

## Section V.2

**Remark V.2.1** Recall these definitions from last class:

- A **linear combination** of vectors is given by adding scalar multiples of those vectors, such as:

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- The **span** of a set of vectors is the collection of all linear combinations of that set, such as:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

**Activity V.2.2** (*~15 min*) The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when there exists a

solution to the vector equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

*Part 2:* Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.

*Part 3:* Given this solution set, does  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belong to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ ?

**Fact V.2.3** A vector  $\mathbf{b}$  belongs to  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  if and only if the linear system corresponding to  $[\mathbf{v}_1 \dots \mathbf{v}_n \mid \mathbf{b}]$  is consistent.

Put another way,  $\mathbf{b}$  belongs to  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  exactly when  $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_n \mid \mathbf{b}]$  doesn't have a row  $[0 \dots 0 \mid 1]$  representing the contradiction  $0 = 1$ .

**Activity V.2.4** (*~10 min*) Determine if  $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \\ 2 \end{bmatrix} \right\}$  by row-reducing an appropriate matrix.

**Activity V.2.5** (*~5 min*) Determine if  $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  by row-reducing an appropriate matrix.

**Activity V.2.6** ( $\sim 10$  min) Does the third-degree polynomial  $3y^3 - 2y^2 + y + 5$  in  $\mathcal{P}^3$  belong to  $\text{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$ ?

*Part 1:* Reinterpret this question as an equivalent exercise involving Euclidean vectors in  $\mathbb{R}^4$ . (Hint: What four numbers must you know to write a  $\mathcal{P}^3$  polynomial?)

*Part 2:* Solve this equivalent exercise, and use its solution to answer the original question.

**Activity V.2.7** ( $\sim 5$  min) Does the matrix  $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$  belong to  $\text{span}\left\{\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\right\}$ ?

**Activity V.2.8** ( $\sim 5$  min) Does the complex number  $2i$  belong to  $\text{span}\{-3 + i, 6 - 2i\}$ ?