Section3.1

1. 对无黏、均熵可压缩流动, 证明涡运动方程的表达形式可写为

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right)=\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\mathbf{u}$$

已知：

$$\frac{\partial \mathbf{u}}{\partial t}+\nabla H+\boldsymbol{\omega}\times\mathbf{u}-T\nabla s-\mathbf{e}=0 \tag{1}$$

两边求旋度得：

$$\nabla\times\left(\frac{\partial \mathbf{u}}{\partial t}+\nabla H+\boldsymbol{\omega}\times\mathbf{u}-T\nabla s-\mathbf{e}\right)=0 \tag{2}$$

因为 $\nabla\times\nabla H\equiv 0$，因此有：

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+\nabla\times(\boldsymbol{\omega}\times\mathbf{u})-\nabla T\times\nabla s-\nabla\times\mathbf{e}=0 \tag{3}$$

又因为：

$$\begin{aligned}\nabla\times(\boldsymbol{\omega}\times\mathbf{u})&=\boldsymbol{\omega}(\nabla\cdot\mathbf{u})-\mathbf{u}(\nabla\cdot\boldsymbol{\omega})+(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}\\ \nabla\times\mathbf{e}&=\nu\nabla^2\boldsymbol{\omega}\end{aligned} \tag{4}$$

结合 $\nabla\cdot\boldsymbol{\omega}=\nabla\cdot(\nabla\times\mathbf{u})=0$ 得：

$$\begin{aligned}&\frac{\partial \boldsymbol{\omega}}{\partial t}+\nabla\times(\boldsymbol{\omega}\times\mathbf{u})-\nabla T\times\nabla s-\nabla\times\mathbf{e}\\ =&\frac{\partial \boldsymbol{\omega}}{\partial t}+(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}+\boldsymbol{\omega}(\nabla\cdot\mathbf{u})-(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}\\ &-\mathbf{u}(\nabla\cdot\boldsymbol{\omega})-\nabla T\times\nabla s-\nu\nabla^2\boldsymbol{\omega}\\ =&\frac{D\boldsymbol{\omega}}{Dt}+\boldsymbol{\omega}(\nabla\cdot\mathbf{u})-(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}-\nabla T\times\nabla s-\nu\nabla^2\boldsymbol{\omega}\\ =&0\end{aligned} \tag{5}$$

根据连续方程，有：

$$\frac{\boldsymbol{\omega}}{\rho^2}\left(\frac{D\rho}{Dt}+\rho(\nabla\cdot\mathbf{u})\right)=\frac{\boldsymbol{\omega}}{\rho^2}\frac{D\rho}{Dt}+\frac{\boldsymbol{\omega}}{\rho}(\nabla\cdot\mathbf{u})=0 \tag{6}$$

因此有：

$$\begin{aligned}&\frac{1}{\rho}\frac{D\boldsymbol{\omega}}{Dt}-\left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{u}-\frac{1}{\rho}\nabla T\times\nabla s-\frac{\nu}{\rho}\nabla^2\boldsymbol{\omega}-\frac{\boldsymbol{\omega}}{\rho^2}\frac{D\rho}{Dt}\\ =&\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right)-\left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{u}-\frac{1}{\rho}\nabla T\times\nabla s-\frac{\nu}{\rho}\nabla^2\boldsymbol{\omega}\\ =&0\end{aligned} \tag{7}$$

对于无粘、等熵流动，有：

$$\frac{1}{\rho}\nabla T \times \nabla s = 0 \tag{8}$$

$$\frac{\nu}{\rho}\nabla^2 \boldsymbol{\omega} = 0 \tag{9}$$

因此有：

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla\right)\mathbf{u} = \frac{\boldsymbol{\omega}}{\rho} \cdot \nabla\mathbf{u} \tag{10}$$

原式得证。

2. 对无黏正压流体的可压缩运动, 证明涡运动方程的表达形式可写为

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \boldsymbol{\omega}(\nabla \cdot \mathbf{u})$$

根据第 1 题中的推导，已知：

$$\frac{D\boldsymbol{\omega}}{Dt} + \boldsymbol{\omega}(\nabla \cdot \mathbf{u}) - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \nabla T \times \nabla s - \nu\nabla^2\boldsymbol{\omega} = 0 \tag{11}$$

对于无粘流动，有：

$$\frac{\nu}{\rho}\nabla^2 \boldsymbol{\omega} = 0 \tag{12}$$

对于正压流动，有：

$$\nabla T \times \nabla s = 0 \tag{13}$$

因此有：

$$\frac{D\boldsymbol{\omega}}{Dt} + \boldsymbol{\omega}(\nabla \cdot \mathbf{u}) - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = 0 \tag{14}$$

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \boldsymbol{\omega}(\nabla \cdot \mathbf{u}) \tag{15}$$

原式得证。