

Name:
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Dr. Clontz

MIDTERM EXAM

Math 237 – Linear Algebra

Version 3

Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

Standard E1.	Mark:
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Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned}x_1 + 4x_3 &= 1 \\x_2 - x_3 &= 7 \\x_1 - x_2 + 3x_3 &= -1\end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array} \right]$$

□

Standard E2.	Mark:
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Put the following matrix in reduced row echelon form.

$$\left[\begin{array}{ccc} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{array} \right]$$

Solution:

$$\begin{aligned} \left[\begin{array}{ccc} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{array} \right] &\sim \left[\begin{array}{ccc} -1 & 0 & -1 \\ 3 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{array} \right] \sim \left[\begin{array}{ccc} 1 & 0 & 1 \\ 3 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{array} \right] \\ &\sim \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{array} \right] \sim \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \\ 0 & 2 & 6 \end{array} \right] \sim \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \end{aligned}$$

□

Standard E3.	Mark:
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Solve the system of equations

$$\begin{aligned} -3x + y &= 2 \\ -8x + 2y - z &= 6 \\ 2y + 3z &= -2 \end{aligned}$$

Solution:

$$\text{RREF} \left(\left[\begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array} \right] \right) = \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The solutions are

$$\left\{ \left[\begin{array}{c} -1 - \frac{c}{2} \\ -1 - \frac{3c}{2} \\ c \end{array} \right] \mid c \in \mathbb{R} \right\} = \left\{ \left[\begin{array}{c} c-1 \\ 3c-1 \\ -2c \end{array} \right] \mid c \in \mathbb{R} \right\}$$

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Standard E4.	Mark:
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Find a basis for the solution set of the system of equations

$$\begin{aligned} x + 2y + 3z + w &= 0 \\ 3x - y + z + w &= 0 \\ 2x - 3y - 2z &= 0 \end{aligned}$$

Solution:

$$\text{RREF} \left(\left[\begin{array}{cccc} 1 & -2 & 3 & 1 \\ 3 & -1 & 1 & 1 \\ 2 & -3 & -2 & 0 \end{array} \right] \right) = \left[\begin{array}{cccc} 1 & 0 & \frac{5}{7} & \frac{3}{7} \\ 0 & 1 & \frac{8}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Then the solution set is

$$\left\{ \left[\begin{array}{c} -\frac{5}{7}a - \frac{3}{7}b \\ -\frac{8}{7}a - \frac{2}{7}b \\ a \\ b \end{array} \right] \mid a, b \in \mathbb{R} \right\}$$

So a basis for the solution set is $\left\{ \left[\begin{array}{c} -\frac{5}{7} \\ -\frac{8}{7} \\ 1 \\ 0 \end{array} \right], \left[\begin{array}{c} -\frac{3}{7} \\ -\frac{2}{7} \\ 0 \\ 1 \end{array} \right] \right\}$, or $\left\{ \left[\begin{array}{c} 5 \\ 8 \\ -7 \\ 0 \end{array} \right], \left[\begin{array}{c} 3 \\ 2 \\ 0 \\ -7 \end{array} \right] \right\}$.

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Standard V1.	Mark:
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Let V be the set of all real numbers with the operations, for any $x, y \in V$, $c \in \mathbb{R}$,

$$x \oplus y = \sqrt{x^2 + y^2}$$

$$c \odot x = cx$$

- (a) Show that the vector **addition** \oplus is **associative**: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.
 (b) Determine if V is a vector space or not. Justify your answer.

Solution: Let $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned}
 (x \oplus y) \oplus z &= \sqrt{x^2 + y^2} \oplus z \\
 &= \sqrt{(\sqrt{x^2 + y^2})^2 + z^2} \\
 &= \sqrt{x^2 + y^2 + z^2} \\
 &= \sqrt{x^2 + (\sqrt{y^2 + z^2})^2} \\
 &= x \oplus \sqrt{y^2 + z^2} \\
 &= x \oplus (y \oplus z)
 \end{aligned}$$

However, this is not a vector space, as there is no zero vector.

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Standard V2.	Mark:
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Determine if $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}$.

Solution:

$$\text{RREF} \left(\left[\begin{array}{ccc|c} 2 & 1 & -3 & 1 \\ 3 & -1 & -2 & 4 \\ -1 & 0 & 5 & 3 \end{array} \right] \right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Since this system has a solution, $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is a linear combination of the three vectors.

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Standard V3.	Mark:
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Does $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$?

Solution: Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 4 & 2 \\ 4 & -9 & -3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has a zero row, the vectors fail to span \mathbb{R}^3 .

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Standard V4.	Mark:
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Let W be the set of all complex numbers that are purely real (i.e of the form $a + 0i$) or purely imaginary (i.e. of the form $0 + bi$). Determine if W is a subspace of \mathbb{C} .

Solution: No, because 1 is purely real and i is purely imaginary, but the linear combination $1 + i$ is neither.

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Standard S1.	Mark:
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Determine if the set of vectors $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 6 \\ 5 \end{bmatrix} \right\}$ is linearly dependent or linearly independent.

Solution:

$$\text{RREF} \left(\begin{bmatrix} 3 & 1 & 3 \\ -1 & 2 & -8 \\ 0 & -2 & 6 \\ 4 & 1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced row echelon form has a nonpivot column, the vectors are linearly dependent.

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Standard S2.	Mark:
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Determine if the set $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis of \mathbb{R}^4 .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.

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Standard S3.	Mark:
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Let $W = \text{span} \left(\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\} \right)$. Find a basis for W .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}$.

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Standard S4.	Mark:
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Let $W = \text{span} \left(\left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$. Compute the dimension of W .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$.

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Additional Notes/Marks	
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