

1. 在 FW-H 方程中, $f = 0$ 的面在运动过程中形状能发生改变吗?

不能。FW-H 方程的前提假设为刚体运动, $f = 0$ 的面不能发生形变。

2. 如果 $|\nabla f| \neq 1$, 试推导 FW-H 方程, 并求其积分表达式。

对于表面, 有:

$$\begin{aligned}\frac{DH(f)}{Dt} &= \frac{\partial H(f)}{\partial t} + v_j \frac{\partial H(f)}{\partial x_j} = 0 \\ \frac{\partial H(f)}{\partial x_j} &= \frac{\partial H(f)}{\partial f} |\nabla f| = |\nabla f| \delta(f)\end{aligned}\quad (1)$$

由上式可得:

$$\frac{\partial H(f)}{\partial t} = -v_j \frac{\partial H(f)}{\partial x_j} = -v_j |\nabla f| \delta(f) \quad (2)$$

于是有:

$$\begin{aligned}\frac{\partial [\phi H(f)]}{\partial t} &= H(f) \frac{\partial \phi}{\partial t} + \phi \frac{\partial H(f)}{\partial t} = H(f) \frac{\partial \phi}{\partial t} - \phi v_j |\nabla f| \delta(f) \\ \frac{\partial [\phi H(f)]}{\partial x_i} &= H(f) \frac{\partial \phi}{\partial x_i} + \phi \frac{\partial H(f)}{\partial x_i} = H(f) \frac{\partial \phi}{\partial x_i} + \phi |\nabla f| \delta(f)\end{aligned}\quad (3)$$

代入连续方程有:

$$\begin{aligned}\frac{\partial [\rho' H(f)]}{\partial t} + \frac{\partial [\rho u_j H(f)]}{\partial x_j} &= \rho u_j |\nabla f| \delta(f) - \rho' v_j |\nabla f| \delta(f) \\ &= [\rho (u_j - v_j) + \rho_0 v_j] |\nabla f| \delta(f)\end{aligned}\quad (4)$$

代入动量方程有:

$$\begin{aligned}\frac{\partial [H(f) \rho u_i]}{\partial t} + c_0^2 \frac{\partial [H(f) \rho']}{\partial x_i} &= -H(f) \frac{\partial T_{ij}}{\partial x_j} + (c_0^2 \rho' \delta_{ij} - \rho u_i v_j) |\nabla f| \delta(f) \\ &= -\frac{\partial [H(f) T_{ij}]}{\partial x_j} + (T_{ij} + c_0^2 \rho' \delta_{ij} - \rho u_i v_j) |\nabla f| \delta(f) \\ &= -\frac{\partial [H(f) T_{ij}]}{\partial x_j} + [\rho u_i (u_j - v_j) + p_{ij}] |\nabla f| \delta(f)\end{aligned}\quad (5)$$

于是有:

$$\frac{\partial^2 [\rho' H(f)]}{\partial t^2} - c_0^2 \frac{\partial^2 [H(f) \rho']}{\partial x_i^2} = \frac{\partial^2 [H(f) T_{ij}]}{\partial x_i \partial x_j} - \frac{\partial [F_i \delta(f)]}{\partial x_i} + \frac{\partial [Q \delta(f)]}{\partial t} \quad (6)$$

其中,

$$\begin{aligned}Q &= [\rho (u_j - v_j) + \rho_0 v_j] |\nabla f| \\ F_i &= [\rho u_i (u_j - v_j) + p_{ij}] |\nabla f|\end{aligned}$$

其积分表达式可以表示为:

$$H(f) c_0^2 \rho'(\mathbf{x}, t) = \int_V \int_{-\infty}^{+\infty} G \left\{ \frac{\partial^2 [H(f) T_{ij}]}{\partial y_i \partial y_j} - \frac{\partial [F_i \delta(f)]}{\partial y_i} + \frac{\partial [Q \delta(f)]}{\partial \tau} \right\} d\tau d^3 \mathbf{y} \quad (7)$$

对于四极子项:

$$\begin{aligned}
G \frac{\partial^2 [T_{ij}H(f)]}{\partial y_i \partial y_j} &= [T_{ij}H(f)] \frac{\partial^2 G}{\partial y_i \partial y_j} \\
&+ \frac{\partial}{\partial y_i} \left(G \frac{\partial [T_{ij}H(f)]}{\partial y_j} \right) \\
&- \frac{\partial}{\partial y_j} \left([T_{ij}H(f)] \frac{\partial G}{\partial y_i} \right)
\end{aligned} \quad (8)$$

其中,

$$\begin{aligned}
\int_{\Sigma+\Omega} \int_{-\infty}^{+\infty} \frac{\partial}{\partial y_i} \left(G \frac{\partial [T_{ij}H(f)]}{\partial y_j} \right) d\tau d^3\mathbf{y} &= \int_{S_\infty} \int_{-\infty}^{+\infty} G \frac{\partial [T_{ij}H(f)]}{\partial y_j} n_i d\tau d^2\mathbf{y} = 0 \\
\int_{\Sigma+\Omega} \int_{-\infty}^{+\infty} \frac{\partial}{\partial y_j} \left(T_{ij}H(f) \frac{\partial G}{\partial y_i} \right) d\tau d^3\mathbf{y} &= \int_{S_\infty} \int_{-\infty}^{+\infty} T_{ij}H(f) \frac{\partial G}{\partial y_j} n_j d\tau d^2\mathbf{y} = 0
\end{aligned}$$

因此有:

$$\int_V \int_{-\infty}^{+\infty} G \frac{\partial^2 [H(f)T_{ij}]}{\partial y_i \partial y_j} d\tau d^3\mathbf{y} = \int_V \int_{-\infty}^{+\infty} H(f)T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} d\tau d^3\mathbf{y} \quad (9)$$

对于偶极子项:

$$\begin{aligned}
\int_V \int_{-\infty}^{+\infty} G \frac{\partial [F_i \delta(f)]}{\partial y_i} d\tau d^3\mathbf{y} &= \int_V \int_{-\infty}^{+\infty} \frac{\partial [GF_i \delta(f)]}{\partial y_i} d\tau d^3\mathbf{y} \\
&- \int_V \int_{-\infty}^{+\infty} F_i \delta(f) \frac{\partial G}{\partial y_i} d\tau d^3\mathbf{y}
\end{aligned} \quad (10)$$

对于无边界区域, 有:

$$\int_V \frac{\partial [GF_i \delta(f)]}{\partial y_i} d^3\mathbf{y} = \int_{\Sigma+\Omega} \frac{\partial [GF_i \delta(f)]}{\partial y_i} d^3\mathbf{y} = \int_{S_\infty} GF_i \delta(f) n_i d^2\mathbf{y} = 0 \quad (11)$$

因此有:

$$\int_V \int_{-\infty}^{+\infty} G \frac{\partial [F_i \delta(f)]}{\partial y_i} d\tau d^3\mathbf{y} = - \int_V \int_{-\infty}^{+\infty} F_i \delta(f) \frac{\partial G}{\partial y_i} d\tau d^3\mathbf{y} \quad (12)$$

对于单极子项:

$$\begin{aligned}
\int_V \int_{-\infty}^{+\infty} G \frac{\partial [Q \delta(f)]}{\partial \tau} d\tau d^3\mathbf{y} &= \int_V \int_{-\infty}^{+\infty} \frac{\partial [GQ \delta(f)]}{\partial \tau} d\tau d^3\mathbf{y} \\
&- \int_V \int_{-\infty}^{+\infty} Q \delta(f) \frac{\partial G}{\partial \tau} d\tau d^3\mathbf{y}
\end{aligned} \quad (13)$$

根据 $G = \frac{\partial G}{\partial \tau} = 0 (t < \tau)$, 有:

$$\int_{-\infty}^{+\infty} \frac{\partial [GQ \delta(f)]}{\partial \tau} d\tau = GQ \delta(f) \Big|_{\tau=-\infty}^{\tau=+\infty} = 0 \quad (14)$$

因此有：

$$\int_V \int_{-\infty}^{+\infty} G \frac{\partial [Q\delta(f)]}{\partial \tau} d\tau d^3\mathbf{y} = - \int_V \int_{-\infty}^{+\infty} Q\delta(f) \frac{\partial G}{\partial \tau} d\tau d^3\mathbf{y} \quad (15)$$

综上，FW-H 方程的积分表达式可表示为：

$$\begin{aligned} H(f)c_0^2\rho'(\mathbf{x},t) = & \int_V \int_{-\infty}^{+\infty} H(f)T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} d\tau d^3\mathbf{y} \\ & + \int_V \int_{-\infty}^{+\infty} F_i \delta(f) \frac{\partial G}{\partial y_i} d\tau d^3\mathbf{y} \\ & - \int_V \int_{-\infty}^{+\infty} Q\delta(f) \frac{\partial G}{\partial \tau} d\tau d^3\mathbf{y} \end{aligned} \quad (16)$$

3. 如果 $f = 0$ 的面不是固体表面，而是流体区域任意选择的可穿透封闭面，FW-H 方程还成立吗？

成立。FW-H 方程仅假设 $f = 0$ 为移动的刚体表面，并没有假设表面是否可穿透。因此 FW-H 方程对可穿透表面成立。对于不可穿透表面，FW-H 方程可以进一步简化，简化后的方程对可穿透表面不成立。