## Application Activities - Module E Part 1 - Class Day 3

**Definition 3.1** A linear equation is an equation of the variables  $x_i$  of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b.$$

A solution for a linear equation is expressed in terms of the Euclidean vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \dots + a_ns_n = b.$$

**Observation 3.2** The linear equation 3x - 5y = -2 may be graphed as a line in the xy plane.



The linear equation x + 2y - z = 4 may be graphed as a plane in xyz space.

**Remark 3.3** In previous classes you likely assumed  $x = x_1$ ,  $y = x_2$ , and  $z = x_3$ . However, since this course often deals with equations of four or more variables, we will almost always write our variables as  $x_i$ .

**Definition 3.4** A system of linear equations (or a linear system for short) is a collection of one or more linear equations.

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

A solution

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \dots + a_{in}s_n = b_i$$

for  $1 \le i \le m$  (that is, the solution satisfies all equations in the system).

**Remark 3.5** When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$x_1 + 3x_3 = 3$$
$$3x_1 - 2x_2 + 4x_3 = 0$$
$$-x_2 + x_3 = -2$$

$$1x_1 + 0x_2 + 3x_3 = 3$$
$$3x_1 - 2x_2 + 4x_3 = 0$$
$$0x_1 - 1x_2 + 1x_3 = -2$$

$$x_1 + 3x_3 = 3$$
  

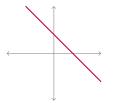
$$3x_1 - 2x_2 + 4x_3 = 0$$
  

$$- x_2 + x_3 = -2$$

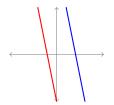
**Definition 3.6** A linear system is **consistent** if there exists a solution for the system. Otherwise it is inconsistent.

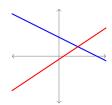
Fact 3.7 All linear systems are either consistent with one solution, consistent with infinitely-many solutions, or inconsistent.

Activity 3.8 Consider the following graphs representing linear systems of two variables. Label each graph with consistent with one solution, consistent with infinitely-many solutions, or inconsistent.









Activity 3.9 All inconsistent linear systems contain a logical contradiction. Find a contradiction in this system.

$$-x_1 + 2x_2 = 5$$

$$2x_1 - 4x_2 = 6$$

**Activity 3.10** Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

Part 1: Find three different solutions  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$  for this system.

Part 2: Let  $x_2 = a$  where a is an arbitrary real number, then find an expression for  $x_1$  in terms of a. Use this to describe all solutions (the **solution set**)  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$  for the linear system in terms of a.

Activity 3.11 Consider the following linear system.

$$x_1 + 2x_2 \quad - \quad x_4 = \quad 3$$

$$x_3 + 4x_4 = -2$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix} + a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$$

to the linear system by setting  $x_2 = a$  and  $x_4 = b$ , and then solving for  $x_1$  and  $x_3$ .

**Observation 3.12** Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.

Remark 3.13 The only important information in a linear system are its coefficients and constants.

Original linear system:

Verbose standard form:

Coefficients/constants:

$$x_1 + 3x_3 = 3$$
  $1x_1 + 0x_2 + 3x_3 = 3$   $1 \quad 0 \quad 3 \mid 3$   
 $3x_1 - 2x_2 + 4x_3 = 0$   $3x_1 - 2x_2 + 4x_3 = 0$   $3 - 2 \quad 4 \mid 0$   
 $-x_2 + x_3 = -2$   $0x_1 - 1x_2 + 1x_3 = -2$   $0 - 1 \quad 1 \mid -2$ 

**Definition 3.14** A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

**Definition 3.15** Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution:  $(x_1, x_2) = (1, 1)$ .

$$3x_1 - 2x_2 = 1$$
  $3x_1 - 2x_2 = 1$   $4x_1 + 4x_2 = 5$   $4x_1 + 2x_2 = 6$ 

Therefore these augmented matrices are equivalent:

$$\begin{bmatrix} 3 & -2 & 1 \\ 1 & 4 & 5 \end{bmatrix} \qquad \begin{bmatrix} 3 & -2 & 1 \\ 4 & 2 & 6 \end{bmatrix}$$

Activity 3.16 Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as valid, and label the procedures that would change the solution set of the corresponding linear system as invalid.

- a) Swap two rows.
- b) Swap two columns.
- c) Add a constant to every term in a row.
- d) Multiply a row by a nonzero constant.
- e) Add a constant multiple of one row to another row.
- f) Replace a column with zeros.