1. 在 FW-H 方程中，$f=0$ 的面在运动过程中形状能发生改变吗？

   不能。FW-H 方程的前提假设为刚体运动，$f=0$ 的面不能发生形变。

2. 如果 $|\nabla f| \neq 1$，试推导 FW-H 方程，并求其积分表达式。

   对于表面，有：

$$\begin{aligned} \frac{\mathrm{D}H(f)}{\mathrm{D}t} &= \frac{\partial H(f)}{\partial t} + v_j \frac{\partial H(f)}{\partial x_j} = 0 \\ \frac{\partial H(f)}{\partial x_j} &= \frac{\partial H(f)}{\partial f} |\nabla f| n_j = |\nabla f| n_j \delta(f) \end{aligned} \tag{1}$$

   由上式可得：

$$\frac{\partial H(f)}{\partial t} = -v_j \frac{\partial H(f)}{\partial x_j} = -v_j |\nabla f| n_j \delta(f) \tag{2}$$

   于是有：

$$\begin{aligned} \frac{\partial[\phi H(f)]}{\partial t} &= H(f)\frac{\partial \phi}{\partial t} + \phi \frac{\partial H(f)}{\partial t} = H(f)\frac{\partial \phi}{\partial t} - \phi v_j |\nabla f| n_j \delta(f) \\ \frac{\partial[\phi H(f)]}{\partial x_i} &= H(f)\frac{\partial \phi}{\partial x_i} + \phi \frac{\partial H(f)}{\partial x_i} = H(f)\frac{\partial \phi}{\partial x_i} + \phi |\nabla f| n_i \delta(f) \end{aligned} \tag{3}$$

   代入连续方程有：

$$\begin{aligned} \frac{\partial\left[\rho' H(f)\right]}{\partial t} + \frac{\partial\left[\rho u_j H(f)\right]}{\partial x_j} &= \rho u_j |\nabla f| n_j \delta(f) - \rho' v_j |\nabla f| n_j \delta(f) \\ &= \left[\rho\left(u_j - v_j\right) + \rho_0 v_j\right] |\nabla f| n_j \delta(f) \end{aligned} \tag{4}$$

   代入动量方程有：

$$\begin{aligned} \frac{\partial\left[H(f)\rho u_i\right]}{\partial t} + c_0^2 \frac{\partial\left[H(f)\rho'\right]}{\partial x_i} &= -H(f)\frac{\partial T_{ij}}{\partial x_j} + \left(c_0^2 \rho' \delta_{ij} - \rho u_i v_j\right) |\nabla f| n_j \delta(f) \\ &= -\frac{\partial\left[H(f)T_{ij}\right]}{\partial x_j} + \left(T_{ij} + c_0^2 \rho' \delta_{ij} - \rho u_i v_j\right) |\nabla f| n_j \delta(f) \\ &= -\frac{\partial\left[H(f)T_{ij}\right]}{\partial x_j} + \left[\rho u_i\left(u_j - v_j\right) + p_{ij}\right] |\nabla f| n_j \delta(f) \end{aligned} \tag{5}$$

   于是有：

$$\frac{\partial^2\left[\rho' H(f)\right]}{\partial t^2} - c_0^2 \frac{\partial^2\left[H(f)\rho'\right]}{\partial x_i^2} = \frac{\partial^2\left[H(f)T_{ij}\right]}{\partial x_i \partial x_j} - \frac{\partial\left[F_i \delta(f)\right]}{\partial x_i} + \frac{\partial[Q\delta(f)]}{\partial t} \tag{6}$$

   其中，

$$\begin{aligned} Q &= \left[\rho\left(u_j - v_j\right) + \rho_0 v_j\right] |\nabla f| n_j \\ F_i &= \left[\rho u_i\left(u_j - v_j\right) + p_{ij}\right] |\nabla f| n_j \end{aligned}$$

   其积分表达式可以表示为：

$$H(f)c_0^2 \rho'(\mathbf{x}, t) = \int_V \int_{-\infty}^{+\infty} G\left\{\frac{\partial^2\left[H(f)T_{ij}\right]}{\partial y_i \partial y_j} - \frac{\partial\left[F_i \delta(f)\right]}{\partial y_i} + \frac{\partial[Q\delta(f)]}{\partial \tau}\right\} \mathrm{d}\tau \mathrm{d}^3\mathbf{y} \tag{7}$$

对于四极子项：

$$
\begin{aligned}
G\frac{\partial^2\left[T_{ij}H(f)\right]}{\partial y_i\partial y_j} = & \left[T_{ij}H(f)\right]\frac{\partial^2 G}{\partial y_i\partial y_j} \\
& +\frac{\partial}{\partial y_i}\left(G\frac{\partial\left[T_{ij}H(f)\right]}{\partial y_j}\right) \\
& -\frac{\partial}{\partial y_j}\left(\left[T_{ij}H(f)\right]\frac{\partial G}{\partial y_i}\right)
\end{aligned} \tag{8}
$$

其中，

$$
\int_{\Sigma+\Omega}\int_{-\infty}^{+\infty}\frac{\partial}{\partial y_i}\left(G\frac{\partial\left[T_{ij}H(f)\right]}{\partial y_j}\right)\mathrm{d}\tau\mathrm{d}^3\mathbf{y}=\int_{S_\infty}\int_{-\infty}^{+\infty}G\frac{\partial\left[T_{ij}H(f)\right]}{\partial y_j}n_i\ \mathrm{d}\tau\mathrm{d}^2\mathbf{y}=0
$$

$$
\int_{\Sigma+\Omega}\int_{-\infty}^{+\infty}\frac{\partial}{\partial y_j}\left(T_{ij}H(f)\frac{\partial G}{\partial y_i}\right)\mathrm{d}\tau\mathrm{d}^3\mathbf{y}=\int_{S_\infty}\int_{-\infty}^{+\infty}T_{ij}H(f)\frac{\partial G}{\partial y_j}n_j\ \mathrm{d}\tau\mathrm{d}^2\mathbf{y}=0
$$

因此有：

$$
\int_V\int_{-\infty}^{+\infty}G\frac{\partial^2\left[H(f)T_{ij}\right]}{\partial y_i\partial y_j}\ \mathrm{d}\tau\mathrm{d}^3\mathbf{y}=\int_V\int_{-\infty}^{+\infty}H(f)T_{ij}\frac{\partial^2 G}{\partial y_i\partial y_j}\ \mathrm{d}\tau\mathrm{d}^3\mathbf{y} \tag{9}
$$

对于偶极子项：

$$
\begin{aligned}
\int_V\int_{-\infty}^{+\infty}G\frac{\partial\left[F_i\delta(f)\right]}{\partial y_i}\ \mathrm{d}\tau\mathrm{d}^3\mathbf{y} = & \int_V\int_{-\infty}^{+\infty}\frac{\partial\left[GF_i\delta(f)\right]}{\partial y_i}\ \mathrm{d}\tau\mathrm{d}^3\mathbf{y} \\
& -\int_V\int_{-\infty}^{+\infty}F_i\delta(f)\frac{\partial G}{\partial y_i}\ \mathrm{d}\tau\mathrm{d}^3\mathbf{y}
\end{aligned} \tag{10}
$$

对于无边界区域，有：

$$
\int_V\frac{\partial\left[GF_i\delta(f)\right]}{\partial y_i}\ \mathrm{d}^3\mathbf{y}=\int_{\Sigma+\Omega}\frac{\partial\left[GF_i\delta(f)\right]}{\partial y_i}\ \mathrm{d}^3\mathbf{y}=\int_{S_\sim}GF_i\delta(f)n_i\ \mathrm{d}^2\mathbf{y}=0 \tag{11}
$$

因此有：

$$
\int_V\int_{-\infty}^{+\infty}G\frac{\partial\left[F_i\delta(f)\right]}{\partial y_i}\ \mathrm{d}\tau\mathrm{d}^3\mathbf{y}=-\int_V\int_{-\infty}^{+\infty}F_i\delta(f)\frac{\partial G}{\partial y_i}\ \mathrm{d}\tau\mathrm{d}^3\mathbf{y} \tag{12}
$$

对于单极子项：

$$
\begin{aligned}
\int_V\int_{-\infty}^{+\infty}G\frac{\partial[Q\delta(f)]}{\partial\tau}\mathrm{d}\tau\mathrm{d}^3\mathbf{y} = & \int_V\int_{-\infty}^{+\infty}\frac{\partial[GQ\delta(f)]}{\partial\tau}\mathrm{d}\tau\mathrm{d}^3\mathbf{y} \\
& -\int_V\int_{-\infty}^{+\infty}Q\delta(f)\frac{\partial G}{\partial\tau}\mathrm{d}\tau\mathrm{d}^3\mathbf{y}
\end{aligned} \tag{13}
$$

根据 $G=\frac{\partial G}{\partial\tau}=0(t<\tau)$, 有：

$$
\int_{-\infty}^{+\infty}\frac{\partial[GQ\delta(f)]}{\partial\tau}\mathrm{d}\tau=GQ\delta(f)|_{\tau=-\infty}^{\tau=+\infty}=0 \tag{14}
$$

因此有：

$$\int_V \int_{-\infty}^{+\infty} G\frac{\partial[Q\delta(f)]}{\partial\tau}\mathrm{d}\tau\mathrm{d}^3\mathbf{y} = -\int_V \int_{-\infty}^{+\infty} Q\delta(f)\frac{\partial G}{\partial\tau}\mathrm{d}\tau\mathrm{d}^3\mathbf{y} \tag{15}$$

综上，FW-H 方程的积分表达式可表示为：

$$\begin{aligned} H(f)c_0^2\rho'(\mathbf{x},t) = &\int_V \int_{-\infty}^{+\infty} H(f)T_{ij}\frac{\partial^2 G}{\partial y_i \partial y_j}\ \mathrm{d}\tau\mathrm{d}^3\mathbf{y} \\ &+ \int_V \int_{-\infty}^{+\infty} F_i\delta(f)\frac{\partial G}{\partial y_i}\ \mathrm{d}\tau\mathrm{d}^3\mathbf{y} \\ &- \int_V \int_{-\infty}^{+\infty} Q\delta(f)\frac{\partial G}{\partial \tau}\mathrm{d}\tau\mathrm{d}^3\mathbf{y} \end{aligned} \tag{16}$$

3. 如果 $f = 0$ 的面不是固体表面，而是流体区域任意选择的可穿透封闭面，FW-H 方程还成立吗？

   成立。FW-H 方程仅假设 $f = 0$ 为移动的刚体表面，并没有假设表面是否可穿透。因此 FW-H 方程对可穿透表面成立。对于不可穿透表面，FW-H 方程可以进一步简化，简化后的方程对可穿透表面不成立。