

Module A

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# Module A: Algebraic properties of linear maps

# How can we understand linear maps algebraically?

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At the end of this module, students will be able to...

- ① **Linear map verification.** ... determine if a map between vector spaces of polynomials is linear or not.
- ② **Linear maps and matrices.** ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.
- ③ **Kernel and Image.** ... compute a basis for the kernel and a basis for the image of a linear map, and verify that the rank-nullity theorem holds for a given linear map.
- ④ **Injectivity and surjectivity.** ... determine if a given linear map is injective and/or surjective.

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## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans  $\mathbb{R}^n$  **V3**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **V5**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **V6,V7**.
- Find a basis of the solution space to a homogeneous system of linear equations **V10**.

# Module A Section 1

## Definition A.1

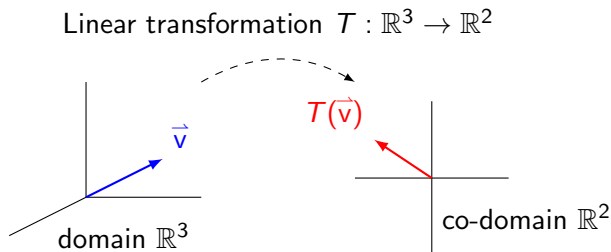
A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if  $V$  and  $W$  are vector spaces, a map  $T : V \rightarrow W$  is called a linear transformation if

- ①  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$  for any  $\vec{v}, \vec{w} \in V$ .
- ②  $T(c\vec{v}) = cT(\vec{v})$  for any  $c \in \mathbb{R}, \vec{v} \in V$ .

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

**Definition A.2**

Given a linear transformation  $T : V \rightarrow W$ ,  $V$  is called the **domain** of  $T$  and  $W$  is called the **co-domain** of  $T$ .



**Example A.3**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that  $T$  is linear, we must verify...

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = T \left( \begin{bmatrix} x + u \\ y + v \\ z + w \end{bmatrix} \right) = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) + T \left( \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

And also...

$$T \left( c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T \left( \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \text{ and } cT \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = c \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore  $T$  is a linear transformation.



**Example A.4**

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

To show that  $T$  is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different,  $T$  is not a linear transformation.

**Fact A.5**

A map between Euclidean spaces  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear exactly when every component of the output is a linear combination of the variables of  $\mathbb{R}^n$ .

For example, the following map is definitely linear because  $x - z$  and  $3y$  are linear combinations of  $x, y, z$ :

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because  $x^2$ ,  $y + 3$ , and  $y - 2^x$  are not linear combinations (even though  $x + y$  is):

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

**Activity A.6** ( $\sim 5$  min) Recall the following rules from calculus, where  $D : \mathcal{P} \rightarrow \mathcal{P}$  is the derivative map defined by  $D(f(x)) = f'(x)$  for each polynomial  $f$ .

$$D(f + g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

- a)  $\mathcal{P}$  is not a vector space
- b)  $D$  is a linear map
- c)  $D$  is not a linear map

**Activity A.7** (*~10 min*) Let the polynomial maps  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  and  $T : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute  $S(x^4 + x)$ ,  $S(x^4) + S(x)$ ,  $T(x^4 + x)$ , and  $T(x^4) + T(x)$ . Which of these maps is definitely not linear?

**Fact A.8**

If  $L : V \rightarrow W$  is linear, then  $L(\vec{z}) = L(0\vec{v}) = 0L(\vec{v}) = \vec{z}$  where  $\vec{z}$  is the additive identity of the vector spaces  $V, W$ .

Put another way, an easy way to prove that a map like  $T(f(x)) = f'(x) + x^3$  can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

## Observation A.9

Showing  $L : V \rightarrow W$  is not a linear transformation can be done by finding an example for any one of the following.

- Show  $L(\vec{z}) \neq \vec{z}$  (where  $\vec{z}$  is the additive identity of  $L$  and  $W$ ).
- Find  $\vec{v}, \vec{w} \in V$  such that  $L(\vec{v} + \vec{w}) \neq L(\vec{v}) + L(\vec{w})$ .
- Find  $\vec{v} \in V$  and  $c \in \mathbb{R}$  such that  $L(c\vec{v}) \neq cL(\vec{v})$ .

Otherwise,  $L$  can be shown to be linear by proving the following in general.

- For all  $\vec{v}, \vec{w} \in V$ ,  $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$ .
- For all  $\vec{v} \in V$  and  $c \in \mathbb{R}$ ,  $L(c\vec{v}) = cL(\vec{v})$ .

Note the similarities between this process and showing that a subset of a vector space is/isn't a subspace.

**Activity A.10** (*~15 min*) Continue to consider  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

**Activity A.10** (*~15 min*) Continue to consider  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

*Part 1:* Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to  $S(f(x)) + S(g(x))$  for all polynomials  $f, g$ .



**Activity A.10** (*~15 min*) Continue to consider  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

*Part 1:* Verify that

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is equal to  $S(f(x)) + S(g(x))$  for all polynomials  $f, g$ .

*Part 2:* Verify that  $S(cf(x))$  is equal to  $cS(f(x))$  for all real numbers  $c$  and polynomials  $f$ .

**Activity A.10** (*~15 min*) Continue to consider  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

*Part 1:* Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to  $S(f(x)) + S(g(x))$  for all polynomials  $f, g$ .

*Part 2:* Verify that  $S(cf(x))$  is equal to  $cS(f(x))$  for all real numbers  $c$  and polynomials  $f$ .

*Part 3:* Is  $S$  linear?

**Activity A.11** (*~20 min*) Let the polynomial maps  $S : \mathcal{P} \rightarrow \mathcal{P}$  and  $T : \mathcal{P} \rightarrow \mathcal{P}$  be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

**Activity A.11** ( $\sim 20$  min) Let the polynomial maps  $S : \mathcal{P} \rightarrow \mathcal{P}$  and  $T : \mathcal{P} \rightarrow \mathcal{P}$  be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

*Part 1:* Note that  $S(0) = 0$  and  $T(0) = 0$ . So instead, show that  $S(x+1) \neq S(x) + S(1)$  to verify that  $S$  is not linear.

**Activity A.11** ( $\sim 20$  min) Let the polynomial maps  $S : \mathcal{P} \rightarrow \mathcal{P}$  and  $T : \mathcal{P} \rightarrow \mathcal{P}$  be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

*Part 1:* Note that  $S(0) = 0$  and  $T(0) = 0$ . So instead, show that  $S(x+1) \neq S(x) + S(1)$  to verify that  $S$  is not linear.

*Part 2:* Prove that  $T$  is linear by verifying that  $T(f(x) + g(x)) = T(f(x)) + T(g(x))$  and  $T(cf(x)) = cT(f(x))$ .

# Module A Section 2

**Remark A.12**

Recall that a linear map  $T : V \rightarrow W$  satisfies

- ①  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$  for any  $\vec{v}, \vec{w} \in V$ .
- ②  $T(c\vec{v}) = cT(\vec{v})$  for any  $c \in \mathbb{R}, \vec{v} \in V$ .

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

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**Activity A.13** (*~5 min*) Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \text{ Compute } T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right).$$

a  $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

b  $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

c  $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

d  $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$



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**Activity A.14** ( $\sim 5$  min) Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \text{ Compute } T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

a  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

b  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

c  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

d  $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

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**Activity A.15** ( $\sim 5$  min) Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \text{ Compute } T\left(\begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix}\right).$$

a  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

b  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

c  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

d  $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

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**Activity A.16** (*~5 min*) Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know

$$T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

What piece of information would help you compute  $T \left( \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \right)$ ?

a The value of  $T \left( \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} \right)$ .

b The value of  $T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$ .

c The value of  $T \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$ .

d Any of the above.

**Fact A.17**

Consider any basis  $\{\vec{b}_1, \dots, \vec{b}_n\}$  for  $V$ . Since every vector  $\vec{v}$  can be written as a linear combination of basis vectors,  $x_1\vec{b}_1 + \dots + x_n\vec{b}_n$ , we may compute  $T(\vec{v})$  as follows:

$$T(\vec{v}) = T(x_1\vec{b}_1 + \dots + x_n\vec{b}_n) = x_1T(\vec{b}_1) + \dots + x_nT(\vec{b}_n).$$

Therefore any linear transformation  $T : V \rightarrow W$  can be defined by just describing the values of  $T(\vec{b}_i)$ .

Put another way, the images of the basis vectors **determine** the transformation  $T$ .

## Definition A.18

Since linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is determined by the standard basis  $\{\vec{e}_1, \dots, \vec{e}_n\}$ , it's convenient to store this information in the  $m \times n$  **standard matrix**  $[T(\vec{e}_1) \cdots T(\vec{e}_n)]$ .

For example, let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear map determined by the following values for  $T$  applied to the standard basis of  $\mathbb{R}^3$ .

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad T(\vec{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Then the standard matrix corresponding to  $T$  is

$$[T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3)] = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix}.$$

**Activity A.19** ( $\sim 3$  min) Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be the linear transformation given by

$$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \quad T(\vec{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \quad T(\vec{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \quad T(\vec{e}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix  $[T(\vec{e}_1) \cdots T(\vec{e}_n)]$  for  $T$ .

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**Activity A.20** (*~5 min*) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

**Activity A.20** (*~5 min*) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

*Part 1:* Compute  $T(\vec{e}_1)$ ,  $T(\vec{e}_2)$ , and  $T(\vec{e}_3)$ .



**Activity A.20** ( $\sim 5$  min) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

*Part 1:* Compute  $T(\vec{e}_1)$ ,  $T(\vec{e}_2)$ , and  $T(\vec{e}_3)$ .

*Part 2:* Find the standard matrix for  $T$ .

**Fact A.21**

Because every linear map  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  has a linear combination of the variables in each component, and thus  $T(\vec{e}_i)$  yields exactly the coefficients of  $x_i$ , the standard matrix for  $T$  is simply an ordered list of the coefficients of the  $x_i$ :

$$T \left( \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \quad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

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**Activity A.22** (*~5 min*) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

**Activity A.22** (*~5 min*) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

*Part 1:* Compute  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$ .

**Activity A.22** ( $\sim 5$  min) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

*Part 1:* Compute  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$ .

*Part 2:* Compute  $T \left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$ .

**Fact A.23**

To quickly compute  $T(\vec{v})$  from its standard matrix  $A$ , multiply and add the entries of each row of  $A$  with the vector  $\vec{v}$ . For example, if  $T$  has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$$

then for  $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  we will write

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 2y + 3z \\ 0x + 1y - 2z \\ 2x - 1y + 0z \end{bmatrix}$$

and for  $\vec{v} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$  we will write

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(0) + 3(-2) \\ 0(3) + 1(0) - 2(-2) \\ 2(3) - 1(0) + 0(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}.$$

**Activity A.24** ( $\sim 15$  min) Compute the following linear transformations of vectors given their standard matrices.

$$T_1 \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \text{ for the standard matrix } A_1 = \begin{bmatrix} 4 & 3 \\ 0 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$$

$$T_2 \left( \begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix} \right) \text{ for the standard matrix } A_2 = \begin{bmatrix} 4 & 3 & 0 & -1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

$$T_3 \left( \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right) \text{ for the standard matrix } A_3 = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & 3 \\ 5 & 1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

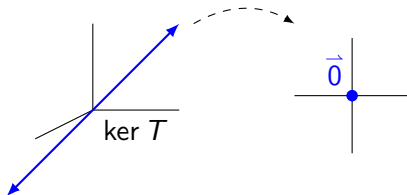
# Module A Section 3



**Definition A.25**

Let  $T : V \rightarrow W$  be a linear transformation. The **kernel** of  $T$  is an important subspace of  $V$  defined by

$$\ker T = \{\vec{v} \in V \mid T(\vec{v}) = \vec{z}\}$$



**Activity A.26** ( $\sim 5$  min) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of  $\mathbb{R}^2$  describes  $\ker T$ , the set of all vectors that transform into  $\vec{0}$ ?

- a)  $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$
- b)  $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
- c)  $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

**Activity A.27** ( $\sim 5$  min) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of  $\mathbb{R}^3$  describes  $\ker T$ , the set of all vectors that transform into  $\vec{0}$ ?

a)  $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

b)  $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \mid a \in \mathbb{R} \right\}$

c)  $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d)  $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

**Activity A.28** (*~10 min*) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

**Activity A.28** ( $\sim 10$  min) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Set  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  to find a linear system of equations whose solution set is the kernel.

**Activity A.28** ( $\sim 10$  min) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Set  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  to find a linear system of equations

whose solution set is the kernel.

*Part 2:* Use RREF( $A$ ) to solve this homogeneous system of equations and find a basis for the kernel of  $T$ .

**Activity A.29** (*~10 min*) Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \right) = \begin{bmatrix} 2x + 4y + 2z - 4w \\ -2x - 4y + z + w \\ 3x + 6y - z - 4w \end{bmatrix}.$$

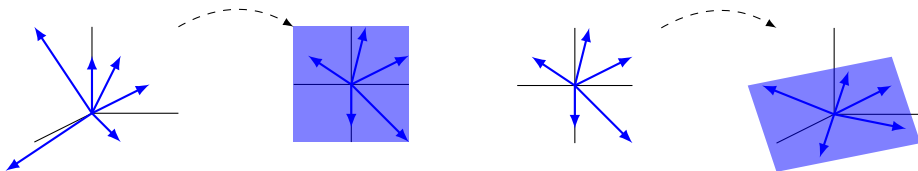
Find a basis for the kernel of  $T$ .

**Definition A.30**

Let  $T : V \rightarrow W$  be a linear transformation. The **image** of  $T$  is an important subspace of  $W$  defined by

$$\text{Im } T = \{ \vec{w} \in W \mid \text{there is some } \vec{v} \in V \text{ with } T(\vec{v}) = \vec{w} \}$$

In the examples below, the left example's image is all of  $\mathbb{R}^2$ , but the right example's image is a planar subspace of  $\mathbb{R}^3$ .





**Activity A.31** ( $\sim 5$  min) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of  $\mathbb{R}^3$  describes  $\text{Im } T$ , the set of all vectors that are the result of using  $T$  to transform  $\mathbb{R}^2$  vectors?

a)  $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

c)  $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

b)  $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$

d)  $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

**Activity A.32** (*~5 min*) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of  $\mathbb{R}^2$  describes  $\text{Im } T$ , the set of all vectors that are the result of using  $T$  to transform  $\mathbb{R}^3$  vectors?

- a)  $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$
- b)  $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
- c)  $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

**Activity A.33** ( $\sim 5$  min) Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3) \quad T(\vec{e}_4)].$$

Since  $T(\vec{v}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4)$ , the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

- a) spans  $\text{Im } T$
- b) is a linearly independent subset of  $\text{Im } T$
- c) is a basis for  $\text{Im } T$

**Observation A.34**

Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}.$$

Since the set  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$  spans  $\text{Im } T$ , we can obtain a basis for

$\text{Im } T$  by finding RREF  $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  and only using the vectors

corresponding to pivot columns:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

**Fact A.35**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation with standard matrix  $A$ .

- The kernel of  $T$  is the solution set of the homogeneous system given by the augmented matrix  $\left[ A \mid \vec{0} \right]$ . Use the coefficients of its free variables to get a basis for the kernel.
- The image of  $T$  is the span of the columns of  $A$ . Remove the vectors creating non-pivot columns in RREF  $A$  to get a basis for the image.

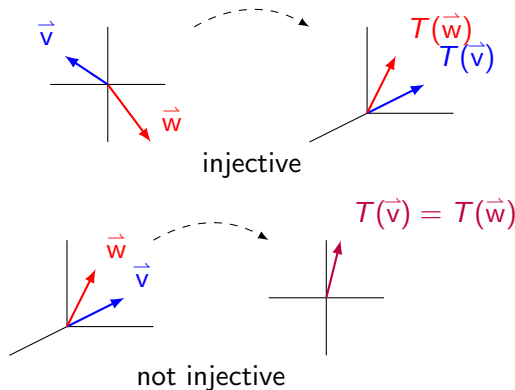
**Activity A.36** ( $\sim 10$  min) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of  $T$ .

**Definition A.37**

Let  $T : V \rightarrow W$  be a linear transformation.  $T$  is called **injective** or **one-to-one** if  $T$  does not map two distinct vectors to the same place. More precisely,  $T$  is injective if  $T(\vec{v}) \neq T(\vec{w})$  whenever  $\vec{v} \neq \vec{w}$ .



**Activity A.38** ( $\sim 3$  min) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is  $T$  injective?

- a) Yes, because  $T(\vec{v}) = T(\vec{w})$  whenever  $\vec{v} = \vec{w}$ .
- b) Yes, because  $T(\vec{v}) \neq T(\vec{w})$  whenever  $\vec{v} \neq \vec{w}$ .
- c) No, because  $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$
- d) No, because  $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$



**Activity A.39** ( $\sim 2$  min) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

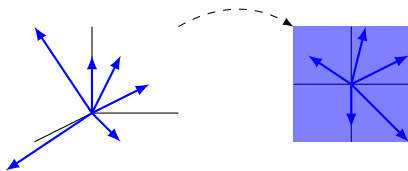
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is  $T$  injective?

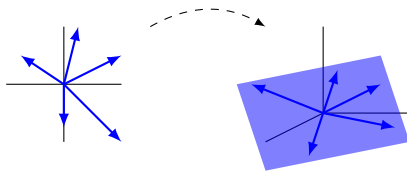
- a) Yes, because  $T(\vec{v}) = T(\vec{w})$  whenever  $\vec{v} = \vec{w}$ .
- b) Yes, because  $T(\vec{v}) \neq T(\vec{w})$  whenever  $\vec{v} \neq \vec{w}$ .
- c) No, because  $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$
- d) No, because  $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$

**Definition A.40**

Let  $T : V \rightarrow W$  be a linear transformation.  $T$  is called **surjective** or **onto** if every element of  $W$  is mapped to by an element of  $V$ . More precisely, for every  $\vec{w} \in W$ , there is some  $\vec{v} \in V$  with  $T(\vec{v}) = \vec{w}$ .



surjective



not surjective

**Activity A.41** ( $\sim 3$  min) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is  $T$  surjective?

- a) Yes, because for every  $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ , there exists  $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$  such that  $T(\vec{v}) = \vec{w}$ .
- b) No, because  $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$  can never equal  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .
- c) No, because  $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$  can never equal  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .

**Activity A.42** ( $\sim 2$  min) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is  $T$  surjective?

- a) Yes, because for every  $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ , there exists  $\vec{v} = \begin{bmatrix} x \\ y \\ 42 \end{bmatrix} \in \mathbb{R}^3$  such that  $T(\vec{v}) = \vec{w}$ .
- b) Yes, because for every  $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ , there exists  $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3$  such that  $T(\vec{v}) = \vec{w}$ .
- c) No, because  $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$  can never equal  $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ .

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**Section 4**

# Module A Section 4

**Observation A.43**

As we will see, it's no coincidence that the RREF of the injective map's standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has all pivot columns. Similarly, the RREF of the surjective map's standard matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has a pivot in each row.

## Observation A.44

Let  $T : V \rightarrow W$ . We have previously defined the following terms.

- The **kernel** of  $T$  is the set of all vectors in  $V$  that are mapped to  $\vec{0} \in W$ . It is a subspace of  $V$ .
- The **image** of  $T$  is the set of all vectors in  $W$  that are mapped to by something in  $V$ . It is a subspace of  $W$ .
- $T$  is called **injective** or **one-to-one** if  $T$  always maps distinct vectors to different places.
- $T$  is called **surjective** or **onto** if every element of  $W$  is mapped to by some element of  $V$ .

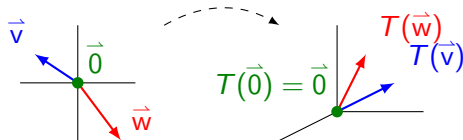
**Activity A.45** ( $\sim 5$  min) Let  $T : V \rightarrow W$  be a linear transformation where  $\ker T$  contains multiple vectors. What can you conclude?

- a  $T$  is injective
- b  $T$  is not injective
- c  $T$  is surjective
- d  $T$  is not surjective



**Fact A.46**

A linear transformation  $T$  is injective **if and only if**  $\ker T = \{\vec{0}\}$ . Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

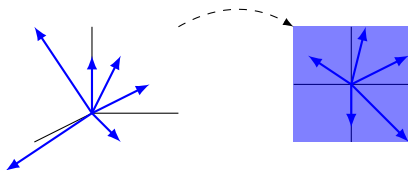


**Activity A.47** ( $\sim 5$  min) Let  $T : V \rightarrow \mathbb{R}^5$  be a linear transformation where  $\text{Im } T$  is spanned by four vectors. What can you conclude?

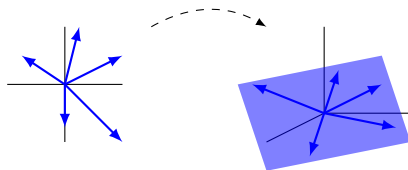
- a  $T$  is injective
- b  $T$  is not injective
- c  $T$  is surjective
- d  $T$  is not surjective

**Fact A.48**

A linear transformation  $T : V \rightarrow W$  is surjective **if and only if**  $\text{Im } T = W$ . Put another way, a surjective linear transformation may be recognized by its identical codomain and image.



surjective,  $\text{Im } T = \mathbb{R}^2$



not surjective,  $\text{Im } T \neq \mathbb{R}^3$

**Activity A.49** ( $\sim 15$  min) Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with standard matrix  $A$ . Sort the following claims into two groups of *equivalent* statements: one group that means  $T$  is **injective**, and one group that means  $T$  is **surjective**.

- a The kernel of  $T$  is trivial, i.e.  $\ker T = \{\vec{0}\}$ .
- b The columns of  $A$  span  $\mathbb{R}^m$ .
- c The columns of  $A$  are linearly independent.
- d Every column of  $\text{RREF}(A)$  has a pivot.
- e Every row of  $\text{RREF}(A)$  has a pivot.
- f The image of  $T$  equals its codomain, i.e.  $\text{Im } T = \mathbb{R}^m$ .
- g The system of linear equations given by the augmented matrix  $\left[ A \mid \vec{b} \right]$  has a solution for all  $\vec{b} \in \mathbb{R}^m$ .
- h The system of linear equations given by the augmented matrix  $\left[ A \mid \vec{0} \right]$  has exactly one solution.

## Observation A.50

The easiest way to show that the linear map with standard matrix  $A$  is injective is to show that  $\text{RREF}(A)$  has a pivot in each column.

The easiest way to show that the linear map with standard matrix  $A$  is surjective is to show that  $\text{RREF}(A)$  has a pivot in each row.

**Activity A.51** ( $\sim 3$  min) What can you conclude about the linear map

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  with standard matrix  $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ ?

- a) Its standard matrix has more columns than rows, so  $T$  is not injective.
- b) Its standard matrix has more columns than rows, so  $T$  is injective.
- c) Its standard matrix has more rows than columns, so  $T$  is not surjective.
- d) Its standard matrix has more rows than columns, so  $T$  is surjective.

**Activity A.52** ( $\sim 2$  min) What can you conclude about the linear map

$T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  with standard matrix  $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ ?

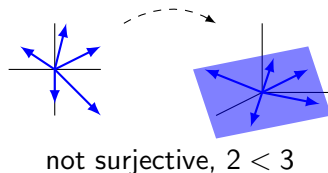
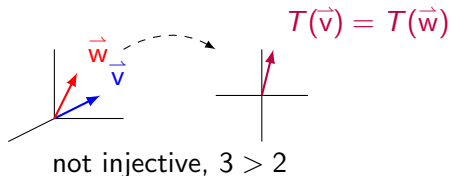
- a) Its standard matrix has more columns than rows, so  $T$  is not injective.
- b) Its standard matrix has more columns than rows, so  $T$  is injective.
- c) Its standard matrix has more rows than columns, so  $T$  is not surjective.
- d) Its standard matrix has more rows than columns, so  $T$  is surjective.

**Fact A.53**

The following are true for any linear map  $T : V \rightarrow W$ :

- If  $\dim(V) > \dim(W)$ , then  $T$  is not injective.
- If  $\dim(V) < \dim(W)$ , then  $T$  is not surjective.

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase dimension from its domain to its image.



But dimension arguments **cannot** be used to prove a map **is** injective or surjective.



**Activity A.54** (*~5 min*) Suppose  $T : \mathbb{R}^n \rightarrow \mathbb{R}^4$  with standard matrix

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$  is both injective and surjective (we call such maps **bijjective**).

**Activity A.54** (*~5 min*) Suppose  $T : \mathbb{R}^n \rightarrow \mathbb{R}^4$  with standard matrix

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$  is both injective and surjective (we call such maps **bijjective**).

*Part 1:* How many pivot rows must RREF  $A$  have?

**Activity A.54** (*~5 min*) Suppose  $T : \mathbb{R}^n \rightarrow \mathbb{R}^4$  with standard matrix

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$  is both injective and surjective (we call such maps **bijjective**).

*Part 1:* How many pivot rows must RREF  $A$  have?

*Part 2:* How many pivot columns must RREF  $A$  have?

**Activity A.54** ( $\sim 5$  min) Suppose  $T : \mathbb{R}^n \rightarrow \mathbb{R}^4$  with standard matrix

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$  is both injective and surjective (we call such maps **bijjective**).

*Part 1:* How many pivot rows must RREF  $A$  have?

*Part 2:* How many pivot columns must RREF  $A$  have?

*Part 3:* What is RREF  $A$ ?

## Module A

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**Activity A.55** ( $\sim 5$  min) Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a bijective linear map with standard matrix  $A$ . Label each of the following as true or false.

- a RREF( $A$ ) is the identity matrix.
- b The columns of  $A$  form a basis for  $\mathbb{R}^n$
- c The system of linear equations given by the augmented matrix  $\left[ A \mid \vec{b} \right]$  has exactly one solution for each  $\vec{b} \in \mathbb{R}^n$ .

## Observation A.56

The easiest way to show that the linear map with standard matrix  $A$  is bijective is to show that  $\text{RREF}(A)$  is the identity matrix.

**Activity A.57** ( $\sim 3$  min) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

- a  $T$  is neither injective nor surjective
- b  $T$  is injective but not surjective
- c  $T$  is surjective but not injective
- d  $T$  is bijective.

**Activity A.58** ( $\sim 3$  min) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

- a  $T$  is neither injective nor surjective
- b  $T$  is injective but not surjective
- c  $T$  is surjective but not injective
- d  $T$  is bijective.



**Activity A.59** ( $\sim 3$  min) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

- a  $T$  is neither injective nor surjective
- b  $T$  is injective but not surjective
- c  $T$  is surjective but not injective
- d  $T$  is bijective.