

## Module E

### Standard E1

**E1.** Write an augmented matrix corresponding to the following system of linear equations.

$$x + 3y - 4z = 5$$

$$3x + 9y + z = 0$$

$$x - z = 1$$

*Solution.*

$$\left[ \begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{array} \right]$$

□

**E1.** Write an augmented matrix corresponding to the following system of linear equations.

$$x_1 + 4x_3 = 1$$

$$x_2 - x_3 = 7$$

$$x_1 - x_2 + 3x_3 = -1$$

*Solution.*

$$\left[ \begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array} \right]$$

□

**E1.** Write an augmented matrix corresponding to the following system of linear equations.

$$x_1 + 4x_3 = 1$$

$$x_2 - x_3 = 7$$

$$x_1 - x_2 + 3x_4 = -1$$

*Solution.*

$$\left[ \begin{array}{cccc|c} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & -1 & 0 & 7 \\ 1 & -1 & 0 & 3 & -1 \end{array} \right]$$

□

**E1.** Write an augmented matrix corresponding to the following system of linear equations.

$$x_1 + 3x_2 - 4x_3 + x_4 = 5$$

$$3x_1 + 9x_2 + x_3 - 7x_4 = 0$$

$$x_1 - x_3 + x_4 = 1$$

*Solution.*

$$\left[ \begin{array}{cccc|c} 1 & 3 & -4 & 1 & 5 \\ 3 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{array} \right]$$

□

**E1.** Write a system of linear equations corresponding to the following augmented matrix.

$$\left[ \begin{array}{cccc|c} 3 & -1 & 0 & 1 & 5 \\ -1 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 0 & -3 \end{array} \right]$$

*Solution.*

$$\begin{aligned}3x_1 - x_2 + x_4 &= 5 \\ -x_1 + 9x_2 + x_3 - 7x_4 &= 0 \\ x_1 - x_3 &= -3\end{aligned}$$

□

**E1.** Write a system of linear equations corresponding to the following augmented matrix.

$$\left[ \begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 4 & 1 & -7 \\ 1 & 2 & -1 & 0 \end{array} \right]$$

*Solution.*

$$\begin{aligned}2x_1 - x_2 &= 1 \\ -x_1 + 4x_2 + x_3 &= -7 \\ x_1 + 2x_2 - x_3 &= 0\end{aligned}$$

□

**E1.** Write a system of linear equations corresponding to the following augmented matrix.

$$\left[ \begin{array}{ccc|c} -4 & -1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{array} \right]$$

*Solution.*

$$\begin{aligned}-4x_1 - x_2 + 3x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 0 \\ -x_1 + 4x_2 + x_3 &= 4\end{aligned}$$

□

**E1.** Write a system of linear equations corresponding to the following augmented matrix.

$$\left[ \begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array} \right]$$

*Solution.*

$$\begin{aligned}x_1 + 4x_3 &= 1 \\ x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= -1\end{aligned}$$

□

## Standard E2

**E2.** Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

*Solution.*

$$\begin{aligned} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix} &\sim \begin{bmatrix} -1 & 0 & -1 \\ 3 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

□

**E2.** Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix}$$

*Solution.*

$$\begin{aligned} \begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ -3 & 5 & 2 & 0 \\ 1 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & 6 \\ 0 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

□

**E2.** Put the following matrix in reduced row echelon form.

$$\left[ \begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array} \right]$$

*Solution.*

$$\begin{aligned} \left[ \begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array} \right] &\sim \left[ \begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & -\frac{2}{3} & -1 & \frac{2}{3} \\ 0 & 2 & 3 & -2 \end{array} \right] \\ &\sim \left[ \begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 2 & 3 & -2 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

□

**E2.** Find the reduced row echelon form of the matrix below.

$$\left[ \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right]$$

*Solution.*

$$\begin{aligned} & \left[ \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right] \sim \left[ \begin{array}{cccc|c} -1 & 0 & 5 & 0 & -1 \\ 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \end{array} \right] \\ & \sim \left[ \begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & -1 & 15 & -2 & -3 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & 0 & 24 & -2 & 0 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{5}{12} & 1 \\ 0 & 1 & 0 & \frac{3}{4} & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{array} \right] \end{aligned}$$

□

**E2.** Find RREF  $A$ , where

$$A = \left[ \begin{array}{cccc|c} 3 & -2 & 1 & 8 & -5 \\ 2 & 2 & 0 & 6 & -2 \\ -1 & 1 & 1 & -4 & 6 \end{array} \right]$$

*Solution.*

$$\text{RREF } A = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \end{array} \right]$$

□

**E2.** Find RREF  $A$ , where

$$A = \left[ \begin{array}{cc|c} 2 & -7 & 4 \\ 1 & -3 & 2 \\ 3 & 0 & 3 \end{array} \right]$$

*Solution.*

$$\text{RREF } A = \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

□

**E2.** Find RREF  $A$ , where

$$A = \left[ \begin{array}{ccc|c} 2 & -1 & 5 & 4 \\ -1 & 0 & -2 & -1 \\ 1 & 3 & -1 & -5 \end{array} \right]$$

*Solution.*

$$\text{RREF } A = \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

□

**E2.** Find RREF  $A$ , where

$$A = \left[ \begin{array}{cccc|c} 2 & 2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 4 & 5 \\ 3 & 3 & -1 & -2 & 1 \end{array} \right]$$

*Solution.*

$$\text{RREF } A = \left[ \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

□

### Standard E3

**E3.** Solve the system of equations

$$x + 3y - 4z = 5$$

$$3x + 9y + z = 2$$

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 2 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

So the solution set is

$$\left\{ \left[ \begin{array}{c} 1 - 3c \\ c \\ -1 \end{array} \right] \mid c \in \mathbb{R} \right\}$$

□

**E3.** Solve the system of equations

$$-3x + y = 2$$

$$-8x + 2y - z = 6$$

$$2y + 3z = -2$$

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The solutions are

$$\left\{ \left[ \begin{array}{c} -1 - \frac{c}{2} \\ -1 - \frac{3c}{2} \\ c \end{array} \right] \mid c \in \mathbb{R} \right\} = \left\{ \left[ \begin{array}{c} c - 1 \\ 3c - 1 \\ -2c \end{array} \right] \mid c \in \mathbb{R} \right\}$$

□

**E3.** Solve the system of linear equations.

$$2x + y - z + w = 5$$

$$3x - y - 2w = 0$$

$$-x + 5z + 3w = -1$$

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right] \right) = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{12} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{array} \right]$$

So the solutions are

$$\left\{ \left[ \begin{array}{c} 1 + a \\ 3 - 21a \\ -7a \\ 12a \end{array} \right] \mid a \in \mathbb{R} \right\}$$

□

**E3.** Solve the system of linear equations.

$$2x + y - z + w = 5$$

$$3x - y - 2w = 0$$

$$-x + 5z + 3w = -1$$

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right] \right) = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{4} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{array} \right]$$

So the solutions are

$$\left\{ \left[ \begin{array}{c} 1+a \\ 3-21a \\ -7a \\ 12a \end{array} \right] \mid a \in \mathbb{R} \right\}$$

□

**E3.** Find the solution set for the following system of linear equations.

$$2x_1 - 2x_2 + 6x_3 - x_4 = -1$$

$$3x_1 + 6x_3 + x_4 = 5$$

$$-4x_1 + x_2 - 9x_3 + 2x_4 = -7$$

*Solution.* Let  $A = \left[ \begin{array}{cccc|c} 2 & -2 & 6 & -1 & -1 \\ 3 & 0 & 6 & 1 & 5 \\ -4 & 1 & -9 & 2 & -7 \end{array} \right]$ , so  $\text{RREF } A = \left[ \begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right]$ . It follows that the

solution set is given by  $\left[ \begin{array}{c} 2-2a \\ 3+a \\ a \\ -1 \end{array} \right]$  for all real numbers  $a$ .

□

**E3.** Find the solution set for the following system of linear equations.

$$2x_1 + 3x_2 - 5x_3 + 14x_4 = 8$$

$$x_1 + x_2 - x_3 + 5x_4 = 3$$

*Solution.* Let  $A = \left[ \begin{array}{cccc|c} 2 & 3 & -5 & 14 & 8 \\ 1 & 1 & -1 & 5 & 3 \end{array} \right]$ , so  $\text{RREF } A = \left[ \begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -3 & 4 & 2 \end{array} \right]$ . It follows that the solution set

is given by  $\left[ \begin{array}{c} 1-2a-b \\ 2+3a-4b \\ a \\ b \end{array} \right]$  for all real numbers  $a, b$ .

□

**E3.** Solve the following linear system.

$$4x_1 + 4x_2 + 3x_3 - 6x_4 = 5$$

$$-2x_3 - 4x_4 = 3$$

$$2x_1 + 2x_2 + x_3 - 4x_4 = -1$$

*Solution.* Let  $A = \left[ \begin{array}{cccc|c} 4 & 4 & 3 & -6 & 5 \\ 0 & 0 & -2 & -4 & 3 \\ 2 & 2 & 1 & -4 & -1 \end{array} \right]$ , so  $\text{RREF } A = \left[ \begin{array}{cccc|c} 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$ . It follows that the system is inconsistent with no solutions (since the bottom row implies the contradiction  $0 = 1$ ).

□

**E3.** Solve the following linear system.

$$3x + 2y + z = 7$$

$$x + y + z = 1$$

$$-2x + 3z = -11$$

*Solution.* Let  $A = \left[ \begin{array}{ccc|c} 3 & 2 & 1 & 7 \\ 1 & 1 & 1 & 1 \\ -2 & 0 & 3 & 11 \end{array} \right]$ , so  $\text{RREF } A = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array} \right]$ . It follows that the system has exactly one solution:  $\begin{bmatrix} 4 & -2 & -1 \end{bmatrix}$  □

### Standard E4

**E4.** Find a basis for the solution set of the system of equations

$$\begin{aligned}x + 3y + 3z + 7w &= 0 \\x + 3y - z - w &= 0 \\2x + 6y + 3z + 8w &= 0 \\x + 3y - 2z - 3w &= 0\end{aligned}$$

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution set is

$$\left\{ \begin{bmatrix} -3a - b \\ a \\ -2b \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

So a basis for the solution set is

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

□

**E4.** Find a basis for the solution set of the system of equations

$$\begin{aligned}x + 2y + 3z + w &= 0 \\3x - y + z + w &= 0 \\2x - 3y - 2z &= 0 \\-x + 2z + 5w &= 0\end{aligned}$$

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & -2 & 3 & 1 \\ 3 & -1 & 1 & 1 \\ 2 & -3 & -2 & 0 \\ -1 & 0 & 2 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution set is

$$\left\{ \begin{bmatrix} a \\ 2a \\ -2a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$$

So a basis for the solution set is  $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$ .

□

**E4.** Find a basis for the solution set of the system of equations

$$\begin{aligned}x + 2y + 3z + w &= 0 \\3x - y + z + w &= 0 \\2x - 3y - 2z &= 0\end{aligned}$$



*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & -2 & 3 & 1 \\ 3 & -1 & 1 & 1 \\ 2 & -3 & -2 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & \frac{5}{7} & \frac{3}{7} \\ 0 & 1 & \frac{8}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution set is

$$\left\{ \begin{bmatrix} -\frac{5}{7}a - \frac{3}{7}b \\ -\frac{8}{7}a - \frac{2}{7}b \\ a \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

So a basis for the solution set is  $\left\{ \begin{bmatrix} -\frac{5}{7} \\ -\frac{8}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{7} \\ -\frac{2}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$ , or  $\left\{ \begin{bmatrix} 5 \\ 8 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ -7 \end{bmatrix} \right\}$ . □

**E4.** Find a basis for the solution set to the system of equations

$$x + 2y - 3z = 0$$

$$2x + y - 4z = 0$$

$$3y - 2z = 0$$

$$x - y - z = 0$$

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -4 \\ 0 & 3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\frac{5}{3} \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the solution set is

$$\left\{ \begin{bmatrix} \frac{5}{3}a \\ \frac{3}{5}a \\ \frac{2}{3}a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$$

So a basis is  $\left\{ \begin{bmatrix} \frac{5}{3} \\ \frac{3}{5} \\ \frac{2}{3} \\ 1 \end{bmatrix} \right\}$  or  $\left\{ \begin{bmatrix} 5 \\ 2 \\ 3 \\ 3 \end{bmatrix} \right\}$ . □

**E4.** Find a basis for the solution set to the homogeneous system of equations

$$2x_1 + 3x_2 - 5x_3 + 14x_4 = 0$$

$$x_1 + x_2 - x_3 + 5x_4 = 0$$

*Solution.* Let  $A = \begin{bmatrix} 2 & 3 & -5 & 14 & | & 0 \\ 1 & 1 & -1 & 5 & | & 0 \end{bmatrix}$ , so  $\text{RREF } A = \begin{bmatrix} 1 & 0 & 2 & 1 & | & 1 \\ 0 & 1 & -3 & 4 & | & 2 \end{bmatrix}$ . It follows that the basis for the

solution set is given by  $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$ . □

**E4.** Find a basis for the solution set to the homogeneous system of equations

$$4x_1 + 4x_2 + 3x_3 - 6x_4 = 0$$

$$-2x_3 - 4x_4 = 0$$

$$2x_1 + 2x_2 + x_3 - 4x_4 = 0$$

*Solution.* Let  $A = \left[ \begin{array}{cccc|c} 4 & 4 & 3 & -6 & 0 \\ 0 & 0 & -2 & -4 & 0 \\ 2 & 2 & 1 & -4 & 0 \end{array} \right]$ , so  $\text{RREF } A = \left[ \begin{array}{cccc|c} 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$ . It follows that the basis for

the solution set is given by  $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ . □

**E4.** Find a basis for the solution set to the homogeneous system of equations given by

$$3x + 2y + z = 0$$

$$x + y + z = 0$$

*Solution.* Let  $A = \left[ \begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right]$ , so  $\text{RREF } A = \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$ . It follows that the basis for the solution set is given by  $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ . □

**E4.** Find a basis for the solution set to the homogeneous system of equations given by

$$2x_1 - 2x_2 + 6x_3 - x_4 = 0$$

$$3x_1 + 6x_3 + x_4 = 0$$

$$-4x_1 + x_2 - 9x_3 + 2x_4 = 0$$

*Solution.* Let  $A = \left[ \begin{array}{cccc|c} 2 & -2 & 6 & -1 & 0 \\ 3 & 0 & 6 & 1 & 0 \\ -4 & 1 & -9 & 2 & 0 \end{array} \right]$ , so  $\text{RREF } A = \left[ \begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$ . It follows that the basis for the solution set is given by  $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ . □

# Module V

## Standard V1

**V1.** Let  $V$  be the set of all real numbers together with the operations  $\oplus$  and  $\odot$  defined by, for any  $x, y \in V$  and  $c \in \mathbb{R}$ ,

$$\begin{aligned}x \oplus y &= x + y - 3 \\c \odot x &= cx - 3(c - 1)\end{aligned}$$

- (a) Show that **scalar multiplication** is **associative**:  $a \odot (b \odot x) = (ab) \odot x$ .  
(b) Determine if  $V$  is a vector space or not. Justify your answer

*Solution.* Let  $x, y \in V$ ,  $c, d \in \mathbb{R}$ . To show associativity:

$$\begin{aligned}c \odot (d \odot x) &= c \odot (dx - 3(d - 1)) \\&= c(dx - 3(d - 1)) - 3(c - 1) \\&= cdx - 3(cd - 1) \\&= (cd) \odot x\end{aligned}$$

We verify the remaining 7 properties to see that  $V$  is a vector space.

- 1) Real addition is associative, so  $\oplus$  is associative.
- 2)  $x \oplus 3 = x + 3 - 3 = x$ , so 3 is the additive identity.
- 3)  $x \oplus (6 - x) = x + (6 - x) - 3 = 3$ , so  $6 - x$  is the additive inverse of  $x$ .
- 4) Real addition is commutative, so  $\oplus$  is commutative.
- 5) Associativity shown above
- 6)  $1 \odot x = x - 3(1 - 1) = x$
- 7)

$$\begin{aligned}c \odot (x \oplus y) &= c \odot (x + y - 3) \\&= c(x + y - 3) - 3(c - 1) \\&= cx - 3(c - 1) + cy - 3(c - 1) - 3 \\&= (c \odot x) \oplus (c \odot y)\end{aligned}$$

8)

$$\begin{aligned}(c + d) \odot x &= (c + d)x - 3(c + d - 1) \\&= cx - 3(c - 1) + dx - 3(d - 1) - 3 \\&= (c \odot x) \oplus (d \odot x)\end{aligned}$$

Therefore  $V$  is a vector space. □

**V1.** Let  $V$  be the set of all points on the parabola  $y = x^2$  with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$\begin{aligned}(x_1, y_1) \oplus (x_2, y_2) &= (x_1 + x_2, y_1 + y_2 + 2x_1x_2) \\c \odot (x_1, y_1) &= (cx_1, c^2y_1)\end{aligned}$$

- (a) Show that the vector **addition**  $\oplus$  is **associative**:  
 $(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) = ((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3).$

- (b) Determine if  $V$  is a vector space or not. Justify your answer.

**V1.** Let  $V$  be the set of all points on the parabola  $y = x^2$  with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V, c \in \mathbb{R}$ ,

$$\begin{aligned}(x_1, y_1) \oplus (x_2, y_2) &= (x_1 - x_2, y_1 + y_2 - 2x_1x_2) \\ c \odot (x_1, y_1) &= (cx_1, c^2y_1)\end{aligned}$$

- (a) Show that scalar multiplication **distributes scalars** over vector addition:  
 $c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2).$

- (b) Determine if  $V$  is a vector space or not. Justify your answer.

*Solution.*

$$\begin{aligned}c \odot ((x_1, y_1) \oplus (x_2, y_2)) &= c \odot (x_1 - x_2, y_1 + y_2 - 2x_1x_2) = (c(x_1 - x_2), c^2(y_1 + y_2 - 2x_1x_2)) \\ c \odot (x_1, y_1) \oplus c \odot (x_2, y_2) &= (cx_1, c^2y_1) \oplus (cx_2, c^2y_2) = (cx_1 - cx_2, c^2y_1 + c^2y_2 - 2(cx_1)(cx_2))\end{aligned}$$

Not a vector space as addition is not commutative. □

**V1.** Let  $V$  be the set of all points on the line  $x + y = 2$  with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V, c \in \mathbb{R}$ ,

$$\begin{aligned}(x_1, y_1) \oplus (x_2, y_2) &= (x_1 + x_2 - 1, y_1 + y_2 - 1) \\ c \odot (x_1, y_1) &= (cx_1 - (c - 1), cy_1 - (c - 1))\end{aligned}$$

- (a) Show that this vector space has an **additive identity** element  $\mathbf{0}$  satisfying  $(x, y) \oplus \mathbf{0} = (x, y).$   
(b) Determine if  $V$  is a vector space or not. Justify your answer.

*Solution.* Let  $(x_1, y_1) \in V$ ; then  $(x_1, y_1) \oplus (1, 1) = (x_1, y_1)$ , so  $(1, 1)$  is an additive identity element. Now we will show the other seven properties. Let  $(x_1, y_1), (x_2, y_2) \in V$ , and let  $c, d \in \mathbb{R}$ .

- 1) Since real addition is associative,  $\oplus$  is associative.
- 2) Since real addition is commutative,  $\oplus$  is commutative.
- 3) The additive identity is  $(1, 1).$
- 4)  $(x_1, y_1) \oplus (2 - x_1, 2 - y_1) = (1, 1)$ , so  $(2 - x_1, 2 - y_1)$  is the additive inverse of  $(x_1, y_1).$
- 5)

$$\begin{aligned}c \odot (d \odot (x_1, y_1)) &= c \odot (dx_1 - (d - 1), dy_1 - (d - 1)) \\ &= (c(dx_1 - (d - 1)) - (c - 1), c(dy_1 - (d - 1))) \\ &= (cdx_1 - cd + c - (c - 1), cdy_1 - cd + c - (c - 1)) \\ &= (cdx_1 - (cd - 1), cdy_1 - (cd - 1)) \\ &= (cd) \odot (x_1, y_1)\end{aligned}$$

$$6) 1 \odot (x_1, y_1) = (x_1 - (1 - 1), y_1 - (1 - 1)) = (x_1, y_1)$$

7)

$$\begin{aligned}c \odot ((x_1, y_1) \oplus (x_2, y_2)) &= c \odot (x_1 + y_1 - 1, x_2 + y_2 - 1) \\ &= (c(x_1 + y_1 - 1) - (c - 1), c(x_2 + y_2 - 1) - (c - 1)) \\ &= (cx_1 + cy_1 - 2c + 1, cx_2 + cy_2 - 2c + 1) \\ &= (cx_1 - (c - 1), cy_1 - (c - 1)) \oplus (cx_2 - (c - 1), cy_2 - (c - 1)) \\ &= c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)\end{aligned}$$

8)

$$\begin{aligned}
(c+d) \odot (x_1, y_1) &= ((c+d)x_1 - (c+d-1), (c+d)y_1 - (c+d-1)) \\
&= (cx_1 - (c-1), cy_1 - (c-1)) \oplus (dx_1 - (d-1), dy_1 - (d-1)) \\
&= c \odot (x_1, y_1) \oplus d \odot (x_1, y_1)
\end{aligned}$$

Therefore  $V$  is a vector space. □

**V1.** Let  $V$  be the set of all pairs of real numbers with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V, c \in \mathbb{R}$ ,

$$\begin{aligned}
(x_1, y_1) \oplus (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\
c \odot (x_1, y_1) &= (0, cy_1)
\end{aligned}$$

(a) Show that scalar multiplication **distributes vectors** over scalar addition:

$$(c+d) \odot (x, y) = c \odot (x, y) \oplus d \odot (x, y).$$

(b) Determine if  $V$  is a vector space or not. Justify your answer.

*Solution.* Let  $(x_1, y_1) \in V$ , and let  $c, d \in \mathbb{R}$ . Then

$$(c+d) \odot (x_1, y_1) = (0, (c+d)y_1) = (0, cy_1) \oplus (0, dy_1) = c \odot (x_1, y_1) \oplus d \odot (x_1, y_1).$$

However,  $V$  is not a vector space, as  $1 \odot (x_1, y_1) = (0, y_1) \neq (x_1, y_1)$ . □

**V1.** Let  $V$  be the set of all pairs of real numbers with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V, c \in \mathbb{R}$ ,

$$\begin{aligned}
(x_1, y_1) \oplus (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\
c \odot (x_1, y_1) &= (c^2 x_1, c^3 y_1)
\end{aligned}$$

(a) Show that scalar multiplication **distributes scalars** over vector addition:

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2).$$

(b) Determine if  $V$  is a vector space or not. Justify your answer.

*Solution.* Let  $(x_1, y_1), (x_2, y_2) \in V$  and let  $c \in \mathbb{R}$ .

$$\begin{aligned}
c \odot ((x_1, y_1) \oplus (x_2, y_2)) &= c \odot (x_1 + x_2, y_1 + y_2) \\
&= (c^2(x_1 + x_2), c^3(y_1 + y_2)) \\
&= (c^2 x_1, c^3 y_1) \oplus (c^2 x_2, c^3 y_2) \\
&= c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)
\end{aligned}$$

However,  $V$  is not a vector space, as the other distributive law fails:

$$(c+d) \odot (x_1, y_1) = ((c+d)^2 x_1, (c+d)^3 y_1) \neq ((c^2 + d^2)x_1, (c^3 + d^3)y_1) = c \odot (x_1, y_1) \oplus d \odot (x_1, y_1).$$

□

**V1.** Let  $V$  be the set of all polynomials with the operations, for any  $f, g \in V, c \in \mathbb{R}$ ,

$$\begin{aligned}
f \oplus g &= f' + g' \\
c \odot f &= cf'
\end{aligned}$$

(here  $f'$  denotes the derivative of  $f$ ).

(a) Show that scalar multiplication **distributes scalars** over vector addition:  $c \odot (f \oplus g) = c \odot f \oplus c \odot g$ .

(b) Determine if  $V$  is a vector space or not. Justify your answer.

*Solution.* Let  $f, g \in \mathcal{P}$ , and let  $c \in \mathbb{R}$ .

$$c \odot (f \oplus g) = c \odot (f' + g') = c(f' + g')' = cf'' + cg'' = cf' \oplus cg' = c \odot f \oplus c \odot g.$$

However, this is not a vector space, as there is no zero vector. Additionally,  $1 \odot f \neq f$  for any nonzero polynomial  $f$ . □

**V1.** Let  $V$  be the set of all real numbers with the operations, for any  $x, y \in V$ ,  $c \in \mathbb{R}$ ,

$$\begin{aligned} x \oplus y &= \sqrt{x^2 + y^2} \\ c \odot x &= cx \end{aligned}$$

(a) Show that the vector **addition**  $\oplus$  is **associative**:  $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ .

(b) Determine if  $V$  is a vector space or not. Justify your answer.

*Solution.* Let  $x, y, z \in \mathbb{R}$ . Then

$$\begin{aligned} (x \oplus y) \oplus z &= \sqrt{x^2 + y^2} \oplus z \\ &= \sqrt{(\sqrt{x^2 + y^2})^2 + z^2} \\ &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{x^2 + (\sqrt{y^2 + z^2})^2} \\ &= x \oplus \sqrt{y^2 + z^2} \\ &= x \oplus (y \oplus z) \end{aligned}$$

However, this is not a vector space, as there is no zero vector. □

## Standard V2

**V2.** Determine if  $\begin{bmatrix} 0 \\ -1 \\ 2 \\ 6 \end{bmatrix}$  can be written as a linear combination of the vectors  $\begin{bmatrix} 3 \\ -1 \\ -1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ .

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{cc|c} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{array} \right] \right) = \left[ \begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Since this system has a solution,  $\begin{bmatrix} 0 \\ -1 \\ 2 \\ 6 \end{bmatrix}$  can be written as a linear combination of the vectors  $\begin{bmatrix} 3 \\ -1 \\ -1 \\ 0 \end{bmatrix}$  and

$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ , namely

$$\begin{bmatrix} 0 \\ -1 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

□

**V2.** Determine if  $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$  can be written as a linear combination of the vectors  $\begin{bmatrix} 5 \\ 2 \\ -3 \\ 2 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ , and  $\begin{bmatrix} 8 \\ 3 \\ 5 \\ -1 \end{bmatrix}$ .

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{ccc|c} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

The system has no solution, so  $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$  is not a linear combination of the three other vectors.

□

**V2.** Determine if  $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$  can be written as a linear combination of the vectors  $\begin{bmatrix} -1 \\ -9 \\ 15 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 5 \\ -5 \end{bmatrix}$ .

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{cc|c} -1 & 1 & 0 \\ -9 & 5 & 0 \\ 15 & -5 & 2 \end{array} \right] \right) = \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

Since this system has no solution,  $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$  cannot be written as a linear combination of the vectors  $\begin{bmatrix} -1 \\ -9 \\ 15 \end{bmatrix}$  and

$\begin{bmatrix} 1 \\ 5 \\ -5 \end{bmatrix}$ .

□

**V2.** Determine if  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ , and  $\begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}$ .

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{ccc|c} 2 & 1 & -3 & 1 \\ 3 & -1 & -2 & 4 \\ -1 & 0 & 5 & 3 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Since this system has a solution,  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is a linear combination of the three vectors. □

**V2.** Determine if  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ , and  $\begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$ .

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{ccc|c} 3 & 1 & 5 & 1 \\ 0 & -1 & 1 & 4 \\ -1 & 4 & -6 & 3 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

So  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is not a linear combination of the three vectors. □

**V2.** Determine if  $\begin{bmatrix} 0 \\ -1 \\ 6 \\ -7 \end{bmatrix}$  belongs to the span of the set  $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \\ 3 \end{bmatrix} \right\}$ .

*Solution.* Since

$$\text{RREF} \left( \left[ \begin{array}{cc|c} 2 & 4 & 0 \\ 0 & -1 & -1 \\ -1 & 4 & 6 \\ 5 & 3 & -7 \end{array} \right] \right) = \left[ \begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

does not contain a contradiction,  $\begin{bmatrix} 0 \\ -1 \\ 6 \\ -7 \end{bmatrix}$  is a linear combination of the three vectors. □

**V2.** Determine if  $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$  belongs to the span of the set  $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \\ 3 \end{bmatrix} \right\}$ .

*Solution.* Since

$$\text{RREF} \left( \left[ \begin{array}{cc|c} 2 & 4 & 4 \\ 0 & -1 & -1 \\ -1 & 4 & 6 \\ 5 & 3 & -7 \end{array} \right] \right) = \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

contains the contradiction  $0 = 1$ ,  $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$  is not a linear combination of the three vectors. □

**V2.** Determine if  $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$  belongs to the span of the set  $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ .



*Solution.* Since

$$\text{RREF} \left( \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & -2 \\ -3 & -6 & 0 & 4 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

contains the contradiction  $0 = 1$ ,  $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$  is not a linear combination of the three vectors. □

### Standard V3

**V3.** Determine if the vectors  $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ , and  $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$  span  $\mathbb{R}^3$

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix has only two pivot columns, the vectors do not span  $\mathbb{R}^3$ . □

**V3.** Determine if the vectors  $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}$ , and  $\begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix}$  span  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are zero rows, they do not span. Alternatively, by inspection  $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ , so the set is linearly dependent, so it spans a subspace of dimension at most 3, therefore it does not span  $\mathbb{R}^4$ . □

**V3.** Determine if the vectors  $\begin{bmatrix} 8 \\ 21 \\ -7 \end{bmatrix}$ ,  $\begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ , and  $\begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}$  span  $\mathbb{R}^3$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 8 & -3 & -1 & 4 \\ 21 & -8 & -3 & 11 \\ -7 & 3 & 2 & -5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the rank is less than 3, they do not span  $\mathbb{R}^3$ . □

**V3.** Determine if the vectors  $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$  span  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, the vectors do not span  $\mathbb{R}^4$ . □

**V3.** Determine if the vectors  $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 1 \\ 0 \\ -3 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}$ , and  $\begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$  span  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since every row contains a pivot, the vectors span  $\mathbb{R}^4$ . □

**V3.** Does  $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$ ?

*Solution.* Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 4 & 2 \\ 4 & -9 & -3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has a zero row, the vectors fail to span  $\mathbb{R}^3$ . □

**V3.** Does  $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$ ?

*Solution.* Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 2 & 2 \\ 4 & -9 & 3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

lacks a zero row, the vectors span  $\mathbb{R}^3$ . □

**V3.** Does  $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 5 \\ 1 \\ -3 \end{bmatrix} \right\} = \mathbb{R}^5$ ?

*Solution.* Since there are only three vectors, they cannot span  $\mathbb{R}^5$ . □

## Standard V4

**V4.** Determine if the set of all lattice points, i.e.  $\{(x, y) \mid x \text{ and } y \text{ are integers}\}$  is a subspace of  $\mathbb{R}^2$ .

*Solution.* This set is closed under addition, but not under scalar multiplication so it is not a subspace.  $\square$

**V4.** Determine if  $\left\{ \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$  a subspace of  $\mathbb{R}^4$ .

*Solution.* It is closed under addition and scalar multiplication, so it is a subspace. Alternatively, it is the image of the linear transformation from  $\mathbb{R}^3 \rightarrow \mathbb{R}^4$  given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix}.$$

$\square$

**V4.** Let  $W$  be the set of all polynomials of even degree. Determine if  $W$  is a subspace of the vector space of all polynomials.

*Solution.*  $W$  is closed under scalar multiplication, but not under addition. For example,  $x - x^2$  and  $x^2$  are both in  $W$ , but  $(x - x^2) + (x^2) = x \notin W$ .  $\square$

**V4.** Let  $W$  be the set of all polynomials of the form  $ax^3 + bx$ . Determine if  $W$  is a subspace of  $\mathcal{P}^3$ .

*Solution.* Yes because  $s(a_1x^3 + b_1x) + t(a_2x^3 + b_2x) = (sa_1 + ta_2)x^3 + (sb_1 + tb_2)x$  also belongs to  $W$ . Alternately, yes because  $W$  is isomorphic to  $\mathbb{R}^2$ .  $\square$

**V4.** Let  $W$  be the set of all  $\mathbb{R}^3$  vectors  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  satisfying  $x + y + z = 1$  (this forms a plane). Determine if  $W$

is a subspace of  $\mathbb{R}^3$ .

*Solution.* No, because  $\mathbf{0}$  does not belong to  $W$ .  $\square$

**V4.** Let  $W$  be the set of all  $\mathbb{R}^3$  vectors  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  satisfying  $x + y + z = 0$  (this forms a plane). Determine if  $W$

is a subspace of  $\mathbb{R}^3$ .

*Solution.* Yes, because  $z = -x - y$  and  $a \begin{bmatrix} x_1 \\ y_1 \\ -x_1 - y_1 \end{bmatrix} + b \begin{bmatrix} x_2 \\ y_2 \\ -x_2 - y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \\ -(ax_1 + bx_2) - (ay_1 + by_2) \end{bmatrix}$ .

Alternately, yes because  $W$  is isomorphic to  $\mathbb{R}^2$ .  $\square$

**V4.** Let  $W$  be the set of all complex numbers that are purely real (i.e of the form  $a + 0i$ ) or purely imaginary (i.e. of the form  $0 + bi$ ). Determine if  $W$  is a subspace of  $\mathbb{C}$ .

*Solution.* No, because 1 is purely real and  $i$  is purely imaginary, but the linear combination  $1 + i$  is neither.  $\square$

**V4.** Let  $W$  be the set of all complex numbers  $a + bi$  satisfying  $a = 2b$ . Determine if  $W$  is a subspace of  $\mathbb{C}$ .

*Solution.* Yes, because  $c(2b_1 + b_1i) + d(2b_2 + b_2i) = 2(cb_1 + db_2) + (cb_1 + db_2)i$  belongs to  $W$ . Alternately, yes because  $W$  is isomorphic to  $\mathbb{R}$ .  $\square$

## Module S

### Standard S1

**S1.** Determine if the vectors  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$  are linearly dependent or linearly independent

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since each column is a pivot column, the vectors are linearly independent.  $\square$

**S1.** Determine if the set of vectors  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}$  is linearly dependent or linearly independent

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, the set is linearly dependent.  $\square$

**S1.** Determine if the set of polynomials  $\{x^2 + x, x^2 + 2x - 1, x^2 + 3x - 2\}$  is linearly dependent or linearly independent

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, the set is linearly dependent.  $\square$

**S1.** Determine if the set of vectors  $\left\{ \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$  is linearly dependent or linearly independent

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} -3 & 1 & 0 \\ 8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every column is a pivot column, therefore the set is linearly independent.  $\square$

**S1.** Determine if the set of vectors  $\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$  is linearly dependent or linearly independent

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

This has a non pivot column, therefore the set is linearly dependent.  $\square$

**S1.** Determine if the set of polynomials  $\{-3x^3 - 8x^2, x^3 + 2x^2 + 2, -x^2 + 3\}$  is linearly dependent or linearly independent

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has a non pivot column, therefore the set is linearly dependent.  $\square$

**S1.** Determine if the set of vectors  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 6 \\ 5 \end{bmatrix} \right\}$  is linearly dependent or linearly independent.

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 3 & 1 & 3 \\ -1 & 2 & -8 \\ 0 & -2 & 6 \\ 4 & 1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced row echelon form has a nonpivot column, the vectors are linearly dependent.  $\square$

**S1.** Determine if the set of matrices  $\left\{ \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -8 \\ 6 & 5 \end{bmatrix} \right\}$  is linearly dependent or linearly independent.

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 3 & 1 & 3 \\ -1 & 2 & -8 \\ 0 & -2 & 6 \\ 4 & 1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced row echelon form has a nonpivot column, the vectors are linearly dependent.  $\square$

## Standard S2

**S2.** Determine if the set  $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^3$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

**S2.** Determine if the set  $\{x^2 + x - 1, 3x^2 - x + 1, 2x^2 - 2\}$  is a basis of  $\mathcal{P}^2$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

**S2.** Determine if the set  $\{x^3 - x, x^2 + x + 1, x^3 - x^2 + 2, 2x^2 - 1\}$  is a basis of  $\mathcal{P}^3$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

**S2.** Determine if the set  $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis. □

**S2.** Determine if the set  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^3$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 4 \\ 2 & 2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

**S2.** Determine if the set  $\{2x^2 - x + 3, 2x^2 + 2, -x^2 + 4x + 1\}$  is a basis of  $\mathcal{P}^2$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 2 & 2 & -1 \\ -1 & 0 & 4 \\ 3 & 2 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

**S2.** Determine if the set  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 5 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 3 & 2 & 1 & -1 \\ -1 & 0 & -1 & 3 \\ 2 & 2 & 0 & 0 \\ 3 & 4 & -1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

**S2.** Determine if the set  $\left\{ \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ -1 & 8 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^{2 \times 2}$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 3 & 2 & 1 & -1 \\ -1 & 0 & 4 & 3 \\ 2 & 2 & -1 & 0 \\ 3 & 4 & 8 & 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

**S2.** Determine if the set  $\left\{ \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 9 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^{2 \times 2}$  or not.

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis. □

**S2.** Determine if the set  $\{x^3 - 3x^2 + 2x + 2, -x^3 + 4x^2 - x + 1, -x^3 + 2x + 1, 3x^2 + 3x + 9\}$  is a basis of  $\mathcal{P}^3$  or not.

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis. □



### Standard S3

**S3.** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\} \right)$ . Find a basis for  $W$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis is  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}$ . □

**S3.** Let  $W$  be the subspace of  $\mathcal{P}^3$  given by  $W = \text{span} (\{x^3 + x^2 + 2x + 1, 3x^3 + 3x^2 + 6x + 3, 3x^3 - x^2 + 3x - 2, 7x^3 - x^2 + 8x - 3\})$ . Find a basis for  $W$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis is  $\{x^3 + x^2 + 2x + 1, 3x^3 - x^2 + 3x - 2\}$ . □

**S3.** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\} \right)$ . Find a basis for  $W$ .

*Solution.* Let  $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$ , and compute  $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$ . Since the first two columns are

pivot columns,  $\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$  is a basis for  $W$ . □

**S3.** Let  $W$  be the subspace of  $\mathcal{P}^2$  given by  $W = \text{span} (\{-3x^2 - 8x, x^2 + 2x + 2, -x + 3\})$ . Find a basis for  $W$ .

*Solution.* Let  $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$ , and compute  $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$ . Since the first two columns are

pivot columns,  $\{-3x^2 - 8x, x^2 + 2x + 2\}$  is a basis for  $W$ . □

**S3.** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$ . Find a basis of  $W$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then  $\left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$  is a basis of  $W$ . □

**S3.** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix} \right\} \right)$ . Find a basis of  $W$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then  $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix} \right\}$  is a basis for  $W$ . □

**S3.** Let  $W = \text{span} \left\{ \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\}$ . Find a basis for this vector space.

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus  $\left\{ \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$  is a basis of  $W$ . □

**S3.** Let  $W = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -8 \\ -1 \end{bmatrix} \right\}$ . Find a basis for this vector space.

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & -8 \\ 1 & 1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus  $\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$  is a basis of  $W$ . □

## Standard S4

**S4.** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\} \right)$ . Find the dimension of  $W$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has two pivot columns, so  $W$  has dimension 2. □

**S4.** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\} \right)$ . Compute the dimension of  $W$ .

*Solution.* Let  $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$ , and compute  $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$ . Since there are two pivot columns,  $\dim W = 2$ . □

**S4.** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$ . Compute the dimension of  $W$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so  $\dim(W) = 3$ . □

**S4.** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix} \right\} \right)$ . Compute the dimension of  $W$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so  $\dim(W) = 3$ . □

**S4.** Let  $W$  be the subspace of  $\mathcal{P}_3$  given by

$W = \text{span} (\{x^3 - x^2 + 3x - 3, 2x^3 + x + 1, 3x^3 - x^2 + 4x - 2, x^3 + x^2 + x - 7\})$ . Compute the dimension of  $W$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so  $\dim(W) = 3$ . □

**S4.** Let  $W$  be the subspace of  $\mathbb{R}^{2 \times 2}$  given by  $W = \text{span} \left( \left\{ \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\} \right)$ .

Compute the dimension of  $W$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so  $\dim(W) = 3$ . □

**S4.** Let  $W = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -8 \\ -1 \end{bmatrix} \right\}$ . Find the dimension of  $W$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & -8 \\ 1 & 1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since it has two pivot columns, its dimension is 2. □

**S4.** Let  $W = \text{span} \{2x^2 - x + 3, 2x^2 + 2, -x^2 + 4x + 1\}$ . Find the dimension of  $W$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 2 & 2 & -1 \\ -1 & 0 & 4 \\ 3 & 2 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since it has three pivot columns, its dimension is 3. □

## Module A

### Standard A1

**A1.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \\ 0 \end{bmatrix}.$$

Write the matrix for  $T$  with respect to the standard bases of  $\mathbb{R}^3$  and  $\mathbb{R}^4$ .

*Solution.*

$$\begin{bmatrix} 3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

□

**A1.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 7x + 2y + 3z \\ 0 \end{bmatrix}.$$

Write the matrix for  $T$  with respect to the standard bases of  $\mathbb{R}^3$  and  $\mathbb{R}^4$ .

*Solution.*

$$\begin{bmatrix} 3 & 1 & 0 \\ -8 & 2 & -1 \\ 7 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

□

**A1.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \\ 7x \end{bmatrix}.$$

Write the matrix for  $T$  with respect to the standard bases of  $\mathbb{R}^3$  and  $\mathbb{R}^4$ .

*Solution.*

$$\begin{bmatrix} 3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \\ 7 & 0 & 0 \end{bmatrix}$$

□

**A1.** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - x_3 \end{bmatrix}.$$

Write the matrix for  $T$  with respect to the standard bases of  $\mathbb{R}^4$  and  $\mathbb{R}^2$ .

*Solution.*

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix}$$

□

**A1.** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - 5x_3 \end{bmatrix}.$$

Write the matrix for  $T$  with respect to the standard bases of  $\mathbb{R}^4$  and  $\mathbb{R}^2$ .

*Solution.*

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & -5 & 0 \end{bmatrix}$$

□

**A1.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the linear transformation given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = [x_2 + 3x_3].$$

Write the matrix for  $T$  with respect to the standard bases of  $\mathbb{R}^3$  and  $\mathbb{R}$ .

*Solution.*

$$[0 \quad 1 \quad 3]$$

□

**A1.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the linear transformation given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = [x_3 + 3x_1].$$

Write the matrix for  $T$  with respect to the standard bases of  $\mathbb{R}^3$  and  $\mathbb{R}$ .

*Solution.*

$$[3 \quad 0 \quad 1]$$

□

## Standard A2

**A2.** Determine if  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} e^x \\ e^y \end{bmatrix}$  is a linear transformation.

*Solution.* It is not linear. For example,

$$\begin{bmatrix} e^2 \\ 1 \end{bmatrix} = T \left( \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) \neq 2T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2e \\ 1 \end{bmatrix}$$

□

**A2.** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by  $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ \sqrt{x} + \sqrt{y} \end{bmatrix}$ . Determine if  $T$  is a linear transformation.

*Solution.*

$$T \left( \begin{bmatrix} 0 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

So  $T$  is not a linear transformation.

□

**A2.** Determine if  $D : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$  given by  $D \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$  is a linear transformation or not.

*Solution.*  $D(I) = 1$  but  $D(2I) = 4 \neq 2D(I)$ , so  $D$  is not linear.

□

**A2.** Determine if  $D : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$  given by  $D \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a - 3c$  is a linear transformation or not.

**A2.** Determine if the map  $T : \mathcal{P}^6 \rightarrow \mathcal{P}^6$  given by  $T(f) = f(x) - f(0)$  is a linear transformation or not.

**A2.** Determine if the map  $T : \mathcal{P}^6 \rightarrow \mathcal{P}^7$  given by  $T(f) = xf(x) - f(1)$  is a linear transformation or not.

**A2.** Determine if the map  $T : \mathcal{P} \rightarrow \mathcal{P}$  given by  $T(f) = f' - f''$  is a linear transformation or not.

**A2.** Determine if the map  $T : \mathcal{P}^3 \rightarrow \mathcal{P}^4$  given by  $T(f(x)) = xf(x) - f(x)$  is a linear transformation or not.

### Standard A3

**A3.** Determine if the following linear maps are injective (one-to-one) and/or surjective (onto).

(a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by  $S \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 3x + 2y \\ x - y \\ x + 4y \end{bmatrix}$

(b)  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y + z \\ 2y + 3z \\ x - y - 2z \end{bmatrix}$

*Solution.*

(a)

$$\text{RREF} \left( \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column,  $T$  is not injective. Since there is a zero row,  $T$  is not surjective.

(b)

$$\text{RREF} \left( \begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since all columns are pivot columns,  $S$  is injective. Since there is a zero row,  $S$  is not surjective.

□

**A3.** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a)  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y + z \\ 2y + 3z \\ x - y - 2z \end{bmatrix}$

(b)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by  $S \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 3x + 2y \\ x - y \\ x + 4y \end{bmatrix}$

*Solution.*

(a)

$$\text{RREF} \left( \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column,  $T$  is not injective. Since there is a zero row,  $T$  is not surjective.

(b)

$$\text{RREF} \left( \begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since all columns are pivot columns,  $S$  is injective. Since there is a zero row,  $S$  is not surjective.

□

**A3.** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).



(a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ .

(b)  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  given by the standard matrix  $\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 3 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix}$

*Solution.*

(a)  $\text{RREF} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Since each column is a pivot column,  $S$  is injective. Since there is no zero row,  $S$  is surjective.

(b) Since  $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$ ,  $T$  is not injective.

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 3 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -5 & -\frac{5}{2} \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are only two pivot columns,  $T$  is not surjective.

□

**A3.** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ .

(b)  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  given by the standard matrix  $\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 4 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix}$

*Solution.*

(a)  $\text{RREF} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Since each column is a pivot column,  $S$  is injective. Since there is no zero row,  $S$  is surjective.

(b) Since  $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$ ,  $T$  is not injective.

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 4 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since there are no zero rows,  $T$  is surjective.

□

**A3.** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  given by the standard matrix  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$ .

(b)  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  given by the standard matrix  $\begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 11 & -1 & 5 \end{bmatrix}$

*Solution.*

$$(a) \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ Since each column is a pivot column, } S \text{ is injective. Since there is no zero row, } S \text{ is not surjective.}$$

(b) Since  $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$ ,  $T$  is not injective.

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 11 & -1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\frac{4}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row,  $T$  is not surjective.

□

**A3.** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

$$(a) S : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \text{ given by the standard matrix } \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}.$$

$$(b) T : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \text{ given by the standard matrix } \begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 7 & -1 & 5 \end{bmatrix}$$

*Solution.*

$$(a) \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ Since each column is a pivot column, } S \text{ is injective. Since there is no zero row, } S \text{ is not surjective.}$$

(b) Since  $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$ ,  $T$  is not injective.

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 7 & -1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Since there is not a zero row,  $T$  is surjective.

□

**A3.** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

$$(a) S : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \text{ where } S(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } S(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

$$(b) T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \text{ where } T(\vec{e}_1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

*Solution.*

- (a)  $\text{RREF} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ . The map is injective since every column has a pivot, but is not surjective because there is a row without a pivot.
- (b)  $\text{RREF} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}$ . The map is not injective since there is a column without a pivot, but it is surjective because every row has a pivot.

□

**A3.** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

- (a)  $S : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  where  $S(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ ,  $S(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ,  $S(\vec{e}_3) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ , and  $S(\vec{e}_4) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ ,
- (b)  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  where  $T(\vec{e}_1) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ ,  $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ , and  $T(\vec{e}_3) = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ .

*Solution.*

- (a)  $\text{RREF} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ . The map is not injective since it has a column without pivot, but it is surjective because every row has a pivot.
- (b)  $\text{RREF} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ . The map is not injective since there is a column without a pivot, and it is not surjective because there is a row without a pivot.

□

### Standard A4

**A4.** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \right) = \begin{bmatrix} x + 3y + 3z + 7w \\ x + 3y - z - w \\ 2x + 6y + 3z + 8w \\ x + 3y - 2z - 3w \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis for the kernel is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

and a basis for the image is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}$$

□

**A4.** Let  $T : \mathcal{P}^3 \rightarrow \mathcal{P}^3$  be the linear transformation given by

$$T(ax^3 + bx^2 + cx + d) = (a + 3b + 3c + 7d)x^3 + (a + 3b - c - d)x^2 + (2a + 6b + 3c + 8d)x + (a + 3b - 2c - 3d)$$

Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis for the kernel is

$$\{-3x^3 + x^2, -x^3 - 2x + 1\}$$

and a basis for the image is

$$\{x^3 + x^2 + 2x + 1, 3x^3 - x^2 + 3x - 2\}$$

□

**A4.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.* Let  $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$ , and compute  $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$ . Then a basis for the image is its columns,

$$\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

And the kernel is the solution set of  $AX = 0$ , so a basis would be

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}$$

□

**A4.** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be the linear map given by  $T \left( \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \right) = \begin{bmatrix} 8x - 3y - z + 4w \\ y + 3z - 4w \\ -7x + 3y + 2z - 5w \end{bmatrix}$ . Compute a basis

for the kernel and a basis for the image of  $T$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 8 & -3 & -1 & 4 \\ 0 & 1 & 3 & -4 \\ -7 & 3 & 2 & -5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus  $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$  is a basis for the kernel. □

**A4.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear map given by  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 8x - 3y - z \\ y + 3z \\ -7x + 3y + 2z \end{bmatrix}$ . Compute a basis for the

kernel and a basis for the image of  $T$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 8 & -3 & -1 \\ 0 & 1 & 3 \\ -7 & 3 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus  $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$  is a basis for the kernel. □

**A4.** Let  $T : \mathbb{R}^{2 \times 3} \rightarrow \mathbb{R}^3$  be the linear map given by  $T \left( \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} \right) = \begin{bmatrix} a + x \\ b + y \\ c + z \end{bmatrix}$ . Compute a basis for the

kernel and a basis for the image of  $T$ .

*Solution.* Rewrite as  $T' \left( \begin{bmatrix} a \\ b \\ c \\ x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} a + x \\ b + y \\ c + z \end{bmatrix}$ .

$$\text{RREF} \left( \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Thus  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \right\}$  is a basis for the kernel.  $\square$

**A4.** Let  $T : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^3$  be the linear map given by  $T \left( \begin{bmatrix} x & y \\ z & w \end{bmatrix} \right) = \begin{bmatrix} 8x - 3y - z + 4w \\ y + 3z - 4w \\ -7x + 3y + 2z - 5w \end{bmatrix}$ . Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 8 & -3 & -1 & 4 \\ 0 & 1 & 3 & -4 \\ -7 & 3 & 2 & -5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus  $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} -1 & -3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \right\}$  is a basis for the kernel.  $\square$

**A4.** Let  $T : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^3$  be the linear map given by  $T \left( \begin{bmatrix} a & b \\ x & y \end{bmatrix} \right) = \begin{bmatrix} a + x \\ 0 \\ b + y \end{bmatrix}$ . Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.* Rewrite as  $T' \left( \begin{bmatrix} a \\ b \\ x \\ y \end{bmatrix} \right) = \begin{bmatrix} a + x \\ 0 \\ b + y \end{bmatrix}$ .

$$\text{RREF} \left( \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Thus  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \right\}$  is a basis for the kernel.  $\square$

## Module M

### Standard M1

**M1.** Let

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 0 & 7 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $CA$  is the only one that can be computed, and

$$CA = \begin{bmatrix} 3 & 9 & 11 & 1 \\ 0 & 0 & 7 & 2 \\ -2 & -6 & -5 & 0 \end{bmatrix}$$

□

**M1.** Let

$$A = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 3 & 7 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $CA$  is the only one that can be computed, and

$$CA = \begin{bmatrix} 2 & 6 & 11 & 1 \\ 1 & 3 & 7 & 2 \\ -1 & -3 & -5 & 0 \end{bmatrix}$$

□

**M1.** Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $AC$  is the only one that can be computed, and

$$AC = \begin{bmatrix} 3 & -5 & 14 \\ 1 & -1 & 2 \end{bmatrix}$$

□

**M1.** Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $AC$  is the only one that can be computed, and

$$AC = \begin{bmatrix} 9 & -2 & 14 \\ 1 & 0 & 2 \end{bmatrix}$$

□

**M1.** Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $AB$  is the only ones that can be computed, and

$$AB = \begin{bmatrix} -3 & -5 & 6 & 14 \\ 0 & 0 & 7 & 35 \end{bmatrix}$$

□

**M1.** Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $BC$  is the only one that can be computed, and

$$BC = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

□

**M1.** Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $CB$  is the only one that can be computed, and

$$CB = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

□

**M1.** Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & -1 & 3 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $CB$  is the only one that can be computed, and

$$CB = \begin{bmatrix} 3 & 3 & -5 & 7 \\ 4 & -4 & 12 & -12 \\ 7 & 2 & 0 & 3 \end{bmatrix}$$

□



## Standard M2

**M2.** Determine if the matrix  $\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}$  is invertible.

*Solution.* The second column is a multiple of the first, so it is not invertible. □

**M2.** Determine if the matrix  $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$  is invertible.

*Solution.*

$$\text{RREF} \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since it is not equivalent to the identity matrix, it is not invertible. □

**M2.** Determine if the matrix  $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & 0 \\ -1 & -1 & 5 \end{bmatrix}$  is invertible.

*Solution.*

$$\text{RREF} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & 0 \\ -1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since it is not equivalent to the identity matrix, it is not invertible. □

**M2.** Determine if the matrix  $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$  is invertible.

*Solution.* This matrix is row equivalent to the identity matrix, so it is invertible. □

**M2.** Determine if the matrix  $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & -2 & 0 & 3 \end{bmatrix}$  is invertible.

*Solution.*

$$\text{RREF} \begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & -2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is not row equivalent to the identity matrix, so it is not invertible. □

**M2.** Determine if the matrix  $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  is invertible.

*Solution.* It is row equivalent to the identity matrix, so it is invertible. □

**M2.** Determine if the matrix  $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix}$  is invertible.

*Solution.*

$$\text{RREF} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since it is not row equivalent to the identity matrix, it is not invertible.

□

**M2.** Determine if the matrix  $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 0 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix}$  is invertible.

*Solution.*

$$\text{RREF} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 0 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since it is row equivalent to the identity matrix, it is invertible.

□

### Standard M3

**M3.** Find the inverse of the matrix  $\begin{bmatrix} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$ .

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{cccc|cccc} 8 & 5 & 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 5 & -3 & 1 & -2 & 0 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \right) = \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -5 & 12 \\ 0 & 1 & 0 & 0 & 1 & 1 & -4 & -9 \\ 0 & 0 & 1 & 0 & -4 & -7 & 20 & 47 \\ 0 & 0 & 0 & 1 & -1 & 0 & 3 & 7 \end{array} \right]$$

So the inverse is  $\begin{bmatrix} 1 & 2 & -5 & 12 \\ 1 & 1 & -4 & -9 \\ -4 & -7 & 20 & 47 \\ -1 & 0 & 3 & 7 \end{bmatrix}$ .

□

**M3.** Compute the inverse of the matrix  $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ .

*Solution.*

$$\text{RREF}(A|I) = \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -11 & 37 \\ 0 & 1 & 0 & 0 & 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

So the inverse is  $\begin{bmatrix} 1 & 2 & -11 & 37 \\ 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

□

**M3.** Find the inverse of the matrix  $\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}$ .

*Solution.*

$$\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 4 & -3 \\ 8 & -13 & 10 \\ 13 & -24 & 18 \end{bmatrix}$$

□

**M3.** Find the inverse of the matrix  $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ .

*Solution.*

$$\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & \frac{3}{2} & -\frac{3}{2} \\ 1 & -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

□

**M3.** Find the inverse of the matrix  $\begin{bmatrix} 4 & -1 & -8 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ .

*Solution.*  $\left[ \begin{array}{ccc|ccc} 4 & -1 & -8 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 5 \\ 0 & 1 & 0 & -5 & 24 & -28 \\ 0 & 0 & 1 & 1 & -5 & 6 \end{array} \right]$ . Thus the inverse is  $\left[ \begin{array}{ccc} 1 & -4 & 5 \\ -5 & 24 & -28 \\ 1 & -5 & 6 \end{array} \right]$ .  $\square$

**M3.** Find the inverse of the matrix  $\left[ \begin{array}{ccc} 1 & -4 & 5 \\ -5 & 24 & -28 \\ 1 & -5 & 6 \end{array} \right]$ .

*Solution.*  $\left[ \begin{array}{ccc|ccc} 1 & -4 & 5 & 1 & 0 & 0 \\ -5 & 24 & -28 & 0 & 1 & 0 \\ 1 & -5 & 6 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -8 \\ 0 & 1 & 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 & 4 \end{array} \right]$ . Thus the inverse is  $\left[ \begin{array}{ccc} 4 & -1 & -8 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{array} \right]$ .  $\square$

**M3.** Find the inverse of the matrix  $\left[ \begin{array}{ccc} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{array} \right]$ .

*Solution.*  $\left[ \begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 2 & -1 & -6 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -3 \\ 0 & 1 & 0 & -14 & 9 & 24 \\ 0 & 0 & 1 & 3 & -2 & -5 \end{array} \right]$ . Thus the inverse is  $\left[ \begin{array}{ccc} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{array} \right]$ .  $\square$

**M3.** Find the inverse of the matrix  $\left[ \begin{array}{ccc} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{array} \right]$ .

*Solution.*  $\left[ \begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 2 & -1 & -6 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -3 \\ 0 & 1 & 0 & -14 & 9 & 24 \\ 0 & 0 & 1 & 3 & -2 & -5 \end{array} \right]$ . Thus the inverse is  $\left[ \begin{array}{ccc} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{array} \right]$ .  $\square$

**M3.** Find the inverse of the matrix  $\left[ \begin{array}{ccc} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{array} \right]$ .

*Solution.*  $\left[ \begin{array}{ccc|ccc} 2 & -1 & -3 & 1 & 0 & 0 \\ -14 & 9 & 24 & 0 & 1 & 0 \\ 3 & -2 & -5 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3 \\ 0 & 1 & 0 & 2 & -1 & -6 \\ 0 & 0 & 1 & 1 & 1 & 4 \end{array} \right]$ . Thus the inverse is  $\left[ \begin{array}{ccc} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{array} \right]$ .  $\square$

## Module G

### Standard G1

**G1.** Compute the determinant of the matrix  $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$ .

*Solution.*

$$\det \begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix} + (-2) \det \begin{bmatrix} 3 & 0 & 4 \\ 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = -1(-4) + (-2)(20) = -36$$

□

**G1.** Compute the determinant of the matrix  $\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix}$ .

*Solution.*

$$\begin{aligned} \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} &= 2 \det \begin{bmatrix} 3 & 0 & -1 \\ 1 & 3 & 1 \\ -3 & -2 & -1 \end{bmatrix} - (-1) \det \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 1 & -3 & -2 \end{bmatrix} \\ &= 2 \left( 3 \det \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} + (-1) \det \begin{bmatrix} 1 & 3 \\ -3 & -2 \end{bmatrix} \right) + \left( 1 \det \begin{bmatrix} 1 & 3 \\ -3 & -2 \end{bmatrix} - 3 \det \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} \right) \\ &= 2(3(-1) + (-1)(7)) + ((1)(7) - 3(-3)) \\ &= 2(-10) + 16 \\ &= -4 \end{aligned}$$

□

**G1.** Compute the determinant of the matrix  $\begin{bmatrix} 2 & 3 & 0 & 1 \\ -1 & 3 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 1 & -1 & 3 & 5 \end{bmatrix}$ .

*Solution.*  $-60$ .

□

**G1.** Compute the determinant of the matrix  $\begin{bmatrix} 3 & -1 & 0 & 7 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ .

*Solution.*  $2$

□

**G1.** Compute the determinant of the matrix  $\begin{bmatrix} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$ .

*Solution.*  $-1$ .

□

**G1.** Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 0 & -4 & 0 \\ -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

*Solution.* 15.

□

**G1.** Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ -2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

*Solution.* -15.

□

**G1.** Compute the determinant of the matrix

$$\begin{bmatrix} 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 0 \end{bmatrix}.$$

*Solution.* 55.

□

**G1.** Compute the determinant of the matrix

$$\begin{bmatrix} 0 & -4 & 1 & 1 \\ -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 0 & -4 & 0 \end{bmatrix}.$$

*Solution.* -55.

□

## Standard G2

**G2.** Let  $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$ . List the eigenvalues of  $A$  along with their algebraic multiplicities.

*Solution.*

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -3 - \lambda & 1 & 0 \\ -8 & 2 - \lambda & -1 \\ 0 & 2 & 3 - \lambda \end{bmatrix} \\ &= (-3 - \lambda) \det \begin{bmatrix} 2 - \lambda & -1 \\ 2 & 3 - \lambda \end{bmatrix} - (1) \det \begin{bmatrix} -8 & -1 \\ 0 & 3 - \lambda \end{bmatrix} \\ &= (-3 - \lambda) ((2 - \lambda)(3 - \lambda) + 2) - (-8(3 - \lambda)) \\ &= (-3 - \lambda)(8 - 5\lambda + \lambda^2) + 24 - 8\lambda \\ &= -\lambda^3 + 2\lambda^2 + 7\lambda - 24 + 24 - 8\lambda \\ &= -\lambda^3 + 2\lambda^2 - \lambda \\ &= -\lambda(\lambda^2 - 2\lambda + 1) \\ &= -\lambda(\lambda - 1)^2 \end{aligned}$$

So  $A$  has eigenvalues 0 (with multiplicity 1) and 1 (with algebraic multiplicity 2). □

**G2.** Compute the eigenvalues, along with their algebraic multiplicities, of the matrix  $\begin{bmatrix} 8 & -3 & -1 \\ 21 & -8 & -3 \\ -7 & 3 & 2 \end{bmatrix}$ .

*Solution.*

$$\begin{aligned} \det(A - \lambda I) &= (8 - \lambda) \det \begin{bmatrix} -8 - \lambda & -3 \\ 3 & 2 - \lambda \end{bmatrix} - (-3) \det \begin{bmatrix} 21 & -3 \\ -7 & 2 - \lambda \end{bmatrix} + (-1) \det \begin{bmatrix} 21 & -8 - \lambda \\ -7 & 3 \end{bmatrix} \\ &= (8 - \lambda)(\lambda^2 + 6\lambda - 7) + 3(-21\lambda + 21) - (-7\lambda + 7) \\ &= (\lambda - 1)((8 - \lambda)(\lambda + 7) - 63 + 7) \\ &= (\lambda - 1)(\lambda - \lambda^2) \\ &= -\lambda(\lambda - 1)^2 \end{aligned}$$

So the eigenvalues are 0 (with algebraic multiplicity 1) and 1 (with algebraic multiplicity 2). □

**G2.** Compute the eigenvalues, along with their algebraic multiplicities, of the matrix  $\begin{bmatrix} 9 & -3 & 2 \\ 19 & -6 & 5 \\ -11 & 4 & -2 \end{bmatrix}$ .

*Solution.* 1 (with algebraic multiplicity 2), and -1 (with algebraic multiplicity 1). □

**G2.** Compute the eigenvalues, along with their algebraic multiplicities, of the matrix  $\begin{bmatrix} 9 & -3 & 2 \\ 19 & -6 & 5 \\ -5 & 2 & 0 \end{bmatrix}$ .

*Solution.* 1 (with algebraic multiplicity 3) □

**G2.** Compute the eigenvalues, along with their algebraic multiplicities, of the matrix  $\begin{bmatrix} 9 & -3 & 2 \\ 23 & -8 & 5 \\ -1 & 0 & 0 \end{bmatrix}$ .

*Solution.* 1 (with algebraic multiplicity 2), and -1 (with algebraic multiplicity 1). □

**G2.** Compute the eigenvalues, along with their algebraic multiplicities, of the matrix  $\begin{bmatrix} 9 & -3 & 2 \\ 23 & -8 & 5 \\ 2 & -1 & 1 \end{bmatrix}$ .

*Solution.* The eigenvalues are  $-1$ ,  $1$ , and  $2$  (each with algebraic multiplicity 1). □

**G2.** Compute the eigenvalues, along with their algebraic multiplicities, of the matrix  $\begin{bmatrix} 8 & -3 & 2 \\ 15 & -5 & 5 \\ -3 & 2 & 1 \end{bmatrix}$ .

*Solution.* The eigenvalues are  $0$  (with algebraic multiplicity 1), and  $2$  (with algebraic multiplicity 2). □

**G2.** Compute the eigenvalues, along with their algebraic multiplicities, of the matrix  $\begin{bmatrix} 8 & -3 & 2 \\ 23 & -9 & 5 \\ -7 & 2 & -3 \end{bmatrix}$ .

*Solution.* The eigenvalues are  $0$  (with algebraic multiplicity 1) and  $-2$  (with algebraic multiplicity 2). □

**G2.** Compute the eigenvalues, along with their algebraic multiplicities, of the matrix  $\begin{bmatrix} 2 & -3 & 2 \\ 8 & -9 & 5 \\ 8 & -7 & 3 \end{bmatrix}$ .

*Solution.* The eigenvalues are  $0$  (with algebraic multiplicity 1) and  $-2$  (with algebraic multiplicity 2). □



### Standard G3

**G3.** Compute the eigenspace of the eigenvalue  $-1$  in the matrix  $\begin{bmatrix} 4 & -2 & -1 \\ 15 & -7 & -3 \\ -5 & 2 & 0 \end{bmatrix}$ .

*Solution.*

$$\text{RREF}(A + I) = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenspace is spanned by  $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue  $1$  in the matrix  $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -\frac{1}{4} \\ -1 \\ 1 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue  $1$  in the matrix  $A = \begin{bmatrix} 8 & -3 & -1 \\ 21 & -8 & -3 \\ -7 & 3 & 2 \end{bmatrix}$

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} \frac{3}{7} \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} \frac{1}{7} \\ 0 \\ 1 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue  $-1$  in the matrix  $A = \begin{bmatrix} 9 & -3 & 2 \\ 19 & -6 & 5 \\ -11 & 4 & -2 \end{bmatrix}$

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -\frac{5}{7} \\ -\frac{12}{7} \\ 1 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue  $1$  in the matrix  $A = \begin{bmatrix} 9 & -3 & 2 \\ 19 & -6 & 5 \\ -11 & 4 & -2 \end{bmatrix}$

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue  $2$  in the matrix  $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 11 & -6 & 1 & -1 \\ -9 & 5 & 1 & 3 \end{bmatrix}$ .

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue  $2$  in the matrix  $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$ .

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue 2 in the matrix  $A = \begin{bmatrix} 0 & -2 & -1 & 0 \\ -4 & -2 & -2 & 0 \\ 14 & 12 & 10 & 2 \\ -13 & -10 & -8 & -1 \end{bmatrix}$ .

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue 3 in the matrix  $A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ -4 & -1 & -2 & 0 \\ 14 & 12 & 11 & 2 \\ -14 & -10 & -9 & -1 \end{bmatrix}$ .

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue 2 in the matrix  $A = \begin{bmatrix} 8 & -3 & 2 \\ 15 & -5 & 5 \\ -3 & 2 & 1 \end{bmatrix}$

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue  $-2$  in the matrix  $A = \begin{bmatrix} 8 & -3 & 2 \\ 23 & -9 & 5 \\ -7 & 2 & -3 \end{bmatrix}$

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue  $-2$  in the matrix  $A = \begin{bmatrix} 2 & -3 & 2 \\ 8 & -9 & 5 \\ 8 & -7 & 3 \end{bmatrix}$

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} \frac{1}{4} \\ 1 \\ 1 \end{bmatrix}$ . □

**G3.** Find the eigenspace associated to the eigenvalue 2 in the matrix  $\begin{bmatrix} -1 & 1 & 0 \\ -9 & 5 & 0 \\ 15 & -5 & 2 \end{bmatrix}$ .

*Solution.* The eigenspace is the solution space of the system  $(B - 2I)X = 0$ .

$$\text{RREF}(B - 2I) = \text{RREF} \left( \begin{bmatrix} -3 & 1 & 0 \\ -9 & 3 & 0 \\ 15 & -5 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the system simplifies to  $x - \frac{y}{3} = 0$ , or  $3x = y$ . Thus the eigenspace is

$$E_2 = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$$

□

## Standard G4

**G4.** Compute the geometric multiplicity of the eigenvalue  $-1$  in the matrix  $\begin{bmatrix} 4 & -2 & -1 \\ 15 & -7 & -3 \\ -5 & 2 & 0 \end{bmatrix}$ .

*Solution.*

$$\text{RREF}(A + I) = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the geometric multiplicity is 2. □

**G4.** Compute the geometric multiplicity of the eigenvalue 2 in the matrix  $\begin{bmatrix} -1 & 1 & 0 \\ -9 & 5 & 0 \\ 15 & -5 & 2 \end{bmatrix}$ .

*Solution.* The eigenspace is the solution space of the system  $(B - 2I)X = 0$ .

$$\text{RREF}(B - 2I) = \text{RREF} \left( \begin{bmatrix} -3 & 1 & 0 \\ -9 & 3 & 0 \\ 15 & -5 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the geometric multiplicity is 2. □

**G4.** Compute the geometric multiplicity of the eigenvalue 1 in the matrix  $A = \begin{bmatrix} 8 & -3 & -1 \\ 21 & -8 & -3 \\ -7 & 3 & 2 \end{bmatrix}$

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} \frac{3}{7} \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} \frac{1}{7} \\ 0 \\ 1 \end{bmatrix}$ , so the geometric multiplicity is 2. □

**G4.** Compute the geometric multiplicity of the eigenvalue  $-1$  in the matrix  $A = \begin{bmatrix} 9 & -3 & 2 \\ 19 & -6 & 5 \\ -11 & 4 & -2 \end{bmatrix}$

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -\frac{5}{7} \\ -\frac{12}{7} \\ 1 \end{bmatrix}$ , so the geometric multiplicity is 1. □

**G4.** Compute the geometric multiplicity of the eigenvalue 1 in the matrix  $A = \begin{bmatrix} 9 & -3 & 2 \\ 19 & -6 & 5 \\ -11 & 4 & -2 \end{bmatrix}$

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ , so the geometric multiplicity is 1. □

**G4.** Compute the geometric multiplicity of the eigenvalue 2 in the matrix  $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 11 & -6 & 1 & -1 \\ -9 & 5 & 1 & 3 \end{bmatrix}$ .

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ , so the geometric multiplicity is 2. □

**G4.** Compute the geometric multiplicity of the eigenvalue 2 in the matrix  $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$ .

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ , so the geometric multiplicity is 3.  $\square$

**G4.** Compute the geometric multiplicity of the eigenvalue 2 in the matrix  $A = \begin{bmatrix} 0 & -2 & -1 & 0 \\ -4 & -2 & -2 & 0 \\ 14 & 12 & 10 & 2 \\ -13 & -10 & -8 & -1 \end{bmatrix}$ .

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ , so the geometric multiplicity is 2.  $\square$

**G4.** Compute the geometric multiplicity of the eigenvalue 3 in the matrix  $A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ -4 & -1 & -2 & 0 \\ 14 & 12 & 11 & 2 \\ -14 & -10 & -9 & -1 \end{bmatrix}$ .

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ , so the geometric multiplicity is 2.  $\square$

**G4.** Compute the geometric multiplicity of the eigenvalue 2 in the matrix  $A = \begin{bmatrix} 8 & -3 & 2 \\ 15 & -5 & 5 \\ -3 & 2 & 1 \end{bmatrix}$

*Solution.* The eigenspace is spanned by  $\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$ , so the geometric multiplicity is 1.  $\square$