

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Module V: Vector Spaces

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

What is a vector space?

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

At the end of this module, students will be able to...

- V1. Vector property verification.** ... show why an example satisfies a given vector space property, but does not satisfy another given property.
- V2. Vector space identification.** ... list the eight defining properties of a vector space, infer which of these properties a given example satisfies, and thus determine if the example is a vector space.
- V3. Linear combinations.** ... determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors.
- V4. Spanning sets.** ... determine if a set of Euclidean vectors spans \mathbb{R}^n .
- V5. Subspaces.** ... determine if a subset of \mathbb{R}^n is a subspace or not.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems
E1,E2,E3.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Academy):
<http://bit.ly/2y8A0wa>
- Linear combinations of Euclidean vectors (Khan Academy):
<http://bit.ly/2nK3wne>
- Adding and subtracting complex numbers (Khan Academy):
<http://bit.ly/1PE3ZMQ>
- Adding and subtracting polynomials (Khan Academy):
<http://bit.ly/2d5SLGZ>

Module V Section 0

Activity V.0.1 (~ 20 min)

Consider each of the following vector properties. Label each property with \mathbb{R}^1 , \mathbb{R}^2 , and/or \mathbb{R}^3 if that property holds for Euclidean vectors/scalars \mathbf{u} , \mathbf{v} , \mathbf{w} of that dimension.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

1 Addition associativity.

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

2 Addition commutivity.

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

3 Addition identity.

There exists some \mathbf{z} where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.

4 Addition inverse.

There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.

5 Addition midpoint uniqueness.

There exists a unique \mathbf{m} where the distance from \mathbf{u} to \mathbf{m} equals the distance from \mathbf{m} to \mathbf{v} .

6 Scalar multiplication associativity.

$$a(b\mathbf{v}) = (ab)\mathbf{v}.$$

7 Scalar multiplication identity.

$$1\mathbf{v} = \mathbf{v}.$$

8 Scalar multiplication relativity.

There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.

9 Scalar distribution.

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

10 Vector distribution.

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

11 Orthogonality.

There exists a non-zero vector \mathbf{n} such that \mathbf{n} is orthogonal to both \mathbf{u} and \mathbf{v} .

12 Bidimensionality.

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} \text{ for some value of } a, b.$$

Definition V.0.2

A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V , and let a, b be scalar numbers.

- **Addition associativity.**
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition commutativity.**
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Addition identity.**
There exists some \mathbf{z} where
 $\mathbf{v} + \mathbf{z} = \mathbf{v}.$
- **Addition inverse.**
There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}.$
- **Scalar multiplication associativity.**
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **Scalar multiplication identity.**
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar distribution.**
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Vector distribution.**
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

Definition V.0.3

The most important examples of vector spaces are the **Euclidean vector spaces** \mathbb{R}^n , but there are other examples as well.

Module V Section 1

Definition V.1.1

A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V , and let a, b be scalar numbers.

- **Addition associativity.**
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition commutativity.**
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Addition identity.**
There exists some \mathbf{z} where
 $\mathbf{v} + \mathbf{z} = \mathbf{v}.$
- **Addition inverse.**
There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}.$
- **Scalar multiplication associativity.**
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **Scalar multiplication identity.**
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar distribution.**
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Vector distribution.**
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

Remark V.1.2

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- \mathbb{R}^n : Euclidean vectors with n components.
- \mathbb{R}^∞ : Sequences of real numbers (v_1, v_2, \dots) .
- $\mathbb{R}^{m \times n}$: Matrices of real numbers with m rows and n columns.
- \mathbb{C} : Complex numbers.
- \mathcal{P}^n : Polynomials of degree n or less.
- \mathcal{P} : Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity V.1.3 (~ 25 min)

Consider the following set that models motion along the curve $y = e^x$. Let

$V = \{(x, y) : y = e^x\}$. Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$.

Activity V.1.3 (~ 25 min)

Consider the following set that models motion along the curve $y = e^x$. Let

$V = \{(x, y) : y = e^x\}$. Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$.

Part 1: Which of the vector space properties are satisfied by V paired with these operations?

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some \mathbf{z} where

$$\mathbf{v} \oplus \mathbf{z} = \mathbf{v}.$$

- **Addition inverse.**

There exists some $-\mathbf{v}$ where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{z}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

Activity V.1.3 (~ 25 min)

Consider the following set that models motion along the curve $y = e^x$. Let

$V = \{(x, y) : y = e^x\}$. Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$.

Part 1: Which of the vector space properties are satisfied by V paired with these operations?

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some \mathbf{z} where

$$\mathbf{v} \oplus \mathbf{z} = \mathbf{v}.$$

- **Addition inverse.**

There exists some $-\mathbf{v}$ where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{z}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

Part 2: Is V a vector space?

Activity V.1.4 (~ 10 min)

Let $V = \{(a, b) \mid a, b \in \mathbb{R}\}$, where $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \oplus \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 + a_2 + b_2 \\ a_1^2 + b_1^2 \end{bmatrix}$ and

$c \odot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1^c \\ a_2 + c \end{bmatrix}$. Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some \mathbf{z} where

$$\mathbf{v} \oplus \mathbf{z} = \mathbf{v}.$$

- **Addition inverse.**

There exists some $-\mathbf{v}$ where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{z}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

Definition V.1.5

A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$ for any choice of scalar multiples c_1, c_2, \dots, c_m .

For example, we say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Definition V.1.6

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m \mid c_i \text{ is a real number}\}$$

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.1.7 (~ 10 min)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.1.7 (~ 10 min)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Part 1: Sketch $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.1.7 (~ 10 min)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Part 1: Sketch $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part 2: Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ in the xy plane.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.1.8 (~ 10 min)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.1.8 (~ 10 min)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Part 1: Sketch the following linear combinations in the xy plane: $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

$0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.1.8 (~ 10 min)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Part 1: Sketch the following linear combinations in the xy plane: $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

$0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Part 2: Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ in the xy plane.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.1.9 (~ 5 min)

Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ in the xy plane.

Module V Section 2

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.2.1 (*~15 min*)

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.2.1 (*~15 min*)

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.2.1 (*~15 min*)

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (Remember, you should use a calculator to help find RREF.)

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.2.1 (*~15 min*)

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (Remember, you should use a calculator to help find RREF.)

Part 3: Given this solution, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Fact V.2.2

A vector \mathbf{b} belongs to $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$ is consistent.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Remark V.2.3

To determine if \mathbf{b} belongs to $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, find $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.2.4 (*~10 min*)

Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.2.5 (*~5 min*)

Determine if $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Observation V.2.6

So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space \mathbb{R}^n ; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.2.7 (*~5 min*)

We previously checked that $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}$ does not belong to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \\ 2 \end{bmatrix} \right\}$.

Does $f(x) = 3x^3 - 2x^2 + x + 5$ belong to $\text{span}\{x^3 - 3x + 2, -x^3 - 3x^2 + 2 + 2x + 2\}$?

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.2.8 (*~10 min*)

Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to $\text{span} \left\{ \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \right\}$?

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.2.9 (~ 5 min)

Does the complex number $2i$ belong to $\text{span}\{-3 + i, 6 - 2i\}$?

Module V Section 3

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.3.1 (~ 5 min)

How many vectors are required to span \mathbb{R}^2 ? Sketch a drawing in the xy plane to support your answer.

- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) Infinitely Many

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.3.2 (*~5 min*)

How many vectors are required to span \mathbb{R}^3 ?

- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) Infinitely Many

Module V

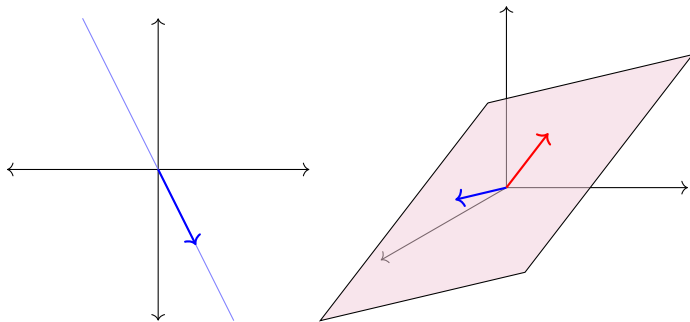
Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Fact V.3.3At least n vectors are required to span \mathbb{R}^n .

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.3.4 (~ 10 min)

Choose a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in \mathbb{R}^3 that is not in $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by ensuring

$$\left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]. \quad (\text{Why does this work?})$$

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Fact V.3.5

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ fails to span all of \mathbb{R}^n exactly when $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_m]$ has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & | & a \\ -1 & 0 & | & b \\ 0 & 1 & | & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix}$$

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.3.6 (~ 5 min)

Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does

 $\mathbb{R}^4 = \text{span } S$?

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.3.7 (*~10 min*)

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2\}$$

Does $\mathcal{P}^3 = \text{span } S$?

Module V Section 4

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Definition V.4.1

A subset of a vector space is called a **subspace** if it is itself a vector space.

Remark V.4.2

To prove that a subset S is a subspace of a vectorspace V , you need only verify that the operations on V restrict to the subset S ; that is you must check two things:

- The set is **closed under addition**: i.e. for any $\mathbf{x}, \mathbf{y} \in S$, $\mathbf{x} + \mathbf{y}$ is also in S .
- The set is **closed under scalar multiplication**: i.e. for any $\mathbf{x} \in S$ and scalar $c \in \mathbb{R}$, the product $c\mathbf{x}$ is also in S .

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.4.3 (*~15 min*)

$$\text{Let } S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}.$$

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.4.3 (~ 15 min)

$$\text{Let } S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}.$$

Part 1: Let $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. Show that if $\mathbf{v}, \mathbf{w} \in S$, then $\mathbf{v} + \mathbf{w} \in S$ as well.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.4.3 (~ 15 min)

$$\text{Let } S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}.$$

Part 1: Let $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. Show that if $\mathbf{v}, \mathbf{w} \in S$, then $\mathbf{v} + \mathbf{w} \in S$ as well.

Part 2: Let $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and let $c \in \mathbb{R}$. Show that if $\mathbf{v} \in S$, then $c\mathbf{v} \in S$ as well.

Therefore S is a subspace of \mathbb{R}^3

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.4.4 (*~10 min*)

Prove that $P = \{ax^2 + b \mid a, b \in \mathbb{R}\}$ is a subspace of the vector space of all degree-two polynomials by showing it is closed under addition and scalar multiplication.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.4.5 (*~10 min*)

Let P be the set of all positive real numbers. Determine if P is a subspace of \mathbb{R} or not.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Remark V.4.6

Since 0 is a scalar and $0\mathbf{v} = \mathbf{0}$ for any vector \mathbf{v} , a set that is closed under scalar multiplication must contain the zero vector.

Therefore, if a set does **not** contain the zero vector, it is **not** a subspace.

Module V

Section V.0

Section V.1

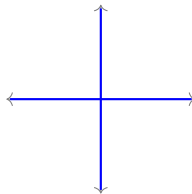
Section V.2

Section V.3

Section V.4

Activity V.4.7 (*~10 min*)

Consider the subset of \mathbb{R}^2 where at least one coordinate of each vector is 0.



Determine if this is a subspace of \mathbb{R}^2 or not.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.4.8 (*~5 min*)

Show that the set of 2×2 matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of $\mathbb{R}^{2 \times 2}$.

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Activity V.4.9 (~ 10 min)

Let W be a subspace of a vector space V . How are $\text{span } W$ and W related?

- (a) $\text{span } W$ is bigger than W
- (b) $\text{span } W$ is the same as W
- (c) $\text{span } W$ is smaller than W

Module V

Section V.0

Section V.1

Section V.2

Section V.3

Section V.4

Fact V.4.10

If S is a subset of a vector space V , then $\text{span } S$ is a subspace of V . In fact, it is the smallest subspace of V containing S .