Sample Assessment Exercises

This document contains one exercise and solution for each standard. The goal is to give you an idea of what the exercises might look like, and what the expectations for a complete solution are.

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & 1 & 1 \\ -1 & -4 & 1 & -7 & 0 \\ 0 & 1 & -1 & 0 & -2 \end{array}\right]$$

Solution:

$$\begin{aligned} 3x_1 + 2x_2 \phantom{{}+x_3} + x_4 &= 1 \\ -x_1 - 4x_2 + x_3 - 7x_4 &= 0 \\ x_2 - x_3 \phantom{{}-7x_4} &= -2 \end{aligned}$$

□

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 2 & -1 & -3 \\ 2 & 4 & -1 & -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 2 & -1 & -3 \\ 2 & 4 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 3 & 1 & 2 \\ 2 & 4 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{13}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{13}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

□

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x + 4y + z &= 5 \\ x + 2y \phantom{{}+z} &= 3 \end{aligned}$$

Solution:

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} 2 & 4 & 1 & 5 \\ 1 & 2 & 0 & 3 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This corresponds to the system

$$\begin{aligned} x + 2y \phantom{{}+z} &= 3 \\ z &= -1 \end{aligned}$$

Since the y-column is a non-pivot column, it is a free variable, so we let $y = a$; then we have

$$\begin{aligned} x + 2y \phantom{{}+z} &= 3 \\ y &= a \\ z &= -1 \end{aligned}$$

and thus

$$\begin{aligned} x &= 3 - 2a \\ y &= a \\ z &= -1 \end{aligned}$$

So the solution set is

$$\left\{ \begin{bmatrix} 3-2a \\ a \\ -1 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$$

$\square$

V1. Let V be the set of all polynomials, together with the operations $\oplus$ and $\odot$ defined by the following for all polynomials $f(x), g(x)$ and scalars $c \in \mathbb{R}$:

$$\begin{aligned} f(x) \oplus g(x) &= xf(x) + xg(x) \\ c \odot f(x) &= cf(x) \end{aligned}$$

(a) Show that distribution property

$$c \odot (f(x) \oplus g(x)) = c \odot f(x) \oplus c \odot g(x)$$

holds.

(b) Show why V is not a vector space.

Solution:

(a) Compute

$$\begin{aligned} c \odot (f(x) \oplus g(x)) &= c \odot (xf(x) + xg(x)) \\ &= c\,(xf(x) + xg(x)) \\ &= cxf(x) + cxg(x) \end{aligned}$$

and

$$\begin{aligned} c \odot f(x) \oplus c \odot g(x) &= (cf(x)) \oplus (cg(x)) \\ &= xcf(x) + xcg(x) \end{aligned}$$

Since these are the same, we have shown that $c \odot (f(x) \oplus g(x)) = c \odot f(x) \oplus c \odot g(x)$ holds.

(b) Suppose $f(x) = 1$, $g(x) = 2$, and $h(x) = 3$. Then

$$\begin{aligned} (f(x) \oplus g(x)) \oplus h(x) &= (x + 2x) \oplus 3 \\ &= 3x \oplus 3 \\ &= 3x^2 + 3x \end{aligned}$$

and

$$\begin{aligned} f(x) \oplus (g(x) \oplus h(x)) &= 1 \oplus (2x + 3x) \\ &= 1 \oplus 5x \\ &= x + 5x^2 \end{aligned}$$

Since $3x^2 + 3x \neq x + 5x^2$, we have shown that the vector property $(f(x) \oplus g(x)) \oplus h(x) = f(x) \oplus (g(x) \oplus h(x))$ fails.

$\square$

V2. Determine if $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Solution:
We compute

$$\mathrm{RREF}\left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 2 & 1 & -1 \\ 1 & -1 & -1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Since this corresponds to an inconsistent system of equations, $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

$\square$

V3. Determine if the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ span $\mathbb{R}^3$.

Solution:
We compute

$$\mathrm{RREF}\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since the last row lacks a pivot, the vectors do not span $\mathbb{R}^3$.

$\square$

V4. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y = 3z \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y = 3z + 2 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.

Solution: First consider $\vec{\mathbf{0}}$. Since $0 + 0 = 3(0)$, we see that $\vec{\mathbf{0}} \in W$. But since $0 + 0 \neq 3(0) + 2$, we see that $\vec{\mathbf{0}} \notin U$. Therefore U is not a subspace.

To show that W is a subspace, let $\vec{\mathbf{v}} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in W$ and $\vec{\mathbf{w}} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in W$, so we know $x_1 + y_1 = 3z_1$ and $x_2 + y_2 = 3z_2$. Consider

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

Since

$$(x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) = 3z_1 + 3z_2 = 3(z_1 + z_2)$$

we see that $\vec{\mathbf{v}} + \vec{\mathbf{w}} \in W$, so W is closed under vector addition.

Now consider

$$c\vec{\mathbf{v}} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}.$$

Since

$$cx_1 + cx_2 = c(x_1 + x_2) = c(3z_1) = 3(cz_1)$$

we see that $c\vec{\mathbf{v}} \in W$, so W is closed under scalar multiplication. Therefore W is a subspace of $\mathbb{R}^3$.

$\square$

V5. Determine if the vectors $\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 5 \\ 1 \\ 5 \end{bmatrix}$ are linearly dependent or linearly independent.

Solution: Compute

$$\text{RREF}\begin{bmatrix} 3 & -1 & 0 & 2 \\ 2 & 1 & 1 & 5 \\ 1 & 2 & -1 & 1 \\ 0 & 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the fourth column is not a pivot column, the vectors are linearly dependent.

$\square$

V6. Determine if the set

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \\ 5 \end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^4$ or not.

Solution: Compute

$$\text{RREF}\begin{bmatrix} 3 & -1 & 0 & 2 \\ 2 & 1 & 1 & 5 \\ 1 & 2 & -1 & 1 \\ 0 & 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the fourth column is not a pivot column, the vectors are linearly dependent and thus not a basis. (Alternate solutions: Since the fourth row not a pivot row, the vectors do not span $\mathbb{R}^4$ and thus are not a basis. Or since the resulting matrix is not the identity matrix, the vectors do not form a basis.)

$\square$

V7. Find a basis for W, the subspace of $\mathbb{R}^4$ given by

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Solution: Observe that

$$\text{RREF}\begin{bmatrix} 1 & 1 & 3 & 1 & 2 \\ -3 & 0 & -6 & 6 & 3 \\ -1 & 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has pivot columns in the first, second, and fourth columns, and therefore

$$\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix},\begin{bmatrix}1\\0\\1\\-2\end{bmatrix},\begin{bmatrix}1\\6\\1\\-1\end{bmatrix}\right\}$$

is a basis of W.

$\square$

V8. Find the dimension of W, the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix},\begin{bmatrix}1\\0\\1\\-2\end{bmatrix},\begin{bmatrix}3\\-6\\-1\\2\end{bmatrix},\begin{bmatrix}1\\6\\1\\-1\end{bmatrix},\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}.$$

Solution: Observe that

$$\mathrm{RREF}\begin{bmatrix}1 & 1 & 3 & 1 & 2\\-3 & 0 & -6 & 6 & 3\\-1 & 1 & -1 & 1 & 0\\2 & -2 & 2 & -1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 2 & 0 & 1\\0 & 1 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

has three pivot columns, and therefore $\dim W = 3$.

$\square$

V9. Find a basis for the subspace

$$W = \operatorname{span}\left\{3x^3 + 2x^2 + x, -x^3 + x^2 + 2x + 3, x^2 - x + 1, 2x^3 + 5x^2 + x + 5\right\}$$

of $\mathcal{P}^3$.

Solution: This question is equivalent to finding a basis for the subspace

$$W' = \operatorname{span}\left\{\begin{bmatrix}3\\2\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\2\\3\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix},\begin{bmatrix}2\\5\\1\\5\end{bmatrix}\right\}$$

of Euclidean vectors.
Compute

$$\mathrm{RREF}\begin{bmatrix}3 & -1 & 0 & 2\\2 & 1 & 1 & 5\\1 & 2 & -1 & 1\\0 & 3 & 1 & 5\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\end{bmatrix}.$$

Since the fourth column is not a pivot column, a basis for W' is given by

$$\left\{\begin{bmatrix}3\\2\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\2\\3\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix},\right\}$$

Thus a basis for W is given by

$$\left\{3x^3 + 2x^2 + x, -x^3 + x^2 + 2x + 3, x^2 - x + 1\right\}$$

$\square$

V10. Find a basis for the solution set of the homogeneous system of equations

$$\begin{aligned} x_1 + \ x_2 + 3x_3 + \ x_4 + 2x_5 &= 0 \\ -3x_1 \qquad - 6x_3 + 6x_4 + 3x_5 &= 0 \\ -x_1 + \ x_2 - \ x_3 + \ x_4 \qquad &= 0 \\ 2x_1 - 2x_2 + 2x_3 - \ x_4 + \ x_5 &= 0. \end{aligned}$$

Solution: Observe that

$$\text{RREF}\left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 2 & 0 \\ -3 & 0 & -6 & 6 & 3 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -2 & 2 & -1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Letting $x_3 = a$ and $x_5 = b$ (since those correspond to the non-pivot columns), this is equivalent to the system

$$\begin{aligned} x_1 \quad + 2x_3 \quad + x_5 &= 0 \\ x_2 + \ x_3 \qquad &= 0 \\ x_3 \qquad &= a \\ x_4 + x_5 &= 0 \\ x_5 &= b \end{aligned}$$

Thus, the solution set is

$$\left\{ \begin{bmatrix} -2a-b \\ -a \\ a \\ -b \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

Since we can write

$$\begin{bmatrix} -2a-b \\ -a \\ a \\ -b \\ b \end{bmatrix} = a \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

a basis for the solution space is

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$\square$

A1. Consider the following maps of polynomials $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ defined by

$$S(f(x)) = f(x) - 3x \text{ and } T(f(x)) = f(x) - 3f'(x).$$

Show that one of these maps is a linear transformation, and that the other map is not.

Solution: S is not a linear transformation because $S(0) = -3x \neq 0$. (Alternate reason: $S(x+1) = 1-2x$ but $S(x) + S(1) = 1 - 5x$.)
As for T,

$$\begin{aligned}
T(f(x)+g(x)) &= (f(x)+g(x)) - 3(f(x)+g(x))' = f(x) - 3f'(x) + g(x) - 3g'(x) \\
T(f(x)) + T(g(x)) &= (f(x) - 3f'(x)) + (g(x) - 3g'(x)) = f(x) - 3f'(x) + g(x) - 3g'(x) \\
T(cf(x)) &= (cf(x)) - 3(cf(x))' = cf(x) - 3cf'(x) \\
cT(f(x)) &= c(f(x) - 3f'(x)) = cf(x) - 3cf'(x)
\end{aligned}$$

Since T preserves both addition and scalar multiplication, T is a linear transformation.

□

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x+y \\ -x+3y-z \\ 7x+y+3z \\ 0 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right)$

Solution:

(a) Since

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \\ 7 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 3 & -1 \\ 7 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $T\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -(-2)+(0) \\ -(-2)+3(0)-(3) \\ 7(-2)+(0)+3(3) \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5 \\ 0 \end{bmatrix}$

Alternatively, $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 3 & -1 \\ 7 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1(-2)+1(0)+0(3) \\ -1(-2)+3(0)-1(3) \\ 7(-2)+1(0)+3(3) \\ 0(-2)+0(0)+0(3) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5 \\ 0 \end{bmatrix}$.

□

A3. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+3y+2z-3w \\ 2x+4y+6z-10w \\ x+6y-z+3w \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of T.

Solution: First, we note the standard matrix

$$A = \begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 4 & 6 & -10 \\ 1 & 6 & -1 & 3 \end{bmatrix}$$

and compute

$$\mathrm{RREF}\,(A) = \begin{bmatrix} 1 & 0 & 5 & -9 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The kernel is given by solution set of the corresponding homogeneous system of equations

$$\ker T = \left\{ \begin{bmatrix} -5a+9b \\ a-2b \\ a \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

so a basis for the kernel is

$$\left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A basis for the image is given by the pivot columns, namely

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \right\}.$$

$\square$

A4. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the standard matrix $\begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 4 & 6 & -10 \\ 1 & 6 & -1 & 3 \end{bmatrix}$

Solution:

(a) RREF $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since each column is a pivot column, S is injective. Since there is no zero row, S is surjective. (Alternatively, since the result is the identity matrix, S is bijective.)

(b)

$$\mathrm{RREF} \begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 4 & 6 & -10 \\ 1 & 6 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 & -9 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeroes, the span of the columns does not equal $\mathbb{R}^3$, so T is not surjective. Since there are non-pivot columns, T is not injective either. (Alternatively, since $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$, T is not injective.)

$\square$

M1. Let

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 3 \\ 1 & -2 & 5 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and show how to compute it.

Solution: AC is the only one that can be computed, since A is 2×2 and C is 2×3. Thus AC will be the 2×3 matrix given by

$$\begin{aligned} AC\left(\vec{\mathbf{e}}_1\right) &= A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\ AC\left(\vec{\mathbf{e}}_2\right) &= A\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix} \\ AC\left(\vec{\mathbf{e}}_3\right) &= A\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 5 \end{bmatrix} \end{aligned}$$

Thus

$$AC = \begin{bmatrix} -3 & 7 & -12 \\ 1 & -2 & 5 \end{bmatrix}.$$

□

M2. Consider the two row operations $R_2 - 3R_1 \to R_2$ and $2R_2 \to R_2$ applied as follows to show $A \sim B$:

$$\begin{aligned} A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} &\sim \begin{bmatrix} 1 & 2 & 3 \\ 4-3(1) & 5-3(2) & 6-3(3) \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & -3 \\ 7 & 8 & 9 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 2 & 3 \\ 2(1) & 2(-1) & 2(-3) \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & -6 \\ 7 & 8 & 9 \end{bmatrix} = B \end{aligned}$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A.

Solution: Each row operation may be applied to the identity matrix I:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and then left-multiplied with A to obtain the solution:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

□

M3. Determine if the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 6 \\ 1 & 6 & -1 \end{bmatrix}$ is invertible or not.

Solution: We compute

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 6 \\ 1 & 6 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since its RREF is not the identity matrix, the matrix is not invertible.

□

M4. Show how to compute the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution:

$$\text{RREF}\left(\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -11 & 32 \\ 0 & 1 & 0 & 0 & 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 1 & 2 & -11 & 32 \\ 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

□

G1. Consider the row operation $R_2 - 4R_1 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 4-4(1) & 5-4(2) & 6-4(3) \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = 12$, find the determinant of RC.

Solution:

1. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0-4(1) & 1-4(0) & 0-4(0) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R$

2. $\det(RC) = \det(R)\det(C) = (1)(12) = 12$.

□

G2. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix}$$

Solution: Here is one possible solution, first applying a single row operation, and then performing Laplace/cofactor expansions to reduce the determinant to a linear combination of 2×2 determinants:

$$\det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} = (-1)\det \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 3 \\ -3 & 1 & -5 \end{bmatrix} + (1)\det \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} &= (-1)\left((1)\det \begin{bmatrix} 1 & 3 \\ 1 & -5 \end{bmatrix} - (1)\det \begin{bmatrix} 3 & -1 \\ 1 & -5 \end{bmatrix} + (-3)\det \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\right) + \\ &\quad (1)\left((1)\det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - (3)\det \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}\right) \\ &= (-1)(-8+14-30) + (1)(1-15) \\ &= 10 \end{aligned}$$

Here is another possible solution, using row and column operations to first reduce the determinant to a 3×3 matrix and then applying a formula:

$$\begin{aligned} \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} &= \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ -3 & 1 & 2 & -7 \end{bmatrix} \\ &= -\det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & -7 \end{bmatrix} = -\det \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 2 \\ -3 & 1 & -7 \end{bmatrix} \\ &= -((-7-18-1)-(3+2-21)) \\ &= 10 \end{aligned}$$

□

G3. Find the eigenvalues of the matrix $\begin{bmatrix} -2 & -2 \\ 10 & 7 \end{bmatrix}$.

Solution: Compute

$$\det(A-\lambda I) = \det \begin{bmatrix} -2-\lambda & -2 \\ 10 & 7-\lambda \end{bmatrix} = (-2-\lambda)(7-\lambda)+20 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$$

The eigenvalues are the roots of the characteristic polynomial, 2 and 3.

□

G4. Find a basis for the eigenspace associated to the eigenvalue 3 in the matrix

$$\begin{bmatrix} -7 & -8 & 2 \\ 8 & 9 & -1 \\ \frac{13}{2} & 5 & 2 \end{bmatrix}.$$

Solution:

$$\mathrm{RREF}(A-3I) = \mathrm{RREF} \begin{bmatrix} -7-3 & -8 & 2 \\ 8 & 9-3 & -1 \\ \frac{13}{2} & 5 & 2-3 \end{bmatrix} = \mathrm{RREF} \begin{bmatrix} -10 & -8 & 2 \\ 8 & 6 & -1 \\ \frac{13}{2} & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenspace associated to 3 is the kernel of $A-3I$, namely

$$\left\{ \begin{bmatrix} -a \\ \frac{3}{2}a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$$

which has a basis of $\left\{ \begin{bmatrix} -1 \\ \frac{3}{2} \\ 1 \end{bmatrix} \right\}$.

$\square$