Esercizio :

$$f(x) = \begin{cases} \sin x & -\pi < x \leq 0 \\ \cos x & 0 < x \leq \pi \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} \sin x + \int_{0}^{\pi} \cos x\,dx$$

$$= \frac{1}{\pi} \Big[-\cos x\Big]_{-\pi}^{0} + \cancel{\Big[\sin x\Big]_{0}^{\pi}}$$

$$= -\frac{1}{\pi}(1+1) = -\frac{2}{\pi}$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} \sin x \cos x\,dx + \int_{0}^{\pi} \cos x \cos x\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} \sin x \cos x\,dx + \int_{0}^{\pi} \cos^2 x\,dx$$

$$\int_{-\pi}^{0} \sin x \cos x\,dx = -\frac{1}{2} \cos^2(x) \Big|_{-\pi}^{0} = 0$$

$$\int_{0}^{\pi} \cos^2 x\,dx = \frac{\pi}{2}$$

$$a_1 = \frac{1}{\pi}\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{0} \sin x \sin(2x)\,dx + \int_{0}^{\pi} \cos x \sin(2x)\,dx$$

$$\int_{-\pi}^{0} \sin x \sin(2x)\,dx = \frac{1}{2}\sin x - \frac{1}{6}\sin(3x)\Big|_{-\pi}^{0} = 0$$

$$\int_{0}^{\pi} \cos x \sin(2x)\,dx = -\frac{1}{2}\cos x - \frac{1}{6}\cos(3x)\Big|_{0}^{\pi} = \frac{4}{3}$$

$$b_2 = \frac{1}{\pi}\,\frac{4}{3} = \frac{4}{3}\pi$$

Disuguaglianza di Bessel

Consideriamo:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

dove:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

Considero il seguente integrale:

$$\int_{-\pi}^{\pi} |f(x) - s_n(x)^2|^2\,dx =$$

$$= \int_{-\pi}^{\pi} \left[f(x)^2 + s_n(x)^2 - 2f(x)\,s_n(x)\right] dx$$

Per $s_n(x)^2$:

$$\int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos(kx) + b_k\sin(kx)\right)^2 dx =$$

$$= \frac{a_0^2}{4}2\pi + \sum_{k=1}^{n} a_k^2\pi + \sum_{k=1}^{n} b_k^2\pi =$$

$$= \frac{a_0^2}{4}\pi + \pi\sum_{k=1}^{n} a_k^2 + b_k^2$$

Per $2f(x)\,s_n(x)$:

$$-2\int_{-\pi}^{\pi} f(x)\left[\frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos(kx) + b_k\sin(kx)\right] dx$$

$$-a_0\overbrace{\int_{-\pi}^{\pi} f(x)\,dx}^{a_0\pi} - 2\sum_{k=1}^{n} a_k \underbrace{\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx}_{a_k}$$

$$\sum^{n} \int^{\pi} f(x)\sin(kx)\,dx$$

$$+2 \sum_{k=1} b_k \underbrace{\int_{-\pi} f(x)\sin(kx)\,dx}_{b_k} =$$

$$= -a_0^2\,\pi - 2\pi \sum_{k=1}^{n} a_k^2 + b_k^2$$

Ottengo:

$$\int_{-\pi}^{\pi} f^2(x)\,dx - \frac{a_0^2}{2}\pi - \pi \sum_{k=1}^{n} (a_k^2 + b_k^2) \geq 0$$

$$\boxed{\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)} \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,dx$$

questa serie converge

segue (dalla condizione necessaria):

$$\lim_{k \to +\infty} a_k = 0 \quad , \quad \lim_{k \to +\infty} b_k = 0$$

ossia:

$$\lim_{k \to +\infty} \int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = \lim_{k \to +\infty} \int_{-\pi}^{\pi} f(x)\sin(kx) = 0$$

Nota: Mandando $k \to +\infty$, i coefficienti di fourier tendono a 0.

Def Una funzione regolare a tratti e' una funzione che presenta dei salti, ossia e' regolare a tratti se esistono un numero finito di punti tali che

f è derivabile con derivata continua.

Nucleo di Dirichlet ("dirisclè")

Definiamo una funzione:

$$d_n = \frac{1}{2} + \sum_{k=1}^{n} \cos(kx)$$

si ha:

$$d_n = \frac{1}{2} + \sum_{k=1}^{n} \cos(kx) = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)}$$

inoltre:

$$\frac{1}{\pi}\int_0^{\pi} d_n(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} d_n(t)\,dt = \frac{1}{2}$$