Module M: Understanding Matrices Algebraically

What algebraic structure do matrices have?

At the end of this module, students will be able to...

M1. **Matrix Multiplication.** ... multiply matrices.

M2. **Invertible Matrices.** ... determine if a square matrix is invertible or not.

M3. **Matrix inverses.** ... compute the inverse matrix of an invertible matrix.

M4. **Row operations as multiplication.** ... describe the row reduction of a matrix as matrix multiplication.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers.
- Identify the domain and codomain of linear transformations.
- Find the matrix corresponding to a linear transformation and compute the image of a vector given a standard matrix **A2**
- Determine if a linear transformation is injective and/or surjective **A3**
- Interpret the ideas of injectivity and surjectivity in multiple ways.

The following resources will help you prepare for this module.

- Function composition (Khan Academy): `http://bit.ly/2wkz7f3`
- Domain and codomain: `https://www.youtube.com/watch?v=BQMyeQOLvpg`
- Interpreting injectivity and surjectivity in many ways: `https://www.youtube.com/watch?v=WpUv72Y6D10`

Module M Section 1

Activity M.1.1 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity M.1.2 *(∼3 min)*

Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be given by the 2×3 standard matrix $B=\begin{bmatrix}2 & 1 & -3\\ 5 & -3 & 4\end{bmatrix}$ and $S:\mathbb{R}^2\to\mathbb{R}^4$ be given by the 4×2 standard matrix $A=\begin{bmatrix}1 & 2\\ 0 & 1\\ 3 & 5\\ -1 & -2\end{bmatrix}$.

What is the codomain of the composition map $S\circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity M.1.3 *(∼2 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What size will the standard matrix of $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$ be? (Rows $\times$ Columns)

(a) 4×3

(b) 4×2

(c) 3×4

(d) 3×2

(e) 2×4

(f) 2×3

Activity M.1.4 *(~15 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Activity M.1.4 *(∼15 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Activity M.1.4 *(~15 min)*

Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be given by the 2×3 standard matrix $B=\begin{bmatrix}2 & 1 & -3\\ 5 & -3 & 4\end{bmatrix}$ and $S:\mathbb{R}^2\to\mathbb{R}^4$ be given by the 4×2 standard matrix $A=\begin{bmatrix}1 & 2\\ 0 & 1\\ 3 & 5\\ -1 & -2\end{bmatrix}$.

Part 1: Compute

$$(S\circ T)(\vec{\mathbf{e}}_1)=S(T(\vec{\mathbf{e}}_1))=S\left(\begin{bmatrix}2\\5\end{bmatrix}\right)=\begin{bmatrix}?\\?\\?\\?\end{bmatrix}.$$

Part 2: Compute $(S\circ T)(\vec{\mathbf{e}}_2)$.

Activity M.1.4 *(∼15 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{\mathbf{e}}_2)$.
Part 3: Compute $(S \circ T)(\vec{\mathbf{e}}_3)$.

Activity M.1.4 *(∼15 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{\mathbf{e}}_2)$.
Part 3: Compute $(S \circ T)(\vec{\mathbf{e}}_3)$.
Part 4: Find the 4×3 standard matrix of $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$.

Definition M.1.5

We define the **product** AB of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For the previous activity, S had a 4×2 matrix and T had a 2×3 matrix, so $S \circ T$ had a 4×3 standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= [(S \circ T)(\vec{\mathbf{e}}_1)(S \circ T)(\vec{\mathbf{e}}_2)(S \circ T)(\vec{\mathbf{e}}_3)] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

Activity M.1.6 *(∼10 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

Find the standard matrix AB of $S \circ T$.

Activity M.1.7 *(∼5 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

Find the standard matrix BA of $T \circ S$.

Activity M.1.8 *(∼10 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be given by the matrix $B = \begin{bmatrix} 3 & 2 & 5 & -4 \\ -1 & -3 & 1 & 2 \end{bmatrix}$ and let $S : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ -4 & 2 \end{bmatrix}$. Compute AB, the standard matrix of the composition $S \circ T$.

Observation M.1.9

Note that an $\mathbb{R}^n$ vector acts exactly the same as an $n \times 1$ matrix, so we will use them interchangablely, as follows.

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix} \qquad X = \vec{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \vec{\mathbf{b}} = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$$

So we may study the linear system

$$\begin{aligned} 3x + y - z &= 5 \\ 2x + 4z &= -7 \\ -x + 3y + 5z &= 2 \end{aligned}$$

as both a vector equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and a matrix equation $AX = B$:

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$$

Module M Section 2

Observation M.2.1

Recall that if $T : \mathbb{R}^n \to \mathbb{R}^k$ is a linear map with standard matrix $B \in M_{k,n}$ and $S : \mathbb{R}^k \to \mathbb{R}^m$ is a linear map with standard matrix $A \in M_{m,k}$, the product matrix $AB \in M_{m,n}$ is defined to be the standard matrix of the composition map

$$S \circ T : \mathbb{R}^n \to \mathbb{R}^m.$$

Activity M.2.2 *(~5 min)*

Matrix multiplication only makes sense if the first matrix has as many columns as the second matrix has rows. Label each of these matrices with **rows** $\times$ **columns**, and then figure out which of the products AB, AC, BA, BC, CA, CB can actually be computed.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Activity M.2.3 *(∼10 min)*

Let $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Compute the product BA.

Activity M.2.4 *(∼5 min)*

Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix I such that $IA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Definition M.2.5

The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

Fact M.2.6

For any square matrix A, $IA = AI = A$:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 7 & -1\\0 & 3 & 2\\1 & 1 & -1\end{bmatrix} = \begin{bmatrix}2 & 7 & -1\\0 & 3 & 2\\1 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 7 & -1\\0 & 3 & 2\\1 & 1 & -1\end{bmatrix}$$

Activity M.2.7 *($\sim$20 min)*

Each row operation can be interpreted as a type of matrix multiplication.

Activity M.2.7 *(∼20 min)*

Each row operation can be interpreted as a type of matrix multiplication.

Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Activity M.2.7 *(∼20 min)*

Each row operation can be interpreted as a type of matrix multiplication.

Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Activity M.2.7 *(∼20 min)*

Each row operation can be interpreted as a type of matrix multiplication.

Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5(1) & 7+5(1) & -1+5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Fact M.2.8

If R is the result of applying a row operation to I, then RA is the result of applying the same row operation to A.

This means that for any matrix A, we can find a series of matrices $R_1, \ldots, R_k$ corresponding to the row operations such that

$$R_1 R_2 \cdots R_k A = \mathrm{RREF}(A).$$

That is, row reduction can be thought of as the result of matrix multiplication.

Module M Section 3

Activity M.3.1 *(∼15 min)*

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into three groups of statements: a group that means T is **injective**, a group that means T is **surjective**, and a group that means T is **bijective**.

(a) $AX = B$ has a solution for all $m \times 1$ matrices B

(b) $AX = B$ has a unique solution for all $m \times 1$ matrices B

(c) $AX = 0$ has a unique solution.

(d) The columns of A span $\mathbb{R}^m$

(e) The columns of A are linearly independent

(f) The columns of A are a basis of $\mathbb{R}^m$

(g) Every column of $\mathrm{RREF}(A)$ has a pivot

(h) Every row of $\mathrm{RREF}(A)$ has a pivot

(i) $m = n$ and $\mathrm{RREF}(A) = I$

Definition M.3.2

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with standard matrix A.

- If T is a bijection and B is any $\mathbb{R}^n$ vector, then $T(X) = AX = B$ has a unique solution X.
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by setting $T^{-1}(B) = X$ to be this unique solution.
- Let A^{-1} be the standard matrix for T^{-1}. We call A^{-1} the **inverse matrix** of A, so we also say that A is **invertible**.

Activity M.3.3 *(∼20 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Activity M.3.3 *(~20 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Part 1: Write an augmented matrix representing the system of equations given by $T(X) = \vec{\mathbf{e}}_1$ (or in matrix form, $AX = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$).

Activity M.3.3 *($\sim$20 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Part 1: Write an augmented matrix representing the system of equations given by $T(X) = \vec{\mathbf{e}}_1$ (or in matrix form, $AX = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$).

Part 2: Solve $T(X) = \vec{\mathbf{e}}_1$ to find $T^{-1}(\vec{\mathbf{e}}_1)$.

Activity M.3.3 *(∼20 min)*

Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A=\begin{bmatrix}2 & -1 & -6\\ 2 & 1 & 3\\ 1 & 1 & 4\end{bmatrix}.$$

Part 1: Write an augmented matrix representing the system of equations given by

$T(X)=\vec{\mathbf{e}}_1$ (or in matrix form, $AX=\begin{bmatrix}1\\0\\0\end{bmatrix}$).

Part 2: Solve $T(X)=\vec{\mathbf{e}}_1$ to find $T^{-1}(\vec{\mathbf{e}}_1)$.

Part 3: Solve $T(X)=\vec{\mathbf{e}}_2$ to find $T^{-1}(\vec{\mathbf{e}}_2)$.

Activity M.3.3 *($\sim$20 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Part 1: Write an augmented matrix representing the system of equations given by

$$T(X) = \vec{\mathbf{e}}_1 \text{ (or in matrix form, } AX = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{).}$$

Part 2: Solve $T(X) = \vec{\mathbf{e}}_1$ to find $T^{-1}(\vec{\mathbf{e}}_1)$.

Part 3: Solve $T(X) = \vec{\mathbf{e}}_2$ to find $T^{-1}(\vec{\mathbf{e}}_2)$.

Part 4: Solve $T(X) = \vec{\mathbf{e}}_3$ to find $T^{-1}(\vec{\mathbf{e}}_3)$.

Activity M.3.3 *(∼20 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(X) = \vec{\mathbf{e}}_1$ (or in matrix form, $AX = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$).

Part 2: Solve $T(X) = \vec{\mathbf{e}}_1$ to find $T^{-1}(\vec{\mathbf{e}}_1)$.

Part 3: Solve $T(X) = \vec{\mathbf{e}}_2$ to find $T^{-1}(\vec{\mathbf{e}}_2)$.

Part 4: Solve $T(X) = \vec{\mathbf{e}}_3$ to find $T^{-1}(\vec{\mathbf{e}}_3)$.

Part 5: Compute A^{-1}, the standard matrix for T^{-1}.

Observation M.3.4

We could have solved these three systems simultaneously by row reducing the matrix $[A\,|\,I]$ at once.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right]$$

Activity M.3.5 *(∼5 min)*

Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A \,|\, I]$.

Activity M.3.6 *(∼5 min)*

Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

Observation M.3.7

An $n \times n$ matrix A is invertible if and only if $\mathrm{RREF}(A) = I_n$.

Activity M.3.8 *(∼10 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Activity M.3.8 *(∼10 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

Activity M.3.8 *(∼10 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

Part 2: If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, find the 2×2 matrix

$$A^{-1}A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

Observation M.3.9

$T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation T. Therefore $A^{-1}A = AA^{-1} = I$ is the identity matrix for any invertible matrix A.