Module E

Standard E1

E1. Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x + 3y - 4z &= 5 \\ 3x + 9y + z &= 0 \\ x - z &= 1 \end{aligned}$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{array}\right]$$

□

E1. Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x_1 + 4x_3 &= 1 \\ x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= -1 \end{aligned}$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array}\right]$$

□

E1. Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x_1 + 4x_3 &= 1 \\ x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_4 &= -1 \end{aligned}$$

Solution.

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & -1 & 0 & 7 \\ 1 & -1 & 0 & 3 & -1 \end{array}\right]$$

□

E1. Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x_1 + 3x_2 - 4x_3 + x_4 &= 5 \\ 3x_1 + 9x_2 + x_3 - 7x_4 &= 0 \\ x_1 - x_3 + x_4 &= 1 \end{aligned}$$

Solution.

$$\left[\begin{array}{cccc|c} 1 & 3 & -4 & 1 & 5 \\ 3 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{array}\right]$$

□

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{cccc|c} 3 & -1 & 0 & 1 & 5 \\ -1 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 0 & -3 \end{array}\right]$$

Solution.

$$\begin{aligned} 3x_1 - x_2 + x_4 &= 5 \\ -x_1 + 9x_2 + x_3 - 7x_4 &= 0 \\ x_1 - x_3 &= -3 \end{aligned}$$

□

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 4 & 1 & -7 \\ 1 & 2 & -1 & 0 \end{array}\right]$$

Solution.

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 4x_2 + x_3 &= -7 \\ x_1 + 2x_2 - x_3 &= 0 \end{aligned}$$

□

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} -4 & -1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{array}\right]$$

Solution.

$$\begin{aligned} -4x_1 - x_2 + 3x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 0 \\ -x_1 + 4x_2 + x_3 &= 4 \end{aligned}$$

□

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array}\right]$$

Solution.

$$\begin{aligned} x_1 + 4x_3 &= 1 \\ x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= -1 \end{aligned}$$

□

Standard E2

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 \\ 3 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\square$

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ -3 & 5 & 2 & 0 \\ 1 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & 6 \\ 0 & -1 & -1 & -3 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\square$

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & -\frac{2}{3} & -1 & \frac{2}{3} \\ 0 & 2 & 3 & -2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 2 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\square$

E2. Find the reduced row echelon form of the matrix below.

$$\begin{bmatrix} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 5 & 0 & -1 \\ 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 1 \\ 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & -1 & 15 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & 0 & 24 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{12} & 1 \\ 0 & 1 & 0 & \frac{3}{4} & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{bmatrix}$$

□

E2. Find RREF A, where

$$A = \begin{bmatrix} 3 & -2 & 1 & 8 & -5 \\ 2 & 2 & 0 & 6 & -2 \\ -1 & 1 & 1 & -4 & 6 \end{bmatrix}$$

Solution.

$$\text{RREF}\, A = \begin{bmatrix} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \end{bmatrix}$$

□

E2. Find RREF A, where

$$A = \begin{bmatrix} 2 & -7 & 4 \\ 1 & -3 & 2 \\ 3 & 0 & 3 \end{bmatrix}$$

Solution.

$$\text{RREF}\, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

E2. Find RREF A, where

$$A = \begin{bmatrix} 2 & -1 & 5 & 4 \\ -1 & 0 & -2 & -1 \\ 1 & 3 & -1 & -5 \end{bmatrix}$$

Solution.

$$\text{RREF}\, A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

E2. Find RREF A, where

$$A = \begin{bmatrix} 2 & 2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 4 & 5 \\ 3 & 3 & -1 & -2 & 1 \end{bmatrix}$$

Solution.

$$\text{RREF}\, A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

□

Standard E3

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} x + 3y - 4z &= 5 \\ 3x + 9y + z &= 2 \end{aligned}$$

Solution.

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 2 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

So the solution set is

$$\left\{ \begin{bmatrix} 1-3c \\ c \\ -1 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$\square$

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} -3x + y &= 2 \\ -8x + 2y - z &= 6 \\ 2y + 3z &= -2 \end{aligned}$$

Solution.

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the solution set is

$$\left\{ \begin{bmatrix} -1-\frac{c}{2} \\ -1-\frac{3c}{2} \\ c \end{bmatrix} \middle| \, c \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} c-1 \\ 3c-1 \\ -2c \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$\square$

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x + y - z + w &= 5 \\ 3x - y - 2w &= 0 \\ -x + 5z + 3w &= -1 \end{aligned}$$

Solution.

$$\mathrm{RREF}\left(\left[\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array}\right]\right) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{4} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{array}\right]$$

So the solution set is

$$\left\{ \begin{bmatrix} 1+a \\ 3-21a \\ -7a \\ 12a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$$

$\square$

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x + y - z + w &= 5 \\ 3x - y - 2w &= 0 \\ -x + 5z + 3w &= -1 \end{aligned}$$

Solution.

$$\operatorname{RREF}\left(\left[\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array}\right]\right) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{4} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{array}\right]$$

So the solution set is

$$\left\{ \begin{bmatrix} 1+a \\ 3-21a \\ -7a \\ 12a \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

□

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x_1 - 2x_2 + 6x_3 - x_4 &= -1 \\ 3x_1 + 6x_3 + x_4 &= 5 \\ -4x_1 + x_2 - 9x_3 + 2x_4 &= -7 \end{aligned}$$

Solution. Let $A = \left[\begin{array}{cccc|c} 2 & -2 & 6 & -1 & -1 \\ 3 & 0 & 6 & 1 & 5 \\ -4 & 1 & -9 & 2 & -7 \end{array}\right]$, so RREF $A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$. It follows that the solution set is given by $\left\{ \begin{bmatrix} 2-2a \\ 3+a \\ a \\ -1 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$. □

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x_1 + 3x_2 - 5x_3 + 14x_4 &= 8 \\ x_1 + x_2 - x_3 + 5x_4 &= 3 \end{aligned}$$

Solution. Let $A = \left[\begin{array}{cccc|c} 2 & 3 & -5 & 14 & 8 \\ 1 & 1 & -1 & 5 & 3 \end{array}\right]$, so RREF $A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -3 & 4 & 2 \end{array}\right]$. It follows that the solution set is given by $\left\{ \begin{bmatrix} 1-2a-b \\ 2+3a-4b \\ a \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$. □

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} 4x_1 + 4x_2 + 3x_3 - 6x_4 &= 5 \\ -2x_3 - 4x_4 &= 3 \\ 2x_1 + 2x_2 + x_3 - 4x_4 &= -1 \end{aligned}$$

Solution. Let $A = \left[\begin{array}{cccc|c} 4 & 4 & 3 & -6 & 5 \\ 0 & 0 & -2 & -4 & 3 \\ 2 & 2 & 1 & -4 & -1 \end{array}\right]$, so RREF $A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$. It follows that the system is inconsistent with no solutions (since the bottom row implies the contradiction $0 = 1$), so its solution set is $\emptyset$. □

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} 3x + 2y + z &= 7 \\ x + y + z &= 1 \\ -2x + 3z &= -11 \end{aligned}$$

Solution. Let $A = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 7 \\ 1 & 1 & 1 & 1 \\ -2 & 0 & 3 & 11 \end{array}\right]$, so RREF $A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right]$. It follows that solution set is $\left\{\begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}\right\}$ □

Module V

Standard V1

V1. Let V be the set of all real numbers together with the operations $\oplus$ and $\odot$ defined by, for any $x, y \in V$ and $c \in \mathbb{R}$,

$$\begin{aligned} x \oplus y &= x + y \\ c \odot x &= cx - 3(c-1) \end{aligned}$$

(a) Show that scalar multiplication is associative: $a \odot (b \odot x) = (ab) \odot x$ for all scalars $a, b \in \mathbb{R}$ and $x \in V$.

(b) Explain why V nonetheless isn't a vector space.

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, x_2 + y_2 + 2x_1y_1) \\ c \odot (x_1, x_2) &= (cx_1, cx_2) \end{aligned}$$

(a) Show that the vector addition $\oplus$ is associative:
$(x_1, x_2) \oplus ((y_1, y_2) \oplus (z_1, z_2)) = ((x_1, x_2) \oplus (y_1, y_2)) \oplus (z_1, z_2)$ for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in V$.

(b) Explain why V nonetheless isn't a vector space.

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1 - 1, x_2 + y_2 - 1) \\ c \odot (x_1, x_2) &= (cx_1, cx_2) \end{aligned}$$

(a) Show that this vector space has an additive identity element: there exists $\vec{z} \in V$ satisfying
$(x, y) \oplus \vec{z} = (x, y)$ for every $(x, y) \in V$.

(b) Explain why V nonetheless isn't a vector space.

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ c \odot (x_1, x_2) &= (0, cx_2) \end{aligned}$$

(a) Show that scalar multiplication distributes over scalar addition:
$(c + d) \odot (x_1, x_2) = c \odot (x_1, x_2) \oplus d \odot (x_1, x_2)$ for every $c, d \in \mathbb{R}$ and $(x_1, x_2) \in V$.

(b) Explain why V nonetheless isn't a vector space.

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ c \odot (x_1, x_2) &= (c^2x_1, c^3x_2) \end{aligned}$$

(a) Show that scalar multiplication distributes over vector addition:
$c \odot ((x_1, x_2) \oplus (y_1, y_2)) = c \odot (x_1, x_2) \oplus c \odot (y_1, y_2)$ for all $c \in \mathbb{R}$ and $(x_1, x_2), (y_1, y_2) \in V$.

(b) Explain why V nonetheless isn't a vector space.

V1. Let V be the set of all real numbers with the operations, for any $x, y \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} x \oplus y &= \sqrt{x^2 + y^2} \\ c \odot x &= cx \end{aligned}$$

(a) Show that the vector addition $\oplus$ is associative: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ for all $x, y, z \in V$.

(b) Explain why V nonetheless isn't a vector space.

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, x_2 y_2) \\ c \odot (x_1, x_2) &= (cx_1, cx_2) \end{aligned}$$

(a) Show that there is an additive identity element: there exists an element $\vec{z} \in V$ such that $(x_1, x_2) \oplus \vec{z} = (x_1, x_2)$ for any $(x_1, x_2) \in V$.

(b) Explain why V nonetheless isn't a vector space.

Standard V2

V3. Determine if $\begin{bmatrix}0\\-1\\2\\6\end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix}3\\-1\\-1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}$.

Solution.

$$\text{RREF}\left(\left[\begin{array}{cc|c}3&-1&0\\-1&0&-1\\-1&1&2\\0&2&6\end{array}\right]\right)=\left[\begin{array}{cc|c}1&0&1\\0&1&3\\0&0&0\\0&0&0\end{array}\right]$$

Since this system has a solution, $\begin{bmatrix}0\\-1\\2\\6\end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix}3\\-1\\-1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}$, namely

$$\begin{bmatrix}0\\-1\\2\\6\end{bmatrix}=\begin{bmatrix}3\\-1\\-1\\0\end{bmatrix}+3\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}.$$

□

V3. Determine if $\begin{bmatrix}0\\1\\-2\\1\end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix}5\\2\\-3\\2\end{bmatrix}$, $\begin{bmatrix}3\\1\\1\\0\end{bmatrix}$, and $\begin{bmatrix}8\\3\\5\\-1\end{bmatrix}$.

Solution.

$$\text{RREF}\left(\left[\begin{array}{ccc|c}8&5&3&0\\3&2&1&1\\5&-3&1&-2\\-1&2&0&1\end{array}\right]\right)=\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]$$

The system has no solution, so $\begin{bmatrix}0\\1\\-2\\1\end{bmatrix}$ is not a linear combination of the three other vectors. □

V3. Determine if $\begin{bmatrix}0\\0\\2\end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix}-1\\-9\\15\end{bmatrix}$ and $\begin{bmatrix}1\\5\\-5\end{bmatrix}$.

Solution.

$$\text{RREF}\left(\left[\begin{array}{cc|c}-1&1&0\\-9&5&0\\15&-5&2\end{array}\right]\right)=\left[\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&1\end{array}\right]$$

Since this system has no solution, $\begin{bmatrix}0\\0\\2\end{bmatrix}$ cannot be written as a linear combination of the vectors $\begin{bmatrix}-1\\-9\\15\end{bmatrix}$ and $\begin{bmatrix}1\\5\\-5\end{bmatrix}$.

□

V3. Determine if $\begin{bmatrix}1\\4\\3\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}2\\3\\-1\end{bmatrix}$, $\begin{bmatrix}1\\-1\\0\end{bmatrix}$, and $\begin{bmatrix}-3\\-2\\5\end{bmatrix}$.

Solution.

$$\text{RREF}\left(\left[\begin{array}{ccc|c}2 & 1 & -3 & 1\\ 3 & -1 & -2 & 4\\ -1 & 0 & 5 & 3\end{array}\right]\right) = \left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{array}\right]$$

Since this system has a solution, $\begin{bmatrix}1\\4\\3\end{bmatrix}$ is a linear combination of the three vectors. □

V3. Determine if $\begin{bmatrix}1\\4\\3\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}3\\0\\-1\end{bmatrix}$, $\begin{bmatrix}1\\-1\\4\end{bmatrix}$, and $\begin{bmatrix}5\\1\\-6\end{bmatrix}$.

Solution.

$$\text{RREF}\left(\left[\begin{array}{ccc|c}3 & 1 & 5 & 1\\ 0 & -1 & 1 & 4\\ -1 & 4 & -6 & 3\end{array}\right]\right) = \left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]$$

Since the corresponding system has no solutions, $\begin{bmatrix}1\\4\\3\end{bmatrix}$ is not a linear combination of the three vectors. □

V3. Determine if $\begin{bmatrix}0\\-1\\6\\-7\end{bmatrix}$ belongs to the span of the set $\left\{\begin{bmatrix}2\\0\\-1\\5\end{bmatrix}, \begin{bmatrix}4\\-1\\4\\3\end{bmatrix}\right\}$.

Solution. Since

$$\text{RREF}\left(\left[\begin{array}{cc|c}2 & 4 & 0\\ 0 & -1 & -1\\ -1 & 4 & 6\\ 5 & 3 & -7\end{array}\right]\right) = \left[\begin{array}{cc|c}1 & 0 & -2\\ 0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{array}\right]$$

does not require a contradiction, $\begin{bmatrix}0\\-1\\6\\-7\end{bmatrix}$ is a linear combination of the three vectors. □

V3. Determine if $\begin{bmatrix}4\\-1\\6\\-7\end{bmatrix}$ belongs to the span of the set $\left\{\begin{bmatrix}2\\0\\-1\\5\end{bmatrix}, \begin{bmatrix}4\\-1\\4\\3\end{bmatrix}\right\}$.

Solution. Since

$$\text{RREF}\left(\left[\begin{array}{cc|c}2 & 4 & 4\\ 0 & -1 & -1\\ -1 & 4 & 6\\ 5 & 3 & -7\end{array}\right]\right) = \left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{array}\right]$$

requires the contradiction $0 = 1$, $\begin{bmatrix}4\\-1\\6\\-7\end{bmatrix}$ is not a linear combination of the three vectors. □

V3. Determine if $\begin{bmatrix}3\\-2\\4\end{bmatrix}$ belongs to the span of the set $\left\{\begin{bmatrix}1\\2\\-3\end{bmatrix}, \begin{bmatrix}2\\4\\-6\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$.

Solution. Since

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & -2 \\ -3 & -6 & 0 & 4 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

requires the contradiction $0 = 1$, $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ is not a linear combination of the three vectors. $\square$

Standard V3

V4. Determine if the vectors $\begin{bmatrix}-3\\1\\1\end{bmatrix}$, $\begin{bmatrix}5\\-1\\-2\end{bmatrix}$, $\begin{bmatrix}2\\0\\-1\end{bmatrix}$, and $\begin{bmatrix}0\\2\\-1\end{bmatrix}$ span $\mathbb{R}^3$

Solution.

$$\text{RREF}\left(\begin{bmatrix}-3&5&2&0\\1&-1&0&2\\1&-2&-1&-1\end{bmatrix}\right)=\begin{bmatrix}1&0&1&5\\0&1&1&3\\0&0&0&0\end{bmatrix}$$

Since the resulting matrix has a zero row, the vectors do not span $\mathbb{R}^3$. □

V4. Determine if the vectors $\begin{bmatrix}1\\1\\2\\1\end{bmatrix}$, $\begin{bmatrix}3\\3\\6\\3\end{bmatrix}$, $\begin{bmatrix}3\\-1\\3\\-2\end{bmatrix}$, and $\begin{bmatrix}7\\-1\\8\\-3\end{bmatrix}$ span $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}1&3&3&7\\1&3&-1&-1\\2&6&3&8\\1&3&-2&-3\end{bmatrix}\right)=\begin{bmatrix}1&3&0&1\\0&0&1&2\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Since there are zero rows, they do not span. □

V4. Determine if the vectors $\begin{bmatrix}8\\21\\-7\end{bmatrix}$, $\begin{bmatrix}-3\\-8\\3\end{bmatrix}$, $\begin{bmatrix}-1\\-3\\2\end{bmatrix}$, and $\begin{bmatrix}4\\11\\-5\end{bmatrix}$ span $\mathbb{R}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}8&-3&-1&4\\21&-8&-3&11\\-7&3&2&-5\end{bmatrix}\right)=\begin{bmatrix}1&0&1&-1\\0&1&3&-4\\0&0&0&0\end{bmatrix}$$

Since there is a zero row, they do not span $\mathbb{R}^3$. □

V4. Determine if the vectors $\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}$, $\begin{bmatrix}3\\1\\3\\6\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\2\\0\\1\end{bmatrix}$ span $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}2&3&0&1\\0&1&0&2\\-2&3&1&0\\0&6&1&1\end{bmatrix}\right)=\begin{bmatrix}1&0&0&-\frac{5}{2}\\0&1&0&2\\0&0&1&-11\\0&0&0&0\end{bmatrix}$$

Since there is a zero row, the vectors do not span $\mathbb{R}^4$. □

V4. Determine if the vectors $\begin{bmatrix}1\\0\\2\\1\end{bmatrix}$, $\begin{bmatrix}3\\1\\0\\-3\end{bmatrix}$, $\begin{bmatrix}0\\3\\0\\-2\end{bmatrix}$, and $\begin{bmatrix}-1\\1\\-1\\-1\end{bmatrix}$ span $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}1&3&0&-1\\0&1&3&1\\2&0&0&-1\\1&-3&-2&-1\end{bmatrix}\right)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Since every row contains a pivot, the vectors span $\mathbb{R}^4$. □

V4. Does $\operatorname{span}\left\{\begin{bmatrix}2\\-1\\4\end{bmatrix},\begin{bmatrix}3\\12\\-9\end{bmatrix},\begin{bmatrix}1\\4\\-3\end{bmatrix},\begin{bmatrix}-4\\2\\-8\end{bmatrix}\right\}=\mathbb{R}^3$?

Solution. Since

$$\mathrm{RREF}\begin{bmatrix}2 & 3 & 1 & -4\\-1 & 12 & 4 & 2\\4 & -9 & -3 & -8\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & -2\\0 & 1 & 1/3 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

has a zero row, the vectors fail to span $\mathbb{R}^3$. □

V4. Does $\operatorname{span}\left\{\begin{bmatrix}2\\-1\\4\end{bmatrix},\begin{bmatrix}3\\12\\-9\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}-4\\2\\-8\end{bmatrix}\right\}=\mathbb{R}^3$?

Solution. Since

$$\mathrm{RREF}\begin{bmatrix}2 & 3 & 1 & -4\\-1 & 12 & 2 & 2\\4 & -9 & 3 & -8\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & -2\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$$

lacks a zero row, the vectors span $\mathbb{R}^3$. □

V4. Does $\operatorname{span}\left\{\begin{bmatrix}2\\-1\\4\\2\\1\end{bmatrix},\begin{bmatrix}-1\\3\\5\\2\\0\end{bmatrix},\begin{bmatrix}1\\0\\5\\1\\-3\end{bmatrix}\right\}=\mathbb{R}^5$?

Solution. Since there are only three vectors, they cannot span $\mathbb{R}^5$. (Or, since RREF must contain a zero row, so they cannot span $\mathbb{R}^5$.) □

Standard V4

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x, y \text{ are integers} \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x = y \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.
Solution. W is not a subspace; for example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ belongs to W while $\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does not.
U is a subspace. □

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ 1 \\ z \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^4$, and that one of the sets is not.
Solution. U is not a subspace; for example, it doesn't contain the zero vector.
W is a subspace. □

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + y + z = 1 \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + y + z = 0 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.
Solution. W is not a subspace; for example, it doesn't contain the zero vector.
U is a subspace. □

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + y = 3z \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + y = 3, z \in \mathbb{R} \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.
Solution. U is not a subspace; for example, it doesn't contain the zero vector.
W is a subspace. □

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x + y = 0 \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| xy = 0 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.
Solution. U is not a subspace; for example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ belong to U while $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ does not.
W is a subspace. □

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x = y \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| |x| = |y| \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.

Solution. U is not a subspace; for example, $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ belong to U while $\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\-1\end{bmatrix}$ does not.
W is a subspace. $\square$

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix}x\\y\end{bmatrix} \middle| \, y = 2x \right\} \qquad U = \left\{ \begin{bmatrix}x\\y\end{bmatrix} \middle| \, y = x^2 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.
Solution. U is not a subspace; for example, $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\4\end{bmatrix}$ belong to U while $\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}2\\4\end{bmatrix}$ does not.
W is a subspace. $\square$

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} \middle| \, z = 2xy \right\} \qquad U = \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} \middle| \, z = 2x + y \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.
Solution. W is not a subspace; for example, $\begin{bmatrix}1\\1\\2\end{bmatrix}$ belongs to W while $\begin{bmatrix}2\\2\\4\end{bmatrix}$ does not.
U is a subspace. $\square$

Standard V5

V5. Determine if the vectors in the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$ are linearly dependent or linearly independent.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since each column is a pivot column, the vectors are linearly independent. □

V5. Determine if the vectors in the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}$ are linearly dependent or linearly independent

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, the set is linearly dependent. □

V5. Determine if the vectors in the set $\left\{ \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$ are linearly dependent or linearly independent

Solution.

$$\text{RREF}\left(\begin{bmatrix} -3 & 1 & 0 \\ 8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every column is a pivot column, therefore the set is linearly independent. □

V5. Determine if the vectors in the set $\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$ are linearly dependent or linearly independent

Solution.

$$\text{RREF}\left(\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

This has a non pivot column, therefore the set is linearly dependent. □

V5. Determine if the vectors in the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 6 \\ 5 \end{bmatrix} \right\}$ are linearly dependent or linearly independent.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 3 & 1 & 3 \\ -1 & 2 & -8 \\ 0 & -2 & 6 \\ 4 & 1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced row echelon form has a nonpivot column, the vectors are linearly dependent. □

V5. Determine if the vectors in the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 6 \\ 5 \end{bmatrix} \right\}$ are linearly dependent or linearly independent.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 3 & 1 & 3 \\ -1 & 2 & -8 \\ 1 & -2 & 6 \\ 4 & 1 & 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced row echelon form has only pivot columns, the vectors are linearly independent. $\square$

V5. Determine if the vectors in the set $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \end{bmatrix} \right\}$ are linearly dependent or linearly independent.

Solution.

$$\text{RREF}\begin{bmatrix} 2 & -1 & 2 \\ 1 & -3 & -4 \\ -1 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the third column is not a pivot column, the set is linearly dependent. $\square$

V5. Determine if the vectors in the set $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right\}$ are linearly dependent or linearly independent.

Solution.

$$\text{RREF}\begin{bmatrix} 2 & -1 & 2 & 1 \\ 1 & -3 & -4 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the third column is not a pivot column, the set is linearly dependent. $\square$

Standard V6

V6. Determine if the set $\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}3\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\-2\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}1 & 3 & 2\\1 & -1 & 0\\-1 & 1 & -2\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

V6. Determine if the set $\left\{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\-1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}0 & 1 & 1 & 0\\1 & -1 & 0 & 2\\1 & 0 & -1 & 0\\1 & 2 & 0 & -1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis. □

V6. Determine if the set $\left\{\begin{bmatrix}3\\-1\\2\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}, \begin{bmatrix}1\\4\\-1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}3 & 2 & 1\\-1 & 0 & 4\\2 & 2 & -1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

V6. Determine if the set $\left\{\begin{bmatrix}3\\-1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\0\\2\\4\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\3\\0\\5\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}3 & 2 & 1 & -1\\-1 & 0 & -1 & 3\\2 & 2 & 0 & 0\\3 & 4 & -1 & 5\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

V6. Determine if the set $\left\{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\3\\6\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\3\\-2\end{bmatrix}, \begin{bmatrix}7\\-1\\8\\-3\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}1 & 3 & 3 & 7\\1 & 3 & -1 & -1\\2 & 6 & 3 & 8\\1 & 3 & -2 & -3\end{bmatrix}\right) = \begin{bmatrix}1 & 3 & 0 & 1\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

V6. Determine if the set $\left\{\begin{bmatrix}8\\21\\-7\end{bmatrix}, \begin{bmatrix}-3\\-8\\3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 8 & -3 & -1 \\ 21 & -8 & -3 \\ -7 & 3 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

V6. Determine if the set $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Solution. Since

$$\text{RREF}\begin{bmatrix} 2 & 3 & -4 \\ -1 & 12 & 2 \\ 4 & -9 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

V6. Determine if the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since this is the identity matrix, this set is a basis. □

Standard V7

V7. Let $W = \operatorname{span}\left(\left\{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\3\\6\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\3\\-2\end{bmatrix}, \begin{bmatrix}7\\-1\\8\\-3\end{bmatrix}\right\}\right)$. Find a basis for W.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix}1&3&3&7\\1&3&-1&-1\\2&6&3&8\\1&3&-2&-3\end{bmatrix}\right) = \begin{bmatrix}1&3&0&1\\0&0&1&2\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Thus a basis is given by its pivot columns: $\left\{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\3\\-2\end{bmatrix}\right\}$. □

V7. Let $W = \operatorname{span}\left(\left\{\begin{bmatrix}-3\\-8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}\right\}\right)$. Find a basis for W.

Solution. Let $A = \begin{bmatrix}-3&1&0\\-8&2&-1\\0&2&3\end{bmatrix}$, and compute $\mathrm{RREF}(A) = \begin{bmatrix}1&0&\frac{1}{2}\\0&1&\frac{3}{2}\\0&0&0\end{bmatrix}$. Since the first two columns are pivot columns, $\left\{\begin{bmatrix}-3\\-8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right\}$ is a basis for W. □

V7. Let $W = \operatorname{span}\left(\left\{\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\3\\6\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\1\end{bmatrix}\right\}\right)$. Find a basis of W.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix}2&3&0&1\\0&1&0&2\\-2&3&1&0\\0&6&1&1\end{bmatrix}\right) = \begin{bmatrix}1&0&0&-\frac{5}{2}\\0&1&0&2\\0&0&1&-11\\0&0&0&0\end{bmatrix}$$

Then its pivot columns $\left\{\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\3\\6\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$ form a basis of W. □

V7. Let $W = \operatorname{span}\left(\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\0\\1\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\4\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}\right)$. Find a basis of W.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix}1&2&3&1\\-1&0&-1&1\\3&1&4&1\\-3&1&-2&-7\end{bmatrix}\right) = \begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

Thus its pivot columns $\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}$ form a basis for W. □

V7. Let $W = \operatorname{span}\left\{\begin{bmatrix}2\\0\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\-8\\-1\end{bmatrix}\right\}$. Find a basis for this vector space.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix}2 & 3 & 0\\0 & 1 & 2\\2 & -1 & -8\\1 & 1 & -1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & -3\\0 & 1 & 2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Thus its pivot columns $\left\{\begin{bmatrix}2\\0\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\-1\\1\end{bmatrix}\right\}$ form a basis of W. □

V7. Let $W = \operatorname{span}\left\{\begin{bmatrix}2\\-1\\4\end{bmatrix}, \begin{bmatrix}3\\12\\-9\end{bmatrix}, \begin{bmatrix}1\\4\\-3\end{bmatrix}, \begin{bmatrix}-4\\2\\-8\end{bmatrix}\right\}$. Find a basis for this vector space.

Solution.

$$\mathrm{RREF}\begin{bmatrix}2 & 3 & 1 & -4\\-1 & 12 & 4 & 2\\4 & -9 & -3 & -8\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & -2\\0 & 1 & 1/3 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

Thus its pivot columns $\left\{\begin{bmatrix}2\\-1\\4\end{bmatrix}, \begin{bmatrix}3\\12\\-9\end{bmatrix}\right\}$ form a basis of W. □

V7. Let $W = \operatorname{span}\left\{\begin{bmatrix}2\\-1\\4\\2\end{bmatrix}, \begin{bmatrix}-4\\2\\-8\\-4\end{bmatrix}, \begin{bmatrix}3\\12\\-9\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\\2\end{bmatrix}\right\}$. Find a basis for this vector space.

Solution.

$$\mathrm{RREF}\begin{bmatrix}2 & -4 & 3 & 1\\-1 & 2 & 12 & 2\\4 & -8 & -9 & 3\\2 & -4 & 2 & 2\end{bmatrix} = \begin{bmatrix}1 & -2 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

Thus its pivot columns $\left\{\begin{bmatrix}2\\-1\\4\\2\end{bmatrix}, \begin{bmatrix}3\\12\\-9\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\\2\end{bmatrix}\right\}$ form a basis of W. □

Standard V8

V8. Let $W = \text{span}\left(\left\{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\3\\6\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\3\\-2\end{bmatrix}, \begin{bmatrix}7\\-1\\8\\-3\end{bmatrix}\right\}\right)$. Find the dimension of W.

Solution.

$$\text{RREF}\left(\begin{bmatrix}1 & 3 & 3 & 7\\1 & 3 & -1 & -1\\2 & 6 & 3 & 8\\1 & 3 & -2 & -3\end{bmatrix}\right) = \begin{bmatrix}1 & 3 & 0 & 1\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

This has two pivot columns, so W has dimension 2. □

V8. Let $W = \text{span}\left(\left\{\begin{bmatrix}-3\\-8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}\right\}\right)$. Compute the dimension of W.

Solution. Let $A = \begin{bmatrix}-3 & 1 & 0\\-8 & 2 & -1\\0 & 2 & 3\end{bmatrix}$, and compute $\text{RREF}(A) = \begin{bmatrix}1 & 0 & \frac{1}{2}\\0 & 1 & \frac{3}{2}\\0 & 0 & 0\end{bmatrix}$. Since there are two pivot columns, $\dim W = 2$. □

V8. Let $W = \text{span}\left(\left\{\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\3\\6\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\1\end{bmatrix}\right\}\right)$. Compute the dimension of W.

Solution.

$$\text{RREF}\left(\begin{bmatrix}2 & 3 & 0 & 1\\0 & 1 & 0 & 2\\-2 & 3 & 1 & 0\\0 & 6 & 1 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0 & -\frac{5}{2}\\0 & 1 & 0 & 2\\0 & 0 & 1 & -11\\0 & 0 & 0 & 0\end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$. □

V8. Let $W = \text{span}\left(\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\0\\1\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\4\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}\right)$. Compute the dimension of W.

Solution.

$$\text{RREF}\left(\begin{bmatrix}1 & 2 & 3 & 1\\-1 & 0 & -1 & 1\\3 & 1 & 4 & 1\\-3 & 1 & -2 & -7\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$. □

V8. Let $W = \text{span}\left\{\begin{bmatrix}2\\0\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\-8\\-1\end{bmatrix}\right\}$. Find the dimension of W.

Solution.

$$\text{RREF}\left(\begin{bmatrix}2 & 3 & 0\\0 & 1 & 2\\2 & -1 & -8\\1 & 1 & -1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & -3\\0 & 1 & 2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Since it has two pivot columns, its dimension is 2. □

V8. Let $W = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\}$. Find the dimension of W.

Solution.

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 4 & 2 \\ 4 & -9 & -3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since it has two pivot columns, its dimension is 2. $\square$

V8. Let $W = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}$. Find the dimension of W.

Solution.

$$\text{RREF} \begin{bmatrix} 2 & -4 & 3 & 1 \\ -1 & 2 & 12 & 2 \\ 4 & -8 & -9 & 3 \\ 2 & -4 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since it has three pivot columns, its dimension is 3. $\square$

Standard V9

V9. Find a basis for the subspace

$$W = \operatorname{span}\left\{x^2+x, x^2+2x-1, x^2+3x-2\right\}$$

of $\mathcal{P}^2$.

Solution. $\left\{x^2+x, x^2+2x-1\right\}$ □

V9. Find a basis for the subspace

$$W = \operatorname{span}\left\{-3x^3-8x^2, x^3+2x^2+2, -x^2+3\right\}$$

of $\mathcal{P}^2$.

Solution. $\left\{-3x^3-8x^2, x^3+2x^2+2\right\}$ □

V9. Find a basis for the subspace

$$W = \operatorname{span}\left\{\begin{bmatrix}1 & -3\\ 2 & 2\end{bmatrix}, \begin{bmatrix}-1 & 4\\ -1 & 1\end{bmatrix}, \begin{bmatrix}0 & 3\\ 3 & 9\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 2 & 1\end{bmatrix}\right\}$$

of $M_{2,2}$.

Solution.

$$\left\{\begin{bmatrix}1 & -3\\ 2 & 2\end{bmatrix}, \begin{bmatrix}-1 & 4\\ -1 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 2 & 1\end{bmatrix}\right\}$$

□

V9. Find a basis for the subspace

$$W = \operatorname{span}\left\{x^3-3x^2+2x+2, -x^3+4x^2-x+1, 3x^2+3x+9, -x^3+2x+1\right\}$$

of $\mathcal{P}^3$.

Solution.

$$\left\{x^3-3x^2+2x+2, -x^3+4x^2-x+1, -x^3+2x+1\right\}$$

□

V9. Find a basis for the subspace

$$W = \operatorname{span}\left\{x^3-x, x^2+x+1, x^3-x^2+2, 2x^2-1\right\}$$

of $\mathcal{P}^3$.

Solution.

$$\left\{x^3-x, x^2+x+1, x^3-x^2+2\right\}$$

□

V9. Let W be the subspace of $\mathcal{P}^3$ given by

$$W = \operatorname{span}\left(\left\{x^3+x^2+2x+1, 3x^3+3x^2+6x+3, 3x^3-x^2+3x-2, 7x^3-x^2+8x-3\right\}\right).$$

Find a basis for W.

Solution.

$$\operatorname{RREF}\left(\begin{bmatrix}1 & 3 & 3 & 7\\ 1 & 3 & -1 & -1\\ 2 & 6 & 3 & 8\\ 1 & 3 & -2 & -3\end{bmatrix}\right) = \begin{bmatrix}1 & 3 & 0 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Then a basis is $\left\{x^3+x^2+2x+1, 3x^3-x^2+3x-2\right\}$. □

V9. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\}$. Find a basis for this vector space.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis of W. $\square$

V9. Let W be the subspace of $\mathcal{P}^2$ given by $W = \operatorname{span}\left(\left\{-3x^2 - 8x, x^2 + 2x + 2, -x + 3\right\}\right)$. Find a basis for W.

Solution. Let $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$, and compute $\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$. Since the first two columns are pivot columns, $\left\{-3x^2 - 8x, x^2 + 2x + 2\right\}$ is a basis for W. $\square$

Standard V10

V10. Find a basis for the solution space of the homogeneous system of equations

$$\begin{aligned} x + 3y + 3z + 7w &= 0 \\ x + 3y - z - w &= 0 \\ 2x + 6y + 3z + 8w &= 0 \\ x + 3y - 2z - 3w &= 0 \end{aligned}$$

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution space is

$$\left\{ \begin{bmatrix} -3a - b \\ a \\ -2b \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

So a basis for the solution space is

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

□

V10. Find a basis for the solution space of the homogeneous system of equations

$$\begin{aligned} x + 2y + 3z + w &= 0 \\ 3x - y + z + w &= 0 \\ 2x - 3y - 2z &= 0 \\ -x + 2z + 5w &= 0 \end{aligned}$$

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 1 & -2 & 3 & 1 \\ 3 & -1 & 1 & 1 \\ 2 & -3 & -2 & 0 \\ -1 & 0 & 2 & 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution space is

$$\left\{ \begin{bmatrix} a \\ 2a \\ -2a \\ a \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

So a basis for the solution space is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$. □

V10. Find a basis for the solution space of the homogeneous system of equations

$$\begin{aligned} x + 2y + 3z + w &= 0 \\ 3x - y + z + w &= 0 \\ 2x - 3y - 2z &= 0 \end{aligned}$$

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix}1 & -2 & 3 & 1\\ 3 & -1 & 1 & 1\\ 2 & -3 & -2 & 0\end{bmatrix}\right)=\begin{bmatrix}1 & 0 & \frac{5}{7} & \frac{3}{7}\\ 0 & 1 & \frac{8}{7} & \frac{2}{7}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Then the solution space is

$$\left\{\begin{bmatrix}-\frac{5}{7}a-\frac{3}{7}b\\ -\frac{8}{7}a-\frac{2}{7}b\\ a\\ b\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$$

So a basis for the solution space is $\left\{\begin{bmatrix}-\frac{5}{7}\\ -\frac{8}{7}\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-\frac{3}{7}\\ -\frac{2}{7}\\ 0\\ 1\end{bmatrix}\right\}$, or $\left\{\begin{bmatrix}5\\ 8\\ -7\\ 0\end{bmatrix},\begin{bmatrix}3\\ 2\\ 0\\ -7\end{bmatrix}\right\}$. □

V10. Find a basis for the solution space to the system of equations

$$\begin{aligned}x+2y-3z&=0\\ 2x+y-4z&=0\\ 3y-2z&=0\\ x-y-z&=0\end{aligned}$$

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix}1 & 2 & -3\\ 2 & 1 & -4\\ 0 & 3 & -2\\ 1 & -1 & -1\end{bmatrix}\right)=\begin{bmatrix}1 & 0 & -\frac{5}{3}\\ 0 & 1 & -\frac{2}{3}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Then the solution space is

$$\left\{\begin{bmatrix}\frac{5}{3}a\\ \frac{2}{3}a\\ a\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$$

So a basis is $\left\{\begin{bmatrix}\frac{5}{3}\\ \frac{2}{3}\\ 1\end{bmatrix}\right\}$ or $\left\{\begin{bmatrix}5\\ 2\\ 3\end{bmatrix}\right\}$. □

V10. Find a basis for the solution space to the homogeneous system of equations

$$\begin{aligned}2x_1+3x_2-5x_3+14x_4&=0\\ x_1+x_2-x_3+5x_4&=0\end{aligned}$$

Solution. Let $A=\left[\begin{array}{cccc|c}2 & 3 & -5 & 14 & 0\\ 1 & 1 & -1 & 5 & 0\end{array}\right]$, so $\mathrm{RREF}\,A=\left[\begin{array}{cccc|c}1 & 0 & 2 & 1 & 0\\ 0 & 1 & -3 & 4 & 0\end{array}\right]$. It follows that the basis for the solution space is given by $\left\{\begin{bmatrix}-2\\ 3\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-1\\ -4\\ 0\\ 1\end{bmatrix}\right\}$. □

V10. Find a basis for the solution space to the homogeneous system of equations

$$\begin{aligned}4x_1+4x_2+3x_3-6x_4&=0\\ -2x_3-4x_4&=0\\ 2x_1+2x_2+x_3-4x_4&=0\end{aligned}$$

Solution. Let $A = \left[\begin{array}{cccc|c} 4 & 4 & 3 & -6 & 0 \\ 0 & 0 & -2 & -4 & 0 \\ 2 & 2 & 1 & -4 & 0 \end{array}\right]$, so $\operatorname{RREF} A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. It follows that the basis for the solution space is given by $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$. □

V10. Find a basis for the solution space to the homogeneous system of equations given by

$$\begin{aligned} 3x + 2y + z &= 0 \\ x + y + z &= 0 \end{aligned}$$

Solution. Let $A = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$, so $\operatorname{RREF} A = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$. It follows that the basis for the solution space is given by $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$. □

V10. Find a basis for the solution space to the homogeneous system of equations given by

$$\begin{aligned} 2x_1 - 2x_2 + 6x_3 - x_4 &= 0 \\ 3x_1 + 6x_3 + x_4 &= 0 \\ -4x_1 + x_2 - 9x_3 + 2x_4 &= 0 \end{aligned}$$

Solution. Let $A = \left[\begin{array}{cccc|c} 2 & -2 & 6 & -1 & 0 \\ 3 & 0 & 6 & 1 & 0 \\ -4 & 1 & -9 & 2 & 0 \end{array}\right]$, so $\operatorname{RREF} A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$. It follows that the basis for the solution space is given by $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. □

Module A

Standard A1

A1. Consider the following maps of polynomials $S : \mathcal{P}^6 \to \mathcal{P}^6$ and $T : \mathcal{P}^6 \to \mathcal{P}^6$ defined by

$$S(f(x)) = f(x) + 3 \text{ and } T(f(x)) = f(x) + f(3).$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. T is linear, S is not. ☐

A1. Consider the following maps of polynomials $S : \mathcal{P}^4 \to \mathcal{P}^5$ and $T : \mathcal{P}^4 \to \mathcal{P}^5$ defined by

$$S(f(x)) = xf(x) - f(1) \text{ and } T(f(x)) = xf(x) - x.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. ☐

A1. Consider the following maps of polynomials $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ defined by

$$S(f(x)) = f'(x) - f''(x) \text{ and } T(f(x)) = f(x) - (f(x))^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. ☐

A1. Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^4$ and $T : \mathcal{P}^2 \to \mathcal{P}^4$ defined by

$$S(f(x)) = x^2 f(x) \text{ and } T(f(x)) = (f(x))^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. ☐

A1. Consider the following maps of polynomials $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ defined by

$$S(f(x)) = (f(x))^2 + 1 \text{ and } T(f(x)) = (x^2 + 1)f(x).$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. T is linear, S is not. ☐

A1. Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^2$ and $T : \mathcal{P}^2 \to \mathcal{P}^2$ defined by

$$S(ax^2 + bx + c) = cx^2 + bx + a \text{ and } T(ax^2 + bx + c) = a^2x^2 + b^2x + c^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. ☐

A1. Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^1$ and $T : \mathcal{P}^2 \to \mathcal{P}^1$ defined by

$$S(ax^2 + bx + c) = 2ax + b \text{ and } T(ax^2 + bx + c) = a^2x + b.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. ☐

A1. Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^3$ and $T : \mathcal{P}^2 \to \mathcal{P}^3$ defined by

$$S(ax^2 + bx + c) = ax^3 + bx^2 + cx \text{ and } T(ax^2 + bx + c) = abcx^3.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. ☐

Standard A2

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x+y \\ -8x+2y-z \\ 2y+3z \\ 0 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation given by the requirements

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 3 \end{bmatrix} \text{ and } T(\vec{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x+y \\ -8x+2y-z \\ 7x+2y+3z \\ 0 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x+y \\ -8x+2y-z \\ 2y+3z \\ 7x \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix}-2\\1\\3\end{bmatrix}\right)$

Solution.

$$\begin{bmatrix}3 & 1 & 0\\-8 & 2 & -1\\0 & 2 & 3\\7 & 0 & 0\end{bmatrix}$$

□

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$\begin{bmatrix}3 & 1 & 0\\-8 & 2 & -1\\0 & 2 & 3\\7 & 0 & 0\end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix}-2\\1\\3\end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$\begin{bmatrix}1 & 0 & 3 & 0\\0 & 3 & -5 & 0\end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix}x\\y\\z\\w\end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix}1\\3\\-1\\2\end{bmatrix}\right)$

Solution. □

A2. Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation given by the following

$$T(\vec{e}_1) = \begin{bmatrix}1\\0\\3\\0\end{bmatrix} \qquad T(\vec{e}_2) = \begin{bmatrix}0\\3\\-5\\0\end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix}2\\-3\end{bmatrix}\right)$

Solution. $\square$

A2. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - x_3 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - 5x_3 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 + 3x_3 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}\right)$

Solution.

$$\begin{bmatrix} 0 & 1 & 3 \end{bmatrix}$$

$\square$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 0 & 1 & 3 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by the following

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\3\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\0\\0\end{bmatrix}$$

(a) Compute $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix}1\\3\end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_3 + 3x_1\end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix}-1\\7\\3\end{bmatrix}\right)$

Solution.

$$\begin{bmatrix}3 & 0 & 1\end{bmatrix}$$

□

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by the standard matrix

$$\begin{bmatrix}3 & 0 & 1\end{bmatrix}$$

(a) Compute $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix}-2\\4\\3\end{bmatrix}\right)$

Standard A3

A4. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+3y+3z+7w \\ x+3y-z-w \\ 2x+6y+3z+8w \\ x+3y-2z-3w \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of T.
Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis for the kernel is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

and a basis for the image is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}$$

□

A4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x+y \\ -8x+2y-z \\ 2y+3z \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of T.

Solution. Let $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$, and compute $\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$. Then a basis for the image is its columns,

$$\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

And the kernel is the solution set of $AX = 0$, so a basis would be

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}$$

□

A4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map given by $T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 8x-3y-z+4w \\ y+3z-4w \\ -7x+3y+2z-5w \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 8 & -3 & -1 & 4 \\ 0 & 1 & 3 & -4 \\ -7 & 3 & 2 & -5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{\begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}\right\}$ is a basis for the image, and $\left\{\begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for the kernel. □

A4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 8x - 3y - z \\ y + 3z \\ -7x + 3y + 2z \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 8 & -3 & -1 \\ 0 & 1 & 3 \\ -7 & 3 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{\begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}\right\}$ is a basis for the image, and $\left\{\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}\right\}$ is a basis for the kernel. □

A4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by the standard matrix $\begin{bmatrix} 1 & 2 & 5 \\ 0 & -2 & -4 \\ -3 & 1 & -1 \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 2 & 5 \\ 0 & -2 & -4 \\ -3 & 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}$ is a basis for the image, and $\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\}$ is a basis for the kernel. □

A4. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear map given by the standard matrix $\begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & -2 & -4 & 2 \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & -2 & -4 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

Thus $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}\right\}$ is a basis for the image, and $\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for the kernel. □

A4. Let $T : \mathbb{R}^2 \to \mathbb{R}^5$ be the linear map given by the standard matrix $\begin{bmatrix} 3 & 1 \\ 0 & 0 \\ -6 & -2 \\ 1 & \frac{1}{3} \\ 9 & 3 \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 3 & 1 \\ 0 & 0 \\ -6 & -2 \\ 1 & \frac{1}{3} \\ 9 & 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus $\left\{\begin{bmatrix} 3 \\ 0 \\ -6 \\ 1 \\ 9 \end{bmatrix}\right\}$ is a basis for the image, and $\left\{\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right\}$ is a basis for the kernel. □

A4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map given by the standard matrix $\begin{bmatrix} 2 & 1 & 0 & 3 \\ -4 & -2 & 0 & 5 \\ -2 & -1 & 0 & 0 \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 2 & 1 & 0 & 3 \\ -4 & -2 & 0 & 5 \\ -2 & -1 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{\begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}\right\}$ is a basis for the image, and $\left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis for the kernel. □

Standard A4

A3. Determine if the following linear maps are injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y \\ x - y \\ x + 4y \end{bmatrix}$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ 2y + 3z \\ x - y - 2z \end{bmatrix}$

Solution.

(a)

$$\text{RREF}\left(\begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 1 & 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since all columns are pivot columns, S is injective. Since there is a zero row, S is not surjective.

(b)

$$\text{RREF}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, T is not injective. Since there is a zero row, T is not surjective.

□

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ 2y + 3z \\ x - y - 2z \end{bmatrix}$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y \\ x - y \\ x + 4y \end{bmatrix}$

Solution.

(a)

$$\text{RREF}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, S is not injective. Since there is a zero row, S is not surjective.

(b)

$$\text{RREF}\left(\begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 1 & 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since all columns are pivot columns, T is injective. Since there is a zero row, T is not surjective.

□

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S:\mathbb{R}^2\to\mathbb{R}^2$ given by the standard matrix $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$.

(b) $T:\mathbb{R}^4\to\mathbb{R}^3$ given by the standard matrix $\begin{bmatrix}2 & 3 & -1 & -2\\ 0 & 1 & 3 & 1\\ 2 & 1 & -7 & -4\end{bmatrix}$

Solution.

(a) RREF $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Since each column is a pivot column, S is injective. Since there is no zero row, S is surjective.

(b) Since $\dim\mathbb{R}^4 > \dim\mathbb{R}^3$, T is not injective.

$$\text{RREF}\left(\begin{bmatrix}2 & 3 & -1 & -2\\ 0 & 1 & 3 & 1\\ 2 & 1 & -7 & -4\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & -5 & -\frac{5}{2}\\ 0 & 1 & 3 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Since there are only two pivot columns, T is not surjective.

□

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S:\mathbb{R}^2\to\mathbb{R}^2$ given by the standard matrix $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$.

(b) $T:\mathbb{R}^4\to\mathbb{R}^3$ given by the standard matrix $\begin{bmatrix}2 & 3 & -1 & -2\\ 0 & 1 & 4 & 1\\ 2 & 1 & -7 & -4\end{bmatrix}$

Solution.

(a) RREF $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Since each column is a pivot column, S is injective. Since there is no zero row, S is surjective.

(b) Since $\dim\mathbb{R}^4 > \dim\mathbb{R}^3$, T is not injective.

$$\text{RREF}\left(\begin{bmatrix}2 & 3 & -1 & -2\\ 0 & 1 & 3 & 1\\ 2 & 1 & -7 & -4\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0 & -\frac{5}{2}\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$$

Since there are no zero rows, T is surjective.

□

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S:\mathbb{R}^2\to\mathbb{R}^4$ given by the standard matrix $\begin{bmatrix}2 & 1\\ 1 & 2\\ 0 & 1\\ 3 & -3\end{bmatrix}$.

(b) $T:\mathbb{R}^4\to\mathbb{R}^3$ given by the standard matrix $\begin{bmatrix}2 & 3 & -1 & 1\\ -1 & 1 & 1 & 1\\ 4 & 11 & -1 & 5\end{bmatrix}$

Solution.

(a) $\begin{bmatrix}2 & 1\\1 & 2\\0 & 1\\3 & -3\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{bmatrix}$. Since each column is a pivot column, S is injective. Since there a no zero row, S is not surjective.

(b) Since $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$, T is not injective.

$$\mathrm{RREF}\left(\begin{bmatrix}2 & 3 & -1 & 1\\-1 & 1 & 1 & 1\\4 & 11 & -1 & 5\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & -\frac{4}{5} & -\frac{2}{5}\\0 & 1 & \frac{1}{5} & \frac{3}{5}\\0 & 0 & 0 & 0\end{bmatrix}$$

Since there is a zero row, T is not surjective.

□

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^4$ given by the standard matrix $\begin{bmatrix}2 & 1\\1 & 2\\0 & 1\\3 & -3\end{bmatrix}$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the standard matrix $\begin{bmatrix}2 & 3 & -1 & 1\\-1 & 1 & 1 & 1\\4 & 7 & -1 & 5\end{bmatrix}$

Solution.

(a) $\begin{bmatrix}2 & 1\\1 & 2\\0 & 1\\3 & -3\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{bmatrix}$. Since each column is a pivot column, S is injective. Since there a no zero row, S is not surjective.

(b) Since $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$, T is not injective.

$$\mathrm{RREF}\left(\begin{bmatrix}2 & 3 & -1 & 1\\-1 & 1 & 1 & 1\\4 & 7 & -1 & 5\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0 & 2\\0 & 1 & 0 & 0\\0 & 0 & 1 & 3\end{bmatrix}$$

Since there is not a zero row, T is surjective.

□

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^3$ where $S(\vec{e}_1) = \begin{bmatrix}2\\1\\0\end{bmatrix}$ and $S(\vec{e}_2) = \begin{bmatrix}1\\2\\1\end{bmatrix}$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ where $T(\vec{e}_1) = \begin{bmatrix}2\\2\end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix}1\\0\end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix}1\\1\end{bmatrix}$.

Solution.

(a) RREF $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. The map is injective since every column has a pivot, but is not surjective because there is a row without a pivot.

(b) RREF $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$. The map is not injective since there is a column without a pivot, but it is surjective because every row has a pivot.

□

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^4 \to \mathbb{R}^3$ where $S(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $S(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $S(\vec{e}_3) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, and $S(\vec{e}_4) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$,

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(\vec{e}_1) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Solution.

(a) RREF $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The map is not injective since it has a column without pivot, but it is surjective because every row has a pivot.

(b) RREF $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The map is not injective since there is a column without a pivot, and it is not surjective because there is a row without a pivot.

□

Module M

Standard M1

M1. Let

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 0 & 7 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.
Solution. CA is the only one that can be computed, and

$$CA = \begin{bmatrix} 3 & 9 & 11 & 1 \\ 0 & 0 & 7 & 2 \\ -2 & -6 & -5 & 0 \end{bmatrix}$$

□

M1. Let

$$A = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 3 & 7 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.
Solution. CA is the only one that can be computed, and

$$CA = \begin{bmatrix} 2 & 6 & 11 & 1 \\ 1 & 3 & 7 & 2 \\ -1 & -3 & -5 & 0 \end{bmatrix}$$

□

M1. Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & -1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.
Solution. AC is the only one that can be computed, and

$$AC = \begin{bmatrix} 3 & -5 & 14 \\ 1 & -1 & 2 \end{bmatrix}$$

□

M1. Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.
Solution. AC is the only one that can be computed, and

$$AC = \begin{bmatrix} 9 & -2 & 14 \\ 1 & 0 & 2 \end{bmatrix}$$

□

M1. Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. AB is the only ones that can be computed, and

$$AB = \begin{bmatrix} -3 & -5 & 6 & 14 \\ 0 & 0 & 7 & 35 \end{bmatrix}$$

□

M1. Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. BC is the only one that can be computed, and

$$BC = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

□

M1. Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. CB is the only one that can be computed, and

$$CB = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

□

M1. Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & -1 & 3 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. CB is the only one that can be computed, and

$$CB = \begin{bmatrix} 3 & 3 & -5 & 7 \\ 4 & -4 & 12 & -12 \\ 7 & 2 & 0 & 3 \end{bmatrix}$$

□

Standard M2

M2. Consider the two row operations $R_2 - 4R_1 \to R_2$ and $\frac{1}{3}R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 4-4(1) & 5-4(2) & 6-4(3) \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ -\frac{1}{3}(0) & -\frac{1}{3}(-3) & -\frac{1}{3}(-6) \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 7 & 8 & 9 \end{bmatrix} = B$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A.

Solution.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

□

M2. Consider the two row operations $R_1 \leftrightarrow R_2$ and $R_3 + R_1 \to R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 2 & 3 \\ -1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ -1 & 4 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ -1+1 & 4+2 & 5+3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ 0 & 6 & 8 \end{bmatrix} = B$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A.

Solution.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

□

M2. Consider the two row operations $\frac{1}{2}R_3 \to R_3$ and $R_1 \leftrightarrow R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} -3 & 4 & 0 \\ -7 & 2 & 3 \\ 2 & -8 & 6 \end{bmatrix} \sim \begin{bmatrix} -3 & 4 & 0 \\ -7 & 2 & 3 \\ \frac{1}{2}(2) & \frac{1}{2}(-8) & \frac{1}{2}(6) \end{bmatrix} = \begin{bmatrix} -3 & 4 & 0 \\ -7 & 2 & 3 \\ 1 & -4 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 3 \\ -7 & 2 & 3 \\ -3 & 4 & 0 \end{bmatrix} = B$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A.

Solution.

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} A$$

□

M2. Consider the two row operations $R_1 \leftrightarrow R_3$ and $3R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} -3 & 4 & 0 \\ -7 & 2 & 3 \\ 2 & -8 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -8 & 6 \\ -7 & 2 & 3 \\ -3 & 4 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & -8 & 6 \\ 3(-7) & 3(2) & 3(3) \\ -3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -8 & 6 \\ -21 & 6 & 9 \\ -3 & 4 & 0 \end{bmatrix} = B$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A.

Solution.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

□

M2. Consider the two row operations $-2R_1 \to R_1$ and $R_3 + 2R_1 \to R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} -\frac{1}{2} & 0 & 3 \\ -4 & 2 & 3 \\ -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} -2(-\frac{1}{2}) & -2(0) & -2(3) \\ -4 & 2 & 3 \\ -2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ -4 & 2 & 3 \\ -2 & 5 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & -6 \\ -4 & 2 & 3 \\ -2+2(1) & 5+2(0) & 0+2(-6) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ -4 & 2 & 3 \\ 0 & 5 & -12 \end{bmatrix} = B$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A.

Solution.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

□

M2. Consider the two row operations $R_2 - 4R_3 \to R_2$ and $R_1 \leftrightarrow R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 0 & 3 & -1 \\ -4 & 2 & 3 \\ -1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & -1 \\ -4-4(-1) & 2-4(4) & 3-4(5) \\ -1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -1 \\ 0 & -14 & -17 \\ -1 & 4 & 5 \end{bmatrix}$$
$$\sim \begin{bmatrix} -1 & 4 & 5 \\ 0 & -14 & -17 \\ 0 & 3 & -1 \end{bmatrix} = B$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A.

Solution.

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} A$$

□

Standard M3

M3. Determine if the matrix $\begin{bmatrix}1 & 3 & 3 & 7\\ 1 & 3 & -1 & -1\\ 2 & 6 & 3 & 8\\ 1 & 3 & -2 & -3\end{bmatrix}$ is invertible.

Solution. The second column is a multiple of the first, so it is not invertible. □

M3. Determine if the matrix $\begin{bmatrix}-3 & 1 & 0\\ -8 & 2 & -1\\ 0 & 2 & 3\end{bmatrix}$ is invertible.

Solution.

$$\text{RREF}\begin{bmatrix}-3 & 1 & 0\\ -8 & 2 & -1\\ 0 & 2 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{3}{2}\\ 0 & 0 & 0\end{bmatrix}$$

Since it is not equivalent to the identity matrix, it is not invertible. □

M3. Determine if the matrix $\begin{bmatrix}1 & 3 & -1\\ 2 & 7 & 0\\ -1 & -1 & 5\end{bmatrix}$ is invertible.

Solution.

$$\text{RREF}\begin{bmatrix}1 & 3 & -1\\ 2 & 7 & 0\\ -1 & -1 & 5\end{bmatrix} = \begin{bmatrix}1 & 0 & -7\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}$$

Since it is not equivalent to the identity matrix, it is not invertible. □

M3. Determine if the matrix $\begin{bmatrix}3 & -1 & 0 & 4\\ 2 & 1 & 1 & -1\\ 0 & 1 & 1 & 3\\ 1 & -2 & 0 & 0\end{bmatrix}$ is invertible.

Solution. This matrix is row equivalent to the identity matrix, so it is invertible. □

M3. Determine if the matrix $\begin{bmatrix}3 & -1 & 0 & 4\\ 2 & 1 & 1 & 1\\ 0 & 1 & 1 & -1\\ 1 & -2 & 0 & 3\end{bmatrix}$ is invertible.

Solution.

$$\text{RREF}\begin{bmatrix}3 & -1 & 0 & 4\\ 2 & 1 & 1 & 1\\ 0 & 1 & 1 & -1\\ 1 & -2 & 0 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

This matrix is not row equivalent to the identity matrix, so it is not invertible. □

M3. Determine if the matrix $\begin{bmatrix}3 & -1 & 0\\ 2 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}$ is invertible.

Solution. It is row equivalent to the identity matrix, so it is invertible. □

M3. Determine if the matrix $\begin{bmatrix}2 & 1 & 0 & 3\\ 1 & -1 & 3 & 1\\ 3 & 2 & -1 & 7\\ 4 & 1 & 2 & 0\end{bmatrix}$ is invertible.

Solution.

$$\text{RREF}\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since it is not row equivalent to the identity matrix, it is not invertible. □

M3. Determine if the matrix $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 0 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix}$ is invertible.

Solution.

$$\text{RREF}\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 0 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since it is row equivalent to the identity matrix, it is invertible. □

Standard M4

M4. Show how to find the inverse of the matrix $\begin{bmatrix} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$.

Solution.

$$\text{RREF}\left(\left[\begin{array}{cccc|cccc} 8 & 5 & 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 5 & -3 & 1 & -2 & 0 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -5 & 12 \\ 0 & 1 & 0 & 0 & 1 & 1 & -4 & -9 \\ 0 & 0 & 1 & 0 & -4 & -7 & 20 & 47 \\ 0 & 0 & 0 & 1 & -1 & 0 & 3 & 7 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 1 & 2 & -5 & 12 \\ 1 & 1 & -4 & -9 \\ -4 & -7 & 20 & 47 \\ -1 & 0 & 3 & 7 \end{bmatrix}$. □

M4. Compute the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution.

$$\text{RREF}(A|I) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -11 & 37 \\ 0 & 1 & 0 & 0 & 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 1 & 2 & -11 & 37 \\ 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ □

M4. Show how to find the inverse of the matrix $\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 4 & -3 \\ 8 & -13 & 10 \\ 13 & -24 & 18 \end{bmatrix}$$

□

M4. Show how to find the inverse of the matrix $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & \frac{3}{2} & -\frac{3}{2} \\ 1 & -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

□

M4. Show how to find the inverse of the matrix $\begin{bmatrix} 4 & -1 & -8 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Solution. $\left[\begin{array}{ccc|ccc} 4 & -1 & -8 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 5 \\ 0 & 1 & 0 & -5 & 24 & -28 \\ 0 & 0 & 1 & 1 & -5 & 6 \end{array}\right]$. Thus the inverse is $\begin{bmatrix} 1 & -4 & 5 \\ -5 & 24 & -28 \\ 1 & -5 & 6 \end{bmatrix}$. □

M4. Show how to find the inverse of the matrix $\begin{bmatrix} 1 & -4 & 5 \\ -5 & 24 & -28 \\ 1 & -5 & 6 \end{bmatrix}$.

Solution. $\left[\begin{array}{ccc|ccc} 1 & -4 & 5 & 1 & 0 & 0 \\ -5 & 24 & -28 & 0 & 1 & 0 \\ 1 & -5 & 6 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -8 \\ 0 & 1 & 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 & 4 \end{array}\right]$. Thus the inverse is $\begin{bmatrix} 4 & -1 & -8 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$. □

M4. Show how to find the inverse of the matrix $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$.

Solution. $\left[\begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 2 & -1 & -6 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -3 \\ 0 & 1 & 0 & -14 & 9 & 24 \\ 0 & 0 & 1 & 3 & -2 & -5 \end{array}\right]$. Thus the inverse is $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$. □

M4. Show how to find the inverse of the matrix $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$.

Solution. $\left[\begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 2 & -1 & -6 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -3 \\ 0 & 1 & 0 & -14 & 9 & 24 \\ 0 & 0 & 1 & 3 & -2 & -5 \end{array}\right]$. Thus the inverse is $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$. □

M4. Show how to find the inverse of the matrix $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$.

Solution. $\left[\begin{array}{ccc|ccc} 2 & -1 & -3 & 1 & 0 & 0 \\ -14 & 9 & 24 & 0 & 1 & 0 \\ 3 & -2 & -5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3 \\ 0 & 1 & 0 & 2 & -1 & -6 \\ 0 & 0 & 1 & 1 & 1 & 4 \end{array}\right]$. Thus the inverse is $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$. □

Module G

Standard G1

G1. Consider the row operation $R_1 + 5R_3 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1+5(7) & 2+5(8) & 3+5(9) \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = 4$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (1)(4) = 4$.

□

G1. Consider the row operation $R_2 - 4R_3 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 4-4(7) & 5-4(8) & 6-4(9) \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = 7$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (1)(7) = 7$.

□

G1. Consider the row operation $R_3 - 2R_1 \to R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7-2(1) & 8-2(2) & 9-2(3) \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = -8$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (1)(-8) = -8$.

□

G1. Consider the row operation $4R_3 \to R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ (4)7 & (4)8 & (4)9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = -12$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (4)(-12) = -48$.

□

G1. Consider the row operation $-8R_1 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} (-8)1 & (-8)2 & (-8)3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = -2$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} -8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (-8)(-2) = 16$.

□

G1. Consider the row operation $5R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ (5)4 & (5)5 & (5)6 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = 3$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (5)(3) = 15$.

$\square$

G1. Consider the row operation that swaps R_1 and R_2 applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = 3$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
2. $\det(RC) = \det(R)\det(C) = (-1)(3) = -3$.

$\square$

G1. Consider the row operation that swaps R_3 and R_2 applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = -7$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.
2. $\det(RC) = \det(R)\det(C) = (-1)(-7) = 7$.

$\square$

G1. Consider the row operation that swaps R_3 and R_1 applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = -11$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.
2. $\det(RC) = \det(R)\det(C) = (-1)(-11) = 11$.

$\square$

Standard G2

G2. Compute the determinant of the matrix $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$.

Solution.

$$\det \begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix} + (-2)\det \begin{bmatrix} 3 & 0 & 4 \\ 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = -1(-4) + (-2)(20) = -36$$

□

G2. Compute the determinant of the matrix $\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix}$.

Solution.

$$\begin{aligned} \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} &= 2\det \begin{bmatrix} 3 & 0 & -1 \\ 1 & 3 & 1 \\ -3 & -2 & -1 \end{bmatrix} - (-1)\det \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 1 & -3 & -2 \end{bmatrix} \\ &= 2\left(3\det \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} + (-1)\det \begin{bmatrix} 1 & 3 \\ -3 & -2 \end{bmatrix}\right) + \left(1\det \begin{bmatrix} 1 & 3 \\ -3 & -2 \end{bmatrix} - 3\begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}\right) \\ &= 2\left(3(-1) + (-1)(7)\right) + \left((1)(7) - 3(-3)\right) \\ &= 2(-10) + 16 \\ &= -4 \end{aligned}$$

□

G2. Compute the determinant of the matrix $\begin{bmatrix} 2 & 3 & 0 & 1 \\ -1 & 3 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 1 & -1 & 3 & 5 \end{bmatrix}$.

Solution. -60. □

G2. Compute the determinant of the matrix $\begin{bmatrix} 3 & -1 & 0 & 7 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 3 & 0 & 0 & 1 \end{bmatrix}$.

Solution. 2. □

G2. Compute the determinant of the matrix $\begin{bmatrix} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$.

Solution. -1. □

G2. Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 0 & -4 & 0 \\ -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution. 15. $\square$

G2. Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ -2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution. -15. $\square$

G2. Compute the determinant of the matrix

$$\begin{bmatrix} 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 0 \end{bmatrix}.$$

Solution. 55. $\square$

G2. Compute the determinant of the matrix

$$\begin{bmatrix} 0 & -4 & 1 & 1 \\ -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 0 & -4 & 0 \end{bmatrix}.$$

Solution. -55. $\square$

Standard G3

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 5 & 1 \\ -24 & -6 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$, yielding the eigenvalues -3 and 2. $\square$

G3. Find the eigenvalues of the matrix $\begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 6\lambda + 5 = (\lambda + 1)(\lambda + 5)$, yielding the eigenvalues -1 and -5. $\square$

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$, yielding the eigenvalues 1 and 5. $\square$

G3. Find the eigenvalues of the matrix $\begin{bmatrix} -2 & 5 \\ 5 & -2 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 4\lambda - 21 = (\lambda + 7)(\lambda - 3)$, yielding the eigenvalues -7 and 3. $\square$

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 10 & -8 \\ 4 & -2 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - 8\lambda + 12 = (\lambda - 6)(\lambda - 2)$, yielding the eigenvalues 6 and 2. $\square$

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 6 & -4 \\ 11 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2)$, yielding the eigenvalues -5 and 2. $\square$

G3. Find the eigenvalues of the matrix $\begin{bmatrix} -6 & -11 \\ 4 & 9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$, yielding the eigenvalues 5 and -2. $\square$

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 1 & -5 \\ 4 & -8 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 7\lambda + 12 = (\lambda + 3)(\lambda + 4)$, yielding the eigenvalues -3 and -4. $\square$

G3. Find the eigenvalues of the matrix $\begin{bmatrix} -2 & 2 \\ 15 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 11\lambda - 12 = (\lambda + 12)(\lambda - 1)$, yielding the eigenvalues -12 and 1. $\square$

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 10 & 2 \\ -39 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3)$, yielding the eigenvalues -3 and 4. $\square$

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 8 & 2 \\ -33 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$, yielding the eigenvalues -3 and 2. $\square$

Standard G4

G4. Find a basis of the eigenspace associated to the eigenvalue -1 for the matrix $\begin{bmatrix} 4 & -2 & -1 & 1 \\ 15 & -7 & -3 & 1 \\ -5 & 2 & 0 & 1 \\ 10 & -4 & -2 & 0 \end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} \frac{2}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 1 for the matrix $A = \begin{bmatrix} -3 & 1 & 0 & 0 \\ -8 & 2 & 0 & -1 \\ 0 & 2 & 1 & 2 \\ 4 & 1 & 0 & 3 \end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 1 for the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 8 & -3 & -1 \\ 2 & 21 & -8 & -3 \\ 3 & -7 & 3 & 2 \end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} 0 \\ \frac{3}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 1 for the matrix
$A = \begin{bmatrix} 9 & -3 & -5 & 2 \\ 19 & -6 & -12 & 5 \\ 1 & 1 & -1 & 3 \\ -11 & 4 & 7 & -2 \end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 2 for the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 11 & -6 & 1 & -1 \\ -9 & 5 & 1 & 3 \end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 2 for the matrix $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 2 for the matrix
$A = \begin{bmatrix}0 & -2 & -1 & 0\\-4 & -2 & -2 & 0\\14 & 12 & 10 & 2\\-13 & -10 & -8 & -1\end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{\begin{bmatrix}-1\\\frac{1}{2}\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}\right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 3 for the matrix
$A = \begin{bmatrix}1 & -2 & -1 & 0\\-4 & -1 & -2 & 0\\14 & 12 & 11 & 2\\-14 & -10 & -9 & -1\end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{\begin{bmatrix}-1\\\frac{1}{2}\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}\right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue -2 for the matrix $A = \begin{bmatrix}0 & 0 & 0 & 2\\1 & 2 & -3 & 3\\1 & 8 & -9 & 6\\1 & 8 & -7 & 4\end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{\begin{bmatrix}-1\\\frac{1}{4}\\1\\1\end{bmatrix}\right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 2 for the matrix $\begin{bmatrix}-1 & 0 & 1 & 0\\6 & 2 & -2 & 0\\-9 & 0 & 5 & 0\\15 & 0 & -5 & 2\end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}\frac{1}{3}\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$. □