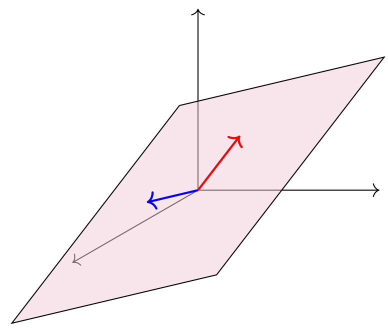
Section V.4

Definition V.4.1 A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space $\mathbb{R}^3$.



Fact V.4.2 Any sub**set** S of a vector space V satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a sub**space**, we need to check that addition and multiplication still make sense using only vectors from S. So we need to check two things:

- The set is **closed under addition**: for any $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in S$, the sum $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ is also in S.
- The set is **closed under scalar multiplication**: for any $\vec{\mathbf{x}} \in S$ and scalar $c \in \mathbb{R}$, the product $c\vec{\mathbf{x}}$ is also in S.

Activity V.4.3 *(~15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, x + 2y + z = 0 \right\}$.

Part 1: Let $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\vec{\mathbf{v}} + \vec{\mathbf{w}} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Part 2: Let $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\vec{\mathbf{v}}$ also belongs to S for any $c \in \mathbb{R}$.

Part 3: Is S is a subspace of $\mathbb{R}^3$?

Activity V.4.4 *(~10 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, x + 2y + z = 4 \right\}$. Choose a vector $\vec{\mathbf{v}} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = \ ?$, and show that $c\vec{\mathbf{v}}$ isn't in S. Is S a subspace of $\mathbb{R}^3$?

Remark V.4.5 Since 0 is a scalar and $0\vec{\mathbf{v}} = \vec{\mathbf{z}}$ for any vector $\vec{\mathbf{v}}$, a set that is closed under scalar multiplication must contain the zero vector $\vec{\mathbf{z}}$ for that vector space.

Put another way, an easy way to check that a subset isn't a subspace is to show it doesn't contain $\vec{\mathbf{0}}$.

Activity V.4.6 *(∼10 min)* Consider these two subsets of $\mathbb{R}^4$:

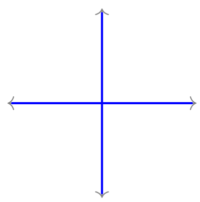
$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\} \qquad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\}$$

Part 1: Which set is not a subspace of $\mathbb{R}^4$?
Part 2: Is the set of polynomials

$$S = \left\{ ax^3 + bx^2 + (b-1)x + (a-1) \,\middle|\, a, b \text{ are real numbers} \right\}$$

a subspace of $\mathcal{P}^3$?

Activity V.4.7 *(∼10 min)* Consider the subset A of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.



This set contains $\vec{\mathbf{0}}$, and it's not hard to show that for every $\vec{\mathbf{v}}$ in A and scalar $c \in \mathbb{R}$, $c\vec{\mathbf{v}}$ is also in A. Is A a subspace of $\mathbb{R}^2$? Why?

Activity V.4.8 *(∼5 min)* Let W be a subspace of a vector space V. How are $\operatorname{span} W$ and W related?

(a) $\operatorname{span} W$ is bigger than W

(b) $\operatorname{span} W$ is the same as W

(c) $\operatorname{span} W$ is smaller than W

Fact V.4.9 If S is any subset of a vector space V, then since $\operatorname{span} S$ collects all possible linear combinations, $\operatorname{span} S$ is automatically a subspace of V.

In fact, $\operatorname{span} S$ is always the smallest subspace of V that contains all the vectors in S.