1. 定义任意时域函数 $f(t)$ 和 $h(t)$，通过 Fourier 变换得到的频域函数分别为 $\tilde{f}(\omega)$ 和 $\tilde{h}(\omega)$，利用 Fourier 变换定义证明下述关系式成立:

(1) 如果 $f(t)=\int_{-\infty}^{\infty}h(\tau)G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau$，则有 $\tilde{f}(\omega)=\tilde{h}(\omega)\tilde{G}(\mathbf{x},\mathbf{y},\omega)$。

根据 Fourier 变换，有:

$$\begin{aligned}
\tilde{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{i\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau e^{i\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} G(\mathbf{x},\mathbf{y},t-\tau)e^{i\omega(t-\tau)}\mathrm{d}t\right]e^{i\omega\tau}\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\tilde{G}(\mathbf{x},\mathbf{y},\omega)e^{i\omega\tau}\mathrm{d}\tau \\
&= \tilde{G}(\mathbf{x},\mathbf{y},\omega)\int_{-\infty}^{\infty} h(\tau)e^{i\omega\tau}\mathrm{d}\tau \\
&= \tilde{h}(\omega)\tilde{G}(\mathbf{x},\mathbf{y},\omega)
\end{aligned} \tag{1}$$

原式得证。

(2) 如果 $f(t)=\int_{-\infty}^{\infty}h(\tau)\frac{\partial G(\mathbf{x},\mathbf{y},t-\tau)}{\partial\tau}\mathrm{d}\tau$，则有 $\tilde{f}(\omega)=-i\omega\tilde{h}(\omega)\tilde{G}(\mathbf{x},\mathbf{y},\omega)$。

根据分步积分，有:

$$\begin{aligned}
f(\omega) &= \int_{-\infty}^{\infty} h(\tau)\frac{\partial G(\mathbf{x},\mathbf{y},t-\tau)}{\partial\tau}\mathrm{d}\tau \\
&= -h(\tau)G(\mathbf{x},\mathbf{y},t-\tau)|_{\tau=-\infty}^{\tau=\infty} - \int_{-\infty}^{\infty} -\frac{\partial h(\tau)}{\partial\tau}G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau
\end{aligned} \tag{2}$$

根据 t 与 τ 的因果关系，有:

$$-h(\tau)G(\mathbf{x},\mathbf{y},t-\tau)|_{\tau=-\infty}^{\tau=\infty}=0 \tag{3}$$

因此有:

$$f(\omega)=\int_{-\infty}^{\infty}\frac{\partial h(\tau)}{\partial\tau}G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau \tag{4}$$

根据 Fourier 变换，有:

$$\begin{aligned}
\tilde{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{i\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial h(\tau)}{\partial\tau}G(\mathbf{x},\mathbf{y},t-\tau)\mathrm{d}\tau e^{i\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{\partial h(\tau)}{\partial\tau}\left[\int_{-\infty}^{\infty} G(\mathbf{x},\mathbf{y},t-\tau)e^{i\omega(t-\tau)}\mathrm{d}t\right]e^{i\omega\tau}\mathrm{d}\tau \\
&= \tilde{G}(\mathbf{x},\mathbf{y},\omega)\int_{-\infty}^{\infty} \frac{\partial h(\tau)}{\partial\tau}e^{i\omega\tau}\mathrm{d}\tau \\
&= -i\omega\tilde{h}(\omega)\tilde{G}(\mathbf{x},\mathbf{y},\omega)
\end{aligned} \tag{5}$$

原式得证。

2. 根据波动方程的时域解，证明频域积分解可以写为

$$\tilde{p}'(\boldsymbol{x},\omega)=\int_S i\omega\rho_0\tilde{u}_n(\boldsymbol{y},\omega)\tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)\mathrm{d}S-\int_S\tilde{p}'(\boldsymbol{y},\omega)\frac{\partial\tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)}{\partial\boldsymbol{n}}\mathrm{d}S.$$

已知声学波动方程的时域解:

$$\begin{aligned}p'(\boldsymbol{x},t)=&-\int_{-\infty}^{+\infty}\int_S\rho_0\frac{\partial u_n(\boldsymbol{y},\tau)}{\partial\tau}G(\boldsymbol{x},\boldsymbol{y},t-\tau)\mathrm{d}S\ \mathrm{d}\tau\\&-\int_{-\infty}^{+\infty}\int_S p'(\boldsymbol{y},\tau)\frac{\partial G(\boldsymbol{x},\boldsymbol{y},t-\tau)}{\partial\boldsymbol{n}}\ \mathrm{d}S\ \mathrm{d}\tau\end{aligned}\tag{6}$$

对上式进行 Fourier 变换:

$$\begin{aligned}\tilde{p}'(\boldsymbol{x},\omega)=&\int_{-\infty}^{+\infty}\left[-\int_{-\infty}^{+\infty}\int_S\rho_0\frac{\partial u_n(\boldsymbol{y},\tau)}{\partial\tau}G(\boldsymbol{x},\boldsymbol{y},t-\tau)\mathrm{d}S\ \mathrm{d}\tau\right.\\&\left.-\int_{-\infty}^{+\infty}\int_S p'(\boldsymbol{y},\tau)\frac{\partial G(\boldsymbol{x},\boldsymbol{y},t-\tau)}{\partial\boldsymbol{n}}\ \mathrm{d}S\ \mathrm{d}\tau\right]e^{i\omega t}\mathrm{d}\\=&-\int_S\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_0\frac{\partial u_n(\boldsymbol{y},\tau)}{\partial\tau}G(\boldsymbol{x},\boldsymbol{y},t-\tau)e^{i\omega t}\mathrm{d}\tau\ \mathrm{d}t\right]\mathrm{d}S\\&-\int_S\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}p'(\boldsymbol{y},\tau)\frac{\partial G(\boldsymbol{x},\boldsymbol{y},t-\tau)}{\partial\boldsymbol{n}}e^{i\omega t}\mathrm{d}\tau\ \mathrm{d}t\right]\mathrm{d}S\end{aligned}\tag{7}$$

由第一题中的结论，式(4)、(5)可得:

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_0\frac{\partial u_n(\boldsymbol{y},\tau)}{\partial\tau}G(\boldsymbol{x},\boldsymbol{y},t-\tau)e^{i\omega t}\mathrm{d}\tau\ \mathrm{d}t\\=&-i\omega\rho_0\tilde{u}_n(\boldsymbol{y},\omega)\tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)\end{aligned}\tag{8}$$

有第一题中的结论，式(1)可得:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}p'(\boldsymbol{y},\tau)\frac{\partial G(\boldsymbol{x},\boldsymbol{y},t-\tau)}{\partial\boldsymbol{n}}e^{i\omega t}\mathrm{d}\tau\ \mathrm{d}t=\tilde{p}'(\boldsymbol{y},\omega)\frac{\partial\tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)}{\partial\boldsymbol{n}}\tag{9}$$

将式(8)、(9)代入式(7)得:

$$\tilde{p}'(\boldsymbol{x},\omega)=\int_S i\omega\rho_0\tilde{u}_n(\boldsymbol{y},\omega)\tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)\mathrm{d}S-\int_S\tilde{p}'(\boldsymbol{y},\omega)\frac{\partial\tilde{G}(\boldsymbol{x},\boldsymbol{y},\omega)}{\partial\boldsymbol{n}}\mathrm{d}S\tag{10}$$

原式得证。