1. 证明自由空间格林函数的偏导数关系：

$$\frac{\partial G_0}{\partial y_i}=\frac{x_i-y_i}{r}\left[\frac{1}{4\pi r c_0}\frac{\partial}{\partial \tau}\delta\left(t-\tau-r/c_0\right)+\frac{\delta\left(t-\tau-r/c_0\right)}{4\pi r^2}\right].$$

已知自由空间格林函数：

$$G_0=\frac{1}{4\pi r}\delta\left(t-\tau-\frac{r}{c_0}\right) \tag{1}$$

对 r 求偏导，得：

$$\begin{aligned}\frac{\partial G_0}{\partial r}&=-\frac{1}{4\pi r^2}\delta\left(t-\tau-\frac{r}{c_0}\right)+\frac{1}{4\pi r}\frac{\partial\delta\left(t-\tau-\frac{r}{c_0}\right)}{\partial r}\\&=-\frac{1}{4\pi r^2}\delta\left(t-\tau-\frac{r}{c_0}\right)+\frac{1}{4\pi r}\frac{\partial\delta\left(t-\tau-\frac{r}{c_0}\right)}{\partial \tau}\frac{\partial\tau}{\partial r}\end{aligned} \tag{2}$$

由 τ 与 r 的关系式 $\tau=t-\frac{r}{c_0}$ 可得：

$$\frac{\partial\tau}{\partial r}=-\frac{1}{c_0} \tag{3}$$

代入式(2), 得

$$\frac{\partial G_0}{\partial r}=-\frac{1}{4\pi r^2}\delta\left(t-\tau-\frac{r}{c_0}\right)-\frac{1}{4\pi r c_0}\frac{\partial\delta\left(t-\tau-\frac{r}{c_0}\right)}{\partial \tau} \tag{4}$$

根据 r 对 y_i 的偏导数：

$$\frac{\partial r}{\partial y_i}=\frac{\partial\sqrt{\sum(x_i-y_i)^2}}{\partial y_i}\quad=-\frac{x_i-y_i}{r} \tag{5}$$

结合式(4),(5), 得：

$$\begin{aligned}\frac{\partial G_0}{\partial y_i}&=\frac{\partial G_0}{\partial r}\frac{\partial r}{\partial y_i}\\&=\frac{x_i-y_i}{r}\left[\frac{1}{4\pi r c_0}\frac{\partial}{\partial \tau}\delta\left(t-\tau-r/c_0\right)+\frac{\delta\left(t-\tau-r/c_0\right)}{4\pi r^2}\right]\end{aligned} \tag{6}$$

原式得证。

2. 利用上述自由空间格林函数的偏导数关系式证明

$$p'(\mathbf{x},t) = -\int_{-\infty}^{+\infty}\int_S \rho_0 \frac{\partial u_n(\mathbf{y},\tau)}{\partial \tau} G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}S\ \mathrm{d}\tau - \int_{-\infty}^{+\infty}\int_S p'(\mathbf{y},\tau)\frac{\partial G(\mathbf{x},\mathbf{y},t-\tau)}{\partial y_i} n_i\ \mathrm{d}S\ \mathrm{d}\tau$$

可以改写为

$$p'(\mathbf{x},t) = -\int_S \left[\rho_0 \frac{\partial u_n}{\partial \tau}\right]_\tau \frac{\mathrm{d}S(\mathbf{y})}{4\pi r} - \int_S \left[\frac{\partial p'}{\partial \tau} n_i + \frac{p' n_i c_0}{r}\right]_\tau \frac{(x_i - y_i)\,\mathrm{d}S(\mathbf{y})}{4\pi r^2 c_0}.$$

原式右侧第一项代入自由格林函数，并对 τ 求积分：

$$\begin{aligned}
\text{右侧第一项} &= -\int_{-\infty}^{+\infty}\int_S \rho_0 \frac{\partial u_n(\mathbf{y},\tau)}{\partial \tau} G_0(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}S\ \mathrm{d}\tau \\
&= -\int_s \int_{-\infty}^{+\infty} \rho_0 \frac{\partial u_n(\mathbf{y},\tau)}{\partial \tau}\frac{1}{4\pi r}\delta(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau\ \mathrm{d}s \\
&= -\int_S \left[\rho_0 \frac{\partial u_n}{\partial \tau}\right]_\tau \frac{\mathrm{d}S(\mathbf{y})}{4\pi r}
\end{aligned} \tag{7}$$

原式右侧第二项代入自由格林函数偏导数关系式（式(6)）：

$$\begin{aligned}
\text{右侧第二项} = &-\int_{-\infty}^{+\infty}\int_S p'(\mathbf{y},\tau)\frac{x_i - y_i}{r}\left[\frac{1}{4\pi r c_0}\frac{\partial}{\partial \tau}\delta\left(t-\tau-\frac{r}{c_0}\right) + \frac{\delta\left(t-\tau-\frac{r}{c_0}\right)}{4\pi r^2}\right] n_i\ \mathrm{d}S\ \mathrm{d}\tau \\
= &-\int_S \left[\int_{-\infty}^{+\infty} p'(\mathbf{y},\tau)\frac{\partial}{\partial \tau}\delta\left(t-\tau-\frac{r}{c_0}\right) n_i \mathrm{d}\tau\right]\frac{(x_i-y_i)\,\mathrm{d}S}{4\pi r^2 c_0} \\
&-\int_S \left[\int_{-\infty}^{+\infty} p'(\mathbf{y},\tau)\frac{c_0}{r}\delta\left(t-\tau-\frac{r}{c_0}\right) n_i \mathrm{d}\tau\right]\frac{(x_i-y_i)\,\mathrm{d}S}{4\pi r^2 c_0}
\end{aligned} \tag{8}$$

其中：

$$\begin{aligned}
\int_{-\infty}^{+\infty} p'(\mathbf{y},\tau)\frac{\partial}{\partial \tau}\delta\left(t-\tau-\frac{r}{c_0}\right) n_i \mathrm{d}\tau = &-p'(\mathbf{y},\tau)\delta(t-\tau-\frac{r}{c_0})n_i\Big|_{\tau=-\infty}^{\tau=\infty} \\
&-\int_{-\infty}^{+\infty} -\frac{\partial p'(\mathbf{y},\tau)}{\partial \tau}\delta(t-\tau-\frac{r}{c_0})\mathrm{d}\tau
\end{aligned} \tag{9}$$

根据 t 与 τ 的因果关系，有：

$$-p'(\mathbf{y},\tau)\delta(t-\tau-\frac{r}{c_0})n_i\Big|_{\tau=-\infty}^{\tau=\infty} = 0 \tag{10}$$

因此有：

$$\begin{aligned}
\int_{-\infty}^{+\infty} p'(\mathbf{y},\tau)\frac{\partial}{\partial \tau}\delta\left(t-\tau-\frac{r}{c_0}\right) n_i \mathrm{d}\tau &= -\int_{-\infty}^{+\infty} -\frac{\partial p'(\mathbf{y},\tau)}{\partial \tau}\delta(t-\tau-\frac{r}{c_0})\mathrm{d}\tau \\
&= \left[\frac{\partial p'}{\partial \tau} n_i\right]_\tau
\end{aligned} \tag{11}$$

同时，式(8)中：

$$\int_{-\infty}^{+\infty} p'(\mathbf{y},\tau)\frac{c_0}{r}\delta\left(t-\tau-\frac{r}{c_0}\right)n_i\mathrm{d}\tau=\left[\frac{p'n_ic_0}{r}\right]_\tau \tag{12}$$

将式(11),(12)代入式(8)，得：

$$\begin{aligned}\text{右侧第二项} &= -\int_S\left[\frac{\partial p'}{\partial\tau}n_i\right]_\tau\frac{(x_i-y_i)\,\mathrm{d}S}{4\pi r^2c_0}-\int_S\left[\frac{p'n_ic_0}{r}\right]_\tau\frac{(x_i-y_i)\,\mathrm{d}S}{4\pi r^2c_0}\\ &= -\int_S\left[\frac{\partial p'}{\partial\tau}n_i+\frac{p'n_ic_0}{r}\right]_\tau\frac{(x_i-y_i)\,\mathrm{d}S}{4\pi r^2c_0}\end{aligned} \tag{13}$$

结合式(7),(13), 得：

$$p'(\mathbf{x},t)=-\int_S\left[\rho_0\frac{\partial u_n}{\partial\tau}\right]_\tau\frac{\mathrm{d}S(\mathbf{y})}{4\pi r}-\int_S\left[\frac{\partial p'}{\partial\tau}n_i+\frac{p'n_ic_0}{r}\right]_\tau\frac{(x_i-y_i)\,\mathrm{d}S(\mathbf{y})}{4\pi r^2c_0} \tag{14}$$

原式得证。