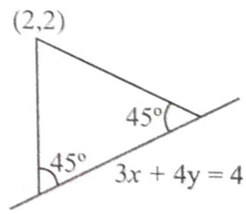
Find the equation to the straight lines passing through the given point (2, 2) and making equal angles of 45° with the given straight line $3x + 4y - 4 = 0$.

Slope of the line $3x + 4y - 4 = 0$ is $m_1 = -\frac{3}{4}$

$$\Rightarrow \tan 45° = \pm\frac{m-m_1}{1+m_1 m}$$

$$\text{i.e., } 1 = \pm\frac{m+3/4}{1-\frac{3}{4}m}$$

$m_A = \frac{1}{7}$, and $m_B = -7$



Hence the required equations of the two lines are

$y - 2 = m_A(x - 2)$

and $y - 2 = m_B(x - 2)$

$\Rightarrow$ $7y - x - 12 = 0$

and $7x + y = 16$.

Concept Application

23. Find the equations of the two lines, each passing through (5, 6) and each making an acute angle of 45° with the line $2x - y + 1 = 0$.

24. If $P(-2, 1)$, $Q(2, 3)$ and $R(-2, -4)$ are three points. Find the angle between the straight lines PQ and QR.

DISTANCE BETWEEN POINT & LINE AND TWO PARALLEL LINES

1. Length of the Perpendicular from a Point on a Line:

The length of the perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

The length of the perpendicular from origin on $ax + by + c = 0$ is $\left|\frac{c}{\sqrt{a^2 + b^2}}\right|$

2. The distance between two parallel lines:

(*i*) When two straight lines are parallel their slopes are equal. Thus any line parallel to $y = mx + c$ is of the type $y = mx + d$, where 'd' is a parameter.

(*ii*) Two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$.

Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$, where k is a parameter.

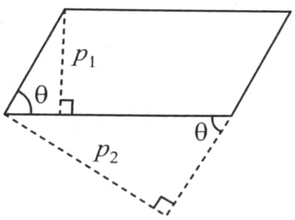
(*iii*) The distance between two parallel lines with equations $ax + by + c_1 = 0$

and $ax + by + c_2 = 0$ is $= \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$.

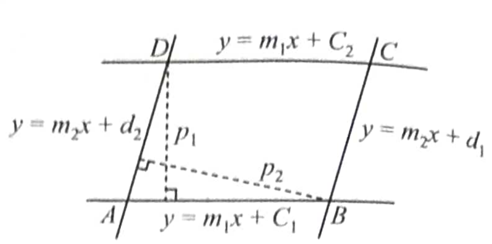
Note that coefficients of x & y in both the equations must be same.

3. Area of parallelogram with given sides :

The area of the parallelogram $= \frac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides.



Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.



4. Condition of parallelogram as shown becomes a rhombus:

$$p_1 = p_2 \Rightarrow \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right| = \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2}}\right|$$

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Example 23: Find the point on y-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3

Sol. Let the point on y-axis be $P(0, k)$

Distance of $P(0, k)$ from $4x - 3y - 12 = 0$ is

$$\left|\frac{4(0) - 3(k) - 12}{\sqrt{4^2 + (-3)^2}}\right| = 3$$

$\Rightarrow \ |3k + 12| = 15$

$\Rightarrow \ |k + 4| = 5$

$\Rightarrow \ k = 1, -9$

Hence the points are (0, 1) & (0, –9)

Example 24: Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form three sides of two squares find the equations to the fourth sides of squares.

Sol. Distance between the lines $x + 2y + 3 = 0$ & $x + 2y - 7 = 0$ is

$$d = \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right| = \left|\frac{3 - (-7)}{\sqrt{1 + 4}}\right| = \frac{10}{\sqrt{5}}$$

The fourth side is parallel to $2x - y - 4 = 0$

Let the fourth side be $2x - y + k = 0$

Distance between two sides

$2x - y - 4 = 0$ and $2x - y + k = 0$ should be $\frac{10}{\sqrt{5}}$

$$\left|\frac{k + 4}{\sqrt{4 + 1}}\right| = \frac{10}{\sqrt{5}}$$

$\Rightarrow \ |k + 4| = 10 \Rightarrow k = 6 \Rightarrow k = -14$

Hence the 4th sides of squares are

$2x - y + 6 = 0$ or $2x - y - 14 = 0$

Example 25: Two mutually perpendicular lines are drawn through the point (a, b) and enclose an isosceles triangle together with the line $x \cos\alpha + y \sin\alpha = P$. Find the area of triangle.

Sol. ΔABC is right angled at A.

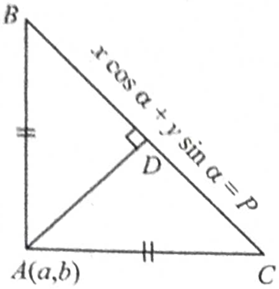
$AB = AC$

AD perpendicular BC

Length of perpendicular from

$$A(a, b) = \left|\frac{a\cos\alpha + b\sin\alpha - P}{\sqrt{\cos^2\alpha + \sin^2\alpha}}\right|$$

$= |a \cos \alpha + b \sin \alpha - P|$



Since ΔABC is isosceles

$\therefore \quad AD = BD = DC$

$\quad BC = 2\,(AD) = 2\,|(a\cos\alpha + b\sin\alpha - P)|$

Area of ΔABC

$= \frac{1}{2} \times |(a\cos\alpha + b\sin\alpha - P)| \cdot 2|(a\cos\alpha + b\sin\alpha - P)|$

$= (a\cos\alpha + b\sin\alpha - P)^2$

Concept Application

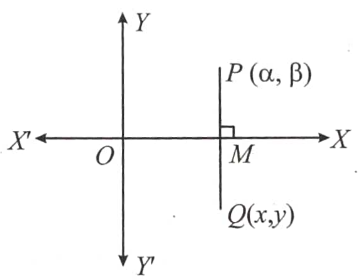
25. Find the perpendicular distance between the lines $3x + 4y - 5 = 0$ and $6x + 8y - 45 = 0$.

26. Find the distance of $P(-2, 3)$ from the line $x - y = 5$.

27. If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

IMAGE OR REFLECTION OF A POINT IN DIFFERENT CASES

1. The image or reflection of a point with respect to x-axis: Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about x-axis, then (M is the mid point of P and Q)

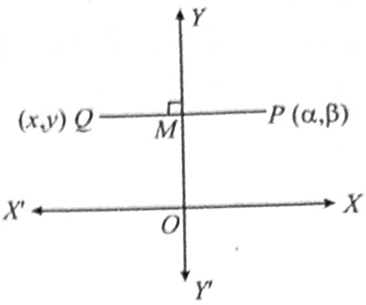


$x = \alpha;\ y = -\beta$

$\therefore\ Q \equiv (\alpha, -\beta)$ i.e., Sign change of ordinate.

Note : The image of the line $ax + by + c = 0$ about x-axis is $ax - by + c = 0$

2. The image or reflection of a point with respect to y-axis: Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about y-axis, then (M is the mid point of PQ)

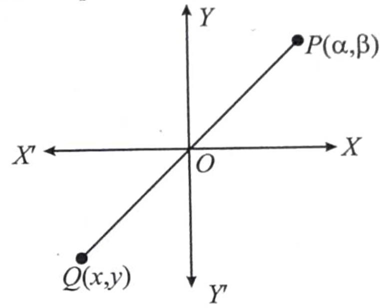


$x = -\alpha$ and $y = \beta$

$\therefore\ Q \equiv (-\alpha, \beta)$ i.e., sign change of abscissae.

Note: The image of the line $ax + by + c = 0$ about y-axis is $-ax + by + c = 0$

3. The image or reflection of a point with respect to origin: Let $P(\alpha, \beta)$ be any point and $Q(x, y)$ be its image about the origin (O is the mid point of PQ), then



$x = -\alpha$ and $y = -\beta$

$\therefore\ Q \equiv (-\alpha, -\beta)$

i.e., sign change of abscissae and ordinate.

4. Foot of the perpendicular from a point (x_1, y_1) on the line $ax + by + c = 0$

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

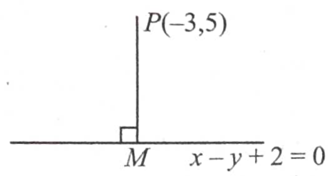
5. The image of a point (x_1, y_1) about the line $ax + by + c = 0$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

Train Your Brain

Example 26: Find the foot of perpendicular of the line drawn from $P(-3, 5)$ on the line $x - y + 2 = 0$.

Sol. Slope of $PM = -1$



$\therefore$ Equation of PM is

$x + y - 2 = 0$ ...(i)

Solving equation (i) with $x - y + 2 = 0$, we get co-ordinates of $M(0, 2)$

Second Method: Here,

$$\frac{x+3}{1} = \frac{y-5}{-1} = -\frac{(1\times(-3)+(-1)\times 5+2)}{(1)^2+(-1)^2}$$

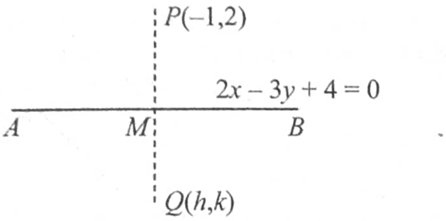
$\Rightarrow \frac{x+3}{1} = \frac{y-5}{-1} = 3 \quad \Rightarrow \quad x+3 = 3$

$\Rightarrow x = 0$ and $y - 5 = -3 \quad \Rightarrow \quad y = 2$

$\therefore \quad M$ is $(0, 2)$

Example 27: Find the image of the point $P(-1, 2)$ in the line mirror $2x - 3y + 4 = 0$.

Sol. Let image of P is Q.



$\therefore \quad PM = MQ$ & $PQ \perp AB$

Let Q is (h, k)

$\therefore \quad M$ is $\left(\frac{h-1}{2}, \frac{k+2}{2}\right)$

It lies on $2x - 3y + 4 = 0$.

$\therefore \quad 2\left(\frac{h-1}{2}\right) - 3\left(\frac{k+2}{2}\right) + 4 = 0.$

or $\quad 2h - 3k = 0 \qquad \ldots (i)$

Slope of $PQ = \frac{k-2}{h+1}$

$PQ \perp AB$

$\therefore \quad \frac{k-2}{h+1} \times \frac{2}{3} = -1 \quad \Rightarrow \quad 3h + 2k - 1 = 0. \qquad \ldots (ii)$

Soving (i) & (ii), we get $h = \frac{3}{13}$, $k = \frac{2}{13}$

$\therefore$ Image of $P(-1, 2)$ is $Q\left(\frac{3}{13}, \frac{2}{13}\right)$

unchanged under the translation of co-ordinate axes

2nd Method: The image of $P(-1, 2)$ about the line $2x - 3y + 4 = 0$ is $\frac{x+1}{2} = \frac{y-2}{-3} = -2\frac{[2(-1)-3(2)+4]}{2^2+(-3)^2}$

$\frac{x+1}{2} = \frac{y-2}{-3} = \frac{8}{13}$

$\Rightarrow \quad 13x + 13 = 16$

$\Rightarrow \quad x = \frac{3}{13}$ and $13y - 26 = -24$

$\Rightarrow \quad y = \frac{2}{13} \qquad \therefore$ image is $\left(\frac{3}{13}, \frac{2}{13}\right)$

Example 28: Find the equation of the straight line that has y-intercept 4 and is parallel to the straight line $2x - 3y = 7$.

Sol. Given line is $2x - 3y = 7 \qquad \ldots (1)$

$\Rightarrow \quad 3y = 2x - 7$

$\Rightarrow \quad y = \frac{2}{3}x - \frac{7}{3}$

$\therefore \quad$ Slope of (1) is 2/3

The required line is parallel to (1), so its slope is also 2/3, y-intercept of required line = 4

$\therefore$ By using $y = mx + c$ form, the equation of the required line is

$y = \frac{2}{3}x + 4$

or $\quad 2x - 3y + 12 = 0$

Concept Application

28. Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ and $2x - y - 4 = 0$ form 3 sides of two squares. Find the equation of remaining sides of these squares.

29. Find the foot of the perpendicular drawn from the point (2, 3) to the line $3x - y + 5 = 0$. Also, find the image of (2, 3) in the given line.

BISECTORS OF THE ANGLES BETWEEN TWO GIVEN LINES

Angle bisector is the locus of a point which moves in such a way so that its distance from two intersecting lines remains same.

The equation of the two bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$.

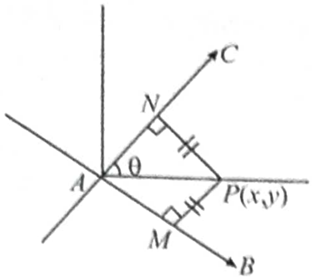
If the two given lines are not perpendicular i.e. $a_1a_2 + b_1b_2 \neq 0$, then one of these equation is the equation of the bisector of the acute angle and the other that of the obtuse angle.

Note: Whether both lines are perpendicular or not but the angular bisectors of these lines will always be mutually perpendicular.

(i) A Line which is equally inclined to given two lines is parallel to the angle bisector of the given lines.

The bisectors of the acute and the obtuse angles: Take one of the lines angle let its slope be m_1 and take one of the bisectors and let its slope be m_2. If θ be the acute angle between them, then find $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$.

If $\tan\theta > 1$ then the bisector taken is the bisector of the obtuse angle and the other one will be the bisector of the acute angle. If $0 < \tan\theta < 1$ then the bisector taken is the bisector of the acute angle and the other one will be the bisector of the obtuse angle.



Another approach

If two lines are $a_1x + b_1y + c_1 = 0$

and $a_2x + b_2y + c_2 = 0$,

then $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

will represent the equation of the bisector of the acute or obtuse angle between the lines according as c_1c_2 $(a_1a_2 + b_1b_2)$ is negative or positive.

The equation of the bisector of the angle containing the origin

(*ii*) Write the equations of the two lines so that the constants c_1 and c_2 become positive. Then the equation $\dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}$ $= \dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ is the equation of the bisector containing the origin.

Notes:

(*i*) If $a_1a_2 + b_1b_2 < 0$, then the origin will lie in the acute angle and if $a_1a_2 + b_1b_2 > 0$, then origin will lie in the obtuse angle.

(*ii*) The note (*i*) is helpful in finding the equation of bisector of the obtuse angle or acute angle directly.

The equation of the bisector of the angle which contains a given point

The equation of the bisector of the angle between the two lines containing the point (α, β) is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = +\left(\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right)$$

or $$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\left(\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right)$$

According as $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ are of the same sings or of opposite signs.

Train Your Brain

Example 29: For the straight line $4x+3y-6=0$ and $5x+12y+9=0$, find the equation of the

(*i*) bisector of the obtuse angle between them.

(*ii*) bisector of the acute angle between them.

(*iii*) bisector of the angle which contains (1, 2).

Sol. Equations of bisectors of the angles between the given lines are $\dfrac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm\dfrac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 9x-7y-41=0$ and $7x+9y-3=0$.

If θ is the acute angle between the line $4x+3y-6=0$ and the bisector $9x-7y-41=0$, then

$$\tan\theta = \left|\frac{-\frac{4}{3}-\frac{9}{7}}{1+\left(\frac{-4}{3}\right)\frac{9}{7}}\right| = \frac{11}{3} > 1$$

Hence

(*i*) The bisector of the obtuse angle is $9x-7y-41=0$

(*ii*) The bisector of the acute angle is $7x+9y-3=0$

(*iii*) The bisector of the angle containing the origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

$\Rightarrow \quad 7x+9y-3=0$

(*i*) For the point (1, 2),

$4x + 3y - 6$

$= 4 \times 1 + 3 \times 2 - 6 > 0$

$\Rightarrow 5x + 12y + 9$

$= 5 \times 1 + 12 \times 2 + 9 > 0$

Hence equation of the bisector of the angle containing the point (1, 2) is

$$\frac{4x+3y-6}{5} = \frac{5x+12y+9}{13} \Rightarrow 9x-7y-41=0$$

Concept Application

30. Let $P = (-1, 0)$ $Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three points. Then the equation of the bisector of the angle PQR is

31. The vertices of a triangle are $A(-1, 7)$, $B(5, 1)$ and $C(1, 4)$. Then equation of bisector of ΔABC?

CONDITION OF CONCURRENCY

Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Alternatively : If three constants A, B & C (not all zero) can be found such that

$A(a_1x+b_1y+c_1)+B(a_2x+b_2y+c_2)+C(a_3x+b_3y+c_3) \equiv 0$, then the three straight lines are concurrent.

OPTICS BASED PROBLEM

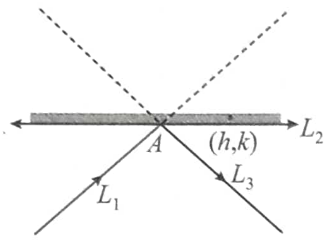
The equation of reflected ray:

Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ be the incident ray in the line mirror $L_2 \equiv a_2x + b_2y + c_2 = 0$

Let L_3 be the reflected ray from the line L_2. Clearly L_2 will be one of the bisectors of the angles between L_1 and L_3. Since L_3 passes through A, so $L_3 \equiv L_1 + \lambda L_2 = 0$.

Let (h, k) be a point on L_2. Then,

$$\frac{|a_1h + b_1k + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_1h + b_1k + c_1 + \lambda(a_2h + b_2k + c_2)|}{\sqrt{(a_1 + \lambda a_2)^2 + (b_1 + \lambda b_2)^2}}$$



Since (h, k) lies on L_2, $a_2h + b_2k + c_2 = 0$

$$\Rightarrow \quad a_1^2 + a_2^2\lambda^2 + 2a_1a_2\lambda + b_1^2 + b_2^2\lambda^2 + 2b_1b_2\lambda + a_1^2 + b_1^2$$

$$\Rightarrow \quad \lambda = 0 \text{ or } \lambda = \frac{-2(a_1a_2 + b_1b_2)}{a_2^2 + b_2^2}$$

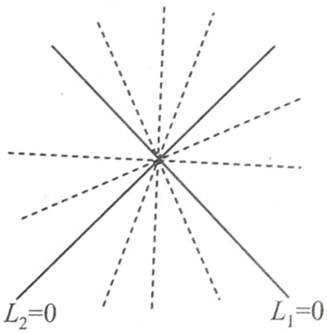
But $\lambda = 0$ gives $L_3 = L_1$.

Hence $L_3 = L_1 - \dfrac{2(a_1a_2 + b_1b_2)}{a_2^2 + b_2^2} L_2 = 0$.

Note: Some times the reflected ray L_3 is also called the mirror image of L_1 in L_2.

FAMILY OF LINES

The general equation of the family of lines through the point of intersection of two given lines is $L + \lambda L' = 0$, where $L = 0$ and $L' = 0$ are the two given lines, and λ is a parameter.



Conversely, any line of the form $L_1 + \lambda L_2 = 0$ passes through a fixed point which is the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$.

The family of lines perpendicular to a given line $ax + by + c = 0$ is given by, $bx - ay + k = 0$, where k is a parameter. The family of lines parallel to a given line $ax + by + c = 0$ is given by $ax + by + k = 0$, where k is a parameter.

One Parameter Family of Straight Lines

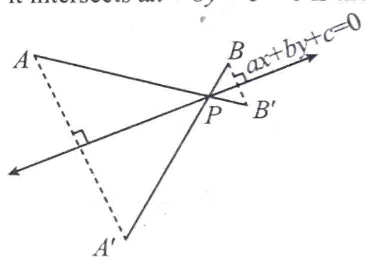
If a linear expression L_1 contains an unknown coefficient, then the line $L_1 = 0$ can not be a fixed line. Rather it represents a family of straight lines known as one parameter family of straight lines. e.g. family of lines parallel to the x-axis i.e. $y = c$ and family of straight lines passing through the origin i.e. $y = mx$.

Each member of the family passes a fixed point. We have two methods to find the fixed point.

Method (1) : Let the family of straight lines of the form $ax + by + c = 0$ where a, b, c are variable parameters satisfying the condition $al + bm + cn = 0$, where l, m, n, are given and $n \neq 0$. Rewriting the condition as $a\left(\frac{l}{n}\right) + b\left(\frac{m}{n}\right) + c = 0$ and comparing with the given family of straight lines, we find that each member of it passes through the fixed point $\left(\frac{l}{n}, \frac{m}{n}\right)$

OPTIMIZATION (MINIMIZATION OR MAXIMIZATION)

1. **Minimization:** Let A and B are two given points on the same side of $ax + by + c = 0$ Suppose we want to determine a point P on $ax + by + c = 0$ such that $PA + PB$ is minimum. Then find the image of A or B about the line $ax + by + c = 0$ (say A' or B') then join B' with A or A' with B wherever it intersects $ax + by + c = 0$ is the required point.



$\therefore \quad PA + PB = PA + PB'$

or $\quad PA + PB = PA' + PB$

Note: By triangle in equality

Sum of two sides of a triangle > Third side

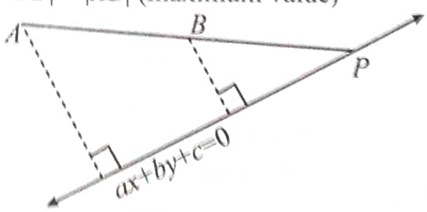
i.e., $|PA + PB| = |PA + PB'| = |AB'|$ (minimum value).

2. **Maximization :** Let A and B are two given points on the same side of $ax + by + c = 0$. Suppose we want to determine a point P on $ax + by + c = 0$ such that $|PA - PB|$ is maximum, then find the equation of line AB wherever it intersects $ax + by + c = 0$ is the required point.

Note: By triangle inequality

Difference of two sides of a triangle < Third side

i.e., $|PA - PB| = |AB|$ (maximum value)



Train Your Brain

Example 30: Prove that the straight lines $4x + 7y = 9$, $5x - 8y + 15 = 0$ and $9x - y + 6 = 0$ are concurrent.

Sol. Given lines are

$4x + 7y - 9 = 0$ ... (1)

$5x - 8y + 15 = 0$ ... (2)

and $9x - y + 6 = 0$ ... (3)

$$\Delta = \begin{vmatrix} 4 & 7 & -9 \\ 5 & -8 & 15 \\ 9 & -1 & 6 \end{vmatrix}$$

$= 4(-48 + 15) - 7(30 - 135) - 9$

$(-5 + 72) = -132 + 735 - 603 = 0$

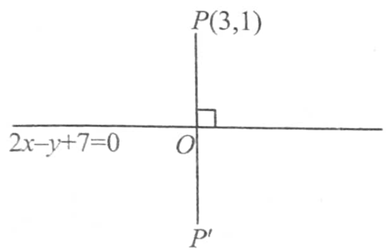
Hence lines (1), (2) and (3) are concurrent.

Example 31: Find the image of (3, 1) across the line $y = 2x + 7$.

Sol. Let the point $P(3, 1)$ has image P' across the line $2x - y + 7 = 0$

Now PP' is perpendicular to $2x - y + 7 = 0$

Slope of $PP' = -\dfrac{1}{2}$



Equation of PP' is $y - 1 = -\dfrac{1}{2}(x - 3)$

$2y - 2 = -x + 3 \Rightarrow x + 2y = 5$

Point of intersection of lines $2x - y + 7 = 0$ and $x + 2y = 5$ is $O\left(\dfrac{-9}{5}, \dfrac{17}{5}\right)$

O is the mid point of PP'

Let $P'(h, k)$, $\dfrac{h+3}{2} = \dfrac{-9}{5}$, $\dfrac{k+1}{2} = \dfrac{17}{5}$

$h = \dfrac{-33}{5}$, $k = \dfrac{29}{5} \Rightarrow$ Image $\left(-\dfrac{33}{5}, \dfrac{29}{5}\right)$

Example 32: If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Sol. Let n given points be (x_i, y_i) where $i = 1, 2 \dots n$ and the variable line is $ax + by + c = 0$, Given that

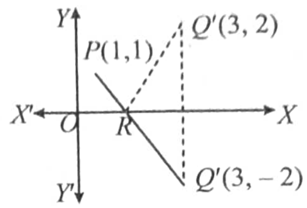
$$\sum_{i=1}^{n}\left(\frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}}\right) = 0.$$

$$\Rightarrow a\Sigma x_i + b\Sigma y_i + cn = 0 \Rightarrow a\frac{\sum x_i}{n} + b\frac{\sum y_i}{n} + c = 0.$$

Hence the variable straight line always passes through the fixed point $\left(\dfrac{\sum x_i}{n}, \dfrac{\sum y_i}{n}\right)$.

Example 33: Find a point R on the x-axis such that $PR + RQ$ is the minimum when $P = (1, 1)$ and $Q = (3, 2)$.

Sol.



Since P and Q lie on the same side of x-axis.

The image of $Q(3, 2)$ about x-axis is $Q'(3, -2)$ then the equation of line PQ' is

$$y - 1 = \frac{-2-1}{3-1}(x - 1) \Rightarrow 3x + 2y - 5 = 0$$

This line meets x-axis at $R\left(\dfrac{5}{3}, 0\right)$ which is the required point.

Concept Application

32. A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line mirror $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

33. Prove that each member of the family of straight lines $(3 \sin\theta + 4 \cos\theta)x + (2 \sin\theta - 7\cos\theta)y + (\sin\theta + 2 \cos\theta) = 0$ (θ is a parameter) passes through a fixed point.

34. Find the equation of the straight line which passes through the point $(2, -3)$ and the point of intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$.

35. Find a point P on the line $3x + 2y + 10 = 0$ such that $|PA - PB|$ is maximum where A is (4, 2) and B is (2, 4).

36. The equation of the line through the intersection of the lines $2x - 3y = 0$ and $4x - 5y = 2$ and

	Column C_1		Column C_2
A.	through the point (2, 1) is	p.	$2x - y = 4$
B.	perpendicular to the line $x + 2y + 1 = 0$ is	q.	$x + y - 5 = 0$
C.	parallel to the line $3x - 4y + 5 = 0$ is	r.	$x - y - 1 = 0$
D.	equally inclined to the axes is	s.	$3x - 4y - 1 = 0$

PAIR OF STRAIGHT LINES

The combined equation of pair of straight lines $L_1 = a_1x + b_1y + c_1 = 0$ and $L_2 = a_2x + b_2y + c_2 = 0$ is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$ i.e. $L_1L_2 = 0$. Opening the brackets and comparing the terms with the terms of general equation of 2^{nd} degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we can get all the following results for a pair of straight lines.

The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ and $h^2 \geq ab$.

$\Rightarrow \; abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $h^2 \geq ab$.

The homogeneous second degree equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines through the origin if $h^2 \geq ab$.

If the lines through the origin whose joint equation is $ax^2 + 2hxy + by^2 = 0$, are $y = m_1x$ and $y = m_2x$, then $y^2 - (m_1 + m_2) xy + m_1m_2x^2 = 0$ and $y^2 + \frac{2h}{b}xy + \frac{a}{b}x^2 = 0$ are identical, so that $m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$

Angle between pair of straight lines: If θ be the angle between two lines, through the origin, then

$$\tan\theta = \pm\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2} = \pm\frac{2\sqrt{h^2 - ab}}{a+b}$$

Lines are perpendicular if $a + b = 0$ coincident if $h^2 = ab$.

In the more general case, the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be perpendicular if $a + b = 0$, parallel if the terms of second degree make a perfect square i.e. $ax^2 + 2hxy + by^2$ gets converted into $(l_1x \pm m_1y)^2$, coincident if the whole equation makes a perfect square i.e. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ can be written as $(lx + my + n)^2$.

Note: Point of intersection of the two lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is obtained by solving the equations $\frac{\partial f}{\partial x} = ax + hy + g = 0$ and $\frac{\partial f}{\partial y} = hx + by + f = 0$ where $\frac{\partial f}{\partial x}$ denotes the derivative of f with respect to y, keeping x constant. The fact can be used in splitting $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ into equations of two straight lines. With the above method, the point of intersection can be found. Now only the slopes need to be determined.

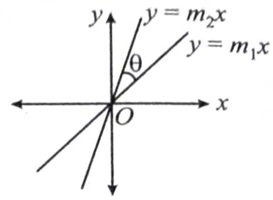
It should be noted that the line $ax + hy + g = 0$ and $hx + by + f = 0$ are not the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. These are the lines concurrent with the lines represented by given equation.

A Pair of Straight Lines Through Origin:

(i) A homogeneous equation of degree two, "$ax^2 + 2hxy + by^2 = 0$" always represents a pair of straight lines passing through the origin if :

(a) $h^2 > ab \Rightarrow$ lines are real & distinct.

(b) $h^2 = ab \Rightarrow$ lines are coincident.



(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0)

This equation is obtained by multiplying the two equations of lines $(m_1x - y)(m_2x - y) = 0$

$\Rightarrow m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$$m_1 + m_2 = -\frac{2h}{b} \;\&\; m_1 m_2 = \frac{a}{b}.$$

(iii) If θ is the acute angle between the pair of straight lines represented by,

$ax^2 + 2hxy + by^2 = 0$, then $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

(iv) The condition that these lines are :

(a) at right angles to each other is $a + b = 0$.
i.e. co–efficient of x^2 + co–efficient of $y^2 = 0$.

(b) coincident is $h^2 = ab$.

(c) equally inclined to the axis of x is $h = 0$.i.e. coeff. of $xy = 0$.

(v) The equation to the pair of straight lines bisecting the angles between the straight lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

Note that a homogeneous equation of degree n represents n straight lines passing through origin.

Notes:

(i) If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines, then the homogeneous equation $ax^2 + 2hxy + by^2 = 0$ also represents pair of lines passing through origin and parallel to the lines $S = 0$.

(ii) If $h^2 = ab$, then $ax^2 + 2hxy + by^2 = 0$ represents pair of coincidental lines so that in this case, $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines.

(iii) Procedure to Find the lines Represented by the Second-Degree General Equation

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Step 1: Factorise the homogeneous part $ax^2 + 2hxy + by^2$ and suppose $ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$

Step 2: $S \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$

Step 3: Equate the corresponding coefficients of x and y and also the constant terms on both sides and solve for n_1 and n_2.

Train Your Brain

Example 34: Find the equations to the pair of lines through the origin which are perpendicular to the lines represented by $2x^2 - 7xy + 3y^2 = 0$.

Sol. We have $2x^2 - 7xy + 3y^2 = 0$.

$\Rightarrow\ 2x^2 - 6xy - xy + 3y^2 = 0$

$\Rightarrow\ 2x(x - 3y) - y(x - 3y) = 0$

$\Rightarrow\ (x - 3y)(2x - y) = 0 \Rightarrow x - 3y = 0$ or $2x - y = 0$

Thus the given equation represents the lines $x - 3y = 0$ and $2x - y = 0$. The equations of the lines passing through the origin and perpendicular to the given lines are $y - 0 = -3(x - 0)$ and $y - 0 = -\frac{1}{2}(x - 0)$ [$\because$ (Slope of $x - 3y = 0$) is $1/3$ and (Slope of $2x - y = 0$) is 2]

$\Rightarrow\ y + 3x = 0$ and $2y + x = 0$

Example 35: Find the angle between the pair of straight lines $4x^2 + 24xy + 11y^2 = 0$

Sol. Given equation is $4x^2 + 24xy + 11y^2 = 0$

Here a = coeff. of $x^2 = 4$, b = coeff. of $y^2 = 11$

and $2h$ = coeff. of $xy = 24$

$\therefore\ h = 12$

Now $\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\dfrac{2\sqrt{144 - 44}}{4 + 11}\right| = \dfrac{4}{3}$

Where θ is the acute angle between the lines.

$\therefore$ acute angle between the lines is $\tan^{-1}(4/3)$ and obtuse angle between them is $\pi - \tan^{-1}(4/3)$

Concept Application

37. Show that the equation $Bx^2 - 2Hxy + Ay^2 = 0$ represents a pair of straight lines which are at right angle to the pair given by the equation $Ax^2 + 2Hxy + By^2 :0$.

38. Show that the following equation represents a pair of lines and find the acute angle between them.
$2x^2 + 7xy + 3y^2 - 5x - 5y + 2 = 0$

39. Show that the two straight lines : $x^2(\tan^2\theta + \cos^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ represented by the equation are such that the difference of their slopes is 2.

40. The equation $x - y = 4$ and $x^2 + 4xy + y^2 = 0$ represent the sides of
(*a*) an equilateral triangle (*b*) a right-angled triangle
(*c*) an isosceles triangle (*d*) none of these

41. Find the centroid of the triangle the equation of whose side $12x^2 - 20xy + 7y^2 = 0$ and $2x - 3y + 4 = 0$.

42. Find the equation of the line pair throught the point (2, 3) and perpendicular to the lines $3x^2 - 8xy + 5y^2 = 0$.

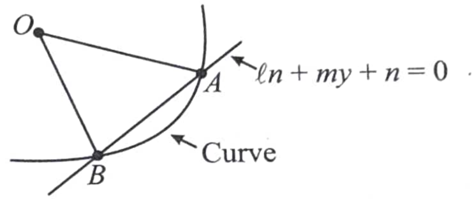
HOMOGENIZATION

This method is used to write the joint equation of two lines connecting origin to the points of intersection of a given line and a given second degree curve.

The equation of a pair of straight lines joining origin to the points of intersection of the line $L \equiv \ell x + my + n = 0$ and a second degree curve $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $ax^2 + 2hxy + by^2 + 2gx$

$$\left(\frac{\ell x + my}{-n}\right) + 2fy\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0.$$

The equation is obtained by homogenizing the equation of curve with the help of equation of line.



Note : (*i*) Here we have written 1 as $\dfrac{\ell x + my}{-n}$ and converted all terms of the curve to second degree expressions

(*ii*) Equation of any curve passing through the points of intersection of two curves $C_1 = 0$ and $C_2 = 0$ is given by $\lambda C_1 + \mu C_2 = 0$, where λ & μ are parameters.

BISECTORS OF THE ANGLE BETWEEN THE LINES GIVEN BY A HOMOGENEOUS EQUATION

OR

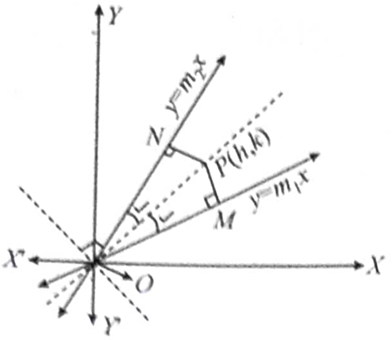
COMBINED EQUATION OF ANGLE BISECTOR

Theorem: The joint equation of the bisectors of the angles between the lines represented by the equation

$$ax^2 + 2hxy + by^2 = 0 \text{ is } \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

Proof : Let the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$, then

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b} \quad \ldots (1)$$



Since the bisectors of the angles between the lines are the locus of a point which is equidistant from the two given lines.

Let $P(h, k)$ be a point on a bisector of the angle between the given lines. Then $PM = PN$

$$\Rightarrow \frac{|k-m_1h|}{\sqrt{(1+m_1^2)}} = \frac{|k-m_2h|}{\sqrt{(1+m_2^2)}}$$

or $$\frac{(k-m_1h)}{\sqrt{(1+m_1^2)}} = \pm\frac{(k-m_2h)}{\sqrt{(1+m_2^2)}}$$

Hence the locus of a $P(h, k)$ is

$$\frac{(y-m_1x)}{\sqrt{(1+m_1^2)}} = \pm\frac{(y-m_2x)}{\sqrt{1+m_2^2)}}$$

$\therefore$ The pair of bisectors is

$$\left(\frac{(y-m_1x)}{\sqrt{(1+m_1^2)}} + \frac{(y-m_2x)}{\sqrt{(1+m_2^2)}}\right)\left(\frac{(y-m_1x)}{\sqrt{(1+m_1^2)}} - \frac{(y-m_2x)}{\sqrt{(1+m_2^2)}}\right) = 0$$

$$\Rightarrow \frac{(y-m_1x)^2}{(1+m_1^2)} - \frac{(y-m_2x)^2}{(1+m_2^2)} = 0$$

$$\Rightarrow (1+m_2^2)(y^2+m_1^2x^2-2m_1xy) - (1+m_1^2)(y^2+m_2^2x^2-2m_2xy) = 0$$

$$\Rightarrow (m_2^2-m_1^2)y^2 - (m_2^2-m_1^2)x^2 + 2xy(m_2-m_1) - 2m_1m_2(m_2-m_1)xy = 0$$

$$\Rightarrow (m_2+m_1)(y^2-x^2) + 2xy - 2m_1m_2xy = 0 \qquad (\because m_1 - m_2 \neq 0)$$

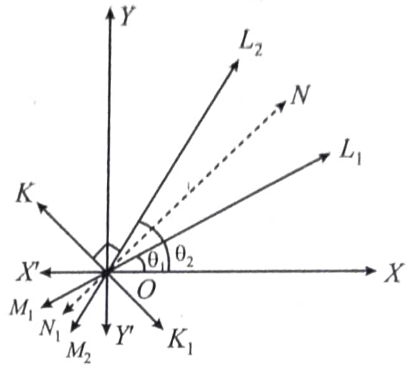
$$\Rightarrow (x^2-y^2)\left(-\frac{2h}{b}\right) = 2xy\left(1-\frac{a}{b}\right) \qquad \left(\begin{aligned}\because m_1+m_2 &= -\frac{2h}{b}\\ m_1m_2 &= \frac{a}{b}\end{aligned}\right)$$

$$\Rightarrow \frac{x^2-y^2}{a-b} = \frac{xy}{h} \qquad (b \neq 0)$$

Alternative Method: Let the equation $ax^2 + 2hxy + by^2 = 0$ represent two lines L_1OM_1 and L_2OM_2 making angles θ_1 and θ_2 with the positive direction of x-axis.

If slopes of L_1OM_1 and L_2OM_2 are m_1 and m_2, then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

and $m_1 + m_2 = -\dfrac{2h}{b}$, $m_1m_2 = \dfrac{a}{b}$ ...(1)



Let NON_1 and KOK_1 are the required bisectors,

Since $\angle NOL_1 = \angle NOL_2 = \dfrac{\theta_2-\theta_1}{2}$

$$\angle NOX = \theta_1 + \frac{\theta_2-\theta_1}{2} = \frac{\theta_1+\theta_2}{2}$$

Since, $\angle NOK = \dfrac{\pi}{2}$

$$\therefore \quad \angle KOX = \frac{\pi}{2} + \angle NOX = \frac{\pi}{2} + \left(\frac{\theta_1+\theta_2}{2}\right)$$

Equation of bisectors are $y = x\tan\left(\dfrac{\theta_1+\theta_2}{2}\right)$

or $\quad y - x\tan\left(\dfrac{\theta_1+\theta_2}{2}\right) = 0$ ...(2)

and $y = x\tan\left(\dfrac{\pi}{2}+\dfrac{\theta_1+\theta_2}{2}\right)$

or $\quad y = -x\cot\left(\dfrac{\theta_1+\theta_2}{2}\right)$

or $\quad y + x\cot\left(\dfrac{\theta_1+\theta_2}{2}\right) = 0$ ...(3)

$\therefore$ Pair of bisectors

$$\left(y - x\tan\left(\frac{\theta_1+\theta_2}{2}\right)\right)\left(y + x\cot\left(\frac{\theta_1+\theta_2}{2}\right)\right) = 0$$

$$\Rightarrow y^2 - x^2 + xy\left(\cot\left(\frac{\theta_1+\theta_2}{2}\right) - \tan\left(\frac{\theta_1+\theta_2}{2}\right)\right) = 0$$

$$\Rightarrow x^2 - y^2 = xy\left(\frac{1+\tan^2\left(\frac{\theta_1+\theta_2}{2}\right)}{\tan\left(\frac{\theta_1+\theta_2}{2}\right)}\right)$$

$$\Rightarrow x^2 - y^2 = xy\left(\frac{2}{\tan(\theta_1+\theta_2)}\right)$$

$$\Rightarrow x^2 - y^2 = 2xy\left(\frac{1-\tan\theta_1\tan\theta_2}{\tan\theta_1+\tan\theta_2}\right)$$

$$\Rightarrow x^2 - y^2 = 2xy\left(\frac{1-m_1m_2}{m_1+m_2}\right)$$

$\Rightarrow\ x^2 - y^2 = 2xy\left(\dfrac{1-a/b}{-2h/b}\right)$

$\Rightarrow\ \dfrac{x^2-y^2}{a-b} = \dfrac{xy}{h}$

Note: The joint equation of the bisectors is

$\dfrac{x^2-y^2}{a-b} = \dfrac{xy}{h}$

or $\quad hx^2 - (a-b)xy - hx^2 = 0$

i.e., coeffi. of x^2 + coeffi. of $y^2 = 0$

Hence the bisectors of the angle between the lines are always perpendicular to each other.

Corollary:

(*i*) If $a = b$, the bisectors are $x^2 - y^2 = 0$

i.e., $x - y = 0,\ x + y = 0$

(*ii*) If $h = 0$, the bisectors are $xy = 0$

i.e., $x = 0,\ y = 0$

Train Your Brain

Example 36: Prove that the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}\dfrac{2\sqrt{2}}{3}$.

Sol. Equation of the given curve is

$x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0 \qquad \text{... (1)}$

and equation of the given straight line is $y - 3x = 2$;

$\therefore\ \dfrac{y-3x}{2} = 1 \qquad \text{... (2)}$

Making equation (1) homogeneous equation of the second degree in x and y with the help of (1), we have

$x^2 + 2xy + 3y^2 + 4x\left(\dfrac{y-3x}{2}\right) + 8y\left(\dfrac{y-3x}{2}\right) - 11\left(\dfrac{y-3x}{2}\right)^2 = 0$

or $\quad x^2 + 2xy + 3y^2 + \dfrac{1}{2}(4xy + 8y^2 - 12x^2 - 24xy) - \dfrac{11}{4}(y^2 - 6xy + 9x^2) = 0$

or $\quad 4x^2 + 8xy + 12y^2 + 2(8y^2 - 12x^2 - 20xy) - 11(y^2 - 6xy + 9x^2) = 0$

or $\quad -119x^2 + 34xy + 17y^2 = 0$

or $\quad 119x^2 - 34xy - 17y^2 = 0$

or $\quad 7x^2 - 2xy - y^2 = 0 \qquad \text{...(3)}$

This is the equation of the lines joining the origin to the points of intersection of (1) and (2).

Comparing equation (3) with the equation

$ax^2 + 2hxy + by^2 = 0$

We have $a = 7$, $b = -1$ and $2h = -2$ i.e. $h = -1$

If θ be the acute angle between pair of lines (3), then

$\tan\theta = \left|\dfrac{2\sqrt{h^2-ab}}{a+b}\right| = \left|\dfrac{2\sqrt{1+7}}{7-1}\right| = \dfrac{2\sqrt{8}}{6} = \dfrac{2\sqrt{2}}{3}$

$\therefore\ \theta = \tan^{-1}\dfrac{2\sqrt{2}}{3}$.

Example 37: Find the equation of the bisectors of the angle between the lines represented by $3x^2 - 5xy + 4y^2 = 0$.

Sol. Given equation is $3x^2 - 5xy + 4y^2 = 0$

Comparing it with the equation

$ax^2 + 2hxy + by^2 = 0 \qquad \text{... (2)}$

then $a = 3,\ h = -\dfrac{5}{2},\ b = 4$

Hence equation of bisectors of the angle between the pair of the lines (1) is

$\dfrac{x^2-y^2}{3-4} = \dfrac{xy}{-5/2} \Rightarrow \dfrac{x^2-y^2}{-1} = \dfrac{2xy}{-5}$

$\Rightarrow\ 5x^2 - 2xy - 5y^2 = 0$

Example 38: Show that the line $y = mx$ bisects the angle between the lines $ax^2 - 2hxy + by^2 = 0$

$h(1 - m^2) + m(a - b) = 0$.

Sol. Equation of pair of bisectors of angles between lines $ax^2 - 2hxy + by^2 = 0$ is

$\dfrac{x^2-y^2}{a-b} = \dfrac{xy}{-h}$

$\Rightarrow\ -h(x^2 - y^2) = (a - b)xy \qquad \text{... (1)}$

but $y = mx$ is one of these lines, then it will satisfy it.

Substituting $y = mx$ in (1)

$-h(x^2 - m^2x^2) = (a - b)x \cdot mx$

Dividing by x^2, $h(1 - m^2) + m(a - b) = 0$

Concept Application

43. The chord $\sqrt{6}y = \sqrt{8}px + \sqrt{2}$ of the curve $py^2 + 1 = 4x$ subtends a right angle at origin, then find the value of p.

44. Find the condition such that the straight lines joining the origin to the points of intersection of the straight line $hx + ky = Zhk$ and the curve $(x - k)^2 + (y - h)^2 = C^2$ are at right angles if

(*a*) $h^2 + k^2 = C^2$ (*b*) $h^2/4 + k^2/6 = C^2$

(c) $(h/2)^2 + (k/3)^2 = C^2$ (*d*) $2h^2 + 3k^2 = C^2$

Short Notes

- **Distance Formula:** $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

- **Section Formula:** $x = \dfrac{mx_2 \pm nx_1}{m \pm n}$; $y = \dfrac{my_2 \pm ny_1}{m \pm n}$

- **Centroid, Incentre & Excentre:**

Centroid $G\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$,

Incentre $I\left(\dfrac{ax_1 + bx_2 + cx_3}{a+b+c}, \dfrac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

Excentre $I_1\left(\dfrac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \dfrac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$

Remarks:

(*i*) If the triangle is equilateral, then centroid, incentre, orthocenter, circumcenter, coincides.

(*ii*) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining. Orthocentre and circumcentre in the ratio 2 : 1.

(*ii*) In a isosceles triangle centroid, orthocentre, incentre, circumcentre lies on the same line.

Area of Triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}\right|$$

Equation of Straight Line

(*a*) Equation of *a* line parallel to *x*-axis at *a* distance a is $y = a$ or $y = -a$.

(*b*) Equation of *x*-axis is $y = 0$.

(*c*) Equation of line parallel to *y*-axis at a distance b is $x = b$ or $x = -b$.

(*d*) Equation of *y*-axis is $x = 0$.

If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line $AB = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Standard Forms of Equations of a Straight Line

(*a*) **Slope Intercept form :** Let m be the slope of a line and c its intercept on *y*-axis, then the equation of this straight line is written as : $y = mx + c$.

(*b*) **Point Slope form :** If m be the slope of a line and it passes through a point (x_1, y_1), then its equation is written as : $y - y_1 = m(x - x_1)$.

(*c*) **Two point form :** Equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is written as :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(*d*) **Intercept form :** If a and b are the intercepts made by a line on the axes of x and y, its equation is written as : $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(*e*) **Normal form :** If p is the length of perpendicular on a line from the origin and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as :

$x \cos \alpha + y \sin \alpha = p$ (p is always positive), where $0 \leq \alpha < 2\pi$.

(*f*) **Parametric form :** $\dfrac{x - h}{\cos\theta} = \dfrac{y - k}{\sin\theta} = r$ is the equation.

(*g*) **General form :** We know that a first degree equation in x and y, $ax + by + c = 0$ always represents a straight line. This form is known as general form of straight line.

(*i*) Slope of this line $= \dfrac{-a}{b} = \dfrac{\text{coefficient of } x}{\text{coefficient of } y}$

(*ii*) Intercept by this line on x-axis $= -\dfrac{c}{a}$ and intercept by this line on y-axis $= -\dfrac{c}{b}$.

(*iii*) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2 + b^2}$.

Angle Between Two Lines

(*a*) If θ be the angle between two lines : $y = m_1x + c_1$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\dfrac{m_1 - m_2}{1 + m_1m_2}\right)$.

(*b*) If equation of lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then these line are–

(*i*) Parallel $\Leftrightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

(*ii*) Perpendicular $\Leftrightarrow a_1a_2 + b_1b_2 = 0$

(iii) Coincident $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iv) Intersecting $\Leftrightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Length of Perpendicular from a Point on a Line

Length of perpendicular from a point (x_1, y_1) on the line $ax + by + c = 0$ is

$$= \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|.$$

In particular the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \frac{|c|}{\sqrt{a^2 + b^2}}$.

Distance Between two Parallel Lines

(i) The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.

(**Note :** The coefficients of x & y in both equations should be same).

(ii) The area of the parallelogram $= \frac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distance between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1, y = m_1x + c_2$ and $y = m_2x + d_1, y = m_2x + d_2$ is given $\left| \frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2} \right|$.

Equation of lines Parallel and Perpendicular to a Given Line

(i) Equation of line parallel to line $ax + by + c = 0$.

$$ax + by + \lambda = 0$$

(ii) Equation of line perpendicular to line $ax + by + c = 0$.

$$bx - ay + k = 0$$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

Straight Line Making a given Angle with a Line

Equations of lines passing through a poing (x_1, y_1) and making an angle α, with the line $y = mx + c$ is written as :

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Position of Two Points with Respect to a Given Line

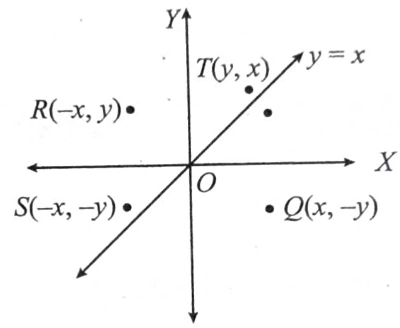
Let the given line be $ax + by + c = 0$ and $P(x_1, y_1), Q(x_2, y_2)$ be two points. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

Concurrency of Lines

Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

Reflection of a Point

Let $P(x, y)$ be any point, then its image with respect to



(i) x-axis is $Q(x, -y)$

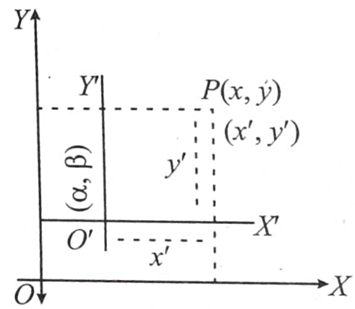
(ii) y-axis is $R(-x, y)$

(iii) origin is $S(-x, -y)$

(iv) line $y = x$ is $T(y, x)$

Transformation of Axes

(a) **Shifting of origin without rotation of axes :** If coordinates of any point $P(x, y)$ with respect to new origin (a, b) will be (x', y')

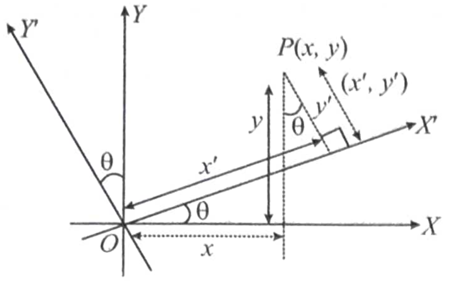


then $x = x' + \alpha, \quad y = y' + \beta$

or $x' = x - \alpha, \quad y' = y - \beta$

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.

(b) **Rotation of axes without shifting the origin :** Let O be the origin. Let $P \equiv (x, y)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes OX' and OY', where $\angle X'OX = \angle YOY' = \theta$



then $x = x' \cos\theta - y' \sin\theta$

$y = x' \cos\theta - y' \cos\theta$

or $y' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

New \ Old	$x\downarrow$	$y\downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

Equation of Bisectors of Angles between Two Lines

If equation of two intersecting lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then equation of bisectors of the angles between these lines are written are:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad ...(1)$$

(a) Equation of bisector of angle containing origin : If the equation of the lines are written with constant terms c_1 and c_2 positive, then the equation of bisectors of the angle containing the origin is obtained by taking sign in (1).

(b) Equation of bisector of acute/obtuse angles : See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by -1 to make the constant terms positive.

Determinate the sign of $a_1a_2 + b_1b_2$

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sing in eq. (1)	use – sign in eq. (1)
–	use – sign in eq. (1)	use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Family of Lines

If equation of two lines be $P = a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x + b_2y + c_2 = 0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional information given in the problem.

General Equation and Homogeneous Equation of Second Degree

(a) A general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of straight lines if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

(b) If θ be the angle between the lines, then $\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a+b}$.

Obviously these lines are

(i) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$.

(ii) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(c) Homogeneous equation of 2nd degree $ax^2 + 2hxy + by^2 = 0$ always represent a pair of straight lines whose equations are

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x \equiv y = m_1x \text{ and } y = m_2x$$

and $m_1 + m_2 = -\frac{2h}{b}$; $m_1m_2 = \frac{a}{b}$

These straight lines passes through the origin and for finding the angle between these lines same formula as given for general equation is used.

The condition that these lines are :

(i) At right angles to each other is $a + b = 0$. i.e. co-efficient of x^2 + co-efficient of $y^2 = 0$.

(ii) Coincident is $h^2 = ab$.

(iii) Equally inclined to the axis of x is $h = 0$. i.e. coefficient of $xy = 0$.

(d) The combined equation of angle bisectors between the lines represented by homogeneous equation of 2nd degree is given by

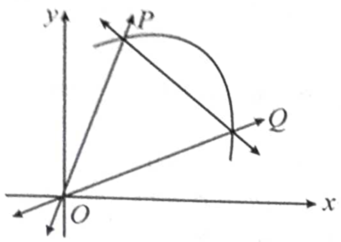
$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}, \ a \neq b, \ h \neq 0.$$

(e) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

(f) If lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel then distance between them is $= 2\sqrt{\frac{g^2 - ac}{a(a + b)}}$.

Equations of Lines Joining the Points of Intersection of a Line and a Curve to the Origin

Let the equation of curve be:



$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad ...(i)$$

and straight line be

$$lx + my + n = 0 \qquad ...(ii)$$

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$$

STANDARD RESULTS

(*i*) Area of rhombus formed by lines $a\,|x| + b\,|y| + c = 0$ or $\pm ax \pm by + c = 0$ is $\frac{2c^2}{|ab|}$.

(*ii*) Area of triangle formed by line $ax + by + c = 0$ and axes is $\frac{c^2}{2|ab|}$.

(*iii*) Co-ordinate of foot of perpendicular (h, k) from (x_1, y_1) to the line $ax+by+c=0$ is given by $\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-(ax_1+by_1+c)}{a^2+b^2}$.

(*iv*) Image of point (x_1, y_1) w.r.t. to the line $ax + by + c = 0$ is given by

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-2(ax_1+by_1+c)}{a^2+b^2}.$$

Chart

(*i*) $\boxed{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$

$\Delta = 0$ ↓

Pair of straight line

→ $h^2 = ab, g^2 = ac, f^2 = bc$ (Pair of coincident line)

→ $h^2 = ab, g^2 \neq ac, f^2 \neq bc$ (Pair of coincident line)

→ $h^2 = \neq ab$ (Pair of straight line not passing through origin)

(*ii*) $\boxed{ax^2 + 2hxy + by^2 = 0}$

- $h^2 = ab$ → Pair of coincident lines passing through origin
- $a + b = 0$ → Pair of perpendicular lines passing through origin
- $h^2 \neq ab$ → Pair of lines passing through origin

Solved Examples

1. The distance of the point (2, 3) from the line $2x - 3y + 9 = 0$ measured along a line $x - y + 1 = 0$, is – [Alternate sold]

(*a*) $\sqrt{2}$ (*b*) $4\sqrt{2}$

(*c*) $\sqrt{8}$ (*d*) $3\sqrt{2}$

Sol. (*b*) The slope of the line $x - y + 1 = 0$ is 1. So it makes an angle of 45° with x-axis.

The equation of a line passing through (2, 3) and making an angle of 45° is

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r$$

$$\left[\text{Using } \frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r\right]$$

co-ordinates of any point on this line are $(2 + r\cos 45^\circ, 3 + r\sin 45^\circ)$ or

$$\left(2+\frac{r}{\sqrt{2}},\ 3+\frac{r}{\sqrt{2}}\right)$$

If this point lies on the line $2x - 3y + 9 = 0$,

then $4 + r\sqrt{2} - 9 - \frac{3r}{\sqrt{2}} + 9 = 0$

$\Rightarrow r = 4\sqrt{2}$.

So the required distance $= 4\sqrt{2}$.

2. If $x + 2y = 3$ is a line and $A(-1, 3)$; $B(2, -3)$; $C(4, 9)$ are three points, then -

(*a*) A is on one side and B, C are on other side of the line

(*b*) A, B are on one side and C is on other side of the line

(*c*) A, C are on one side and B is on other side of the line

(*d*) All three points are on one side of the line

Sol. (*c*) Substituting the coordinates of points A, B and C in the expression $x + 2y - 3$, we get

The value of expression for A is

$= -1 + 6 - 3 = 2 > 0$

The value of expression for B is

$= 2 - 6 - 3 = -7 < 0$

The value of expression for C is

$= 4 + 18 - 3 = 19 > 0$

$\because$ Signs of expressions for A, C are same while for B, the sign of expression is different

$\therefore$ A, C are on one side and B is on other side of the line

3. If $A(-2, 1)$, $B(2, 3)$ and $C(-2, -4)$ are three points, then the angle between BA and BC is -

(*a*) $\tan^{-1}\left(\frac{3}{2}\right)$ (*b*) $\tan^{-1}\left(\frac{2}{3}\right)$

(*c*) $\tan^{-1}\left(\frac{7}{4}\right)$ (*d*) $\tan^{-1}\left(\frac{3}{2}\right)$

Sol. (*b*) Let m_1 and m_2 be the slopes of BA and BC respectively. Then

$$m_1 = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2} \text{ and } m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$$

Let θ be the angle between BA and BC. Then

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}}\right| = \left|\frac{\frac{10}{8}}{\frac{15}{8}}\right| = \pm\frac{2}{3}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

4. The area of the parallelogram formed by the lines $4y - 3x = 1$, $4y - 3x - 3 = 0$, $3y - 4x + 1 = 0$, $3y - 4x + 2 = 0$ is -

(*a*) 3/8 (*b*) 2/7

(*c*) 1/6 (*d*) 3/4

Sol. (*b*) Let the equation of sides AB, BC, CD and DA of parallelogram $ABCD$ are respectively

$$y = \frac{3}{4}x + \frac{1}{4} \quad ...(1); \qquad y = \frac{3}{4}x + \frac{3}{4} \quad ...(2)$$

$$y = \frac{4}{3}x - \frac{1}{3} \quad ...(3); \qquad y = \frac{4}{3}x - \frac{2}{3} \quad ...(4)$$

Here $m = \frac{3}{4}, n = \frac{4}{3}, a = \frac{1}{4}, b = \frac{3}{4}, c = -\frac{1}{3}, d = -\frac{2}{3}$

$\therefore$ Area of parallelogram $ABCD$

$$= \left|\frac{(a-b)(c-d)}{m-n}\right| = \left|\frac{\left(\frac{1}{4} - \frac{3}{4}\right)\left(-\frac{1}{3} + \frac{2}{3}\right)}{\frac{3}{4} - \frac{4}{3}}\right|$$

$$= \left|\frac{-\frac{1}{2} \times \frac{1}{3}}{-\frac{7}{12}}\right| = \frac{2}{7} \text{ sq-units}$$

5. If the lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent, then -

(*a*) $a - b - c = 0$ (*b*) $a + b + c = 0$

(*c*) $b + c - a = 0$ (*d*) $a + b - c = 0$

Sol. (*b*) If the lines are concurrent, then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

$\Rightarrow 3abc - a^3 - b^3 - c^3 = 0$

$\Rightarrow (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow (a + b + c)[(a - b)2 + (b - c)^2 + (c - a)^2] = 0$

$\Rightarrow a + b + c = 0$

$[\therefore (a - b)^2 + (b - c)^2 + (c - a)2 \neq 0]$

6. The vertices of ΔOBC are respectively $(0, 0)$, $(-3, -1)$ and $(-1, -3)$. The equation of line parallel to BC and at a distance 1/2 from O which intersects OB and OC is-

(*a*) $2x + 2y + \sqrt{2} = 0$ (*b*) $2x - 2y + \sqrt{2} = 0$

(*c*) $2x + 2y - \sqrt{2} = 0$ (*d*) $2x - 2y - \sqrt{2} = 0$

Sol. (*a*) Slope of $BC = \frac{-3+1}{-1+3} = -1$

Now equation of line parallel to BC is

$y = -x + k \Rightarrow y + x = k$

Now length of perpendicular from O on this line

$$= \pm\frac{k}{\sqrt{2}} = \frac{1}{2} \Rightarrow k = -\frac{\sqrt{2}}{2}$$

$\therefore$ Equation of required line is

$2x + 2y + \sqrt{2} = 0$

7. Find the number of integer value of m which makes the x coordinates of point of intersection of lines. $3x + 4y = 9$ and $y = mx + 1$ integer.

(*a*) 2 (*b*) 0 (*c*) 4 (*d*) 1

Sol. (*a*) $3x + 4y = 9$

$mx - y = -1$

$$x = \frac{5}{3 + 4m}$$

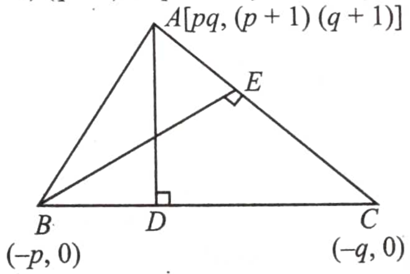
$m = -1, -2$

8. The locus of the orthocenter of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$, and $y = 0$, where $p \neq q$, is

(*a*) a hyperbola (*b*) a parabola

(*c*) an ellipse (*d*) a straight line

Sol. (*d*) Intersection points of given lines are $(-p, 0)$, $(-q, 0)$, $[pq, (p + 1)(q + 1)]$ respectively



Now equation of altitudes AD and BE are $x = pq$, and $qx + (q+1)y + pq = 0$

Their point of intersection is $(pq, -pq)$

so $h = pq$, $k = -pq$

so locus is $h = -k$

$h + k = 0$

$\Rightarrow x + y = 0$ which is a straight line

9. Two lines are given by $(x - 2y)^2 + k(x - 2y) = 0$. The value of k, so that the distance between them is 3 is

(*a*) $k = 0$ (*b*) $k = \pm 3\sqrt{5}$

(*c*) $k = -5$ (*d*) $k = 3$

Sol. (*b*) The lines $x - 2y = 0$ and $x - 2y + k = 0$ are parallel. The distance between these two lines =

$$\left|\frac{k}{\sqrt{1+(-2)^2}}\right| = 3 \Rightarrow k = \pm 3\sqrt{5}$$

10. Let P and Q be any two points on the lines represented by $2x - 3y = 0$ and $2x + 3y = 0$ respectively. If the area of triangle OPQ (where O is origin) is 5, then the equation of the locus of midpoint of PQ, can be equal to

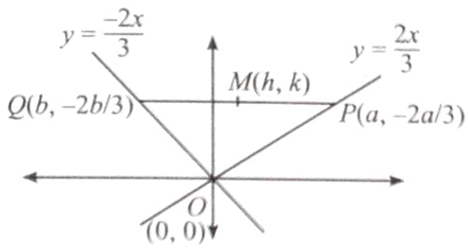
(*a*) $4x^2 - 9y^2 + 30 = 0$ (*b*) $9x^2 - 4y^2 + 30 = 0$

(*c*) $9x^2 - 4y^2 - 30 = 0$ (*d*) $4x^2 - 9y^2 - 30 = 0$

Sol. (*a*, *d*) We have

$$\text{Area } (\Delta OPQ) = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a & \frac{2a}{3} & 1 \\ b & \frac{-2b}{3} & 1 \end{vmatrix} = 5$$

(Given)



$\Rightarrow \frac{4ab}{3} = \pm 10$

So, $4ab = \pm 30$ ...(1)

Also $2h = a + b$ ...(2)

$2k = \frac{2a - 2b}{3} \Rightarrow a - b = 3k$ ...(3)

As $4ab = (a+b)^2 - (a-b)^2$

$\Rightarrow \pm 30 = 4h^2 - 9k^2$ [Using (1), (2) and (3)]

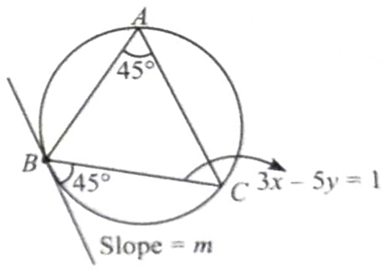
So, required locus can be $4x^2 - 9y^2 = \pm 30$, which is hyperbola.

11. Let A, B, C be the vertices of a triangle ABC and B, C lies on the line $3x - 5y = 1$. If $\angle A = \frac{\pi}{4}$, then the slope of tangent at B to the circumcircle of ΔABC, can be

(*a*) $\frac{-1}{4}$ (*b*) -4 (*c*) 4 (*d*) $\frac{1}{4}$

Sol. (*a*, *c*) Let the slope of tangent at B be m and it makes 45° with $3x - 5y = 1$, so $\tan 45° = \left|\dfrac{m - \frac{3}{5}}{1 + \frac{3m}{5}}\right|$

$\Rightarrow m = 4, \frac{-1}{4}$.



12. In an acute triangle ABC, if the coordinates of orthocentre 'H' are (4, b), centroid 'G' are $(b, 2b - 8)$ and circumcentre 'S' are $(-4, 8)$, then 'b' cannot be

(*a*) 4 (*b*) 8 (*c*) 12 (*d*) -12

Sol. (*a*, *b*, *c*, *d*)

As H, G and C are collinear

$$\therefore \begin{vmatrix} 4 & b & 1 \\ b & 2b-8 & 1 \\ -4 & 8 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 4 & b & 1 \\ b-4 & b-8 & 0 \\ -(b+4) & 16-2b & 0 \end{vmatrix} = 0$$

$\Rightarrow (b-4)(16-2b) + (b+4)(b-8) = 0$

$\Rightarrow 2(b-4)(8-b) + (b+4)(b-8) = 0$

$\Rightarrow (8-b)[(2b-8) - (b+4)] = 0$

$\Rightarrow (8-b)(b-12) = 0$

Hence $b = 8$ or 12, which is wrong because collinearity does not explain centroid, orthocentre and circumcentre.

Now H ——2—— G ——1—— S

(4,b) (b, 2b–8) (–4, 8)

$\therefore \frac{-8+4}{3} = b \Rightarrow b = \frac{-4}{3}$

and $\frac{16+b}{3} = 2b - 8 \Rightarrow b = 8$

But no common value of 'b' is possible

$\Rightarrow a, b, c, d$

13. Consider the lines $L_1 : 3x - 4y + 2 = 0$ and $3y - 4x - 5 = 0$, then which of the following statements is correct?

(*a*) The line $x + y + 3 = 0$ bisects the acute angle between L_1 and L_2 containing the origin.

(*b*) The line $x - y + 1 = 0$ bisects the obtuse angle between L_1 and L_2 not containing the origin.

(*c*) The line $x + y + 3 = 0$ bisects the obtuse angle between L_1 and L_2 containing the origin.

(*d*) The line $x - y + 1 = 0$ bisects the acute angle between L_1 and L_2 not containing the origin.

Sol. (*c*, *d*) $3x - 4y + 2 = 0$

$4x - 3y + 5 = 0$

Bisectors are $3x - 4y + 2 = \pm (4x - 3y + 5)$

Positive sign $3x - 4y + 2 = 4x - 3y + 5$

$x + y + 3 = 0$ (containing origin)

Negative sign $3x - 4y + 2 = -4x + 3y - 5$

$7x - 7y + 7 = 0$

$x - y + 1 = 0$ (not containing origin)

Acute/Obtuse

$a_1a_2 + b_1b_2 = 0$

$3(4) + (-4)(-3) = 0$

$(12) + (12) = 24 > 0$

Positive Sign obtuse angle bisector

Negative Sign acute angle bisector

$x + y + 3 = 0$ bisects the obtuse angle containing the origin and

$x - y + 1 = 0$ bisects the acute angle containing the origin.

14. Let 'α' be the solution of the equation

$$e^{\tan^2\theta} + (1 - \cos\theta)(3 - \cos\theta) = \cos 4\theta \text{ in } [-2\pi, 2\pi].$$

If $f(x) = -2 + \cos\frac{x}{2} + \tan x$ then possible equation of straight line passing through the points (0, 0) and $(\alpha, f(\alpha))$ is(are)

(*a*) $x + y = 0$
(*b*) $x = 0$
(*c*) $3x - 2\pi y = 0$
(*d*) $3x + 2\pi y = 0$

Sol. (*b*, *c*, *d*)

$$e^{\tan^2\theta} + 3 - 4\cos\theta + \cos^2\theta = 2\cos^2 2\theta - 1$$

$$\underbrace{e^{\tan^2\theta} + (\cos\theta - 2)^2}_{\geq 2} = \underbrace{2\cos^2\theta}_{\leq 2}$$

$\therefore \tan^2\theta = 0, \cos\theta = 1, \cos^2 2\theta = 1$

$\Rightarrow \theta = \tan^2 n\pi \Rightarrow \theta = \cos 2n\pi \cos 2\theta = \pm 1$

$2\theta = n\pi, \theta = \frac{n\pi}{2}$

$\therefore \alpha = 0, -2\pi, 2\pi$

$(0, -1), (-2\pi, -3)(2\pi, -3)$

Possible equations of straight line are $x = 0$

$y = \frac{3}{2\pi}x \Rightarrow 3x - 2\pi y = 0$

$y = \frac{-3}{2\pi}x \Rightarrow 3x + 2\pi y = 0.$

15. Show that all the chords of the curve $3x^2 - y^2 - 2x + 4y = 0$, which subtend a right angle at the origin pass through a fixed point. Find that the point.

Sol. Let the equation of chord be $lx + my = 1$. So equation of pair of straight line joining origin to the points of intersection of chord and curve.

$3x^2 - y^2 - 2x(lx + my) + 4y(lx + my) = 0$, which subtends right angle at origin.

$\Rightarrow (3 - 2l + 4m - 1) = 0$

$\Rightarrow l = 2m + 1.$

Hence chord becomes $(2m + 1)x + my = 1$

$x - 1 + m(2x + y) = 0$

$\quad L_1 \qquad L_2$

Which will pass through point of intersection of $L_1 = 0$ and $L_2 = 0$.

$\Rightarrow \quad x = 1, y = -2$. Hence fixed point is $(1, -2)$.

16. Locus of all point $P(x, y)$ satisfying $x^3 + y^3 + 3xy = 1$ consists of union of

(*a*) a line and an isolated point
(*b*) a line pair and an isolated point
(*c*) a line and a circle
(*d*) a circle and an isolated point

Sol. (*b*) $(x + y - 1) = 0$ or $x = y = -1$

17. Suppose *ABCD* is a trapezium whose sides and height are integers and *AB* is parallel to *CD*. If the area of *ABCD* is 12 and the sides are distinct, then $|AB - CD|$.

(*a*) is 2
(*b*) is 4
(*c*) is 8
(*d*) cannot be determined from the data

Sol. (*b*) Area = 12

$\Rightarrow (1/2)(a + b) \times h = 12$

$\Rightarrow (a + b) \times h = 24$

18. Suppose *P* is an interior point of a ΔABC such that the ratio

$$\frac{d(A, BC)}{d(P, BC)} = \frac{d(B, CA)}{d(P, CA)} = \frac{d(C, AB)}{d(P, AB)} = \frac{p_1}{p_2}, \text{ then}$$

(*a*) *P* is orthocenter and $p_1 : p_2 = 4 : 1$
(*b*) *P* is orthocenter and $p_1 : p_2 = 3 : 1$
(*c*) *P* is centroid and $p_1 : p_2 = 3 : 1$
(*d*) *P* is circumcentre and $p_1 : p_2 = 4 : 1$

Sol. Let $B(0, 0)$ $C(a, 0)$, $A(h, k)$ and $P(\alpha, \beta)$.

$$\frac{d(A, BC)}{d(P, BC)} = \frac{k}{\beta}$$

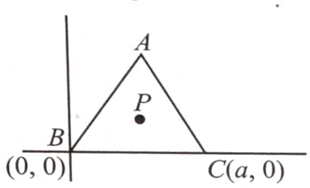
Equation of *AC* is $kx - (h - a)y - ka = 0$

$$\therefore \frac{d(B, AC)}{d(P, AC)} = \frac{ka}{k\alpha - (h - a)\beta - ka}$$

Equation *AB* is $kx - hy = 0$

$$\frac{d(C, AB)}{d(P, AB)} = \frac{ka}{k\alpha - h\beta}$$

Solving, $\alpha = \frac{h + a}{3}, \beta = \frac{k}{3}$



19. Two particles start from the same point $(2, -1)$, one moving 2 units along the line $x + y = 1$ and the other 5 units along $x - 2y = 4$. If the particles move towards increasing *y*, then their new position will be

(a) $(2-\sqrt{2}, \sqrt{2}-1)$

(b) $(2\sqrt{5}+2, \sqrt{5}-1)$

(c) $(2+\sqrt{2}, \sqrt{2}+1)$

(d) $(2\sqrt{5}-2, \sqrt{5}-1)$

Sol. Let $P(2, -1)$ goes 2 units along $x+y=1$ upto A and 5 units along $x-2y=4$ up to B.

Slope of $PA = -1 = \tan 135°$, slope of $PB = \frac{1}{2} = \tan\theta$

$\therefore \sin\theta = \frac{1}{\sqrt{5}}, \cos\theta = \frac{2}{\sqrt{5}}$

$\therefore A = (x_1 + r\cos 135°, y_1 = r\sin 135°)$

$= \left(2+2x-\frac{1}{\sqrt{2}}, -1+2\times\frac{1}{\sqrt{2}}\right) = \left(2-\sqrt{2}, \sqrt{2}-1\right)$

$B = (x_1 + r\cos\theta, y_1 + r\sin\theta)$

$= \left(2+5\times\frac{2}{\sqrt{5}}, -1+\frac{5}{\sqrt{5}}\right) = \left(2\sqrt{5}+2, \sqrt{5}-1\right)$

20. A diagonal of rhombus $ABCD$ is a member of both the family of lines $(x+y-1)+\lambda_1(2x+3y-2)=0$ and $(x-y+2)+\lambda_2(2x-3y+5)$ where $\lambda_1, \lambda_2 \in R$ and one of the vertex of the rhombus is $(3, 2)$. If area of the rhombus is $12\sqrt{5}$ square units then find the length of semi longer diagonal of the rhombus ________

Sol. Since diagonal is a member of both the families so it will pass through $(1, 0)$ and $(-1, 1)$. Equation of diagonal AC is $x+2y-1=0$. Since one of the vertex $(3, 2)$ which does not be on AC, so equation of BD is $2x-y=4$.

Point of intersection of AC and BD is $P\left(\frac{9}{5}, -\frac{2}{5}\right)$

If vertex B is $(3, 2)$ then vertex D is

$\left(\frac{3}{5}, -\frac{14}{5}\right)$ also $BD = \frac{12\sqrt{5}}{5}$ (say d_1)

Area of rhombus $= \frac{1}{2}\times d_1 \times d_2$

$\frac{1}{2}\times\left(\frac{12\sqrt{5}}{5}\right)\times d_2 = 12\sqrt{5}$

$d_2 = 10$

21. Prove that equation $2x^2+5xy+3y^2+6x+7y+4=0$ represents a pair of straight lines. Find the coordinates of their point of intersection:

Sol. Given equation is

$2x^2+5xy+3y^2+6x+7y+4=0$

Writing the equation (1) as a quadratic equation is x we have

$2x^2+(5y+6)x+3y^2+7y+4=0$

$$\therefore x = \frac{-(5y+6)\pm\sqrt{(5y+6)^2-4.2(3y^2+7y+4)}}{4}$$

$$\therefore \frac{-(5y+6)\pm\sqrt{y^2+4y+4}}{4}$$

or $4x+4y+4=0$ and $4x+6y+8=0$

or $x+y+1=0$ and $2x+3y+4=0$

Solving these two equations, the required point of intersection is $(1, -2)$

Method-2

$2x^2+5xy+3y^2 = (2x+3y)(x+y)$

So, $(2x+3y+c_1)(x+y+c_2) = 2x^2+5xy+3y^2+6x+7y+4$

$c_1+2c_2 = 6, c_1+3c_2 = 7, c_1c_2 = 4,$

So, $c_2 = 1, c_1 = 4$

22. A line $4x+y=1$ through the point $A(2, -7)$ meets the line BC whose equation is $3x-4y+1=0$ at a point B. Find the equation of the line AC so that $AB = AC$.

Sol. $52x+89y+519=0$

Method-1:

Let m be the slope of AC, the AB and AC are equally inclined to BC.

$$\therefore \frac{3/4-(-4)}{1+(3/4)(-4)} = \frac{\pm(3/4-m)}{1+(3/4)m}$$

$\therefore m = -52/89, -4$

$\therefore$ Equation of AC through $A(2, -7)$ is

$$y+7 = \frac{-52}{89}(x-2)$$

or $89y+623 = -52x+104$ or $52x+89y+519=0$

Method-2:

Let AB makes an angle α with BC

$$\Rightarrow \tan\alpha = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{19}{8}\right|$$

Let the gradient of AC be m

$$\left|\frac{m-\frac{3}{4}}{1+\frac{3m}{4}}\right| \Rightarrow \frac{19}{8} \Rightarrow m = -4 \text{ or } -\frac{52}{89}$$

Since the gradient of AB is -4

$\therefore$ Gradient of AC is $= -\frac{52}{89}$

23. Equation of a line is given by $y+2at = t(x-at^2)$, t being the parameter. Find the locus of the point of intersection of the lines which are at right angles.

Sol. $y+2at = ta-at^3$

Slope is $= t$

Let is passes through $P(h, k)$

$\therefore k+2at = th-at^3$

$\Rightarrow at_3 + t(2a-h)+k=0$ ......(1)

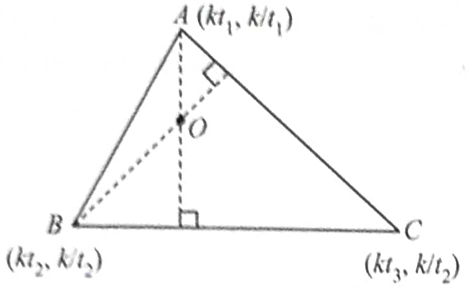
$t_1t_2t_3 = -\frac{k}{a}\{t_1t_2=-1\}$; $t_3 = \frac{k}{a}$

Substituting t_3 in (1) we can get the locus

$y_2 = a(x-3a)$

24. Find orthocenter of ΔABC where $A \equiv (kt_1, k/t_1)$, $B \equiv (kt_2, k/t_2)$ and $C \equiv (kt_3, k/t_3)$.

Sol.



Let O be (p, q)

m_{AO} = slope of line joining A and O

$$\left(m_{AO} = \frac{q - k/t_1}{p - kt_1}\right)$$

$m_{AO} \cdot m_{BC} = -1$ ...(i)

$$m_{BC} = \frac{-1}{t_2 t_3}$$

$m_{OC} \cdot m_{AB} = -1$ ...(ii)

$$\frac{q - k/t_1}{p - kt_1} \cdot \frac{1}{t_2 t_3} = 1 \qquad \left(m_{OC} = \frac{q - k/t_3}{p - kt_3}\right)$$

$$\Rightarrow q - \frac{k}{t_1} = (p - kt_1)t_2 t_3 \qquad m_{AB} = \frac{-1}{t_1 t_2}$$

$$q - pt_2t_3 = \frac{k}{t_1} - kt_1t_2t_3 \quad \text{...(iii)}$$

$$q - pt_1t_2 = \frac{k}{t_3} - kt_1t_2t_3 \quad \text{...(iv)}$$

(iii) – (iv)

$$p = \left(\frac{k}{t_1} - \frac{k}{t_3}\right)\frac{1}{(t_1t_2 - t_2t_3)} = \frac{-k}{t_1t_2t_3}$$

$$q = -kt_1t_2t_3$$

25. If the equation $ax^2 - 6xy + y^2 + 2gx + 2fy + c = 0$ represents a pair of lines whose slopes are m and m^2 then sum of all possible values of a is

(a) 17

(b) –19

(c) 19

(d) –17

Sol. The given lines will be parallel to lines $ax^2 - 6xy + y^2 = 0$

$$\text{So, } \left(\frac{y}{x}\right)^2 - 6\left(\frac{y}{x}\right) + a = 0$$

$m + m^2 = 6$

Which gives $m = -3$ or 2 .....(1)

and $mm^2 = a$ .....(2)

From (2) $a = -27$ or 8

Hence sum of all possible value of $a = -19$

26. The line parallel to the x-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$, where $(a, b) \neq (0, 0)$ is

(a) below the x-axis at a distance of 3/2 from it

(b) below the x-axis at a distance of 2/3 from it

(c) above the x-axis at a distance of 3/2 from it

(d) above the x-axis at a distance of 2/3 from it

Sol. (a) $(a + b\lambda)x + (2b - 2a\lambda)y + (3b - 3\lambda a) = 0$

$$\therefore a + b\lambda = 0 \Rightarrow = -\frac{a}{b}$$

$$\therefore y = -\frac{3}{2}$$

Exercise-1 (Topicwise)

COORDINATE SYSTEM, DISTANCE FORMULA AND ITS APPLICATION

1. The triangle formed by the lines $x+y-4=0$, $3x+y=4$, $x+3y=4$ is
   (a) Isosceles (b) Equilateral
   (c) Right angled (d) Right angled Isosceles

SECTION FORMULA AND ITS AND COLLINEARITY

2. The equation of the line parallel to the line $2x-3y=1$ and passing through the middle point of the line segment joining the points (1, 3) and (1, – 7), is
   (a) $2x-3y+8=0$ (b) $2x-3y=8$
   (c) $2x-3y+4=0$ (d) $2x-3y=4$
3. A line passes through the point of intersection of $2x+y=5$ and $x+3y+8=0$ and parallel to the line $3x+4y=7$ is
   (a) $3x+4y+3=0$ (b) $3x+4y=0$
   (c) $4x-3y+3=0$ (d) $4x-3y=3$
4. The ratio in which the line joining the points (3, –4) and (–5, 6) is divided by x-axis
   (a) 2 : 3 (b) 6 : 4
   (c) 3 : 2 (d) 3 : 4
5. In what ratio the line $y-x+2=0$ divides the line joining the points (3, –1) and (8, 9)?
   (a) 1: 2 (b) 2: 1
   (c) 2: 3 (d) 3: 4

Locus

6. A point moves so that square of its distance from the point (3, – 2) is numerically equal to its distance from the line $5x-12y=13$. The equation of the locus of the point is
   (a) $13x^2+13y^2-83x+64y+182=0$
   (b) $x^2+y^2-11x+16y+26=0$
   (c) $x^2+y^2-11x+16y=0$
   (d) $x^2+y^2+11x+16y+26=0$
7. Locus of the points which are at equal distance from $3x+4y-11=0$ and $12x+5y+2=0$ and which is near the origin is
   (a) $21x-77y+153=0$ (b) $99x+77y-133=0$
   (c) $7x-11y=19$ (d) $99x+77y+133=0$
8. A point moves such that the area of the triangle formed by it with the points (1, 5) and (3, –7) is 21sq. unit. The locus of the point is
   (a) $6x+y-32=0$ (b) $6x-y+32=0$
   (c) $x+6y-32=0$ (d) $6x-y-32=0$
9. If $A(\cos\alpha, \sin\alpha)$, $B(\sin\alpha, -\cos\alpha)$, $C(1, 2)$ are the vertices of a ΔABC, then as α varies, the locus of its centroid is-
   (a) $x^2+y^2-2x-4y+3=0$
   (b) $x^2+y^2-2x-4y+1=0$
   (c) $3(x^2+y^2)-2x-4y+1=0$
   (d) $3(x^2-y^2)-2x-4y+1=0$
10. The locus of the mid-point of the intercept between the axes of the variable line $x\cos\alpha+y\sin\alpha=p$, where p is constant, is
   (a) $x^2+y^2=4p^2$ (b) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{p^2}$
   (c) $x^2+y^2=\frac{4}{p^2}$ (d) $\frac{1}{x^2}-\frac{1}{y^2}=\frac{2}{p^2}$
11. A stick of length 10 units rests against the floor along x-axis and a wall of a room along y-axis. If the stick begins to slide on the floor then the locus of its middle point is
   (a) $x^2+y^2=2.5$ (b) $x^2+y^2=25$
   (c) $x^2+y^2=100$ (d) $x^2-y^2=25$

Different forms of a Straight Line, Parametric form

12. The equations of the lines which cuts off an intercept – 1 from y-axis are equally inclined to the axes are
   (a) $x-y+1=0,\ x+y+1=0$
   (b) $x-y-1=0,\ x+y-1=0$
   (c) $x-y-1=0,\ x+y+1=0$
   (d) $x+y+1=0, x+2y+1=0$
13. A line L is perpendicular to the line $5x-y=1$ and the area of the triangle formed by the line L and coordinate axes is 5. The equation of the line L is
   (a) $x+5y=5$ (b) $x+5y=\pm5\sqrt{2}$
   (c) $x-5y=5$ (d) $x-5y=5\sqrt{2}$
14. The equation of the straight line passing through the point (3, 2) and perpendicular to the line $y=x$ is
   (a) $x-y=5$ (b) $x+y=5$
   (c) $x+y=1$ (d) $x-y=1$
15. The equation of the lines which passes through the point (3, – 2) and are inclined at 60° to the line $\sqrt{3}x+y=1$
   (a) $y+2=0,\ \sqrt{3}x-y-2-3\sqrt{3}=0$
   (b) $x-2=0,\ \sqrt{3}x-y+2+3\sqrt{3}=0$
   (c) $\sqrt{3}x-y-2-3\sqrt{3}=0$
   (d) $\sqrt{3}x+y+2+3\sqrt{3}=0$

16. The intercept cut off from y–axis is twice that from x-axis by the line and line is passes through (1, 2) then its equation is
(a) $2x+y=4$ (b) $2x+y+4=0$
(c) $2x-y=4$ (d) $2x-y+4=0$

17. Equation of a straight line on which length of perpendicular from the origin is four units and the line makes an angle of 120° with the x–axis in counterclockwise sense, is
(a) $x\sqrt{3}+y+8=0$ (b) $x\sqrt{3}-y=8$
(c) $x\sqrt{3}-y=8$ (d) $x-\sqrt{3}\,y+8=0$

18. The equation to the straight line passing through the point of intersection of the lines $5x-6y-1=0$ and $3x+2y+5=0$ and perpendicular to the line $3x-5y+11=0$ is
(a) $5x+3y+8=0$ (b) $3x-5y+8=0$
(c) $5x+3y+11=0$ (d) $3x-5y+11=0$

19. A line passes through (2, 2) and is perpendicular to the line $3x+y=3$. Its y–intercept is
(a) 1/3 (b) 2/3
(c) 1 (d) 4/3

20. A straight line moves so that the sum of the reciprocals of its intercepts on two perpendicular lines is constant, then the line passes through
(a) A fixed point (b) A variable point
(c) Origin (d) Two variable point

21. If the middle points of the sides BC, CA and AB of the triangle ABC be (1, 3), (5, 7) and (–5, 7), then the equation of the side AB is
(a) $x-y-2=0$ (b) $x-y+12=0$
(c) $x+y-12=0$ (d) $x+y-2=0$

22. If he coordinates of the vertices of the triangle ABC be (–1, 6), (–3, –9), and (5, –8) respectively, then the equation of the median through C is
(a) $13x-14y-47=0$ (b) $13x-14y+47=0$
(c) $13x+14y+47=0$ (d) $13x+14y-47=0$

23. The equation of the line which cuts off an intercept 3 units on OX and an intercept –2 unit on OY, is
(a) $\frac{x}{3}-\frac{y}{2}=1$ (b) $\frac{x}{3}+\frac{y}{2}=1$
(c) $\frac{x}{2}+\frac{y}{3}=1$ (d) $\frac{x}{2}-\frac{y}{3}=1$

24. If we reduce $3x+3y+7=0$ to the form $x\cos\alpha+y\sin\alpha=p$, then the value of p is
(a) $\frac{7}{2\sqrt{3}}$ (b) $\frac{7}{3}$
(c) $\frac{3\sqrt{7}}{2}$ (d) $\frac{7}{3\sqrt{2}}$

25. A vertex of square is (3, 4) and diagonal $x+2y=1$, then the second diagonal which passes through given vertex will be
(a) $2x-y+2=0$ (b) $x+2y=11$
(c) $2x-y=2$ (d) $2x+y=2$

Angle between Lines

26. If the coordinates of the vertices A, B, C of the triangle ABC be (–4, 2), (12, –2) and (8, 6) respectively, then $\angle B=$
(a) $\tan^{-1}\left(\frac{3}{4}\right)$ (b) $\tan^{-1}\left(\frac{6}{7}\right)$
(c) $\tan^{-1}\left(\frac{5}{6}\right)$ (d) $\tan^{-1}\left(\frac{7}{6}\right)$

27. If $\frac{1}{ab'}+\frac{1}{ba'}=0$, then lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b'}+\frac{y}{a'}=1$ are
(a) Parallel
(b) Inclined at 60° to each other
(c) Perpendicular to each other
(d) Inclined at 30° to each other

28. The line passing through the points (3, – 4) and (–2, 6) and a line passing through (–3,6) and (9, –18) are
(a) Perpendicular
(b) Parallel
(c) Makes an angle 60° with each other
(d) Make an angle 30° with each other

29. If vertices of a parallelogram are respectively (0, 0), (1, 0), (2, 2) and (1, 2), then angle between diagonals is
(a) $\pi/3$ (b) $\pi/2$
(c) $3\pi/2$ (d) $\pi/4$

30. The acute angle between the lines $y=3$ and $y=\sqrt{3}x+9$ is
(a) 30° (b) 60°
(c) 45° (d) 90°

31. The angle between the lines $y=(2-\sqrt{3})x+5$ and $y=(2+\sqrt{3})x-7$ is
(a) 30° (b) 60°
(c) 45° (d) 90°

32. The number of different slopes of straight lines which are equally inclined to both the axes is
(a) 4 (b) 2
(c) 3 (d) 1

POINTS AND A LINE, POSITION AND DISTANCE OF A POINT FORM THE LINE

33. The point on the line $x+y=4$ which lie at a unit distance from the line $4x+3y=10$, are
(a) (3, 1), (–7, 11) (b) (3, 1), (7, 11)
(c) (–3, 1), (–7, 11) (d) (1, 3), (–7, 11)

34. The perpendicular distance of the straight line $12x+5y=7$ from the origin is given by
(a) $\frac{7}{13}$ (b) $\frac{12}{13}$
(c) $\frac{5}{13}$ (d) $\frac{1}{13}$

35. The equation of the base of an equilateral triangle is $x+y=2$ and the vertex is $(2,-1)$. The length of the side of the triangle is

(a) $\sqrt{3/2}$ (b) $\sqrt{2}$
(c) $\sqrt{2/3}$ (d) $\sqrt{1/3}$

36. Choose the correct statement which describe the position of the point $(-6, 2)$ relative to straight lines $2x+3y-4=0$ and $6x+9y+8=0$

(a) Below both the lines (b) Above both the lines
(c) In between the lines (d) Both (a) & (b)

37. The image of a point $A(3, 8)$ in the line $x+3y=0$, is

(a) $(-1,-4)$ (b) $(-3,-8)$
(c) $(1,-4)$ (d) $(3, 8)$

38. The length of perpendicular from $(3, 1)$ on line $4x+3y+20=0$, is

(a) 6 (b) 7 (c) 5 (d) 8

39. The position of the point $(8, -9)$ with respect to the lines $2x+3y-4=0$ and $6x+9y+8=0$ is

(a) Point lies on the same side of the lines
(b) Point lies on different sides of the line
(c) Point lies on one of the lines
(d) Point lies on the both lines

40. The length of the perpendicular from the point $(a\cos\alpha, a\sin\alpha)$ upon the straight line $y=x\tan\alpha+c, c>0$

(a) $c\cos\alpha$ (b) $c\sin^2\alpha$
(c) $c\sec^2\alpha$ (d) $c\cos^2\alpha$

41. The coordinates of the foot of the perpendicular from the point $(2, 3)$ on the line $y=3x+4$ are given by

(a) $\left(\frac{37}{10},-\frac{1}{10}\right)$ (b) $\left(-\frac{1}{10},\frac{37}{10}\right)$
(c) $\left(\frac{10}{37},-10\right)$ (d) $\left(\frac{2}{3},-\frac{1}{3}\right)$

42. The point $P(a.b)$ lies on the straight line $3x+2y=13$ and the point $Q(a, b)$ lies on the straight line $4x-y=5$ then the equation of line PQ is

(a) $x-y=5$ (b) $x+y=5$
(c) $x+y=-5$ (d) $x-y=-5$

43. The vertex of an equilateral triangle is $(2.-1)$ and the equation of its base is $x+2y=1$. The length of its side is

(a) $4/\sqrt{15}$ (b) $2/\sqrt{15}$
(c) $4/3\sqrt{3}$ (d) $1/\sqrt{5}$

DISTANCE BETWEEN LINES OF CONCURRENCY OF LINES

44. The distance between the lines $3x+4y=9$ and $6x+8y=15$ is

(a) 3/2 (b) 3/10
(c) 6 (d) 12

45. If the lines $ax+by+c=0$, $bx+cy+a=0$ and $cx+ay+b=0$ be concurrent, then

(a) $a^3+b^3+c^3+3abc=0$
(b) $a^3+b^3+c^3-abc=0$
(c) $a^3+b^3+c^3-3abc=0$
(d) $a^3+b^3+c^3+2abc=0$

46. The straight lines $4ax+3by+c=0$ where $a+b+c=0$, will be concurrent at point

(a) (4, 3) (b) (1/4, 1/3)
(c) (1/2, 1/3) (d) (1/3, 1/2)

FAMILY OF LINES

47. The equation of straight line passing through point of intersection of the straight lines $3x-y+2=0$ and $5x-2y+7=0$ and having infinite slope is

(a) $x=2$ (b) $x+y=3$
(c) $x=3$ (d) $x=4$

48. If a and b are two arbitrary constants, then the straight line $(a-2b)x+(a+3b)y+3a+4b=0$ will pass through

(a) $(-1,-2)$ (b) $(1, 2)$
(c) $(-2,-3)$ (d) $(2, 3)$

49. The line $(p+2q)x+(p-3q)y=p-q$ for different values of p and q passes through a fixed point whose co-ordinates are

(a) $\left(\frac{3}{2},\frac{5}{2}\right)$ (b) $\left(\frac{2}{5},\frac{2}{5}\right)$
(c) $\left(\frac{3}{5},\frac{3}{5}\right)$ (d) $\left(\frac{2}{5},\frac{3}{5}\right)$

50. Given the family of lines, a $(3x+4y+6)+b(x+y+2)=0$. The line of the family situated at the greatest distance from the point $P(2,3)$ has equation

(a) $4x+3y+8=0$ (b) $5x+3y+10=0$
(c) $15x+8y+30=0$ (d) $3x+3y+8=0$

ANGLE BISECTORS

51. The equation of the line which bisects the obtuse angle between the lines $x-2y+4=0$ and $4x-3y+2=0$, is

(a) $(4-\sqrt{5})x-(3-2\sqrt{5})y+(2-4\sqrt{5})=0$
(b) $(4+\sqrt{5})x-(3+2\sqrt{5})y+(2+4\sqrt{5})=0$
(c) $(4+\sqrt{5})x+(3+2\sqrt{5})y+(2+4\sqrt{5})=0$
(d) $(4-\sqrt{5})x+(3-2\sqrt{5})y-(2+4\sqrt{5})=0$

52. The equation of the bisector of the acute angle between the lines $3x-4y+7=0$ and $12x+5y-2=0$ is

(a) $21x+77y-101=0$ (b) $11x-3y+9=0$
(c) $31x+77y+101=0$ (d) $11x-3y-9=0$

53. Equation of angle bisectors between x and y-axes are

(a) $y=\pm x$ (b) $y=\pm 2x$
(c) $y=\pm\frac{1}{\sqrt{2}}x$ (d) $y=\pm 3x$

54. The combined equation of the bisectors of the angle between the lines represented by $(x^2 + y^2)\sqrt{3} = 4xy$ is
(a) $y^2 - x^2 = 0$
(b) $xy = 0$
(c) $x^2 + y^2 = 2xy$
(d) $\frac{x^2 - y^2}{\sqrt{3}} = \frac{xy}{2}$

OPTICS PROBLEM

55. A ray of light coming from the point (1, 2) is reflected at a point A on the x-axis and then passes through the point (5, 3). The coordinates of the point A are
(a) (13/5, 0)
(b) (5/13, 0)
(c) (– 7, 0)
(d) (–5/ 13, 0)

56. The point (4, 1) undergoes the following two successive transformations
(i) Reflection about the line $y = x$
(ii) Translation through a distance 2 units along the positive x-axis
Then the final coordinates of the point are
(a) (4, 3)
(b) (3, 4)
(c) (1, 4)
(d) $\left(\frac{7}{2}, \frac{7}{2}\right)$

57. If (–2, 6) is the image of the point (4, 2) with respect to line $L = 0$, then $L =$
(a) $3x - 2y + 5 = 0$
(b) $3x - 2y + 10 = 0$
(c) $2x + 3y - 5 = 0$
(d) $6x - 4y - 7 = 0$

SPECIAL POINTS

58. The circumcentre of the triangle with vertices (0,0), (3, 0) and (0, 4) is-
(a) (1, 1)
(b) (2, 3/2)
(c) (3/2, 2)
(d) (1/2, 2)

59. The mid points of the sides of a triangle are (5, 0), (5,12) and (0, 12), then orthocentre of this triangle is -
(a) (0, 0)
(b) (0, 24)
(c) (10, 0)
(d) $\left(\frac{13}{3}, 8\right)$

AREA OF TRIANGLE

60. Area of a triangle whose vertices are ($a \cos \theta$, $b \sin \theta$), ($-a \sin \theta$, $b \cos \theta$) and ($-a \cos \theta$, $-b \sin \theta$) is-
(a) $a\, b \sin \theta \cos \theta$
(b) $a \cos \theta \sin \theta$
(c) $\frac{1}{2}ab$
(d) ab

61. The point A divides the join of the points (–5, 1) and (3, 5) in the ratio k : 1 and coordinates of points B and C are (1, 5) and (7, –2) respectively. If the area of ΔABC be 2 units, then k equals-
(a) 7,9
(b) 6,7
(c) 7, 31/9
(d) 9, 31/9

62. The points with the co-ordinates $(2a, 3a)$, $(3b, 2b)$ and (c, c) are collinear
(a) for no value of a, b, c
(b) for all values of a, b, c
(c) If a, $\frac{c}{5}$, b are in H.P.
(d) If a, $\frac{2}{5}$ c, b are in H.P.

63. If A and B are two points on the line $3x + 4y + 15 = 0$ such that $OA = OB = 9$ units, then the area of the triangle OAB is
(a) 18 sq. units
(b) $18\sqrt{2}$ sq. units
(c) $9\sqrt{2}$ sq. units
(d) $18\sqrt{3}$ sq. units

PAIR OF STRAIGHT LINES

64. A second degree homogenous equation in x and y always represents
(a) A pair of straight lines passing through origin
(b) A circle
(c) A parabola
(d) A ellipse

65. If $6x^2 + 11xy - 10y^2 + x + 31y + k = 0$ represents a pair of straight lines, then $k =$
(a) –15
(b) 6
(c) –10
(d) –4

66. If $4ab = 3h^2$, then the ratio of slopes of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ will be
(a) $\sqrt{2} : 1$
(b) $\sqrt{3} : 1$
(c) 2: 1
(d) 1: 3

67. If the equation $ax^2 + 2hxy + by^2$ represents two lines $y = m_1x$ and $y = m_2x$, then
(a) $m_1 + m_2 = \frac{-2h}{b}$ and $m_1m_2 = \frac{a}{b}$
(b) $m_1 + m_2 = \frac{2h}{b}$ and $m_1m_2 = \frac{-a}{b}$
(c) $m_1 + m_2 = \frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$
(d) $m_1 + m_2 = \frac{2h}{b}$ and $m_1m_2 = -ab$

68. The nature of straight lines represented by the equation $4x^2 + 12xy + 9y^2 = 0$ is
(a) Real and coincident
(b) Real and different
(c) Imaginary and different
(d) Imaginary and coincident

69. If the point (2, – 3) lies on $kx^2 - 3y^2 + 2x + y - 2 = 0$, then k is equal to
(a) $\frac{1}{7}$
(b) 16
(c) 7
(d) 12

70. Angle between the lines represented by the equation $x^2 + 2xy \sec\theta + y^2 = 0$ is

(a) θ (b) 2θ

(c) $\frac{\theta}{2}$ (d) None of these

71. The equation of pair of straight lines joining the point of intersection of the curve $x^2 + y^2 = 4$ and $y - x = 2$ to the origin, is

(a) $x^2 + y^2 = (y - x)^2$

(b) $x^2 + y^2 + (y - x)^2 = 0$

(c) $x^2 + y^2 = 4(y - x)^2$

(d) $x^2 + y^2 + 4(y - x)^2 = 0$

72. Equation of angle bisector between the lines $3x + 4y - 7 = 0$ and $12x + 5y + 17 = 0$ are

(a) $\frac{3x+4y-7}{\sqrt{25}} = \pm\frac{12x+5y+17}{\sqrt{169}}$

(b) $\frac{3x+4y+7}{\sqrt{25}} = \frac{12x+5y+17}{\sqrt{169}}$

(c) $\frac{3x+4y+7}{\sqrt{25}} = \pm\frac{12x+5y+17}{\sqrt{169}}$

(d) $\frac{3x+4y-7}{\sqrt{25}} = \pm\frac{12x-5y+17}{\sqrt{169}}$

73. Which of the following is not always inside a triangle?

(a) Incentre

(b) Centroid

(c) Intersection of altitudes

(d) Intersection of medians

74. If $x = x_1 \pm r\cos\theta, y = y_1 \pm r\sin\theta$ be the equation of straight line then, the parameter in this equation is

(a) θ

(b) x_1

(c) y_1

(d) r

Exercise-2 (Learning Plus)

1. The orthocentre of the triangle ABC is 'B' and the circumcentre is 'S' (a, b). If A is the origin, then the co–ordinates of C are :

(a) $(2a, 2b)$ (b) $\left(\frac{a}{2}, \frac{b}{2}\right)$

(c) $\left(\sqrt{a^2 + b^2}, 0\right)$ (d) none

2. The mid points of the sides of a triangle are (5, 0), (5, 12) and (0, 12), then orthocentre of the triangle is

(a) (0, 0) (b) (0, 24)

(c) (10, 0) (d) $\left(\frac{13}{3}, 8\right)$

3. Given the points A (0, 4) and B (0, – 4), the equation of the locus of the point $P(x, y)$ such that $|AP - BP| = 6$ is:

(a) $9x^2 - 7y^2 + 63 = 0$ (b) $9x^2 - 7y^2 - 63 = 0$

(c) $7x^2 - 9y^2 + 63 = 0$ (d) $7x^2 - 9y^2 - 63 = 0$

4. A variable straight line passes through a fixed point (a, b) intersecting the co–ordinates axes at A & B. If 'O' is the origin, then the locus of the centroid of the triangle OAB is :

(a) $bx + ay - 3xy = 0$

(b) $bx + ay - 2xy = 0$

(c) $ax + by - 3xy = 0$

(d) $ax + by - 2xy = 0$

5. If $P(1, 0)$; $Q(-1, 0)$ & $R(2, 0)$ are three given points, then the locus of the points S satisfying the relation, $SQ^2 + SR^2 = 2\,SP^2$ is

(a) A straight line parallel to x-axis

(b) A circle passing through the origin

(c) A circle with the centre at the origin

(d) A straight line parallel to y-axis

6. In a triangle ABC, co-ordinates of A are (1, 2) and the equations to the medians through B and C are $x + y = 5$ and $x = 4$ respectively. Then the co-ordinates of B and C will be

(a) (–2, 7), (4, 3) (b) (7, – 2), (4, 3)

(c) (2, 7), (– 4, 3) (d) (2, – 7), (3, – 4)

7. The line joining two points A (2, 0) and B (3, 1) is rotated about A in the anticlock wise direction through an angle of 15°. The equation of the line in the new position is :

(a) $x - \sqrt{3}\,y - 2 = 0$ (b) $x - 2y - 2 = 0$

(c) $\sqrt{3}\,x - y - 2\sqrt{3} = 0$ (d) $\sqrt{3}x + y + 2\sqrt{3} = 0$

8. The distance of the point (2, 3) from the line $2x - 3y + 9 = 0$ measured along a line $x - y + 1 = 0$ is :

(a) $5\sqrt{3}$ (b) $4\sqrt{2}$

(c) $3\sqrt{2}$ (d) $2\sqrt{2}$

9. The lines $ax + by + c = 0$, where $3a + 2b + 4c = 0$, are concurrent at the point :

(a) $\left(\frac{1}{2}, \frac{3}{4}\right)$ (b) (1, 3)

(c) (3, 1) (d) $\left(\frac{3}{4}, \frac{1}{2}\right)$

10. The base BC of a triangle ABC is bisected at the point (p, q) and the equation to the side AB & AC are $px + qy = 1$ & $qx + py = 1$. The equation of the median through A is
(a) $(p - 2q)x + (q - 2p)y + 1 = 0$
(b) $(p + q)x + y - 2 = 0$
(c) $(2pq - 1)(px + qy - 1) = (p^2 + q^2 - 1)(qx + py - 1)$
(d) None

11. The equation of the internal bisector of $\angle BAC$ of ΔABC with vertices $A(5, 2)$, $B(2, 3)$ and $C(6, 5)$ is
(a) $2x + y + 12 = 0$
(b) $x + 2y - 12 = 0$
(c) $2x + y - 12 = 0$
(d) $x - 2y - 12 = 0$

12. The straight lines joining the origin to the points of intersection of the line $2x + y = 1$ and curve $3x^2 + 4xy - 4x + 1 = 0$ include an angle:
(a) $\frac{\pi}{2}$
(b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$
(d) $\frac{\pi}{6}$

13. The equations of the lines through the point of intersection of the lines $x - y + 1 = 0$ and $2x - 3y + 5 = 0$ and whose distance from the point (3,2) is
(a) $3x - 4y - 6 = 0$ and $4x + 3y + 1 = 0$
(b) $3x - 4y + 6 = 0$ and $4x - 3y - 1 = 0$
(c) $3x - 4y + 6 = 0$ and $4x - 3y + 1 = 0$
(d) $3x - 4y - 6 = 0$ and $4x - 3y + 1 = 0$

14. The equation of the straight line joining the point (a,b) to the point of intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ is
(a) $a^2y - b^2x = ab(a - b)$
(b) $a^2y + b^2x = ab(a + b)$
(c) $a^2y + b^2x = ab$
(d) $a^2x + b^2y = ab(a - b)$

15. One diagonal of a square is along the line $8x - 15y = 0$ and one of its vertex is (1.2). Then the equation of the sides of the square passing through this vertex, are
(a) $2\ 3x + 7y = 9$, $7x + 23y = 5\ 3$
(b) $23x - 7y + 9 = 0$, $7x + 23y + 53 = 0$
(c) $23x - 7y - 9 = 0$, $7x + 23y - 53 = 0$
(d) None of these

16. If $\frac{1}{ab'}+\frac{1}{ba'}=0$, then lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b'}+\frac{y}{a'}=1$ are
(a) Parallel
(b) Inclined at 60° to each other
(c) Perpendicular to each other
(d) Inclined at 30° to each other

17. The values of k for which lies $kx + 2y + 2 = 0$, $2x + ky + 3 = 0.3x + 3y + k = 0$.
(a) $\{2, 3, 5\}$
(b) $\{2, 3, -5\}$
(c) $\{3, -5\}$
(d) $\{-5\}$

18. If the length of the perpendicular drawn from the origin to the line whose intercepts on the axes are a and b be p, then
(a) $a^2 + b^2 = p^2$
(b) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{2}{p^2}$
(c) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{2}{p^2}$
(d) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}$

19. If p and p' be the distances of origin from the lines $x \sec\alpha + y \operatorname{cosec}\alpha = k$ and $x\cos\alpha - y\sin\alpha = k\cos 2\alpha$, then $4p^2 + p'^2 =$
(a) k
(b) $2k$
(c) k^2
(d) $2k^2$

20. The points $(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(a, 0)$ will be collinear, if
(a) $t_1t_2 = 1$
(b) $t_1t_2 = -1$
(c) $t_1 + t_2 = 1$
(d) $t_1 + t_2 = -1$

21. The locus of the image of the point (2, 3) in the line $(x - 2y + 3) + x(2x - 3y + 4) = 0$ is $(\lambda \in R)$
(a) $x^2 + y^2 - 3x - 4y - 4 = 0$
(b) $2x^2 + 3y^2 + 2x + 4y - 7 = 0$
(c) $x^2 + y^2 - 2x - 4y + 4 = 0$
(d) $2x^2 + 3y^2 - 2x - 4y + 7 = 0$

22. A light ray coming along the line $3x + 4y = 5$ gets reflected from the line $ax + by = 1$ and goes along the line $5x - 12y = 10$. Then,
(a) $a = 64/115$, $b = 112/15$
(b) $a = 14/15$, $b = -8/115$
(c) $a = 64/115$, $b = -8/115$
(d) $a = 64/15$, $b = 14/15$

23. A beam of light is sent along the line $x - y = 1$, which after refracting from the x-axis enters the opposite side by turning through 30° away from the normal at the point of incidence on the x-axis. Then the equation of the refracted ray is
(a) $\left(2-\sqrt{3}\right)x - y = 2+\sqrt{3}$
(b) $\left(2+\sqrt{3}\right)x - y = 2+\sqrt{3}$
(c) $\left(2-\sqrt{3}\right)x + y = \left(2+\sqrt{3}\right)$
(d) $y = \left(2-\sqrt{3}\right)(x-1)$

24. If the slope of one line of the pair of lines represented by $ax^2 + 4xy + y^2 = 0$ is 3 times the slope of the other line, then a is
(a) 1
(b) 2
(c) 3
(d) 4

25. The image of the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ by the line mirror $y = 0$ is
(a) $ax^2 - 2h\,xy + by^2 = 0$
(b) $bx^2 - 2h\,xy + ay^2 = 0$
(c) $bx^2 + 2h\,xy + ay^2 = 0$
(d) $ax^2 - 2h\,xy - by^2 = 0$

26. If the lines represented by the equation $ax^2 - bxy - y^2 = 0$ make angles α and β with the x-axis, then $\tan(\alpha + \beta) =$

(a) $\frac{b}{1+a}$ (b) $\frac{-b}{1+a}$

(c) $\frac{a}{1+b}$ (d) $\frac{a}{1-b}$

27. The equation $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ represents a pair of straight lines. The distance between them is

(a) $7/\sqrt{5}$ (b) $7/2\sqrt{5}$

(c) $\sqrt{7}/5$ (d) $7/3\sqrt{5}$

28. Start with an equilateral triangle of side length i. Construct a second triangle by connecting the a midpoints of the sides of the first triangle. Construct a third triangle by connecting the midpoints of the edges of the second triangle. Continue this process indefinitely. The sum of the areas of all the triangles, is

(a) $\sqrt{3}$ (b) $\frac{2}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{3}}$ (d) ∞

29. A triangle ABC is formed by the lines $2x - 3y - 6 = 0$; $3x - y + 3 = 0$ and $3x + 4y - 12 = 0$. If the points $P(\alpha, 0)$ and $Q(0, \beta)$ always lie on or inside the & ABC, then

(a) $\alpha \in [-1, 2]$ and $\beta \in [-2, 3]$

(b) $\alpha \in [-1, 3]$ and $\beta \in [-2, 4]$

(c) $\alpha \in [-2, 4]$ and $\beta \in [-3, 4]$

(d) $\alpha \in [-1, 3]$ and $\beta \in [-2, 3]$

30. Area of a parallelogram two of whose sides are given by $2x^2 - 5xy + 2y^2 = 0$ and one of its diagonal is $5x + 2y = 1$, is

(a) $\frac{1}{6}$ (b) $\frac{1}{12}$

(c) $\frac{1}{20}$ (d) $\frac{1}{36}$

31. The algebraic sum of distances of the lines $ax + by + 2 = 0$ from (1, 2), (2, 1) and (3, 5) is zero and the lines $ax - by + 4 = 0$ and $3x + 4y + 5 = 0$ cuts the coordinate axes at concyclic points. Then

(a) $a + b = \frac{-2}{7}$

(b) area of the triangle formed by the line $ax + by + 2 = 0$ with coordinates axes is $\frac{14}{5}$

(c) line $ax + by + 3 = 0$ always passes through the point $(-1, 1)$

(d) $\max\{a, b\} = \frac{5}{7}$

32. If $(\alpha.\ \alpha^2)$ lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0.5x - 6y - 1 = 0$ then

(a) $2\alpha + 3\alpha^2 - 1 > 0$ (b) $\alpha + 2\alpha^2 - 3 < 0$

(c) $\alpha + 2\alpha^2 + 3 < 0$ (d) $6\alpha^2 - 5\alpha + 1 > 0$

33. If t_1, t_2 and t_3 are distinct the points $(t_1, 2at_1 + at_1^3)$, $(t_2, 2at_2 + at_2^3)$ and $(t_3, 2at_3 + at_3^3)$ are collinear, then $\frac{4^n \cdot 6^n \cdot 8^n}{2^n \cdot 3^n \cdot 4^n}$ is equal to :

(a) 0 (b) 4

(c) 2 (d) 1

34. Let ABC be a triangle with $\angle C = 90°$. Draw CD perpendicular to AB. Choose points M and N on sides AC and BC respectively such that DM is parallel to BC and DN is parallel to AC. If $DM = 5$, $DN = 4$, then AC and BC are respectively equal to

(a) 41/4, 41/5 (b) 39/4, 39/5

(c) 38/4, 38/5 (d) 37/4, 37/5

35. $OPQR$ is a square and M, N are the mid-points of the sides PQ and QR respectively. If the ratio of the areas of the square and the triangle OMN is $k : 6$, then $\frac{k}{4}$ is equal to

(a) 2 (b) 4

(c) 12 (d) 16

36. A ray moving along the line $px + qy = 1$, after striking a mirror placed along the line $y = 3$, is reflected along the line

(a) $px - qy = 1$ (b) $px - qy = 1 - q$

(c) $px - qy = 1 - 6q$ (d) $px - qy = 1 - 4q$

37. If the coordinates of vertices of a triangle is always rational then the triangle cannot be

(a) Scalene (b) Isosceles

(c) Rightangle (d) Equilateral

38. $y = 10^x$ is the reflection of $y = \log_{10} x$ in the line whose equation is

(a) $y = x$ (b) $y = -x$

(c) $y = 10^x$ (d) $y = -10x$

39. If the pairs of lines $x^2 + 2xy + ay^2 = 0$ and $ax^2 + 2xy + y^2 = 0$ have exactly one line in common then the joint equation of the other two lines, is given by:

(a) $3x^2 + 8xy - 3y^2 = 0$ (b) $3x^2 + 10xy - 3y^2 = 0$

(c) $y^2 + 2xy - 3x^2 = 0$ (d) $x^2 + 2xy - 3y^2 = 0$

40. The distance between the two lines represented by the equation $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$ is

(a) 8/5 (b) 6/5

(c) 11/5 (d) none of these

41. The gradient of one of the lines $x^2 + hyx + 2y^2 = 0$ is twice that of the other, then $h =$

(a) ± 3 (b) $\pm\frac{3}{2}$

(c) ± 2 (d) ± 1

42. The equation of the line joining origin to the points of intersection of the curve $x^2 + y^2 = a^2$ and $x^2 + y^2 - ax - ay = 0$ is

(a) $x^2 - y^2 = 0$ (b) $xy = 0$

(c) $xy - x^2 = 0$ (d) $y^2 + xy = 0$

43. The area of the triangle formed by the line $x + y = 2$ and angle bisectors of the pair of straight lines $x^2 - y^2 + 2y = 1$ is a and coordinate of orthocentre of triangle is (b, c), then $4a + b + c$ is equal to:

(a) 0 (b) 4

(c) 2 (d) 3

44. The orthocenter, circumcenter, centroid, and incenter of the triangle formed by the line $y = x + a$ with the coordinate axes lie on

(a) $x^2 + y^2 = 1$ (b) $y = x$

(c) $y = 2x$ (d) $y = 3x$

45. The family of straight lines $2x \sin^2\theta + y \cos 2\theta = 2\cos^2\theta$ ($\theta \in R$) passes through a fixed point (a,b) then value of $a + b$ is

(a) 1 (b) 2

(c) 3 (d) 4

46. If the coordinates of vertices of a triangle is always rational, then coordinates of which of the following points will not be always rational

(a) centroid (b) circumcenter

(c) orthocenter (d) incenter

47. One side of a rectangle lies along the line $4x + 7y + 5 = 0$. Two of its vertices are $(-3, 1)$ and $(1, 1)$. Then the equations of other sides are :

(a) $7x - 4y + 25 = 0$ (b) $7x + 4y + 25 = 0$

(c) $7x - 4y - 3 = 0$ (d) $4x + 7y = 11$

48. The area of a triangle is 5. Two of its vertices are $(2, 1)$ & $(3, -2)$. The third vertex lies on $y = x + 3$. Find the third vertex.

(a) $\left(\frac{7}{2}, -\frac{13}{2}\right)$ (b) $\left(\frac{-3}{2}, \frac{3}{2}\right)$

(c) $\left(\frac{7}{2}, \frac{13}{2}\right)$ (d) $\left(\frac{3}{2}, \frac{3}{2}\right)$

49. The sides of a triangle are the straight line $x + y = 1$, $7y = x$ and $\sqrt{3}\,y + x = 0$. Then which of the following is an interior points of triangle ?

(a) circumcentre (b) centroid

(c) incentre (d) orthocentre

50. Equation of a straight line passing through the point $(4, 5)$ and equally inclined to the lines $3x = 4y + 7$ and $5y = 12x + 6$ is

(a) $9x - 7y = 1$ (b) $9x + 7y = 71$

(c) $7x + 9y = 73$ (d) $7x - 9y + 17 = 0$

51. If the image of the $m(\lambda, \lambda^2)$ in the line $x + y = \lambda^2$ is $(0,2)$ then λ can be

(a) -1 (b) 2

(c) 1 (d) -2

52. A and B are two feed points whose coordinates are $(3, 2)$ and $(5, 4)$ respectively. The co-ordinates of a point P if ABP is an equilateral triangle, is/are

(a) $\left(4-\sqrt{3}, 3+\sqrt{3}\right)$ (b) $\left(4+\sqrt{3}, 3-\sqrt{3}\right)$

(c) $\left(3-\sqrt{3}, 4+\sqrt{3}\right)$ (d) $\left(3+\sqrt{3}, 4-\sqrt{3}\right)$

53. The point $(4, 1)$ undergoes the following transformations, then the match the correct alternatives

Column-I		Column-II	
A.	Reflection about the line $y = -x$	p.	$(-4, -1)$
B.	Reflection about origin is	q.	$(-8/5, 19/5)$
C.	Reflection about the line $y = 2x$ is	r.	$(-4, 1)$
D.	Reflection about the line $4x + 3y - 5 = 0$ is	s.	$\left(-\frac{12}{25}, -\frac{59}{25}\right)$

(a) A-(q); B-(p); C-(s); D-(r)

(b) A-(p); B-(q); C-(r); D-(s)

(c) A-(s); B-(r); C-(p); D-(q)

(d) A-(q); B-(p); C-(r); D-(s)

54. The ordinate of a point P on the line $6x + y = 9$, which is closest to the point $(-3, 1)$ can be expressed in the form a/b. Where $a, b \in N$ and are in lowest form, then find the value $(a + b)$.

55. If the equation of base of an equilateral triangle is $2x - y = 1$ and the vertex is $(-1, 2)$, then the length of the side of the triangle is $\sqrt{b/a}$, then find value of $b + a$

56. Consider two points $A(4, 3)$ & $B(6,0)$. Let P be a point on the line $x + y = 4$,such that $|PA - PB|$ is minimum and area of ΔAPB is $\frac{m}{n}$, (where m and n are co-n primes), then $(m - n)$ is equal to

57. Determine the ratio in which the point $P(3, 5)$ divides the join of $A(1, 3)$ & $B(7, 9)$. Find the harmonic conjugate of P w.r.t. A & B.

58. If the line $y - \sqrt{3}x + 3 = 0$ cuts the curve $y^2 = x + 2$ at A and B, then find the value of $PA.PB$ {where $P \equiv (\sqrt{3}, 0)$}

59. A man starts from the point $P(-3, 4)$ and reaches the point $Q(0, 1)$ after touching the line $2x + y = 7$ at R. Find R on the line so that he travels along the shortest path.

60. The equation of two equal sides AB and AC of an isosceles triangle ABC are $x + y = 5$ & $7x - y = 3$ respectively. Find the equations of the side BC if the area of the triangle of ABC is 5 units.

61. The equation of line through the intersection of lines $x - y = 1$ and $2x - 3y + 1 = 0$ and parallel to $3x + 4y = 12$ is

62. The equations of the altitudes AD, BE, CF of a triangle ABC are $x + y = 0$, $x - 4y = 0$ and $2x - y = 0$ respectively. The coordinates of A are $(t\, -t)$. Find coordinates of B & C. Prove that if t varies the locus of the centroid of the triangle ABC is $x + 5y = 0$.

63. The vertices of a triangle are $A(a,\ a\tan\alpha)$, $B(b,\ b\tan\beta)$, $C(c,\ c\tan\gamma)$. If the circumcentre of ΔABC coincides with origin and $H(\bar{x},\bar{y})$ is the orthocenter, then show that $\dfrac{\bar{y}}{\bar{x}}=\left(\dfrac{\sin\alpha+\sin\beta+\sin\gamma}{\cos\alpha+\cos\beta+\cos\gamma}\right)$.

64. Through what angle should the axes be rotated so that the equation $9x^2-2\sqrt{3}xy+7y^2=10$ may be changed to $3x^2+5y^2-5=0$?

65. If the slope of one line is double the slope of another line and the combined equation of the pair of lines is $(x^2/a)+(2xy/h)+(y^2/b)=0$, then find the ratio $ab:h^2$.

66. Two sides of an isosceles triangle are given by the equations $7x-y+3=0$ and $x+y-3=0$ and its third side passes through the point $(1,-10)$. Determine the equation of the third side.

67. P is the point $(-1,2)$, a variable line through P cuts the x & y axes at A & B respectively Q is the point on AB such that PA, PQ, PB are H.P. Show that the locus of Q is the line $y=2x$.

Exercise-3 (JEE Advanced Level)

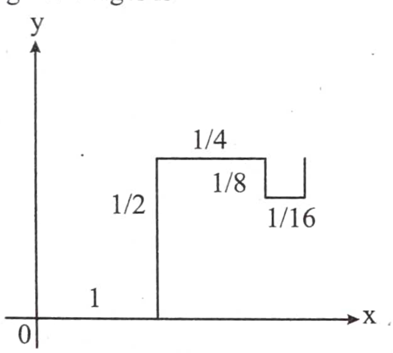
MULTIPLE CORRECT TYPE QUESTIONS

1. Let the co-ordinates of the two points A and B be $(1, 2)$ and $(7, 5)$ respectively. The line AB is rotated through $45°$ in anti-clockwise direction about the point of trisection of AB which is nearer to B. The equation of the line in new position is
   (a) $2x-y-6=0$ (b) $x-y-1=0$
   (c) $3x-y-11=0$ (d) $3x-y-9=0$

2. If a,b,c form an $A.P.$ with common difference $d(\neq 0)$ and x, y, z form a $G.P.$ with common ratio $r\ (\neq 1)$, then the area of the triangle with vertices (a,x), (b,y) and (c,z) is independent of
   (a) a (b) b (c) x (d) r

3. If variable line $9ax+4by=5(a,b>0)$ always passes through $(1, 1)$ then the maximum value of $\left(3\sqrt{a}+2\sqrt{b}\right)$ is equal to
   (a) $\sqrt{5}$ (b) $\sqrt{10}$ (c) $\sqrt{13}$ (d) $\sqrt{15}$

4. A particle begins at the origin and moves successively in the following manner as shown: 1 unit to the right, 1/2 unit up, 1/4 unit to the right, 1/8 unit down, 1/16 unit to the right etc. The length of each move is half the length of the previous move and movement continues in the 'zigzag' manner indefinitely. The co-ordinates of the point to which the 'zigzag' converges is



   (a) $(4/3, 2/3)$ (b) $(4/3, 2/5)$
   (c) $(3/2, 2/3)$ (d) $(2, 2/5)$

5. If $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ are the vertices of a triangle, then the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}+\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0 \text{ represents}$$

   (a) The median through A
   (b) The altitude through A
   (c) The perpendicular bisector of BC
   (d) The line joining the centroid with a vertex

6. Find the number of triangles formed by the lines represented by $x^3-x^2-x-2=0$ and $xy^2+2xy+4x-2y^2-4y^2-4y-8=0$
   (a) one (b) two
   (c) three (d) zero

7. The condition that one of the straight lines given by the equation $ax^2+2hxy+by^2=0$ may coincide with one of those given by the equation $ax^2+2h'xy+b'y^2=0$ is
   (a) $(ab'-a'b)^2=4(ha'-h'a)(bh'-b'h)$
   (b) $(ab'-a'b)=(ha'-h'a)(bh'-b'h)$
   (c) $(ha'-ha)=4(ab'-a'b)(bh'-b'h)$
   (d) $(bh'-b'h)=4(ab'-a'b)(ha'-h'a)$

8. In the xy plane, the line 'ℓ_1' passes through the point $(1, 1)$ and the line 'ℓ_2' passes through the point $(-1, 1)$. If the difference of the slopes of the lines is 2. Find the locus of the point of intersection of the lines ℓ_1 and ℓ_2.
   (a) $y=-x^2$ (b) $y=2+x^2$
   (c) $y=2-x^2$ (d) $y=x^2$

9. Straight lines $2x+y=5$ and $x-2y=3$ intersect at the point A. Points B and C are chosen on these two lines such that $AB=AC$. Then the equation of a line BC passing through the point $(2, 3)$ is
   (a) $3x-y-3=0$ (b) $x+3y-11=0$
   (c) $3x+y-9=0$ (d) $x-3y+7=0$

10. A line segment is divided so that the lesser part is to the greater part is equal to the greater part is to the whole. If R is the ratio of the lesser part to the greater part, then

(a) $R^2 + R - 1 = 0$ (b) $R^2 + R^{-1} = 2$

(c) $R^{(R^2 + R^{-1})} = 2$ (d) $R^2 + R + 1 = 0$

COMPREHENSION BASED QUESTIONS

Comprehension (Q. 11 to 13): Let ABC be an acute angled triangle and AD, BE and CF are its medians, where E and F are the points (3, 4) and (1, 2) respectively and centroid of ΔABC is $G(3, 2)$, then answer the following questions:

11. The equation of side AB is

(a) $2x + y = 4$ (b) $x + y - 3 = 0$

(c) $4x - 2y = 0$ (d) None of these

12. Co-ordinates of D are

(a) (7, –4) (b) (5, 0)

(c) (7, 4) (d) (– 3, 0)

13. Height of altitude drawn from point A is (in units)

(a) $4\sqrt{2}$ (b) $3\sqrt{2}$

(c) $6\sqrt{2}$ (d) $2\sqrt{3}$

Comprehension (Q. 14 to 16): Given two straight lines AB and AC whose equations are $3x + 4y = 5$ and $4x - 3y = 15$ respectively. Then the possible equation of line BC through (1, 2), such that ΔABC is isosceles, is $L_1 = 0$ or $L_2 = 0$, then answer the following questions

14. If $L_1 \equiv ax + by + c = 0$ & $L_2 \equiv dx + ey + f = 0$ where $a, b, c, d, e, f \in I$, and $a, d > 0$, then $c + f =$

(a) 1 (b) 2

(c) 3 (d) 4

15. A straight line through $P(2, c + f - 1)$, inclined at an angle of 60° with positive Y-axis in clockwise direction. The co-ordinates of one of the points on it at a distance $(c + f)$ units from point P is (c, f obtained from previous question)

(a) $(2 + 2\sqrt{3}, 5)$ (b) $(3 + 2\sqrt{3}, 3)$

(c) $(2 + 3\sqrt{2}, 4)$ (d) $(2 + 3\sqrt{2}, 3)$

16. If (a, b) is the co-ordinates of the point obtained in previous question, then the equation of line which is at the distance $|b - 2a - 1|$ units from origin and make equal intercept on co-ordinate axes in first quadrant, is

(a) $x + y + 4\sqrt{6} = 0$ (b) $x + y + 2\sqrt{6} = 0$

(c) $x + y - 4\sqrt{6} = 0$ (d) $x + y - 2\sqrt{6} = 0$

Comprehension (Q. 17 to 19):

A. The equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is a constant

B The equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ. is a constant

17. The number of lines that are parallel to $2x + 6y + 7 = 0$ and have an intercept of length 10 between the coordinate axes is

(a) 1 (b) 2

(c) 4 (d) infinitely many

18. A line passes through (2, 2) and is perpendicular to the line $3x + y = 3$. Its y-intercept is

(a) 1/3 (b) 2/3 (c) 1 (d) 4/3

19. If a line is perpendicular to the line $5x - y = 0$ and forms a triangle with coordinate axes of area 5 sq. units, then its equation is

(a) $x + 5y \pm 5\sqrt{2} = 0$ (b) $x - 5y \pm 4\sqrt{2} = 0$

(c) $5x + y \pm 5\sqrt{2} = 0$ (d) $5x + y \pm 4\sqrt{2} = 0$

Comprehension (Q. 20 to 22): Consider two points $A = (1, 2)$ and $B = (3, -1)$. Let M be a point on the straight line $L \equiv x + y = 0$.

20. If M be a point on the line $L = 0$ such that $AM + BM$ is minimum, then the reflection of M in the line $x = y$ is

(a) (1,–1) (b) (–1,1)

(c) (2, –2) (d) (–2, 2)

21. If M be a point on the line $L = 0$ such that $|AM - BM|$ is maximum, then the distance of M from .N ≡ (1, 1) is

(a) $5\sqrt{2}$ (b) 7

(c) $3\sqrt{5}$ (d) 10

22. If M be a point on the line $L = 0$ such that $|AM = BM|$ is minimum, then the area of ΔAMB equals

(a) $\frac{13}{4}$ (b) $\frac{13}{2}$

(c) $\frac{13}{6}$ (d) $\frac{13}{8}$

MATCH THE COLUMN TYPE QUESTIONS

23. Let ABC be a triangle such that the coordinates of A are (–3, 1). Equation of the median through B is $2x + y - 3 = 0$ and equation of the angular bisector of C is $7x - 4y - 1 = 0$. Then match the entries of column-I with their corresponding correct entries of column-II.

Column I		Column II	
A	Equation of the line AB is	p	$2x + y - 3 = 0$
B	Equation of the line BC is	q	$2x - 3y + 9 = 0$
C	Equation of the line CA is	r	$4x + 7y + 5 = 0$
		s	$18x - y - 49 = 0$

(a) A-(q); B-(p); C-(s) (b) A-(p); B-(q); C-(r)

(c) A-(s); B-(r); C-(p) (d) A-(p); B-(s); C-(q)

24. Consider the lines given by

$L_1 = x + 3y - 5 = 0$

$L_2 = 3x - ky - 1 = 0$

$L_3 = 5x + 2y - 12 = 0$

Match the statements/expression in **Column-I** with the statements/expression in **Column-II** and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in OMR.

	Column I		Column II
A	L_1, L_2, L_3 are concurrent, if	p	$k=-9$
B	One of L_1, L_2, L_3 is parallel to at least one of the other two, if	q	$k=-\frac{6}{5}$
C	L_1, L_2, L_3 form a triangle, if	r	$k=\frac{5}{6}$
D	L_1, L_2, L_3 do not form a triangle, if	s	$k=5$

(*a*) A-(s); B-(r,q); C-(q); D-(r,q,s)
(*b*) A-(s); B-(p,q); C-(r); D-(p,q,s)
(*c*) A-(p); B-(p,r); C-(s); D-(p,r,s)
(*d*) A-(q); B-(s,q); C-(p); D-(p,q,r)

25. Matching column Type

	Column I		Column II
A	A straight line with negative slope passing through (1, 4) meets the cording axes at A $OA + OB$, O being the origin, is	p	$5\sqrt{2}$
B	If the point Q is symmetric to the point P (4, –1) with respect to the bisector of the first quadrant, then the length of PQ is	q	$3\sqrt{2}$
C	On the portion of the straight line $x + y = 2$ between the axis a square is constructed away from the origin, with this portion as one of its sides. If 'd' denotes the perpendicular distance of a side of this square from the origin then the maximum value of 'd' is	r	$\frac{9}{2}$
D	If the parametric equation of a line is given by $x = 4+\frac{\lambda}{\sqrt{2}}$ and $y = -1+\sqrt{2}\lambda$ where λ is the parameter, then the intercept made by the line on the x-axis is	s	9

(*a*) A-(s); B-(p); C-(q); D-(r)
(*b*) A-(p); B-(q); C-(r); D-(s)
(*c*) A-(s); B-(r); C-(p); D-(q)
(*d*) A-(q); B-(p); C-(r); D-(s)

NUMERICAL BASED QUESTIONS

26. A line which is parallel to $y = x$ is rotated about the point (2, 0) through angle 15° anticlockwise direction, then y intercept of the line passing through the point of intersection of new line with $y = x$ and at right angle to new line, is $a + b\sqrt{3}$, then find value of $a + b$ equals

27. If the points of intersection of curves $C_1 = 4y^2 - \lambda x^2 - 2xy - 9x + 3$ and $C_2 = 2x^2 + 3y^2 - 4xy + 3x - 1$ subtends a right angle at origin, then find the the value of λ.

28. The portion of the line $ax + by - 1 = 0$, intercepted between the lines $ax + y + 1 = 0$ and $x + by = 0$ subtends a right angle at the origin and the condition in a and b is $\lambda a + b + b^2 = 0$, then find value of λ.

29. For all real values of a and b , lines $(2a + b)x + (a + 3b)y + (d - 3a) = 0$ and nix $+ 2y + 6 = 0$ are concurrent. Then $|m|$ is equal to

30. Triangle ABC with $AB = 13$, $BC = 5$, and $AC = 12$ slides on the coordinates axes with wA and B on the positive x -axis and positive y -axis respectively. The locus of vertex c is a line $12x - ky = 0$. Then the value of k is __________.

31. If $(0, p)$ lies inside the triangle formed by the lines $y + 3x + 2 = 0$, $3y - 2x - 5 = 0$ and $4y + x - 14 = 0$ then find the number of integral values of p.

32. Prove that equation of chord connecting the points $(a \sec \theta, b \tan \theta)$ and $(a \sec \phi\ b \tan \phi)$ is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right) \quad ...(i)$$

33. A variable line is drawn through O, to cut two fixed straight lines L_1 and L_2 in A_1 and A_2, respectively.

A point A is taken on the variable line such that $\frac{m+n}{OA}=\frac{m}{OA_1}+\frac{n}{OA_2}$.

Show that the locus of A is a straight line passing through the point of intersection of L_1 and L_2 where O is being the origin.

34. A straight line through $P(-2, -3)$ cuts the pair of straight lines $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ in Q and R. Find the equation of the line if $PQ \cdot PR = 20$.

35. If a, b, c are all different and the points $\left(\frac{r^3}{r-1}, \frac{r^2-3}{r-1}\right)$ where $r = a, b, c$ are collinear, then prove that $3(a + b + c) = ab + bc + ca - abc$.

36. A line through $A(-5, -4)$ meets the line $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at the points B, C & D respectively, If $\left(\frac{15}{AB}\right)^2+\left(\frac{10}{AC}\right)^2=\left(\frac{6}{AD}\right)^2$. Find the equation of the line.

37. The ends A, B of a straight line segment of constant length 'c' slide upon the fixed rectangular axes OX & OY respectively. If the rectangle $OAPB$ be completed then show that the locus of the foot of the perpendicular drawn from P and AB is $x^{2/3}\ y^{2/3} = c^{2/3}$.

38. The line segment joining $A(3, 0)$ and $B(5, 2)$ is rotated about A in the anti-clockwise direction through an angle of 45° so that B goes to C. If $D(x, y)$ is the image of C with respect to y-axis, find the value of $x + y + 7$.

39. A triangle is formed by the lines whose equations are $AB : x + y - 5 = 0$, $BC : x + 7y - 7 = 0$ and $CA : 7x + y + 14 = 0$. Find

(i) Type of triangle

(ii) the bisector of the interior angle at B

(iii) the bisector of the exterior angle at C

(iv) nature of the interior angle at A

40. The equations of the perpendicular bisectors of the sides AB & AC of a triangle ABC are $x - y + 5 = 0$ & $x + 2y = 0$, respectively. If the point A is $(1, -2)$, find the equation of the line BC.

41. Let ABC be a triangle with $AB = AC$. If D is the mid point of BC, E the foot of the perpendicular from D to AC and F the midpoint of DE, prove analytically that AF is perpendicular to BE.

42. If $x \cos \alpha + y \sin \alpha = p$, where $p = -\dfrac{\sin^2 \alpha}{\cos \alpha}$ be a straight line, prove that the perpendiculars on this straight line from the points $(m^2, 2m)$, $(mm', m + m')$, $(m'^2, 2m')$ form a G.P.

43. Find the area of the triangle formed by the straight lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$, and $y = m_3x + c_3$.

44. The straight lines $(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0$ form a Δ with the line $Ax + By + C = 0$, then prove that

(i) Area of Δ is $\dfrac{C^2}{\sqrt{3}(A^2+B^2)}$

(ii) Δ is equilateral

(iii) The orthocentre of Δ does not lie on one of its vertexs

45. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, represents a pair of straight lines, prove that the third pair of straight lines (excluding $xy = 0$) passing through the points where these meet the axes is $ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \dfrac{4fg}{c} xy = 0$.

46. If the lines $ax^2 + 2hxy + by^2 = 0$ from two sides of a parallelogram and the line $lx + my = 1$ is one diagonal, prove that the equation of the other diagonal is, $y(bl - hm) = x(am - hl)$.

47. A triangle has two sides $y = m_1x$ and $y = m_2x$ where m_1 and m_2 are the roots of the equation $b\alpha^2 + 2h\alpha + a = 0$. If (a, b) be the orthocentre of the triangle, then find the equation of the third side in terms of a, b and h.

48. A point moves so that the distance between the feet of the perpendiculars from it on the lines $bx^2 + 2hxy + ay^2 = 0$ is a constant $2d$. Show that the equation to its locus is $(x^2 + y^2)(h^2 - ab) = d^2 \{(a - b)^2 + 4h^2\}$

49. Let ABC be a triangle having orthocentre and circumcentre at $(9, 5)$ and $(0, 0)$ respectively. If the equation of side BC is $2x - y = 10$, then find the possible coordinates of vertex A.

50. If the sides of a triangle are $L_r \equiv x \cos \alpha_r + y \sin \alpha_r - p_r = 0$, $(r = 1, 2, 3)$ then Prove that

(i) Any line through the intersection of lines L_1 and L_2 and perpendicular to L_3 is

$$\frac{L_1}{\cos\alpha_1 \cos\alpha_3 + \sin\alpha_1 \sin\alpha_3} = \frac{L_2}{\cos\alpha_2 \cos\alpha_3 + \sin\alpha_2 \sin\alpha_3}$$

(ii) the orthocentre is given by $L_1 \cos(\alpha_2 - \alpha_3) = L_2 \cos(\alpha_1 - \alpha_3) = L_3 \cos(\alpha_1 - \alpha_2)$

51. The vertices of a triangle are

$[at_1t_2, a(t_1 + t_2)]$ $[at_2t_3, a(t_2 + t_3)]$, $[at_3t_1, a(t_3 + t_1)]$. Find the coordinates of its orthocentre.

52. Straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at the point A. Points B and C are chosen on these two lines such that $AB = AC$. Determine the possible equations of the line BC passing through the point $(1, 2)$

53. The base of a triangle passes through a fuced point (f, g), its sides are respectively bisected at right angles by the lines $y^2 - 8xy - 9x^2 = 0$. Determine the locus of its vertex.

54. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, represents a pair of straight lines, prove that the third pair of straight lines (excluding $xy = 0$) passing through the points where these meet the axes is $ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \dfrac{4fg}{c} \cdot xy = 0$.

55. Find the acute angle between two straight lines passing through the point $M(-6, -8)$ and the points in which the line segment $2x + y + 10 = 0$ enclosed between the co-ordinate axes is divided in the ratio $1 : 2$ in the direction from the point of its intersection with the x-axis to the point of intersection with the y-axis.

56. Through the origin O a straight line is drawn to cut the lines $y = m_1x + C_1$ and $y = m_2x + C_2$ at Q and R respectively. Find the locus of the point P on this variable line, such that OP is the geometric mean of OQ and OR.

Exercise-4 (Past Year Questions)

JEE MAIN

1. Two sides of a rhombous are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at $(-1, -2)$, then which one of the following is a vertex of this rhombous? **(2016)**
   (a) $(-3, -9)$ (b) $(-3, -8)$
   (c) $\left(\frac{1}{3}, -\frac{8}{3}\right)$ (d) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$

2. Let k be an integer such that triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq. units. Then the orthocentre of this triangle is at the point: **(2017)**
   (a) $\left(2, \frac{1}{2}\right)$ (b) $\left(2, -\frac{1}{2}\right)$
   (c) $\left(1, \frac{3}{4}\right)$ (d) $\left(1, -\frac{3}{4}\right)$

3. A straight line through a fixed point $(2, 3)$ intersects the coordinate axes at distinct points P and Q. If O is the origin and the rectangle $OPRQ$ is completed, then the locus of R is: **(2018)**
   (a) $2x + 3y = xy$ (b) $3x + 2y = xy$
   (c) $3x + 2y = 6xy$ (d) $3x + 2y = 6$

4. Consider the set of all lines $px + qy + r = 0$ such that $3p + 2q + 4r = 0$. Which one of the following statements is true ? **(2019)**
   (a) The lines are concurrent at the point $\left(\frac{3}{4}, \frac{1}{2}\right)$
   (b) Each line passes through the origin
   (c) The lines are all parallel
   (d) The lines are not concurrent

5. Let the equation of two sides of a triangle be $3x - 2y + 6 = 0$ and $4x + 5y - 20 = 0$. If the orthocentre of this triangle is at $(1, 1)$, then the equation of its third side is: **(2019)**
   (a) $122y - 26x - 1675 = 0$ (b) $26y - 61x + 1675 = 0$
   (c) $122y + 26x + 1675 = 0$ (d) $26x - 122y - 1675 = 0$

6. A point P moves on the line $2x - 3y + 4 = 0$. If $Q(1, 4)$ and $R(3, -2)$ are fixed points, then the locus of the centroid of ΔPQR is a line: **(2019)**
   (a) With slope 3/2 (b) Parallel to x-axis
   (c) With slope 2/3 (d) Parallel to y-axis

7. If the line $3x + 4y - 24 = 0$ intersects the x-axis at the point A and the y-axis at the point B, then the incentre of the triangle OAB, where O is the origin, is: **(2019)**
   (a) $(3, 4)$ (b) $(2, 2)$
   (c) $(4, 3)$ (d) $(4, 4)$

8. Two sides of a parallelogram are along the lines, $x + y = 3$ and $x - y + 3 = 0$. If its diagonals intersect at $(2, 4)$, then one of its vertex is: **(2019)**
   (a) $(3, 5)$ (b) $(2, 1)$
   (c) $(2, 6)$ (d) $(3, 6)$

9. Two vertices of a triangle are $(0, 2)$ and $(4, 3)$. If its orthocenter is at the origin, then its third vertex lies in which quadrant? **(2019)**
   (a) third (b) second
   (c) first (d) fourth

10. If in a parallelogram $ABCD$, the coordinates of A, B and C are respectively $(1, 2)$, $(3, 4)$ and $(2, 5)$, then the equation of the diagonal AD is: **(2019)**
    (a) $5x - 3y + 1 = 0$ (b) $5x + 3y - 11 = 0$
    (c) $3x - 5y + 7 = 0$ (d) $3x + 5y - 13 = 0$

11. If the straight line, $2x - 3y + 17 = 0$ is perpendicular to the line passing through the points $(7, 17)$ and $(15, \beta)$, then β equals: **(2019)**
    (a) $\frac{35}{3}$ (b) -5
    (c) $-\frac{35}{3}$ (d) 5

12. If a straight line passing through the point $P(-3, 4)$ is such that its intercepted portion between the coordinate axes is bisected at P, then its equation is: **(2019)**
    (a) $3x - 4y + 25 = 0$ (b) $4x - 3y + 24 = 0$
    (c) $x - y + 7 = 0$ (d) $4x + 3y = 0$

13. A point on the straight line, $3x + 5y = 15$ which is equidistant from the coordiante axes will lie only in: **(2019)**
    (a) 1st and 2nd quadrants
    (b) 4th quadrant
    (c) 1st, 2nd and 4th quadrant
    (d) 1st quadrant

14. Let $O(0, 0)$ and $A(0, 1)$ be two fixed points. Then the locus of a point P such that the perimeter of ΔAOP is 4, is : **(2019)**
    (a) $8x^2 - 9y^2 + 9y = 18$ (b) $9x^2 + 8y^2 - 8y = 16$
    (c) $8x^2 + 9y^2 - 9y = 18$ (d) $9x^2 - 8y^2 + 8y = 16$

15. Suppose that the points (h, k), $(1, 2)$ and $(-3, 4)$ lie on the line L_1. If a line L_2 passing through the points (h, k) and $(4, 3)$ is perpendicular to L_1, then $\frac{k}{h}$ equals : **(2019)**
    (a) 3 (b) $-\frac{1}{7}$ (c) $\frac{1}{3}$ (d) 0

16. Slope of a line passing through $P(2, 3)$ and intersecting the line, $x + y = 7$ at a distance of 4 units from P, is **(2019)**

(a) $\frac{\sqrt{5}-1}{\sqrt{5}+1}$ (b) $\frac{1-\sqrt{5}}{1+\sqrt{5}}$

(c) $\frac{1-\sqrt{7}}{1+\sqrt{7}}$ (d) $\frac{\sqrt{7}-1}{\sqrt{7}+1}$

17. If the two lines $x + (a - 1)y = 1$ and $2x + a^2y = 1$ $(a \in R - \{0, 1\})$ are perpendicular, then the distance of their point of intersection from the origin is: **(2019)**

(a) $\frac{2}{5}$ (b) $\frac{2}{\sqrt{5}}$

(c) $\frac{\sqrt{2}}{5}$ (d) $\sqrt{\frac{2}{5}}$

18. A rectangle is inscribed in a circle with a diameter lying along the line $3y = x + 7$. If the two adjacent vertices of the rectangle are $(-8, 5)$ and $(6, 5)$, then the area of the rectangle (in sq. units) is: **(2019)**

(a) 72 (b) 84 (c) 98 (d) 56

19. Lines are drawn parallel to the line $4x - 3y + 2 = 0$, at a distance 3/5 from the origin. Then which one of the following points lies on any of these lines ? **(2019)**

(a) $\left(-\frac{1}{4}, \frac{2}{3}\right)$ (b) $\left(\frac{1}{4}, \frac{1}{3}\right)$

(c) $\left(-\frac{1}{4}, -\frac{2}{3}\right)$ (d) $\left(\frac{1}{4}, -\frac{1}{3}\right)$

20. The region represented by $|x - y| \le 2$ and $|x + y| \le 2$ is bounded by a; **(2019)**

(a) square of side length $2\sqrt{2}$ units

(b) rhombus of side length 2 units

(c) square of area 16 sq. units

(d) rhombus of area $8\sqrt{2}$ sq. units

21. A straight line L at a distance of 4 units from the origin makes positive intercepts on the coordinate axes and the perpendicular from the origin to this line makes an angle of 60° with the line $x + y = 0$. Then an equation of the line L is: **(2019)**

(a) $(\sqrt{3}+1)x + (\sqrt{3}-1)y = 8\sqrt{2}$

(b) $(\sqrt{3}-1)x + (\sqrt{3}+1)y = 8\sqrt{2}$

(c) $\sqrt{3}x + y = 8$

(d) $x + \sqrt{3}y = 8$

22. A triangle has a vertex at $(1, 2)$ and the mid points of the two sides through it are $(-1, 1)$ and $(2, 3)$. Then the centroid of this triangle is: **(2019)**

(a) $\left(\frac{1}{3}, 1\right)$ (b) $\left(\frac{1}{3}, 2\right)$

(c) $\left(1, \frac{7}{3}\right)$ (d) $\left(\frac{1}{3}, \frac{5}{3}\right)$

23. Let $A(1, 0)$, $B(6, 2)$ and $C\left(\frac{3}{2}, 6\right)$ be the vertices of triangle ABC. If P is a point inside the triangle ABC su[ch] that the triangles APC, APB and BPC have equal area[s], then the length of the line segment PQ, where Q is the po[int] $\left(\frac{-7}{6}, \frac{-1}{3}\right)$, is ______. (202

24. The locus of the mid-point of the perpendiculars drawn fro[m] points on the line, $x = 2y$ to the line $x = y$ is: (202

(a) $3x - 2y = 0$ (b) $3x - 3y = 0$

(c) $5x - 7y = 0$ (d) $7x - 5y = 0$

25. Let two points be $A(1, -1)$ and $B(0, 2)$. If a poi[nt] $P(x', y')$ be such that the area of $\Delta PAB = 5$ square units a[nd] it lies on the line, $3x + y - 4\lambda = 0$, then a value of λ is (202

(a) 3 (b) -3

(c) 1 (d) 4

26. Let C be the centroid of the triangle with vertic[es] $(3, -1)$, $(1, 3)$ and $(2, 4)$. Let P be the point of intersecti[on] of the lines $x + 3y - 1 = 0$ and $3x - y + 1 = 0$. Then the li[ne] passing through the points C and P also passes through th[e] point (202

(a) $(7, 6)$ (b) $(-9, -7)$

(c) $(9, 7)$ (d) $(-9, -6)$

27. The set of all possible values of θ in the interval $(0, \pi)$ f[or] which the points $(1, 2)$ and $(\sin\theta, \cos\theta)$ lie on the same si[de] of the line $x + y = 1$ is: (202

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

(c) $\left(0, \frac{\pi}{4}\right)$ (d) $\left(0, \frac{3\pi}{4}\right)$

28. If a ΔABC has vertices $A(-1, 7)$, $B(-7, 1)$ and $C(5, -$ then its orthocentre has coordinates. (202

(a) $(-3, 3)$ (b) $\left(-\frac{3}{5}, \frac{3}{5}\right)$

(c) $(3, -3)$ (d) $\left(-\frac{3}{5}, \frac{3}{5}\right)$

29. If the perpendicular bisector of the line segment joini[ng] the points $P(1, 4)$ and $Q(k, 3)$ has y-intercept equal [to] -4, then a value of k is: (202

(a) $\sqrt{14}$ (b) $\sqrt{15}$

(c) -4 (d) -2

30. A triangle ABC lying in the first quadrant has two vertices $A(1, 2)$ and $B(3, 1)$. If $\angle BAC = 90°$, and ar$(\Delta ABC) = 5\sqrt{}$ sq. units, then the abscissa of the vertex C is : (202

(a) $1 + \sqrt{5}$ (b) $1 + 2\sqrt{5}$

(c) $2\sqrt{5} - 1$ (d) $2 + \sqrt{5}$

31. If the line, $2x - y + 3 = 0$ is at a distance $\frac{1}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$ from the lines $4x - 2y + \alpha = 0$ and $6x - 3y + \beta = 0$, respectively, then the sum of all possible values of α and β is ______. **(2020)**

32. A ray of light coming from the point $(2, 2\sqrt{3})$ is incident at an angle 30° on the line $x = 1$ at the point A. The ray gets reflected on the line $x = 1$ and meets x-axis at the point B. Then, the line AB passes through the point : **(2020)**

(a) $\left(3, -\frac{1}{\sqrt{3}}\right)$ (b) $(3, -\sqrt{3})$

(c) $(4, -\sqrt{3})$ (d) $\left(4, -\frac{\sqrt{3}}{2}\right)$

33. Consider a triangle having vertices $A(-2, 3)$, $B(1, 9)$ and $C(3, 8)$. If a line L passing through the circumcentre of triangle ABC, bisects line BC, and intersects y-axis at point $\left(0, \frac{\alpha}{2}\right)$, then the value of real number α is **(2021)**

34. Let $\tan\alpha, \tan\beta$ and $\tan\gamma, \alpha, \beta, \gamma \neq \frac{(2n-1)\pi}{2}, n \in \mathbb{N}$ be the slopes of three line segments OA, OB and OC, respectively, where O is origin. If circumcentre of ΔABC coincides with origin and its orthocentre lies on y axis, then the value of $\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha \cos\beta \cos\gamma}\right)^2$ is equal to **(2021)**

35. Let $A(-1, 1)$, $B(3, 4)$ and $C(2, 0)$ be given three a points. A line $y = mx$, $m > 0$, intersects line AC and BC at a point P and Q respectively. Let A_1 and A_2 be the areas of ΔABC and ΔPQC respectively, such that $A_1 = 3A_2$, then the value of m is equal to **(2021)**

(a) $\frac{4}{15}$ (b) 1

(c) 2 (d) 3

36. In a triangle PQR, the co-ordinates of the point P and Q are $(-2, 4)$ and $(4, -2)$ respectively. If the equation of th perpendicular bisector PR is $2x - y + 2 = 0$, then the centre of the circumcircle of the 'PQR' is **(2021)**

(a) $(-1, 0)$ (b) $(-2, -2)$

(c) $(0, 2)$ (d) $(1, 4)$

37. Let the tangent to the circle $x^2 + y^2 = 25$ at the point $P(3, 4)$ meet x-axis and y-axis at point P and Q, respectively. If r is the radius of the circle passing through the origin O and having centre at the incentre of the triangle OPQ, then r^2 is equal to **(2021)**

(a) $\frac{529}{64}$ (b) $\frac{125}{72}$

(c) $\frac{625}{72}$ (d) $\frac{585}{66}$

38. The number of integer values of m so that the abscissa of point of intersection of lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer, is **(2021)**

(a) 1 (b) 2

(c) 3 (d) 0

39. The equation of one of the straight lines which passes through the point $(1, 3)$ and makes an angles $\tan^{-1}(\sqrt{2})$ with the straight line $y + 1 = \sqrt{2}x$ **(2021)**

(a) $4\sqrt{2}x + 5y - (15 + 4\sqrt{2}) = 0$

(b) $5\sqrt{2}x + 4y - (15 + 4\sqrt{2}) = 0$

(c) $4\sqrt{2}x + 5y - 4\sqrt{2} = 0$

(d) $4\sqrt{2}x - 5y - (5 + 4\sqrt{2}) = 0$

40. Let the centroid of an equilateral triangle ABC be at the origin. Let one of the sides of the equilateral triangle be along the straight line $x + y = 3$. If R and r be the radius of circumcircle and incircle respectively of ΔABC, then $(R + r)$ is equal to **(2021)**

(a) $\frac{9}{\sqrt{2}}$ (b) $7\sqrt{2}$

(c) $2\sqrt{2}$ (d) $3\sqrt{2}$

41. A man is walking on a straight line. The arithmetic mean of the reciprocals of the intercepts of this line on the coordinate axis is $\frac{1}{4}$. Three stones, A, B and C are placed at the point $(1, 1)$, $(2, 2)$ and $(4, 4)$ respectively. Then which of these stones is/are on the path of the man **(2021)**

(a) B only (b) A only

(c) All of three (d) C only

42. The image of tthe point $(3, 5)$ in the line $x - y + 1 = 0$, lies on **(2021)**

(a) $(x - 2)^2 + (y - 4)^2 = 4$ (b) $(x - 4)^2 + (y + 2)^2 = 16$

(c) $(x - 4)^2 + (y - 4)^2 = 8$ (d) $(x - 2)^2 + (y - 2)^2 = 12$

43. The intersection of three lines $x - y = 0$, $x + 2y = 3$ and $2x + y = 6$ is a **(2021)**

(a) Equilateral triangle (b) Right angled triangle

(c) Isosceles triangle (d) None of these

44. If the locus of the mid-point of the line segment from the point $(3, 2)$ to a point on a circle, $x^2 + y^2 = 1$ is a circle of the radius r, then r is equal to **(2021)**

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$

(c) 1 (d) $\frac{1}{3}$

45. Let m_1, m_2 be the slopes of two adjacent sides of a square of side a such that $a^2 + 11a + 3(m_1^2 + m_2^2) = 220$. If one vertex of the square is $(10(\cos\alpha - \sin\alpha), 10(\sin\alpha + \cos\alpha))$, where $\alpha \in \left(0, \frac{\pi}{2}\right)$ and the equation of one diagonal is $(\cos\alpha - \sin\alpha)x + (\sin\alpha + \cos\alpha)y = 10$, then $72(\sin^4\alpha + \cos^4\alpha) + a^2 - 3a + 13$ is equal to : **(2022)**

(a) 119 (b) 128 (c) 145 (d) 155

46. Let $A(\alpha, -2)$, $B(\alpha, 6)$ and $C(\alpha/4, -2)$ be vertices of a ΔABC. If $(5, \alpha/4)$ is the circumcentre of ΔABC, then which of the following is NOT correct about ΔABC? **(2022)**

(a) area is 24 (b) perimeter is 25

(c) circumradius is 5 (d) inradius is 2

47. A point P moves so that the sum of squares of its distances from the points $(1, 2)$ and $(-2, 1)$ is 14. Let $f(x, y) = 0$ be the locus of P, which intersects the x-axis at the points A, B and the y-axis at the points C, D. Then the area of the quadrilateral $ACBD$ is equal to: **(2022)**

(a) $\frac{9}{2}$ (b) $\frac{3\sqrt{17}}{2}$

(c) $\frac{3\sqrt{17}}{4}$ (d) 9

48. The equations of the sides AB, BC and CA of a triangle ABC are $2x + y = 0$, $x + py = 39$ and $x - y = 3$ respectively and $P(2, 3)$ is its circumcentre. Then which of the following is NOT true? **(2022)**

(a) $(AC)^2 = 9p$

(b) $(AC)^2 + p^2 = 136$

(c) $32 <$ area $(\Delta ABC) < 36$

(d) $34 <$ area $(\Delta ABC) < 38$

49. Let $A(1, 1)$, $B(-4, 3)$, $C(-2, -5)$ be vertices of a triangle ABC, P be a point on side BC, and Δ_1 and Δ_2 be the areas of triangles APB and ABC, respectively. If $\Delta_1 : \Delta_2 = 4 : 7$, then the area enclosed by the lines AP, AC and the x-axis is **(2022)**

(a) $\frac{1}{4}$ (b) $\frac{3}{4}$

(c) $\frac{1}{2}$ (d) 1

50. A line, with the slope greater than one, passes through the point $A(4, 3)$ and intersects the line $x - y - 2 = 0$ at the point B. If the length of the line segment AB is $\frac{\sqrt{29}}{3}$, then B also lies on the line: **(2022)**

(a) $2x + y = 9$ (b) $3x - 2y = 7$

(c) $x + 2y = 6$ (d) $2x - 3y = 3$

51. The equations of the sides AB, BC and CA of a triangle ABC are $2x + y = 0$, $x + py = 15a$ and $x - y = 3$ respectively. If its orthocentre is $(2, a)$, $-\frac{1}{2} < a < 2$, then p is equal to **(2022)**

52. Let α_1, α_2 $(\alpha_1 < \alpha_2)$ be the values of α of the points $(\alpha, -3)$, $(2, 0)$ and $(1, \alpha)$ to be collinear. Then the equation of the line, passing through (α_1, α_2) and making an angle of $\frac{\pi}{3}$ with the positive direction of the x-axis, is : **(2022)**

(a) $x - \sqrt{3}y - 3\sqrt{3} + 1 = 0$

(b) $\sqrt{3}x - y + \sqrt{3} + 3 = 0$

(c) $x - \sqrt{3}y + 3\sqrt{3} + 1 = 0$

(d) $\sqrt{3}x - y + \sqrt{3} - 3 = 0$

53. The distance of the origin from the centroid of the triangle whose two sides have the equations $x - 2y + 1 = 0$ and $2x - y - 1 = 0$ and whose orthocenter is $\left(\frac{7}{3}, \frac{7}{3}\right)$ is: **(2022)**

(a) $\sqrt{2}$ (b) 2 (c) $2\sqrt{2}$ (d) 4

54. The distance between the two points A and A' which lie on $y = 2$ such that both the line segments AB and $A'B$ (where B is the point $(2, 3)$) subtend angle $\frac{\pi}{4}$ at the origin, is equal to: **(2022)**

(a) 10 (b) $\frac{48}{5}$ (c) $\frac{52}{5}$ (d) 3

55. Let a triangle be bounded by the lines $L_1 : 2x + 5y = 10$; $L_2 : -4x + 3y = 12$ and the line L_3, which passes through the point $P(2, 3)$, intersects L_2 at A and L_1 at B. If the point P divides the line-segment AB, internally in the ratio $1 : 3$, then the area of the triangle is equal to : **(2022)**

(a) $\frac{110}{13}$ (b) $\frac{132}{13}$

(c) $\frac{142}{13}$ (d) $\frac{151}{13}$

56. If two straight lines whose direction cosines are given by the relations $l + m + n = 0$, $3l^2 + m^2 + cnl = 0$ are parallel, then the positive value of c is: **(2022)**

(a) 6 (b) 4 (c) 3 (d) 2

57. Let R be the point $(3, 7)$ and let P and Q be two points on the line $x + y = 5$ such that PQR is an equilateral triangle. Then the area of ΔPQR is: **(2022)**

(a) $\frac{25}{4\sqrt{3}}$ (b) $\frac{25\sqrt{3}}{2}$

(c) $\frac{25}{\sqrt{3}}$ (d) $\frac{25}{2\sqrt{3}}$

58. Let the area of the triangle with vertices $A(1, \alpha)$, $B(\alpha, 0)$ and $C(0, \alpha)$ be 4 sq. units. If the points $(\alpha - \alpha)$, $(-\alpha, \alpha)$ and (α^2, β) are collinear, then β is equal to **(2022)**

(a) 64 (b) -8

(c) -64 (d) 512

59. A ray of light passing through the point $P(2, 3)$ reflects on the x-axis at point A and the reflected ray passes through the point $Q(5, 4)$. Let R be the point that divides the line segment AQ internally into the ratio $2 : 1$. Let the co-ordinates of the foot of the perpendicular M from R on the bisector of the angle PAQ be (α, β). Then, the value of $7\alpha + 3\beta$ is equal to **(2022)**

60. Let $A\left(\frac{3}{\sqrt{a}}, \sqrt{a}\right)$, $a > 0$, be a fixed point in the xy-plane. The image of A in y-axis be B and the image of B in x-axis be C. If $D(3\cos\theta, a\sin\theta)$ is a point in the fourth quadrant such that the maximum area of ΔACD is 12 square units, then a is equal to **(2022)**

61. Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv lx + my + n = 0$ intersect at the point P and makes an angle θ with each other. Find the equation of a line L different from L_2 which passes through P and makes the same angle θ with L_1. **(1988)**

62. Let ABC be a triangle with $AB = AC$. If D is mid point of BC, the foot of the perpendicular drawn from D to AC and F the mid-point of DE. Prove that AF is perpendicular to BE. **(1989)**

63. A line cuts the X-axis at $A(7, 0)$ and the Y-axis at $B(0, -5)$. A variable line PQ is drawn perpendicular to AB cutting the X-axis in P and the Y-axis in Q. If AQ and BP intersect at R, find the locus of R. **(1990)**

64. For $a > b > c > 0$, the distance between $(1, 1)$ and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$. Then **(2013)**
    (a) $a+b-c>0$
    (b) $a-b+c<0$
    (c) $a-b+c>0$
    (d) $a+b-c<0$

65. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distance of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \le d_1(P) + d_2(P) \le 4$, is **(2014)**

66. Let $a, \lambda, \mu \in \mathbb{R}$. Consider the system of linear equations **(2016)**

$$ax + 2y = \lambda$$
$$3x + 2y = \mu$$

Which of the following statement(s) is(are) correct?
    (a) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ.
    (b) If $a \ne -3$, then the system has a unique solutions for all values of λ and μ.
    (c) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$
    (d) If $\lambda + \mu \ne 0$, then the system has no solution for $a = -3$

Question Stem for Question Nos. 67 and 68

Consider the lines L_1 and L_2 defined by
$L_1; x\sqrt{2} + y - 1 = 0$ and $L_2; x\sqrt{2} - y + 1 = 0$

For a fixed constant λ, let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2. The line $y = 2x + 1$ meets C at two points R and S, where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R' and S'. Let D be the square of the distance between R' and S'

67. The value of λ^2 is________ .

68. The value of D is________ .

ANSWER KEY

CONCEPT APPLICATION

2. (*c*) **3.** $[(i)\ (x, y) = (2, 7), (ii)\ (\bar{x}, \bar{y}) = (-52, -11)]$ **4.** [1 : 2] **5.** [5 : 9] **6.** $\left[\left(\frac{5}{2}, \frac{5}{2}\right)\right]$

7. [(2/3, 4)] **8.** [1/2] **10.** $[x = \pm 2]$ **11.** $[2x - y - 3 = 0]$ **12.** $[2x + 3y + 3 = 0]$ **13.** $\left[\left(\frac{3}{4}, -2\right)\right]$

15. $[2x^2 + y^2 - 1 = 0]$ **16.** $[3\pi / 4]$ **17.** [–3] **18.** (*b*) **19.** $\left[x\sqrt{3} + y = 10\right]$ **20.** (*a*) **21.** (*b*)

22. (square) **23.** $[x - 3y + 13 = 0]$ **24.** $\left[\tan^{-1}\frac{2}{3}\right]$ **25.** $\left[\left(\frac{6x+8y-10}{10}\right) - \left(\frac{6x+8y-45}{10}\right) = \frac{35}{10} = \frac{7}{2}\right]$ **26.** $[5\sqrt{2}$

27. $\left[\left(\frac{\Sigma x_i}{n}, \frac{\Sigma y_i}{n}\right)\right]$ **28.** $[2x - y + 6 = 0, 2x - y - 14 = 0]$ **29.** $\left[\left(\frac{53}{25}, \frac{71}{25}\right), \left(\frac{56}{25}, \frac{67}{25}\right)\right]$ **30.** $\left[\sqrt{3}x + y = 0\right]$

31. $[x - 7y + 2 = 0]$ **32.** $[2y - 29x + 31 = 0]$ **33.** $\left[\left(\frac{-11}{29}, \frac{2}{29}\right)\right]$ **34.** $[-8x + 4y + 28 = 0, 2x - y - 7 = 0]$

35. [(–22, 28)] **36.** (A → (r); B → (p); C → (s); D → (q)) **38.** [π/4] **40.** (*a*) **41.** [8/3, 8/3]

42. $[5x^2 + 8xy + 3y^2 - 44x - 34y + 95 = 0]$ **43.** $\left[\frac{-9 \pm \sqrt{33}}{8}\right]$ **44.** (*a*)

EXERCISE-1 (TOPICWISE)

1. (*a*)	**2.** (*b*)	**3.** (*a*)	**4.** (*a*)	**5.** (*c*)	**6.** (*a*)	**7.** (*b*)	**8.** (*a*)	**9.** (*c*)	**10.** (*b*)
11. (*b*)	**12.** (*c*)	**13.** (*b*)	**14.** (*b*)	**15.** (*a*)	**16.** (*a*)	**17.** (*a*)	**18.** (*a*)	**19.** (*d*)	**20.** (*a*)
21. (*b*)	**22.** (*c*)	**23.** (*a*)	**24.** (*d*)	**25.** (*c*)	**26.** (*d*)	**27.** (*c*)	**28.** (*d*)	**29.** (*d*)	**30.** (*b*)
31. (*b*)	**32.** (*b*)	**33.** (*a*)	**34.** (*a*)	**35.** (*c*)	**36.** (*a*)	**37.** (*a*)	**38.** (*b*)	**39.** (*a*)	**40.** (*a*)
41. (*b*)	**42.** (*b*)	**43.** (*b*)	**44.** (*b*)	**45.** (*c*)	**46.** (*b*)	**47.** (*c*)	**48.** (*a*)	**49.** (*d*)	**50.** (*d*)
51. (*a*)	**52.** (*b*)	**53.** (*a*)	**54.** (*a*)	**55.** (*a*)	**56.** (*b*)	**57.** (*a*)	**58.** (*c*)	**59.** (*a*)	**60.** (*d*)
61. (*c*)	**62.** (*d*)	**63.** (*b*)	**64.** (*a*)	**65.** (*a*)	**66.** (*d*)	**67.** (*a*)	**68.** (*a*)	**69.** (*c*)	**70.** (*a*)
71. (*a*)	**72.** (*a*)	**73.** (*c*)	**74.** (*d*)						

EXERCISE-2 (LEARNING PLUS)

1. (*a*)	**2.** (*a*)	**3.** (*a*)	**4.** (*a*)	**5.** (*d*)	**6.** (*b*)	**7.** (*c*)	**8.** (*b*)	**9.** (*d*)	**10.** (*c*)
11. (*c*)	**12.** (*a*)	**13.** (*c*)	**14.** (*a*)	**15.** (*c*)	**16.** (*c*)	**17.** (*b*)	**18.** (*d*)	**19.** (*c*)	**20.** (*b*)
21. (*c*)	**22.** (*c*)	**23.** (*d*)	**24.** (*c*)	**25.** (*a*)	**26.** (*b*)	**27.** (*b*)	**28.** (*c*)	**29.** (*d*)	**30.** (*d*)
31. (*c*)	**32.** (*c*)	**33.** (*d*)	**34.** (*a*)	**35.** (*b*)	**36.** (*c*)	**37.** (*d*)	**38.** (*c*)	**39.** (*b*)	**40.** (*a*)
41. (*a*)	**42.** (*b*)	**43.** (*d*)	**44.** (*b*)	**45.** (*a*)	**46.** (*d*)	**47.** (*a,c,d*)	**48.** (*c,d*)	**49.** (*b,c*)	**50.** (*a,c*)
51. (*a,b*)	**52.** (*a,b*)	**53.** (*a*)	**54.** [100]	**55.** [23]	**56.** [9]	**57.** [1:2 ; *Q* (–5, –3)]		**58.** $[\frac{4}{3}(2+\sqrt{3})]$	

59. [42/25, 91/25] **60.** $x - 3y - 1 = 0$; $x - 3y - 21 = 0$; $3x + y - 2 = 0$; $3x + y - 12 = 0$ **61.** $[3x + 4y - 24 = 0]$

62. $[B\left(\frac{-2t}{3}, \frac{-t}{6}\right); C\left(\frac{t}{2}, t\right)]$ **64.** [π/3] **65.** [9:8] **66.** $[3x + y + 7 = 0$ or $x - 3y - 31 = 0]$

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (*c*)	**2.** (*a*)	**3.** (*b*)	**4.** (*d*)	**5.** (*a*)	**6.** (*d*)	**7.** (*a*)	**8.** (*c,d*)	**9.** (*a,b*)	**10.** (*a,b,c*)
11. (*a*)	**12.** (*b*)	**13.** (*c*)	**14.** (*d*)	**15.** (*a*)	**16.** (*c*)	**17.** (*b*)	**18.** (*d*)	**19.** (*a*)	**20.** (*b*)
21. (*d*)	**22.** (*a*)	**23.** (*d*)	**24.** (*b*)	**25.** (*a*)	**26.** [2]	**27.** [19]	**28.** [2]	**29.** [2]	**30.** [5]

31. [2] **34.** [$x - y = 1$; $3x - y + 3 = 0$] **36.** [$2x + 3y + 22 = 0$] **38.** [$4 + 2\sqrt{2}$]

39. [(i) isosceles, (ii) $3x + 6y = 16$, (iii) $8x + 8y + 7 = 0$, (iv) acute] **40.** [$14x + 23y = 40$]

42. [$\Delta' = \frac{1}{2}\left(\frac{(c_2 - c_1)^2}{m_1 - m_2} + \frac{(c_3 - c_2)^2}{m_2 - m_3} + \frac{(c_3 - c_1)^2}{m_3 - m_1}\right)$] **43.** $\left|\frac{1}{2}\left[m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2)\right]\right|$

47. [$(a + b)(ax + by) = ab(a + b - 2h)$] **49.** [(1, 9)] **51.** [$x = -a$; $y = a(t_1 + t_2 + t_3 + t_1t_2t_3)$]

52. [$x - 7y + 13 = 0$; $7x + y - 9 = 0$] **53.** [$4(x^2 - y^2) + (4g + 5f)x + (4f - 5g) = 0$] **55.** [$\pi/4$]

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (*c*)	**2.** (*a*)	**3.** (*b*)	**4.** (*a*)	**5.** (*d*)	**6.** (*c*)	**7.** (*b*)	**8.** (*d*)	**9.** (*b*)	**10.** (*a*)
11. (*d*)	**12.** (*b*)	**13.** (*a*)	**14.** (*b*)	**15.** (*c*)	**16.** (*c*)	**17.** (*d*)	**18.** (*b*)	**19.** (*a*)	**20.** (*a*)
21. (*a*)	**22.** (*b*)	**23.** [5]	**24.** (*c*)	**25.** (*a*)	**26.** (*d*)	**27.** (*a*)	**28.** (*a*)	**29.** (*c*)	**30.** (*b*)
31. [30]	**32.** (*b*)	**33.** [9]	**34.** [144]	**35.** (*b*)	**36.** (*b*)	**37.** (*c*)	**38.** (*b*)	**39.** (*a*)	**40.** (*a*)
41. (*a*)	**42.** (*a*)	**43.** (*c*)	**44.** (*b*)	**45.** (*b*)	**46.** (*b*)	**47.** (*b*)	**48.** (*b*)	**49.** (*c*)	**50.** (*c*)
51. (*c*)	**52.** (*b*)	**53.** (*c*)	**54.** (*c*)	**55.** (*b*)	**56.** (*a*)	**57.** (*d*)	**58.** (*c*)	**59.** [31]	**60.** [8]

JEE Advanced

61. [$2(al + bm)(ax + by + c) - (a^2 + b^2)(lx + my + n) = 0$] **63.** [$x^2 + y^2 - 7x + 5y = 0$] **64.** (*a,c*) **65.** [6]

66. (*b,c,d*)

CHAPTER

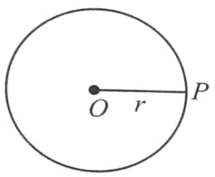
14 Circle

INTRODUCTION

A circle is a round-shaped figure that has no corners or edges. The word circle is derived from the Greek word kirkos, meaning hoop or ring. You could think of a circle as a hula hoop. It is a 2-D geometrical curve & one of the conic sections.

DEFINITION

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point remains constant. The fixed point is called the centre and the constant distance is called the radius of the circle. The given figure consists of a circle with centre O and radius equal to r units.



BASIC THEOREMS AND RESULTS OF CIRCLES

Theorem 1:

(*i*) If two arcs of a circle (or of congruent circles) are congruent, the corresponding chords are equal and vice versa also true.

(*ii*) Equal chords of a circle (or of congruent circles) subtend equal angles at the centre and vice versa also true.

Theorem 2:

(*i*) The perpendicular from the centre of a circle to a chord bisects the chord and vice versa also true.

(*ii*) Perpendicular bisectors of two chords of a circle intersect at its centre.

Theorem 3:

(*i*) There is one and only one circle passing through three non collinear points.

(*ii*) If two circles intersects in two points, then the line joining the centres is perpendicular bisector of common chords.

Theorem 4:

(*i*) Equal chords of a circle (or of congruent circles) are equidistant from the centre and vice versa also true.

(*ii*) If two equal chords are drawn from a point on the circle, then the centre of circle will lie on angle bisector of these two chords.

(*iii*) Of any two chords of a circle larger will be near to centre.

Theorem 5:

(*i*) The degree measure of an arc or angle subtended by an arc at the centre is double the angle subtended by it at any point of alternate segment.

(*ii*) Angle in the same segment of a circle are equal.

(*iii*) The angle in a semi circle is a right angle.

Converse: The arc of a circle subtending a right angle in alternate segment is semi circle.

Theorem 6: An angle subtended by a minor arc in the alternate segment is acute and any angle subtended by a major arc in the alternate segment is obtuse.

Theorem 7: If a line segment joining two points subtends equal angles at two other points lying on the same side of the line segment, the four points are concyclic, i.e. lie on the same circle.

Cyclic Quadrilaterals: A quadrilateral is called a cyclic quadrilateral if its all vertices lie on a circle.

Theorem 1: The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.

OR

The opposite angles of a cyclic quadrilateral are supplementary.

Converse: If the sum of any pair of opposite angle of a quadrilateral is 180°, then the quadrilateral is cyclic.

Theorem 2: If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle.

Theorem 3: The quadrilateral formed by angle bisectors of a cyclic quadrilateral is also cyclic.

Theorem 4: If two sides of cyclic quadrilateral are parallel then the remaining two sides are equal and the diagonals are also equal.

OR

A cyclic trapezium is isosceles and its diagonals are equal.

Converse: If two non-parallel sides of a trapezium are equal then it is cyclic.

OR

An isosceles trapezium is always cyclic.

Theorem 5: The bisectors of the angles formed by producing the opposite sides of a cyclic quadrilateral (provided they are not parallel), intersect at right angle.

TANGENTS TO A CIRCLE

Theorem 1: A tangent to a circle is perpendicular to the radius through the point of contact.

Converse: A line drawn through the end point of a radius and perpendicular to it is a tangent to the circle.

Theorem 2: If two tangents are drawn to a circle from an external point, then:

(*i*) they are equal

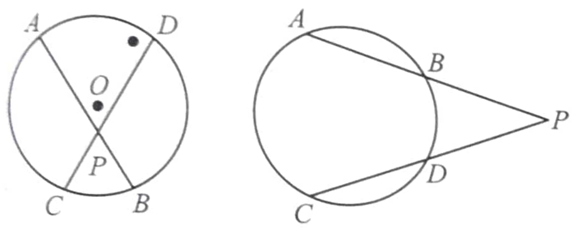
(*ii*) the subtend equal angles at the centre,

(*iii*) they are equally inclined to the segment, joining the centre to that point.

Theorem 3: (Intersecting Chord Theorem.)

If two chords of a circle intersect inside or outside the circle when produced, the rectangle formed by the segments of one chord is equal in area to the rectangle formed by the two segments of the other chord

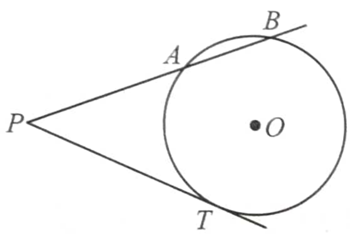
$PA \times PB = PC \times PD$



Theorem 4: (Tangent Secant Theorem.)

If PAB is a secant to a circle intersecting the circle at A and B and PT is tangent segment, the $PA \times PB = PT^2$

OR

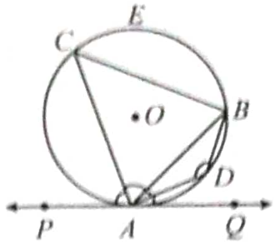


Area of the rectangle formed by the two segments of a chord is equal to the area of the square of side equal to the length of the tangent from the point on the circle.

Theorem 5: (Alternate Segment Theorem.)

If a chord is drawn through the point of contact of tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.

$\angle BAQ = \angle ACB$ and $\angle BAP = \angle ADB$

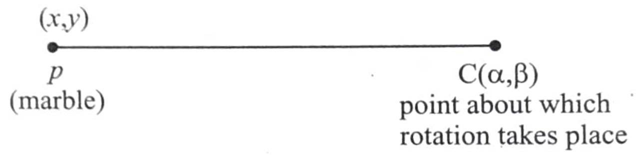


Converse: If a line is drawn through an end point of a chord of a circle so that the angle formed with the chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

EQUATION OF CIRCLE IN VARIOUS FORMS

From our day-to-day observation we know that a marble tied to the end of a string and rotated by 360° describes a circular path. Looking at the above observation mathematically, we assign coordinates to the marble and the point about which it is rotated as (x, y) and (α, β) respectively.

Diagrammatically,



Taking a constant length of the string, r, we have by distance formula,

$$\sqrt{(x-\alpha)^2+(y-\beta)^2} = r$$

$$\Rightarrow \quad (x-\alpha)^2+(y-\beta)^2 = r^2 \qquad \dots (i)$$

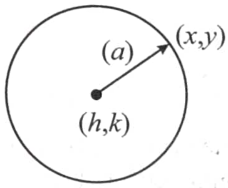
Equation (*i*) gives us the required equation of the circle with centre (α, β) and radius r.

A point moving in a plane such that its distance from a fixed point is always constant describes a circle. The fixed point is called centre and the constant distance is called radius.

We present below the equation of circle in various forms:

(*i*) **Circle with Centre at the point (h, k) and radius a:** The equation of the circle is $(x-h)^2+(y-k)^2 = a^2$.

Particular Case: The equation of a circle with centre at the origin and radius a is $x^2 + y^2 = a^2$.



Train Your Brain

Example 1: If the lines $x + y = 6$ and $x + 2y = 4$ are diameters of the circle which passes through the point (2, 6), then find its equation.

Sol. Here centre will be the point of intersection of the diameter,
i.e., C(8, –2).
Also, the circle passes through the point P(2, 6). Then radius is CP 10.
Hence, the required equation is $(x - 8)^2 + (y + 2)^2 = 10^2$
$= x^2 + y^2 - 16x + 4y - 32 = 0$

Example 2: If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then the radius of the circle is

(*a*) 3/2 (*b*) 3/4
(*c*) 1/10 (*d*) 1/20

Sol. The diameter of the circle is perpendicular distance between the parallel lines (tangents)

$3x - 4y + 4 = 0$ and $3x - 4y - \frac{7}{2} = 0$ and so it is equal to $\frac{4+7/2}{\sqrt{9+16}} = \frac{3}{2}$.

Hence radius is $\frac{3}{4}$.

Example 3: If $y = 2x + m$ is a diameter to the circle $x^2 + y^2 + 3x + 4y - 1 = 0$, then find m

Sol. Centre of circle = (–3/2 , –2). This lies on diameter $y = 2x + m$
$\Rightarrow -2 = (-3/2) \times 2 + m \quad \Rightarrow \quad m = 1$

Example 4: Two straight lines rotate about two fixed points. If they start from their position of coincidence such that one rotates at the rate double that of the other. Prove that the locus of their point of intersection is a circle.

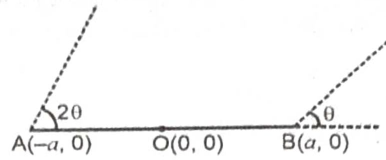
Sol. Let A $(-a, 0)$ and B $(a, 0)$ be two fixed points.
Let one line which rotates about B an angle θ with the x-axis at any time t and at that time the second line which rotates about A make an angle 2θ with x-axis.
Now equation of line through B and A are respectively

$y - 0 = \tan\theta\,(x - a)$ ... (*i*)
and $y - 0 = \tan 2\theta\,(x + a)$ ...(*ii*)

From (*ii*), $y = \frac{2\tan\theta}{1-\tan^2\theta}(x+a)$

$$= \left\{\frac{\frac{2y}{(x-a)}}{1-\frac{y^2}{(x-a)^2}}\right\}(x+a)$$

(from (*i*))



$$\Rightarrow \quad y = \frac{2y(x-a)(x+a)}{(x-a)^2 - y^2}$$

$\Rightarrow \quad (x - a)^2 - y^2 = 2(x^2 - a^2)$
or $\quad x^2 + y^2 + 2ax - 3a^2 = 0$ which is the required locus.

Example 5. Find the centre and radius of the circle
(*a*) $4x^2 + 4y^2 - 4x - 5y - 1 = 0$
(*b*) $2x^2 + 2y^2 + \lambda xy + (\lambda - 2)\,x + (\lambda + 8)\,y - 8 = 0$

Sol. (*a*) We can write the given equation of circle as

$$x^2 + y^2 - x - \frac{5y}{4} - \frac{1}{4} = 0$$

On compairing with standard equation of circle

$$g = -\frac{1}{2}, f = -\frac{5}{8}, c = -\frac{1}{4}$$

Centre is $(-g, -f) \Rightarrow \left(\frac{1}{2}, \frac{5}{8}\right)$

radius is $\sqrt{g^2 + f^2 - c}$

$$\Rightarrow \quad \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{-5}{8}\right)^2 + \frac{1}{4}} = \frac{\sqrt{57}}{8}$$

(*b*) We can write the given equation as

$$x^2 + y^2 + \frac{\lambda}{2}xy + \left(\frac{\lambda-2}{2}\right)x + \left(\frac{\lambda+8}{2}\right)y - 4 = 0$$

Since there is no term of xy in the equation of circle $\Rightarrow \lambda = 0$
By substituting the value of λ, the given equation reduces to $x^2 + y^2 - x + 4y - 4 = 0$

Centre is $\left(\frac{1}{2}, -2\right)$ and radius is $\sqrt{\frac{1}{4} + 4 + 4} = \frac{\sqrt{33}}{2}$

Concept Application

1. Find the equation of the circle with radius 5 whose enter lies on the x-axis and passed through the point (2, 3).
2. Find the locus of midpoint of the circle $x^2 + y^2 = a^2$ which subtends a right at the poin $(c, 0)$.

3. A circle touches the line $y = x$ at a point P such that $OP = 4\sqrt{2}$, where o is the origin. The circle contains the point $(-10, 2)$ in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$. Determine the equation of the circle.

4. If one end of the diameter is $(1, 1)$ and other end lies on the line $x + y = 3$, then locus of centre of circle is
(a) $x + y = 1$ (b) $2(x - y) = 5$
(c) $2x + 2y = 5$ (d) $2x + 2y = 1$

5. If the line $x + 2by + 7 = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$, then $b =$
(a) 3 (b) -5 (c) -1 (d) 5

(ii) **General equation of a circle:** General equation of second degree in x and y is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

The equation of a circle with centre (α, β) and radius r is

$(x - \alpha)^2 + (y - \beta)^2 = r^2$

$x^2 + y^2 - 2\alpha x - 2\beta y + (\alpha^2 + \beta^2 - r^2) = 0$ ...(ii)

Comparing (i) and (ii),

$$\frac{a}{1} = \frac{b}{1} = \frac{h}{0} = \frac{g}{-\alpha} = \frac{f}{-\beta} = \frac{c}{(\alpha^2 + \beta^2 - r^2)}$$

$\Rightarrow a = b$ i.e., coefficient of x^2 = coefficient of y^2.

$h = 0$ i.e., coefficient of $xy = 0$

The general equation of a circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(iii)

where g, f and c are constants.

To find the centre and radius.

Equation (iii) can be written as

$(x + g)^2 + (y + f)^2 = [\sqrt{(g^2 + f^2 - c)}]^2$

Comparing with the equation of the circle,

$\alpha = -g$, $\beta = -f$ and $r = \sqrt{(g^2 + f^2 - c)}$.

$\therefore$ Coordinates of the centre are $(-g, -f)$ and radius $= \sqrt{(g^2 + f^2 - c)}$.

(iii) **Important remarks**

(a) If $g^2 + f^2 - c > 0$, equation represents real circle with centre $(-g, -f)$.

(b) If $g^2 + f^2 - c = 0$, the equation represents a circle whose centre is $(-g, -f)$ and radius is zero i.e., the circle coincides with the centre and so it represents a point $(-g, -f)$. It is, therefore, called a point circle.

(c) If $g^2 + f^2 < c$, the radius of the circle is imaginary. In this case the equation does not represent any real geometrical locus. It is better not to say that the circle does not exist, but to say that it is a circle with a real centre and an imaginary radius.

(d) Dependence of the circle on three unknown parameters. The equation i.e., $x^2 + y^2 + 2gx + 2fy + c = 0$ contains three unknown quantities g, f, c. Hence for determining the equation of a circle, three conditions are required.

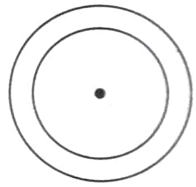
Notes:

1. Circle may exist in any quadrant hence for general cases use $\pm$ sign before h and k.

2. A general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ in x, y represent a circle if
   (i) coefficient of x^2 = coefficient of y^2, i.e. $a = b$.
   (ii) coefficient of xy is zero, i.e. $h = 0$
   (iii) $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$
   However for a point circle (Whose radius is zero), $\Delta = 0$

3. Rule to find the centre and radius of a circle whose equation is given:
   (i) Make the coefficients of x^2 and y^2 equal to 1 and right hand side equal to zero.
   (ii) Then co-ordinates of centre will be (α, β) where
   $$\alpha = -\frac{1}{2}\text{(coefficient of } x)$$
   $$\text{and } \beta = -\frac{1}{2}\text{(coefficient of } y).$$
   (iii) Radius $= \sqrt{(\alpha^2 + \beta^2 - \text{constant term})}$

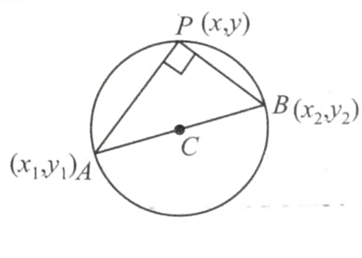
4. (i) Two circles are said to be concentric if they have the same centre.



   (ii) Equations of two concentric circles differ by a constant only.
   (iii) The equation of the circle concentric with the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is of the form $x^2 + y^2 + 2gx + 2fy + k = 0$, k is an unknown constant.
   (iv) **Diameter form i.e., circle with the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter:** Let $P(x, y)$ be any point on the circle with AB as diameter then $\angle APB = \pi/2$ (angle in semicircle). i.e., AP is $\perp$ to BP.

   $$\therefore \left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

   i.e., $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$, is the required equation.



Note:

(i) Equation of circle with least radius passess through (x_1, y_1) and (x_2, y_2) is diametric form with those points as extermeties of diameter.

(ii) If (x_1, y_1) and (x_2, y_2) are extremeties of diameter of a circle then equation of circle $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ can be written as $(x^2 - x(x_1 + x_2) + x_1x_2) + (y^2 - y(y_1 + y_2) + y_1y_2) = 0$ where x_1 and x_2 are roots of $x^2 - x(x_1 + x_2) + x_1x_2 = 0$ and y_1 and y_2 are roots of $y^2 - y(y_1 + y_2) + y_1y_2 = 0$

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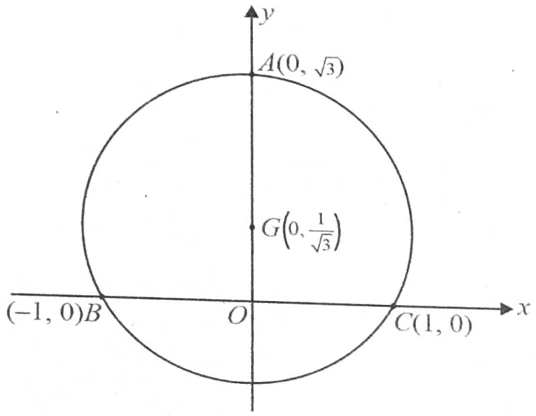
Example 6: Two vertices of an equilateral triangle are (–1, 0) and (1, 0) and its third vertex lies above the x-axis. Find equation of circumcircle triangle.

Sol. We have equilateral triangle ABC.



Let $B = (-1, 0)$, $C = (1, 0)$

Then A lies on y-axis.

$\therefore OA = AB \sin 60° = \sqrt{3}$

$\therefore A = (0, \sqrt{3})$

In an equilateral triangle, circumcentre coincides with centroid.

Thus, circumcentre of triangle ABC is

$$G = \left(\frac{-1+1+0}{3}, \frac{\sqrt{3}+0+0}{3}\right) = \left(0, \frac{1}{\sqrt{3}}\right)$$

$$\text{Circumradius} = AG = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

Therefore, equation of circumcircle of triangle ABC is

$$(x-0)^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{2}{\sqrt{3}}\right)^2$$

Example 7: Find the equations of circles each having radius 5 and touching the line $3x + 4y - 11 = 1$ at point $(1, 2)$.

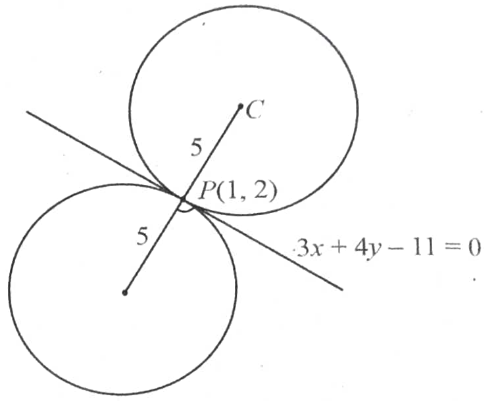
Sol. As shown in the figure, the two circles touch the line $3x + 4y - 11 = 0$ at point $P(1, 2)$.

$\therefore$ Centre lies on the line perpendicular to the tangent at P at distance 5 from it on either side.

Slope of given tangent line is $\frac{-3}{4}$.

Thus, the slope of CP is $\frac{4}{3}$.

$\therefore \tan\theta = \frac{4}{3}$



$\therefore$ Centre, $C = (1 \pm 5\cos\theta, 2 \pm \sin\theta)$

$\therefore \quad C = \left(1 \pm 5\frac{3}{5}, 2 \pm 5\frac{4}{5}\right)$

$\therefore \quad C = (4, 6)$ or $(-2, -2)$

Therefore, equations of circles are

$(x - 4)^2 + (y - 6)^2 = 25$

or $\quad (x + 2)^2 + (y - 2)^2 = 25$

Example 8: Prove that for all values of θ, the locus of the point of intersection of the lines $x\cos\theta + y\sin\theta = a$ and $x\sin\theta - y\cos\theta = b$ is a circle.

Sol. Since the point of intersection satisfies both the given lines we can find the locus by eliminating θ from the given equation. Therefore, by squaring and adding, we get equation $x^2 + y^2 = a^2 + b^2$ which is the equation of cricle.

Example 9: Prove that the maximum number of points with rational coordinates on a circle whose center is $(\sqrt{3}, 0)$ is two.

Sol. There cannot be three points on the circle with rational coordinates as for then the center of the circle, being the circumcenter of a triangle whose vertices have rational coordinates, must have rational coordinates (since the coordinates will be obtained by solving two linear equations in x, y having rational coefficients). But the point does not have rational coordiantes,

Also, the equation of the circle is $(x - \sqrt{3})^2 + y^2 = r^2$

or $x = \sqrt{3} \pm \sqrt{r^2 - y^2}$

For suitable r and x, where x is rational, y may have two rational values.

For example, $r = 2, x = 0, y = 1, -1$ satisfy

$x = \sqrt{3} \pm \sqrt{r^2 - y^2}$

So, we get two points (0, 1) and (0, –1) which have rational coordinates.

Concept Application

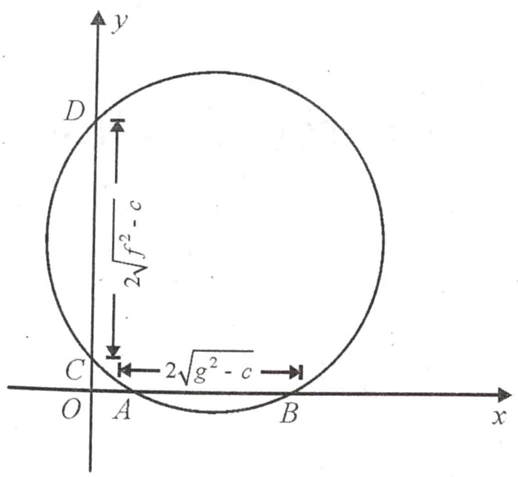
6. Find the image of the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ in the line $2x - 3y + 5 = 0$.
7. If $(m_i, 1/m_i)$, $m_i > 0$, $i = 1, 2, 3, 4$, are four distinct points on a circle, then show that $m_1 m_2 m_3 m_4 = 1$.
8. Prove that the locus of the centroid of the traingle whose vertices are $(a \cos t, a \sin t)$, $(b \sin t, -b \cos t)$, and $(1, 0)$, where t is a parameter, is circles,
9. If the equation $px^2 + (2 - q)xy + 3y^2 - 6qx + 30y + 6q = 0$ represents a circle, then find the values of p and q.
10. Equation of circle passing through two points (2, 0) and (0, 2) and having least area is
    (a) $x^2 + y^2 - 2x - 2y = 0$ (b) $x^2 + y^2 + 2x + 2y = 0$
    (c) $x^2 + y^2 - x - y = 0$ (d) $x^2 + y^2 + x + y = 0$

Intercept of circle on axes

One of the methods to find the intercepts of circle on axes is solving equation with the axes.

Another method uses the standard formula for intercepts consider the following circle:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(1)$$



To, solve this circle with x-axis, we put $y = 0$.

$\therefore\ x^2 + 2gx + c = 0$

$|x_1 - x_2| = 2\sqrt{g^2 - c}$

Similarly, points of intersection of circle with y-axis are $C(0, -f - \sqrt{f^2 - c})$ and $D(0, -f + \sqrt{f^2 - c})$ and intercept on y-axis is $CD = 2\sqrt{f^2 - c}$

EQUATION OF CIRCLE PASSING THROUGH THREE POINTS

The general equation of circle, i.e., $x^2 + y^2 + 2gx + 2fy + c = 0$ contains three indepndent constants g, f and c.

Hence, for determining the equation of a circle, three points or three conditions are required to form three equations in g, f and c. If circle passes through points (x_i, y_i), $i = 1, 2, 3$, then we have three equations

$$x_i^2 + y_i^2 + 2gx_i + 2fy_1 + c = 0, i = 1, 2, 3$$

Solving these three equations, we get values of g, f and c.

1. **Second method :** Find the circum-centre and radius as in the lesson on 'Points and Straight lines'. We recall that:

   If (α, β) be the circum-centre, then $(x_1 - \alpha)^2 + (y_1 - \beta)^2 = (x_2 - a)^2 + (y_2 - \beta)^2 = (x_3 - \alpha)^2 + (y_2 - \beta)^2$.

   Solving these, we find α, β. Also radius

   $\sqrt{(x_1 - \alpha)^2 + (y_1 - \beta)^2} = r$ (say).

   Then equation of the circle is $(x - \alpha)^2 + (y - \beta)^2 = r^2$.

Train Your Brain

Example 10: Find the length of intercept which the circle $x^2 + y^2 + 10x - 6y + 9 = 0$ makes on the axes.

Sol. Comparing the given equation with $x^2 + y^2 + 2gx + 2fy + c = 0$ we get

$g = 5, f = -3$ and $c = 9$

$\therefore$ Length of intercept on x-axis

$= 2\sqrt{g^2 - c} = 2\sqrt{(5)^2 - 9} = 8$

Length of intercept on y-axis

$2\sqrt{f^2 - c} = 2\sqrt{(-3)^2 - 9} = 0$

Thus, circle touches the y-axis.

Alternatively, putting $y = 0$ inthe equation of circle, we get

$x^2 + 10x + 9 = 0$

or $(x + 1)(x + 9) = 0$

Thus, points of intersection with x-axis are $A(-1, 0)$ and $B(-9, 0)$.

Thus, $AB = 8$.

Putting $x = 0$ in the equationof circle, we get

$y^2 - 6y + 9 = 0$

or $(y-3)^2 = 0$

Thus, circle touches the y-axis.

So, length of intercept on y-axis is 0.

Example 11: If the intercepts of the variable circle on the x-and y-axes are 2 units and 4 units, respectively, the find locus of the centre of the variable circle.

Sol. Let the equation of variable circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

We have to find the locus of the centre $C(-g, -f)$. According to the question, we have

$2\sqrt{g^2 - c} = 2$ and $2\sqrt{f^2 - c} = 4$

$\therefore$ $g^2 - c = 1$ and $f^2 - c = 4$

Eliminating c, we get

$f^2 - g^2 = 3$ or $(-f)^2 - (-g)^2 = 3$

Therefore, required locus is $y^2 - x^2 = 3$.

Example 12: Find the equation of the circle which passes through the points $(1, -2)$, $(4, -3)$ and whose center lies on the line $3x + 4y = 7$

Sol. Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

If (1) passes through the points $(1, -2)$ and $(4, -3)$, then

$5 + 2g - 4f + c = 0$ ...(2)

and $25 + 8g - 6f + c = 0$ ...(3)

Since the center $(-g, -f)$ lies on the line $3x + 4y = 7$, we have

$-3g - 4f = 7$ ...(4)

Solving (2), (3) and (4) we get

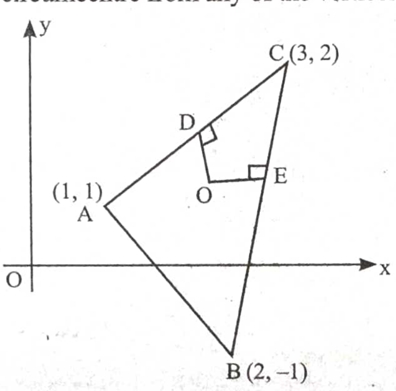
$g = -\frac{47}{15}, f = \frac{3}{5}$, and $c = \frac{11}{3}$

Substituting in (1) the equation of the circle is

$15x^2 + 15y^2 - 94x + 18y + 55 = 0$

Example 13: Find the equation of the circle passing through the three non-collinear points $(1, 1)$, $(2, -1)$ and $(3, 2)$.

Sol. The centre of the circumcircle is the point of intersection of the perpendicular bisectors of the sides of the triangle and the radius is the distance of the circumcentre from any of the vertices of the triangle.



Let D and E are the mid-points of AC and BC, then

$D \equiv \left(2, \frac{3}{2}\right)$ and $E \equiv \left(\frac{5}{2}, \frac{1}{2}\right)$

Solpe of $AC = \frac{2-1}{3-1} = \frac{1}{2}$

Slope of $OD = -2$

Equation of OD, $\left(y - \frac{3}{2}\right) = -2(x-2)$

$\Rightarrow 2y - 3 = -4x + 8$

$\Rightarrow 4x + 2y - 11 = 0$

Slope of $BC = \frac{2-(-1)}{3-2} = \frac{3}{1}$

Slope of $OE = \frac{-1}{3}$

Equation of OE, $\left(y - \frac{1}{2}\right) = \frac{-1}{3}\left(x - \frac{5}{2}\right)$

$3y - \frac{3}{2} = -x + \frac{5}{2}$

$3y + x - 4 = 0$ ...(ii)

On solving equation (i) and equation (ii) we get $x = \frac{5}{2}$ and $y = \frac{1}{2}$

$\therefore$ Circumcentre is $\left(\frac{5}{2}, \frac{1}{2}\right)$

and radius $= \sqrt{\left(1 - \frac{5}{2}\right)^2 + \left(1 - \frac{1}{2}\right)^2}$

$= \sqrt{\frac{9}{4} + \frac{1}{4}} = \sqrt{\frac{10}{4}} = \sqrt{\frac{5}{2}}$

$\therefore$ Equation of circle is $\left(x - \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{2}$

$\Rightarrow x^2 + y^2 - 5x - y + 4 = 0$

Concept Application

11. Find the equation of a circle which passes through the point $(2, 0)$ and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$ as $c \to 1$.

12. Find the equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$.

13. The circle $x^2 + y^2 - 4x - 8y + 16 = 0$ rolls up the tangent to it at $(2 + \sqrt{3}, 3)$ by 2 units, find the equation of the circle in the new position.

14. Let L_1 be a straight line through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then which of the following equations can represent L_1?

(a) $x + y = 0$ (b) $x - y = 0$
(c) $x + 7y = 0$ (d) $x - 7y = 0$

Parametric Equation of a Circle

For the ease of understanding let us first consider a circle with centre at origin and then generalize it to consider the case when the centre is at (h, k).

The equation of circle shown in the figure is $x^2 + y^2 = r^2$.

Take any point P on the circle and draw a line joining $P(x, y)$ to the centre $(0, 0)$ such that it makes an angle θ with the positive direction of x-axis.

Drop a perpendicular from P on the x-axis.

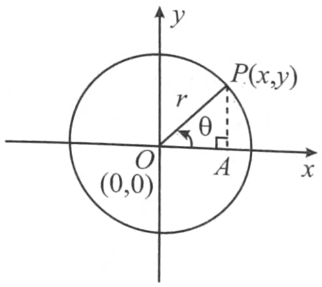
In the right angles ΔPOA

$x = r\cos\theta$, $y = r\sin\theta$ where, $0 \le \theta < 2\pi$

The point $P(r\cos\theta, r\sin\theta)$ satisfies the equation of the circle.

$\therefore$ $x = r\cos\theta$

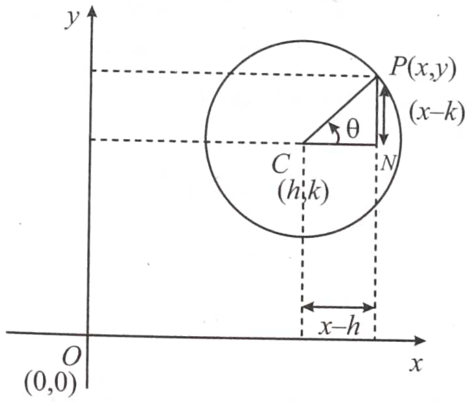
$y = r\sin\theta$ are said to be the parametric equations of the circle.



Now lets consider in the figure below. The equation of the circle is

$(x - h)^2 + (y - k)^2 = r^2$ ...(i)

where, r is the radius of the circle.



From, ΔCPN,

$(x - h) = r\cos\theta$

$(y - k) = r\sin\theta$

$\Rightarrow$ $x = h + r\cos\theta$ ...(ii)

$y = k + r\sin\theta$ ...(iii)

where, $0 \le \theta < 2\pi$

$P(h + r\cos\theta, k + r\sin\theta)$ satisfies equation (i)

$\therefore$ (ii) and (iii) represent the parametric equations of the circle.

(i) The parametric equations of a circle $x^2 + y^2 = a^2$ are $x = a\cos\theta$ and $y = a\sin\theta$.

(ii) The parametric equations of circle $(x - h)^2 + (y - k)^2 = a^2$ are $x = h + a\cos\theta$, and $y = k + a\sin\theta$, where θ is a parameter.

Note: Parametric equation of circle is very useful when we need to write the equation for part of the circle only

Eg: If we have to find the equation of quarter of the circle then we have to restrict the range of θ from $[\alpha, \alpha + \pi/2]$ where value of α is according to the problem $(0 \le \alpha \le 3\pi/2)$.

Aliter: If the triangle ABC be completed the perpendicular bisectors of the sides of the triangle will intersect at the centre of the circle passing through A, B, C and the distance of either of the vertices from this centre will be equal to the radius of the circle.

Train Your Brain

Example 14: Express a point lying on the circle $x^2 + y^2 + 10x + 2y + 1 = 0$ in parametric form.

Sol. The centre (h, k) is $(-5, -1)$ and radius a is $\sqrt{25+1-1} = 5$.

So, the parametric form is

$x = h + a\cos\theta = -5 + 5\cos\theta$

$y = k + a\sin\theta = -1 + 5\sin\theta$

$\therefore$ Any point in this circle can be taken as

$(-5 + 5\cos\theta, -1 + 5\sin\theta)$

Example 15: A circle has radius equal to 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through $(7, 3)$.

Sol. Let the centre of the circle be (α, β). It lies on the line $y = x - 1$

$\Rightarrow$ $\beta = \alpha - 1$. Hence the centre is $(\alpha, \alpha - 1)$.

$\Rightarrow$ The equation of the circle is $(x - \alpha)^2 + (y - \alpha + 1)^2 = 9$

It passes through $(7, 3)$

$\Rightarrow$ $(7 - \alpha)^2 + (4 - \alpha)^2 = 9$

$\Rightarrow$ $2\alpha^2 - 22\alpha + 56 = 0$ $\Rightarrow$ $\alpha^2 - 11\alpha + 28 = 0$

$\Rightarrow$ $(\alpha - 4)(\alpha - 7) = 0$ $\Rightarrow$ $\alpha = 4, 7$

Hence the required equations are

$x^2 + y^2 - 8x - 6y + 16 = 0$ and

$x^2 + y^2 - 14x - 12y + 76 = 0$.

Concept Application

15. The parametric coordinates of any point on the circle $x^2 + y^2 - 4x - 4y = 0$ are

(a) $(-2 + 2\cos\alpha, -2 + 2\sin\alpha)$

(b) $(2 + 2\cos\alpha, 2 + 2\sin\alpha)$

(c) $(2 + 2\sqrt{2}\cos\alpha, 2 + 2\sqrt{2}\sin\alpha)$

(d) $(2 - 2\cos\alpha, 2 - 2\sin\alpha)$

CIRCLE IN DIFFERENT CASES

(i) Circle with centre at the point (h, k) and which touches the axis of x.

If the circle touches the axis of x, then its radius is equal to the numerical value of y-coordinate of the centre.

$\therefore$ Equation of the circle is

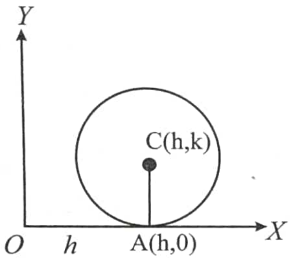
$(x - h)^2 + (y - k)^2 = |k|^2 = k^2$

or $x^2 + y^2 - 2hx - 2ky + h^2 = 0$ ...(i)

This circle meets the x-axis, at the points where $y = 0$.

$\therefore$ Putting $y = 0$ in (i), we get

$x^2 - 2hx + h^2 = 0$ or $(x - h)^2 = 0$.



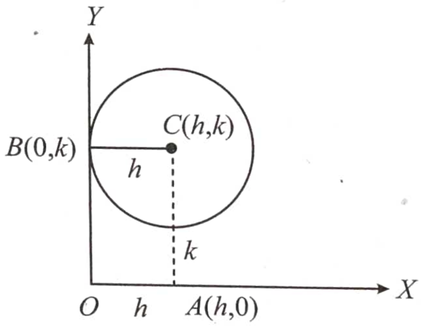
which gives two identical values of x $(= h)$.

Thus the circle touches the x-axis at the point $(h, 0)$.

Important to remember: If a circle touches the x-axis at the point $(h, 0)$, then putting $y = 0$, the equation of the circle will reduce to the form $(x - h)^2 = 0$.

(ii) Circle with centre at the point (h, k) and which touches the axis of y.

If the circle touches the axis of y, then its radius is equal to the numerical value of x-coordinate of the centre.



$\therefore$ Equation of the circle is given by

$(x - h)^2 + (y - k)^2 = h^2$.

or $x^2 + y^2 - 2hx - 2ky + k^2 = 0$. ... (i)

This circle meets the y-axis at the points where $x = 0$

$\therefore$ Putting $x = 0$ in (i), we get $y^2 - 2ky + k^2 = 0$ or $(y - k)^2 = 0$

which gives two identical values of y $(= k)$.

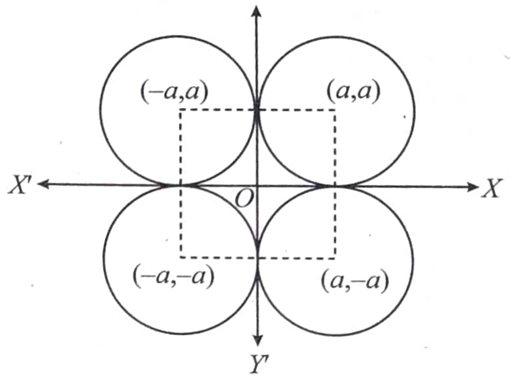
Thus the circle touches the y-axis at the point $(0, k)$.

Important to remember: If a circle touches the y-axis at the point $(0, k)$, then putting $x = 0$, the equation of the circle will reduce to the form $(y - k)^2 = 0$

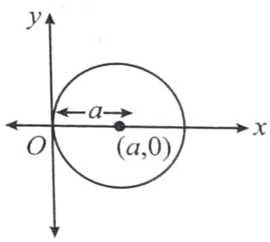
(iii) Circle which touches both the coordinate axes.

Let a be the radius of the circle. Since the centre of the circle may be in any of the four quadrants, therefore it will be any one of the four points $(\pm a, \pm a)$. Thus there are four circles of radius a touching both the coordinate axes, and their equations are

$(x \pm a)^2 + (y \pm a)^2 = a^2$ or $x^2 + y^2 \pm 2ax \pm 2ay + a^2 = 0$



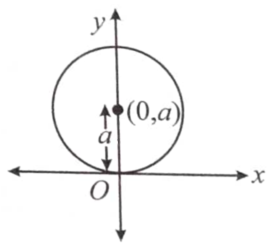
(iv) Circle passing through origin and centre lies on x-axis (Circle touching y-axis at origin)



As centre lies on x-axis so let cordinates of centre by $c(a, 0)$ then radius of circle is a units.

Equation of circle is $(x - a)^2 + y^2 = a^2 \Rightarrow x^2 + y^2 - 2ax = 0$

(v) Circle touching x-axis at origin



As centre of the circle lies on Y-axis, so co-ordinates of centre are $(0, a)$, and radius of circle is a units.

$\therefore$ Equation of circle is $x^2 + (y - a)^2 = a^2$

$\Rightarrow x^2 + y^2 - 2ay = 0$

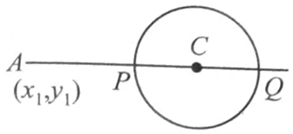
POSITION OF A POINT W.R.T. CIRCLE

(i) Let the circle is $x^2+y^2+2gx+2fy+c=0$ and the point is (x_1, y_1) then point (x_1, y_1) lies out side the circle or on the circle or inside the circle according as

$\Rightarrow x_1^2+y_1^2+2gx_1+2fy_1+c >, =, < 0$

or $S_1 >, =, < 0$

Where, $S_1 = x_1^2+y_1^2+2gx_1+2fy_1+c$



(ii) The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ and $AC - r$ respectively.

Train Your Brain

Example 16: A circle is drawn touching the x-axis and centre at the point which is the reflection of (a, b) in the line $y - x = 0$. The equation of the circle is

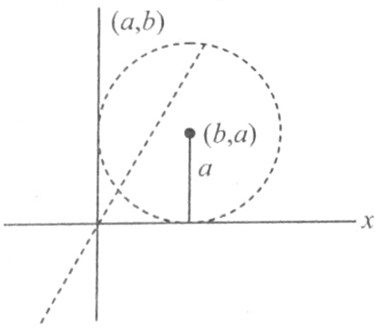
(a) $x^2 + y^2 - 2bx - 2ay + a^2 = 0$

(b) $x^2 + y^2 - 2bx - 2ay + b^2 = 0$

(c) $x^2 + y^2 - 2ax - 2by + b^2 = 0$

(d) $x^2 + y^2 - 2ax - 2by + a^2 = 0$

Sol. (b) Reflection of (a, b) in $y - x = 0$ is (b, a) centre (b, a) touching x-axis.



$r = Q$

$(x - b)^2 + (y - a)^2 = a^2$

$x^2 + y^2 - 2bx - 2ay + b^2 = 0$

Example 17: The equation of circles passing through (3, –6) touching both the axes is

(a) $x^2 + y^2 - 6x + 6y + 9 = 0$

(b) $x^2 + y^2 + 6x - 6y + 9 = 0$

(c) $x^2 + y^2 + 30x - 30y + 225 = 0$

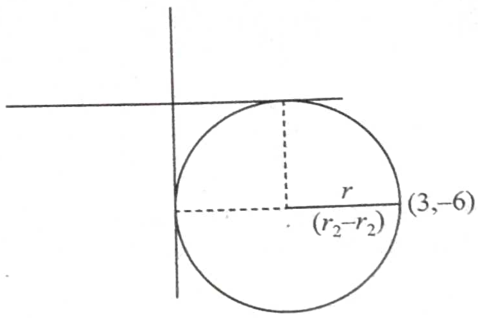
(d) $x^2 + y^2 - 30x + 30y + 225 = 0$

Sol. (a, d) Now

$(r - 3)^2 + (-r + 6)^2 = r^2$

$r^2 - 18r + 45 = 0$

$\Rightarrow r = 3, 15$



Hence circle

$(x - 3)^2 + (y + 3)^2 = 3^2$

$x^2 + y^2 - 6x + 6y + 9 = 0$

$(x - 15)^2 + (y + 15)^2 = (15)^2$

$\Rightarrow x^2 + y^2 - 30x + 30y + 225 = 0$

Concept Application

16. Equations of circles which pass through the points (1, –2) and (3, – 4) and touch the x-axis is

(a) $x^2 + y^2 + 6x + 2y + 9 = 0$

(b) $x^2 + y^2 + 10x + 20y + 25 = 0$

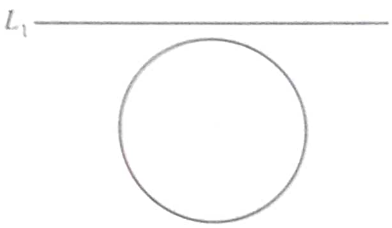
(c) $x^2 + y^2 - 6x + 4y + 9 = 0$

(d) $x^2 + y^2 - 10x - 20y - 25 = 0$

CIRCLE AND A LINE

We consider 3 cases:

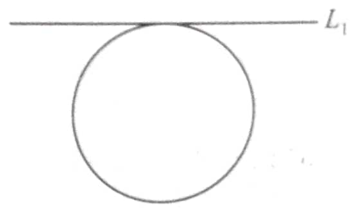
Case I:



Line L_1 does not intersect the circle

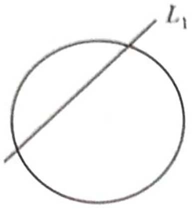
Case II:

Line L_1 touches the circle



Case III:

Line L_1 intersects the circle in two points.



Now we proceed to derive the conditions of the three cases. For the ease of calculations we take the circle as, $x^2 + y^2 = a^2$ ... (*i*)

and the equation of the line as

$y = mx + c$ ... (*ii*)

substituting the value of y in equation (*i*),

$x^2 + (mx + c)^2 = a^2$

$\Rightarrow x^2(1 + m^2) + 2mcx + (c^2 - a^2) = 0$ ... (*iii*)

Equation (*iii*) represents a quadratic equation in x. The discriminant of the equation is

$\Delta = 4m^2c^2 - 4(1 + m^2)(c^2 - a^2)$

Case I:

There is no real point of intersection.

i.e., $\Delta < 0$

$\Rightarrow 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) < 0$

$\Rightarrow a^2(1 + m^2) < c^2$

$\Rightarrow a^2 < \dfrac{c^2}{(1+m^2)} \quad \Rightarrow a < \left|\dfrac{c}{\sqrt{1+m^2}}\right|$

Case II:

There are two coincident points of intersection.

i.e., $\Delta = 0$

$4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0 \Rightarrow a^2(1 + m^2) = c^2$

$\Rightarrow a = \left|\dfrac{c}{\sqrt{1+m^2}}\right|$

$\Rightarrow c = \pm a\sqrt{1+m^2}$

This is the required condition of tangency.

$\therefore$ the equation of the tangents are

$y = mx \pm a\sqrt{1+m^2}$

Case III:

There are two different real points of intersection.

i.e., $\Delta > 0$

$\Rightarrow 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) > 0.$

$\therefore a^2(1 + m^2) > c^2$ or $a > \left|\dfrac{c}{\sqrt{1+m^2}}\right|$

Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$

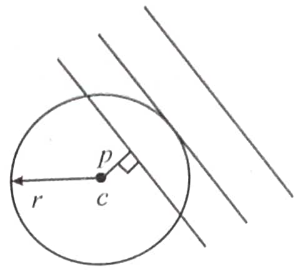
and $L = lx + my + n = 0.$

Let r be the radius of the circle and p be the length of the perpendicular drawn from the centre $(-g, -f)$ on the line L. Then.

(*i*) line intersect the circle in two distinct points if $p < r$.

(*ii*) line touch the circle if $p = r$.

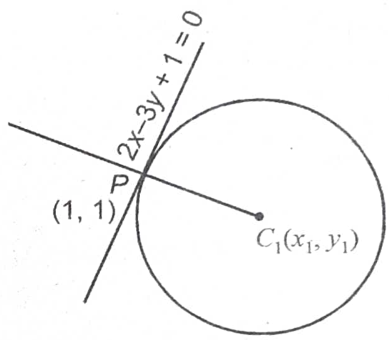
(*iii*) line neither intersects nor touches the circle i.e., passes outside the circle if $p > r$.



Train Your Brain

Example 18. Find the equations of circles which have radius $\sqrt{13}$ and which touch the line $2x - 3y + 1 = 0$ at (1, 1).

Sol. Let one of the circles have centre $C_1(x_1, y_1)$ and let the point P have coordinates (1, 1). Since $C_1P \perp 2x - 3y + 1 = 0$. The equation of C_1P is $3x + 2y = 5$



Since $C_1(x_1, y_1)$ lies on it, we have : $3x_1 + 2y_1 = 5$ ...(1)

Also, $C_1P = \sqrt{13} \Rightarrow \left|\dfrac{2x_1 - 3y_1 + 1}{\sqrt{13}}\right| = \sqrt{13}$

$\therefore 2x_1 - 3y_1 + 1 = \pm 13$

Taking '+' sign : $2x_1 - 3y_1 = 12$ ...(2)

Taking '–' sign : $2x_1 - 3y_1 = -14$ ...(3)

Solving (1) and (2) we get : $x_1 = 3$ and $y_1 = -2$

$\therefore$ Equation of one of the circles is

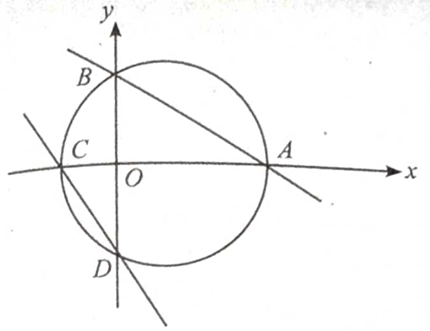
$(x - 3)^2 + (y + 2)^2 = 13$

Solving (1) and (3), we get : $x_1 = -1; y_1 = 4$

$\therefore$ Equation of the other circle is

$(x + 1)^2 + (y - 4)^2 = 13$

Example 19. If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points, show that $a_1a_2 = b_1b_2$.



Sol. Let the straight line $a_1x + b_1y + c_1 = 0$ cut the coordinate axes in the points A and B respectively; then, A and B have coordinates $A\left(\frac{-c_1}{a_1}, 0\right)$ and $B\left(0, \frac{-c_1}{b_1}\right)$ respectively.

Let the line $a_2x + b_2y + c_2 = 0$ cut the axes in points C and D respectively; then, C and D have coordinates $\left(\frac{-c_2}{a_2}, 0\right)$ and $\left(0, \frac{-c_2}{b_2}\right)$ respectively.

By geometry, since A, B, C and D are concyclic, we have $|(OA)\cdot(OC)| = |(OB)\cdot(OD)|$

$$\Rightarrow \left|\frac{-c_1}{a_1}\right|\cdot\left|\frac{-c_2}{a_2}\right| = \left|\frac{-c_1}{b_1}\right|\cdot\left|\frac{-c_2}{b_2}\right| \Rightarrow a_1a_2 = b_1b_2$$

Example 20. Find the equation of the circles which pass through the origin and cut off chords of length 'a' from each of the lines $y = x$ and $y = -x$.

Sol. Let $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

be the equation of such circle. We note that '$c = 0$' as (1) passes through the origin.

Let OP and OQ denote the chords intercepted by (1) on $y = x$ and $y = -x$ respectively where O is the origin.

If the coordinates of P are (α, α) then "$a = OP$"

$$\Rightarrow \quad \alpha = \pm\frac{a}{\sqrt{2}}$$

The possible coordinates of Q are $(-\alpha, \alpha)$ or $(\alpha, -\alpha)$

Since P and Q lie on (1), we get : $g = 0$ and $f = -\alpha$ or $g = -\alpha$ and $f = 0$

according as Q is $(-\alpha, \alpha)$ or $(\alpha, -\alpha)$ and where P is (α, α)

$\therefore$ Equation of the required circles are :

when $\alpha = \frac{a}{\sqrt{2}}$: $x^2 + y^2 - \sqrt{2}\,ay = 0$

or $x^2 + y^2 - \sqrt{2}\,ax = 0$

when $\alpha = -\frac{a}{\sqrt{2}}$: $x^2 + y^2 + \sqrt{2}ay = 0$

or $x^2 + y^2 + \sqrt{2}a\cdot x = 0$

Example 21. A triangle has two of its sides along the coordinate axes; its third side touches the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$. Prove that the locus of the circumcentre of the triangle is

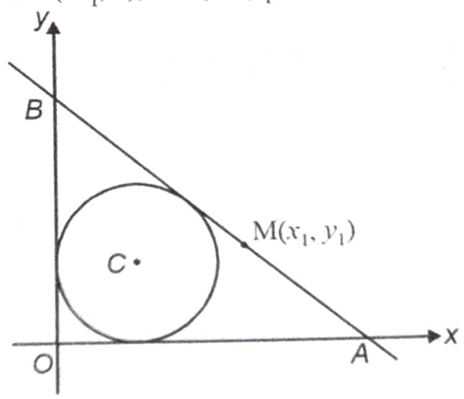
$a^2 - 2a(x + y) + 2xy = 0$ where $a > 0$

Sol. $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ ...(1)

Eq. (1) has center $C(a, a)$ and radius $r = a$.

Let OAB be the required triangle and let $M(x_1, y_1)$ be any point on the locus; then, M is the mid-point of segment AB.

$A \equiv (2x_1, 0)$; $B \equiv (0, 2y_1)$



Equation of the straight line AB is $\frac{x}{2x_1} + \frac{y}{2y_1} = 1$

i.e., $xy_1 + yx_1 - 2x_1y_1 = 0$

Since AB is a tangent to the circle, the length of the perpendicular from C on AB is equal to radius of the circle

$$\therefore \quad \left|\frac{ay_1 + ax_1 - 2x_1y_1}{\sqrt{x_1^2 + y_1^2}}\right| = a$$

$$\therefore \quad (ay_1 + ax_1 - 2x_1y_1)^2 = a^2(x_1^2 + y_1^2)$$

$$4x_1^2y_1^2 + 2a^2x_1y_1 - 4ax_1y_1(x_1 + y_1) = 0$$

$$\therefore \quad 2x_1y_1 + a^2 - 2a(x_1 + y_1) = 0 \quad (\because\ x_1 \neq 0;\ y_1 \neq 0)$$

Equation of locus is $a^2 - 2a(x + y) + 2xy = 0$

Concept Application

17. If $P(2, 8)$ is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$ which neither touches nor intersects the axes, then set for p is

18. If the line $3x - 4y = \lambda$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$, then λ is equal to

(*a*) $-35, -15$ (*b*) $-35, 15$

(*c*) $35, 15$ (*d*) $35, -15$

19. If the line $x-2y=k$ cuts off a chord of length 2 from the circle $x^2+y^2=3$, then $k=$

(a) 0 (b) ± 1

(c) $\pm\sqrt{10}$ (d) $\pm\sqrt{5}$

20. If a circle passes through points of intersection of co-ordinate axes with the lines $\lambda x-y+1=0$ and $x-2y+3=0$ then value of λ is

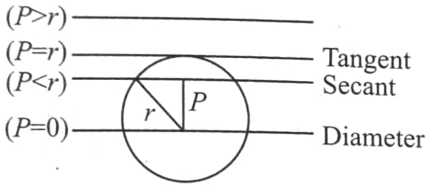
(a) 3 (b) 1

(c) 2 (d) 4

TANGENT LINE OF CIRCLE

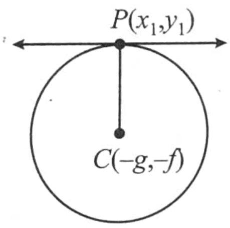
When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(a) Condition of Tangency : The line $L=0$ touches the circle $S=0$ if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. $P=r$.



For the Circle $x^2+y^2+2gx+2fy+c=0$

Consider a point $P(x_1, y_1)$ on the circle, where the tangent touches the circle



Slope of CP $\dfrac{(y_1+f)}{(x_1+g)}$

Tangent being perpendicular to the radius, slope of the tangent $=\dfrac{-(x_1+g)}{(y_1+f)}$.

Equation of the tangent at P is,

$$(y-y_1)=\frac{-(x_1+g)}{(y_1+f)}(x-x_1)$$

$\Rightarrow$ $(y-y_1)(y_1+f)+(x_1+g)(x-x_1)=0$

$\Rightarrow$ $yy_1-y_1^2+yf-y_1f+xx_1+xg-x_1^2-x_1g=0$

$\Rightarrow$ $xx_1+yy_1+xg+fy=x_1^2+y_1^2+x_1g+y_1f$

Adding (gx_1+y_1f+c) to both sides, we have

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c$

$=x_1^2+y_1^2+2x_1g+2fy_1+c$ ... (i)

$\because$ the point $P(x_1, y_1)$ lies on the circle,

$\therefore$ $x_1^2+y_1^2+2gx_1+2fy_1+c=0$

$\therefore$ Equation (i) reduces to,

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.

Note: We conclude that in order to write the equation of the tangent to the circle

$x^2+y^2+2gx+2fy+c=0$,

we put,

- xx_1 in place of x^2
- yy_1 in place of y^2
- $\dfrac{(x+x_1)}{2}$ in place of x
- $\dfrac{(y+y_1)}{2}$ in place of y

Note : Let $P(x_1, y_1)$ be any point. If we replace x^2 by xx_1, y^2 by yy_1, x by $\frac{1}{2}(x+x_1)$ and y by $\frac{1}{2}(y+y_1)$ in the left hand side of the equation of circle, we get $T=0$.

For example, for the circle $x^2+y^2=a^2$, $T\equiv xx_1+yy_1-a^2=0$ and for the circle $x^2+y^2+2gx+2fy+c=0$, $T\equiv xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.

Train Your Brain

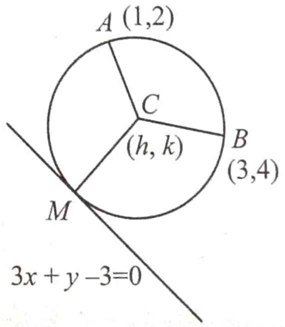
Example 22. Find the equation of the circles passing through the points $A(1, 2)$, $B(3, 4)$ and touching the line $3x+y-3=0$.

Sol. To determine the coordinates of the centre $C(h, k)$, we know that $CA = CB$ and $CA = CM$ or $(h-1)^2+(k-2)^2=(h-3)^2+(k-4)^2$ ... (i)

and $(h-1)^2+(k-2)^2=\dfrac{(3h+k-3)^2}{10}$ ...(ii)

Let $\alpha=h-1$ and $\beta=k-2$.

Then (i) can be written as $\alpha^2+\beta^2=(\alpha-2)^2+(\beta-2)^2$ or $\alpha+\beta=2$ ...(iii)



(ii) can be written as $10(\alpha^2+\beta^2)=(3\alpha+\beta+2)^2$

i.e., $10[(\alpha+\beta)^2-2\alpha\beta]=[3(\alpha+\beta)-2\beta+2]^2$

i.e., $10(4-2\alpha\beta)=(8-2\beta)^2$, using (iii)

i.e., $5[4-2\beta(2-\beta)]=2(4-\beta)^2$, again using (iii)

i.e., $8\beta^2-4\beta-12=0$ or $2\beta^2-\beta-3=0$

or $(\beta+1)(2\beta-3)=0$

or $\beta=-1$ or $\frac{3}{2}$ and $\alpha=3$ or $\frac{1}{2}$

$\Rightarrow \alpha=3, \beta=-1$ and $\alpha=\frac{1}{2}, \beta=\frac{3}{2}$

Correspondingly, $h=4, k=1$ and $h=\frac{3}{2}, k=\frac{7}{2}$

we have $(x-4)^2+(y-1)^2=10$

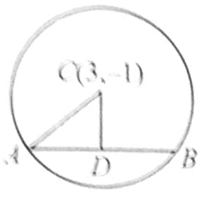
and $\left(x-\frac{3}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\frac{10}{4}$

Expanding these equations, we get, $x^2+y^2-8x-2y+7=0$ and $x^2+y^2-3x-7y+12=0$

Example 23. The equation of a circle whose centre is $(3,-1)$ and which cuts off a chord of length 6 on the line $2x-5y+18=0$.

Sol. Let $AB\ (=6)$ be the chord intercepted by the line $2x-5y+18=0$ from the circle and let CD be the perpendicular drawn from centre $(3,-1)$ to the chord AB.

i.e., $AD=3$, $CD=\frac{2.3-5(-1)+18}{\sqrt{2^2+5^2}}=\sqrt{29}$

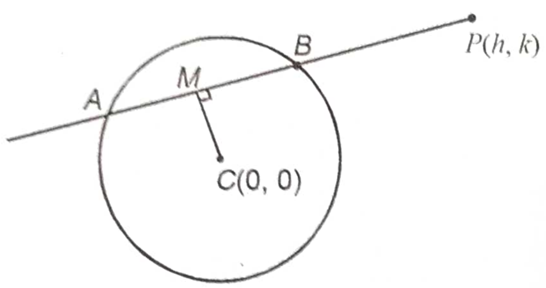


Therefore, $CA^2=3^2+(\sqrt{29})^2=38$.

Hence required equation is $(x-3)^2+(y+1)^2=38$

Example 24. Through a fixed point (h,k) secants are drawn to the circle $x^2+y^2=r^2$. Show that locus of mid points of portions of secants intercepted by the circle is $x^2+y^2=hx+ky$.

Sol.



$x^2+y^2=r^2$ ...(1)

Eq. (1) has centre C(0, 0) and radius r.

Let $P\equiv(h,k)$ and let $M\equiv(x_1,y_1)$ be any point on the locus; then, M is the mid point of the chord AB which is part of the secant drawn from $P(h,k)$ to (1).

Since $CM\perp AB$, we have: $\frac{y_1-0}{x_1-0}\cdot\frac{y_1-k}{x_1-h}=-1$

$\therefore\ y_1(y_1-k)+x_1(x_1-h)=0$

$x_1^2+y_1^2=hx_1+ky_1$

Equation of locus is $x^2+y^2=hx+ky$

Exampe 25. Two circles, each of radius 5 units, touch each other at the point (1, 2). If the equation of their common tangent is $4x+3y=10$, find the equation of the circles.

Sol. Let C_1 and C_2 denote the centres of the two circles. The equation of the common tangent at $P(1,2)$ is

$4x+3y=10$ ...(1)

$\therefore$ The equation of the common normal at (1, 2) is

$3x-4y+5=0$ ...(2)

$\tan\theta=$ slope of (2) $=\frac{3}{4}$;

$\therefore\ \cos\theta=\frac{4}{5}$ and $\sin\theta=\frac{3}{5}$

Also, equation of normal in parametric form is

$x-1=r\cos\theta;\ y-2=r\sin\theta$ ...(3)

Since units, the coordinates of C_1 and C_2 are obtained by putting $r=5$ and $r=-5$ successively in (3). Hence, $C_1(5,5)$ and $C_2(-3,-1)$ are the coordinates of the centres. The equation of the two circles are:

$(x-5)^2+(y-5)^2=25$ and $(x+3)^2+(y+1)^2=25$

Example 26. A tangent at a point on the circle $x^2+y^2=a^2$ intersects a concentric circle C at two points P and Q. The tangents to the circle C at P and Q meet at a point on the circle $x^2+y^2=b^2$. Find the equation to the circle C.

Sol. Let the equation of circle C be $x^2+y^2=r^2$ ...(1)

Let $A(h,k)$ be a point on the circle $x^2+y^2=a^2$ ...(2)

Tangent at A cuts the circle (1) at P and Q.

Let the tangents at P and Q meet at $B(b\cos\theta, b\sin\theta)$ on the circle $x^2+y^2=b^2$ ...(3)

Equation of chord of contact PQ of tangents drawn from B to (1) is $bx\cos\theta+by\sin\theta=r^2$ ...(4)

Since (4) is a tangent at A to (2)

$\therefore\ a=\left|\frac{r^2}{\sqrt{b^2\cos^2\theta+b^2\sin^2\theta}}\right|$

$\therefore\ r^2=ab$ and eq. (1) becomes $x^2+y^2=ab$

Example 27. The tangents to $x^2+y^2=a^2$ having inclinations α and β intersect at P. If $\cot\alpha+\cot\beta=0$, then find the locus of P.

Sol. Let the coordinates of P be (h,k). Let the equation of a tangent from $P(h,k)$ to the circle $x^2+y^2=a^2$ be

$y = mx + a\sqrt{1+m^2}$.

Since $P(h, k)$ lies on $y = mx + a\sqrt{1+m^2}$.

$\therefore \quad k = mh + a\sqrt{1+m^2}$

$\Rightarrow \quad (k - mh)^2 = a\sqrt{(1+m^2)}$

$\Rightarrow \quad m^2(h^2 - a^2) - 2mkh + k^2 - a^2 = 0$

This is a quadratic in m. Let the two roots be m_1 and m_2.

Then, $m_1 + m_2 = \dfrac{2hk}{h^2 - a^2}$

But $\tan\alpha = m_1$, $\tan\beta = m_2$ and it is given that $\cot\alpha + \cot\beta = 0$

$$\therefore \quad \frac{1}{m_1} + \frac{1}{m_2} = 0 \Rightarrow m_1 + m_2 = 0$$

$$\Rightarrow \quad \frac{2hk}{h^2 - a^2} = 0 \Rightarrow hk = 0$$

Hence, the locus of (h, k) is $xy = 0$.

Exampe 28. A circle has radius 2 units and its centre lies on the line $y = x + 1$. Find the equation of the circle if it passes through $(2, 5)$.

Sol. Let the centre of the circle be (h, k).

Since centre lies on the line $y = x + 1$, so (h, k) satisfy $y = x + 1$.

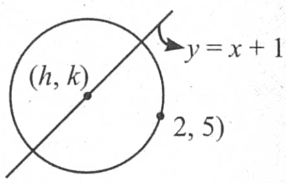
$k = h + 1$ ...(*i*)

Since, the circle passes through the point $(2, 5)$. Therefore distance of this point from the centre is the radius of the circle.

$r = \sqrt{(h-2)^2 + (k-5)^2}$ or

$2 = \sqrt{(h-2)^2 + (h+1-5)^2}$

$\Rightarrow \sqrt{(h-2)^2 + (h-4)^2}$



$\Rightarrow 2 = \sqrt{2h^2 - 12h + 20}$

or $4 = 2h^2 - 12h + 20$

or $h^2 - 6h + 8 = 0$

or $(h - 2)(h - 4) = 0$

or $h = 2$ and $h = 4$

for $h = 2$, we get $k = 3$, from equation (*i*), for $h = 4$, we get $k = 5$, from equation (*i*).

Hence, there are two circles which satisfy the given condition.

$(x - 2)^2 + (y - 3)^2 = 4$ or $x^2 + y^2 - 4x - 6y + 9 = 0$

and $(x - 4)^2 + (y - 5)^2 = 4$

or $x^2 + y^2 - 8x - 10y + 37 = 0$

Concept Application

21. With respect to circle $x^2 + y^2 + 4x - 6y + 8 = 0$, find the tangent passing through point $(4, -2)$.

22. $lx + my + n = 0$ is a tangent line to the circle $x^2 + y^2 = r^2$, if

(*a*) $l^2 + m^2 = n^2 r$

(*b*) $l^2 + m^2 = n^2 + r^2$

(*c*) $n^2 = r^2(l^2 + m^2)$

(*d*) None of these

23. If $y = c$ is a tangent to the circle $x^2 + y^2 - 2x + 2y - 2 = 0$ at $(1, 1)$, then the value of c is

(*a*) 1 (*b*) 2

(*c*) -1 (*d*) -2

24. Line $3x + 4y = 25$ touches the circle $x^2 + y^2 = 25$ at the point

(*a*) $(4, 3)$

(*b*) $(3, 4)$

(*c*) $(-3, -4)$

(*d*) None of these

25. The equations of the tangents drawn from the point $(0, 1)$ to the circle $x^2 + y^2 - 2x + 4y = 0$ are

(*a*) $2x - y + 1 = 0, x + 2y - 2 = 0$

(*b*) $2x - y - 1 = 0, x + 2y - 2 = 0$

(*c*) $2x - y + 1 = 0, x + 2y + 2 = 0$

(*d*) $2x - y - 1 = 0, x + 2y + 2 = 0$

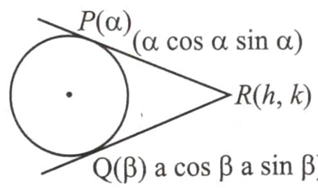
Equation of the tangent (T = 0)

(*i*) Tangent at the point (x_1, y_1) on the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(*ii*) The tangent at the point (a cos t, asin t) on the circle $x^2 + y^2 = a^2$ is $x \cos t + y \sin t = a$

(*iii*) Student generally use the formula that equation of tangent at on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1)\ c = 0$. But they forget that in this case (x_1, y_1) lying on the circle.

2. The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is (h, k) where $h = \dfrac{a\cos\dfrac{\alpha+\beta}{2}}{\cos\dfrac{\alpha-\beta}{2}}, k = \dfrac{a\sin\dfrac{\alpha+\beta}{2}}{\cos\dfrac{\alpha-\beta}{2}}$



(iii) The equation of tangent at the point (x_1, y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

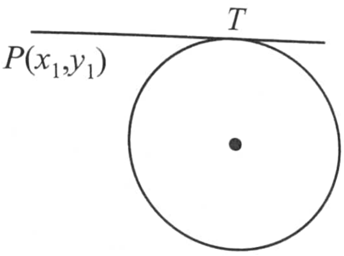
(iv) If line $y = mx + c$ is a straight line touching the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$ and point of contacts are $\left(\mp\frac{am}{\sqrt{1+m^2}}, \mp\frac{a}{\sqrt{1+m^2}}\right)$ or $\left(\mp\frac{a^2m}{c}, \pm\frac{a^2}{c}\right)$ and equation of tangent is $y = mx \pm a\sqrt{1+m^2}$

(v) The equation of tangent with slope m of the circle $(x-h)^2 + (y-k)^2 = a^2$ is

$$(y-k) = m(x-h) \pm a\sqrt{1+m^2}$$

Note: To get the equation of tangent at the point (x_1, y_1) on any curve we replace xx_1 in place of x^2, yy_1 in place of y^2, $\frac{x+x_1}{2}$ in place of x, $\frac{y+y_1}{2}$ in place of y, $\frac{xy_1+yx_1}{2}$ in place of xy and c in place of c.

(c) Length of tangent ($\sqrt{S_1}$): The length of tangent draw from point (x_1, y_1) out side the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is,

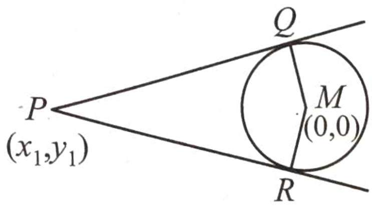


$$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Note: When we use this formula the coefficient of x^2 and y^2 must be 1.

(d) Equation of Pair to tangents ($SS_1 = T^2$) : Let the equation of circle $X \equiv x^2 + y^2 = a^2$ and $P(x_1, y_1)$ is any point outside the circle. From the point we can draw two real and distinct tangent PQ & PR and combine equation of pair of tangents is $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$.

or $SS_1 = T^2$.



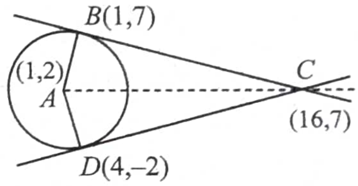
Train Your Brain

Example 29. Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ and $B(1, 7)$ and $D(4, -2)$ are points on the circle then, if tangents be drawn at B and D, which meet at C, then area of quadrilateral $ABCD$ is

Sol. $x.1 + y.7 - 1(x+1) - 2(y+7) - 20 = 0$ or $y = 7$ ...(i)

Tangent at $D(4, -2)$ is $3x - 4y - 20 = 0$ ...(ii)

Solving (i) and (ii), C is (16, 7)



Area $ABCD = AB \times BC = 5 \times 15 = 75$ units.

Example 30. Find the equation of the pair of tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1).

Sol. Given circle is $S \equiv x^2 + y^2 - 2x + 4y = 0$ ...(i)

Let $P \equiv (0, 1)$

For point P, $S_1 = 0^2 + 1^2 - 2.0 + 4.1 = 5 > 0$

Point P lies outside the circle

and $T \equiv x \cdot 0 + y \cdot 1 - (x + 0) + 2(y + 1)$

i.e., $T = -x + 3y + 2$

Now equation of the pair of tangents from $P(0, 1)$ to circle (i) is

$SS_1 = T^2$

or $5(x^2 + y^2 - 2x + 4y) = (-x + 3y + 2)^2$

or $5x^2 + 5y^2 - 10x + 20y = x^2 + 9y^2 + 4 - 6xy - 4x + 12y$

or $4x^2 - 4y^2 - 6x + 8y + 6xy - 4 = 0$

or $2x^2 - 2y^2 + 3xy - 3x + 4y - 2 = 0$ ...(ii)

From (ii), $2x^2 + 3(y - 1)x - (2y^2 - 4y + 2) = 0$

$\therefore \quad x = \frac{-3(y-1) \pm \sqrt{9(y-1)^2 + 8(2y^2 - 4y + 2)}}{4}$

or $4x + 3y - 3 = \pm\sqrt{25y^2 - 50y + 25} = \pm 5(y-1)$

$\therefore$ Separate equations of tangents are

$4x - 2y + 2 = 0$ i.e., $2x - y + 1 = 0$

and $4x + 8y - 8 = 0$ i.e., $x + 2y - 2 = 0$.

Concept Application

26. The length of the tangent drawn from the point (2, 3) to the circles $2(x^2 + y^2) - 7x + 9y - 11 = 0$.

(*a*) 18 (*b*) 14

(*c*) $\sqrt{14}$ (*d*) $\sqrt{28}$

27. Tangents are drawn from (4, 4) to the circle $x^2 + y^2 - 2x - 2y - 7 = 0$ to meet the circle at A and B. The length of the chord *AB* is

(*a*) $2\sqrt{3}$ (*b*) $3\sqrt{2}$

(*c*) $2\sqrt{6}$ (*d*) $6\sqrt{2}$

28. Pair of tangents are drawn from every point on the line $3x + 4y = 12$ on the circle $x^2 + y^2 = 4$. Their variable chord of contact always passes through a fixed point whose co-ordinates are

(*a*) $\left(\frac{4}{3}, \frac{3}{4}\right)$ (*b*) $\left(\frac{3}{4}, \frac{3}{4}\right)$

(*c*) (1, 1) (*d*) $\left(1, \frac{4}{3}\right)$

29. The angle between the two tangents from the origin to the circle $(x-7)^2 + (y+1)^2 = 25$ is

(*a*) 0 (*b*) $\frac{\pi}{3}$

(*c*) $\frac{\pi}{6}$ (*d*) $\frac{\pi}{2}$

30. The number of tangents which can be drawn from the point (1, 2) to the circle $x^2 + y^2 - 2x - 4y + 4 = 0$ are

(*a*) 0 (*b*) 1

(*c*) 2 (*d*) 3

31. Number of tangents of the circle $x^2 + y^2 = 1$ which are parallel to the *x*-axis

(*a*) 2 (*b*) 1

(*c*) 3 (*d*) 4

32. Find the length of the Tangent from any point on the circle

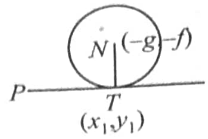
$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + \lambda = 0$ to the circle

$S_2 \equiv x^2 + y^2 + 2gx + 2gx + 2fy + \mu = 0$ is

NORMAL OF CIRCLE

Normal at a point on the circle is the straight line which is perpendicular to the tangent at the point of contact and passes through the centre of circle.

(*a*) Equation of normal at point (x_1, y_1) on circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is



$$y - y_1 = \left(\frac{y_1 + f}{x_1 + g}\right)(x - x_1)$$

(*b*) The equation of normal on any point (x_1, y_1) of circle $x^2 + y^2 = a^2$ is $\left(\frac{y}{x} = \frac{y_1}{x_1}\right)$

(*c*) If $x^2 + y^2 = a^2$ is the equation of the circle then at any point '*t*' of this circle (a cos t, a sin t), the equation of normal is $x \sin t - y \cos t = 0$.

Train Your Brain

Example 31. Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).

Sol. Since normal to the circle always passes through the centre so equation of the normal will be the line passing through (5, 6) and $\left(\frac{5}{2}, -1\right)$ i.e. $y + 1 = \frac{7}{5/2}\left(x - \frac{5}{2}\right)$

$\Rightarrow 5y + 5 = 14x - 35 \Rightarrow 14x - 5y - 40 = 0$

Example 32. If the straight line $ax + by = 2$; $a, b \neq 0$ touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, then the values of a and b are respectively

Sol. Given $x^2 + y^2 - 2x = 3$

$\therefore$ centre is (1, 0) and radius is 2 and for circle $x^2 + y^2 - 4y = 6$, centre is (0, 2) and radius is $\sqrt{10}$. Since line $ax + by = 2$ touches the first circle

$\therefore \left|\frac{a(1) + b(0) - 2}{\sqrt{a^2 + b^2}}\right| = 2$ or $(a - 2) = [2\sqrt{a^2 + b^2}]$ ...(*i*)

Also the given line is normal to the second circle. Hence it will pass through the centre of the second circle.

$\therefore$ $a(0) + b(2) = 2$ or $2b = 2$ or $b = 1$

Putting this value in equation (*i*) we get

$a - 2 = 2\sqrt{a^2 + 1^2}$ or $(a - 2)^2 = 4(a^2 + 1)$

or $a^2 + 4 - 4a = 4a^2 + 4$ or $3a^2 + 4a = 0$

or $a(3a + 4) = 0$ or $a = 0, -\frac{4}{3}$

$\therefore$ values of a and b are $\left(-\frac{4}{3}, 1\right)$ respectively according to the given choices.

Concept Application

33. The normal at the point (3, 4) on a circle cuts the circle at the point (–1, –2). Then the equation of the circle is

(*a*) $x^2 + y^2 + 2x - 2y - 13 = 0$

(*b*) $x^2 + y^2 - 2x - 2y - 11 = 0$

(*c*) $x^2 + y^2 - 2x + 2y + 12 = 0$

(*d*) $x^2 + y^2 - 2x - 2y + 14 = 0$

34. The equation of normal to the circle $x^2 + y^2 - 4x + 4y - 17 = 0$ which passes through (1, 1) is

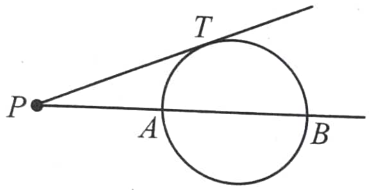
(*a*) $3x + y - 4 = 0$ (*b*) $x - y = 0$

(*c*) $x + y = 0$ (*d*) None of these

POWER OF THE POINT

Square of the length of the tangent from the point P is defined as power of the point 'P' with respect to given circle $\Rightarrow PT^2 = S_1$

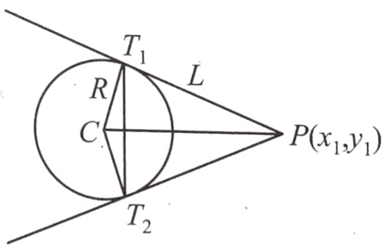
Note: Power of a point remains constant with respect to a circle $PA \cdot PB = PT^2$



CHORD OF CONTACT

A line joining the two points of contacts of two tangents drawn from a point out side the circle, is called chord of contact of that point.

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is : $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (i.e. $T = 0$ same as equation of tangent).



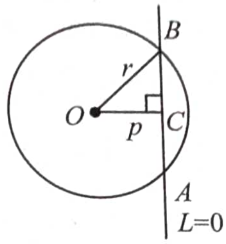
Note:

❖ For the existence of chord of contact points should be outside the circle.

❖ The above result is valid for all second degree conic.

LENGTH OF THE CHORD

Let O be the centre of the circle $S = 0$ and the line of the chord AB be $L = 0$.

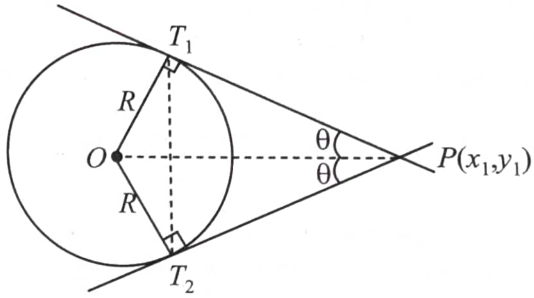


Let p = length of $\perp$ from centre to the line $L = 0$

Then AB = length of the chord

$= 2\sqrt{(r^2 - p^2)}$ (by Pythagoras theorem)

Remember:



(*a*) Angle between the pair of tangents from $P(x_1, y_1)$ is equal to 2θ

where, $\tan\theta = \dfrac{R}{L}$ so, $2\theta = \tan^{-1}\left(\dfrac{2RL}{L^2 - R^2}\right)$

(*b*) Length of chord of contact, $T_1T_2 = 2L\sin\theta$ (where, $\tan\theta = \dfrac{R}{L}$)

so, $T_1T_2 = \dfrac{2LR}{\sqrt{R^2 + L^2}}$.

(*c*) Area of the triangle formed by the pair of the tangents and its chord of contact

area $= \dfrac{1}{2} \times T_1T_2 \times L\cos\theta$

so, area $= \dfrac{RL^3}{R^2 + L^2}$

(*d*) Equation of the circle circumscribing the triangle PT_1T_2 or quadrilateral CT_1PT_2 is :

$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

(since P, T_1, O, T_2 are concyclic)

(*e*) The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.

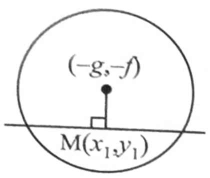
Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$;

$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT ($T = S_1$)

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$. This on simplification can be put in the form $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.



Note:

1. The shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.
2. Chord bisected at point (x_1, y_1) is the farthest from centre among all the chords passing through the point (x_1, y_1). Also for such chord, the length of the chord is minimum.

EQUATION OF CHORD WHEN END POINTS OF CHORD GIVEN IN PARAMETRIC FORM

The equation of chord joining $P(\alpha) = (a\cos\alpha, a\sin\alpha)$ and $Q(\beta) = (a\cos\beta, a\sin\beta)$ on the circle $x^2 + y^2 = a^2$ is

$$x\cos\left(\frac{\alpha+\beta}{2}\right) + y\sin\left(\frac{\alpha+\beta}{2}\right) = a\cos\left(\frac{\alpha-\beta}{2}\right)$$

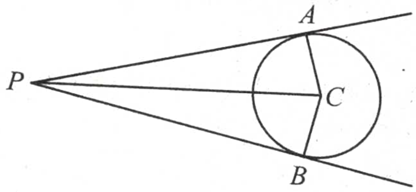
Train Your Brain

Example 33. Find the area of the quadrilateral formed by a pair of tangents from the point (4, 5) to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ and a pair of its radii.

Sol. Given circle is $S \equiv x^2 + y^2 - 4x - 2y - 11 = 0$ ...(*i*)

Let C be its centre and a be its radius; then $C \equiv (2, 1)$ and $a = 4$.

Let $P \equiv (4, 5)$



Now length of tangent PA or PB from P to circle

$= \sqrt{4^2 + 5^2 - 4.4 - 2.5 - 11} = 2$

and radius $CA = 4$

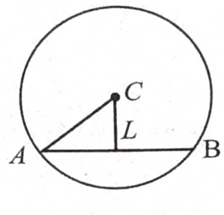
$\therefore$ area of $\Delta PAC = \frac{1}{2}PA.AC = \frac{1}{2}2.4 = 4$

$\therefore$ area of quadrilateral $PACB = 2$ area of $\Delta PAC = 8$ square units.

Examle 34. Find the length of the chord $4x - 3y = 5$ of the circle $x^2 + y^2 + 3x - y - 10 = 0$.

Sol. Given line is $4x - 3y - 5 = 0$ ... (*i*)

and given circle is $x^2 + y^2 + 3x - y - 10 = 0$ ... (*ii*)



Let C be the centre and a the radius of circle (*ii*),

then $C \equiv \left(-\frac{3}{2}, \frac{1}{2}\right)$ and $a^2 = \frac{9}{4} + \frac{1}{4} + 10 = \frac{25}{2}$

CL = length of perpendicular from C to line (*i*)

$$= \frac{\left|4\left(-\frac{3}{2}\right) - 3\left(\frac{1}{2}\right) - 5\right|}{\sqrt{4^2 + (-3)^2}} = \frac{5}{2}.$$

Let line (*i*) cut the circle (*ii*) at A and B, then,

$$AB = 2AL = 2\sqrt{a^2 - CL^2} = 2\sqrt{\frac{25}{2} - \frac{25}{4}} = 5 \text{ units.}$$

Example 35. Let a circle be given by $2x(x - a) + y(2y - b) = 0$ $(a \neq 0, b \neq 0)$. Find the condition on a and b if two chords, each bisected by the x-axis, can be drawn to the circle from $(a, b/2)$.

Sol. The given circle is $2x(x - a) + y(2y - b) = 0$

or $x^2 + y^2 - ax - by/2 = 0$

Let AB be the chord which is bisected by x-axis at a point M. Let its co-ordinates be $M(h, 0)$.

and $S \equiv x^2 + y^2 - ax - by/2 = 0$

$\therefore$ Equation of chord AB is $T = S_1$

$$hx + 0 - \frac{a}{2}(x + h) - \frac{b}{4}(y + 0) = h^2 + 0 - ah - 0$$

Since its passes through $(a, b/2)$ we have $ah - \frac{a}{2}(a + h) - \frac{b^2}{8} = h^2 - ah \Rightarrow h^2 - \frac{3ah}{2} + \frac{a^2}{2} + \frac{b^2}{8} = 0$

Now there are two chords bisected by the x-axis, so there must be two distinct real roots of h.

$\therefore$ $B^2 - 4AC > 0$

$$\Rightarrow \left(\frac{-3a}{2}\right)^2 - 4.1.\left(\frac{a^2}{2} + \frac{b^2}{8}\right) > 0$$

$\Rightarrow$ $a^2 > 2b^2$.

Concept Application

35. The locus of the centres of the circles such that the point (2, 3) is the mid point of the chord $5x + 2y = 16$ is

(a) $2x - 5y + 11 = 0$ (b) $2x + 5y - 11 = 0$

(c) $2x + 5y + 11 = 0$ (d) None of these

36. The equation of the circle having the lines $y^2 - 2y + 4x - 2xy = 0$ as its normals and passing through the point (2, 1) is

(a) $x^2 + y^2 - 2x - 4y + 3 = 0$

(b) $x^2 + y^2 - 2x + 4y - 5 = 0$

(c) $x^2 + y^2 + 2x + 4y - 13 = 0$

(d) None of these

37. A circle is drawn touching the x-axis and centre at the point which is the reflection of (a, b) in the line $y - x = 0$. The equation of the circle is

(a) $x^2 + y^2 - 2bx - 2ay + a^2 = 0$

(b) $x^2 + y^2 - 2bx - 2ay + b^2 = 0$

(c) $x^2 + y^2 - 2ax - 2by + b^2 = 0$

(d) $x^2 + y^2 - 2ax - 2by + a^2 = 0$

38. The co-ordinates of the point from where the tangents are drawn to the circles $x^2 + y^2 = 1$, $x^2 + y^2 + 8x + 15 = 0$ and $x^2 + y^2 + 10y + 24 = 0$ are of same length, are

(a) $\left(2, \frac{5}{2}\right)$ (b) $\left(-2, -\frac{5}{2}\right)$

(c) $\left(-2, \frac{5}{2}\right)$ (d) $\left(2, -\frac{5}{2}\right)$

39. The locus of the middle points of chords of the circle $x^2 + y^2 - 2x - 6y - 10 = 0$ which passes through the origin, is

(a) $x^2 + y^2 + x + 3y = 0$ (b) $x^2 + y^2 - x + 3y = 0$

(c) $x^2 + y^2 + x - 3y = 0$ (d) $x^2 + y^2 - x - 3y = 0$

DIRECTOR CIRCLE

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let $P(h, k)$ is the point of intersection of two tangents drawn on the circle $x^2 + y^2 = a^2$. Then the equation of the pair of tangents is $SS_1 = T^2$.

i.e. $(x^2 + y^2 - a^2)(h^2 + k^2 - a^2) = (hx + ky - a^2)^2$

As lines are perpendicular to each other then, coefficient of x^2 + coefficient of $y^2 = 0$.

$\Rightarrow$ $[(h^2 + k^2 - a^2) - h^2] + [(h^2 + k^2 - a^2) - k^2] = 0$

$\Rightarrow$ $h^2 + k^2 = 2a^2$

$\therefore$ locus of (h, k) is $x^2 + y^2 = 2a^2$ which is the equation of the director circle.

Method 2: As we know, equation of tangent with slope m of the circle $x^2 + y^2 = a^2$ is

$$y = mx \pm a\sqrt{1+m^2}$$

point (h, k) will lie on tangent.

$$\therefore\ k = mh \pm a\sqrt{1+m^2}$$

By solving:

$$m^2(h^2 - a^2) - 2hkm + k^2 - a^2 = 0$$

As tangents are perpendicular,

$$m_1m_2 = -1$$

$$\frac{k^2 - a^2}{h^2 - a^2} = -1$$

$$h^2 + k^2 = 2a^2$$

$$x^2 + y^2 = 2a^2$$

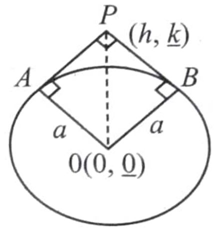
Method 3: Let (h, k) be the point of interrection of two tangents from figure it is abuious that $OAPB$ is a square with side a diagonal of $\Delta OAPB = \sqrt{2}a$

$$OP = \sqrt{2}a$$

$$\sqrt{(h-0)^2 + (k-0)^2} = \sqrt{2}a$$

$$h^2 + k^2 = 2a^2$$

$$x^2 + y^2 = 2a^2$$



Note: Director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

So, the director circle of $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$

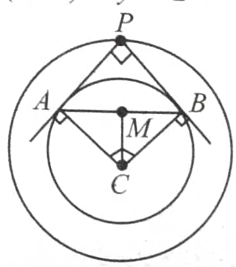
Train Your Brain

Example 36. Let P be any moving point on the circle $x^2 + y^2 - 2x = 1$, from this point chord of contact is drawn w.r.t the circle $x^2 + y^2 - 2x = 0$. Find the locus of the circumcentre of the triangle CAB, C being centre of the circle.

Sol. The two circles are

$(x - 1)^2 + y^2 = 1$ ... (i)

$(x - 1)^2 + y^2 = 2$ ...(ii)



So the second circle is the director circle of the first. So $\angle APB = \pi/2$. Also $\angle ACB = \pi/2$

Now circumcentre of the right angled triangle CAB would lie on the mid point of AB

So let the point be $M \equiv (h, k)$

Now, $CM = CB \sin 45^\circ = \frac{1}{\sqrt{2}}$

So, $(h-1)^2 + k^2 = \left(\frac{1}{\sqrt{2}}\right)^2$

So, locus of M is $(x-1)^2 + y^2 = \frac{1}{2}$.

Concept Application

40. The angle between the two tangents from the origin to the circle $(x-7)^2 + (y+1)^2 = 25$ equals

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

41. Tangents are drawn from the point (2, 3) to the circle $x^2 + y^2 = 9$, then

Statement I: Tangents are mutually perpendicular.

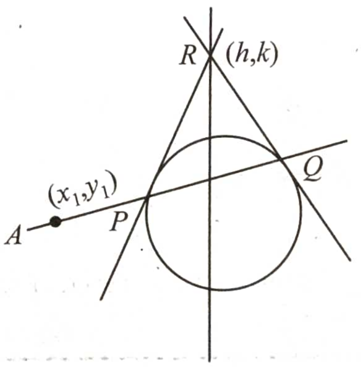
Statement II: Locus of point of intersection of perpendicular tangents is $x^2 + y^2 = 18$.

Given statements are:

(a) TT (b) TF (c) FT (d) FF

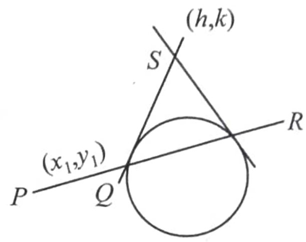
POLE AND POLAR

Let any straight line through the given point $A(x_1, y_1)$ intersect the given circle $S = 0$ in two points P and Q and if the tangent of the circle at P and Q meet at the point R then locus of point R is called polar of the point A and point A is called the pole, with respect to the given circle.



(a) The equation of the polar of point (x_1, y_1) w.r.t. circle $x^2 + y^2 = a^2$.

Let PQR is a chord which passes through the point $P(x_1, y_1)$ which intersects the circle at point Q and R and the tangents are drawn at points Q and R meet at point $S(h, k)$ then equation of QR the chord of contact is $x_1h + y_1k = a^2$



$\therefore$ locus of point $S(h, k)$ is $xx_1 + yy_1 = a^2$ which is the equation of the polar.

Note:

(i) The equation of the polar is, $T = 0$, so the polar of point (x_1, y_1) w.r.t. circle
$x^2 + y^2 + 2gx + 2fy + c = 0$ is
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

(ii) If point is outside the circle then equation of polar and chord of contact is same. So the chord of contact is polar.

(iii) If point is inside the circle then chord of contact does not exist but polar exists.

(iv) If point lies on the circle then polar, chord of contact and tangent on that point are same.

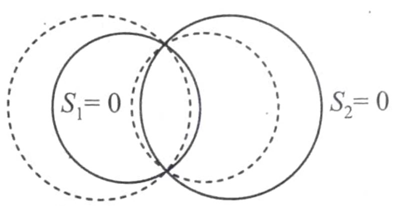
(v) If the polar of P w.r.t. a circle passes through the point Q, then the polar of point Q will pass through P and hence P and Q are conjugate points of each other w.r.t. the given circle.

(vi) If pole of a line w.r.t. a circle lies on second line. Then pole of second line lies on first line and hence both lines are conjugate lines of each other w.r.t. the given circle.

(b) Pole of a given line with respect to a circle.

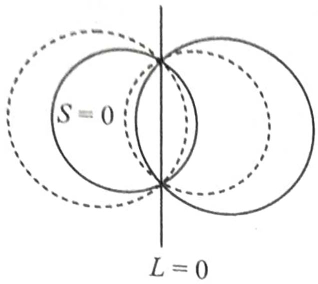
To find the pole of a line we assume the coordinates of the pole then from these coordinates we find the polar. This polar and given line represent the same line. Then by comparing the coefficients of similar terms we can get the coordinates of the pole. The pole of $\ell x + my + n = 0$.

w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\frac{-\ell a^2}{n}, \frac{-ma^2}{n}\right)$

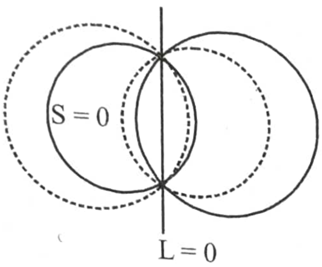


FAMILY OF CIRCLES

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ and $S_2 = 0$ is ; $S_1 + KS_2 = 0$ $(K \neq -1)$.



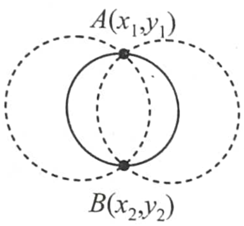
(b) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ and line $L = 0$ is given by $S + KL = 0$,



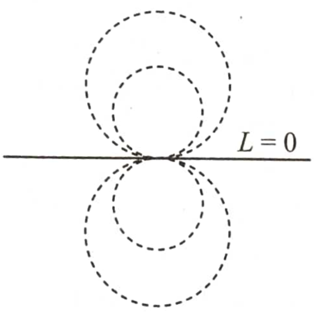
(c) The equation of a family of circles passing through two given points (x_1, y_1) and (x_2, y_2) can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

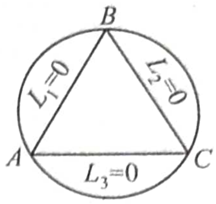
where K is a parameter.



(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

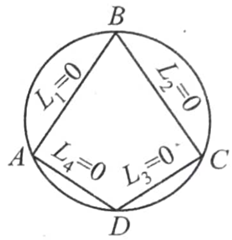


(e) The equation of the family of circles which touch the line $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any value of m is $(x - x_1)^2 + (y - y_1)^2 + \lambda[(y - y_1) - m(x - x_1)] = 0$. If m is infinite, the equation is $(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1)$ 0.



(f) Family of circles cicumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is given by; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ and coefficient of $x^2 =$ coefficient of y^2

(g) Equation of circle circumscribing a quadrilateral whose side in order are represented by the line $L_1=0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ are $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.



(h) Equation of circle (if exist) passing through point of intersection of two straight lines $L_1 = 0$ and $L_2 = 0$ with co-ordinate axes in $L_1L_2 + \lambda xy = 0$

(i) Equation of circle passing through four points of intersection of two second degree conics $S = 0$ and $S' = 0$ is $S + \lambda SS' = 0$

Train Your Brain

Example 37. The equation of the circle through the points of intersection of $x^2 + y^2 - 1 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and touching the line $x + 2y = 0$, is

Sol. Family of circle is

$$x^2 + y^2 - 2x - 4y + 1 + \lambda(x^2 + y^2 - 1) = 0$$

$$(1+\lambda)x^2 + (1+\lambda)y^2 - 2x - 4y + (1-\lambda) = 0$$

or $$x^2 + y^2 - \frac{2}{1+\lambda}x - \frac{4}{1+\lambda}y + \frac{1-\lambda}{1+\lambda} = 0$$

Centre is $\left[\frac{1}{1+\lambda}, \frac{2}{1+\lambda}\right]$ and

$$\text{radius} = \sqrt{\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 - \frac{1-\lambda}{1+\lambda}} = \frac{\sqrt{4+\lambda^2}}{1+\lambda}.$$

Since it touches the line $x + 2y = 0$, hence,

Radius = Perpendicular distance from centre to the line.

i.e.,

$$\left|\frac{\frac{1}{1+\lambda}+2\frac{2}{1+\lambda}}{\sqrt{1^2+2^2}}\right| = \frac{\sqrt{4+\lambda^2}}{1+\lambda} \Rightarrow \sqrt{5} = \sqrt{4+\lambda^2}$$

$\Rightarrow \quad \lambda = \pm 1$

$\lambda = -1$ cannot be possible in case of circle. So $\lambda = 1$. Thus, we get the equation of circle $x^2 + y^2 - x - 2y = 0$.

Example 38. The length of the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$, is

(*a*) $\frac{9}{2}$ (*b*) $2\sqrt{2}$ (*c*) $3\sqrt{2}$ (*d*) $\frac{3}{2}$

Sol. Equation of common chord is $2x + 1 = 0$

$\Rightarrow x = -1/2$

This cuts the circle at points

$$\left(-\frac{1}{2}\right)^2 + y^2 + 2\left(-\frac{1}{2}\right) + 3y + 1 = 0$$

$$\Rightarrow \frac{1}{4} + y^2 - 1 + 3y + 1 = 0$$

$$\Rightarrow 4y^2 + 12y + 1 = 0$$

$$\Rightarrow y = \frac{-12 \pm 8\sqrt{2}}{8} = \frac{-3 \pm 2\sqrt{2}}{2}$$

$\therefore$ length $= 2\sqrt{2}$.

Hence (*b*) is the correct answer.

Example 39. The circles $x^2 + y^2 + 6x + 6y = 0$ and $x^2 + y^2 - 12x - 12y = 0$

(*a*) Touch each other internally

(*b*) Touch each other externally

(*c*) Intersect in two points

(*d*) Cut orthogonally

Sol. Centres of circles are $(-3, -3)$ and $(6, 6)$ and radii are $3\sqrt{2}$ and $6\sqrt{2}$.

So $C_1C_2 = r_1 + r_2$

Hence (*b*) is the correct answer.

Examle 40. Equation of a circle with centre (4, 3) touching the circle $x^2 + y^2 = 1$ is

(*a*) $x^2 + y^2 - 8x - 6y - 9 = 0$

(*b*) $x^2 + y^2 - 8x - 6y + 11 = 0$

(*c*) $x^2 + y^2 - 8x - 6y - 11 = 0$

(*d*) $x^2 + y^2 - 8x - 6y + 9 = 0$

Sol. Let the circle touching the circle $x^2 + y^2 = 1$, be $x^2 + y^2 - 8x - 6y + k = 0$,

The equation of the common tangent is $S_1 - S_2 = 0$

i.e. $8x + 6y - 1 - k = 0$

This is a tangent to the circle $x^2 + y^2 = 1$.

$$\left|\frac{-1-k}{\sqrt{8^2+6^2}}\right| = 1$$

$\Rightarrow k + 1 = \pm 10 \Rightarrow k = -11$ or 9

Therefore the circles are $x^2 + y^2 - 8x - 6y + 9 = 0$ and $x^2 + y^2 - 8x - 6y - 11 = 0$

Hence (*c*) and (*d*) are the correct answers.

Examle 41. Find the circle whose diameter is the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.

Sol. Given circles are

$S \equiv x^2 + y^2 + 2x + 3y + 1 = 0$

and $S' \equiv x^2 + y^2 + 4x + 3y + 2 = 0$

Hence, their common chord is $S - S' = 0$

$\Rightarrow -2x - 1 = 0$ or $2x + 1 = 0$ ...(*i*)

Now, the required circle must pass through the point of intersection of S and S'.

Hence, its equation is $S + \lambda S' = 0$

$\Rightarrow (x^2 + y^2 + 2x + 3y + 1) + \lambda(x^2 + y^2 + 4x + 3y + 2) = 0$

$\Rightarrow x^2(1 + \lambda) + y^2(1 + \lambda) + 2x(1 + 2\lambda) + 3y(1 + \lambda) + (1 + 2\lambda) = 0 + 3y + (1 + 2\lambda) = 0$

or $x^2 + y^2 + 2x\frac{(1+2\lambda)}{(1+\lambda)} + 3y + \frac{(1+2\lambda)}{(1+\lambda)} = 0$ ...(*ii*)

Its centre is $\left(-\frac{1+2\lambda}{1+\lambda}, -\frac{3}{2}\right)$

But from Eq. (*i*), $2x + 1 = 0$ is a diameter of this circle. Hence, its centre must lie on this line

$\Rightarrow \quad -2 - 4\lambda + 1 + \lambda = 0 \quad \Rightarrow \quad -1 - 3\lambda = 0$

$\therefore \quad \lambda = -\frac{1}{3}$

Hence, from Eq. (*ii*), the required circle is

$2x^2 + 2y^2 + 2x + 6y + 1 = 0$

Concept Application

42. The two circles $x^2 + y^2 - 2x + 6y + 6 = 0$ and $x^2 + y^2 - 5x + 6y + 15 = 0$ touch each other. The equation of their common tangent is

(*a*) $x = 3$ (*b*) $y = 6$

(*c*) $7x - 12y - 21 = 0$ (*d*) $7x + 12y + 21 = 0$

43. Circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$

(*a*) Touch each other internally

(*b*) Touch each other externally

(*c*) Cuts each other at two points

(*d*) Cuts each other more than two points

DIRECT AND TRANSVERSE COMMON TANGENTS

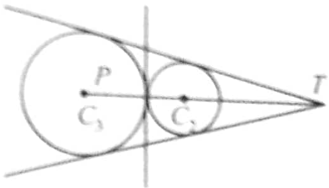
Let two circles having centre c_1 and c_2 and radii, r_1 and r_2 and c_1c_2 is the distance between their centres then

Direct common tangent : The common tangent having centre of both circle on same side

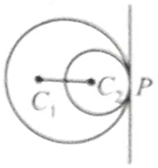
Transverse common tangents: The common tangent having centre of both circle on opposite side.

(a) Both circles will touch:

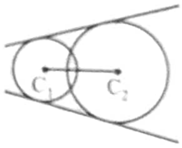
(i) **Externally:** If $c_1c_2 = r_1 + r_2$ i.e the distance between their centres is equal to sum of their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ (internally). In this case there will be **three common tangents.**



(ii) **Internally:** If $C_1C_2 = |r_1 - r_2|$ i.e. the distance between their centres is equal to difference between their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be **only one common tangent.**



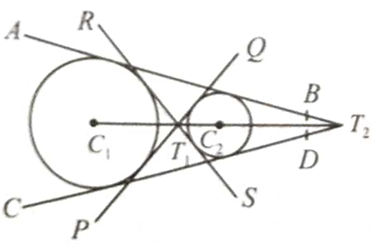
(b) The circles will intersect: When $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there will be **two common tangents.**



(c) The circles will not intersect:

(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$ In this case there will be **no common tangent.**

(ii) When circle are apart from each other the $C_1C_2 > r_1 + r_2$ and in this case there will be four common tangents. Lines PQ and RS are called **transverse or indirect or internal common tangents**.



(a) Direct Common Tangent (D.C.T.) meet at a point which divides the line joining the centre of circles externally in the ratio of their radius.

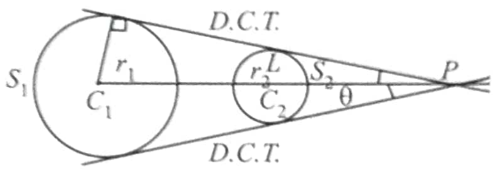
(b) Transverse Common Tangent (T.C.T.) meets at a point which divides the line joining the centres of circles internally in the ratio of thir radius.

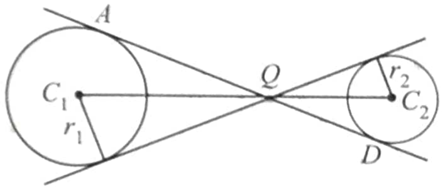
Proof $\frac{PC_1}{PC_2} = \frac{r_1}{r_2}$ (since for $S_2 = 0$, C_2P is the angle bisector and also for $S_1 = 0$, C_1P is the angle bisector.)

$\therefore$ C_1, C_2, P will lie in a same line

$$\frac{C_1Q}{C_2Q} = \frac{r_1}{r_2}$$

Note: $C_1 : Q, C_2, P$ constitute a harmonic range i.e. C_1Q, C_1C_2, C_1P are in H.P.





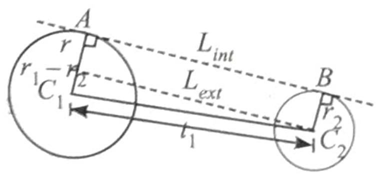
Note: The length of common tangent (both external and intenal common tangent)

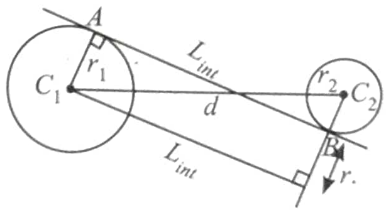
From figure: $L^2_{ext} + (r_1 - r_2)^2 = d^2$

$\Rightarrow L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}$

$\therefore d^2 = L^2_{int} + (r_1 + r_2)^2$

$\therefore L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$





Train Your Brain

Example 42. Tangents to the circle $x^2 + y^2 = a^2$ cut the circle $x^2 + y^2 = 2a^2$ at P and Q. The tangents at P and Q to the circle $x^2 + y^2 = 2a^2$ intersect at

(*a*) Right angles (*b*) 60°

(*c*) Can't be determined (*d*) None of the above.

Sol. Equation of tangent at any point $(a\cos\theta, a\sin\theta)$ is

$x\cos\theta + y\sin\theta = a$ ...(1)

Let the point of intersection of the tangents at P and Q be (h, k). If the tangents at P and Q intersect at right angles, then locus of (h, k) will be director circle of $x^2 + y^2 = 2a^2$ i.e. $x^2 + y^2 = 4a^2$. PQ is chord of contact of the circle $x^2 + y^2 = 2a^2$ w.r.t. the point (h, k) i.e. equation of PQ is $hx + ky = 2a^2$ ...(2)

(1) and (2) are same equation

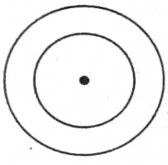
$$\frac{\cos\theta}{h} = \frac{\sin\theta}{k} = \frac{a}{2a^2} = \frac{1}{2a}$$

$\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow h^2 + k^2 = 4a^2$

$\therefore$ locus of (h, k) is $x^2 + y^2 = 4a^2$ which is director circle to $x^2 + y^2 = 2a^2$

Hence (*a*) is the correct answer.



Example 43. The number of common tangents that can be drawn to the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

Sol. The two circles are $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$

Centre: $C_1 \equiv (2, 3)$, $C_2 \equiv (-1, -1)$

radii: $r_1 = 4$, $r_2 = 1$

We have, $C_1C_2 = 5 = r_1 + r_2$, therefore there are 3 common tangents to the given circles.

Hence (*c*) is the correct answer.

Example 44. If the circles $x^2 + y^2 - 8x + 2y + 8 = 0$ and $x^2 + y^2 - 2x - 6y + 10 - a^2 = 0$ have exactly two common tangents, then

(*a*) $1 < |a| < 8$ (*b*) $2 < |a| < 8$

(*c*) $3 < |a| < 8$ (*d*) $4 < |a| < 8$

Sol. Centres of the circles are (4, –1) and (1, 3). Their radii are 3 and $|a|$ respectively. They will have exactly two common tangents if they meet in two distinct points. That means

$\Rightarrow |3 - |a|| < 5 < 3 + |a|$

$|3 - |a|| < \sqrt{3^2 + 4^2} < 3 + |a|$

From $|3 - |a|| < 5$ we get $a \in (-\infty, -2) \cup (2, \infty)$

From $|3 + |a|| < 5$ we get $a \in (-8, 8)$

$a \in (-8, -2) \cup (2, 8)$ i.e. $2 < |a| < 8$

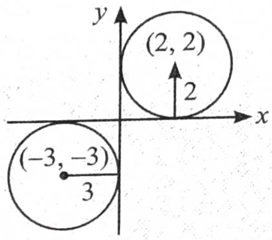
Hence (*b*) is the correct answer.

Example 45. Find equation of transverse common tangent of

$(x - 2)^2 + (y - 2)^2 = 2^2$,

$(y + 3)^2 + (x + 3)^2 = 3^2$

Sol. Equation of transverse common tangent is $x = 0$ and $y = 0$ from figure.

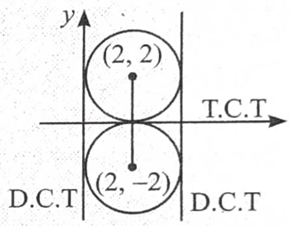


Example 46. Find the equation of D.C.T. and T.C.T.

$(x - 2)^2 + (y - 2)^2 = 2^2$,

$(y - 2)^2 + (x + 2)^2 = 2^2$

Sol. The figure is one of the equation of D.C.T. is $x = 0$ and T.C.T. is $y = 0$



Concept Application

44. The length of the common chord of circles $x^2 + y^2 - 6x - 16 = 0$ and $x^2 + y^2 - 8y - 9 = 0$ is

(*a*) $10\sqrt{3}$ (*b*) $5\sqrt{3}$

(*c*) $5\sqrt{3}/2$ (*d*) None of these

45. If the two circles, $x^2 + y^2 + 2g_1x + 2f_1y = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y = 0$ touches each other, then

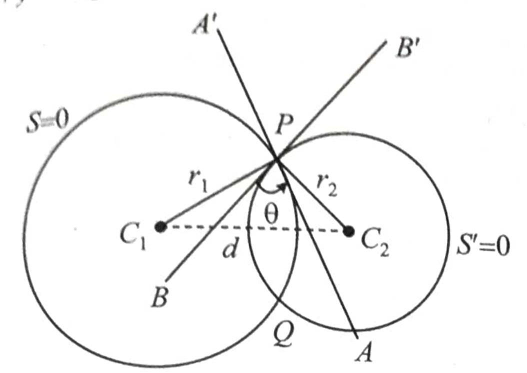
(*a*) $f_1g_1 = f_2g_2$ (*b*) $\frac{f_1}{g_1} = \frac{f_2}{g_2}$

(*c*) $f_1f_2 = g_1g_2$ (*d*) None of these

46. Prove that if a line L = 0 is a tangents to the circle S = 0 then it will also be a tangent to the circle $S + \lambda L = 0$.

ANGLE OF INTERSECTION OF TWO CIRCLES

Let the two circles $S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$



Intersect each other at the points P and Q. The angle θ between two circles S = 0 and S′ = 0 is defined as the angle between the tangents to the two circles at the points of intersection.

C_1, C_2 and r_1, r_2 are centres and radius of circles $S = 0$ and $S' = 0$ then

$C_1 \equiv (-g_1, -f_1)$ and $C_2 \equiv (-g_2, -f_2)$

and $r_1 = \sqrt{(g_1^2 + f_1^2 - c_1)}$ and $r_2 \sqrt{(g_2^2 + f_2^2 - c_2)}$

Let $d = |C_1C_2|$ = Distance between their centres

$$= \sqrt{(-g_1 + g_2)^2 + (-f_1 + f_2)^2}$$

$$= \sqrt{(g_1^2 + f_1^2 + g_2^2 + f_2^2 - 2g_1g_2 - 2f_1f_2)}$$

Now in ΔC_1PC_2, $\cos\alpha = \left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$ ... (*i*)

($\because \alpha + \theta + 90^\circ + 90^\circ = 360^\circ$)

or $\cos(180^\circ - \theta) = \left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$ ($\therefore \alpha = 180^\circ - \theta$)

If the angle between the circles is 90°, i.e., θ = 90°, then the circles are said to be orthogonal circles or we say that the circles cut each other ORTHOGONALLY.

Then from (*i*)

$$0 = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} \text{ or } r_1^2 + r_2^2 - d^2 = 0 \text{ or } r_1^2 + r_2^2 = d^2$$

$$\Rightarrow g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2 = g_1^2 + f_1^2 + g_2^2 + f_2^2 - 2g_1g_2 - 2f_1f_2$$

or **Condition of orthogonality**

$$\boxed{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$$

Train Your Brain

Example 47. If the circles $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is

(*a*) 2 or $-\dfrac{3}{2}$ (*b*) –2 or –

(*c*) 2 or (*d*) –2 or

Sol. Apply $2gg_1 + 2ff_1 = c_1 + c_2$

$\Rightarrow 2(0) + 2kk = 6 + k$

or $2k^2 - k - 6 = 0 \Rightarrow k = 2, -3/2$

Hence (*a*) is the correct answer.

Concept Application

47. The two circles $x^2 + y^2 - 2x + 22y + 5 = 0$ and $x^2 + y^2 + 14x + 6y + k = 0$ intersect orthogonally provided k is equal to

(*a*) 47 (*b*) – 47

(*c*) 49 (*d*) – 49

48. Let 'θ' is the angle of intersection of two circles with centres C_1 and C_2 and radius r_1 and r_2 respectively then the circles touch each other.

(*a*) $\cos\theta = \pm 1$ (*b*) $\cos\theta = 1/2$

(*c*) $\cos\theta = -1/2$ (*d*) $\cos\theta = 1/3$

COMMON CHORD OF TWO CIRCLES

The common chord joining the point of intersection of two given circles is called their common chord.

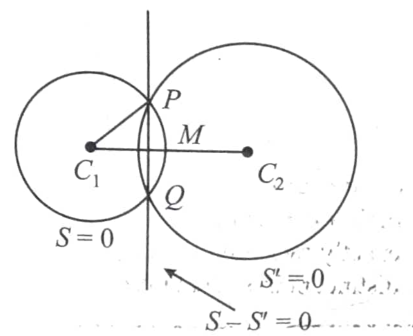
If $S = 0$ and $S' = 0$ be two intersecting circles, the equation of their common chord is

$S - S' = 0$

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 10$

and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 10$

be two circles intersecting at P and Q.



Then PQ is their common chord.

$\therefore \quad S - S' = 0$

$\Rightarrow \quad 2(g - g')x + 2(f - f')y + c - c' = 0$

is the common chord of two circles $S = 0$ and $S' = 0$.

Length of the Common Chord

$$PQ = 2(PM) = 2\sqrt{\{(C_1P)^2 - (C_1M)^2\}}$$

where C_1P = radius of the circle $S = 0$ and C_1M is the length of perpendicular from C_1 on common Chord PQ.

Notes:

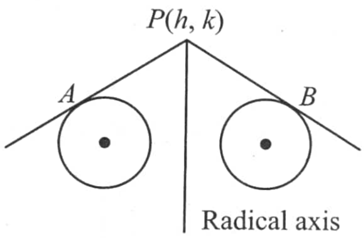
1. The length of common chord PQ of two cirlces is maximum when it is a diameter of the smaller circle.
2. IF circle is described on the common chord as a diameter then centre of the circle passing through P and Q lie on the common chord of two circles i.e., $S - S' = 0$.
3. If the length of common chord is zero, then the two circles touch each other and the common chord becomes the common tangent to the two circles at the common point of contact.

RADICAL AXIS AND RADICAL CENTRE

(*a*) **Radical axis:** The locus of a point, which moves in such a way that the whose power with respect to the circles are equal, is called the radical axis. If two circles are:

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$



Let $P(h, k)$ be the point, then from definition :

$$h^2 + k^2 + 2g_1h + 2f_1k + c_1 = h^2 + k^2 + 2g_2h + 2f_2k + c_2$$

or $2(g_1 - g_2)h + 2(f_1 - f_2)k + c_1 - c_2 = 0$

$\therefore$ locus of (h, k)

$\Rightarrow 2x(g_1 - g_2) + 2y(f_1 - f_2)k + c_1 - c_2 = 0$

$\Rightarrow \boldsymbol{S_1 - S_2 = 0}$

which is the equation of radical axis.

Note:

(*i*) To get the equation of the radical axis first of all make the coefficient of x^2 and $y^2 = 1$

(*ii*) If circles touch each other then Radical axis is the common tangent to both the circles.

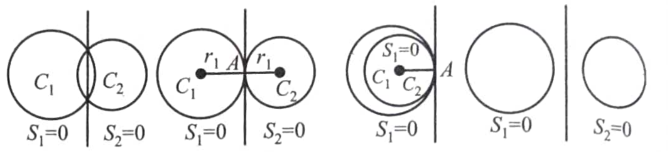
(*iii*) When the two circles intersect on real points then common chord is the Radical axis of the two circles.

(*iv*) The Radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.

(*v*) If circles are concentric then the Radical axis does not always exist but if circles are not concentric then Radical axis always exists.

(*vi*) If two circles are orthogonal to the third circle then Radical axis of both circles passes through the centre of the third circle.

(*vii*) A system of circle, every pair of which have the same radical axis, is called a coaxial system of circles.



Common chord and radical axis | Common tangent and radical axis | Only radical axis

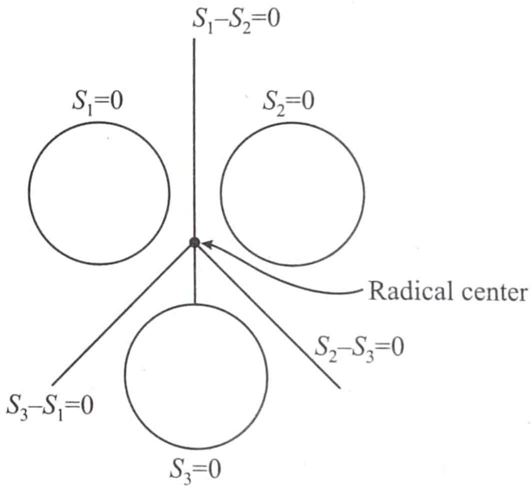
(*b*) **Radical centre:** The Radical axis of three circles (Taking two at a time) meet on a point. This point of intersection of radical axes is knonw as radical centre.

Radical Axis of S_1 and S_2 :

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$$

Radical Axis of S_2 and S_3 :

$$2(g_2 - g_3)x + 2(f_2 - f_3)y + c_2 - c_3 = 0$$



Radical Axis of S_3 and S_1 :

$$2(g_3 - g_1)x + 2(f_3 - f_1)y + c_3 - c_1 = 0$$

$$D = \begin{vmatrix} g_1 - g_2 & f_1 - f_2 & c_1 - c_2 \\ g_2 - g_3 & f_2 - f_3 & c_2 - c_3 \\ g_3 - g_1 & f_3 - f_1 & c_3 - c_1 \end{vmatrix} = 0;$$

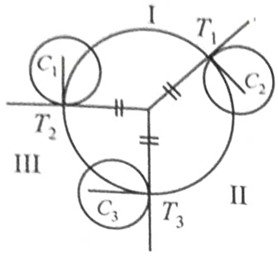
Use $R_1 \to R_1 + R_2 + R_3$

$\Rightarrow D = 0$

(solve any 2 radical axes to get radical centre which is a point from which tangent to the three circles are equal.)

Note:

(*i*) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.



(*ii*) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocentre will be its radical centre.

(*iii*) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles.

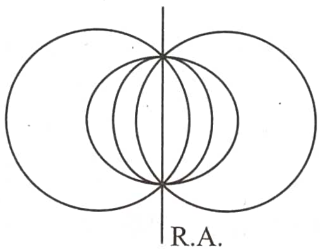
(*iv*) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles, $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent is a circle which is orthogonal to all the three circles.

AN IMPORTANT RESULT ABOUT RADICAL AXES

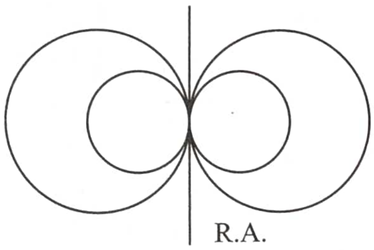
Let $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$ be three circles. Then their radical axes taken in pairs are either parallel or concurrent. Radical axes are parallel if the centres of the circles are collinear.

COAXIAL SYSTEM OF CIRCLES

Definition : A system of circles, every 2 of which have the same radical axis, is called Coaxial system of circles.



or



$$x^2 + y^2 - 4x - 6y - 12 + \lambda(-10x - 10y) = 0$$

Train Your Brain

Example 48. Prove that the circle $x^2 + y^2 - 6x - 4y + 9 = 0$ bisects the circumference of the circle $x^2 + y^2 - 8x - 6y + 23 = 0$.

Sol. Given circles are

$S_1 \equiv x^2 + y^2 - 6x - 4y + 9 = 0$ ... (*i*)

and $S_2 \equiv x^2 + y^2 - 8x - 6y + 23 = 0$ ... (*ii*)

Equation of common chord of circles (*i*) and (*ii*) which is also the radical axis of circles (*i*) and (*ii*) is

$S_1 - S_2 = 0$ or $2x + 2y - 14 = 0$ or $x + y - 7 = 0$ ... (*iii*)

Centre of circle (*ii*) is (4, 3). Clearly line (*iii*) passes through the point (4, 3) and hence line (*iii*) is the diameter of circle (*ii*). Hence circle (*i*) bisects circumference of circle (*ii*).

Example 49. Find the equation of the circle through the points of intersection of the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ and cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Sol. The equation of the circle through the intersection of the given circles is

$x^2 + y^2 - 4x - 6y - 12 + \lambda(-10x - 10y) = 0$ ...(*i*)

where $(-10x - 10y = 0)$ is the equation of radical axis for the circle $x^2 + y^2 - 4x - 6y - 12 = 0$

and $x^2 + y^2 + 6x + 4y - 12 = 0$

Equation (*i*) can be rearranged as

$x^2 + y^2 - x(10\lambda + 4) - y(10\lambda + 6) - 12 = 0$

It cuts the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Hence $2gg_1 + 2ff_1 = c + c_1$

$\Rightarrow 2(5\lambda + 2)(1) + 2(5\lambda + 3)(0) = -12 - 4 \Rightarrow \lambda = -2$

Hence the required circle is

$x^2 + y^2 - 4x - 6y - 12 - 2(-10x - 10y) = 0$

i.e., $x^2 + y^2 + 16x + 14y - 12 = 0$

Example 50. Find the radical centre of circle

$x^2 + y^2 + 3x + 2y + 1 = 0$

$x^2 + y^2 - x + 6y + 5 = 0$ and

$x^2 + y^2 + 5x - 8y + 15 = 0$.

Also find the equation of the circle cutting them orthogonally.

Sol. Equation of two radical axes are

$S_1 - S_2 \equiv 4x - 4y - 4 = 0$ or $x - y - 1 = 0$

and $S_2 - S_3 \equiv -6x - 14y - 10 = 0$ or $3x + 7y + 5 = 0$

Solving them the radical centre is (3, 2) also, if r is the length of the tangent drawn from the radical centre (3, 2) to any one of the given circles, S_1, we have $r = \sqrt{S_1} = \sqrt{3^2 + 2^2 + 3.3 + 2.2 + 1} = \sqrt{27}$

Hence (3, 2) is the centre and $\sqrt{27}$ is the radius of the circle intersecting them orthogonally.

$\therefore$ Its equation is $(x-3)^2 + (y-2)^2 = r^2 = 27$

$\Rightarrow x^2 + y^2 - 6x - 4y - 14 = 0$

Example 51. Find the equation of the circle which cuts the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and the lines $x = -g$ and $y = -f$ orthogonally

Sol. $x = -g$, $y = -f$ cuts the circle orthogonally mean these lines one normal to required circle.

Centre of required circle will be $(-g, -f)$

So equation of circle will be $x^2 + y^2 + 2gx + 2fy + c' = 0$

Now it cut the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ orthogonally then

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$2g^2 + 2f^2 = c + c' \quad \Rightarrow \quad c' = 2g^2 + 2f^2 - c$

Equation of required circle will be

$x^2 + y^2 + 2gx + 2fy + 2g^2 + 2f^2 - c = 0$

Example 52. Prove that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ will bisect the circumference of the circle $x^2 + y^2 + 2g'x + 2f'y + c' = 0$.

Sol. $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ ... (*i*)

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ ... (*ii*)

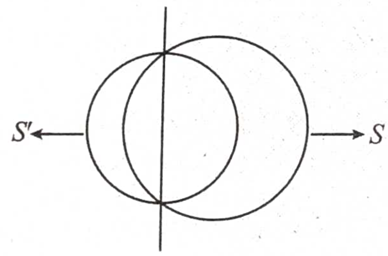
Now equation of radical axis is

$S - S' \equiv 2(g - g')x + 2(f - f')y + c - c' = 0$ ... (*iii*)

If S bisect circumference of S' then radical axis passes through point $(-g', -f')$

So equation (*iii*) will become

$-2(g - g')g' - 2(f - f')f' + c - c' = 0$



$\Rightarrow 2g'(g - g') + 2(f - f')f' = c - c'$

Example 53. Find the locus of the centres of circles which bisect the circumference of circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 2x + 6y + 1 = 0$.

Sol. Let the equation of circle is

$S_1 : x^2 + y^2 + 2gx + 2fy + c = 0$

$S_2 : x^2 + y^2 = 4 \qquad S_3 : x^2 + y^2 - 2x + 6y + 1 = 0$

Radical axis of S_1 and S_2 is $2gx + 2fy + c + 4 = 0$

Radical axis passes through centre of $x^2 + y^2 = 4$

i.e. $(0, 0) \Rightarrow c = -4$

Radical axis of S_1 and S_3 is $(2g + 2)x + (2f - 6)y + c - 1 = 0$

it passes through centre of S_3 i.e. $(1, -3)$

so $2g + 2 - 6f + 18 + c - 1 = 0$ also $c = -4$

$\Rightarrow 2g - 6f + 15 = 0$

Now centre of circle is $(-g, -f)$, $h = -g$, $k = -f$

$\Rightarrow -2h + 6k + 15 = 0$ locus is $2x - 6y - 15 = 0$

Concept Application

49. The equation of the common chord of the circles $(x-a)^2 + (y-b)^2 = c^2$ and $(x-b)^2 + (y-a)^2 = c^2$ is

(*a*) $x - y = 0$ (*b*) $x + y = 0$

(*c*) $x + y = a^2 + b^2$ (*d*) $x - y = a^2 - b^2$

50. Length of the common chord of the circles $x^2 + y^2 + 5x + 7y + 9 = 0$ and $x^2 + y^2 + 7x + 5y + 9 = 0$ is

(*a*) 9 (*b*) 8

(*c*) 7 (*d*) 6

51. The equation of radical axis of the circles $2x^2 + 2y^2 - 7x = 0$ and $x^2 + y^2 - 4y - 7 = 0$ is

(*a*) $7x + 8y + 14 = 0$ (*b*) $7x - 8y + 14 = 0$

(*c*) $7x - 8y - 14 = 0$ (*d*) $-7x - 8y + 14 = 0$

52. The radical centre of the circles $x^2 + y^2 - 16x + 60 = 0$, $x^2 + y^2 - 12x + 27 = 0$, $x^2 + y^2 - 12y + 8 = 0$ is

(*a*) (13, 33/4) (*b*) (33/4, –13)

(*c*) (33/4, 13) (*d*) (33/4, 20/3)

Short Notes

Standard Equations of The Circle

(a) Central Form: If (h, k) is the centre and r is the radius of the circle then its equation is $(x - h)^2 + (y - k)^2 = r^2$.

(b) General equation of circle: $x^2 + y^2 + 2gx + 2fy + c = 0$, where g, f, c are constants and centre is $(-g, -f)$

i.e. $\left(-\dfrac{\text{coefficient of } x}{2}, -\dfrac{\text{coefficient of } y}{2}\right)$

and radius $r = \sqrt{g^2 + f^2 - c}$

Intercepts cut by the circle on axes

The intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on:

(i) x-axis $= 2\sqrt{g^2 - c}$

(ii) y-axis $= 2\sqrt{f^2 - c}$

Diameter form of circle

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

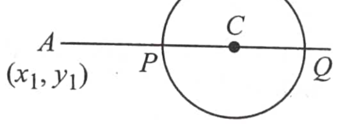
The parametric forms of the circle

(i) The parametric equation of the circle $x^2 + y^2 = r^2$ are $x = r\cos\theta$, $y = r\sin\theta$; $\theta \in [0, 2\pi]$.

(ii) The parametric equation of the circle $(x - h)^2 + (y - k)^2 = r^2$ is $x = h + r\cos\theta$, $y = k + r\sin\theta$ where θ is parameter.

Position of a point w.r.t. Circle

(a) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1, y_1) then:



Point (x_1, y_1) lies outside the circle or on the circle or inside the circle according as

$\Rightarrow \; x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$ or $S_1 >, =, < 0$

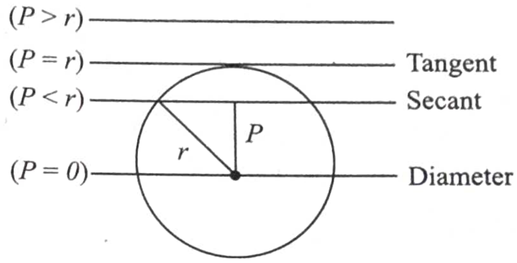
(b) The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $AC - r$ respectively.

(c) The power of point is given by S_1.

Tangent Line of Circle

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(a) Condition of Tangency: The line $L = 0$ touches the circle $S = 0$ if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. $P = r$.



(b) Equation of the tangent (T = 0)

(i) Tangent at the point (x_1, y_1) on the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(ii) (1) The tangent at the point $(a\cos t, a\sin t)$ on the circle $x^2 + y^2 = a^2$ is $x\cos t + y\sin t = a$.

(2) The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is $\left(\dfrac{a\cos\dfrac{\alpha+\beta}{2}}{\cos\dfrac{\alpha-\beta}{2}}, \dfrac{a\sin\dfrac{\alpha+\beta}{2}}{\cos\dfrac{\alpha-\beta}{2}}\right)$

(iii) The equation of tangent at the point (x_1, y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(iv) If line $y = mx + c$ is a straight line touching the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1 + m^2}$ and contact points are $\left(\pm\dfrac{am}{\sqrt{1+m^2}}, \pm\dfrac{1}{\sqrt{1+m^2}}\right)$ or $\left(\pm\dfrac{a^2 m}{c}, \pm\dfrac{a^2}{c}\right)$ and

equation of tangent is $y = mx \pm a\sqrt{1 + m^2}$

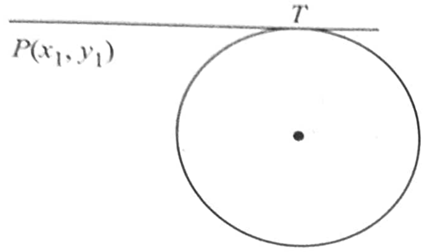
(v) The equation of tangent with slope m of the circle $(x - h)^2 + (y - k)^2 = a^2$ is $(y - k) = m(x - h) \pm a\sqrt{1 + m^2}$

Note: To get the equation of tangent at the point (x_1, y_1) on any curve we replace xx_1 in place x^2, yy_1 in place of y^2, $\dfrac{x + x_1}{2}$ in place of x, $\dfrac{y + y_1}{2}$ in place of y, $\dfrac{xy_1 + yx_1}{2}$ in place of xy and c in place of c.

(c) Length of tangent ($\sqrt{S_1}$): The length of tangent drawn from point (x_1, y_1) out side the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is,

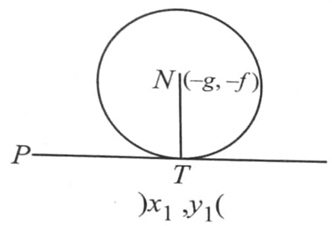
$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$



(*d*) **Equation of Pair of tangents ($SS_1 = T^2$):** $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$ or $SS_1 = T^2$.

Normal of Circle

(*a*) Equation of normal at point (x_1, y_1) of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is



$$y - y_1 = \left(\frac{y_1 + f}{x_1 + g}\right)(x - x_1)$$

(*b*) The equation of normal on any point (x_1, y_1) of circle $x^2 + y^2 = a^2$ is $\left(\frac{y}{x} = \frac{y_1}{x_1}\right)$.

chord of Contact

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (i.e. $T = 0$ same as equation of tangent).

Equation of the chord with A Given Middle Point (T = S$_1$)

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.

Director Circle

Equation of director circle is $x^2 + y^2 = 2a^2$.

$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

Note: The director circle of

$x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$.

Pole and Polar

The equation of the polar is the $T = 0$, so the polar of point (x_1, y_1) w.r.t. circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Pole of a given line with respect to a circle

Similar terms we can get the coordinates of the pole. The pole of

$lx + my + n = 0$

w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\frac{-la^2}{n}, \frac{-ma^2}{n}\right)$.

Family of Circles

(*a*) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K S_2 = 0$ $(K \neq -1)$.

(*b*) The equation of the family of cirlces passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.

(*c*) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form:

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \text{ where}$$

K is a parameter.

(*d*) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

(*e*) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ & coefficient of x^2 = coefficient of y^2.

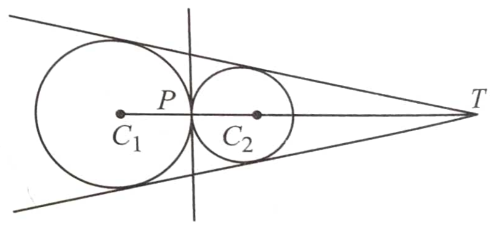
(*f*) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

Direct and transverse common tangents

Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then :

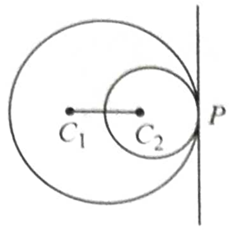
(*a*) **Both circles will touch**

(*i*) **Externally** if $C_1C_2 = r_1 + r_2$, point P divides C_1C_2 in the ratio $r_1 : r_2$ (internally).



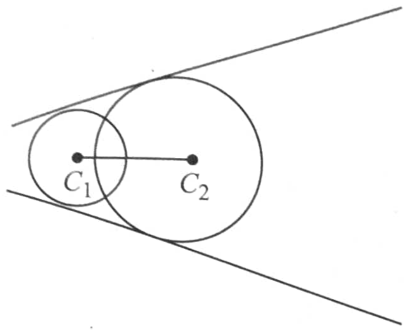
In this case there are **three common tangents**.

(*ii*) **Internally** if $C_1C_2 = |r_1 - r_2|$, point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be only **one common tangent**.

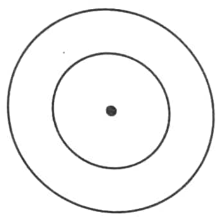


(*b*) **The circles will intersect**

when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents.**

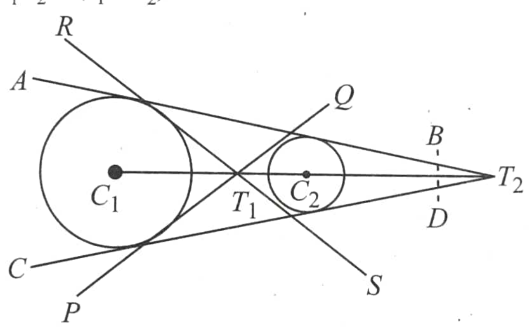


(*c*) **The circles will not intersect**



(*i*) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$. In this case there will be no common tangent.

(*ii*) $C_1C_2 > (r_1 + r_2)$



Note: Length of direct common tangent $= \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

Length of transverse common tangent $= \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

The angle of intersection of two circles

$\cos\theta =$

$$\frac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}} \text{ or } \boxed{\cos\theta = \left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)}$$

the circles to be orthogonal is

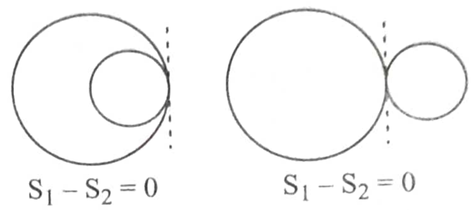
$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

Radical axis of the two circles ($S_1 - S_2 = 0$)

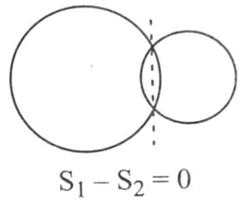
Then the equation of radical axis is given by $S_1 - S_2 = 0$.

Notes:

(*i*) If two circles touches each other then common tangent is radical axis.



(*ii*) If two circles cuts each other then common chord is radical axis.



(*iii*) If two circles cuts third circle orthogonally then radical axis of first two is locus of centre of third circle.

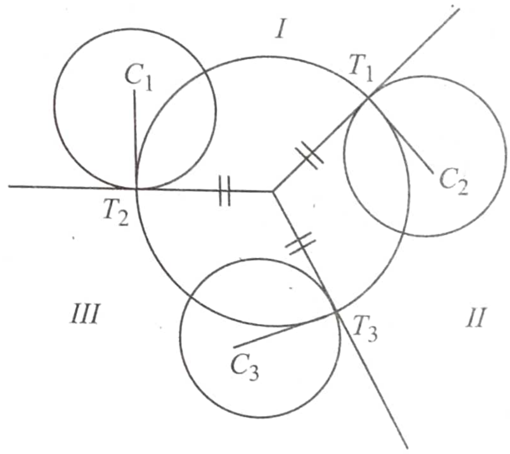
(*iv*) The radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.

Radical centre

The radical centre of three circles is the point from which length of tangents on three circles are equal i.e. the point of intersection of radical axis of the circles is the radical centre of the circles.

Notes:

(*i*) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.



(*ii*) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocenter will be its radical centre.

Solved Examples

1. If the straight line $ax + by = 2$; $a, b \neq 0$ touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, then the values of a and b are respectively

   (a) 1, −1 (b) 1, 2 (c) $-\frac{4}{3}, 1$ (d) 2, 1

Sol. (c) Given $x^2 + y^2 - 2x = 3$

$\therefore$ Centre is (1, 0) and radius is 2

Given $x^2 + y^2 - 4y = 6$

$\therefore$ Centre is (0, 2) and radius is $\sqrt{10}$. Since line $ax + by = 2$ touches the first circle.

$$\therefore \frac{|a(1)+b(0)-2|}{\sqrt{a^2+b^2}} = 2 \quad \text{or} \quad |(a-2)| = [2\sqrt{a^2+b^2}]$$

Also the given line is normal to the second circle. Hence it will pass through the centre of the second circle.

$\therefore a(0) + b(2) = 2$ or $2b = 2$ or $b = 1$

Putting this value in equation (i) we get

$|a - 2| = 2\sqrt{a^2+1^2}$ or $(a - 2)^2 = 4(a^2 + 1)$

or $a^2 + 4 - 4a = 4a^2 + 4$ or $3a^2 + 4a = 0$

or $a(3a + 4) = 0$ or $a = 0, -\frac{4}{3} (a \neq 0)$

$\therefore$ Values of a and b are $\left(-\frac{4}{3}, 1\right)$.

2. Find all the common tangents to the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$.

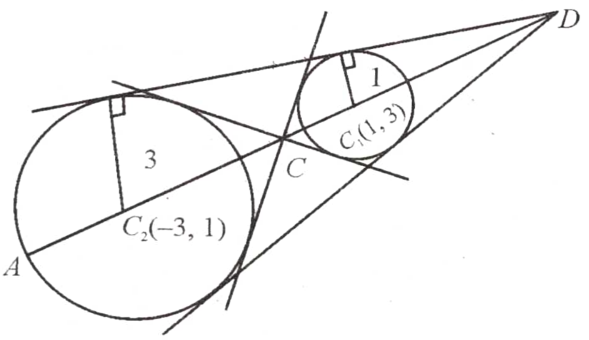
Sol. The given circles are $x^2 + y^2 - 2x - 6x + 9 = 0$

$\Rightarrow (x - 1)^2 + (y - 3) = 1$ ...(i)

and $x^2 + y^2 + 6x - 2y + 1 = 0$

$\Rightarrow (x + 3)^2 + (y - 1)^2 = 9$ ...(ii)

Centres and radii of circles Eq. (i) and Eq. (ii) are



$C_1(1, 3)$, $r_1 = 1$

and $C_2(-3, 1)$, $r_2 = 3$ respectively.

$\therefore C_1C_2 = \sqrt{(16+4)} = 2\sqrt{5}$

$\therefore C_1C_2 > r_1 + r_2$

Hence, the circles do not intersect to each other.

The direct common tangents meet AB produced at D, then point D will divide C_2C_1 in the ratio 3 : 1 (externally).

Coordinates of D are $\left(\frac{3(1)-1(-3)}{3-1}, \frac{3(3)-1(1)}{3-1}\right)$ or (3, 4)

and the point C divide C_2C_1, in the ratio 3 : 1 (internally) then coordinates of C are

$$\left(\frac{3(1)+1(-3)}{3+1}, \frac{3(3)+1(1)}{3+1}\right) \text{or} \left(0, \frac{5}{2}\right)$$

Direct tangents: Any line through (3, 4) is

$y - 4 - m(x - 3)$

$\Rightarrow mx - y + 4 - 3m = 0$ ...(i)

Apply the usual condition of tangency to any of the circle

$$\frac{m-3+4-3m}{\sqrt{m^2+1}} = \pm 1$$

$\Rightarrow (-2m + 1)^2 = m^2 + 1$

$\Rightarrow 3m^2 - 4m = 0$

$\Rightarrow m = 0, m = 4/3$

$\therefore$ Equations of direct common tangents are

$y = 4$ and $4x - 3y = 0$

Transverse tangents: Any line through $C(0, 5/2)$ is

$y - 5/2 = mx$

or $mx - y + 5/2 = 0$ ...(ii)

Apply the usual condition of tangency to any of the circle

$$\therefore \frac{m \cdot 1 - 3 + \frac{5}{2}}{\sqrt{m^2+1}} = \pm 1$$

$$\Rightarrow m^2 + \frac{1}{4} - m = m^2 + 1$$

$$\Rightarrow 0 \cdot m^2 - m - \frac{3}{4} = 0$$

$\therefore m = \infty$ and $m = -3/4$.

Hence, equations of transverse tangents are $x = 0$ and $3x + 4y - 10 = 0$

3. Show that the common tangents to the circles $x^2 + y^2 - 6x = 0$ and $x^2 + y^2 + 2x = 0$ form an equilateral triangle.

Sol. The given circles are $x^2 + y^2 - 6x = 0$

or $(x - 3)^2 + (y - 0)^2 = 9$ ...(i)

and $x^2 + y^2 + 2x = 0$

or $(x + 1)^2 + (y - 0)^2 = 1$ ...(ii)

Centres and raddii of circles Equations (i) and (ii) are $C_1(3, 0), r_1 = 3$ and $C_2(-1, 0), r_2 = 1$, respectively.

$\because C_1C_2 = \sqrt{[3-(-1)]^2 + 0} = 4$

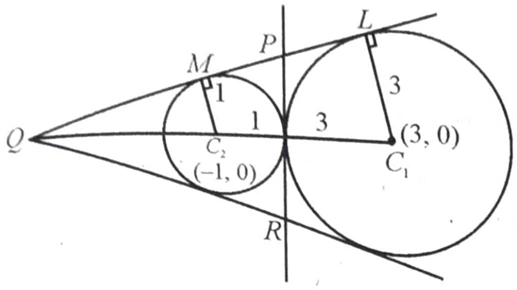
$\therefore C_1C_2 = r_1 + r_2$

Hence, the two circles touch each other externally. therefore, there will be three common tangents. Equation of the common tangent at the point of contact is $S_1 - S_2 = 0$

$\Rightarrow (x^2 + y^2 - 6x) - (x^2 + y^2 + 2x) = 0$

$\Rightarrow -8x = 0$

$\therefore x = 0$



Let the coordinates of Q be (h, k), then

$$\frac{QC_2}{QC_1} = \frac{C_2M}{C_1L} = \frac{1}{3}$$

$\therefore QC_2 : QC_1 = 1 : 3$

$$\therefore \quad h = \frac{1\cdot(3) - 3\cdot(-1)}{1-3} = -3 \text{ and } k = 0$$

$\therefore Q = (-3, 0)$

Equation of line passing through $Q(-3, 0)$ is

$y - 0 = m(x + 3)$

or $mx - y + 3m = 0$ ...(iii)

where, m is the slope of direct tangents since Eq. (iii) is the common tangent (direct) of the circles Eqs. (i) and (ii), then length of perpendicular from centre of Eq. (ii) i.e. $(-1, 0)$ to the Eq. (iii) = radius of circle Eq. (ii).

$$\Rightarrow \frac{|-m - 0 + 3m|}{\sqrt{m^2+1}} = 1 \text{ or } 4m^2 = m^2 + 1$$

$\Rightarrow 3m^2 = 1$

$$\therefore \quad m = \pm\frac{1}{\sqrt{3}}$$

From Eq. (iii), common tangents are (direct)

$$y = \frac{x}{\sqrt{3}} + \sqrt{3} \text{ and } y = -\frac{x}{\sqrt{3}} - \sqrt{3}$$

Hence, all common tangents are $x = 0$ ...(iv)

$$y = \frac{x}{\sqrt{3}} + \sqrt{3} \quad \text{...}(v)$$

$$\text{and } y = -\frac{x}{\sqrt{3}} - \sqrt{3} \quad \text{...}(vi)$$

Let P, Q, R be the point of intersection of lines Eqs.(iv), (v); (v), (vi) and (iv), (vi) respectively, then

$P = (0, \sqrt{3}); Q = (-3, 0)$ and $R = (0, -\sqrt{3})$

Now, $PQ = QR = RP = 2\sqrt{3}$

Hence, ΔPQR is and equilateral triangle thus common tangents form an equilateral triangle.

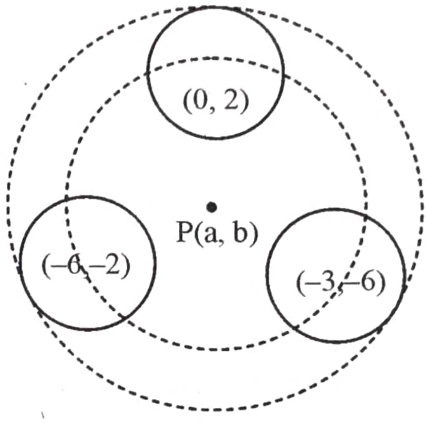
4. Find the equation of the circle of minimum radius which contains the three circles.

$S_1 \equiv x^2 + y^2 - 4y - 5 = 0$

$S_2 \equiv x^2 + y^2 + 12x + 4y + 31 = 0$

$S_3 \equiv x^2 + y^2 + 6x + 12y + 36 = 0$

Sol.



For S_1, centre = (0, 2) and radius = 3

For S_2, centre (–6, –2) and radius = 3

For S_3, centre = (–3, –6) and radius = 3

Let $P(a, b)$ be the centre of the circle passing through the centres of the three given circles, then

$(a - 0)^2 + (b - 2)^2 = (a + 6)^2 + (b + 2)^2$

$\Rightarrow (a + 6)^2 - a^2 = (b - 2)^2 - (b + 2)^2$

$(2a + 6)6 = 2b(-4)$

$$b = \frac{2\times 6(a+3)}{-8} = -\frac{3}{2}(a+3)$$

again $(a - 0)^2 + (b - 2)^2 = (a + 3)^2 + (b + 6)^2$

$\Rightarrow (a + 3)^2 - a^2 = (b - 2)^2 - (b + 6)^2$

$(2a + 3)3 = (2b + 4)(-8)$

$$(2a+3)3 = -16\left[-\frac{3}{2}(a+3)+2\right]$$

$6a + 9 = -8(-3a - 5)$

$6a + 9 = 24a + 40$

$18a = -31$

$$a = -\frac{31}{18},\ b = -\frac{23}{12}$$

Radius of the required circle

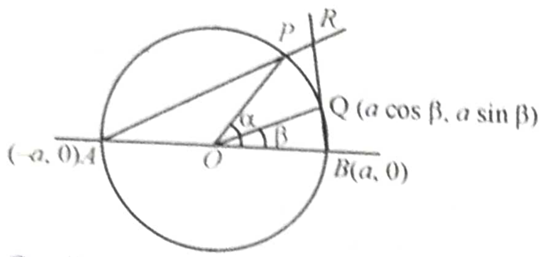
$$= 3 + \sqrt{\left(-\frac{31}{18}\right)^2 + \left(-\frac{23}{12} - 2\right)^2} = 3 + \frac{5}{36}\sqrt{949}$$

$\therefore$ equation of the required circle is

$$\left(x + \frac{31}{18}\right)^2 + \left(y + \frac{23}{12}\right)^2 = \left(3 + \frac{5}{36}\sqrt{949}\right)^2$$

5. A circle with centre at the origin and radius equal to a meets the axis of x at A and B. P (α) and $Q(B)$ are two points on the circle so that $\alpha - \beta = 2\gamma$, where γ is a constant. Find the locus of the point of intersection of AP and BQ.

Sol.



Coordinates of A are $(-a, 0)$ and of P are $(a\cos\alpha, a\sin\alpha)$

$\therefore$ Equation of AP is $y = \dfrac{a\sin\alpha}{a(\cos\alpha+1)}(x+a)$

or $y = \tan(\alpha/2)(x+a)$ ...(*i*)

Similarly equation of BQ is

$y = \dfrac{a\sin\beta}{a(\cos\beta-1)}(x-a)$

or $y = -\cot(\beta/2)(x-a)$ ...(*ii*)

We now eliminate α, β from equation (*i*) and (*ii*)

$\therefore\ \tan\left(\dfrac{\alpha}{2}\right) = \dfrac{y}{a+x},\ \tan\left(\dfrac{\beta}{2}\right) = \dfrac{a-x}{y}$

Now $\alpha - \beta = 2\gamma$

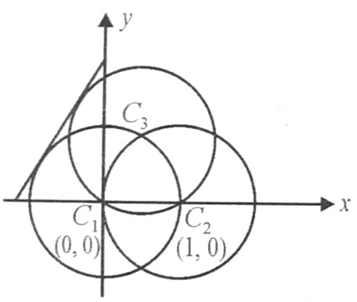
$$\Rightarrow \tan\gamma = \frac{\tan\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\beta}{2}\right)}{1+\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)} = \frac{\frac{y}{a+x}-\frac{a-x}{y}}{1+\frac{y}{a+x}\cdot\frac{a-x}{y}}$$

$$\Rightarrow \tan\gamma = \frac{y^2-(a^2-x^2)}{(a+x)y+(a-x)y} = \frac{x^2+y^2-a^2}{2ay}$$

$\Rightarrow\ x^2 + y^2 - 2ay\tan\gamma = a^2$.

6. C_1 and C_2 are circles of unit radius with centres at (0, 0) and (1, 0), respectively. C_3 is a circle of unit radius, passes through the centres of the circles C_1 and C_2 and have its centre above x-axis. Find the equation of the common tangent to C_1 and C_3 which does not pass through C_2.

Sol.



Equation of any circle throguh (0, 0) and (1, 0) is

$$(x-1)(x-0)+(y-0)(y-0)+\lambda\begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1\end{vmatrix} = 0$$

$\Rightarrow x^2 + y^2 - x + \lambda y = 0$

If it refpresents C_3, its radius = 1

$\Rightarrow 1 = (1/4) + (\lambda^2/4)$

$\Rightarrow \lambda = \pm\sqrt{3}$

As the centre of C_3, lies above the x-axis, we take $\lambda = -\sqrt{3}$ and thus an equation of C_3 is

$x^2 + y^2 - x - \sqrt{3}y = 0$.

Since C_1 and C_3 intersect and are of unit radius, their common tangents are parallel to the line joining their centres (0, 0) and $(1/2, \sqrt{3}/2)$.

So, let the equation of a common tangent be

$\sqrt{3}x - y + k = 0$.

It will touch C_1, if $\dfrac{|k|}{\sqrt{3+1}} = 1 \Rightarrow k = \pm 2$

From the figure, we observe that the required tangent makes positive intercept on the y-axis and negative on the x-axis and hence its equation is $\sqrt{3}x - y + 2 = 0$.

7. Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, if $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$.

Sol. Given circles are $x^2 + y^2 + 2ax + c^2 = 0$ ...(*i*)

and $x^2 + y^2 + 2by + c^2 = 0$ ...(*ii*)

Let C_1 and C_2 be the centre of circle (*i*) and (*ii*), respectively and r_1 and r_2 be their radii, then

$C_1 = (-a, 0),\ C_2 = (0, -b)$,

$r_1 = \sqrt{a^2-c^2},\ r_2 = \sqrt{b^2-c^2}$

Here we find the two circles touch each other internally or externally.

For touch, $|C_1C_2| = |r_1 \pm r_2|$

or $\sqrt{(a^2+b^2)} = |\sqrt{(a^2-c^2)} \pm \sqrt{(b^2-c^2)}|$

On squaring

$a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{(a^2-c^2)}\sqrt{(b^2-c^2)}$

or $c^2 = \pm\sqrt{a^2b^2 - c^2(a^2+b^2) + c^4}$

Again squaring, $c^4 = a^2b^2 - c^2(a^2+b^2) + c^4$

or $c^2(a^2+b^2) = a^2b^2$ or $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$

8. The equation of a circle of radius 1 touching the circles $x^2 + y^2 - 2|x| = 0$ is

(*a*) $x^2 + y^2 + 2\sqrt{3}x - 2 = 0$

(*b*) $x^2 + y^2 - 2\sqrt{3}y + 2 = 0$

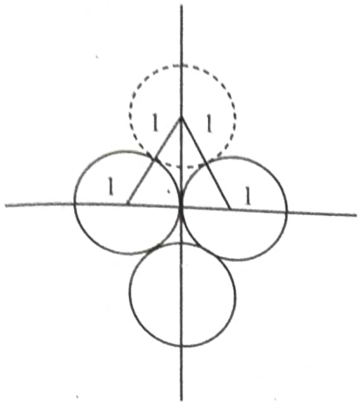
(*c*) $x^2 + y^2 + 2\sqrt{3}y + 2 = 0$

(*d*) $x^2 + y^2 + 2\sqrt{3}y + 2 = 0$

Approach: First simplify the equation of the given circle and then plot its graph.

Sol. The given circles are $x^2 + y^2 - 2x = 0, x > 0$ and $x^2 + y^2 + 2x = 0, x < 0$

From the figure the centre of the required circle will be $(0, \sqrt{3})$ and $(0, -\sqrt{3})$.



Hence (b, c) is the correct answer.

9. A circle touches the lines $y = \frac{x}{\sqrt{3}}, y = \sqrt{3}x$ and has unit radius. If the centre of this lies in the first quadrant then one possible equation of this circle is

(a) $x^2 + y^2 - 2x(\sqrt{3}+1) - 2y(\sqrt{3}+1) + 8 + 4\sqrt{3} = 0$

(b) $x^2 + y^2 - 2x(\sqrt{3}+1) - 2y(\sqrt{3}+1) + 5 + 4\sqrt{3} = 0$

(c) $x^2 + y^2 - 2x(\sqrt{3}+1) - 2y(\sqrt{3}+1) + 7 + 4\sqrt{3} = 0$

(d) $x^2 + y^2 - 2x(\sqrt{3}+1) - 2y(\sqrt{3}+1) + 6 + 4\sqrt{3} = 0$

Sol. (c) Angle between lines is $60° - 30° = 30°$. This equation of their acute angle bisector is $y \tan(30° + 15°)$. x i.e., $y = x$. let $C =$

(h, h) then $\frac{|h - h\sqrt{3}|}{2} = 1 \Rightarrow (\sqrt{3}-1)h = 2$

$$\Rightarrow h = \frac{2}{(\sqrt{3}-1)} = \frac{2(\sqrt{3}+1)}{2} = (\sqrt{3}+1)$$

thus equation of circle is

$$(x - (\sqrt{3}+1))^2 + (y - (\sqrt{3}+1))^2 = 1$$

$$\Rightarrow x^2 + y^2 - 2x(\sqrt{3}+1) - 2y(\sqrt{3}+1) + 7 + 4\sqrt{3} = 0$$

10. If the point $P(2a+1, a-1)$ is an interior point of the smaller segment of the circle $x^2 + y^2 - 2x - 4y = 4$ made by the chord $x + y - 2 = 0$, then set of values of a is containced in or equal to

(a) $\left(0, \frac{1}{2}\right)$ (b) $\left(0, \frac{1}{\sqrt{2}}\right)$

(c) $\left(0, \frac{3}{4}\right)$ (d) $\left(\frac{1}{2}, \frac{2}{3}\right)$

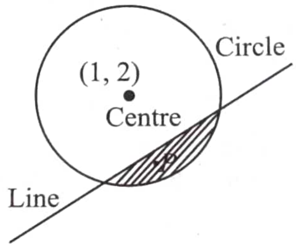
Sol. (b, d) Since $B(2a + 1, a - 1)$ is interior point of circle, So $(2a + 1)^2 + (a - 1)^2 - 2(2a + 1) - 4(a - 1) - 4 < 0$

$$\Rightarrow 0 < a < \frac{6}{5} \quad ...(i)$$

Also, given point $(2a + 1, a - 1)$ lies on smaller segment made by the chord $x + y - 2 = 0$ on circle, so $(2a + 1, a - 1)$ and centre of circle $(1, 2)$ will be on opposite side of the line.

$$\therefore (2a+1) + (a-1) - 2 < 0 \Rightarrow a < \frac{2}{3} \quad ...(ii)$$

$\therefore$ From (i) and (ii), we conclude that $a \in \left(0, \frac{2}{3}\right)$



11. If $\frac{x}{a} + \frac{y}{b} = 1$ touches the circle $x^2 + y^2 = r^2$, then find locus of the point $\left(\frac{1}{a}, \frac{1}{b}\right)$.

Sol. $\frac{x}{a} + \frac{y}{b} = 1$ touches $x^2 + y^2 = r^2$

$$\Rightarrow \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = r \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{r^2}$$

$\Rightarrow \left(\frac{1}{a}, \frac{1}{b}\right)$ lies on $x^2 + y^2 = \frac{1}{r^2}$, which is a circle.

Exercise-1 (Topicwise)

DIFFERENT FORMS OF A CIRCLE

1. The centres of the circles $x^2 + y^2 - 6x - 8y - 7 = 0$ and $x^2 + y^2 - 4x - 10y - 3 = 0$ are the ends of the diameter of the circle
   (a) $x^2 + y^2 - 5x - 9y + 26 = 0$
   (b) $x^2 + y^2 + 5x - 9y + 14 = 0$
   (c) $x^2 + y^2 + 5x - y - 14 = 0$
   (d) $x^2 + y^2 + 5x + y + 14 = 0$

2. The intercepts made by the circle $x^2 + y^2 - 5x - 13y - 14 = 0$ on the x-axis and y-axis are respectively
   (a) 9, 13 (b) 5, 13
   (c) 9, 15 (d) 5, 9

3. Equation of line passing through mid point of intercepts made by circle $x^2 + y^2 - 4x - 6y = 0$ on co-ordinate axis is
   (a) $3x + 2y - 12 = 0$ (b) $3x + y - 6 = 0$
   (c) $3x + 4y - 12 = 0$ (d) $3x + 2y - 6 = 0$

4. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq. units. The equation of the circle is
   (a) $x^2 + y^2 - 2x - 2y = 47$
   (b) $x^2 + y^2 - 2x - 2y = 62$
   (c) $x^2 + y^2 - 2x + 2y = 47$
   (d) $x^2 + y^2 - 2x + 2y = 62$

5. If a be the radius of a circle which touches x-axis at the origin, then its equation is
   (a) $x^2 + y^2 + ax = 0$ (b) $x^2 + y^2 \pm 2ya = 0$
   (c) $x^2 + y^2 \pm 2xa = 0$ (d) $x^2 + y^2 + ya = 0$

6. The equation of the circle passing through (3, 6) and whose centre is (2, –1) is
   (a) $x^2 + y^2 - 4x + 2y = 45$
   (b) $x^2 + y^2 - 4x - 2y + 45 = 0$
   (c) $x^2 + y^2 + 4x - 2y = 45$
   (d) $x^2 + y^2 - 4x + 2y + 45 = 0$

7. The equation of a circle which passes through the three points (3, 0) (1, –6), (4, –1) is
   (a) $2x^2 + 2y^2 + 5x - 11y + 3 = 0$
   (b) $x^2 + y^2 - 5x + 11y - 3 = 0$
   (c) $x^2 + y^2 + 5x - 11y + 3 = 0$
   (d) $2x^2 + 2y^2 - 5x + 11y - 3 = 0$

8. B and C are fixed point having co-ordinates (3, 0) and (–3, 0) respectively. If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation
   (a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 = 2$
   (c) $9(x^2 + y^2) = 1$ (d) $9(x^2 + y^2) = 4$

9. The length of intercept on y-axis, by a circle whose diameter is the line joining the points (–4, 3) and (12, –1) is
   (a) $3\sqrt{2}$ (b) $\sqrt{13}$
   (c) $4\sqrt{13}$ (d) $2\sqrt{13}$

10. The equation of the circle having the lines $y^2 - 2y + 4x - 2xy = 0$ as its normals & passing through the point (2, 1) is
    (a) $x^2 + y^2 - 2x - 4y + 3 = 0$
    (b) $x^2 + y^2 - 2x + 4y - 5 = 0$
    (c) $x^2 + y^2 + 2x + 4y - 13 = 0$
    (d) $x^2 + y^2 - 2x + 4y + 5 = 0$

11. Equation of a circle is $x^2 + y^2 - 12x - 16y + 19 = 0$ find radius of the circle.
    (a) 5 (b) 9 (c) 7 (d) 3

12. (6, 0), (0, 6) and (7, 7) are the vertices of a triangle. The circle inscribed in the triangle has the equation
    (a) $x^2 + y^2 - 9x + 9y + 36 = 0$
    (b) $x^2 + y^2 - 9x - 9y + 36 = 0$
    (c) $x^2 + y^2 + 9x - 9y + 36 = 0$
    (d) $x^2 + y^2 - 9x - 9y - 36 = 0$

13. The radius of the circle passing through the points (1, 2), (3, –4) and (5, –6) is
    (a) 13 (b) 10 (c) 15 (d) 20

POSITION OF A POINT

14. Find the co-ordinates of a point p on line $x + y = -13$, nearest to the circle $x^2 + y^2 + 4x + 6y - 5 = 0$
    (a) (– 6, – 7) (b) (– 15, 2)
    (c) (– 5, – 6) (d) (– 7, – 6)

15. The greatest distance of the point P (10, 7) from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is
    (a) 13 (b) 10
    (c) 15 (d) 20

LINE AND CIRCLE

16. Two lines through (2, 3) from which the circle $x^2 + y^2 = 25$ intercepts chords of length 8 units have equations
    (a) $2x + 3y = 13, x + 5y = 17$
    (b) $y = 3, 12x + 5y = 39$
    (c) $x = 2, 9x - 11y = 51$
    (d) $x = 2, 9x + 11y = 51$

17. The line $3x + 5y + 9 = 0$ w.r.t. the circle $x^2 + y^2 - 4x + 6y + 5 = 0$ is
(a) Chord (b) Diameter
(c) Tangent (d) Secant

18. The co-ordinates of the middle point of the chord cut off on $2x - 5y + 18 = 0$ by the circle $x^2 + y^2 - 6x + 2y - 54 = 0$ are
(a) (1, 4) (b) (2, 4)
(c) (4, 1) (d) (1, 1)

19. The radius of the circle passing through the point (6, 2) two of whose diameters are $x + y = 6$ and $x + 2y = 4$ is $\sqrt{a}$, then find value of a
(a) 21 (b) 10
(c) 14 (d) 20

20. Two diameters of the circle $3x^2 + 3y^2 - 6x - 18y - 7 = 0$ are along the lines $3x + y = c_1$ and $x - 3y = c_2$. Then, the value of c_1c_2 is
(a) 42 (b) –48
(c) 48 (d) –42

21. The line segment joining the points (4, 7) and (–2, –1) is a diameter of a circle. If the circle intersects the x– axis at A and B, then AB is equal to
(a) 4 (b) 12 (c) 8 (d) 16

22. **Statement I:** Two points A(10, 0) and O(0, 0) are given and a circle $x^2 + y^2 - 6x + 8y - 11 = 0$. The circle always cuts the line segments OA.
Statement II: The centre of the circle, point A and the point O are not collinear.
Given statements are:
(a) TT (b) TF (c) FT (d) FF

TANGENT AND NORMAL

23. $y = \sqrt{3}x + c_1$ & $y = \sqrt{3}x + c_2$ are two parallel tangents of a circle of radius 2 units, then $|c_1 - c_2|$ is equal to
(a) 8 (b) 4
(c) 2 (d) 1

24. The number of tangents that can be drawn from the point (8, 6) to the circle $x^2 + y^2 - 100 = 0$ is
(a) 0 (b) 1
(c) 2 (d) 3

25. The gradient of the tangent line at the point ($a \cos \theta$, $a \sin \theta$) to the circle $x^2 + y^2 = a^2$, is
(a) $\tan(\pi - \theta)$ (b) $\tan \theta$
(c) $\cot \theta$ (d) $-\cot \theta$

26. The tangent lines to the circle $x^2 + y^2 - 6x + 4y = 12$ which are parallel to the line $4x + 3y + 5 = 0$ are given by:
(a) $4x + 3y - 7 = 0, 4x + 3y + 15 = 0$
(b) $4x + 3y - 31 = 0, 4x + 3y + 19 = 0$
(c) $4x + 3y - 17 = 0, 4x + 3y + 13 = 0$
(d) None of these

27. The tangent to the circle $x^2 + y^2 = 5$ at the point (1, –2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ at
(a) (–2, 1) (b) (–3, 0)
(c) (–1, –1) (d) (3, –1)

28. If $y = c$ is a tangent to the circle $x^2 + y^2 - 2x + 2y - 2 = 0$ at (1, 1), then the value of c is
(a) 1 (b) 2 (c) –1 (d) –2

29. Line $3x + 4y = 25$ touches the circle $x^2 + y^2 = 25$ at the point
(a) (4, 3) (b) (3, 4)
(c) (–3, –4) (d) (4, –3)

30. The equations of the tangents drawn from the point (0, 1) to the circle $x^2 + y^2 - 2x + 4y = 0$ are
(a) $2x - y + 1 = 0, x + 2y - 2 = 0$
(b) $2x - y - 1 = 0, x + 2y - 2 = 0$
(c) $2x - y + 1 = 0, x + 2y + 2 = 0$
(d) $2x - y - 1 = 0, x + 2y + 2 = 0$

31. The equation of the normal to the circle $x^2 + y^2 = 9$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is
(a) $x - y = \frac{\sqrt{2}}{3}$ (b) $x + y = 0$
(c) $x - y = 0$ (d) $x + y = \frac{\sqrt{2}}{3}$

32. The equation of the diameter of the circle $(x - 2)^2 + (y + 1)^2 = 16$ which bisects the chord cut off by the circle on the line $x - 2y - 3 = 0$ is
(a) $x + 2y = 0$ (b) $2x + y - 3 = 0$
(c) $3x + 2y - 4 = 0$ (d) $2x - y + 3 = 0$

33. Slope of tangent to the circle $(x - r)^2 + y^2 = r^2$ at the point (x, y) lying on the circle is
(a) $\frac{x}{y - r}$ (b) $\frac{r - x}{y}$
(c) $\frac{y^2 - x^2}{2x}$ (d) $\frac{y^2 + x^2}{2xy}$

LENGTH OF TANGENT AND POWER OF A POINT

34. A line segment through a point P cuts a given circle in 2 points A & B, such that $PA = 16$ & $PB = 9$, find the length of tangent from points to the circle
(a) 7 (b) 25
(c) 12 (d) 10

35. The length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + p = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + q = 0$ is:
(a) $\sqrt{q - p}$ (b) $\sqrt{p - q}$
(c) $\sqrt{q + p}$ (d) $2\sqrt{q + p}$

36. The length of the tangent drawn from the point (2, 3) to the circles $2(x^2 + y^2) - 7x + 9y - 11 = 0$.

(a) 18 (b) 14

(c) $\sqrt{14}$ (d) $\sqrt{28}$

27. The polar of the point (5, –1/2) w.r.t circle $(x - 2)^2 + y^2 = 4$ is

(a) $5x - 10y + 2 = 0$ (b) $6x - y - 20 = 0$

(c) $10x - y - 10 = 0$ (d) $x - 10y - 2 = 0$

DIRECTOR CIRCLE & CHORD OF CONTACT

38. The distance between the chords of contact of tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and from the point (g, f) is

(a) $\sqrt{g^2 + f^2}$ (b) $\frac{\sqrt{g^2 + f^2 - c}}{2}$

(c) $\frac{g^2 + f^2 - c}{2\sqrt{g^2 + f^2}}$ (d) $\frac{\sqrt{g^2 + f^2 + c}}{2\sqrt{g^2 + f^2}}$

CHORD WITH A GIVEN MID POINT

39. The locus of the mid-points of the chords of the circle $x^2 + y^2 - 2x - 4y - 11 = 0$ which subtend 60° at the centre is

(a) $x^2 + y^2 - 4x - 2y - 7 = 0$

(b) $x^2 + y^2 + 4x + 2y - 7 = 0$

(c) $x^2 + y^2 - 2x - 4y - 7 = 0$

(d) $x^2 + y^2 + 2x + 4y + 7 = 0$

40. The locus of the centres of the circles such that the point (2, 3) is the mid point of the chord $5x + 2y = 16$ is

(a) $2x - 5y + 11 = 0$ (b) $2x + 5y - 11 = 0$

(c) $2x + 5y + 11 = 0$ (d) $2x - 5y - 11 = 0$

41. The equation of chord of the circle $x^2 + y^2 - 6x + 10y - 9 = 0$, which is bisected at (–2, 4) must be

(a) $5x + 9y = 36$ (b) $5x + 9y = 46$

(c) $5x - 9y = 36$ (d) $5x - 9y = 46$

PAIR OF TANGENTS

42. Number of common tangents of the circles $(x + 2)^2 + (y - 2)^2 = 49$ and $(x - 2)^2 + (y + 1)^2 = 4$ is:

(a) 0 (b) 1

(c) 2 (d) 3

43. The equation of the common tangent to the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ at their point of contact is

(a) $12x + 5y + 19 = 0$ (b) $5x + 12y + 19 = 0$

(c) $5x - 12y + 19 = 0$ (d) $12x - 5y + 19 = 0$

44. How many common tangents can be drawn to the following circles $x^2 + y^2 = 6x$ and $x^2 + y^2 + 6x + 2y + 1 = 0$.

(a) 0 (b) 1

(c) 2 (d) 4

45. A pair of tangents are drawn from the origin to the circle $x^2 + y^2 + 20(x + y) + 20 = 0$. The equation of the pair of tangents is

(a) $x^2 + y^2 + 10xy = 0$ (b) $x^2 + y^2 + 5xy = 0$

(c) $2x^2 + 2y^2 + 5xy = 0$ (d) $2x^2 + 2y^2 - 5xy = 0$

46. The equation of pair of tangents to the circle $x^2 + y^2 - 2x + 4y + 3 = 0$ from (6, –5), is

(a) $7x^2 + 23y^2 + 30xy + 66x + 50y - 73 = 0$

(b) $7x^2 + 23y^2 + 30xy - 66x - 50y - 73 = 0$

(c) $7x^2 + 23y^2 - 30xy - 66x - 50y + 73 = 0$

(d) $7x^2 + 23y^2 + 30xy - 66x + 50y + 73 = 0$

POSITION OF TWO CIRCLE AND ANGLE OF INTERSECTION OF TWO CIRCLES

47. The number of common tangents of the circles $x^2 + y^2 - 2x - 1 = 0$ and $x^2 + y^2 - 2y - 7 = 0$

(a) 1 (b) 3 (c) 2 (d) 4

48. If the circle $x^2 + y^2 = 9$ touches the circle $x^2 + y^2 + 6y + c = 0$, then c is equal to

(a) –27 (b) 36 (c) –36 (d) 27

RADICAL AXIS AND RADICAL CENTRE

49. The locus of the centre of the circle which bisects the circumferences of the circles $x^2 + y^2 = 4$ & $x^2 + y^2 - 2x + 6y + 1 = 0$ is:

(a) A straight line (b) A circle

(c) A pair of straight lines (d) A parabola

50. The point from which the tangents to the circles

$x^2 + y^2 - 8x + 40 = 0$

$5x^2 + 5y^2 - 25x + 80 = 0$

$x^2 + y^2 - 8x + 16y + 160 = 0$

are equal in length is

(a) $\left(8, \frac{15}{2}\right)$ (b) $\left(-8, \frac{15}{2}\right)$

(c) $\left(8, -\frac{15}{2}\right)$ (d) $\left(-8, -\frac{15}{2}\right)$

FAMILY OF CIRCLES AND COAXIAL SYSTEM

51. Tangents are drawn to the circle $x^2 + y^2 = 1$ at the points where it is met by the circles. $x^2 + y^2 - (\lambda + 6)x + (8 - 2\lambda)y - 3 = 0$, λ being the variable. The locus of the point of intersection of these tangents is

(a) $2x - y + 10 = 0$ (b) $x + 2y - 10 = 0$

(c) $x - 2y + 10 = 0$ (d) $2x + y - 10 = 0$

52. For the circles $S_1 \equiv x^2 + y^2 - 4x - 6y - 12 = 0$ and $S_2 \equiv x^2 + y^2 + 6x + 4y - 12 = 0$ and the line $L \equiv x + y = 0$ then which of the following statement is false

(a) L is common tangent of S_1 and S_2
(b) L is common chord of S_1 and S_2
(c) L is radical axis of S_1 & S_2
(d) L is Perpendicular to the line joining the centre of S_1 and S_2

POLE AND POLAR AND ORTHOGONALITY

53. Equation of the circle cutting orthogonally the three circles $x^2+y^2-2x+3y-7=0$, $x^2+y^2+5x-5y+9=0$ and $x^2+y^2+7x-9y+29=0$ is
(a) $x^2+y^2-16x-18y-4=0$
(b) $x^2+y^2-7x+11y+6=0$
(c) $x^2+y^2+2x-8y+9=0$
(d) $x^2+y^2+2x+y+9=0$

54. The locus of the centers of the circles which cut the circles $x^2+y^2+4x-6y+9=0$ and $x^2+y^2-5x+4y-2=0$ orthogonally is
(a) $9x+10y-7=0$
(b) $x-y+2=0$
(c) $9x-10y+11=0$
(d) $9x+10y+7=0$

Exercise-2 (Learning Plus)

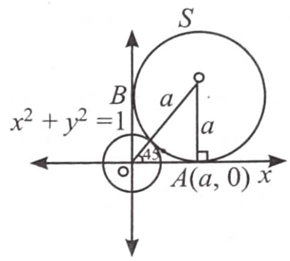
1. The equation of the circle whose radius is 5 and which touches the circle $x^2+y^2-2x-4y-20=0$ externally at the point (5, 5), is
(a) $x^2+y^2-18x-16y-120=0$
(b) $x^2+y^2-18x-16y+120=0$
(c) $x^2+y^2+18x+16y-120=0$
(d) $x^2+y^2+18x-16y+120=0$

2. Circle C_1 and C_2 of radii r and R respectively, touch each other as shown in figure. The line l_1 which is parallel to the line joining the centres of C_1 and C_2 is tangent to C_1 at P and intersects C_2 at A, B. If $R^2 = 2r^2$, then $\angle AOB$ equals
(a) $22\frac{1}{2}^\circ$
(b) $45°$
(c) $60°$
(d) $67\frac{1}{2}^\circ$

3. The locus of the centre of a circle which touches externally the circle $x^2+y^2-6x-6y+14=0$ and also touches the y-axis, is given by the equation
(a) $x^2-6x-10y+14=0$
(b) $x^2-10x-6y+14=0$
(c) $y^2-6x-10y+14=0$
(d) $y^2-10x-6y+14=0$

4. If p and q be the longest and the shortest distances respectively of the point $(-7, 2)$ from any point (α, β) on the curve whose equation is $x^2+y^2-10x-14y-51=0$ then G.M. of p and q is
(a) $2\sqrt{11}$
(b) $5\sqrt{5}$
(c) 13
(d) 16

5. Let S be the circle in xy-plane which touches the x-axis at point A, the y-axis at point B and unit circle $x^2+y^2=1$ at point C externally. If O denotes the origin, then the angle OCA equals



(a) $5\pi/8$
(b) $\pi/2$
(c) $3\pi/4$
(d) $3\pi/5$

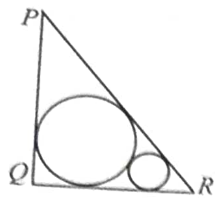
6. The tangents are drawn from the point (4, 5) to the circle $x^2+y^2-4x-2y-11=0$. The area of quadrilateral formed by these tangents is
(a) 15 *sq. units*
(b) 75 *sq. units*
(c) 8 *sq. units*
(d) 4 *sq. units*

7. Equation of circle that cuts the circle $x^2+y^2+2gx+2fy+c=0$, lines $x=-g$ and $y=-f$ orthogonally, is
(a) $x^2+y^2+2gx+2fy+g^2+f^2-c=0$
(b) $x^2+y^2+2gx+2fy+g^2+f^2+c=0$
(c) $x^2+y^2+2gx+2fy-g^2-f^2-c=0$
(d) $x^2+y^2-2gx+2fy-g^2-f^2-c=0$

8. The common chord of the circle $x^2+y^2+4x+1=0$ and $x^2+y^2+6x+2y+3=0$ is
(a) $x+y+1=0$
(b) $5x+y+2=0$
(c) $2x+2y+5=0$
(d) $3x+y+3=0$

9. The length of common chord of the circles $(x-a)^2+y^2=a^2$ and $x^2+(y-b)^2=b^2$ is
(a) $2\sqrt{a^2+b^2}$
(b) $\frac{ab}{\sqrt{a^2+b^2}}$
(c) $\frac{2ab}{\sqrt{a^2+b^2}}$
(d) $\frac{ab}{\sqrt{a^2-b^2}}$

10. The right-angled triangle has two circles touching its sides as shown. If the angle at R is 60° and the radius of the smaller circle is 1, then the radius of the larger circle is



(a) $2\sqrt{3}$ (b) 2

(c) $2\sqrt{2}$ (d) 3

11. Middle point of the chord of the circle $x^2+y^2=25$ intercepted on the line $x-2y=2$ is

(a) $\left(\frac{3}{5},\frac{4}{5}\right)$ (b) $(-2,-2)$

(c) $\left(\frac{2}{5},-\frac{4}{5}\right)$ (d) $\left(\frac{8}{3},\frac{1}{3}\right)$

12. The equation of a circle passing through points of intersection of the circles $x^2+y^2+13x-3y=0$ and $2x^2+2y^2+4x-7y-25=0$ and point (1, 1) is

(a) $4x^2+4y^2-30x-10y-25=0$

(b) $4x^2+4y^2+30x-13y-25=0$

(c) $4x^2+4y^2-17x-10y+25=0$

(d) $4x^2+4y^2+30x+10y-25=0$

13. The locus of centre of a circle passing through (a, b) and cuts orthogonally to circle $x^2+y^2=p^2$, is

(a) $2ax+2by-(a^2+b^2+p^2)=0$

(b) $2ax+2by-(a^2-b^2+p^2)=0$

(c) $x^2+y^2-3ax-4by+(a^2+b^2-p^2)=0$

(d) $x^2+y^2-2ax-3by+(a^2-b^2-p^2)=0$

14. The equation of the circle having the lines $x^2+2xy+3x+6y=0$ as its normals and having size just sufficient to contain the circle $x(x-4)+y(y-3)=0$ is

(a) $x^2+y^2+3x-6y-40=0$

(b) $x^2+y^2+6x-3y-45=0$

(c) $x^2+y^2+8x+4y-20=0$

(d) $x^2+y^2+4x+8y+20=0$

15. A square is inscribed in the circle $x^2+y^2-2x+4y+3=0$. Its sides are parallel to the coordinate axes. Then one vertex of the square is

(a) $(1+\sqrt{2},-2)$ (b) $(1-\sqrt{2},-2)$

(c) $(1,-2+\sqrt{2})$ (d) None of these

16. The value of 'c' for which the set, $\{(x, y)|x^2+y^2+2x\le 1\}\cap\{(x, y)|x-y+c\ge 0)\}$ contains only one point in common is

(a) $(-\infty,-1]\cup[3,\infty)$ (b) $\{-1, 3\}$

(c) $\{-3\}$ (d) $\{-1\}$

17. The centre of a circle C lies on the line $2x-2y+9=0$ and the circle C cuts orthogonally the circle $x^2+y^2=4$. The circle C passes through fixed points

(a) $(-3, 3)$ (b) $\left(\frac{-1}{2},\frac{1}{2}\right)$

(c) $(-4, 4)$ (d) $(-2, 2)$

18. The axes are translated so that the new equation of the circle $x^2+y^2-5x+2y-5=0$ has no first degree terms and the new equation $x^2+y^2=\frac{\lambda}{4}$, then find the value of λ.

(a) 48 (b) 40

(c) 49 (d) 98

19. Let A be the centre of the circle $x^2+y^2-2x-4y-20=0$. Suppose that the tangents at the points $B(1, 7)$ and $D(4,-2)$ on the circle meet at the point C. Find the area of the quadrilateral $ABCD$.

(a) 30 (b) 60

(c) 105 (d) 75

20. The greatest distance of the point $P(10, 7)$ from the circle $x^2+y^2-4x-2y-20=0$ is

(a) 5 (b) 15

(c) 10 (d) None of these

21. Consider two circles $x^2+y^2=1$ and $(x-1)^2+(y-1)^2=3$ then the point circle which is orthogonal to the given two circles can be

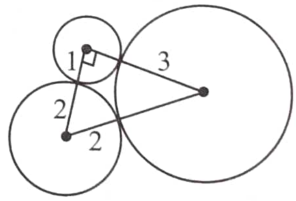
(a) $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=0$

(b) $\left(x-\frac{1}{\sqrt{2}}\right)^2+\left(y-\frac{1}{\sqrt{2}}\right)^2=0$

(c) $(x-1)^2+(y-1)^2=0$

(d) $(x-1)^2+(y+1)^2=0$

22. The circle of radii 1, 2 and 3 units respectively touch each other externally in the plane. The circumradius of the triangle formed by joining the centers of the circles is



(a) 1.5 (b) 2

(c) 2.5 (d) 3

23. Equation of circle touching the lines $|x-2|+|y-3|=4$ will be

(a) $(x-2)^2+(y-3)^2=12$

(b) $(x-2)^2+(y-3)^2=4$

(c) $(x-2)^2+(y-3)^2=4$

(d) $(x-2)^2+(y-3)^2=8$

24. P is a point (a, b) in the first quadrant. If the two circles which pass through P and touch both the co-ordinate axes cut at right angles, then

(a) $a^2 - 6ab + b^2 = 0$ (b) $a^2 + 2ab - b^2 = 0$
(c) $a^2 - 4ab + b^2 = 0$ (d) $a^2 - 8ab + b^2 = 0$

25. The angle between the tangents from (a, b) to the circle $x^2 + y^2 = a^2$ is

(a) $\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right)$ (b) $2\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right)$

(c) $2\tan^{-1}\left(\frac{\sqrt{S_1}}{a}\right)$ (d) $3\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right)$

26. If the circles

$x^2 + y^2 + 2ax + 2by + c = 0$

and $x^2 + y^2 + 2bx + 2ay + c = 0$

where c > 0, have exactly one point in common then the value of $\frac{(a+b)^2}{2c}$ is

(a) 1 (b) $\sqrt{2}$
(c) 2 (d) 1/2

27. Let a circle S whose centre is (h, k), $k > 0$ touches a line pair $4xy - 3y^2 + 12 - h = 0$ at A and B.

If 'r' is the radius of the circle 'S' then $(r + k)$ is equal to

(a) 6 (b) 12
(c) 24 (d) 48

28. Consider the curves

$$C_1 : \frac{x}{\cos\alpha} = \frac{y}{\sin\alpha} = 2$$

$$C_2 : \frac{x-2}{\cos\beta} = \frac{y-3}{\sin\beta} = 3$$

$$C_3 : \frac{x+2}{\cos\lambda} = \frac{y-1}{\sin\lambda} = 1$$

where α, β, γ are parameters then answer the following.

The locus of the point from which pair of perpendicular tangents can be drawn to curve C_3 is

(a) $(x+1)^2 + (y-1)^2 - \sqrt{2}$

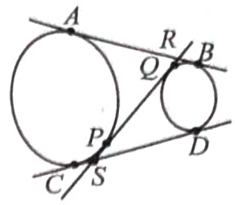
(b) $\frac{x+1}{\cos\theta} = \frac{y-1}{\sin\theta} = \sqrt{2}$

(c) $\frac{x+1}{\cos\phi} = \frac{y-1}{\sin\phi} = 2$

(d) $(x+1)^2 + (y-1)^2 = 4$

(where and are parameters)

29. Suppose S_1 and S_2 are two unequal circles; AB and CD are the direct common tangents to these circles. A transverse common tangent PQ cuts AB in R and CD in S. If $AB = 10$, then RS is.



(a) 8 (b) 9
(c) 10 (d) 11

30. Consider the circle $c : x^2 + y^2 = 1$ and the line $L : y = m(x + 2)$. If L intersect c at P and Q, then locus of middle point of PQ is

(a) $(x + 1)^2 + y^2 = 1$ (b) $x^2 + (y - 1)^2 = 1$
(c) $(x - 1)^2 + y^2 = 1$ (d) $x^2 + (y + 1)^2 = 1$

31. The equation $(1 + a^2)x^2 + 2a^2x + a^2 + b^2 - 1 = 0$ has roots of opposite sign if (a, b) lies

(a) On a straight line $x + y = 1$
(b) Inside a circle of centre (0, 0) and radius 1
(c) On a parabola of vertex (0, 0) and focal length 1
(d) On a straight line $x - y = 1$

32. If a circle passes through the points of intersection of the co-ordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$ and $x - 2y + 3 = 0$. then the value of λ is

(a) 2 (b) 4
(c) 6 (d) 3

33. If a circle having centre at (α, β) cut the circles $x^2 + y^2 - 2x - 2y - 7 = 0$ and $x^2 + y^2 + 4x - 6y - 3 = 0$ orthogonally, then $\left|\frac{3}{4}\alpha - \frac{\beta}{2}\right|$ is equal to

(a) 1 (b) 1/2
(c) 1/4 (d) 0

34. Sum of the x and y intercepts of the circle described on the line segment joining (–2, 1) and (1, 2) as diameter, is

(a) 1 (b) 2
(c) 3 (d) 4

35. A variable circle touches the x-axis and also touches the circle with centre at (0, 3) and radius 2. The locus of centre of the vaiable circle is

(a) An ellipse (b) A circle
(c) A hyperbola (d) A parabola

36. A point $P(\alpha, \beta)$ is called rational point if both α and β are rational numbers and if both α and β are integers, then point $P(\alpha, \beta)$ is called lattice point.

Number of lattice points on the circumference of circle $x^2 + y^2 = 25$ is

(a) 14 (b) 8 (c) 12 (d) 10

37. A circle passing through the points $(3, \sqrt{7/2})$ thouches the pair of lines $x^2 - y^2 - 2x + 1 = 0$. The centre of the circle is.

(a) (4, 0) (b) (6, 0)
(c) (0, 4) (d) (5, 0)

38. Tangents are drawn to the circle $x^2 + y^2 = 50$ from a point 'P' lying on the x-axis. These tangents meet the y-axis at points 'P_1' and 'P_2'. Possible coordinates of 'P' so that area of triangle PP_1P_2 is minimum, is/are

(a) (10, 0) (b) $(10\sqrt{2}, 0)$

(c) (−10, 0) (d) $(-10\sqrt{2}, 0)$

39. The equation of the circle which touches both the axes and the line $\frac{x}{3} + \frac{y}{4} = 1$ and lies in the first quadrant is $(x - c)^2 + (y - c)^2 = c^2$ where c is

(a) 1 (b) 2

(c) 4 (d) 6

40. $x^2 + y^2 + 6x = 0$ and $x^2 + y^2 - 2x = 0$ are two circles, then

(a) They touch each other externally

(b) They touch each other internally

(c) Area of triangle formed by their common tangents is $3\sqrt{3}$ sq. units.

(d) Their common tangents do not form any triangle.

41. The centre(s) of the circle(s) passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is/are

(a) $\left(\frac{3}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{3}{2}\right)$

(c) $\left(\frac{1}{2}, 2^{1/2}\right)$ (d) $\left(\frac{1}{2}, -2^{1/2}\right)$

42. Two equation of the circle which touches the axes of coordinates and the line $\frac{x}{3} + \frac{y}{4} = 1$ and whose centre lies in the first quadrant is $x^2 + y^2 - 2cx - 2cy + c^2 = 0$ where c is

(a) 1 (b) 2

(c) 3 (d) 6

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. Curves $ax^2 + 2hxy + by^2 - 2gx - 2fy + c = 0$ and $a'x^2 - 2hxy + (a' + a - b)y^2 - 2g'x - 2f'y + c = 0$ intersect at four concyclic points A, B, C and D. If P is the point $\left(\frac{g'+g}{a'+a}, \frac{f'+f}{a'+a}\right)$, the vlaue of $\frac{PA^2 + PB^2 + PC^2}{PD^2}$ is

(a) 1 (b) 2 (c) 3 (d) 4

2. Let C be the circle $x^2 + y^2 = 1$ in the xy-plane. For each $t \geq 0$, let L_t be the line passing through (0, 1) and (t, 0). Note that L_t intersects C in two points, one of which is (0, 1). Let Q_t be the other point. As t varies between 1 and $1+\sqrt{2}$, the collection of points Q_t sweeps out an arc on C. The angle subtended by this arc at (0, 0) is

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{3\pi}{8}$

3. Let $A \equiv (0, 0)$, $B \equiv (4, 0)$ and on segment AB is given a point M. On the same side of AB squares $AMCD$ and $BMFE$ are constructed above AB. The circumcircles S_1 and S_2 of two squares $AMCD$ and $BMFE$ respectively whose centres are P and Q, intersect in M and another point N. For all positions of M varying along the segment AB, the lines MN passes through the fixed point $R(a, b)$, then $a + b =$

(a) 0 (b) 1

(c) $\sqrt{3}$ (d) 2

4. A circle is inscribed into a rhombus $ABCD$ with one angle 60°. The distance from the centre of the circle to the nearest vertex is equal to 1. If P is any point of the cirlce, then $|PA|^2 + |PB|^2 + |PC|^2 + |PD|^2$ is equal to

(a) 12 (b) 11 (c) 9 (d) 8

5. Let $ABCD$ be a square of side length l, and a circle passing through B and C, and touching AD. The radius of is

(a) $\frac{3}{8}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{5}{8}$

6. Two circles with radii 'r_1' and 'r_2', $r_1 > r_2 \geq 2$, touch each other externally. If 'θ' be the angle between the direct common tangents, then

(a) $\theta = \sin^{-1}\left(\frac{r_1 + r_2}{r_1 - r_2}\right)$ (b) $\theta = 2\sin^{-1}\left(\frac{r_1 - r_2}{r_1 + r_2}\right)$

(c) $\theta = \sin^{-1}\left(\frac{r_1 - r_2}{r_1 + r_2}\right)$ (d) $\theta = 2\sin^{-1}\left(\frac{r_1 + r_2}{r_1 - r_2}\right)$

7. Two circles with radius $\sqrt{2}$ cm and 1 cm meet at a point A, whose centres lie on x-axis. The distance between their centres is 2 cm. The chord AC of the larger circle cuts the smaller circle at a point B and is bisected by that point. The length of chord AC is.

(a) $\frac{\sqrt{5}}{2}$ cm (b) $\sqrt{\frac{7}{2}}$ cm

(c) $\frac{\sqrt{3}}{2}$ cm (d) $\sqrt{\frac{5}{2}}$ cm

8. A circle C is tangent to the x and y-axis in first quadrant at the points P and Q respectively. BC and AD are parallel tangents to the circle with slope -1. If the points A and B are on the y-axis while C and D are on x-axis and area of figure $ABCD$ is $900\sqrt{2}$ sq. units. The radius of the circle is

(a) $5\sqrt{2}$ (b) $6\sqrt{2}$

(c) 30 (d) 15

9. Locus of the centre of a circle touching the circle $x^2+y^2-4y-2x=\frac{7}{4}$ internally and tangents on which from (1, 2) is making an angle 60° with each other is director circle (C). The equation of the director circle (C) is

(a) $(x-2)^2+(y-1)^2=\sqrt{3}$

(b) $(x-1)^2+(y-2)^2=3$

(c) $(x-1)^2+(y-2)^2=\sqrt{3}$

(d) $(x-1)^2+(y+2)^2=3$

10. Let ABC be a triangle whose vertices are $A(-5, 5)$ and $B(7, -1)$. If vertex C has on a circle whose director circle has equation $x^2+y^2=100$, then locus of the orthocentre of triangle ABC is equal to

(a) $x^2+y^2+4x-8x-30=0$

(b) $x^2+y^2-4x+8y-30=0$

(c) $x^2+y^2+4x-8y-30=0$

(d) $x^2+y^2+4x+8y-30=0$

11. A circle passes through $A(0, 4)$ and $B(8, 0)$ has its centre on x-axis. If point C lies on the circumference of the circle and m is the greatest area of triangle ABC, then m is equal to

(a) $10(\sqrt{5}-1)$ (b) $10(\sqrt{5}+1)$

(c) $20(\sqrt{5}+1)$ (d) $20(\sqrt{5}-1)$

12. Consider a curve $ax^2+2hxy+by^2=1$ and a point P not on the curve. A line is drawn from the point P intersects the curve at points Q & R. If the product PQ. PR is independent of the slope of the line, then the curve is

(a) Parallel lines (b) A line

(c) A circle (d) Perpendicular lines

13. Two fixed points P and Q are 4 units apart and are on same side of a variable line L. Let PM and QN are perpendicular distance of P and Q from line L satisfy equation $PM+3QN=4$, then line L always touches a circle C

(a) the centre of circle C lies on line PQ

(b) the radius of circle is 1

(c) the radius of circle is 2

(d) the centre of the circle C lies on perpendicular bisector of PQ

14. The circle $x^2+y^2-2x-3ky-2=0$ passes through two fixed points, (k is the parameter), whose coordinates are:

(a) $(1+\sqrt{3},0)$ (b) $(-1+\sqrt{3},0)$

(c) $(-\sqrt{3}-1,0)$ (d) $(1-\sqrt{3},0)$

15. For $C_1: x^2+y^2=4$, $C_2:(x-10)^2+y^2=16$ direct common tangents of these circles touch them at P, Q, R, S. Another circle of radius 'λ' is drawn passing through P, Q, R, S. Then

(a) Midpoint of C_1C_2 is centre of the circle passing through P, Q, R, S.

(b) Centre of the circle passing through P, Q, R, S divides C_1C_2 in the ration 1 : 2

(c) $\lambda^2=33$

(d) $\lambda^2=35$

COMPREHENSION TYPE

Comprehension-1 (No. 16 to 18): Consider the relation $4l^2-5m^2+6l+1=0$, where $l, m \in R$, and the line $lx+my+1=0$ touches a fixed circle S.

16. Centre and radius of circle S are

(a) (2, 0), 3 (b) $(-2, 0), \sqrt{5}$

(c) $(3, 0), \sqrt{5}$ (d) (–3, 0), 3

17. Tangents PA and PB are drawn to the circle S from the points P on the line $x+y=1$. The chord of contact AB passes through the fixed point

(a) $\left(\frac{-1}{2},\frac{3}{2}\right)$ (b) $\left(\frac{1}{2},\frac{-5}{2}\right)$

(c) $\left(\frac{1}{2},\frac{5}{2}\right)$ (d) $\left(\frac{1}{2},\frac{-3}{2}\right)$

18. If the line $x-2y+c=0$ intersects the fixed circle S orthogonally, then c equals

(a) –3 (b) –2 (c) 2 (d) 3

Comprehension-2 (No. 19 to 21): Let C_1, C_2 are two circles each of radius 1 touching internally the sides of triangles POA_1, PA_1A_2 respectively where $P\equiv(0, 4)$ O is origin, A_1, A_2 are the points on positive x-axis.

On the basis of above passage, answer the following questions:

19. Angle subtended by circle C_1 at P is

(a) $\tan^{-1}\frac{2}{3}$ (b) $2\tan^{-1}\frac{2}{3}$

(c) $\tan^{-1}\frac{3}{4}$ (d) $2\tan^{-1}\frac{3}{4}$

20. Centre of circle C_2 is

(a) (3, 1) (b) $\left(3\frac{1}{2},1\right)$

(c) $\left(3\frac{3}{4},1\right)$ (d) None of these

21. Length of tangent from P to circle C_2

(a) 4 (b) $\frac{9}{2}$ (c) 5 (d) $\frac{19}{4}$]

Comprehension-3 (No. 22 to 24): A circle C whose radius is 1 unit, touches the x-axis at point A. The centre Q of C lies in first quadrant. The tangent from origin O to the circle touches it at T and a point P lies on it such that ΔOAP is a right angled triangle at A and its perimeter is 8 units.

22. The length of QP is

(a) 1/2
(b) 4/3
(c) 5/3
(d) 2/3

23. Equation of circle C is

(a) $\{x-(2+\sqrt{3})\}^2+(y-1)^2=1$
(b) $\{x-(\sqrt{3}+\sqrt{2})\}^2+(y-1)^2=1$
(c) $(x-2)^2+(y-1)^2=1$
(d) $(x-\sqrt{3})^2+(y+2)^2=1$

24. If tangent OT cuts the two parallel tangents (one of them is OA) at O and R, then equation of circle circumscribing the ΔORQ is

(a) $x^2+y^2-\sqrt{3}x-y=0$
(b) $x^2+y^2-\sqrt{3}x-2y=0$
(c) $\left(x-\dfrac{3}{4}\right)^2+(y-1)^2=\dfrac{25}{16}$
(d) $x^2+y^2+2\sqrt{3}x-2y=0$

Comprehension-4 (No. 25 to 26): The line $y = ax + b$ intersects the curve $c : x^2 + y^2 + 6x + 10y + 1 = 0$ at the points A and B. If the lines segment AB subtends a right angle at origin then the locus of the point (a, b) is the curve $g(x, y) = 0$.

25. The equation of curve $g(x, y) = 0$ is

(a) $x^2 + 2y^2 - 6xy + 10y + 1 = 0$
(b) $x^2 + 2y^2 - 6xy - 10y + 1 = 0$
(c) $x^2 - 2y^2 - 6xy + 10y + 1 = 0$
(d) $x^2 - 2y^2 - 6xy - 10y + 1 = 0$

26. The slope of tangent to the curve $g(x, y) = 0$ at the point where the line $y = 1$ intersects it in first quadrant is

(a) 1/2
(b) 1/3
(c) 1/4
(d) 1/6

Comprehension-5 (No. 27 to 29): Two circles $S_1 : x^2 + y^2 - 2x - 2y - 7 = 0$ and $S_2 : x^2 + y^2 - 4x - 4y - 1 = 0$ intersects each other at A and B.

On the basis of above passage, answer the following questions:

27. Length of AB is

(a) 6
(b) $\sqrt{33}$
(c) $\sqrt{34}$
(d) $\sqrt{35}$

28. Equation of circle passing through A and B whose AB is diameter

(a) $x^2 + y^2 - 3x - 3y - 5 = 0$
(b) $x^2 + y^2 - 3x - 3y - 4 = 0$
(c) $x^2 + y^2 + 3x + 3y - 4 = 0$
(d) $x^2 + y^2 + 3x + 3y - 5 = 0$

29. Mid point of AB is

(a) $\left(\dfrac{5}{2}, \dfrac{1}{2}\right)$
(b) $\left(\dfrac{3}{2}, \dfrac{3}{2}\right)$
(c) (2, 1)
(d) (1, 2)

Comprehension-6 (No. 30 to 31): To the circle $x^2 + y^2 = 4$ two tangents are drawn from P(–4, 0), which touches the circle at A and B and a rhombus PA P'B is completed.

On the basis of above passage, answer the following questions.

30. Circumcentre of the triangle PAB is at

(a) (–2, 0)
(b) (2, 0)
(c) $\left(\dfrac{\sqrt{3}}{2}, 0\right)$
(d) None of these

31. Ratio of the area of triangle PAP′ to that of P′AB is

(a) 2 : 1
(b) 1 : 2
(c) $\sqrt{3}$: 2
(d) None of these

Comprehension-7 (No. 32 to 34): P is variable point on the line $L = 0$. Tangents are drawn to the circle $x^2 + y^2 = 4$ from P to touch Q and R. The parallelogram $PQSR$ is complicated.

32. If $L \equiv 2x + y = 6$, then the locus of circumcentre of ΔPQR is

(a) $2x - y = 4$
(b) $y + 2x = 3$
(c) $x - 2y = 4$
(d) $x + 2y = 3$

33. If $L \equiv y = 4$, then the locus of circumcentre of ΔPQR is

(a) $y = 2$
(b) $x = 2$
(c) $y = 4$
(d) $x = -4$

34. If $P = (2, 3)$, then the circumcentre of ΔQRS is

(a) $\left(\dfrac{2}{13}, \dfrac{7}{26}\right)$
(b) $\left(\dfrac{2}{13}, \dfrac{3}{16}\right)$
(c) $\left(\dfrac{4}{13}, \dfrac{5}{26}\right)$
(d) None of these

MATCH THE COLUMN TYPE QUESTIONS

35. Match the column

	Column-I		Column-II
A.	Number of values of a for which the common chord of the circles $x^2 + y^2 = 8$ and $(x - a)^2 + y^2 = 8$ subtends a right angle at the origin is	p.	4
B.	A chord of the circle $(x - 1)^2 + y^2 = 4$ lies along the line $y = 22\sqrt{3}(x - 1)$. The length of the chord is equal to	q.	2
C.	The number of circles touching all the three lines $3x + 7y = 2$, $21x + 49y = 5$ and $9x + 21y = 0$ are	r.	0
D.	If radii of the smallest and largest circle passing through the point $(\sqrt{3}, \sqrt{2})$ and touching the circle $x^2 + y^2 - 2\sqrt{2}y - 2 = 0$ are r_1 and r_2 respectively, then the mean of r_1 and r_2 is	s.	1

(a) A – q, B – p, C – r, D – s
(b) A – p, B – q, C – r, D – s
(c) A – q, B – p, C – r, D – s
(d) A – r, B – q, C – s, D – p

36. Triangles ABC are described on a given base BC and of a given vertical angle α.

	Column-I		Column-II
A.	The locus of orthocentre of ΔABC is	p.	Part of circle such that BC subtend angle the $\pi - \alpha$ its circumference
B.	The locus of incentre of ΔABC is	q.	Part of circle such that BC subtend angle $\frac{\pi}{2}+\frac{\alpha}{2}$ at its circumference
C.	The locus of the excentre corresponding to vertex opposite to base of ΔABC is	r.	Part of circle such that BC subtend angle $\frac{\pi}{2}-\frac{\alpha}{2}$ at its circumference
D.	The locus of centroid of ΔABC is	s.	Part of circle such that PQ subtend angle α at its circumference where P, Q are points of trisection of segment BC.

(*a*) A – p, B – r, C – q, D – s
(*b*) A – p, B – q, C – r, D – s
(*c*) A – q, B – p, C – r, D – s
(*d*) A – r, B – q, C – s, D – p

37. Match the column

	Column-I		Column-II
A.	Number of common tangents of the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x - 6y + 2 = 0$ is	p.	(*p*) 1
B.	Number of transverse common tangents of the circles $x^2 + y^2 - 4x - 10y + 4 = 0$ and $x^2 + y^2 - 6x - 12y - 55 = 0$ is	q.	(*q*) 2
C.	Number of common tangents of the circles $x^2 + y^2 - 2x - 4y = 0$ & $x^2 + y^2 - 8y - 4 = 0$ is	r.	(*r*) 3
D.	Number of direct common tangents of the circles $x^2 + y^2 + 2x - 8y + 13 = 0$ and $x^2 + y^2 - 6x - 2y + 6 = 0$ is	s.	(*s*) 0

(*a*) A – p, B – r, C – q, D – s
(*b*) A – p, B – q, C – r, D – s
(*c*) A – r, B – s, C – p, D – q
(*d*) A – r, B – q, C – s, D – p

38. Let $S = \{(x\,.\,y): x^2 + y^2 - 6x - 8y + 21 \le 0\}$

	Column-I		Column-II
A.	$\max\left\{\frac{12x}{7}-\frac{5y}{7};(x,y)\in S\right\}$	p.	3
B.	$\min\left\{\frac{1}{2}(x^2+y^2+1)+(x-y);\ (x,y)\in S\right\}$	q.	4
C.	$\max\left\{\frac{3x}{7}+\frac{4y}{7};(x,y)\in S\right\}$	r.	5
D.	$\min\left\{\frac{\sqrt{3}y+\lvert x-3\rvert}{\lvert x-3\rvert};(x,y)\in S\right\}$	s.	6
		t.	7

(*a*) A – s, B – q, C – r, D – q
(*b*) A – q, B – s, C – p, D – r
(*c*) A – s, B – p, C – p, D – q
(*d*) A – r, B – q, C – r, D – q

NUMERICAL BASED QUESTION

39. Obtain the equations of the straight lines passing through the point $A(2, 0)$ & making $45°$ angle with the tangent at A to the circle $(x + 2)^2 + (y - 3)^2 = 25$. Find the equations of the circles each of radius 3 whose centres are on these straight lines at a distance of $5\sqrt{2}$ from A.

40. The number of such points $(a+1, \sqrt{3}a)$, where a is any integer, lying inside the region bounded by the circles $x^2+y^2-2x-3=0$ and $x^2+y^2-2x-15=0$, is

41. The area of region in xy plane consisting of all points (p, q) such that quadratic equation $px^2 + 2(p + q - 7)x + 2q = 0$ has less than 2 real solutions is ________, where ($\pi = 3.14$).

42. Two circles whose radii are equal to 4 and 8 intersect at right angles. The length of their common chord is $\frac{\lambda}{\sqrt{5}}$, then find λ.

43. If the line $x \sin \alpha - y + \alpha \sec \alpha = 0$ touches the circle with radius 'a' and centre at the origin then find the most general values of 'α' and sum of the values of 'α' lying in $[0, 100\pi]$.

44. The circles, which cut the family of circles passing through the fixed points $A \equiv (2, 1)$ and $B \equiv (4, 3)$ orthogonally, pass through two fixed points (x_1, y_1) and (x_2, y_2), which may be real or imaginary. Find the value of $(x_1^3 + x_2^3 + y_1^3 + y_2^3)$.

45. Find the intervals of the values of a for which the line $y+c=0$ bisects two chords drawn from the point $\left(\frac{1+\sqrt{2}a}{2},\frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2+2y^2-(1+\sqrt{2}a)x-(1-\sqrt{2}a)y=0$.

46. Consider a family of circles passing through two fixed points $A(3, 7)$ & $B(6, 5)$. The the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

47. Let E, F, G, H be 4 distinct points inside square $ABCD$ whose area is 1 square units such that $\angle EDC = \angle ECD = \angle HDA = \angle HAD = \angle GAB = \angle GBA = \angle FBC = \angle FCB = 15°$

(*i*) If $\angle AEB = \frac{\pi}{k}$; then find k

(*ii*) Find the radius of circle circumscribing the ΔAHD

48. A circle touches the line $y = x$ at a point P such that $OP = 4\sqrt{2}$, where O is the origin. The circle contains the point $(-10, 2)$ in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$. Determine the equation of the circle.

49. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be 3 distinct points lying on circle $S : x^2 + y^2 = 1$, such that

$$x_1x_2 + y_1y_2 + x_2x_3 + y_2y_3 + x_3x_1 + y_3y_1 = -\frac{3}{2}$$

(*i*) Let P be any arbitrary point lying on S, then find the value of $(PA)^2 + (PB)^2 + (PC)^2$

(*ii*) Let I and G represent incenter and centroid of ΔABC respectively, then find the value of $IA + IB + IC + GA + GB + GC$

50. An isosceles right angled triangle whose sides are 1, 1, $\sqrt{2}$ lies entirely in the first quadrant with the ends of the hypotenuse on the coordinate axes. If it slides prove that the locus of its centroid is $(3x - y)^2 + (x - 3y)^2 = \frac{32}{9}$.

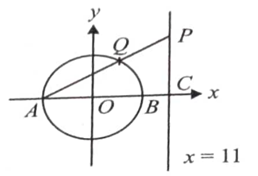
51. Given that a right angled trapezium has an inscribed circle. Prove that the length of the right angled leg is the Harmonic mean of the lengths of bases.

52. The circle $C : x^2 + y^2 + kx + (1 + k)y - (k + 1) = 0$ passes through the same two points for every real number k. Find

(*i*) the coordinates of these two points.

(*ii*) the minimum value of the radius of a circle C.

53. In the given figure, the circle $x^2 + y^2 = 25$ intersects x-axis at points A and B. The line $x = 11$ intersects x-axis, at point C. Point P moves along the line $x = 11$ above and on the x-axis and AP intersects the circle at Q.



The co-ordinates of P if $|AP - BP|$ is maximum

54. Let K denotes the square of the diameter of the circle whose diameter is the common chord of the two circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$ and
W denotes the sum of the abscissa and ordinates of a point P where all variable chords of the curve $y^2 = 8x$ subtending right angles at the origin, are concurrent and
H denotes the square of the length of the tangent from the point $(3, 0)$ on the circle $2x^2 + 2y^2 + 5y - 16 = 0$.
Find the value of KWH.

55. The line $lx + my + n = 0$ intersects the curve $ax^2 + 2hxy + by^2 = 1$ at the point P and Q. The circle on PQ as diameter passes through the origin. Prove that $n^2(a + b) = l^2 + m^2$.

56. Let the variable line $ax + by + c = 0$, where a, b, c are in arithmetic progression be normal to a family or circles. If r be the radius of the circle of the family which intersects the circle $x^2 + y^2 - 4x - 4y = 1 = 0$ orthogonally, then find the value of r^2.

57. Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from the origin O to a circle of radius 3, with center in the first quadrant. If A is one of the points of contact find the length of OA.

58. The value of k for which the pint $(2k + 1, k - 1)$ is an interior point of the smaller segment of the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ w.r.t. the chord $x + y - 2 = 0$ is

Exercise-4 (Past Year Questions)

JEE MAIN

1. The centers of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x- axis, lie on: **(2016)**

(*a*) A circle
(*b*) An ellipse which is not a circle
(*c*) A hyperbola
(*d*) A parabola

2. If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at $(-3, 2)$ then the radius of S is: **(2016)**

(*a*) $5\sqrt{2}$ (*b*) $5\sqrt{3}$
(*c*) 5 (*d*) 10

3. Let the orthocentre and centroid of a triangle be $A(-3, 5)$ and $B(3, 3)$ respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is **(2018)**

(a) $2\sqrt{10}$ (b) $3\sqrt{\frac{5}{2}}$

(c) $\frac{3\sqrt{5}}{2}$ (d) $\sqrt{10}$

4. If the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ then the value of c is **(2018)**

(a) 185 (b) 85

(c) 95 (d) 195

5. Three circles of radii a, b, c ($a < b < c$) touch each other externally. If they have x–axis as a common tangent, then **(2019)**

(a) $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$ (b) $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$

(c) a, b, c are in A.P. (d) $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are in A.P

6. If the circles $x^2 + y^2 - 16x - 20y + 164 = r^2$ and $(x - 4)^2 + (y - 7)^2 = 36$ intersect at two distinct points, then **(2019)**

(a) $0 < r < 1$ (b) $1 < r < 11$

(c) $r > 11$ (d) $r = 11$

7. If a circle C passing through the point (4, 0) touches the circle $x^2 + y^2 + 4x - 6y = 12$ externally at the point (1, –1), then the radius of C is **(2019)**

(a) $2\sqrt{5}$ (b) 4

(c) 5 (d) $\sqrt{57}$

8. If the area of an equilateral triangle inscribed in the circle, $x^2 + y^2 + 10x + 12y + c = 0$ is $27\sqrt{3}$ sq. units then c is equal to **(2019)**

(a) 13 (b) 20

(c) –25 (d) 25

9. A circle cuts a chord of length $4a$ on the x -axis and passes through a point on the y – axis, distance $2b$ form the origin. Then the locus of the centre of this circle, is **(2019)**

(a) A hyperbola (b) An ellipse

(c) A straight line (d) A parabola

10. A square is inscribed in the circle $x^2 + y^2 - 6x + 8y - 103 = 0$ with its sides parallel to the coordinate axes. Then the distance of the vertex of the square which is nearest to the origin is **(2019)**

(a) 6 (b) $\sqrt{137}$

(c) $\sqrt{41}$ (d) 13

11. Two circles with equal radii intersecting at the points (0, 1) and (0, –1). The tangent at the point (0, 1) to one of the circles passes through the centre of the other circle. Then the distance between the centres of these circles is **(2019)**

(a) 1 (b) 2

(c) $2\sqrt{2}$ (d) $\sqrt{2}$

12. The straight line $x + 2y = 1$ meets the coordinate axes at A and B. A circle is drawn through A, B and the origin. Then the sum of perpendicular distances from A and B on the tangent to the circle at the origin is **(2019)**

(a) $\frac{\sqrt{5}}{2}$ (b) $2\sqrt{5}$

(c) $\frac{\sqrt{5}}{4}$ (d) $4\sqrt{5}$

13. Let C_1 and C_2 be the centres of the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and $x^2 + y^2 - 6x - 6y + 14 = 0$ respectively. If P and Q are the points of intersection of these circles, then the area (in sq. units) of the quadrilateral PC_1QC_2 is **(2019)**

(a) 8 (b) 6

(c) 9 (d) 4

14. If a variable line, $3x + 4y - \lambda = 0$ is such that the two circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 18x - 2y + 78 = 0$ are on its opposite sides, then the set of all values of λ is the interval **(2019)**

(a) (2, 17) (b) [13, 23]

(c) [12, 21] (d) (23, 31)

15. Let z_1 and z_2 be two complex numbers satisfying $|z_1| = 9$ and $|z_2 - 3 - 4i| = 4$. Then the minimum value of $|z_1 - z_2|$ is **(2019)**

(a) 0 (b) $\sqrt{2}$

(c) 1 (d) 2

16. If a circle of radius R passes through the origin O and intersects the coordinate axes at A and B, then the locus of the foot of perpendicular from O on AB is **(2019)**

(a) $(x^2 + y^2)^2 = 4R^2x^2y^2$

(b) $(x^2 + y^2)^3 = 4R^2x^2y^2$

(c) $(x^2 + y^2)^2 = 4Rx^2y^2$

(d) $(x^2 + y^2)(x + y) = R^2xy$

17. The sum of the squares of the lengths of the chords intercepted on the circle, $x^2 + y^2 = 16$, by the lines, $x + y = n$, $n \in N$, where N is the set of all natural numbers, is **(2019)**

(a) 320 (b) 160

(c) 105 (d) 210

18. The tangent and the normal lines at the point $(\sqrt{3}, 1)$ to the circle $x^2 + y^2 = 4$ and the x-axis form a triangle. The area of this triangle (in square units) is **(2019)**

(a) $\frac{1}{3}$ (b) $\frac{4}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{3}}$

19. If a tangent to the circle $x^2 + y^2 = 1$ intersects the corrdinate axes at distinct points P and Q, then the locus of the mid-point of PQ is **(2019)**

(a) $x^2 + y^2 - 2xy = 0$ (b) $x^2 + y^2 - 16x^2y^2 = 0$

(c) $x^2 + y^2 - 4x^2y^2 = 0$ (d) $x^2 + y^2 - 2x^2y^2 = 0$

20. The common tangent to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 + 6x + 8y - 24 = 0$ also passes through the point **(2019)**

(a) $(-4, 6)$ (b) $(6, -2)$
(c) $(-6, 4)$ (d) $(4, -2)$

21. If the circles $x^2 + y^2 + 5Kx + 2y + K = 0$ and $2(x^2 + y^2) + 2Kx + 3y - 1 = 0$, $(K \in R)$, intersect at the points P and Q, then the line $4x + 5y - K = 0$ passes through P and Q for **(2019)**

(a) exactly two values of K
(b) exactly one value of K
(c) no value of K.
(d) infinitely many values of K

22. The locus of the centres of the circles, which touch the circle, $x^2 + y^2 = 1$ externally, also touch the y-axis and lie in the first quadrant, is **(2019)**

(a) $y = \sqrt{1+4x}$, $x \geq 0$ (b) $x = \sqrt{1+4y}$, $y \geq 0$
(c) $x = \sqrt{1+2y}$, $y \geq 0$ (d) $y = \sqrt{1+2x}$, $x \geq 0$

23. The line $x = y$ touches a circle at the point $(1, 1)$. If the circle also passes through the point $(1, -3)$, then its radius is **(2019)**

(a) $3\sqrt{2}$ (b) 3
(c) $2\sqrt{2}$ (d) 2

24. If the angle of intersection at a point where the two circles with radii 5 cm and 12 cm intersect is 90°, then the length (in cm) of their common chord is **(2019)**

(a) $\frac{60}{13}$ (b) $\frac{120}{13}$
(c) $\frac{13}{2}$ (d) $\frac{13}{5}$

25. A circle touching the x-axis at (3,0) and making an intercept of length 8 on the y-axis passes through the point **(2019)**

(a) (3, 10) (b) (2, 3)
(c) (1, 5) (d) (3, 5)

26. If a line, $y = mx + c$ is a tangent to the circle, $(x - 3)^2 + y^2 = 1$ and it is perpendicular to a line L_1, where L_1 is the tagnent to the circle, $x^2 + y^2 = 1$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; then **(2020)**

(a) $c^2 + 7c + 6 = 0$ (b) $c^2 + 6c + 7 = 0$
(c) $c^2 - 6c + 7 = 0$ (d) $c^2 - 7c + 6 = 0$

27. A circle touches the y-axis at the point (0, 4) and passes through the point (2, 0). Which of the following lines is not a tangent to this circle? **(2020)**

(a) $4x - 3y + 17 = 0$ (b) $3x - 4y - 24 = 0$
(c) $3x + 4y - 6 = 0$ (d) $4x + 3y - 8 = 0$

28. If the curves, $x^2 - 6x + y^2 + 8 = 0$ and $x^2 - 8y + y^2 + 16 - k = 0$, $(k > 0)$ touch each other at a point, then the largest value of k is **(2020)**

29. The number of integral values of k for which the line, $3x + 4y = k$ intersects the circle, $x^2 + y^2 - 2x - 4y + 4 = 0$ at two distinct points is ______. **(2020)**

30. Let the latus ractum of the parabola $y^2 = 4x$ be the common chord to the circles C_1 and C_2 each of them having radius $2\sqrt{5}$. Then, the distance between the centres of the circles C_1 and C_2 is **(2020)**

(a) 8 (b) 12
(c) $8\sqrt{5}$ (d) $4\sqrt{5}$

31. The diameter of the circle, whose centre lies on the line $x + y = 2$ in the first quadrant and which touches both the lines $x = 3$ and $y = 2$, is ______. **(2020)**

32. Let PQ be a diameter of the circle $x^2 + y^2 = 9$. If α and β are the lengths of the perpendiculars from P and Q on the straight li ne, $x + y = 2$ respectively, then the maximum value of $\alpha\beta$ is______. **(2020)**

33. The circle passing through the intersection of the circles, $x^2 + y^2 - 6x = 0$ and $x^2 + y^2 - 4y = 0$, having its centre on the line, $2x - 3y + 12 = 0$, also passes through the point **(2020)**

(a) $(-3, 6)$ (b) $(-1, 3)$
(c) $(-3, 1)$ (d) $(1, -3)$

34. If the length of the chord of the circle, $x^2 + y^2 = r^2$ $(r > 0)$ along the line, $y - 2x = 3$ is r then r^2 is equal to **(2020)**

35. Let B be the centre of the circle $x^2 + y - 2x + 2y + 1 = 0$. Let the tangents at two points P and Q on the circle intersect at the point $A(3, 1)$. Then $8 \cdot \left(\frac{\text{area } \Delta APQ}{\text{area } \Delta BPQ}\right)$ is equal to **(2021)**

36. Numerical
If the variable line $3x + 4y = \alpha$ lies between the two circle $(x - 1)^2 + (y - 1)^2 = 1$ and $(x - 9)^2 + (y - 1)^2 = 4$, without intercepting a chord on either circle, then the sum of all the integral values of is .... **(2021)**

37. Two circles each of radius 5 units touch each other at the point (1, 2). If the equation of their common tangent is $4x + 3y = 10$, and $C_1(\alpha, \beta)$ and $C_2(\gamma, \delta)$, $C_1 \neq C_2$ are their centres then $|(\alpha + \beta)(\gamma + \delta)|$ equal to **(2021**

38. Let the equation $x^2 + y^2 + px + (1-p)y + 5 = 0$ represen circles of varying radius $r \in (0,5]$. Then the number o elements in the set $S = \{q : q = p^2$ and is an integer is **(202**

39. The locus of a point, which moves such that the sum of squar of its distances from the points (0, 0), (1, 0) (0, 1), (1, 1) is units, is a circle of diameter d. Then d^2 is equal to **(202**

40. Let a circle C : $(x - h)^2 + (y - k)^2 = r^2$, $k > 0$, touch the x-ax at (1, 0). If the line $x + y = 0$ intersects the circle C at P a Q such that the length of the chord PQ is 2, then the val of $h + k + r$ is equal to ________ . **(202**

41. Let a circle C touch the lines $L_1 : 4x - 3y + K_1 = 0$ a $L_2 : 4x - 3y + K_2 = 0$, $K_1, K_2 \in R$. If a line passing throu

the centre of the circle C intersects L_1 at (–1, 2) and L_2 at (3, –6), then the equation of the circle C is **(2022)**

(a) $(x-1)^2+(y-2)^2=4$ (b) $(x+1)^2+(y-2)^2=4$

(c) $(x-1)^2+(y+2)^2=16$ (d) $(x-1)^2+(y-2)^2=16$

42. A circle touches both the y-axis and the line $x+y=0$. Then the locus of its center is **(2022)**

(a) $y=\sqrt{2}x$ (b) $x=\sqrt{2}y$

(c) $y^2-x^2=2xy$ (d) $x^2-y^2=2xy$

43. Let the tangents at two A and B on the circle $x^2+y^2-4x+3=0$ meet at origin $O(0,0)$. Then the area of the triangle OAB is : **(2022)**

(a) $\frac{3\sqrt{3}}{2}$ (b) $\frac{3\sqrt{3}}{4}$ (c) $\frac{3}{2\sqrt{3}}$ (d) $\frac{3}{4\sqrt{3}}$

44. For $t\in(0,2\pi)$, if ABC is an equilateral triangle with vertices A (sin$t$, – cos$t$), B(cos$t$, sin$t$) and C (a, b) such that its othocentre lies on a cricle with centre (1, 1/3), then (a^2-b^2) is equal to **(2022)**

(a) 8/3 (b) 8

(c) 77/9 (d) 80/9

45. Let C be the centre of the circle $x^2+y^2-x+2y=\frac{11}{4}$ and P be a point on the circle. A line passes through the point C, makes an angle of $\pi/4$ with the line CP and intersects the circle at the points Q and R. Then the area of the triangle PQR (in unit^2) is: **(2022)**

(a) 2 (b) $2\sqrt{2}$

(c) $8\sin\left(\frac{\pi}{8}\right)$ (d) $8\cos\left(\frac{\pi}{8}\right)$

46. A circle C_1 passes through the origin O and has diameter 4 on the positive x-axis. The line $y=2x$ gives a chord OA of circle C_1. Let C_2 be the circle with OA as a diameter. If the tangent to C_2 at the point A meets the x-axis at P and y-axis at Q, then $QA : AP$ is equal to : **(2022)**

(a) 1 : 4 (b) 1 : 5

(c) 2 : 5 (d) 1 : 3

47. If the circle $x^2+y^2-2gx+6y-19c=0, g,c\in\mathbb{R}$ passes throught the point (6, 1) and its centre lies on the line $x-2cy=8$, then the length of intercept made by the circle on x-axis is **(2022)**

(a) $\sqrt{11}$ (b) 4

(c) 3 (d) $2\sqrt{23}$

48. Let the abscissae of the two points P and Q on a circle be the roots of $x^2-4x-6=0$ and the ordinates of P and Q be the roots of $y^2+2y-7=0$. If PQ is a diameter of the circle $x^2+y^2+2ax+2by+c=0$, then the value of $(a+b-c)$ is .... **(2022)**

(a) 12 (b) 13 (c) 14 (d) 16

49. Consider three circles: **(2022)**

$C_1 : x^2+y^2=r^2$

$C_2 : (x-1)^2+(y-1)^2=r^2$

$C_3 : (x-2)^2+(y-1)^2=r^2$

If a line $L : y=mx+c$ be a common tangent to C_1, C_2 and C_3 such that C_1 and C_3 lie on one side of line L while C_2 lies on other side, then the value of $20(r^2+c)$ is euqal to :

(a) 23 (b) 15

(c) 12 (d) 6

50. Let a triangle ABC be inscribed in the circle $x^2-\sqrt{2}(x+y)+y^2=0$ such that $\angle BAC=\frac{\pi}{2}$. If the length of side AB is $\sqrt{2}$, then the are of the ΔABC is equal to : **(2022)**

(a) 1 (b) $\frac{(\sqrt{6}+\sqrt{3})}{2}$

(c) $\frac{(3+\sqrt{3})}{4}$ (d) $\frac{(\sqrt{6}+2\sqrt{3})}{4}$

51. Let the tangent to the circle $C_1 x^2+y^2=2$ at the point $M(-1, 1)$ intersect the circle $C_2 : (x-3)+(y-2)^2=5$, at two distinct points A and B. If the tangents to C_2 at the points A and B intersect at N, then the area of the triangle ANB is equal to: **(2022)**

(a) 1/2 (b) 2/3

(c) 1/6 (d) 5/3

52. If the tangents drawn at the points $O(0,0)$ and $P\left(1+\sqrt{5},2\right)$ on the cirlce $x^2+y^2-2x-4y=0$ intersect at the point Q, then the ara of the triangle OPQ is equal to **(2022)**

(a) $\frac{3+\sqrt{5}}{2}$ (b) $\frac{4+2\sqrt{5}}{2}$

(c) $\frac{5+3\sqrt{5}}{2}$ (d) $\frac{7+3\sqrt{5}}{2}$

53. The set of values of k, for which the circle $C : 4x^2+4y^2-12x+8y+k=0$ lies inside the fourth quadrant and the point $\left(1,-\frac{1}{3}\right)$ lie on or inside the circle C, is. **(2022)**

(a) An empty set (b) $\left(6,\frac{65}{9}\right]$

(c) $\left[\frac{80}{9},10\right)$ (d) $\left(9,\frac{92}{9}\right]$

54. Let C be a circle passing through the points $A(2, 1)$and $B(3, 4)$. The line segment ABs not a diameter of C. If r is the radius of C and its centre lies on the circle $(x-5)^2+(y-1)^2=\frac{13}{2}$, then r^2 is equal to : **(2022)**

(a) 32 (b) 65/2 (c) 61/2 (d) 30

55. A circletouchesboth the y-axis and the line $x + y = 0$.Then the locus of its center is : **(2022)**

(a) $y=\sqrt{2}x$ (b) $x=\sqrt{2}y$
(c) $y^2-x^2=2xy$ (d) $x^2-y^2=2xy$

56. Let a circle C touch the lines **(2022)**
$L_1: 4x - 3y + K_1 = 0$ and
$L_2: 4x - 3y + K_2 = 0$, $K_1, K_2 \in R$. If a line passing through the centre of the circle C intersects L_1 at $(-1, 2)$ and L_2 at $(3, -6)$, then the equation of the cirlce C is :

(a) $(x-1)^2+(y-2)^2=4$ (b) $(x+1)^2+(y-2)^2=4$
(c) $(x-1)^2+(y+2)^2=16$ (d) $(x-1)^2+(y-2)^2=16$

57. Let a,b and c be the length of sides of a triangle ABC such that $\frac{a+b}{7}=\frac{b+c}{8}=\frac{c+a}{9}$. If r and R are the radius of incircle and radius of circumcircle of the triangle ABC, respectively, then the value of R/r is equal to : **(2022)**

(a) 5/2 (b) 2 (c) 3/2 (d) 1

JEE ADVANCED

58. The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. The equation of the locus of the point P is **(1996)**

(a) $x^2 + y^2 + 4x - 6y + 4 = 0$
(b) $x^2 + y^2 + 4x - 6y - 9 = 0$
(c) $x^2 + y^2 + 4x - 6y - 4 = 0$
(d) $x^2 + y^2 + 4x - 6y + 9 = 0$

59. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2+y^2-x+3y=0$ on L_1 and L_2 are equal, then which of the following equation can represent L_1? **(1999)**

(a) $x+y=0$ (b) $x-y=0$
(c) $x+7y=0$ (d) $x-7y=0$

60. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where, $pq \neq 0$) are bisected by the X-axis, then **(1999)**

(a) $p^2 = q^2$ (b) $p^2 = 8q^2$
(c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

61. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is **(2012)**

(a) $20(x^2 + y^2) - 36x + 45y = 0$
(b) $20(x^2 + y^2) + 36x - 45y = 0$
(c) $36(x^2 + y^2) - 20x + 45y = 0$
(d) $36(x^2 + y^2) + 20x - 45y = 0$

Comprehension (No. 62 to 63): A tangent PT is drawn to the circle $x^2+y^2=4$ at the point $P(\sqrt{3}, 1)$. A straight line L, perpendicular to PT is a tangent to the circle $(x-3)^2+y^2=1$. **(2012)**

62. A common tangent of the two circles is

(a) $x=4$ (b) $y=2$
(c) $x+\sqrt{3}y=4$ (d) $x+2\sqrt{2}y=6$

63. A possible equation of L is

(a) $x-\sqrt{3}y=1$ (b) $x+\sqrt{3}y=1$
(c) $x-\sqrt{3}y=-1$ (d) $x+\sqrt{3}y=5$

64. Circle(s) touching x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on y-axis is (are) **(2013)**

(a) $x^2+y^2-6x+8y+9=0$
(b) $x^2+y^2-6x+7y+9=0$
(c) $x^2+y^2-6x-8y+9=0$
(d) $x^2+y^2-6x-7y+9=0$

65. A circle S passes through the point $(0, 1)$ and is orthogonal to the circles $(x-1)^2+y^2=16$ and $x^2+y^2=1$. Then **(2014)**

(a) Radius of S is 8 (b) Radius of S is 7
(c) Centre of S is $(-7, 1)$ (d) Centre of S is $(-8, 1)$

66. Let RS be the diameter of the circle $x^2+y^2=1$, where S is the point $(1, 0)$. Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. Then the locus of E passes through the point(s) **(2016)**

(a) $\left(\frac{1}{3},\frac{1}{\sqrt{3}}\right)$ (b) $\left(\frac{1}{4},\frac{1}{2}\right)$ (c) $\left(\frac{1}{3},-\frac{1}{\sqrt{3}}\right)$ (d) $\left(\frac{1}{4},-\frac{1}{2}\right)$

67. For how many values of p, the circle $x^2+y^2+2x+4y-p=0$ and the coordinate axes have exactly three common points? **(2017)**

Comprehension (No. 68 to 69): Let S be the circle in the xy-plane defiend by the equation$x^2+y^2=4$.

68. let E_1E_2 and F_1F_2 be the chord of S passing through the point $P_0(1, 1)$ and parallel to the x-axis and the y-axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slope -1. Let the tangents to S at E_1 and E_2 meet at E_3, the tangents of S at F_1 and F_2 meet at F_3, and the tangents to S at G_1 and G_2 meet at G_3. Then, the point E_3, F_3 and G_3 lie on the curve **(2018)**

(a) $x+y=4$ (b) $(x-4)^2+(y-4)^2=16$
(c) $(x-4)(y-4)=4$ (d) $xy=4$

69. Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N. Then, the mid-point of the line segment MN must lie on the curve:

(a) $(x+y)^2=3xy$ (b) $x^{2/3}+y^{2/3}=2^{4/3}$
(c) $x^2+y^2=2xy$ (d) $x^2+y^2=x^2y^2$

70. Let T be the line passing through the points $P(-2, 7)$ and $Q(2, -5)$. Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangents to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point $R(1, 1)$ be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statemebt(s) is (are) TRUE? **(2018)**

(*a*) The point $(-2,7)$ lies in E_1

(*b*) The point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does NOT lie in E_2

(*c*) The point $\left(\frac{1}{2}, 1\right)$ lies in E_2

(*d*) The point $\left(0, \frac{3}{2}\right)$ does NOT lie in E_2

71. A line $y = mx + 1$ intersects the circle $(x-3)^2 + (y+2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ? **(2019)**

(*a*) $6 \le m < 8$

(*b*) $2 \le m < 4$

(*c*) $4 \le m < 6$

(*d*) $-3 \le m < -1$

72. Let the point B be the reflection of the point $A(2, 3)$ with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is _____ **(2019)**

Comprehension (No. 73 to 74): Answer the following by appropriately matching the lists based on the information given in the paragraph

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^3 + (y-4)^2 = 16$, intersect at the points X and Y. **(2019)**

Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions

(*i*) Centre of C_3 is collinear with the centres of C_1 and C_2

(*ii*) C_1 and C_2 both lie inside C_3, and

(*iii*) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expression given in the List-I whose values are given in List-II below :

	List -I		List-II
A.	$2h + k$	p.	6
B.	$\frac{\text{Length of } ZW}{\text{Length of } XY}$	q.	$\sqrt{6}$
C.	$\frac{\text{Area of triangle } MZN}{\text{Area of triangle } ZMW}$	r.	$\frac{5}{4}$
D.	α	s.	$\frac{21}{5}$
		t.	$2\sqrt{6}$
		u.	$\frac{10}{3}$

73. Which of the following is the only INCORRECT combination?

Options

(*a*) $D \to s$ (*b*) $D \to u$ (*c*) $C \to r$ (*d*) $A \to p$

74. Which of the following is the only CORRECT combination?

Options **(2019)**

(*a*) $B \to t$ (*b*) $A \to s$ (*c*) $A \to u$ (*d*) $B \to q$

Comprehension (No. 75 to 76): Let $M = \{(x, y)\ R \times R: x2 + y2 \le r2\}$, where $r > 0$. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}, n = 1, 2, 3, ...$ Let $S0 = 0$ and for $n \ge 1$, let Sn denote the sum of the first n terms of this progression. For $n \ge 1$, let Cn denote the circle with center $(Sn-1, 0)$ and radius an, and Dn denote the circle with center $(Sn-1, Sn-1)$ and radius an. **(2021)**

75. Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M. Let 1 be the maximum possible number of circles among these k circles such that no two circles intersect. Then

(*a*) $k + 21 = 22$ (*b*) $2k + 1 = 26$

(*c*) $2k + 31 = 34$ (*d*) $3k + 21 = 40$

76. Consider M with $r = \frac{(2^{199} - 1)\sqrt{2}}{2^{198}}$. The number of all those circles Dn that are inside M is

(*a*) 198 (*b*) 199 (*c*) 200 (*d*) 201

77. Let G be a circle of radius $R > 0$. Let $G_1, G_2, ..., G_n$ be n circles of equal radius $r > 0$. Suppose each of the n circles $G_1, G_2, ..., G_n$ touches the circle G externally. Also, for $i = 1, 2, ..., n-1$, the circle G_i touches G_{i+1} externally, and G_n touches G_1 externally. Then, which of the following statements is/are TRUE? **(2022)**

(*a*) If $n = 4$, then $(\sqrt{2} - 1)r < R$

(*b*) If $n = 5$, then $r < R$

(*c*) If $n = 8$, then $(\sqrt{2} - 1)r < R$

(*d*) If $n = 12$, then $\sqrt{2}(\sqrt{3} + 1)r > R$

ANSWER KEY

CONCEPT APPLICATION

1. $x^2+y^2+4x-21=0$ or $x^2+y^2-12x+11=0$ **2.** $2(x^2+y^2)-2cx+c^2-a^2=0$ **3.** $x^2+y^2+18x-2y+32=0$ **4.** (*c*) **5.** (*d*) **6.** $(x+3)^2+(y-4)^2=9$ **9.** p = 3 and q = 2 **10.** (*a*) **11.** $25x^2+25y^2-20x+2y-60=0$ **12.** $(x+16)^2+(y+2)^2=25$ **13.** $(x-3)^2+(y-4-\sqrt{3})^2=4$ **14.** (*b*, *c*) **15.** (*c*) **16.** (*b*, *c*) **17.** $[p \in \phi]$ **18.** (*b*) **19.** (*c*) **20.** (*c*) **21.** $y=\left(\dfrac{-30\pm 2\sqrt{70}}{31}\right)(x-4)-2$ **22.** (*c*) **23.** (*a*) **24.** (*b*) **25.** (*a*) **26.** (*d*) **27.** (*b*) **28.** (*a*) **29.** (*d*) **30.** (*a*) **31.** (*a*) **32.** $=\sqrt{\mu-\lambda}]$ **33.** (*b*) **34.** (*a*) **35.** (*a*) **36.** (*a*) **37.** (*b*) **38.** (*b*) **39.** (*d*) **40.** (*c*) **41.** (*d*) **42.** (*a*) **43.** (*a*) **44.** (*b*) **45.** (*b*) **47.** (*a*) **48.** (*a*) **49.** (*a*) **50.** (*d*) **51.** (*c*) **52.** (*d*)

EXERCISE-1 (TOPICWISE)

1. (*a*) **2.** (*c*) **3.** (*d*) **4.** (*c*) **5.** (*b*) **6.** (*a*) **7.** (*d*) **8.** (*a*) **9.** (*c*) **10.** (*a*) **11.** (*b*) **12.** (*b*) **13.** (*b*) **14.** (*a*) **15.** (*c*) **16.** (*b*) **17.** (*b*) **18.** (*a*) **19.** (*d*) **20.** (*b*) **21.** (*c*) **22.** (*d*) **23.** (*a*) **24.** (*b*) **25.** (*d*) **26.** (*d*) **27.** (*d*) **28.** (*a*) **29.** (*b*) **30.** (*a*) **31.** (*c*) **32.** (*b*) **33.** (*b*) **34.** (*c*) **35.** (*a*) **36.** (*c*) **37.** (*b*) **38.** (*c*) **39.** (*c*) **40.** (*a*) **41.** (*d*) **42.** (*b*) **43.** (*b*) **44.** (*d*) **45.** (*c*) **46.** (*a*) **47.** (*a*) **48.** (*a*) **49.** (*a*) **50.** (*c*) **51.** (*a*) **52.** (*a*) **53.** (*a*) **54.** (*c*)

EXERCISE-2 (LEARNING PLUS)

1. (*b*) **2.** (*b*) **3.** (*d*) **4.** (*a*) **5.** (*a*) **6.** (*c*) **7.** (*d*) **8.** (*a*) **9.** (*c*) **10.** (*d*) **11.** (*c*) **12.** (*b*) **13.** (*a*) **14.** (*b*) **15.** (*d*) **16.** (*b*) **17.** (*b*) **18.** (*c*) **19.** (*d*) **20.** (*b*) **21.** (*b*) **22.** (*c*) **23.** (*d*) **24.** (*c*) **25.** (*b*) **26.** (*a*) **27.** (*b*) **28.** (*b*) **29.** (*c*) **30.** (*a*) **31.** (*b*) **32.** (*a*) **33.** (*b*) **34.** (*d*) **35.** (*d*) **36.** (*c*) **37.** (*a*, *b*) **38.** (*a*, *c*) **39.** (*a*, *d*) **40.** (*a*, *c*) **41.** (*c*, *d*) **42.** (*a*, *d*)

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (*c*) **2.** (*b*) **3.** (*a*) **4.** (*b*) **5.** (*d*) **6.** (*b*) **7.** (*b*) **8.** (*d*) **9.** (*b*) **10.** (*c*) **11.** (*b*) **12.** (*c*) **13.** (*a*, *b*) **14.** (*a*, *d*) **15.** (*a*, *c*) **16.** (*c*) **17.** (*b*) **18.** (*a*) **19.** (*c*) **20.** (*b*) **21.** (*b*) **22.** (*c*) **23.** (*c*) **24.** (*c*) **25.** (*b*) **26.** (*d*) **27.** (*c*) **28.** (*b*) **29.** (*b*) **30.** (*a*) **31.** (*d*) **32.** (*b*) **33.** (*a*) **34.** (*d*) **35.** (*a*) **36.** (*b*) **37.** (*c*) **38.** (*d*) **39.** [32] **40.** [0] **41.** [153.86] **42.** [16] **43.** [5050π] **44.** [40] **45.** $(-\infty,-2)\cup(2,\infty)$ **46.** (2, 23/3) **47.** (*i*) → 3, (ii) → 1 **48.** $x^2+y^2+18x-2y+32=0$ **49.** (*i*) → 6, (ii) → 6 **50.** **51.** **52.** (*i*) → (1,0) and (1/2, 1/2) (ii) → $\dfrac{1}{2\sqrt{2}}$ **53.** (11,0) **54.** [64] **55.** **56.** [8] **57.** $9+3\sqrt{10}$ **58.** $(6/5,\infty)$

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (*d*)	**2.** (*b*)	**3.** (*b*)	**4.** (*c*)	**5.** (*a*)	**6.** (*b*)	**7.** (*c*)	**8.** (*d*)	**9.** (*d*)	**10.** (*c*)
11. (*b*)	**12.** (*a*)	**13.** (*d*)	**14.** (*c*)	**15.** (*a*)	**16.** (*b*)	**17.** (*d*)	**18.** (*d*)	**19.** (*c*)	**20.** (*b*)
21. (*c*)	**22.** (*d*)	**23.** (*c*)	**24.** (*b*)	**25.** (*a*)	**26.** (*b*)	**27.** (*d*)	**28.** [36]	**29.** [9]	**30.** (*a*)
31. [3]	**32.** [7]	**33.** (*a*)	**34.** [12/5]	**35.** [18]	**36.** [165]	**37.** [40]	**38.** [61]	**39.** (16)	**40.** [7]
41. (*c*)	**42.** (*d*)	**43.** (*d*)	**44.** (*b*)	**45.** (*b*)	**46.** (*a*)	**47.** (*d*)	**48.** (*a*)	**49.** (*d*)	**50.** (*a*)
51. (*c*)	**52.** (*c*)	**53.** (*d*)	**54.** (*b*)	**55.** (*d*)	**56.** (*c*)	**57.** (*a*)			

JEE Advanced

58. (*d*)	**59.** (*b,c*)	**60.** (*d*)	**61.** (*a*)	**62.** (*d*)	**63.** (*a*)	**64.** (*a, c*)	**65.** (*b,c*)	**66.** (*a,c*)	**67.** [2]
68. (*a*)	**69.** (*d*)	**70.** (*d*)	**71.** (*b*)	**72.** [10]	**73.** (*a*)	**74.** (*d*)	**75.** (*d*)	**76.** (*b*)	**77.** (*c,d*)

CHAPTER

15 Parabola

INTRODUCTION

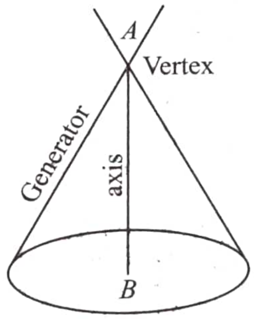
The Rainbow, surface of a concave mirror, dish antennas, headlights are taking the shape of a parabola. A projectile is anybody which is thrown or jumped into the air. Once it has left the ground it will follow a flight path called a parabola until it once more comes back down to earth. This applies to balls, javelins, discus, long jumpers etc.

CONIC SECTION

Conic sections are the curve generated by intersection of a plane with one or two right circular cone.

Section of Right Circular Cone by Different Planes

A right circular cone is as shown in the figure–1



Circular Base

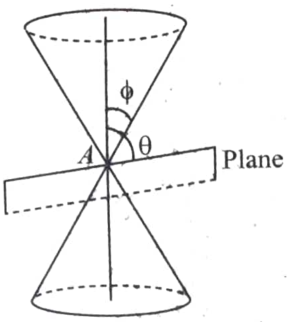
Case-1: When plane is passing through A

(*i*) When $\theta > \phi$: Two imaginary lines

(*ii*) When $\theta = \phi$: Two real and coincident lines

(*iii*) When $\theta < \phi$: Two real and distinct lines

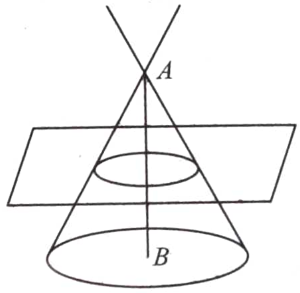
Section of a right circular cone by a plane passing through its vertex is a pair of straight lines passing through the vertex as shown in the figure



Case-2 : When plane is not passing through A

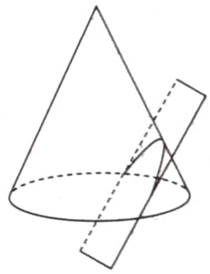
(*i*) When $\theta = 90°$: Circle

Section of a right circular cone by a plane parallel to its base is a circle as shown in figure.

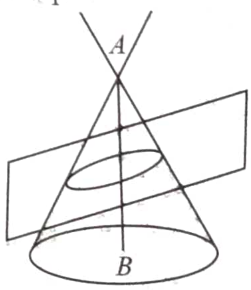


(*ii*) When $\theta = \phi$: Parabola

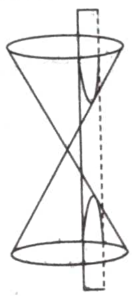
Section of a right circular cone by a plane parallel to a generator of the cone is a parabola as shown in figure.



(*iii*) When $\theta < \phi < 90°$: Ellipse



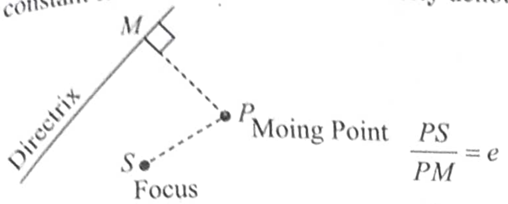
(*iv*) When $0 \leq \theta < \phi$: Hyperbola



Mathematical Definition

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

- The fixed point is called the **Focus.**
- The fixed straight line is called the **Directrix.**
- The constant ratio is called the **Eccentricity** denoted by e.



- The line passing through the focus & perpendicular to the directrix is called the **Axis.**
- A point of intersection of a conic with its axis is called a **Vertex.**

If S is (p, q) & directrix is $\ell x + my + n = 0$

then $PS = \sqrt{(x-p)^2 + (y-q)^2}$

& $PM = \dfrac{|\ell x + my + n|}{\sqrt{\ell^2 + m^2}}$

$\dfrac{PS}{PM} = e \Rightarrow (\ell^2 + m^2)\,[(x-p)^2 + (y-q)^2] = e^2\,(\ell x + my + n)^2$

Which is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is

$$(l^2 + m^2)\,[(x-p)^2 + (y-q)^2] = e^2\,(lx + my + n)^2$$
$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

DISTINGUISHING BETWEEN THE CONIC

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (*i*) When the focus lies on the directrix

In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if :

$e > 1$ the lines will be real & distinct intersecting at S.

$e = 1$ the lines will be coincident.

$e < 1$ the lines will be imaginary.

Case (*ii*) When the focus does not lie on the directrix

The conic represents :

A parabola	An ellipse	A hyperbola	A rectangular hyperbola
$e = 1; D \neq 0$	$0 < e < 1; D \neq 0$	$D \neq 0; e > 1;$	$e > 1 ; D \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab;$ $a + b = 0$

Note: (*i*) For pair of straight lines $e \to \infty$

(*ii*) All second degree terms in parabola form a perfect square

Identifying a Conic Section Without Completing the Square

A non-degenerate conic section of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

in which A and C are not both zero, is

- a circle is $A = C$,
- a parabola if $AC = 0$,
- an ellipse if $A \neq C$ and $AC > 0$, and
- a hyperbola if $AC < 0$.

DEFINITION OF VARIOUS TERMS RELATED TO A CONIC

1. **Focus:** The fixed point is called a focus of the conic.
2. **Directrix:** The fixed line is called a directrix of the conic.
3. **Axis:** The line passing through the focus and perpendicular to the directrix is called the axis of the conic.
4. **Vertex:** The points of intersection of the conic and the axis are called vertices of the conic.
5. **Centre:** The point which bisects every chord of the conic passing through it, is called the centre of the conic.
6. **Latus-rectum:** The latus-rectum of a conic is the chord passing through the focus and perpendicular to the axis.
7. **Double ordinate:** A chord which is perpendicular to the axis of parabola or parallel to its directrix.

Train Your Brain

Example 1: What conic does $13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$ represent ?

Sol. Compare the given equation with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\therefore a = 13, h = -9, b = 37, g = 1, f = 7, c = -2$

then $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= (13)(37)(-2) + 2(7)(1)(-9) - 13(7)^2 - 37(1)^2 + 2(-9)^2$

$= -962 - 126 - 637 - 37 + 162$

$= -1600 \neq 0$

and also $h^2 = (-9)^2 = 81$ and $ab = 13 \times 37 = 481$

Here $h^2 - ab < 0$

So we have $h^2 - ab < 0$ and $\Delta \neq 0$. Hence the given equation represents an ellipse.

Example 2: For what value of λ the equation of conic $2xy + 4x - 6y + \lambda = 0$ represents two real intersecting straight lines? if $\lambda = 17$ then this equation represents?

Sol. Comparing the given equation of conic with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\therefore a = 0, b = 0, h = 1, g = 2, f = -3, c = \lambda$

For two intersecting real lines

$h^2 - ab = 1$ and $\Delta = 0$

here $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 0 + 2 \times (-3) \times 2 \times 1 - 0 - 0 - \lambda (1)^2$

$= -12 - \lambda = 0$

$\therefore \lambda = -12$

and $h^2 - ab = 1$

hence for $\lambda = -12$ above equation always represent real intersecting lines.

if $\lambda = 17$ then $\Delta \neq 0$

and $h^2 - ab > 0$

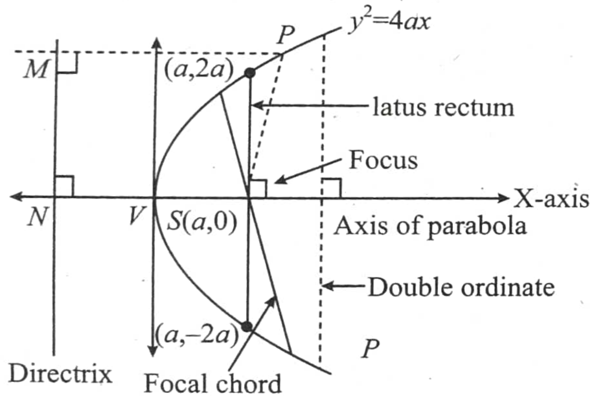
So we have $\Delta \neq 0$ and $h^2 - ab > 0$. Hence the given equation represents a Hyperbola.

Concept Application

1. If the equation $ax^2 + 4xy + y^2 + 2x + 2y + 2ay + 3 = 0$ represents a parabola, then 'a' is equal to:
   (a) 1 (b) 2
   (c) 4 (d) 3
2. If the coordinates of the extremities of latus rectum are given, then the maximum number of parabolas, which can be drawn is:
   (a) 1 (b) 2
   (c) 4 (d) Infinite

PARABOLA

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

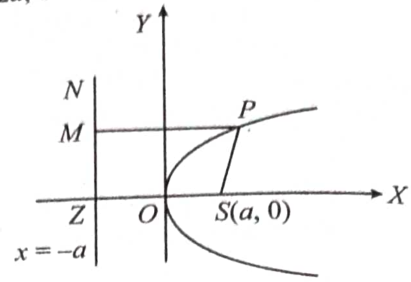


STANDARD EQUATION OF A PARABOLA

Let S be the focus and ZN is the directrix of the parabola. From S, draw SZ perpendicular to the directrix.

Let O be the middle point of ZS. Take O as the origin and OS as x-axis and OY perpendicular to OS as the y-axis.

Let $ZS = 2a$, then $ZO = OS = a$



Now, $S \equiv (a, 0)$ and the equation of ZN is $x = -a$ or $x + a = 0$.

Let $P(x, y)$ be any point on the parabola.

$\therefore$ $PS = PM$ (by definition of parabola).

$\Rightarrow \sqrt{(x-a)^2+(y-0)^2} = \dfrac{|x+a|}{\sqrt{1^2+0}}$

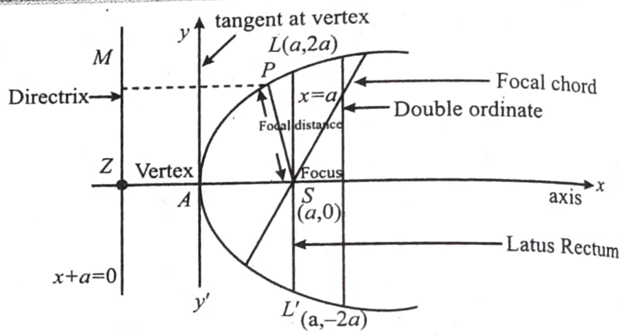
$\Rightarrow \sqrt{(x-a)^2+y^2} = |x+a|$

or $(x-a)^2 + y^2 = (x+a)^2$

or $x^2 - 2ax + a^2 + y^2 = x^2 + 2xa + a^2$

or $y^2 = 4ax$ which is the required equation.

TERMS RELATED TO PARABOLA



1. **Axis:** A straight line passes through the focus and perpendicular to the directrix is called the axis of parabola. For the parabola $y^2 = 4ax$, x-axis is the axis.
   Since equation has even power of y therefore the parabola is symmetric about x-axis i.e. about its axis.
2. **Vertex:** The point of intersection of a parabola and its axis is called the vertex of the Parabola. For the parabola $y^2 = 4ax$, $O(0, 0)$ is the vertex.
   The vertex is the middle point of the focus and the point of intersection of axis and directrix.
3. **Focal Distance:** The distance of any point $P(x, y)$ on the parabola from the focus is called the focal length (distance) of point P.
   The focal distance of P = the perpendicular distance of the point P from the directrix.
4. **Double Ordinate:** The chord which is perpendicular to the axis of Parabola or parallel to Directrix is called double ordinate of the Parabola.

5. **Focal Chord:** Any chord of the parabola passing through the focus is called Focal chord.

6. **Latus Rectum:** If a double ordinate passes through the focus of parabola then it is called as latus rectum. The extremities of the latus rectum are L $(a, 2a)$ and $L'(a, -2a)$. Since $LS = L'S = 2a$, therefore length of the latus rectum $LL' = 4a$.

7. **Parametric Equation of Parabola:** The parametric equation of Parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$.

Hence any point on this parabola is $(at^2, 2\,at)$ which is also called as 't' point.

Note: (*i*) The length of the latus rectum = 2 × perpendicular distance of focus from the directrix.

(*ii*) If $y^2 = lx$ then length of the latus rectum $= l$.

(*iii*) Two parabolas are said to be equal if they have same latus rectum.

(*iv*) The ends of a double ordinate of a parabola can be taken as $(at^2, 2\,at)$ and $(at^2, -2at)$.

(*v*) Parabola has no centre, but circle, ellipse, hyperbola have centre.

(*vi*) Perpendicular distance from focus on directrix = half the latus rectum.

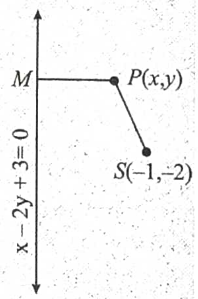
(*vii*) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(*viii*) Two parabolas are laid to be equal if they have the same latus rectum.

Train Your Brain

Example 3: Find the equation of the parabola whose focus is at $(-1, -2)$ and the directrix is $x - 2y + 3 = 0$.

Sol. Let $P(x, y)$ be any point on the parabola whose focus is $S(-1, -2)$ and the directrix $x - 2y + 3 = 0$. Draw PM perpendicular to directrix $x - 2y + 3 = 0$. Then by definition,



$SP = PM$

$\Rightarrow SP^2 = PM^2$

$\Rightarrow (x + 1)^2 + (y + 2)^2$

$= \left(\dfrac{x-2y+3}{\sqrt{1+4}}\right)^2$

$\Rightarrow 5\,[(x + 1)^2 + (y + 2)^2]$

$= (x - 2y + 3)^2$

$\Rightarrow 5(x^2+y^2+2x+4y+5) = (x^2+4y^2+9 - 4xy + 6x - 12y)$

$\Rightarrow 4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$

This is the equation of the required parabola.

Example 4: The length of the latus rectium of the parabola whose focus is $(3, 3)$ are directorix is $3x - 4y - 2 = 0$

Sol. Length of the latus rectum = 2[Perpendicualr distance from focus to directrix]

$$= 2\left|\frac{3(3)-4(3)-2}{\sqrt{3^2+y^2}}\right| = 2$$

Concept Application

3. The focus of a parabola is $(2, 3)$ and the directrix is $3x - 4y + 3 = 0$, the length of its latus rectum is:

(*a*) $\dfrac{4}{5}$ (*b*) $\dfrac{3}{5}$

(*c*) $\dfrac{6}{5}$ (*d*) $\dfrac{12}{5}$

4. The equation of parabola whose focus is $(5,3)$ and directrix is $3x - 4y + 1 = 0$

(*a*) $(4x+3y)^2 - 256x - 142y + 849 = 0$

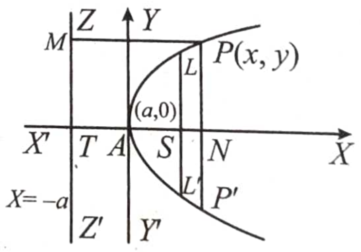
(*b*) $(4x-3y)^2 - 256x - 142y + 849 = 0$

(*c*) $(3x+4y)^2 - 142x - 256y + 849 = 0$

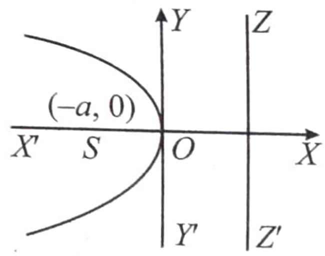
(*d*) $(3x-4y)^2 - 256x - 142y + 849 = 0$

TYPE OF PARABOLA

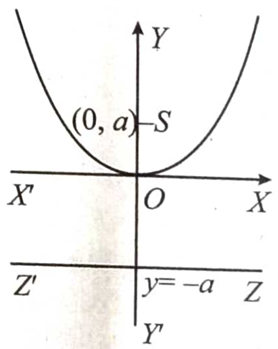
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$



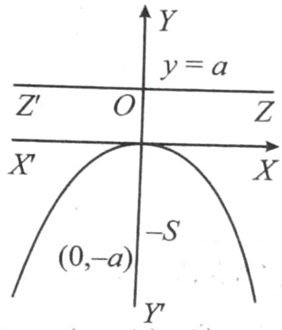
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Parametric eqution	Focal length
$y^2 = 4ax$	$(0, 0)$	$(a, 0)$	$y = 0$	$x = -a$	$4a$	$(a, \pm 2a)$	$(at^2, 2at)$	$x + a$
$y^2 = -4ax$	$(0, 0)$	$(-a, 0)$	$y = 0$	$x = a$	$4a$	$(-a, \pm 2a)$	$(-at^2, 2at)$	$x - a$
$x^2 = +4ay$	$(0, 0)$	$(0, a)$	$x = 0$	$y = -a$	$4a$	$(\pm 2a, a)$	$(2at, at^2)$	$y + a$
$x^2 = -4ay$	$(0, 0)$	$(0, -a)$	$x = 0$	$y = a$	$4a$	$(\pm 2a, -a)$	$(2at, -at^2)$	$y - a$
$(y - k)^2 = 4a(x - h)$	(h, k)	$(h + a, k)$	$y = k$	$x + a - h = 0$	$4a$	$(h + a, k \pm 2a)$	$(h + at^2, k + 2at)$	$x - h + a$
$(x - p)^2 = 4b(y - q)$	(p, q)	$(p, a + q)$	$x = p$	$y + a - q = 0$	$4a$	$(p \pm 2a, q + a)$	$(p + 2at, q + at^2)$	$y - q + a$

REDUCTION TO GENERALIZED EQUATION OF PARABOLA

If the equation of a parabola is either in the form $x = \ell y^2 + my + n$ or $y = \ell x^2 + mx + n$ then it can be reduced into generalised form. For this we change the given equation into the following forms-

$(y - k)^2 = 4a\,(x - h)$ or $(x - h)^2 = 4a\,(y - k)$

And then we compare from the standard equation of parabola to find all its parameters.

(A) When the equation of parabola is:

$(y - k)^2 = 4a\,(x - h)$ ...(*i*)

Equation (*i*) is of the form $Y^2 = 4aX$

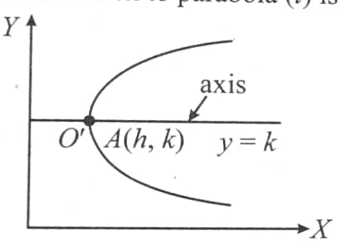
where $Y = y - k$ and $X = x - h$

1. Axis of parabola is $Y = 0$, i.e., $y - k = 0 \Rightarrow y = k$
2. Coordinates of vertex of parabola are given by

   $X = 0$ and $Y = 0$

   i.e. $x - h = 0$ and $y - k = 0$

   $\therefore$ Vertex is (h, k)
3. Tangent at the vertex to parabola (*i*) is given by



   $X = 0$, i.e., $x - h = 0$

   Therefore, tangent at the vertex is $x = h$.
4. Coordinates of focus of parabola are given by

   $X = a$ and $Y = 0$

   i.e. by $x - h = a$ and $y - k = 0$

   $\therefore$ Focus of parabola is $(a + h, k)$.
5. Equation of directrix of parabola is

   $X = -a$ i.e., $x - h = -a$

   Therefore, directrix of parabola is $x = h - a$
6. Length of latus rectum of parabola is $|\,4a\,|$.
7. Coordinates of ends of latus rectum of parabola are given by

   $X = a$ & $Y = \pm 2a$

   i.e., by $x - h = a$, $y - k = \pm 2a$ i.e. coordinate of latus rectum is $(a + h, k \pm 2a)$.
8. Parametric equation is $x = h + at^2$ and $y = k + 2at$.

(B) When the equation of parabola is:

$(x - h)^2 = 4a\,(y - k)$ ...(*i*)

Equation (*i*) is of the form $X^2 = 4aY$

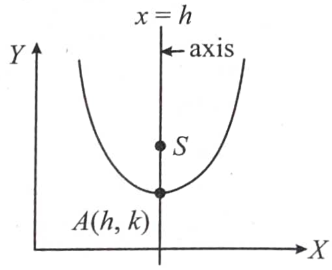
where $X = x - h$ and $Y = y - k$

1. Axis of parabola is $X = 0$, i.e., $x - h = 0$
2. Coordinates of vertex of parabola is given by

   $X = 0$ and $Y = 0$

   i.e., by $x - h = 0$ and $y - k = 0$

   $\therefore$ $x = h$ and $y = k$



   Hence vertex of parabola is (h, k)
3. Equation of tangent at the vertex to parabola is

   $Y = 0$ i.e., $y - k = 0$

   or $y = k$
4. Coordinates of focus of parabola are given by

   $X = 0$

   and $Y = a$

   i.e., by $x - h = 0$

   and $y - k = a$

   $\therefore$ Focus of parabola is $(h, k + a)$.
5. Equation of directrix of parabola (*i*) is given by

   $Y = -a$

   or $y - k = -a$

   or $y = k - a$
6. Length of latus rectum of parabola is $|\,4a\,|$.
7. Coordinates of ends of latus rectum of parabola are given by

   $Y = a$, $X = \pm 2a$

   i.e., $y - k = a$, $X - h = \pm 2a$

   $\therefore$ Ends of latus rectum are $(h \pm 2a, k + a)$
8. Parametric equation is $x = h + 2at$ and $y = k + at^2$.

Equation of Parabola	Vertex	Axis	Focus	Directrix	Parametric equation
$(y-k)^2 = 4a(x-h)$	(h, k)	$y = k$	$(h + a, k)$	$x + a - h = 0$	$x = h + at^2, y = k + 2at$
$(x-h)^2 = 4a(y-k)$	(h, k)	$x = h$	$(h, a + k)$	$y + a - k = 0$	$x = h + 2at, y = k + at^2$

Note: (i) For the parabola $y = Ax^2 + Bx + C$, the length of latus rectum is $\frac{1}{|A|}$ and axis is parallel to y-axis. If A is positive then it is concave up parabola, if A is negative then it is concave down parabola.

(ii) For the parabola $x = Ay^2 + By + C$, the length of latus rectum is $\frac{1}{|A|}$ and axis is parallel to x-axis. If A is positive then it is opening right and if A is negative then it is opening left parabola.

Train Your Brain

Example 5: Find the parametric equation of the parabola $(x-1)^2 = -12(y-2)$

Sol. $\because 4a = -12 \Rightarrow a = -3$,

parametric equation is $y - 2 = -3t^2$

$x - 1 = -6t \Rightarrow x = 1 - 6t, y = 2 - 3t^2$

Example 6: The parametric equation of the curve $(y-2)^2 = 12(x-4)$ are-

(a) $6t, 3t^2$ (b) $2 + 3t, 4 + t^2$

(c) $4 + 3t^2, 2 + 6t$ (d) $3t, 3t^2$

Sol. Here $a = 3$

$x - 4 = at^2 \Rightarrow x = 4 + 3.t^2 = 4 + 3t^2$

$y - 2 = 2at \Rightarrow y = 2 + 2.3t = 2 + 6t$

Example 7: Find the equation of parabola, whose axis is parallel to y-axis and which passes through points (0, 2), (– 1, 0) and (1, 6).

Sol. General equation of such parabola is

$y = Ax^2 + Bx + C$ ...(i)

Since it passes throught (0, 2) (– 1, 0) and (1, 6), then we have

$C = 2$ ...(ii)

$A - B + C = 0$ ...(iii)

$A + B + C = 6$ ...(iv)

from (ii), (iii) and (iv),

$A = 1, B = 3, C = 2$

$\therefore$ equation of parabola is $y = x^2 + 3x + 2$.

Example 8: Find the value of λ if the equation $\sqrt{(x-1)^2 + (y-2)^2} = \sqrt{\lambda}\,(x + y + 3)$ represent parabola. Find its focus, directrix, axis and equation of vertex.

Sol. $(x-1)^2 + (y-2)^2 = \lambda (x + y + 3)^2$ are given equation

$$(x-1)^2 + (y-2)^2 = 2\lambda \left(\frac{x+y+3}{\sqrt{2}}\right)^2$$

$\Rightarrow$ Above equation represents parabola if $2\lambda = 1$

$\Rightarrow \lambda = \frac{1}{2}$

In this case focus is (1, 2) and directrix is $x + y + 3 = 0$

Line perpendicular to directrix is $x - y + k = 0$ and passes through (1, 2) gives axis of parabola

$\Rightarrow 1 - 2 + k = 0 \Rightarrow k = 1$

$\therefore$ axis is $x - y + 1 = 0$

Now axis and directrix meet at (– 2, – 1) (called foot of directrix)

Thus vertex is the mid point of foot of directrix and the focus.

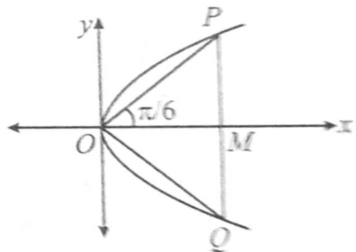
i.e. vertex $\left(\frac{1-2}{2}, \frac{2-1}{2}\right)$ i.e., $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Example 9: An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, such that one vertex of this triangle coincides with the vertex of the parabola. Then find the side length of this triangle.

Sol. If ΔOPQ is an equilateral triangle then

$OP = OQ = PQ = k \Rightarrow \angle POM = \frac{\pi}{6}$

$\therefore OM = k\cos\frac{\pi}{6}$ and $PM = k\sin\frac{\pi}{6}$



$\therefore P\left(\frac{\sqrt{3}}{2}k, \frac{k}{2}\right)$ lie on the parabola $y^2 = 4ax$

$\Rightarrow \left(\frac{k}{2}\right)^2 = 4a \cdot \frac{\sqrt{3}}{2}k \Rightarrow k = 8\sqrt{3}a$

Example 10: The parametric equation of a parabola is $x = t^2 + 1$, $y = 2t + 1$. Then find the coordinate of vertex and length of latus -rectum.

Sol. Eliminate t from parametric equation, we get equation of parabola. Hence

$$x = \left(\frac{y-1}{2}\right)^2 + 1 \text{ or } (y-1)^2 = 4(x-1)$$

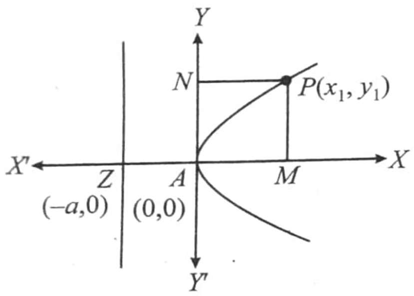
$\therefore$ vertex is (1, 1) and length of latus rectum = 4.

Concept Application

5. Equation of the directrix of the parabola $y^2 - 3x - 2y + 7 = 0$ is:
   (*a*) $4x - 5 = 0$ (*b*) $x - 2 = 0$
   (*c*) $4x - 11 = 0$ (*d*) $4y - 1 = 0$
6. Coordinates of the vertex are (–2, 2)and the directrix has the equation $x + y - 4 = 0$. The latus rectum of the parabola is:
   (*a*) $2\sqrt{2}$ (*b*) $4\sqrt{2}$
   (*c*) $8\sqrt{2}$ (*d*) $16\sqrt{2}$

EQUATION OF PARABOLA WITH RESPECT TO TWO PERPENDICULAR LINES

Let $P(x_1, y_1)$ is any point on the parabola then equation of parabola $y^2 = 4ax$ is consider as



$y_1^2 = 4ax_1$

$(PM)^2 = 4a\,(PN)$

$$\begin{Bmatrix}\text{Perpendicular distance}\\ \text{from a point P to axis}\end{Bmatrix}^2 = \begin{Bmatrix}\text{length of the}\\ \text{latusrectum}\end{Bmatrix} \cdot \begin{Bmatrix}\text{Perpendicular distance from a}\\ \text{point P to tangent at vertex}\end{Bmatrix}$$

Train Your Brain

Example 11: Find the equation of the parabola whose latus rectum is 4 units, axis is the line $3x + 4y - 4 = 0$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$.

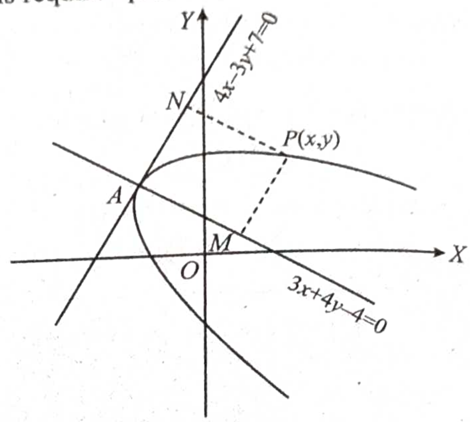
Sol. Let $P(x, y)$ be any point on the parabola and let PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively then

$(PM)^2 = (\text{latus rectum})\,(PN)$

$$\Rightarrow \left(\frac{3x+4y-4}{\sqrt{3^2+4^2}}\right)^2 = 4\left(\frac{4x-3y+7}{\sqrt{4^2+(-3)^2}}\right)$$

$$\Rightarrow (3x + 4y - 4)^2 = 20\,(4x - 3y + 7)$$

which is required parabola.



Example 12: Find the equation of the parabola whose focus is (1, –1) and directirx is $x + y + 7 = 0$

Sol. Distance of a fixed point and a fixed line are equal.

$$\therefore \sqrt{(x-1)^2 + (y+1)^2} = \left|\frac{x+y+7}{\sqrt{1+1}}\right|$$

$$\Rightarrow y^2 - 2xy - 18x - 10y - 45 = 0$$

Concept Application

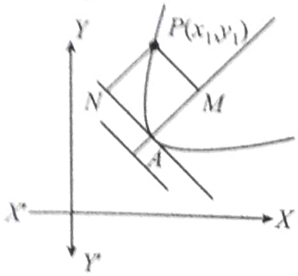
7. The vertex is (2, 1)and the directrix is $y = x + 1$. Equation of the parabola is:
   (*a*) $x^2 + 2xy + y^2 - 14x + 2y + 17 = 0$
   (*b*) $x^2 - 2xy + y^2 + 14x - 2y + 17 = 0$
   (*c*) $x^2 - 2xy + y^2 + 24x + 8y + 16 = 0$
   (*d*) $x^2 + 2xy + y^2 - 24x - 8y + 16 = 0$
8. Find the equation of parabola whose latus rectum is 4 units and axis is $3x + 4y = 0$ and tangent at vertex is $4x - 3y + 2 = 0$
   (*a*) $(3x+4y)^2 = 20(4x-3y+2)$
   (*b*) $(4x-3y+2)^2 = 20(3x+4y)$
   (*c*) $(3x+4y)^2 = (4x-3y+2)$
   (*d*) $(3x+4y)^2 = 4(4x-3y+2)$

OBLIQUE PARABOLA

$AM = \ell x + my + n = 0$ is axis of parabola

$AN = mx - \ell y + k = 0$ is tangent at vertex

$P(x_1, y_1)$ is any point on parabola



$(PM)^2 = 4a\,(PN)$

$$\left\{\begin{matrix}\text{Perpendicular distance}\\ \text{from a point P to axis}\end{matrix}\right\}^2 = \left\{\begin{matrix}\text{Length of the}\\ \text{latusrectum}\end{matrix}\right\}\left\{\begin{matrix}\text{Perpendicular distance from}\\ \text{a point P to tangent at vertex}\end{matrix}\right\}$$

$$\left(\frac{\ell x_1 + my_1 + n}{\sqrt{\ell^2 + m^2}}\right)^2 = 4a\left(\frac{mx_1 - \ell y_1 + k}{\sqrt{\ell^2 + m^2}}\right)$$

Comparing with $Y^2 = 4aX$

(i) Vertex $(X = 0,\ Y = 0)$

$$\left(\frac{mx - \ell y + k}{\sqrt{\ell^2 + m^2}} = 0,\ \frac{\ell x + my + n}{\sqrt{\ell^2 + m^2}} = 0\right)$$

(ii) Focus $(X = a,\ Y = 0)$

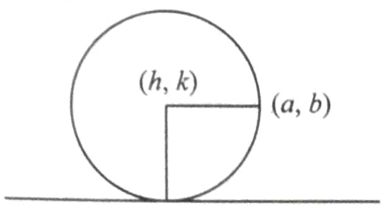
$$\left(\frac{mx - \ell y + k}{\sqrt{\ell^2 + m^2}} = a,\ \frac{\ell x + my + n}{\sqrt{\ell^2 + m^2}} = 0\right)$$

(iii) Directrix $X + a = 0$

$$\frac{mx - \ell y + k}{\sqrt{\ell^2 + m^2}} + a = 0$$

SPECIAL CASE OF PARABOLA

The locus of centres of a circle which passes through a fixed point and tangential to a line, is a parabola whose focus is fixed point, directrix is tangent of a circle.

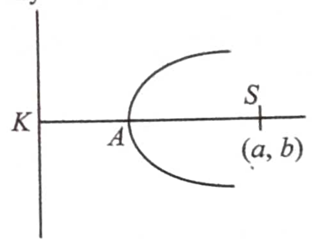


$\ell x + \text{my} + n$

$$\sqrt{(h-a)^2 + (k-b)^2} = \left|\frac{\ell h + mk + n}{\sqrt{\ell^2 + m^2}}\right|$$

Locus of centre of circle $\sqrt{(x-a)^2 + (y-b)^2} = \left|\frac{\ell x + my + n}{\sqrt{\ell^2 + m^2}}\right|$

LENGTH OF THE LATUSRECTUM

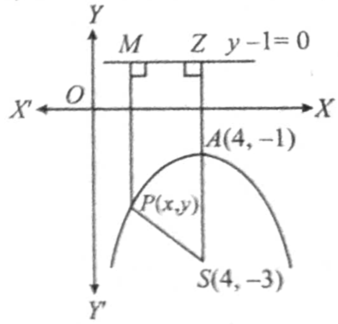


$$= 2(KS) = 2\left|\frac{\ell a + mb + n}{\sqrt{\ell^2 + m^2}}\right|$$

Train Your Brain

Example 13: Find the equation of the parabola whose focus is (4, –3) and vertex is (4, –1)

Sol. Let $A(4, -1)$ be the vertex and $S(4, -3)$ be the focus.



$\therefore$ Slope of $AS = \dfrac{-3+1}{4-4} = \infty$

Which is parallel to y-axis

$\therefore$ Directrix parallel to x-axis.

Let $Z(x_1, y_1)$ be any point on the directrix, then A is the midpoint of SZ

$\therefore\quad 4 = \dfrac{x_1 + 4}{2} \Rightarrow x_1 = 4$ and $-1 = \dfrac{y_1 - 3}{2} \Rightarrow y_1 = 1$

$\therefore Z = (4, 1)$

Also directrix is parallel to x-axis and passes through $Z(4, 1)$ so equation of directrix is

$y = 1$ or $y - 1 = 0$

now let $P(x, y)$ be any point on the parabola, Join SP and draw PM perpendicular to the directrix, Then by definition

$\Rightarrow\quad SP = PM$

$(SP)^2 = (PM)^2$

$$\Rightarrow\quad (x-4)^2 + (y+3)^2 = \left(\frac{|y-1|}{\sqrt{1^2}}\right)^2$$

$$\Rightarrow\quad (x-4)^2 + (y+3)^2 = (y-1)^2$$

or $\quad x^2 - 8x + 8y + 24 = 0$

Example 14: The length of latus rectum of the parabola whose focus is (2,3) and directirx is line $x - 4y + 3 = 0$ Also, find equation of parabola

Sol. Distance of any point on the parabola from the focus is equal to its disatance from disrectirx

$$\sqrt{(x-2)^2+(y-3)^2} = \left|\frac{x-4y+3}{\sqrt{1^2+4^2}}\right|$$

$$\Rightarrow \quad 16x^2+y^2+8xy-74x-78y+212=0$$

Latus rectum = 2 × distance from focus andirectrix

$$= 2\left|\frac{2.1-4\cdot 3+3}{\sqrt{1^2+4^2}}\right| = \frac{14}{\sqrt{17}}$$

Concept Application

9. Length of the latus rectum of the parabola $169[(x-1)^2+(y-3)^2] = (5x-12y+17)^2$ is:

(*a*) 14/13 (*b*) 28/13 (*c*) 56/13 (*d*) 4

10. If the lengths of the two segments of a focal chord intercepted by its axis are 3 and 5, then the latus rectum of the parabola will be:

(*a*) 15/8 (*b*) 15/4 (*c*) 15/2 (*d*) 15

Position of a Point with Respect to a Parabola $y^2 = 4ax$:

Let $P(x_1, y_1)$ be a point. From P draw $PM \perp AX$ (on the axis of parabola) meeting the parabola $y^2 = 4ax$ at $Q\ (x_1, y_2)$ where $Q\ (x_1, y_2)$ lie on the parabola therefore

$y_2^2 = 4ax_1$ ... (1)

Now, P will be outside, on or inside the parabola $y^2 = 4ax$ acoording as

$PM >, =$, or $< QM$

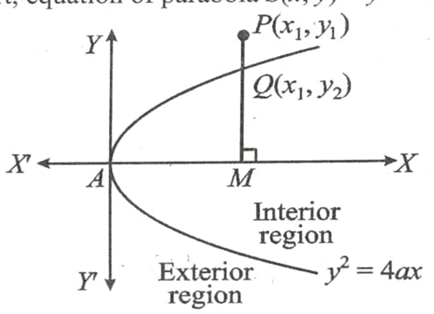
$\Rightarrow$ $(PM)^2 >, =$, or $< (QM)^2$

$\Rightarrow$ $y_1^2 >, =$, or $< y_2^2$

$\Rightarrow$ $y_1^2 >, =$, or $< 4ax_1$ (from(1))

Hence $y_1^2 - 4ax_1 >, =$, or < 0

Hence in short, equation of parabola $S(x, y) = y^2 - 4ax$.



(*i*) If $S(x_1, y_1) > 0$ then $P(x_1, y_1)$ lie outside the parabola.

(*ii*) If $S(x_1, y_1) < 0$ then $P(x_1, y_1)$ lie inside the parabola.

(*iii*) If $S(x_1, y_1) = 0$ then $P(x_1, y_1)$ lie on the parabola.

This result holds true for circle, parabola and ellipse.

Train Your Brain

Example 15: Find the value of α for which the point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$.

Sol. ∵ Point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$

$\therefore\ y_1^2 - 4x_1 < 0 \quad \Rightarrow \quad \alpha^2 - 4(\alpha - 1) < 0$

$\Rightarrow \alpha^2 - 4\alpha + 4 < 0 \quad \Rightarrow \quad (\alpha - 2)^2 < 0 \Rightarrow \alpha \in \phi$

Example 16: Show that the point (2, 3) lies outside the parabola $y^2 = 2x$.

Sol. Let $S(x, y) \equiv y^2 - 2x$

$\therefore\ S(2, 3) = 9 - 2.2 = 5 =$ positive

$\Rightarrow$ (2, 3) lie outside the parabola

Example 17: Find the position of the point (– 2, 2) with respect to the parabola $y^2 - 4y + 9x + 11 = 0$.

Sol. Let $S(x, y) = y^2 - 4y + 9x + 11$

$\therefore\ S(-2, 2) = -11 =$ negative

$\Rightarrow$ (–2, 2) lie inside the parabola

Note:

(*i*) The length of focal chord having parameters t_1 and t_2 for its end points is $a(t_2 - t_1)^2$.

(*ii*) $\because \left|t + \frac{1}{t}\right| \geq 2$ for all $t \neq 0$ $\quad (\because AM \geq GM)$

$$\therefore\ a\left(t + \frac{1}{t}\right)^2 \geq 4a$$

$\Rightarrow$ Length of focal chord ≥ latus rectum

i.e., The length of smallest focal chord of the parabola is $4a$, which is the latus rectum of a parabola.

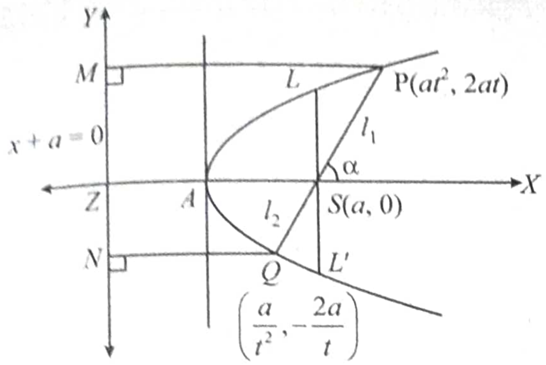
Example 18: Prove that the semi latus rectum of the parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord of the parabola.

Sol. Let parabola be $y^2 = 4ax$

If PQ be the focal chord then

$$P = (at^2, 2at) \text{ and } Q = \left(\frac{a}{t^2}, \frac{-2a}{t}\right)$$

If segment of focal chord are l_1 and l_2



then $l_1 = SP = PM = a + at^2 = a(1 + t^2)$

and $l_2 = SQ = QN = a + \dfrac{a}{t^2} = \dfrac{a(1+t^2)}{t^2}$

$\therefore$ Harmonic mean of l_1 and l_2

$$= \frac{2l_1 l_2}{l_1 + l_2} = \frac{2}{\frac{1}{l_2} + \frac{1}{l_1}} = \frac{2}{\frac{t^2}{a(1+t^2)} + \frac{1}{a(1+t^2)}}$$

$$= \frac{2}{1/a} = 2a = \text{semi latus rectum.}$$

Note: If l_1 and l_2 are the length of segments of a focal chord of a parabola, then its latus rectum is $\dfrac{4l_1l_2}{l_1 + l_2}$.

Concept Application

11. Set of exhaustive values of a so that the point $(a + 1, 2a)$ lies inside the parabola $4x - 2y - x^2 = 0$ is:

(a) $(-1, 3)$ (b) $(-\infty, -1) \cup (3, \infty)$

(c) $(-3, 1)$ (d) $(-\infty, -3) \cup (1, \infty)$

12. Set of all real values of 'a' so that the point $(-2a, a + 1)$ becomes an interior point of the region bounded by the circle $x^2 + y^2 = 4$ and the parabola $y^2 = 4x$ is:

(a) $\left(-1, \dfrac{3}{5}\right)$

(b) $(-5 - 2\sqrt{6}, -5 + 2\sqrt{6})$

(c) $\left(-5 + 2\sqrt{6}, \dfrac{3}{5}\right)$

(d) $(-1, 5 - 5 + 2\sqrt{6})$

13. Let the equation of a parabola be $x^2 + 6x + 4y + 1 = 0$ and coordinates of the points A and B are $(0, 1)$ and $(-1, 0)$ respectively, Then:

(a) A and B lie outside of the parabola

(b) A lies inside and B lies outside of the parabola

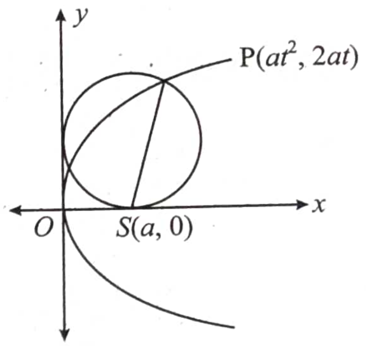
(c) A and B lie inside of the parabola

(d) A lies outside and B lies inside of the parabola

LENGTH OF FOCAL CHORD

If point P is $(at^2, 2at)$, then length of focal chord PQ is $a\left(t + \dfrac{1}{t}\right)^2$

- The length of the focal chord which makes an angle θ with positive direction of x-axis is $\mathbf{4a\,cosec^2\theta}$.
- If ℓ_1 and ℓ_2 are the lengths of the segments of a focal of a parabola, then its latusrectum is $\dfrac{4\ell_1\ell_2}{\ell_1 + \ell_2}$
- **Circle described on the focal as diameter touches the tangent at vertex**



Train Your Brain

Example 19: Circles are drawn with diameter being any focal chord of the parabola $y^2 - 4x - y - 4 = 0$ will always touch a fixed line, find its equation.

Sol. $y^2 - 4x - y - 4 = 0$

$$\Rightarrow y^2 - y + \frac{1}{4} = 4x + \frac{17}{4}$$

$$\Rightarrow \left(y - \frac{1}{2}\right)^2 = 4\left(x + \frac{17}{16}\right)$$

Circle drawn with diameter being any focal chord of the parabola always touches the directrix of the parabola.

Thus, circle will touch the line $x + \dfrac{17}{16} = -1$,

i.e., $16x + 33 = 0$.

Example 20: Through the vertex O of a parabola $y^2 = 4x$ chords OP and OQ are drawn at right angles to one another. Show that for all position of P, PQ cuts the axis of the parabola at a fixed point.

Sol. The given parabola is $y^2 = 4x$ ...(i)

Let $P \equiv (t_1^2, 2t_1)$, $Q \equiv (t_2^2, 2t_2)$.

Slope of $OP = \dfrac{2t_1}{t_1^2} = \dfrac{2}{t_1}$ and slope of $OQ = \dfrac{2}{t_2}$

Since $OP \perp OQ$, $\dfrac{4}{t_1t_2} = -1$ or $t_1t_2 = -4$ ...(ii)

The equation of PQ is $y(t_1 + t_2) = 2(x + t_1t_2)$

$\Rightarrow y\left(t_1 - \frac{4}{t_1}\right) = 2(x-4)$ [from (ii)]

$\Rightarrow 2(x-4) - y\left(t_1 - \frac{4}{t_1}\right) = 0$

$\Rightarrow L_1 + \lambda L_2 = 0$

$\therefore$ variable line PQ passes through a fixed point which is point of intersection of $L_1 = 0$ and $L_2 = 0$ i.e. (4, 0)

Concept Application

14. Length of the chord of the parabola $y^2 = 4ax$ passing through the vertex and inclined at an inclined at an angle ϕ with the axis is:

(a) $4a \operatorname{cosec}^2 \theta$ (b) $4a \cot\theta \operatorname{cosec}^2 \theta$

(c) $4a \operatorname{cosec}^3 \theta$ (d) $4a \cos\theta \operatorname{cosec}^2 \theta$

15. If (2, –8) is an end of a focal chord of the parabola $y^2 = 32x$, then the coordinates of the other end are:

(a) (32, 32) (b) (2, 8)

(c) (1/2, 4) (d) (8, 16)

16. If (16, 16) is an end of a focal chord of the parabola $x^2 = 16y$, then the coordinates of the other end are:

(a) (–16, 16) (b) (–12, 9)

(c) (–4, 1) (d) (–2, 1/4)

LINE & A PARABOLA

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a \gtreqless cm \Rightarrow$ condition of tangency is, $c = a/m$.

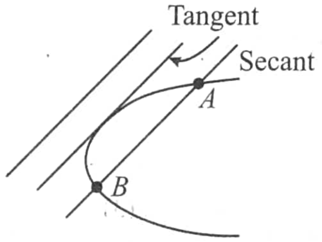
Note: (a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as

$a >=< cm \Rightarrow$ condition of tangency is, $c = \frac{a}{m}$.

(b) Line $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $c = -am^2$

(c) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line $y = mx + c$ is

$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$



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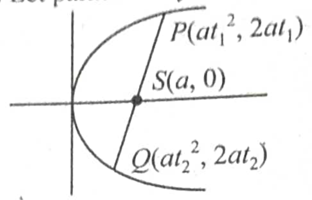
Example 21: If t_1, t_2 are end points of a focal chord then show that $t_1 t_2 = -1$.

Sol. Let parabola is $y^2 = 4ax$



since P, S and Q are collinear

$\therefore \quad m_{PQ} = m_{PS}$

$\Rightarrow \quad \frac{2}{t_1 + t_2} = \frac{2t_1}{t_1^2 - t_1} \Rightarrow t_1^2 - 1 = t_1^2 + t_1t_2 \Rightarrow t_1t_2 = -1$

Example 22: If (9, 6) is a point on parabola $y^2 = 4x$ then find other point on focal chord

Sol. Let $(9, 6) = (t_1^2, 2t_1)$

$\therefore \ t_1 = 3$

As we know for focal chord

$t_1t_2 = -1$

$\therefore \ t_2 = \frac{-1}{3}$

Other point is $[t_2^2, -2t_2]$ i.e. $\left(\frac{1}{9}, \frac{-2}{3}\right)$

Concept Application

17. The straight line L is $8y = 4x - 9a$ and the parabola P is $y^2 = 4ax$. Then:

(a) L represents is focal chord of P

(b) L represents a tangent to P

(c) L represents a normal to P

(d) L represents a co normal

18. The equation of a line L is $x + my + am^2 = 0$ and the equation of a parabola P is $y^2 = 4ax$ Then:

(a) L is a tangent to P at the point $(am^2, -2am)$

(b) Lis a normal to P at the point $(am^2, -2am)$

(c) L is a focal chord of P

(d) L is normal to P at the point $(a/m^2, 2a/m)$

EQUATION OF TANGENT

(a) Point form : Equation of parabola is

$y^2 = 4ax$ ... (1)

Let $P \equiv (x_1, y_1)$ and $Q = (x_2, y_2)$ be any two points on parabola (1), then

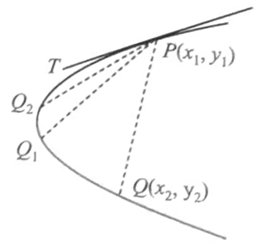
$y_1^2 = 4ax_1$ ... (2)

and $y_2^2 = 4ax_2$ ... (3)

Subtracting (2) from (3) then

$y_2^2 - y_1^2 = 4a(x_2 - x_1)$

or $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4a}{y_2 + y_1}$ ... (4)



Equation of PQ is $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ ...(5)

From (4) and (5), then $y - y_1 = \dfrac{4a}{y_2 + y_1}(x - x_1)$ ...(6)

Now for tangent at P, $Q \to P$, i.e., $x_2 \to x_1$ and $y_2 \to y_1$ then equation (6) becomes

$y - y_1 = \dfrac{4a}{2y_1}(x - x_1)$

or $yy_1 - y_1^2 = 2ax - 2ax_1$

or $yy_1 = 2ax + y_1^2 - 2ax_1$

or $yy_1 = 2ax + 4ax_1 - 2ax_1$ [From (2)]

or $yy_1 = 2ax + 2ax_1$

which is the required equation of tangent at (x_1, y_1).

The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\dfrac{x + x_1}{2}$, y by $\dfrac{y + y_1}{2}$ and xy by $\dfrac{xy_1 + yx_1}{2}$ and without changing the constant (if any) in the equation of curve. This method (standard substitution) is apply to all conic when point lie on the conic.

Equation of tangent of standard parabola:

Equation of Parabolas	Tangent at (x_1, y_1)
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

(b) Slope form

The equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$yy_1 = 2a(x + x_1)$ ... (1)

Since m is the slope of the tangent then

$m = \dfrac{2a}{y_1}$ or $y_1 = \dfrac{2a}{m}$

Since (x_1, y_1) lies on $y^2 = 4ax$ therefore

$y_1^2 = 4ax_1$ or $\dfrac{4a^2}{m^2} = 4ax_1 \Rightarrow x_1 = \dfrac{a}{m^2}$.

Substituting the values of x_1 and y_1 in (1), we get

$y = mx + \dfrac{a}{m}$ ...(2)

Thus, $y = mx + \dfrac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ for all values of m, where m is the slope of the tangent and the co-ordinates of the point of contact is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

Thus $y = mx + c$ is the tangent of $y^2 = 4ax$ for all values of m if only if $c = \dfrac{a}{m}$ and $(y - k) = m(x - h) + \dfrac{a}{m}$ is tangent to the parabola$(y - k)^2 = 4a(x - h)$.

The equation of tangent, condition of tangency and point of contact in terms of slope (m) of standard parabolas are shown below in the table.

Equation of parabolas	Point of contact in terms of slope (m)	Equation of tangent in terms of slope (m)	Condition of tangnecy
$y^2 = 4ax$	$\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$	$y = mx + \dfrac{a}{m}$	$c = \dfrac{a}{m}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$

2nd Method:

Let the parabola be $y^2 = 4ax$ ... (1)

and the given line by $y = mx + c$ ... (2)

Putting the value of y from (2) in (1), we get

$m^2x^2 + 2x(mc - 2a) + c^2 = 0$ ... (3)

The line $y = mx + c$ is a tangent to parabola $y^2 = 4ax$ if the roots of equation (3) are equal. The condition for this is $4(mc - 2a)^2 - 4m^2c^2 = 0$ (Discriminant of the quadratic equation = 0)

or $-4mca + 4a^2 = 0$ or $c = \dfrac{a}{m}$, which is the required condition of tangency.

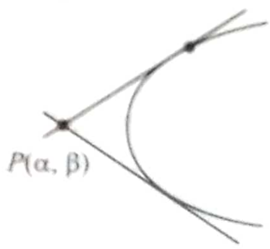
Two tangents can be drawn from a point $P(\alpha, \beta)$ to a parabola if P lies outside the parabola :

Let the parabola be $y^2 = 4ax$ ...(1)

Let $P(\alpha, \beta)$ be the given point

The equation of a tangent to parabola (1) is

$y = mx + \dfrac{a}{m}$ ...(2)



If line (2) passes through $P(\alpha, \beta)$, then

$\beta = m\alpha + \dfrac{a}{m}$

or $m^2\alpha - \beta m + a = 0$ ... (3)

There will be two tangents to parabola (1) from $P(\alpha, \beta)$ if roots of equation (3) are real and distinct i.e., $D > 0$ i.e. if $\beta^2 - 4\alpha a > 0 \Rightarrow P(\alpha, \beta)$ lies outside parabola (1).

We can also find the angle between two tangents from point $P(\alpha, \beta)$ using the formula $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

(c) Parametric form: We have to find the equation of tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ or 't'

Since the equation of tangent of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \quad \text{... (1)}$$

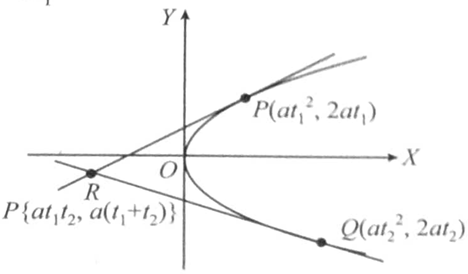
replacing x_1 by at^2 and y_1 by $2at$, then (1) becomes

$$y(2at) = 2a(x + at^2) \quad \Rightarrow \quad ty = x + at^2$$

Point of Intersection of Tangents at any Two Points on the Parabola

Let the given parabola be $y^2 = 4ax$ and two points on the parbaola are $P = (at_1^2, 2at_1)$ and $Q = (at_2^2, 2\,at_2)$

Equation of tangents at $P\,(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are $t_1 y = x + at_1^2$ ... (1)



and $t_2 y = x + at_2^2$ ... (2)

Solving these equations we get $x = at_1t_2$, $y = a(t_1 + t_2)$

Thus, the co-ordinates of the point of intersection of tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are $R(at_1t_2, a(t_1 + t_2))$.

Note : (*i*) The Arithmetic mean of the y-co-ordinates of P and Q $\left(\text{i.e., } \dfrac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y - co-ordinate of the point of intersection of tangents at P and Q on the parabola.

(*ii*) The Geometric mean of the x-co-ordinates of P and Q $\left(\text{i.e., } \sqrt{at_1^2 \times at_2^2} = at_1t_2\right)$ is the x co-ordinate of the point of intersection of tangents at P and Q on the parabola.

DIRECTOR CIRCLE

Locus of the point of intersection of the perpendicular tangents to the parabola is called the director circle. In case of parabola its own directrix representation director circle.

For standard parabola, $y^2 = 4ax$ equation of directrix is $x + a = 0$.

Proof: Equation of tangent in slop form

$$y = mx + \frac{a}{m}$$

$$y = \frac{m^2x + a}{m}$$

$$my = m^2x + 9$$

$$m^2x - my + a = 0$$

It passes h, k

$$m^2h - mk + a = 0$$

$$m_1 \cdot m_2 = a/h$$

We know that tangents are perpendicular to each other.

$$-1 = a/x$$

$$\boxed{x + a = 0}$$

Train Your Brain

Example 23: If the line $2x - 3y = k$ touches the parabola $y^2 = 6x$, then find the value of k.

Sol. Line $y = \dfrac{2}{3}x - \dfrac{k}{3}$ touches is $y^2 = 6x$, then use the condition of tangency $c = \dfrac{a}{m} \Rightarrow -\dfrac{k}{3} = \dfrac{9}{4} \Rightarrow k = -\dfrac{27}{4}$

Example 24: If the tangents at P and Q on a parabola (whose focus is S) meet in the point T, then prove that SP, ST and SQ are in geometric progression.

Sol. Let $P\,(at_1^2, 2at_1)$ and $Q\,(at_2^2, 2at_2)$ be any two points on the parabola $y^2 = 4ax$, then point of intersection of tangents at P and Q will be

$$T = [at_1t_2, a(t_1 + t_2)]$$

Now $\quad SP = a(t_1^2 + 1),\ SQ = a(t_2^2 + 1)$

$$ST = a\sqrt{(t_1^2 + 1)(t_2^2 + 1)}$$

$$\therefore \quad ST^2 = SP \cdot SQ$$

$\therefore \quad SP$, ST and SQ are in $G.P.$

Example 25: If two tangents are drawn from the pont (h, k) to the parabola $y^2 = 4x$ such that the slope of one tangent is double of the other, then prove that $9h = 2k^2$.

Sol. Tangent to parabola $y^2 = 4x$ is

$y = mx + \dfrac{1}{m}$ and it passes through (h, k).

so $\quad k = mh + \dfrac{1}{m}$ i.e. $hm^2 - km + 1 = 0$

Its roots are m_1 and $2m_1$

$$\therefore \quad m_1 + 2m_1 = \frac{k}{h} \Rightarrow 3m_1 = \frac{k}{h} \quad \text{... (i)}$$

$$m_1 \cdot 2m_1 = \frac{1}{h} \quad \Rightarrow \quad 2m_1^2 = \frac{1}{h} \quad \text{... (ii)}$$

from (*i*) and (*ii*) eliminate m, we get $9h = 2k^2$.

Example 26: The angle between the tangents drawn from a point $(-a, 2a)$ to $y^2 = 4ax$ is

Sol. The given point $(-a, 2a)$ lies on the directrix $x = -a$ of the parabola $y^2 = 4ax$. Thus, the tangents are at right angle.

Example 27: The circle drawn with variable chord $x + ay - 5 = 0$ (a being a parameter) of the parabola $y^2 = 20x$ as diameter will always touch the line

Sol. Clearly $x + ay - 5 = 0$ will always pass through the focus of $y^2 = 20x$ i.e. $(5, 0)$. Thus the drawn circle will always touch the directrix of the parabola i.e., the line $x + 5 = 0$.

Concept Application

19. Equation of the tangent to parabola $y^2 = 4x$ at the point $(1, -2)$ is:

(a) $x + y + 1 = 0$

(b) $x - y + 3 = 0$

(c) $3x + y - 1 = 0$

(d) $3x - y - 5 = 0$

20. Number of distinct tangents that can be drawn from the point $(-2, 2)$, to the parabola $y^2 - 4y + x + 13 = 0$ is:

(a) 0 (b) 1 (c) 2 (d) 3

21. The straight line $lx + my + n = 0$ is a tangent to the parabola $y^2 = 4a(x - b)$ if:

(a) $al^2 + bm^2 = ln^2$

(b) $am^2 + bl^2 + ln = 0$

(c) $am^2 = bl^2 + ln$

(d) $al^2 = bm^2 + ln$

22. The equation of al line L is $x + my + am^2 = 0$ and the equation of a parabola P is $y^2 = 4ax$. Then:

(a) L is a tangent to P at the point $(am^2, -2am)$

(b) L is a normal to P at the point $(am^2, -2am)$

(c) L is a focal chord of P

(d) L is normal to P at the point $(a/m^2, 2a/m)$

23. Equation of a tangent to the parabola $x^2 + 2y = 0$ which is parallel to the line $x - 2y = 4$ is:

(a) $8y = 4x - 1$ (b) $4y = 2x - 1$

(c) $8y = 4x + 1$ (d) $8x = 4y + 1$

24. The tangents to the parabola $y^2 = 4ax$ at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ intersect on its axis, then:

(a) $t_1 t_2 = 1$

(b) $t_1 t_2 = -1$

(c) $t_1 + t_2 = 0$

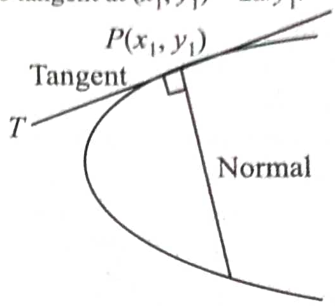
(d) $t_1 + t_2 = 1$

NORMAL TO THE PARABOLA: Y2 = 4AX

(a) Point form: Since the equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \quad \text{... (1)}$$

The slope of the tangent at $(x_1, y_1) = 2a/y_1$.



$\therefore$ Slope of the normal at $(x_1, y_1) = -y_1/2a$

Hence the equation of normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

(b) Slope form: The equation of normal to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad \text{... (1)}$$

Since m is the slope of the normal

then $m = -\frac{y_1}{2a}$ or $y_1 = -2am$

Since (x_1, y_1) lies on $y^2 = 4ax$ therefore

$y_1^2 = 4ax_1$ or $4a^2m^2 = 4ax_1$

$\therefore$ $x_1 = am^2$

Substituting the values of x_1 and y_1 in (1) we get

$$y + 2am = m(x - am^2) \quad \text{... (2)}$$

Thus, $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$ where m is the slope of the normal. The co-ordinates of the point of contact are $(am^2, -2am)$.

Hence $y = mx + c$ will be normal to parabola. If and only if $c = -2am - am^3$

Note :

Equation of parabolas	Point of contact in terms of slope (m)	Equation of normals in terms of slope (m)	Condition of normality
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = 2a + \frac{a}{m^2}$

(c) Parametric form: Equation of normal of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad \text{...(1)}$$

Replacing x_1 by at^2 and y_1 by $2at$ then (1) becomes

$y - 2at = -t(x - at)$

or $y = -tx + 2at + at^3$

Three Supplementry Results

(a) Point of intersection of normals at any two points on the parabola :

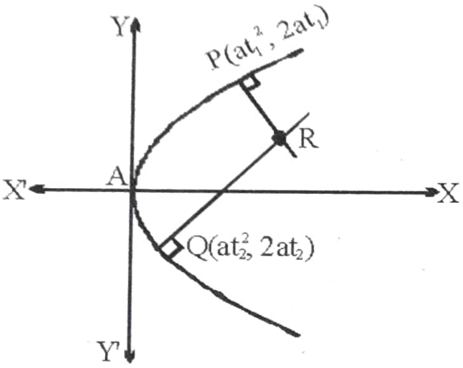
Let the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ lie on the parabola $y^2 = 4ax$

The equations of the normals at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$$y = -t_1x + 2at_1 + at_1^3 \qquad \text{... (1)}$$

and $$y = -t_2x + 2at_2 + at_2^3 \qquad \text{... (2)}$$

Hence point of intersection of above normals will be obtained by solving (1) and (2), we get



$$x = 2a + a(t_1^2 + t_2^2 + t_1t_2)$$
$$y = -at_1t_2(t_1 + t_2)$$

If R is the point of intersection then it is

$$R = [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$$

(b) Relation between t_1 and t_2 if normal at t_1 meets the parabola again at t_2 :

Let the parabola be $y^2 = 4ax$, equation of normal at $P(at_1^2, 2at_1)$ is

$$y = -t_1x + 2at_1 + at_1^3 \qquad \text{... (1)}$$

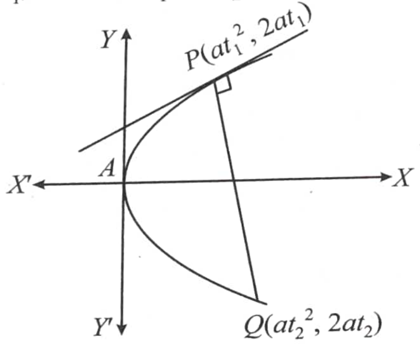
Since normal meet the parabola again at $Q(at_2^2, 2at_2)$

$\therefore\ 2at_2 = -at_1t_2^2 + 2at_1 + at_1^3$

$\Rightarrow 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$

$\Rightarrow a(t_2 - t_1)[2 + t_1(t_2 + t_1)] = 0$

$\because\ a(t_2 - t_1) \neq 0 \quad (\because t_1 \text{ and } t_2 \text{ are different})$



$\therefore\ 2 + t_1(t_2 + t_1) = 0$

$\therefore\ t_2 = -t_1 - \dfrac{2}{t}$

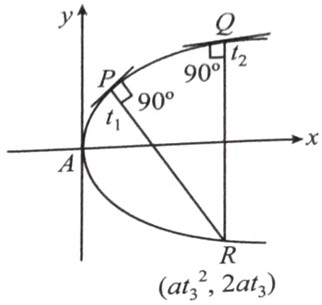
(c) If normal to the parabola $y^2 = 4ax$ drawn at any point $(at^2, 2at)$ meet the parabola at t_3 then

$$t_3 = -t - \frac{2}{t}$$

$\Rightarrow t^2 + tt_3 + 2 = 0 \qquad \text{... (i)}$

It has two roots t_1 and t_2. Hence there are two such point $P(t_1)$ and $Q(t_2)$ on the parabola from where normals are drawn and which meet parabola at $R(t_3)$

$\Rightarrow t_1 + t_2 = -t_3 \quad \text{and} \quad t_1t_2 = 2$



Thus the line joining $P(t_1)$ and $Q(t_2)$ meet x-axis at $(-2a, 0)$

Train Your Brain

Example 28: Find the equation of a normal at the parabola $y^2 = 4x$ which passes through (3, 0).

Sol. Equation of Normal $y = mx - 2am - am^3$

Here $a = 1$ and it passes through (3, 0)

$0 = 3m - 2m - m^3$

$\Rightarrow m^3 - m = 0$

$\Rightarrow m = 0, \pm 1$

for $m = 0 \Rightarrow y = 0$

$m = 1 \Rightarrow y = x - 3$

$m = -1 \Rightarrow y = -x + 3$

Example 29: Show that normal to the parabola $y^2 = 8x$ at the point (2, 4) meets it again at (18, – 12). Find also the length of the normal chord.

Sol. Comparing the given parabola with $y^2 = 4ax$

$\therefore\ 4a = 8$

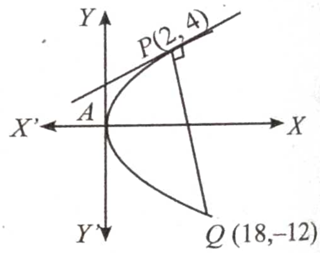
$\therefore\ a = 2$

$P(at_1^2, 2at_1) \equiv (2, 4)$

$\Rightarrow t_1 = 1$

$\Rightarrow$ parameter at $Q(t_2)$

$= -t_1 - \dfrac{2}{t_1} = -3$



$\therefore\ Q(2(-3)^2, 2 \times 2(-3))$

i.e. $Q(18, -12)$

$\therefore$ Length of normal chord PQ = Distance between points P and Q.

$= PQ = \sqrt{(18-2)^2 + (-12-4)^2} = 16\sqrt{2}$

Concept Application

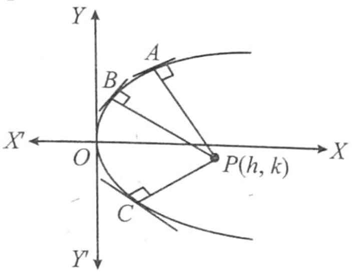
25. If the normals at the points (p_1, q_1) and (p_2, q_2) on the parabola $y^2 = 4ax$ intersect at the parabola, then $p_1 p_2$ is equal to:
(a) 2 (b) $4a^2$
(c) $8a^2$ (d) 8

26. Equation of the normal to parabola $8y^2 = x$ at the point (2, 1/2) is:
(a) $2y = x - 1$ (b) $8y = x + 2$
(c) $4y + 4 = 3x$ (d) $8x + y = 33/2$

27. Equation of the normal to parabola $y^2 = 4x$ at the point (1, – 2) is:
(a) $x + y + 1 = 0$ (b) $3y + x - 7 = 0$
(c) $y - x + 3 = 0$ (d) $3x - y - 5 = 0$.

CO-NORMAL POINTS

Maximum three normals can be drawn from a point to a parabola and their feet (points where the normal meet the parabola) are called co-normal points.



Let $P(h, k)$ be any given point and
$y^2 = 4ax$ be a parabola.
The equation of any normal to $y^2 = 4ax$ is
$y = mx - 2am - am^3$
If it passes through (h, k) then
$k = mh - 2am - am^3$

$$\Rightarrow am^3 + m(2a - h) + k = 0 \quad \text{... (i)}$$

This is a cubic equation in m, so it has three roots, say m_1, m_2 and m_3.

$$\therefore \quad m_1 + m_2 + m_3 = 0, \quad \text{... (ii)}$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a-h)}{a}, \quad \text{... (iii)}$$

$$m_1m_2m_3 = -\frac{k}{a} \quad \text{... (iv)}$$

Hence for any given point $P(h, k)$, (i) has three real or imaginary roots. Correspoinding to each of these three roots, we have each normal passing through $P(h, k)$. Hence we have three normals PA, PB and PC drawn through P to the parabola.
Points A, B, C in which the three normals from $P(h, k)$ meet the parabola are called co-normal points.

Properties of Co-normal Points:

1. The algebraic sum of the slopes of three concurrent normals is zero. This follows from equation (ii).
2. The algebraic sum of ordinates of the feets of three normals drawn to a parabola from a given point is zero.
Let the ordiantes of A, B, C be y_1, y_2, y_3 respectively then
$y_1 = -2\,am_1,\ y_2 = -2am_2$ and $y_3 = -2am_3$
$\therefore$ Algebraic sum of these ordinates is
$y_1 + y_2 + y_3 = -2am_1 - 2am_2 - 2am_3$
$= -2a(m_1 + m_2 + m_3)$
$= -2a \times 0 = 0$ {from equation (ii)}
3. If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) is real then $h > 2a$.
When normals are real, then all the three roots of equation (i) are real and in that case
$m_1^2 + m_2^2 + m_3^2 > 0$ (for any values of m_1, m_2, m_3)
$\Rightarrow (m_1 + m_2 + m_3)^2 - 2\ (m_1m_2 + m_2m_3 + m_3m_1) > 0$
$\Rightarrow (0)^2 - \dfrac{2(2a-h)}{a} > 0 \Rightarrow h - 2a > 0$ or $h > 2a$
4. The centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola.
If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of ΔABC, then its centroid is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) = \left(\frac{x_1+x_2+x_3}{3}, 0\right)$$

Since $y_1 + y_2 + y_3 = 0$ (from result-2). Hence the centroid lies on the x-axis, which is the axis of the parabola also.

Now $\dfrac{x_1+x_2+x_3}{3} = \dfrac{1}{3}(am_1^2 + am_2^2 + am_3^2)$

$$= \frac{a}{3}(m_1^2 + m_2^2 + m_3^2)$$

$$= \frac{a}{3}\{(m_1+m_2+m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1)\}$$

$$= \frac{a}{3}\left\{(0)^2 - 2\left\{\frac{2a-h}{a}\right\}\right\} = \frac{2h-4a}{3}$$

$\therefore$ Centroid of ΔABC is $\left(\dfrac{2h-4a}{3}, 0\right)$

Train Your Brain

Example 30: If from point $P(h, k)$ three normals are drawn to the parabola $(y - 1)^2 = 8(x - 2)$ then find the condition

Sol. Here $a = 2$ and abscissa of point from where three normals are drawn must be greater than $2a$.
i.e. $x - 2 > 2a$ i.e. $x > 6$
Hence $h > 6$.

Example 31: Find the locus of points through which three normals to parabola $y^2 = 4ax$ passes and two of them are perperdicular to each other.

Sol. Let $P(h, k)$ be the point of intersection of three normals to be parabola $y^2 = 4ax$ then it will be

$y = mx - 2am - am^3$ and it passes through $P(h, k)$

$k = mh - 2am - am^3$

$\therefore \quad am^3 + m(2a - h) + k = 0 \quad \ldots (i)$

It has three roots m_1, m_2 and m_3

$m_1 + m_2 + m_3 = 0 \quad \ldots (ii)$

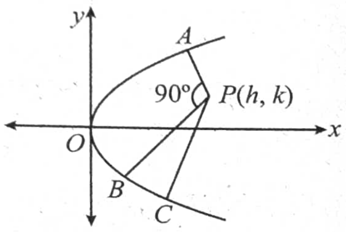
$m_1 m_2 m_3 = \frac{-k}{a} \quad \ldots (iii)$

Given condition, $m_1 m_2 = -1 \quad \ldots (iv)$

as given in figure.

from (iii) and (iv)

$m_3 = \frac{k}{a}$



$\because \ m_3$ is also a root of equation (1) therefore it will satify equation (i)

$$\therefore \quad a\left(\frac{k}{a}\right)^3 + (2a-h)\left(\frac{k}{a}\right) + k = 0$$

$\Rightarrow \ k^3 + (2a - h)\,ka + ka^2 = 0$

$\Rightarrow \ k^2 + 3a^2 - ah = 0$

$\therefore$ locus of $P(h, k)$ is $y^2 + 3a^2 - ax = 0$.

Concept Application

28. If the normals at the points (p_1, q_1) and (p_2, q_2) on the parabola $y^2 = 4ax$ intersect at the parabola, then $q_1 q_2$ is equal to:

(a) 2 (b) 4

(c) $4a^2$ (d) $8a^2$

29. The normals at three distinct points on the parabola $y^2 = 4ax$ pass through the point (h, k). The sum of the intercepts which the normals cut off from the axis is:

(a) $h + 2a$ (b) $2(h + a)$

(c) $2h + a$ (d) $2(h + k + a)$

30. The feet of the normals drawn from the point (h, k) to the parabola $y^2 = 4ax$ lie on the curve:

(a) $x - (h - 2a)y - 2ak = 0$

(b) $xy - (h - 2a)y - 2ak = 0$

(c) $y - (h - 2a)x + 2ak = 0$

(d) $xy - (h - 2a)x + 2ak = 0$

PAIR OF TANGENTS

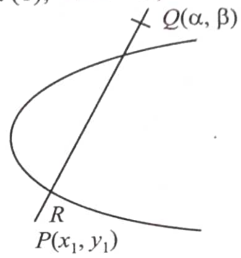
Let the parabola be $y^2 = 4ax \quad \ldots (1)$

Let $P(x_1, y_1)$ be a point outside the parabola.

Let a chord of the parabola through the point $P(x_1, y_1)$ cut the parabola at R and let $Q(\alpha, \beta)$ be an arbitrary point on line PR. Let R divide PQ in the ratio $\lambda : 1$,

Then $R = \left(\frac{\lambda\alpha + x_1}{\lambda + 1}, \frac{\lambda\beta + y_1}{\lambda + 1}\right)$.

Since R lies on parabola (1), therefore,



$$\left(\frac{\lambda\beta + y_1}{\lambda + 1}\right)^2 - 4a\left(\frac{\lambda\alpha + x_1}{\lambda + 1}\right) = 0$$

or $\quad (\lambda\beta + y_1)^2 - 4a(\lambda\alpha + x_1)(\lambda + 1) = 0$

or $\quad (\beta^2 - 4a\alpha)\lambda^2 + 2[\beta y_1 - 2a(\alpha + x_1)]\lambda + (y_1^2 - 4ax_1) = 0 \quad \ldots (2)$

Line PQ will become tangent to parabola (1) if roots of equation (2) are equal or if

$4[\beta y_1 - 2a(\alpha + x_1)]^2 = 4(\beta^2 - 4a\alpha)(y_1^2 - 4ax_1)$

Hence, locus of $Q(\alpha, \beta)$ i.e. equation of pair of tangents from $P(x_1, y_1)$ is

$[yy_1 - 2a(x + x_1)]^2 = (y^2 - 4ax)(y_1^2 - 4ax_1)$

$\Rightarrow \ \boldsymbol{SS_1 = T^2}$

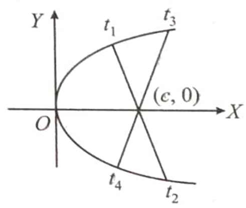
where S, S_1 and T have usual meanings.

The same result holds true for circle, ellipse and hyperbola also.

CHORD JOINING TWO POINTS

Line joining any two points on the parabola is called its chords.

Let the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$, lie on the parabola then equation of chord is



$$(y - 2at_1) = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$

$$\Rightarrow \ y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

$\Rightarrow \ (t_1 + t_2)\,y = 2x + 2at_1 t_2$

If this chord meet the x-axis at point $(c, 0)$ then from above equation $\quad c + at_1 t_2 = 0$ i.e. $t_1 t_2 = -c/a$.

Note: (*i*) If the chord joining t_1, t_2 and t_3, t_4 pass through a point $(c, 0)$ on the axis, then $t_1t_2 = t_3t_4 = -c/a$.

(*ii*) If PQ is a focal chord then $t_1t_2 = -1$ or $t_2 = -\frac{1}{t_1}$, which is required relation.

Hence if one extremity of a focal chord is $(at^2, 2at)$ then the other extremity will be $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

CHORD OF CONTACT

Two tangents PA and PB are drawn to parabola, then line joining AB is called the chord of contact to the parabola with respect to point P.

Let the parabola be $y^2 = 4ax$ ... (1)

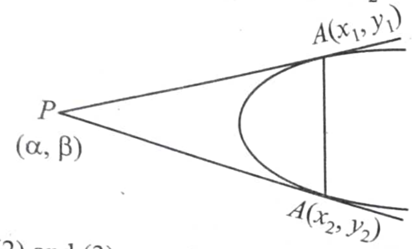
Let $P(\alpha, \beta)$ be a point outside the parabola.

Let PA and PB be the two tangents from $P(\alpha, \beta)$ to parabola (1).

Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$

Equation of the tangent PA is $yy_1 = 2a(x + x_1)$ ... (2)

Equation of the tangent PB is $yy_2 = 2a(x + x_2)$ ... (3)



Since lines (2) and (3) pass through $P(\alpha, \beta)$, therefore

$\beta y_1 = 2a(a + x_1)$ ... (4)

and $\beta y_2 = 2a(a + x_2)$ ... (5)

Now we consider the equation

$y\beta = 2a(x + \alpha)$ ... (6)

From (4) and (5), it follows that line (6) passes through $A(x_1, y_1)$ and $B(x_2, y_2)$.

Hence (6) is the equation of line AB which is the chord of contact of point $P(\alpha, \beta)$ with respect to parabola (1) i.e, chord of contact is $y\beta = 2a(x + \alpha)$

The same result holds true for circle, ellipse and hyperbola also.

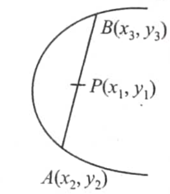
CHORD WITH A GIVEN MIDDLE POINT

Equation of the parabola is $y^2 = 4ax$ ... (1)

Let AB be a chord of the parabola whose middle point is $P(x_1, y_1)$.

Equation of chord AB is $y - y_1 = m(x - x_1)$ ... (2)

where m = slope of AB



Let $A = (x_2, y_2)$ and $B = (x_3, y_3)$.

Since A and B lie on parabola (1)

$\therefore\ y_2^2 = 4ax_2$ and $y_3^2 = 4ax_3$

$\therefore\ y_2^2 - y_3^2 = 4a(x_2 - x_3)$

or $\frac{y_2 - y_3}{x_2 - x_3} = \frac{4a}{y_2 + y_3}$ ...(3)

But $P(x_1, y_1)$ is the middle point of AB $y_2 + y_3 = 2y_1$

$\therefore$ From (3), $\frac{y_2 - y_3}{x_2 - x_3} = \frac{4a}{2y_1} = \frac{2a}{y_1}$

$\therefore$ Slope of AB i.e., $m = \frac{2a}{y_1}$ ... (4)

From (2), equation of chord AB is $y - y_1 = \frac{2a}{y_1}(x - x_1)$

or $yy_1 - y_1^2 = 2ax - 2ax_1$ or $yy_1 - 2ax = y_1^2 - 2ax_1$

or $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

[Subtracting $2ax_1$ from both sides] ...(5)

(5) is the required equation. In usual notations, equation (5) can be written as $T = S_1$.

The same result holds true for circle, ellipse and hyperbola also.

Train Your Brain

Example 32: Write the equation of pair of tangents to the parabola $y^2 = 4x$ drawn from a point $P(-1, 2)$

Sol. We know the equation of pair of tangents are given by $SS_1 = T^2$

$\therefore\ (y^2 - 4x)(4 + 4) = (2y - 2(x - 1))^2$

$\Rightarrow 8y^2 - 32x = 4y^2 + 4x^2 + 4 - 8xy + 8y - 8x$

$\Rightarrow y^2 - x^2 + 2xy - 6x - 2y = 1$

Example 33: Find the locus of the point P from which tangents are drawn to parabola $y^2 = 4ax$ having slopes m_1, m_2 such that

(*i*) $m_1 + m_2 = m_0$ (const) (*ii*) $\theta_1 + \theta_2 = \theta_0$ (const)

Sol. Equation of tangent to $y^2 = 4ax$, is

$$y = mx + \frac{a}{m}$$

Let it passes through $P(h, k)$

$\therefore\ m^2h - mk + a = 0$

(*i*) $m_1 + m_2 = m_0 = \frac{k}{h} \Rightarrow y = m_0x$

(*ii*) $\tan\theta_0 = \frac{m_1 + m_2}{1 - m_1m_2} = \frac{k/h}{1 - a/h}$

$\Rightarrow y = (x - a)\tan\theta_0$

Example 34: Tangent are drawn to parabola $y^2 = 4ax$ at point where the line $lx + my + n = 0$ meets the parabola. Find the point of intersection of these tangents.

Sol. Let the tangent intersect at $P(h, k)$, then $lx + my + n = 0$ will be the chord of contact of P. That means $lx + my + n = 0$ and $yk - 2ax - 2ah = 0$ will represent the same line. Thus,

$$\frac{k}{m} = \frac{-2a}{1} = \frac{-2ah}{n} \Rightarrow h = \frac{n}{1},\ k = -\frac{2am}{1}$$

Example 35: Find the length of chord of contact of the tangents drawn from point (x_1, y_1) to the parabola $y^2 = 4ax$.

Sol. Let tangent at $P(t_1)$ and $Q(t_2)$ meet at (x_1, y_1)

$\therefore \quad at_1t_2 = x_1$ and $a(t_1 + t_2) = y_1$

$$\because \quad PQ = \sqrt{(at_1^2 - at_2^2)^2 + (2a(t_1 - t_2))^2}$$

$$= a\sqrt{((t_1 + t_2)^2 - 4t_1t_2)((t_1 + t_2)^2 + 4)}$$

$$= \sqrt{\frac{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}{a^2}}$$

Example 36: If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P and Q, then find the point of intersection of tangents at P and Q.

Sol. Let (h, k) be point of intersection of tangents then chord of contact is

$yk = 4(x + h)$

$4x - yk + 4h = 0 \quad \text{... (i)}$

But given is $x - y - 1 = 0$

$$\therefore \quad \frac{4}{1} = \frac{-k}{-1} = \frac{4h}{-1}$$

$\Rightarrow \quad h = -1, k = 4$

$\therefore \quad$ point $\equiv (-1, 4)$

Example 37: Find the locus of point whose chord of contact w.r.t. to the parabola $y^2 = 4bx$ is the tangent of the parabola $y^2 = 4ax$.

Sol. Equation of tangent to

$$y^2 = 4ax \text{ is } y = mx + \frac{a}{m} \quad \text{... (i)}$$

Let it is chord of contact for parabola $y^2 = 4bx$ w.r.t. the point $P(h, k)$

$\therefore \quad$ Equation of chord of contact is $yk = 2b(x + h)$

$$\Rightarrow y = \frac{2b}{k}x + \frac{2bh}{k} \quad \text{... (ii)}$$

From (i) and (ii), $m = \frac{2b}{k}, \frac{a}{m} = \frac{2bh}{k} \Rightarrow a = \frac{4b^2h}{k^2}$

$\therefore \quad$ Locus of P is $y^2 = \frac{4b^2}{a}x$.

Example 38: Find the locus of middle of the chord of the parabola $y^2 = 4ax$ which pass through a given (p, q).

Sol. Let $P(h, k)$ be the mid point of chord of the parabola $y^2 = 4ax$.

So equation of chord is $yk - 2a(x + h) = k^2 - 4ah$. Since it passes through (p, q)

$\therefore \quad qk - 2a(p + h) = k^2 - 4ah$

$\therefore \quad$ Required locus is $y^2 - 2ax - qy + 2ap = 0$.

Concept Application

31. The locus of the point of intersection of the perpendicular drawn to the parabola $x^2 - 8x + 2y + 2 = 0$ is:

(a) $2y - 15 = 0$ (b) $2x + 9 = 0$

(c) $2y + 15 = 0$ (d) $2x - 9 = 0$

32. If P be $(1, 0)$ and Q be any point on the parabola $y^2 = 8x$, then the locus of the mid-point of PQ is:

(a) $y^2 - 4x + 2 = 0$ (b) $y^2 + 4x + 2 = 0$

(c) $x^2 - 4y + 2 = 0$ (d) $x^2 + 4y + 2 = 0$

33. Tangents are drawn to the parabola $y^2 = 4ax$ from a point P. If the chord of contact passes through the point $(a, 4a)$, then the locus of P is:

(a) $x - 2y + a = 0$ (b) $2x - y + a = 0$

(c) $x^2 = 2(y + a)$ (d) $y^2 = 2(x + a)$

34. If the tangent to a parabola at a point Q meets the tangent at the vertex in R and S is the focus of the parabola, then $\angle QRS$ is equal to:

(a) $\pi/4$ (b) $\pi/2$

(c) $\pi/3$ (d) $\pi/6$

AN IMPORTANT CONCEPT

If a family of straight lines can be represented by an equation $\lambda^2 P + \lambda Q + R = 0$ where λ is a parameter and P, Q, R are linear functions of x and y then the family of lines will be tangent to the curve $Q^2 = 4PR$.

POLAR AND POLE

Let P be any point inside or outside a parabola. Suppose in a straight line drawn through P intersect the parabola at Q and R. Then the locus of point of intersection of the tangents to the parabola at Q and R is called the polar of given point P with respect to the parabola and point P is called the pole of the polar.

Equation of the Polar of the point $P(x_1, y_1)$ *w.r.t.* the parabola $y^2 = 4ax$ is, $yy_1 = 2a(x + x_1)$

The pole of the line $lx + my + n = 0$ w.r.t. the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$.

Properties of Polar:

(i) The polar of the focus of the parabola is the directrix.

(ii) When the point (x_1, y_1) lies on the parabola the equation to its polar is the same as the equation to the chord of contact of tangents drawn from (x_1, y_1) when (x_1, y_1) is on the parabola the polar is the same as the tangent at the point.

(iii) Two straight lines are said to be conjugated to each other w.r.t. a parabola when the pole of one lies on the other.

Similarly two points P and Q are said to be conjugate points if polar of P passes through Q and vice versa.

(iv) Polar of a given point P w.r.t. any Conic is the locus of the harmonic conjugate of P w.r.t. the two points is which any line through P cuts the conic.

DIAMETER OF A PARABOLA

Diameter of a conic is the locus of middle points of a series of its parallel chords.

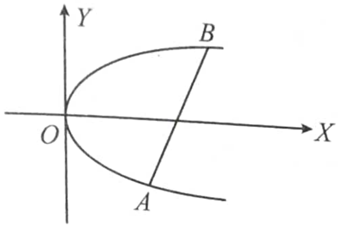
Equation of diameter of a parabola:

Let the parabola be $y^2 = 4ax$ ... (1)

Let AB be one of the chords of a series of parallel chords having slope m.

Let $P(\alpha, \beta)$ be the middle point of chord AB, then equation of AB will be $T = S_1$.

or $y\beta - 2a(x + \alpha) = \beta^2 - 4\alpha a$ ... (2)

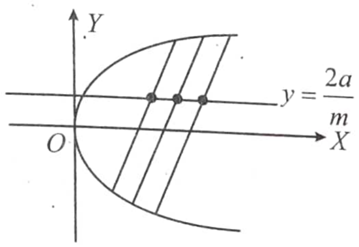


Slope of line (2) $= \dfrac{2a}{\beta}$

but slope of line (1) i.e. line AB is m.

$\therefore \dfrac{2a}{\beta} = m$ or $\beta = \dfrac{2a}{m}$

Hence locus of $P(\alpha, \beta)$ i.e. equation of diameter (which is the locus of a series of a parallel chords having slope m) is



$$y = \frac{2a}{m} \quad \text{... (3)}$$

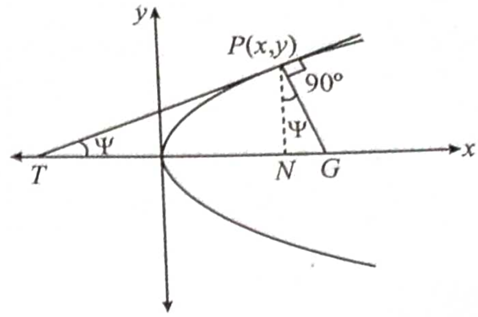
Clearly line (3) is parallel to the axis of the parabola. Thus a diameter of a parabola is parallel to its axis.

LENGTH OF TANGENT, SUBTANGNET, NORMAL AND SUB-NORMAL

Let the parabola is $y^2 = 4ax$. Let the tangent and normal at any point

$P(x, y)$ meet the axis of parabola at T and G respectively and tangent makes an angle ψ with x-axis.

$\therefore \tan\psi = \left(\dfrac{dy}{dx}\right)_{P(x,y)}$ and $PN = y$



$\therefore$ PT = length of tangent = PN cosec ψ = y cosec ψ

PG = length of normal = $y \sec\psi$

TN = length of sub-tangent = $PN \cot\psi = y \cot\psi$

NG = length of sub-normal = $y \tan\psi$

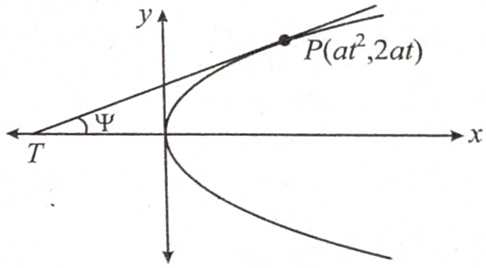
Train Your Brain

Example 39: Find the length of tangent, sub-tangent, normal and sub-normal to $y^2 = 4ax$ at $(at^2, 2at)$.

Sol. $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a$$

$$\therefore \left(\frac{dy}{dx}\right)_{(at^2, 2at)} = \frac{1}{t} = \tan\psi \Rightarrow \cot\psi = t$$



$\therefore$ l (tangent) $= y \operatorname{cosec}\psi = 2at\sqrt{1+\cot^2\psi}$

$= 2at\sqrt{1+t^2}$

l (normal) $= y\sec\psi = 2at\sqrt{1+\tan^2\psi}$

$= 2a\sqrt{1+t^2}$

l (sub-tangent) $= y\cot\psi = 2at \cdot t = 2at^2$

l (sub-normal) $= y\tan\psi = 2at \cdot \dfrac{1}{t} = 2a$

Thus length of sub-normal of parabola is constant and equal to semi-latus rectum.

Example 40: If the equation $m^2(x + 1) + m(y - 2) + 1 = 0$ represents a family of lines, where 'm' is parameter then find the equation of the curve to which these lines will always be tangents.

Sol. $m^2(x+1)+m(y-2)+1=0$. The equation of the curve to which above lines will always be tangents can be obtained by equating its discriminant to zero.

$\therefore \quad (y-2)^2-4(x+1)=0$

$\Rightarrow y^2-4y+4-4x-4=0 \quad \Rightarrow \quad y^2=4(x+y)$

Concept Application

35. Circle drawn with a focal chord of the parabola $y^2 = 4x + y + 4$ as diameter will always touch the line $16x + k = 0$, where k equals to:

(*a*) 8 (*b*) 16

(*c*) 33 (*d*) 36

36. A ray of light moving parallel to the X-axis gets reflected from a cancave parabolic mirror whose equation is $y^2 + 4x = 4$. The reflected ray must pass through the point:

(*a*) (1, 0) (*b*) (–1, 0)

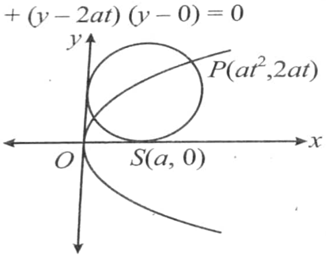
(*c*) (0, 0) (*d*) (–2, 0)

IMPORTANT HIGHLIGHTS OF PARABOLA

1. Circle described on the focal length (distance) as diameter touches the tangent at the vertex.

Equation of the circle described on SP as diameter is

$(x-at^2)(x-a)+(y-2at)(y-0)=0$



Solve it with y-axis i.e. $x = 0$, we get

$y^2-2aty+a^2t^2=0 \quad \Rightarrow \quad (y-at)^2=0$

circle touches y-axis at $(0, at)$.

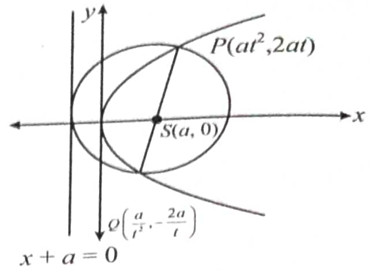
2. Circle described on the focal chord as diameter touches directrix

Equation of the circle described on PQ as diameter is

$$(x-at^2)\left(x-\frac{a}{t^2}\right)+(y-2at)\left(y+\frac{2a}{t}\right)=0$$

Solving it with $x=-a$

$$(-a-at^2)\left(-a-\frac{a}{t^2}\right)+(y-2at)\left(y+\frac{2a}{t}\right)=0$$



$$\Rightarrow y^2-2a\left(t-\frac{1}{t}\right)y+a^2\left(t-\frac{1}{t}\right)^2=0$$

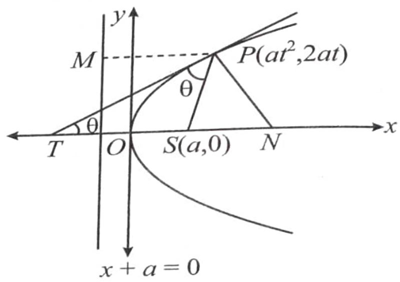
$$\Rightarrow \left[y-a\left(t-\frac{1}{t}\right)\right]^2=0$$

$\Rightarrow$ circle tocuhes the directrix.

3. Tangent at P is $yt = x + at^2$, meet x-axis at T, then $T(-at^2, 0)$. Normal at P is $y + xt = 2at + at^3$, meet x-axis at N,

then $N(2a + at^2, 0)$

$\Rightarrow ST = SN = a + at^2 = PM = PS$



$\Rightarrow \angle PTS = \angle TPS = \theta$

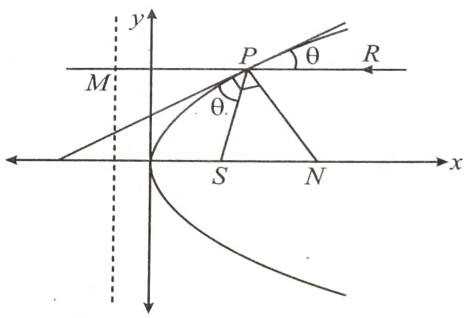
$\because \quad TS = PS = PM \Rightarrow \angle TPM = \theta$

Tangent and Normal at any point P bisect the angle between PS and PM internally and externally. This property leads to the *reflection property of parabola*.

Circle circumscribing the triangle formed by any tangent, normal and x-axis, has its centre at focus.

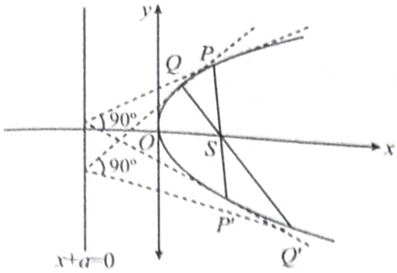
If we extend MP, then from figure

$\angle RPN = \angle SPN = 90 - \theta$



Thus ray parallel axis meet parabola at P and after reflection from P it passes through the focus.

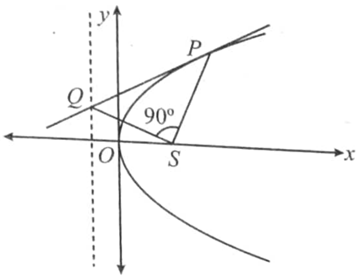
4. The tangents at the extremities of a focal chord intersect at right angles on the directrix.



5. The portion of tangent to the parabola intercepted between the directrix and the curve subtends a right angle at the focus.
Tangent at $P(at^2, 2at)$ is $yt = x + at^2$ meet the directrix at $x = -a \Rightarrow Q\left(-a, \frac{at^2 - a}{t}\right)$ and $S(a, 0)$.

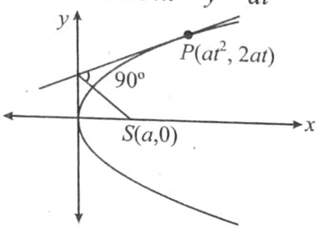
$$\text{Slope at } SP = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1} = m_1$$

$$\text{Slope at } SQ = \frac{\frac{at^2 - a}{t} - 0}{-a - a} = \frac{t^2 - 1}{-2t} = m_2.$$



$\Rightarrow m_1 m_2 = -1$
$\Rightarrow SP \perp SQ$
$\Rightarrow \angle PSQ = 90°$

6. Tangent at P is $yt = x + at^2$ ... (*i*)
Line perpendicular to above line is $xt + y = \lambda$
and passes through $(a, 0)$ gives $\lambda = at$
$\therefore$ Perpendicular line will be $xt + y = at$ ... (*ii*)



Solve (*i*) and (*ii*), we get $x = 0$
i.e., these two line intersect at y-axis i.e. tangnet at the vertex.
The foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at vertex.

7. **Tangents and Normals** at the extremities of the latus rectum of a parabola $y^2 = 4ax$ **constitute a square**, their points of intersection being $(-a, 0)$ and $(3a, 0)$.

8. The **circle circumscribing the triangle** formed by **any three tangents** to a parabola passes through the focus.

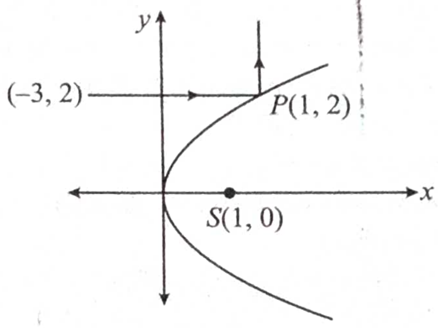
9. The **orthocentre** of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix & has the co–ordinates $(-a, a(t_1 + t_2 + t_3 + t_1t_2t_3))$.

10. The area of the triangle formed by three points on a parabola is **twice the area** of the triangle formed by the tangents at these points.

Train Your Brain

Example 41: If incident ray from point $(-3, 2)$ parallel to the axis of parabola $y^2 = 4x$ strike the parabola, then find the equation of reflected ray.

Sol. Since incident ray strikes parabola at $P(1, 2)$ i.e. extremity of latus rectum and it will pass through the focus of parabola therefore reflected ray will be parallel to y-axis and its equation will be $x = 1$.



Example 42: A ray of light moving parallel to the x-axis get reflected from a parabolic mirror $(y - 2)^2 = 4(x + 1)$. Find the point on the axis of parabola through which the ray must pass after reflection.

Sol. Axis of parabola is $y = 2$ i.e., parallel to x-axis. As we know if incident ray is parallel to x-axis then after reflextion it will pass through the focus of parabola and focus is $(0, 2)$.

Concept Application

37. Let S is the focus of parabola $y^2 = 4ax$ and X is the foot of the directrix, PP' is a double ordinate of the curve and PX meets the curve again at Q. $P'Q$ always passes through

(*a*) $(a, 0)$
(*b*) $\left(\frac{a}{4}, 0\right)$
(*c*) $\left(\frac{2}{a}, 0\right)$
(*d*) $\left(\frac{a}{2}, \frac{a}{2}\right)$

38. Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S. The ratio of the area of the triangle PQS and PQR is

(*a*) $1 : \sqrt{2}$
(*b*) $1 : 2$
(*c*) $1 : 4$
(*d*) $1 : 8$

Short Notes

General Equation of A Conic : Focal Directrix Property

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is:

$$(l^2 + m^2)\,[(x-p)^2 + (y-q)^2] = e^2\,(lx + my + n)^2$$

$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Case (i) When the focus lines on the directrix

In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if:

$e > 1$, $h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1$, $h^2 = ab$ the lines will coincident.

$e < 1$, $h^2 < ab$ the lines will be imaginary.

When the focus does not lie on the directix

The conic represents:

a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1; D \neq 0$ $h^2 = ab$	$0<e<1; D\neq 0$ $h^2 < ab$	$D \neq 0;\ e > 1$ $h^2 > ab$	$e > 1;\ D \neq 0$ $h^2 > ab;\ a + b = 0$

Standard equation of a parabola is $y^2 = 4ax$. For this parabola:

(i) Vertex is $(0, 0)$
(ii) Focus is $(a, 0)$
(iii) Axis is $y = 0$
(iv) Directrix is $x + a = 0$

Latus Rectum

A focal chord perpendicular to the axis of a parabola is called the LATUS RECTUM. For $y^2 = 4ax$.

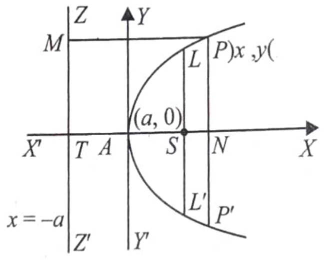
(i) Length of the latus rectum $= 4a$.
(ii) Length of the semi latus rectum $= 2a$.
(iii) Ends of the latus rectum are $L(a, 2a)$ & $L'(a, -2a)$.

Parametric Representation

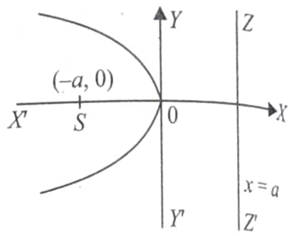
The simplest & the best form of representing the co-ordinates of a point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$. The equation $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

Types of Parabola

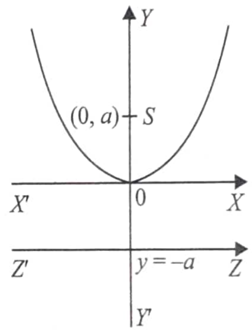
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$.



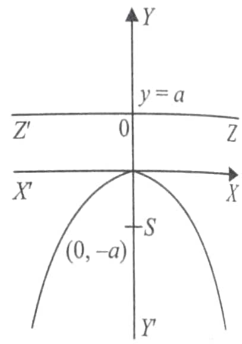
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Para-metric equation	Focal length
$y^2 = 4ax$	$(0, 0)$	$(a, 0)$	$y = 0$	$x = -a$	$4a$	$(a, \pm 2a)$	$(at^2, 2at)$	$x + a$
$y^2 = -4ax$	$(0, 0)$	$(-a, 0)$	$y = 0$	$x = a$	$4a$	$(-a, \pm 2a)$	$(-at^2, 2at)$	$x - a$
$x^2 = +4ay$	$(0, 0)$	$(0, a)$	$x = 0$	$y = -a$	$4a$	$(\pm 2a, a)$	$(2at, at^2)$	$y + a$
$x^2 = -4ay$	$(0, 0)$	$(0, -a)$	$x = 0$	$y = a$	$4a$	$(\pm 2a, -a)$	$(2at, -at^2)$	$y - a$
$(y-k)^2 = 4a(x-h)$	(h, k)	$(h+a, k)$	$y = k$	$x + a - h = 0$	$4a$	$(h + a, k \pm 2a)$	$(h + at^2, k + 2at)$	$x - h + a$
$(x-p)^2 = 4b(y-q)$	(p, q)	$(p, b+q)$	$x = p$	$y + b - q = 0$	$4b$	$(p \pm 2a, q + a)$	$(p + 2at, q + at^2)$	$y - q + b$

Position of a Point Relative to a Parabola

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

Chord Joining Two Points

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.

Note:

(i) If PQ is focal chord then $t_1t_2 = -1$.
(ii) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\dfrac{a}{t^2}, \dfrac{-2a}{t}\right)$.
(iii) If $t_1t_2 = k$ then chord always passes a fixed point $(-ka, 0)$.

Line & A Parabola

(a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a > = < cm$

$\Rightarrow$ condition of tangency is, $c = \frac{a}{m}$.

Note: Line $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $c = -am^2$.

(b) Length of the chord intercepted by the parabola $y^2=4ax$ on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

Note: length of the focal chord making an angle α with the x-axis is $4a \operatorname{cosec}^2 \alpha$.

Tangent to the Parabola $y^2 = 4ax$:

(a) **Point form:** Equation of tangent to the given parabola at its point (x_1, y_1) is $yy_1 = 2a(x + x_1)$.

(b) **Slope form:** Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(c) **Parametric form:** Equation of tangent to the given parabola at its point $P(t)$, is–

$$ty = x + at^2$$

Note: Point of intersection of the tangents at the point t_1 & t_2 is $[at_1 t_2, a(t_1 + t_2)]$. (i.e. G.M. and A.M. of abscissae and ordinates of the points).

Normal to the Parabola $y^2 = 4ax$

(a) **Point form:** Equation of normal to the given parabola at its point (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

(b) **Slope form:** Equation of normal to the given parabola whose slope is 'm', is $y = mx - 2am - am^3$. foot of the normal is $(am^2, -2am)$.

(c) **Parametric form:** Equation of normal to the given parabola at its point $P(t)$, is $y + tx = 2at + at^3$.

Note:

(i) Point of intersection of normals at t_1 & t_2 is $(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))$.

(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

Chord of Contact

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.

Remember that the area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{2a}$. Also note that the chord of contact exists only if the point P is not inside.

Chord with A Given Middle Point

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1)$.

Diameter

The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola is $y = 2a/m$, where m = slope of parallel chords.

Conormal Points

Foot of the normals of three concurrent normals are called conormals point.

(i) Algebraic sum of the slopes of three concurrent normals of parabola $y^2 = 4ax$ is zero.

(ii) Sum of ordinates of the three conormal points on the parabola $y^2 = 4ax$ is zero.

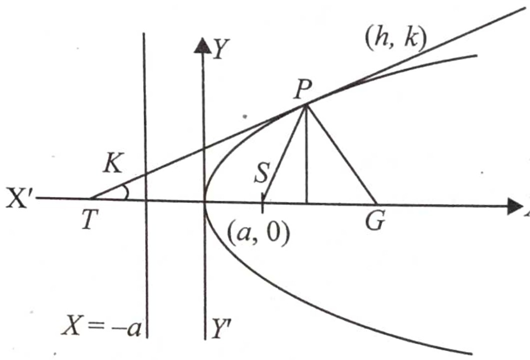
(iii) Centroid of the triangle formed by three co-normal points lies on the axis of parabola.

(iv) If $27ak^2 < 4(h - 2a)^3$ satisfied then three real and distinct normal are drawn from point (h, k) on parabola $y^2 = 4ax$.

(v) If three normals are drawn from point $(h, 0)$ on parabola $y^2 = 4ax$, then $h > 2a$ and one the normal is axis of the parabola and other two are equally inclined to the axis of the parabola.

Important Highlights

(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then $ST = SG = SP$ where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.



(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix**, and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point $P(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of a length a $\sqrt{1+t^2}$ on a normal at the point P.

(*d*) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(*e*) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean beetween segments of any focal chord

i.e. $2a = \dfrac{2bc}{b+c}$ or $\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a}$.

(*f*) Image of the focus lies on directrix with respect to any tangent of parabola $y^2 = 4ax$.

Solved Examples

1. Tangent drawn at point $P(1, 3)$ of a parabola intersects its tangent at vertex at $M(-1, 5)$ and cuts the axis of parabola at T. If $R(-5, 5)$ is a point on SP; where S is focus of the parabola, then

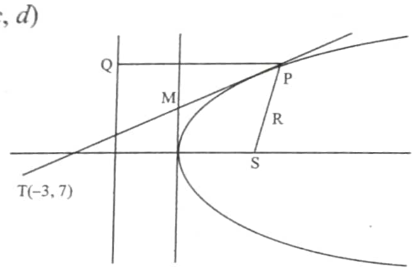
(*a*) slope of axis is -3

(*b*) radius of circumcircle of $DSMP$ is $\sqrt{\dfrac{5}{2}}$ units

(*c*) $(ST)^2 - (SM)^2 = (PM)^2$

(*d*) tangent cuts the axis of parabola at $T(-3, 7)$

Sol. (*a, b, c, d*)



Equation of tangent at P is $x + y = 4$

Clearly mirror image of $R(-5, 5)$ lies on line PQ.

Now mirror image R' or R

$\Rightarrow \dfrac{\alpha'+5}{1} = \dfrac{\beta'-5}{1} = \dfrac{-2(-5+5-4)}{2} = 4$

$\Rightarrow (\alpha', \beta') \equiv (-1, 9)$

Let PM cuts the axis at T; as M is midpoint of PT

$\Rightarrow T$ is $(-3, 7)$

We know that $SP = ST$ and $\angle SMP = \dfrac{\pi}{2}$

Equation of $SP \equiv y - 3 = -\dfrac{1}{3}(x-1)$

$\Rightarrow x + 3y - 10 = 0$

Let $S \equiv (10 - 3\beta, \beta)$

Again $TS \parallel PQ \Rightarrow \dfrac{\beta-7}{13-3\beta} = \dfrac{9-3}{-1-1} = -3$

$\therefore$ focus is $(-2, 4)$

2. If the line $2x - \lambda y - 8 = 0$, $\lambda \in R$ intersects the parabola $y^2 = 16x$ at A and B. S be the focus of the parabola, then

(*a*) a circle which 'BS' as diameter always touches $x = -2$

(*b*) a circle which 'AB' as diameter always touches $x = -4$

(*c*) a circle which 'AS' as diameter always touches $x = 0$

(*d*) normal at A and B are always perpendicular to each other.

Sol. (*b, c, d*)

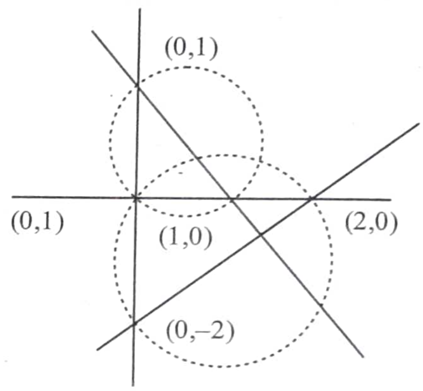
The line $2x - \lambda y - 8 = 0$ is a focal chord.

3. If the coordinates of focus of a parabola which touches $x = 0$, $y = 0$, $x + y = 1$ and $y = x - 2$ are (α, β), then

(*a*) $\alpha + 2\beta = 2$ (*b*) $2\alpha - \beta = 2$

(*c*) $\alpha + \beta = 0$ (*d*) $2\alpha - 3\beta = 0$

Sol. (*a, b*)



Circle through point of intersection of 3 tangents passes through focus.

So focus is point of intersection $x(x-1) + y(y-1) = 0$ and $x(x-2) + (y+2)y = 0$

Which is $\left(\dfrac{6}{5}, \dfrac{2}{5}\right)$.

4. Suppose that a parabola(P) $y = ax^2 + bx + c$, where $a > 0$ and $(a+b+c)$ is an integer has vertex $\left(\dfrac{1}{4}, \dfrac{-9}{8}\right)$. If the minimum possible value of 'a' can be written as $\dfrac{p}{q}$ where p and q are relatively prime positive integers, then

(*a*) $p + q = 11$

(*b*) $p - q = 7$

(*c*) Parabola (P) passes through $(1, -2)$

(*d*) Parabola (P) passes through $(1, -1)$

Sol. (*a*, *d*)

$$A = -2b,\ b + 8c = -9 \text{ and } a + b + c = \frac{9a - 18}{16} = -1$$

(minimum) achieved at $a = 2/9$.

5. A parabola C whose focus is $S(0, 0)$ and passing through $P(3, 4)$. Equation of tangent at P to parabola is $3x + 4y - 25 = 0$. A chord through S parallel to tangent at P intersects the parabola at A and B

(*a*) length of AB is 20 units

(*b*) area of triangle APS is 40 square units

(*c*) Latus rectum of parabola is 20 units

(*d*) area of triangle PAB is 80 square units

Sol. (*a*, *c*)

The foot of the perpendicular from S to $3x + 4y - 25 = 0$ is P.

The tangent $3x + 4y - 25 = 0$ tangent at vertex and axis is $4x - 3y = 0$

So, latus rectum $= AB = 20$

6. A variable circle passes through the point $A(2, 1)$ and touches the x-axis. Locus of the other end of the diameter through A is a parabola. Which of the following statements is/are correct.

(*a*) The length of the latus rectum of the parabola is 4

(*b*) Equation of axis of parabola is $x = 2$.

(*c*) Equation of directrix is $x = 1$

(*d*) Focus is (1, 0)

Sol. (*a*, *b*)

Equation of the variable circle

$(x - h)(x - 2) + (y - k)(y - 1) = 0$

$x^2 + y^2 - (2 + h)x - (k + 1)y + k + 2h = 0$

As x-intercept $= 0$

$\Rightarrow g^2 = C$

$\Rightarrow (h - 2)^2 = 4k$

7. Consider the parabola $y^2 = 4ax$ and $x^2 = 4by$. The straight line $b^{1/3}y + a^{1/3}x + a^{2/3}b^{2/3} = 0$

(*a*) touches $y^2 = 4ax$

(*b*) touches $x^2 = 4by$

(*c*) intersects both parabolas in real and distinct points

(*d*) touches $y^2 = 4ax$ and intersect $x^2 = 4by$ at two real and distinct points

Sol. (*a*, *b*)

The equation of the line be written in the slope form as

$$y = -\frac{a^{1/3}}{b^{1/3}}x + \frac{a}{\left(-a^{1/3}/b^{1/3}\right)} \text{ i.e. } y = mx + \frac{a}{m}$$

Where $m = -\dfrac{a^{1/3}}{b^{1/3}}$

So it touches the parabola $y^2 = 4ax$

The equation of the line can also be written in the form

$$x = -\frac{b^{1/3}}{a^{1/3}}y + \left(\frac{b}{-b^{1/3}/a^{1/3}}\right)$$

i.e. $x = my + \dfrac{b}{m}$ where $m = \dfrac{-b^{1/3}}{a^{1/3}}$

So it touches the parabola $x^2 = 4by$ also.

8. If the point $P(4, -2)$ is the one end of the focal chord PQ of the parabola $y^2 = x$, then

(*a*) the slope of the normal at Q is $-\dfrac{1}{4}$

(*b*) harmonic mean of PS and QS (S being focus) is $\dfrac{1}{2}$

(*c*) tangents at P and Q to the $y^2 = x$ are perpendicular

(*d*) tangent intersects at $x = -\dfrac{1}{4}$

Sol. (*a*, *b*, *c*, *d*)

This section contains 8 multiple choice questions. Each question has four choices (*A*), (*B*), (*C*) and (*D*) out of which ONE option is correct.

9. Through the vertex O of the parabola $y^2 = 4ax$, a perpendicular is drawn to any tangent meeting it at P and the parabola at Q, then

(*a*) $(OP)^2 + (OQ)^2 =$ constant

(*b*) $OP + OQ =$ constant

(*c*) $(OP)\cdot(OQ) =$ constant

(*d*) $\dfrac{OP}{OQ} =$ constant

Sol. (*c*) Any tangent to the parabola $y = mx + \dfrac{a}{m}$ ...(i)

Any line through vertex O and perpendicular to tangent is $y = \dfrac{-1}{m}x$ ...(ii)

$$OP = \frac{a}{m\sqrt{1+m^2}}$$

Line (2) meets the parabola $y^2 = 4ax$ at Q

$$\therefore \left(\frac{-1}{m}x\right)^2 = 4ax$$

$x = 4am^2$ and $y = 4am$

$x = 4am^2$ and $y = -4am$

$\therefore OQ = 4am\sqrt{m^2+1}$

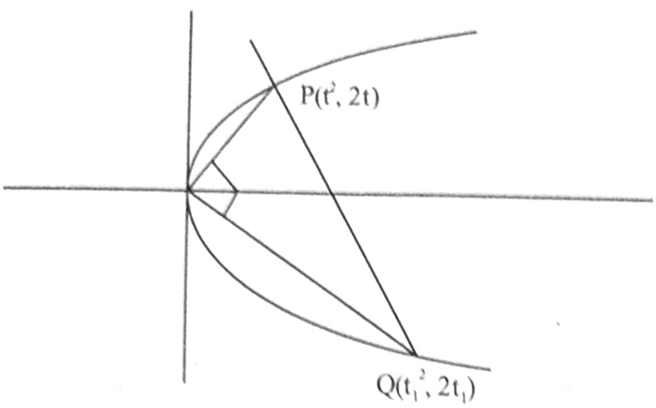
$\therefore (OP)(OQ) = 4a^2$

10. The length of normal chord which subtend an angle of 90° at the vertex of the parabola $y^2 = 4x$ is

(*a*) $6\sqrt{3}$ (*b*) $7\sqrt{2}$

(*c*) $8\sqrt{3}$ (*d*) $4\sqrt{2}$

Sol. (*a*)



$t_1 = -t - \dfrac{2}{t}$

Slope of OP × Slope of $OQ = -1$

$\dfrac{2t}{t^2} \cdot \dfrac{2t_1}{t_1^2} = -1 \Rightarrow tt_1 = -4$

$\Rightarrow t_1 = -\dfrac{4}{t} \Rightarrow -\dfrac{4}{t} = -\dfrac{2}{t} - t_1$

$t = \sqrt{2}$

$\Rightarrow t_1 = -2\sqrt{2} \Rightarrow Q \equiv (8, -4\sqrt{2}), P \equiv (2, 2\sqrt{2})$

$\Rightarrow PQ = 6\sqrt{3}$

11. Minimum distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$ is

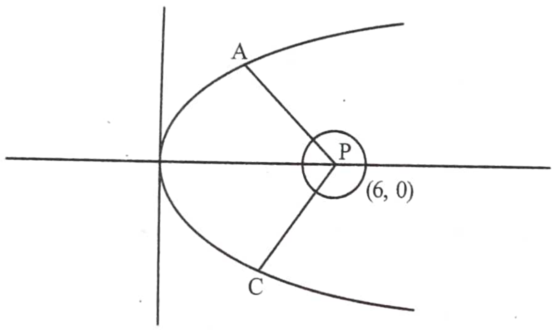
(*a*) $\sqrt{5}$ (*b*) $\sqrt{21}$

(*c*) $\sqrt{28} - \sqrt{5}$ (*d*) $\sqrt{21} - \sqrt{5}$

Sol. (*a*) Centre and radius of the given circle is P(6, 0) and $\sqrt{5}$ respectively.

Equation of normal for $y^2 = 4x$ at $(t^2, 2t)$ is $y = -tx + 2t + t^3$, it must pass through (6, 0) in order that it gives minimum distance between the two curves.

$\therefore 0 = t^3 - 4t \Rightarrow t = 0$ or $t = \pm 2$



$\therefore A(4, 4)$ and $C(4, -4)$

$PA = PC = \sqrt{20} = 2\sqrt{5}$

$\therefore$ Required minimum distance $= 2\sqrt{5} - \sqrt{5} = \sqrt{5}$

12. If the normals at three distinct points $(p^2, 2p)$, $(q^2, 2q)$ and $(r^2, 2r)$ of the parabola $y^2 = 4x$ are concurrent then

(*a*) $p + r = 2q$

(*b*) $p + r = q$

(*c*) $p + 2q + 3r = 0$

(*d*) $p + q + r = 0$

Sol. (*d*) Equation of three normal are

$px + y - (p^3 + 2p) = 0$

$qx + y - (q^3 + 2q) = 0$

$rx + y - (r^3 + 2r) = 0$

Since these lines are concurrent.

$$\begin{vmatrix} p & 1 & p^3 + 2p \\ q & 1 & q^3 + 2q \\ r & 1 & r^3 + 2r \end{vmatrix} = 0$$

$\Rightarrow (p + q + r)(p - q)(q - r)(r - p) = 0$

$\Rightarrow p + q + r = 0.$

13. Maximum number of common normal of $y^2 = 4by$ may be equal to

(*a*) 2 (*b*) 4

(*c*) 5 (*d*) 0

Sol. (*c*) Equation of normal to $y^2 = 4ax$ and $x^2 = 4by$ in terms of m are given by

$y = mx - 2am - am^3$ and $y = mx + 2b + \dfrac{b}{m^2}$.

For common normal $2b + \dfrac{b}{m^2} + 2am + am^3 = 0$

$\Rightarrow am^5 + 2am^3 + 2bm^2 + b = 0$

So, a maximum of 5 normals are possible.

14. The mirror image of the parabola $y^2 = 4x$ in the tangent to the parabola at the point (1, 2) is

(*a*) $(x - 1)^2 = 4(y + 1)$

(*b*) $(x + 1)^2 = 4(y - 1)$

(*c*) $(x + 1)^2 = 4(y - 1)$

(*d*) $(x - 1)^2 = 4(y - 1)$

Sol. (*c*) Any point on the given parabola is $(t^2, 2t)$. The equation of the tangent at (1, 2) is $x - y + 1 = 0$

The image (h, k) of the point $(t^2, 2t)$ in $x - y + 1 = 0$ is given by $\dfrac{h - t^2}{1} = \dfrac{k - 2t}{-1} = -\dfrac{2(t^2 - 2t + 1)}{1 + 1}$

$h = t^2 - t^2 + 2t - 1 = 2t - 1$ and $k = 2t + t^2 - 2t + 1 = t^2 + 1$

Eliminating t from $h = 2t - 1$ and $k = t^2 + 1$

We get $(h + 1)^2 = 4(k - 1)$

The required equation of reflection is $(x + 1)^2 = 4(y - 1)$.

This section contains four questions, each having two matching lists. Choices for the correct combination of elements from List–I and List–II are given as option (*A*), (*B*), (*C*), (*D*), out of which one is correct.

15.

	List-I		List-II
A.	Parabola $y^2 = 4x$ and the circle having its centre at (6, 5) intersects at right angle, at the point (a, a) then one value of a is equal to	p.	16
B.	The angle between the tangents drawn to $(y-2)^2 = 4(x+3)$ at the point where it is intersected by the line $3x - y + 8 = 0$ is $\frac{4\pi}{p}$, then p has the value equal to	q.	8
C.	Two perpendicular tangents PA and PB are drawn to parabola $y^2 = 16x$, then minimum value of AB is	r.	$10\sqrt{5}$
D.	Length of the normal chord of the parabola $y^2 = 8x$ at the point where abscissa and ordinate are equal is	s.	4

(*a*) A→q; B→s; C→s; D→q
(*b*) A→s; B→q; C→p; D→r
(*c*) A→q; B→r; C→s; D→q
(*d*) A→r; B→p; C→s; D→q

Sol. (*b*)
P. Equation of tangent of parabola $y^2 = 4x$ at (a, a)
$a.y = 2(x + a)$ is normal to the circle
$5a = 2(6 + a)$
$a = 4$

16. Consider the parabola $(x-1)^2 + (y-2)^2 = \frac{(12x-5y+3)^2}{169}$

	List-I		List-II
A.	Locus of point of intersection of perpendicular tangent	p.	$12x - 5y - 2 = 0$
B.	Locus of foot of perpendicular from focus upon any tangent	q.	$5x + 12y - 29 = 0$
C	Line along which minimum length of focal chord occurs	r.	$12x - 5y + 3 = 0$
D.	Line about which parabola is symmetrical	s.	$24x - 10y + 1 = 0$

(*a*) A→r; B→s; C→p; D→q
(*b*) A→r; B→p; C→s; D→q
(*c*) A→s; B→q; C→r; D→q
(*d*) A→r; B→p; C→s; D→q

Sol. (*a*) Locus of point of intersection of perpendicular tangent is directrix which is $12x - 5y + 3 = 0$.

Locus of foot of perpendicular from focus upon any tangent is tangent at the vertex, which is parallel to directrix and equidistant form directrix and latus rectum line, i.e. $12x - 5y + \lambda = 0$.

Where $\frac{|\lambda - 3|}{\sqrt{12^2+5^2}} = \frac{|\lambda+2|}{\sqrt{12^2+5^2}} \Rightarrow \lambda = \frac{1}{2}$

Hence, equation of tangent at vertex is $24x - 10y + 1 = 0$.

Minimum length of focal chord occurs along the latus rectum line, which is a line passing through the focus and parallel to directrix, i.e. $12x - 5y - 2 = 0$.

Parabola is symmetrical about its axis, which is a line passing through the focus (1, 2) and perpendicular to the directrix, which has equation $5x + 12y - 29 = 0$

17. Consider the parabola $y^2 = 12x$

	List-I		List-II
A.	Equation of tangent can be	p.	$2x + y - 6 = 0$
B.	Equation of normal can be	q.	$3x - y + 1 = 0$
C.	Equation of chord of contact w.r.t. any point on the directrix can be	r.	$x - 2y - 12 = 0$
D.	Equation of chord which subtends right angle at the vertex can be	s.	$2x - y - 36 = 0$

(*a*) A→r; B→p; C→s; D→q
(*b*) A→q; B→s; C→p; D→r
(*c*) A→q; B→r; C→s; D→q
(*d*) A→r; B→p; C→s; D→q

Sol. (*b*) Equation of tangent having slope m is $y = mx + \frac{3}{m}$.

Line $3x - y + 1 = 0$ is tangent for $m = 3$.
Equation of normal having slope m is $y = mx - 6m - 3m^3$.
Line $2x - y - 36 = 0$ is normal for $m = 2$
Chord of contact w.r.t. any point of the directrix is the focal chord which passes through the focus (3, 0).
Line $2x + y - 6 = 0$ passes through the focus.
Chords which subtends right angle at the vertex are concurrent at point $(4 \times 3, 0)$ or (12, 0).
Line $x - 2y - 12 = 0$ passes through the point (12, 0).

18.

	List-I		List-II
A.	The locus of the vertices of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is $xy = k$, then $[k]$, where $[\cdot]$ represent Greatest Integer Function	p.	3
B.	Find equation of parabola whose focus is (–6, –6) and vertex is (–2, –2). This equation consist constant term of $2n$ type. Find value of $n(n \in N)$.	q.	15

C.	A water jet from fountain reaches its maximum height of 4m at a distance 0.5m from vertical passing through point O of water outlet. Find height of jet above horizontal OX at distance of 0.75m from point O.	r.	7
D.	From a point A common tangents are drawn to circle $x^2 + y^2 = 2$ and parabola $y^2 = 8x$. Then area of quadrilateral formed by common tangents, chord of contact of point A, with respect to circle and parabola.	s.	1

(*a*) A→q; B→s; C→r; D→p

(*b*) A→r; B→s; C→s; D→q

(*c*) A→s; B→r; C→p; D→q

(*d*) A→r; B→p; C→s; D→q

Sol. (*c*)

P. $\dfrac{y}{a} = \left(\dfrac{ax}{\sqrt{3}} + \dfrac{\sqrt{3}}{4}\right)^2 - \dfrac{3}{16} - 2$

or $\left(x + \dfrac{3}{4a}\right)^2 = \dfrac{a^2}{3a}\left(y + \dfrac{35}{16}a\right)$

Vertex is $x = \dfrac{-3}{4a}, y = \dfrac{-35a}{16}$

Locus of the vertex is $xy = \dfrac{105}{64}$

Q. Focus (–6, –6) and vertex is (–2, –2)

Equation of SA is $y = x$

Equation of directrix $x + y = \lambda$

$2 + 2 = \lambda$

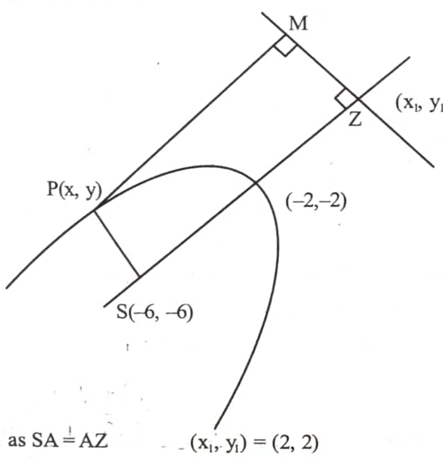
$\lambda = 4$

$SP = PM$

$\sqrt{(x+6)^2 + (y+6)^2} = \dfrac{|x + y - 4|}{\sqrt{2}}$

$(x - y)^2 + 32(x + y) + 128 = 0$

$128 = 27$ so $n = 7$



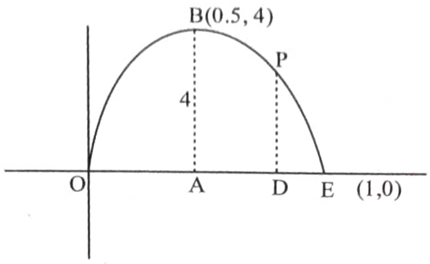
R. $c = 0$

$\dfrac{a}{4} + \dfrac{b}{2} = 4, a + b = 0$

$a = -16, b = 16$

$y = -16x^2 + 16x$

$y = -16\left(\dfrac{3}{4}\right)^2 + 16 \times \dfrac{3}{4} = 3m$



$y = ax^2 + bx + c$

$AE = OA = .5m$

$OE = 1m$

S. Here centre of circle to vertex of parabola and both circle and parabola are symmetrical about axis of parabola. In this care point of intersection of common tangents must lie on directrix and axis of parabola i.e. $A(-2, 0)$

Chord of contact wrt. $A(-a, 0)$ is $x(-2) + y(0) = 2$

$x = -1$

∴ Coordinates of R is (–1, 1) and chord of contact wrt (–2, 0)

$X = 2$

Coordinate of $p = (2, 4)$

Area = 2 (ar ΔPAS – ar ΔRAN)

$= 2\left[\dfrac{1}{2} \times 4 \times 4 - \dfrac{1}{2} \times 1 \times 1\right] = 15$

This section contains 8 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive)

19. Minimum area of circle which touches the parabola's $y = x^2 + 1$ and $y^2 = x - 1$ is A. Evaluate $\dfrac{32}{\pi}A$

Sol. [9] $y = x^2 + 1$ and $y^2 = x - 1$ are inverse relations of each other, their graphs are symmetric about $y = x$ and shortest distance between these occur along common normal i.e., a line ⊥ to parallel tangent of both curves i.e. at the point where tangent is parallel to $y = x$

∴ PQ is ⊥ to $y = x \Rightarrow$ slope of tangent at P) = 1

∴ Diameter of circle ⇒ $P\left(\dfrac{5}{4}, \dfrac{1}{2}\right), Q\left(\dfrac{1}{2}, \dfrac{5}{4}\right)$

$\Rightarrow r = \dfrac{3\sqrt{2}}{8}$

∴ Area of curve $\pi\left(\dfrac{3\sqrt{2}}{8}\right)^2 = \dfrac{9\pi}{32}$ square units.

20. The ends of a line segment are $P(1, 3)$ and $Q(1, 1)$. R is a point on the line segment PQ such that $PR : QR = 1 : \lambda$ $(\lambda > 0)$. If R is an interior point of the parabola $y^2 = 4x$, then evaluate $[\lambda]$ (where $[\cdot]$ denotes the greatest integer function)

Sol. [0] $R\left(1, \frac{1+3\lambda}{\lambda+1}\right)$. Since R is an interior point.

Therefore $S_1 < 0$

$$\Rightarrow \left(\frac{1+3\lambda}{\lambda+1}\right)^2 - 4 < 0$$

$$\Rightarrow \left(\lambda+\frac{3}{5}\right)(\lambda-1) < 0 \Rightarrow \lambda \in (0,1) \quad (\because \lambda > 0)$$

21. A tangent is drawn to the parabola $y^2 = 4x$ at the point 'P' whose abscissa lies in the interval $[1, 4]$. Maximum possible area of the triangle formed by the tangent at 'P', ordinate of the point 'P' and the x-axis is A. Evaluate $\left(\frac{A}{4}\right)$.

Sol. [4] Equation of tangent to parabola at $P(t)$ is given by

$ty = x + t^2$, $\tan\theta = \frac{1}{t}$

$\therefore$ Area of $\Delta APN = \Delta = \frac{1}{2}(AN)(PN) = \frac{1}{2}(2t^2)(2t)$

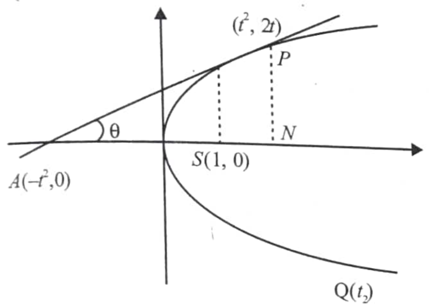
$\Delta = 2t^3 = 2(t^2)^{3/2}$

$\because t^2 \in [1, 4] \Rightarrow \Delta_{max}$ when $t^2 = 4$

$\Rightarrow \Delta_{max} = 16$

The maximum area of Δ is 16 square units

$\left(\frac{A}{4}\right) = 4$



22. If the normal to a parabola $y^2 = 4ax$ at P meets the curve again in Q and if PQ and the normal at Q makes angles α and β respectively with the x-axis then $\tan\alpha(\tan\alpha + \tan\beta) = -N$. Find N.

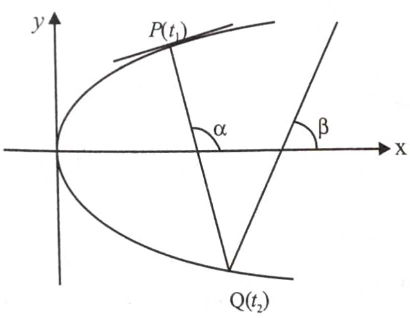
Sol. [2] $\tan\alpha = -t_1$ and $\tan\beta = -t_2$

also $t_2 = -t_1 - \frac{2}{t_1}$

$t_1t_2 + t_1^2 = -2$

$\tan\alpha\tan\beta + \tan^2\alpha = -2$

Thus $N = 2$.



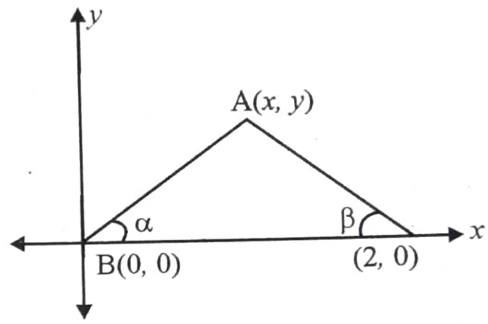
23. If on a given base BC, $B(0, 0)$ and $C(2, 0)$) a triangle be described such that the sum of the tangents of the base angles is 4 , then equation of locus of opposite vertex A is parabola whose directrix is $y = k$, then the value of k is

Sol. [8] Given $\tan\alpha + \tan\beta = 4$

$$\Rightarrow \frac{y}{x} + \frac{y}{2-x} = 4 \Rightarrow y = 2x(2-x)$$

$$\Rightarrow -\frac{y}{2} = x^2 - 2x = (x-1)^2 - 1 \Rightarrow (x-1)^2 = -\frac{1}{2}(y-2) \Rightarrow$$

Directrix $y - 2 = \frac{1}{8} \Rightarrow y = \frac{17}{8}$



24. An equilateral triangle ABC is inscribed in the parabola $y = x^2$ and one of the side of the equilateral triangle has the gradient 2. If the sum of x-coordinates of the vertices of the triangle is a rational in the form p/q where p and q are coprime, the find the value of $\frac{(p+q)}{2}$.

Sol. [7] $y = x^2$

To find $t_1 + t_2 + t_3 = ?$

$$m_1 = \frac{t_2^2 - t_1^2}{t_2 - t_1} = t_2 + t_1$$

Similarly, $m_2 = t_2 + t_3$ and $m_3 = t_3 + t_1$

Hence, $\sum t_i = \frac{m_1 + m_2 + m_3}{2}$

Now, $\tan 60° = \left|\frac{m-2}{1+2m}\right| \Rightarrow \pm\sqrt{3}(1+2m) = m-2$

Taking positive sign, we get $\sqrt{3}(1+2m) = m-2$

$$\Rightarrow m(2\sqrt{3}-1) = -(2+\sqrt{3}) \Rightarrow m = \frac{-(2+\sqrt{3})}{2\sqrt{3}-1}$$

Taking negative sign, we get $m - 2 = -2\sqrt{3}m - \sqrt{3}$

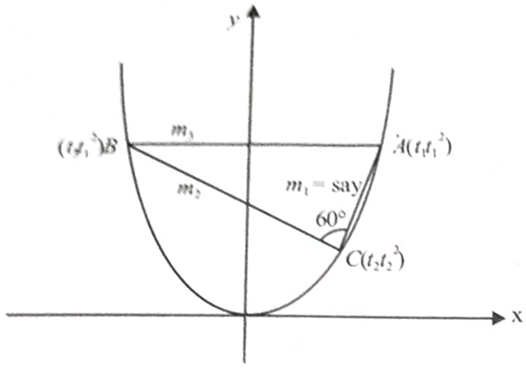
$$\Rightarrow m(2\sqrt{3}+1) = 2-\sqrt{3} \Rightarrow m = \frac{2-\sqrt{3}}{2\sqrt{3}+1}$$

$$\therefore m_1 = \frac{-(2+\sqrt{3})}{2\sqrt{3}-1},\ m_2 = \frac{2-\sqrt{3}}{2\sqrt{3}+1} \text{ and } m_3 = 2$$

$$\therefore \sum_{i=1}^{3} t_i = \frac{m_1+m_2+m_3}{2} = \frac{\frac{-(2+\sqrt{3})}{2\sqrt{3}-1} + \frac{2-\sqrt{3}}{2\sqrt{3}+1} + 2}{2}$$

$$= \frac{-4\sqrt{3}-2-6-\sqrt{3}+4\sqrt{3}-6-2+\sqrt{3}+22}{\frac{11}{2}}$$

$$= \frac{6}{22} = \frac{3}{11} = \frac{p}{q} \Rightarrow \frac{(p+q)}{2} = \frac{14}{2} = 7$$



25. Tangents are drawn from the point $(-1, 2)$ on the parabola $y^2 = 4x$. The length, these tangents will intercept on the line $x = 2$ is $A\sqrt{B}$. Find $A + B$.

Sol. [8] $SS_1 = T^2$

$(y^2 - 4x)\ (y_1^2 - 4x_1) = (yy_1 - 2(x + x_1))^2$

$(y^2 - 4x)\ (4 + 4) = [2y - 2(x - 1)]^2 = 4(y - x + 1)^2$

$2(y^2 - 4x) = (y - x + 1)^2$; solving with the line $x = 2$ we get,

$2(y^2 - 8) = (y - 1)^2$ or $2(y^2 - 8) = y^2 - 2y + 1$

Or $y^2 + 2y - 17 = 0$

Where $y_1 + y_2 = -2$ and $y_1y_2 = -17$

Now $|y_1 - y_2|^2 = (y_1 + y_2)^2 - 4y_1y_2$

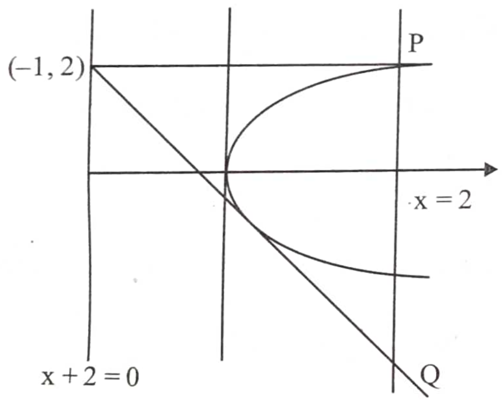
Or $|y_1 - y_2|^2 = 4 - 4(-17) = 72$

$\therefore\ (y_1 - y_2) = \sqrt{72} = 6\sqrt{2}$

Thus, $A = 6$

$B = 2$

$\Rightarrow A + B = 6 + 2 = 8$



26. Line $y = 2x - b$ cuts the parabola $y = x^2 - 4x$ at points A and B. Then the value of b for which the $\angle AOB$ (where O is the origin) is a right angle is.

Sol. [7] Line $y = 2x - b \Rightarrow 1 = \frac{2x - y}{b}$

Homogenising parabola with line

$$x^2 - 4x\left(\frac{2x-y}{b}\right) - y\left(\frac{2x-y}{b}\right) = 0$$

Since $\angle AOB = 90°$

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow 1 - \frac{8}{b} + \frac{1}{b} = 0 \Rightarrow b = 7.$$

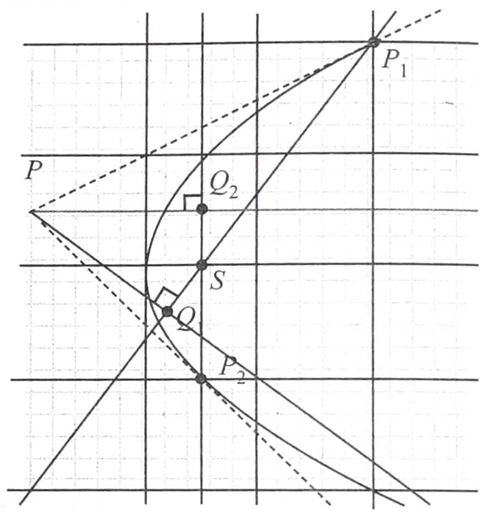
27. Consider the parabola $y^2 = 4x$. Let S be the focus of parabola. A pair of tangents drawn to the parabola from the point $P = (-2, 1)$ meet the parabola at P_1 and P_2. Let Q_1 and Q_2 be points on the lines SP_1 and SP_2 respectively such that PQ_1 is perpendicular to SP_1 and PQ_2 is perpendicular to SP_2. Then which of the following is/are TRUE?

(a) $SQ_1 = 2$ (b) $Q_1Q_2 = \frac{3\sqrt{10}}{5}$

(c) $PQ_1 = 3$ (d) $SQ_2 = 1$

Sol. (*b*, *c*, *d*)

$y^2 = 4x$ So $a = 1$



Let P_1 and P_2 be $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$

let P be pt. of intersection of tangents

$\Rightarrow P \equiv (t_1t_2, t_1 + t_2) \equiv (-2, 1)$

let t_1, t_2 be roots of $t^2 - t - 2 = 0$

So $t = 1, 2$

$\Rightarrow P_1(4, 4)$ and $P_2\,(1, -2)$

Slop of $SP_1 = \frac{4}{3}$ slope of SP_2 = WD

Equation of SP_1

$4x - 3y - 4 = 0$

Equation of PQ_1

$$y - 1 = -\frac{4}{3}(x + 2) \Rightarrow 3x + 4y + 2 = 0 \therefore Q_1\left(\frac{2}{5}, -\frac{4}{5}\right)$$

Equation of SP_2 $\quad x = 1$

Equation of PQ_2 $\quad y = 1 \quad \therefore Q_2(1, 1)$

$\therefore SQ_1 = 1, Q_1Q_2 = \dfrac{3\sqrt{10}}{5}$

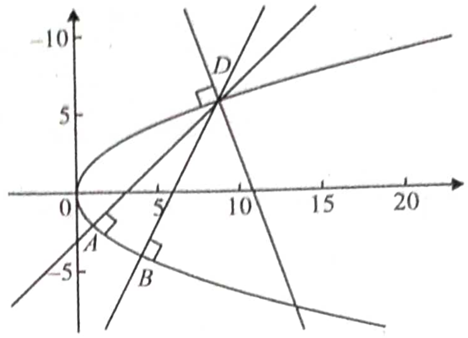
$$PQ_1 = \sqrt{\left(-2-\frac{2}{5}\right)^2+\left(1+\frac{4}{5}\right)^2} = \sqrt{\frac{144}{25}+\frac{81}{25}} = 3$$

and $SQ_2 = \sqrt{(1-1)^2+(1-0)^2} = 1$

28. $y^2 = 4x$, if A and B are $(1, -2)$ and $(4, -4)$ then, prove that normal drawn at A & B meet the parabola $y^2 = 4x$ and find C, D; where C, D are points of intersection of tangents and normals drawn at A and B respectively.

Sol. For $A(t_1)$, $t_1 = -1$,

$B(t_2)$, $t_2 = -2$ so $t_1t_2 = 2$, $a = 1$



So in figure D (9, 6) is the point of intersection of normals at the parabola and

$(at_1t_2, a(t_1 + t_2)) = (2, -3)$

So coordinates of C are (2, –3).

Exercise-1 (Topicwise)

DIFFERENT FORMS OF PARABOLA

1. The focus of the parabola $x^2 = -16y$ is
(a) (4, 0) (b) (0, 4)
(c) (-4, 0) (d) (0, -4)

2. If the vertex of a parabola be at origin and directrix be $x + 5 = 0$, then its latus rectum is
(a) 5 (b) 10
(c) 20 (d) 40

3. The latus rectum of a parabola whose directrix is $x + y - 2 = 0$ and focus is $(3, -4)$, is
(a) $-3\sqrt{2}$ (b) $3\sqrt{2}$
(c) $-3/\sqrt{2}$ (d) $3/\sqrt{2}$

4. If (2, 0) is the vertex and y-axis the directrix of a parabola, then its focus is
(a) (2, 0) (b) (–2, 0) (c) (4, 0) (d) (-4, 0)

5. If the parabola $y^2 = 4ax$ passes through (–3, 2), then length of its latus rectum is
(a) 2/3 (b) 1/3 (c) 4/3 (d) 4

6. The ends of latus rectum of parabola $x^2 + 8y = 0$ are
(a) (–4, –2) and (4, 2) (b) (4, –2) and (–4, 2)
(c) (–4, –2) and (4, –2) (d) (4, 2) and (–4, 2)

7. Focus and directrix of the parabola $x^2 = -8ay$ are
(a) $(0, -2a)$ and $y = 2a$ (b) $(0, 2a)$ and $y = -2a$
(c) $(2a, 0)$ and $x = -2a$ (d) $(-2a, 0)$ and $x = 2a$

8. The equation of latus rectum of a parabola is $x + y = 8$ and the equation of the tangent at the vertex is $x + y = 12$, then length of the latus rectum is
(a) $4\sqrt{2}$ (b) $2\sqrt{2}$ (c) 8 (d) $8\sqrt{2}$

9. The equation of the parabola whose vertex is (–1, –2), axis is vertical and which passes through the point (3, 6), is
(a) $x^2 + 2x - 2y - 3 = 0$ (b) $2x^2 = 3y$
(c) $x^2 - 2x - y + 3 = 0$ (d) $x^2 = 3y$

10. Axis of the parabola $x^2 - 4x - 3y + 10 = 0$ is
(a) $y + 2 = 0$ (b) $x + 2 = 0$
(c) $y - 2 = 0$ (d) $x - 2 = 0$

11. The equation of the latus rectum of the parabola represented by equation $y^2 + 2Ax + 2By + C = 0$ is
(a) $x = \dfrac{B^2 + A^2 - C}{2A}$ (b) $x = \dfrac{B^2 - A^2 + C}{2A}$
(c) $x = \dfrac{B^2 - A^2 - C}{2A}$ (d) $x = \dfrac{A^2 - B^2 - C}{2A}$

12. The equation of the locus of a point which moves so as to be at equal distances from the point $(a, 0)$ and the y-axis is
(a) $y^2 - 2ax + a^2 = 0$ (b) $y^2 + 2ax + a^2 = 0$
(c) $x^2 - 2ay + a^2 = 0$ (d) $x^2 + 2ay + a^2 = 0$

13. PQ is a double ordinate of the parabola $y^2 = 4ax$. The locus of the points of trisection of PQ is
(a) $9y^2 = 4ax$ (b) $9x^2 = 4ay$
(c) $9y^2 + 4ax = 0$ (d) $9x^2 + 4ay = 0$

14. The equation of the directrix of parabola $y^2 + 4y + 4x + 2 = 0$ is $x = a$ then find the value of a
(a) 2/3 (b) 5/3
(c) 3/2 (d) 5/2

15. If the vertex of the parabola $y = x^2 - 16x + k$ lies on x-axis, then the value of k is
(a) 32 (b) 53
(c) 64 (d) 52

16. Find the value of λ for which the equation $\lambda x^2 + 4xy + y^2 + \lambda x + 3y + 2 = 0$ represents a parabola
(a) 2 (b) 3 (c) 4 (d) 5

17. The points on the parabola $y^2 = 12x$ whose focal distance is 4, are
(a) $(2, \sqrt{3}), (2, -\sqrt{3})$ (b) $(1, 2\sqrt{3}), (1, -2\sqrt{3})$
(c) (1, 2) (d) (2, 3)

PARAMETRIC FORM

18. The parametric equation of the curve $y^2 = 8x$ are
(a) $x = t^2, y = 2t$ (b) $x = 2t^2, y = 4t$
(c) $x = 2t, y = 4t^2$ (d) $x = 4t, y = 2t^2$

19. The point $P(9/2, 6)$ lies on the parabola $y^2 = 4ax$, then parameter of the point P is
(a) 2/3 (b) 1/3
(c) 3/2 (d) 1/6

Focal Chord and Chord Passing Through Fixed Point

20. If 'a' and 'c' are the segments of a focal chord of a parabola and b the semi-latus rectum, then
(a) a, b, c are in $A.P.$ (b) a, b, c are in $G.P.$
(c) a, b, c are in $H.P.$ (d) a, b, c are in $A.G.P.$

21. The focal distance of a point P on the parabola $y^2 = 12x$, if the ordinate of P is 6, is
(a) 3 (b) 5
(c) 12 (d) 6

Tangent

22. The equation of the common tangent of the parabolas $x^2 = 108y$ and $y^2 = 32x$, is
(a) $2x+3y=36$ (b) $2x+3y+36=0$
(c) $3x+2y=36$ (d) $3x+2y+36=0$

23. The equation of a tangent to the parabola $y^2 = 4ax$ making an angle θ with x-axis is
(a) $y = x\cot\theta + a\tan\theta$
(b) $x = y\tan\theta + a\cot\theta$
(c) $y = x\tan\theta + a\cot\theta$
(d) $x = y\cot\theta + a\tan\theta$

24. The equation of the tangent to the parabola $y^2 = 4x + 5$ parallel to the line $y = 2x + 7$ is
(a) $2x-y-3=0$ (b) $2x-y+3=0$
(c) $2x+y+3=0$ (d) $x-2y+6=0$

25. The point of the contact of the tangent to the parabola $y^2 = 4ax$ which makes an angle of 60° with x-axis, is
(a) $\left(\frac{a}{3}, \frac{2a}{\sqrt{3}}\right)$ (b) $\left(\frac{2a}{\sqrt{3}}, \frac{a}{3}\right)$
(c) $\left(\frac{a}{\sqrt{3}}, \frac{2a}{3}\right)$ (d) $\left(\frac{2a}{3}, \frac{a}{\sqrt{3}}\right)$

26. The equation of the tangent at a point $P(t)$ where 't' is any parameter to the parabola $y^2 = 4ax$, is
(a) $yt = x + at^2$ (b) $y = xt + at^2$
(c) $y = xt + \frac{a}{t}$ (d) $y = tx$

27. The focus of the parabola is (1, 1) and the tangent at the vertex has the equation $x + y = 1$. Then, which is not true:
(a) Equation of the parabola is $(x-y)^2 = 2(x+y-1)$.
(b) Equation of the parabola is $(x-y)^2 = 4(x+y-1)$.
(c) The co-ordinates of the vertex are (1/2, 1/2).
(d) Length of the latus rectum is $2\sqrt{2}$.

28. Find the latus rectum of the parabola whose focus is (3, 4) and whose tangent at vertex has the equation $x + y = 7 + 5\sqrt{2}$.
(a) 10 (b) 20
(c) 30 (d) 40

Normal

29. The point on the parabola $y^2 = 8x$ at which the normal is inclined at 60° to the x-axis has the co-ordinates
(a) $(6, -4\sqrt{3})$ (b) $(6, 4\sqrt{3})$
(c) $(-6, -4\sqrt{3})$ (d) $(-6, 4\sqrt{3})$

30. The equation of normal to the parabola $y^2 = 4ax$ at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$, is
(a) $y = m^2x - 2mx - am^3$
(b) $m^3y = m^2x - 2am^2 - a$
(c) $m^3y = 2am^2 - m^2x + a$
(d) $y = m^2x + 2mx + am^2$

31. Equation of any normal to the parabola $y^2 = 4a(x-a)$ is
(a) $y = mx - 2am - am^3$
(b) $y = m(x+a) - 2am - am^3$
(c) $y = m(x-a) + \frac{a}{m}$
(d) $y = m(x-a) - 2am - am^3$

32. The normal to the parabola $y^2 = 8x$ at the point (2, 4) meets the parabola again at the point
(a) {–18, –12} (b) {–18, 12}
(c) {18, 12} (d) (18, –12)

33. The number of normal drawn to the parabola $y^2 = 4x$ from the point (1,0) is
(a) 3 (b) 0 (c) 2 (d) 1

34. At what point on the parabola $y^2 = 4x$ the normal makes equal angles with the axes?
(a) (4, 4) (b) (9, 6) (c) (4, – 1) (d) (1, 2)

35. The line $2x + y + \lambda = 0$ is a normal to the parabola $y^2 = -8x$, then λ is
(a) 12 (b) – 12 (c) 24 (d) – 24

Tangents from External Point

36. From the point (–1, –60) two tangents are drawn to the parabola $y^2 = 4x$. Then the angle between the two tangents is
(a) 30° (b) 45° (c) 60° (d) 90°

37. Tangent to the parabola $y = x^2 + 6$ at (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at the point
(a) (–6, –9) (b) (–13, –9)
(c) (–6, –7) (d) (13, 7)

38. The line $4x - 7y + 10 = 0$ intersects the parabola, $y^2 = 4x$ at the points A & B. The co-ordinates of the point of intersection of the tangents drawn at the points A & B are:
(a) $\left(\frac{7}{2}, \frac{5}{2}\right)$ (b) $\left(-\frac{5}{2}, -\frac{7}{2}\right)$
(c) $\left(\frac{5}{2}, \frac{7}{2}\right)$ (d) $\left(-\frac{7}{2}, -\frac{5}{2}\right)$

Chord of Contact and Common Chord

39. The length of chord of contact of the tangents drawn from the point (2, 5) to the parabola $y^2 = 8x$, is
(a) $\frac{1}{2}\sqrt{41}$ (b) $\sqrt{41}$
(c) $\frac{3}{2}\sqrt{41}$ (d) $2\sqrt{41}$

40. If the segment intercepted by the parabola $y^2 = 4ax$ with the line $lx + my + n = 0$ subtends a right angle at the vertex, then

(a) $4al + n = 0$ (b) $4al + 4am + n = 0$

(c) $4am + n = 0$ (d) $al + n = 0$

41. The focal chord to $y^2 = 16x$ is tangent to $(x-6)^2 + y^2 = 2$, then the possible value of the slope of this chord, are

(a) $\{-1, 1\}$ (b) $\{-2, 2\}$

(c) $\{-2, 1/2\}$ (d) $\{2, -1/2\}$

Chord with Given Mid Point

42. The locus of the middle points of the chords of the parabola $y^2 = 4ax$ which passes through the origin

(a) $y^2 = ax$ (b) $y^2 = 2ax$

(c) $y^2 = 4ax$ (d) $x^2 = 4ay$

43. If one end of a focal chord of the parabola $y^2 = 4x$ is (1, 2), the other end does not lies on

(a) $x^2 y + 2 = 0$ (b) $xy + 2 = 0$

(c) $xy - 2 = 0$ (d) $x^2 + xy - y - 1 = 0$

Locus

44. Locus of the point of intersection of the perpendicular tangents of the curve $y^2 + 4y - 6x - 2 = 0$ is

(a) $2x - 1 = 0$ (b) $2x + 3 = 0$

(c) $2y + 3 = 0$ (d) $2x + 5 = 0$

45. The locus of the middle points of the focal chords of the parabola, $y^2 = 4x$ is:

(a) $y^2 = x - 1$ (b) $y^2 = 2(x - 1)$

(c) $y^2 = 2(1 - x)$ (d) $y^2 = 2(x + 1)$

46. If S be the focus of a parabola and PQ be the focal chord, such that $SP = 3$ and $SQ = 6$, then the length of latus rectum of the parabola is

(a) 4 (b) 2

(c) 8 (d) 16

47. The length of a focal chord of the parabola $y^2 = 4ax$ making an angle θ with the axis of the parabola is

(a) $4a \operatorname{cosec}^2 \theta$ (b) $4a \sec^2 \theta$

(c) $a \operatorname{cosec}^2 \theta$ (d) None of these

Exercise-2 (Learning Plus)

1. The equation of the parabola whose focus is $(-3, 0)$ and the directrix is, $x + 5 = 0$ is:

(a) $y^2 = 4(x-4)$ (b) $y^2 = 2(x+4)$

(c) $y^2 = 4(x-3)$ (d) $y^2 = 4(x+4)$

2. Length of the latus rectum of the parabola $25[(x-2)^2 + (y-3)^2] = (3x - 4y + 7)^2$ is:

(a) 4 (b) 2

(c) 1/5 (d) 2/5

3. A parabola is drawn with its focus at (3, 4) and vertex at the focus of the parabola $y^2 - 12x - 4y + 4 = 0$. The equation of the parabola is:

(a) $x^2 - 6x - 8y + 25 = 0$

(b) $y^2 - 8x - 6y + 25 = 0$

(c) $x^2 - 6x + 8y - 25 = 0$

(d) $x^2 + 6x - 8y - 25 = 0$

4. The point of intersection of the curves whose parametric equations are $x = t^2 + 1, y = 2t$ and $x = 2s, y = 2/s$ is given by

(a) (1, –3) (b) (2, 2)

(c) (–2, 4) (d) (1, 2)

5. The length of the chord of the parabola, $y^2 = 12x$ passing through the vertex & making an angle of 60° with the axis of x is:

(a) 8 (b) 4

(c) 16/3 (d) 2

6. If $(t^2, 2t)$ is one end of a focal chord of the parabola $y^2 = 4x$ then the length of the focal chord will be

(a) $\left(t+\frac{1}{t}\right)^2$ (b) $\left(t+\frac{1}{t}\right)\sqrt{\left(t^2+\frac{1}{t^2}\right)}$

(c) $\left(t-\frac{1}{t}\right)\sqrt{\left(t^2+\frac{1}{t^2}\right)}$ (d) $\left(t-\frac{1}{t}\right)^2$

7. The normal chord of a parabola $y^2 = 4ax$ at the point $P(x_1, x_1)$ subtends a right angle at the

(a) focus (b) vertex

(c) end of the latus rectum (d) directrix

8. The latus rectum of a parabola whose focal chord is PSQ such that $SP = 3$ and $SQ = 2$ is given by:

(a) 24/5 (b) 12/5

(c) 6/5 (d) 9/5

9. The equation of the tangent to the parabola $y = (x-3)^2$ parallel to the chord joining the points (3, 0) and (4, 1) is:

(a) $2x - 2y + 6 = 0$ (b) $2y - 2x + 6 = 0$

(c) $4y - 4x + 11 = 0$ (d) $4x - 4y = 13$

10. An equation of a tangent common to the parabolas $y^2 = 4x$ and $x^2 = 4y$ is

(a) $x - y + 1 = 0$ (b) $x + y - 1 = 0$

(c) $x + y + 1 = 0$ (d) $y = 0$

11. Length of the normal chord of the parabola, $y^2 = 4x$, which makes an angle of $\frac{\pi}{4}$ with the axis of x is

(a) 8 (b) $8\sqrt{2}$
(c) 4 (d) $4\sqrt{2}$

12. The combined equation of pair of tangents from $(-2, 0)$ to the parabola, $y^2 = 8x$ is:
(a) $y = \pm(x-2)$ (b) $x = \pm(y-2)$
(c) $y = \pm(x+2)$ (d) $x = \pm(y+2)$

13. The tangents to the parabola $x = y^2 + c$ from origin are perpendicular then c is equal to
(a) 1/2 (b) 1 (c) 2 (d) 1/4

14. The locus of a point such that two tangents drawn from it to the parabola $y^2 = 4ax$ are such that the slope of one is double the other is
(a) $y^2 = \frac{9}{2}ax$ (b) $y^2 = \frac{9}{4}ax$
(c) $y^2 = 9ax$ (d) $x^2 = 4ay$

15. Tangents are drawn from the points on the line $x - y + 3 = 0$ to parabola $y^2 = 8x$. Then the variable chords of contact pass through a fixed point whose coordinates are
(a) (3, 2) (b) (2, 4)
(c) (3, 4) (d) (4, 1)

16. TP & TQ are tangents to the parabola, $y^2 = 4ax$ at P & Q. If the chord PQ passes through the fixed point $(-a, b)$ then the locus of T is
(a) $ay = 2b(x - b)$ (b) $bx = 2a(y - a)$
(c) $by = 2a(x - a)$ (d) $ax = 2b(y - b)$

17. If the tangent at the point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4a(x + b)$ at Q & R, then the mid point of QR is
(a) $(x_1 + b, y_1 + b)$ (b) $(x_1 - b, y_1 - b)$
(c) (x_1, y_1) (d) $(x_1 + b, y_1)$

18. From the focus of the parabola $y^2 = 8x$ as centre, a circle is described so that a common chord of the curves is equidistant from the vertex and focus of the parabola. The equation of the circle is
(a) $(x-2)^2 + y^2 = 3$ (b) $(x-2)^2 + y^2 = 9$
(c) $(x+2)^2 + y^2 = 9$ (d) $(x+2)^2 - y^2 = 9$

19. The feet of the perpendicular drawn from focus upon any tangent to the parabola, $y = x^2 - 2x - 3$ lies on
(a) $y + 4 = 0$ (b) $y = 0$
(c) $y = -2$ (d) $y + 1 = 0$

20. The equation of the circle drawn with the focus of the parabola $(x - 1)^2 - 8y = 0$ as its centre and touching the parabola at its vertex is
(a) $x^2 + y^2 - 4y = 0$
(b) $x^2 + y^2 - 4y + 1 = 0$
(c) $x^2 + y^2 - 2x - 4y = 0$
(d) $x^2 + y^2 - 2x - 4y + 1 = 0$

21. Two parabolas $y^2 = 4a(x - l_1)$ and $x^2 = 4a(y - l_2)$ always touch one another, the quantities l_1 and l_2 are both variable. Locus of their point of contact has the equation
(a) $xy = a^2$ (b) $xy = 2a^2$
(c) $xy = 4a^2$ (d) $xy = 3a^2$

22. Through the vertex O of the parabola $y^2 = 8x$, a perpendicular is drawn to any tangent meeting it at P and the parabola at Q. Find the value of $OP \cdot OQ$.
(a) 18 (b) 14
(c) 16 (d) 12

23. A tangent is drawn to the circle $x^2 + y^2 + 4x + 6y - 7 = 0$ at the point $(2, -1)$. The image of the tangent in the line $y = x + 2$ touches the parabola
(a) $y^2 = 5x$ (b) $5y^2 = x$
(c) $y^2 = -5x$ (d) $5y^2 = -x$

24. Coordinates of the point on the straight line $x + y = 4$, which is nearest to the parabola $y^2 = 4(x - 10)$ is
(a) $\left(\frac{17}{2}, -\frac{9}{2}\right)$ (b) (2, 2)
(c) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (d) (3, 5)

25. Which one of the following equations represented parametrically, represents equation to a parabolic profile?
(a) $x = 3\cot t;\ y = 4\sin t$
(b) $x^2 - 2 = -2\cos t;\ y = 4\cos^2\frac{t}{2}$
(c) $\sqrt{x} = \tan t; \sqrt{y} = \sec t$
(d) $x = \sqrt{1 - \sin t};\ y = \sin\frac{t}{2} + \cos\frac{t}{2}$

26. The condition that the line $\frac{x}{p} + \frac{y}{q} = 1$ be a normal to the parabola $y^2 = 4ax$ is
(a) $p^3 = 2ap^2 + aq^2$ (b) $p^3 = 2aq^2 + ap^2$
(c) $q^3 = 2ap^2 + aq^2$ (d) None of these

27. If $2x + y + k = 0$ is a normal to the parabola $y^2 = -8x$, then the value of k is
(a) −16 (b) −8
(c) −24 (d) 24

28. If $(t^2, 2t)$ is one end of a focal chord of the parabola, $y^2 = 4x$ then the length of the focal chord will be
(a) $\left(t + \frac{1}{t}\right)^2$ (b) $\left(t + \frac{1}{t}\right)\sqrt{\left(t^2 + \frac{1}{t^2}\right)}$
(c) $\left(t - \frac{1}{t}\right)\sqrt{\left(t^2 + \frac{1}{t^2}\right)}$ (d) None of these

29. A line passing through the point (21, 30) and normal to the curve $y = 2\sqrt{x}$ can have the slope

(a) –5 (b) 3

(c) –2 (d) 2

30. A parabola touches two given straight lines originating from a given point. The locus of mid-point of portion of any tangent. Which is intercepted between the given straight line is

(a) parabola (b) ellipse

(c) straight line (d) hyperbola

31. If $(a^2, a - 2)$ be a point interior to the region of the parabola $y^2 = 2x$ bounded by the chord joining the points (2, 2) and (8, –4), then the set of all possible real values of a is

(a) $a \in [-2, -2+\sqrt{2}]$

(b) $a \in (-2+2\sqrt{2}, 2)$

(c) $a \in (-\sqrt{2}, 0)$

(d) None of these

32. If (x_1, y_1); $r = 1, 2, 3, 4$ be the points of intersection of the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then

(a) $y_1 + y_2 + y_3 + y_4 = 0$

(b) $y_1 + y_2 - y_3 - y_4 = 0$

(c) $y_1 - y_2 + y_3 - y_4 = 0$

(d) $y_1 - y_2 - y_3 + y_4 = 0$

33. The curve represented by the equation $\sqrt{px} + \sqrt{qy} = 1$, where $p, q \in R, p, q > 0$ is

(a) A circle (b) A parabola

(c) An ellipse (d) A hyperbola

34. Suppose $OABC$ is a rectangle in the xy-plane where O is the origin and A, B lie on the parabola $y = x^2$. Then C must lie on the curve

(a) $y = x^2 + 2$ (b) $y = 2x^2 + 1$

(c) $y = -x^2 + 2$ (d) $y = -2x^2 + 1$

35. Let A be the vertex and L the length of the latus rectum of the parabola, $y^2 - 2y - 4x - 7 = 0$. The equation of the parabola with A as vertex $2L$ the length of the latus rectum and the axis at right angles to that of the given curve is:

(a) $x^2 + 4x + 8y - 4 = 0$

(b) $x^2 + 4x - 8y + 12 = 0$

(c) $x^2 + 4x + 8y + 12 = 0$

(d) $x^2 + 8x - 4y + 8 = 0$

36. The locus of the mid point of the focal radii of a variable point moving on the parabola, $y^2 = 4ax$ is a parabola whose

(a) Latus rectum is half the latus rectum of the original parabola.

(b) Vertex is $(a/2, 0)$.

(c) Directrix is y-axis.

(d) Focus has the co-ordinates $(a, 0)$.

37. Equation of a tangent to the parabola $y^2 = 12x$ which make an angle of 45° with line $y = 3x + 77$ is

(a) $2x - 4y + 3 = 0$ (b) $x - 2y + 12 = 0$

(c) $4x + 2y + 3 = 0$ (d) $2x + y - 12 = 0$

38. Two parabolas have the same focus. If their directrices are the x–axis & the y–axis respectively, then the slope of their common chord is:

(a) 1 (b) –1 (c) 4/3 (d) 3/4

39. Consider a circle with its centre lying on the focus of the parabola, $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle & the parabola is:

(a) $\left(\frac{p}{2}, p\right)$ (b) $\left(\frac{p}{2}, -p\right)$

(c) $\left(-\frac{p}{2}, p\right)$ (d) $\left(-\frac{p}{2}, -p\right)$

40. P is a point on the parabola $y^2 = 4ax$ $(a > 0)$ whose vertex is A. PA is produced to meet the directrix in D and M is the foot of the perpendicular from P on the directrix. If a circle is described on MD as a diameter then it intersects the x–axis at a point whose co–ordinates are:

(a) $(-3a, 0)$ (b) $(-a, 0)$

(c) $(-2a, 0)$ (d) $(a, 0)$

41. AB is a chord of the parabola $y^2 = 4ax$ joining $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$. Match the following

	Column-I		Column-II
A.	AB is a normal chord	p.	$t_2 = -t_1 + 2$
B.	AB is a focal chord	q.	$t_2 = -\frac{4}{t_1}$
C.	AB subtends 90° at (0, 0)	r.	$t_2 = -\frac{1}{t_1}$
D.	AB is inclined at 45° to the axis of parabola	s.	$t_2 = -t_1 - \frac{2}{t_1}$

(a) A → s; B → r; C → q; D → p

(b) A → r; B → q; C → s; D → p

(c) A → s; B → p; C → r; D → q

(d) A → p; B → q; C → s; D → r

42. The locus of midpoint of chord of the parabola $y^2 = 4ax$

	Column – I		Column – II
A.	which passes through focus, is	p.	$y = 0$
B.	which is normal, is	q.	$4a^2 = (4ax - y^2)(y^2 + 4a^2)$
C.	for which line joining origin to the extremeties of chord are equally inclined to the axis of the parabola is	r.	$y^2 = 2a(x - a)$
D.	whose length is 2, is	s.	$y^4 + 2a(2a - x)y^2 + 8a^4 = 0$

(a) A → s; B → r; C → q; D → p
(b) A → r; B → s; C → p; D → q
(c) A → r; B → p; C → s; D → q
(d) A → p; B → q; C → s; D → r

43. Minimum distance between the curves $y^2 = x - 1$ and $x^2 = y - 1$ is

44. If $y = 2$ is directrix and (0, 1) be the vertex of parabola $x^2 + \lambda y + \mu = 0$ then; $(\lambda - \mu) = ?$

45. Three normals are drawn from the point (7, 14) to the parabola $x^2 - 8x - 16y = 0$. Find the coordinates of the feet of the normals.

46. The normals at the points $(4a, -4a)$, $(9a, -6a)$ of the parabola $y^2 = 4ax$ meet in P. Find the equation of the third normals from P.

47. The circle $x^2 + y^2 + 2\lambda x = 0$, $\lambda \in R$, touches the parabola $y^2 = 4x$ externally. Then

48. A line AB makes intercepts of length a and b on the coordinate axes. Find the equation of the parabola passing through A, B and the origin, if AB is the shortest focal chord of the parabola.

49. Prove that the locus of the middle points of normal chords of the parabola $y^2 = 4ax$ is $\frac{y^2}{2a} + \frac{4a^2}{y^2} = x - 2a$.

50. Show that the common tangents to the parabola $y^2 = 4x$ and the circle $x^2 + y^2 + 2x = 0$ form an equilateral triangle.

51. Prove that the normal at $(am^2, -2am)$ to the parabola $y^2 = 4ax$ intersects the parabola again at an angle $\tan^{-1}\left|\frac{m}{2}\right|$.

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. If the straight line $3x + 4y = 24$ intersects the axes at A and B and the straight line $4x + 3y = 24$ intersects the axis at C and D then the number of parabolas possible to draw through A, B, C and D is
(a) 0 (b) 1
(c) 2 (d) 4

2. The image of the parabola $y^2 = x$ in the line $x + y + 4 = 0$ is
(a) $x^2 = -(y + 4)$ (b) $(x + 4)^2 = -(y + 4)$
(c) $(x + 4)^2 = -y$ (d) $(x - 4)^2 = y$

3. If at $x = 1$, $y = 2x$ is tangent to the parabola $y = ax^2 + bx + c$, then respective values of a, b, c are
(a) $\frac{1}{2}, 1, \frac{1}{2}$ (b) $1, \frac{1}{2}, \frac{1}{2}$
(c) $\frac{1}{2}, \frac{1}{2}, 1$ (d) $1, 1, \frac{1}{2}$

4. Two triangles ABC and DEF are formed by three tangents and the corresponding three normals to a prabola $y^2 = 4ax$ respectively. The orthocenter of ΔABC has coordinates (–5, 4). Then, distance of orthocenter of ΔDEF from axis of parabola is
(a) 5 units (b) 2 units
(c) 4 units (d) Cannot be determined

5. The points on the axis of the parabola $x^2 + 2x + 4y + 13 = 0$ from the where three distinct normals can be drawn are given by
(a) $(-1, k)$, $k \in (-1, 2)$ (b) (–1, 6)
(c) $(-1, k)$, $k \in (-\infty, -5)$ (d) (–1, 1)

6. $\min\left[(x_1 - x_2)^2 + \left(5 + \sqrt{1 - x_1^2} - \sqrt{4x_2}\right)^2\right]$ $\forall x_1, x_2 \in R$ is
(a) $4\sqrt{5} + 1$ (b) $4\sqrt{5} - 1$
(c) $\sqrt{5} + 1$ (d) $\sqrt{5} - 1$

7. Normals at two point (x_1, y_1) and (x_2, y_2) of parabola $y^2 = 4x$ meet again on the parabola where $x_1 + x_2 = 4$, then $|y_1 + y_2|$ equals to
(a) $\sqrt{2}$ (b) $2\sqrt{2}$
(c) $4\sqrt{2}$ (d) None of these

8. The end points of two normal chords of a parabola are concyclic, then the tangents at the feet of the normals will intersect at
(a) Tangent at vertex of the parabola
(b) Axis of the parabola
(c) Directrix of the parabola
(d) None of these

9. The equation of the circle touching the line $2x + 3y + 1 = 0$ at the point (1, –1) and passing through the focus of the parabola $y^2 = 4x$ is
(a) $3x^2 + 3y^2 - 8x + 3y + 5 = 0$
(b) $3x^2 + 3y^2 + 8x - 3y + 5 = 0$
(c) $x^2 + y^2 - 3x + y + 6 = 0$
(d) None of these

10. The tangent and normal at the point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meet the x-axis in T and G respectively, then the angle at which the tangent at P to the parabola is inclined to the tangent at P to the parabola is inclined to the tangent at P to the circle through P, T, G is

(a) $\tan^{-1}(t^2)$ (b) $\cot^{-1}(t^2)$
(c) $\tan^{-1}(t)$ (d) $\cot^{-1}(t)$

11. The chord $x + y = 1$ cute the curve $y^2 = 12x$ in points A and B. The normal at A and B intersect at C. A third line from C cuts the $y^2 = 12x$ normally at D. Then which of the following is/are correct
(a) Sum of slopes of normals at A and B is 3
(b) Coordinates of D are (12, 12)
(c) Tangent at D intersect y-axis at (0, 6)
(d) Coordinates of D are (12, –12)

12. If a parabola touches the lines $y = x$ and $y = -x$ at $A(3, 3)$ and $B(1, -1)$ respectively, then
(a) Equation of axis of parabola is $2x + y = 0$
(b) Slope of tangent at vertex is $\frac{1}{2}$
(c) Focus is $\left(\frac{6}{5}, -\frac{3}{5}\right)$
(d) Directrix passes through (1, –2)

13. P is a point on the parabola $y^2 = 4x$ where abscissa and ordinate are equal. Equation of a circle passing through the focus and touching the parabola at P is:
(a) $x^2 + y^2 - 13x + 2y + 12 = 0$
(b) $x^2 + y^2 - 3x - 18y + 2 = 0$
(c) $x^2 + y^2 + 13x - 2y - 14 = 0$
(d) $x^2 + y^2 - x = 0$

14. For a given parabola $y^2 = 4ax$ two variable chords PQ and RS at right angles are drawn through the fixed point $A(x_1, y_1)$ inside the parabola, making variable angles θ and α with x-axis. If r_1, r_2, r_3, r_4 are distances of P, Q, R and S from A, then the value of $\frac{1}{r_1r_2} + \frac{1}{r_3r_4}$
(a) Independent of θ
(b) Independent of α
(c) Depends upon both θ and α
(d) Is a constant

15. A circle 'S' is described on the focal chord of the parabola $y^2 = 4x$ as diameter. If the focal chord is inclined at an angle of 45° with axis of x, then which of the following is/are true
(a) Radius of the circle is 4
(b) Centre of the circle is (3, 2)
(c) The line $x + 1 = 0$ touches the circle
(d) The circle $x^2 + y^2 + 2x - 6y + 3 = 0$ is orthogonal to 'S'

16. PQ is a double ordinate of the parabola $y^2 = 4ax$. If the normal at P intersect the line passing through Q and parallel to x-axis at G then locus of G is a parabola with
(a) Length of latus rectum equal to $4a$
(b) Vertex at $(4a, 0)$
(c) Directrix as the line $x - 3a = 0$
(d) Focus at $(5a, 0)$

COMPREHENSION BASED QUESTIONS

Comprehension (Q. 17 to 19): Let $L_1 : 5x - y - 3 = 0$ and $L_2 : x + 5y - 11 = 0$ are tangents to a parabola which meets the parabola at A and B. Also normals at A and B intersect at $M(3, 4)$, where M lies on axis of the parabola.

17. The equation of the directrix of the parabola is
(a) $x + 2y = 5$ (b) $x + y = 3$
(c) $2x - y = 0$ (d) $3x - y = 1$

18. The length of latus rectum of the parabola is
(a) $2\sqrt{2}$ (b) 4 (c) $4\sqrt{2}$ (d) 8

19. The equation of the circle circumscribing the ΔAMB is
(a) $x^2 + y^2 - 3x - 5y + 8 = 0$
(b) $x^2 + y^2 - 5x - 6y + 14 = 0$
(c) $x^2 + y^2 - 4x - 6y + 11 = 0$
(d) $x^2 + y^2 - 6x - 7y + 20 = 0$

Comprehension (Q. 20 to 21): Let $<a_n>$ & $<b_n>$ be the arithmetic sequences each with common difference 2 such that $a_1 < b_1$ & let $c_n = \sum_{k=1}^{n} a_n, d_n = \sum_{k=1}^{n} b_k$: Suppose that the point $A_n(a_n, c_n)$, $B_n(b_n, d_n)$ are all lying on the parabola $C: y = px^2 + qx + r$ where p, q, r constants.

20. The value of p equals
(a) 1/4 (b) 1/3 (c) 1/2 (d) 2

21. If $r = 0$, then the value of a_1 & b_1 are
(a) $\frac{1}{2}$ & 1 (b) 1 & $\frac{3}{2}$ (c) 0 & 2 (d) $\frac{1}{2}$ & 2

MATCH THE COLUMN TYPE QUESTIONS

22. Match the following List-I with List-II.

	Column-I		Column-II
A.	The normal chord at a point t on the parabola $y^2 = 4x$ subtends a right angle at the vertex, then t^2 is	p	4
B.	The area of the triangle inscribed in the curve $y^2 = 4x$, the parameter of coordinates whose vertices are 1, 2 and 4 is	q	2
C.	The number of distinct normal possible from $\left(\frac{11}{4}, \frac{1}{4}\right)$ to the parabola $y^2 = 4x$ is	r	3/4
D.	The normal at $(a, 2a)$ on $y^2 = 4ax$ meets the curve again at $(at^2, 2at)$, then the value of $\|t - 1\|$ is	s	6

(a) A → q; B → r; C → q; D → p
(b) A → r; B → q; C → s; D → p
(c) A → s; B → p; C → r; D → q
(d) A → p; B → q; C → s; D → r

NUMERICAL BASED QUESTION

23. Find the locus of the points of intersection of tangents drawn at the ends of all normal chords of the parabola $y^2 = 8(x-1)$.

24. Prove that Ortho-centres of triangles formed by three tangents and corresponding normals to a parabola are equidstant from axis of parabola.

25. TP and TQ are tangents to the parabola $y^2 = 4dx$ and the normals at P and Q meet at a point R on the curve; prove that the centre of the circle circumscribing the triangle TPQ lies on the parabola $2y^2 = a(x-a)$.

26. A parabola is drawn to pass through A and B the ends of a diameter of a given circle of radius a, and to have as directrix a tangent to a concentric circle of radius b; then axes being AB and a perpendicular diameter, prove that the locus of the focus of the parabola is $\frac{x^2}{b^2}+\frac{y^2}{b^2-a^2}=1$.

27. A variable tangent to the parabola $y^2 = 4ax$ meets the circle $x^2+y^2=r^2$ at P&Q. Prove that the locus of the mid point of PQ is $x(x^2+y^2)+ay^2=0$.

28. Prove that the locus of intersections of tangents to the parabola $y^2 = 4ax$ which intercept a fixed length 'd' on the directrix is $(y^2-4ax)(x+a)^2 = d^2x^2$.

Exercise-4 (Past Year Questions)

JEE MAIN

1. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is: **(2014)**

   (a) $\frac{1}{8}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

2. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is **(2015)**

   (a) $x^2 = y$ (b) $y^2 = x$
   (c) $y^2 = 2x$ (d) $x^2 = 2y$

3. Tangent and normal aee drawn at $P(16, 16)$ $sq.$ units the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan\theta$ **(2018)**

   (a) 2 (b) 3
   (c) $\frac{4}{3}$ (d) $\frac{1}{2}$

4. Axis of a parabola lies along x-axis. If its vertex and focus are at distance 2 and 4 respectively from the origin, on the positive x-axis then which of the following points does not lie on it? **(2019)**

   (a) $(5, 2\sqrt{6})$ (b) $(8, 6)$
   (c) $(6, 4\sqrt{2})$ (d) $(4, -4)$

5. Equation of a common tangent to the circle, $x^2+y^2-6x=0$ and the parabola, $y^2 = 4x$, is: **(2019)**

   (a) $2\sqrt{3}y = 12x+1$ (b) $\sqrt{3}y = x+3$
   (c) $2\sqrt{3}y = -x-12$ (d) $\sqrt{3}y = 3x+1$

6. Let $A(4, -4)$ and $B(9, 6)$ be points on the parabola $y^2 = 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of ΔACB is maximum. Then, the area (in sq. units) of ΔACB, is: **(2019)**

   (a) $31\frac{3}{4}$ (b) 32
   (c) $30\frac{1}{2}$ (d) $31\frac{1}{4}$

7. The shortest distance between the point $\left(\frac{3}{2}, 0\right)$ and the curve $y = \sqrt{x}\,(x > 0)$, is: **(2019)**

   (a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{3}}{2}$
   (c) $\frac{3}{2}$ (d) $\frac{5}{4}$

8. If the parabolas $y^2 = 4b(x-c)$ and $y^2 = 8ax$ have three common normals, then which one of the following is a valid choice for the ordered triad (a, b, c)? **(2019)**

   (a) $\left(\frac{1}{2}, 2, 3\right)$ (b) $(1, 1, 3)$
   (c) $\left(\frac{1}{2}, 2, 0\right)$ (d) $(1, 1, 0)$

9. The length of the chord of the parabola $x^2 = 4y$ having equation $x - \sqrt{2}y + 4\sqrt{2} = 0$ is: **(2019)**

   (a) $3\sqrt{2}$ (b) $2\sqrt{11}$
   (c) $8\sqrt{2}$ (d) $6\sqrt{3}$

10. If the area of the triangle whose one vertex is the vertex of the parabola, $y^2 + 4(x - a^2) = 0$ and the other two vertices are the points of intersection of the parabola and y – axis, is 250 sq. units, then a value of 'a' is **(2019)**

    (a) $5\sqrt{5}$ (b) $5(2^{1/3})$
    (c) $(10)^{2/3}$ (d) 5

11. Equation of a common tangent to the parabola $y^2 = 4x$ and the hyperbola $xy = 2$ is **(2019)**
(a) $x + y + 1 = 0$ (b) $x - 2y + 4 = 0$
(c) $x + 2y + 4 = 0$ (d) $4x + 2y + 1 = 0$

12. The equation of a tangent to the parabola, $x^2 = 8y$, which makes an angle θ with the positive direction of x-axis, is **(2019)**
(a) $y = x\tan\theta + 2\cot\theta$ (b) $y = x\tan\theta - 2\cot\theta$
(c) $x = y\cot\theta + 2\tan\theta$ (d) $x = y\cot\theta - 2\tan\theta$

13. The shortest distance between the line $y = x$ and the curve $y^2 = x - 2$ is : **(2019)**
(a) $\frac{7}{4\sqrt{2}}$ (b) $\frac{7}{8}$
(c) $\frac{11}{4\sqrt{2}}$ (d) 2

14. The tangent to the parabola $y^2 = 4x$ at the point where it intersects the circle $x^2 + y^2 = 5$ in the first quadrant, passes through the point: **(2019)**
(a) $\left(-\frac{1}{3}, \frac{4}{3}\right)$ (b) $\left(-\frac{1}{4}, \frac{1}{2}\right)$
(c) $\left(\frac{3}{4}, \frac{7}{4}\right)$ (d) $\left(\frac{1}{4}, \frac{3}{4}\right)$

15. If one end of a focal chord of the parabola, $y^2 = 16x$ is at (1, 4), then the length of this focal chord is **(2019)**
(a) 25 (b) 24
(c) 20 (d) 22

16. If the tangent to the parabola $y^2 = x$ at a point (α, β), $(\beta > 0)$ is also a tangent to the ellipse, $x^2 + 2y^2 = 1$, then a is equal to: **(2019)**
(a) $2\sqrt{2}+1$ (b) $\sqrt{2}-1$
(c) $\sqrt{2}+1$ (d) $2\sqrt{2}-1$

17. The area (in sq. units) of the smaller of the two circles that touch the parabola, $y^2 = 4x$ at the point (1, 2) and the x-axis is :- **(2019)**
(a) $4\pi(2-\sqrt{2})$ (b) $8\pi(3-2\sqrt{2})$
(c) $4\pi(3+\sqrt{2})$ (d) $4\pi(3+\sqrt{2})$

18. If the line $ax + y = c$, touches both the curves $x^2 + y^2 = 1$ and $y^2 = 4x$, then $|c|$ is equal to: **(2019)**
(a) 1/2 (b) 2
(c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

19. Let P be the point of intersection of the common tangents to the parabola $y^2 = 12x$ and the hyperbola $8x^2 - y^2 = 8$. If S and denote the foci of the hyperbola where S lies on the positive x-axis then P divides SS' in a ratio: **(2019)**
(a) 5 : 4 (b) 14 : 13
(c) 2 : 1 (d) 13 : 11

20. The tangents to the curve $y = (x - 2)^2 - 1$ at its points of intersection with the line $x - y = 3$, intersect at the point: **(2019)**
(a) $\left(-\frac{5}{2}, 1\right)$ (b) $\left(-\frac{5}{2}, 1\right)$.
(c) $\left(\frac{5}{2}, -1\right)$ (d) $\left(\frac{5}{2}, 1\right)$

21. The equation of a common tangent to the curves, $y^2 = 16x$ and $xy = -4$ is: **(2019)**
(a) $x + y + 4 = 0$ (b) $x - 2y + 16 = 0$
(c) $2x - y + 2 = 0$ (d) $x - y + 4 = 0$

22. If $y = mx + 4$ is a tangent to both the parabolas, $y^2 = 4x$ and $x^2 = 2by$, then b is equal to **(2020)**
(a) -64 (b) -32
(c) -128 (d) 128

23. The locus of a point which divides the line joining the point $(0, -1)$ and a point on the parabola, $x^2 = 4y$, internally in the ratio 1:2, is **(2020)**
(a) $9x^2 - 12y = 8$ (b) $x^2 - 3y = 2$
(c) $9x^2 - 3y = 2$ (d) $4x^2 - 3y = 2$

24. Let a line $y = mx$ $(m > 0)$ intersect the parabola, $y^2 = x$ at a point P, other than the origin. Let the tangent to it at P meet the x-axis at the point Q. If area $(\Delta\, OPQ) = 4$ sq. units, then m is equal to **(2020)**

25. If one end of a focal chord AB of the parabola $y^2 = 8x$ is at $A\left(\frac{1}{2}, -2\right)$, then the equation of the tangent to it at B is: **(2020)**
(a) $x - 2y + 8 = 0$ (b) $x + 2y + 8 = 0$
(c) $2x - y - 24 = 0$ (d) $2x + y - 24 = 0$

26. The area (in sq. units) of an equalateral triangle inxcribed in the parabola $y^2 = 8x$, with one of its vertices on the vertex of this parabola, is : **(2020)**
(a) $64\sqrt{3}$ (b) $256\sqrt{3}$
(c) $128\sqrt{3}$ (d) $192\sqrt{3}$

27. If the tangent to the curve, $y = e^x$ at a point (c, e^c) and the normal to the parabola, $y^2 = 4x$ at the point (1, 2) intersect at the same point on the x-axis, then the value of c is ______ . **(2020)**

28. Let P be a point on the parabola, $y^2 = 12x$ and N be the foot of the perpendicular drawn from P on the axis of the parabola. A line is now drawn through the mid-point M of PN, parallel to its axis which meets the parabola at Q. If the y-intercept of the line NQ is 4/3 then: **(2020)**
(a) $MQ = \frac{1}{4}$ (b) $PN = 3$
(c) $PN = 4$ (d) $MQ = \frac{1}{3}$

29. If the common tangent to the parabolas, $y^2 = 4x$ and $x^2 = 4y$ also touches the circle, $x^2 + y^2 = c^2$, then c is equal to : **(2020)**

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$

(c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}}$

30. The centre of the circle passing through the point (0, 1) and touching the parabola $y = x^2$ at the point (2, 4) is : **(2020)**

(a) $\left(\frac{-53}{10}, \frac{16}{5}\right)$ (b) $\left(\frac{6}{5}, \frac{53}{10}\right)$

(c) $\left(\frac{-16}{5}, \frac{53}{10}\right)$ (d) $\left(\frac{3}{10}, \frac{16}{5}\right)$

31. Let L_1 be a tangent to the parabola $y^2 = 4(x + 1)$ and L_2 be a tangent to the parabola $y^2 = 8(x + 2)$ such that L_1 and L_2 intersect at right angles. Then L_1 and L_2 meet on the straight line: **(2020)**

(a) $2x + 1 = 0$ (b) $x + 3 = 0$

(c) $x + 2y = 0$ (d) $x + 2 = 0$

32. The locus of the mid-point of the line segment joining the focus of the parabola $y^2 = 4ax$ to a moving point of the parabola, is another parabola whose directrix is: **(2021)**

(a) $x = a$ (b) $x = 0$

(c) $x = -\frac{a}{2}$ (d) $\frac{a}{2}$

33. If P is a point on the parabola $y = x^2 + 4$ which is closest to the straight line $y = 4x - 1$, then the co-ordinates of P are: **(2021)**

(a) (–2, 8) (b) (1, 5)

(c) (3, 13) (d) (2, 8)

34. A tangent is drawn to the parabola $y^2 = 6x$ which is perpendicular to the line $2x + y = 1$. Which of the following points does NOT lie on it? **(2021)**

(a) (0, 3) (b) (–6, 0)

(c) (4, 5) (d) (5, 4)

35. A line is a common tangent to the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$. If the two points of contact (a, b) and (c, d) are distinct and lie in the first quadrant, then $2(a + c)$ is equal to **(2021)**

36. If the three normals drawn to the parabola, $y^2 = 2x$ pass through the point $(a, 0)$ $a \neq 0$, then 'a' must be greater than: **(2021)**

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) –1 (d) 1

37. Let $x^2 + y^2 + Ax + By + C = 0$ be a circle passing through (0, 6) and touching the parabola $y = x^2$ at (2, 4). Then $A + C$ is equal to ________. **(2022)**

(a) 16 (b) 88/5

(c) 72 (d) –8

38. If $y = m_1x + c_1$ and $y = m_2x + c_2$, $m_1 \neq m_2$ are two common tangents of circle $x^2 + y^2 = 2$ and parabola $y^2 = x$, then the value of $8|m_1m_2|$ is equal to **(2022)**

(a) $3+4\sqrt{2}$ (b) $-5+6\sqrt{2}$

(c) $-4+3\sqrt{2}$ (d) $7+6\sqrt{2}$

39. Let $x = 2t$, $y = \frac{t^2}{3}$ be a conic. Let S be the focus and B be the point on the axis of the conic such that $SA \perp BA$, where A is any point on the conic. If k is the ordinate of the centroid of ΔSAB, then $\lim_{t\to 1} k$ is equal to **(2022)**

(a) $\frac{17}{18}$ (b) $\frac{19}{18}$

(c) $\frac{11}{18}$ (d) $\frac{13}{18}$

40. If the line $y = 4 + kx$, $k > 0$, is the tangent to the parabola $y = x - x^2$ at the point P and V is the vertex of the parabola, then the slope of the line through P and V is **(2022)**

(a) $\frac{3}{2}$ (b) $\frac{26}{9}$

(c) $\frac{5}{2}$ (d) $\frac{23}{6}$

41. Let the normal at the point P on the parabola $y^2 = 6x$ pass through the point (5, –8). If the tangent at P to the parabola intersects its directrix at the point Q, then the ordinate of the point Q is **(2022)**

(a) –3 (b) $-\frac{9}{4}$

(c) $-\frac{5}{2}$ (d) –2

42. Let P_1 be a parabola with vertex (3, 2) and focus (4, 4) and P_2 be its mirror image with respect to the line $x + 2y = 6$. Then the directrix of P_2 is $x + 2y =$ ______. **(2022)**

43. A circle of radius 2 units passes through the vertex and the focus of the parabola $y^2 = 2x$ and touches the parabola $y = \left(x - \frac{1}{4}\right)^2 + \alpha$, where $\alpha > 0$. Then $(4\alpha - 8)^2$ is equal to ________. **(2022)**

44. Let l be a line which is normal to the curve $y = 2x^2 + x + 2$ at a point P on the curve. If the point $Q(6, 4)$ lies on the line l and O is origin, then the area of the triangle OPQ is equal to ________. **(2022)**

45. The sum of diameters of the circles that touch (i) the parabola $75x^2 = 64(5y - 3)$ at the point $\left(\frac{8}{5}, \frac{6}{5}\right)$ and (ii) the y-axis, is equal to ______. **(2022)**

46. Two tangent lines L_1 and L_2 are drawn from the point (2, 0) to the parabola $2y^2 = -x$. If the lines L_1 and L_2 are also tangent to the circle $(x - 5)^2 + y^2 = r$, then $17r$ is equal to ______. **(2022)**

JEE ADVANCED

47. Through the vertex O of parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles to one another. Show that for all positions of P, PQ cuts the axis of the parabola at a fixed point. Also, find the locus of the middle point of PQ. **(1994)**

48. Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4ax$ internally in the ratio 1 : 2 is a parabola. Find the vertex of this parabola. **(1995)**

49. From a point A common tangents are drawn to the circle $x^2 + y^2 = \frac{a^2}{2}$ and parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola. **(1996)**

50. Let S be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is. **(2012)**

Comprehension (Q. 51 to 53): Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$. **(2013)**

51. Length of chord PQ is

(a) $7a$ (b) $5a$ (c) $2a$ (d) $3a$

52. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta =$

(a) $\frac{2}{3}\sqrt{7}$ (b) $\frac{-2}{3}\sqrt{7}$ (c) $\frac{2}{3}\sqrt{5}$ (d) $\frac{-2}{3}\sqrt{5}$

53. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral $PQRS$ is **(2014)**

(a) 3 (b) 6
(c) 9 (d) 15

Comprehension (Q. 54 to 56): Let a, r, s, t be nonzero real numbers. Let $P(at^2, 2at)$, Q, R $(ar^2, 2ar)$ and $(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$ **(2014)**

54. The value of r is

(a) $-\frac{1}{t}$ (b) $\frac{t^2+1}{t}$ (c) $\frac{1}{t}$ (d) $\frac{t^2-1}{t}$

55. If $st = 1$, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

(a) $\frac{(t^2+1)^2}{2t^3}$ (b) $\frac{a(t^2+1)^2}{2t^3}$ (c) $\frac{a(t^2+1)^2}{t^3}$ (d) $\frac{a(t^2+2)^2}{t^3}$

56. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k) then which of the following is (are) possible value(s) of p, h and k? **(2017)**

(a) $p = -1, h = 1, k = -3$
(b) $p = 2, h = 3, k = -4$
(c) $p = -2, h = 2, k = -4$
(d) $p = 5, h = 4, k = -3$

Give the answer Q.57, Q.58 and Q.59 by appropriately matching the information given in the three columns of the following table.
Columns I, II and III contain conics, equations of tangents to the conics and points of contact, repectively. **(2017)**

Column-I	Column-II	Column-III
(I) $x^2 + y^2 = a^2$	(i) $my = m^2 x + a$	(P) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

57. For $a =$, if a tangent is drawn to a suitable conic (Column-I) at the point of contact $(-1, 1)$, then which of the following options is the only CORRET combination for obtaining its equation ?

(a) (I) (ii) (Q)
(b) (I) (i) (P)
(c) (III) (i) (P)
(d) (II) (ii) (Q)

58. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only CORRECT combination?

(a) (IV) (iv) (S) (b) (II) (iv) (R)
(c) (IV) (iii) (S) (d) (II) (iii) (R)

59. If a tangent to a suitable conic (Column 1) is found to be $y = x + 8$ and its point of contact is (8, 16), then which of the following options is the only CORRECT combination?

(a) (III) (i) (P)
(b) (I) (ii) (Q)
(c) (II) (iv) (R)
(d) (III) (ii) (Q)

60. Let E denote the parabola $y^2 = 8x$. Let $P = (-2, 4)$ and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is(are) True? **(2021)**

(a) The triangle PFQ is a right-angled triangle.
(b) The triangle QPQ' is a right-angle triangle.
(c) The distance between P and F is.
(d) F lies on the line joining Q and Q'.

61. Consider the parabola $y^2 = 4x$. Let S be the focus of the parabola. A pair of tangents drawn to the parabola from the point $P = (-2, 1)$ meet the parabola at P_1 and P_2. Let Q_1 and Q_2 be points on the lines SP_1 and SP_2 respectively such that PQ_1 is perpendicular to SP_1 and PQ_2 is perpendicular to SP_2. Then, which of the following is/are TRUE? **(2022)**

(a) $SQ_1 = 2$
(b) $Q_1Q_2 = \frac{3\sqrt{10}}{5}$
(c) $PQ_1 = 3$
(d) $SQ_2 = 1$

ANSWER KEY

CONCEPT APPLICATION

1. (*c*)	**2.** (*b*)	**3.** (*c*)	**4.** (*a*)	**5.** (*a*)	**6.** (*c*)	**7.** (*a*)	**8.** (*a*)	**9.** (*b*)	**10.** (*c*)
11. (*c*)	**12.** (*d*)	**13.** (*d*)	**14.** (*d*)	**15.** (*a*)	**16.** (*c*)	**17.** (*c*)	**18.** (*a*)	**19.** (*a*)	**20.** (*a*)
21. (*c*)	**22.** (*a*)	**23.** (*c*)	**24.** (*c*)	**25.** (*b*)	**26.** (*d*)	**27.** (*c*)	**28.** (*d*)	**29.** (*b*)	**30.** (*b*)
31. (*a*)	**32.** (*a*)	**33.** (*a*)	**34.** (*b*)	**35.** (*c*)	**36.** (*c*)	**37.** (*a*)	**38.** (*c*)		

EXERCISE-1 (TOPICWISE)

1. (*d*)	**2.** (*c*)	**3.** (*b*)	**4.** (*c*)	**5.** (*c*)	**6.** (*c*)	**7.** (*a*)	**8.** (*d*)	**9.** (*a*)	**10.** (*d*)
11. (*c*)	**12.** (*a*)	**13.** (*a*)	**14.** (*b*)	**15.** (*c*)	**16.** (*c*)	**17.** (*b*)	**18.** (*b*)	**19.** (*c*)	**20.** (*c*)
21. (*d*)	**22.** (*b*)	**23.** (*c*)	**24.** (*b*)	**25.** (*a*)	**26.** (*a*)	**27.** (*a*)	**28.** (*b*)	**29.** (*a*)	**30.** (*c*)
31. (*d*)	**32.** (*d*)	**33.** (*d*)	**34.** (*d*)	**35.** (*c*)	**36.** (*d*)	**37.** (*c*)	**38.** (*c*)	**39.** (*c*)	**40.** (*a*)
41. (*a*)	**42.** (*b*)	**43.** (*c*)	**44.** (*d*)	**45.** (*b*)	**46.** (*c*)	**47.** (*a*)			

EXERCISE-2 (LEARNING PLUS)

1. (*d*)	**2.** (*d*)	**3.** (*a*)	**4.** (*b*)	**5.** (*a*)	**6.** (*a*)	**7.** (*a*)	**8.** (*d*)	**9.** (*d*)	**10.** (*c*)
11. (*b*)	**12.** (*c*)	**13.** (*d*)	**14.** (*a*)	**15.** (*c*)	**16.** (*c*)	**17.** (*c*)	**18.** (*b*)	**19.** (*a*)	**20.** (*d*)
21. (*c*)	**22.** (*c*)	**23.** (*c*)	**24.** (*a*)	**25.** (*b*)	**26.** (*a*)	**27.** (*d*)	**28.** (*a*)	**29.** (*a*)	**30.** (*c*)
31. (*b*)	**32.** (*a*)	**33.** (*b*)	**34.** (*a*)	**35.** (*a, b*)	**36.** (*a, b, c, d*)		**37.** (*b, c*)	**38.** (*a, b*)	**39.** (*a, b*)

40. (*a, d*) **41.** (*a*) **42.** (*b*) **43.** $\left[\frac{3\sqrt{2}}{4}\right]$ **44.** [8] **45.** [(0,0), (–4, 3), (16, 8)] **46.** $5x + y = 135a$

47. [$\lambda > 0$] **48.** $\left[\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\frac{\left(\frac{x}{a}+\frac{y}{b}\pm\frac{(a\mp b)^2}{2ab}\right)^2}{\frac{1}{a^2}+\frac{1}{b^2}}\right]$

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (*b*)	**2.** (*b*)	**3.** (*a*)	**4.** (*c*)	**5.** (*c*)	**6.** (*b*)	**7.** (*c*)	**8.** (*b*)	**9.** (*a*)	**10.** (*c*)
11. (*b, c*)	**12.** (*c, d*)	**13.** (*a, d*)	**14.** (*a, b, d*)	**15.** (*a, c*)	**16.** (*a, b, c, d*)		**17.** (*b*)	**18.** (*a*)	**19.** (*c*)
20. (*a*)	**21.** (*c*)	**22.** (*a*)	**23.** $(x + 3)y^2 + 32 = 0$						

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (*c*)	**2.** (*d*)	**3.** (*a*)	**4.** (*b*)	**5.** (*b*)	**6.** (*d*)	**7.** (*a*)	**8.** [2]	**9.** (*d*)	**10.** (*d*)
11. (*c*)	**12.** (*c*)	**13.** (*a*)	**14.** (*c*)	**15.** (*a*)	**16.** (*c*)	**17.** (*b*)	**18.** (*c*)	**19.** (*a*)	**20.** (*c*)
21. (*d*)	**22.** (*c*)	**23.** (*a*)	**24.** (0.5)	**25.** (*a*)	**26.** (*d*)	**27.** (*d*)	**28.** (*a*)	**29.** (*d*)	**30.** (*c*)
31. (*b*)	**32.** (*b*)	**33.** (*d*)	**34.** (*d*)	**35.** (9)	**36.** (*d*)	**37.** [1]	**38.** [3]	**39.** [4]	**40.** [3]
41. [2]	**42.** [10]	**43.** [63]	**44.** [13]	**45.** [10]	**46.** [9]				

JEE Advanced

47. [$y^2 = 2x - 8$] **48.** $\left[\left(\frac{2}{9},\frac{8}{9}\right)\right]$ **49.** $\left[\frac{15a^2}{4}\right]$ **50.** (*d*) **51.** (*b*) **52.** (*d*) **53.** (*d*)

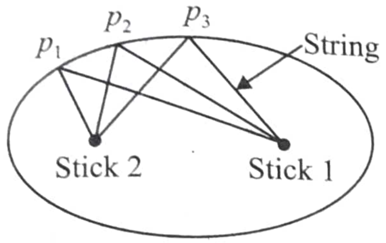
54. (*d*) **55.** (*b*) **56.** (*b*) **57.** (*a*) **58.** (*b*) **59.** (*a*) **60.** (*a, b, d*) **61.** (*a, b, c, d*)

CHAPTER

16 Ellipse

INTRODUCTION

It is an oval or a compressed circle. An ellipse can be drawn in the sand. In the sand, insert two sticks. Create a loop out of a piece of string large enough to fit around the two sticks and still leave some extra length. Pick up a moving third stick, hook it inside the string loop, tighten the loop by drawing the stick away from the first two sticks, and then drag that third stick as far as the loop will allow into the sand. An ellipse is the final shape that is traced in the sand.

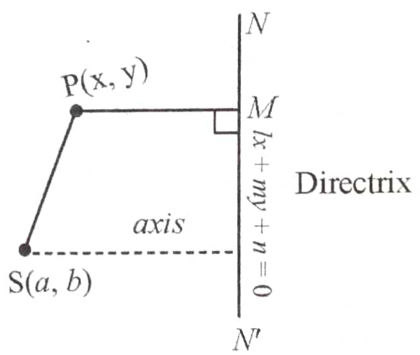


DEFINITION

An ellipse is the locus of the point which moves in a plane such that the ratio of its distance from a fixed point (focus) to fixed straight line (directrix) is always constant (called eccentricity) and is less than 1.

In the given figure, S is the focus and NN' is the directrix. Let P be a point on the ellipse, then

$\frac{PS}{PM} = e, \ e < 1$ (for ellipse)



GENERAL EQUATION OF AN ELLIPSE

We can find the equation of an ellipse when the coordinates of its focus S(a, b), equation of the directrix $lx + my + n = 0$ and eccentricity (e) are given.

Let $P(x, y)$ be any point on the ellipse. Then by definition.

$\Rightarrow \quad SP = e\,PM$ (e is the eccentricity)

$$\Rightarrow \quad (x-a)^2 + (y-b)^2 = e^2\frac{(lx+my+n)^2}{(l^2+m^2)}$$

$\Rightarrow \quad (l^2 + m^2)\ \{(x - a)^2 + (y - b)^2\} = e^2\{lx + my + n\}^2$ reduces to $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, in which $\Delta \neq 0$, $h^2 < ab$, if ($h = 0$, $a \neq b$)

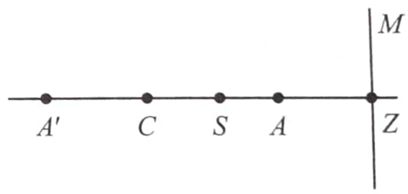
STANDARD EQUATION

Let S be the focus and ZM is the directrix of an ellipse. Draw perpendicular from S to the directrix which meet it at Z. A moving point is on the ellipse such that

$PS = ePM$

then there is point lies on the line SZ and which divide SZ internally at A and externally at A' in the ratio of $e : 1$.

Therefore, $\quad SA = e\,AZ$ ... (*i*)



$SA' = eA'Z$ ... (*ii*)

Let $AA' = 2a$ and take C as mid point of AA'

$\therefore \quad CA = CA' = a$

Add (*i*) and (*ii*)

$SA + SA' = e\,(AZ + A'Z)$

$\Rightarrow \quad AA' = e[CZ - CA + CA' + CZ]$

$2a = 2eCZ$

$\Rightarrow \quad CZ = \frac{a}{e}$ ... (iii)

Subtract (ii) and (i), we get

$SA' - SA = e(A'Z - AZ)$

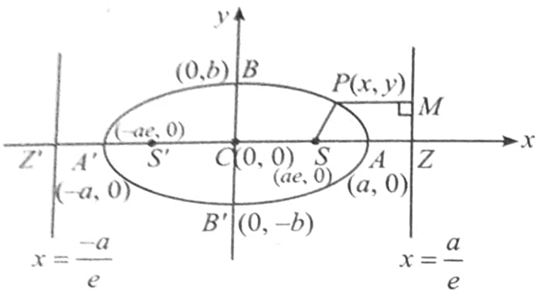
$\Rightarrow \quad (CA' + CS) - (CA - CS) = e[(CA' + CZ) - (CZ - CA)]$

$\Rightarrow \quad 2CS = 2e\,CA$

$\therefore \quad CS = ae$ ... (iv)

Result (iii) and (iv) are independent of axis.

Consider CZ line as x-axis, C as origin and perpendicular to this line and passes through C is considered as y-axis.



Let $P(x, y)$ is a moving point, then

By defintion of ellipse.

$PS = ePM$

$\Rightarrow \quad (PS)^2 = e^2 (PM)^2$

$\Rightarrow \quad (x - ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} - x\right)^2$

$\Rightarrow \quad (x - ae)^2 + y^2 = (a - ex)^2$

$\Rightarrow \quad x^2 + a^2e^2 - 2xae + y^2 = a^2 + e^2x^2 - 2xae$

$\Rightarrow \quad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$ or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

where $b^2 = a^2(1 - e^2)$

FACTS ABOUT AN ELLIPSE

1. By the symmetry of equation of ellipse, if we take second focus $S'(-ae, 0)$ and second directrix $x = -\frac{a}{e}$ and perform same calculation, we get same equation of ellipse, therefore there are two focii and two directrix of an ellipse. The two focii of ellipse are $(ae, 0)$ and $(-ae, 0)$ and the two corresponding directrices are lines $x = \frac{a}{e}$ and $x = \frac{-a}{e}$. If focus of the ellipse is taken as $(ae, 0)$, then corresponding directrix is $x = \frac{a}{e}$ and if focus is $(-ae, 0)$, then corresponding directrix is $x = \frac{-a}{e}$.
2. Distance between foci $SS' = 2ae$ and distance between directrix $ZZ' = 2\frac{a}{e}$.
3. Eccentricity is also known as Degree of flatness of an ellipse and written as

   $e = \frac{CS}{CA} = \frac{\text{Distance from centre to focus}}{\text{Distance from centre to vertex}}$

   $e^2 = 1 - \frac{b^2}{a^2}$

   If $e \to 0$

   $\Rightarrow b \to a$

   $\Rightarrow$ foci becomes closer and move towards centre and ellipse becomes circle.

   If $e \to 1$

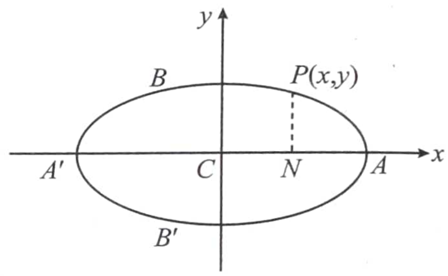
   $\Rightarrow b \to 0 \Rightarrow$ ellipse get thinner and thinner
4. Two ellipse are said to be similar if they have same eccentricity.
5. Distance of focus from the extremity of minor axis is equal to 'a' because $a^2e^2 + b^2 = a^2$
6. Let $P(x, y)$ be any point on the ellipse.

   $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

   $\therefore \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

   $\Rightarrow \quad \frac{y^2}{b^2} = \frac{(a-x)(a+x)}{a^2}$

   $\Rightarrow \quad \frac{PN^2}{b^2} = \frac{AN \cdot A'N}{a^2}$

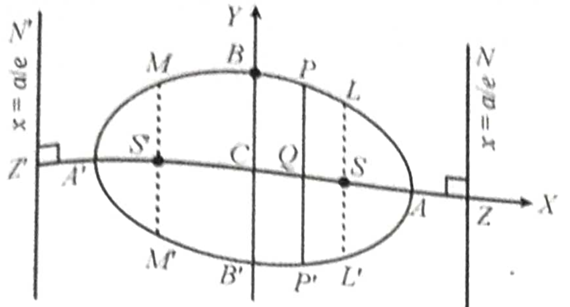


   $\Rightarrow \quad \frac{PN^2}{AN \cdot A'N} = \frac{b^2}{a^2}$
7. By definition of ellipse, the distance of any point P on the ellipse from focus = e (the distance of point P from the corresponding directrix).

BASIC TERMS RELATED TO AN ELLIPSE

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i)

1. **Centre:** In the figure, C is the centre of the ellipse. All chords passing through C are called diameter and bisected at C.
2. **Foci:** S and S' are the two foci of the ellipse and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively.



The line containing two foci are called the focal axis and the distance between S & S' the focal length

3. **Directrices:** ZN and $Z'N'$ are the two directrices of the ellipse and their equations are $x = \frac{a}{e}$ and $x = \frac{-a}{e}$ respectively. Here Z and Z' are called foot of directrix.

4. **Axes:** The line segments $A'A$ and $B'B$ are called the major and minor axes respectively of the ellipse.

   The point of intersection of major and minor axis is called **centre** of the ellipse. Major and minor axis together are called **principal axis** of ellipse.

   Here Semi-major axis are $CA = CA' = a$

   and Semi-minor axis are $CB = CB' = b$

5. **Vertex:** The points where major axis meet the ellipse is called its vertices. In the given figure, A' and A are the vertices of the ellipse.

6. **Ordinate and Double Ordinates:** Let P be a point on the ellipse. From P we draw PQ perpendicular to major axis of the ellipse. Produce PQ to meet the ellipse at P', then PQ is called an ordinate and PQP' is called the double ordinate of the point P. It is also defined as any chord perpendicular to major axis is called its double ordinate.

7. **Latus Rectum:** When double ordinate passes through focus then it is called the Latus rectum.

   Here LL' and MM' are called latus rectum.

   Let $L'L = 2k$, then $LS = k$ so $L = (ae, k)$.

   Since $L\ (ae, k)$ lies on the ellipse (1), therefore $\frac{a^2e^2}{a^2} + \frac{k^2}{b^2} = 1$

   or $\frac{k^2}{b^2} = 1 - e^2$

   or $k^2 = b^2(1 - e^2)$

   $= b^2 \cdot \frac{b^2}{a^2} = \frac{b^4}{a^2}$ $[\because b^2 = a^2(1 - e^2)]$

   $\therefore k = \frac{b^2}{a}$

   Hence length of semi latus rectum LS $= \frac{b^2}{a} = MS'$

   i.e. length of the latus rectum LL'

   or $MM' = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}}$

   $= 2a(1 - e^2)$

   $= 2e$ (distance from focus to the corresponding directrix).

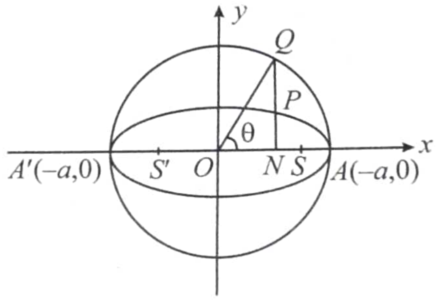
   and the end points of latus rectum are $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, \frac{-b^2}{a}\right)$, $M\left(-ae, \frac{b^2}{a}\right)$ and $M'\left(-ae, -\frac{b^2}{a}\right)$

8. **Focal Chord:** A chord of the ellipse passing through its focus is called a focal chord.

AUXILIARY CIRCLE & ECCENTRIC ANGLE

A circle described on major axis of ellipse as diameter is called the **auxiliary circle**.

Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that line through Q perpendicular to the x–axis on the way intersects the ellipse at P, then P and Q are called as the **Corresponding Points** on the ellipse and the auxiliary circle respectively. 'θ' is called the **Eccentric Angle** of the point P on the ellipse $(-\pi < \theta \leq \pi)$.



$Q \equiv (a\cos\theta, a\sin\theta)$

$P \equiv (a\cos\theta, b\sin\theta)$

Note that:

$$\frac{\ell(PN)}{\ell(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$$

Note:

If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle.

Here $x = a\cos\theta$, $y = b\sin\theta$ is called as **parametric equation** of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Train Your Brain

Example 1: Find the equation to the ellipse whose focus is the point $(-1, 1)$ whose directrix is the straight line $x - y + 3 = 0$ and eccentricity is $1/2$.

Sol. Let $P \equiv (h, k)$ be moving point $e = \frac{PS}{PM} = \frac{1}{2}$

$$\Rightarrow (h+1)^2+(k-1)^2=\frac{1}{4}\left(\frac{h-k+3}{\sqrt{2}}\right)^2$$

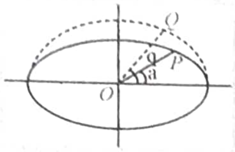
$\Rightarrow$ locus of $P(h, k)$ is

$8(x^2+y^2+2x-2y+2)=(x^2+y^2-2xy+6x-6y+9)$

$7x^2+7y^2+2xy+10x-10y+7=0$

Example 2: Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1=1$ whose radius makes angle α with x-axis.

Sol. Let $P\equiv(a\cos\theta, b\sin\theta)$



$\therefore\ m_{(op)}=\frac{b}{a}\tan\theta=\tan\alpha$

$\Rightarrow \tan\theta=\frac{a}{b}\tan\alpha$

$$OP=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$$

$$=\sqrt{\frac{a^2+b^2\tan^2\theta}{\sec^2\theta}}$$

$$=\sqrt{\frac{a^2+b^2\tan^2\theta}{1+\tan^2\theta}}=\sqrt{\frac{a^2+b^2\times\frac{a^2}{b^2}\tan^2\alpha}{1+\frac{a^2}{b^2}\tan^2\alpha}}$$

$$OP=\frac{ab}{\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}}$$

Example 3: Find the equation of the ellipse (in standard form) having latus rectum 5 and eccentriciy 2/3

Sol. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a>b$.

Latus rectum $=5=\frac{2b^2}{a}\Rightarrow 2b^2=5a$, ...(*i*)

Also $b^2=a^2(1-e^2)=a^2\left(1-\frac{4}{9}\right)=\frac{5a^2}{9}$

$\Rightarrow \frac{5a}{2}=\frac{5a^2}{9}\Rightarrow a=\frac{9}{2}$ and hence $b^2=\frac{5}{2}a=\frac{45}{4}$.

The equation of the ellipse, in the standard form is

$\frac{x^2}{81/4}+\frac{y^2}{45/4}=1.$

Concept Application

1. Find the equation of the ellipse whose focus is (1, 0) the directrix is $x+y+1=0$ and eccentricity is $\frac{1}{\sqrt{2}}$.

2. If the eccentricity of the ellipse $\frac{x^2}{\alpha^2+1}+\frac{y^2}{\alpha^2+2}=1$ is $\frac{1}{\sqrt{6}}$, then the latus rectum of the ellipse is

   (*a*) $\frac{5}{\sqrt{6}}$ (*b*) $\frac{10}{\sqrt{6}}$

   (*c*) $\frac{8}{\sqrt{6}}$ (*d*) None of these

3. $x^2+2\lambda xy+y^2+2x+2y+4=0$ represents an ellipse if

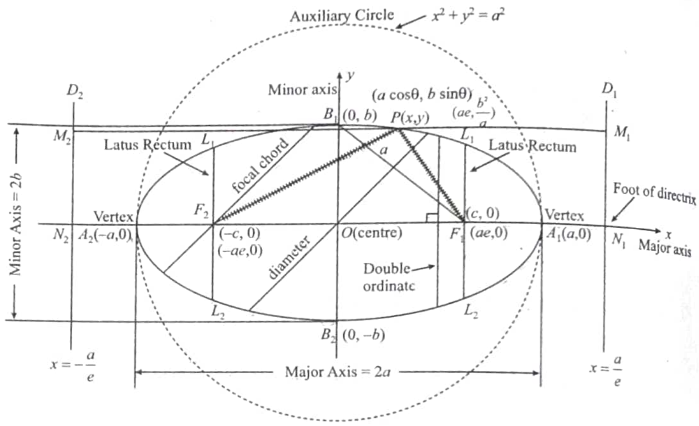
   (*a*) $-1\le l\le 1$ (*b*) $-1<\lambda<1$

   (*c*) $\lambda<-1$ or $\lambda>1$ (*d*) $\lambda\in(-1,1)-\left\{\frac{1}{2}\right\}$

4. The positive difference between the lengths of the latus recta of $3y=x^2+4x-9$ and $x^2+4y^2-6x+16y=24$ is

   (*a*) 1/2 (*b*) 2 (*c*) 3/2 (*d*) 5/2

ELLIPSE AT A GLANCE

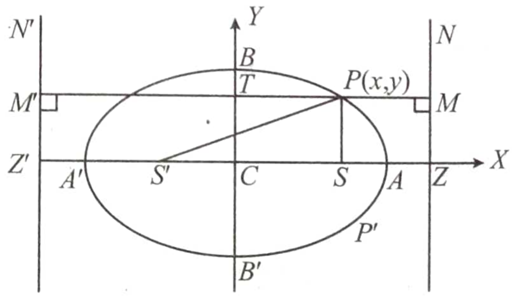


FOCAL DISTANCE OF A POINT

Let $P(x, y)$ be any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad ...(i)$$

Then by definition of ellipse,



$SP = ePM = e(MT - PT)$

$= e\left(\dfrac{a}{e} - x\right) = a - ex$

and $S'P = ePM' = e(M'T + PT)$

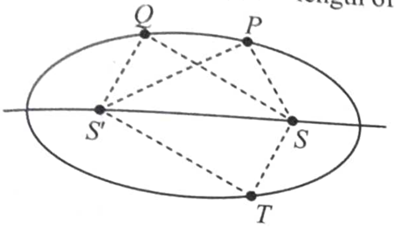
$= e\left(\dfrac{a}{e} + x\right) = a + ex$

Hence $SP + S'P = 2a$

Because of the above property, ellipse is also defined as the locus of a point which moves in a plane such that the sum of its distance from two fixed points (called foci) is a constant (Length of major axis).

This defintion is called the physical definition of the ellipse.

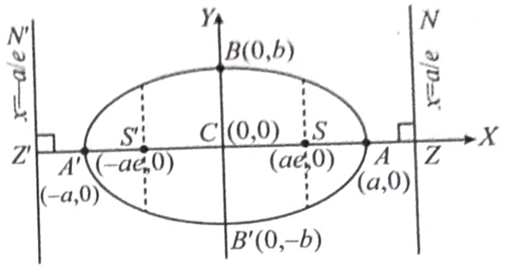
Hence $PS + PS' = QS + QS' = TS + TS'$ = length of major axis



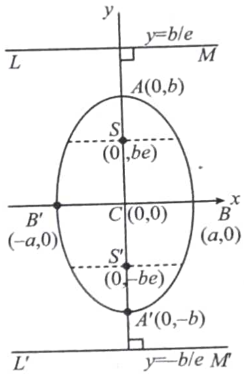
TWO STANDARD FORMS OF ELLIPSE

There are two standard forms of ellipse with centre at the origin and axes along coordinate axes. The foci of the ellipse are either on the x-axis or on the y-axis.

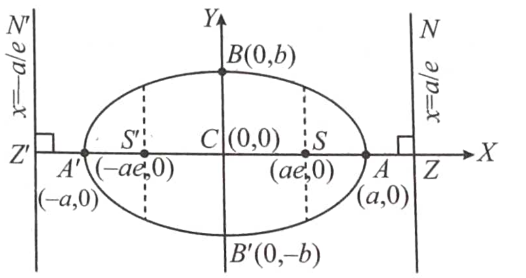
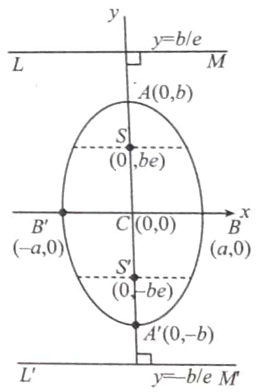
1. **Major axis along x–axis:** The equation of this type of ellipse is of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > b > 0$ and $b = a\sqrt{1-e^2}$.



2. **Major axis along y–axis:** The equation of this type of ellipse is of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. where $0 < a < b$ and $a = b\sqrt{1-e^2}$.



COMPARISON CHART BETWEEN STANDARD ELLIPSE

Basic Elements	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	
	$a > b$	$a < b$
		
Centre	(0, 0)	(0, 0)
Vertex	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$

Equation of directrix	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Relation among a, b, and e	$b^2 = a^2(1 - e^2)$	$a^2 = b^2(1 - e^2)$
Equation of major axis	$y = 0$	$x = 0$
Equation of minor axis	$x = 0$	$y = 0$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
End of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Focal distances of $P(x_1, y_1)$	$a \pm ex_1$	$b \pm ey_1$
$SP + SP'$	$2a$	$2b$
Distance between foci	$2ae$	$2be$
Distance between directrix	$\frac{2a}{e}$	$\frac{2b}{e}$
Parametric equation	$(a\cos\theta, b\sin\theta)\ (0 < \theta < 2\pi)$	$(a\cos\theta, b\sin\theta)$

If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and nothing is mentioned then the rule is to assume that $a > b$.

To find the Various Parameter of an Ellipse

Equation of an ellipse whose axis are parallel to coordiante axis & its centre is (h, k). The foci of the ellipse are either on x-axis or on the y-axis.

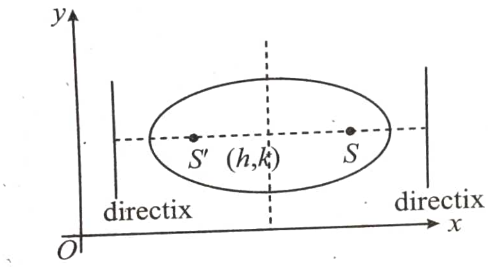
(I) Major axis Parallel to x–axis

Here the equation of ellipse is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$,

where $a > b$ and $b^2 = a^2(1 - e^2)$

Here the equation of the ellipse is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $X = x - h$ and $Y = y - k$.

1. Equation of major axis is $Y = 0$, i.e., $y - k = 0$



Equation of minor axis is $X = 0$, i.e., $x - h = 0$

2. Coordinate of centre of the ellipse are given by $X = 0$ and $Y = 0$ i.e, $x - h = 0$ and $y - k = 0$

∴ Centre of the ellipse is (h, k)

3. Coordinate of foci of the ellipse are given by $X = \pm ae$, $Y = 0$ i.e., $x - h = \pm ae$ and $y - k = 0$

∴ Hence foci of the ellipse are $(h \pm ae, k)$

4. Equation of the directrices of the ellipse are $X = \pm\frac{a}{e}$, i.e., $x - h = \pm\frac{a}{e}$.

Thus directrices are $x = h \pm \frac{a}{e}$

5. Coordinate of ends of latus rectum are given by $X = \pm ae$, $Y = \pm\frac{b^2}{a}$ i.e. $x - h = \pm ae$, $y - k = \pm\frac{b^2}{a}$

Therefore ends of latus rectum are given by $\left(h \pm ae, k \pm \frac{b^2}{a}\right)$

6. Coordinate of vertices of the ellipse are given by $X = \pm a$, $Y = 0$ i.e., $x - h = \pm a$, $y - k = 0$.

Hence vertices are $(h \pm a, k)$

(II) Major axis Parallel to y-axis:

Here the equation of ellipse is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$,

where $a < b$ and $a^2 = b^2(1 - e^2)$

Equation (1) is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $X = x - h$ and $Y = y - k$

1. Equation of major axis is $X = 0$, i.e., $x - h = 0$

Equation of minor axis is $Y = 0$, i.e., $y - k = 0$

2. Coordinate of centre of the ellipse are given by $X = 0$ and $Y = 0$

$\Rightarrow \quad x - h = 0$ and $y - k = 0$

$\therefore$ Centre of the ellipse is (h, k)

3. Coordinate of foci of the ellipse are given by $X = 0$, $Y = \pm$ be

$x - h = 0$ and $y - k = \pm be$

$x = h$ and $y = k \pm be$

$\therefore$ Foci are $(h, k \pm be)$

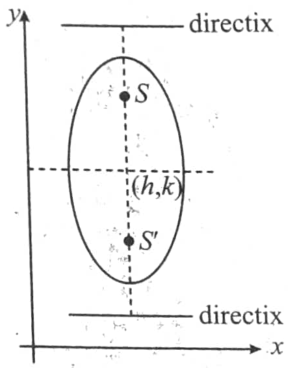
4. Equation of the directrices of the ellipse are $Y = \pm\frac{b}{e}$,

i.e,. $y - k = \pm\frac{b}{e}$

Thus directrices are $y = k \pm \frac{b}{e}$

5. Coordinate of ends of latera recta are given by

$X = \pm\frac{a^2}{b}$ and $Y = \pm ae$,



i.e. $x - h = \pm\frac{a^2}{b}$ and $y - k = \pm be$,

or $\quad x = h \pm \frac{a^2}{b}$ and $y = k \pm be$

$\therefore$ Coordinates of ends of latera recta are given by $\left(h \pm \frac{a^2}{b}, k \pm be\right)$

6. Coordinates of vertices of the ellipse is given by $X = 0$ and $Y = \pm b$ i.e., $x - h = 0$ and $y - k = \pm b$ therefore coordinates of vertex are $(h, k \pm b)$

Comparison Chart between above two Ellipse

	Basic Elements	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	
		$a > b$	$a < b$
1.	Length of major axis	$2a$	$2b$
	Length of minor axis	$2b$	$2a$
2.	Equation of major axis	$y - k = 0$	$x - h = 0$
	Equation of minor axis	$x - h = 0$	$y - k = 0$
3.	Centre of ellipse	(h, k)	(h, k)
4.	Eccentricity	$e = \sqrt{1-\frac{b^2}{a^2}}$	$e = \sqrt{1-\frac{a^2}{b^2}}$
5.	Foci	$(h \pm ae, k)$	$(h, k \pm be)$
6.	Equation of directrix	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{b}{e}$
7.	Extremities of latus rectum	$\left(h \pm ae, k \pm \frac{b^2}{a}\right)$	$\left(h \pm \frac{a^2}{b}, k \pm be\right)$
8.	Vertices of an ellipse	$(h \pm a, k)$	$(h, k \pm b)$
9.	Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$

Train Your Brain

Example 4: If latus rectum of an ellipse is half of its minor axis, then its eccentricity is

Sol. As given $\frac{2b^2}{a} = b \Rightarrow 2b = a \Rightarrow 4b^2 = a^2$

$\Rightarrow 4a^2(1 - e^2) = a^2 \Rightarrow 1 - e^2 = 1/4 \therefore e = \sqrt{3}/2$

Example 5: Find the equation of the ellipse whose foci are $(2, 3)$, $(-2, 3)$ and whose semi minor axis is of length $\sqrt{5}$.

Sol. Here S is $(2, 3)$ and S' is $(-2, 3)$ and $b = \sqrt{5}$

$\Rightarrow SS' = 4 = 2ae \Rightarrow ae = 2$

but $b^2 = a^2(1 - e^2) \Rightarrow 5 = a^2 - 4 \Rightarrow a = 3$.

Hence the equation to major axis is $y = 3$

Centre of ellipse is midpoint of SS' i.e. $(0, 3)$

$\therefore$ Equation to ellipse is $\frac{x^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$

or $\quad \frac{x^2}{9} + \frac{(y-3)^2}{5} = 1$

Example 6: Find the equation to the ellipse whose centre is origin, axes are the axes of co-ordinate and passes through the points $(2, 2)$ and $(3, 1)$.

Sol. Let the equation to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since it passes through the points $(2, 2)$ and $(3, 1)$

$\therefore \quad \frac{4}{a^2} + \frac{4}{b^2} = 1 \quad \ldots (i)$

and $\frac{9}{a^2} + \frac{1}{b^2} = 1 \quad \ldots (ii)$

from $(i) - 4\,(ii)$, we get

$$\frac{4-36}{a^2} = 1-4 \Rightarrow a^2 = \frac{32}{3}$$

from (i), we get

$$\frac{1}{b^2} = \frac{1}{4} - \frac{3}{32} = \frac{8-3}{32} \Rightarrow b^2 = \frac{32}{5}$$

$\therefore$ Ellipse is $3x^2 + 5y^2 = 32$

Example 7: Find the equation of the ellipse whose focii are (4, 0) and (– 4, 0) and eccentricity is 1/3

Sol. Since both focus lies on x-axis, therefore x-axis is major axis and mid point of focii is origin which is centre and a line perpendicular to major axis and passes through centre is minor axis which is y-axis.

Let equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\because$ $ae = 4$ and $e = \frac{1}{3}$ (Given)

$\therefore$ $a = 12$ and $b^2 = a^2(1 - e^2)$

$$\Rightarrow b^2 = 144\left(1 - \frac{1}{9}\right) \Rightarrow b^2 = 16 \times 8 \Rightarrow b = 8\sqrt{2}$$

Equation of ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$

Example 8: Find the equation of axes, directrix, co-ordinate of focii, centre, vertices, length of latus-rectum and eccentricity of an ellipse $\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$

Sol. Let $x - 3 = X$, $y - 2 = Y$, so equation of ellipse becomes as $\frac{X^2}{5^2} + \frac{Y^2}{4^2} = 1$.

Equation of major axis is $Y = 0 \Rightarrow y = 2$.

equation of minor axis is $X = 0 \Rightarrow x = 3$.

Centre $(X = 0, Y = 0) \Rightarrow x = 3, y = 2$ $C \equiv (3, 2)$

Length of semi-major axis $a = 5$

Length of major axis $2a = 10$

Length of semi-minor axis $b = 4$

Length of minor axis $= 2b = 8$.

Let 'e' be eccentricity

$\therefore$ $b^2 = a^2(1 - e^2)$

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{25-16}{25}} = \frac{3}{5}.$$

Length of latus rectum $= LL' = \frac{2b^2}{a} = \frac{2\times16}{5} = \frac{32}{5}$

Co-ordinates focii are $X = \pm ae$, $Y = 0$

$\Rightarrow$ $S \equiv (X = 3, Y = 0)$ and $S' \equiv (X = -3, Y = 0)$

$\Rightarrow$ $S \equiv (6, 2)$ and $S' \equiv (0, 2)$

Extremities of major axis

$A \equiv (X = a, Y = 0)$ and $A' \equiv (X = -a, Y = 0)$

$\Rightarrow$ $A \equiv (x = 8, y = 2)$ and $A' = (x = -2, 2)$

$A \equiv (8, 2)$ and $A' \equiv (-2, 2)$

Extremities of minor axis $B \equiv (X = 0, Y = b)$ and $B' \equiv (X = 0, Y = -b)$

$B \equiv (x = 3, y = 6)$ and $B' \equiv (x = 3, y = -2)$

$B \equiv (3, 6)$ and $B' \equiv (3, -2)$

Equation of directrix $X = \pm \frac{a}{e}$

$$x - 3 = \pm\frac{25}{3} \Rightarrow x = \frac{34}{3} \;\&\; x = -\frac{16}{3}$$

Concept Application

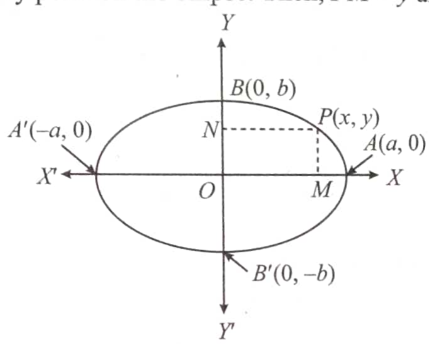
5. Find the equation of the ellipse, whose foci are (2, 3) (–2, 3) and whose semi minor axis is of length $\sqrt{5}$.
6. Find the equation of the ellipse having centre at (1, 2), one focus at (6, 2) and passing through the point (4, 6).
7. An ellipse passes through the point (–3, 1) and its eccentricity is $\sqrt{\frac{2}{5}}$. The equation of the ellipse is
   (a) $3x^2 + 5y^2 = 32$ (b) $3x^2 + 5y^2 = 25$
   (c) $3x^2 + y^2 = 4$ (d) $3x^2 + y^2 = 9$

OBLIQUE ELLIPSES

Equation of Ellipse Referred to Two Perpendicular Line

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as shown in the figure.

Let $P(x, y)$ be any point on the ellipse. Then, $PM = y$ and $PN = x$.



$$\Rightarrow \frac{PN^2}{a^2} + \frac{PM^2}{b^2} = 1$$

It follows from this, that if perpendicular distance p_1 and p_2 of a moving point $P(x, y)$ from two mutually perpendicular coplanar straight line $L_1 = a_1x + b_1y + c_1 = 0$, $L_2 = b_1x - a_1y + c_2 = 0$, respectivley, satisfy the equation

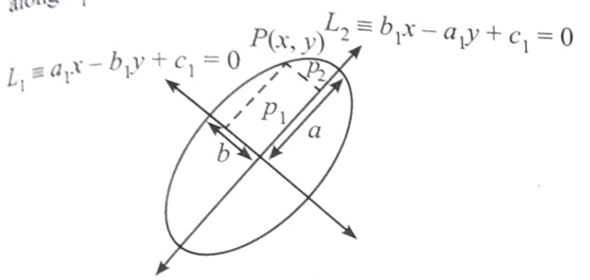
$$\frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} = 1$$

i.e., $$\frac{\left(\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}\right)^2}{a^2} + \frac{\left(\dfrac{b_1x - a_1y + c_2}{\sqrt{b_1^2 + a_1^2}}\right)^2}{b^2} = 1$$

Then locus of the point P is an ellipse in the plane of the given lines such that

(i) The centre of the ellipse is the point of the intersection of the lines $L_1 = 0$ and $L_2 = 0$

(ii) The major axis lies along $L_2 = 0$ and the minor axis lies along $L_1 = 0$, if $a > b$



Train Your Brain

Example 9: Find the major axis, minor axis, centre and eccentricity of the ellipse

$$4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 180$$

Sol. We have,

$$4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 180$$

$$\Rightarrow \frac{(x-2y+1)^2}{45} + \frac{(2x+y+2)^2}{20} = 1$$

$$\Rightarrow \frac{\left(\dfrac{x-2y+1}{\sqrt{1+4}}\right)^2}{3^2} + \frac{\left(\dfrac{2x+y+2}{\sqrt{4+1}}\right)^2}{2^2} = 1$$

$$\Rightarrow \frac{X^2}{3^2} + \frac{Y^2}{2^2} = 1$$

We have

$$X = \frac{x-2y+1}{\sqrt{5}} \text{ and } Y = \frac{2x+y+2}{\sqrt{5}}$$

It follows from this that:

Length of the major axis = $2 \times 3 = 6$

Length of the minor axis = $2 \times 2 = 4$

Equation of major axis is $Y = 0$ i.e., $2x + y + 2 = 0$

Equation of the minor axis is $X = 0$ i.e., $x - 2y + 1 = 0$

Center of the ellipse is the point of intersection of the lines $2x + y + 2 = 0$ and $x - 2y + 1 = 0$ i.e., $(-1, 0)$.

Let e be the eccentricity of the ellipse. Then,

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$$

Example 10: Find the equation the ellipse whose axis are of length 6 and $2\sqrt{6}$ and their equations are $x - 3y + 3 = 0$ and $3x + y - 1 = 0$.

Sol. Equation of ellipse will be

$$\frac{\left(\dfrac{x-3y+3}{\sqrt{10}}\right)^2}{(\sqrt{6})^2} + \frac{\left(\dfrac{3x+y-1}{\sqrt{10}}\right)^2}{(3)^2} = 1$$

Concept Application

8. Consider the ellipse $4(x + y - 5)^2 + 9(x - y + 9)^2 = 36$. Then which statement about the foci is correct
   (a) Both of them lie in third quadrant
   (b) Both of them lie in second quadrant
   (c) One lies in third quadrant and another in first quadrant
   (d) One lies in second quadrant and another in first quadrant

9. If A and B are foci of ellipse $(x - 2y + 3)^2 + (8x + 4y + 4)^2 = 20$ and P is any point on it, then $PA + PB =$
   (a) 2 (b) 4 (c) $\sqrt{2}$ (d) $2\sqrt{2}$

10. An ellipse of major axis $20\sqrt{3}$ and minor axis 20 slides along the coordinate axes and always remains confined in the 1st quadrant. The locus of the centre of the ellipse therefore describes the arc of a circle. The length of this arc is
    (a) 5π (b) 20π
    (c) $\frac{5\pi}{3}$ (d) $\frac{10\pi}{3}$

EQUATION OF CHORD OF AN ELLIPSE

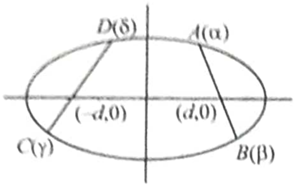
Equation of a chord of an ellipse joining two points $P(\alpha)$ and $Q(\beta)$ on it is equal to

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

(use formula of line joining points $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$)

If this particular chord passes through $(d, 0)$ then we have

$$\frac{d}{a}\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right);$$



$$\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a}{d}$$

Using componendo and dividendo rule

$$\frac{\cos\left(\frac{\alpha+\beta}{2}\right)-\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)+\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a-d}{a+d}$$

or $\quad -\dfrac{2\sin\alpha/2\ \sin\beta/2}{2\cos\alpha/2\ \cos\beta/2}=\dfrac{a-d}{a+d}$

i.e., $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=\dfrac{d-a}{d+a}$

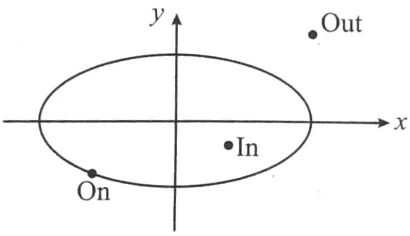
if $d=\pm ae$ i.e. PQ is a focal chord then

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}\ \text{ or }\ \frac{e+1}{e-1}$$

POSITION OF A POINT W.R.T AN ELLIPSE

Let $S(x, y)=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1$ be the given ellipse and $P(x_1, y_1)$ is the given point.

(*i*) If $S(x_1, y_1)>0$ then $P(x_1, y_1)$ lie outside the ellipse.



(*ii*) If $S(x_1, y_1)<0$ then $P(x_1, y_1)$ lie inside the ellipse.

(*iii*) If $S(x_1, y_1)=0$ then $P(x_1, y_1)$ lie on the ellipse. This result holds true for circle and parabola also.

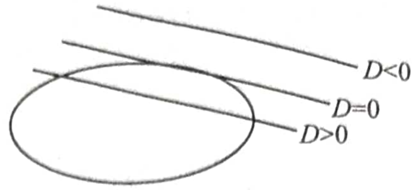
LINE AND AN ELLIPSE

Let the equations of the line is $y = mx + c$ ...(*i*)

and equation of ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ...(*ii*)

The points of intersection of the line and the ellipse can be obtained by solving the two equations simultaneously.

Hence by eliminating y from (*i*) & (*ii*), we get



$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

i.e. $(b^2+a^2m^2)x^2+2a^2cmx+a^2(c^2-b^2)=0$ ...(*iii*)

Let x_1, x_2 be the roots of the quadratic equation (*iii*). The line meets the ellipse in real and distinct points if the roots x_1 and x_2 are real and different. The line is a tangent to the ellipse if $x_1 = x_2$ and the line does not meet the ellipse if the roots x_1 and x_2 are imaginary. All these will be decided by the discriminant of quadratic equation (*iii*).

$D\ = 4a^4m^2c^2-4(b^2+a^2m^2)a^2(c^2-b^2)$

$\quad = 4a^2b^2\{a^2m^2+b^2-c^2\}$

If $D = 0$, then $y = mx + c$ is tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ i.e. $c^2=a^2m^2+b^2$ known as **condition of tangency**

Thus, $y=mx\pm\sqrt{a^2m^2+b^2}$ is always a tangent to ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Train Your Brain

Example 11: Check whether the point $P(3, 2)$ lies inside, on or outside of the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$.

Sol. $S(3,2)\equiv\dfrac{9}{25}+\dfrac{4}{16}-1=\dfrac{9}{25}+\dfrac{1}{4}-1<0$

$\therefore$ Point $P(3, 2)$ lies inside the ellipse.

Example 12: For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Sol. $\because$ Equation of ellipse is $9x^2 + 16y^2 = 144$ or $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$

Comparing this with $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ then we get $a^2 = 16$ and $b^2 = 9$ and comparing the line $y = x + \lambda$ with $y = mx + c$

$\therefore m = 1$ and $c = \lambda$

If the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$, then $c^2 = a^2m^2 + b^2$

$\Rightarrow \lambda^2 = 16\times 1^2+9 \Rightarrow \lambda^2 = 25 \qquad \therefore \lambda=\pm 5$

Example 13: Write the equation of chord of an ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ joining two points $P\left(\frac{\pi}{4}\right)$ and $Q\left(\frac{5\pi}{4}\right)$.

Sol. Equation of chord is

$$\frac{x}{5}\cos\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2}+\frac{y}{4}\cdot\sin\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2}=\cos\frac{\left(\frac{\pi}{4}-\frac{5\pi}{4}\right)}{2}$$

$$\frac{x}{5}\cos\left(\frac{3\pi}{4}\right)+\frac{y}{4}\sin\left(\frac{3\pi}{4}\right)=0$$

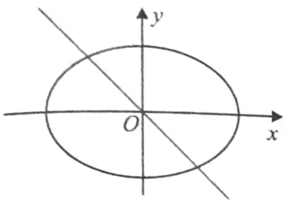
$$-\frac{x}{5}+\frac{y}{4}=0 \Rightarrow 4x=5y$$

Example 14: Find the set of value(s) 'α' for which the point $P(\alpha,-\alpha)$ lies inside the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

Sol. **M#1:** $y = x$

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

$x = 12/5$ or $-12/5$



M#2: If $P(\alpha, -\alpha)$ lies inside the ellipse

$\therefore S_1 < 0$

$$\Rightarrow \frac{\alpha^2}{16}+\frac{\alpha^2}{9}-1<0$$

$$\Rightarrow \frac{25}{144}\cdot\alpha^2<1$$

$$\Rightarrow \alpha^2<\frac{144}{25}$$

$$\therefore \alpha\in\left(-\frac{12}{5},\frac{12}{5}\right)$$

Concept Application

11. The number of values of c such that straight line $y = 4x + c$ touches the curve $\frac{x^2}{4}+y^2=1$ is

(a) 0 (b) 1
(c) 2 (d) Infinite

12. Write the equation of chord of an ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ joining two points $P\left(\frac{\pi}{4}\right)$ and $Q\left(\frac{5\pi}{4}\right)$.

ANGLE OF INTERSECTION OF TANGENTS AND DIRECTOR CIRCLE

Locus of the point of intersection of the tangents which meet at right angles is called the Director Circle.

Let equation of any tangent is $y = mx + \sqrt{a^2m^2+b^2}$

If it passes through (h, k) then

$$k = mh \pm \sqrt{a^2m^2+b^2}$$

$$(k - mh)^2 = a^2m^2 + b^2$$

$$(h^2 - a^2)m^2 - 2khm + k^2 - b^2 = 0 \quad \text{...(iii)}$$

Equation (*iii*) has two roots m_1 and m_2

$$m_1+m_2=\frac{2hk}{h^2-a^2} \quad \text{...(iv)}$$

$$m_1m_2=\frac{k^2-b^2}{h^2-a^2} \quad \text{...(v)}$$

Hence passing through a given point there can be a maximum of two tangents.

Equation (*iii*) can be used to determine the locus of the point of intersection of two tangents enclosing.

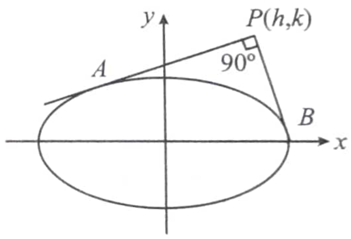
If from any point $P(h, k)$ pair of tangents are drawn to the ellipse which include an angle α, then

$$\tan\alpha=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}$$

By putting value of $m_1 + m_2$ and m_1m_2 in above equation we will get the angle between pair of tangents.

If $PA \perp PB$ then $m_1m_2 = -1$

i.e. $m_1m_2=\frac{k^2-b^2}{h^2-a^2}=-1$

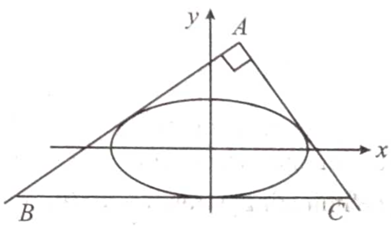


i.e. $k^2 - b^2 = a^2 - h^2$;

i.e. $x^2 + y^2 = a^2 + b^2$

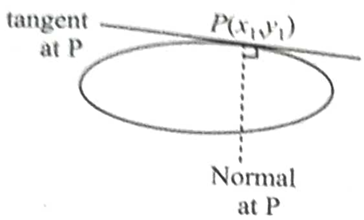
which is the director circle of the ellipse. Hence director circle of an ellipse is a circle whose centre is the centre of ellipse and whose radius is the length of the line joining the ends of the major and minor axis.

Note: If a right triangle, right angled at A circumscribes an ellipse then locus of the point A is the director circle of the ellipse.



Tangent to the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) Point form: $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ is tangent to the ellipse at (x_1, y_1).



Since point (x_1, y_1) lie on the curve therefore we can use standard substitution **T = 0** to obtain the equation of tangent.

(b) Slope form: Let the given line is $y = mx + c$ and given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

If line touch the ellipse then by solving the two equations simultaneously (by eliminating y from (*i*) and (*ii*), we get $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$.

i.e. $(b^2 + a^2m^2)\,x^2 + 2a^2 cmx + a^2(c^2 - b^2) = 0$ ... (*iii*)

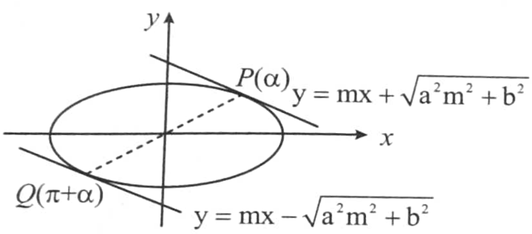
Since line is tangent to the ellipse therefore its $D = 0$

$4a^4c^2m^2 - 4(b^2 + a^2m^2)\cdot a^2(c^2 - b^2) = 0$

or $4a^2[a^2c^2m^2 - b^2c^2 - a^2c^2m^2 + b^4 + a^2b^2m^2] = 0$

or $b^2(-c^2 + b^2 + a^2m^2) = 0$

or $c^2 = b^2 + a^2m^2$ or $c = \pm\sqrt{a^2m^2+b^2}$



which is the required **condition of tangency**.

Substituting this value of c in $y = mx + c$, we have

$y = mx + \sqrt{a^2m^2+b^2}$ or $y = mx - \sqrt{a^2m^2+b^2}$, which are tangents to the ellipse for all values of m.

Here ± sign represents two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

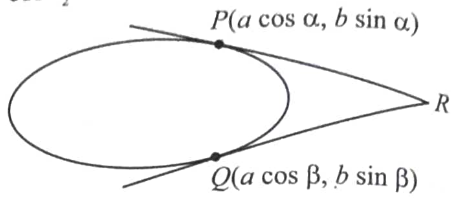
The equation of any tangent to the ellipse $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

$(y-k) = m(x-h) \pm \sqrt{a^2m^2+b^2}$

(c) Parametric form: $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ is tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$.

Note: (*i*) Point of intersection of these tangents at the point α & β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$ can be deduced by comparing chord joining $P(\alpha)$ and $Q(\beta)$ with C.O.C. of the pair of tangents from $R(x_1, y_1)$ on the ellipse, where

$x_1 = a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$; $y_1 = b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$

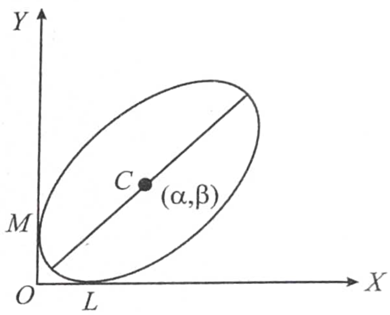


(*ii*) The eccentric angles of point of contact of two parallel tangents differ by π. Conversely if the difference between the eccentric angles of two points is π then the tangents at these points are parallel.

Train Your Brain

Example 15: An ellipse slides between two lines at right angles to another. Show that the locus of its centre is a circle.

Sol. Let the two given perpendicular lines be taken as the x and y axes respectivley.



Let $C(\alpha, \beta)$ be the centre of the ellipse in any position. Here the position of centre C changes as the ellipse slides.

Let a and b be the semimajor and minor axes of the elipse.

Equation of the director circle of the ellipse is

$(x - \alpha)^2 + (y - \beta)^2 = a^2 + b^2$ ...(*i*)

Since OX and OY are mutually perpendicular tangents to sliding ellipse for all its positions, therefore, $O(0, 0)$ will lie on its director circle (1)

$\therefore\ \alpha^2 + \beta^2 = a^2 + b^2$

Hence locus of $C(a, b)$ is $x^2 + y^2 = a^2 + b^2$ ...(*ii*)

Example 16: A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

Sol. Given ellipse are $\frac{x^2}{4}+\frac{y^2}{1}=1$ ... (i)

and, $\frac{x^2}{6}+\frac{y^2}{3}=1$ ... (ii)

any tangent to (i) is $\frac{x\cos\theta}{2}+\frac{y\sin\theta}{1}=1$ ...(iii)

It cuts (ii) at P and Q, and suppose tangent at P and Q meet at (h, k) Then equation of chord of contact of (h, k) with respect to ellipse (ii) is $\frac{hx}{6}+\frac{ky}{3}=1$ ...(iv)

comparing (iii) and (iv), we get $\frac{\cos\theta}{h/3}=\frac{\sin\theta}{k/3}=1$

$\Rightarrow \cos\theta=\frac{h}{3}$ and $\sin\theta=\frac{k}{3} \Rightarrow h^2+k^2=9$ locus of the point (h, k) is $x^2+y^2=9$

$\Rightarrow\ x^2+y^2=6+3=a^2+b^2$

i.e. director circle of second ellipse. Hence the tangents are at right angles.

Example 17: Find the equations of the tangents to the ellipse $x^2 + 16y^2 = 16$ each one of which makes an angle of $60°$ with the x- axis.

Sol. We have, $x^2 + 16y^2 = 16 \Rightarrow \frac{x^2}{4^2}+\frac{y^2}{1^2}=1$

This is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a^2 = 16$ and $b^2 = 1$

So, the equations of the tangents are

$y=mx\pm\sqrt{a^2m^2+b^2}$

i.e. $y=\sqrt{3}x\pm\sqrt{16\times3+1} \Rightarrow y=\sqrt{3}x\pm7$

Example 18: Prove that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angle differ by a constant α is an ellipse.

Sol. Let $P(h, k)$ be the point of intersection of tangents at $A(\theta)$ and $B(\beta)$ to the ellipse.

$$\therefore\ h=\frac{a\cos\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)} \text{ and } k=\frac{b\sin\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)}$$

$$\Rightarrow\ \left(\frac{h}{a}\right)^2+\left(\frac{k}{b}\right)^2=\sec^2\left(\frac{\theta-\beta}{2}\right)$$

but given that $\theta-\beta=\alpha$

$$\therefore\ \text{locus is } \frac{x^2}{a^2\sec^2\left(\frac{\alpha}{2}\right)}+\frac{y^2}{b^2\sec^2\left(\frac{\alpha}{2}\right)}=1$$

Example 19: Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Sol. Slope of the given line $= -2$

Slope of tangent $= 1/2$

Given ellipse is $\frac{x^2}{4}+\frac{y^2}{3}=1$

Equation of tangent whose slope is 'm' is y

$=mx\pm\sqrt{4m^2+3}$

$\because\ m=\frac{1}{2}\quad \therefore\ y=\frac{1}{2}x\pm\sqrt{1+3}$

$2y=x\pm4$

Concept Application

13. If the tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, then the locus of the mid points of the intercept made by the tangents between the coordinate axes is

14. If CF is perpendicular from the centre C of the ellipse $\frac{x^2}{49}+\frac{y^2}{25}=1$ on the tangent at any point P and G is the point where the normal at P meets the minor axis, then $(CF \cdot PG)^2$ is equal to

15. A tangent is drawn at the point $(3\sqrt{3}\cos\theta, \sin\theta)$ for $0<\theta<\pi/2$ of an ellipse $\frac{x^2}{27}+\frac{y^2}{1}=1$. The least value of the sum of the intercepts on the coordinate axis by this tangent is attained at θ equal to

(a) $\frac{\pi}{6}$ (b) $\frac{2\pi}{3}$

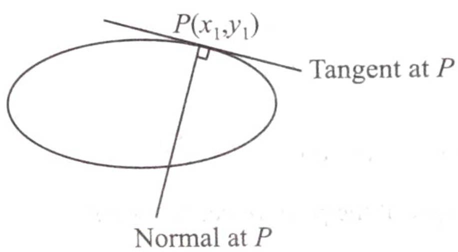
(c) $\frac{3\pi}{8}$ (d) $\frac{3\pi}{4}$

16. Find the equation of common tangent to the parabola $y^2 = 16\sqrt{3}x$ and ellipse $2x^2 + y^2 = 4$.

Normal to the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) Point form: Equation of the tangent to the ellipse at (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

The slope of the tangent at $(x_1, y_1)=\frac{-x_1}{a^2}\times\frac{b^2}{y_1}$



$\therefore$ Slope of the normal at $(x_1, y_1) = \frac{a^2}{x_1} \times \frac{y_1}{b^2} = \frac{a^2 y_1}{b^2 x_1}$

Hence the equation of the normal at (x_1, y_1) is $y - y_1$

$= \frac{a^2 y_1}{b^2 x_1}(x - x_1)$

or $\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{\frac{y_1}{b^2}}$

(b) Slope form: Equation of a normal to the given ellipse whose slope is 'm' is $y = mx \mp \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2 m^2}}$.

Note: Maximum four normals can be drawn to an ellipse from a given point.

(c) Parametric form: Equation of the normal to the given ellipse at the point $(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\operatorname{cosec}\theta = (a^2 - b^2)$.

REMARK

Conormal Points Lie on a Fixed Curve

Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ be the conormal points. Let normals at P, Q, R, S meet in $T(h, k)$.

Then equation of normal at $P(x_1, y_1)$ is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

The point $T(h, k)$ lies on it

$(a^2 - b^2)x_1y_1 + b^2kx_1 - a^2hy_1 = 0$ ... (i)

Similarly, for points Q, R and S in equation (i), x_1, y_1 are just replaced by (x_2, y_2), (x_3, y_3), (x_4, y_4) respectively.

$\therefore$ P, Q, R, S all lie on the curve $(a^2 - b^2)xy + b^2kx - a^2hy = 0$, which is called Apollonius a rectangular hyperbola.

Note: (i) **Condition of conormal points:** If $A(\alpha)$, $B(\beta)$, $C(\gamma)$ and $D(\delta)$ are conormal points then sum of their eccentric angles is odd multiple of π. i.e. $\alpha + \beta + \gamma + \delta = \mathbf{(2n + 1)\pi}$.

Proof: Normal at P $(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2$$

If it passes through point (h, k), then.

$$\frac{ah}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2$$

$$\Rightarrow \frac{ah\left(1 + \tan^2\frac{\theta}{2}\right)}{1 - \tan^2\frac{\theta}{2}} - \frac{bk\left(1 + \tan^2\frac{\theta}{2}\right)}{2\tan\frac{\theta}{2}} = a^2 - b^2$$

$$\Rightarrow \frac{(1 + t^2)ah}{(1 - t^2)} - \frac{(1 + t^2)bk}{2t} = a^2 - b^2 \text{ where } t = \tan\frac{\theta}{2}$$

$$\Rightarrow 2t(1 + t^2)ah - (1 - t^4)bk = 2t(1 - t^2)(a^2 - b^2)$$

$$\Rightarrow bkt^4 + \left[2ah + 2(a^2 - b^2)\right]t^3 + \left[2ah - 2(a^2 - b^2)\right]t - bk = 0$$

This equation has four roots, i.e. $\tan\frac{\alpha}{2}, \tan\frac{\beta}{2}\tan\frac{\gamma}{2}, \tan\frac{\delta}{2}$, where α, b, γ, δ are eccentric angles of feet of normals on the ellipse.

Now $s_1 = \Sigma\tan\frac{\alpha}{2}$

$$s_2 = \Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = 0$$

$$s_3 = \Sigma\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$$

$$s_4 = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2} = -1$$

Now $\tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = \frac{s_1 - s_3}{1 - s_2 + s_4}$

But $1 - s_2 + s_4 = 0$

$$\Rightarrow \left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = (2n + 1)\frac{\pi}{2}, n \in Z$$

$$\Rightarrow \alpha + \beta + \gamma + \delta = (2n + 1)\pi, n \in Z$$

(ii) **Condition of concyclic points:** If $A(\alpha)$, $B(\beta)$, $C(\gamma)$ and $D(\delta)$ are four concyclic point them sum of their eccentric angles is even multiple of π. i.e. $\alpha + \beta + \gamma + \delta = \mathbf{2n\pi}$.

Proof: Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in four points P, Q, R, S.

Solving circle and ellipse ($x = a\cos\theta$, $y = b\sin\theta$); we have $a^2\cos^2\theta + b^2\sin^2\theta + 2ag\cos\theta + 2bf\sin\theta + c = 0$

$$\Rightarrow a^2\left(\frac{1 - t^2}{1 + t^2}\right)^2 + b^2\left(\frac{2t}{1 + t^2}\right)^2 + 2ag\left(\frac{1 - t^2}{1 + t^2}\right)$$

$$+2bf\left(\frac{2t}{1 + t^2}\right) + c = 0, \text{ where } t = \tan\frac{\theta}{2}$$

$\Rightarrow$ $a^2(1 - t^2)^2 + 4b^2t^2 + 2ag(1 - t^2)(1 + t^2) + 4bft(1 + t^2) + c(1 + t^2)^2 = 0$

$\Rightarrow$ $(a^2 - 2ag + c)t^4 + 4bft^3 + (-2a^2 + 4b^2 + 2c)t^2 + 4bft + (a^2 + 2ag + c) = 0$

Roots of the equation are $\tan\frac{\alpha}{2}, \tan\frac{\beta}{2}, \tan\frac{\gamma}{2}\tan\frac{\delta}{2}$, where $\alpha, \beta, \gamma, \delta$ are eccentric angles of P, Q, R, S, respectively.

Also $S_1 = S_3$

$$\Rightarrow \tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = \frac{s_1 - s_3}{1 - S_2 + S_4} = 0$$

$$\Rightarrow \frac{\alpha+\beta+\gamma+\delta}{2} = n\pi, n \in Z$$

$$\Rightarrow \alpha+\beta+\gamma+\delta = 2n\pi, n \in Z$$

Train Your Brain

Example 20: Find the condition that the line $lx + my = n$ may be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol. Equation of normal to the given ellipse at $(a\cos\theta, b\sin\theta)$ is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ ... (i)

If the line $\ell x+my=n$ is also normal to the ellipse then there must be a value of θ for which line (i) and line $lx+my=n$ are identical. For the value of θ we have

$$\frac{l}{\left(\frac{a}{\cos\theta}\right)}=\frac{m}{-\left(\frac{b}{\sin\theta}\right)}=\frac{n}{(a^2-b^2)}$$

or $\cos\theta=\frac{an}{\ell(a^2-b^2)}$ ... (iii)

and $\sin\theta=\frac{-bn}{m(a^2-b^2)}$ ...(iv)

Squaring and adding (iii) and (iv), we get

$$1=\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{\ell^2}+\frac{b^2}{m^2}\right)$$

which is the required condition.

Example 21: If the normal at an end of a latus-rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by $e=\sqrt{\frac{\sqrt{5}-1}{2}}$

Sol. The co-ordinates of an end of the latus-rectum are $(ae, b^2/a)$. The equation of normal at $P(ae, b^2/a)$ is

$$\frac{a^2x}{ae}-\frac{b^2(y)}{b^2/a}=a^2-b^2 \text{ or } \frac{ax}{e}-ay=a^2-b^2$$

It passes through one extremity of the minor axis whose co-ordinates are $(0, -b)$

$\therefore \quad 0+ab=a^2-b^2$

$\Rightarrow \quad (a^2b^2)=(a^2-b^2)^2$

$\Rightarrow \quad a^2.a^2(1-e^2)=(a^2e^2)^2$

$\Rightarrow \quad 1-e^2-e^4=0$

$\Rightarrow \quad e^4+e^2-1=0$

$\Rightarrow \quad (e^2)^2+e^2-1=0$

$\therefore \quad e^2=\frac{-1\pm\sqrt{1+4}}{2} \Rightarrow e=\sqrt{\frac{\sqrt{5}-1}{2}}$

(taking positive sign)

Example 22: If the line $3y=3x+1$ is a normal to the ellipse $\frac{x^2}{5}+\frac{y^2}{b^2}=1$, then find out the length of the minor axis of the ellipse.

Sol. Equation of normal with slope m to the given ellipse is

$$y=mx\pm\frac{m(5-b^2)}{\sqrt{b^2m^2+5}}$$

$\Rightarrow \quad y=x\pm\frac{(5-b^2)}{\sqrt{b^2+5}}$ (as $m=1$) ... (i)

$y=x+\frac{1}{3}$ ...(ii)

both (i) and (ii) represents same line

so $\pm\frac{(5-b^2)}{\sqrt{b^2+5}}=\frac{1}{3} \Rightarrow (5-b^2)^2=\frac{1}{9}(5+b^2)$

$\Rightarrow 9b^4-91b^2+220=0 \Rightarrow b^2=4$ or $55/9$

so length of minor axis is either 4 or $\frac{2}{3}\sqrt{55}$.

Concept Application

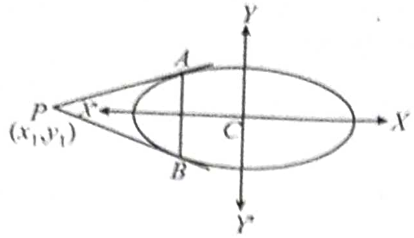
17. A point on the ellipse $x^2+3y^2=37$ where the normal is parallel to the line $6x-5y=2$ is
    (a) $(5, -2)$ (b) $(2, 5)$
    (c) $(-5, 2)$ (d) $(-5, -2)$
18. An ellipse passes through the point $(2, 3)$ and its axes along the coordinate axes. $3x+2y-1=0$ is a tangent to the ellipse, then the equation of the ellipse is

PAIR OF TANGENTS & CHORD OF CONTACT

If $P(x_1, y_1)$ be any point lies outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, , and a pair of tangents PA, PB can be drawn to it from P.

Then the equation of pair of tangents of PA and PB is $SS_1=T^2$

where $S=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1,\ T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$



i.e. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$

Chord of Contact: Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$T = 0$, where

$$T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

Equation of chord with mid point (X_1, Y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose mid-point be (x_1, y_1) is $T = S_1$

where $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1,\ S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

i.e. $\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$

To find the length of the chord intercepted by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on the straight line $y = mx + c$.

Points of intersectin of the ellipse and the line are given by

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

i.e $(a^2m^2+b^2)x^2+2a^2cmx+a^2(c^2-b^2)=0$. ...(*i*)

Therefore the straight line meets the ellipse in two points (real, concident or imaginary).

If (x_1, y_1) and (x_2, y_2) be the points of intersection, the length of the chord is

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\sqrt{1+m^2}\,|x_1-x_2|, \qquad ...(ii)$$

(since $y_1-y_2=m(x_1-x_2)$)

where x_1 and x_2 are the roots of the equation (*i*), and

$$x_1+x_2=-\frac{2a^2cm}{a^2m^2+b^2},\ x_1x_2=\frac{a^2(c^2-b^2)}{a^2m^2+b^2} \text{ so that}$$

$$(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$$

$$=\frac{4a^4c^2m^2}{(a^2m^2+b^2)^2}-\frac{4a^2(c^2-b^2)}{(a^2m^2+b^2)}=\frac{4a^2b^2(a^2m^2+b^2-c^2)}{(a^2m^2+b^2)}.$$

Hence the length of the chord is

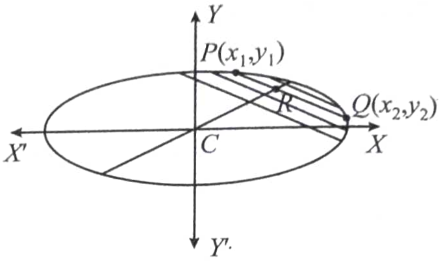
$$\sqrt{\frac{(1+m^2)4a^2b^2(a^2m^2+b^2-c^2)}{(a^2m^2+b^2)}}$$

i.e. $\frac{2ab}{a^2m^2+b^2}\sqrt{(1+m^2)(a^2m^2+b^2-c^2)}$.

DIAMETER

The focus of the middle point of a system of parallel chords of an ellipse is called a diameter and the point where the diameter intersect the ellipse is called the vertex of the diameter.

Let $y = mx + c$ be system of parallel chords to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for different chords c varies, m remains constant.



Let (h, k) be the middle point of the chord $y = mx + c$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then

$$T=S_1 \Rightarrow \frac{xh}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$$

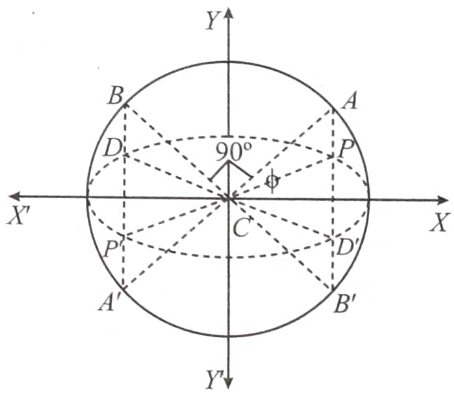
$$\therefore \quad \text{Slope}=-\frac{b^2h}{a^2k}=m \quad \Rightarrow \quad k=-\frac{b^2h}{a^2m}$$

Hence locus of the mid point is $y=-\frac{b^2x}{a^2m}$

CONJUGATE DIAMETERS

Two diameters are said to be conjugate when each bisects all chords parallel to the other. If $y = mx$ and $y = m_1x$ be two conjugate diameters of an ellipse then

$$mm_1=\frac{b^2}{a^2}$$



Conjugate diameters of circle i.e., AA' and BB' are perpendicular to each other. Hence conjugate diameters of ellipse are PP' and DD'

Hence angle between conjugate diameters of ellipse $\neq 90°$

Now the co-ordinates of the four extremities of two conjugate diameters are

$P(a\cos\phi, b\sin\phi)$, $P'(-a\cos\phi, -b\sin\phi)$,

$D(-a\sin\phi, b\cos\phi)$, $D'(a\sin\phi, -b\cos\phi)$

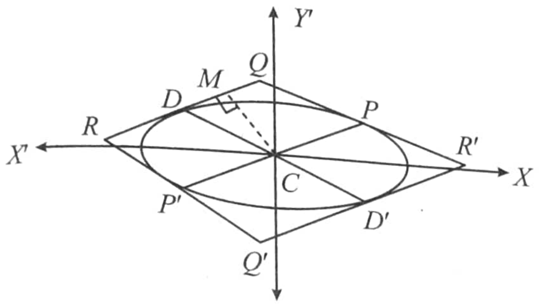
Properties of Conjugate Diameters

Prop. 1: The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle.

Prop. 2: The sum of the squares of any two conjugate semi diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes of the ellipse i.e., $CP^2 + CD^2 = a^2 + b^2$

Prop. 3: The product of the focal distances of a point on an ellipse is equal to the square of the semi diameter which is conjugate to the diameter through the point.

Prop. 4: The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to product of the axes.



Prop. 5: The polar of any point with respect to ellipse is parallel to the diameter to the one on which the point lies. Hence obtain the equation of the chord whose mid point is (h, k).

EQUI-CONJUGATE DIAMETERS

Two conjugate diameters are called equi-conjugate if their lengths are equal. In such cases therefore.

$(CP)^2 = (CD)^2$

$\therefore\ a^2\cos^2\phi + b^2\sin^2\phi = a^2\sin^2\phi + b^2\cos^2\phi$

$\Rightarrow\ a^2(\cos^2\phi - \sin^2\phi) - b^2(\cos^2\phi - \sin^2\phi) = 0$

$\Rightarrow\ (a^2 - b^2)\cos^2\phi = 0$

$\because\ (a^2 - b^2) \neq 0$

$\therefore\ \cos^2\phi = 0$

$\therefore\ \phi = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$

$\therefore\ (CP) = (CD) = \sqrt{\dfrac{(a^2+b^2)}{2}}$

Train Your Brain

Example 23: How many real tangents can be drawn from the point (4, 3) to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. Find the equation of these tangents and angle between them.

Sol. Given point $P \equiv (4, 3)$ ellipse $S \equiv \dfrac{x^2}{16} + \dfrac{y^2}{9} - 1 = 0$

$\because\ S_1 \equiv \dfrac{16}{16} + \dfrac{9}{9} - 1 = 1 > 0$

$\Rightarrow$ Point $P \equiv (4, 3)$ lies outside the ellipse.

$\therefore$ Two tangents can be drawn from the point $P(4, 3)$.

Equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{16} + \frac{y^2}{9} - 1\right)\cdot 1 = \left(\frac{4x}{16} + \frac{3y}{9} - 1\right)^2$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} - 1 = \frac{x^2}{16} + \frac{y^2}{9} + 1 + \frac{xy}{6} - \frac{x}{2} - \frac{2y}{3}$$

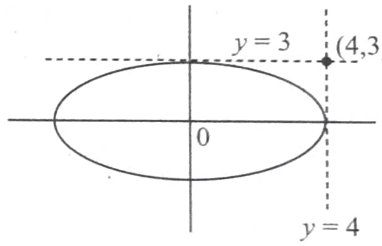
$\Rightarrow\ -xy + 3x + 4y - 12 = 0$

$\Rightarrow\ (4 - x)(y - 3) = 0$

$\Rightarrow\ x = 4$ and $y = 3$

and angle between them $= \dfrac{\pi}{2}$

2nd Method: By direct observation



$x = 4$, $y = 3$ are tangents.

Example 24: If tangents to the parabola $y^2 = 4ax$ intersect the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Sol. Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

$\therefore$ equation of chord of contact AB is

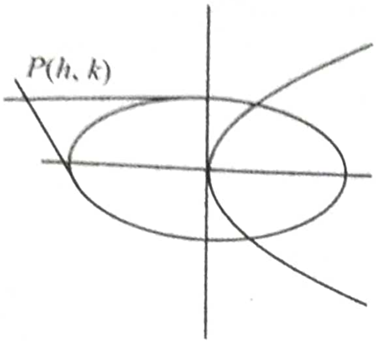
$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1 \qquad \text{... }(i)$$

which touches the parabola equation of tangent to parabola $y^2 = 4ax$

$$y = mx + \frac{a}{m}$$

$$\Rightarrow \quad mx - y = -\frac{a}{m} \qquad ...(ii)$$

equation (i) and (ii) as must be same



$$\therefore \quad \frac{m}{\left(\frac{h}{a^2}\right)} = \frac{-1}{\left(\frac{k}{b^2}\right)} = \frac{-\frac{a}{m}}{1}$$

$$\Rightarrow \quad m = -\frac{h}{k}\frac{b^2}{a^2} \ \& \ m = \frac{ak}{b^2}.$$

$$\therefore \quad -\frac{hb^2}{ka^2} = \frac{ak}{b^2} \Rightarrow \text{locus of } P \text{ is } y^2 = -\frac{b^4}{a^3}\cdot x$$

Example 25: Find the locus of the mid - point of focal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Sol. Let $P = (h, k)$ be the mid-point

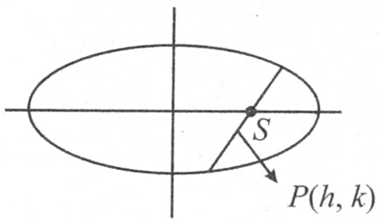
$\therefore$ Equation of chord whose mid-point is given

$$\frac{xh}{a^2}+\frac{yk}{b^2}-1 = \frac{h^2}{a^2}+\frac{k^2}{b^2}-1$$

Since it is a focal chord, therefore it passes through focus, either $(ae, 0)$ or $(-ae, 0)$

If it passes through $(ae, 0)$

$$\therefore \quad \text{locus is } \frac{ex}{a} = \frac{x^2}{a^2}+\frac{y^2}{b^2}$$

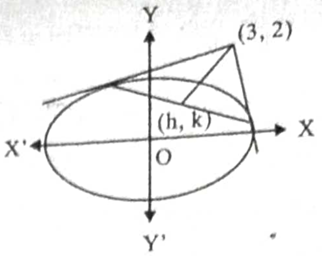


If it passes through $(-ae, 0)$

$$\therefore \quad \text{locus is } -\frac{ex}{a} = \frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Example 26: Tangents are drawn from the point (3, 2) to the ellipse $x^2 + 4y^2 = 9$. Find the equation to their chord of contact and the middle point of this chord of contact.

Sol. $x^2 + 4y^2 = 9$



Equation of the chord of contact of the pair of tangents from (3, 2)

$$3x + 8y = 9 \qquad ...(i)$$

This must be the same as chord whose middle point is (h, k)

$$T = S_1$$

$$\frac{hx}{9}+\frac{ky}{9/4} = \frac{h^2}{9}+\frac{k^2}{9/4}$$

$$\Rightarrow \quad hx + 4ky = h^2 + 4k^2 \qquad ...(ii)$$

Equations (i) and (ii) represent same straight lines. Comparing coefficient of (i) and (ii), we get

$$\frac{h}{3} = \frac{4k}{8} = \frac{h^2+4k^2}{9}$$

$$\Rightarrow \quad 2h = 3k \text{ and } 3h = h^2 + 4k^2$$

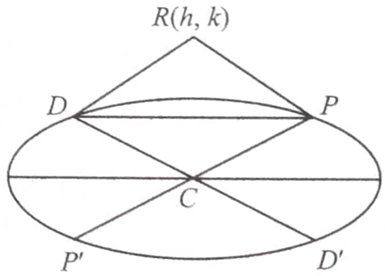
$$\Rightarrow \quad 3h = h^2 + 4\times\frac{4h^2}{9}$$

$$\Rightarrow \quad \frac{25h^2}{9} = 3h$$

$$\Rightarrow \quad h = \frac{27}{25} \text{ and } k = \frac{18}{25}$$

Example 27: Show that the tangents at the ends of conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$ intersect on the ellipse $x^2/a^2 + y^2/b^2 = 2$.

Sol. Let CP and CD be two semi-conjugate diameters, so that if eccentric angle of P is ϕ then eccentric angle of D is $\frac{\pi}{2}+\phi$



$\therefore$ Co-ordinates of P and D are $(a\cos\phi, b\sin\phi)$ and $\left(a\cos\left(\frac{\pi}{2}+\phi\right), b\sin\left(\frac{\pi}{2}+\phi\right)\right)$ respectively

$\therefore$ Equation of (PD) is

$$\frac{x}{a}\cos\left(\frac{\phi+\frac{\pi}{2}+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\phi+\frac{\pi}{2}+\phi}{2}\right)=\cos\left(\frac{\frac{\pi}{2}+\phi-\phi}{2}\right)$$

$$\Rightarrow \frac{x}{a}\cos\left(\frac{\pi}{4}+\phi\right)+\frac{y}{b}\sin\left(\frac{\pi}{4}+\phi\right)=\frac{1}{\sqrt{2}} \quad ...(i)$$

If its pole or point of intersection of tangents at its extremities be (h, k) then its equation is the same as that of the polar or the chord of contact of (h, k)

i.e., $\frac{hx}{a^2}+\frac{ky}{b^2}=1$ ...(ii)

Since equation (i) and (ii) are identical, comparing

$$\frac{h}{a\cos\left(\frac{\pi}{4}+\phi\right)}=\frac{k}{b\sin\left(\frac{\pi}{4}+\phi\right)}=\sqrt{2}$$

or $\sqrt{2}\cos\left(\frac{\pi}{4}+\phi\right)=\frac{h}{a}$ ...(iii)

or $\sqrt{2}\sin\left(\frac{\pi}{4}+\phi\right)=\frac{k}{b}$ ...(iv)

Squaring and adding equations (iii) and (iv), then

$$\frac{h^2}{a^2}+\frac{k^2}{b^2}=2\left(\cos^2\left(\frac{\pi}{4}+\phi\right)+\sin^2\left(\frac{\pi}{4}+\phi\right)\right)$$

$$\therefore \frac{h^2}{a^2}+\frac{k^2}{b^2}=2$$

Hence locus of (h, k) is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$

Alternative Method: Equation of tangents at P and D are

$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$ ...(i)

and $\frac{x}{a}\cos\left(\frac{\pi}{2}+\phi\right)+\frac{y}{b}\sin\left(\frac{\pi}{2}+\phi\right)=1$

i.e., $-\frac{x}{a}\sin\phi+\frac{y}{b}\cos\phi=1$ ...(ii)

Squaring and adding equations (i) and (ii), we get $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$. Which is required locus.

Concept Application

19. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are at right angles, then find the value of $\frac{x_1x_2}{y_1y_2}$

20. If α and β are the eccentric angles of the extremities of a focal chord of an ellipse, then prove that the eccentricity of the ellipse is a $\frac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)}$

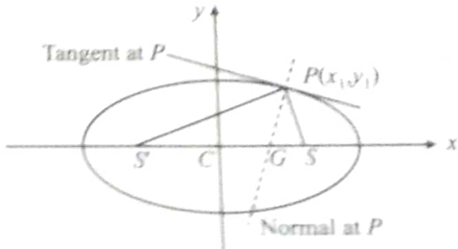
IMPORTANT HIGHLIGHTS

1. Tangent and normal at any point P bisect the external and internal angles between the focal distances of SP and $S'P$.

Let the equation of an ellipse is

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (i)

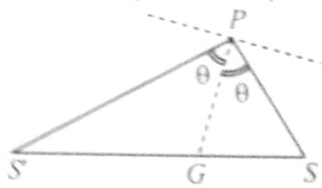
& Normal at $P(x_1, y_1)$ is



$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$ ... (ii)

The normal meet x-axis at $G \Rightarrow$ Put $y = 0$ in equation (ii) we get $CG = x = \left(\frac{a^2-b^2}{a^2}\right)x_1 = e^2x_1$

$\therefore SG = CS - CG = ae - e^2x_1 = e(a - ex_1) = eSP$

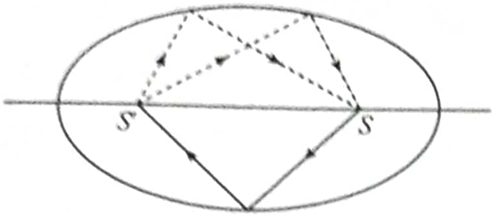


Similarly $S'G = eS'P$

$\therefore \frac{SG}{S'G}=\frac{eSP}{eS'P}=\frac{SP}{S'P}$

$\Rightarrow$ PG is bisector of angle $\angle P$ in $\Delta S'PS$.

This lead to reflection property of ellipse.

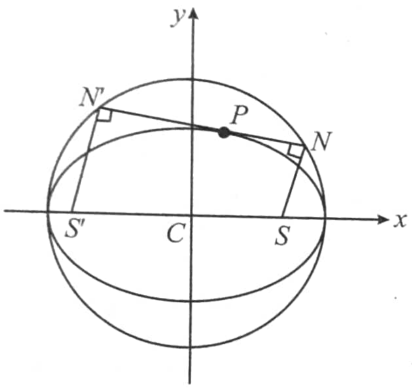


If incoming light ray passes through focus S'(or S), strike the concave side of ellipse then after reflection it will pass through other focus.

2. The locus of the point of interseciton of feet of perpendicular from foci on any tangent to an ellipse is the auxiliary circle.

Let ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ... (*i*)

its tangent is $y-mx=\sqrt{a^2m^2+b^2}$ ... (*ii*)



Equation of perpendicular to above the passes through focus $(ae, 0)$ is

$my+x=ae$ ... (*iii*)

Eliminate m from (*ii*) and (*iii*) we will get focus of intersection point.

For that square and add (*ii*) and (*iii*) we will get an answer

$\therefore\ y^2+m^2x^2-2mxy+x^2+my^2+2mxy=a^2m^2+b^2+a^2e^2$

$\Rightarrow x^2(1+m^2)+y^2(1+m^2)=a^2m^2+a^2$

$\Rightarrow x^2+y^2=a^2$ which is the auxilairy circle

3. The product of perpendicular distance from the foci to any tangent of an ellipse is equal to square of the semi minor axis.

From previous equation of any tangent is

$mx-y+\sqrt{a^2m^2+b^2}=0$

$$SN=\left|\frac{\sqrt{a^2m^2+b^2}+ame}{\sqrt{1+m^2}}\right|$$

$$\&\ S'N'=\left|\frac{\sqrt{a^2m^2+b^2}-ame}{\sqrt{1+m^2}}\right|$$

$$SN\cdot S'N'=\frac{(a^2m^2+b^2)-a^2m^2e^2}{(1+m^2)}$$

$$=\frac{a^2m^2+b^2-(a^2-b^2)m^2}{1+m^2}=b^2 \qquad (\because a^2e^2=a^2-b^2)$$

4. Tangents at the extremities of latus-rectum of an ellipse intersect on the foot of corresponding directrix.

Equation of tangent at $L\left(ae,+\dfrac{b^2}{a}\right)$ is

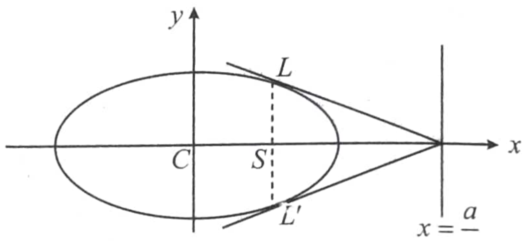
$$\frac{x(ae)}{a^2}+\frac{y\left(\frac{b^2}{a}\right)}{b^2}=1 \Rightarrow xe+y=a \qquad \text{... }(i)$$

& equation of tangent at $L'\left(ae,-\dfrac{b^2}{a}\right)$ is

$ex-y=a$ ... (*ii*)

Solve (*i*) & (*ii*) we get $x=\dfrac{a}{e}$ & $y=0$

i.e. at the foot of directrix of ellipse.



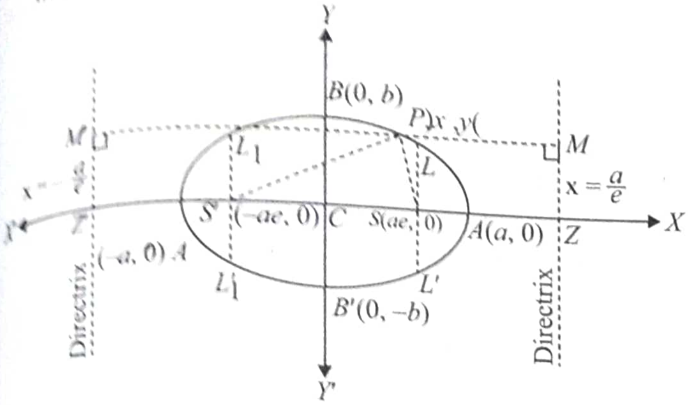
5. If P be any point on the ellipse with S and S' as its foci then $\ell(SP)+\ell(S'P)=2a$.
6. If the normal at any point P on the ellipse with centre C meet the major and minor axes in G and g respectively and if CF be perpendicular upon this normal then
   (*i*) $PF\cdot PG=b^2$ (*ii*) $PF\cdot Pg=a^2$
7. The portion of the tangent to an ellipse between the point of contact and the directrix subtends a right angle at the corresponding focus.
8. Area enclosed by an ellipse having length of major and minor axes as $2a$ and $2b$ is given by πab.

Short Notes

...the co-ordinate axis is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Where $a > b$ and $b^2 = a^2(1 - e^2)$

$\Rightarrow a^2 - b^2 = a^2e^2$,

where e = eccentricity $(0 < e < 1)$.



FOCI : S ≡ (ae, 0) and S′ ≡ (– ae, 0).

(j) **Latus Rectum:** The focal chord perpendicular to the major axis is called the **latus rectum.**

(i) Length of latus rectum

$$(\text{LL}') = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1 - e^2)$$

(ii) Equation of latus rectum : $x = \pm\, ae$.

(iii) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, -\frac{b^2}{a}\right)$,

$L_1\left(-ae, \frac{b^2}{a}\right)$ and $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

(k) **Focal Radii:** $SP = a - ex$ and $S'P = a + ex$

$\Rightarrow SP + S'P = 2a$ = Major axis.

(l) **Eccentricity:** $e = \sqrt{1 - \frac{b^2}{a^2}}$

Position of a Point W.r.t. an Ellipse

The point $P(x_1, y_1)$ lines outside, inside or on the ellipse according as; $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 >< \text{or} = 0$.

Paramatric Representation

The equations $x = a\cos\theta$ and $y = b\sin\theta$ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where θ is a parameter (eccentric angle).

Note that if $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then ; $Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

Line and an Ellipse

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two real points, coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$.

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α and β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$.

Tangent to the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(a) **Point form:** Equation of tangent to the given ellipse at its point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

(b) **Slope form:** Equation of tangent to the given ellipse whose slope is 'm', $y = mx \pm \sqrt{a^2m^2 + b^2}$.

Point of contact are $\left(\frac{\pm a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 + b^2}}\right)$

(c) **Parametric form:** Equation of tangent to the given ellipse at its point $(a\cos\theta, b\sin\theta)$, is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.

Normal to the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(a) **Point form:** Equation of the normal to the given ellipse at (x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$.

(b) **Slope form:** Equation of a normal to the given ellipse whose slope is 'm' is $y = mx \pm \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

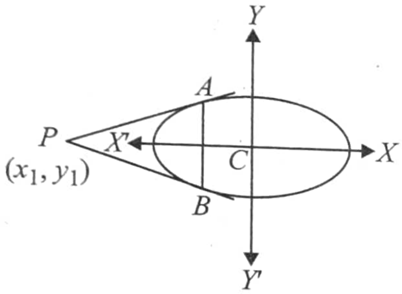
(c) Parametric form: Equation of the normal to the given ellipse at the point ($a\cos\theta$, $b\sin\theta$) is ax. sec θ – by. cosec $\theta = (a^2 - b^2)$.

Chord of Contact

If PA and PB be the tangents from point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the chord of contact AB is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ or $T=0$ at (x_1, y_1).

Pair or Tangents

If $P(x_1, y_1)$ be any point lies outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and a pair of tangents PA, PB can be drawn to it from P.



Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$, $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

$$\text{i.e., } \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

Director Circle

$x^2+y^2=a^2+b^2$ i.e. a circle whose centre is the centre of the ellipse and whose radius is the length of the line joining the ends of the major and minor axis.

Euqation of Chord with Mid Point (x_1, y_1)

$$\text{i.e. } \left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$$

Important Highlights

1. If P be any point on the ellipse with S and S' as its foci then $\ell(SP)+\ell(S'P)=2a$.
2. The locus of the point of interseciton of feet of perpendicular from foci on any tangent to an ellipse is the auxiliary circle.
3. The product of perpendicular distance from the foci to any tangent of an ellipse is equal to square of the semi minor axis.
4. Tangents at the extremities of latus-rectum of an ellipse intersect on the foot of corresponding directrix.
5. The portion of the tangent to an ellipse between the point of contact and the directrix subtends a right angle at the corresponding focus.
6. Tangent and normal at any point P bisect the external and internal angles between the focal distances of SP and $S'P$.
7. If the normal at any point P on the ellipse with centre C meet the major and minor axes in G and g respectively and if CF be perpendicular upon this normal then
   (i) $PF \cdot PG = b^2$ (ii) $PF \cdot Pg = a^2$
8. Area enclosed by an ellipse having length of major and minor axes as $2a$ and $2b$ is given by πab.

Solved Examples

1. Tangents are drawn from a point on the circle $x^2+y^2=9$ to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$.. If the angle between the tangents is θ then $\sin 2\theta$ is:

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{7}}{3}$

(c) $\frac{\sqrt{3}}{2}$ (d) 0

Sol. (d) Angle between tangent from point on director circle is $\frac{\pi}{2}$

2. If the variable line $y = kx + 2h$ is tangent to an ellipse $2x^2 + 3y^2 = 6$, then locus of $P(h, k)$ is a conic C whose eccentricity equals:

(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{7}}{3}$

(c) $\frac{\sqrt{7}}{2}$ (d) $\frac{\sqrt{7}}{4}$

Sol. (d) By using condition of tangency, we get

$4h^2 = 3k^2 + 2$

$\therefore$ Locus of $P(h, k)$ is $4x^2 - 3y^2 = 2$ (which is hperbola).

Hence $e^2 = 1+\frac{4}{3} \Rightarrow e=\sqrt{\frac{7}{3}}$

3. If the line $lx + my + n = 0$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at points whose eccentric angles differ by $\frac{\pi}{2}$, then find the value of $\frac{a^2\ell^2+b^2m^2}{n^2}$

Sol. Let the points of intersection of the line and the ellipse be $(a\cos\theta, b\sin\theta)$ and $\left(a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right)$. Since they lie on the given line $lx + my + n = 0$

$la\cos\theta + mb\sin\theta + n = 0$

$\Rightarrow la\cos\theta + mb\sin\theta = -n$

and $-la\sin\theta + mb\cos\theta + n = 0$

$\Rightarrow la\sin\theta - mb\cos\theta = n$

Squaring and adding, we get

$a^2l^2 + b^2m^2 = 2n^2$

$$\Rightarrow \frac{a^2\ell^2+b^2m^2}{n^2}=2$$

4. An ellipse passes through the point $(4, -1)$ and touches the line $x + 4y - 10 = 0$. Find its equation if its axes coincide with coordinate axes.

Sol. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(*i*)

It passes through $(4, -1)$ so $a^2 + 16b^2 = a^2b^2$ ...(*ii*)

Also $x + 4y - 10 = 0$ touches the ellipse

$\Rightarrow y = (-1/4)x + (5/2)$

$\Rightarrow \frac{25}{4}=\frac{a^2}{16}+b^2$

$\Rightarrow a^2 + 16b^2 = 100$ ...(*iii*)

From (*ii*) and (*iii*), we get

$a^2b^2 = 100$ or $ab = 10$ ...(*iv*)

Solving (*ii*) and (*iv*), we have

$(a = 4\sqrt{5}, b = \sqrt{5}/2)$ or $(a = 2\sqrt{5}, b = \sqrt{5})$

Hence there are two ellipse satisfying the given conditions, i.e.,

$$\frac{x^2}{80}+\frac{4y^2}{5}=1 \text{ and } \frac{x^2}{20}+\frac{y^2}{5}=1$$

5. If the normal at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the axes in G and g, respectively, then find the ratio $PG : Pg$.

Sol. Let $P(a\cos\theta, b\sin\theta)$ be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Then the equation of the normal at P is $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$

It meets the axes at $G\left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$ and

$g\left(0, -\frac{a^2-b^2}{a}\sin\theta\right)$

$$\therefore PG^2 = \left(a\cos\theta - \frac{a^2-b^2}{a}\cos\theta\right)^2 + b^2\sin^2\theta$$

$$= \frac{b^2}{a^2}\left(b^2\cos^2\theta + a^2\sin^2\theta\right)$$

and $Pg^2 = \frac{a^2}{b^2}\left(b^2\cos^2\theta + a^2\sin^2\theta\right)$

$\therefore PG : Pg = b^2 : a^2$

6. Prove that the locus of the mid-points of the intercepts of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, intercepted between the axes, is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$.

Sol. The tangent to the ellipse at any point $(a\cos\theta, b\sin\theta)$ is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

Let it meet the axes in P and Q, so that P is $(a\sec\theta, 0)$ and Q is $(0, b\operatorname{cosec}\theta)$. If (h, k) is the mid-point of PQ, then $h = \frac{a\sec\theta}{2}$

$\Rightarrow \cos\theta = \frac{a}{2h}$ and $k = \frac{b\operatorname{cosec}\theta}{2} \Rightarrow \sin\theta = \frac{b}{2k}$.

Squaring and adding, we get $\frac{a^2}{4h^2}+\frac{b^2}{4k^2}=1$.

Hence the locus of (h, k) is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$.

7. The tangents at any point $P(a\cos f, b\sin f)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle at two points which subtend a right angle at the centre. Prove that the eccentricity of the ellipse is $\frac{1}{\sqrt{1+\sin^2\phi}}$.

Sol. Tangent to the ellipse is $\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1$ ...(*i*)

Homogenizing the equating of auxiliary circle $x^2 + y^2 = a^2$ with the help of (1), we get

$$x^2+y^2=a^2\left(\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi\right)^2$$

$$\Rightarrow (1-\cos^2\phi)x^2 - \frac{2a}{b}\sin\phi\cos\phi\, xy$$

$$+\left(1-\frac{a^2}{b^2}\sin^2\phi\right)y^2=0.$$

It represents a pair of perpendicular lines if

$$(1-\cos^2\phi)+\left(1-\frac{a^2}{b^2}\sin^2\phi\right)=0$$

$$\Rightarrow \frac{b^2}{a^2}=\frac{\sin^2\phi}{1+\sin^2\phi}\Rightarrow 1-e^2=\frac{\sin^2\phi}{1+\sin^2\phi}$$

$$\therefore e^2=\frac{1}{1+\sin^2\phi} \text{ or } e=\frac{1}{\sqrt{1+\sin^2\phi}}$$

8. If the tangent drawn at a point $(k^2, 2k)$ on the parabola $y^2=4x$ is the same as the normal drawn at a point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2+5y^2=20$, find the values of k and q.

Sol. Equation of the tangent at $(k^2, 2k)$ to the parabola $y^2=4x$ is

$x-ky+k^2=0.$ ...(*i*)

Equation of the normal at $(\sqrt{5}\cos\theta, 2\sin\theta)$ to the ellipse $\frac{x^2}{5}+\frac{y^2}{4}=1$ is

$$\sqrt{5}\sec\theta\, x-2\operatorname{cosec}\theta\, y=1$$

or $x-\frac{2}{\sqrt{5}}\cot\theta,\ y-\frac{1}{\sqrt{5}}\cos\theta=0.$ ...(*ii*)

Since (*i*) and (*ii*) represent the same line, $1=\frac{2\cos\theta}{\sqrt{5}k}=\frac{-\cos\theta}{\sqrt{5}k^2}.$

Eliminating t from these equations, we get

$4\cot^2\theta=-\sqrt{5}\cos\theta \Rightarrow \cos\theta=0$ or $\sqrt{5}\cos^2\theta-4\cos\theta-\sqrt{5}=0$

$$\Rightarrow \theta=\frac{\pi}{2},\frac{3\pi}{2} \text{ or } \cos\theta=-\frac{1}{\sqrt{5}} \text{ i.e. } \theta=\pi\pm\cos^{-1}\frac{1}{\sqrt{5}}$$

So that, $k=0, 0, \frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}.$

9. The tangent and normal to the ellipse $x^2+4y^2=4$ at a point $P(\theta)$ on it meets the major axes in Q and R respectively. If $QR=2$, then $\cos\theta$ is equal to

(*a*) $\frac{1}{\sqrt{3}}$ (*b*) $\frac{2}{3}$

(*c*) $\frac{1}{3}$ (*d*) $\frac{1}{\sqrt{2}}$

Sol. Ellipse $\equiv \frac{x^2}{4}+y^2=1$

let $P(\theta)\equiv(2\cos\theta, \sin\theta)$

Equation of the tangent at P: $\frac{x\cos\theta}{2}+y\sin\theta=1$ ...(*i*)

$\Rightarrow Q\equiv(2\sec\theta, 0).$

Equation of the normal at P : $2x\sec\theta-y\operatorname{cosec}\theta=3$

$\Rightarrow R\equiv\left(\frac{3}{2}\cos\theta, 0\right).$

Therefore $QR=$

$$\left|2\sec\theta-\frac{3}{2}\cos\theta\right|=2\Rightarrow\left|\frac{4-3\cos^2\theta}{2\cos\theta}\right|=2$$

$16+9\cos^4\theta-24\cos^2\theta=16\cos^2\theta$

or $9\cos^4\theta-40\cos^2\theta+16=0$

or $9\cos^4\theta-36\cos^2\theta-4\cos^2\theta+16=0$

or $(9\cos^2\theta-4)(\cos^2\theta-4)=0$

$\Rightarrow \cos^2\theta=\frac{4}{9}$

$\Rightarrow \cos\theta=\pm\frac{2}{3}$

Hence (*b*) is the correct answer.

10. From a point P tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ If the chord of contact is normal to the ellipse, the find the locus of P.

Sol. Taking $P(x_1, y_1)$ the chord of contact is $S_1=0$

i.e., $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ...(*i*)

A normal to the ellipse at any point $(a\cos\theta, b\sin\theta)$ is

$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ ...(*ii*)

(*i*) and (*ii*) imply, $\frac{x_1\cos\theta}{a^3}=\frac{-y_1\sin\theta}{b^3}=\frac{1}{a^2-b^2}$

Eliminating θ we get $\frac{a^6}{x_1}+\frac{b^6}{y_1}=(a^2-b^2)^2$

Hence the locus of P is $\frac{a^6}{x_2}+\frac{b^6}{y_2}=(a^2-b^2)^2.$

11. From a point P tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ If the chord of contact touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ then find the locus of P.

Sol. Let $P(x_1, y_1)$, the chord of contact is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

or $y=\left(\frac{-b^2}{a^2}\cdot\frac{x_1}{y_1}\right)x+\frac{b_2}{y_1}$

If it is a tangent to $\frac{x^2}{a^2}+\frac{y^2}{d^2}=1$. Then $\frac{b^4}{y_1^2}=c^2\frac{b^4x_1^2}{a^4y_1^2}+d^2$

Hence the locus of P is $\frac{a^2x^2}{a^4}+\frac{d^2y^2}{b^4}=1$.

12. Find a point on the curve $x^2+2y^2=6$, whose distance from the line $x+y=7$ is minimum.

Sol. The given equation can be written as $\frac{x^2}{6}+\frac{y^2}{3}=1$, which represents an ellipse.

Any point on this ellipse is $P(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$. The shortest distance between the ellipse and the given line is along the common normal to both.

Slope of the normal at P is

$\frac{\sqrt{6}\sec\theta}{\sqrt{3}\operatorname{cosec}\theta}=\sqrt{2}\tan\theta = 1=$ slope of normal to the line $x+y=7$.

Hence $\cos\theta=\sqrt{\frac{2}{3}}, \sin\theta=\frac{1}{\sqrt{3}}$ so that P is (2, 1).

13. If PQ be a chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which subtends right angle at the centre then its distance from the centre is equal to

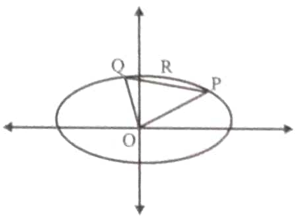
(a) $\frac{ab}{\sqrt{a^2+b^2}}$

(b) $\sqrt{a^2+b^2}$

(c) $\sqrt{ab}$

(d) Depends on the slope of the chord

Sol. Let $x\cos\alpha+y\sin\alpha=p$ be the chord PQ, then p is the desired distance. Homogenizing the equation of the ellipse with the help of this equation, we get the combined equation of OP and OQ,



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{x\cos\alpha+y\sin\alpha}{p}\right)^2$$

$$\Rightarrow \left(\frac{1}{a^2}-\frac{\cos^2\alpha}{p^2}\right)x^2$$

$$+\left(\frac{1}{b^2}-\frac{\sin^2\alpha}{p^2}\right)y^2-\frac{2xy\sin\alpha\cos\alpha}{p^2}=0$$

As OP $\perp$ OQ, so, $\frac{1}{a^2}-\frac{\cos^2\alpha}{p^2}+\frac{1}{b^2}-\frac{\sin^2\alpha}{p^2}=0$

$$\Rightarrow \quad p=\frac{ab}{\sqrt{a^2+b^2}}$$

Hence (a) is the correct answer.

Comprehension (Q. 14 to 15): Let O be the centre of an ellipse $(E): \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ and having AB and CD as its major and minor axes respectively. If S_1 be one of the foci of the ellipse, radius of incircle of ΔOCS_1 be 1 unit and $OS_1=6$ unit, then

14. The area of ellipse (E) is

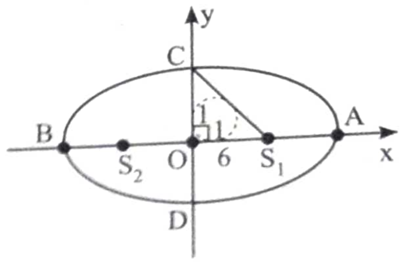
(a) $\frac{65\pi}{4}$

(b) 16π

(c) $\frac{25\pi}{4}$

(d) 30π

Sol. (a)



$OS_1=ae=6$, $OC=b$ (suppose)

Also, suppose $CS_1=a$

$\because$ Area of $\Delta OCS_1=\frac{1}{2}(OS_1)\times(OC)=3b$

$=\frac{1}{2}(6+a+b)$

$\therefore$ In radius of $\Delta OCS_1=1 \Rightarrow \frac{3b}{\frac{1}{2}(6+a+b)}=1$

Area of ellipse $=\pi ab=\frac{65\pi}{4}$

15. Perimeter of ΔOCS_1 is

(a) 10 unit (b) 12 unit (c) 15 unit (d) 18 unit

Sol. (c) Perimeter of $\Delta OCS_1=6+a+b$

$=6+13/2+5/2=15$ unit.

16. The values of m for which it is possible to draw the chord $y=\sqrt{m}x+1$ to the ellipe $x^2+(2+\sin^2\alpha)y^2=1$, which subtends of right angle at orogin for some value of α is/are

(a) $m=2$

(b) $m=\frac{3}{2}$

(c) $m=3$

(d) $m=\frac{7}{2}$

Sol. (a, b, c) Combined equation of apri fo lines through the origin joining the points of intersection of lien $y=\sqrt{m}x+1$ with the given curve is $x^2 + (2 + \sin\alpha)y^2 - (y - mx)^2 = 0$ for the chord to subtend a right angle at the origin $(1 - m) + (2 + \sin^2\alpha - 1) = 0$ (as sum of the coefficient of $x^2 + y^2 = 0$)

$\Rightarrow \sin^2\alpha = m - 2$

$\Rightarrow 0 \le m - 2 \le 1$

$\Rightarrow 2 \le m \le 3$

17. Find the set of value(s) of 't' for which the point $P(4t, 2t^2)$ does not lie inside the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{2} = 1$.

Sol. $\dfrac{x^2}{16} + \dfrac{y^2}{2} = 1 \; P(4t, 2t^2)$

Curve C_1: $8y = x^2$ $\{(2at, at^2)\, x^2 = 4ay\}$

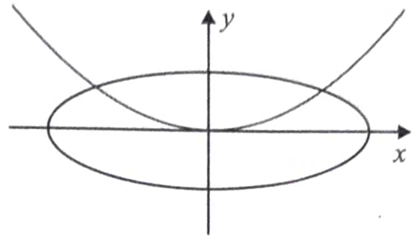
Curve C_2: $\dfrac{x^2}{16} + \dfrac{y^2}{2} = 1$

Pt. of intersection of C_1 and C_2 are $P = (2\sqrt{2}, 1), Q(-2\sqrt{2}, 1)$

So $4t \nsubseteq (-2\sqrt{2}, 2\sqrt{2})$

$\Rightarrow |t| \nsubseteq 2\sqrt{2}$

$\Rightarrow |t| \nsubseteq \dfrac{1}{\sqrt{2}}$



Alternative: $S_1 \ge 0$

$\Rightarrow t^2 + 2t^4 \ge 1$

$\Rightarrow (t^2 + 1)(2t^2 - 1) \ge 0$

$\Rightarrow 2t^2 - 1 \ge 0 \quad \Rightarrow |t| \ge \dfrac{1}{\sqrt{2}}$

Exercise-1 (Topicwise)

EQUATION OF ELLIPSE

1. The equation to the ellipse (referred to its axes as the axes of x and y respectively) whose foci are $(\pm 2, 0)$ and eccentricity 1/2, is

(a) $\frac{x^2}{12}+\frac{y^2}{16}=1$ (b) $\frac{x^2}{16}+\frac{y^2}{12}=1$

(c) $\frac{x^2}{16}+\frac{y^2}{8}=1$ (d) $\frac{x^2}{8}+\frac{y^2}{16}=1$

2. The eccentricity of the ellipse $9x^2+5y^2-30y=0$ is:

(a) 1/3 (b) 2/3

(c) 3/4 (d) 1/2

3. The distance between the directrices of the ellipse $\frac{x^2}{36}+\frac{y^2}{20}=1$ is

(a) 8 (b) 12 (c) 18 (d) 24

4. The distance between the foci of the ellipse $3x^2+4y^2=48$ is

(a) 2 (b) 4 (c) 6 (d) 8

5. The equation of the ellipse whose one of the vertices is (0,7), centre at (0, 0) and the corresponding directrix is $y = 12$, is

(a) $95x^2+144y^2=4655$ (b) $144x^2+95y^2=4655$

(c) $95x^2+144y^2=13680$ (d) $144x^2+95y^2=13680$

6. The foci of $16x^2+25y^2=400$ are

(a) $(\pm 3, 0)$ (b) $(0, \pm 3)$

(c) $(3, -3)$ (d) $(-3, 3)$

7. The locus of a variable point whose distance from $(-2, 0)$ is $\frac{2}{3}$ times its distance from the line $x=-\frac{9}{2}$, is

(a) Ellipse (b) Parabola

(c) Hyperbola (d) Circle

8. If the eccentricity of the two ellipse $\frac{x^2}{169}+\frac{y^2}{25}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are equal, then the value of $|a/b|$ (where $|a|>|b|$) is

(a) 5/13 (b) 6/13 (c) 13/5 (d) 13/6

9. The centre of the ellipse $4x^2+9y^2-16x-54y+61=0$ is

(a) (1,3) (b) (2, 3) (c) (3, 2) (d) (3, 1)

10. Latus rectum of ellipse $4x^2+9y^2-8x-36y+4=0$ is

(a) 8/3 (b) 4/3 (c) $\frac{\sqrt{5}}{3}$ (d) 16/3

11. The equation of the ellipse whose centre is (2, –3), one of the foci is (3, –3) and the corresponding vertex is (4, –3) is

(a) $\frac{(x-2)^2}{3}+\frac{(y+3)^2}{4}=1$ (b) $\frac{(x-2)^2}{4}+\frac{(y+3)^2}{3}=1$

(c) $\frac{x^2}{3}+\frac{y^2}{4}=1$ (d) $\frac{(x-3)^2}{4}+\frac{(y-2)^2}{3}=1$

12. The equation $14x^2-4xy+11y^2-44x-58y+71=0$ represents

(a) A circle (b) An ellipse

(c) A hyperbola (d) A rectangular hyperbola

13. The foci of the ellipse $25(x+1)^2+9(y+2)^2=225$ are at

(a) (–1, 2) and (–1, –6) (b) (–1, 2) and (6, 1)

(c) (1, –2) and (1, –6) (d) (–1, –2) and (1, 6)

14. The co-ordinates of the foci of the ellipse $3x^2+4y^2-12x-8y+4=0$ are

(a) (1, 2), (3, 4) (b) (1, 4), (3, 1)

(c) (1, 1), (3, 1) (d) (2, 3), (5, 4)

15. For the ellipse $25x^2+9y^2-150x-90y+225=0$ the eccentricity $e =$

(a) 2/5 (b) 3/5 (c) 4/5 (d) 1/5

16. The eccentric angles of the extremities of latus recta of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by ($|a|>|b|$)

(a) $\tan^{-1}\left(\pm\frac{ae}{b}\right)$ (b) $\tan^{-1}\left(\pm\frac{be}{a}\right)$

(c) $\tan^{-1}\left(\pm\frac{b}{ae}\right)$ (d) $\tan^{-1}\left(\pm\frac{a}{be}\right)$

17. If the foci of an ellipse are $(\pm\sqrt{5}, 0)$ and its eccentricity is $\frac{\sqrt{5}}{3}$, then the equation of the ellipse is

(a) $9x^2+4y^2=36$ (b) $4x^2+9y^2=36$

(c) $36x^2+9y^2=4$ (d) $9x^2+36y^2=4$

18. The distance between directrix of the ellipse $(4x-8)^2+16y^2=(x+\sqrt{3}y+10)^2$ is

(a) 12 (b) 16 (c) 20 (d) 24

PARAMATRIC EQUATION

19. The parametric representation of a point on the ellipse whose foci are (– 1, 0) and (7, 0) and eccentricity 1/2 is:

(a) $(3+8\cos\theta, 4\sqrt{3}\sin\theta)$

(b) $(8\cos\theta, 4\sqrt{3}\sin\theta)$

(c) $(3+4\sqrt{3}\cos\theta, 8\sin\theta)$

(d) $(8+\sqrt{3}\sin\theta, 8\cos\theta)$

20. Suppose x and y are real numbers and that $x^2+9y^2-4x+6y+4=0$ then find the maximum value of $(4x-9y)$.

(a) 14 (b) 16

(c) 18 (d) 20

AUXILLARY CIRCLE, POINT AND ELLIPSE, CHORD JOINING TWO POINTS

21. The position of the point (4, –3) with respect to the ellipse $2x^2+5y^2=20$ is

(a) Outside the ellipse (b) On the ellipse

(c) On the major axis (d) Inside the ellipse

TANGENT AND NORMAL

22. The equation of the tangents to the ellipse $4x^2+3y^2=5$ which are parallel to the line $y=3x+7$ are

(a) $y=3x\pm\sqrt{\frac{155}{3}}$ (b) $y=3x\pm\sqrt{\frac{155}{12}}$

(c) $y=3x\pm\sqrt{\frac{95}{12}}$ (d) $y=3x\pm\sqrt{\frac{135}{12}}$

23. Find the equations of tangents to the ellipse $9x^2+16y^2=144$ which pass through the point (2, 3).

(a) $y=3$ and $y=-x+5$ (b) $y=5$ and $y=-x+3$

(c) $y=3$ and $y=x-5$ (d) $y=5$ and $y=x+3$

24. The area of the quadrilateral formed by the tangents at the end points of latus rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$, is

(a) 27/4 sq. unit (b) 9 sq. unit

(c) 27/2 sq. unit (d) 27 sq. unit

25. The line $lx+my-n=0$ will be tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if

(a) $a^2l^2+b^2m^2=n^2$ (b) $al^2+bm^2=n^2$

(c) $a^2l+b^2m=n$ (d) $al^2+bm^2=n$

26. The angle between the tangents drawn to the ellipse $3x^2+2y^2=5$ from the point (1, 2), is

(a) $\tan^{-1}\left(\frac{12}{5}\right)$ (b) $\tan^{-1}(6\sqrt{5})$

(c) $\tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$ (d) $\tan^{-1}(12\sqrt{5})$

27. The locus of the point of intersection of the perpendicular tangents to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is

(a) $x^2+y^2=9$ (b) $x^2+y^2=4$

(c) $x^2+y^2=13$ (d) $x^2+y^2=5$

28. The equation of the tangents drawn at the ends of the major axis of the ellipse $9x^2+5y^2-30y=0$, are

(a) $y=\pm 3$ (b) $x=\pm\sqrt{5}$

(c) $y=0, y=6$ (d) $x^2+y^2=5$

29. The equation of normal at the point (0, 3) of the ellipse $9x^2+5y^2=45$ is

(a) $y-3=0$ (b) $y+3=0$

(c) x-axis (d) y-axis

30. The equation of tangent and normal at point (3, –2) of ellipse $4x^2+9y^2=36$ are

(a) $\frac{x}{3}-\frac{y}{2}=1, \frac{x}{2}+\frac{y}{3}=\frac{5}{6}$ (b) $\frac{x}{3}+\frac{y}{2}=1, \frac{x}{2}-\frac{y}{3}=\frac{5}{6}$

(c) $\frac{x}{2}+\frac{y}{3}=1, \frac{x}{3}-\frac{y}{2}=\frac{5}{6}$ (d) $\frac{x}{2}+\frac{y}{3}=1, \frac{x}{3}+\frac{y}{2}=\frac{5}{6}$

31. The value of λ, for which the line $2x-\frac{8}{3}\lambda y=-3$ is a normal to the conic $x^2+\frac{y^2}{4}=1$ is

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $\frac{3}{2}$ (d) $\frac{3}{8}$

PROPERTIES OF ELLIPSE

32. If radius of the director circle of the ellipse

$$\frac{(3x+4y-2)^2}{100}+\frac{(4x-3y+5)^2}{625}=1 \text{ is :}$$

(a) 6 (b) $\sqrt{34}$ (c) $\sqrt{29}$ (d) $\sqrt{26}$

33. An ellipse has foci at (9, 20) and (49, 55) in the xy-plane and is tangent to the x-axis. The length of its major axis is

(a) 85 (b) 75 (c) 65 (d) 55

34. If the normals at α, β, γ, and δ on an ellipse are concurrent, then the value of $(\Sigma\cos\alpha)(\Sigma\sec\alpha)$ is

(a) 2 (b) 4

(c) 6 (d) 8

35. Find a and b for the ellipse $b^2x^2+a^2y^2=a^2b^2$ if the distance between the directrices is $5\sqrt{5}$ and the distance between the foci is $4\sqrt{5}$.

(a) $a=\pm 5, b=\pm\sqrt{5}$ (b) $a=\pm\sqrt{5}, b=\pm 5$

(c) $a=\pm 5\sqrt{5}, b=\pm 5\sqrt{5}$ (d) None of these

36. The equation of the ellipse whose focus is (1, –1), directrix is the line $x-y-3=0$ and the eccentricity is $\frac{1}{2}$, is

(a) $7x^2+2xy+7y^2-10x+10y+7=0$

(b) $7x^2+2xy+7y^2+7=0$

(c) $7x^2+2xy+7y^2+10x-10y-7=0$

(d) None of these

37. The eccentricity of the ellipse $4x^2 + 9y^2 + 8x + 36y + 4 = 0$ is

(a) $\frac{5}{6}$ (b) $\frac{3}{5}$

(c) $\frac{\sqrt{2}}{3}$ (d) $\frac{\sqrt{5}}{3}$

38. If distance between the directrices be thrice the distance between the focii, then eccentricity of ellipse is

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$

(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{4}{5}$

39. The equation $\frac{x^2}{2-r} + \frac{y^2}{r-5} + 1 = 0$ represents an ellipse, if

(a) $r > 2$ (b) $2 < r < 5$

(c) $r > 5$ (d) $r \in \{2, 5\}$

40. The curve represented by $x = 3(\cos t + \sin t)$, $y = 4(\cos t - \sin t)$, is

(a) Ellipse (b) Parabola

(c) Hyperbola (d) Circle

41. The position of the point (1, 3) with respect to the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$

(a) Outside the ellipse (b) On the ellipse

(c) On the major axis (d) On the minor axis

42. The angle between the tangents drawn from the point (2, 2) to the ellipse, $3x^2 + 5y^2 = 15$ is:

(a) $\pi/6$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

43. If F_1 and F_2 are the feet of the perpendiculars from the focii S_1 & S_2 of an ellipse $\frac{x^2}{5} + \frac{y^2}{3} = 1$ on the tangent at any point P on the ellipse, then $(S_1F_1).(S_2F_2)$ is equal to :

(a) 2 (b) 3 (c) 4 (d) 5

44. If the distance between the focii of an ellipse is equal to the length of its latus rectum, the eccentricity of the ellipse is :

(a) $\frac{\sqrt{5}+1}{2}$ (b) $\frac{\sqrt{5}-1}{2}$

(c) $\frac{\sqrt{5}+2}{2}$ (d) $\frac{2}{\sqrt{5}+1}$

45. Let P be any point on ellipse $3x^2 + 4y^2 = 12$ and S, S' are its foci then the locus of the centroid of triangle PSS' is a conic C which is incorrect?

(a) Length of latus rectum equals 1.

(b) Locus of point of intersection of perpendicular tangents is $x^2 + y^2 = \frac{7}{9}$

(c) Equation of auxiliary circle is $x^2 + y^2 = \frac{4}{9}$

(d) Area of quadrilateral formed by tangents at the extremities of latus-rectum equals $\frac{16}{3}$.

46. The length of major axis of the ellipse $(5x - 10)^2 + (5y + 15)^2 = \frac{(3x-4y+7)^2}{4}$ is:

(a) $\frac{20}{3}$ (b) $\frac{15}{3}$

(c) $\frac{10}{3}$ (d) $\frac{25}{3}$

47. The distance between the directrices of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is $\frac{a}{\sqrt{b}}$, then a/b equals

(a) $\frac{9}{\sqrt{5}}$ (b) $\frac{18}{\sqrt{5}}$

(c) $\frac{12}{\sqrt{5}}$ (d) $\frac{21}{\sqrt{5}}$

Exercise-2 (Learning Plus)

1. If the eccentricity of an ellipse be 5/8 and the distance between its foci be 10, then its latus rectum is

(a) $\frac{39}{4}$ (b) 12

(c) 15 (d) $\frac{37}{2}$

2. An ellipse having foci at (3, 3) and (–4, 4) and passing through the origin has eccentricity equal to

(a) $\frac{3}{7}$ (b) $\frac{2}{7}$

(c) $\frac{5}{7}$ (d) $\frac{3}{5}$

3. A ladder 12 units long slides in a vertical plane with its ends in contact with a vertical wall and a horizontal floor along x–axis. The locus of a point on the ladder 4 units from its foot has the equation:

(a) $\frac{x^2}{4} + y^2 = 1$ (b) $\frac{x^2}{16} + \frac{y^2}{64} = 1$

(c) $\frac{x^2}{64} + \frac{y^2}{16} = 1$ (d) $x^2 + \frac{y^2}{4} = 1$

4. The combined equation of directrices of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is (1,1) and minor axis is $x + y = 4$ is:

(a) $x^2 + y^2 + 2xy + 8(x + y) + 48 = 0$

(b) $x^2 + y^2 + 2xy - 8(x + y) + 48 = 0$

(c) $x^2 + y^2 + 2xy - 8(x + y) - 48 = 0$

(d) $x^2 + y^2 + 2xy + 8(x + y) - 48 = 0$

5. Let P be a variable point on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ with focii at S and S'. If A be the area of triangle PSS', then the maximum value of A is

(a) 24 sq. units (b) 12 sq. units
(c) 36 sq. units (d) 6 sq. units

6. If the distance of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ from the centre is 2, then the eccentric angle is

(a) $\pi/3$ (b) $\pi/4$
(c) $\pi/6$ (d) $\pi/2$

7. The eccentricity of the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ is decreasing at the rate of 0.1/second due to change in semi minor axis only. The time at which ellipse become auxiliary circle is

(a) 2 seconds (b) 3 seconds
(c) 4 seconds (d) 5 seconds

8. If $\tan\theta_1 \cdot \tan\theta_2 = -\frac{a^2}{b^2}$ then the chord joining two points θ_1 & θ_2 on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will subtend a right angle at

(a) Focus (b) Centre
(c) End of the major axis (d) End of the minor axis

9. If $\frac{x}{a}+\frac{y}{b}=\sqrt{2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point P, then eccentric angle of P is

(a) 0 (b) 45°
(c) 60° (d) 90°

10. If the line $y = 2x + c$ be a tangent to the ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$, then c is equal to

(a) ± 4 (b) ± 6
(c) ± 1 (d) ± 8

11. If the line $3x + 4y = -\sqrt{7}$ touches the ellipse $3x^2 + 4y^2 = 1$ then, the point of contact is

(a) $\left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$ (b) $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

(c) $\left(\frac{1}{\sqrt{7}}, \frac{-1}{\sqrt{7}}\right)$ (d) $\left(\frac{-1}{\sqrt{7}}, \frac{-1}{\sqrt{7}}\right)$

12. If the line $x\cos\alpha + y\sin\alpha = p$ be normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then

(a) $p^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = a^2 - b^2$
(b) $p^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = (a^2 - b^2)^2$
(c) $p^2(a^2\sec^2\alpha + b^2\operatorname{cosec}^2\alpha) = a^2 - b^2$
(d) $p^2(a^2\sec^2\alpha + b^2\operatorname{cosec}^2\alpha) = (a^2 - b^2)^2$

13. The equation to the locus of the middle point of the portion of the tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ included between the co-ordinate axes is the curve

(a) $9x^2 + 16y^2 = 4x^2y^2$ (b) $16x^2 + 9y^2 = 4x^2y^2$
(c) $3x^2 + 4y^2 = 4x^2y^2$ (d) $9x^2 + 16y^2 = x^2y^2$

14. Which of the following is the common tangent to the ellipses $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ & $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$?

(a) $ay = bx + \sqrt{a^4 - a^2b^2 + b^4}$
(b) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$
(c) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$
(d) $by = ax - \sqrt{a^4 - a^2b^2 + b^4}$

15. Equation of one of the common tangent of $y^2 = 4x$ and $\frac{x^2}{4}+\frac{y^2}{3}=1$ is:

(a) $x + 2y + 4 = 0$ (b) $x + 2y - 4 = 0$
(c) $x - 2y - 4 = 0$ (d) $2x + y - 4 = 0$

16. The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the positive end of latus rectum is

(a) $x + ey + e^2a = 0$ (b) $x - ey - e^3a = 0$
(c) $x - ey - e^2a = 0$ (d) $x + ey - e^3a = 0$

17. The point of intersection of the tangents at the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its corresponding point Q on the auxiliary circle, lies on the line :

(a) $x = a/e$ (b) $x = 0$
(c) $y = 0$ (d) $y = a/e$

18. Equation of the line passing through centre and bisecting the chord $7x + y = 20$ of the ellipse $x^2 + \frac{y^2}{7} = 1$, is:

(a) $x + y = 5$ (b) $x + y = 0$
(c) $x - y = 0$ (d) $x - y = 5$

19. The eccentric angle of the point where the line, $5x - 3y = 8\sqrt{2}$ is a normal to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ is

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) $\tan^{-1} 2$

20. PQ is a double ordinate of the ellipse $x^2 + 9y^2 = 9$, the normal at P meets the diameter through Q at R, then the locus of the mid point of PR is:

(a) A circle (b) A parabola

(c) An ellipse (d) A hyperbola

21. The equation of the chord of the ellipse $2x^2 + 5y^2 = 20$ which is bisected at the point (2, 1) is

(a) $4x + 5y + 13 = 0$ (b) $4x + 5y = 13$

(c) $5x + 4y + 13 = 0$ (d) $5x + 4y = 13$

22. Q is a point on the auxiliary circle of an ellipse. P is the corresponding point on ellipse. N is the foot of perpendicular from focus S, to the tangent of auxiliary circle at Q. Then

(a) $SP = SN$ (b) $SP = PQ$

(c) $PN = SP$ (d) $NQ = SP$

23. A ray passing through $\left(\sqrt{\frac{5}{6}}, 0\right)$ strikes the surface of the curve $2x^2 + 3y^2 = 5$ and gets reflected then reflected ray will pass through

(a) $\left(-\sqrt{\frac{5}{6}}, 0\right)$ (b) $\left(0, \sqrt{\frac{5}{6}}\right)$

(c) (5, 6) (d) None of these

24. If (x, y) lies on the ellipse $x^2 + 2y^2 = 2$, then maximum value of $x^2 + y^2 + \sqrt{2}\,xy - 1$ is

(a) $\frac{\sqrt{5}+1}{2}$ (b) $\frac{\sqrt{5}-1}{2}$

(c) $\frac{\sqrt{5}+1}{4}$ (d) $\frac{\sqrt{5}-1}{4}$

25. The minimum value of $\{(r+5-4|\cos\theta|)^2+(r-3|\sin\theta|)^2\}$ $\forall r, \theta \in R$ is

(a) 0 (b) 2 (c) 3 (d) 4

26. How many tangents to the circle $x^2 + y^2 = 3$ are normal to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$?

(a) 3 (b) 2 (c) 1 (d) 0

27. Number of integral values of 'α' for which the point $\left(7-\frac{5}{4}\alpha, \alpha\right)$ lies inside the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ is

(a) 0 (b) 1 (c) 2 (d) 3

28. If the normal at $P\left(2, \frac{3\sqrt{3}}{2}\right)$ meets the major axis of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at Q and S' and S are foci of the given ellipse then $SQ : S'Q$ (P is nearer to S) is

(a) $\frac{4-\sqrt{7}}{4+\sqrt{7}}$ (b) $\frac{8+\sqrt{7}}{8-\sqrt{7}}$ (c) $\frac{4+\sqrt{7}}{4-\sqrt{7}}$ (d) $\frac{8-\sqrt{7}}{8+\sqrt{7}}$

29. The equation, $3x^2 + 4y^2 - 18x + 16y + 43 = C$.

(a) Cannot represent a real pair of straight lines for any value of C

(b) Represents an ellipse, if $C > 0$

(c) No locus, if $C < 0$

(d) A point, if $C = 0$

30. If P is a point of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose focii are S and S'. Let $\angle PSS' = \alpha$ and $\angle PS'S = \beta$, then

(a) $PS + PS' = 2a$, if $a > b$

(b) $PS + PS' = 2b$, if $a < b$

(c) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e}$

(d) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{\sqrt{a^2-b^2}}{b^2}\left[a-\sqrt{a^2-b^2}\right]$ when $a > b$

31. Point/points, from which tangents to the ellipse $5x^2 + 4y^2 = 20$ are perpendicular, is/are :

(a) $(1, 2\sqrt{2})$ (b) $(2\sqrt{2}, 1)$

(c) $(2, \sqrt{5})$ (d) $(\sqrt{5}, 2)$

32. Identify the statements which are True.

(a) The equation of the director circle of the ellipse, $5x^2 + 9y^2 = 45$ is $x^2 + y^2 = 14$.

(b) The sum of the focal distances of the point (0, 6) on the ellipse $\frac{x^2}{25}+\frac{y^2}{36}=1$ is 10.

(c) The point of intersection of any tangent to a parabola & the perpendicular to it from the focus lies on the tangent at the vertex.

(d) The line through focus and $(at_1^2, 2at_1)$ on $y^2 = 4ax$, meets it again in the point $(at_2^2, 2at_2)$ if $t_1t_2 = -1$.

33. Identify correct statement(s) about conic

$$\sqrt{(x-5)^2+(y-7)^2}+\sqrt{(x+1)^2+(y+1)^2}=12$$

(a) Centre of conic is (2, 3)

(b) Conic is hyperbola with foci (5, 7) and (–1, –1)

(c) Conic is ellipse with major axis $4x - 3y + 1 = 0$

(d) Eccentricity of conic is $\frac{5}{7}$

34. If the tangent to the ellipse $x^2 + 4y^2 = 16$ at $P(4\cos\theta, 2\sin\theta)$ is also normal to circle $x^2 + y^2 = 8x + 4y$, then θ can be

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) 0 (d) $\frac{7\pi}{4}$

35. P and Q are two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentric angles are differ by 90°, then:

(a) Locus of point of intersection of tangents at P and Q is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$

(b) Locus of mid-point (P, Q) is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$

(c) Product of slopes of OP and OQ where O is the centre is $\frac{-b^2}{a^2}$

(d) Max. area of ΔOPQ is $\frac{1}{2}$ ab

36. Match the column:

	Column-I		Column-II
A.	If the mid point of a chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is (0, 3), then length of the chord is $\frac{4k}{5}$, then k is	p.	6
B.	Eccentric angle of one of the points on the ellipse $x^2+3y^2=6$ at a distance 2 units from the centre of the ellipse is $\frac{k\pi}{4}$, then k is	q.	8
C.	If the distance between a focus and corresponding directix of an ellipse be 8 and the eccentricity be $\frac{1}{2}$, then length of the minor axis is $\frac{k}{\sqrt{3}}$, then k is	r.	3
D.	Sum of distances of a point on the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ from the focii	s.	16

(*a*) A → q; B → r; C → s; D → p
(*b*) A → p; B → q; C → r; D → s
(*c*) A → r; B → q; C → s; D → p
(*d*) A → q; B → r; C → p; D → s

37. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ intersects it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to

38. Tangents are drawn from the points on the line $x-y-5=0$ to $x^2+4y^2=4$. Then all the chords of contact pass through a fixed point, find its coordinates.

39. An ellipse has foci at $F_1(9, 20)$ and $F_2(49, 55)$ in the xy-plane and is tangent to the x-axis. Find the length of its major axis.

40. If the normals at $P(\theta)$ and $Q\left(\frac{\pi}{2}+\theta\right)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1; a>b$ meet the major axis at G and g, respectively, then the value of PG^2+Qg^2 is

41. If normal at any point P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$ meets the axes at M and N so that at $\frac{PM}{PN}=\frac{2}{3}$, then value of eccentricity e is

42. The tangent and normal at a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the minor axis is A and B. S and S' are the foci of the ellipse, then find $\angle APB+\angle ASB+\angle AS'B$.

43. A normal chord to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ make an angle of 45° with the axis. Find the square of its length

44. Prove that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angle differ by a constant α is an ellipse.

45. A point moves such that the sum of the square of the distances from two fixed straight lines intersecting at angle 2α is a constant. Prove that the locus is an ellipse of eccentricity

$$\frac{\sqrt{\cos 2\alpha}}{\cos\alpha} \text{ if } \alpha<\frac{\pi}{4} \text{ and } \frac{\sqrt{-\cos 2\alpha}}{\sin\alpha} \text{ if } \alpha>\frac{\pi}{4}.$$

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. If the reflection of the ellipse $\frac{(x-4)^2}{16}+\frac{(y-3)^2}{9}=1$ in the mirror line $x-y-2=0$ is $k_1x^2+k_2y^2-160x-36y+292=0$, then $\frac{k_1+k_2}{5}$ is equal to

(*a*) 4 (*b*) 5
(*c*) 6 (*d*) 7

2. A circle has the same centre as an ellipse & passes through the foci F_1 and F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the foci is less than

(*a*) 12 (*b*) 13
(*c*) 14 (*d*) 15

3. Point 'O' is the centre of the ellipse with major axis AB and minor axis CD. Point F is one focus of the ellipse. If $OF=6$ and the diameter of the inscribed circle of triangle OCF is 2, then the product $(AB)(CD)$ is equal to

(*a*) 65 (*b*) 52 (*c*) 78 (*d*) 87

4. The locus of feet of perpendicular from either foci of the ellipse $(x-y+1)^2+(2x+2y-6)^2=20$ on any tangent will be

(a) $x^2+y^2+2x+4y+5=0$
(b) $x^2+y^2+2x+4y-5=0$
(c) $x^2+y^2-2x-4y-5=0$
(d) $x^2+y^2-2x-4y+5=0$

5. In an ellipse, its foci and the ends of its major axis are equally spaced. If the length of its semi-minor axis is $2\sqrt{2}$, then the length of its semi-major axis is

(a) 4 (b) $2\sqrt{3}$
(c) $\sqrt{10}$ (d) 3

6. An ellipse in the first quadrant is tangent to co-ordinate axes. If one focus is $F_1(3, 7)$ and other focus is $F_2(d, 7)$ then the value of $\sqrt{3d}$ is equal to

(a) 5 (b) 7
(c) 14 (d) 49

7. In triangle ABC, $a = 4$ and $b = c = 2\sqrt{2}$. A point P moves within the triangle such that the square of its distance from BC is half the area of rectangle contained by its distances from the other two sides. If D be the centre of locus of then

(a) Locus of P is an ellipse with eccentricity $\sqrt{\frac{2}{3}}$
(b) Locus of P is a hyperbola with eccentricity $\sqrt{\frac{3}{2}}$
(c) Area of the quadrilateral $ABCD = \frac{16}{3}$ sq. units
(d) Area of the quadrilateral $ABCD = \frac{32}{3}$ sq. units

8. The tangent at any point 'P' on the standard ellipse with focii as S and S' meets the tangents at the vertices A and A' in the points V and V', then :

(a) $(AV)(A'V') = b^2$
(b) $(AV)(A'V') = a^2$
(c) $\angle V'SV = 90°$
(d) $V'S'SV$ is a cyclic quadrilateral

9. AB and CD are two equal and parallel chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Tangents to the ellipse at A and B intersect at P and tangents at C and D at Q. The line PQ

(a) Passes through the origin
(b) Is bisected at the origin
(c) Cannot pass through the origin
(d) Is not bisected at the origin

10. If one of the member of the orthogonal trajectory of $y^2 = kx$ where k is a parameter, is a conic 'c' which passes through the point (2, 0), then identify the correct statement(s) about 'c'.

(a) Eccentricity of 'c' is $\frac{1}{\sqrt{2}}$
(b) Director circle of the conic 'c' is $x^2 + y^2 = 12$
(c) Focal length of 'c' is 4
(d) Latus rectum of the conic 'c' is $2\sqrt{2}$

11. If ends of latus rectum of parabola $(x + 1)^2 = 4(y + 1)$ are vertices of an ellipse whose eccentricity is half of eccentricity of parabola, then for ellipse,

(a) Distance between foci is 2.
(b) Foci are (0, 0) and (2, 0).
(c) Length of latus rectum is 3.
(d) Combined equation of directrices are $(x-3)(x+5)=0$.

12. A line ℓ is tangent to both the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ and the circle $x^2+y^2=16$ and θ is the acute angle made by the line ℓ with the major axis of the ellipse, then

(a) $\theta = \text{arc sin } \frac{3}{4}$
(b) $\theta = \text{arc cos } \frac{3}{4}$
(c) Length of common tangent is $\frac{7\sqrt{3}}{4}$
(d) Length of common tangent is $\frac{3\sqrt{7}}{4}$

13. The normal at a variable point P on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a > b)$ meets axes of the ellipse in Q and R. Then which of the following is/are correct statement(s)

(a) The locus of mid-point of QR is ellipse
(b) The locus of mid-point of QR is conic whose eccentricity is same as of given ellipse
(c) The locus of mid-point of QR is hyperbola
(d) The locus of mid-point of QR is conic with its eccentricity half of the eccentricity of given ellipse

14. Three points A, B, C are taken on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with eccentric angles θ, $\theta+\alpha$, $\theta+2\alpha$ is, then

(a) The area of ΔABC is independent of θ
(b) The area of ΔABC is independent of α
(c) The maximum value of area is $\frac{3\sqrt{3}}{4}ab$
(d) The maximum value of area is $\frac{3\sqrt{3}}{2}ab$

15. If $x^2 + 4y^2 - 4 = 0$ and minimum and maximum values of $x^2 + y^2 - xy$ are a and b respectively then

(a) $a = \frac{5-\sqrt{13}}{2}$ (b) $b = \frac{5+\sqrt{13}}{2}$
(c) $a = \frac{-5-\sqrt{13}}{2}$ (d) $b = \frac{5\sqrt{13}}{2}$

COMPREHENSION BASED TYPE QUESTIONS

Comprehension-1 (Q. 16 to 17):

$\frac{(3x-4y+10)^2}{2}+\frac{(4x+3y-15)^2}{3}=1$ is an ellipse

16. Eccentricity of the ellipse is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{\sqrt{3}}{5}$

(c) $\frac{\sqrt{2}}{5}$ (d) $\frac{2}{\sqrt{2}}$

17. Centre of the ellipse is

(a) (0, 0) (b) $\left(\frac{6}{5},\frac{17}{5}\right)$

(c) $\left(\frac{\sqrt{2}}{5},\frac{\sqrt{3}}{5}\right)$ (d) $\left(\frac{17}{5},\frac{6}{5}\right)$

Comprehension-2 (Q. 18 to 19): Two tangents PA and PB are drawn from a point $P(h, k)$ to the ellipse $E:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$. Angle of the tangents with the positive x-axis are θ_1 and θ_2. Normals at A and B are intersecting at Q point.

On the basis of above information answer the following questions.

18. Circumcentre of ΔQAB is

(a) Mid point of AB (b) Mid point of PQ

(c) Orthocentre of ΔPAB (d) Can't say

19. Locus of P, if $\cot\theta_1+\cot\theta_2=\lambda$, is

(a) $2xy=\lambda(y^2-b^2)$ (b) $2xy-\lambda(b^2-y^2)=0$

(c) $xy=\lambda$ (d) $x^2+xy=\lambda$

Comprehension-3 (Q. 20 to 21): Consider an ellipse $E:\frac{x^2}{16}+\frac{y^2}{12}=1$ and a parabola P whose vertex is $(-\sqrt{3},0)$ and focus is the origin.

20. The area of the triangle formed by the normals to the ellipse and the parabola at their point of intersection and the x-axis is:

(a) 2 sq. units (b) 4 sq. units

(c) 6 sq. units (d) $8\sqrt{3}$ sq units

21. If the parabola P divides the ellipse E into two regions whose areas are A_1 and A_2 $(A_1<A_2)$ then $\frac{A_1}{A_2}$ equals:

(a) $\frac{\pi\sqrt{3}-1}{\pi\sqrt{3}+1}$ (b) $\frac{\pi\sqrt{3}-2}{\pi\sqrt{3}+2}$

(c) $\frac{\pi\sqrt{3}-8}{\pi\sqrt{3}+8}$ (d) $\frac{\pi\sqrt{3}-4}{\pi\sqrt{3}+4}$

Comprehension-4 (Q. 22 to 24): Common tangents are drawn to $C_1: x^2+y^2=16$ and $C_2: 4x^2+25y^2=100$.

22. The x-intercept of common tangent of curves C_1 and C_2 having negative gradient in the first quadrant is:

(a) $2\sqrt{7}$ (b) $\frac{-4\sqrt{7}}{\sqrt{3}}$ (c) $-2\sqrt{7}$ (d) $\frac{4\sqrt{7}}{\sqrt{3}}$

23. Area of quadrilateral formed by common tangents between C_1 and C_2 is:

(a) $\frac{142\sqrt{3}}{3}$ (b) $\frac{112}{3}\sqrt{3}$ (c) $\frac{92}{3}\sqrt{3}$ (d) $\frac{62}{3}\sqrt{3}$

24. If tangents are drawn from any point on director circle of C_1 to auxiliary circle of C_2, then locus of mid-points of chords of contact is

(a) $x^2+y^2=\frac{625}{12}$ (b) $x^2+y^2=\frac{625}{24}$

(c) $x^2+y^2=\frac{625}{16}$ (d) $x^2+y^2=\frac{625}{32}$

Comprehension-5 (Q. 25 to 27): For the ellipse $2x^2-2xy+4y^2-(3+\sqrt{2})=0$.

25. The inclination of major axis of it with x-axis is

(a) $\pi/12$ (b) $\pi/8$ (c) $3\pi/8$ (d) $5\pi/8$

26. If the given ellipse is rotated so that it has its major axis coincident with x-axis, then its equation becomes

(a) $3(x^2+y^2-1)=(y^2-x^2+1)\sqrt{2}$

(b) $3(x^2+y^2-1)=(y^2-x^2-1)\sqrt{2}$

(c) $3(x^2+y^2-1)=(y^2-x^2-1)\sqrt{2}$

(d) $\sqrt{2}(x^2-y^2+1)=3(y^2+x^2-1)$

27. The maximum area of the circle lying inside the given ellipse is

(a) π (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/6$

Comprehension-6 (Q. 28 to 29): Let the equation $ax^2+2hxy+by^2-1=0$ represent an ellipse, then $h^2-ab<0$. The equation of the given ellipse can be changed to $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$.

Choose the correct answer:

28. $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ equals

(a) $\frac{1}{a^2}+\frac{1}{b^2}$ (b) $\frac{1}{a}+\frac{1}{b}$

(c) $a+b$ (d) $\frac{1}{a}-\frac{1}{b}$

29. The square of eccentricity of the ellipse $ax^2+2hxy+by^2-1=0$ is

(a) $\frac{\sqrt{(a-b)^2+4h^2}}{(a+b)}$ (b) $\frac{2\sqrt{(a-b)^2+4h^2}}{a+b+\sqrt{(a-b)^2+4h^2}}$

(c) $\frac{\sqrt{(a-b)^2+4h^2}}{a+b+\sqrt{(a-b)^2+4h^2}}$ (d) $\frac{1}{\sqrt{2}}$

MATCH THE COLUMN TYPE QUESTIONS

30. A tangent having slope $-\frac{4}{3}$ touches the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$ at point P and intersects the major and minor axes at A & B respectively, O is the centre of the ellipse

	Column-I		Column-II
A.	Distance between the parallel tangents having slopes $-\frac{4}{3}$, is	p.	24
B.	Area of ΔAOB is	q.	7/24
C.	If the tangent in first quadrant touches the ellipse at (h, k) then value of hk is	r.	48/5
D.	If equation of the tangent intersecting positive axes is $\ell x + my = 1$, then $\ell + m$ is equal to	s.	12

(a) A → r; B → p; C → s; D → q
(b) A → p; B → q; C → r; D → s
(c) A → p; B → r; C → s; D → q
(d) A → r; B → p; C → q; D → s

31. Let m_1, m_2, m_3 $(m_1 < m_2 < m_3)$ be the slopes of the distinct normal to $\frac{x^2}{16}+\frac{y^2}{12}=1$, passing through $\left(\frac{1}{2\sqrt{2}}, \frac{\sqrt{2}}{2\sqrt{3}}\right)$, then

	Column-I		Column-II
A.	$m_1+m_2+m_3$ is equal to	p.	$2\sqrt{3}$
B.	$m_1m_2m_3$ is equal to	q.	$-\frac{8}{3\sqrt{3}}$
C.	$m_1+m_2-m_3$ is equal to	r.	$4-\frac{2}{\sqrt{3}}$
D.	$m_3-m_1-m_2$ is equal to	s.	$4+\frac{2}{\sqrt{3}}$

(a) A → r; B → p; C → s; D → q
(b) A → p; B → q; C → r; D → s
(c) A → p; B → r; C → s; D → q
(d) A → r; B → p; C → q; D → s

32. Match the columns:

	Column-I		Column-II
A.	Find the locus of foot of perpendicular drawn from centre to any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.	p.	$y^2=-\frac{b^4}{a^3}\cdot x$
B.	Find the locus of the foot of the perpendicular from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on the chord joining two points whose eccentric angles differ by $\frac{\pi}{2}$.	q.	$x^2+\frac{(a+b)^2}{b^2}y^2=1$
C.	From a point Q on the circle $x^2+y^2=1^2$, perpendicular QM is drawn to x-axis, find the locus of point 'P' dividing QM in ratio $a:b$.	r.	$2(x^2+y^2)^2=a^2x^2+b^2y^2$
D.	If tangents to the parabola $y^2=4ax$ intersect the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at A and B, the find the locus of point of intersection of tangents at A and B.	s.	Locus is $(x^2+y^2)^2=a^2x^2+b^2y^2$

(a) A → r; B → p; C → s; D → q
(b) A → p; B → q; C → r; D → s
(c) A → s; B → r; C → q; D → p
(d) A → r; B → p; C → q; D → s

33. Match the columns:

	Column-I		Column-II
A.	Find the set of value(s) of 'λ' for which the line $3x-4y+\lambda=0$ intersect the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at two distinct points.	p.	$\left(\frac{12}{5},\frac{16}{5}\right)$
B.	Find the set of value(s) of α for which the point $\left(7-\frac{5}{4}\alpha,\alpha\right)$ lies inside the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.	q.	$(-12\sqrt{2},12\sqrt{2})$
C.	Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B. The orthocenter of the triangle PAB is	r.	$\left(\frac{2}{5},-\frac{1}{5}\right)$
D.	On the ellipse, $4x^2+9y^2=1$, the points at which the tangents are parallel to the line $8x=9y$ are (in 4th quadrant)	s.	$\left(\frac{11}{5},\frac{8}{5}\right)$

(a) A → r; B → p; C → s; D → q
(b) A → q; B → p; C → s; D → r
(c) A → s; B → r; C → q; D → p
(d) A → r; B → p; C → q; D → s

NUMERICAL BASED QUESTION

34. The eccentricity of the ellipse which meets the straight line $\frac{x}{7}+\frac{y}{2}=1$ on the axis of x and the straight line $\frac{x}{3}-\frac{y}{5}=1$ on the axis of y and whose axes lie along the axes of coordinates is $\frac{2\sqrt{6}}{\lambda}$. Find λ

35. An ellipse is drawn with major and minor axes of lengths 10 and 8 respectively. Using one focus as centre, a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. Find the radius of the circle.

36. Given the equation of the ellipse $\frac{(x-3)^2}{16}+\frac{(y+4)^2}{49}=1$, a parabola is such that its vertex is the lowest point of the ellipse and it passes through the ends of the minor axis of the ellipse. The equation of the parabola is in the form 16y $16y= a(x-h)^2-k$. Determine the value of $(a+h+k)$.

37. For an ellipse the product of perpendiculars from foci upon any tangent is 1 and the area of the parallelogram formed by the tangents at the ends of conjugate diameters is 8. If the eccentricity of ellipse is e then the value of $\frac{2e}{\sqrt{3}}$ is

38. The straight line $\frac{x}{4}+\frac{y}{3}=1$ intersects the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at two points A and B, there is a point P on this ellipse such that the area of ΔPAB is equal to $6(\sqrt{2}-1)$, the number of such points (P) is

39. If θ is acute angle of intersect between two curves $x^2+5y^2=6$ and $y^2=x$, then $\cot\theta$ is equal to

40. Let (α, β) be a point from which two perpendicular tangents can be drawn to the ellipse $4x^2+5y^2=20$. If $F=4\alpha+3\beta$, then the maximum value of F

41. If N is the foot of the perpendicular drawn from any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 (a>b)$ to its major axis AA then $\frac{PN^2}{AN.AN}$ is equal to

42. If tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercept on the x-axis constant length c, find the locus of the point of intersection of tangents

43. A ray emanating from the point $(-4, 0)$ is incident on the ellipse $9x^2+25y^2=225$ at the point P with abscissa 3. Find the equation of the reflected ray after first reflection.

44. Consider the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$. Let $H(\alpha, 0), 0<\alpha<2$, be a point. A straight line drawn through H parallel to y-axis crosses the ellipse and its audiliary circle at points E and F respectively, in the first quadrant. The tangents to the ellipse at the point E intersects the positive x-axis at a point G. Suppose the straight line joining F and the origin makes an angle ϕ with the positive x-axis.

(*i*) If $\phi=\frac{\pi}{4}$, then find the area of the triangle FGH

(*ii*) If $\phi=\frac{\pi}{12}$, then find the area of the triangle FGH

45. A straight line PQ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2+y^2=r^2 (b<r<a)$. RS is a focal chord of the ellipse. If RS is parallel to PQ and meets the circle at points R and S. Find the length of RS.

46. $C_1, P_1, E_1; C_2, P_2, E_2; \ldots$ is the series of circle C_i, parabola P_i and ellipse E_i drawn in the positive direction such that the size of each curve remain unchanged and centre of C_i is the vertex of P_i, focus of P_i is the centre of E_i and also of C_{i+1} and so on. If $C_1 \equiv x^2+y^2=a^2$, $P_1 \equiv y^2-4ax$, $E_1 \equiv \frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1$, where $(b<a)$, then find the equation of C_n, P_n, E_n and also find the value of λ such that the point (λ, b) lies in the area bounded by C_n, P_n and E_n.

47. A tangent is drawn at any fixed point P on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and if chord of contact of the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ with respect to any point on this tangent passes through a fixed point, then prove that the line joining this fixed point to the point P never subtends right angle at the origin.

48. If (x_1, y_1) & (x_2, y_2) are two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the tangents at which meet in (h, k) & the normals in (p, q), prove that $a^2p=e^2hx_1x_2$ and $b^4q=-e^2ky_1y_2a^2$ where 'e' is the eccentricity.

Exercise-4 (Past Year Questions)

JEE MAIN

1. The eccentricity of an ellipse whose centre is at the origin is 1/2. If one of its directices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is: **(2017)**

(a) $x + 2y = 4$ (b) $2y - x = 2$
(c) $4x - 2y = 1$ (d) $4x + 2y = 7$

2. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is: **(2018)**

(a) $\frac{7}{9}$ (b) 4
(c) $\frac{9}{2}$ (d) 6

3. Let $S = \left\{(x, y) \in R^2 : \frac{y^2}{1+r} - \frac{x^2}{1-r} = 1\right\}$, where $r \neq \pm 1$. Then S represents. **(2019)**

(a) A hyperbola whose eccentricity is $\frac{2}{\sqrt{1-r}}$, when $0 < r < 1$
(b) An ellipse whose eccentricity is $\sqrt{\frac{2}{r+1}}$, when $r > 1$.
(c) A hyperbola whose eccentricity is $\frac{2}{\sqrt{r+1}}$, when $0 < r < 1$.
(d) An ellipse whose eccentricity is $\frac{1}{\sqrt{r+1}}$, when $r > 1$.

4. Let the length of the latus rectum of an ellipse with its major axis long x-axis and center at the origin, be 8. If the distance between the foci of this ellipse is equal to the length of the length of its minor axis, then which one of the following points lies on it? **(2019)**

(a) $(4\sqrt{2}, 2\sqrt{2})$ (b) $(4\sqrt{3}, 2\sqrt{2})$
(c) $(4\sqrt{3}, 2\sqrt{3})$ (d) $(4\sqrt{2}, 2\sqrt{3})$

5. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$ at all points on the ellipse other than its four vertices than the mid points of the tangents intercepted between the coordinate axes lie on the curve: **(2019)**

(a) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$ (b) $\frac{x^2}{4} + \frac{y^2}{2} = 1$
(c) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (d) $\frac{x^2}{2} + \frac{y^2}{4} = 1$

6. Let S and S' be the foci of an ellipse and B be any one of the extremities of its minor axis. If $\Delta S'BS$ is a right angled triangle with right angle at B and area $(\Delta S'BS) = 8$ *sq.* units, then the length of a latus rectum of the ellipse is : **(2019)**

(a) 4 (b) $2\sqrt{2}$
(c) $4\sqrt{2}$ (d) 2

7. If the tangents on the ellipse $4x^2 + y^2 = 8$ at the points (1, 2) and (a, b) are perpendicular to each other, then a^2 is equal to: **(2019)**

(a) $\frac{64}{17}$ (b) $\frac{2}{17}$ (c) $\frac{128}{17}$ (d) $\frac{4}{17}$

8. In an ellipse, with centre at the origin, if the difference of the lengths of major axis and minor axis is 10 and one of the foci is at $(0, 5\sqrt{3})$, then the length of its latus rectum is: **(2019)**

(a) 10 (b) 8
(c) 5 (d) 6

9. The tangent and normal to the ellipse $3x^2 + 5y^2 = 32$ at the point $P(2, 2)$ meet the x-axis at Q and R, respectively. Then the area (in sq. units) of the triangle PQR is: **(2019)**

(a) $\frac{14}{3}$ (b) $\frac{16}{3}$
(c) $\frac{68}{15}$ (d) $\frac{34}{15}$

10. If the line $x - 2y = 12$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $\left(3, \frac{-9}{2}\right)$, then the length of the latus rectum of the ellipse is: **(2019)**

(a) 9 (b) $8\sqrt{3}$
(c) $12\sqrt{2}$ (d) 5

11. If the normal to the ellipse $3x^2 + 4y^2 = 12$ at a point P on it is parallel to the line, $2x + y = 4$ and the tangent of the ellipse at P passes through $Q(4,4)$ then PQ is equal to : **(2019)**

(a) $\frac{\sqrt{221}}{2}$ (b) $\frac{\sqrt{157}}{2}$
(c) $\frac{\sqrt{61}}{2}$ (d) $\frac{5\sqrt{5}}{2}$

12. An ellipse, with foci at (0, 2) and (0, –2) and minor axis of length 4, passes through which of the following points? **(2019)**

(a) $(1, 2\sqrt{2})$ (b) $(2, \sqrt{2})$

(c) $(2, 2\sqrt{2})$ (d) $(\sqrt{2}, 2)$

13. If the distance between the foci of an ellipse is 6 and the distance between its directrices is 12, then the length of its latus rectum is **(2020)**

(a) $\sqrt{3}$ (b) $3\sqrt{2}$

(c) $\frac{3}{\sqrt{2}}$ (d) $2\sqrt{3}$

14. If $3x + 4y = 12\sqrt{2}$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{9} = 1$ for some $a \in R$ then the distance between the foci of the ellipse is: **(2020)**

(a) $2\sqrt{2}$ (b) $2\sqrt{7}$

(c) 4 (d) $2\sqrt{5}$

15. Let the line $y = mx$ and the ellipse $2x^2 + y^2 = 1$ intersect at a point P in the first quadrant. If the normal to this ellipse at P meets the co-ordinate axes at $\left(\frac{-1}{3\sqrt{2}}, 0\right)$ and $(0, \beta)$, then β is equal to **(2020)**

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{2}{3}$ (d) $\frac{2\sqrt{2}}{3}$

16. The length of the minor axis (along y-axis) of an ellipse in the standard form is $\frac{4}{\sqrt{3}}$. If this ellipse touches the line, $x + 6y = 8$; then its eccentricity is: **(2020)**

(a) $\frac{1}{3}\sqrt{\frac{11}{3}}$ (b) $\sqrt{\frac{5}{6}}$

(c) $\frac{1}{2}\sqrt{\frac{11}{3}}$ (d) $\frac{1}{2}\sqrt{\frac{5}{3}}$

17. Let $x = 4$ be a directrix to an ellipse whose centre is at the origin and its eccentricity is $\frac{1}{2}$. If $P(1, \beta), \beta > 0$ is a point on this ellipse, then the equation of the normal to it at P is **(2020)**

(a) $7x - 4y = 1$ (b) $4x - 2y = 1$

(c) $4x - 3y = 2$ (d) $8x - 2y = 5$

18. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$ be a given ellipse, length of whose latus rectum is 10. If its eccentricity is the maximum value of the function, $\phi(t) = \frac{5}{12} + t - t^2$, then $a^2 + b^2$ is equal to: **(2020)**

(a) 135 (b) 116 (c) 126 (d) 145

19. If the point P on the curve, $4x^2 + 5y^2 = 20$ is farthest from the point $Q(0, -4)$, then PQ^2 is equal to: **(2020)**

(a) 29 (b) 48 (c) 21 (d) 36

20. If the co-ordinates of two points A and B are $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$ respectively and P is any point on the conic, $9x^2 + 16y^2 = 144$, then $PA + PB$ is equal to: **(2020)**

(a) 9 (b) 16

(c) 6 (d) 8

21. If the normal at an end of a latus rectum of an ellipse passes through an extremity of the minor axis, then the eccentricity e of the ellipse satisfies: **(2020)**

(a) $e^2 + 2e - 1 = 0$ (b) $e^2 + e - 1 = 0$

(c) $e^4 + 2e^2 - 1 = 0$ (d) $e^4 + e^2 - 1 = 0$

22. Which of the following points lies on the locus of the foot of perpendicular drawn upon any tangent to the ellipse, $\frac{x^2}{4} + \frac{y^2}{2} = 1$ from any of its foci? **(2020)**

(a) $(1, 2)$ (b) $(-2, \sqrt{3})$

(c) $(-1, \sqrt{3})$ (d) $(-1, \sqrt{2})$

23. If the curve $x^2 + 2y^2 = 2$ intersects the line $x + y = 1$ at two points P and Q, then the angle subtended by the line segment PQ at the origin is: **(2021)**

(a) $\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{4}\right)$ (b) $\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{4}\right)$

(c) $\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{3}\right)$ (d) $\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{3}\right)$

24. Let L be a common tangent line to the curves $4x^2 + 9y^2 = 3$[6] and $(2x)^2 + (2y)^2 = 31$. Then the square of the slope of the line L is **(2021)**

25. If the point of intersections of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = 4b, b > 4$ lie on the curve $y^2 = 3x^2$, then b is equal to: **(2021)**

(a) 12 (b) 5

(c) 6 (d) 10

26. Let L be a tangent line to the parabola $y^2 = 4x - 20$ at $(6, 2)$. If L is also a tangent to the ellipse $\frac{x^2}{2} + \frac{y^2}{b} = 1$, then the value of b is equal to; **(2021)**

(a) 11 (b) 14 (c) 16 (d) 20

27. The acute angle between the pair of tangents drawn to the ellipse $2x^2 + 3y^2 = 5$ from the point $(1, 3)$ is **(2021)**

(a) $\tan^{-1}\left(\frac{16}{7\sqrt{5}}\right)$ (b) $\tan^{-1}\left(\frac{24}{7\sqrt{5}}\right)$

(c) $\tan^{-1}\left(\frac{32}{7\sqrt{5}}\right)$ (d) $\tan^{-1}\left(\frac{3+8\sqrt{5}}{35}\right)$

28. If the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the line $\frac{x}{7} + \frac{y}{2\sqrt{6}} = 1$ on the x-axis and the line $\frac{x}{7} - \frac{y}{2\sqrt{6}} = 1$ on the y-axis, then the eccentricity of the ellipse is **(2021)**

(a) $\frac{5}{7}$ (b) $\frac{2\sqrt{6}}{7}$ (c) $\frac{3}{7}$ (d) $\frac{2\sqrt{5}}{7}$

29. Let the eccentricity of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$, be $\frac{1}{4}$. If this ellipse passes through the point $\left(-4\sqrt{\frac{2}{5}}, 3\right)$, then $a^2 + b^2$ is equal to: **(2022)**

30. If m is the slope of a common tangent to the curves $\frac{x^2}{16}+\frac{y^2}{9}=1$ and $x^2+y^2=12$, then $12m^2$ is equal to: **(2022)**

(a) 6 (b) 9 (c) 10 (d) 12

31. Let the maximum area of the triangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{4}=1, a>2$, having one of its vertices at one end of the major axis of the ellipse and one of its sides parallel to the y-axis, be $6\sqrt{3}$. Then the exxentricity of the ellipse is: **(2022)**

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{4}$

32. The line $y=x+1$ meets the ellipse $\frac{x^2}{4}+\frac{y^2}{2}=1$ at two points P and Q. If r is the radius of the circle with PQ as diameter then $(3r)^2$ is equal to **(2022)**

(a) 20 (b) 12

(c) 11 (d) 8

33. Let the common tangents to the curves $4(x^2+y^2)=9$ and $y^2=4x$ intersect at the point Q. Let an ellipse, centered at the origin O, has lengths of semi-minor and semi-major axes equal to OQ and 6, respectively. If e and l respectively denote the eccentricity and the length of the latus rectum of this ellipse, then $\frac{l}{e^2}$ is equal to ________. **(2022)**

JEE ADVANCED

34. Let d be the perpendicular distance from the centre of the ellipse $x^2/a^2 + y^2/b^2 = 1$ to the tangent drawn at a point P on the ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that $(PF_1 - PF_2)^2 = 4a^2\left(1-\frac{b^2}{d^2}\right)$ **(1995)**

35. An ellipse has eccentricity 1/2 and one focus at the point $P\left(\frac{1}{2},1\right)$. Its one directrix is the common tangent, nearer to the point P, to the circle $x^2+y^2=1$ and the hyperbola $x^2-y^2=1$. The equation of the ellipse, in the standard form is **(1996)**

36. A tangent to the ellipse $x^2+4y^2=4$ meets the ellipse $x^2+2y^2=6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2+2y^2=6$ are at right angles. **(1997)**

37. The ellipse $E_1 : \frac{x^2}{9}+\frac{y^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is **(2012)**

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

38. Consider two straight lines, each of which is tangent to both the circle $x^2+y^2=\frac{1}{2}$ and the parabola $y^2=4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin $O(0, 0)$ and whose semi-major axis is OQ. If the length of the minor axis of this elipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE? **(2018)**

(a) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectrum is 1

(b) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(c) The area of the region bounded by the ellipse between the lines $x=\frac{1}{\sqrt{2}}$ and $x=1$ is $\frac{1}{4\sqrt{2}}(\pi-2)$

(d) The area of the region bounded by the ellipse between the lines $x=\frac{1}{\sqrt{2}}$ and $x=1$ is $\frac{1}{16}(\pi-2)$

39. Define the collections $\{E_1, E_2, E_3,......\}$ of ellipses and $\{R_1, R_2, R_3,.....\}$ of rectangles as follows :

$E_1 : \frac{x^2}{9}+\frac{y^2}{4}=1;$

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ; **(2019)**

E_n : ellipse $\frac{x^2}{a_n^2}+\frac{y^2}{b_n^2}=1$ of largest area inscribed in R_{n-1}, $n>1$;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, $n>1$.

Then which of the following options is/are correct?

(a) The eccentricities of E_{18} and E_{19} are NOT equal

(b) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

(c) The length of latus rectum of E_9 is $\frac{1}{6}$

(d) $\sum_{n=1}^{N}$(area of R_n) < 24, for each positive integer N

40. Let E be the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. For any three distinct points P, Q and Q' on E, let $M(P, Q)$ be the mid-point of the line segment joining P and Q, and $M(P, Q')$ be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between $M(P, Q)$ and $M(P, Q')$, as P, Q and Q' vary on E, is ______ **(2021)**

41. Consider the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$.

Let $H(a, 0)$, $0<\alpha<2$, be a point. A straight line drawn through H parallel to the y-axis crosses the ellipse and its auxilary circle at points E and F respectively, in the first quadrant. The tangent to the ellipse at the point E intersects the positive x-axis at a point G suppose the straight line joining F and the origin makes an angle ϕ with the positive x-axis. **(2022)**

	List-I		List-II
A.	If $\phi=\frac{\pi}{4}$, then the area of the triangle FGH is	p.	$\frac{(\sqrt{3}-1)^4}{8}$
B.	If $\phi=\frac{\pi}{3}$, then the area of the triangle FGH is	q.	1
C.	If $\phi=\frac{\pi}{6}$, then the area of the triangle FGH is	r.	$\frac{3}{4}$
D.	If $\phi=\frac{\pi}{12}$, then the area of the triangle FGH is	s.	$\frac{1}{2\sqrt{3}}$
		t.	$\frac{3\sqrt{3}}{2}$

(*a*) $A\to r$; $B\to s$; $C\to q$; $D\to p$
(*b*) $A\to r$; $B\to t$; $C\to s$; $D\to p$
(*c*) $A\to r$; $B\to t$; $C\to s$; $D\to p$
(*d*) $A\to r$; $B\to s$; $C\to q$; $D\to p$

ANSWER KEY

CONCEPT APPLICATIO

1. $3x^2-2xy+3y^2-10x-2y+3=0$ 2. (b) 3. (d) 4. (a) 5. $5x^2+9y^2-5y+36=0$ 6. $\frac{(x-1)^2}{45}+\frac{(y-2)^2}{20}=1$
7. (a) 8. (b) 9. (b) 10. (d) 11. (c) 12. $4x=5y$ 13. $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ 14. [2401] 15. (a)
16. $y=\pm 2x\pm 2\sqrt{3}$ 17. (d) 18. No such ellipse exists 19. $-a^4/b^4$

EXERCISE-1 (TOPICWISE)

1. (b)	2. (b)	3. (c)	4. (b)	5. (b)	6. (a)	7. (a)	8. (c)	9. (b)	10. (a)
11. (b)	12. (b)	13. (a)	14. (c)	15. (c)	16. (c)	17. (b)	18. (b)	19. (a)	20. (b)
21. (a)	22. (b)	23. (a)	24. (d)	25. (a)	26. (c)	27. (c)	28. (c)	29. (d)	30. (a)
31. (a)	32. (c)	33. (a)	34. (b)	35. (a)	36. (a)	37. (d)	38. (c)	39. (b)	40. (a)
41. (c)	42. (d)	43. (b)	44. (b)	45. (d)	46. (a)	47. (b)			

EXERCISE-2 (LEARNING PLUS)

1. (a)	2. (c)	3. (c)	4. (c)	5. (b)	6. (b)	7. (d)	8. (b)	9. (b)	10. (b)
11. (d)	12. (d)	13. (a)	14. (b)	15. (a)	16. (b)	17. (c)	18. (c)	19. (b)	20. (c)
21. (b)	22. (a)	23. (a)	24. (a)	25. (a)	26. (d)	27. (b)	28. (d)	29. (a,b,c,d)	
30. (a,b,c)	31. (a,b,c,d)	32. (a,c,d)	33. (a,c)	34. (a,b)	35. (a,b,c,d)	36. (a)	37. [−2/3]	38. [4/5, −1/5]	

39. [85] 40. $a^2(1-e^2)(2-e^2)$) 41. $\left[\sqrt{\frac{1}{3}}\right]$ 42. [3π/2] 43. $\frac{32a^4b^4}{(a^2+b^2)^3}$ 44. $\frac{x^2}{a^2}\cos^2\frac{\alpha}{2}+\frac{y^2}{b^2}\cos^2\frac{\alpha}{2}=1$

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (b)	2. (c)	3. (a)	4. (c)	5. (d)	6. (b)	7. (a,c)	8. (a,c,d)	9. (a,b)	10. (a,b,c,d)
11. (a,c,d)	12. (b,d)	13. (a,b)	14. (a,c)	15. (a,b)	16. (a)	17. (b)	18. (b)	19. (a)	20. (c)
21. (b)	22. (a)	23. (b)	24. (d)	25. (b)	26. (d)	27. (a)	28. (c)	29. (b)	30. (a)
31. (b)	32. (c)	33. (b)	34. [7]	35. [2]	36. [186]	37. [1]	38. [3]	39. [9/7]	40. [15]

41. $[b^2/a^2]$ 42. $4y^2[b^2c^2+a^2y^2-a^2b^2)=c^2(y^2-b^2)^2$ 43. $12x+5y=48$ 44. (i)[1] (ii) $\left(\frac{\sqrt{3}-1}{8}\right)^4$

45. [2b] 46. $\frac{b^2}{4a}+(n-1)a<\lambda<\sqrt{a^2-b^2}+(n-1)a$

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (c)	2. (c)	3. (b)	4. (b)	5. (c)	6. (a)	7. (b)	8. (c)	9. (c)	10. (a)
11. (d)	12. (d)	13. (b)	14. (b)	15. (b)	16. (c)	17. (b)	18. (c)	19. (d)	20. (d)
21. (d)	22. (c)	23. (a)	24. [3]	25. (a)	26. (b)	27. (b)	28. (a)	29. (b)	30. (b)
31. (a)	32. (a)	33. [4]							

JEE Advanced

35. $\frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}}+\frac{(y-1)^2}{\frac{1}{12}}=1$ 37. (c) 38. (a, c) 39. (c, d) 40. [4] 41. (c)

CHAPTER

17 Hyperbola

INTRODUCTION

A hyperbola, a type of smooth curve lying in a plane, has two pieces, called branches, that are mirror images of each other. A hyperbola is a set of points whose difference of distances from two foci is a constant value. A guitar is an example of hyperbola as its sides form hyperbola. Hyperbolas are also used by the astronomers to predict the path of the satellite such that the required adjustments can be made to set the satellite in its destination.

DEFINITION

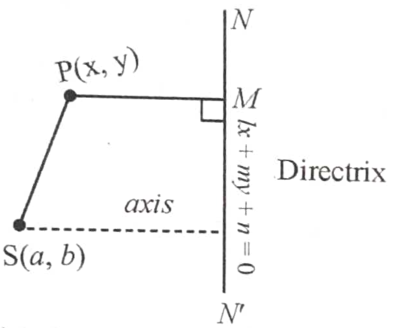
A hyperbola is the locus of a point which moves in a plane such that the ratio of its distance from a fixed point and a given straight line is always constant and is greater than 1.

The fixed point is called the focus, the fixed line is called the directrix and the constant ratio is called the eccentricity of the hyperbola and denoted by e.

In the given figure, S is the focus and $N'N$ the directrix.

Let P be any point on the hyperbola, then

$$\frac{PS}{PM} = e,\ e > 1.\text{(for hyperbola)}$$



Equation of a hyperbola can be obtained if the coordinates of its focus, equation of its directrix and eccentricity are given.

GENERAL EQUATION OF HYPERBOLA

We can find the equation of an ellipse when the coordinates of its focus S(a, b), equation of the directrix $lx + my + n = 0$ and eccentricity (e) are given.

Let $P(x, y)$ be any point on the hyperbola. Then by defination.

$\Rightarrow$ $SP = e\,PM$ ($e > 1$, is the eccentricity)

$$\Rightarrow\quad (x-a)^2 + (y-b)^2 = e^2\frac{(lx+my+n)^2}{(l^2+m^2)}$$

$$\Rightarrow\quad (l^2+m^2)\{(x-a)^2 + (y-b)^2\} = e^2\{lx+my+n\}^2$$

reduces to $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, in which $\Delta \neq 0$, $h^2 > ab$.

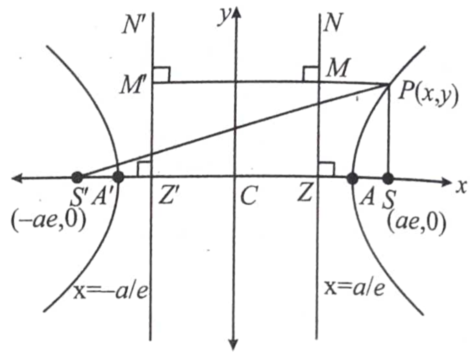
STANDARD EQUATION & DEFINITION(S)

Let S be the focus and ZN is the directrix of an ellipse. Draw perpendicular from S to the directrix which meet it at Z. A moving point is on the hyperbola such that

$PS = ePM$

then there is point lies on the line SZ and which divide SZ internally at A and externally at A' in the ratio of $e : 1$.

Therefore $\quad SA = e\,AZ$ ...(i)



$SA' = eA'Z$ ...(ii)

Let $AA' = 2a$ and take C as mid point of AA'

$\therefore\quad CA = CA' = a$

Add (i) and (ii)

$SA + SA' = e\,(AZ + A'Z)$

$(CS - CA) + (CA' + CS) = e\,[CA - CZ + CA' + CZ]$

$2CS = 2e \cdot CA$

$CS = ae$

Subtract (ii) and (i), we get

$SA' - SA = e\,(A'Z - AZ)$

$(CA' + CS) - (CS - CA) = e\,[(CA' + CZ) - (CA - CZ)]$

$2CA = 2e \cdot CZ \Rightarrow CZ = \dfrac{a}{e}$.

Consider CZ line as x-axis, C as origin and perpendicular to this line and passes through C is considered as y-axis. Now represent important parameters on coordinates plane. Let $P(x, y)$ is a moving point, then

By definition of hyperbola

$PS = ePM$

$\Rightarrow (PS)^2 = e^2 (PM)^2$

$\Rightarrow (x - ae)^2 + (y - 0)^2 = e^2\left(x - \frac{a}{e}\right)^2$

$\Rightarrow (x - ae)^2 + y^2 = (a - ex)^2$

$\Rightarrow x^2 + a^2e^2 - 2xae + y^2 = a^2 + e^2x^2 - 2xae$

$\Rightarrow x^2 (1 - e^2) + y^2 = a^2 (1 - e^2)$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$

or $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$

Hence equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

where $b^2 = a^2 (e^2 - 1)$

FACTS ABOUT THE HYPERBOLA: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

1. By symmetry of equation of hyperbola, if we take second focus $(-ae, 0)$ and second directrix $x = -\frac{a}{e}$ and perform same calculation then we get same equation of hyperbola. This suggest that their are two foci are $(ae, 0)$ and $(-ae, 0)$ and the two corresponding directrices as $x = \frac{a}{e}$ and $x = -\frac{a}{e}$.
2. By definition of hyperbola, the distance of any point P on the hyperbola from focus = e(the distance of P from the corresponding directrix)
3. Distance between foci SS' = $2ae$ and distance between directrix $ZZ' = 2\frac{a}{e}$.
4. Two hyperbola are said to be similar if they have same eccentricity.
5. The hyperbola has two branches neither of them cut the y-axis (conjugate axis).
6. Since the fundamental equation to the hyperbola only differs from that to the ellipse is having $-b^2$ instead of b^2. It is observed that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.
7. Eccentricity $e = \sqrt{1+\frac{b^2}{a^2}}$. Also, $b^2 = a^2 (e^2 - 1)$.

The smaller the e, the smaller will be the value of b for a given a. Therefore, as e decreases for a given a, the branches of the hyperbola would be bending towards x-axis. As e increases, the branches open up.

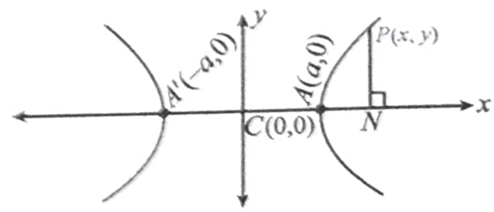
8. Equation of hyperbola when its transverse and conjugate axes are x and y-axes respectively is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \text{ or } x^2 - \frac{y^2}{(e^2-1)} = a^2$$

If e kept constant and $a \to 0$, then hyperbola will tend to pair of straight lines $x^2 - \frac{y^2}{(e^2-1)} = 0$, both passing through the origin.

Thus in this situation limiting case of a hyperbola is a pair of straight lines.

9. Since $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \therefore \frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 = \frac{(x-a)(x+a)}{a^2}$



From figure, $AN = CN - CA = x - a$

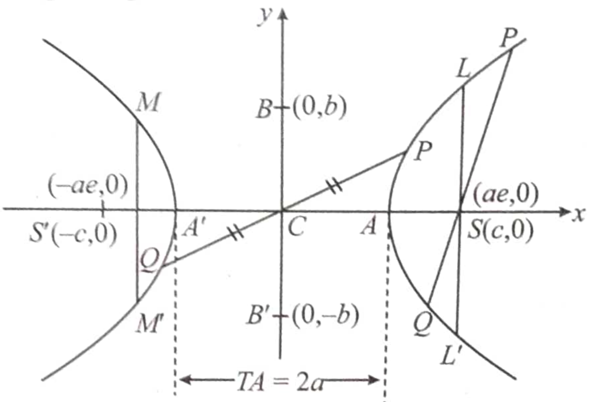
$A'N = CN + CA = x + a$ and PN = y

$\therefore \frac{(PN)^2}{A'N \cdot AN} = \frac{b^2}{a^2}$.

10. The directrix is a straight line that runs parallel to the hyperbola's conjugate axis.
11. The Transverse axis is always perpendicular to the directrix.
12. The foci and the vertices lie on the transverse axis.
13. At the vertices, the tangent line is always parallel to the directrix of a hyperbola.

TERMS RELATED TO HYPERBOLA

1. **Centre:** In the figure, C is the centre of the ellipse. All chords passing through C are called diameter and bisected at C.



2. **Foci:** $S(ae, 0)$ and $S'(-ae, 0)$ are two foci of hyperbola. Line containing the fixed points S and S' (called Foci) is called Transverse Axis (TA) or Focal Axis and the distance between S and S' is called Focal Length.

3. **Axes:** The line AA' is called transverse axis and the line BB' is perpendicular to it and passes through the centre $(0, 0)$ of the hyperbola is called conjugate axis.

The length of transverse and conjugate axes are taken as $2a$ and $2b$ respectively.

The transverse and conjugate axes together are called principal axes of hyperbola and their intersection point is called the centre of hyperbola.

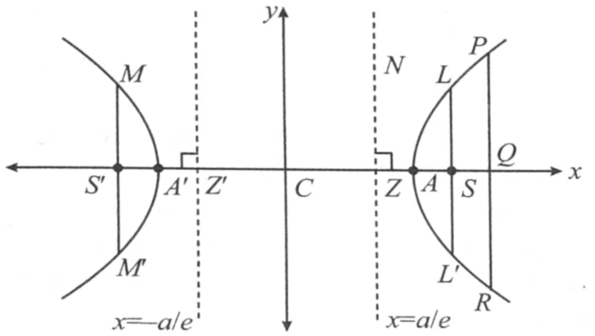
The points of intersection of the directrix with the transverse axis are known as Foot of the directrix (Z and Z').

4. **Vertex:** The points of intersection (A, A') of the curve with the transverse axis are called Vertices of the hyperbola.
5. **Double ordinate:** Any chord perpendicular to the Transverse axis is called a Double Ordinate.
6. **Latus-rectum:** When double ordinates passes through the focus of parabola then it is called the latus rectum. In the given figure LL' and MM' are the latus-rectums of the hyperbola.

Let $LL' = 2k$ then $LS = L'S = k$

Let $L(ae, k)$ lie on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$$\therefore \frac{a^2e^2}{a^2} - \frac{k^2}{b^2} = 1$$



or $k^2 = b^2(e^2 - 1) = b^2\left(\dfrac{b^2}{a^2}\right)$ $\quad [\because b^2 = a^2(e^2 - 1)]$

$$\therefore \quad k = \frac{b^2}{a} \qquad (\because k > 0)$$

$$\therefore \quad 2k = \frac{2b^2}{a} = LL'$$

Length of latus rectum

$$= LL' = \frac{2b^2}{a} = \frac{(CA)^2}{TA} = 2a(e^2 - 1) = 2e\left(ae - \frac{a}{e}\right)$$

= $(2e)$ (distance between the focus and the foot of the corresponding directrix)

End points of latus-rectums are

$$L = \left(ae, \frac{b^2}{a}\right), L' = \left(ae, -\frac{b^2}{a}\right); M = \left(-ae, \frac{b^2}{a}\right);$$

$M' = \left(-ae, -\dfrac{b^2}{a}\right)$ respectively.

7. **Focal chord:** A chord of hyperbola passing through its focus is called a focal chord.
8. **Asymptotes:** Asymptotes of hyperbola is a straight line which tends to touch hyperbola at infinity.

To find the asymptote of the hyperbola :

Let $y = mx + c$ is the asymptote of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Solving these two we get the quadratic as $(b^2 - a^2m^2)\, x^2 - 2a^2\, mcx - a^2(b^2 + c^2) = 0$ ... (1)

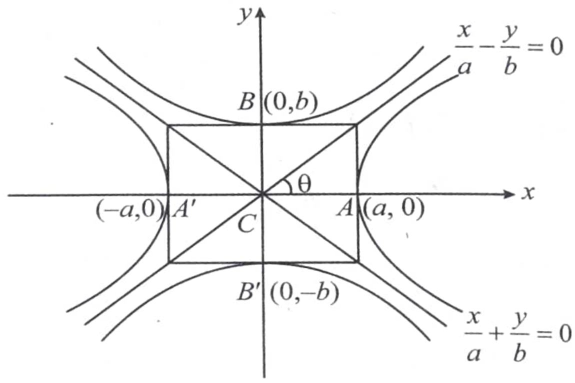
In order that $y = mx + c$ be an asymptote, both roots of equation (1) must approach infinity, the conditions for which are: coeff of $x^2 = 0$ & coeff of $x = 0$.

$$\Rightarrow \quad b^2 - a^2m^2 = 0$$

or $\quad m = \pm \dfrac{b}{a}$ & $a^2\, mc = 0 \quad \Rightarrow \quad c = 0.$

$\therefore$ Equations of asymptote are $\dfrac{x}{a} + \dfrac{y}{b} = 0$ and $\dfrac{x}{a} - \dfrac{y}{b} = 0$

Combined equation to the asymptotes $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0.$



ECCENTRICITY

For the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ we have $b^2 = a^2(e^2 - 1)$

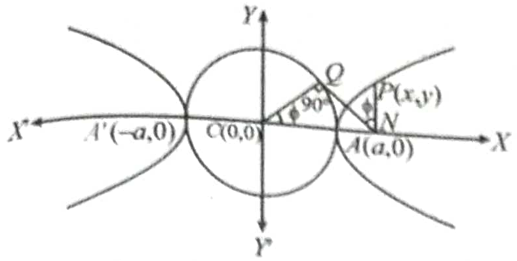
$$\Rightarrow e = \sqrt{1 + \left(\frac{b^2}{a^2}\right)} \quad \Rightarrow e = \sqrt{\left\{1 + \frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}\right\}}$$

Eccentricity defines the curvature of the hyperbola and is mathematically spelled as:

$$e = \frac{\text{distance from centre to focus}}{\text{distance from centre to vertex}}$$

AUXILIARY CIRCLE & PARAMETRIC EQUATIONS OF HYPERBOLA

Let $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ be the hyperbola with centre C and transverse axis $A'A$. Therefore circle drawn with centre C and segment $A'A$ as a diameter is called auxiliary circle of the hyperbola.



$\therefore$ Equation of the auxiliary circle is

$x^2 + y^2 = a^2$

Let $P(x, y)$ be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Draw PN perpendicular to x-axis.

Let NQ be a tangent to the auxiliary circle $x^2 + y^2 = a^2$. Join CQ and let $\angle QCN = \phi$ then P and Q are the corresponding points of the hyperbola and the auxiliary circle. Here ϕ is the eccentric angle of P. $(0 \le \phi < 2\pi)$.

Since $Q = (a \cos \phi, a \sin \phi)$

we get its corresponding point on hyperbola P(asec ϕ, $\pm$b tanϕ)

The equations of $x = a \sec \phi$ and $y = b \tan \phi$ are known as the parametric equations of the hyperbola. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Position of points Q an auxiliary circle and corresponding point P which describes the hyperbola are shown below in the table. Here $0 \le \phi < 2\pi$.

ϕ varies from	Q ($a \cos \phi$, $a \sin \phi$)	P ($a \sec \phi$, $b \tan \phi$)
0 to $\frac{\pi}{2}$	I	I
$\frac{\pi}{2}$ to π	II	II
π to $\frac{3\pi}{2}$	III	III
$\frac{3\pi}{2}$ to 2π	IV	IV

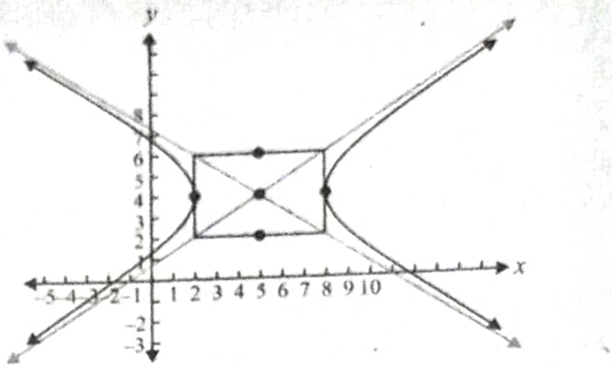
Train Your Brain

Example 1: Graph: $\frac{(x-5)^2}{9} - \frac{(y-4)^2}{4} = 1$.

Sol. In this case, the expression involving x has a positive leading coefficient; therefore, the hyperbola opens left and right. Here $a = 3$ and $b = 2$.

The center $(5, 4)$

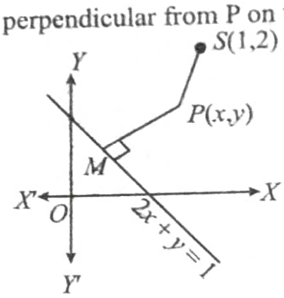
The lines through the corners of this rectangle define the asymptotes.



Example 2: Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

Sol. Let $P(x, y)$ be any point on the hyperbola.

Draw PM perpendicular from P on the directrix.



Then by definition $SP = e\,PM$

$\Rightarrow (SP)^2 = e^2 (PM)^2$

$\Rightarrow (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2$

$\Rightarrow 5\,(x^2 + y^2 - 2x - 4y + 5\}$

$= 3\,(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

which is the required hyperbola.

Example 3: If foci of a hyperbola are foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. If the eccentricity of the hyperbola be 2, then its equation is

Sol. For ellipse $e = \frac{4}{5}$, so foci $= (\pm 4, 0)$

For hyperbola $e = 2$,

so $a = \frac{ae}{e} = \frac{4}{2} = 2$, $b = 2\sqrt{4-1} = 2\sqrt{3}$

Hence equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{12} = 1$

Example 4: Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis.

Sol. Let the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Then transverse axis $= 2a$ and latus–rectum $= \frac{2b^2}{a}$

According to question $\frac{2b^2}{a} = \frac{1}{2}(2a)$

$\Rightarrow 2b^2 = a^2 (\because b^2 = a^2(e^2 - 1))$

$\Rightarrow 2a^2(e^2 - 1) = a^2 \Rightarrow 2e^2 - 2 = 1$

$\Rightarrow e^2 = \frac{3}{2} \quad \therefore \quad e = \sqrt{\frac{3}{2}}$

Hence the required eccentricity is $\sqrt{\frac{3}{2}}$.

Concept Application

1. Eccentricity of the hyperbola conjugate to the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$

   (a) $\frac{2}{\sqrt{3}}$ (b) 2

   (c) $\sqrt{3}$ (d) $\frac{4}{3}$

2. The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Then the value of b^2 is-

   (a) 5 (b) 7 (c) 9 (d) 4

3. The equation $\frac{x^2}{29-p} + \frac{y^2}{4-p} = 1 (p \neq 4, 29)$ represent-

   (a) An ellipse if p is any constant greater than 4

   (b) A hyperbola if p is any constant between 4 and 29

   (c) A rectangular hyperbola if p is any constant greater than 29.

   (d) No real curve is p is less than 29.

4. The eccentricity of the conjugate hyperbola of the hyperbola $x^2 - 3y^2 = 1$ is:

   (a) 2 (b) $\frac{2}{\sqrt{3}}$ (c) 4 (d) $\frac{4}{5}$

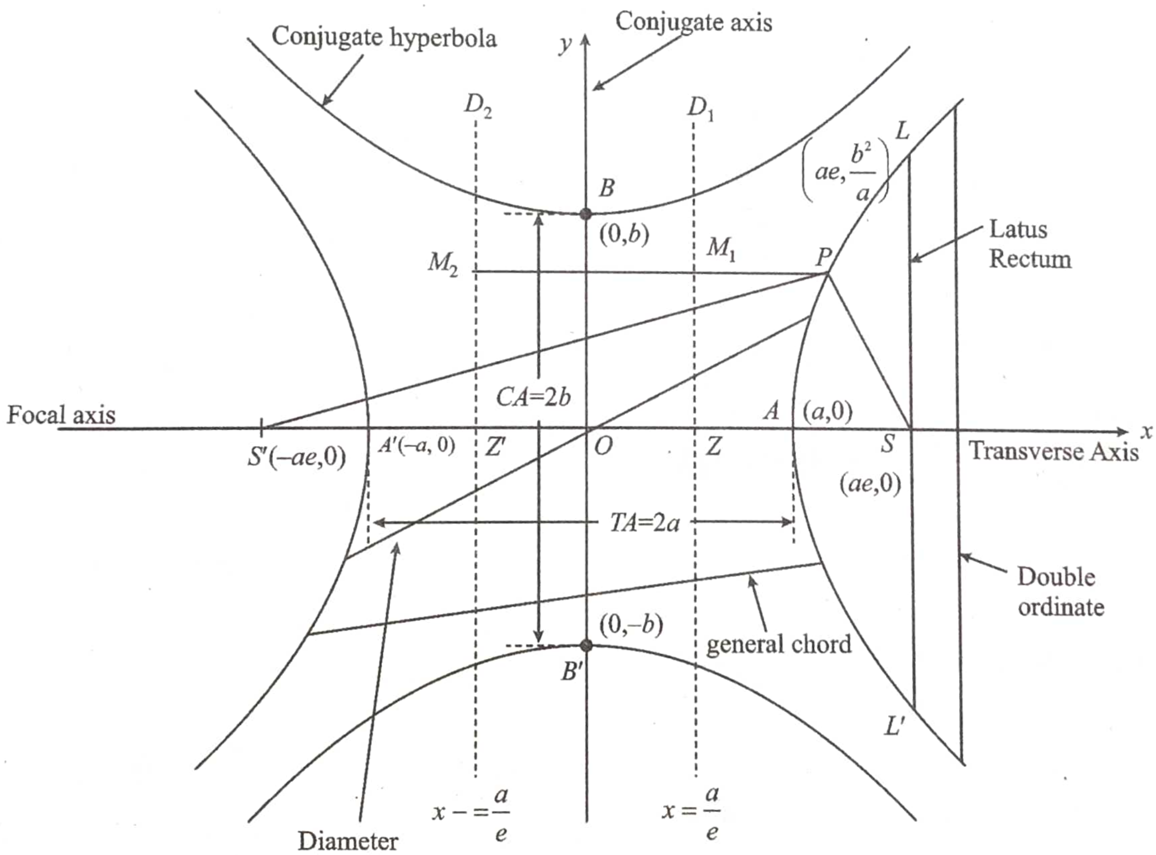
5. Equation $(2 + \lambda)x^2 - 2\lambda xy + (\lambda - 1)y^2 - 4x - 2 = 0$ represents a hyperbola if

   (a) $\lambda = 4$ (b) $\lambda = 1$

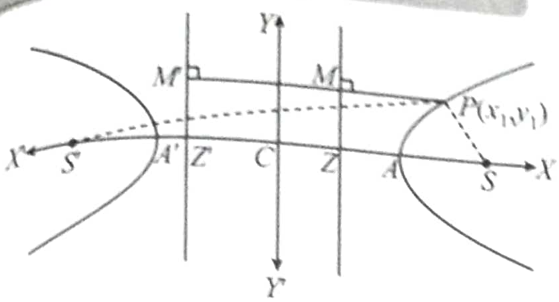
   (c) $\lambda = 4/3$ (d) $\lambda = -1$

HYPERBOLA AT A GLANCE

Parametric coordinates $x = a \sec \theta$ and $y = b \tan \theta$



FOCAL DISTANCE OF A POINT ON HYPERBOLA



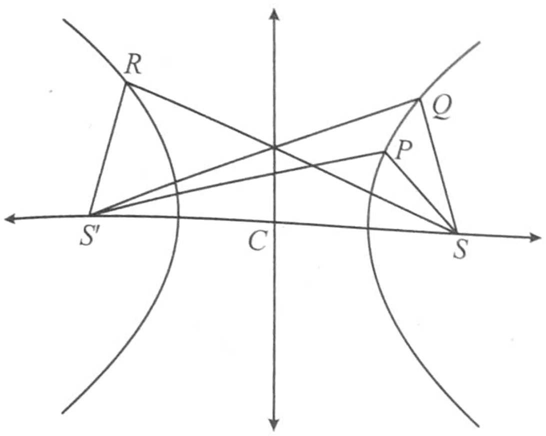
The hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ... (1)

The foci S and S' are $(ae, 0)$ and $(-ae, 0)$ & corresponding directrix are $x=\frac{a}{e}$ and $x=-\frac{a}{e}$ respectively.

Let $P(x_1, y_1)$ be any point on (1).

Now $SP = ePM = e\left(x_1-\frac{a}{e}\right)=ex_1-a$

and $S'P = ePM' = e\left(x_1+\frac{a}{e}\right)=ex_1+a$



$\therefore\ S'P-SP=(ex_1+a)-(ex_1-a)=2a$

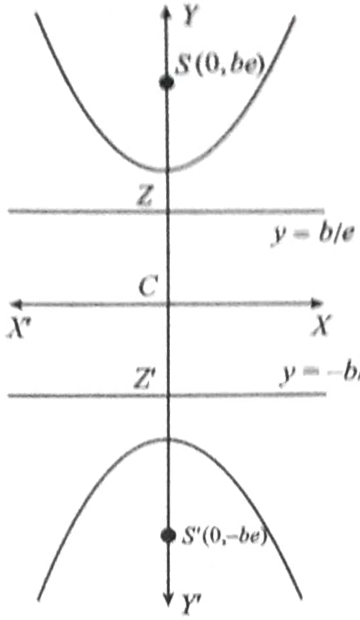
$= AA'$ = Transverse axis

Thus hyperbola is the locus of a point which moves in a plane such that the difference of its distances from two fixed point (foci) is constant and always equal to transverse axis.

Hence, in the given figure $PS'-PS=QS'-QS=RS-RS'$ = length of transverse axis.

CONJUGATE HYPERBOLA

Corresponding to every hyperbola there exist a hyperbola such that, the conjugate axis and transverse axis of one is equal to the transverse axis and conjugate axis of other, such hyperbolas are known as conjugate to each other.



Hence for the hyperbola,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad ...(1)$$

the conjugate hyperbola is,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1 \quad ...(2)$$

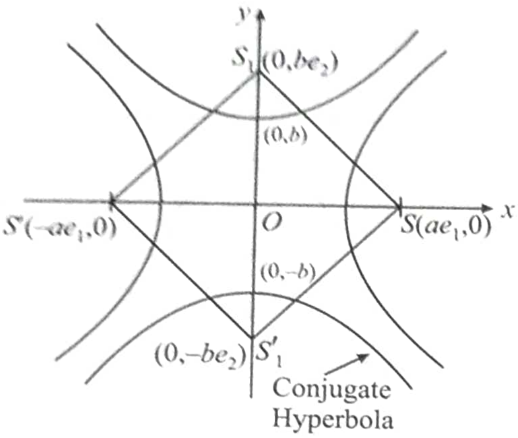
Table: Comparison between Hyperbola and its Conjugate Hyperbola

Basic Elements	Hyperbola	Conjugate Hyperbola
	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	$(0, 0)$	$(0, 0)$
Length of transeverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Eccentricity	$e=\sqrt{1+\frac{b^2}{a^2}}$	$e=\sqrt{1+\frac{a^2}{b^2}}$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrix	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametric coordinate	$(a\sec\phi, b\tan\phi)$	$(a\tan\phi, b\sec\phi)$
	$0\le\phi<2\pi$	$0\le\phi<2\pi$
Focal distances	$ex_1\pm a$	$ey_1\pm b$
Diffference of focal distances	$2a$	$2b$

Equation of transverse axis	$y = 0$	$x = 0$
Equation of conjugates	$x = 0$	$y = 0$
Tangent at vertices	$x = \pm a$	$y = \pm b$

Note:

1. The foci of a hyperbola and its conjugate are concylic and form the vertices of a square.



2. If e_1 and e_2 are the eccentrcities of the hyperbola and its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

Proof: The eccentricity e_1 of the given hyperbola is obtained from

$b^2 = a^2(e_1^2 - 1)$ ...(i)

The eccentricity e_2 of the conjugate hyperbola is given by

$a^2 = b^2(e_2^2 - 1)$ ...(ii)

Multiply (i) and (ii)

We get $1 = (e_1^2 - 1)(e_2^2 - 1)$

$\Rightarrow 0 = e_1^2 e_2^2 - e_1^2 - e_2^2 \Rightarrow e_1^{-2} + e_2^{-2} = 1$.

FOCAL DISTANCE OF A POINT ON HYPERBOLA

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

When equation of the hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

This equation is the form of $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$, where $X = x - h$ and $Y = y - k$

1. Length of semi-transverse axis $= a$, length of semi-conjugate axis $= b$
2. Equation of transverse axis is $Y = 0$, i.e, $y - k = 0$
   Equation of conjugate axis is $X = 0$, i.e, $x - h = 0$
3. Coordinates of centre is given by $X = 0$ and $Y = 0$, i.e., $x - h = 0$ and $y - k = 0$
   Therefore, centre is (h, k)
4. Eccentricity of the hyperbola $e = \sqrt{1 + \frac{b^2}{a^2}}$
5. Coordinates of vertices of the hyperbola are given by $X = \pm a$, $Y = 0$ i.e., $x - h = \pm a$, $y - k = 0$.
   Hence vertices are $(h \pm a, k)$.
6. Coordinate of foci are given by $X = \pm ae$, $Y = 0$ i.e., $x - h = \pm ae$, $y - k = 0$.
   Hence foci are $(h \pm ae, k)$
7. Equation of directrices of the hyperbola are
   $X = \pm \frac{a}{e}$, i.e., $x - h = \pm \frac{a}{e}$.
   Hence directrices are $x = h \pm \frac{a}{e}$
8. Length of latus rectum $= \frac{2b^2}{a}$
9. Coordinate of ends of latus rectum are given by $X = \pm ae$
   $Y = \pm \frac{b^2}{a}$ i.e. $x - h = \pm ae$, $y - k = \pm \frac{b^2}{a}$
   $\therefore$ end if LR is $\left(h \pm ae, k \pm \frac{b^2}{a}\right)$

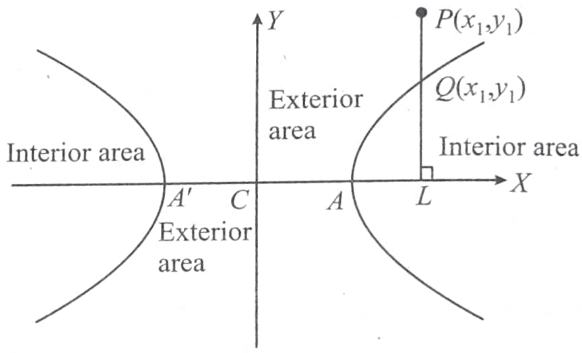
POSITION OF A POINT 'P' W.R.T. A HYPERBOLA

Let $S(x, y) \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the given hyperbola and $P(x_1, y_1)$ is the given point.

(i) If $S(x_1, y_1) > 0$ then $P(x_1, y_1)$ lie inside the ellipse.

(ii) If $S(x_1, y_1) < 0$ then $P(x_1, y_1)$ lie outside the ellipse.

(iii) If $S(x_1, y_1) = 0$ then $P(x_1, y_1)$ lie on the ellipse.



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Example 5: Find the coordinates of foci, the eccentricity and latus-rectum, equations of directrices for the hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.

Sol. Equation can be rewritten as

$$\frac{(x-4)^2}{4^2} - \frac{(y-3)^2}{3^2} = 1 \text{ so } a = 4, b = 3$$

$b^2 = a^2(e^2 - 1)$ given $e = \frac{5}{4}$

Foci: $X = \pm ae$, $Y = 0$ gives the foci as (9, 3), (−1, 3)

Centre: $X = 0$, $Y = 0$ i.e. (4, 3)

Directrices: $X = \pm \frac{a}{e}$ i.e. $x - 4 = \pm \frac{16}{5}$

$\therefore$ directrices are $5x - 36 = 0$; $5x - 4 = 0$

Latus-rectum $= \frac{2b^2}{a} = 2 \cdot \frac{9}{4} = \frac{9}{2}$

Example 6: The eccentricity of the hyperbola $4x^2 - 9y^2 - 8x = 32$ is

Sol. $4x^2 - 9y^2 - 8x = 32 \Rightarrow 4(x-1)^2 - 9y^2 = 36$

$$\Rightarrow \frac{(x-1)^2}{9} - \frac{y^2}{4} = 1$$

Here $a^2 = 9$, $b^2 = 4$

$$\therefore \text{ eccentricity } e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{4}{9}} = \frac{\sqrt{13}}{3}$$

Example 7: The eccentricity of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is

Sol. Equation of the conjugate hyperbola to the hyperbola

$x^2 - 3y^2 = 1$ is $-x^2 + 3y^2 = 1 \Rightarrow \frac{-x^2}{1} + \frac{y^2}{1/3} = 1$

Here $a^2 = 1$, $b^2 = 1/3$

$$\therefore \text{ eccentricity } e = \sqrt{1 + a^2/b^2} = \sqrt{1+3} = 2$$

Example 8: Find the lengths of transverse axis, conjugate axis, eccentricity, the co-ordinates of foci, vertices, lengths of the latus-rectum and equations of the directrices of the hyperbola $16x^2 - 9y^2 = -144$.

Sol. The equation $16x^2 - 9y^2 = -144$ can be written as $\frac{x^2}{9} - \frac{y^2}{16} = -1$.

$a = 3$, $b = 4$, This conjugate hyperbola

The length of transverse axis $= 2b = 8$.

The length of conjugate axis $= 2a = 6$.

$$\text{Eccentricity } e = \sqrt{\left(1 + \frac{a^2}{b^2}\right)} = \sqrt{\left(1 + \frac{9}{16}\right)} = \frac{5}{4}$$

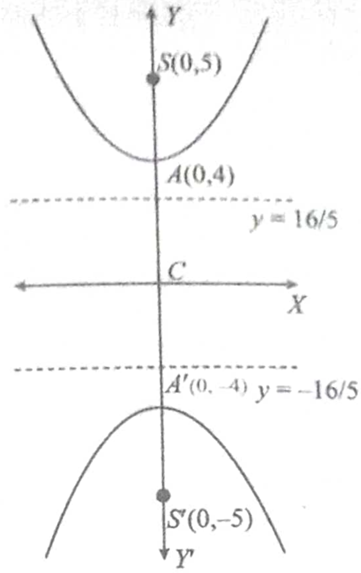
The co-ordinates of the foci are $(0, \pm be)$ i.e., $(0, \pm 5)$

The co-ordinates of the vertices are $(0, \pm b)$ i.e., $(0, \pm 4)$.

$$\text{The length of latus-rectum} = \frac{2a^2}{b} = \frac{2\,(3)^2}{4} = \frac{9}{2}$$

The equation of directrices are

$$y = \pm \frac{b}{e} = \pm \frac{4}{(5/4)} \Rightarrow y = \pm \frac{16}{5}.$$



Example 9: The equation $9x^2 - 16y^2 - 18x + 32y - 151 = 0$ represent a hyperbola-

(*a*) The length of the transverse axes is 4

(*b*) Length of latus rectum is 9

(*c*) Equation of directrix is $x = \frac{21}{5}$ and $x = -\frac{11}{5}$

(*d*) None of these

Sol. We have $9x^2 - 16y^2 - 18x + 32y - 151 = 0$

$9(x^2 - 2x) - 16(y^2 - 2y) = 151$

$9(x^2 - 2x + 1) - 16(y^2 - 2y + 1) = 144$

$9(x-1)^2 - 16(y-1)^2 = 144$

$$\frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1$$

Shifting the origin at (1, 1) without rotating the axes

$$\frac{X^2}{16} - \frac{Y^2}{9} = 1$$

where $X = x - 1$ and $Y = y - 1$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

where $a^2 = 16$ and $b^2 = 9$ so

The length of the transverse axes

$= 2a = 8$; $2b = 6$.

The length of the letus rectum

$$= \frac{2b^2}{a} = \frac{2 \cdot 9}{4} = \frac{9}{2} \text{ and } e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$$

The equaiton of the directrix $X = \pm \frac{a}{e}$

$$x - 1 = \pm \frac{16}{5} \Rightarrow x = \pm \frac{16}{5} + 1$$

$$x = \frac{21}{5} \text{ and } x = -\frac{11}{5}$$

Equation of directrix is $x = \frac{11}{5}$

and $x = -\frac{11}{5}$ **Ans. [c]**

Example 10: Obtain the equation of a hyperbola with co-ordinate axes as principal axes and given that the distances of one of its vertices from the foci are 9 and 1 units.

Sol. Let equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.. ... (1)

If vertices are A $(a, 0)$ and A' $(-a, 0)$ and foci are $S\,(ae, 0)$ and $S'\,(-ae, 0)$

Given $(S'A) = 9$ and $(SA) = 1$

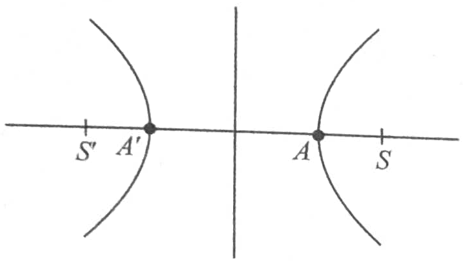
$\Rightarrow$ $a + ae = 9$ and $ae - a = 1$

or $a\,(1 + e) = 9$ and $a\,(e - 1) = 1$

$\therefore \frac{a\,(1+e)}{a\,(e-1)} = \frac{9}{1} \Rightarrow 1 + e = 9e - 9 \Rightarrow e = \frac{5}{4}$

$\because a\,(1 + e) = 9$

$\therefore a\left(1+\frac{5}{4}\right) = 9 \Rightarrow a = 4$



$b^2 = a^2\,(e^2 - 1) = 16\left(\frac{25}{16} - 1\right) \Rightarrow b^2 = 9$

From (1) equation of hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Example 11: Find the position of the point (5, – 4) relative to the hyperbola $9x^2 - y^2 = 1$.

Sol. Here $S(x, y) \equiv 9x^2 - y^2 - 1$

and $S\,(5, -4) = 9\,(5)^2 - (-4)^2 - 1$

$= 225 - 16 - 1 = 208 > 0$

So the point (5, – 4) inside the hyperbola $9x^2 - y^2 = 1$.

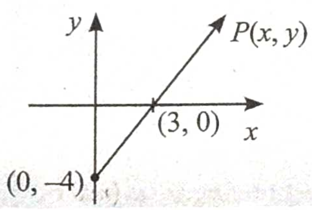
Example 12: The locus of $P(x, y)$ such that

$\sqrt{x^2 + y^2 + 8y + 16} - \sqrt{x^2 + y^2 - 6x + 9} = 5,$

(*a*) hyperbola (*b*) circle

(*c*) finite line segement (*d*) infinite ray

Sol. (*d*)



We have $\sqrt{x^2 + (y+4)^2} - \sqrt{(x-3)^2 + y^2} = 5$. The distance between the two points (3, 0) and (0, –4) is 5.

Locus of P is the part of line $\frac{x}{3} - \frac{y}{4} = 1$

Example 13: If the foci of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{3} = 1$ coincide, the value of a is

(*a*) 3 (*b*) 2

(*c*) $\frac{1}{\sqrt{3}}$ (*d*) $\sqrt{3}$

Sol. (*a*) Foci of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ are $(\pm\sqrt{12}, 0)$

Foci of $\frac{x^2}{a^2} - \frac{y^2}{3} = 1$ are $(\pm\sqrt{a^2 + 3}, 0)$

Given $a^2 + 3 = 12 \Rightarrow a^2 = 9 \Rightarrow a = 3$

Concept Application

6. The equation of the hyperbola whose foci are (–2, 0) and (2, 0) and eccentricity is 2 is given by

(*a*) $-3x^2 + y^2 = 3$ (*b*) $x^2 - 3y^2 = 3$

(*c*) $3x^2 - y^2 = 3$ (*d*) $-x^2 + 3y^2 = 3$

7. AB is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that ΔAOB (where 'O' is the origin) is an equilateral triangle, then the eccentricity e of the hyperbola satisfies.

(*a*) $e > \sqrt{3}$ (*b*) $1 < e < \frac{2}{\sqrt{3}}$

(*c*) $e = \frac{2}{\sqrt{3}}$ (*d*) $e > \frac{2}{\sqrt{3}}$

8. The equation of one of the directrices of a hyperbola is $2x + y = 1$, the corresponding focus is (1, 2) and $e = \sqrt{3}$. Find the equation of the hyperbola and coordinates of the centre and focus.

9. Show that the equation $7y^2 - 9x^2 + 54x - 28y - 116 = 0$ represent a hyperbola. Find the coordinates of the centre, length of transverse and conjugate axis, eccentricity, latus rectum, coordinate of foci, vertices, and equations of the directrices of the hyperbola.

10. Find the coordinates of the foci and the centre of the hyperbola

$$\frac{(3x - 4y - 12)^2}{100} - \frac{(4x + 3y - 12)^2}{225} = 1$$

11. The equation of the hyperbola whose foci are (6, 5), (–4, 5) and eccentricity 5/4 is

(*a*) $\frac{(x-1)^2}{16}-\frac{(y-5)^2}{9}=1$

(*b*) $\frac{x^2}{16}-\frac{y^2}{9}=1$

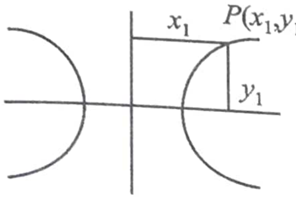
(*c*) $\frac{(x-1)^2}{16}-\frac{(y-5)^2}{9}=1$

(*d*) None of these

MEANING OF: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

x_1 = perpendicular distance of P to TA

y_1 = perpendicular distance of P to CA



$$\therefore \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$$

$$\frac{\begin{Bmatrix}\text{Perpendicular distance of } P \\ \text{from conjugate axis}\end{Bmatrix}^2}{\{\text{Length of semi transverse axis}\}^2}-\frac{\begin{Bmatrix}\text{Perpendicular distance of } P \\ \text{from transverse axis}\end{Bmatrix}^2}{\{\text{Length of semi conjugate axis}\}^2}=1$$

CHORD OF HYPERBOLA

The **equation to the chord** of the hyperbola joining the two points $P(\alpha)$ and $Q(\beta)$ is given by

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}$$

where $P(\alpha)$ is $(a \sec\alpha, b\tan\alpha)$ and $Q(\beta)$ is $(a\sec\beta, b\tan\beta)$

Remark

(*i*) If chord passes through $(d, 0)$ then

$$\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}=\frac{a-d}{a+d}$$

(*ii*) If $d = ae$ i.e. PQ is a focal chord, then

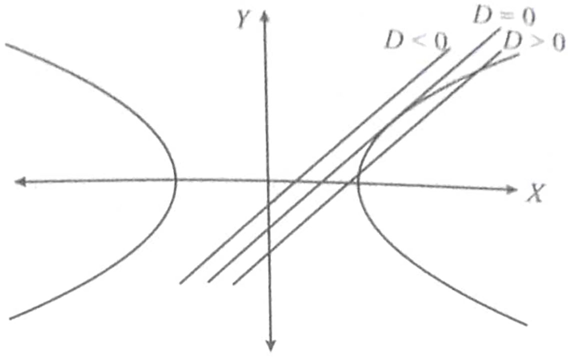
$$\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}=\frac{1-e}{1+e}$$

(*iii*) If $d = -ae$ i.e. PQ is a focal chord, then

$$\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}=\frac{1+e}{1-e}$$

LINE AND A HYPERBOLA

Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1)



and the given line be $y = mx + c$ ... (2)

The point of intersection of line and hyperbola can be obtained by solving (1) and (2), therefore

Eliminating y from (1) and (2), we get $\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$

$\Rightarrow b^2x^2 - a^2(mx+c)^2 = a^2b^2$

$\Rightarrow (a^2m^2 - b^2)x^2 + 2mca^2x + a^2(b^2+c^2) = 0$ ... (3)

Above equation is a quadratic in x and gives two values of x. It shows that every straight line will cut the hyperbola in two points, may be real, coincident or imaginary according as discriminant of (3) $>, =, < 0$.

i.e., $4m^2c^2a^4 - 4(a^2m^2-b^2)a^2(b^2+c^2) >, =, < 0$

or $-a^2m^2 + b^2 + c^2 >, =, < 0$

or $c^2 >, =, < a^2m^2 - b^2$

Clearly, $y = mx \pm \sqrt{a^2m^2-b^2}$ is a tangent to the hyperbola iff $m^2 > \frac{b^2}{a^2}$ i.e. $m \in \left(-\infty, \frac{-b}{a}\right) \cup \left(\frac{b}{a}, \infty\right)$

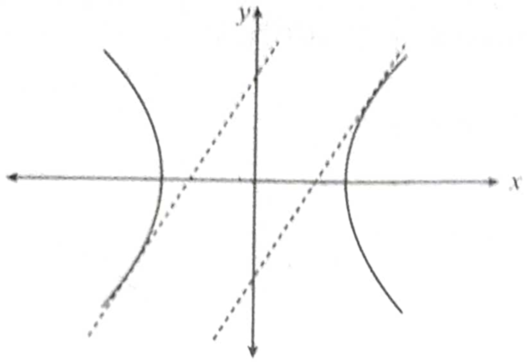
TANGENT TO THE HYPERBOLA: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(*a*) Point form: Equation of the tangent to the given hyperbola at the point (x_1, y_1) is T = 0 i.e. $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

(*b*) Slope form: Given hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ... (1)

and the given line $y = mx + c$ touches hyperbola then solve (1) and (2), we get

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1 \quad \text{or} \quad b^2x^2 - a^2(mx+c)^2 = a^2b^2$$



or $(b^2 - a^2m^2)x^2 - 2a^2 mcx - a^2(c^2 + b^2) = 0$ ... (3)

Since line (2) will be tangent to hyperbola (1) if roots of equation (3) are equal i.e. $D = 0$

$4a^4m^2c^2 + 4a^2(b^2 - a^2m^2)(c^2 + b^2) = 0$

or $a^2m^2c^2 + b^2c^2 - a^2c^2m^2 + b^4 - a^2b^2m^2 = 0$

or $b^2c^2 + b^4 - a^2b^2m^2 = 0$

or $c^2 + b^2 - a^2m^2 = 0$

or $c^2 = a^2m^2 - b^2$ or $c = \pm\sqrt{a^2m^2 - b^2}$. This is the required **condition of tangency**

Note:

(*i*) Equation of any tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be taken as $y = mx + \sqrt{a^2m^2 - b^2}$ or $y = mx - \sqrt{a^2m^2 - b^2}$. The co-ordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{(a^2m^2 - b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m^2 - b^2)}}\right)$$

(*ii*) The equation of any tangent to the hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ may be taken as $(y - k) = m(x - h) \pm \sqrt{a^2m^2 - b^2}$.

(c) Parametric form: Equation of the tangent to the given hyperbola at the point

$(a \sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

Note:

(*i*) Point of intersection of the tangents at θ_1 & θ_2 is

$$x = a\frac{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}, y = b\tan\left(\frac{\theta_1 + \theta_2}{2}\right)$$

(*ii*) Slope of tangent at $P(\theta)$ is $\frac{b}{a}$ cosec θ.

Tangents drawn from outside point:

$y = mx \pm \sqrt{a^2m^2 - b^2}$ is a tangent to the standard hyperbola....(1)

If above tangent passes through (h, k) then

$(k - mh)^2 = a^2m^2 - b^2$

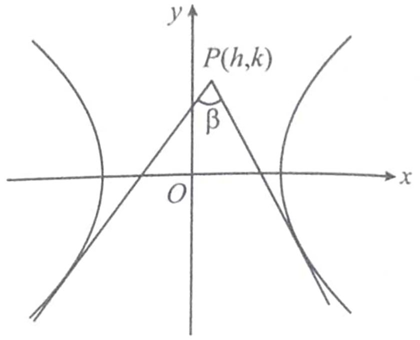
$(h^2 - a^2)m^2 - 2kmh + k^2 + b^2 = 0$ ...(2)

Above equation is quadratic in m therefore it has two roots m_1 and m_2.

Hence passing through a given point (h, k) there is a maximum of two tangents can be drawn to the hyperbolam they are $y - k = m_1(x - h)$ and $y - k = m_2(x - h)$,

If $D < 0$, then no tangent can be drawn from (h, k) to the hyperbola.

Also from equation (2), $m_1 + m_2 = \frac{2kh}{h^2 - a^2}$ ...(3)



$m_1m_2 = \frac{k^2 + b^2}{h^2 - a^2}$ ... (4)

Equations (3) and (4) are used to find the locus of the point of intersection of a pair of tangents which intersect at an angle β then use $\tan\beta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

Note: The two tangents can be made on;

A. The first branch only

B. The second branch only

C. One on the first and the other on the second branch.

We determine this by simply making asymptotes from the centre of the hyperbola. This divides the plane into four regions. Now,

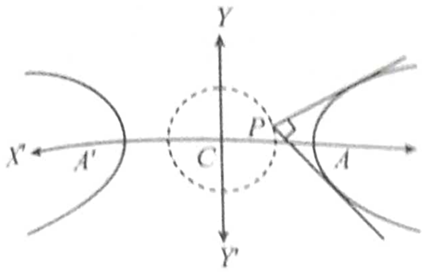
I. If your external point lies in the region in which the first branch lies, both the tangents will be drawn on the **first branch.**

II. If your external point lies in the region in which the second branch lies, both the tangents will be drawn on the **second branch.**

III. If your external point lies in the two regions in which no branch lies, **one** tangent will be drawn on the **first branch** and **one** tangent will be drawn on the **second branch.**

DIRECTOR CIRCLE

The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which are perpendicular to each other, is called the director circle.



pair of perpendicular tangents are drawn from $P(h, k)$ using uation (4), $m_1 m_2 = \dfrac{k^2+b^2}{h^2-a^2} = -1$.

$\Rightarrow$ $h^2 + k^2 = a^2 - b^2$

$\therefore$ locus of $P(h, k)$ is $x^2 + y^2 = a^2 - b^2$.

ote: (*i*) If $b^2 < a^2$ this circle is real

(*ii*) If $b^2 = a^2$ the radius of the circle is zero and it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

(*iii*) If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle and so no tangents at right angle can be drawn to the curve.

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Example 14: Prove that the straight line $\ell x + my + n = 0$ touches the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ if $a^2\ell^2 - b^2m^2 = n^2$.

Sol. The given line is $\ell x + my + n = 0$

or $y = -\dfrac{\ell}{m}x - \dfrac{n}{m}$

Comparing this line with $y = Mx + c$

$\therefore M = -\dfrac{\ell}{m}$ and $c = -\dfrac{n}{m}$ ... (1)

This line (1) will touch the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ if

$c^2 = a^2M^2 - b^2$

$\Rightarrow \dfrac{n^2}{m^2} = \dfrac{a^2\ell^2}{m^2} - b^2$ or $a^2\ell^2 - b^2m^2 = n^2$

Example 15: Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ if $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

Sol. The given line is $x\cos\alpha + y\sin\alpha = p$

$\Rightarrow y\sin\alpha = -x\cos\alpha + p \Rightarrow y = -x\cot\alpha + p\operatorname{cosec}\alpha$

Comparing this line with $y = mx + c$

where $m = -\cot\alpha$, $c = p\operatorname{cosec}\alpha$

Since the given line touches the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

then $c^2 = a^2m^2 - b^2$

$\Rightarrow p^2\operatorname{cosec}^2\alpha = a^2\cot^2\alpha - b^2$

or $p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$

Example 16: Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Sol. Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$\therefore m \times 1 = -1 \Rightarrow m = -1$

Since $x^2 - 4y^2 = 36$ or $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$

Comparing this with $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$\therefore a^2 = 36$ and $b^2 = 9$

So the equation of tangents are

$y = (-1)x \pm \sqrt{36\times(-1)^2 - 9}$

$y = -x + \sqrt{27} \Rightarrow x + y + 3\sqrt{3} = 0$

Example 17: The locus of the point of intersection of two tangents of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ if the product of their slopes is c^2, will be

Sol. Equation of any tangent of the hyperbola with slope m is

$y = mx \pm \sqrt{a^2m^2 - b^2}$

If it passes through (x_1, y_1) then $(y_1 - mx_1)^2 = a^2m^2 - b^2$

$\Rightarrow (x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 + b^2) = 0$

If $m = m_1, m_2$ then as given $m_1m_2 = c^2$

$\Rightarrow \dfrac{y_1^2 + b^2}{x_1^2 - a^2} = c^2$. Hence required locus will be $y^2 + b^2 = c^2(x^2 - a^2)$

Example 18: A common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$ is

Sol. $\dfrac{x^2}{16} - \dfrac{y^2}{9} - 1$, $x^2 + y^2 = 9$

Equation of tangent $y = mx + \sqrt{16m^2 - 9}$ (for hyperbola)

Equation of tangent $y = m'x + 3\sqrt{1 + m'^2}$ (circle)

For common tangent $m = m'$ and

$3\sqrt{1 + m'^2} = \sqrt{16m^2 - 9}$ or $9 + 9m^2 = 16m^2 - 9$

or $7m^2 = 18 \Rightarrow m = \pm 3\sqrt{\dfrac{2}{7}}$

$\therefore$ required equation is $y = \pm 3\sqrt{\dfrac{2}{7}}\,x + 3\sqrt{1 + \dfrac{18}{7}}$

or $y = \pm 3\sqrt{\dfrac{2}{7}}x + \dfrac{15}{\sqrt{7}}$

Example 19: The absolute value of slope of common tangents to parabola $y^2 = 8x$ and hyperbola $3x^2 - y^2 = 3$ is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

Sol. (*b*) Tangent to $y^2 = 8x$ is $y = mx + \frac{2}{m}$

Tangent to $\frac{x^2}{1} - \frac{y^2}{3} = 1$ is $y = mx \pm \sqrt{m^2 - 3}$ on comparing, we get $m = \pm 2$

Example 20: The director circle of a hyperbola is $x^2 + y^2 - 4y = 0$. One end of the major axis is (2, 0) then a focus is

(*a*) $(\sqrt{3}, 2-\sqrt{3})$ (*b*) $(-\sqrt{3}, 2+\sqrt{3})$

(*c*) $(\sqrt{6}, 2-\sqrt{6})$ (*d*) $(-\sqrt{6}, 2+\sqrt{6})$

Sol. (*c,d*)

Radius of director circle $\sqrt{a^2 - b^2} = 2$

Axis of hyperbola is line joining hte center $C(0, 2)$ and $A(2, 0)$ end of major axis)

$\therefore\ a = CA = 2\sqrt{2}$

$\therefore\ (2\sqrt{2})^2 - b^2 = 4$

$\therefore\ b^2 = 4$

$\therefore\ e = \frac{\sqrt{3}}{\sqrt{2}}$

Center of the hyperbola is center of the director circle (0, 2)

Focus lies on this line at distance ae from center

$\therefore$ If foci are (x, y) then $\frac{x-0}{-\frac{1}{\sqrt{2}}} = \frac{y-2}{\frac{1}{\sqrt{2}}} = \pm 2\sqrt{3}$

$(x, y) = (\sqrt{6}, 2-\sqrt{6})$ or $(-\sqrt{6}, 2+\sqrt{6})$

Example 21: The points on the ellipse $\frac{x^2}{2} + \frac{y^2}{10} = 1$ from which perpendicular tangents can be drawn to the hyperbola $\frac{x^2}{5} - \frac{y^2}{1} = 1$ is/are 5

(*a*) $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{5}{2}}\right)$ (*b*) $\left(\sqrt{\frac{3}{2}}, -\sqrt{\frac{5}{2}}\right)$

(*c*) $\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{5}{2}}\right)$ (*d*) $\left(\sqrt{\frac{5}{2}}, \sqrt{\frac{3}{2}}\right)$

Sol. (*a, b, c*)

Required points will lie on the intersection of ellipse $\frac{x^2}{2} + \frac{y^2}{10} = 1$ with director circle of hyperbola $\frac{x^2}{5} - \frac{y^2}{1} = 1$ i.e. on $x^2 + y^2 = 4$

$\Rightarrow\ (\sqrt{2}\cos\theta)^2 + (\sqrt{10}\sin\theta)^2 = 4$

Solving, we get

$\sin\theta = \pm\frac{1}{2}, \cos\theta = \pm\frac{\sqrt{3}}{2}$

$\therefore$ Points are $\left(\pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{5}{2}}\right)$

Concept Application

12. A tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ meets x-axis at P and y-axis at Q. Lines PR and QR are drawn such that $OPRQ$ is a rectangle (where O is the origin). Then R lies on:

(*a*) $\frac{2}{x^2} - \frac{4}{y^2} = 1$ (*b*) $\frac{4}{x^2} - \frac{2}{y^2} = 1$

(*c*) $\frac{4}{x^2} + \frac{2}{y^2} = 1$ (*d*) $\frac{2}{x^2} + \frac{4}{y^2} = 1$

13. Find the equation of tangents to the curve $4x^2 - 9y^2 = 1$, which is parallel to $4y = 5x + 7$

14. The straight line $x + y = \sqrt{2}P$ will touch the hyperbola $4x^2 - 9y^2 = 36$, if

(*a*) $P^2 = 2$ (*b*) $P^2 = 5$

(*c*) $5P^2 = 2$ (*d*) $2P^2 = 5$

15. If the line $y = mx + \sqrt{a^2m^2 - b^2}$ touches the hyperbola $\frac{x^2}{a^2} \frac{-y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$. Show that $\theta = \sin^{-1}\left(\frac{b}{am}\right)$

16. Find the value of M for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$

17. The tangents from $(1, 2\sqrt{2})$ to the hyperbola $16x^2 - 25y^2 = 400$ include between them an angle equal to:

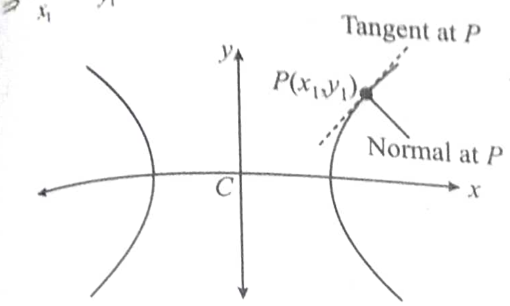
(*a*) $\frac{\pi}{6}$ (*b*) $\frac{\pi}{4}$ (*c*) $\frac{\pi}{3}$ (*d*) $\frac{\pi}{2}$

NORMAL TO THE HYPERBOLA: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(*a*) Point form: Equation of normal to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point (x_1, y_1) is

$$\frac{x-x_1}{x_1/a^2} = \frac{y-y_1}{y_1/(-b^2)}$$

$$\Rightarrow \frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$



(b) Slope form: The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$ foot of normal are $\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}}, \mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)$

(c) Parametric form: The equation of the normal at the point P $(a \sec\theta, b\tan\theta)$ to the given hyperbola is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$.

Note: In general, maximum four normals can be drawn to a hyperbola from a point and if $\alpha, \beta, \gamma, \delta$ are the eccentric angles of these four co-normal points, then $\alpha + \beta + \gamma + \delta$ is an odd multiple of π i.e. $\alpha + \beta + \gamma + \delta = (2n + 1)\pi$.

Train Your Brain

Example 22: Line $x \cos\alpha + y \sin\alpha = p$ is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if

Sol. Equation of a normal to the hyperbola is $ax \cos\theta + by \cot\theta = a^2 + b^2$ comparing it with the given line equation

$$\frac{a\cos\theta}{\cos\alpha} = \frac{b\cot\theta}{\sin\alpha} = \frac{a^2+b^2}{p}$$

$$\Rightarrow \sec\theta = \frac{ap}{\cos\alpha(a^2+b^2)}$$

$$\tan\theta = \frac{bp}{\sin\alpha(a^2+b^2)}$$

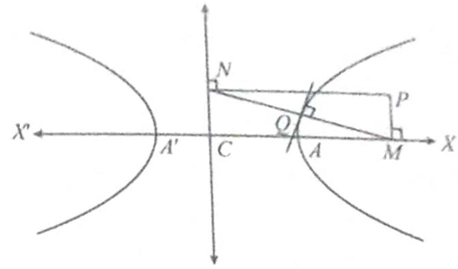
Eliminating θ, we get

$$\frac{a^2p^2}{\cos^2\alpha(a^2+b^2)^2} - \frac{b^2p^2}{\sin^2\alpha(a^2+b^2)^2} - 1$$

$$\Rightarrow a^2\sec^2\alpha - b^2\operatorname{cosec}^2\alpha = \frac{(a^2+b^2)^2}{p^2}$$

Example 23: The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes. Prove that the locus of P is hyperbola $(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

Sol. Equation of normal at any point Q is $ax \cos\theta + by \cot\theta = a^2 + b^2$



$$\therefore M \equiv \left(\frac{a^2+b^2}{a}\sec\theta, 0\right), N \equiv \left(0, \frac{a^2+b^2}{b}\tan\theta\right)$$

$\therefore$ Let $P \equiv (h, k)$

$$\Rightarrow h = \frac{a^2+b^2}{a}\sec\theta, \quad k = \frac{a^2+b^2}{b}\tan\theta$$

$$\Rightarrow \frac{a^2h^2}{(a^2+b^2)^2} - \frac{b^2k^2}{(a^2+b^2)^2} = \sec^2\theta - \tan^2\theta = 1$$

$\therefore$ locus of P is $(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

Concept Application

18. If the tangent and the normal to a rectangular hyperbola at a point cut off intercepts a_1, a_2, on one axis and b_1, b_2 on the other axis, then $a_1a_2 + b_1b_2$ is equal to

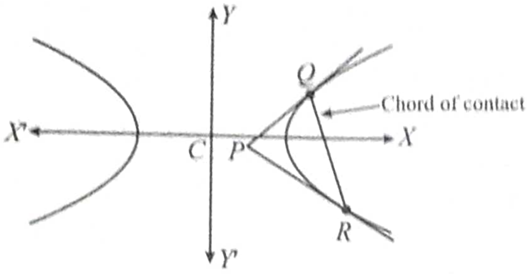
(a) -1 (b) 0 (c) 1 (d) $\sqrt{2}$

19. Normal is drawn at one of the extremities of the latus rectum of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which meets the axis at points. A and B. Then find the area of triangle OAB (O being the origin)

20. Prove that the line $lx + my - n = 0$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$

CHORD OF CONTACT

If the tangents from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ touch the hyperbola at Q and R, then the equation of the chord of contact QR is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$, which is $T = 0$.



PAIR OF TANGENTS

The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$, lying outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$$

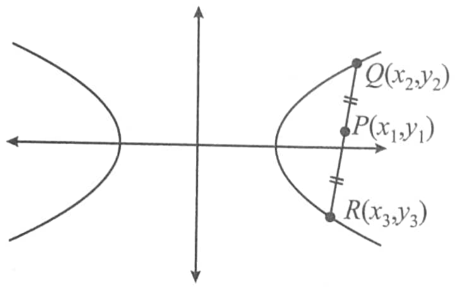
or $SS_1 = T^2$

where $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$; $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ and $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

CHORD WITH A GIVEN MIDDLE POINT (X_1, Y_1)

The equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, bisected at the point $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$



or $T=S_1$, where $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

and $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

DIAMETER

The locus of the middle points of a system of parallel chords of a hyperbola is called a diameter and the point where the diameter intersects the hyperbola is called the vertex of the diameter.

Theorem: The equation of a diameter bisecting a system of parallel chords of slope m of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $y=\frac{b^2}{a^2m}x$

CONJUGATE DIAMETERS

Two diameters are said to be conjugate when each bisects all chords parallel to the others.

If $y = mx$, $y = m_1x$ be conjugate diameters, then $mm_1 = \frac{b^2}{a^2}$

PARTICULAR CASE

(*i*) The asymptotes passes through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(*ii*) Asymptotes are the tangent to the hyperbola from the centre.

(*iii*) A hyperbola and its conjugate have the same asymptote.

(*iv*) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

$$H:\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0,\ A:\frac{x^2}{a^2}-\frac{y^2}{b^2}=0,$$

$$C:\frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0,$$

So, $A-H=C-A$.

(*vi*) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(*vii*) A simple method to find the co-ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as: Let $f(x, y) = 0$ represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ gives the centre of the hyperbola.

Train Your Brain

Example 24: Find the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$. Find also the general equation of all the hyperbolas having the same set of asymptotes.

Sol. Let $2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$ be asymptotes.

Use $\Delta = 0$

This will represent pair of straight line so

$$4\lambda + 25 - \frac{25}{2} - 8 - \frac{25}{4}\lambda = 0$$

$\Rightarrow \lambda = 2$

$\Rightarrow 2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$ are asymptotes

$\Rightarrow (2x+y+2)=0$ and $(x+2y+1)=0$ are asymptotes and $2x^2 + 5xy + 2y^2 + 4x + 5y + c = 0$ is general equation of hyperbola.

ample 25: Find the hyperbola whose asymptotes are $2x - y = 3$ and $3x + y - 7 = 0$ and which passes through the int $(1, 1)$.

ol. The equation of the hyperbola differs from the equation of the asymptotes by a constant

$\Rightarrow$ The equation of the hyperbola with asymptotes $3x + y - 7 = 0$ and $2x - y = 3$ is

$(3x + y - 7)(2x - y - 3) + k = 0$. It passes through $(1, 1)$

$\Rightarrow k = -6$.

Hence the equation of the hyperbola is

$(2x - y - 3)(3x + y - 7) = 6$.

ample 26: Find the equation of the hyperbola whose ymptotes are $x + 2y + 3 = 0$ and $3x + 4y + 5 = 0$ and which sses through the point $(1, -1)$. Find also the equation of e conjugate hyperbola.

Sol. Combined equation of asymptotes is

$(x + 2y + 3)(3x + 4y + 5) = 0$ or

$3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$ ...(1)

(Also we know that the equation of the hyperbola differs from that of asymptotes by a constant.

Let the equation of the hyperbola be

$3x^2 + 10xy + 8y^2 + 14x + 22y + \lambda = 0$ ...(2)

Since it passes through $(1, -1)$ then

$3(1)^2 + 10(1)(-1) + 8(-1)^2 + 14(1) + 22(-1) + \lambda = 0$

$\Rightarrow 3 - 10 + 8 + 14 - 22 + \lambda = 0$

$\therefore \lambda = 7$

From (2), equation of hyperbola is

$3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$ ...(3)

But we know that equation of conjugate hyperbola = 2 (Combined equation of asymptotes) – (Equation of hyperbola)

$\Rightarrow 6x^2 + 20xy + 16y^2 + 28x + 44y + 30 - 3x^2 - 10xy - 8y^2 - 14x - 22y - 7 = 0$

or $3x^2 + 10xy + 8y^2 + 14x + 22y + 23 = 0$.

Concept Application

21. The asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ form with any tangent to the hyperbola a triangle whose area is $a^2 \tan \lambda$ in magnitude, then its eccentricity is-

(a) $\sec \lambda$ (b) $\operatorname{cosec} \lambda$

(c) $\sec^2 \lambda$ (d) $\operatorname{cosec}^2 \lambda$

22. Find the equation of the hyperbola which has $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ as its asymptotes and which passes through the origin.

23. Show that the acute angle between the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 (a^2 > b^2)$ is $2\cos^{-1}\left(\frac{1}{e}\right)$ which is the eccentricity of the hyperbola.

24. Find the angle between the asymptotes of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

25. Find the asymptotes of the hyperbola $xy - 3y - 2x = 0$

26. From any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, tangents are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$. The area cut-off by the chord of contact on the asymptotes is equal to:

(a) $\frac{a}{2}$ (b) ab (c) $2ab$ (d) $4ab$

27. The asymptotes of the hyperbola $xy = hx + ky$ are :

(a) $x - k = 0$ & $y - h = 0$ (b) $x + h = 0$ & $y + k = 0$

(c) $x - k = 0$ & $y + h = 0$ (d) $x + k = 0$ & $y - h = 0$

28. The asymptotes of the hyperbola $xy - 3x - 2y = 0$ are

(a) $x - 2 = 0$ and $y - 3 = 0$

(b) $x - 3 = 0$ and $y - 2 = 0$

(c) $x + 2 = 0$ and $y + 3 = 0$

(d) $x + 3 = 0$ and $y + 2 = 0$

29. If 'e' is the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and θ is angle between the asymptotes, then $\cos(\theta/2)$ is:

(a) $\frac{1-e}{e}$ (b) $\frac{\sqrt{e^2-1}}{e}$ (c) $\frac{1}{e}$ (d) $\frac{1}{1+e}$

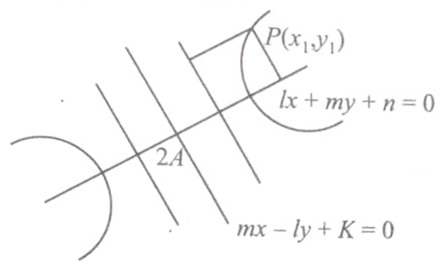
OBLIQUE HYPERBOLA

Let $lx + my + n = 0$ is Transverse axis

$my - ly + K = 0$ is conjugate axis

Length of transvese axis is $2A$

Length of conjugate axis is $2B$



$$\frac{\left(\frac{mx_1 - ly_1 + K}{\sqrt{m^2 + l^2}}\right)^2}{A^2} - \frac{\left(\frac{lx_1 + my_1 + n}{\sqrt{l^2 + m^2}}\right)^2}{B^2} = 1$$

General Note: Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

RECTANGULAR HYPERBOLA

A hyperbola is said to be rectangular hyperbola if its transverse axis is equal to its conjugate axis i.e. $b = a \Rightarrow e = \sqrt{2}$

In rectangular hyperbola asymptotes are perpendicular to each other, the equation of rectangular hyperbola is $x^2 - y^2 = a^2$ and its asymptotes are $x - y = 0$ and $x + y = 0$. Since asymptotes are inclined at 45° and 135° to the x-axis respectively.

If we rotate the axes through $\theta = -45°$ without changing the origin. Thus when we replace (x, y) by

$[x \cos(-45°) - y \sin(-45°), x \sin(-45)° + y \cos(-45°)]$

i.e., $\left(\dfrac{x+y}{\sqrt{2}}, \dfrac{-x+y}{\sqrt{2}}\right)$

then equation $x^2 - y^2 = a^2$ reduce to

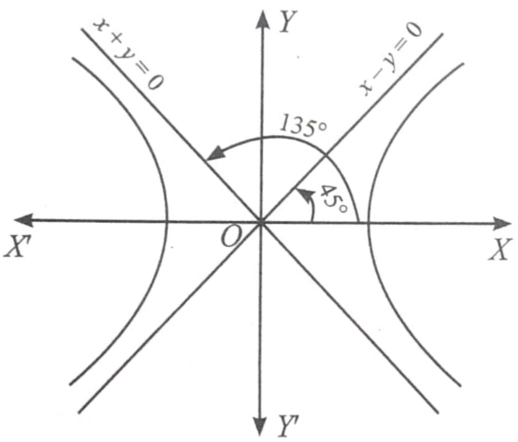
$$\left(\frac{x+y}{\sqrt{2}}\right)^2 - \left(\frac{-x+y}{\sqrt{2}}\right)^2 = a^2$$

or $\dfrac{1}{2}\{(x+y)^2 - (-x+y)^2\} = a^2$

$\Rightarrow \dfrac{1}{2}(2y)(2x) = a^2$

or $xy = \dfrac{a^2}{2} = \left(\dfrac{a}{\sqrt{2}}\right)^2 = c^2$ (say)

or $xy = c^2$



Note:

1. The equations of the asymptotes and the conjugate hyperbola of the rectangular hyperbola $xy = c^2$, where the axes are the asymptotes, are $xy = 0$ and $xy = -c^2$ respectively.
2. The equation of a rectangular hyperbola having co-ordinate axes as its asymptotes is $xy = c^2$. If the asymptotes of a rectangular hyperbola are $x = \alpha$, $y = \beta$ then its equation is $(x - \alpha)(y - \beta) = c^2$ or $xy - \alpha y - \beta x + \lambda = 0$.
3. Since $x = ct$, $y =$ satisfies $xy = c^2$

   $\therefore (x, y) = \left(ct, \dfrac{c}{t}\right) (t \neq 0)$ is called a 't' point on the rectangular hyperbola. The set $\left\{x = ct, y = \dfrac{c}{t}\right\}$ represents its parametric equations with parameter 't'

   For the hyperbola $xy = c^2$

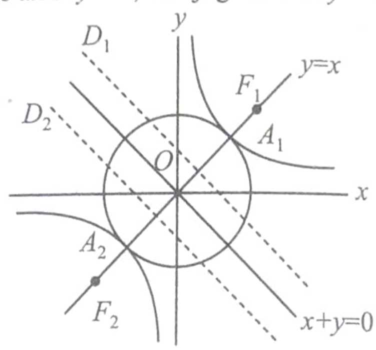
   (*i*) Vertices: (c, c) and $(-c, -c)$.

   (*ii*) Foci: $(\sqrt{2}c, \sqrt{2}c)$ & $(-\sqrt{2}c, -\sqrt{2}c)$

   (*iii*) Directrices: $x + y = \pm\sqrt{2}c$

   (*iv*) Latus rectum: $l = 2\sqrt{2}c = T.A = C.A$

   (*v*) Transvers axis: $y = x$, Conjugate axis: $y = -x$



PROPERTIES OF RECTANGULAR HYPERBOLA XY = C²

(*i*) Equation of the chord joining 't_1' and 't_2' is $x + yt_1t_2 - c(t_1 + t_2) = 0$

(*ii*) Equation of tangent at (x, y) is $xy_1 + x_1y = 2c^2$

(*iii*) Equation of tangent at 't' is $\dfrac{x}{t} + yt = 2c$

(*iv*) Point of intersection of tangents at 't_1' and 't_2' is $\left(\dfrac{2ct_1t_2}{t_1+t_2}, \dfrac{2c}{t_1+t_2}\right)$

(*v*) Equation of normal at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$

(*vi*) Equation of normal at 't' is $xt^3 - yt - ct + c = 0$

(*vii*) Point of intersection of normals at 't_1' and 't_2' is

$$\left(\frac{c\{t_1t_2(t_1^2 + t_1t_2 + t_2^2) - 1\}}{t_1t_2(t_1+t_2)}, \frac{c\{t_1^3t_2^3 + (t_1^2 + t_1t_2 + t_2^2)\}}{t_1t_2(t_1+t_2)}\right)$$

at the point 't_1' to the rectangular hyperbola $xy = c^2$ meets it.

CONCYCLIC POINTS ON THE HYPERBOLA XY = C²

Let a circle and the rectangular hyperbola $xy = c$ meet at the four points t_1, t_2, t_3 and t_4,

1. Then, $t_1t_2t_3t_4 = 1$.

Proof: Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + d = 0$ ...(*i*)

Solving (*i*) and the equation of hyperbola, we have

$$x^2+\frac{c^4}{x^2}+2gx+2f\frac{c^2}{x}+d=0$$

or $x^4+2gx^3+dx^2+2fc^2x+c^4=0$ ...(*ii*)

From (*i*),

$x_1x_2x_3x_4=c^4$

$c^4[t_1t_2t_3t_4]=c^4$

or $t_1t_2t_3t_4=1$

2. The center of the mean position of the four points bisects the distance between the centers of the two curves.

Proof: Again, the center of the mean position of the four points of intersection is $(\Sigma x_i/4, \Sigma y_i/4)$.

Now, from (*i*),

$$x_1+x_2+x_3+x_4=-2g \qquad ...(iii)$$

or $\dfrac{\Sigma x_i}{4}=-\dfrac{g}{2}$

Using $xy=c^2$, we have

$$y_1+y_2+y_3+y_4=c^2\left[\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right]$$

$$=\frac{c^2}{x_1x_2x_3x_4}\Sigma x_1x_2x_3$$

$$=\frac{c^2}{c^4}(-2fc^2)=-2f$$

or $\dfrac{\Sigma y_1}{4}=-\dfrac{f}{2}$

Hence, $\left(\dfrac{\Sigma x_i}{4},\dfrac{\Sigma y_i}{4}\right)=\left(-\dfrac{g}{2},-\dfrac{f}{2}\right)$

Train Your Brain

Example 27: C is the centre of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$. The tangent at any point P on this hyperbola meets the straight lines $bx-ay=0$ and $bx+ay=0$ in the points Q and R respectively. Show that $CQ.CR=a^2+b^2$.

Sol. P is $(a\sec\theta, b\tan\theta)$

Tangent at P is $\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1$.

It meets $bx-ay=0$ i.e. $\dfrac{x}{a}=\dfrac{y}{b}$ in Q

$\therefore$ Q is $\left(\dfrac{a}{\sec\theta-\tan\theta},\dfrac{b}{\sec\theta-\tan\theta}\right)$

It meets $bx+ay=0$ i.e. $\dfrac{x}{a}=\dfrac{-y}{b}$ in R.

$\therefore$ R is $\left(\dfrac{a}{\sec\theta+\tan\theta},\dfrac{-b}{\sec\theta+\tan\theta}\right)$

$$\therefore\ CQ.CR=\frac{\sqrt{(a^2+b^2)}}{\sec\theta-\tan\theta}\cdot\frac{\sqrt{(a^2+b^2)}}{\sec\theta+\tan\theta}=a^2+b^2$$

$\therefore\ \sec^2\theta-\tan^2\theta=1$

Example 28: Chords of the circle $x^2+y^2=a^2$ touches the hyperbola $x^2/a^2-y^2/b^2=1$. Prove that locus of their middle point is the curve $(x^2+y^2)^2=a^2x^2-b^2y^2$.

Sol. Let (h, k) be the mid-point of the chord of the circle $x^2+y^2=a^2$, so that its equation by $T=S_1$ is

$$hx+ky=h^2+k^2 \text{ or } y=-\frac{h}{k}x+\frac{h^2+k^2}{k}$$

i.e. of the form $y=mx+c$. It will touch the hyperbola if $c^2=a^2m^2-b^2$

$$\therefore\ \left(\frac{h^2+k^2}{k}\right)^2=a^2\left(-\frac{h}{k}\right)^2-b^2$$

or $(h^2+k^2)^2=a^2h^2-b^2k^2$

Generalizing, the locus of mid-point (h, k) is $(x^2+y^2)^2=a^2x^2-b^2y^2$

Example 29: The number of normal(s) of a rectangular hyperbola which can touch its conjugate is equal to

(*a*) 0 (*b*) 2 (*c*) 4 (*d*) 8

Sol. (*c*) Normal to hyperbola $xy=c^2$ at $\left(ct,\dfrac{c}{t}\right)$ is $y-\dfrac{c}{t}=t^2(x-ct)$

Solving with $xy=-c^2$, we get

$$\Rightarrow\ x\left\{\frac{c}{t}+t^2(x-ct)\right\}+c^2=0$$

$$\Rightarrow\ t^2x^2+\left(\frac{c}{t}-ct^3\right)x+c^2=0$$

Since line touches the curve, above equation has equal roots

$$\therefore\ \Delta=0,\left(\frac{1}{t}-t^3\right)^2-4t^2=0$$

$\Rightarrow (1-t^4)^2-4t^4=0$

$\Rightarrow t^4-2t^2-1=0$ or $t^4+2t^2-1=0$

$$\Rightarrow\ t^4=\frac{2\pm\sqrt{8}}{2}\text{ or }\frac{-2\pm\sqrt{8}}{2}\Rightarrow t^2=1+\sqrt{2}\text{ or }-1+\sqrt{2}$$

Thus four such values of t are possible.

Example 30: Write the following equation of a Rotated Conic Section in Standard Form

$$7x^2-6\sqrt{3}xy+13y^2-16=0$$

Sol. Rewrite the equation

$$\cot 2\theta=\frac{A-C}{B}=\frac{1}{\sqrt{3}}\cdot\ \{A=7, B=-6\sqrt{3}, C=13\}$$

so the equation in the standard from, is

$$\frac{x'^2}{4}+\frac{y'^2}{1}=1$$

Concept Application

30. If the normal at the point 't_1' to the rectangular hyperbola $xy = c^2$ meets it again at the point 't_2'. Prove that $t^3_1 t_2 = -1$

31. PQ and RS are two perpendicular chords of the rectangular hyperbola $xy = c^2$. If C is the centre of the rectangular hyperbola, then find the value of product of the slopes of CP, CQ, CR and CS.

32. If the normal to the rectangular hyperbola $xy = c^2$ at the point 't' meets the curve again at 't_1' then $t^3 t_1$ has the value equal to

(a) 1 (b) −1
(c) 0 (d) None

33. The normal at t of the rectangular hyperbola $xy = 16$ meets the x-axis at

(a) $\left(\frac{4}{t}-4t^3,0\right)$ (b) $\left(0,\frac{4}{t}-4t^3\right)$

(c) $\left(4t-\frac{4}{t^3},0\right)$ (d) $\left(\frac{4}{t^3}-4t,0\right)$

IMPORTANT HIGHLIGHTS OF HYPERBOLA

❖ A **hyperbola** is symmetric along the conjugate axis, and shares many similarities with the ellipse. Concepts like foci, directrix, latus rectum, eccentricity, apply to a hyperbola.

❖ Sometimes students get confused with parabola and hyperbola, so it is important to understand that

✦ A parabola is a locus that contains all points with the same distance from a focus and a directrix. On the other hand, a hyperbola is a locus of all the points where the distance between two foci is constant.

✦ A hyperbola is an open curve with two branches and two foci and directrices, whereas a parabola is an open curve with one focus and directrix.

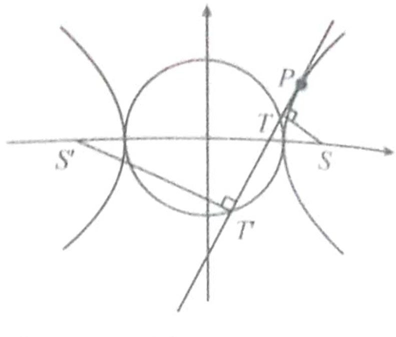
✦ A parabola's eccentricity is one, whereas a hyperbola's eccentricity is larger than one.

✦ When a plane intersects a cone at its slant height, a parabola is generated. On the other hand, a hyperbola is generated when a plane hits a cone at its perpendicular height.

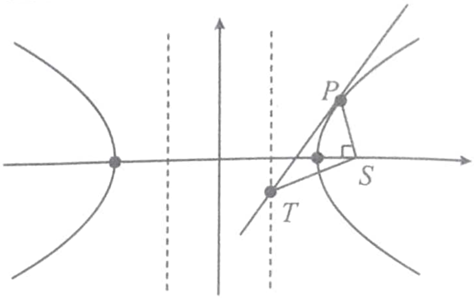
❖ Example of hyperbola include the path followed by the tip of the shadow of a sundial.

❖ Difference of focal distances is a constant, i.e. $|PS - PS'| = 2a$

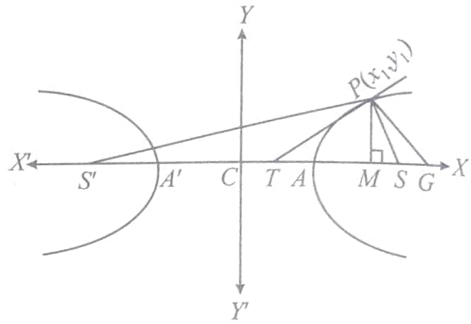
❖ Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ and the product of these perpendiculars is b^2.



❖ The portion of the tangent between the point of contact and the directrix subtends a right angle at the corresponding focus.



Prop. 3: If the normal at P meets the transverse axis in G, then $SG = e \cdot SP$. Prove also that the tangent and normal bisect the angle between the focal distances of P.



Let the co-ordinates of P be (x_1, y_1). The equation of normal at P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$ ... (1)

The normal (1) meets the x-axis i.e., $y = 0$ in (1) then co-ordinates of G are $\left(\frac{(a^2+b^2)}{a^2}x_1,0\right)$ or $(e^2x_1, 0)$

$\therefore \quad CG = e^2x_1$

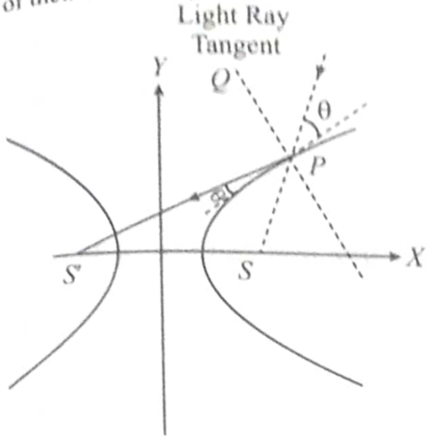
Now $SG = CG - CS = e^2x_1 - ae = e(ex_1 - a) = e \cdot SP$

Similarly, $S'G = e \cdot S'P$

$$\therefore \quad \frac{SG}{S'G}=\frac{SP}{S'P}$$

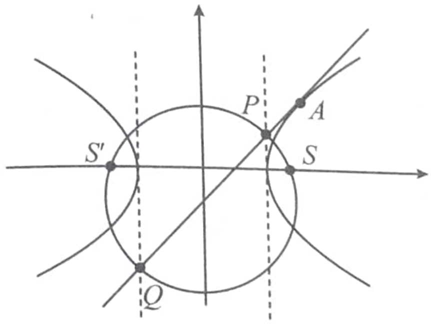
This relation shows that the normal PG is the external bisector of the angle SPS'. The tangent PT being perpendicular to PG is therefore the internal bisector of the angle SPS'.

❖ The tangent and normal at any point of a hyperbola bisect the angle between the focal radii. This explains the reflection property of the hyperbola as **"An incoming light ray"** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.

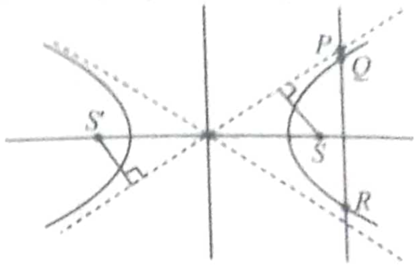


❖ Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1$ $(a>k>b>0)$ are confocal and therefore orthogonal.

❖ The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.



❖ If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point and the curve is always equal to the square of the semi conjugate axis.



❖ Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix and the common points of intersection lie on the auxiliary circle.

❖ If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is 2θ then the eccentricity of the hyperbola is $\sec\theta$.

❖ Product of perpendicular distances from andy point on the hyperbola (or its conjugate hyperbola) on its asymptote is $\frac{a^2b^2}{a^2+b^2}$.

❖ A rectangular hyperbola circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i,\frac{c}{t_i}\right)$ i $=1, 2, 3$ be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1\,t_2\,t_3},-c\,t_1\,t_2\,t_3\right)$.

❖ If a circle and the rectangular hyperbola $xy=c^2$ meet in the four points t_1, t_2, t_3 and t_4, then

(*a*) $t_1\,t_2\,t_3\,t_4=1$

(*b*) the centre of the mean position of the four points bisects the distance between the centres of the two curves.

(*c*) the centre of the circle through the points t_1, t_2 and t_3 is :

$$\left\{\frac{c}{2}\left(t_1+t_2+t_3+\frac{1}{t_1\,t_2\,t_3}\right),\frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+t_1\,t_2\,t_3\right)\right\}$$

Short Notes

❖ Standard equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $b^2=a^2(e^2-1)$

or $a^2e^2=a^2+b^2$ *i.e.* $e^2=1+\frac{b^2}{a^2}=1+\left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$

(*a*) Foci:

$S\equiv(ae,0)$ & $S'\equiv(-ae,0)$.

(*b*) Equations of Directrices:

$x=\frac{a}{e}$ & $x=-\frac{a}{e}$.

(*c*) Vertices:

$A\equiv(a,0)$ & $A'\equiv(-a,0)$.

(*d*) Latus Rectum:

(*i*) Equation: $x=\pm ae$

(*ii*) Length $=\frac{2b^2}{a}=\frac{(\text{Conjugate Axis})}{(\text{Transverse Axis})}=2a(e^2-1)$

$=2e$(distance from focus to directrix)

(*iii*) Ends: $\left(ae,\frac{b^2}{a}\right),\left(ae,\frac{-b^2}{a}\right);\left(-ae,\frac{b^2}{a}\right),\left(-ae,\frac{-b^2}{a}\right)$

(e) Focal Property:

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis *i.e.* $||PS| - |PS'|| = 2a$. The distance SS' = focal length.

(f) Focal Distance:

Distance of any point $P(x, y)$ on hyperbola from foci $PS = ex - a$ & $PS' = ex + a$.

Conjugate Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each.

Auxillary Circle: $x^2 + y^2 = a^2$.

Parametric Representation: $x = a \sec\theta$ & $y = b \tan\theta$

Position of A point 'P' w.r.t. A Hyperbola:

$S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 >= \text{or} < 0$ according as the point (x_1, y_1) lines inside, on

or outside the curve.

Tangents

(i) Slope Form: $y = m \times \pm \sqrt{a^2m^2 - b^2}$

(ii) Point Form: at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(iii) Parametric Form: $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

❖ **Normal to The Hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(a) Point form: Equation of the normal to the given hyperbola at the point $P(x_1, y_1)$ on it is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2$.

(b) Slope form: The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{(a^2 - m^2b^2)}}$ foot of normal are

$$\left(\pm\frac{a^2}{\sqrt{(a^2 - m^2b^2)}}, \mp\frac{mb^2}{\sqrt{(a^2 - m^2b^2)}}\right).$$

(c) Parametric form: The equation of the normal at the point P ($a \sec\theta$, $b \tan\theta$) to the given hyperbola is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$.

Director Circle

Equation to the director circle is: $x^2 + y^2 = a^2 - b^2$.

Chord of Contact

If PA and PB be the tangents from point $P(x_1, y_1)$ to the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the equation of the chord of contact AB is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ or $T = 0$ at (x_1, y_1).

Equation of Chord with mid Point (x_1, y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, whose mid-point be (x_1, y_1) is $T = S_1$ where T

$$= \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1,\ S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

i.e. $\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right) = \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right)$.

Asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

Reflection property of the hyperbola: An incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.

Rectangular or Equilateral Hyperbola: $xy = c^2$, eccentricity is $\sqrt{2}$.

Vertices: $(\pm c \pm c)$; Focii : $(\pm\sqrt{2}c, \pm\sqrt{2}c)$. Directrices : $x + y = \pm\sqrt{2}\,c$.

Latus Rectum (l): $= l = 2\sqrt{2}\,c = T.A. = C.A.$

Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$

Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at $P(t)$ is $\frac{x}{t} + ty = 2c$.

Equation of the normal at $P(t)$ is $xt^3 - yt = c(t^4 - 1)$.

Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

Solved Examples

1. In X-Y plane, the path defined by the equation $\frac{1}{x^m} + \frac{1}{y^m} + \frac{k}{(x+y)^n} = 0$, is

(a) a parabola if $m = \frac{1}{2}$, $k = -1$, $n = 0$

(b) a hyperbola if $m = 1$, $k = -1$, $n = 0$

(c) a pair of lines if $m = k = n = 1$

(d) a pair of lines if $m = k = -1$, $n = 1$

Sol. (a,b,c,d)

(a) $\sqrt{x} + \sqrt{y} = 1$

$\Rightarrow x + y + 2\sqrt{xy} = 1$

$\Rightarrow 4xy = (1 - x - y)^2$

(*b*) $\frac{1}{x}+\frac{1}{y}=1 \Rightarrow xy - x - y = 0$ is a hyperbola

(*c*) $\frac{1}{x}+\frac{1}{y}+\frac{1}{x+y}=0 \Rightarrow x^2+3xy+y^2=0$, which is a pair of lines.

(*d*) $x+y-\frac{1}{x+y}=0$

$\Rightarrow (x+y)^2 = 1$

$\Rightarrow x + y = \pm 1$ which is a pair of lines.

2. A point moves such that the sum of the squares of its distances from the two sides of length '*a*' of a rectangle is twice the sum of the squares of its distances from the other two sides of length '*b*'. The locus of the point can be

(*a*) a circle (*b*) an ellipse

(*c*) a hyperbola (*d*) a pair of lines

Sol. (*c*, *d*)

Let the two sides of the rectangle lie along *x*-axis and *y*-axis.

Let point *P* be (h, k)

Given that

$(PA)^2 + (PB)^2 = 2(PC^2 + PD^2)$

$\Rightarrow k^2 + (k - b)^2 = 2(h^2 + (a - h)^2)$

$\Rightarrow 2k^2 - 2kb + b^2 = 4h^2 - 4ab + 2a^2$

Replacing *h* by *x* and *k* by *y*

$\Rightarrow 2y^2 - 2by + b^2 = 4x^2 - 4ax + 2a^2$

or $2(y^2 - by) + b^2 = 4(x^2 - ax) + 2a^2$

or $4\left(x-\frac{a}{2}\right)^2-2\left(y-\frac{b}{2}\right)^2=\frac{b^2}{2}-a^2$

Hence it is a hyperbola if $\frac{b^2}{2}-a^2 \neq 0$

or pair of lines if $\frac{b^2}{2}-a^2=0$

3. A tangent to the hyperbola $y=\frac{x+9}{x+5}$ passing through the origin is

(*a*) $x + 25y = 0$ (*b*) $5x + y = 0$

(*c*) $5x - y = 0$ (*d*) $x - 25y = 0$

Sol. (*a*) $y=\frac{x+9}{x+5}=1+\frac{4}{x+5}$

$\frac{dy}{dx}$ at $(x_1, y_1)=\frac{-4}{(x_1+5)^2}$

$\therefore$ Equation of tangent

$y-y_1=\frac{-4}{(x_1+5)^2}(x-x_1)$

$y-1-\frac{4}{x_1+5}=\frac{-4}{(x_1+5)^2}(x-x_1)$

Since it passes through (0, 0)

$(x_1+5)^2+4(x_1+5)+4x_1=0$

$x_1=-15$ or $x_1=-3$. So equations are $x+25y=0$ or $x+y=0$.

4. If two tangents can be drawn the different branches of hyperbola $\frac{x^2}{1}-\frac{y^2}{4}=1$ from (α, α^2), then

(*a*) $\alpha \in (-2, 0)$ (*b*) $\alpha \in (0, 2)$

(*c*) $\alpha \in (-\infty, -2)$ (*d*) $\alpha \in (2, 0)$

Sol. (*c*,*d*)

(α, α^2) must lie between the asymptotes of hyperbola $\frac{x^2}{1}-\frac{y^2}{4}=1$ in I and II quadrant. Asymptotes are $y = \pm 2x$; $\Rightarrow 2\alpha < \alpha^2 \Rightarrow \alpha < 0$ or $\alpha > 2$.

and $-2\alpha < \alpha^2 \Rightarrow \alpha < -2$ or $\alpha > 0$

$\Rightarrow \alpha \in (-\infty, -2) \cup (2, \infty)$

5. If *m* is the slope of a tangent to the hyperbola

$\frac{x^2}{a^2-b^2}-\frac{y^2}{a^3-b^3}=1$ where $a > b > 1$ then

(*a*) $(a + b)m^2 + ab \geq (a + b)^2$

(*b*) $(a + b)^2 m + ab \geq (a + b)$

(*c*) $abm^2 + (a + b) \geq (a + b)^2$

(*d*) $(a + b)m^2 + a^2b^2 \geq (a + b)^2$

Sol. (*a*) For hyperbola $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$ equation of tangent having slope *m* is

$\Rightarrow y = mx \pm \sqrt{A^2m^2 - B^2}$

where $A^2m^2 - B^2 \geq 0$

$\Rightarrow m^2 \geq \frac{B^2}{A^2} \Rightarrow m^2 \geq \frac{a^3-b^3}{a^2-b^2}$

$\Rightarrow (a + b)m^2 \geq a^2 + ab + b^2$

$\Rightarrow (a + b)m^2 + ab \geq (a + b)^2$

6. Two tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{9}=1$, having slopes 2 and *m* where $(m \neq 2)$ cuts the axes at four concyclic points then the slope *m* is/are

(*a*) $-\frac{1}{2}$ (*b*) -2 (*c*) $\frac{1}{2}$ (*d*) 2

Sol. (*c*) Tangents having slope '2', $y = 2x \pm \sqrt{100-9}$

$= 2x \pm \sqrt{91}$

It meets the axis at $A \equiv \left(\mp\frac{\sqrt{91}}{2}, 0\right) B(0, \pm\sqrt{91})$

Tangents having slope '*m*', $y = mx \pm \sqrt{25m^2-9}$

It meets the axis at $D\left(\pm\frac{\sqrt{25m^2-9}}{m}, 0\right)$,

$C \equiv (0, \mp\sqrt{25m^2-9})$

$ABCD$ is cyclic quadrilateral

$\Rightarrow OA \times OD = OB \times OC$

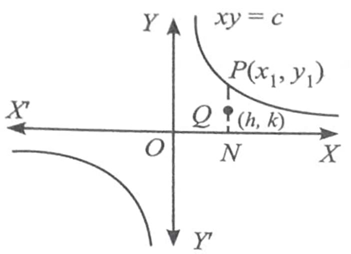
$$\Rightarrow \frac{\sqrt{91}(\sqrt{25m^2-9})}{2m} = \sqrt{91}(\sqrt{25m^2-9})$$

$$\Rightarrow 2m = 1 \Rightarrow m = \frac{1}{2}$$

7. If PN is the perpendicular from a point on a rectangular hyperbola to its asymptotes, then find the locus of the midpoint of PN.

Sol. Let $xy = c^2$ be the rectangular hyperbola and let $P(x, y)$ be the point on it.

Let $Q(h, k)$ be the midpoint of PN. Then the coordinates of Q are $(x_1, y_1/2)$. Therefore,



$x_1 = h$ and $\frac{y_1}{2} = k$

or $x_1 = h$ and $y_1 = 2k$

But (x_1, y_1) lies on $xy = c^2$.

Therefore,

$h(2k) = c^2$

or $hk = \frac{c^2}{2}$

Hence, the locus of (h, k) is $xy = c^2/2$ which is a rectangular hyperbola.

8. If the pair of asymptotes of a hyperbola H is $xy - 2x - y + 2 = 0$. Its conjugate hyperbola H' passes through (3, 0) then

(*a*) eccentricity of H' is $\sqrt{2}$

(*b*) focus of H' is $\left(2+\frac{1}{\sqrt{2}}, 2+\frac{1}{\sqrt{2}}\right)$

(*c*) focus of H is $(\sqrt{2}, -\sqrt{2})$

(*d*) equation of transverse axis of H is $x - y + 1 = 0$

Sol. (*a, d*) $\because (x-1)(x-2) = 4$

conjugate hyperbola $(x-1)(x-2) = -4$

$\therefore \frac{a^2}{2} = 4 \therefore a = 2\sqrt{2}$

focus of $H' \equiv (-2\sqrt{2}, 2\sqrt{2})$ & $(2\sqrt{2}, -2\sqrt{2})$

focus of $H \equiv (1-2\sqrt{2}, 2+2\sqrt{2})$ & $(1+2\sqrt{2}, 2-2\sqrt{2})$

Equation of transverse axis $y - 2 = x - 1 \Rightarrow x - y + 1 = 0$.

9. The coordinate plane is constrained by the hyperbola $x^2 - y^2 = 4$ so that only the region exterior of the hyperbola is accessible what are the possible values of the ordinates of the center of a circular disc of radius 4 where centres lies on the y-axis?

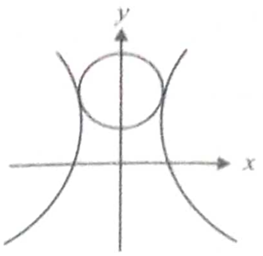
(*a*) $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$ (*b*) $(-\infty, -2\sqrt{3}] \cup [2\sqrt{3}, \infty)$

(*c*) $(-\infty, -\sqrt{6}] \cup [\sqrt{6}, \infty)$ (*d*) $(-\infty, -2\sqrt{6}] \cup [2\sqrt{6}, \infty)$

Sol. Let the equations of circle be

$x^2 + (y - \lambda)^2 = 16$

$\lambda = \pm 2\sqrt{6}$



10. Tangents are drawn to a hyperbola from any point on one of the branches of the conjugate hyperbola. Show that their chord of contant will touch the other branch of the conjugate hyperbola.

Sol. Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. So its conjugate hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$. Let any point on it be (a tan θ, b sec θ). Now equation of chord of contant will

$\frac{x}{a}\tan\theta - \frac{y}{b}\sec\theta = 1 \Rightarrow \frac{x}{a}(-\tan\theta) - \frac{y}{b}(-\sec\theta) = -1.$

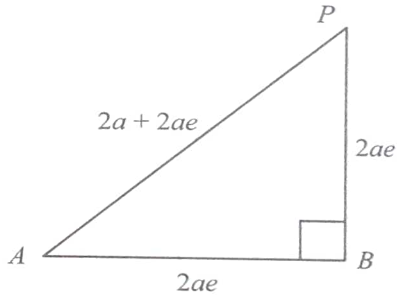
So, it is a tangent to the conjugate hyperbola at point ($-a$ tan θ, $-b$ sec θ) which will obviously on the other branch.

11. A and B are focus of hyperbola $25[(x-1)^2 + (y+1)^2 = k^2] - k^2(3x + 4y + 5)^2$, $k > 1$ and P is a variable point on hyperbola such that ΔAPB is right angled isosceles triangle. If $k = \tan \alpha$, $a \in [0, \pi]$

Sol. When, AP $= 2a + 2ae$

$\Rightarrow AP^2 = AB^2 + PB^2$

$\Rightarrow 4a^2 + a^2e^2 + 8a^2e = 4a^2e^2 + 4a^2e^2$



$\Rightarrow e^2 - 2e - 1 = 0$

$\Rightarrow e = \frac{2+2\sqrt{2}}{2}$ or $\frac{2-2\sqrt{2}}{2}$ reject

$\Rightarrow e = 1 \pm \sqrt{2} \Rightarrow k = 1 + \sqrt{2}$

$a = 67\frac{1°}{2}$

12. Let C_1 be the graph of $xy = 1$ and the reflection of C_1 in the line $y = 2x$ is C_2. If the equation of C_2 is expressed as $12x^2 + bxy + cy^2 + d = 0$ then (bc) is equal to

Sol. [84] Any point on hyperbola $xy = 1$ is $\left(t, \frac{1}{t}\right)$.

Now image of $\left(t, \frac{1}{t}\right)$ by the line mirror $y = 2x$,

$$\frac{x-t}{2} = \frac{y-\frac{1}{t}}{-1} = -2\left(\frac{2t-\frac{1}{t}}{5}\right)$$

$$\Rightarrow 5x = \frac{4}{t} - 3t \text{ and } 5y = 4t + \frac{3}{t}$$

Now eliminating 't' from above two equations we get,

$$12x^2 - 12y^2 - 7xy + 25 = 0$$

$$\Rightarrow bc = (-7)(-12) = 84$$

13. A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

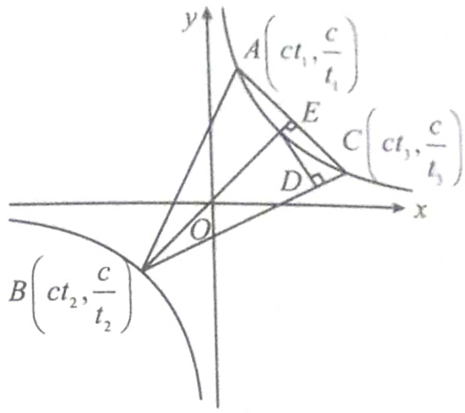
Sol. Let t_1, t_2 and t_3 are the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.

$\therefore$ co-ordinates of A, B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively

Now slope of BC is $\frac{\frac{c}{t_3}-\frac{c}{t_2}}{ct_3 - ct_2} = -\frac{1}{t_2 t_3}$

$\therefore$ Slope of AD is $t_2 t_3$

Equation of altitude AD is $y - \frac{c}{t_1} = t_2 t_3(x - ct_1)$



or $t_1 y - c = xt_1t_2t_3 - ct_1^2t_2t_3$ ... (i)

Similarly equation of altitude BE is

$t_2 y - c = xt_1t_2t_3 - ct_1^2t_2t_3$ ... (ii)

Solving (i) and (ii), we get the orthocentre $\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$ which lies on $xy = c^2$.

Exercise-1 (Topicwise)

STANDARD FORM AND EQUATION OF HYPERBOLA

1. The equation $\frac{x^2}{10-a}+\frac{y^2}{4-a}=1$ represents an hyperbola ($a>0$) if
(a) $a<4$
(b) $a>4$
(c) $4<a<10$
(d) $a>10$

2. A directrix of the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ is:
(a) $x=9/\sqrt{13}$
(b) $y=9/\sqrt{13}$
(c) $x=6/\sqrt{13}$
(d) $y=6/\sqrt{13}$

3. The equation of the hyperbola whose conjugate axis is 5 and the distance between the foci is 13, can be
(a) $25x^2-144y^2=900$
(b) $144x^2-25y^2=900$
(c) $144x^2+25y^2=900$
(d) $25x^2+144y^2=900$

4. If (4, 0) and (–4, 0) be the vertices and (6, 0) and (–6, 0) be the foci of a hyperbola, then its eccentricity is
(a) 5/2
(b) 2
(c) 3/2
(d) $\sqrt{2}$

5. The equation of the transverse and conjugate axis of the hyperbola $16x^2-y^2+64x+4y+44=0$ are
(a) $x=2,\ y+2=0$
(b) $x=2,\ y=2$
(c) $y=2,\ x+2=0$
(d) $x+2=0,\ y+2=0$

6. If (0, ±4) and (0, ±2) be the foci and vertices of a hyperbola, then its equation is
(a) $\frac{x^2}{4}-\frac{y^2}{12}=1$
(b) $\frac{x^2}{12}-\frac{y^2}{4}=1$
(c) $\frac{y^2}{4}-\frac{x^2}{12}=1$
(d) $\frac{y^2}{12}-\frac{x^2}{4}=1$

7. The equation of the hyperbola whose directrix is $x+2y-1=0$, focus (2, 1) and eccentricity 2 will be
(a) $x^2-16xy-11y^2-12x+6y+21=0$
(b) $3x^2+16xy+15y^2-4x-14y-1=0$
(c) $x^2+16xy+11y^2-12x-6y+21=0$
(d) $3x^2-16xy+15y^2-4x+14y+1=0$

8. The equation $16x^2-3y^2-32x+12y-44=0$ represents a hyperbola
(a) The length of whose transverse axis is $4\sqrt{3}$
(b) The length of whose conjugate axis is 4
(c) Whose center is (–1, 2)
(d) Whose eccentricity is $\sqrt{\frac{19}{3}}$

9. The co-ordinates of a focus of the hyperbola $9x^2-16y^2+18x+32y-151=0$ are
(a) (–1, 1)
(b) (6, 1)
(c) (4, –1)
(d) (– 6, 1)

10. If the foci of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ coincide with the foci of $\frac{x^2}{25}+\frac{y^2}{9}=1$ and the eccentricity of the hyperbola is 2, then which of the following is false:
(a) $a^2+b^2=16$.
(b) There is no director circle to the hyperbola.
(c) The center of the director circle is (0, 0).
(d) The length of latus rectum of the hyperbola is 12.

11. The equation $(x-\alpha)^2+(y-\beta)^2=k(lx+my+n)^2$ represents:
(a) A parabola for $k<(l^2+m^2)^{-1}$
(b) An ellipse for $0<k<(l^2+m^2)^{-1}$
(c) A hyperbola for $K>(l^2+m^2)^{-1}$
(d) A point circle for $k=0$

12. A hyperbola passes through the points (3, 2) and (–17, 12) and has its centre at origin and transverse axis is along x-axis. Then find the length of its transverse axis.
(a) 1
(b) 2
(c) 3
(d) 5

13. If the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide. Find the value of b^2.
(a) 5
(b) 6
(c) 7
(d) 8

14. If e and e' are the eccentricities of the ellipse $5x^2+9y^2=45$ and the hyperbola $5x^2-4y^2=45$ respectively. Find the value of ee'.
(a) 1
(b) 3/2
(c) 5/4
(d) 3

15. The equation $\frac{x^2}{1-r}-\frac{y^2}{1+r}=1$, $r>1$ represents
(a) A circle
(b) A parabola
(c) An ellipse
(d) None of these

16. The parametric equations for the conic section $x^2 - 8x - 4y^2 - 16y - 4 = 0$ are:
(a) $x = -4 + 2\sec\theta$ $y = 2 + \tan\theta$
(b) $x = -4 + 2\tan\theta$ $y = 2 + \sec\theta$
(c) $x = -4 + 2\tan\theta$ $y = 2 + \sec\theta$
(d) $x = 4 + 2\sec\theta$ $y = -2 + \tan\theta$

17. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2\alpha + y^2 = 25$ then a value of α is:
(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$

18. Which of the following equations in parametric form can not represent a hyperbola, where 't' is a parameter?
(a) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ and $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$
(b) $\frac{tx}{a} - \frac{y}{b} + t = 0$ and $\frac{x}{a} + \frac{ty}{b} - 1 = 0$
(c) $x = e^t + e^{-1}$ and $y = e^t - e^{-1}$
(d) $x^2 - 6 = 2\cos t$ and $y^2 + 2 = 4\cos^2\frac{t}{2}$

POSITION OF POINT & AUXILIARY CIRCLE

19. The equation of auxiliary circle of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is
(a) $x^2 + y^2 = a^2$ (b) $x^2 + y^2 = b^2$
(c) $x^2 + y^2 = a^2 + b^2$ (d) $x^2 + y^2 = a^2 - b^2$

LINE & HYPERBOLA, TANGENT & NORMAL

20. If the line $y = 2x + \lambda$ be a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{144} = 1$, then λ equal to:
(a) 16 (b) -16
(c) ± 16 (d) ± 4

21. The line $3x - 4y = 5$ is a tangent to the hyperbola $x^2 - 4y^2 = 5$. The point of contact is
(a) (3, 1) (b) (2, 1/4)
(c) (1, 3) (d) (1, 2)

22. The equation of the tangents to the conic $3x^2 - y^2 = 3$ perpendicular to the line $x + 3y = 2$ is
(a) $y = 3x \pm \sqrt{6}$ (b) $y = 6x \pm \sqrt{3}$
(c) $y = x \pm \sqrt{6}$ (d) $y = 3x \pm 6$

23. If the distance between two parallel tangents drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{49} = 1$ is 2, then their slope is equal to:
(a) $\pm 5/2$ (b) $\pm 4/5$
(c) $\pm 7/2$ (d) None of these

24. The equation of the normal at the point $(a\sec\theta, b\tan\theta)$ of the curve $b^2x^2 - a^2y^2 = a^2b^2$ is
(a) $\frac{ax}{\cos\theta} + \frac{by}{\sin\theta} = a^2 + b^2$ (b) $\frac{ax}{\tan\theta} + \frac{by}{\sec\theta} = a^2 + b^2$
(c) $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ (d) $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 - b^2$

25. The equation of the normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the point $(8, 3\sqrt{3})$ is
(a) $\sqrt{3}x + 2y = 25$ (b) $x + y = 25$
(c) $y + 2x = 25$ (d) $2x + \sqrt{3}y = 25$

26. The value of m, for which the line $y = mx + \frac{25\sqrt{3}}{3}$, is a normal to the conic $\frac{x^2}{16} - \frac{y^2}{9} = 1$, is
(a) $\sqrt{3}$ (b) $-\frac{2}{\sqrt{3}}$
(c) $-\frac{\sqrt{3}}{2}$ (d) 1

27. The equation of common tangents to the parabola $y^2 = 8x$ and hyperbola $3x^2 - y^2 = 3$, is
(a) $2x \pm y + 1 = 0$ (b) $2x \pm y - 1 = 0$
(c) $x \pm 2y + 1 = 0$ (d) $x \pm 2y - 1 = 0$

28. If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes in G and g and C is the centre of the hyperbola, then which of the following is false
(a) $PG = PC$ (b) $Pg = PC$
(c) $PG = Pg$ (d) $Gg = PC$

29. The equation of the tangent lines to the hyperbola $x^2 - 2y^2 = 18$ which are perpendicular to the line $y = x$ are:
(a) $y = -x + 7$ (b) $y = -x + 3$
(c) $y = -x - 4$ (d) $y = x - 3$

30. If $y = mx + c$ is a tangent to the hyperbola $x^2 - 3y^2 = 1$, then
(a) $c^2 = m^2$ (b) $c^2 = m^2 + 1$
(c) $c^2 > m^2$ (d) $c^2 < m^2$

CHORD OF CONTACTS, CHORD WITH GIVEN MID POINT

31. What will be equation of that chord of hyperbola $25x^2 - 16y^2 = 400$, whose mid point is (5, 3)
(a) $115x - 117y = 17$ (b) $125x - 48y = 481$
(c) $127x + 33y = 341$ (d) $15x + 121y = 105$

32. Equation of the chord of hyperbola $5x^2 - 3y^2 = 45$, which is bisected at (5, –3) is
(a) $9x + 25y = 88$ (b) $25x + 9y = 88$
(c) $9x + 9y = 88$ (d) $25x + 9y = 98$

33. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are at right angles, then $\frac{x_1x_2}{y_1y_2}$ is equal to

(a) $-\frac{a^2}{b^2}$ (b) $-\frac{b^2}{a^2}$

(c) $-\frac{b^4}{a^4}$ (d) $-\frac{a^4}{b^4}$

34. The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2$ is

(a) $3x - 4y = 4$

(b) $3y - 4x + 4 = 0$

(c) $4x - 4y = 3$

(d) $3x - 4y = 2$

ASYMPTOTES

35. The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$

(a) $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$

(b) $2x^2 + 5xy + 2y^2 = 0$

(c) $2x^2 + 5xy + 2y^2 - 4x + 5y + 2 = 0$

(d) $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

36. A hyperbola passes through (2, 3) and has asymptotes $3x - 4y + 5 = 0$ and $12x + 5y - 40 = 0$. Then, the equation of its transverse axis is:

(a) $77x - 21y - 265 = 0$

(b) $21x - 77y + 265 = 0$

(c) $21x - 77y - 265 = 0$

(d) $21x + 77y - 265 = 0$

RECTANGULAR HYPERBOLA

37. The coordinates of the foci of the rectangular hyperbola $xy = c^2$ are

(a) $(\pm c, \pm c)$

(b) $(\pm c\sqrt{2}, \pm c\sqrt{2})$

(c) $\left(\pm\frac{c}{\sqrt{2}}, \pm\frac{c}{\sqrt{2}}\right)$

(d) $\left(\pm\frac{c}{2}, \pm\frac{c}{2}\right)$

38. The locus of the point of intersection of lines $(x + y)t = a$ and $x - y = at$, where t is the parameter, is

(a) A circle

(b) An ellipse

(c) A rectangular hyperbola

(d) A parabola

39. The length of the transverse axis of the rectangular hyperbola $xy = 18$ is:

(a) 6 (b) 12

(c) 18 (d) 9

40. A tangent to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ intercepts a length of unity from each of the co-ordinate axes, then the point (a, b) lies on the rectangular hyperbola

(a) $x^2 - y^2 = 2$ (b) $x^2 - y^2 = 1$

(c) $x^2 - y^2 = -1$ (d) $x^2 - y^2 = -2$

41. If $5x^2 + \lambda y^2 = 20$ represents a rectangular hyperbola, then λ equals

(a) 5 (b) 4

(c) –5 (d) –4

42. The distance between the directrices of a rectangular hyperbola is 10 units, then distance between its foci is

(a) $10\sqrt{2}$ (b) 5

(c) $5\sqrt{2}$ (d) 20

43. If a circle cuts a rectangular hyperbola $xy = c^2$ at A, B, C, D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively. Then

(a) $t_1t_2 = t_3t_4$ (b) $t_1t_2t_3t_4 = 1$

(c) $t_1 = t_2$ (d) $t_3 = t_4$

44. Find the area of triangle formed by tangent to the hyperbola $xy = 16$ *at* (16, 1) and co-ordinate axes equals.

(a) 30 (b) 16

(c) 32 (d) 8

45. Equation of a rectangular hyperbola whose asymptotes are $x = 3$ and $y = 5$ and passing through (7, 8) is

(a) $xy - 3y + 5x + 3 = 0$

(b) $xy + 3y + 5x + 3 = 0$

(c) $xy - 3y + 5x - 3 = 0$

(d) $xy - 3y - 5x + 3 = 0$

46. The equation of the chord of contact of tangents from (2, 3) to the rectangular hyperbola $xy = 9$ is

(a) $3x + 2y = 18$ (b) $2x - 3y = 18$

(c) $2x + 3y = 9$ (d) $3x + 2y = 9$

47. Asymptotes of the hyperbola $xy = 4x + 3y$ are

(a) $x = 3, y = 4$ (b) $x = 4, y = 3$

(c) $x = 2, y = 6$ (d) $x = 6, y = 2$

48. If the conics $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ and $x^2 - y^2 = c^2$ cut orthogonally then

(a) $a^2 + b^2 = 2c^2$

(b) $b^2 - a^2 = 2c^2$

(c) $a^2b^2 = 2c^2$

(d) $a^2 - b^2 = 2c^2$

Exercise-2 (Learning Plus)

1. The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is
   (a) 1 (b) $\sqrt{2}$ (c) 2 (d) 1/2

2. Which of the following pair, may represent the eccentricities of two conjugate hyperbolas, for all $\alpha \in (0, \pi/2)$?
   (a) $\sin\alpha, \cos\alpha$ (b) $\tan\alpha, \cot\alpha$
   (c) $\sec\alpha, \operatorname{cosec}\alpha$ (d) $1 + \sin\alpha, 1 + \cos\alpha$

3. The equation of a common tangent with positive slope to the circle as well as to the hyperbola is
   (a) $2x - \sqrt{5}y - 20 = 0$ (b) $2x - \sqrt{5}y + 4 = 0$
   (c) $3x - 4y + 8 = 0$ (d) $4x - 3y + 4 = 0$

4. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci, is:
   (a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$
   (c) $\frac{2}{\sqrt{3}}$ (d) None of these

5. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}k = 0$ and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$ for different values of k is
   (a) Ellipse (b) Parabola
   (c) Circle (d) Hyperbola

6. If the latus rectum of an hyperbola be 8 and eccentricity be $\frac{3}{\sqrt{5}}$ then the equation of the hyperbola (with axes along co-ordinate axes) can be
   (a) $4x^2 - 5y^2 = 100$ (b) $5x^2 - 4y^2 = 100$
   (c) $4x^2 + 5y^2 = 100$ (d) $5x^2 + 4y^2 = 100$

7. If the centre, vertex and focus of a hyperbola be (0, 0), (4, 0) and (6, 0) respectively, then the equation of the hyperbola is
   (a) $4x^2 - 5y^2 = 8$ (b) $4x^2 - 5y^2 = 80$
   (c) $5x^2 - 4y^2 = 80$ (d) $5x^2 - 4y^2 = 8$

8. The vertices of a hyperbola are at (0, 0) and (10, 0) and one of its foci is at (18, 0). The equation of the hyperbola is
   (a) $\frac{x^2}{25} - \frac{y^2}{144} = 1$ (b) $\frac{(x-5)^2}{25} - \frac{y^2}{144} = 1$
   (c) $\frac{x^2}{25} - \frac{(y-5)^2}{144} = 1$ (d) $\frac{(x-5)^2}{25} - \frac{(y-5)^2}{144} = 1$

9. The equation of the transverse axis of the hyperbola $(x - 3)^2 + (y + 1)^2 = (4x + 3y)^2$ is:
   (a) $x + 3y = 0$ (b) $4x + 3y = 9$
   (c) $3x - 4y = 13$ (d) $4x + 3y = 0$

10. The length of the transverse axis of a hyperbola is 7 and it passes through the point (5, −2). The equation of the hyperbola (if axes are along co-ordinate axes)is
   (a) $\frac{4}{49}x^2 - \frac{196}{51}y^2 = 1$ (b) $\frac{49}{4}x^2 - \frac{51}{196}y^2 = 1$
   (c) $\frac{4}{49}x^2 - \frac{51}{196}y^2 = 1$ (d) $\frac{49}{2}x^2 - \frac{51}{98}y^2 = 1$

11. The equations of the transverse and conjugate axis of a hyperbola are, respectively, $x + 2y - 3 = 0$ and $2x - y + 4 = 0$, and their respective lengths are $\sqrt{2}$ and $\frac{2}{\sqrt{3}}$. The equation of the hyperbola is:
   (a) $\frac{2}{5}(x+2y-3)^2 - \frac{3}{5}(2x-y+4)^2 = 1$
   (b) $\frac{2}{5}(2x-y+4)^2 - \frac{3}{5}(x+2y-3)^2 = 1$
   (c) $2(2x-y+4)^2 - 3(x+2y-3)^2 = 1$
   (d) $2(x+2y-3)^2 - 3(2x-y+4)^2 = 1$

12. If the eccentricity of the hyperbola $x^2 - y^2\sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\alpha + y^2 = 25$, then a value of α is
   (a) $\pi/6$ (b) $\pi/4$
   (c) $\pi/3$ (d) $\pi/2$

13. Let LL' be the latus rectum through the focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and A' be the farther vertex. If $\Delta A'LL'$ is equilateral, then the eccentricity of the hyperbola is (axis are coordinate axis)
   (a) $\sqrt{3}$ (b) $\sqrt{3}+1$
   (c) $\frac{(\sqrt{3}+1)}{\sqrt{2}}$ (d) $\frac{(\sqrt{3}+1)}{\sqrt{3}}$

14. The asympote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from with any tangent to the hyperbola a triangle whose area is $a^2\tan\lambda$ in magnitude then its eccentricity is:
   (a) $\sec\lambda$ (b) $\operatorname{cosec}\lambda$ (c) $\sec^2\lambda$ (d) $\operatorname{cosec}^2\lambda$

15. The ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ have the same foci then the equation of the circle through the points of intersection of two conics is
   (a) $x^2 + y^2 = 5$
   (b) $\sqrt{5}(x^2 + y^2) - 3x - 4y = 0$
   (c) $\sqrt{5}(x^2 + y^2) + 3x + 4y = 0$
   (d) $x^2 + y^2 = 25$

16. If $P(\sqrt{2}\sec\theta, \sqrt{2}\tan\theta)$ is a point on the hyperbola whose distance from the origin is $\sqrt{6}$ where P is in the first quadrant then $\theta =$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

17. If a variable line has its intercepts on the coordinate axis e and e' where $\frac{e}{2}$ and $\frac{e'}{2}$ are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle $x^2 + y^2 = r^2$, where $r =$

(a) 1 (b) 2

(c) 3 (d) Cannot be decided

18. AB is a double ordinate of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that ΔAOB (where 'O' is the origin) is an equilateral triangle, then the eccentricity e of the hyperbola satisfies

(a) $e > \sqrt{3}$ (b) $1 < e < \frac{2}{\sqrt{3}}$

(c) $e = \frac{2}{\sqrt{3}}$ (d) $e > \frac{2}{\sqrt{3}}$

19. The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$ is:

(a) Zero (b) 1

(c) 2 (d) 4

20. The equation to the common tangents to the two hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ are

(a) $y = \pm x \pm \sqrt{b^2-a^2}$ (b) $y = \pm x \pm (a^2 - b^2)$

(c) $y = \pm x \pm \sqrt{a^2-b^2}$ (d) $y = \pm x \pm \sqrt{a^2+b^2}$

21. If the values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are the roots of the equation $x^2 - (a + b)x - 4 = 0$, then the value of $(a + b)$ is equal to:

(a) 2 (b) 4

(c) Zero (d) 8

22. Locus of the feet of the perpendiculars drawn from either foci on a variable tangent to the hyperbola $16y^2 - 9x^2 = 1$ is

(a) $x^2 + y^2 = 9$

(b) $x^2 + y^2 = 1/9$

(c) $x^2 + y^2 = 7/144$

(d) $x^2 + y^2 = 1/16$

23. The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is

(a) $3x - 4y = 4$ (b) $3y - 4x + 4 = 0$

(c) $4x - 4y = 3$ (d) $3x - 4y = 2$

24. The chords passing through $L(2, 1)$ intersects the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at P and Q. If the tangents at P and Q intersects at R then Locus of R is

(a) $x - y = 1$ (b) $9x - 8y = 72$

(c) $x + y = 3$ (d) $9x + 8y = 72$

25. The locus of a point whose chord of contact with respect to the circle $x^2 + y^2 = 4$ is a tangent to the hyperbola $xy = 1$ is a/an:

(a) Ellipse (b) Circle

(c) Hyperbola (d) Parabola

26. A normal to the hyperbola $\frac{x^2}{4}-\frac{y^2}{1}=1$ has equal intercepts on the positive x-axis and y-axis. If this normal touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then $a^2 + b^2$ is equal to:

(a) 5 (b) 25

(c) 16 (d) 25/9

27. If the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at any point P $(a\sec\theta, b\tan\theta)$ meets the transverse and conjugate axes in G and g respectively and if 'f' is the foot of perpendicular to the normal at P from the centre 'C', then minimum of PG is

(a) $\frac{b^2}{a}$ (b) $\left|\frac{a}{b}(a+b)\right|$

(c) $\left|\frac{b}{a}(a-b)\right|$ (d) $\left|\frac{a}{b}(a-b)\right|$

28. The number of points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=3$ from which mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$ is/are:

(a) 0 (b) 2 (c) 3 (d) 4

29. If the angle between the asymptotes of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is 120° and the product of perpendiculars drawn from the foci upon its any tangent is 9, then the locus of the point of intersection of perpendicular tangents of the hyperbola can be:

(a) $x^2 + y^2 = 6$ (b) $x^2 + y^2 = 9$

(c) $x^2 + y^2 = 3$ (d) $x^2 + y^2 = 18$

30. The center of a rectangular hyperbola lies on the line $y = 2x$. If one of the asymptotes is $x + y + c = 0$, then the other asymptote is:

(a) $6x + 3y - 4c = 0$

(b) $3x + 6y - 5c = 0$

(c) $3x - 6y - c = 0$

(d) $3x - 3y - c = 0$

31. If the foci of a hyperbola lie on $y = x$ and one of the asymptotes is $y = 2x$, then the equation of the hyperbola, given that it passes through (3, 4), is:

(a) $x^2 - y^2 - \frac{5}{2}xy + 5 = 0$

(b) $2x^2 - 2y^2 + 5xy + 5 = 0$

(c) $2x^2 + 2y^2 - 5xy + 10 = 0$

(d) None of these

32. If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ are four concyclic points on the rectangular hyperbola $xy = c^2$, the coordinates of orthocentre of the ΔPQR are

(a) (x_4, y_4) (b) $(x_4, -y_4)$

(c) $(-x_4, -x_4)$ (d) $(-x_4, -y_4)$

33. A rectangular hyperbola circumscribe a triangle ABC, then it will always pass through its

(a) Orthocenter (b) Circum centre

(c) Centroid (d) Incentre

34. If the sum of the slopes of the normal from a point P to the hyperbola $xy = c^2$ is equal to $\lambda(\lambda \in R^+)$, then the locus of point P is:

(a) $x^2 = \lambda c^2$ (b) $y^2 = \lambda c^2$

(c) $xy = \lambda c^2$ (d) $y^2 = \lambda c^3$

35. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is:

(a) $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$

(b) $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$

(c) $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$

(d) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

36. If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again at t', then

(a) $t' = -\frac{1}{t^3}$ (b) $t' = \frac{1}{t}$

(c) $t' = \frac{1}{t^2}$ (d) $t'^2 = -\frac{1}{t^2}$

37. Locus of the middle points of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is:

(a) $y + mx = 0$ (b) $y - mx = 0$

(c) $my - x = 0$ (d) $my + x = 0$

38. The locus of the foot of the perpendicular from the centre of the hyperbola $xy = c^2$ on a variable tangent is

(a) $(x^2 - 2)^2 = 4c^2 xy$ (b) $(x^2 + y^2)^2 = 2c^2xy$

(c) $(x^2 - y^2)^2 = 2c^2xy$ (d) $(x^2 + y^2)^2 = 4c^2xy$

39. The ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $a^2x^2 - y^2 = 4$ intersect at right angles. Then the equation of the circle through the points of intersection of two conics is

(a) $x^2 + y^2 = 5$

(b) $\sqrt{5}(x^2 + y^2) - 3x - 4y = 0$

(c) $\sqrt{5}(x^2 + y^2) + 3x + 4y = 0$

(d) $x^2 + y^2 = 25$

40. The angle between the lines joining the origin to the points of intersection of the line $\sqrt{3}x + y = 2$ and the curve $y^2 - x^2 = 4$ is:

(a) $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (b) $\pi/6$

(c) $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (d) $\pi/2$

41. The point of contact of $5x + 12y = 19$ and $x^2 - 9y^2 = 9$ will lie on

(a) $4x + 15y = 0$ (b) $7x + 12y = 19$

(c) $4x + 15y + 1 = 0$ (d) $7x - 12y = 19$

42. If with standard notations t_1, t_2, t_3, t_4 are four co-normal points on the hyperbola $xy = c^2$ then the orthocentre of the triangle formed by joining the points t_1, t_2, t_3 is given by

(a) (0, 0)

(b) $\left(ct_4, \frac{c}{t_4}\right)$

(c) $\left(c(t_2 + t_2 + t_3), \frac{c}{t_1 + t_2 + t_3}\right)$

(d) $\left(\frac{c}{t_4}, ct_4\right)$

43. If the tangent at the point $(a \sec \alpha, b \tan \alpha)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis at T, then distence of T form focus of hyperbola is

(a) $a(e \pm \cos \alpha)$ (b) ae

(c) $b(e + \cos \alpha)$ (d) $\sqrt{a^2e^2 + b^2 \cot^2 \alpha}$

44. The values of 'm' for which a line with slope m is common tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 (a \neq b)$ and parabola $y^2 = 4ax$ be

(a) $(-\infty, -1) \cup (1, \infty) - \left\{+\sqrt{\frac{2+\sqrt{5}}{2}}\right\}$

(b) $(-\infty, -1) \cup (1, \infty) - \left\{\pm\sqrt{\frac{1+\sqrt{5}}{2}}\right\}$

(c) $(-\infty, -1) \cup (1, \infty)$

(d) $(-\infty, -2) \cup (4, \infty) - \left\{+\sqrt{\frac{2+\sqrt{5}}{2}}\right\}$

45. Locus of the middle point of the parallel chords with gradient m of the rectangular hyperbola
(a) $y + mx = 0$
(b) $y - mx = 0$
(c) $mx - x = 0$
(d) $mx + x = 0$

46. Number of common tangent with finite slope to the curves $xy = c^2$ and $y^2 = 4ax$ is
(a) 0 (b) 1 (c) 2 (d) 4

47. If e_1 is the eccentricity of the hyperbola $xy = 1$ and e_2 is the eccentricity of the ellipse which is confocal with the hyperbola and $e_1e_2 = 1$, then
(a) Equation of the ellipse is $3x^2 + 3y^2 - 2xy - 16 = 0$
(b) Equation of the ellipse is $3x^2 + y^2 + 2xy - 12 = 0$
(c) Equation of the ellipse is $3x^2 + 3y^2 - 2xy - 8 = 0$
(d) Equation of the ellipse is $3x^2 + y^2 + 2xy - 6 = 0$

48. S_1: Perpendicular tangents can be drawn to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ only if $e > \sqrt{2}$.

S_2: Locus of point of intersection of perpendicular tangents to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2 + y^2 = a^2 + b^2$

State, in order, whether S_1, S_2 are true or false
(a) TF (b) FT (c) FF (d) TT

49. At the point of intersection of the rectangular hyperbola $xy = c^2$ and the parabola $y^2 = 4ax$ tangents to the rectangular hyperbola and the parabola make an angle θ and ϕ respectively with the axis of X, then-
(a) $\theta = \tan^{-1}(-2\tan\phi)$
(b) $\phi = \tan^{-1}(-2\tan\phi)$
(c) $\theta = \frac{1}{2}\tan^{-1}(-\tan\phi)$
(d) $\phi = \frac{1}{2}\tan^{-1}(-1\tan\phi)$

50. If the product of the length of perpendiculars drawn from any point on the hyperbola $x^2 - 2y^2 - 2 = 0$ to its asymptotes is $\frac{k}{3}$ then find k.
(a) 1 (b) 2 (c) 0 (d) 3

51. Total number of tangents of the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, that are perpendicular to the line $5x + 2y - 3 = 0$, is/are
(a) 1
(b) 2
(c) 0
(d) 3

52. Tangents are drawn to $3x^2 - 2y^2 = 6$ from a point P. If these tangents intersect the coordinate axes at concyclic points then the locus of P is $x^2 - y^2 = q$. Find the value of $q/2$
(a) 1
(b) 2.5
(c) 0
(d) 3

53. If the distance between two parallel tangents having slope m drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{49}=1$ is 2, then the value of $2|m|$ is:
(a) 1 (b) 5 (c) 0 (d) 3

54. If area of the triangle formed by latus rectum and tangents at the end points of latus rectum of $\frac{x^2}{16}-\frac{y^2}{9}=1$ is A, then 80 A is:
(a) 100
(b) 324
(c) 240
(d) 160

55. Let the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then
(a) The equation of the hyperbola is $\frac{x^2}{3}-\frac{y^2}{2}=1$
(b) A focus of the hyperbola is (2, 0)
(c) The eccentricity of the hyperbola is $\frac{2}{\sqrt{3}}$
(d) The equation of the hyperbola is $x^2 - 3y^2 = 3$

56. Straight line $Ax + By + D = 0$ would be tangent to $xy = c^2$, if:
(a) $A > 0, B > 0$
(b) $A < 0, B < 0$
(c) $A > 0, B < 0$
(d) $A < 0, B > 0$

57. If $\frac{x}{a}+\frac{y}{b}=1$ is a tangent to the curve $x = 4t$, $y=\frac{4}{t}$, $t \in R$ then
(a) $a > 0, b > 0$
(b) $a > 0, b < 0$
(c) $a < 0, b > 0$
(d) $a < 0, b < 0$

58. The equation(s) to common tangent(s) to the two hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ is/are:
(a) $y = x + \sqrt{a^2-b^2}$
(b) $y = x - \sqrt{a^2-b^2}$
(c) $y = -x + \sqrt{a^2-b^2}$
(d) $y = -x - \sqrt{a^2-b^2}$

59. If (5, 12) and (24, 7) are the foci of a conic passing through the origin then the eccentricity of conic is
(a) $\sqrt{386}/12$
(b) $\sqrt{386}/13$
(c) $\sqrt{386}/25$
(d) $\sqrt{386}/38$

60. If θ is the angle between the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with eccentricity e, then $\sec\theta/2$ can be
(a) e
(b) $e/2$
(c) $e/3$
(d) $\frac{e}{\sqrt{e^2-1}}$

61. The tangent to the hyperbola, $x^2 - 3y^2 = 3$ at the point $(\sqrt{3}, 0)$ when associated with two asymptotes constitutes.
(a) Isosceles triangle
(b) An equilateral triangle
(c) A triangle whose area is $\sqrt{3}$ sq. unit
(d) A right isosceles triangle

62. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$, then:

(a) $x_1 + x_2 + x_3 + x_4 = 0$

(b) $y_1 + y_2 + y_3 + y_4 = 0$

(c) $x_1x_2x_3x_4 = c^4$

(d) $y_1y_2y_3y_4 = c^4$

63. The lines parallel to the normal to the curve $xy = 1$ is/are:

(a) $3x + 4y + 5 = 0$

(b) $3x - 4y + 5 = 0$

(c) $4x + 3y + 5 = 0$

(d) $3y - 4x + 5 = 0$

64. Let the variable line $y = kx + h$ is tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$. If locus of $P(h, k)$ is a conic, then which of the following is true?

(a) Focus of conic lies on y-axis.

(b) Eccentricity of conic is greater than the eccentricity of given hyperbola.

(c) Centre of conic is at origin.

(d) Conic is an ellipse.

65. Let a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The transverse and conjugate axis of this hyperbola coincide with the major and minor axis of the given ellipse. Also, the product of the eccentricities of the given ellipse and hyperbola is 1. Then:

(a) The equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(b) The equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$

(c) The focus of the hyperbola is (5, 0)

(d) The vertex of the hyperbola is $(5\sqrt{3}, 0)$

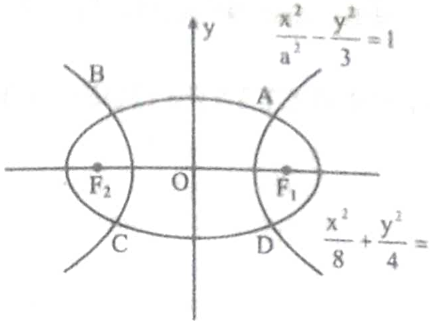
66. Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{3} = 1$ of eccentricity e is confocal (1) with the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$. Let A, B, C and D are t points of intersection of hyperbola and ellipse, then

(a) $e = \frac{5}{2}$

(b) $e = 2$

(c) A, B, C, D are concyclic points

(d) Number of common tangents of hyperbola and ellipse is 2



67. Equation of a tangent passing through (2, 8) to the hyperbola $5x^2 - y^2 = 5$ is:

(a) $3x - y + 2 = 0$

(b) $3x + y - 14 = 0$

(c) $23x - 3y - 22 = 0$

(d) $3x - 23y + 178 = 0$

68. Let a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The transverse and conjugate axes of this hyperbola coincide with the major and minor axes of the given ellipse. Also, the product of the eccentricities of the given ellipse and hyperbola is 1. Then

(a) The equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(b) Ecentricity of the hyperbola is $\frac{5}{3}$

(c) The focus of the hyperbola is (5, 0)

(d) The vertex of the hyperbola is $(5\sqrt{3}, 0)$

69. Which of the following pairs of ellipse and hyperbola are

(a) $\frac{x^2}{100} + \frac{y^2}{64} = 1$ and $\frac{x^2}{19} - \frac{y^2}{17} = 1$

(b) $\frac{x^2}{81} + \frac{y^2}{49} = 1$ and $\frac{x^2}{17} - \frac{y^2}{15} = 1$

(c) $\frac{x^2}{64} + \frac{y^2}{36} = 1$ and $\frac{x^2}{15} - \frac{y^2}{13} = 1$

(d) $\frac{x^2}{49} + \frac{y^2}{25} = 1$ and $\frac{x^2}{13} - \frac{y^2}{11} = 1$

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. The tangent to a curve at a point $P(x, y)$ meets the x-axis at T and y-axis at S while the normal at P meets the x-axis at N and y-axis at M, O is the origin then incorrect statement is
   (a) If $TP = PS$ their locus of point P is Hyperbola.
   (b) If $NM = NP$ their locus of point P is Hyperbola.
   (c) If $TP = OP$ their locus of point P can not be Hyperbola.
   (d) If $NP = OP$ their locus of point P can not be hyperbola.

2. If a hyperbola whose foci are $(-2, 4)$ and $(4, 6)$ touches y-axis then equation of hyperbola is
   (a) $(3x+y-8)^2 - \frac{(x-3y+14)^2}{4} = 20$
   (b) $\frac{(x+3y-7)^2}{2} - \frac{(x-3y+8)^2}{8} = 1$
   (c) $\frac{(3x+y-8)^2}{2} - \frac{(x-3y+14)^2}{8} = 1$
   (d) $\frac{(x-3y+14)^2}{2} - \frac{(3x+y-8)^2}{8} = 20$

3. Through the positive vertex of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ a tangent is drawn, where does it meet the conjugate hyperbola
   (a) At the points $(\pm a\sqrt{2}, b)$
   (b) At the points $(0, 0)$
   (c) At the points $(\pm a, b)$
   (d) At the points $(a, \pm b\sqrt{2})$

4. Consider the hyperbola $\frac{(x-7)^2}{4} - \frac{(y+3)^2}{9} = 1$. A variable point $P(\alpha + 7, \alpha^2 - 4)\ \forall\ \alpha \in R$ exists in the xy plane. Let B_L and B_R be left and right branches of the given hyperbola, then
   (a) The values of α for which 2 distinct real tangents can be drawn to B_L from P, is -1
   (b) The values of α for which no 2 real tangents can be drawn to both B_L and B_R from P, is -1
   (c) The values of α for which only one real tangent can be drawn to B_L only from point P, is -1
   (d) The values of α for which 2 real tangents can be drawn to B_R only from point P, is -1

5. Let p and q be non-zero real numbers. Then the equation $(px^2 + qy^2 + r)(4x^2 + 4y^2 - 8x - 4) = 0$ represents:
   (a) Two straight lines and a circle, when $r = 0$ and p, q are of the opposite sign.
   (b) Two circles, when $p = q$ and r is of sign opposite to that of p.
   (c) A hyperbola and a circle, when p and q are of opposite sign and $r \neq 0$.
   (d) A circle and an ellipse, when p and q are unequal but of same sign and r is of sign opposite to that of p.

6. Two rays aimed towards the point $(5, 0)$, after reflection from outer surface of the hyperbola $9x^2 - 16y^2 = 144$ at points P and Q intersect at R. If abscissa of points P and Q is 8, then:
   (a) Area of ΔPQR is $27\sqrt{3}$ sq. units.
   (b) Area of ΔPQR is $39\sqrt{3}$ sq. units.
   (c) Angle between asymptotes of the given hyperbola is $\tan^{-1}\frac{3}{4}$.
   (d) Angle between asymptotes of the given hyperbola is $\tan^{-1}\frac{24}{7}$.

7. If a variable chord of the hyperbola $x^2 - y^2 = 9$ touches the parabola $y^2 = 12x$, then locus of middle points of these chords is expressed as $x^3 + l_1xy^2 + l_2y^2 = 0$ ($l_1, l_2 \in I$), then:
   (a) $l_1 + l_2 = 2$
   (b) $\lambda_1^2 + \lambda_2^2 = 10$
   (c) Number of divisors of $\left(\lambda_1^2 + \lambda_2^2\right)$ is 4
   (d) Number of divisors of $\left(\lambda_1^2 + \lambda_2^2\right)$ is 6

8. If e_1 is the eccentricity of the hyperbola $xy = 1$ and e_2 is the eccentricity of the ellipse which is confocal with the hyperbola and $e_1e_2 = 1$, then select the correct option(s)
   (a) Equation of the ellipse is $3x^2 + 3y^2 - 2xy - 16 = 0$
   (b) Equation of the ellipse is $3x^2 + y^2 + 2xy - 12 = 0$
   (c) Product of abscissa of the point of intersection of ellipse and hyperbola is 1
   (d) Product of abscissa of the point of intersection of ellipse and hyperbola is 1/3

9. Consider the hyperbola $H : x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are):

(a) $\frac{d\ell}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$

(b) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2 - 1}\right)}$ for $x_1 > 1$

(c) $\frac{d\ell}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$

(d) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

10. Three points A, B and C are taken on rectangular hyperbola $xy = 4$ where $B(-2, -2)$ and $C(6, 2/3)$. The normal at A is parallel to BC, then

(a) Circumcentre of ΔABC is $(2, -2/3)$

(b) Equation of circumcircle of ΔABC is $3x^2 + 3y^2 - 12x + 4y - 40 = 0$

(c) Orthocenter of ΔABC is $\left(\frac{2}{\sqrt{3}}, 2\sqrt{3}\right)$

(d) None of these

11. The normal at one extrimity of latus rectum (in 1st quadrant) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $a > b > 0$ meet the rectengular hyperbola $xy = 9$ at point P and Q then

(a) If P is $\left(6, \frac{3}{2}\right) \Rightarrow Q$ is $\left(\frac{-3\sqrt{2}}{2}, -3\sqrt{2}\right)$

(b) Eccentricity of hyperbola is $\sqrt{2}$

(c) If P is $\left(6, \frac{3}{2}\right) \Rightarrow Q$ is $\left(-\frac{3e}{2}, -\frac{6}{e}\right)$ where e is eccentricity of the given ellipse

(d) If O is origin, then product of slopes of OP and OQ is positive

COMPREHENSION BASED QUESTIONS

Comprehension-1 (Q. 12 to 14): The general second degree equation represent a hyperbola if $h^2 - ab > 0$ and $\Delta \neq 0$ where $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$. Let H and H' be two hyperbolas. They are said to be conjugate to one another if the transverse and conjugate axes of one are respectively the conjugate and transverse axes of the other. If A is the pair of its asymptotes, then equation of H, H', A differ only by constant term. Also $H - A = A - H'$.

Now answer the following questions.

12. Let P be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and PN be the perpendicular on transverse axis. Let A and A' be the vertices of hyperbola, then $\frac{PN^2}{NA \cdot NA'} =$

(a) $\frac{a^2}{b^2}$ (b) $\frac{b^2}{a^2}$

(c) a^2 (d) b^2

13. The director circle of the hyperbola $x^2 - y^2 = a^2$ is

(a) $x^2 + y^2 = ax^2$ (b) $x^2 + y^2 = \frac{a^2}{2}$

(c) $x^2 + y^2 = a^2$ (d) None of these

14. A double ordinate PNP' of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the asymptote at Q and Q' respectively where Q is nearer to P. Then $PQ \cdot PQ' =$

(a) square of semitransverse axis

(b) square of semiconjugate axis

(c) square of transverse axis

(d) square of conjugate axis

Comprehension-2 (Q. 15 to 17): A line is drawn through the point $P(-1, 2)$ meets the hyperbola $xy = c^2$ at the points A and B (Points A and B lie on the same side of P) and Q is a point on the line segment AB.

15. If the point Q is choosen such that PA, PQ and PB are in AP, then locus of point Q is

(a) $x = y(1 + 2x)$ (b) $x = y(1 + x)$

(c) $2x = y(1 + 2x)$ (d) $2x = y(1 + x)$

16. If the point Q is choosen such that PA, PQ and PB are in G.P, then locus of point Q is

(a) $xy - y + 2x - c^2 = 0$

(b) $xy + y - 2x + c^2 = 0$

(c) $xy + y + 2x + c^2 = 0$

(d) $xy - y - 2x - c^2 = 0$

17. If the point Q is choosen such that PA, PQ, PB are in H.P, then locus of Q is

(a) $2x - y = 2c^2$ (b) $x - 2y = 2c^2$

(c) $2x + y + 2c^2 = 0$ (d) $x + 2y = 2c^2$

Comprehension-3 (Q. 18 to 20): Consider a hyperbola $xy = 4$ and a line $y + 2x = 4$, O is the centre of hyperbola. Tangent at any point P of hyperbola intersect the coordinate axes at A and B.

18. Locus of circumcentre of triangle OAB is

(a) An ellipse with eccentricity $\frac{1}{\sqrt{2}}$

(b) An ellipse with eccentricity $\frac{1}{\sqrt{3}}$

(c) A hyperbola with eccentricity $\sqrt{2}$

(d) A circle

19. Shortest distance between the line and hyperbola is

(a) $\frac{8\sqrt{2}}{\sqrt{5}}$ (b) $\frac{4(\sqrt{2}-1)}{\sqrt{5}}$

(c) $\frac{2\sqrt{2}}{\sqrt{5}}$ (d) $\frac{4(\sqrt{2}+1)}{\sqrt{5}}$

20. Let the given line intersecrts the x-axis at R. If a line through R, intersect the hyperbolas at S and T, then minimum value of $RS \times RT$ is

(*a*) 2 (*b*) 4

(*c*) 6 (*d*) 8

Comprehension-4 (Q. 21 to 22): Consider a hyperbola: $\frac{(x-7)^2}{a^2} - \frac{(y+3)^2}{b^2} = 1$. The line $3x - 2y - 25 = 0$, which is not a tangent, intersect the hyperbola at $H\left(\frac{11}{3}, -7\right)$ only. A variable point $P(\alpha + 7, \alpha^2 - 4)\ \forall\ \alpha \in R$ exists in the plane of the given hyperbola.

21. The eccentricity of the hyperbola is

(*a*) $\sqrt{\frac{7}{5}}$ (*b*) $\sqrt{2}$

(*c*) $\frac{\sqrt{13}}{2}$ (*d*) $\frac{3}{2}$

22. Which of the following are not the values of a for which two tangents can be drawn one to each branch of the given hyperbola is

(*a*) $(2, \infty)$ (*b*) $(-\infty, -2)$

(*c*) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (*d*) None of these

Comprehension-5 (Q. 23 to 24): Consider a hyperbola 'H' whose centre is at origin and line $x + y = 2$ touches it at point (1, 1). The tangent $x + y = 2$ intersects the asymptotes of H at point A and B such that length of segment $AB = 6\sqrt{2}$.

23. Equation of pair of directrices of H is:

(*a*) $x^2 + y^2 + 2xy - 1 = 0$

(*b*) $5x^2 + 5y^2 + 10xy - 4 = 0$

(*c*) $5x^2 + 5y^2 + 10xy - 2 = 0$

(*d*) $5x^2 + 5y^2 + 10xy - 6 = 0$

24. Equation of H w.r.t. $x - y$ system referring to transverse axis as x-axis conjugate axis at y-axis respectively is:

(*a*) $x^2 - \frac{y^2}{18} = 1$ (*b*) $\frac{x^2}{2} - \frac{y^2}{18} = 1$

(*c*) $\frac{x^2}{18} - \frac{y^2}{2} - 1$ (*d*) $\frac{x^2}{18} - y^2 = 1$

Comprehension-6 (Q. 25 to 27): The vertices of a ΔABC lies on a rectangular hyperbola such that the orthocentre of the triangle is (3, 2) and the asymptotes of the rectangular hyperbola are parallel to the coordinate axis. If the two perpendicular tangents of the hyperbola intersect at the point (1, 1).

25. The equation of the asymptotes is

(*a*) $xy - 1 = x - y$ (*b*) $xy + 1 = x + y$

(*c*) $2xy = x - y$ (*d*) None of these

26. Equation of the rectangular hyperbola is

(*a*) $xy = 2x + y - 2$ (*b*) $2xy = x + 2y + 5$

(*c*) $xy = x + y + 1$ (*d*) None of these

27. Number of real normals that can drawn from the point (1, 1) to the rectangular hyperbola is

(*a*) 4 (*b*) 0

(*c*) 3 (*d*) 2

MATCH THE COLUMN TYPE QUESTIONS

28. Match the Column:

	Column-I		Column-II
A.	Number of integral values of b for which tangent parallel to 16 line $y = x + 1$ can be drawn to hyperbola $\frac{x^2}{5} - \frac{y^2}{b^2} = 1$ is	p.	6
B.	The equation of the hyperbola with vertices (3, 0) and (–3, 0) and semi-latusrectum 4, is given by is $4x^2 - 3y^2 = 4k$, then $k =$	q.	4
C.	The product of the lengths of the perpendiculars from the two focii on any tangent to the hyperbola $\frac{x^2}{25} - \frac{y^2}{3} = 1$ is $\sqrt{k}$, then k is	r.	3
D.	An equation of a tangent to the hyperbola, $16x^2 - 25y^2 - 96x + 100y - 356 = 0$ which makes an angle $\frac{\pi}{4}$ with the transverse axis is $y = x + l$, ($l > 0$), then $2l$ is	s.	9

(*a*) A→q; B→s; C→s; D→q

(*b*) A→r; B→s; C→s; D→q

(*c*) A→q; B→r; C→s; D→q

(*d*) A→r; B→p; C→s; D→q

29. Match the Column:

	Column-I		Column-II
A.	The foci of the hyperbola $12x^2 - 4y^2 - 24x + 32y - 127 = 0$ are	p.	(10, 5)
B.	The foci of the hyperbola $8x^2 - y^2 - 64x + 10y + 71 = 0$ are	q.	(2, –2)
C.	The foci of the hyperbola $9x^2 - 16y^2 - 36x + 96y + 36 = 0$ are	r.	(–4, 4)
		s.	(–2, 5)
			(6, 4)
			(2, 8)

(*a*) A → q; B → s; C → s

(*b*) A → r, t; B → p, s; C → s

(*c*) A → q, t; B → r; C → s

(*d*) A → r; B → p, s; C → s

30. **Match the following:**

	Column-I		Column-II
A.	The points common to the hyperbola $x^2 - y^2 = 9$ and the circle $x^2 + y^2 = 41$ are	p.	$(-5, -4)$
B.	Tangents are drawn from the point $(0, -9/4)$ to the hyperbola $x^2 - y^2 = 9$. Then the point of tangency may have coordinates	q.	(q) $(5, 4)$
C.	The point which is diametrically opposite to point $(5, 4)$ with respect to the hyperbola $x^2 - y^2 = 9$ is	r.	(r) $(-5, 4)$
D.	If P and Q lie on the hyperbola $x^2 - y^2 = 9$ such that the area of the isosceles triangle PQR, where $PR = QR = 10$ sq. units and $R \equiv (0, -6)$, then P can have the coordinates		(s) $(5, -4)$

(a) $A \to p, q, r, s$; $B \to s$; $C \to p$; $D \to p, s$
(b) $A \to p, q$; $B \to q, s$; $C \to s$; $D \to q$
(c) $A \to p, q, r, s$; $B \to q, r$; $C \to p$; $D \to p, s$
(d) $A \to p, q, r$; $B \to p$; $C \to s$; $D \to p, r$

31. **Match the following:**

	Column-I		Column-II
A.	If z is a complex number such that $\text{Im}(z^2) = 3$, then the eccentricity of the locus is	p.	$\sqrt{3}$
B.	If the latus rectum of a hyperbola through one focus subtends an angle of $60°$ at the other focus, then its eccentricity is	q.	2
C.	If $A \equiv (3, 0)$ and $B \equiv (-3, 0)$ and $PA - PB = 4$, then the eccentricity of conjugate hyperbola is	r.	$\sqrt{2}$
D.	If the angle between the asymptotes of a hyperbola is $\pi/3$, then the eccentricity of its conjugate hyperbola is	s.	$\frac{3}{\sqrt{5}}$

(a) A→q; B→s; C→s; D→q
(b) A→r; B→s; C→s; D→q
(c) A→q; B→r; C→s; D→q
(d) A→r; B→p; C→s; D→q

NUMERICAL BASED QUESTIONS

32. The minimum value of $(\tan A - \cos B)^2 + (\cot A - \sin B)^2$, where A and B are independent variables, is of the form $a - b\sqrt{c}$, where a, b, c are natural numbers. The value of $a + b + c$ is equal to

33. The centre of a circle passing through foci of a hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ and the points at which the tangent at the point $(6, 2\sqrt{3})$ meets the tangents at the vertices is $\left(h, \frac{-k}{\sqrt{3}}\right)$, then k is equal to

34. If e_1 and e_2 are the eccentricities of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate hyperbola, such that $e_1^2 + e_2^2 = k$, then find k

35. If m_1 and m_2 are slopes of the tangents to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ which passes through the point of contact of $3x - 4y = 5$ and $x^2 - 4y^2 = 5$ then $32(m_1 + m_2 - m_1 m_2) = ...$

36. Tangents are drawn from the point $(\alpha, 2)$ to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ & ϕ to the x-axis. If $\tan\theta \cdot \tan\phi = 2$, then the value of $2\alpha^2 - 7$ is

37. If a variable line has its intercepts on the coordinate axis e and e', where $e/2$ and $e'/2$ are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle $x^2 + y^2 = r^2$, where $r =$

38. If the chord $x\cos\alpha + y\sin\alpha = p$ of the hyperbola $\frac{x^2}{16} - \frac{y^2}{18} = 1$ subtends a right angle at the center, and the diameter of the circle, concentric with the hyperbola, to which the given chord is a tangent is d, then the value of $d/4$ is:

39. Tangents are drawn from any point on the hyperbola $4x^2 - 9y^2 = 36$ to the circle $x^2 + y^2 - 9 = 0$.
The locus of mid-point of chord of contact is
$\left(\frac{x^2}{9} - \frac{y^2}{4}\right) = \left(\frac{x^2 + y^2}{k}\right)^2$, where $k \in N$. Find k.

40. Tangent are drawn from any point on the hyperbola $x^2/9 - y^2/4 = 1$ to the circle $x^2 + y^2 = 9$. If the locus of the mid-point of the chord of contact is a $a(x^2 + y^2)^2 = bx^2 - cy^2$, then the value of $a^2 + b^2 + c^2 - 7870$ is equal to ______

41. The number of normal(s) to hyperbola which can touch its conjugate is equal to ______

42. If $x + y = 3$ is tangent at $(2, 1)$ on hyperbola, intersects asymptotes at A and B such that $AB = 8\sqrt{2}$. If centre of hyperbola is $(-1, -1)$, then least possible value of sum of semi-transverse axis and semi-conjugate axis is $p\sqrt{q}$, where $HCF(p, q) = 1$ and $\sqrt{q}$ is irrational number. Then, $(p + q)$ is equal to

43. Five points are selected on a circle of radius r, consider the centres of the five rectangular hyperbola each passing through four of these five points, then these centres lie on a circle of radius $r/2$

Exercise-4 (Past Year Questions)

JEE MAIN

1. The eccentricity of the hyperbola whose length of the latusrectum is equal to 8 and the length of conjugate axis is equal to half of the distance between its foci, is **(2016)**

(a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$

(c) $\frac{2}{\sqrt{3}}$ (d) $\sqrt{3}$

2. A hyperbola passes through the point $p(\sqrt{2},\sqrt{3})$ and has foci at $(\pm 2, 0)$. The tangent to this hyperbola at P also passes through the point. **(2017)**

(a) $(-\sqrt{2},-\sqrt{3})$ (b) $(3\sqrt{2},2\sqrt{3})$

(c) $(2\sqrt{2},3\sqrt{3})$ (d) $(\sqrt{3},\sqrt{2})$

3. Let $0<\theta<\frac{\pi}{2}$. If the eccentricity of the hyperbola $\frac{x^2}{\cos^2\theta}-\frac{y^2}{\sin^2\theta}=1$ is greater than 2, then the length of its latus rectum lies in the interval: **(2019)**

(a) $(3,\infty)$ (b) $\left(\frac{3}{2},2\right]$

(c) $(2, 3]$ (d) $\left(1,\frac{3}{2}\right]$

4. A hyperbola has its centre at the origin, passes through the point (4, 2) and has transverse of axis of length 4 along the x-axis. Then the eccentricity of the hyperbola is: **(2019)**

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{3}{2}$

(c) $\sqrt{3}$ (d) 2

5. The equation of a tangent to the hyperbola $4x^2 - 5y^2 = 20$ parallel to the line $x - y = 2$ is: **(2019)**

(a) $x-y+1=0$ (b) $x-y+7=0$

(c) $x-y+9=0$ (d) $x-y-3=0$

6. If a hyperbola has length of its conjugate axis equal to 5 and the distance between its foci is 13, then the eccentricity of the hyperbola is: **(2019)**

(a) 13/12 (b) 2

(c) 13/6 (d) 13/8

7. If the vertices of a hyperbola be at (–2, 0) and (2, 0) and one of its foci be at (–3, 0), then which one of the following points does not lie on this hyperbola? **(2019)**

(a) $(-6,2\sqrt{10})$ (b) $(2\sqrt{6},5)$

(c) $(4,\sqrt{15})$ (d) $(6,5\sqrt{2})$

8. If the eccentricity of the standard hyperbola passing through the point (4,6) is 2, then the equation of the tangent of the hyperbola at (4,6) is **(2019)**

(a) $2x-y-2=0$ (b) $3x-2y=0$

(c) $2x-3y+10=0$ (d) $x-2y+8=0$

9. If the line $y = mx + 7\sqrt{3}$ is normal to the hyperbola $\frac{x^2}{24}-\frac{y^2}{18}=1$, then a value of m is **(2019)**

(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{3}{\sqrt{5}}$

(c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{15}}{2}$

10. If $5x + 9 = 0$ is the directrix of the hyperbola $16x^2 - 9y^2 = 144$, then its corresponding focus is: **(2019)**

(a) $\left(-\frac{5}{3},0\right)$ (b) (5, 0)

(c) (–5, 0) (d) $\left(\frac{5}{3},0\right)$

11. If a directrix of a hyperbola centred at the origin and passing through the point $(4,-2\sqrt{3})$ is $5x=4\sqrt{5}$ and its eccentricity is e, then: **(2019)**

(a) $4e^4-24e^2+35=0$ (b) $4e^4-8e^2-35=0$

(c) $4e^4-12e^2-27=0$ (d) $4e^4-24e^2+27=0$

12. If a hyperbola passes through the point P (10, 16) and it has vertices at (±6, 0), then the equation of the normal to it at P is **(2020)**

(a) $x+3y=58$ (b) $x+2y=42$

(c) $3x+4y=94$ (d) $2x+5y=100$

13. If e_1 and e_2 are the eccentricities of the ellipse, $\frac{x^2}{18}+\frac{y^2}{4}=1$ and the hyperbola, $\frac{x^2}{9}-\frac{y^2}{4}=1$ respectively and (e_1, e_2) is a point on the ellipse, $15x^2 + 3y^2 = k$, then k is equal to **(2020)**

(a) 14 (b) 15

(c) 16 (d) 17

14. For some $\theta\in\left(0,\frac{\pi}{2}\right)$, if the eccentricity of the hyperbola, $x^2-y^2\sec^2\theta=10$ is $\sqrt{5}$ times the eccentricity of the ellipse, $x^2\sec^2\theta+y^2=5$, then the length of the latus rectum of the ellipse, is: **(2020)**

(a) $2\sqrt{6}$ (b) $\frac{2\sqrt{5}}{3}$

(c) $\frac{4\sqrt{5}}{3}$ (d) $\sqrt{30}$

15. A line parallel to the straight line $2x - y = 0$ is tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ at the point (x_1, y_1). Then $x_1^2 + 5y_1^2$ is equal to: (2020)

(a) 8 (b) 6

(c) 10 (d) 5

16. Let e_1 and e_2 be the eccentricities of the ellipse, $\frac{x^2}{25} + \frac{y^2}{b^2} = 1 (b < 5)$ and the hyperbola, $\frac{x^2}{16} - \frac{y^2}{b^2} = 1$ respectively satisfying $e_1 e_2 = 1$. If α and β are the dist ances between the foci of the ellipse and the foci of the hyperbola respectively, then the ordered pair (α, β) is equal to: (2020)

(a) (8, 10) (b) $\left(\frac{24}{5}, 10\right)$

(c) $\left(\frac{20}{3}, 12\right)$ (d) (8, 12)

17. A hyperbol a having the transverse axis of length $\sqrt{2}$ has the same foci as that of the ellipse of $3x^2 + 4y^2 = 12$, then this hyperbola does not pass through which of the following points? (2020)

(a) $\left(-\sqrt{\frac{3}{2}}, 1\right)$ (b) $\left(\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}}\right)$

(c) $\left(\frac{1}{\sqrt{2}}, 0\right)$ (d) $\left(1, -\frac{1}{\sqrt{2}}\right)$

18. Let $P(3, 3)$ be a point on the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal to it at P intersects the x-axis at $(9, 0)$ and e is its eccentricity, then the ordered pair (a^2, e^2) is equal to: (2020)

(a) (9, 3) (b) $\left(\frac{9}{2}, 3\right)$ (c) $\left(\frac{3}{2}, 2\right)$ (d) $\left(\frac{9}{2}, 2\right)$

19. If the line $y = mx + c$ is a common tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{64} = 1$ and the circle $x^2 + y^2 = 36$, then which one of the following is true ? (2020)

(a) $5m = 4$ (b) $8m + 5 = 0$

(c) $c^2 = 369$ (d) $4c^2 = 369$

20. The locus of the midpoints of the chord of the circle, which is tangent to the hyperbola, $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is: (2021)

(a) $(x^2 + y^2)^2 - 9x^2 + 16y^2 = 0$

(b) $(x^2 + y^2)^2 - 9x^2 + 144y^2 = 0$

(c) $(x^2 + y^2)^2 - 16x^2 + 9y^2 = 0$

(d) $(x^2 + y^2)^2 - 9x^2 - 16y^2 = 0$

21. The locus of the mid points of the chords of the hyperbola $x^2 - y^2 = 4$, which touch the parabola $y^2 = 8x$, is: (2021)

(a) $y^3(x-2) = x^2$ (b) $x^3(x-2) = y^2$

(c) $y^2(x-2) = x^3$ (d) $x^2(x-2) = y^3$

22. The locus of the centroid of the triangle formed by any point P on the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$, and its foci is: (2021)

(a) $16x^2 - 9y^2 + 32x + 36y - 36 = 0$

(b) $9x^2 - 16y^2 + 36x + 32y - 144 = 0$

(c) $16x^2 - 9y^2 + 32x + 36y - 144 = 0$

(d) $9x^2 - 16y^2 + 36x + 32y - 36 = 0$

23. Let a line $L : 2x + y = k, k > 0$ be a tangent to the hyperbola $x^2 - y^2 = 3$. If L is also a tangent to the parabola $y^2 = \alpha x$, then α is equal to: (2021)

(a) 12 (b) −12 (c) 24 (d) −24

24. Consider a hyperbola $H : x^2 - 2y^2 = 4$. Let the tangent at a point $P(4, \sqrt{6})$ meet the x-axis at Q and latus rectum at $R(x_1, y_1), x_1 \geq 0$. If F is a focus of H which is nearer to the point P, then the area of ΔQFR is equal to: (2021)

(a) $\sqrt{6} - 1$ (b) $\frac{7}{\sqrt{6}} - 2$

(c) $4\sqrt{6} - 1$ (d) None of these

25. If the line $x - 1 = 0$, is a directrix of the hyperbola $kx^2 - y^2 = 6$, then the hyperbola passes through the point (2022)

(a) $(-2\sqrt{5}, 6)$ (b) $(-\sqrt{5}, 3)$

(c) $(\sqrt{5}, -2)$ (d) $(2\sqrt{5}, 3\sqrt{6})$

26. Let the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{\alpha} = \frac{1}{25}$ coincide. Then the length of the latus rectum of the hyperbola is: (2022)

(a) $\frac{32}{9}$ (b) $\frac{18}{5}$ (c) $\frac{27}{4}$ (d) $\frac{27}{10}$

JEE ADVANCED

27. Each of the four inequalities given below defines a region in the xy plane. One of these four regions does not have the following property. For any two points (x_1, y_1) and (x_2, y_2) in the region, the point $((x_1 + x_2)/2, (y_1 + y_2)/2)$ is also in the region. The inequality defining this region is (1981)

(a) $x^2 + 2y^2 \leq 1$ (b) $\max \{|x|, |y|\} \leq 1$

(c) $x^2 - 2y^2 \leq 1$ (d) $y^2 - x \leq 0$

28. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is (1999)

(a) $9x^2 - 8y^2 + 18x - 9 = 0$

(b) $9x^2 - 8y^2 - 18x + 9 = 0$

(c) $9x^2 - 8y^2 - 18x - 9 = 0$

(d) $9x^2 - 8y^2 + 18x + 9 = 0$

29. Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta+\phi=\frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h, k) is the point of intersection of the normals at P and Q, then k is equal to **(1999)**

(a) $\frac{a^2+b^2}{a}$ (b) $-\left(\frac{a^2+b^2}{a}\right)$

(c) $\frac{a^2+b^2}{b}$ (d) $-\left(\frac{a^2+b^2}{b}\right)$

30. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line $2x-y=1$. The points of contacts of the tangents on the hyperbola are **(2012)**

(a) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (b) $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

(c) $(3\sqrt{3}, -2\sqrt{2})$ (d) $(-3\sqrt{3}, 2\sqrt{2})$

31. If $2x-y+1=0$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{16}=1$, then which of the following CANNOT be sides of a right angled triangle? **(2017)**

(a) $a, 4, 1$ (b) $2a, 4, 1$

(c) $a, 4, 2$ (d) $2a, 8, 1$

32. Let $H:\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $a>b>0$, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of $60°$ at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$. **(2018)**

	List-I		List-II
A.	The length of the conjugate axis of H is	p.	8
B.	The eccentricity of H is	q.	$\frac{4}{\sqrt{3}}$
C.	The distance between the foci of H is	r.	$\frac{2}{\sqrt{3}}$
D.	The length of the latus rectum of H is	s.	4

(a) A → s; B → q; C → p; D → r

(b) A → s; B → r; C → p; D → q

(c) A → s; B → p; C → r; D → q

(d) A → r; B → s; C → q; D → p

33. Let a and b be positive real numbers such that $a>1$ and $b<a$. Let P be a point in the first quadrant that lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Suppose the tangent to the hyperbola at P passes through the point $(1,0)$, and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinate axes. Let Δ denote the area of the triangle formed by the tangent at P, the normal at P and the x-axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are TRUE? **(2020)**

(a) $1<e<\sqrt{2}$ (b) $\sqrt{2}<e<2$

(c) $\Delta=a^4$ (d) $\Delta=b^4$

34. Consider the hyperbola $\frac{x^2}{100}-\frac{y^2}{64}=1$ with foci at S and S_1, where S lies on the positive x-axis. Let P be a point on the hyperbola, in the first quadrant. Let $\angle SPS_1=\alpha$, with $\alpha<\frac{\pi}{2}$. The straight line passing through the point S and having the same slope as that of the tangent at P to the hyperbola, wintersects the straight line S_1P at P_1. Let δ be the distance of P from the straight line SP_1, and $\beta=S_1P$. Then the greatest integer less than or equal to $\frac{\beta\delta}{9}\sin\frac{\alpha}{2}$ is ______. **(2022)**

35. For the hyperbola $H: x^2-y^2=1$ and the ellipse $E:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b>0$, let the

(1) Eccentricity of E be reciprocal of the eccentricity of H, and

(2) The line $y=\sqrt{\frac{5}{2}}x+K$ be a common tangent of E and H. Then $4(a^2+b^2)$ is equal to ______. **(2022)**

36. A common tangent T to the curves $C_1:\frac{x^2}{4}+\frac{y^2}{9}=1$ and $C_2:\frac{x^2}{42}-\frac{y^2}{143}=1$ does not pass through the fourth quadrant. If T touches C_1 at (x_1, y_1) and C_2 at (x_2, y_2), then $|2x_1+x_2|$ is equal to ______.

ANSWER KEY

CONCEPT APPLICATION

1. (a) 2. (b) 3. (b) 4. (a) 5. (b, d) 6. (c) 7. (d)

8. $7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0$ Centre $\equiv \left(\frac{-4}{5}, \frac{11}{10}\right)$, Second focus $\equiv \left(\frac{-13}{5}, \frac{1}{5}\right)$

9. Centre is [3, 2]
Length of transverse axis = $2b = 6$
Length of conjugate axis = $2\sqrt{7}$
Eccentricity = 4/3
Length of latus rectum = 14/3
Foci : (3, –2) and (3, 6)
Vertices are (3, –1) and (3, 5)
Equation of directrices are : $y = \frac{17}{4}$ and $y = \frac{-1}{4}$

10. Centre $\equiv \left(\frac{84}{25}, \frac{-12}{25}\right)$, foci are $\left(\frac{84+15\sqrt{13}}{25}, \frac{-12-20\sqrt{13}}{25}\right)$ $\left(\frac{84-15\sqrt{13}}{25}, \frac{-12+20\sqrt{13}}{25}\right)$

11. (a) 12. (b) 13. $24y - 30x = \pm\sqrt{161}$ 14. (d) 15. 16. $m = \sqrt{\frac{17}{20}}$ 17. (d) 18. (b)

19. $\frac{1}{2}a^2e^5$ 20. 21. (a) 22. $12x^2 - 7xy - 12y^2 + 31x + 17y = 0$ 23. 24. $2\tan^{-1}\left(\frac{3}{4}\right)$

25. Asymptotes are $x - 3 = 0$ and $y - 2 = 0$

26. (d) 27. (a) 28. (a) 29. (b, c) 30. 31. [1] 32. (b) 33. (c)

EXERCISE-1 (TOPICWISE)

1. (c)	2. (a)	3. (a)	4. (c)	5. (c)	6. (c)	7. (a)	8. (d)	9. (d)	10. (d)
11. (a)	12. (b)	13. (c)	14. (a)	15. (d)	16. (d)	17. (b)	18. (b)	19. (a)	20. (c)
21. (a)	22. (a)	23. (a)	24. (c)	25. (d)	26. (b)	27. (a)	28. (d)	29. (b)	30. (d)
31. (b)	32. (d)	33. (d)	34. (a)	35. (d)	36. (d)	37. (b)	38. (c)	39. (b)	40. (b)
41. (c)	42. (d)	43. (b)	44. (c)	45. (a)	46. (a)	47. (a)	48. (d)		

EXERCISE-2 (LEARNING PLUS)

1. (b)	2. (c)	3. (b)	4. (c)	5. (d)	6. (a)	7. (c)	8. (b)	9. (c)	10. (c)
11. (b)	12. (b)	13. (d)	14. (a)	15. (a)	16. (a)	17. (b)	18. (d)	19. (a)	20. (c)
21. (c)	22. (d)	23. (a)	24. (b)	25. (c)	26. (d)	27. (a)	28. (a)	29. (d)	30. (d)
31. (c)	32. (d)	33. (a)	34. (a)	35. (a)	36. (a)	37. (a)	38. (d)	39. (a)	40. (c)
41. (a)	42. (b)	43. (a)	44. (b)	45. (a)	46. (b)	47. (a)	48. (a)	49. (a)	50. (b)
51. (c)	52. (b)	53. (b)	54. (b)	55. (b, d)	56. (a, b)	57. (a, d)	58. (a, b, c, d)		59. (a, d)
60. (a, d)	61. (b, c)	62. (a, b, c, d)		63. (b, d)	64. (a, b, c)	65. (a, c)	66. (b, c)	67. (a, c)	68. (a, c)
69. (a, b, c, d)									

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (*a*)	**2.** (*a*)	**3.** (*d*)	**4.** (*b*)	**5.** (*a, b, c*)	**6.** (*b, d*)	**7.** (*a, b, c*)	**8.** (*a, c*)	**9.** (*a, b, d*)	**10.** (*a, b, c*)
11. (*b, c, d*)	**12.** (*b*)	**13.** (*c*)	**14.** (*b*)	**15.** (*c*)	**16.** (*b*)	**17.** (*a*)	**18.** (*c*)	**19.** (*b*)	**20.** (*d*)
21. (*c*)	**22.** (*d*)	**23.** (*c*)	**24.** (*b*)	**25.** (*b*)	**26.** (*c*)	**27.** (*d*)	**28.** (*a*)	**29.** (*b*)	**30.** (*c*)
31. (*d*)	**32.** [7]	**33.** [2]	**34.** [1]	**35.** [22]	**36.** [4]	**37.** [2]	**38.** [6]	**39.** [9]	**40.** [3]
41. [4]	**42.** [9]								

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (*c*)	**2.** (*c*)	**3.** (*a*)	**4.** (*a*)	**5.** (*a*)	**6.** (*a*)	**7.** (*d*)	**8.** (*a*)	**9.** (*c*)	**10.** (*c*)
11. (*a*)	**12.** (*d*)	**13.** (*c*)	**14.** (*c*)	**15.** (*b*)	**16.** (*a*)	**17.** (*b*)	**18.** (*b*)	**19.** (*d*)	**20.** (*a*)
21. (*c*)	**22.** (*a*)	**23.** (*d*)	**24.** (*b*)	**25.** (*c*)	**26.** (*d*)				

JEE Advanced

27. (*c*)	**28.** (*b*)	**29.** (*d*)	**30.** (*a, b*)	**31.** (*a, c, d*)	**32.** (*b*)	**33.** (*a, d*)	**34.** [7]	**35.** [3]	**36.** [20]

CHAPTER

18 Complex Numbers-II

VECTOR REPRESENTATION OF A COMPLEX NUMBER

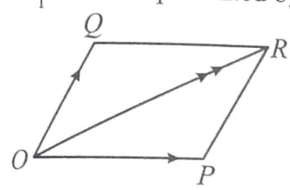
In the Argand's diagram any complex number $Z = x + iy$ can be represented by a point P with coordinates (x, y). The vector $\overrightarrow{OP}$ can also be used to represent Z. The length of the vector $\overrightarrow{OP}$, i.e., OP is the modulus of Z and the angle θ that OP makes with the positive X-axis is the amplitude of Z.

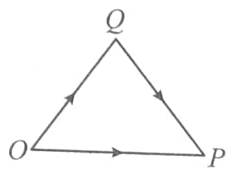
REPRESENTATION OF AN ALGEBRAIC OPERATION ON COMPLEX NUMBERS

Sum: If two complex numbers Z_1 and Z_2 be represented by the points P and Q or by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then the sum $Z_1 + Z_2$ is represented by R or $\overrightarrow{OR}$, where $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$ and OR is the diagonal of the parallelogram with OP and OQ as adjacent sides.

Difference: $Z_1 - Z_2$ will be represented by $\overrightarrow{QP}$ since $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$.

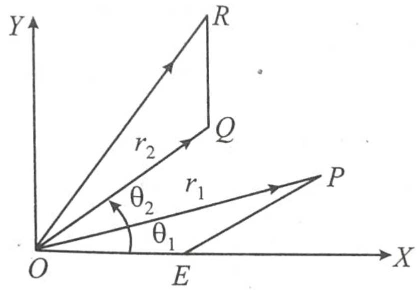
$Z_2 - Z_1$ will be represented by $\overrightarrow{PQ}$





Multiplication: If $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then $Z_1 Z_2 = r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$.

If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent Z_1 and Z_2, construct ΔOQR similar to ΔOEP where $OE = 1$.



$\angle XOR = \angle XOQ + \angle QOR = \angle XOR + \angle EOP = \theta_2 + \theta_1$

and $\dfrac{OR}{OQ} = \dfrac{OP}{OE}$,

$\therefore \quad OR = OP \cdot OQ = r_1 r_2$ { as $OE = 1$}

Hence $\overrightarrow{OR}$ represents the product $Z_1 Z_2$.

Division $\dfrac{Z_1}{Z_2} = \left(\dfrac{r_1}{r_2}\right)\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$

Construct ΔORP similar to ΔOEQ

Now $\dfrac{OR}{OE} = \dfrac{OP}{OQ} \Rightarrow OR = \dfrac{r_1}{r_2}$

and $\angle ROX = \angle ROP - \angle EOP = \angle EOQ - \angle EOP = \theta_2 - \theta_1$

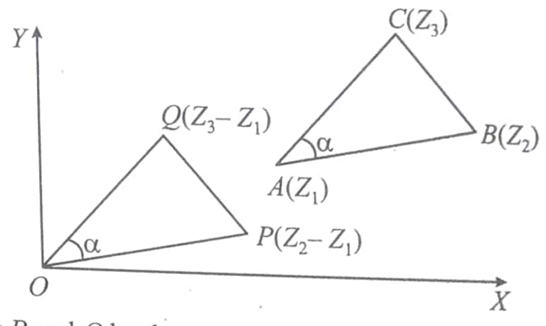
$\therefore \angle XOR = \theta_1 - \theta_2$

Hence, $\overrightarrow{OR}$ represents $\dfrac{Z_1}{Z_2}$.

Corollary 1

If Z_1, Z_2, Z_3 are the vertices of a triangle ABC described in the counter-clockwise direction, then

$\dfrac{Z_3 - Z_1}{Z_3 - Z_1} = \dfrac{CA}{BA}(\cos\alpha + i\sin\alpha)$, where $\alpha = \angle BAC$



Let P and Q be the points representing $Z_2 - Z_1$ and $Z_3 - Z_1$. Then the triangles POQ and BAC are congruent.

$\therefore \dfrac{CA}{BA} = \dfrac{OQ}{OP}$ and $\angle POQ = \angle BAC = \alpha$

Now $\dfrac{Z_3 - Z_1}{Z_2 - Z_1}$ has modulus $\dfrac{OQ}{OP} = \dfrac{CA}{BA}$ and argument $\angle POQ$ $= \alpha$

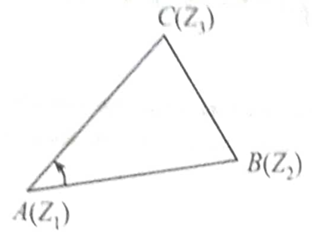
Hence $\dfrac{Z_3 - Z_1}{Z_2 - Z_1} = \left(\dfrac{CA}{BA}\right)(\cos\alpha + i\sin\alpha)$

In particular, if $\alpha = 90°$ and $AB = AC$, then

$$\frac{Z_3 - Z_1}{Z_2 - Z_1} = i \text{ or } (Z_3 - Z_1) = i\,(Z_2 - Z_1)$$

Corollary 2

If Z_1, Z_2, Z_3 are represented by A, B, C, then



$$\arg\left(\frac{Z_3 - Z_1}{Z_2 - Z_1}\right) = \angle BAC$$

$$\arg\left(\frac{Z_2 - Z_3}{Z_1 - Z_3}\right) = \angle ACB \text{ and}$$

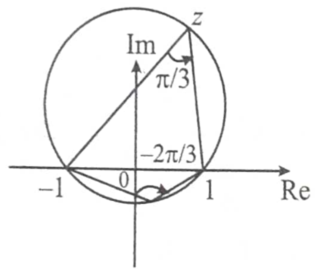
$$\arg\left(\frac{Z_1 - Z_2}{Z_3 - Z_2}\right) = \angle CBA$$

Train Your Brain

Example 1: If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$ then interpret the locus.

Sol. $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$

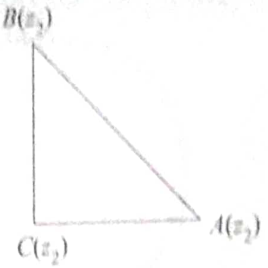
$$\Rightarrow \arg\left(\frac{1-z}{-1-z}\right) = \frac{\pi}{3}$$



Here $\arg\left(\frac{1-z}{-1-z}\right)$ represents the angle between lines joining -1 and z, and 1 and z. As this angle is constant, the locus of z will be a larger segment of circle. (angle in a segment is constant).

Example 2: Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Sol. In the isosceles triangle ABC, $AC = BC$ and $BC \perp AC$. It means that AC is rotated through angle $\pi/2$ to occupy the position BC.



Hence we have,

$$\frac{z_2 - z_3}{z_1 - z_3} = e^{+i\pi/2} = +i \Rightarrow z_2 - z_3 = +i(z_1 - z_3)$$

$$\Rightarrow z_2^2 + z_3^2 - 2z_2z_3 = -(z_1^2 + z_3^2 - 2z_1z_3)$$

$$\Rightarrow z_1^2 + z_2^2 - 2z_1z_2 = 2z_1z_3 + 2z_2z_3 - 2z_3^2$$

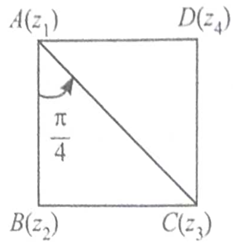
$$= 2(z_1 - z_3)(z_3 - z_2)$$

$$\Rightarrow (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$

Example 3: If the vertices of a square $ABCD$ are z_1, z_2, z_3 and z_4 then find z_3 and z_4 in terms of z_1 & z_2.

Sol. Using vector rotation at angle A

$$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_i}{z_2 - z_1}\right| e^{i\frac{\pi}{4}}$$



$\because |z_3 - z_1| = AC$ and $|z_2 - z_1| = AB$

Also $AC = \sqrt{2}AB$

$\therefore |z_3 - z_1| = \sqrt{2}\,|z_2 - z_1|$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\Rightarrow z_3 - z_1 = (z_2 - z_1)(1 + i)$$

$$\Rightarrow z_3 = z_1 + (z_2 - z_1)(1 + i)$$

Similarly $z_4 = z_2 + (1 + i)(z_1 - z_2)$

Concept Application

1. (*i*) A complex number $z_1 = 3 + 4i$ is rotated about another fixed complex number $z_1 = 1 + 2i$ in anticlockwise direction by 45° angle. Find the complex number represented by new position of z in argand plane.

   (*ii*) If A, B, C are three points in argand plane representing the complex number z_1, z_2, z_3 such that $z_i = \frac{\lambda z_2 + z_3}{\lambda + 1}$, where $\lambda \in R$, then find the distance of point A from the line joining points B and C.

(iii) If $A(z_1), B(z_2), C(z_3)$ are vertices of ΔABC in which $\angle ABC = \frac{\pi}{4}$ and $\frac{AB}{BC} = \sqrt{2}$, then find z_2 in terms of z_1 and z_3.

(iv) If a & b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle then a and b are equal to:-

(a) $a = b = 1/2$ (b) $a = b = 2 - \sqrt{3}$

(c) $a = b = -2 + \sqrt{3}$ (d) $a = b = \sqrt{2} - 1$

(v) If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, find locus of z.

2. If the point represented by the complex number $2-i$ is rotated about origin through an angle $\pi/2$ in the clockwise direction, then find the new position of point?

(a) $1 + 2i$ (b) $-1 - 2i$

(c) $2 + i$ (d) $-1 + 2i$

3. If $z = x + iy$, then area of the triangle whose vertices are points z, iz and $z + iz$ is

(a) $2|z|^2$ (b) $\frac{1}{2}|z|^2$

(c) $|z|^2$ (d) $\frac{3}{2}|z|^2$

BASIC CONCEPT OF GEOMETRY IN COMPLEX NUMBER

1. **Distance Formula:** Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex number represented by points P and Q on the argand plane.

Then the distance $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Also $z_2 - z_1 = (x_2 + iy_2) - (x_1 + iy_1) = (x_2 - x_2) + i(y_2 - y_1)$

$\therefore |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = PQ$

Hence distance between two points z_1 and z_2 is given by $|z_2 - z_1|$

2. **Section formula:** If z_1 and z_2 are affixes of the two points P and Q respectively and point C divides the line segment joining P and Q internally in the ratio $m : n$ then affix z of C is given by

$$z = \frac{mz_2 + nz_1}{m + n} \text{ where } m, n > 0$$

If C divides PQ in the ratio $m : n$ externally then

$$z = \frac{mz_2 - nz_1}{m - n}$$

Note: If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a, b, c are not all simultaneously zero, then the complex numbers z_1, z_2 and z_3 are collinear.

3. **Area of triangle:**

Area of triangle formed by the points z_1, z_2 and z_3 is

$$\left|\frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|$$

4. **Important Centers of Triangle:**

If the vertices A, B, C of a Δ are represented by complex numbers z_1, z_2, z_3 respectively and a, b, c are the length of sides then,

❖ **Centroid** of the $\Delta ABC = \frac{z_1 + z_2 + z_3}{3}$:

❖ **Orthocentre** of the

$$\Delta ABC = \frac{(a \sec A)z_1 + (b \sec B)z_2 + (c \sec C)z_3}{a \sec A + b \sec B + c \sec C}$$

or $$\frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$$

❖ **Incentre** of the $\Delta ABC = (az_1 + bz_2 + cz_3) \div (a + b + c)$.

❖ **Circumcentre** of the $\Delta ABC = (z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

5. **Equation of Ray:**

❖ $\text{amp}(z) = \theta$ is a ray emanating from the origin inclined at an angle θ to the positive x–axis.

❖ $\text{amp}(z - z_0) = \theta$ is a ray emanating from the $z = z_0$ inclined at an angle θ to the positive x–axis.

6. **Perpendicular Bisector:**

$|z - a| = |z - b|$ is the perpendicular bisector of the line joining complex number a and b.

STRAIGHT LINE

1. The equation of a line joining z_1 and z_2 is given by, $z = z_1 + t\,(z_1 - z_2)$ where t is a real parameter.

2. $z = z_1\,(1 + it)$ where t is a real parameter is a line through the point z_1 and perpendicular to the line joining z_1 to the origin.

3. The equation of a line passing through z_1 and z_2 can be expressed in the determinant form as $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$. This is also the condition for three complex numbers z, z_1, z_2 to be collinear. The above equation on manipulating, takes the form $\bar{\alpha}z + \alpha\bar{z} + r = 0$ where r is real and α is a non zero complex constant.

Note: If we replace z by $ze^{i\theta}$ and $\bar{z}$ by $\bar{z}e^{-i\theta}$ then we get equation of a straight line which makes an angle θ with the given straight line.

4. (i) Complex slope of a line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is $\omega = -\frac{\alpha}{\bar{\alpha}}$.

(*ii*) Complex slope of a line joining by the points z_1 and z_2 is $\omega = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

(*iii*) Complex slope of a line making θ angle with real axis $\omega = e^{i(2\theta)}$

5. **Dot and cross product:** Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers [vectors]. The dot product [also called the scalar product] of z_1 and z_2 is defined by

$$z_1 \cdot z_2 = |z_1|\,|z_2| \cos\theta = x_1x_2 + y_1y_2 = \text{Re}\{\bar{z}_1 z_2\}$$
$$= \frac{1}{2}\{\bar{z}_1 z_2 + z_1\bar{z}_2\}$$

where θ is the angle between z_1 and z_2 which lies between 0 and π

If vectors z_1, z_2 are perpendicular then

$$z_1 \cdot z_2 = 0 \;\Rightarrow\; \frac{z_1}{\bar{z}_1} + \frac{z_2}{\bar{z}_2} = 0.$$

i.e. Sum of complex slopes = 0

The cross product of z_1 and z_2 is defined by

$$z_1 \times z_2 = |z_1|\,|z_2| \sin\theta = x_1y_2 - y_1x_2 = \text{Im}\{\bar{z}_1 z_2\}$$
$$= \frac{1}{2i}\{\bar{z}_1 z_2 - z_1\bar{z}_2\}$$

If vectors z_1, z_2 are parallel then

$$z_1 \times z_2 = 0 \;\Rightarrow\; \frac{z_1}{\bar{z}_1} = \frac{z_2}{\bar{z}_2}.$$

i.e. Complex slopes are equal.

Note: ω_1 and ω_2 are the complex slopes of two lines.

(*i*) If lines are parallel then $\omega_1 = \omega_2$

(*ii*) If lines are perpendicular then $\omega_1 + \omega_2 = 0$

6. $\text{Arg}\left(\dfrac{z - z_1}{z - z_2}\right) = \theta$ represent

(*i*) a line segment if $\theta = \pi$

(*ii*) Pair of ray if $\theta = 0$

7. Perpendicular distance of a point z_0 from the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is $\dfrac{|\bar{\alpha}z_0 + \alpha\bar{z}_0 + r|}{2|\alpha|}$

8. **Reflection points for a straight line:** Two given points P & Q are the reflection points for a given straight line if the given line is the right bisector of the segment PQ. Note that the two points denoted by the complex numbers z_1 and z_2 will be the reflection points for the straight line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if; $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.

Train Your Brain

Example 4: If z_1 and z_2 are two fixed points in the argand plane then find the locus of a point z in each of the following

(*i*) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(*ii*) $|z - z_1| = |z - z_2|$

(*iii*) $|z - z_1| - |z - z_2| = |z_1 - z_2|$

Sol. (*i*) Let P and Q be two points represented by z_1 and z_2 in the argand plane and let R be a point having affix z.

Then $PR = |z - z_1|$, $QR = |z - z_2|$ and $PQ = |z_1 - z_2|$

$|z - z_1| + |z - z_2| = |z_1 - z_2|$

$\Rightarrow PR + QR = PQ$

$\Rightarrow R(z)$ lies on the line segment joining $P(z_1)$ and $Q(z_2)$

(*ii*) $|z - z_1| = |z - z_2|$

$\Rightarrow PR = QR$

$\Rightarrow R(z)$ is equidistant from $P(z_1)$ and $Q(z_2)$.

$\Rightarrow R$ lies on the perpendicular bisector of line segment PQ.

(*iii*) $|z - z1| - |z - z_2| = |z_1 - z_2|$

$\Rightarrow PR - QR = PQ$

$\Rightarrow R(z)$ lies on the line joining $P(z_1)$ and $Q(z_1)$ but does not lie between them.

Example 5: Let $A(z_1)$ and $B(z_2)$ represent two complex numbers on the complex plane. Suppose the complex slope of the line joining A and B is defined as $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$. Then the lines l_1 with complex slope ω_1 and l_2 with complex slope ω_2 on the complex plane will be perpendicular to each other if

(*a*) $\omega_1 + \omega_2 = 0$ (*b*) $\omega_1 - \omega_2 = 0$

(*c*) $\omega_1\omega_2 = -1$ (*d*) $\omega_1\omega_2 = 1$

Sol. (*a*) l_1 is perpendicular to l_2

$$\Rightarrow \frac{z_1 - z_2}{z_3 - z_4} \text{ is purely imaginary}$$

$$\frac{z_1 - z_2}{z_3 - z_4} + \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_4} = 0$$

$$\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} + \frac{z_3 - z_4}{\bar{z}_3 - \bar{z}_4} = 0 \Rightarrow \omega_1 + \omega_2 = 0$$

Note: If l_1 parallel to l_2 then

$$\frac{z_1 - z_2}{z_3 - z_4} = \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_4} \Rightarrow \omega_1 = \omega_2$$

Concept Application

4. If $\arg(z - a) = \dfrac{\pi}{4}$, where $a \in R$, then the locus of $z \in C$ is a

(*a*) Hyperbola (*b*) Parabola

(*c*) Ellipse (*d*) Straight line

5. The straight line $(1+2i)z+(2i-1)\bar{z}=10i$ on the complex plane, has intercept on the imaginary axis equal to

(a) 5 (b) $\frac{5}{2}$

(c) $-\frac{5}{2}$ (d) -5

6. If $z=x+iy$ and $\omega=\frac{1-iz}{z-i}$, then $|\omega|=1$, then find the locus of z?

(a) z lies on the y-axis (b) z lies on the real axis

(c) z lies on the $y=1$ (d) z lies on the $x=1$

7. If $|z_1|=|z_2|=|z_3|=1$ and $z_1+z_2+z_3=0$, then area of the triangle whose vertices are z_1, z_2, z_3 is

(a) $\frac{3\sqrt{3}}{4}$ (b) $\frac{\sqrt{3}}{4}$

(c) 1 (d) 2

CIRCLE/PARABOLA/ELLIPSE/HPYERBOLA

Circle

1. The equation of circle having centre z_0 and radius ρ is: $|z-z_0|=\rho$ or $z\bar{z}-z_0\bar{z}-\bar{z}_0z+\bar{z}_0z_0-\rho^2=0$ which is of the form $z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+k=0$, k is real. Centre is $-\alpha$ and radius $=\sqrt{|\alpha|^2-k}$.

   Circle will be real if $|a|^2-k\geq 0$.

2. The equation of the circle described on the line segment joining z_1 & z_2 as diameter is $\arg\frac{z-z_2}{z-z_1}=\pm\frac{\pi}{2}$

   or $(z-z_1)(\bar{z}-\bar{z}_2)+(z-z_2)(\bar{z}-\bar{z}_1)=0$.

3. $\text{Arg}\left(\frac{z-z_1}{z-z_2}\right)=\theta$ represent

   a part of circle, if $0<\theta<\pi$.

4. If $\left|\frac{z-z_1}{z-z_2}\right|=k\neq 1, 0$, then locus of z is circle.

5. Condition for four given points z_1, z_2, z_3 and z_4 to be concyclic is the number $\frac{z_3-z_1}{z_3-z_2}\cdot\frac{z_4-z_2}{z_4-z_1}$ should be real.

   Hence the equation of a circle through 3 non collinear points z_1, z_2 and z_3 can be taken as $\frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$ is real

   $$\Rightarrow \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}=\frac{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}$$

6. **Inverse points w.r.t. a circle:** Two points P and Q are said to be inverse with respect to a circle with centre 'O' and radius ρ, if:

   (i) the point O, P, Q are collinear and P, Q are on the same side of O.

   (ii) $OP\cdot OQ=\rho^2$.

Note:

The two points z_1 and z_2 will be the inverse points w.r.t. the circle $z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+r=0$ if and only if $z_1\bar{z}_2+\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$.

Parabola

If $|z-z_0|=\left|\frac{\bar{\alpha}z+\alpha\bar{z}+r}{2|\alpha|}\right|$ then locus of z is parabola whose focus is z_0 and directrix is the line $\bar{\alpha}z+\alpha\bar{z}+r=0$ (Provided $\bar{\alpha}z_0+\alpha\bar{z}_0+r\neq 0$)

Ellipse

If $|z-z_1|+|z-z_2|=K>|z_1-z_2|$ then locus of z is an ellipse whose focii are z_1 and z_2

Hyperbola

If $||z-z_1|-|z-z_2||=K<|z_1-z_2|$ then locus of z is a hyperbola, whose focii are z_1 and z_2.

PTOLEMY'S THEOREM

It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the products of lengths of the two pairs of its opposite sides.

i.e. $|z_1-z_3||z_2-z_4|=|z_1-z_2||z_3-z_4|+|z_1-z_4||z_2-z_3|$.

Train Your Brain

Example 6: Let α, β be fixed complex numbers and z is a variable complex number such that, $|z-\alpha|^2+|z-\beta|^2=k$. Find out the limits for 'k' such that the locus of z is a circle. Find also the center and radius of the circle

Sol. $(z-\alpha)(\bar{z}-\bar{a})+(z-\beta)(\bar{z}-\bar{\beta})-k=0$

$$z\bar{z}-\alpha\bar{z}-\alpha\bar{z}+\alpha\bar{\alpha}+z\bar{z}-\beta\bar{z}-\bar{\beta}z+\beta\bar{\beta}-k=0$$

$$z\bar{z}+\frac{(-\alpha-\beta)\bar{z}}{2}+\frac{(-\bar{\alpha}-\bar{\beta})z}{2}+\frac{(\alpha\bar{\alpha}+\beta\bar{\beta}-k)}{2}=0 \quad ..(1)$$

compare (1) with $z\bar{z}+\bar{a}z+a\bar{z}+b=0$

$$a=\frac{-\alpha-\beta}{2}, b=\frac{\alpha\bar{\alpha}+\beta\bar{\beta}-k}{2}$$

If (1) is circle then center $= -a$ and radius $= \sqrt{|a|^2 - b}$

for circle $|a|^2 - b \geq 0$

$$\left|\frac{\alpha+\beta}{2}\right|^2 - \frac{\alpha\bar{\alpha}+\beta\bar{\beta}-k}{2} \geq 0$$

$$\Rightarrow \alpha\bar{\alpha}+\beta\bar{\beta}-\alpha\bar{\beta}-\bar{\alpha}\beta \leq 2k$$

hence,

$k \geq \frac{1}{2}|\alpha-\beta|^2$, centre$=\frac{1}{2}(\alpha+\beta)$, radius $= \frac{1}{2}\sqrt{2k-(\alpha-\beta)^2}$

Concept Application

8. For the complex number $w = \dfrac{4z-5i}{2z+1}$

The locus of z, when w is a purely imaginary number is

(*a*) a circle with centre $\left(\frac{1}{2}, -\frac{5}{4}\right)$ passing through origin.

(*b*) a circle with centre $\left(-\frac{1}{4}, \frac{5}{8}\right)$ passing through origin.

(*c*) a circle with centre $\left(\frac{1}{4}, -\frac{5}{8}\right)$ and radius $\frac{\sqrt{29}}{8}$

(*d*) any other circle

9. For the complex number $w = \dfrac{4z-5i}{2z+1}$

The locus of z, when $|w| = 1$ is

(*a*) a circle with $\left(-\frac{5}{8}, \frac{1}{4}\right)$ centre and radius $\frac{1}{2}$

(*b*) a circle with centre $\left(\frac{1}{4}, -\frac{5}{8}\right)$ and radius $\frac{1}{2}$

(*c*) a circle with centre $\left(\frac{5}{8}, -\frac{1}{4}\right)$ and radius $\frac{1}{2}$

(*d*) any other circle

10. Intercept made by the circle $z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+r=0$ on the real axis on complex plane, is

(*a*) $\sqrt{(\alpha+\bar{\alpha})-r}$ (*b*) $\sqrt{(\alpha+\bar{\alpha})^2-2r}$

(*c*) $\sqrt{(\alpha+\bar{\alpha})^2-r}$ (*d*) $\sqrt{(\alpha+\bar{\alpha})^2-4r}$

11. $R(z^2) = 1$ is represented by

(*a*) The parabola $x^2 + y^2 = 1$

(*b*) The hyperbola $x^2 - y^2 = 1$

(*c*) Parabola or a circle

(*d*) Straight line

12. If $|8 + z| + |z - 8| = 16$ where z is a complex number, then the point z will lie on

(*a*) A circle (*b*) An ellipse

(*c*) A straight line (*d*) A parabola

13. Find the centre and radius of circle given by equation, $z\bar{z} + (3 + 4i)\,\bar{z} + (3 - 4i)\,z + 24 = 0$

(*a*) $-3 - 4i$, 1 (*b*) $-3 + 4i$, 1

(*c*) $3 - 4i$, 1 (*d*) $3 + 4i$, 1

14. Find the Cartesian equation of the locus of 'z' in the complex plane satisfying, $|z - 4| + |z + 4| = 16$.

(*a*) ellipse, $\sqrt{(x-4)^2+y^2}+\sqrt{(x+4)^2+y^2}=16$

(*b*) circle, $\sqrt{(x+4)^2+y^2}=16$

(*c*) ellipse, $\sqrt{x^2+(y-4)^2}+\sqrt{(x)^2+(y+4)^2}=16$

(*d*) circle, $\sqrt{(x)^2+(y+4)^2}=16$

15. Find the points representing the complex number z for which $|z + 5|^2 - |z - 5|^2 = 10$?

(*a*) Parabola (*b*) Ellipse

(*c*) Straight line (*d*) Hyperbola

16. The locus represented by the equation $|z + 1| + |z-1| = 2$ is

(*a*) an ellipse having its foci at (1,0) and (–1,0).

(*b*) one of the family of circles of the point of intersection of circles $|z - 1| = 1$ and $|z + 1| = 1$.

(*c*) the radical axis of the circles $|z - 1|=1$ and $|z + 1| = 1$.

(*d*) the portion of the real axis between points (1,0) and (–1,0) including the points

17. When $|z| = 1$, the points $1 + 2z$ lie on a

(*a*) Circle with radius 1, centre (0, 1)

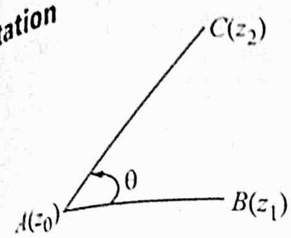
(*b*) Circle with radius 2, centre (1, 0)

(*c*) Straight line

(*d*) Circle with radius 3, centre (0, 0)

Short Notes

Rotation



$$\frac{z_2 - z_0}{|z_2 - z_0|} = \frac{z_1 - z_0}{|z_1 - z_0|} e^{i\theta}$$

Take θ in anticlockwise direction.

Result Related with Triangle

(a) Equilateral triangle:

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

or $$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

(b) Area of triangle ΔABC given by modulus of $\frac{1}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$.

Equation of line Through Points z_1 and z_2

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow z(\bar{z}_1 - \bar{z}_2) + z_1\bar{z}(z_2 - z_1) + \bar{z}_2 - \bar{z}_1 z_2 = 0$$

$$\Rightarrow \quad z(\bar{z}_1 - \bar{z}_2)i + \bar{z}(z_2 - z_1)i + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

Let $(z_2 - z_1)i = a$, then equation of line is $\boxed{\bar{a}z + a\bar{z} + b = 0}$ where $a \in C$ & $b \in R$.

Notes

(i) *Complex slope of line* $\bar{a}z + a\bar{z} + b = 0$ is $-a\frac{1}{\bar{a}}$.

(ii) Two lines with slope μ_1 and μ_2 are parallel or perpendicular if $\mu_1 = \mu_2$ or $\mu_1 + \mu_2 = 0$.

(iii) Length of perpendicular from point $A(\alpha)$ to line $\bar{a}z + a\bar{z} + b = 0$ is $\frac{|\bar{a}\alpha + a\bar{\alpha} + b|}{2|a|}$.

Equation of Circle

(a) Circle whose centre is z_0 and radii $= r$

$$|z - z_0| = r$$

(b) General equation of circle

$$z\bar{z} + a\bar{z} + \bar{a}z + b = 0$$

centre '$-a$' & radii $= \sqrt{|a|^2 - b}$

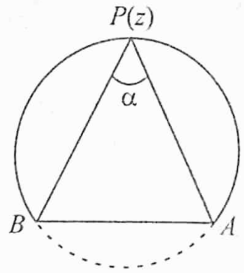
(c) Diameter form $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

or $$\arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\frac{\pi}{2}$$

(d) Equation $\left|\frac{z - z_1}{z - z_2}\right| = k$ represent a circle if $k \neq 1$ and a straight line if $k = 1$.

(e) Equation $|z - z_1|^2 + |z - z_2|^2 = k$

represent circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$



(f) $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha \quad 0 < \alpha < \pi, \alpha \neq \frac{\pi}{2}$

represent a segment of circle passing through $A(z_1)$ and $B(z_2)$.

Standard LOCI

(a) $|z - z_1| + |z - z_2| = 2k$ (a constant) represent

(i) If $2k > |z_1 - z_2| \Rightarrow$ An ellipse

(ii) If $2k = |z_1 - z_2| \Rightarrow$ A line segment

(iii) If $2k < |z_1 - z_2| \Rightarrow$ No solution

(b) Equation $||z - z_1| - |z - z_2|| = 2k$ (a constant) represent

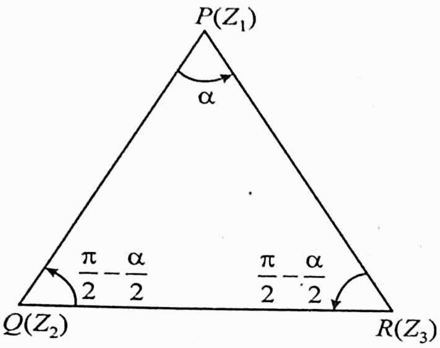
(i) If $2k < |z_1 - z_2| \Rightarrow$ A hyperbola

(ii) If $2k = |z_1 - z_2| \Rightarrow$ Union of two ray

(iii) If $2k > |z_1 - z_2| \Rightarrow$ No solution

Solved Examples

1. The points P, Q and R represent the complex numbers Z_1, Z_2 and Z_3 respectively and the angles of the triangle PQR at Q and R are both $\frac{\pi}{2}-\frac{\alpha}{2}$, Prove that $(Z_3 - Z_2)^2 = 4(Z_3 - Z_1)$ $(Z_1 - Z_2)\sin^2\left(\frac{\alpha}{2}\right)$



Sol. QP is obtained from QR by a rotation counter clockwise the length PQ is different from the length of QR

$$\therefore\ Z_1 - Z_2 = \frac{PQ}{QR}(Z_2 - Z_3)\left\{\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right\}$$

through an angle $\frac{\pi}{2}-\frac{\alpha}{2}$; of course

Similarly

$$\therefore\ Z_1 - Z_3 = \frac{PQ}{QR}(Z_2 - Z_3)\left\{\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right\}$$

Multiplying the two

$$\left[(Z_1 - Z_2)(Z_1 - Z_3) = \frac{PQ\cdot PR}{QR^2}(Z_3 - Z_2)(Z_2 - Z_3)\right.$$

$$\left.\left\{\cos^2\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) + \sin^2\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right\}\right]$$

$$\text{Now } \frac{QR}{\sin\alpha} = \frac{PQ}{\cos\frac{\alpha}{2}} = \frac{PR}{\cos\frac{\alpha}{2}} \qquad \text{(by sine rule)}$$

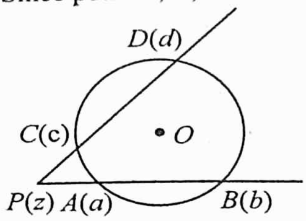
$$\therefore\ \frac{PQ\cdot PR}{QR^2} = \frac{\cos^2\frac{\alpha}{2}}{4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}}$$

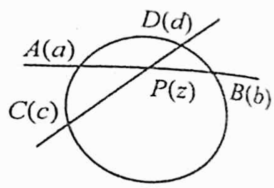
$$\therefore (Z_1 - Z_2)(Z_1 - Z_3)4\sin^2\frac{\alpha}{2} = (Z_3 - Z_2)(Z_2 - Z_3)$$

i.e., $(Z_3 - Z_2)^2 = 4(Z_3 - Z_1)(Z_1 - Z_2)\sin^2\frac{\alpha}{2}$.

2. Two different non parallel lines cut the circle $|z| = r$ in point a, b, c, d respectively. Prove that these lines meet in the point z given by $z = \dfrac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$

Sol. Since point P, A, B are collinear





$$\therefore \begin{vmatrix} z & \bar{z} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix} = 0$$

$$\Rightarrow\ z(\bar{a}-\bar{b}) - \bar{z}(a-b) + (a\bar{b}-\bar{a}b) = 0 \qquad \text{...}(i)$$

Similarly, points P, C, D are collinear, so

$$z(\bar{c}-\bar{d}) - \bar{z}(c-d) + (c\bar{d}-\bar{c}d) = 0 \qquad \text{...}(ii)$$

On applying $(i) \times (c-d) - (ii)(a-b)$, we get

$$\therefore\ z(\bar{a}-\bar{b})(c-d) - z(\bar{c}-\bar{d})(a-b) = (c\bar{d}-\bar{c}d)$$

$$(a-b) - (a\bar{b}-\bar{a}b)(c-d) \qquad \text{...}(iii)$$

$\because\ z\bar{z} = r^2 = k\text{(say)}$

$$\because\ \bar{a} = \frac{k}{a}, \bar{b} = \frac{k}{b}, \bar{c} = \frac{k}{c}, \bar{d} = \frac{k}{d}$$

From equation (iii) we get

$$z\left(\frac{k}{a}-\frac{k}{b}\right)(c-d) - z\left(\frac{k}{c}-\frac{k}{d}\right)(a-b)$$

$$= \left(\frac{ck}{d}-\frac{kd}{c}\right)(a-b) - \left(\frac{ak}{b}-\frac{bk}{a}\right)(c-d)$$

$$\therefore\ z = \frac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$$

3. If z_1, z_2 and z_3 are the affixes of three points A, B and C respectively and satisfy the condition $|z_1 - z_2| = |z_1| + |z_2|$ and $|(2-i)z_1 + iz_3| = |z_1| + |(1-i)z_1 + iz_3|$ then, prove that $\Delta\, ABC$ in a right angled.

Sol. $|z_1 - z_2| = |z_1| + |z_2|$

$\Rightarrow z_1, z_2$ and origin will be collinear and z_1, z_2 will be opposite side of origin.

Similarly, $|(2-i)z_1 + iz_3| = |z_1| + |(1-i)z_1 + iz_3|$

$\Rightarrow z_1$ and $(1-i)z_1 + iz_3 = z_4$ say, are collinear with origin and lies on same side of origin.

Let $z_4 = \lambda z_1$, λ real then, $(1-i)z_1 + iz_3 = \lambda z_1$

$\Rightarrow i(z_3 - z_1) = (\lambda - 1)\, z_1$

$$\Rightarrow \frac{(z_3 - z_1)}{-z_1} = (\lambda - 1)\, i$$

$$\Rightarrow \frac{z_3 - z_1}{0 - z_1} = me^{i\pi/2},\ m = \lambda - 1$$

$\Rightarrow z_3 - z_1$ is perpendicular to the vector $0 - z_1$.

i.e. also z_2 is on line joining origin and z_1 so we can say the triangle formed by z_1, z_2 and z_3 is right angled.

Paragraph (Q. 4 to 6) : Let A, B, C be three sets of complex numbers as defined below.

$A = \{z : |z + 1| \le 2 + Re(z)\}$, $B = \{z : |z - 1| \ge 1\}$ and $C = \left\{z : \left|\frac{z-1}{z+1}\right| \ge 1\right\}$

4. The number of point(s) having integral coordinates in the region $A \cap B \cap C$ is

(a) 4 (b) 5

(c) 6 (d) 10

5. The area of region bounded by $A \cap B \cap C$ is

(a) $2\sqrt{3}$ (b) $\sqrt{3}$

(c) $4\sqrt{3}$ (d) 2

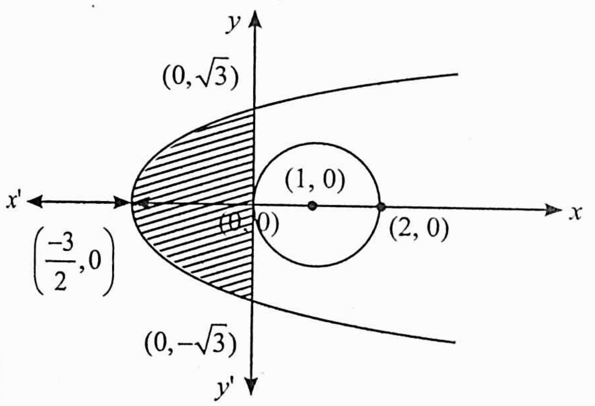
6. The real part of the complex number in the region $A \cap B \cap C$ and having maximum amplitude

(a) -1 (b) $\frac{-3}{2}$

(c) $\frac{1}{2}$ (d) -2

Sol. 4. (b) 5. (a) 6. (b)

For A, $|z + 1| \le 2 + Re(z)$



$\Rightarrow (x + 1)^2 + y^2 \le 4 + 4x + x^2$

$\Rightarrow y^2 \le 3 + 2x$

$$\Rightarrow y^2 \le 2\left(x + \frac{3}{2}\right) \quad ...(1)$$

For B, $|z - 1| \ge 1$

$$\Rightarrow (x - 1)^2 + y^2 \ge 1 \quad ...(2)$$

For C, $|z - 1|^2 \ge |z + 1|^2$

$\Rightarrow (z - 1)(\bar{z} - 1) \ge (z + 1)(\bar{z} + 1)$

$\Rightarrow (z\bar{z} - \bar{z} - z + 1) \ge (z\bar{z} + \bar{z} + z + 1)$

$\Rightarrow z + \bar{z} \le 0$

i.e. $x \le 0$ ...(3)

(i) $(-1,0)$, $(-1, 1)$, $(-1,-1)$, $(0,0)$, $(0,1)$, $(0,-1)$ but $z = -1$ is not in the domain in set C

$\therefore$ Total number of point(s) having integral coordinates in the region $A \cap B \cap C$ is 5.

(ii) Required area $2\int_{\frac{-3}{2}}^{0} \sqrt{2\left(x + \frac{3}{2}\right)}\, dx = 2\sqrt{3}$ (square units)

(iii) Clearly $z = \frac{-3}{2} + i0$ is the complex number in the region $A \cap B \cap C$ and having maximum amplitude.

$\therefore$ Re$(z) = \frac{-3}{2}$

7. Complex The locii of a point $P(z)$ in the complex plane satisfying $\left|z + \frac{1}{z}\right| = 2$ are two circles C_1 and C_2. These circles

(a) have centres on real axis.

(b) cut each other orthogonally.

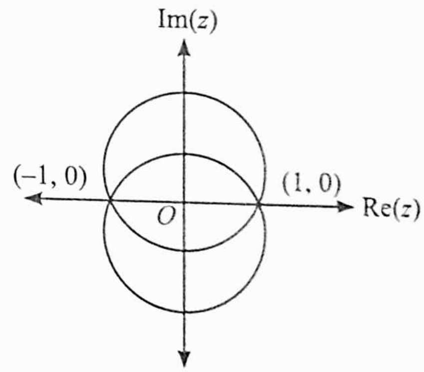
(c) are congruent.

(d) have exactly two common tangents.

Sol. (c) We have $\left|z + \frac{1}{z}\right| = 2. \Rightarrow \left|z + \frac{1}{z}\right|^2 = 4$

$$\Rightarrow \left(z + \frac{1}{z}\right)\left(\bar{z} + \frac{1}{\bar{z}}\right) = 4 \Rightarrow z\bar{z} + \frac{\bar{z}}{z} + \frac{z}{\bar{z}} + \frac{1}{z\bar{z}} = 4$$

$$\Rightarrow z\bar{z} + \frac{\bar{z}^2 + z^2}{z\bar{z}} + \frac{1}{z\bar{z}} = 4 \Rightarrow (z\bar{z})^2 - 4(z\bar{z}) + z^2 + \bar{z}^2 + 1 = 0$$



$\Rightarrow (z\bar{z} - 1)^2 + (z - \bar{z})^2 = 0 \Rightarrow (z\bar{z} - 1)^2 - i^2(z - \bar{z})^2 = 0$

$\Rightarrow (z\bar{z} - 1 + i(z - \bar{z}))(z\bar{z} - 1 - i(z - \bar{z})) = 0$

Each of the above factors represents a circle with centre and radius (in ordered pair) as $(i\sqrt{i\bar{i} + 1})$ and $(-i\sqrt{i\bar{i} + 1})$

i.e. $((0,1), \sqrt{2})$ and $((0,-1), \sqrt{2})$ respectively.

The union of the circle with equations $x^2 + y^2 - 2y - 1 = 0$ and $x^2 + y^2 + 2y - 1 = 0$

Note : These circles cut each other orthogonally.

8. Let A and B be two distinct points denoting the complex numbers α and β respectively. A complex number z lies between A and B where $z \neq \alpha, z \neq \beta$. Which of the following relation(s) hold good?

(*a*) $|\alpha - z| + |z - \beta| = |\alpha - \beta|$

(*b*) $\exists$ a positive real number 't' such that $z = (1 - t)\alpha + t\beta$

(*c*) $\begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0$

(*d*) $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$

Sol. (*a,b,c*)

$AP + PB = AB$

$|z - \alpha| + |\beta - z| = |\alpha - \beta| \Rightarrow A$ is True

Now $z = \alpha + t(\beta - \alpha)$

$= (1 - t)\alpha + t\beta$ where $t \in (0,1) \Rightarrow B$ is True

again $\dfrac{z-\alpha}{\beta-\alpha}$ is real $\Rightarrow \dfrac{z-\alpha}{\beta-\alpha} = \dfrac{\bar{z}-\bar{\alpha}}{\bar{\beta}-\bar{\alpha}}$

$\Rightarrow \begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0$ Ans.

also $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$ if and only if

$\begin{vmatrix} z-\alpha & z-\bar{\alpha} & 0 \\ \alpha & \bar{\alpha} & 1 \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} & 0 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} (z-a) & \bar{z}-\bar{\alpha} \\ \beta-\bar{\alpha} & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0$ Ans.

9. Equation of a straight line on the complex plane passing through a point P denoting the complex number $\overrightarrow{OP}$ and perpendicular to the vector OP where 'O' is the origin can be written as

(*a*) $\text{Im}\left(\dfrac{z-\alpha}{\alpha}\right) = 0$

(*b*) $\text{Re}\left(\dfrac{z-\alpha}{\alpha}\right) = 0$

(*c*) $\text{Re}(\bar{\alpha}z) = 0$

(*d*) $\bar{\alpha}z + \alpha\bar{z} - 2|\alpha|^2 = 0$

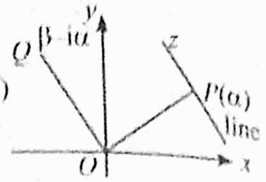
Sol. (*b*) Required line is passing through $P(\alpha)$ and parallel to the vector $\overrightarrow{OQ}$ hence $z = \alpha + i\lambda\alpha, \lambda \in R$

$\dfrac{z-a}{a}$ = purely imaginary

$\Rightarrow \text{Re}\left(\dfrac{z-\alpha}{\alpha}\right) = 0 \Rightarrow (B)$ (multiply Nr and Dr by $\bar{\alpha}$)

$\Rightarrow \text{Re}((z-\alpha)\bar{\alpha}) = 0 \Rightarrow \text{Re}(z\bar{\alpha} - |\bar{\alpha}|) = 0$

(multiply Nr and Dr by $\bar{\alpha}$)



also, $\dfrac{z-\alpha}{\alpha} + \dfrac{\bar{z}-\bar{\alpha}}{\bar{\alpha}} = 0$

$\bar{\alpha}(z-\alpha) + \alpha(\bar{z}-\bar{\alpha}) = 0$

$\bar{\alpha}z + \alpha\bar{z} - 2|\alpha|^2 = 0 \Rightarrow (D)]$

10. Let tangents at $A(z_1)$ and $B(z_2)$ are drawn to the circle $|z| = 2$. Then which of the following is/are **CORRECT?**

(*a*) The equation of tangent at A is given by $\dfrac{z}{z_1} + \dfrac{\bar{z}}{\bar{z}_1} = 2$.

(*b*) If tangents at $A(z_1)$ and $B(z_2)$ intersect at $P(z_p)$, then $z_p = \dfrac{2z_1z_2}{z_1+z_2}$.

(*c*) Slope of tangent at $A(z_1)$ is $\dfrac{1}{i}\left(\dfrac{z_1+\bar{z}_1}{z_1-\bar{z}_1}\right)$

(*d*) If points $A(z_1)$ and $B(z_2)$ on the circle $|z| = 2$ are such that $z_1 + z_2 = 0$, then tangents intersect at $\dfrac{\pi}{2}$.

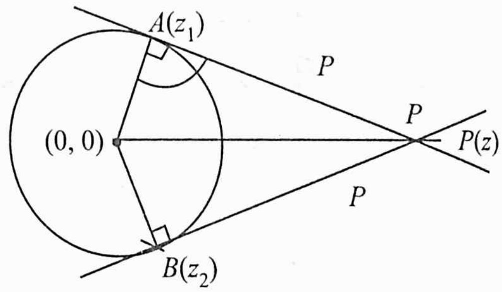
Sol (*a,b,c*)

(*a*) $\text{Arg}\left(\dfrac{z-z_1}{0-z_1}\right) = \dfrac{\pi}{2}$

$\Rightarrow \dfrac{z-z_1}{z_1}$ is purely magmary.

So, $\dfrac{z_1-z}{z_1} + \dfrac{\bar{z}_1-\bar{z}}{\bar{z}_1} = 0$

$\Rightarrow \dfrac{z}{z_1} + \dfrac{\bar{z}}{\bar{z}_1} = 2$



(*b*) By applying rotation, we get $(AP = p)$

$$\frac{z-z_1}{0-z_1} = \frac{p}{r}e^{\frac{i\pi}{2}}. \quad \text{...(1)}$$

$$\text{Also } \frac{0-z_2}{z-z_2} = \frac{r}{p}e^{\frac{i\pi}{2}} \quad \text{...(2)}$$

$\therefore$ On multiplying (1) and (2), we get $z_p = \dfrac{2z_1z_2}{z_1-z_2}$

(c) As equation of tangent at $A(x_1, y_1)$ is $xx_1 + yy_1 = 4$

∴ Slope of tangent

$$= \frac{-x_1}{y_1} = \frac{-2x_1}{2y_1} = -i\left(\frac{z_1+\bar{z}_1}{z_1-\bar{z}_1}\right) = \frac{1}{i}\left(\frac{z_1+\bar{z}_1}{z_1-\bar{z}_1}\right)$$

(d) Clearly tangents are parallel lines.

(As $A(z_1)$ and $B(z_2)$ are ends of diameter of circle.)]

PROBLEM

11. If z_1, z_2, z_3 are the vertices of an equilateral triangle with z_0 as its circum centre, then changing origin to z_0, new vertices become z_1', z_2', z_3', show that $z_1'^2, z_2'^2, z_3'^2 = 0$

Sol. ABC is an equilateral triangle, then by rotation principle

$$\frac{z_1 - z_2}{z_3 - z_2} = e^{i\pi/3} \quad \text{... (1)}$$

and $$\frac{z_1 - z_3}{z_3 - z_2} = e^{-i\pi/3} \quad \text{...(2)}$$

Multiplying equation (1) and (2)

$(z_1 - z_2)(z_1 - z_3) = (z_3 - z_2)(z_2 - z_3)$

$z_1^2 - z_1z_3 - z_1z_2 + z_2z_3 = z_2z_3 - z_3^2 - z_2^2 + z_2z_3$

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3 \quad \text{...(3)}$$

In equilateral triangle circumcenter coincides with centroid

$z_1 + z_2 + z_3 = 3z_0$

Shifting origin to z_0 changes z_1 to $z_1' + z_0$

z_2 to $z_2' + z_0$

z_3 to $z_3' + z_0$

Therefore $z_1' + z_2' + z_3' = 0$

$z_1'^2 + z_2'^2 + z_3'^2 + 2\left(z_1'z_2' + z_1'z_3' + z_2'z_3'\right) = 0$ (on squaring)

Using equation (3), we get

$3\left(z_1'^2 + z_2'^2 + z_3'^2\right) = 0 \Rightarrow z_1'^2 + z_2'^2 + z_3'^2 = 0$

PROBLEM 2

12. If z_1, z_2, z_3 are the roots of the equation $3z^3 + 3az^2 + a^2z + b = 0$, then find valve of $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1}$.

Sol. We have, $z_1 + z_2 + z_3 = -a$

$$z_1z_2 + z_2z_3 + z_3z_1 = \frac{a^2}{3}$$

$\therefore z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$

$$= a^2 - \frac{2a^2}{3} = \frac{a^2}{3} = z_1z_2 + z_2z_3 + z_3z_1$$

⇒ Triangle with vertices z_1, z_2 and z_3 is an equilateral triangle

Let $z_1 - z_2 = \alpha$, $z_2 - z_3 = \beta$, $z_3 - z_1 = \gamma$

$\Rightarrow \alpha + \beta + \gamma = 0$ and $|a| = |b| = |\gamma|$

$\Rightarrow \bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$

$$\Rightarrow \frac{|\alpha|^2}{\alpha} + \frac{|\beta|^2}{\beta} + \frac{|\gamma|^2}{\gamma} = 0$$

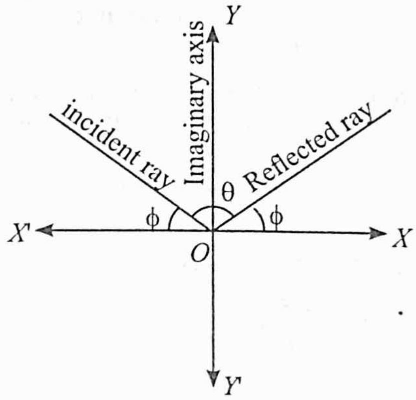
$$\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0 \quad \text{(Since } |a| = |b| = |\gamma|)$$

$$\Rightarrow \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0.$$

13. Prove that angle between the line $\bar{a}z + a\bar{z} = 0$ and its reflection in the real axis is :

$$\theta = \tan^{-1}\left[\frac{2\,\text{Re}(a)\,\text{Im}(a)}{\{\text{Re}(a)\}^2 - \{\text{Im}(a)\}^2}\right]$$

Sol. Let $z = x + iy$, the equation $\bar{a}z + a\bar{z} = 0$ can be written as $(\bar{a} + a)x + i(\bar{a} - a)y = 0$



$$\Rightarrow \left(\frac{a + \bar{a}}{2}\right)x + \left(\frac{a - \bar{a}}{2i}\right)y = 0$$

$$\Rightarrow \{\text{Re}(a)\}x + \{\text{Im}(a)\}y = 0$$

$\therefore$ Slope of the given line $(m) = -\dfrac{\{\text{Re}(a)\}}{\{\text{Im}(a)\}}$

then $\tan\left(180^\circ - \phi\right) = -\dfrac{\{\text{Re}(a)\}}{\{\text{Im}(a)\}}$

$$\Rightarrow -\tan\phi = -\frac{\{\text{Re}(a)\}}{\{\text{Im}(a)\}}$$

$$\Rightarrow \therefore \tan\phi = \frac{\{\text{Re}(a)\}}{\{\text{Im}(a)\}} \quad \text{...}(i)$$

Hence angle between the incident ray and reflected ray = $(\pi - 2\phi)$

$= \tan^{-1} \tan f$

$= \tan^{-1} \tan (180° - 2\phi)$

$= \tan^{-1} (-\tan 2\phi)$

$$= \tan^{-1}\left(-\frac{2\tan\phi}{1 - \tan^2\phi}\right)$$

$$= \tan^{-1}\left(\frac{-2\frac{\{\text{Re}(a)\}}{\{\text{Im}(a)\}}}{1-\frac{\{\text{Re}(a)\}^2}{\{\text{Im}(a)\}^2}}\right)$$

$$= \tan^{-1}\left(\frac{2\,\text{Re}(a)\,\text{Im}(a)}{\{\text{Re}(a)\}^2-\{\text{Im}(a)\}^2}\right)$$

14. Let z be a complex number satisfying $|z-3| \le |z-1|$, $|z-3| \le |z-5|$, $|z-i| \le |z+i|$ and $|z-i| \le |z-5i|$. The area of the region in which z can lie is equal to ______

Sol. $|z-3| \le |z-1| \Rightarrow z+\bar{z} \ge 4 \Rightarrow x \ge 2$ $\quad(\because z+\bar{z}=2x)$

$|z-3| \le |z-5| \Rightarrow z+\bar{z} \le 8 \Rightarrow x \le 4$

$|z-i| \le |z+i| \Rightarrow i(z-\bar{z}) \le 0 \Rightarrow y \ge 0$ $\quad(\because z-\bar{z}=2iy)$

$|z-i| \le |z-5i| \Rightarrow i(z-\bar{z}) \ge -6 \Rightarrow y \le 3$

clearly the region is a rectangle of area $= 2 \times 3 = 6$

15. The number of solution of the simultaneous complex equations $|z-3-i| = 2$ and $|z-2+i| + |z-4-3i| = 6$ is ____

Sol. $|z-3-i| = 2$ represents a circle with centre $3+i$ and radius 2 $|z-2+i| + |z-4-3i| = 6$ represents an ellipse with centre $\left(\frac{2-1+4+3i}{2}\right) = 3+i$ and major axis $= 6$

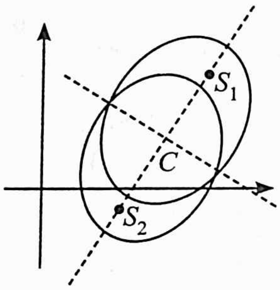
Also, the distance between foci $= 2\sqrt{5}$

$\Rightarrow$ eccentricity $= \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3} < 1$

$\therefore$ Minor axis $= 2\times 3\sqrt{1-\frac{5}{a}} = 4$

$\because$ diameter of circle = minor axis

So, the circle touches the ellipse internally



16. The amplitude of $\sin\frac{\pi}{5} + i\left(1-\cos\frac{\pi}{5}\right)$ is

Sol. $\alpha = \tan^{-1}\left|\frac{1-\cos\frac{\pi}{5}}{\sin\frac{\pi}{5}}\right| \left\{\frac{1-\cos d}{\sin d} = \tan\frac{d}{2}\right\}$

$= \tan^{-1}\left|\tan\frac{\pi}{10}\right|$

$= \tan^{-1}\left|\tan\frac{\pi}{10} = \frac{\pi}{10}\right|$

$\tan^{-1}\tan t = t$ if $t \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\left.\begin{aligned} 1-\cos\frac{\pi}{5} > 0 \\ \sin\frac{\pi}{5} > 0 \end{aligned}\right\}$$

17. A complex number z is rotated in anticlockwise direction by an angle α and we get z' and if the same complex number z is rotated by an angle & in clockwise direction and we get z'' then

(*a*) z', z, z'' are in G.P.

(*b*) $z'^2 + z''^2 = 2z^2 \cos 2\alpha$

(*c*) $z' + z'' = 2z \cos \alpha$

(*d*) z', z, z'' are in H.P.

Sol. (*c*) $z' = ze^{i\alpha}$

$z'' = ze^{-i\alpha}$

$\therefore z'z'' = z^2 \Rightarrow z', z, z''$ are in G.P.

$$\left(\frac{z'}{z}\right)^2 + \left(\frac{z''}{z}\right)^2 = 2\cos 2\alpha$$

$$\Rightarrow z'^2 + z''^2 = 2z^2 \cos 2\alpha$$

$$z' + z'' = 2z\cos\alpha$$

Exercise-1 (Topicwise)

ROTATION THEOREM

1. The vector $z = -4 + 5i$ is turned counter clockwise through an angle of 180° & stretched 1.5 times. The complex number corresponding to the newly obtained vector is :

   (a) $6 - \frac{15}{2}i$ (b) $-6 + \frac{15}{2}i$

   (c) $6 + \frac{15}{2}i$ (d) $-6 - \frac{15}{2}i$

2. (i) Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find $z_3 \in C$ such that triangle z_1, z_2, z_3 is equilateral.

   (a) $\sqrt{3}(1-i)$ (b) $\sqrt{3}(1+i)$

   (c) $2\sqrt{3}(-1+i)$ (d) $\sqrt{3}(-1-i)$

GEOMETRY OF COMPLEX NO

3. If the vertices of a quadrilateral be $A = 1 + 2i$, $B = -3 + i$, $C = -2 - 3i$ and $D = 2 - 2i$, then the quadrilateral is

   (a) Parallelogram (b) Rectangle

   (c) Square (d) Rhombus

4. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is

   (a) 0 (b) 2 (c) 7 (d) 17

5. Write equation of straight line which makes angle 60° and passing through point $z_0 = 2 + i$.

   (a) $\arg(z-2-i) = \frac{\pi}{3}$ (b) $|z-2-i| = \sqrt{3}$

   (c) $\arg(z+2+i) = \frac{\pi}{3}$ (d) $\arg(z-2-i) = -\frac{\pi}{3}$

6. Write equation of line passing through z_1 $(-2-i)$ and z_2 $(3 + i)$.

   (a) $z = (2 + i) + k(5 + 2i)$, $k \in Z$

   (b) $z = (-2 - i) + k(5 + 2i)$, $k \in Z$

   (c) $z = (-2 - i) + k(5 - 2i)$, $k \in Z$

   (d) $z = (2 + i) + k(5 - 2i)$, $k \in Z$

7. Find the equation of circle in argument form whose end points of diameter are given by $(2 - i)$ and $(3 + i)$

   (a) $\arg\left(\frac{z+3-i}{z-2+i}\right) = \pm\frac{\pi}{2}$ (b) $\arg\left(\frac{z-3+i}{z-2+i}\right) = \pm\frac{\pi}{2}$

   (c) $\arg\left(\frac{z-3-i}{z-2+i}\right) = \pm\frac{\pi}{2}$ (d) $\arg\left(\frac{z-3-i}{z+2-i}\right) = \pm\frac{\pi}{2}$

8. The locus of z, for $\arg z = -\pi/3$?

   (a) same as the locus of z for $\arg z = 2\pi/3$

   (b) same as the locus of z for $\arg z = \pi/3$

   (c) the part of the straight line $\sqrt{3}x + y = 0$ with $(y < 0, x > 0)$

   (d) the part of the straight line $\sqrt{3}x + y = 0$ with $(y > 0, x < 0)$

9. $P(z)$ is the point moving in the Argand's plane satisfying $\arg(z-1) - \arg(z+i) = \pi$ then, P is

   (a) a real number, hence lies on the real axis.

   (b) an imaginary number, hence lies on the imaginary axis.

   (c) a point on the hypotenuse of the right angled triangle OAB formed by $O \equiv (0,0)$; $A \equiv (1,0)$; $B \equiv (0,-1)$.

   (d) a point on an arc of the circle passing through $A \equiv (1,0)$; $B \equiv (0,-1)$.

10. If the points $A(z)$, $B(-z)$, $C(1 - z)$ are the vertices of an equilateral triangle ABC, then $\text{Re}(z)$ is

    (a) $\frac{1}{4}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

11. Let $\lambda \in R$, the origin and the non-real roots of $2z^2 + 2z + \lambda = 0$ form the three vertices of an equilateral triangle in the Argand plane then λ is

    (a) 1 (b) $\frac{2}{3}$

    (c) 2 (d) -1

12. If centre of the square $ABCD$ is the origin. Denote by z the vertex A. Then find the centroid of the triangle ABC?

    (a) $\frac{iz}{3}$ (b) $\frac{-iz}{3}$

    (c) $\frac{z}{3}$ (d) $\frac{-z}{3}$

13. $z_1 = \frac{a}{1-i}$; $z_2 = \frac{b}{2+i}$; $z_3 = a - bi$ for $a, b \in R$

    if $z_1 - z_2 = 1$ then find the centroid of the triangle formed by the points z_1, z_2, z_3 in the argand's plane

    (a) $\frac{1-7i}{9}$ (b) $\frac{1+7i}{3}$

    (c) $\frac{i+7}{9}$ (d) $\frac{1+7i}{9}$

14. Let z be such that is equidistant from three distinct points z_1, z_2, z_3 in the Argand plane. If z, z_1 and z_2 are collinear, then find the $\arg((z_3-z_2)/(z_3-z_1))$, ($z_1$, z_2, z_3 are in anti-clockwise sence)?

    (a) $\pi/2$ (b) 0 (c) $-\pi/2$ (d) π

15. The complex number $z = x + iy$ which satisfy the equation $\left|\frac{z-5i}{z+5i}\right| = 1$ lie on:

    (a) the x-axis

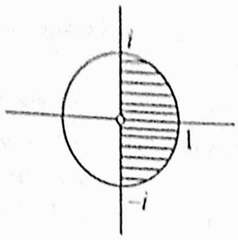
    (b) the straight line $y = 5$

    (c) a circle passing through the origin

    (d) the y-axis

16. If $|z^2-1| = |z|^2+1$, then z lies on :
(*a*) the real axis (*b*) the imaginary axis
(*c*) a circle (*d*) an ellipse

17. The locus of z which lies in shaded region is best represented by



(*a*) $|z| \le 1, \frac{-\pi}{2} \le \arg z \le \frac{\pi}{2}$

(*b*) $|z| = 1, \frac{-\pi}{2} \le \arg z \le 0$

(*c*) $|z| \ge 0, 0 \le \arg z \le \frac{\pi}{2}$

(*d*) $|z| \le 1, \frac{\pi}{2} \le \arg z \le \pi$

18. The equation $|z-1|^2+|z+1|^2=2$ represents
(*a*) a circle of radius '1' (*b*) a straight line
(*c*) the ordered pair (0, 0) (*d*) a circle of radius '$\sqrt{2}$'

19. The region of Argand diagram defined by $|z-1|+|z+1| \le 4$ is :
(*a*) interior of an ellipse
(*b*) exterior of a circle
(*c*) interior and boundary of an ellipse
(*d*) exterior of ellipse

20. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if:
(*a*) $z_1+z_4=z_2+z_3$ (*b*) $z_1+z_3=z_2+z_4$
(*c*) $z_1+z_2=z_3+z_4$ (*d*) $z_1z_3=z_2z_4$

21. Interpret the following locii in $z \in C$. $1<|z-2i|<3$
(*a*) Region between the concentric circles with centre (0, 2) and radii 1 and 3
(*b*) region outside or on the circle centre (1/2, 2) and radius = 1/2
(*c*) semicircle in I and IV quadrants, $x^2+y^2=1$
(*d*) a ray emanating from the point.

22. Interpret the following locii in $z \in C$.
Arg $(z+i)$– Arg $(z-i) = \pi/2$
(*a*) Region between the concentric circles with centre (0, 2) and radii 1 and 3
(*b*) region outside or on the circle centre (1/2, 2) and radius = 1/2
(*c*) semicircle in I and IV quadrants, $x^2+y^2=1$
(*d*) a ray emanating from the point.

23. Interpret the following locii in $z \in C$.
Arg $(z-a) = \pi/3$ where $a = 3+4i$.
(*a*) Region between the concentric circles with centre (0, 2) and radii 1 and 3
(*b*) region outside or on the circle centre (1/2, 2) and radius = 1/2
(*c*) semicircle in I and IV quadrants, $x^2+y^2=1$
(*d*) a ray emanating from the point.

24. The figure formed by four points $1+0i$; $-1+0i$; $3+4i$ and $\frac{25}{-3-4i}$ on the argand plane is:
(*a*) a parallelogram but not a rectangle
(*b*) a trapezium which is not equilateral
(*c*) a cyclic quadrilateral
(*d*) rectangle only

25. If z_1, z_2 are two non-zero complex numbers such that $\frac{z_1}{z_2}+\frac{z_2}{z_1}=1$, then z_1, z_2 and the origin are
(*a*) Collinear
(*b*) Form right angled triangle
(*c*) Form the right angle isosceles tringle
(*d*) Form an equilateral triangle

26. Let $P(e^{i\alpha/1}), Q(e^{i\alpha/2}), R(e^{i\alpha/3})$ be the vertices a of a triangle PQR in the Argand plane, the orthocenter of the triangle PQR is
(*a*) $e^{i(\alpha_1+\alpha_2+\alpha_3)}$ (*b*) $\frac{2}{3}e^{i(\alpha_1+\alpha_2+\alpha_3)}$
(*c*) $e^{i\alpha/1}+e^{i\alpha/2}e^{i\alpha/3}$ (*d*) $\frac{1}{3}e^{i(\alpha_1+\alpha_2+\alpha_3)}$

27. If $z^2+z|z|+|z^2|=0$, then the locus of z is
(*a*) a circle (*b*) a straight line
(*c*) a pair of straight lines (*d*) an ellipse

28. If $z = 3/(2+\cos\theta+i\sin\theta)$, then locus of z is
(*a*) a straight line
(*b*) a circle having centre on y-axis
(*c*) a parabola
(*d*) a circle having centre on x-axis

29. Let $z=1-t+i\sqrt{t^2+t+2}$, where t is a real parameter. The locus of z in the Argand plane is
(*a*) a hyperbola (*b*) an ellipse
(*c*) a straight line (*d*) a circle

30. If z_1 lies on $|z|=1$ and z_2 lies on $|z|=2$, then
(*a*) $3 \le |z_1-2z_2| \le 5$ (*b*) $1 \le |z_1+z_2| \ge 3$
(*c*) $|z_1-3z_2| \le 5$ (*d*) $|z_1-z_2| \ge 1$

31. The locus of the centre of a circle which touches the circles $|z-z_1| = a$, $|z-z_2| = b$ externally (z, z_1, z_2) are complex numbers) will be
(*a*) An ellipse (*b*) A straight line
(*c*) A circle (*d*) A hyperbola

32. If $|z-1|+|z+3| \le 8$, then the range of values of $|z-4|$ is
(*a*) (0, 7) (*b*) (1,8)
(*c*) [1,9] (*d*) [2, 5]

Exercise-2 (Learning Plus)

1. On the Argand plane point 'A' denotes a complex number z_1. If A triangle OBQ is made directly similar to the triangle OAM, where $OM = 1$ as shown in the figure. If the point B denotes the complex number z_2, then find the complex number corresponding to the point 'Q'?

(a) $\frac{z_2}{z_1}$ (b) $\frac{z_1}{z_2}$

(c) $\frac{z_2}{z_1}+i$ (d) $\frac{z_2}{z_1}-i$

2. Interpret the following locii in $z \in C$.

$\text{Re}\left(\frac{z+2i}{iz+2}\right) \le 4 \ (z \ne 2i)$

(a) Region between the concentric circles with centre (0, 2) and radii 1 and 3

(b) region outside or on the circle centre (1/2, 2) and radius = 1/2

(c) semicircle in I and IV quadrants, $x^2 + y^2 = 1$

(d) a ray emanating from the point.

3. If one of the vertices of the square circumscribing the circle $|z-1| = \sqrt{2}$ is $2+\sqrt{3}i$, then find the centroid of the triangle formed by other vertices ?

(a) $\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}}i$ (b) $\frac{2}{\sqrt{3}}-\frac{i}{\sqrt{3}}$

(c) $\frac{2}{3}-\frac{1}{\sqrt{3}}i$ (d) $\frac{2+i}{3}$

4. In the argand's plane, if A is the point representing z_1, B is the point representing z_2 and $z = \frac{\overrightarrow{OA}}{\overrightarrow{OB}}$ then

(a) z is purely real

(b) z is purely imaginary

(c) $|z| = 1$

(d) ΔAOB is a scalene triangle

5. If one vertex of the triangle having maximum area that can be inscribed in the circle $|z-i| = 5$ is $3-3i$, then another vertex of the triangle can be:

(a) $\frac{1}{2}[-3+4\sqrt{3}+(3\sqrt{3}+6)i]$

(b) $\frac{1}{2}[3+4\sqrt{3}-(3\sqrt{3}=-2)i]$

(c) $\frac{1}{2}[3-4\sqrt{3}+(3\sqrt{3}+4)i]$

(d) $\frac{1}{2}[3+4\sqrt{3}-(3\sqrt{3}+2)i]$

6. Let A and B be two points on the circle $|z| = r$ represented by z_1 and z_2 respectively then find the complex number representing the point of intersection of the tangents at A and B?

(a) $\frac{2z_1z_2}{z_1+z_2}$ (b) $\frac{2z_1}{z_1+z_2}$

(c) $\frac{2z_2}{z_1+z_2}$ (d) $\frac{2z_1}{z_2}$

7. If $A(z_1), B(z_2), C(z_3)$ are the vertices of an equilateral triangle ABC, then value of $\arg\left(\frac{z_3+z_2-2z_1}{z_3-z_2}\right)$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

8. Let $0 < \alpha < \frac{\pi}{2}$ be a fixed angle. If $P = (\cos\theta, \sin\theta)$ and $Q = (\cos(\alpha-\theta), \sin(\alpha-\theta))$ then Q is obtained from P by

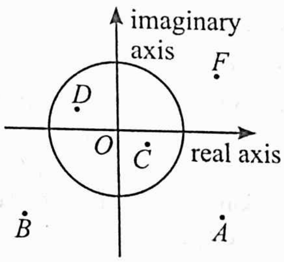
(a) clockwise rotation around origin through angle a

(b) anticlockwise rotation around origin through an angle a

(c) reflection in the line through origin with slope tanq

(d) reflection in the line through origin with slope $\tan(\alpha/2)$

9. The digram shows several numbers in the complex plane. The circle is the unit circle centered at the origin. Then which of the points may be the reciprocal of F?



(a) (0,0) (b) $\left(-\frac{1}{3},0\right)$

(c) $\left(\frac{1}{3},0\right)$ (d) $\left(0,\frac{2}{\sqrt{5}}\right)$

10. Find the equation of the radical axis of the two circles represented by $|z-2| = 3$ and $|z-2-4i| = 4$ on the complex plan

(a) $3y-1=0$ (b) $3y+1=0$

(c) $3x-1=0$ (d) $3x+1=0$

11. All complex numbers 'z' which satisfy the relation $|z-|z+1|| = |z+|z-1||$ on the complex plane lie on the

(*a*) line $y=0$

(*b*) line $x=0$

(*c*) circle $x^2+y^2=1$

(*d*) line $x=0$ or on a line segment joining $(-1,0)$ to $(1,0)$

12. The points $z_1 = 3+\sqrt{3}i$ and $z_2 = 2\sqrt{3}+6i$ are given on a complex plane. The complex number lying on the bisector of the angle formed by the vectors z_1 and z_2 is:

(*a*) $z = \frac{(3+2\sqrt{3})}{2} + \frac{\sqrt{3}+2}{2}i$

(*b*) $z = 5+5i$

(*c*) $z = -1-i$

(*d*) $z = -1+i$

13. Let z_1 and z_2 be roots of the equation $z^2+pz+q=0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, then

(*a*) $|p|^2 = 4|q| \cos^2 \alpha/2$

(*b*) $|p|^2 = 4|q| \sin^2 \alpha/2$

(*c*) $|q|^2 = 4|p| \cos^2 \alpha/2$

(*d*) $|q|^2 = 4|p| \sin^2 \alpha/2$

14. A rectangle of maximum area is inscribed in the circle $|z-3-4i| = 1$. If one vertex of the rectangle is $4+4i$, then find the another adjacent vertex of this rectangle?

(*a*) $3-5i$ or $3-3i$ (*b*) $3-5i$ or $3+3i$

(*c*) $3+5i$ or $3-3i$ (*d*) $3+5i$ or $3+3i$

15. Locus of z if $\arg[z-(1+i)] = \begin{cases} \frac{3\pi}{4} & \text{when } |z| \le |z-2| \\ \frac{-\pi}{4} & \text{when } |z| > |z-2| \end{cases}$ is

(*a*) Straight lines passing through (2, 0)

(*b*) Straight lines passing through (2, 0), (1, 1)

(*c*) a line segment

(*d*) a set of two rays

16. The complex number associated with the vertices A, B, C of ΔABC are $e^{i\theta}$, ω, $\overline{\omega}$ respectively [where ω, $\overline{\omega}$ are the complex cube of unity and $\cos\theta > \text{Re}(\omega)$], then the complex number of the point where angle bisector of A meets the circumcircle of the triangle, is

(*a*) $e^{i\theta}$ (*b*) $e^{-i\theta}$

(*c*) $\omega\overline{\omega}$ (*d*) $\omega+\overline{\omega}$

17. Consider a complex number $w = \frac{z-i}{2z+1}$ where $z = x+iy$, where $x, y \in R$. If the complex number w is purely imaginary then locus of z is

(*a*) a straight line

(*b*) a circle with centre $\left(-\frac{1}{4}, \frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{4}$.

(*c*) a circle with centre $\left(\frac{1}{4}, -\frac{1}{2}\right)$ and passing through origin.

(*d*) neither a circle nor a straight line.

18. If the equation $z^3+(3+i)z^2-3z-(m+i)=0$, where $m \in R$ has at least one real root, then m can have the value equal to

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4

19. If $z_1, z_2, \ldots, z_{100}$ are root of equation $1+z+z^2+\ldots+z^{100} = 0$, then the value of $\sum_{r=1}^{100} \frac{1}{z_r-1}$ is equal to

(*a*) 100 (*b*) -50 (*c*) -25 (*d*) 150

20. Let W_1 and W_2 are two distinct points in an argand plane. If $p|W_1| = q|W_2|$, then the point $\frac{pW_1}{qW_2} + \frac{qW_2}{pW_1}$ is a point on the

(*a*) Line segment $[-2, 2]$ of the real axis

(*b*) Line segment $[-2, 2]$ of the imaginary axis

(*c*) Unit circle $|z| = 1$

(*d*) The line $\arg W = \tan^{-1}2$

21. If the lines $\alpha_1\overline{z}+\overline{\alpha}_1 z+\beta_1 = 0$ and $\alpha_2\overline{z}+\overline{\alpha}_2 z+\beta_2 = 0$, $\beta_1, \beta_2 \in R$ are mutually perpendicular, then

(*a*) $\alpha_1\overline{\alpha}_2+\overline{\alpha}_1\alpha = 0$ (*b*) $\alpha_1\overline{\alpha}_2 = \overline{\alpha}_1\alpha$

(*c*) $\left|\arg\left(\frac{\alpha_1}{\alpha_2}\right)\right| = \frac{\pi}{2}$ (*d*) $\arg\left(\frac{\alpha_1}{\alpha_2}\right) = 0$ or π

22. Let $A(z_1)$, $B(z_2)$, $C(z_3)$, are Point in complex plane such that $z_1|z_2-z_2| - z_2|z_3-z_3| - z_3|z_1-z_2| = 0$ then which of the following may be correct?

(*a*) A, B, C are collinear such that A. lies between B and C

(*b*) A, B, C are collinear such that B lies between A and C

(*c*) $A\ B\ C$ are collinear such that C lies between A and B

(*d*) $0(0)$ is the centre of circle which touches the sides of triangle ABC.

23. $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are four complex numbers representing the vertices of a rhombus which is not a square taken in order on the complex plane, then which one of the following hold(s) good?

(*a*) $\frac{z_1-z_4}{z_2-z_3}$ is purely real

(*b*) $\text{amp}\frac{z_1-z_4}{z_2-z_4} \neq \text{amp}\frac{z_2-z_4}{z_3-z_4}$

(*c*) $\frac{z_1-z_3}{z_2-z_4}$ is purely imaginary

(*d*) $|z_1-z_3| \neq |z_2-z_4|$

24. Equation of a straight line on the complex plane passing through a point P denoting the complex number α and perpendicular to the vector $\overline{OP}$ where 'O' in the origin ?

(a) $\text{Im}\left(\frac{z-\alpha}{\alpha}\right)=0$

(b) $\text{Re}\left(\frac{z-\alpha}{\alpha}\right)=0$

(c) $\text{Re}(\bar{\alpha}z)=0$

(d) $\bar{\alpha}z+\alpha z-2|\alpha|^2=0$

25. The points z_1, z_2, z_3 on the complex plane are the vertices of an equilateral triangle if and only if :

(a) $\Sigma(z_1-z_2)(z_2-z_3)=0$

(b) $z_1^2+z_2^2+z_3^2=2(z_1z_2+z_2z_3+z_3z_1)$

(c) $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

(d) $2(z_1^2+z_2^2+z_3^2)=z_1z_2+z_2z_3+z_3z_1$

26. If from a point P representing the complex number z_1 on the curve $|z|=2$, pair of tangents are drawn to the curve $|z|=1$, meeting at points $Q(z_2)$ and $R(z_3)$, then

(a) The complex number $\frac{z_1+z_2+z_3}{3}$ will lie on the curve $|z|=1$

(b) $\left(\frac{4}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}\right)\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)=9$

(c) $\arg\left(\frac{z_2}{z_3}\right)=\frac{2\pi}{3}$

(d) Orthocentre and circumcentre of ΔPQE will coincide

27. Let A and B be two distinct point denoting the complex number α and β respectively. A complex number z lies between A and B where $z\neq\alpha$, $z\neq\beta$. Which of the following relation(s) hold good?

(a) $|\alpha-z|+|z-\beta|=|\alpha-\beta|$

(b) $\exists$ a positive real number 't' such that $z=(1-t)\alpha+\beta$

(c) $\begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix}=0$

(d) $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix}=0$

28. One vertex of the triangle of maximum area that can be inscribed in the circle $|z-(1+i)|=2\sqrt{2}$ is $-2+2i$, remaining vertices are

(a) $\left(\frac{5-\sqrt{3}}{2}\right)+i\left(\frac{1+3\sqrt{3}}{2}\right)$

(b) $\left(\frac{5-\sqrt{3}}{2}\right)+i\left(\frac{1-3\sqrt{3}}{2}\right)$

(c) $\left(\frac{5+\sqrt{3}}{2}\right)+i\left(\frac{1-3\sqrt{3}}{2}\right)$

(d) $\left(\frac{5+\sqrt{3}}{2}\right)+i\left(\frac{1+3\sqrt{3}}{2}\right)$

29. Let z_1, z_2 be two complex numbers represented by points on the circle $|z|=1$ and $|z|=2$ respectively then

(a) maximum value of $|2z_1+z_2|$ is 4

(b) maximum value of $|z_1-z_2|$ is 1

(c) minimum value of $\left|z_2+\frac{1}{z_1}\right|$ is 3

(d) maximum value of $|z_1-z_2|$ is 2

30. Let z_1, z_2, z_3 be three distinct complex numbers satisfying, $|z_1-1|=|z_2-1|=|z_3-1|=1$. Let A, B & C be the points representing vertices of equilateral triangle in the Argand plane corresponding to z_1, z_2 and z_3 respectively. Which of the following are true

(a) $z_1+z_2+z_3=3$

(b) $z_1^2+z_2^2+z_3^2=3$

(c) area of triangle $=\frac{3\sqrt{3}}{4}$

(d) $z_1z_2+z_2z_3+z_3z_1=1$

31. Let z_1, z_2, z_3 be three distinct non-zero complex number, then.

(a) there always exist real numbers p, q, r such that $pz_1+qz_2+rz_3=0$

(b) If $pz_1+pz_2+rz_3=0$ and $p+q+r=0$ $p, q, r \in R$, then z_1, z_2, z_3 are collinear.

(c) There always exist complex numbers p, q, r such that $pz_1+qz_2+rz_3=0$ and $p+q+r=0$

(d) there always exist real numbers p, q, r such that $pz_1+qz_2+rz_3=0$ and $p+q+r=0$

32. Consider a curve 'C' given by equation $|z-2-2i|=\left|z\cos\left(\frac{\pi}{4}-\arg z\right)\right|$, then

(a) represents a parabola

(b) 'C' represents a ellipse

(c) point $P\left(\frac{7+3i}{2}\right)$ lies on 'C'

(d) Point $P\left(\frac{7+3i}{4}\right)$ lies on 'C'

33. If $|z_1|=|z_2|=|z_3|=1$ and z_1, z_2, z_3 are represented by the vertices of an equilateral triangle then

(a) $z_1+z_2+z_3=0$

(b) $z_1z_2z_3=1$

(c) $z_1z_2+z_2z_3+z_3z_1=0$

(d) $z_2^3+z_3^3=2z_1^3$

Comprehension (Q. 34 to 35) : Consider a complex number $w = \frac{z-i}{2z+1}$ where $z = x + iy$, where $x, y \in R$.

34. If the complex number w is purely real then locus of z is

(*a*) a straight line passing through origin

(*b*) a straight line with gradient 3 and y intercept (-1)

(*c*) a straight line with gradient 2 and y intercept 1.

(*d*) a straight line with gradient 1 and y intercept 2

35. If $|w| = 1$ then the locus of P is

(*a*) a point circle (*b*) an imaginary circle

(*c*) a real circle (*d*) not a circle.

36. If z_r (r = 1, 2,...6) are the vertices of a regular hexagon, if $\sum_{r=1}^{6} z_r^2 = \beta z_0^2$, where z_0 is the circumcentre. then β

37. Let A, B and C represent the complex numbers z_1, z_2, z_3 respectively on the complex plane. If the circumcentre of the triangle ABC lies at the origin, then the orthocentre of triangle ABC is $az_1 + bz_2 + cz_3$. find $a + b + c$

38. If z_1, z_2, z_3, z_4 be the vertices A, B, C, D respectively of a square on the Argand diagram taken in anticlockwise direction then find $a+b$ if

$az_2 = (1+i)\, z_1 + (1-i)\, z_3$ and $bz_4 = (1-i)\, z_1 + (1+i)\, z_3$

39. If tangents drawn to circle $|z| = 2$ at $A(z_1)$ and $B(z_2)$ meet at $P(z_p)$. if $z_p = \frac{k z_1 z_2}{z_1 + z_2}$. then $k^k + i$

40. If A and B represent the complex numbers z_1 and z_2 such that $|z_1+z_2|=|z_1-z_2|$, the circumcentre of the ΔOAB is $\frac{z_1+z_2}{\alpha-5}$, where O is the origin. then α is

41. If perimeter of locus represented by $\arg\left(\frac{z+i}{z-i}\right) = \frac{\pi}{4}$ (wher $i = \sqrt{-1}$) is k, then find the value of $\frac{2k^2}{\pi^2}$

42. If $|z| = \min(|z-1|, |z+1|)$ then find the value of $|z + \bar{z}|$.

43. If $|z-i| \le 2$ and $z_1 = 5+3i$, (where $i = \sqrt{-1}$ then the maximum value of $|iz + z_1|$ is

44. The number of points in the complex plane that satisfying the conditions $|z-2| = 2$, $z(1-i) + \bar{z}(1+i) = 4$, (where $i = \sqrt{-1}$) is

45. If z_1, z_2, z_3 are the vertices of the ΔABC on the complex plane which are also the roots of the equation $z^3 - 3\alpha z^2 + 3\beta z + x = 0$, then prove that $\alpha^2 = \beta$ is the condition for the ΔABC to be equilateral triangle.

46. If z_1, z_2, z_3 be vertices of an isosceles Δ right angled at z_2, then prove that $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.

47. If z_1, z_2, z_3 be three distinct complex numbers satisfying $|z_1-1| = |z_2-1| = |z_3-1|$. Let A,B and C be the points represented in the Argand plane correspodngin to z_1, z_2 and z_3 respectively. Prove that $z_1+z_2+z_3 = 3$ if and only if ΔABC is an equilateral triangle.

48. Let the complex number z_1, z_2, z_3 represents vertices of an equilateral triangle. If z_0 be the circumcentre of the triangle, then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

49. Prove that all roots of the equation $\left(\frac{z+1}{z}\right)^n = 1$ are collinear on the complex plane.

50. If area of the region on complex plane given by complex number z, such that $\frac{\pi}{4} \le \arg\left(\frac{z-2}{z+2}\right) \le \frac{3\pi}{4}$ is $a\pi + b$ then $(b-a)$ equals

51. The number of common roots (complex or real both) in the equation $z^{12} - 1 = 0$, $z^{18} - 1 = 0$ and $z^{24} - 1 = 0$ is

52. If $\alpha = e^{i\left(\frac{2\pi}{7}\right)}$, then form a quadratic equation whose roots are A & B where $A = \alpha + \alpha^2 + \alpha^4$ & $B = \alpha^3 + \alpha^5 + \alpha^6$

53. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. Then prove that all the complex number satisfying $\arg\left(\frac{z-z_1}{z-z_2}\right) = \frac{\pi}{4}$ is a circle and find its radius and center

54. Let z_1, z_2 and z_3 represent the vertices A, B and C of the triangle ABC, respectively, in the Argand plane, such that $|z_1| = |z_2| = |z_3| = 5$. Prove that $z_1 \sin 2A + z_3 \sin 2B + z_3 \sin 2C = 0$.

Exercise-3 (JEE Advanced Level)

MULTIPLE CORRECT TYPE QUESTIONS

1. Which of the following represents a point on an argands' plane, equidistant from the roots of the equation $(z+1)^4 = 16z^4$?

(a) $(0,0)$ (b) $\left(-\frac{1}{3},0\right)$

(c) $\left(\frac{1}{3},0\right)$ (d) $\left(0,\frac{2}{\sqrt{5}}\right)$

2. If z_1 and z_2 be non zero complex numbers satisfying the equation $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$, then find the geometrical nature of the triangle whose vertices are the origin and the points representing z_1 and z_2

(a) an isosceles right angled triangle
(b) equilateral triangle
(c) non- isosceles right angled triangle
(d) scalan triangle

3. Let P denotes a complex number z on the Argand's plane, and Q denotes a complex number $\sqrt{2|z|^2}\ CiS\left(\frac{\pi}{4}+\theta\right)$ where θ = amp z. If 'O' is the origin, then prove that the ΔOPQ is right isosceles.

(a) right angled isosceles triangle
(b) equilateral triangle
(c) non- isosceles right angled triangle
(d) scalan triangle

4. If $z_1 = a + ib$ and $z_2 = c + id$ two complex numbers lying on the circle $x^2 + y^2 = 1$ in the Argand diagram and $\text{Re}(z_1\bar{z}_2) = 0$ then the complex numbers $\omega_1 = (a + ic)$ and $\omega_2 = b + id$ are such that

(a) Only ω_1 lies on the circle $x^2 + y^2 = 1$
(b) Only ω_2 lies on the circle $x^2 + y^2 = 1$
(c) $\text{Re}(\omega_1\bar{\omega}_2) = 0$
(d) $\text{Im}(\omega_1\bar{\omega}_2) = 0$

5. Let a be a complex number such that $|a| < 1$ and z_1, z_2, z_3, ... be the vertices of polygon such that $z_k = 1 + a + a^2 + \ldots.. + a^{k-1}$ for all $k = 1, 2, 3, \ldots..$ then $z_1, z_2, \ldots.$ lie within the circle

(a) $\left|z-\frac{1}{1-a}\right| = \frac{1}{|a-1|}$ (b) $\left|z+\frac{1}{a+1}\right| = \frac{1}{|a+1|}$

(c) $\left|z-\frac{1}{1-a}\right| = |a-1|$ (d) $\left|z+\frac{1}{a+1}\right| = |a+1|$

6. Let C_1 and C_2 are concentric circles of radius 1 and 8/3 respectively having centre at (3,0) on the argand plane. If the complex number z satisfies the inequality, $\log_{1/3}\left(\frac{|z-3|^2+2}{11|z-3|-2}\right) > 1$ then:

(a) z lies outside C_1 but inside C_2
(b) z lies inside of both C_1 and C_2
(c) z lies outside both of C_1 and C_2
(d) none of these.

7. z_1, z_2 and z_3 are three points on a circle centred at origin. A point z is chosen on the circle such that the line joining z and z_1 is perpendicular to the line joining z_2 and z_3. Which of the following is true?

(a) $zz_1 + z_2z_3 = 0$ (b) $z^2 - z_1^2 + z_2z_3 = 0$

(c) $z^2 + z_1^2 + z_2z_3 = 0$ (d) $zz_1 - z_2z_3 = 0$

8. Let z_1, and z_2 be roots of the equation $z^2 + 8(i-1)z + 63 - 16i = 0$ where $i^2 = -1$. The area of triangle formed by o, z_1 and z_2 (where o being origin) is equal to

(a) 24 (b) 26
(c) 28 (d) 30

9. The reflection of the complex number $\frac{2+i}{3-i}$ in the straight line $z(1+i) = \bar{z}(1-i)$ is

(a) $\frac{2+i}{3+i}$ (b) $\frac{-2+i}{-3+i}$

(c) $\frac{2-i}{3-i}$ (d) $\frac{2+i}{-3+i}$

10. Let points $A(z_1)$, $B(z_2)$, $C(z_3)$ are 3 distinct collinear points such that B lies between A & C, then

(a) $\frac{|z_3|-|z_2|}{|z_3-z_2|} \le \frac{|z_3|-|z_2|}{|z_3-z_1|}$

(b) $\frac{|z_3|-|z_2|}{|z_3-z_2|} \ge \frac{|z_3|-|z_2|}{|z_3-z_1|}$

(c) $\frac{|z_3|-|z_1|}{|z_3-z_1|} \ge \frac{|z_3|-|z_2|}{|z_2-z_1|}$

(d) $\frac{|z_3|-|z_1|}{|z_3-z_1|} \le \frac{|z_2|-|z_1|}{|z_2-z_1|}$

11. Each of the circles $|z - 1 - i|$ and $|z - 1 + i| = 1$ touches internally a circle of radius 2. The complex equation of the circle touching all the three circles can be

(a) $3z\bar{z} + z + \bar{z} - 1 = 0$
(b) $3z\bar{z} - 7(z + \bar{z}) + 15 = 0$
(c) $\bar{z}\bar{z} - z - \bar{z} - 3 = 0$
(d) $3z\bar{z} + i(z + \bar{z}) - 1 = 0$

12. Suppose that the complex number z lies on the curve such that $\frac{z-4}{z-2i}$ is purely imaginary. If the complex number z_1 represents the mid-point of chord OA of this curve, O being the origin, then z_1 necessarily satisfy

(a) $\frac{z_1-2}{z_1-i} = ik, k \in R-\{0\}$

(b) $\frac{z_1}{z_1-2-i} = ik, k \in R-\{0\}$

(c) $\frac{z_1-2}{2z_1-i} = k, k \in R-\{0\}$

(d) $|z_1| = \frac{\sqrt{5}}{2}$

13. Let. $\ell(z) = Az + B$, A, $B \in$ complex number. It is given that maximum value of $|\ell(z)|$ on the segment, $-1 \leq x \leq 1$, $y = 0$ of the real line in the complex plane ($z = x + iy$) is M Then, for every $|\ell(z)| < \lambda KM$ (K is the sum of the distances from point $P(z)$ to the points $Q_1(1, 0)$ and $Q_2(-1\ 0)$ where λ. can be

(a) $\frac{1}{4}$ (b) 1

(c) 2 (d) 3

14. Suppose $A(z_1)$, $B(z_2)$ and $C(z_3)$ are vertices of a triangle lying on the unit circle $|z| = 1$. AD is altitude of the ΔABC meeting the unit circle in E.

(a) orthocentre of ΔABC is $z_1 + z_2 + z_3$

(b) affix of E is $-z_2z_3/z_1$

(c) If $z_1^2 = z_2z_3$ and $z_2^2 = z_3z_1$. then ΔABC is equilateral.

(d) If $z_2 + z_3 = 0$, then ΔABC is a right angled.

15. If $\left|\frac{z-\alpha}{z-\beta}\right| = k, k > 0$ where, $z = x + iy$ and $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ are fixed complex numbers. Then which of the following are true

(a) If $k \neq 1$ then locus is a circle whose centre is $\left(\frac{k^2\beta - \alpha}{k^2 - 1}\right)$

(b) If $k \neq 1$ then locus is a circle whose radius is $\left|\frac{k(\alpha - \beta)}{1 - k^2}\right|$

(c) If $k = 1$ then locus is perpendicular bisector of line joining $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$

(d) If $k \neq 1$ then locus is a circle whose centre is $\left(\frac{k^2\alpha - \beta}{k^2 - 1}\right)$

16. Let a, b, c be distinct complex numbers with $|a| = |b| = |c| = 1$ and z_1, z_2 be the roots of the equation $az^2 + bz + c = 0$ with $|z| = 1$. Also let P and Q represents the complex numbers z_1 and z_2 in the complex plane with $\angle POQ = \theta$ where O is the origin then

(a) $b^2 = ac, \theta = \frac{2\pi}{3}$ (b) $\theta = \frac{2\pi}{3}, PQ = \sqrt{3}$

(c) $b^2 = ac, PQ = 2\sqrt{3}$ (d) $2b^2 = ac, \theta = \frac{\pi}{3}$

17. Let $P_k(k = 1, 2, \ldots, n)$ be teh nth root of unit. Let $z = a + ib$ and $A_k = \text{Re}(z)\,\text{Re}(P_k) + i\{\text{Im}(z)\,\text{Im}(P_k)\}$, then which of the following is true

(a) A_k lies on ellipse

(b) A_k lies on hyperbola

(c) If S be the focus of locus of A_k then $\sum_{k=1}^{n} A_kS = na$

(d) If S be the focus of locus of A_k then

$$n\sum_{k=1}^{n}(A_kS)^2 = \frac{n}{2}(3a^2 - b^2)$$

COMPREHENSION BASED QUESTIONS

Comprehension-1 (Q. 18 to 20): Suppose A,B,C are three collinear points corresponding complex numbers $z_1 = z_2 = \frac{1}{2} + bi$, $z_3 = 1 + ci$ (a,b,c being real number), respectively. Consider a curve 'C' whose equation is given by $z = z_1 \cos^4 t + 2\, z_2 \cos^2 t \sin^2 t + z_3 \sin^4 t$, $t \in R$.

18. 'C' in argand plane represents

(a) Straight line (b) Circle

(c) Parabola (d) Ellipse

19. A line bisecting AB and parallel to AC meet 'C' in point P, then point P is given by

(a) $\left(-\frac{1}{2}, \frac{a+c-2b}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{a+c-2b}{4}\right)$

(c) $\left(\frac{1}{2}, \frac{a+c+2b}{2}\right)$ (d) $\left(\frac{1}{2}, \frac{a+c+2b}{4}\right)$

20. Point P lies

(a) Inside ΔABC (b) Outside ΔABC

(c) On the side of ΔABC (d) Nothing can be said

Comprehension-2 (Q. 21 to 24): A regular heptagon (seven sides) is inscribed in a circle of radius 1. Let $A_1A_2 \ldots A_7$ be its vertices, G_1 is centroid of $\Delta A_1A_2A_5$ and G_2 be centroid of $\Delta A_3A_6A_7$. P is centroid of ΔOG_1G_2, where O is centre of circum suribing circle.

21. $\angle POA_1$ is equal to

(a) $\frac{\pi}{7}$ (b) $\frac{2\pi}{7}$

(c) $\frac{5\pi}{7}$ (d) $\frac{6\pi}{7}$

22. OP is equal to

(a) $\frac{10}{9}$ (b) $\frac{8}{9}$ (c) $\frac{1}{9}$ (d) 1

23. G_3 lies on segment OA_4 such that centroid of triangle G_1, G_2, G_3 is O, then

(a) $3OG_3 = OA_4$ (b) $3OG_2 = A_4G_3$

(c) $2OG_3 = OA_4$ (d) $OG_3 = G_3A_4$

24. PA_1 is equal to

(a) $\frac{1}{9}\sqrt{\left(82 - 18\cos\frac{\pi}{7}\right)}$ (b) $\frac{1}{9}\sqrt{\left(82 + 18\cos\frac{\pi}{7}\right)}$

(c) $\frac{1}{9}\sqrt{\left(82 - 18\sin\frac{\pi}{7}\right)}$ (d) $\frac{1}{9}\sqrt{\left(82 + 18\sin\frac{\pi}{7}\right)}$

Comprehension-3 (Q. 25 to 26): In an argand plane z_1, z_2 and z_3 are rescepectively the vertices of an isosceles triangle ABC with $AC = BC$ and $\angle CAB = \theta$. If z_4 is incentre of triangle. Then

25. The value of $\left(\frac{AB}{IA}\right)^2\left(\frac{AC}{AB}\right)$

(a) $\frac{(z_2 - z_1)(z_1 - z_3)}{(z_4 - z_1)^2}$ (b) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)}$

(c) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$ (d) None of these

26. The value of $(z_4 - z_1)^2 (1 + \cos\theta)\sec\theta$ is

(a) $(z_2 - z_1)(z_3 - z_1)$ (b) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)}$

(c) $\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2}$ (d) $(z_2 - z_1)(z_3 - z_1)^2$

Comprehension-4 (Q. 27 to 29): Read the following write up carefully answer the following questions: The complex slope of a line passing through two points represented by complex numbers z_1 and z_2 is defined by $\frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1}$ and we shall denote by ω. If z_0 is complex number and c is a real number, then $\bar{z}_0 z + z_0\bar{z} + c = 0$ represents a straight line. Its complex slope is $-\frac{z_0}{\bar{z}_0}$.

Now consider two lines $\alpha\bar{z} + \bar{\alpha}z + i\beta = 0$ ...(i)

and $a\bar{z} + \bar{a}z + b = 0$ ...(ii)

where α, β and a, b are complex constants and let their complex slopes be denoted by ω_1 and ω_2 respectively.

27. If the lines are inclined at an angle of 120° to each other then

(a) $\omega_2\bar{\omega}_1 = \omega_1\bar{\omega}_1$ (b) $\omega_2\bar{\omega}_1^2 = \omega_1\bar{\omega}_2^2$

(c) $\omega_1^2 = \omega_2^2$ (d) $\omega_1 + 2\omega_2 = 0$

28. Which of the following must be true?

(a) a must be purely imaginary

(b) β must be purely imaginary

(c) a must be real

(d) b must be imaginary

29. If line (i) makes an angle of 45° with real axis, then $(1+i)\left(-\frac{2\alpha}{\bar{\alpha}}\right)$ is

(a) $2\sqrt{2}$ (b) $2\sqrt{2}i$

(c) $2(1 - i)$ (d) $-2(1 + i)$

MATCH THE COLUMN TYPE QUESTIONS

Table (Q. 30 to 32):

	Column-I		Column-II		Column-III
I.	If $S = \{Z : \lvert Z + \bar{Z}\rvert + 2\lvert Z - \bar{Z}\rvert = 4$ and $\lvert Z\rvert$ is minimum$\}$ then A is (where A is area of polygon formed of all points in S taking as vertices)	(i)	If the Point ($\sec\alpha$, $\operatorname{cosec}\alpha$) moves in the plane of circle $x^2 + y^2 = 3$ and the minimum distance of this point from circle is $a - \sqrt{b}\,(a, b \in N)$ then $a + b$	P.	2
II.	Let $S = \{Z \mid Z\bar{Z} - (3-4i)Z - (3+4i)\bar{Z} + 21 = 0\mid\}$ If M and m be maximum value and minimum value of $\frac{Z - \bar{Z}}{i(Z + \bar{Z})}$ then $\frac{1}{M} + \frac{1}{m}$ is	(ii)	To circles $x^2 + y^2 + 2n_1x + 2y + \frac{1}{2} = 0$ and $x^2 + y^2 + n_2x + n_2y + n_1 = \frac{1}{2}$ intersect each other orthogonally where n_1, n_2 are integers then the number of possible ordered pairs (n_1, n_2) is	Q.	3
III.	Let x is the minimum value of $\lvert Z\rvert^2 + \lvert Z - 3\rvert^2 + \lvert Z - 6i\rvert^2$ then $\frac{x}{10}$ is	(iii)	If $a_n = \sqrt{1 + \left(1 + \frac{1}{n}\right)^2} + \sqrt{1 + \left(1 - \frac{1}{n}\right)^2}$, then the value of $\left(\sum_{n=1}^{20}\frac{1}{a_n}\right) - 3$ is	R.	4

IV.	Consider a triangle formed by the points $A\left(\frac{2}{\sqrt{3}}e^{i(\pi/2)}\right), B\left(\frac{2}{\sqrt{3}}e^{-i\left(\frac{5\pi}{6}\right)}\right), \left(\frac{2}{\sqrt{3}}e^{-i\left(\frac{5\pi}{6}\right)}\right)$ Let $P(Z)$ is any point on it's in-circle, then $AP^2 + BP^2 + CP^2$ is	(iv)	The eq^n $9x^3 + 9x^2y - 45x^2 = 4y^3 + 4xy^2 - 20y^2$ represents 3 straight lines two of which pass through origin then $\frac{1}{10}$ (Area of triangle formed by these lines)	S.	5

30. Which of the following options is the only **CORRECT** combination?

(*a*) (I)(ii)(R) (*b*) (II)(iv)(S)

(*c*) (III)(i)(Q) (*d*) (IV)(iv)(S)

31. Which of the following options is the only **CORRECT** combination?

(*a*) (II)(i)(P) (*b*) (I)(iii)(R)

(*c*) (II)(ii)(P) (*d*) (IV)(iii)(Q)

32. Which of the following options is the only **CORRECT** combination?

(*a*) (II)(i)(Q) (*b*) (I)(iii)(P)

(*c*) (II)(ii)(R) (*d*) (iv)(i)(S)

33. Match the Column

	Column-I		Column-II
A.	If $\text{Re}\left(\frac{iz+1}{iz-1}\right) = 2$, then z lies on the curve	p.	$4x^2+4y^2+x-6y+2=0$
B.	$z_1 = 6+i, z_2 = 4-3i$ and z is a complex number such that $\arg\left(\frac{z-z_1}{z_2-z}\right) = \frac{\pi}{2}$, then z lies on	q.	$x^2+y^2+4y+3=0$
C.	If $\text{Im}\left(\frac{2z+1}{1+iz}\right) = 2$, then z lies on	r.	$3(x^2+y^2)-2x-4y=0$
D.	If = 1, then z lies on	s.	$x^2+y^2-x+2y-1=0$
		t.	$(x-5)^2+(y+1)^2=5$

(*a*) A → r; B → p; C → t; D → q

(*b*) A → q; B → r; C → p; D → t

(*c*) A → q; B → t; C → p; D → r

(*d*) A → t; B → q; C → r; D → p

34. If $z \in C$, then the locus of z on an Argand diagram is

	Column-I		Column-II
A.	$\lvert z-2-i\rvert = \lvert z\rvert \left\lvert \sin\left(\frac{\pi}{4} - \arg z\right)\right\rvert$	p.	a pair of straight lines
B.	$(z-3+i)(\bar{z}-3-i) = 5$	q.	circle
C.	$3\lvert z-2+i\rvert = 7$	r.	parabola
D.	$\lvert z-3\rvert = 2$	s.	ellipse

(*a*) A → r; B → p; C → t; D → s

(*b*) A → r; B → q; C → q; D → q

(*c*) A → q; B → t; C → p; D → r

(*d*) A → s; B → q; C → r; D → p

35. Match the statements in Column-I with those in Column-II
[**Note:** Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.]

	Column I		Column II
A.	The set of points z satisfying $\lvert z - i\lvert z\rvert\rvert = \lvert z + i\lvert z\rvert\rvert$ is contained in or equal to	p.	An ellipse with eccentricity $\frac{4}{5}$
B.	The set of points z satisfying $\lvert z+4\rvert + \lvert z-4\rvert = 10$ is contained in or equal to	q.	The set of points z satisfying Im $z = 0$
C.	If $\lvert w\rvert = 2$, then the set of points $z = w - \frac{1}{w}$ is contained in or equal to	r.	The set of points z satisfying $\lvert \text{Re } z\rvert \le 1$
D.	If $\lvert w\rvert = 1$, then the set of points $z = w + \frac{1}{w}$ is contained in or equal to	s.	The set of points z satisfying $\lvert\text{Re } z\rvert \le 2$
		t.	The set of points z satisfying $\lvert z\rvert \le 3$

(*a*) A → s,q; B → p; C → r,t; D → s

(*b*) A → r; B → q; C → q,s; D → q

(*c*) A → q; B → t,q; C → p, D → r

(*d*) A → q; B → p; C → p,t; D → s,q

36. Match the columns:

	Column-I		Column-II
A.	The least value of $8\lvert z-7\rvert + 6\lvert z-5\rvert$; $z \in C$	p.	12
B.	The least value of $\left\lvert\frac{z-7+11i}{\sqrt{2}}\right\rvert + \left\lvert\frac{z+5-i}{\sqrt{2}}\right\rvert + \left\lvert\frac{z+4}{\sqrt{2}}\right\rvert$; $z \in C$ is	q.	13
C.	Find the least value of $\lvert z-3-4i\rvert^2 + \lvert z-i\rvert^2 + \lvert z\rvert^2 + \lvert z-1\rvert^2$; $z \in C$	r.	21

D.	Find the least value of $\lvert z-3\rvert^2+\lvert z-5+2i\rvert^2+\lvert z-1+i\rvert^2$ is	s.	10
		t.	$5+\sqrt{2}$

(a) A → p; B → p; C → r; D → q
(b) A → r; B → q; C → q; D → q
(c) A → q; B → p; C → p; D → r
(d) A → s; B → q; C → r; D → p

NUMERICAL BASED QUESTION

37. Let $z_1, z_2, \ldots, z_n$ be in G.P with first term as unity such that $z_1 + z_2 + \ldots + z_n = 0$. Now if $z_1, z_2, \ldots, z_n$ represent vertices of a polygon, then the distance between the incentre and circumcentre of the polygon is

38. The points A, B, C represent the complex numbers z_1, z_2, z_3 respectively on a complex plane and the angle B and C of the triangle ABC are each equal to $1/2(\pi-\alpha)$.

if $(z_2-z_3)^2 = k(z_3-z_1)(z_1-z_2)\sin^2\frac{\alpha}{2}$. then value of k is

39. If z_1, z_2, z_3 be vertices of an isosceles Δ right angled at z_3, then show that $(z_1-z_2)^\alpha = \alpha(z_1-z_3)(z_3-z_2)$. then value of α

40. If z is a complex number and the minimum value of $|z|+|z-1|+|2z-3|$ is λ and if $y = 2[x]+3 = 3[x-\lambda]$, then find the value of $[x+y]$ (where [.] denotes the greatest integer function)

41. Assume that $A_i (i = 1, 2, \ldots, n)$ are the vertices of a regular polygon inscribed in a circle of radius unity. Find
(i) $|A_1A_2|^2 + |A_1A_3|^2 + \ldots + |A_1A_n|^2$
(ii) $|A_1A_2||A_1A_3|\ldots|A_1A_n|$

42. A, B, C are the points representing the complex numbers z_1, z_2, z_3 respectively and the circumcentre of the triangle ABC lies at the origin. If the altitudes of the triangle through the opposite vertices meets the circumcircle at D, E, F respectively. Find the complex numbers corresponding to the points D, E, F in terms of z_1, z_2, z_3.

43. $\omega_1, \omega_2, \ldots, \omega_n$ be complex numbers. A line L in the argand plane is called a mean line for the points $\omega_1, \omega_2, \ldots, \omega_n$ if L contains points (complex numbers) $z_1, z_2, \ldots, z_n$ such that $\sum_{i=1}^{n}(z_i - \omega_i) = 0$. For the numbers $\omega_1 = 32 + 170i$, $\omega_2 = -7 + 64i$, $\omega_3 = -9 + 200i$, $\omega_4 = 1 + 27i$ and $\omega_5 = -14 + 43i$, there is a unique mean line with y-intercept 3. Then the slope of this mean line is equal to

44. A particle starts to travel from a point P on the curve C_1: $|z-3-4i| = 5$, where $|z|$ is maximum. From P, the particle moves through an angle $\tan^{-1}\frac{3}{4}$ in anticlockwise direction on $|z-3-4i| = 5$ and reaches at point Q. From Q, it comes down parallel to imaginary axis by 2 units and reaches at point R. Find the complex number corresponding to point R in the Argand plane.

45. If $\alpha = e^{i\left(\frac{\pi}{3}\right)}$, then find $\sum_{r=0}^{5}\frac{1}{2-\alpha^r} =$

SUBJECTIVE TYPE

46. Given, $z = \cos\frac{2\pi}{2n+1} + \sin\frac{2\pi}{2n+1}$, '$n$' a positive integer, find the equation whose roots are, $\alpha = z + z^3 + \ldots + z^{2n-1}$ and $\beta = z^2 + z^4 + \ldots + z^{2n}$ is $z^2 + z + \frac{1}{4}\sec^2(2\theta) = 0$, $\theta = \frac{2\pi}{2n+1}$

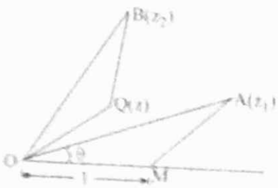
47. Let $A \equiv z_1; B \equiv z_2; C \equiv z_3$ are three complex numbers denoting the vertices of an acute angled triangle. If the origin 'O' is the orthocentre of the triangle, then prove that
$z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1$
hence show that the ΔABC is a right angled triangle
$\Leftrightarrow z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1 = 0$

48. Show that all roots of the equation $a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n = n$, where $|a_i| \le 1$, $i = 0, 1, 2, \ldots, n$ lie outside the circle with centre at the origin and radius $\frac{n-1}{n}$.

49. Let z_1, z_2, z_3 are three pair wise distinct complex numbers and t_1, t_2, t_3 are non-negative real numbers such that $t_1 + t_2 + t_3 = 1$. Prove that the complex number $z = t_1 z_1 + t_2 z_2 + t_3 z_3$ lies inside a triangle with vertices z_1, z_2, z_3 or on its boundry.

50. If P is a point on the Aragand diagram. On the circle with OP as diameter two points Q and R are taken such that $\angle POQ = \angle QOR = \theta$. If '$O$' is the origin and P, Q and R are represented by the complex number Z_1, Z_2 and Z_3 respectively, show that : $Z_2^2\cos 2\theta = Z_1 Z_3\cos^2\theta$.

51. If z_1 and z_2 are two complex number and if $\arg\frac{z_1+z_2}{z_1-z_2} = \frac{\pi}{2}$ but $|z_1+z_2| \ne |z_1-z_2|$ then prove that the figure formed by the 0, z_1, z_2 and z_1+z_2 is a rhombus but not a square



52. z_1, z_2 and z_3 are three non-zero complex numbers such that $z_1 \ne z_2 \ne z_3$, and $a = |z_1|, b = |z_2|, c = |z_3|$. If $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$ then show that $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3-z_1}{z_2-z_1}\right)^2$

53. On the Argand plane z_1, z_2 and z_3 are respectively the vertices of an isosceles triangle ABC with $AC = BC$ and equal angles are θ. If z_4 is the incentre of the triangle then prove that $(z_2-z_1)(z_3-z_1) = (1+\sec\theta)(z_4-z_1)^2$

54. Let $A(z_1), B(z_2), C(z_3)$ be the vertices of a triangle in the Argand plane. Then prove that

$$\cos A = \frac{c}{2b}\left[\frac{z_3-z_1}{z_2-z_1} + \frac{\bar{z}_3-\bar{z}_1}{\bar{z}_1-\bar{z}_1}\right]$$

and $$\sin A = \frac{c}{2b}\left[\frac{z_3-z_1}{z_2-z_1} + \frac{\bar{z}_3-\bar{z}_1}{\bar{z}_2-\bar{z}_1}\right]$$

Exercise-4 (Past Year Questions)

JEE MAIN

1. All the points in the set $S = \left\{\frac{\alpha + i}{\alpha - i} ; \alpha \in R\right\} (i = \sqrt{-1})$ lie on a **(2019)**
   (a) circle whose radius is 1
   (b) straight line whose slope is 1.
   (c) straight line whose slope is -1
   (d) circle whose radius is $\sqrt{2}$

2. The equation $|z - i| = |z - 1|$, $i = \sqrt{-1}$, represents: **(2019)**
   (a) the line through the origin with slope -1.
   (b) a circle of radius 1.
   (c) a circle of radius 1/2.
   (d) the line through the origin with slope 1.

3. If $\text{Re}\left(\frac{z-1}{2z+i}\right) = 1$, where $z = x + iy$, then the point (x, y) lies on a **(2020)**
   (a) straight line whose slope is $\frac{-2}{3}$
   (b) straight line whose slope is $\frac{3}{2}$
   (c) circle whose diameter is $\frac{\sqrt{5}}{2}$
   (d) circle whose centre is at $\left(\frac{-1}{2}, \frac{-3}{2}\right)$

4. If the four complex numbers and z, $\bar{z}$, $\bar{z} - 2\text{Re}(\bar{z})$ represent the vertices of a square of side 4 units in the Argand plane, then $|z|$ is equal to: **(2020)**
   (a) $4\sqrt{2}$ (b) 2
   (c) $2\sqrt{2}$ (d) 4

5. Let the lines $(2 - i)z = (2 + i)\bar{z}$ and $(2 + i)z + (i - 2)\bar{z} - 4i = 0$, (here $i^2 = -1$) be normal to a circle C. If the line $iz + \bar{z} + 1 + i = 0$ is tangent to this circle, then its radius is: **(2021)**
   (a) $\frac{3}{2\sqrt{2}}$ (b) $3\sqrt{2}$
   (c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{3}{\sqrt{2}}$

6. Let z be those complex numbers which satisfy $|z + 5| \leq 4$ and $z(1 + i) + z(1 - i) \geq -10$, $i = \sqrt{-1}$. If maximum value of $|z + 1|^2$ is $\alpha + \beta\sqrt{2}$, then the value of $(\alpha + \beta)$ is **(2021)**

7. The area of the triangle with vertices $A(z)$, $B(iz)$ and $C(z + iz)$ is: **(2021)**
   (a) $(1/2)\,|z + iz|^2$ (b) 1
   (c) 1/2 (d) $(1/2)\,|z|^2$

8. Let S_1, S_2 and S_3 be three sets defined as $S_1 = \{z \in c : |z - 1| \leq \sqrt{2}\}$, $S_2 = \{z \in c : \text{Re}((1 - i)z) \geq 1\}$, $S_3 = \{z \in c : im(z) \leq 1\}$. Then the set $S_1 \cap S_3$ **(2021)**
   (a) has infinitely many elements
   (b) has exactly two elements
   (c) has exactly three elements
   (d) is a singleton

9. For $n \in N$ let
   $$S_n = \left\{z \in C : |z - 3| + 2i| = \frac{n}{4}\right\} \text{ and}$$
   $$T_n = \left\{z \in C : \left|z - 2 + 3i = \frac{1}{n}\right|\right\}$$
   Then the number of elements in the set $\{n \in N : S_n \cap T_n = \phi\}$ is **(2022)**
   (a) 0 (b) 2
   (c) 3 (d) 4

10. Let O be the origin and A be the point $z_1 = 1 + 2i$. If B is the point z_2, $Re(z_2) < 0$, such that OAB is a right angled isosceles triangle with OB as hypotenuse, then which of the following is NOT true? **(2022)**
    (a) $\arg z_2 = \pi - \tan^{-1}3$
    (b) $(\arg)(z_1 - 2z_2) = -\tan^{-1}\frac{4}{3}$
    (c) $|z_2| = \sqrt{10}$
    (d) $|2z_1 - z_2| = 5$

11. $S = \{z = x + iy : |z - 1 + i| \geq |z|, |z| < 2, |z + i| = |z - 1|\}$. Then the set of all values of x, for which $w = 2x + iy \in S$ for some $y \in R$ is **(2022)**
    (a) $\left(-\sqrt{2}, \frac{1}{2\sqrt{2}}\right]$ (b) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{4}\right]$
    (c) $\left(-\sqrt{2}, \frac{1}{2}\right]$ (d) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right]$

12. Let $A = \{z \in C : 1 \leq |z - (1 + i)| \leq 2\}$ and $B = \{z \in A : |z - (1 - i)| = 1\}$. Then, B: **(2022)**
    (a) Is an empty set
    (b) Contains exactly two elements
    (c) Contains exactly three elements
    (d) Is an infinite set

13. Let $S = z \in C : |z - 3| \leq 1$ and $z(4 + 3i) + \bar{z}(4 - 3i) \leq 24$. If $\alpha + i\beta$ is the point in S which is closest to $4i$, then $25(a + b)$ is equal to____. **(2022)**

14. Let $A = \left\{ z \in C : \left| \frac{z-1}{z+1} \right| < 1 \right\}$ and

$B = \left\{ z \in C : \arg\left(\frac{z-1}{z+1} \right) = \frac{2\pi}{3} \right\}$

Then $A \cap B$ is **(2022)**

(*a*) A portion of a circle centred at $\left(0, -\frac{1}{\sqrt{3}}\right)$ that lies in the second and third quadrants only

(b) A portion of a circle centred at $\left(0, -\frac{1}{\sqrt{3}}\right)$ that lies in the second quadrant only

(*c*) An empty set

(*d*) A portion of a circle of radius $\frac{2}{\sqrt{3}}$ that lies in the third quadrant only

JEE ADVANCED

15. The cube roots of unity when represented on Argand diagram form the vertices of an equilateral triangle. **(1988)**

16. $|z| \le 1$, $|\omega| \le 1$, then show that $|z - \omega|^2 \le (|z| - |\omega|)^2 + (\arg z - \arg \omega)^2$ **(1995)**

17. If $0 < \alpha < \frac{\pi}{2}$ is a fixed angle. If $P = (\cos \theta, \sin \theta)$ and $Q = \{\cos(\alpha - \theta), \sin(\alpha - \theta)\}$, then Q is obtained from P by **(2002)**

(*a*) clockwise rotation around origin through an angle α

(*b*) anti-clockwise rotation around origin through an angle α

(*c*) reflection in the line through origin with slope $\tan \alpha$

(*d*) reflection in the line through origin with slope $\tan\frac{\alpha}{2}$

18. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lies on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$ **(2013)**

(*a*) $\frac{1}{\sqrt{2}}$ (*b*) $\frac{1}{2}$ (*c*) $\frac{1}{\sqrt{7}}$ (*d*) $\frac{1}{3}$

19. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \ne 0$. Suppose $S = \left\{ z \in \mathbb{C} : z = \frac{1}{a + ibt}, t \in \mathbb{R}, t \ne 0 \right\}$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in S$, then (x, y) lies on **(2016)**

(*a*) The circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \ne 0$

(*b*) The circle with radius $\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a > 0, b \ne 0$

(*c*) The x-axis for $a \ne 0, b = 0$

(*d*) The y-axis for $a = 0, b \ne 0$

20. For a non-zero complex number z, let $\arg(z)$ denotes the principal argument with $-\pi < \arg(z) < \pi$. Then, which of the following statement(s) is (are) **False** ? **(2018)**

(*a*) $\arg(-1 - i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

(*b*) The function $f : R \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in R$, is continuous at all points of R, where $i = \sqrt{-1}$

(*c*) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(*d*) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}\right) = \pi$, lies on a stright line

21. For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z + \alpha}{z + \beta}\right) = \frac{\pi}{4}$, the ordered pair (x, y) lies on the circle

$$x^2 + y^2 + 5x - 3y + 4 = 0$$

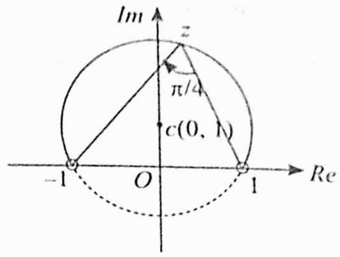
Then which of the following statements is (are) TRUE? **(2021)**

(*a*) $\alpha = -1$ (*b*) $\alpha\beta = 4$

(*c*) $\alpha\beta = -4$ (*d*) $\beta = 4$

ANSWER KEY

CONCEPT APPLICATION



1. (i) $1+\left(2+2\sqrt{2}\right)i$ (ii) 0 (iii) $z_2 = z_3 + i(z_1 - z_3)$ (iv) b

(v) Locus is all the points on the major arc of circle as shown excluding points 1 & −1.

2. (b) 3. (b) 4. (d) 5. (a) 6. (b) 7. (a) 8. (b) 9. (d)
10. (d) 11. (b) 12. (c) 13. (a) 14. (a) 15. (c) 16. (d) 17. (b)

EXERCISE-1 (TOPICWISE)

1. (a) 2. (b) 3. (c) 4. (b) 5. (d) 6. (b) 7. (c) 8. (c) 9. (c) 10. (a)
11. (b) 12. (b) 13. (d) 14. (a) 15. (a) 16. (b) 17. (a) 18. (c) 19. (c) 20. (b)
21. (a) 22. (c) 23. (d) 24. (c) 25. (d) 26. (c) 27. (c) 28. (d) 29. (a) 30. (d)
31. (d) 32. (c)

EXERCISE-2 (LEARNING PLUS)

1. (a) 2. (b) 3. (c) 4. (c) 5. (a) 6. (a) 7. (b) 8. (d) 9. (c) 10. (a)
11. (a) 12. (b) 13. (a) 14. (d) 15. (d) 16. (d) 17. (b) 18. (a) 19. (b) 20. (a)
21. (d) 22. (a,d) 23. (a,c,d) 24. (b,d) 25. (a,c,d) 26. (a,b,c,d) 27. (a,b,c,d) 28. (b,c)
29. (a,b) 30. (a,b,c) 31. (a,b) 32. (a,d) 33. (a,c,d) 34. (c) 35. (c) 36. [6] 37. [3] 38. [4]
39. [5] 40. [7] 41. [9] 42. [1] 43. [7] 44. [2] 50. [−12] 51. [6] 52. [2] 53. $\left(7+9i, 3\sqrt{2}\right)$

EXERCISE-3 (JEE ADVANCED LEVEL)

1. (c) 2. (a) 3. (a) 4. (c) 5. (a) 6. (a) 7. (a) 8. (c) 9. (d) 10. (b,c)
11. (a,b) 12. (a,b) 13. (b,c,d) 14. (a,b,c,d) 15. (a,c) 16. (a,b) 17. (a,c,d) 18. (c) 19. (d) 20. (a)
21. (a) 22. (c) 23. (a) 24. (a) 25. (c) 26. (a) 27. (b) 28. (b) 29. (d) 30. (c)
31. (c) 32. (d) 33. (c) 34. (b) 35. (d) 36. (a) 37. [0] 38. [4] 39. [2] 40. [30]
41. [(i) $2n$ (ii) n] 42. [$\alpha = -z_2 z_3 / z_1$; $\beta = -z_3 z_1 / z_2 i$; $\gamma = -z_1 z_2 / z_3$] 43. [163] 44. [3+7i] 45. [56/15]
51. [6] 52. [2] 53. $\left(7+9i, 3\sqrt{2}\right)$

EXERCISE-4 (PAST YEAR QUESTIONS)

JEE Main

1. (a) 2. (d) 3. (c) 4. (c) 5. (a) 6. [48] 7. (d) 8. (a) 9. (*) 10. (d)
11. (b) 12. (d) 13. [80] 14. (b)

JEE Advanced

17. (d) 18. (c) 19. (a,c,d) 20. (a,b,d) 21. (b,d)