1. 已知三维频域自由空间格林函数为 $G_0(\mathbf{x},\mathbf{y},\omega)=\frac{e^{ikr}}{4\pi r}$，推导 $\frac{\partial G_0}{\partial y_i}$ 和 $\frac{\partial^2 G_0}{\partial y_i \partial y_j}$ 的解析表达式。

   已知，在三维频域下：

$$r=\sqrt{\sum_{i=1}^{n=3}(x_i-y_i)^2} \tag{1}$$

   因此有：

$$\begin{aligned}\frac{\partial G_0}{\partial y_i}&=\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{4\pi r}\right)\frac{\partial r}{\partial y_i}\\&=\frac{ikre^{ikr}-e^{ikr}}{4\pi r^2}\frac{\partial\sqrt{\sum_{i=1}^{n=3}(x_i-y_i)^2}}{\partial y_i}\\&=\left(\frac{ik}{4\pi r}-\frac{1}{4\pi r^2}\right)e^{ikr}\left(-\frac{x_i-y_i}{r}\right)\\&=\frac{x_i-y_i}{r}\left(\frac{e^{ikr}}{4\pi r^2}-\frac{ike^{ikr}}{4\pi r}\right)\end{aligned} \tag{2}$$

   同理有：

$$\begin{aligned}\frac{\partial^2 G_0}{\partial y_i\partial y_j}&=\frac{\partial}{\partial y_j}\left(\frac{\partial G_0}{\partial y_i}\right)\\&=\frac{\partial}{\partial r}\left(\frac{\partial G_0}{\partial y_i}\right)\frac{\partial r}{\partial y_i}\\&=\frac{x_i-y_i}{r^2}\left(-\frac{3e^{ikr}}{4\pi r^2}+\frac{3ike^{ikr}}{4\pi r}+\frac{k^2e^{ikr}}{4\pi}\right)\left(-\frac{x_j-y_j}{r}\right)\\&=\frac{(x_i-y_i)(x_j-y_j)}{r^3}\left(\frac{3e^{ikr}}{4\pi r^2}-\frac{3ike^{ikr}}{4\pi r}-\frac{k^2e^{ikr}}{4\pi}\right)\end{aligned} \tag{3}$$

   综上，

$$\frac{\partial G_0}{\partial y_i}=\frac{x_i-y_i}{r}\left(\frac{e^{ikr}}{4\pi r^2}-\frac{ike^{ikr}}{4\pi r}\right) \tag{4}$$

$$\frac{\partial^2 G_0}{\partial y_i\partial y_j}=\frac{(x_i-y_i)(x_j-y_j)}{r^3}\left(\frac{3e^{ikr}}{4\pi r^2}-\frac{3ike^{ikr}}{4\pi r}-\frac{k^2e^{ikr}}{4\pi}\right) \tag{5}$$

2. 假设静止固体表面是可穿透的，并忽略粘性的贡献，写出 Curle 方程的频域积分公式。

已知忽略粘性贡献的 Curle 方程为：

$$
\begin{aligned}
c_0^2\rho'(\mathbf{x},t) = & \int_V \int_{-\infty}^{+\infty} T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \ \mathrm{d}^3\mathbf{y}\mathrm{d}\tau \\
& - \int_S \int_{-\infty}^{+\infty} \left(\rho u_i u_j + p_{ij}\right) n_j \frac{\partial G}{\partial y_i} \ \mathrm{d}^2\mathbf{y}\mathrm{d}\tau \\
& - \int_S \int_{-\infty}^{+\infty} G \frac{\partial \left(\rho u_j n_j\right)}{\partial \tau} \mathrm{d}^2\mathbf{y}\mathrm{d}\tau
\end{aligned}
\tag{6}
$$

不妨设：

$$
F_i(\mathbf{y},\tau) = \left(\rho u_i u_j + p_{ij}\right) n_j \tag{7}
$$

$$
Q(\mathbf{y},\tau) = \rho u_j n_j \tag{8}
$$

代入自由格林函数 G_0，根据 G_0 的性质：

$$
\frac{\partial G_0}{\partial y_i} = -\frac{\partial G_0}{\partial x_i} \tag{9}
$$

$$
\frac{\partial^2 G_0}{\partial y_i \partial y_j} = \frac{\partial^2 G_0}{\partial x_i \partial x_j} \tag{10}
$$

可以得到：

$$
\begin{aligned}
c_0^2\rho'(\mathbf{x},t) = & \frac{\partial^2}{\partial x_i \partial x_j} \int_{-\infty}^{+\infty} \int_V T_{ij}(\mathbf{y},\tau) G_0 \ \mathrm{d}^3\mathbf{y}\mathrm{d}\tau \\
& + \frac{\partial}{\partial x_i} \int_{-\infty}^{+\infty} \int_S F_i(\mathbf{y},\tau) G_0 \ \mathrm{d}^2\mathbf{y}\mathbf{d}\tau \\
& - \int_{-\infty}^{+\infty} \int_S \frac{\partial Q(\mathbf{y},\tau)}{\partial \tau} G_0 \ \mathrm{d}^2\mathbf{y}\mathrm{d}\tau \\
= & \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left[T_{ij}\left(\mathbf{y}, t - r/c_0\right)\right]_{\tau = t - r/c_0} \frac{\mathrm{d}^3\mathbf{y}}{4\pi r} \\
& + \frac{\partial}{\partial x_i} \int_S \left[F_i\left(\mathbf{y}, t - r/c_0\right)\right]_{\tau = t - r/c_0} \frac{\mathrm{d}^2\mathbf{y}}{4\pi r} \\
& - \int_S \left[\frac{\partial}{\partial \tau} Q\left(\mathbf{y}, t - r/c_0\right)\right]_{\tau = t - r/c_0} \frac{\mathrm{d}^2\mathbf{y}}{4\pi r}
\end{aligned}
\tag{11}
$$

根据 Fourier 变换，可得：

$$
\begin{aligned}
\left(c_0^2\tilde{\rho'}(\mathbf{x},\omega)\right)_{quadrupole} & = \frac{\partial^2}{\partial x_i \partial x_j} \int_V \int_{-\infty}^{+\infty} T_{ij}(\mathbf{y}, t - r/c_0) e^{i\omega t} \mathrm{d}t \frac{\mathrm{d}^3\mathbf{y}}{4\pi r} \\
& = \frac{\partial^2}{\partial x_i \partial x_j} \int_V \widetilde{T_{ij}}(\mathbf{y},\omega) e^{i\omega r/c_0} \frac{\mathrm{d}^3\mathbf{y}}{4\pi r}
\end{aligned}
\tag{12}
$$

同理有：

$$
\begin{aligned}
\left(c_0^2\tilde{\rho}'(\mathbf{x},\omega)\right)_{dipole} &= \frac{\partial}{\partial x_i}\int_S\int_{-\infty}^{+\infty} F_i(\mathbf{y},t-r/c_0)e^{i\omega t}\mathrm{d}t\frac{\mathrm{d}^2\mathbf{y}}{4\pi r} \\
&= \frac{\partial}{\partial x_i}\int_S \widetilde{F}_i(\mathbf{y},\omega)e^{i\omega r/c_0}\frac{\mathrm{d}^2\mathbf{y}}{4\pi r}
\end{aligned}
\tag{13}
$$

根据 Fourier 变换的偏分性质，可得：

$$
\begin{aligned}
\left(c_0^2\tilde{\rho}'(\mathbf{x},\omega)\right)_{monopole} &= \int_S\int_{-\infty}^{+\infty} \frac{\partial}{\partial \tau}\left[Q\left(\mathbf{y},t-r/c_0\right)\right]e^{i\omega t}\mathrm{d}t\frac{\mathrm{d}^2\mathbf{y}}{4\pi r} \\
&= \int_S -i\omega\widetilde{Q}(\mathbf{y},\omega)e^{i\omega r/c_0}\frac{\mathrm{d}^2\mathbf{y}}{4\pi r}
\end{aligned}
\tag{14}
$$

综上，curle 方程的频域积分表达式为：

$$
\begin{aligned}
c_0^2\tilde{\rho}'(\mathbf{x},\omega) &= \frac{\partial^2}{\partial x_i\partial x_j}\int_V \widetilde{T_{ij}}(\mathbf{y},\omega)e^{i\omega r/c_0}\frac{\mathrm{d}^3\mathbf{y}}{4\pi r} \\
&+ \frac{\partial}{\partial x_i}\int_S \widetilde{F}_i(\mathbf{y},\omega)e^{i\omega r/c_0}\frac{\mathrm{d}^2\mathbf{y}}{4\pi r} \\
&+ \int_S i\omega\widetilde{Q}(\mathbf{y},\omega)e^{i\omega r/c_0}\frac{\mathrm{d}^2\mathbf{y}}{4\pi r}
\end{aligned}
\tag{15}
$$