Section1.1

线性声波在密度为 ρ_0 的均匀静止理想流体介质中以速度 c_0 传播，声学压力和声学速度用 p' 和 u' 表示，速度势函数用 ϕ 表示，证明：

(1) 声压可以表示为

$$p' = -\rho_0 \frac{\partial \phi}{\partial t}.$$

已知线化声学动量方程：

$$\rho_0 \frac{\partial u'}{\partial t} + \nabla p' = 0 \tag{1}$$

代入势函数:

$$u' = \nabla \phi \tag{2}$$

得到：

$$\begin{aligned} \nabla p' &= -\rho_0 \frac{\partial u'}{\partial t} \\ &= -\rho_0 \frac{\partial (\nabla \phi)}{\partial t} \\ &= \nabla (-\rho_0 \frac{\partial \phi}{\partial t}) \end{aligned} \tag{3}$$

由上式可得：

$$p' = -\rho_0 \frac{\partial \phi}{\partial t} \tag{4}$$

原式得证。

(2) 势函数和声学速度满足波动方程

$$\frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0, \frac{1}{c_0^2} \frac{\partial^2 u'}{\partial t^2} - \nabla^2 u' = 0.$$

已知声学波动方程：

$$\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0 \tag{5}$$

代入(4)式，得：

$$\begin{aligned} \frac{1}{c_0^2} (-\rho_0) \frac{\partial^2}{\partial t^2} \left(\frac{\partial \phi}{\partial t} \right) - (-\rho_0) \nabla^2 (\frac{\partial \phi}{\partial t}) = 0 \\ \frac{\partial}{\partial t} \left[\frac{1}{\cos^2} \frac{\partial^2 \phi}{\partial t^2} \right] - \frac{\partial}{\partial t} \left[\nabla^2 \phi \right] = 0 \end{aligned} \tag{6}$$

两边对 t 积分，得：

$$\frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0 \tag{7}$$

第一式得证。

将(7)式两边求梯度，得：

$$
\begin{aligned}
\nabla\left(\frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2}\right)-\nabla^3\phi=0 \\
\frac{1}{c_0^2}\frac{\partial^2\left(\nabla\phi\right)}{\partial t^2}-\nabla^2\left(\nabla\phi\right)=0
\end{aligned}
\tag{8}
$$

代入势函数（式(2)），得：

$$
\frac{1}{c_0^2}\frac{\partial^2 u'}{\partial t^2}-\nabla^2 u'=0 \tag{9}
$$

第二式得证。

Section1.2

1. 证明自由空间格林函数的偏导数关系：

$$\frac{\partial G_0}{\partial y_i}=\frac{x_i-y_i}{r}\left[\frac{1}{4\pi r c_0}\frac{\partial}{\partial \tau}\delta\left(t-\tau-r/c_0\right)+\frac{\delta\left(t-\tau-r/c_0\right)}{4\pi r^2}\right].$$

已知自由空间格林函数：

$$G_0=\frac{1}{4\pi r}\delta\left(t-\tau-\frac{r}{c_0}\right) \tag{10}$$

对 r 求偏导，得：

$$\begin{aligned}\frac{\partial G_0}{\partial r}&=-\frac{1}{4\pi r^2}\delta\left(t-\tau-\frac{r}{c_0}\right)+\frac{1}{4\pi r}\frac{\partial\delta\left(t-\tau-\frac{r}{c_0}\right)}{\partial r}\\&=-\frac{1}{4\pi r^2}\delta\left(t-\tau-\frac{r}{c_0}\right)+\frac{1}{4\pi r}\frac{\partial\delta\left(t-\tau-\frac{r}{c_0}\right)}{\partial \tau}\frac{\partial\tau}{\partial r}\end{aligned} \tag{11}$$

由 τ 与 r 的关系式 $\tau=t-\frac{r}{c_0}$ 可得：

$$\frac{\partial\tau}{\partial r}=-\frac{1}{c_0} \tag{12}$$

代入式(11), 得

$$\frac{\partial G_0}{\partial r}=-\frac{1}{4\pi r^2}\delta\left(t-\tau-\frac{r}{c_0}\right)-\frac{1}{4\pi r c_0}\frac{\partial\delta\left(t-\tau-\frac{r}{c_0}\right)}{\partial \tau} \tag{13}$$

根据 r 对 y_i 的偏导数：

$$\frac{\partial r}{\partial y_i}=\frac{\partial\sqrt{\sum(x_i-y_i)^2}}{\partial y_i}\quad=-\frac{x_i-y_i}{r} \tag{14}$$

结合式(13),(14), 得：

$$\begin{aligned}\frac{\partial G_0}{\partial y_i}&=\frac{\partial G_0}{\partial r}\frac{\partial r}{\partial y_i}\\&=\frac{x_i-y_i}{r}\left[\frac{1}{4\pi r c_0}\frac{\partial}{\partial \tau}\delta\left(t-\tau-r/c_0\right)+\frac{\delta\left(t-\tau-r/c_0\right)}{4\pi r^2}\right]\end{aligned} \tag{15}$$

原式得证。

2. 利用上述自由空间格林函数的偏导数关系式证明

$$p'(\mathbf{x},t) = -\int_{-\infty}^{+\infty}\int_S \rho_0 \frac{\partial u_n(\mathbf{y},\tau)}{\partial \tau} G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}S\ \mathrm{d}\tau - \int_{-\infty}^{+\infty}\int_S p'(\mathbf{y},\tau)\frac{\partial G(\mathbf{x},\mathbf{y},t-\tau)}{\partial y_i} n_i\ \mathrm{d}S\ \mathrm{d}\tau$$

可以改写为

$$p'(\mathbf{x},t) = -\int_S \left[\rho_0 \frac{\partial u_n}{\partial \tau}\right]_\tau \frac{\mathrm{d}S(\mathbf{y})}{4\pi r} - \int_S \left[\frac{\partial p'}{\partial \tau} n_i + \frac{p' n_i c_0}{r}\right]_\tau \frac{(x_i - y_i)\,\mathrm{d}S(\mathbf{y})}{4\pi r^2 c_0}.$$

原式右侧第一项代入自由格林函数，并对 τ 求积分：

$$\begin{aligned}
\text{右侧第一项} &= -\int_{-\infty}^{+\infty}\int_S \rho_0 \frac{\partial u_n(\mathbf{y},\tau)}{\partial \tau} G_0(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}S\ \mathrm{d}\tau \\
&= -\int_s \int_{-\infty}^{+\infty} \rho_0 \frac{\partial u_n(\mathbf{y},\tau)}{\partial \tau} \frac{1}{4\pi r}\delta(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau\ \mathrm{d}s \\
&= -\int_S \left[\rho_0 \frac{\partial u_n}{\partial \tau}\right]_\tau \frac{\mathrm{d}S(\mathbf{y})}{4\pi r}
\end{aligned} \tag{16}$$

原式右侧第二项代入自由格林函数偏导数关系式（式(15)）:

$$\begin{aligned}
\text{右侧第二项} = &-\int_{-\infty}^{+\infty}\int_S p'(\mathbf{y},\tau)\frac{x_i - y_i}{r}[\frac{1}{4\pi r c_0}\frac{\partial}{\partial \tau}\delta\left(t-\tau-\frac{r}{c_0}\right) \\
&+ \frac{\delta\left(t-\tau-\frac{r}{c_0}\right)}{4\pi r^2}] n_i\ \mathrm{d}S\ \mathrm{d}\tau \\
= &-\int_S \left[\int_{-\infty}^{+\infty} p'(\mathbf{y},\tau)\frac{\partial}{\partial \tau}\delta\left(t-\tau-\frac{r}{c_0}\right) n_i \mathrm{d}\tau\right] \frac{(x_i - y_i)\,\mathrm{d}S}{4\pi r^2 c_0} \\
&-\int_S \left[\int_{-\infty}^{+\infty} p'(\mathbf{y},\tau)\frac{c_0}{r}\delta\left(t-\tau-\frac{r}{c_0}\right) n_i \mathrm{d}\tau\right] \frac{(x_i - y_i)\,\mathrm{d}S}{4\pi r^2 c_0}
\end{aligned} \tag{17}$$

其中：

$$\begin{aligned}
\int_{-\infty}^{+\infty} p'(\mathbf{y},\tau)\frac{\partial}{\partial \tau}\delta\left(t-\tau-\frac{r}{c_0}\right) n_i \mathrm{d}\tau = &-p'(\mathbf{y},\tau)\delta(t-\tau-\frac{r}{c_0})n_i\Big|_{\tau=-\infty}^{\tau=\infty} \\
&- \int_{-\infty}^{+\infty} -\frac{\partial p'(\mathbf{y},\tau)}{\partial \tau}\delta(t-\tau-\frac{r}{c_0})\mathrm{d}\tau
\end{aligned} \tag{18}$$

根据 t 与 τ 的因果关系，有：

$$-p'(\mathbf{y},\tau)\delta(t-\tau-\frac{r}{c_0})n_i\Big|_{\tau=-\infty}^{\tau=\infty} = 0 \tag{19}$$

因此有：

$$\begin{aligned}\int_{-\infty}^{+\infty} p'(\mathbf{y},\tau)\frac{\partial}{\partial\tau}\delta\left(t-\tau-\frac{r}{c_0}\right)n_i\mathrm{d}\tau &= -\int_{-\infty}^{+\infty} -\frac{\partial p'(\mathbf{y},\tau)}{\partial\tau}\delta(t-\tau-\frac{r}{c_0})\mathrm{d}\tau \\ &= \left[\frac{\partial p'}{\partial\tau}n_i\right]_\tau\end{aligned} \tag{20}$$

同时，式(17)中：

$$\int_{-\infty}^{+\infty} p'(\mathbf{y},\tau)\frac{c_0}{r}\delta\left(t-\tau-\frac{r}{c_0}\right)n_i\mathrm{d}\tau = \left[\frac{p'n_ic_0}{r}\right]_\tau \tag{21}$$

将式(20),(21)代入式(17)，得：

$$\begin{aligned}\text{右侧第二项} &= -\int_S\left[\frac{\partial p'}{\partial\tau}n_i\right]_\tau\frac{(x_i-y_i)\,\mathrm{d}S}{4\pi r^2c_0} - \int_S\left[\frac{p'n_ic_0}{r}\right]_\tau\frac{(x_i-y_i)\,\mathrm{d}S}{4\pi r^2c_0} \\ &= -\int_S\left[\frac{\partial p'}{\partial\tau}n_i+\frac{p'n_ic_0}{r}\right]_\tau\frac{(x_i-y_i)\,\mathrm{d}S}{4\pi r^2c_0}\end{aligned} \tag{22}$$

结合式(16),(22), 得：

$$p'(\mathbf{x},t) = -\int_S\left[\rho_0\frac{\partial u_n}{\partial\tau}\right]_\tau\frac{\mathrm{d}S(\mathbf{y})}{4\pi r} - \int_S\left[\frac{\partial p'}{\partial\tau}n_i+\frac{p'n_ic_0}{r}\right]_\tau\frac{(x_i-y_i)\,\mathrm{d}S(\mathbf{y})}{4\pi r^2c_0} \tag{23}$$

原式得证。

Section1.3

1. 定义任意时域函数 $f(t)$ 和 $h(t)$，通过 Fourier 变换得到的频域函数分别为 $\tilde{f}(\omega)$ 和 $\tilde{h}(\omega)$，利用 Fourier 变换定义证明下述关系式成立:

   (1) 如果 $f(t)=\int_{-\infty}^{\infty}h(\tau)G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau$，则有 $\tilde{f}(\omega)=\tilde{h}(\omega)\tilde{G}(\mathbf{x},\mathbf{y},\omega)$。

   根据 Fourier 变换，有:

$$
\begin{aligned}
\tilde{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{i\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau e^{i\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} G(\mathbf{x},\mathbf{y},t-\tau)e^{i\omega(t-\tau)}\mathrm{d}t\right]e^{i\omega\tau}\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\tilde{G}(\mathbf{x},\mathbf{y},\omega)e^{i\omega\tau}\mathrm{d}\tau \\
&= \tilde{G}(\mathbf{x},\mathbf{y},\omega)\int_{-\infty}^{\infty} h(\tau)e^{i\omega\tau}\mathrm{d}\tau \\
&= \tilde{h}(\omega)\tilde{G}(\mathbf{x},\mathbf{y},\omega)
\end{aligned} \tag{24}
$$

   原式得证。

   (2) 如果 $f(t)=\int_{-\infty}^{\infty}h(\tau)\frac{\partial G(\mathbf{x},\mathbf{y},t-\tau)}{\partial\tau}\mathrm{d}\tau$，则有 $\tilde{f}(\omega)=-i\omega\tilde{h}(\omega)\tilde{G}(\mathbf{x},\mathbf{y},\omega)$。

   根据分步积分，有:

$$
\begin{aligned}
f(\omega) &= \int_{-\infty}^{\infty} h(\tau)\frac{\partial G(\mathbf{x},\mathbf{y},t-\tau)}{\partial\tau}\mathrm{d}\tau \\
&= -h(\tau)G(\mathbf{x},\mathbf{y},t-\tau)|_{\tau=-\infty}^{\tau=\infty} - \int_{-\infty}^{\infty} -\frac{\partial h(\tau)}{\partial\tau}G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau
\end{aligned} \tag{25}
$$

   根据 t 与 τ 的因果关系，有:

$$
-h(\tau)G(\mathbf{x},\mathbf{y},t-\tau)|_{\tau=-\infty}^{\tau=\infty} = 0 \tag{26}
$$

   因此有:

$$
f(\omega) = \int_{-\infty}^{\infty}\frac{\partial h(\tau)}{\partial\tau}G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau \tag{27}
$$

根据 Fourier 变换，有:

$$\begin{aligned}
\tilde{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{i\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial h(\tau)}{\partial \tau} G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau e^{i\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{\partial h(\tau)}{\partial \tau}\left[\int_{-\infty}^{\infty} G(\mathbf{x},\mathbf{y},t-\tau)e^{i\omega(t-\tau)}\mathrm{d}t\right] e^{i\omega\tau}\mathrm{d}\tau \\
&= \tilde{G}(\mathbf{x},\mathbf{y},\omega)\int_{-\infty}^{\infty} \frac{\partial h(\tau)}{\partial \tau} e^{i\omega\tau}\mathrm{d}\tau \\
&= -i\omega\tilde{h}(\omega)\tilde{G}(\mathbf{x},\mathbf{y},\omega)
\end{aligned} \tag{28}$$

原式得证。

2. 根据波动方程的时域解，证明频域积分解可以写为

$$\tilde{p'}(\boldsymbol{x},\omega) = \int_S i\omega\rho_0\tilde{u}_n(\boldsymbol{y},\omega)\tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)\mathrm{d}S - \int_S \tilde{p'}(\boldsymbol{y},\omega)\frac{\partial\tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)}{\partial\boldsymbol{n}}\mathrm{d}S.$$

已知声学波动方程的时域解:

$$\begin{aligned}
p'(\boldsymbol{x},t) = &-\int_{-\infty}^{+\infty}\int_S \rho_0\frac{\partial u_n(\boldsymbol{y},\tau)}{\partial\tau}G(\boldsymbol{x},\boldsymbol{y},t-\tau)\mathrm{d}S\ \mathrm{d}\tau \\
&-\int_{-\infty}^{+\infty}\int_S p'(\boldsymbol{y},\tau)\frac{\partial G(\boldsymbol{x},\boldsymbol{y},t-\tau)}{\partial\boldsymbol{n}}\ \mathrm{d}S\ \mathrm{d}\tau
\end{aligned} \tag{29}$$

对上式进行 Fourier 变换:

$$\begin{aligned}
\tilde{p'}(\boldsymbol{x},\omega) = &\int_{-\infty}^{+\infty}\left[-\int_{-\infty}^{+\infty}\int_S \rho_0\frac{\partial u_n(\boldsymbol{y},\tau)}{\partial\tau}G(\boldsymbol{x},\boldsymbol{y},t-\tau)\mathrm{d}S\ \mathrm{d}\tau\right. \\
&\left.-\int_{-\infty}^{+\infty}\int_S p'(\boldsymbol{y},\tau)\frac{\partial G(\boldsymbol{x},\boldsymbol{y},t-\tau)}{\partial\boldsymbol{n}}\ \mathrm{d}S\ \mathrm{d}\tau\right] e^{i\omega t}\mathrm{d} \\
= &-\int_S\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_0\frac{\partial u_n(\boldsymbol{y},\tau)}{\partial\tau}G(\boldsymbol{x},\boldsymbol{y},t-\tau)e^{i\omega t}\mathrm{d}\tau\ \mathrm{d}t\right]\mathrm{d}S \\
&-\int_S\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}p'(\boldsymbol{y},\tau)\frac{\partial G(\boldsymbol{x},\boldsymbol{y},t-\tau)}{\partial\boldsymbol{n}}e^{i\omega t}\mathrm{d}\tau\ \mathrm{d}t\right]\mathrm{d}S
\end{aligned} \tag{30}$$

由第一题中的结论，式(27)、(28)可得:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_0\frac{\partial u_n(\boldsymbol{y},\tau)}{\partial\tau}G(\boldsymbol{x},\boldsymbol{y},t-\tau)e^{i\omega t}\mathrm{d}\tau\ \mathrm{d}t \\
=&-i\omega\rho_0\tilde{u}_n(\boldsymbol{y},\omega)\tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)
\end{aligned} \tag{31}$$

有第一题中的结论，式(24)可得:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p'(\boldsymbol{y},\tau)\frac{\partial G(\boldsymbol{x},\boldsymbol{y},t-\tau)}{\partial \boldsymbol{n}}e^{i\omega t}\mathrm{d}\tau\ \mathrm{d}t = \tilde{p}'(\boldsymbol{y},\omega)\frac{\partial \tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)}{\partial \boldsymbol{n}} \tag{32}$$

将式(31)、(32)代入式(30)得:

$$\tilde{p}'(\boldsymbol{x},\omega) = \int_S i\omega\rho_0\tilde{u}_n(\boldsymbol{y},\omega)\tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)\mathrm{d}S - \int_S \tilde{p}'(\boldsymbol{y},\omega)\frac{\partial \tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)}{\partial \boldsymbol{n}}\mathrm{d}S \tag{33}$$

原式得证。

Section2.1

1. Lighthill 声比拟方程能直接应用于高 Ma 流动诱发的气动噪声问题吗?
   不能。(1)Lighthill 声比拟方程假设空气介质是均匀静止的，但该条件不能适用于高马赫数流动；(2)Lighthill 声比拟方程没有考虑能量输运作用；(3)Lighthill 声比拟方程仅适用于弱可压缩流动，不适用于高马赫数下的强可压缩流动。

2. 从 Lighthill 声比拟方程出发，详细证明方程的时域积分解为

   根据声比拟方程，有：

$$p'(\mathbf{x},\tau)=\int_{-\infty}^{\infty}\int_V G\frac{\partial^2 T_{ij}(\mathbf{y},\tau)}{\partial y_i\partial y_j}\mathrm{d}V\mathrm{d}\tau \tag{34}$$

   根据分步积分，有：

$$G\frac{\partial^2 T_{ij}}{\partial y_i\partial y_j}=T_{ij}\frac{\partial^2 G}{\partial y_i\partial y_j}+\frac{\partial}{\partial y_i}\left(G\frac{\partial T_{ij}}{\partial y_j}\right)-\frac{\partial}{\partial y_j}\left(T_{ij}\frac{\partial G}{\partial y_i}\right) \tag{35}$$

   因此有：

$$\begin{aligned}\int_{-\infty}^{\infty}\int_V G\frac{\partial^2 T_{ij}}{\partial y_i\partial y_j}\mathrm{d}V\mathrm{d}\tau=&\int_{-\infty}^{\infty}\int_V T_{ij}\frac{\partial^2 G}{\partial y_i\partial y_j}\mathrm{d}V\mathrm{d}\tau\\&+\int_{-\infty}^{\infty}\int_V\frac{\partial}{\partial y_i}\left(G\frac{\partial T_{ij}}{\partial y_j}\right)\mathrm{d}V\mathrm{d}\tau\\&-\int_{-\infty}^{\infty}\int_V\frac{\partial}{\partial y_j}\left(T_{ij}\frac{\partial G}{\partial y_i}\right)\mathrm{d}V\mathrm{d}\tau\end{aligned} \tag{36}$$

   注意到 $T_{ij}=T_{ji}$，因此有：

$$\begin{aligned}\int_{-\infty}^{\infty}\int_V G\frac{\partial^2 T_{ij}}{\partial y_i\partial y_j}\mathrm{d}V\mathrm{d}\tau=&\int_{-\infty}^{\infty}\int_V T_{ij}\frac{\partial^2 G}{\partial y_i\partial y_j}\mathrm{d}V\mathrm{d}\tau\\&+\int_{-\infty}^{\infty}\int_V\frac{\partial}{\partial y_j}\left[G\frac{\partial T_{ij}}{\partial y_i}-T_{ij}\frac{\partial G}{\partial y_i}\right]\mathrm{d}V\mathrm{d}\tau\end{aligned} \tag{37}$$

   应用高斯散度定理，有：

$$\begin{aligned}\int_{-\infty}^{\infty}\int_V G\frac{\partial^2 T_{ij}}{\partial y_i\partial y_j}\mathrm{d}V\mathrm{d}\tau=&\int_{-\infty}^{\infty}\int_V T_{ij}\frac{\partial^2 G}{\partial y_i\partial y_j}\mathrm{d}V\mathrm{d}\tau\\&+\int_{-\infty}^{\infty}\int_S\left[G\frac{\partial T_{ij}}{\partial y_i}-T_{ij}\frac{\partial G}{\partial y_i}\right]n_i\mathrm{d}S\mathrm{d}\tau\end{aligned} \tag{38}$$

对于 Lighthill 声比拟方程，S 为无穷大，因此有：

$$\int_{-\infty}^{\infty}\int_{S}\left[G\frac{\partial T_{ij}}{\partial y_i}-T_{ij}\frac{\partial G}{\partial y_i}\right]n_i\mathrm{d}S\mathrm{d}\tau=0 \tag{39}$$

因此：

$$\begin{aligned}p'(\mathbf{x},\tau)&=\int_{-\infty}^{\infty}\int_{V}G\frac{\partial^2 T_{ij}(\mathbf{y},\tau)}{\partial y_i\partial y_j}\mathrm{d}V\mathrm{d}\tau\\&=\int_{-\infty}^{\infty}\int_{V}T_{ij}(\mathbf{y},\tau)\frac{\partial^2 G}{\partial y_i\partial y_j}\mathrm{d}V\mathrm{d}\tau\end{aligned} \tag{40}$$

代入自由空间格林函数 G_0，有：

$$p'(\mathbf{x},\tau)=\int_{-\infty}^{\infty}\int_{V}T_{ij}(\mathbf{y},\tau)\frac{\partial^2 G_0(\mathbf{x},\mathbf{y},t-\tau)}{\partial y_i\partial y_j}\mathrm{d}V\mathrm{d}\tau \tag{41}$$

自由空间格林函数 G_0 满足：

$$\frac{\partial^2 G_0}{\partial y_i\partial y_j}=\frac{\partial^2 G_0}{\partial x_i\partial x_j} \tag{42}$$

因此有：

$$\begin{aligned}p'(\mathbf{x},\tau)&=\int_{-\infty}^{\infty}\int_{V}T_{ij}(\mathbf{y},\tau)\frac{\partial^2 G_0(\mathbf{x},\mathbf{y},t-\tau)}{\partial x_i\partial x_j}\mathrm{d}^3\mathbf{y}\ \mathrm{d}\tau\\&=\frac{\partial^2}{\partial x_i\partial x_j}\int_{V}\int_{-\infty}^{\infty}T_{ij}(\mathbf{y},\tau)G_0(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau\ \mathrm{d}^3\mathbf{y}\\&=\frac{\partial^2}{\partial x_i\partial x_j}\int_{V}\int_{-\infty}^{\infty}T_{ij}(\mathbf{y},\tau)\frac{\delta(t-\tau-r/c_0)}{4\pi r}\mathrm{d}\tau\ \mathrm{d}^3\mathbf{y}\\&=\frac{1}{4\pi}\frac{\partial^2}{\partial x_i\partial x_j}\int_{V}\frac{T_{ij}\left(\mathbf{y},t-r/c_0\right)}{r}\ \mathrm{d}^3\mathbf{y}\end{aligned} \tag{43}$$

原式得证。

Section2.2

1. 已知三维频域自由空间格林函数为 $G_0(\mathbf{x},\mathbf{y},\omega)=\frac{e^{ikr}}{4\pi r}$，推导 $\frac{\partial G_0}{\partial y_i}$ 和 $\frac{\partial^2 G_0}{\partial y_i \partial y_j}$ 的解析表达式。

   已知，在三维频域下：

$$r=\sqrt{\sum_{i=1}^{n=3}(x_i-y_i)^2} \tag{44}$$

   因此有：

$$\begin{aligned}\frac{\partial G_0}{\partial y_i}&=\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{4\pi r}\right)\frac{\partial r}{\partial y_i}\\&=\frac{ikre^{ikr}-e^{ikr}}{4\pi r^2}\frac{\partial\sqrt{\sum_{i=1}^{n=3}(x_i-y_i)^2}}{\partial y_i}\\&=\left(\frac{ik}{4\pi r}-\frac{1}{4\pi r^2}\right)e^{ikr}\left(-\frac{x_i-y_i}{r}\right)\\&=\frac{x_i-y_i}{r}\left(\frac{e^{ikr}}{4\pi r^2}-\frac{ike^{ikr}}{4\pi r}\right)\end{aligned} \tag{45}$$

   同理有：

$$\begin{aligned}\frac{\partial^2 G_0}{\partial y_i \partial y_j}&=\frac{\partial}{\partial y_j}\left(\frac{\partial G_0}{\partial y_i}\right)\\&=\frac{\partial}{\partial r}\left(\frac{\partial G_0}{\partial y_i}\right)\frac{\partial r}{\partial y_i}\\&=\frac{x_i-y_i}{r^2}\left(-\frac{3e^{ikr}}{4\pi r^2}+\frac{3ike^{ikr}}{4\pi r}+\frac{k^2e^{ikr}}{4\pi}\right)\left(-\frac{x_j-y_j}{r}\right)\\&=\frac{(x_i-y_i)(x_j-y_j)}{r^3}\left(\frac{3e^{ikr}}{4\pi r^2}-\frac{3ike^{ikr}}{4\pi r}-\frac{k^2e^{ikr}}{4\pi}\right)\end{aligned} \tag{46}$$

   综上，

$$\frac{\partial G_0}{\partial y_i}=\frac{x_i-y_i}{r}\left(\frac{e^{ikr}}{4\pi r^2}-\frac{ike^{ikr}}{4\pi r}\right) \tag{47}$$

$$\frac{\partial^2 G_0}{\partial y_i \partial y_j}=\frac{(x_i-y_i)(x_j-y_j)}{r^3}\left(\frac{3e^{ikr}}{4\pi r^2}-\frac{3ike^{ikr}}{4\pi r}-\frac{k^2e^{ikr}}{4\pi}\right) \tag{48}$$

2. 假设静止固体表面是可穿透的，并忽略粘性的贡献，写出 Curle 方程的频域积分公式。

已知忽略粘性贡献的 Curle 方程为：

$$
\begin{aligned}
c_0^2\rho'(\mathbf{x},t) = & \int_V \int_{-\infty}^{+\infty} T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \ \mathrm{d}^3\mathbf{y}\mathrm{d}\tau \\
& - \int_S \int_{-\infty}^{+\infty} \left(\rho u_i u_j + p_{ij}\right) n_j \frac{\partial G}{\partial y_i} \ \mathrm{d}^2\mathbf{y}\mathrm{d}\tau \\
& - \int_S \int_{-\infty}^{+\infty} G \frac{\partial \left(\rho u_j n_j\right)}{\partial \tau} \mathrm{d}^2\mathbf{y}\mathrm{d}\tau
\end{aligned} \tag{49}
$$

不妨设：

$$
F_i(\mathbf{y},\tau) = \left(\rho u_i u_j + p_{ij}\right) n_j \tag{50}
$$

$$
Q(\mathbf{y},\tau) = \rho u_j n_j \tag{51}
$$

代入自由格林函数 G_0，根据 G_0 的性质：

$$
\frac{\partial G_0}{\partial y_i} = -\frac{\partial G_0}{\partial x_i} \tag{52}
$$

$$
\frac{\partial^2 G_0}{\partial y_i \partial y_j} = \frac{\partial^2 G_0}{\partial x_i \partial x_j} \tag{53}
$$

可以得到：

$$
\begin{aligned}
c_0^2\rho'(\mathbf{x},t) = & \frac{\partial^2}{\partial x_i \partial x_j} \int_{-\infty}^{+\infty} \int_V T_{ij}(\mathbf{y},\tau) G_0 \ \mathrm{d}^3\mathbf{y}\mathrm{d}\tau \\
& + \frac{\partial}{\partial x_i} \int_{-\infty}^{+\infty} \int_S F_i(\mathbf{y},\tau) G_0 \ \mathrm{d}^2\mathbf{y}\mathbf{d}\tau \\
& - \int_{-\infty}^{+\infty} \int_S \frac{\partial Q(\mathbf{y},\tau)}{\partial \tau} G_0 \ \mathrm{d}^2\mathbf{y}\mathrm{d}\tau \\
= & \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left[T_{ij}\left(\mathbf{y}, t - r/c_0\right)\right]_{\tau = t - r/c_0} \frac{\mathrm{d}^3\mathbf{y}}{4\pi r} \\
& + \frac{\partial}{\partial x_i} \int_S \left[F_i\left(\mathbf{y}, t - r/c_0\right)\right]_{\tau = t - r/c_0} \frac{\mathrm{d}^2\mathbf{y}}{4\pi r} \\
& - \int_S \left[\frac{\partial}{\partial \tau} Q\left(\mathbf{y}, t - r/c_0\right)\right]_{\tau = t - r/c_0} \frac{\mathrm{d}^2\mathbf{y}}{4\pi r}
\end{aligned} \tag{54}
$$

根据 Fourier 变换，可得：

$$\begin{aligned}\left(c_0^2\tilde{\rho}'(\mathbf{x},\omega)\right)_{quadrupole} &= \frac{\partial^2}{\partial x_i \partial x_j}\int_V\int_{-\infty}^{+\infty} T_{ij}(\mathbf{y},t-r/c_0)e^{i\omega t}\mathrm{d}t\frac{\mathrm{d}^3\mathbf{y}}{4\pi r} \\ &= \frac{\partial^2}{\partial x_i \partial x_j}\int_V \widetilde{T_{ij}}(\mathbf{y},\omega)e^{i\omega r/c_0}\frac{\mathrm{d}^3\mathbf{y}}{4\pi r}\end{aligned} \tag{55}$$

同理有：

$$\begin{aligned}\left(c_0^2\tilde{\rho}'(\mathbf{x},\omega)\right)_{dipole} &= \frac{\partial}{\partial x_i}\int_S\int_{-\infty}^{+\infty} F_i(\mathbf{y},t-r/c_0)e^{i\omega t}\mathrm{d}t\frac{\mathrm{d}^2\mathbf{y}}{4\pi r} \\ &= \frac{\partial}{\partial x_i}\int_S \widetilde{F}_i(\mathbf{y},\omega)e^{i\omega r/c_0}\frac{\mathrm{d}^2\mathbf{y}}{4\pi r}\end{aligned} \tag{56}$$

根据 Fourier 变换的偏分性质，可得：

$$\begin{aligned}\left(c_0^2\tilde{\rho}'(\mathbf{x},\omega)\right)_{monopole} &= \int_S\int_{-\infty}^{+\infty} \frac{\partial}{\partial \tau}\left[Q\left(\mathbf{y},t-r/c_0\right)\right]e^{i\omega t}\mathrm{d}t\frac{\mathrm{d}^2\mathbf{y}}{4\pi r} \\ &= \int_S -i\omega\widetilde{Q}(\mathbf{y},\omega)e^{i\omega r/c_0}\frac{\mathrm{d}^2\mathbf{y}}{4\pi r}\end{aligned} \tag{57}$$

综上，curle 方程的频域积分表达式为：

$$\begin{aligned}c_0^2\tilde{\rho}'(\mathbf{x},\omega) = &\frac{\partial^2}{\partial x_i \partial x_j}\int_V \widetilde{T_{ij}}(\mathbf{y},\omega)e^{i\omega r/c_0}\frac{\mathrm{d}^3\mathbf{y}}{4\pi r} \\ &+ \frac{\partial}{\partial x_i}\int_S \widetilde{F}_i(\mathbf{y},\omega)e^{i\omega r/c_0}\frac{\mathrm{d}^2\mathbf{y}}{4\pi r} \\ &+ \int_S i\omega\widetilde{Q}(\mathbf{y},\omega)e^{i\omega r/c_0}\frac{\mathrm{d}^2\mathbf{y}}{4\pi r}\end{aligned} \tag{58}$$

Section2.3

1. 针对声学远场，证明近似表达式：

$$\frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{T_{ij}(\boldsymbol{y})}{r}\right] \approx \frac{1}{c_0^2}\frac{(x_i-y_i)(x_j-y_j)}{r^3}\left[\frac{\partial^2 T_{ij}(\boldsymbol{y})}{\partial \tau^2}\right].$$

根据偏分法则可以得到：

$$\begin{aligned}\frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{T_{ij}(\boldsymbol{y})}{r}\right] &= \left[\frac{\partial^2 T_{ij}(\mathbf{y})}{\partial \tau^2}\right]\frac{1}{r}\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_j} \\ &\quad + 2\left[\frac{\partial T_{ij}(\mathbf{y})}{\partial \tau}\right]\frac{\partial \tau}{\partial x_i}\frac{\partial (1/r)}{\partial x_i} + [T_{ij}(\mathbf{y})]\frac{\partial (1/r)}{\partial x_i \partial x_j}\end{aligned} \tag{59}$$

对于声学远场，可以将忽略上式中的 r^{-2} 和 r^{-3} 项，因此有：

$$\frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{T_{ij}(\boldsymbol{y})}{r}\right] \approx \left[\frac{\partial^2 T_{ij}(\mathbf{y})}{\partial \tau^2}\right]\frac{1}{r}\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_j} \tag{60}$$

其中，

$$\frac{\partial \tau}{\partial x_i} = \frac{\partial \tau}{\partial r}\frac{\partial r}{\partial x_i} \tag{61}$$

根据 τ 与 r 关系式：

$$\tau = t - \frac{r}{c_0} \tag{62}$$

有：

$$\frac{\partial \tau}{\partial r} = -\frac{1}{c_0} \tag{63}$$

又因为：

$$\frac{\partial r}{\partial x_i} = \frac{\partial \sqrt{\sum (x_i-y_i)^2}}{\partial x_i} = \frac{x_i - y_i}{r} \tag{64}$$

因此有：

$$\frac{\partial \tau}{\partial x_i} = -\frac{1}{c_0}\frac{x_i-y_i}{r} \tag{65}$$

同理：

$$\frac{\partial \tau}{\partial x_j} = -\frac{1}{c_0}\frac{x_j-y_j}{r} \tag{66}$$

代入式(60)，得：

$$\frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{T_{ij}(\boldsymbol{y})}{r}\right] \approx \frac{1}{c_0^2}\frac{(x_i-y_i)(x_j-y_j)}{r^3}\left[\frac{\partial^2 T_{ij}(\boldsymbol{y})}{\partial \tau^2}\right] \tag{67}$$

原式得证。

2. 对于等熵流动，$\frac{\partial^2}{\partial \tau^2}\left(p' - c_0^2 \rho'\right) = 0$ 一定成立吗？

   不一定。$p' = c_0^2 \rho'$ 成立的前提是均匀介质，对于梯度较大的介质，$p' \neq c_0^2 \rho'$，因此，$\frac{\partial^2}{\partial \tau^2}\left(p' - c_0^2 \rho'\right) = 0$ 不一定成立。

3. 参数 p' 和 ρ' 哪一个更适合描述非稳态低速燃烧流动产生的噪声？

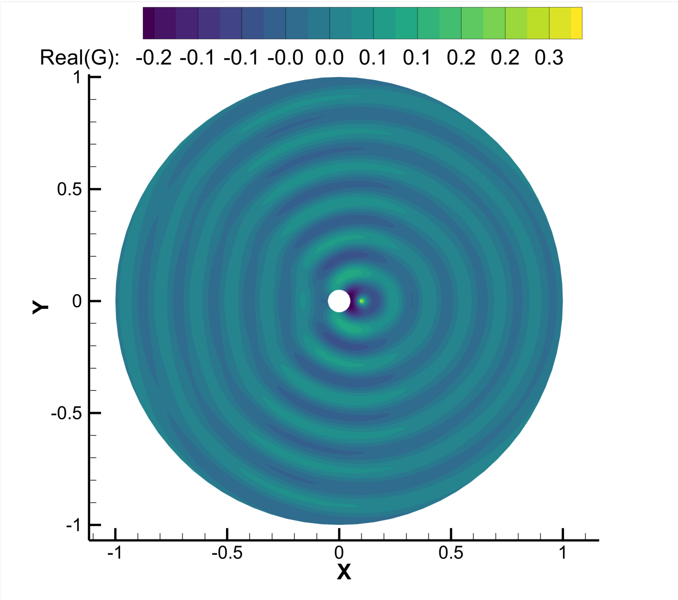
   p' 更适合。非稳态低速燃烧流动涉及到能量方程，而参数 p' 主要就源于能量方程，因此 p' 更适合。

Section2.4

1. 如图所示二维刚性圆柱, 单极子点源声辐射的波数 $k = 40$, 根据精确格林函数 G 的解析解, 编程计算:

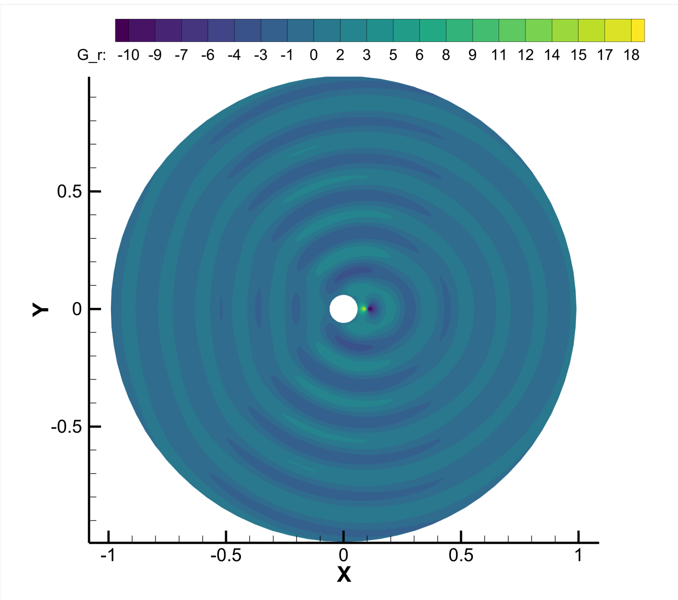
   (1) G 的空间分布 (空间外边界与坐标原点的距离不小于 7 倍波长);

   极坐标系下，G 的空间分布如图所示。



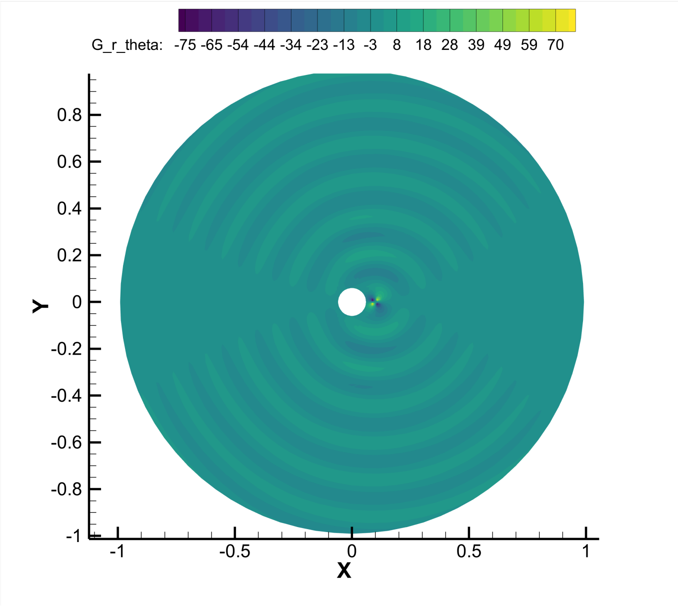
   (2) $\partial G/\partial y_i$ 的空间分布;

   极坐标系下，$\partial G/\partial r$ 的空间分布如图所示。



(3) $\partial^2 G/\partial y_i \partial y_j$ 的空间分布。

极坐标系下，$\partial^2 G/\partial r \partial \theta$ 的空间分布如图所示。



Section2.5

1. 在 FW-H 方程中，$f=0$ 的面在运动过程中形状能发生改变吗？

   不能。FW-H 方程的前提假设为刚体运动，$f=0$ 的面不能发生形变。

2. 如果 $|\nabla f| \neq 1$，试推导 FW-H 方程，并求其积分表达式。

   对于表面，有：

$$\begin{aligned} \frac{\mathrm{D}H(f)}{\mathrm{D}t} &= \frac{\partial H(f)}{\partial t} + v_j \frac{\partial H(f)}{\partial x_j} = 0 \\ \frac{\partial H(f)}{\partial x_j} &= \frac{\partial H(f)}{\partial f}|\nabla f| n_j = |\nabla f| n_j \delta(f) \end{aligned} \tag{68}$$

由上式可得：

$$\frac{\partial H(f)}{\partial t} = -v_j \frac{\partial H(f)}{\partial x_j} = -v_j |\nabla f| n_j \delta(f) \tag{69}$$

于是有：

$$\begin{aligned} \frac{\partial[\phi H(f)]}{\partial t} &= H(f)\frac{\partial \phi}{\partial t} + \phi \frac{\partial H(f)}{\partial t} = H(f)\frac{\partial \phi}{\partial t} - \phi v_j |\nabla f| n_j \delta(f) \\ \frac{\partial[\phi H(f)]}{\partial x_i} &= H(f)\frac{\partial \phi}{\partial x_i} + \phi \frac{\partial H(f)}{\partial x_i} = H(f)\frac{\partial \phi}{\partial x_i} + \phi |\nabla f| n_i \delta(f) \end{aligned} \tag{70}$$

代入连续方程有：

$$\begin{aligned} \frac{\partial\left[\rho' H(f)\right]}{\partial t} + \frac{\partial\left[\rho u_j H(f)\right]}{\partial x_j} &= \rho u_j |\nabla f| n_j \delta(f) - \rho' v_j |\nabla f| n_j \delta(f) \\ &= \left[\rho\left(u_j - v_j\right) + \rho_0 v_j\right] |\nabla f| n_j \delta(f) \end{aligned} \tag{71}$$

代入动量方程有：

$$\begin{aligned} \frac{\partial\left[H(f)\rho u_i\right]}{\partial t} + c_0^2 \frac{\partial\left[H(f)\rho'\right]}{\partial x_i} &= -H(f)\frac{\partial T_{ij}}{\partial x_j} + \left(c_0^2 \rho' \delta_{ij} - \rho u_i v_j\right) |\nabla f| n_j \delta(f) \\ &= -\frac{\partial\left[H(f) T_{ij}\right]}{\partial x_j} + \left(T_{ij} + c_0^2 \rho' \delta_{ij} - \rho u_i v_j\right) |\nabla f| n_j \delta(f) \\ &= -\frac{\partial\left[H(f) T_{ij}\right]}{\partial x_j} + \left[\rho u_i\left(u_j - v_j\right) + p_{ij}\right] |\nabla f| n_j \delta(f) \end{aligned} \tag{72}$$

于是有：

$$\frac{\partial^2\left[\rho' H(f)\right]}{\partial t^2} - c_0^2 \frac{\partial^2\left[H(f)\rho'\right]}{\partial x_i^2} = \frac{\partial^2\left[H(f) T_{ij}\right]}{\partial x_i \partial x_j} - \frac{\partial\left[F_i \delta(f)\right]}{\partial x_i} + \frac{\partial[Q\delta(f)]}{\partial t} \tag{73}$$

其中，

$$Q = [\rho(u_j - v_j) + \rho_0 v_j] |\nabla f| n_j$$

$$F_i = [\rho u_i (u_j - v_j) + p_{ij}] |\nabla f| n_j$$

其积分表达式可以表示为：

$$H(f) c_0^2 \rho'(\mathbf{x}, t) = \int_V \int_{-\infty}^{+\infty} G \left\{ \frac{\partial^2 [H(f) T_{ij}]}{\partial y_i \partial y_j} - \frac{\partial [F_i \delta(f)]}{\partial y_i} + \frac{\partial [Q \delta(f)]}{\partial \tau} \right\} \mathrm{d}\tau \mathrm{d}^3\mathbf{y} \tag{74}$$

对于四极子项：

$$\begin{aligned} G\frac{\partial^2 [T_{ij} H(f)]}{\partial y_i \partial y_j} = & [T_{ij} H(f)] \frac{\partial^2 G}{\partial y_i \partial y_j} \\ & + \frac{\partial}{\partial y_i} \left(G \frac{\partial [T_{ij} H(f)]}{\partial y_j} \right) \\ & - \frac{\partial}{\partial y_j} \left([T_{ij} H(f)] \frac{\partial G}{\partial y_i} \right) \end{aligned} \tag{75}$$

其中，

$$\int_{\Sigma+\Omega} \int_{-\infty}^{+\infty} \frac{\partial}{\partial y_i} \left(G \frac{\partial [T_{ij} H(f)]}{\partial y_j} \right) \mathrm{d}\tau \mathrm{d}^3\mathbf{y} = \int_{S_\infty} \int_{-\infty}^{+\infty} G \frac{\partial [T_{ij} H(f)]}{\partial y_j} n_i \ \mathrm{d}\tau \mathrm{d}^2\mathbf{y} = 0$$

$$\int_{\Sigma+\Omega} \int_{-\infty}^{+\infty} \frac{\partial}{\partial y_j} \left(T_{ij} H(f) \frac{\partial G}{\partial y_i} \right) \mathrm{d}\tau \mathrm{d}^3\mathbf{y} = \int_{S_\infty} \int_{-\infty}^{+\infty} T_{ij} H(f) \frac{\partial G}{\partial y_j} n_j \ \mathrm{d}\tau \mathrm{d}^2\mathbf{y} = 0$$

因此有：

$$\int_V \int_{-\infty}^{+\infty} G \frac{\partial^2 [H(f) T_{ij}]}{\partial y_i \partial y_j} \ \mathrm{d}\tau \mathrm{d}^3\mathbf{y} = \int_V \int_{-\infty}^{+\infty} H(f) T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \ \mathrm{d}\tau \mathrm{d}^3\mathbf{y} \tag{76}$$

对于偶极子项：

$$\begin{aligned} \int_V \int_{-\infty}^{+\infty} G \frac{\partial [F_i \delta(f)]}{\partial y_i} \ \mathrm{d}\tau \mathrm{d}^3\mathbf{y} = & \int_V \int_{-\infty}^{+\infty} \frac{\partial [G F_i \delta(f)]}{\partial y_i} \ \mathrm{d}\tau \mathrm{d}^3\mathbf{y} \\ & - \int_V \int_{-\infty}^{+\infty} F_i \delta(f) \frac{\partial G}{\partial y_i} \ \mathrm{d}\tau \mathrm{d}^3\mathbf{y} \end{aligned} \tag{77}$$

对于无边界区域，有：

$$\int_V \frac{\partial [G F_i \delta(f)]}{\partial y_i} \ \mathrm{d}^3\mathbf{y} = \int_{\Sigma+\Omega} \frac{\partial [G F_i \delta(f)]}{\partial y_i} \ \mathrm{d}^3\mathbf{y} = \int_{S_\sim} G F_i \delta(f) n_i \ \mathrm{d}^2\mathbf{y} = 0 \tag{78}$$

因此有：

$$\int_V \int_{-\infty}^{+\infty} G \frac{\partial\left[F_i \delta(f)\right]}{\partial y_i} \, \mathrm{d}\tau \mathrm{d}^3\mathbf{y} = -\int_V \int_{-\infty}^{+\infty} F_i \delta(f) \frac{\partial G}{\partial y_i} \, \mathrm{d}\tau \mathrm{d}^3\mathbf{y} \tag{79}$$

对于单极子项：

$$\begin{aligned}\int_V \int_{-\infty}^{+\infty} G \frac{\partial[Q\delta(f)]}{\partial \tau} \mathrm{d}\tau \mathrm{d}^3\mathbf{y} = &\int_V \int_{-\infty}^{+\infty} \frac{\partial[GQ\delta(f)]}{\partial \tau} \mathrm{d}\tau \mathrm{d}^3\mathbf{y} \\ &- \int_V \int_{-\infty}^{+\infty} Q\delta(f) \frac{\partial G}{\partial \tau} \mathrm{d}\tau \mathrm{d}^3\mathbf{y}\end{aligned} \tag{80}$$

根据 $G = \frac{\partial G}{\partial \tau} = 0(t < \tau)$, 有：

$$\int_{-\infty}^{+\infty} \frac{\partial[GQ\delta(f)]}{\partial \tau} \mathrm{d}\tau = GQ\delta(f)|_{\tau=-\infty}^{\tau=+\infty} = 0 \tag{81}$$

因此有：

$$\int_V \int_{-\infty}^{+\infty} G \frac{\partial[Q\delta(f)]}{\partial \tau} \mathrm{d}\tau \mathrm{d}^3\mathbf{y} = -\int_V \int_{-\infty}^{+\infty} Q\delta(f) \frac{\partial G}{\partial \tau} \mathrm{d}\tau \mathrm{d}^3\mathbf{y} \tag{82}$$

综上，FW-H 方程的积分表达式可表示为：

$$\begin{aligned}H(f) c_0^2 \rho'(\mathbf{x}, t) = &\int_V \int_{-\infty}^{+\infty} H(f) T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \, \mathrm{d}\tau \mathrm{d}^3\mathbf{y} \\ &+ \int_V \int_{-\infty}^{+\infty} F_i \delta(f) \frac{\partial G}{\partial y_i} \, \mathrm{d}\tau \mathrm{d}^3\mathbf{y} \\ &- \int_V \int_{-\infty}^{+\infty} Q\delta(f) \frac{\partial G}{\partial \tau} \mathrm{d}\tau \mathrm{d}^3\mathbf{y}\end{aligned} \tag{83}$$

3. 如果 $f = 0$ 的面不是固体表面，而是流体区域任意选择的可穿透封闭面，FW-H 方程还成立吗？

   成立。FW-H 方程仅假设 $f = 0$ 为移动的刚体表面，并没有假设表面是否可穿透。因此 FW-H 方程对可穿透表面成立。对于不可穿透表面，FW-H 方程可以进一步简化，简化后的方程对可穿透表面不成立。

Section2.6

1. 对均匀静止介质中以攻数 M_i 运动的声源, 证明

$$\frac{\partial M_r}{\partial \tau}=\frac{1}{r}\left\{r_i\frac{\partial M_i}{\partial \tau}+c_0\left(M_r^2-M^2\right)\right\}, M=\sqrt{M_1^2+M_2^2+M_3^2}$$

已知：

$$M_r=\frac{r_iM_i}{r} \tag{84}$$

根据微分公式，有：

$$\begin{aligned}\frac{\partial M_r}{\partial \tau}&=\frac{\partial}{\partial \tau}(\frac{r_iM_i}{r})\\&=\frac{r_i}{r}\frac{\partial M_i}{\partial \tau}+\frac{M_i}{r}\frac{\partial r_i}{\partial \tau}-\frac{r_iM_i}{r^2}\frac{\partial r}{\partial \tau}\end{aligned} \tag{85}$$

其中：

$$\frac{M_i}{r}\frac{\partial r_i}{\partial \tau}=\frac{M_i}{r}(-v_i)=-\frac{M_i^2c_0}{r} \tag{86}$$

$$\frac{r_iM_i}{r^2}\frac{\partial r}{\partial \tau}=\frac{M_r}{r}(-M_rc_0)=-\frac{M_r^2c_0}{r} \tag{87}$$

带入式(85), 得：

$$\begin{aligned}\frac{\partial M_r}{\partial \tau}&=\frac{r_i}{r}\frac{\partial M_i}{\partial \tau}+\frac{M_i}{r}\frac{\partial r_i}{\partial \tau}-\frac{r_iM_i}{r^2}\frac{\partial r}{\partial \tau}\\&=\frac{r_i}{r}\frac{\partial M_i}{\partial \tau}-\frac{M_i^2c_0}{r}+\frac{M_r^2c_0}{r}\\&=\frac{1}{r}\left\{r_i\frac{\partial M_i}{\partial \tau}+c_0\left(M_r^2-M_i^2\right)\right\}\\&=\frac{1}{r}\left\{r_i\frac{\partial M_i}{\partial \tau}+c_0\left(M_r^2-M^2\right)\right\}, M=\sqrt{M_1^2+M_2^2+M_3^2}\end{aligned} \tag{88}$$

2. 证明偶极子噪声的积分表达式

$$\begin{aligned}\pi p_D(\mathbf{x},t)=&\int_S\left[\frac{r_i}{r^2c_0\left(1-M_r\right)^2}\left\{\frac{\partial F_i}{\partial \tau}+\frac{F_i}{1-M_r}\left(\frac{r_j}{r}\frac{\partial M_j}{\partial \tau}\right)\right\}\right]\mathrm{d}^2\mathbf{y}\\&+\int_S\left[\frac{1}{r^2\left(1-M_r\right)^2}\left\{\frac{F_ir_i}{r}\frac{1-M^2}{1-M_r}-F_iM_i\right\}\right]\mathrm{d}^2\mathbf{y}\end{aligned}$$

已知：

$$\pi p_D(\mathbf{x},t)=-\frac{\partial}{\partial x_i}\int_S\left[\frac{F_i}{r\left(1-M_r\right)}\right]\mathrm{d}^2\mathbf{y} \tag{89}$$

根据微分公式，有：

$$\frac{\partial}{\partial x_i}\left[\frac{F_i}{r\left(1-M_r\right)}\right]=\left[\frac{\partial}{\partial x_i}\left\{\frac{F_i}{r\left(1-M_r\right)}\right\}+\left[\frac{\partial \tau}{\partial x_i}\ \frac{\partial}{\partial \tau}\left\{\frac{F_i}{r\left(1-M_r\right)}\right\}\right]\tag{90}$$

其中：

$$\begin{aligned}\frac{\partial}{\partial x_i}\left\{\frac{F_i}{r\left(1-M_r\right)}\right\}&=-\frac{F_i}{r^2\left(1-M_r\right)^2}\frac{\partial r\left(1-M_r\right)}{\partial x_i}\\&=-\frac{F_i}{r^2\left(1-M_r\right)^2}\left[\frac{\partial r}{\partial x_i}-\frac{\partial r M_r}{\partial x_i}\right]\end{aligned}\tag{91}$$

又因为：

$$\frac{\partial r}{\partial x_i}=\frac{r_i}{r}\tag{92}$$

$$\frac{\partial r M_r}{\partial x_i}=\frac{\partial r_i M_i}{\partial x_i}=M_i\frac{\partial r_i}{\partial x_i}=M_i\tag{93}$$

因此有：

$$\frac{\partial}{\partial x_i}\left\{\frac{F_i}{r\left(1-M_r\right)}\right\}=-\frac{F_i}{r^2\left(1-M_r\right)^2}\left(\frac{r_i}{r}-M_i\right)\tag{94}$$

同时，对于：

$$\frac{\partial \tau}{\partial x_i}\frac{\partial}{\partial \tau}\left\{\frac{F_i}{r\left(1-M_r\right)}\right\}\tag{95}$$

其中，

$$\frac{\partial \tau}{\partial x_i}=\frac{\partial \tau}{\partial g}\frac{\partial g}{\partial x_i}=\frac{1}{1-M_r}\frac{r_i}{r c_0}=\frac{r_i}{r c_0(1-M_r)}\tag{96}$$

$$\begin{aligned}\frac{\partial}{\partial \tau}\left\{\frac{F_i}{r\left(1-M_r\right)}\right\}=&F_i\frac{\partial}{\partial \tau}\left\{\frac{1}{r\left(1-M_r\right)}\right\}+\frac{1}{r\left(1-M_r\right)}\frac{\partial F_i}{\partial \tau}\\=&-\frac{F_i}{r^2\left(1-M_r\right)^2}\left\{(1-M_r)\frac{\partial r}{\partial \tau}-r\frac{\partial M_r}{\partial \tau}\right\}\\&+\frac{1}{r\left(1-M_r\right)}\frac{\partial F_i}{\partial \tau}\\=&\frac{F_i}{r^2\left(1-M_r\right)^2}\left\{(1-M_r)c_0 M_r+r\frac{\partial M_r}{\partial \tau}\right\}\\&+\frac{1}{r\left(1-M_r\right)}\frac{\partial F_i}{\partial \tau}\end{aligned}\tag{97}$$

综上，

$$
\begin{aligned}
\pi p_D(\mathbf{x},t) &= -\frac{\partial}{\partial x_i}\int_S\left[\frac{F_i}{r\left(1-M_r\right)}\right]\mathrm{d}^2\mathbf{y}\\
&= -\int_S\left[\frac{\partial}{\partial x_i}\left\{\frac{F_i}{r\left(1-M_r\right)}\right\}+\left[\frac{\partial\tau}{\partial x_i}\ \frac{\partial}{\partial\tau}\left\{\frac{F_i}{r\left(1-M_r\right)}\right\}\right]\right]\mathrm{d}^2\mathbf{y}\\
&= \int_S\left[\frac{r_i}{r^2c_0\left(1-M_r\right)^2}\left\{\frac{\partial F_i}{\partial\tau}+\frac{F_i}{1-M_r}\left(\frac{r_j}{r}\frac{\partial M_j}{\partial\tau}\right)\right\}\right]\mathrm{d}^2\mathbf{y}\\
&\quad+\int_S\left[\frac{1}{r^2\left(1-M_r\right)^2}\left\{\frac{F_ir_i}{r}\frac{1-M^2}{1-M_r}-F_iM_i\right\}\right]\mathrm{d}^2\mathbf{y}
\end{aligned}
\tag{98}
$$

原式得证。

Section2.7

1. 将 Fourier 变换对定义为 $\tilde{f}(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\ \mathrm{d}t, f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\omega)e^{i\omega t}d\omega$, 证明时域格林函数 $G_0(\boldsymbol{x},\boldsymbol{y},t-\tau)=\frac{\delta(t-\tau-R/c_0)}{4\pi\Re}$ 的频域表达式为 $\tilde{G}_0(\boldsymbol{x},\boldsymbol{y},\omega)=\frac{\exp(-ikR)}{4\pi\Re}$ 。

   根据 Fourier 变换的定义，有:

$$\begin{aligned}\tilde{G}_0(\boldsymbol{x},\boldsymbol{y},\omega)&=\int_{-\infty}^{\infty} G_0(\boldsymbol{x},\boldsymbol{y},t-\tau)e^{-i\omega(t-\tau)}\ \mathrm{d}t-\tau\\&=\int_{-\infty}^{\infty}\frac{\delta\left(t-\tau-R/c_0\right)}{4\pi\Re}e^{-i\omega(t-\tau)}\ \mathrm{d}t-\tau\\&=\frac{1}{4\pi\Re}\int_{-\infty}^{\infty}\delta\left(t-\tau-R/c_0\right)e^{-i\omega(t-\tau-R/c_0)}e^{-i\omega R/c_0}\ \mathrm{d}t-\tau\\&=\frac{e^{-i\omega R/c_0}}{4\pi\Re}\int_{-\infty}^{\infty}\delta\left(t-\tau-R/c_0\right)e^{-i\omega(t-\tau-R/c_0)}\ \mathrm{d}t-\tau\end{aligned}\tag{99}$$

   对于 Dirac Function，有：

$$\int_{-\infty}^{\infty}\delta\left(t\right)e^{-i\omega t}\ \mathrm{d}t=1\tag{100}$$

   因此有：

$$\begin{aligned}\tilde{G}_0(\boldsymbol{x},\boldsymbol{y},\omega)&=\frac{e^{-i\omega R/c_0}}{4\pi\Re}\int_{-\infty}^{\infty}\delta\left(t-\tau-R/c_0\right)e^{-i\omega(t-\tau-R/c_0)}\ \mathrm{d}t-\tau\\&=\frac{e^{-i\omega R/c_0}}{4\pi\Re}\\&=\frac{e^{-ikR}}{4\pi\Re}\end{aligned}\tag{101}$$

   其中，$k=\frac{\omega}{c_0}$。

2. 对均匀平均流中的静止点源 $Q(\boldsymbol{y},\tau)=\exp(i\omega\tau)$, 其辐射的声场用 $\phi(\boldsymbol{x},t)$ 表示, 利用上题中的格林函数, 证明 $\phi(\boldsymbol{x},t)=\frac{\exp[i\omega(t-R/c_0)]}{4\pi\Re}$

$$\begin{aligned}\phi(\boldsymbol{x},t)&=\int_{-\infty}^{\infty} Q(\boldsymbol{y},\tau)G_0(\boldsymbol{x},\boldsymbol{y},\tau)\ \mathrm{d}\tau\\&=\int_{-\infty}^{\infty} e^{i\omega\tau}\frac{\delta\left(t-\tau-R/c_0\right)}{4\pi\Re}\ \mathrm{d}\tau\\&=\frac{e^{i\omega(t-R/c_0)}}{4\pi\Re}\int_{-\infty}^{\infty}\delta\left(t-\tau-R/c_0\right)e^{-i\omega(t-\tau-R/c_0)}\ \mathrm{d}\tau\\&=\frac{e^{i\omega(t-R/c_0)}}{4\pi\Re}\end{aligned}\tag{102}$$

Section2.8

1. 假设亚声速均匀流沿 x_1 轴正向运动, 在 $\boldsymbol{y}$ 点有一静止声源辐射声波, 如果已知观察点 $\boldsymbol{x}$ 的时间 t, 如何确定延迟时间 τ ?

   对于均匀运动介质中的静止声源，根据 Prandtl-Glauert-Lorentz 变换，有：

$$\bar{t} = \beta t + M\frac{x_1}{\beta c_0}, M = \frac{v}{c_\infty} \tag{103}$$

   可以得到延迟时间 $\bar{\tau}$ 为：

$$\bar{\tau} = \frac{Mx_1 - r_1\bar{M}\beta}{\beta c_0} + \tau\beta \tag{104}$$

2. 假设均匀静止介质中有一点源以恒定速度 v (亚声速) 沿 x_1 轴正向运动, 其初始位置为 $\boldsymbol{y}_0$, 如果已知观察点 $\boldsymbol{x}$ 的时间 t, 如何确定延迟时间 τ?

   对于静止介质中的运动声源，有

$$t - \tau = \frac{1}{c_\infty}\sqrt{\left(x_1 - v\tau\right)^2 + x_2^2} \tag{105}$$

   求得：

$$\bar{\tau} = \frac{\left(c_\infty t - Mx_1\right) \pm \sqrt{\left(x_1 - vt\right)^2 + \left(1 - M^2\right)x_2^2}}{c_\infty\left(1 - M^2\right)}, M = \frac{v}{c_\infty} \tag{106}$$

3. 均匀静止介质中, 一强度为 $q(t)$ 的点源以恒定速度 $\boldsymbol{v}$ 亚声速直线运动, 且 $t = 0$ 时刻恰好经过坐标原点, 辐射声场的速度势函数 $\phi(\boldsymbol{x}, t)$ 满足方程 $\frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = q(t)\delta(\boldsymbol{x} - \boldsymbol{v}t)$, 证明

$$\phi(\boldsymbol{x}, t) = \frac{q\left(t - R/c_0\right)}{4\pi R(1 - M\cos\theta)}, \quad M = \frac{|\boldsymbol{v}|}{c_0}$$

   其中, R 为观察点 $\boldsymbol{x}$ 与声源辐射声波时所在位置间的距离, θ 为声源运动方向与声传播方向的夹角。