Section4.3

1. 将当地密度 ρ、速度 $\boldsymbol{u}$ 和压力 p 分解为时均值和脉动值两部分，即

$$\begin{aligned}\rho(\boldsymbol{x},t)&=\rho_0(\boldsymbol{x})+\rho\prime(\boldsymbol{x},t)\\ \boldsymbol{u}(\boldsymbol{x},t)&=\boldsymbol{u}_0(\boldsymbol{x})+\boldsymbol{u}\prime(\boldsymbol{x},t)\\ p(\boldsymbol{x},t)&=p_0(\boldsymbol{x})+p\prime(\boldsymbol{x},t)\end{aligned}$$

对线性小振幅扰动，以 $(\rho\prime,\rho_0\prime\boldsymbol{u}\prime,p\prime)$ 为声学变量，建立线化欧拉方程组。

对于连续方程：

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0 \tag{1}$$

代入声学变量，得：

$$\frac{\partial\rho'}{\partial t}+\nabla\cdot(\rho_0\mathbf{u}_0+\rho'\mathbf{u}_0+\rho_0\mathbf{u}'+\rho'\mathbf{u}')=0 \tag{2}$$

又因为：

$$\nabla\cdot(\rho_0\mathbf{u}_0)=-\frac{\partial\rho_0}{\partial t}=0 \tag{3}$$

得到线化连续方程：

$$\begin{aligned}&\frac{\partial\rho'}{\partial t}+\nabla\cdot(\rho'\mathbf{u}_0+\rho_0\mathbf{u}'+\rho'\mathbf{u}')\\ =&\frac{\partial\rho'}{\partial t}+\mathbf{u}_0\cdot\nabla\rho'+\rho'\nabla\cdot\mathbf{u}_0+\rho_0\nabla\cdot\mathbf{u}'+\mathbf{u}'\cdot\nabla\rho_0\\ =&-\nabla\cdot(\rho'\mathbf{u}')\end{aligned} \tag{4}$$

对于动量方程：

$$\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=0 \tag{5}$$

代入声学变量，得：

$$\begin{aligned}&(\rho_0+\rho\prime)\frac{\partial(\boldsymbol{u}_0+\boldsymbol{u}\prime)}{\partial t}+(\rho_0+\rho\prime)(\boldsymbol{u}_0+\boldsymbol{u}\prime)\cdot\nabla(\boldsymbol{u}_0+\boldsymbol{u}\prime)+\nabla(p_0+p\prime)\\ =&\rho_0\left(\frac{\partial\mathbf{u}'}{\partial t}+\mathbf{u}_0\cdot\nabla\mathbf{u}'\right)+(\rho_0\mathbf{u}_0\cdot\nabla\mathbf{u}_0+\nabla p_0)+(\rho_0\mathbf{u}'+\rho'\mathbf{u}_0)\cdot\nabla\mathbf{u}_0+\nabla p'\\ &+\left[\rho'\left(\frac{\partial\mathbf{u}'}{\partial t}+\mathbf{u}_0\cdot\nabla\mathbf{u}'\right)+\rho\mathbf{u}'\cdot\nabla\mathbf{u}'+\rho'\mathbf{u}'\cdot\nabla\mathbf{u}_0\right]=0\end{aligned} \tag{6}$$

又因为：

$$\rho_0\mathbf{u}_0\cdot\nabla\mathbf{u}_0+\nabla p_0=0 \tag{7}$$
$$\frac{\mathrm{D}_0}{\mathrm{D}t}=\frac{\partial}{\partial t}+\mathbf{u}_0\cdot\nabla \tag{8}$$

得到线化动量方程：

$$\rho_0\frac{\mathrm{D}_0\mathbf{u}'}{\mathrm{D}t}+(\rho_0\mathbf{u}'+\rho'\mathbf{u}_0)\cdot\nabla\mathbf{u}_0+\nabla p'=-\rho'\frac{\mathrm{D}_0\mathbf{u}'}{\mathrm{D}t}-\rho_0\mathbf{u}'\cdot\nabla\mathbf{u}'-\rho'\mathbf{u}'\cdot\nabla\mathbf{u}_0 \tag{9}$$

对于能量方程：

$$\frac{\partial p}{\partial t}+\mathbf{u}\cdot\nabla p+\gamma p\nabla\cdot\mathbf{u}=0 \tag{10}$$

代入声学变量，得：

$$\begin{aligned}
&\frac{\partial(p_0+p\prime)}{\partial t}+(\boldsymbol{u_0}+\boldsymbol{u}\prime)\cdot\nabla(p_0+p\prime)+\gamma(p_0+p\prime)\nabla\cdot(\boldsymbol{u_0}+\boldsymbol{u}\prime)\\
=&\left(\frac{\partial p'}{\partial t}+\mathbf{u}_0\cdot\nabla p'\right)+(\mathbf{u}_0\cdot\nabla p_0+\gamma p_0\nabla\cdot\mathbf{u}_0)+\mathbf{u}'\cdot\nabla p_0+\mathbf{u}'\cdot\nabla p'\\
&+\gamma p_0\nabla\cdot\mathbf{u}'+\gamma p'\nabla\cdot\mathbf{u}'+\gamma p'\nabla\cdot\mathbf{u}_0\\
=&0
\end{aligned} \tag{11}$$

又因为：

$$\mathbf{u}_0\cdot\nabla p_0+\gamma p_0\nabla\cdot\mathbf{u}_0=0 \tag{12}$$

得到线化能量方程：

$$\frac{D_0p'}{Dt}+\mathbf{u}'\cdot\nabla p_0+\gamma p_0\nabla\cdot\mathbf{u}'+\gamma p'\nabla\cdot\mathbf{u}_0=-\mathbf{u}'\cdot\nabla p'-\gamma p'\nabla\cdot\mathbf{u}' \tag{13}$$

综上，得到线化欧拉方程组

$$\begin{aligned}
\frac{\partial\rho'}{\partial t}+\mathbf{u}_0\cdot\nabla\rho'+\rho'\nabla\cdot\mathbf{u}_0+\rho_0\nabla\cdot\mathbf{u}'+\mathbf{u}'\cdot\nabla\rho_0=-\nabla\cdot(\rho'\mathbf{u}')\\
\rho_0\frac{\mathrm{D}_0\mathbf{u}'}{\mathrm{D}t}+(\rho_0\mathbf{u}'+\rho'\mathbf{u}_0)\cdot\nabla\mathbf{u}_0+\nabla p'=-\rho'\frac{\mathrm{D}_0\mathbf{u}'}{\mathrm{D}t}-\rho_0\mathbf{u}'\cdot\nabla\mathbf{u}'-\rho'\mathbf{u}'\cdot\nabla\mathbf{u}_0\\
\frac{D_0p'}{Dt}+\mathbf{u}'\cdot\nabla p_0+\gamma p_0\nabla\cdot\mathbf{u}'+\gamma p'\nabla\cdot\mathbf{u}_0=-\mathbf{u}'\cdot\nabla p'-\gamma p'\nabla\cdot\mathbf{u}'
\end{aligned} \tag{14}$$