Module P: Applications of Linear Algebra

Module P Section 1

Definition P.1.1

In chemistry, we learn that when the two substances

- Hydrochloric acid HCl (formed from 1 H and 1 Cl atom)
- Sodium Na (formed from 1 Na atom)

react, their atoms rearrange to form the substances

- Salt $NaCl$ (formed from 1 Na and 1 Cl atom)
- Hydrogen gas H_2 (formed from 2 H atoms).

This may be represented by the **chemical equation**

$$?\,HCl + ?\,Na \to ?\,NaCl + ?\,H_2$$

where each $?$ represents the amount of that substance before/after the reaction.

Activity P.1.2 *(∼5 min)*

The **law of conservation of mass** states that the quantity of atoms before and after a chemical reaction must remain the same.

Find positive integers so that both sides of the chemical equation represent the same amount of matter:

$$?\,HCl + \,?\,Na \to \,?\,NaCl + \,?\,H_2$$

Definition P.1.3

A chemical equation is **balanced** if the given quantities of each substance before and after the reaction are equal and minimal positive integers:

$$2HCl + 2Na \to 2NaCl + H_2$$

Observation P.1.4

For example, the following equation isn't balanced because all the integers may be divided by three:

$$6HCl + 6Na \to 6NaCl + 3H_2$$

Therefore if a chemical equation can be balanced, there is exactly one correct solution.

Activity P.1.5 *(∼15 min)*
Balance the following chemical equations:

$$? \, Fe + \, ? \, Cl_2 \to \, ? \, FeCl_3$$

$$? \, Ca(OH)_2 + \, ? \, H_3PO_4 \to \, ? \, Ca_3(PO_4)_2 + \, ? \, H_2O$$

$$? \, K_4Fe(CN)_6 + \, ? \, H_2SO_4 + \, ? \, H_2O \to \, ? \, K_2SO_4 + \, ? \, FeSO_4 + \, ? \, (NH_4)_2SO_4 + \, ? \, CO$$

(Note that $(NH_4)_2SO_4$ represents 2 N, 8 H, 1 S, and 4 O.)

Observation P.1.6

For the purposes of balancing chemical equations, the set

$$L = \{\mathbf{A} \mid \mathbf{A} \text{ is combination of elements}\}$$

may be treated as a kind of **vector space**. This means that balancing the chemical equation

$$?\, Fe + \,?\, Cl_2 \to \,?\, FeCl_3$$

may be acheived by finding a solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to the vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl}).$$

Activity P.1.7 *(∼5 min)*

To solve the vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl})$$

we are only concerned with the subspace $W = \operatorname{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ of L. Since the element **Fe** cannot be created from the element **Cl** in a chemical reaction and vice versa, the set $\{\mathbf{Cl}, \mathbf{Fe}\}$:

a) spans W, but is linearly dependent.

b) is linearly independent, but does not span W.

c) is a basis for W.

Observation P.1.8

$W = \operatorname{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ is a two-dimensional subspace of L, so as usual we'd rather work with its isomorphic Euclidean space $\mathbb{R}^2$.

Thus we should assign a transformation of bases such as:

$$\mathbf{Cl} \leftrightarrow \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{Fe} \leftrightarrow \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Activity P.1.9 *(∼10 min)*

Rewrite the $W = \text{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl})$$

using the transformation of bases

$$\mathbf{Cl} \leftrightarrow \mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix} \qquad \mathbf{Fe} \leftrightarrow \mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

and show how it may be simplifed to

$$x\begin{bmatrix}0\\1\end{bmatrix} + y\begin{bmatrix}2\\0\end{bmatrix} - z\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Activity P.1.10 *(∼10 min)*
Consider the Euclidean vector equation

$$x\begin{bmatrix}0\\1\end{bmatrix}+y\begin{bmatrix}2\\0\end{bmatrix}-z\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

Activity P.1.10 *(∼10 min)*
Consider the Euclidean vector equation

$$x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \end{bmatrix} - z \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Part 1: Find its solution set.

Activity P.1.10 *(∼10 min)*
Consider the Euclidean vector equation

$$x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \end{bmatrix} - z \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Part 1: Find its solution set.
Part 2: Find a vector in the solution space that consists of minimal positive integers.

Activity P.1.10 *(∼10 min)*

Consider the Euclidean vector equation

$$x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \end{bmatrix} - z \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Part 1: Find its solution set.

Part 2: Find a vector in the solution space that consists of minimal positive integers.

Part 3: Balance the chemical equation

$$?\, Fe + \,?\, Cl_2 \to \,?\, FeCl_3.$$

Activity P.1.11 *(∼10 min)*
Balance the chemical equation

$$? \, Ca(OH)_2 + \, ? \, H_3PO_4 \to \, ? \, Ca_3(PO_4)_2 + \, ? \, H_2O$$

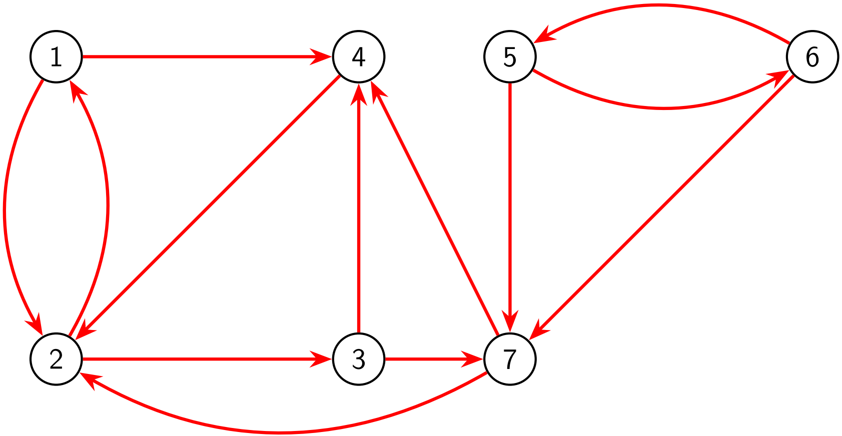
by first converting it into an $\mathbb{R}^4$ vector equation and finding its solution set.

Module P Section 2

Activity P.2.1 *(∼10 min)*

A \$700,000,000,000 Problem:

In the picture below, each circle represents a webpage, and each arrow represents a link from one page to another.



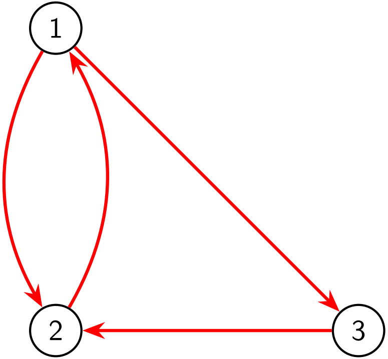
Based on how these pages link to each other, write a list of the 7 webpages in order from most imporant to least important.

Observation P.2.2
The $700,000,000,000 Idea:
Links are endorsements.

1. A webpage is important if it is linked to (endorsed) by important pages.
2. A webpage distributes its importance equally among all the pages it links to (endorses).

Example P.2.3

Consider this small network with only three pages. Let x_1, x_2, x_3 be the importance of the three pages respectively.

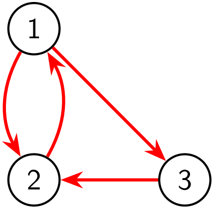


1. x_1 splits its endorsement in half between x_2 and x_3
2. x_2 sends all of its endorsement to x_1
3. x_3 sends all of its endorsement to x_2.

This corresponds to the **page rank system**

$$\begin{aligned} x_2 \quad\quad &= x_1 \\ \frac{1}{2}x_1 \quad + x_3 &= x_2 \\ \frac{1}{2}x_1 \quad\quad &= x_3 \end{aligned}$$

Example P.2.4



$$\begin{aligned} x_2 &= x_1 \\ \frac{1}{2}x_1 + x_3 &= x_2 \\ \frac{1}{2}x_1 &= x_3 \end{aligned}$$

We can summarize the left hand side of the system by putting its coefficients into a **page rank matrix** $A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$, and store the right hand side of the system as the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Thus, computing the imporance of pages on a network is equivalent to solving the matrix equation $A\mathbf{x} = \mathbf{x}$.

Activity P.2.5 *($\sim$5 min)*

A **page rank vector** for a page rank matrix A is a vector $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{x}$. This vector describes the relative importance of webpages on the network described by A.

Thus, the $700,000,000,000 problem is what kind of problem?

(a) A bijection problem

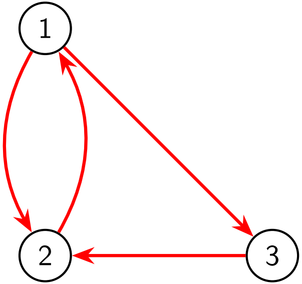
(b) A calculus problem

(c) A determinant problem

(d) An eigenvector problem

Activity P.2.6 *(∼10 min)*

Find a page rank vector $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{x}$ (an eigenvector associated to the eigenvalue 1) for the following network's page rank matrix A.

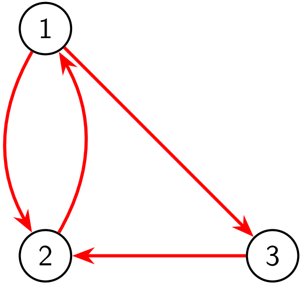


$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Observation P.2.7

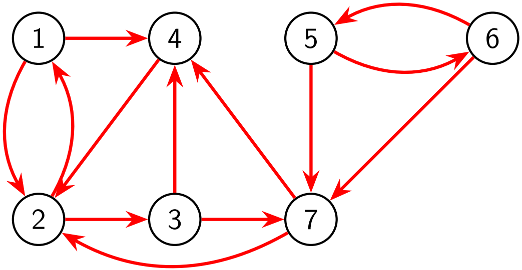
Row-reducing $A - I = \begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ yields the basic eigenvector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Therefore, we may conclude that pages 1 and 2 are equally important, and both pages are twice as important as page 3.



Activity P.2.8 *(∼10 min)*

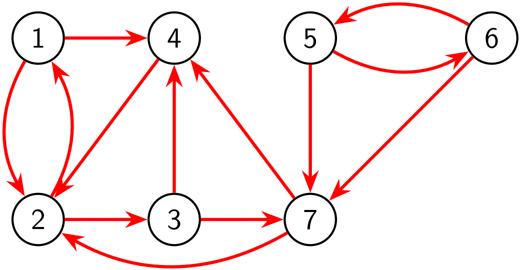
Compute the 7×7 page rank matrix for the following network.



For example, since website 1 distributes its endorsement equally between 2 and 4, the first column is $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity P.2.9 *(∼10 min)*

Find a page rank vector for the transition matrix.

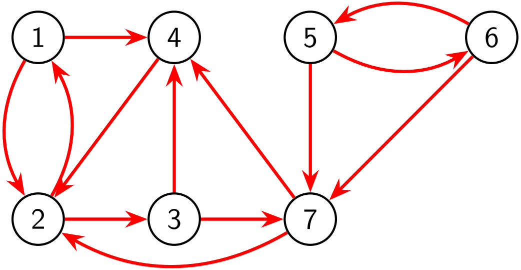


$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Which webpage is most important?

Observation P.2.10

Since a page rank vector for the network is given by $\mathbf{x}$, it's reasonable to consider page 2 as the most important page.

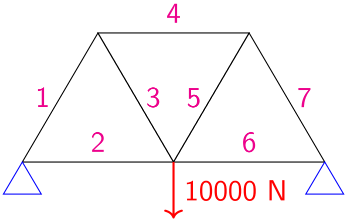


$$\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

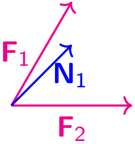
Module P Section 3

Observation P.3.1

Consider the truss pictured below with two fixed anchor points and a 10000 N load (assume all triangles are equilateral).

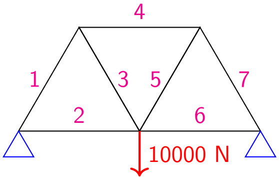


The horizontal and vertical forces must balance at each node. For example, at the bottom left node there are 3 forces acting.

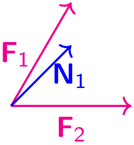


We adhere to the convention that a compression force on a strut is positive, while a negative force represents tension.

Observation P.3.2



We decompose the first node into vertical and horizontal forces:



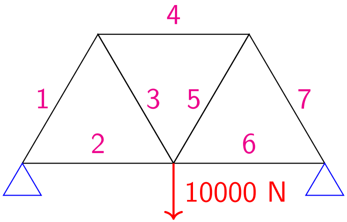
$$\mathbf{F}_1 = F_1 \begin{bmatrix} \cos(60^\circ) \\ \sin(60^\circ) \end{bmatrix}$$

$$\mathbf{N}_1 = \begin{bmatrix} N_{1,h} \\ N_{1,v} \end{bmatrix}$$

$$F_1 \sin(60^\circ) + N_{1,v} = 0$$
$$F_1 \cos(60^\circ) + N_{1,h} + F_2 = 0$$

Activity P.3.3 *(∼10 min)*

Consider the truss pictured below with two fixed anchor points and a 10000 N load (assume all triangles are equilateral).

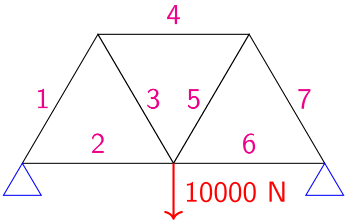


From the bottom left node we obtained 2 equations in the four variables

- F_1 (compression force on strut one)
- $N_{1,v}$ and $N_{1,h}$ (horizontal and vertical components of the normal force from the left anchor)
- F_2 (compression force on strut 2).

Activity P.3.3 *(∼10 min)*

Consider the truss pictured below with two fixed anchor points and a 10000 N load (assume all triangles are equilateral).

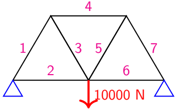


From the bottom left node we obtained 2 equations in the four variables

- F_1 (compression force on strut one)
- $N_{1,v}$ and $N_{1,h}$ (horizontal and vertical components of the normal force from the left anchor)
- F_2 (compression force on strut 2).

Part 1: Determine how many total equations there will be after accounting for all of the nodes, and and list all of the variables. You do not need to actually determine all of the equations.

Activity P.3.4 *(∼10 min)*

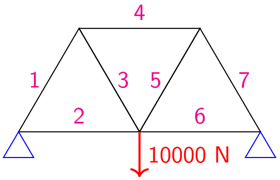


The resulting system is

$$
\begin{aligned}
N_{1,v} + (\sin(60^\circ))F_1 &= 0 \\
N_{1,h} + (\cos(60^\circ))F_1 + F_2 &= 0 \\
-(\sin(60^\circ))F_1 - (\sin(60^\circ)F_3 &= 0 \\
-(\cos(60^\circ))F_1 + (\cos(60^\circ)F_3 + F_4 &= 0 \\
(\sin(60^\circ)F_3 + (\sin(60^\circ))F_5 &= 10000 \\
-F_2 - (\cos(60^\circ)F_3 + (\cos(60^\circ))F_5 + F_6 &= 0 \\
-(\sin(60^\circ))F_5 - (\sin(60^\circ))F_7 &= 0 \\
-F_4 - (\cos(60^\circ))F_5 + (\cos(60^\circ))F_7 &= 0 \\
N_{2,v} + (\sin(60^\circ))F_7 &= 0 \\
N_{2,h} - F_6 - (\cos(60^\circ))F_7 &= 0
\end{aligned}
$$

Solve this system to determine which struts are compressed and which are in tension.

Observation P.3.5



The determined part of the solution is

$$N_{1,v} = N_{2,v} = 5000$$
$$F_1 = F_4 = F_7 = -5882.4$$
$$F_3 = F_5 = 5882.4$$

So struts 1,4,7 are in tension, while struts 3 and 5 are compressed.
The forces on struts 2 and 6 (and the horizontal normal forces) are not strictly determined in this setting.