Module E: Solving Systems of Linear Equations

How can we solve systems of linear equations?

At the end of this module, students will be able to...

1. **Systems as matrices.** ... translate back and forth between a system of linear equations, a vector equation, and the corresponding augmented matrix.
2. **Row reduction.** ... explain why a matrix isn't in reduced row echelon form, and put a matrix in reduced row echelon form.
3. **Systems of linear equations.** ... compute the solution set for a system of linear equations or a vector equation.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.
- Describe sets using set-builder notation, and check if an element is a member of a set described by set-builder notation.

The following resources will help you prepare for this module.

- Systems of linear equations (Khan Academy): http://bit.ly/2l21etm
- Solving linear systems with substitution (Khan Academy): http://bit.ly/1SlMpix
- Set builder notation: https://youtu.be/xnfUZ-NTsCE

Module E Section 0

Definition E.1

A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector that can be plugged into the equation).

Remark E.2

In previous classes you likely used the variables x, y, z in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as x_i, and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

Definition E.3

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \middle| \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

Remark E.4

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Concise standard form:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & 3 \\ 3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & -2 \end{array}$$

Definition E.5

A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**.

Fact E.6

All linear systems are one of the following:

- **Consistent with one solution**: its solution set contains a single vector, e.g. $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$
- **Consistent with infinitely-many solutions**: its solution set contains infinitely many vectors, e.g. $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix} \middle| a \in \mathbb{R}\right\}$
- **Inconsistent**: its solution set is the empty set $\{\} = \emptyset$

Activity E.7 *(∼10 min)* All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is $\emptyset$.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity E.8 *(∼10 min)* Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

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Part 1: Find three different solutions for this system.

Activity E.8 *(∼10 min)* Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions for this system.
Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{ \begin{bmatrix} ? \\ a \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}$ for the linear system.

Activity E.9 *(∼10 min)* Consider the following linear system.

$$\begin{alignedat}{5} x_1 &+ 2x_2 & & & - & x_4 &= & \; 3 \\ & & & x_3 & + & 4x_4 &= & -2 \end{alignedat}$$

Describe the solution set

$$\left\{ \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation E.10

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{aligned}
-2x_1 - 4x_2 + x_3 - 4x_4 &= -8\\
x_1 + 2x_2 + 2x_3 + 12x_4 &= -1\\
x_1 + 2x_2 + x_3 + 8x_4 &= 1
\end{aligned}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.

Module E Section 1

Remark E.11

The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{alignedat}{2} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{alignedat}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition E.12

A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{alignedat}{5}
a_{11}x_1 &+ a_{12}x_2 &+ \ldots + &a_{1n}x_n &= b_1 \\
a_{21}x_1 &+ a_{22}x_2 &+ \ldots + &a_{2n}x_n &= b_2 \\
\vdots & \quad \vdots & & \vdots & \vdots \\
a_{m1}x_1 &+ a_{m2}x_2 &+ \ldots + &a_{mn}x_n &= b_m
\end{alignedat}$$

$$\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right]$$

Example E.13

The corresponding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

Definition E.14

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent, which we denote with $\sim$:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity E.15 *(∼10 min)* Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.
b) Swap two columns.
c) Add a constant to every term in a row.
d) Multiply a row by a nonzero constant.
e) Add a constant multiple of one row to another row.
f) Replace a column with zeros.
g) Replace a row with zeros.

Definition E.16

The following **row operations** produce equivalent augmented matrices:

1. Swap two rows, for example, $R_1 \leftrightarrow R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}\right]$$

2. Multiply a row by a nonzero constant, for example, $2R_1 \to R_1$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 2(1) & 2(2) & 2(3) \\ 4 & 5 & 6 \end{array}\right]$$

3. Add a constant multiple of one row to another row, for example, $R_2 - 4R_1 \to R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4-4(1) & 5-4(2) & 6-4(3) \end{array}\right]$$

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity E.17 *(~10 min)* Consider the following (equivalent) linear systems.

A

$$\begin{aligned} x+2y+z&=3\\ -x-y+z&=1\\ 2x+5y+3z&=7 \end{aligned}$$

B

$$\begin{aligned} 2x+5y+3z&=7\\ -x-y+z&=1\\ x+2y+z&=3 \end{aligned}$$

C

$$\begin{aligned} x-z&=1\\ y+z&=1\\ y+2z&=4 \end{aligned}$$

D

$$\begin{aligned} x+2y+z&=3\\ y+z&=1\\ 2x+5y+3z&=7 \end{aligned}$$

E

$$\begin{aligned} x-z&=1\\ y+z&=1\\ z&=3 \end{aligned}$$

F

$$\begin{aligned} x+2y+z&=3\\ y+z&=1\\ y+2z&=4 \end{aligned}$$

Rank the six linear systems from most complicated to simplest.

Activity E.18 *(∼5 min)* We can rewrite the previous in terms of equivalences of augmented matrices

$$\left[\begin{array}{ccc|c} 2 & 5 & 13 & 7 \\ -1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -1 & -1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

Definition E.19

A matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

Every matrix has a unique reduced row echelon form. If A is a matrix, we write $\mathrm{RREF}(A)$ for the reduced row echelon form of that matrix.

Activity E.20 *(∼15 min)* Recall that a matrix is in **reduced row echelon form (RREF)** if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

A

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

B

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

C

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

D

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

E

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

F

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

For each matrix, circle the leading terms, and label it as RREF or not RREF. For the ones not in RREF, find their RREF.

Module E Section 2

Remark E.21
In practice, if we simply need to convert a matrix into reduced row echelon form, we use technology to do so.

However, it is also important to understand the **Gauss-Jordan elimination** algorithm that a computer or calculator uses to convert a matrix (augmented or not) into reduced row echelon form. Understanding this algorithm will help us better understand how to interpret the results in many applications we use it for in Module V.

Activity E.22 *(∼8 min)* Consider the matrix

$$\begin{bmatrix} 2 & 6 & -1 & 6 \\ 1 & 3 & -1 & 2 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the first move in converting to RREF?

- (a) Add row 3 to row 2 ($R_2 + R_3 \to R_2$)
- (b) Add row 2 to row 3 ($R_3 + R_2 \to R_3$)
- (c) Swap row 1 to row 2 ($R_1 \leftrightarrow R_2$)
- (d) Add -2 row 2 to row 1 ($R_1 - 2R_2 \to R_1$)

Activity E.23 *(∼7 min)* Consider the matrix

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 6 & -1 & 6 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

- a) Add row 1 to row 3 ($R_3 + R_1 \to R_3$)
- b) Add -2 row 1 to row 2 ($R_2 - 2R_1 \to R_2$)
- c) Add 2 row 2 to row 3 ($R_3 + 2R_2 \to R_3$)
- d) Add 2 row 3 to row 2 ($R_2 + 2R_3 \to R_2$)

Activity E.24 *(∼5 min)* Consider the matrix

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

- (a) Add row 1 to row 2 ($R_2 + R_1 \to R_2$)
- (b) Add -1 row 3 to row 2 ($R_2 - R_3 \to R_2$)
- (c) Add -1 row 2 to row 3 ($R_3 - R_2 \to R_3$)
- (d) Add row 2 to row 1 ($R_1 + R_2 \to R_1$)

Activity E.25 *(∼10 min)* Consider the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -1 & 1 \end{bmatrix}.$$

Activity E.25 *(∼10 min)* Consider the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -1 & 1 \end{bmatrix}.$$

Part 1: Perform three row operations to produce a matrix closer to RREF.

Activity E.25 *(∼10 min)* Consider the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -1 & 1 \end{bmatrix}.$$

Part 1: Perform three row operations to produce a matrix closer to RREF.

Part 2: Finish putting it in RREF.

Activity E.26 *(∼10 min)* Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 2 & 3 \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix}.$$

Compute RREF(A).

Activity E.27 *(∼10 min)* Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{bmatrix}.$$

Compute RREF(A).

Remark E.28

A video example of how to perform the Gauss-Jordan Elimination algorithm by hand is available at `https://youtu.be/Cq0Nxk2dhhU`.

Practicing several exercises on your own using this method is strongly recommended.

Activity E.29 *(∼10 min)* Free browser-based technologies for mathematical computation are available online.

- Go to `https://octave-online.net`.
- Type `A=sym([1 3 4 ; 2 5 7])` and press `Enter` to store the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 7 \end{bmatrix}$ in the variable A.
  - The symbolic function `sym` is used to calculate precise answers rather than floating-point approximations.
  - The vertical bar in an augmented matrix does not affect row operations, so the RREF of $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 5 & 7 \end{array}\right]$ may be computed in the same way.
- Type `rref(A)` and press `Enter` to compute the reduced row echelon form of A.

Remark E.30

We will frequently need to know the reduced row echelon form of matrices during class, so feel free to use Octave-Online.net to compute RREF efficiently.

You may alternatively use the calculator you will use during assessments. Be sure to use fractions mode to compute exact solutions rather than floating-point approximations.