

# Linear Algebra

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December 16, 2017

# Module I: Introduction

**Remark I.0.1**

This brief module gives an overview for the course.

# Module I Section 1

### Remark I.1.1

#### What is Linear Algebra?

Linear algebra is the study of **linear maps**.

- In Calculus, you learn how to approximate any function by a linear function.
- In Linear Algebra, we learn about how linear maps behave.
- Combining the two, we can approximate how any function behaves.

## Remark 1.1.2

### What is Linear Algebra good for?

- Linear algebra is used throughout several fields in higher mathematics.
- In computer graphics, linear algebra is used to help represent 3D objects in a 2D grid of pixels.
- Linear algebra is used to approximate differential equation solutions in a vast number of engineering applications (e.g. fluid flows, vibrations, heat transfer) whose solutions are very difficult (or impossible) to find precisely.
- Google's search engine is based on its Page Rank algorithm, which ranks websites by computing an eigenvector of a matrix.

**Remark I.1.3****What will I learn in this class?**

By the end of this class, you will be able to:

- Solve systems of linear equations. (Module E)
- Identify vector spaces and their properties. (Module V)
- Analyze the structure of vector spaces and sets of vectors. (Module S)
- Use and apply the algebraic properties of linear transformations. (Module A)
- Perform fundamental operations in the algebra of matrices. (Module M)
- Use and apply the geometric properties of linear transformations. (Module G)

## Module E: Solving Systems of Linear Equations



**How can we solve systems of linear equations?**

At the end of this module, students will be able to...

- E1. Systems as matrices.** ... translate back and forth between a system of linear equations and the corresponding augmented matrix.
- E2. Row reduction.** ... put a matrix in reduced row echelon form.
- E3. Systems of linear equations.** ... compute the solution set for a system of linear equations.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.

The following resources will help you prepare for this module.

- Systems of linear equations (Khan Academy): <http://bit.ly/2l21etm>
- Solving linear systems with substitution (Khan Academy):  
<http://bit.ly/1SlMpix>

# Module E Section 1

**Definition E.1.1**

A **linear equation** is an equation of the variables  $x_i$  of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

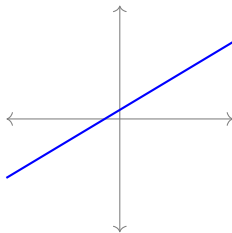
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

**Observation E.1.2**

The linear equation  $3x - 5y = -2$  may be graphed as a line in the  $xy$  plane.



The linear equation  $x + 2y - z = 4$  may be graphed as a plane in  $xyz$  space.

**Remark E.1.3**

In previous classes you likely assumed  $x = x_1$ ,  $y = x_2$ , and  $z = x_3$ . However, since this course often deals with equations of four or more variables, we will almost always write our variables as  $x_i$ .



**Definition E.1.4**

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

**A solution**

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \dots + a_{in}s_n = b_i$$

for  $1 \leq i \leq m$  (that is, the solution satisfies all equations in the system).

**Remark E.1.5**

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2\end{aligned}$$

Verbose standard form:

$$\begin{aligned}1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2\end{aligned}$$

Concise standard form:

$$\begin{aligned}x_1 \quad \quad + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ \quad - x_2 + x_3 &= -2\end{aligned}$$

**Definition E.1.6**

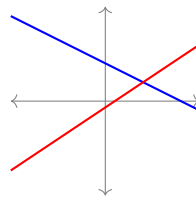
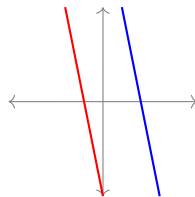
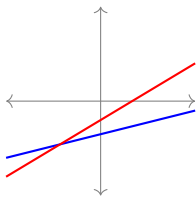
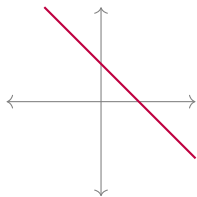
A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

**Fact E.1.7**

All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

**Activity E.1.8** (*~5 min*)

Consider the following graphs representing linear systems of two variables. Label each graph with **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.



**Activity E.1.9** (*~10 min*)

All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system.

$$-x_1 + 2x_2 = 5$$

$$2x_1 - 4x_2 = 6$$

**Activity E.1.10** (*~10 min*)

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

**Activity E.1.10** (*~10 min*)

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

*Part 1:* Find three different solutions  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ ,  $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ ,  $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$  for this system.



**Activity E.1.10** ( $\sim 10$  min)

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

*Part 1:* Find three different solutions  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ ,  $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ ,  $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$  for this system.

*Part 2:* Let  $x_2 = a$  where  $a$  is an arbitrary real number, then find an expression for  $x_1$  in terms of  $a$ . Use this to describe *all* solutions (the **solution set**)  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$  for the linear system in terms of  $a$ .

**Activity E.1.11** (*~10 min*)

Consider the following linear system.

$$\begin{aligned}x_1 + 2x_2 - x_4 &= 3 \\x_3 + 4x_4 &= -2\end{aligned}$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix} + a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$$

to the linear system by setting  $x_2 = a$  and  $x_4 = b$ , and then solving for  $x_1$  and  $x_3$ .

**Observation E.1.12**

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.

**Remark E.1.13**

The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2\end{aligned}$$

Verbose standard form:

$$\begin{aligned}1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2\end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

**Definition E.1.14**

A system of  $m$  linear equations with  $n$  variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\&\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m\end{aligned}$$

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

**Definition E.1.15**

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution:  $(x_1, x_2) = (1, 1)$ .

$$3x_1 - 2x_2 = 1$$

$$x_1 + 4x_2 = 5$$

$$3x_1 - 2x_2 = 1$$

$$4x_1 + 2x_2 = 6$$

Therefore these augmented matrices are equivalent:

$$\left[ \begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array} \right]$$

**Activity E.1.16** (*~10 min*)

Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

- a) Swap two rows.
- b) Swap two columns.
- c) Add a constant to every term in a row.
- d) Multiply a row by a nonzero constant.
- e) Add a constant multiple of one row to another row.
- f) Replace a column with zeros.

## Module E Section 2



### Definition E.2.1

The following **row operations** produce equivalent augmented matrices:

- 1 Swap two rows.
- 2 Multiply a row by a nonzero constant.
- 3 Add a constant multiple of one row to another row.

Whenever two matrices  $A, B$  are equivalent (so whenever we do any of these operations), we write  $A \sim B$ .

**Activity E.2.2** (*~10 min*)

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

**Activity E.2.2** ( $\sim 10$  min)

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

*Part 1:* Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

- 1 Swap  $R_1$  (first row) and  $R_2$  (second row).
- 2 Multiply  $R_2$  by  $\frac{1}{2}$ .
- 3 Add  $R_1$  to  $R_3$ .
- 4 Add  $-3R_1$  to  $R_2$ .
- 5 Add  $-2R_2$  to  $R_3$ .
- 6 Multiply  $R_3$  by  $\frac{1}{3}$ .

**Activity E.2.2** ( $\sim 10$  min)

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

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*Part 1:* Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

- 1 Swap  $R_1$  (first row) and  $R_2$  (second row).
- 2 Multiply  $R_2$  by  $\frac{1}{2}$ .
- 3 Add  $R_1$  to  $R_3$ .
- 4 Add  $-3R_1$  to  $R_2$ .
- 5 Add  $-2R_2$  to  $R_3$ .
- 6 Multiply  $R_3$  by  $\frac{1}{3}$ .

*Part 2:* Which linear system would you rather solve?

### Definition E.2.3

The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix. Examples:

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

**Activity E.2.4** (*~10 min*)

Find your own sequence of row operations to manipulate the matrix

$$\left[ \begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array} \right]$$

into row echelon form. (Note that row echelon form is not unique.)

The most efficient way to do this is by circling **pivot positions** in your matrix:

- 1 Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
- 2 Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
- 3 Repeat these two steps as often as possible.

**Activity E.2.5** (*~10 min*)

Solve this simplified linear system:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

**Observation E.2.6**

The concise standard form of the solution to this linear system corresponds to a simplified row echelon form matrix:

$$\begin{array}{rcl} x_1 & = & -2 \\ x_2 & = & 7 \\ x_3 & = & 2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$



**Definition E.2.7**

A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0. Examples:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

**Activity E.2.8** (*~10 min*)

Show that the following two linear systems:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

$$x_1 = -2$$

$$x_2 = 7$$

$$x_3 = 2$$

are equivalent by converting the first system to an augmented matrix, and then zeroing out all terms above pivot positions (the leading terms).

**Remark E.2.9**

We may verify that  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$  is a solution to the original linear system

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

by plugging the solution into each equation.

**Fact E.2.10**

Every augmented matrix  $A$  reduces to a unique reduced row echelon form matrix. This matrix is denoted as  $\text{RREF}(A)$ .

**Activity E.2.11** (*~10 min*)

Consider the following matrix.

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

**Activity E.2.11** (*~10 min*)

Consider the following matrix.

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

*Part 1:* Find  $\text{RREF}(A)$ .

**Activity E.2.11** (*~10 min*)

Consider the following matrix.

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

*Part 1:* Find  $\text{RREF}(A)$ .

*Part 2:* How many solutions does the corresponding linear system have?

## Module E Section 3



### Definition E.3.1

An algorithm that reduces  $A$  to  $\text{RREF}(A)$  is called **Gauss-Jordan elimination**. For example:

- 1 Circle the cell that (a) is in the top-most row without a pivot position and (b) is in the left-most column with a nonzero term either in that position or below it. This position (not the number inside) is called a **pivot**.
- 2 Change the pivot's value to 1 by using row operations involving only the pivot row and rows below it.
- 3 Add or subtract multiples of the pivot row to zero out above and below the pivot.
- 4 Return to Step 1 and repeat as needed until the matrix is in row reduced echelon form.

**Observation E.3.2**

Here is an example of applying Gauss-Jordan elimination to a matrix:

$$\begin{aligned}
 \left[ \begin{array}{cccc|c} \textcircled{2} & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array} \right] &\sim \left[ \begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ -1 & 1 & 3 & -1 & -3 \\ 2 & -2 & -6 & 1 & 3 \end{array} \right] \sim \left[ \begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ 0 & \textcircled{-1} & 2 & 0 & -1 \\ 0 & 2 & -4 & -1 & -1 \end{array} \right] \\
 &\sim \left[ \begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 2 & -4 & -1 & -1 \end{array} \right] \sim \left[ \begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 1 & 4 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{-1} & -3 \end{array} \right] \\
 &\sim \left[ \begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 1 & 4 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right] \sim \left[ \begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 0 & 1 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right]
 \end{aligned}$$

### Definition E.3.3

The columns of  $\text{RREF}(A)$  without a leading term represent **free variables** of the linear system modeled by  $A$  that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by  $A$ .

**Example E.3.4**

Here,  $x_3$  is the free variable set equal to  $a$  since its column lacks a pivot, and the other bounded variables are put in terms of  $a$ .

$$2x_1 - 2x_2 - 6x_3 + x_4 = 3$$

$$-x_1 + x_2 + 3x_3 - x_4 = -3$$

$$x_1 - 2x_2 - x_3 + x_4 = 1$$

$$x_1 - 5x_3 = 1$$

$$x_2 - 2x_3 = 1$$

$$x_4 = 3$$

 $\Rightarrow$ 

$$x_1 = 1 + 5a$$

$$x_2 = 1 + 2a$$

$$x_3 = a$$

$$x_4 = 3$$

$$\begin{array}{c} \Downarrow \\ \left[ \begin{array}{cccc|c} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array} \right] \end{array} \sim \begin{array}{c} \Uparrow \\ \left[ \begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 0 & 1 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right] \end{array}$$

So the solution set is  $\left\{ \begin{bmatrix} 1 + 5a \\ 1 + 2a \\ a \\ 3 \end{bmatrix} \mid a \in \mathbb{R} \right\}.$

**Activity E.3.5** (*~20 min*)

Solve the system of linear equations, circling the pivot positions in your augmented matrices as you work.

$$-x_1 + x_2 - 3x_3 + 2x_4 = 0$$

$$2x_1 - x_2 + 5x_3 + 3x_4 = -11$$

$$3x_1 + 2x_2 + 4x_3 + x_4 = 1$$

$$x_2 - x_3 + x_4 = 1$$

Remember to find the solution set of the system by setting the free variable (the column without a pivot position) equal to  $a$ , and then express each of the other bounded variables equal to an expression in terms of  $a$ .

**Remark E.3.6**

From now on, unless specified, there's no need to show your work in finding  $\text{RREF}(A)$ , so you may use a calculator to speed up your work.

**Activity E.3.7** (*~10 min*)

Solve the linear system

$$2x_1 - 3x_2 = 17$$

$$x_1 + 2x_2 = -2$$

$$-x_1 - x_2 = 1$$

**Activity E.3.8** (*~5 min*)

Show that all linear systems of the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

are consistent by finding a quickly verifiable solution.



**Definition E.3.9**

A **homogeneous system** is a linear system satisfying  $b_i = 0$ , that is, it is a linear system of the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

**Fact E.3.10**

Because the zero vector is always a solution, the solution set to any homogeneous system with infinitely-many solutions may be generated by multiplying the parameters representing the free variables by a minimal set of Euclidean vectors, and adding these up. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

**Definition E.3.11**

A minimal set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

**Activity E.3.12** (*~10 min*)

Find a basis for the solution set of the following homogeneous linear system.

$$x_1 + 2x_2 \quad - \quad x_4 = 0$$

$$x_3 + 4x_4 = 0$$

$$2x_1 + 4x_2 + x_3 + 2x_4 = 0$$

## Module V: Vector Spaces

# What is a vector space?

At the end of this module, students will be able to...

- V1. Vector property verification.** ... show why an example satisfies a given vector space property, but does not satisfy another given property.
- V2. Vector space identification.** ... list the eight defining properties of a vector space, infer which of these properties a given example satisfies, and thus determine if the example is a vector space.
- V3. Linear combinations.** ... determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors.
- V4. Spanning sets.** ... determine if a set of Euclidean vectors spans  $\mathbb{R}^n$ .
- V5. Subspaces.** ... determine if a subset of  $\mathbb{R}^n$  is a subspace or not.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems

**E1,E2,E3.**



The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Academy):  
<http://bit.ly/2y8A0wa>
- Linear combinations of Euclidean vectors (Khan Academy):  
<http://bit.ly/2nK3wne>
- Adding and subtracting complex numbers (Khan Academy):  
<http://bit.ly/1PE3ZMQ>
- Adding and subtracting polynomials (Khan Academy):  
<http://bit.ly/2d5SLGZ>

# Module V Section 1

**Activity V.1.1** ( $\sim 20$  min)

Consider each of the following vector properties. Label each property with  $\mathbb{R}^1$ ,  $\mathbb{R}^2$ , and/or  $\mathbb{R}^3$  if that property holds for Euclidean vectors/scalars  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  of that dimension.

**1 Addition associativity.**

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

**2 Addition commutivity.**

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

**3 Addition identity.**

There exists some  $\mathbf{0}$  where  $\mathbf{v} + \mathbf{0} = \mathbf{v}$ .

**4 Addition inverse.**

There exists some  $-\mathbf{v}$  where  
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ .

**5 Addition midpoint uniqueness.**

There exists a unique  $\mathbf{m}$  where the  
 distance from  $\mathbf{u}$  to  $\mathbf{m}$  equals the  
 distance from  $\mathbf{m}$  to  $\mathbf{v}$ .

**6 Scalar multiplication associativity.**

$$a(b\mathbf{v}) = (ab)\mathbf{v}.$$

**7 Scalar multiplication identity.**

$$1\mathbf{v} = \mathbf{v}.$$

**8 Scalar multiplication relativity.**

There exists some scalar  $c$  where either  
 $c\mathbf{v} = \mathbf{w}$  or  $c\mathbf{w} = \mathbf{v}$ .

**9 Scalar distribution.**

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

**10 Vector distribution.**

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

**11 Orthogonality.**

There exists a non-zero vector  $\mathbf{n}$  such  
 that  $\mathbf{n}$  is orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$ .

**12 Bidimensionality.**

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} \text{ for some value of } a, b.$$

## Definition V.1.2

A **vector space**  $V$  is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  belong to  $V$ , and let  $a, b$  be scalar numbers.

- **Addition associativity.**  
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition commutativity.**  
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Addition identity.**  
There exists some  $\mathbf{0}$  where  
 $\mathbf{v} + \mathbf{0} = \mathbf{v}.$
- **Addition inverse.**  
There exists some  $-\mathbf{v}$  where  
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$
- **Scalar multiplication associativity.**  
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **Scalar multiplication identity.**  
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar distribution.**  
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Vector distribution.**  
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

**Definition V.1.3**

The most important examples of vector spaces are the **Euclidean vector spaces**  $\mathbb{R}^n$ , but there are other examples as well.

**Activity V.1.4** ( $\sim 25$  min)

Consider the following set that models motion along the curve  $y = e^x$ . Let

$V = \{(x, y) : y = e^x\}$ . Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$ , and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$ .

**Activity V.1.4** ( $\sim 25$  min)

Consider the following set that models motion along the curve  $y = e^x$ . Let

$V = \{(x, y) : y = e^x\}$ . Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$ , and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$ .

*Part 1:* Which of the vector space properties are satisfied by  $V$  paired with these operations?

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some  $\mathbf{0}$  where

$$\mathbf{v} \oplus \mathbf{0} = \mathbf{v}.$$

- **Addition inverse.**

There exists some  $-\mathbf{v}$  where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

**Activity V.1.4** ( $\sim 25$  min)

Consider the following set that models motion along the curve  $y = e^x$ . Let

$V = \{(x, y) : y = e^x\}$ . Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$ , and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$ .

*Part 1:* Which of the vector space properties are satisfied by  $V$  paired with these operations?

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some  $\mathbf{0}$  where

$$\mathbf{v} \oplus \mathbf{0} = \mathbf{v}.$$

- **Addition inverse.**

There exists some  $-\mathbf{v}$  where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

*Part 2:* Is  $V$  a vector space?



## Module V Section 2

**Remark V.2.1**

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$ : Euclidean vectors with  $n$  components.
- $\mathbb{R}^\infty$ : Sequences of real numbers  $(v_1, v_2, \dots)$ .
- $\mathbb{R}^{m \times n}$ : Matrices of real numbers with  $m$  rows and  $n$  columns.
- $\mathbb{C}$ : Complex numbers.
- $\mathcal{P}^n$ : Polynomials of degree  $n$  or less.
- $\mathcal{P}$ : Polynomials of any degree.
- $C(\mathbb{R})$ : Real-valued continuous functions.

**Activity V.2.2** ( $\sim 10$  min)

Let  $V = \{(a, b) : a, b \text{ are real numbers}\}$ , where

$(a_1, b_1) \oplus (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$  and  $c \odot (a, b) = (a^c, b + c)$ .

Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some  $\mathbf{0}$  where

$$\mathbf{v} \oplus \mathbf{0} = \mathbf{v}.$$

- **Addition inverse.**

There exists some  $-\mathbf{v}$  where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

**Definition V.2.3**

A **linear combination** of a set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$  is given by  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$  for any choice of scalar multiples  $c_1, c_2, \dots, c_m$ .

For example, we say  $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

**Definition V.2.4**

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

**Activity V.2.5** ( $\sim 10$  min)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

**Activity V.2.5** ( $\sim 10$  min)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

*Part 1:* Sketch  $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  in the  $xy$  plane for  $c = 1, 3, 0, -2$ .

**Activity V.2.5** ( $\sim 10$  min)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

*Part 1:* Sketch  $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  in the  $xy$  plane for  $c = 1, 3, 0, -2$ .

*Part 2:* Sketch a representation of all the vectors given by  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$  in the  $xy$  plane.



**Activity V.2.6** ( $\sim 10$  min)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

**Activity V.2.6** ( $\sim 10$  min)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

*Part 1:* Sketch the following linear combinations in the  $xy$  plane:  $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ,

$0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ,  $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ,  $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

**Activity V.2.6** ( $\sim 10$  min)

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

*Part 1:* Sketch the following linear combinations in the  $xy$  plane:  $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ,

$0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ,  $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ,  $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

*Part 2:* Sketch a representation of all the vectors given by  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$  in the  $xy$  plane.

**Activity V.2.7** ( $\sim 5$  min)

Sketch a representation of all the vectors given by  $\text{span} \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$  in the  $xy$  plane.

**Activity V.2.8** (*~15 min*)

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

**Activity V.2.8** ( $\sim 15$  min)

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector

equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

**Activity V.2.8** ( $\sim 15$  min)

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector

equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

*Part 2:* Solve this system. (Remember, you should use a calculator to help find RREF.)

**Activity V.2.8** (*~15 min*)

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector

equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

*Part 2:* Solve this system. (Remember, you should use a calculator to help find RREF.)

*Part 3:* Given this solution, does  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belong to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ ?



## Module V Section 3

**Fact V.3.1**

A vector  $\mathbf{b}$  belongs to  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  if and only if the linear system corresponding to  $[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$  is consistent.

**Remark V.3.2**

To determine if  $\mathbf{b}$  belongs to  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ , find  $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$ .

**Activity V.3.3** ( $\sim 5$  min)

Determine if  $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  by row-reducing an appropriate matrix.

**Activity V.3.4** (*~5 min*)

Determine if  $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  by row-reducing an appropriate matrix.

### Observation V.3.5

So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space  $\mathbb{R}^n$ ; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

**Activity V.3.6** (*~5 min*)

We previously checked that  $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$  does not belong to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ .

Does  $f(x) = 3x^2 - 2x + 1$  belong to  $\text{span}\{x^2 - 3, -x^2 - 3x + 2\}$ ?

**Activity V.3.7** ( $\sim 10$  min)

Does the matrix  $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$  belong to  $\text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\}$ ?



**Activity V.3.8** ( $\sim 10$  min)

Does the complex number  $2i$  belong to  $\text{span}\{-3 + i, 6 - 2i\}$ ?

**Activity V.3.9** (*~10 min*)

How many vectors are required to span  $\mathbb{R}^2$ ? Sketch a drawing in the  $xy$  plane to support your answer.

- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) Infinitely Many

**Activity V.3.10** (*~5 min*)

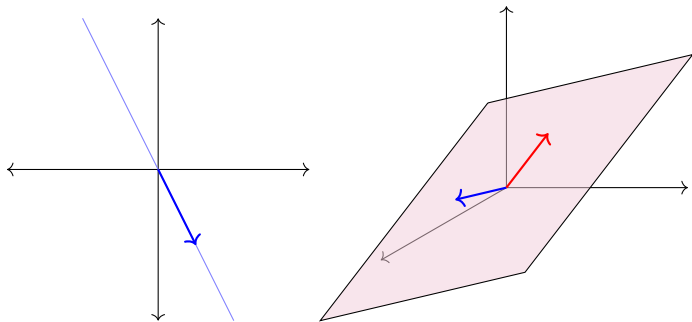
How many vectors are required to span  $\mathbb{R}^3$ ?

- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) Infinitely Many

## Module V Section 4

**Fact V.4.1**

At least  $n$  vectors are required to span  $\mathbb{R}^n$ .



**Activity V.4.2** ( $\sim 10$  min)

Choose a vector  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  in  $\mathbb{R}^3$  that is not in  $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$  by ensuring

$$\left[ \begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]. \text{ (Why does this work?)}$$

**Fact V.4.3**

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$  fails to span all of  $\mathbb{R}^n$  exactly when  $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_m]$  has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & | & a \\ -1 & 0 & | & b \\ 0 & 1 & | & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix}$$

**Activity V.4.4** (*~5 min*)

Consider the set of vectors  $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$ . Does  $\mathbb{R}^4 = \text{span } S$ ?



**Activity V.4.5** (*~10 min*)

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2\}$$

Does  $\mathcal{P}^3 = \text{span } S$ ?

**Definition V.4.6**

A subset of a vector space is called a **subspace** if it is itself a vector space.

**Fact V.4.7**

If  $S$  is a subset of a vector space  $V$ , then  $\text{span } S$  is a subspace of  $V$ .

**Remark V.4.8**

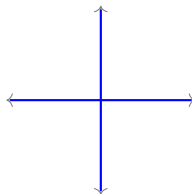
To prove that a subset is a subspace, you need only verify that  $c\mathbf{v} + d\mathbf{w}$  belongs to the subset for any choice of vectors  $\mathbf{v}, \mathbf{w}$  from the subset and any real scalars  $c, d$ .

**Activity V.4.9** ( $\sim 5$  min)

Prove that  $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$  is a subspace of the vector space of all degree-two polynomials by showing that  $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$  belongs to  $P$ .

**Activity V.4.10** (*~10 min*)

Consider the subset of  $\mathbb{R}^2$  where at least one coordinate of each vector is 0.



Find a linear combination  $c\mathbf{v} + d\mathbf{w}$  that does not belong to this subset.

**Fact V.4.11**

Suppose a subset  $S$  of  $V$  is isomorphic to another vector space  $W$ . Then  $S$  is a subspace of  $V$ .

**Activity V.4.12** (*~5 min*)

Show that the set of  $2 \times 2$  matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of  $\mathbb{R}^{2 \times 2}$  by identifying a Euclidean space isomorphic to  $S$ .



## Module S: Structure of vector spaces

# What structure do vector spaces have?

At the end of this module, students will be able to...

- S1. Linear independence.** ... determine if a set of Euclidean vectors is linearly dependent or independent.
- S2. Basis verification.** ... determine if a set of Euclidean vectors is a basis of  $\mathbb{R}^n$ .
- S3. Basis computation.** ... compute a basis for the subspace spanned by a given set of Euclidean vectors.
- S4. Dimension.** ... compute the dimension of a subspace of  $\mathbb{R}^n$ .
- S5. Abstract vector spaces.** ... solve exercises related to standards V3-S4 when posed in terms of polynomials or matrices.
- S6. Basis of solution space.** ... find a basis for the solution set of a homogeneous system of equations.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V2,V3**.

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Academy):  
<http://bit.ly/2y8A0wa>
- Linear combinations of Euclidean vectors (Khan Academy):  
<http://bit.ly/2nK3wne>
- Adding and subtracting complex numbers (Khan Academy):  
<http://bit.ly/1PE3ZMQ>
- Adding and subtracting polynomials (Khan Academy):  
<http://bit.ly/2d5SLGZ>

# Module S Section 1

**Activity S.1.1** (*~15 min*)

In the previous module, we considered

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

and showed that  $\text{span } S \neq \mathbb{R}^4$ . Find two vectors from this set that are linear combinations of the other three vectors.

### Definition S.1.2

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.



**Activity S.1.3** (*~10 min*)

Suppose  $3\mathbf{v}_1 - 5\mathbf{v}_2 = \mathbf{v}_3$ , so the set  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent. Is the vector equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$  consistent with one solution, consistent with infinitely many solutions, or inconsistent?

**Fact S.1.4**

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly dependent if and only if  $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$  is consistent with infinitely many solutions.

**Activity S.1.5** (*~10 min*)

Find

$$\text{RREF} \left[ \begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array} \right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

**Fact S.1.6**

A set of Euclidean vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly dependent if and only if  $\text{RREF} [\mathbf{v}_1 \ \dots \ \mathbf{v}_n]$  has a column without a pivot position.

**Activity S.1.7** ( $\sim 5$  min)

Is the set of Euclidean vectors  $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$  linearly dependent or linearly independent?

**Activity S.1.8** (*~10 min*)

Is the set of polynomials  $\{x^3 + 1, x^2 + 2, 4 - 7x, 2x^3 + x\}$  linearly dependent or linearly independent?

## Module S Section 2

**Activity S.2.1** (*~10 min*)

Last time we saw that  $\{x^3 + 1, x^2 + 2, 4 - 7x, 2x^3 + x\}$  is linearly independent. Show that it spans  $\mathcal{P}^3$ .



**Definition S.2.2**

A **basis** is a linearly independent set that spans a vector space.

**Observation S.2.3**

A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

**Activity S.2.4** (*~15 min*)Which of the following sets are bases for  $\mathbb{R}^4$ ?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

**Activity S.2.5** (*~10 min*)

If  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is a basis for  $\mathbb{R}^4$ , that means  $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$  doesn't have a column without a pivot position, and doesn't have a row of zeros. What is  $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$ ?

**Fact S.2.6**

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$  is a basis for  $\mathbb{R}^n$  if and only if  $m = n$  and

$$\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, a basis for  $\mathbb{R}^n$  must have exactly  $n$  vectors and its square matrix must row-reduce to the **identity matrix** containing all zeros except for a downward diagonal of ones.

**Activity S.2.7** ( $\sim 10$  min)

Consider the set  $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

**Activity S.2.7** ( $\sim 10$  min)

Consider the set  $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

*Part 1:* Use RREF  $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$  to identify which vector may be removed to make the set linearly independent.

**Activity S.2.7** ( $\sim 10$  min)

Consider the set  $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

*Part 1:* Use RREF  $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$  to identify which vector may be removed to make the set linearly independent.

*Part 2:* Find a basis for  $\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$



## Module S Section 3

**Fact S.3.1**

To compute a basis for the subspace  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ , simply remove the vectors corresponding to the non-pivot columns of  $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$ .

**Activity S.3.2** (*~10 min*)

Find all subsets of  $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$  that are a basis for  $\text{span } S$  by changing the order of the vectors in  $S$ .

**Activity S.3.3** (*~10 min*)

Assume  $\mathbf{w}_1 \neq \mathbf{w}_2$  are distinct vectors in  $V$ , which has a basis containing a single vector:  $\{\mathbf{v}\}$ . Could  $\{\mathbf{w}_1, \mathbf{w}_2\}$  be a basis?

**Fact S.3.4**

All bases for a vector space are the same size.

**Definition S.3.5**

The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

**Activity S.3.6** (*~15 min*)Find the dimension of each subspace of  $\mathbb{R}^4$ .

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

**Activity S.3.7** (*~5 min*)

What is the dimension of the vector space of 7th-degree (or less) polynomials  $\mathcal{P}^7$ ?

a) 6

b) 7

c) 8

d) infinite



**Activity S.3.8** (*~5 min*)

What is the dimension of the vector space of all polynomials  $\mathcal{P}$ ?

a) 6

b) 7

c) 8

d) infinite

### Observation S.3.9

Several interesting vector spaces are infinite-dimensional:

- The space of polynomials  $\mathcal{P}$  (consider the set  $\{1, x, x^2, x^3, \dots\}$ ).
- The space of continuous functions  $C(\mathbb{R})$  (which contains all polynomials, in addition to other functions like  $e^x = 1 + x + x^2/2 + x^3/3 + \dots$ ).
- The space of real number sequences  $\mathbb{R}^\infty$  (consider the set  $\{(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, \dots), \dots\}$ ).

**Fact S.3.10**

Every vector space with finite dimension, that is, every vector space with a basis of the form  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is isomorphic to a Euclidean space  $\mathbb{R}^n$ :

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

## Module A: Algebraic properties of linear maps

**How can we understand linear maps algebraically?**

At the end of this module, students will be able to...

- A1. Linear maps and matrices.** ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.
- A2. Linear map verification.** ... determine if a map between vector spaces of polynomials is linear or not.
- A3. Injectivity and surjectivity.** ... determine if a given linear map is injective and/or surjective.
- A4. Kernel and Image.** ... compute a basis for the kernel and a basis for the image of a linear map.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing **E1, E2, E3, E4**.
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace **V3**.
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of vectors is a basis **S2**.

# Module A Section 1



**Definition A.1.1**

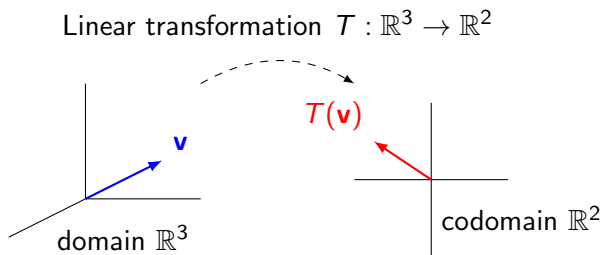
A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if  $V$  and  $W$  are vector spaces, a map  $T : V \rightarrow W$  is called a linear transformation if

- ①  $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$  for any  $\mathbf{v}, \mathbf{w} \in V$
- ②  $T(c\mathbf{v}) = cT(\mathbf{v})$  for any  $c \in \mathbb{R}$ ,  $\mathbf{v} \in V$ .

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.

**Definition A.1.2**

Given a linear transformation  $T : V \rightarrow W$ ,  $V$  is called the **domain** of  $T$  and  $W$  is called the **co-domain** of  $T$ .



**Example A.1.3**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$$

To show that  $T$  is linear, we must verify...

$$T \left( \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = T \left( \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \right) = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

$$T \left( \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right) + T \left( \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 - z_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - z_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

And also...

$$T \left( c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T \left( \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \right) = \begin{bmatrix} cx - cz \\ cy \end{bmatrix} \text{ and } cT \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = c \begin{bmatrix} x - z \\ y \end{bmatrix} = \begin{bmatrix} cx - cz \\ cy \end{bmatrix}$$

Therefore  $T$  is a linear transformation.

**Activity A.1.4** (*~15 min*)

Determine if each of the following maps are linear transformations

**Activity A.1.4** (*~15 min*)

Determine if each of the following maps are linear transformations

*Part 1:*  $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $T_1 \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \sqrt{x^2 + y^2}$ .

**Activity A.1.4** (*~15 min*)

Determine if each of the following maps are linear transformations

*Part 1:*  $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $T_1 \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \sqrt{x^2 + y^2}$ .

*Part 2:*  $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T_2 \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$

**Activity A.1.4** ( $\sim 15$  min)

Determine if each of the following maps are linear transformations

*Part 1:*  $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $T_1 \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \sqrt{x^2 + y^2}$ .

*Part 2:*  $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T_2 \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$

*Part 3:*  $T_3 : \mathcal{P}^d \rightarrow \mathcal{P}^{d-1}$  given by  $T_3(f(x)) = f'(x)$ .

**Activity A.1.4** ( $\sim 15$  min)

Determine if each of the following maps are linear transformations

*Part 1:*  $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $T_1 \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \sqrt{x^2 + y^2}$ .

*Part 2:*  $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T_2 \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$

*Part 3:*  $T_3 : \mathcal{P}^d \rightarrow \mathcal{P}^{d-1}$  given by  $T_3(f(x)) = f'(x)$ .

*Part 4:*  $T_4 : \mathcal{P} \rightarrow \mathcal{P}$  given by  $T_4(f(x)) = f(x) + x^2$



**Activity A.1.5** ( $\sim 5$  min)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear transformation, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and  $T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b)  $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(c)  $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(d)  $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

**Activity A.1.6** ( $\sim 3$  min)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear transformation, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and  $T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b)  $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(c)  $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(d)  $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

**Activity A.1.7** (*~5 min*)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear transformation, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and  $T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c)  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d)  $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

**Activity A.1.8** ( $\sim 2$  min)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear transformation, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and  $T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c)  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d)  $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

**Activity A.1.9** ( $\sim 5$  min)

Suppose  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  is a linear transformation. How many facts of the form  $T(\mathbf{v}_i) = \mathbf{w}_i$  do you need to know in order to be able to compute  $T(\mathbf{v})$  for *any*  $\mathbf{v} \in \mathbb{R}^4$ ?

- (a) 2
- (b) 3
- (c) 4
- (d) 5
- (e) You need infinitely many

(In this situation, we say that the vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  **determine**  $T$ .)

**Fact A.1.10**

Consider any basis  $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  for  $V$ . Since every vector can be written *uniquely* as a linear combination of basis vectors, every linear transformation  $T : V \rightarrow W$  is determined by those basis vectors.

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = x_1T(\mathbf{b}_1) + \cdots + x_nT(\mathbf{b}_n)$$

**Definition A.1.11**

The **standard basis** of  $\mathbb{R}^n$  is the (ordered) basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \dots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Since linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is determined by the values of each  $T(\mathbf{e}_i)$ , it's convenient to store this information in the  $m \times n$  **standard matrix**  $[T(\mathbf{e}_1) \ \dots \ T(\mathbf{e}_n)]$ .

**Example A.1.12**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation determined by the following values for  $T$  applied to the standard basis of  $\mathbb{R}^3$ .

$$T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Then the standard matrix corresponding to  $T$  is

$$\begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix}.$$



**Activity A.1.13** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard basis.

**Activity A.1.14** ( $\sim 5$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \end{bmatrix}.$$

Compute  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$ .

**Activity A.1.15** (*~10 min*)

Let  $D : \mathcal{P}^3 \rightarrow \mathcal{P}^2$  be the derivative map  $D(f(x)) = f'(x)$ . (Earlier we showed this is a linear transformation.)

**Activity A.1.15** ( $\sim 10$  min)

Let  $D : \mathcal{P}^3 \rightarrow \mathcal{P}^2$  be the derivative map  $D(f(x)) = f'(x)$ . (Earlier we showed this is a linear transformation.)

*Part 1:* Write down an equivalent linear transformation  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  by converting  $\{1, x, x^2, x^3\}$  and  $\{D(1), D(x), D(x^2), D(x^3)\}$  into appropriate vectors in  $\mathbb{R}^4$  and  $\mathbb{R}^3$ .

**Activity A.1.15** ( $\sim 10$  min)

Let  $D : \mathcal{P}^3 \rightarrow \mathcal{P}^2$  be the derivative map  $D(f(x)) = f'(x)$ . (Earlier we showed this is a linear transformation.)

*Part 1:* Write down an equivalent linear transformation  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  by converting  $\{1, x, x^2, x^3\}$  and  $\{D(1), D(x), D(x^2), D(x^3)\}$  into appropriate vectors in  $\mathbb{R}^4$  and  $\mathbb{R}^3$ .

*Part 2:* Write the standard matrix corresponding to  $T$ .

## Module A Section 2

**Definition A.2.1**

Let  $T : V \rightarrow W$  be a linear transformation.  $T$  is called **injective** or **one-to-one** if  $T$  does not map two distinct values to the same place. More precisely,  $T$  is injective if  $T(\mathbf{v}) \neq T(\mathbf{w})$  whenever  $\mathbf{v} \neq \mathbf{w}$ .

**Activity A.2.2** (*~5 min*)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of  $T$  is thus  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

Is  $T$  injective?



**Activity A.2.3** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of  $T$  is thus  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ .

Is  $T$  injective?

**Definition A.2.4**

Let  $T : V \rightarrow W$  be a linear transformation.  $T$  is called **surjective** or **onto** if every element of  $W$  is mapped to by an element of  $V$ . More precisely, for every  $\mathbf{w} \in W$ , there is some  $\mathbf{v} \in V$  with  $T(\mathbf{v}) = \mathbf{w}$ .

**Activity A.2.5** (*~5 min*)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of  $T$  is thus  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

Is  $T$  surjective?

**Activity A.2.6** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of  $T$  is thus  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ .

Is  $T$  surjective?

**Definition A.2.7**

Let  $T : V \rightarrow W$  be a linear transformation. The **kernel** of  $T$  is an important subspace of  $V$  defined by

$$\ker T = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

**Activity A.2.8** ( $\sim 5$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the standard matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Find the kernel of  $T$ .

**Activity A.2.9** ( $\sim 5$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ . Find the kernel of  $T$ .

**Activity A.2.10** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$



**Activity A.2.10** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Write a system of equations whose solution set is the kernel.

**Activity A.2.10** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Write a system of equations whose solution set is the kernel.

*Part 2:* Use  $\text{RREF}(A)$  to solve the system of equations and find the kernel of  $T$ .

**Activity A.2.10** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Write a system of equations whose solution set is the kernel.

*Part 2:* Use  $\text{RREF}(A)$  to solve the system of equations and find the kernel of  $T$ .

*Part 3:* Find a basis for the kernel of  $T$ .

**Definition A.2.11**

Let  $T : V \rightarrow W$  be a linear transformation. The **image** of  $T$  is an important subspace of  $W$  defined by

$$\text{Im } T = \{ \mathbf{w} \in W \mid \text{there is some } v \in V \text{ with } T(\mathbf{v}) = \mathbf{w} \}$$

**Activity A.2.12** (*~5 min*)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the standard matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Find the image of  $T$ .

**Activity A.2.13** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ . Find the image of  $T$ .

**Activity A.2.14** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

**Activity A.2.14** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Find a convenient set of vectors  $S \subseteq \mathbb{R}^2$  such that  $\text{span } S = \text{Im } T$ .



**Activity A.2.14** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Find a convenient set of vectors  $S \subseteq \mathbb{R}^2$  such that  $\text{span } S = \text{Im } T$ .

*Part 2:* Find a convenient basis for the image of  $T$ .

**Observation A.2.15**

Let  $T : V \rightarrow W$  be a linear transformation with corresponding matrix  $A$ .

- If  $A$  is a matrix corresponding to  $T$ , the kernel is the solution set of the homogeneous system with coefficients given by  $A$ .
- If  $A$  is a matrix corresponding to  $T$ , the image is the span of the columns of  $A$ .

## Module A Section 3

### Observation A.3.1

Let  $T : V \rightarrow W$ . We have previously defined the following terms.

- $T$  is called **injective** or **one-to-one** if  $T$  does not map two distinct values to the same place.
- $T$  is called **surjective** or **onto** if every element of  $W$  is mapped to by some element of  $V$ .
- The **kernel** of  $T$  is the set of all things that are mapped to **0**. It is a subspace of  $V$ .
- The **image** of  $T$  is the set of all things in  $W$  that are mapped to by something in  $V$ . It is a subspace of  $W$ .

**Activity A.3.2** ( $\sim 5$  min)

Let  $T : V \rightarrow W$  be a linear transformation where  $\ker T = \{\mathbf{0}\}$ . Can you answer either of the following questions about  $T$ ?

(a) Is  $T$  injective?

(b) Is  $T$  surjective?

(Hint: If  $T(\mathbf{v}) = T(\mathbf{w})$ , then what is  $T(\mathbf{v} - \mathbf{w})$ ?)

**Fact A.3.3**

A linear transformation  $T$  is injective **if and only if**  $\ker T = \{\mathbf{0}\}$ . Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

**Activity A.3.4** ( $\sim 5$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear transformation where  $\text{Im } T = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$ .

Can you answer either of the following questions about  $T$ ?

- (a) Is  $T$  injective?
- (b) Is  $T$  surjective?

**Fact A.3.5**

A linear transformation  $T : V \rightarrow W$  is surjective **if and only if**  $\text{Im } T = W$ . Put another way, a surjective linear transformation may be recognized by its same codomain and image.



**Activity A.3.6** ( $\sim 15$  min)

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with standard matrix  $A$ . Sort the following claims into two groups of equivalent statements.

- (a)  $T$  is injective
- (b)  $T$  is surjective
- (c) The kernel of  $T$  is trivial.
- (d) The columns of  $A$  span  $\mathbb{R}^m$
- (e) The columns of  $A$  are linearly independent
- (f) Every column of  $\text{RREF}(A)$  has a pivot.
- (g) Every row of  $\text{RREF}(A)$  has a pivot.
- (h) The image of  $T$  equals its codomain.
- (i) The system of linear equations given by the augmented matrix  $[A \mid \mathbf{b}]$  has a solution for all  $\mathbf{b} \in \mathbb{R}^m$
- (j) The system of linear equations given by the augmented matrix  $[A \mid \mathbf{0}]$  has exactly one solution.

**Definition A.3.7**

If  $T : V \rightarrow W$  is both injective and surjective, it is called **bijjective**.

**Activity A.3.8** ( $\sim 5$  min)

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a bijective linear map with standard matrix  $A$ . Label each of the following as true or false.

- (a) The columns of  $A$  form a basis for  $\mathbb{R}^m$
- (b)  $\text{RREF}(A)$  is the identity matrix.
- (c) The system of linear equations given by the augmented matrix  $[A \mid \mathbf{b}]$  has exactly one solution for all  $\mathbf{b} \in \mathbb{R}^m$ .

**Activity A.3.9** ( $\sim 10$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is bijective.

**Activity A.3.10** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is bijective.

**Activity A.3.11** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y + z \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is bijective.

**Activity A.3.12** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is bijective.

# Module M: Understanding Matrices Algebraically



# What algebraic structure do matrices have?

At the end of this module, students will be able to...

**M1. Matrix Multiplication.** ... multiply matrices.

**M2. Invertible Matrices.** ... determine if a square matrix is invertible or not.

**M3. Matrix inverses.** ... compute the inverse matrix of an invertible matrix.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers
- Solve systems of linear equations **E3**
- Find the matrix corresponding to a linear transformation **A1**
- Determine if a linear transformation is injective and/or surjective **A3**
- Interpret the ideas of injectivity and surjectivity in multiple ways

The following resources will help you prepare for this module.

- Function composition (Khan Academy): <http://bit.ly/2wkz7f3>

# Module M Section 1

**Activity M.1.1** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

What is the domain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
- (b)  $\mathbb{R}^2$
- (c)  $\mathbb{R}^3$
- (d)  $\mathbb{R}^4$

**Activity M.1.2** ( $\sim 2$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

What is the codomain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
- (b)  $\mathbb{R}^2$
- (c)  $\mathbb{R}^3$
- (d)  $\mathbb{R}^4$

**Activity M.1.3** ( $\sim 2$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

The standard matrix of  $S \circ T$  will lie in which matrix space?

- (a)  $4 \times 3$  matrices
- (b)  $4 \times 2$  matrices
- (c)  $3 \times 2$  matrices
- (d)  $2 \times 3$  matrices
- (e)  $2 \times 4$  matrices
- (f)  $3 \times 4$  matrices



**Activity M.1.4** (*~15 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

**Activity M.1.4** (*~15 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

*Part 1:* Compute  $(S \circ T)(\mathbf{e}_1)$

**Activity M.1.4** (*~15 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

*Part 1:* Compute  $(S \circ T)(\mathbf{e}_1)$

*Part 2:* Compute  $(S \circ T)(\mathbf{e}_2)$

**Activity M.1.4** (*~15 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

*Part 1:* Compute  $(S \circ T)(\mathbf{e}_1)$

*Part 2:* Compute  $(S \circ T)(\mathbf{e}_2)$

*Part 3:* Compute  $(S \circ T)(\mathbf{e}_3)$ .

**Activity M.1.4** (*~15 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

*Part 1:* Compute  $(S \circ T)(\mathbf{e}_1)$

*Part 2:* Compute  $(S \circ T)(\mathbf{e}_2)$

*Part 3:* Compute  $(S \circ T)(\mathbf{e}_3)$ .

*Part 4:* Find the standard matrix of  $S \circ T$ .

**Activity M.1.5** ( $\sim 2$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$  and  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given

by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ .

What is the domain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
- (b)  $\mathbb{R}^2$
- (c)  $\mathbb{R}^3$
- (d)  $\mathbb{R}^4$

**Activity M.1.6** ( $\sim 2$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$  and  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given

by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ .

What is the codomain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
- (b)  $\mathbb{R}^2$
- (c)  $\mathbb{R}^3$
- (d)  $\mathbb{R}^4$

**Activity M.1.7** ( $\sim 2$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$  and  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given

by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ .

The standard matrix of  $S \circ T$  will lie in which matrix space?

- (a)  $2 \times 2$  matrices
- (b)  $2 \times 3$  matrices
- (c)  $3 \times 2$  matrices
- (d)  $3 \times 3$  matrices



**Activity M.1.8** (*~10 min*)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$  and  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given

by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ .

Find the standard matrix of  $S \circ T$ .

**Activity M.1.9** ( $\sim 5$  min)

Let  $T : \mathbb{R}^1 \rightarrow \mathbb{R}^4$  be given by the matrix  $B = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}$  and  $S : \mathbb{R}^4 \rightarrow \mathbb{R}^1$  be given by

the matrix  $A = \begin{bmatrix} 2 & 3 & 2 & 5 \end{bmatrix}$ .

Find the standard matrix of  $S \circ T$ .

**Definition M.1.10**

We define the product of a  $m \times n$  matrix  $A$  and a  $n \times k$  matrix  $B$  to be the  $m \times k$  standard matrix (denoted  $AB$ ) of the composition map of the two corresponding linear functions.

**Fact M.1.11**

If  $AB$  is defined,  $BA$  need not be defined, and if it is defined, it is in general different from  $AB$ .

**Activity M.1.12** (*~10 min*)

Let  $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ . Compute  $AB$ .

**Activity M.1.13** (*~5 min*)

Let  $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$  and  $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ . Compute  $AX$

**Observation M.1.14**

Consider the system of equations

$$3x + y - z = 5$$

$$2x + 4z = -7$$

$$-x + 3y + 5z = 2$$

We can interpret this as a **matrix equation**  $AX = B$  where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$$

For this reason, we will swap out the use of Euclidean vectors  $\mathbf{x} \in \mathbb{R}^n$  and  $n \times 1$  matrices  $X$  whenever it is convenient.

## Module M Section 2



**Activity M.2.1** ( $\sim 5$  min)

Let  $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ . Find a  $3 \times 3$  matrix  $I$  such that  $IA = A$ , that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

**Definition M.2.2**

The identity matrix  $I_n$  (or just  $I$  when  $n$  is obvious from context) is the  $n \times n$  matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

**Fact M.2.3**

For any square matrix  $A$ ,  $IA = AI = A$ :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

**Activity M.2.4** ( $\sim 15$  min)

Each row operation can be interpreted as a type of matrix multiplication.

**Activity M.2.4** ( $\sim 15$  min)

Each row operation can be interpreted as a type of matrix multiplication.

*Part 1:* Tweak the identity matrix slightly to create a matrix that doubles the third row of  $A$ :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

**Activity M.2.4** ( $\sim 15$  min)

Each row operation can be interpreted as a type of matrix multiplication.

*Part 1:* Tweak the identity matrix slightly to create a matrix that doubles the third row of  $A$ :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

*Part 2:* Create a matrix that swaps the second and third rows of  $A$ :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

**Activity M.2.4** (*~15 min*)

Each row operation can be interpreted as a type of matrix multiplication.

*Part 1:* Tweak the identity matrix slightly to create a matrix that doubles the third row of  $A$ :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

*Part 2:* Create a matrix that swaps the second and third rows of  $A$ :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

*Part 3:* Create a matrix that adds 5 times the third row of  $A$  to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5 & 7+5 & -1-5 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

**Fact M.2.5**

If  $R$  is the result of applying a row operation to  $I$ , then  $RA$  is the result of applying the same row operation to  $A$ .

This means that for any matrix  $A$ , we can find a series of matrices  $R_1, \dots, R_k$  corresponding to the row operations such that

$$R_1 R_2 \cdots R_k A = \text{RREF}(A).$$

That is, row reduction can be thought of as the result of matrix multiplication.



**Activity M.2.6** ( $\sim 15$  min)

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with standard matrix  $A$ . Sort the following items into groups of statements about  $T$ .

- (a)  $T$  is injective (i.e. one-to-one)
- (b)  $T$  is surjective (i.e. onto)
- (c)  $T$  is bijective (i.e. both injective and surjective)
- (d)  $AX = B$  has a solution for all  $m \times 1$  matrices  $B$
- (e)  $AX = B$  has a unique solution for all  $m \times 1$  matrices  $B$
- (f)  $AX = 0$  has a unique solution.
- (g) The columns of  $A$  span  $\mathbb{R}^m$
- (h) The columns of  $A$  are linearly independent
- (i) The columns of  $A$  are a basis of  $\mathbb{R}^m$
- (j) Every column of  $\text{RREF}(A)$  has a pivot
- (k) Every row of  $\text{RREF}(A)$  has a pivot
- (l)  $m = n$  and  $\text{RREF}(A) = I$

**Activity M.2.7** ( $\sim 5$  min)

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with matrix  $A$ . If  $T$  is injective, which of the following cannot be true?

- (a)  $A$  has strictly more columns than rows
- (b)  $A$  has the same number of rows as columns (i.e.  $A$  is square)
- (c)  $A$  has strictly more rows than columns

**Activity M.2.8** ( $\sim 5$  min)

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with matrix  $A$ . If  $T$  is surjective, which of the following cannot be true?

- (a)  $A$  has strictly more columns than rows
- (b)  $A$  has the same number of rows as columns (i.e.  $A$  is square)
- (c)  $A$  has strictly more rows than columns

**Activity M.2.9** ( $\sim 5$  min)

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with matrix  $A$ . If  $T$  is bijective, which of the following cannot be true?

- (a)  $A$  has strictly more columns than rows
- (b)  $A$  has the same number of rows as columns (i.e.  $A$  is square)
- (c)  $A$  has strictly more rows than columns

## Module M Section 3

### Definition M.3.1

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear map with standard matrix  $A$ .

- If  $T$  is a bijection and  $B$  is any  $\mathbb{R}^n$  vector, then  $T(X) = AX = B$  has a unique solution  $X$ .
- So we may define an **inverse map**  $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by setting  $T^{-1}(B) = X$  to be this unique solution.
- Let  $A^{-1}$  be the standard matrix for  $T^{-1}$ . We call  $A^{-1}$  the **inverse matrix** of  $A$ , so we also say that  $A$  is **invertible**.

**Activity M.3.2** (*~10 min*)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the bijective linear map defined by  $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$ .

It can be shown that  $T$  is bijective and has the inverse map

$$T^{-1} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}.$$

**Activity M.3.2** ( $\sim 10$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the bijective linear map defined by  $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$ .

It can be shown that  $T$  is bijective and has the inverse map

$$T^{-1} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}.$$

*Part 1:* Compute  $(T^{-1} \circ T) \left( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$ .



**Activity M.3.2** ( $\sim 10$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the bijective linear map defined by  $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$ .

It can be shown that  $T$  is bijective and has the inverse map

$$T^{-1} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}.$$

*Part 1:* Compute  $(T^{-1} \circ T) \left( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$ .

*Part 2:* If  $A$  is the standard matrix for  $T$  and  $A^{-1}$  is the standard matrix for  $T^{-1}$ , what must  $A^{-1}A$  be?

**Observation M.3.3**

$T^{-1} \circ T = T \circ T^{-1}$  is the identity map for any bijective linear transformation  $T$ .  
Therefore  $A^{-1}A = AA^{-1} = I$  is the identity matrix for any invertible matrix  $A$ .

**Activity M.3.4** (*~20 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the matrix  $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ .

**Activity M.3.4** ( $\sim 20$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the matrix  $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ .

*Part 1:* Solve  $T(X) = \mathbf{e}_1$  to find  $T^{-1}(\mathbf{e}_1)$ .

**Activity M.3.4** ( $\sim 20$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the matrix  $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ .

*Part 1:* Solve  $T(X) = \mathbf{e}_1$  to find  $T^{-1}(\mathbf{e}_1)$ .

*Part 2:* Solve  $T(X) = \mathbf{e}_2$  to find  $T^{-1}(\mathbf{e}_2)$ .

**Activity M.3.4** ( $\sim 20$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the matrix  $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ .

*Part 1:* Solve  $T(X) = \mathbf{e}_1$  to find  $T^{-1}(\mathbf{e}_1)$ .

*Part 2:* Solve  $T(X) = \mathbf{e}_2$  to find  $T^{-1}(\mathbf{e}_2)$ .

*Part 3:* Solve  $T(X) = \mathbf{e}_3$  to find  $T^{-1}(\mathbf{e}_3)$ .

**Activity M.3.4** ( $\sim 20$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the matrix  $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ .

*Part 1:* Solve  $T(X) = \mathbf{e}_1$  to find  $T^{-1}(\mathbf{e}_1)$ .

*Part 2:* Solve  $T(X) = \mathbf{e}_2$  to find  $T^{-1}(\mathbf{e}_2)$ .

*Part 3:* Solve  $T(X) = \mathbf{e}_3$  to find  $T^{-1}(\mathbf{e}_3)$ .

*Part 4:* Compute  $A^{-1}$ , the standard matrix for  $T^{-1}$ .

**Observation M.3.5**

We could have solved these three systems simultaneously by row reducing the matrix  $[A \mid I]$  at once.

$$A = \left[ \begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array} \right]$$



**Activity M.3.6** (*~10 min*)

Find the inverse  $A^{-1}$  of the matrix  $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$  by row-reducing  $[A \mid I]$ .

**Activity M.3.7** (*~10 min*)

Is the matrix  $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$  invertible? Give a reason for your answer.

**Observation M.3.8**

A matrix  $A \in \mathbb{R}^{n \times n}$  is invertible if and only if  $\text{RREF}(A) = I_n$ .

## Module G: Geometry of Linear Maps

**How can we understand linear maps geometrically?**

At the end of this module, students will be able to...

- G1. Row operations.** ... represent a row operation as matrix multiplication, and compute how the operation affects the determinant.
- G2. Determinants.** ... compute the determinant of a square matrix.
- G3. Eigenvalues.** ... find the eigenvalues of a  $2 \times 2$  matrix.
- G4. Eigenvectors.** ... find a basis for the eigenspace of a square matrix associated with a given eigenvalue.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **A1**.
- Recall and use the definition of a linear transformation **A2**.
- Find all roots of quadratic polynomials (including complex ones), and be able to use the rational root theorem to find all rational roots of a higher degree polynomial.
- Interpret the statement “ $A$  is an invertible matrix” in many equivalent ways in different contexts.

The following resources will help you prepare for this module.

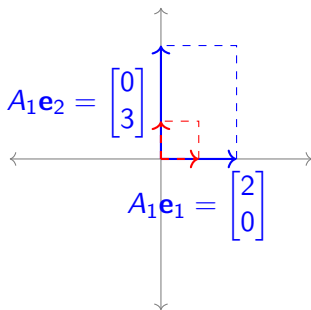
- Finding the area of a parallelogram (Khan Academy):  
<http://bit.ly/2B05iWx>
- Factoring quadratics (Khan Academy): <http://bit.ly/1XjfbV2>
- Finding complex roots of quadratics (Khan Academy):  
<http://bit.ly/1HH3yAA>



# Module G Section 1

**Activity G.1.1** ( $\sim 5$  min)

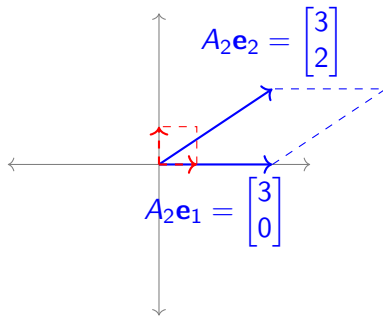
The image below illustrates how the linear transformation  $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$  transforms the unit square.



- (a) What is the area of the transformed unit square?
- (b) Find two vectors that were stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.

**Activity G.1.2** ( $\sim 5$  min)

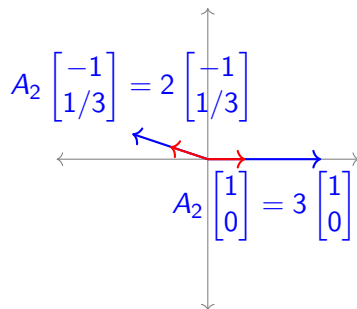
The image below illustrates how the linear transformation  $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $A_2 = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$  transforms the unit square.



- (a) What is the area of the transformed unit square?
- (b) Find at least one vector that was stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.

**Observation G.1.3**

It's possible to find two non-parallel vectors that are stretched by the transformation given by  $A_2$ :



The process for finding such vectors will be covered later in this module.

**Activity G.1.4** ( $\sim 5$  min)

Consider the linear transformation given by the standard matrix  $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ .

- (a) Sketch the transformation of the unit square (the parallelogram given by the columns of the standard matrix).
- (b) Compute the area of the transformed unit square.

**Activity G.1.5** ( $\sim 5$  min)

Consider the linear transformation given by the standard matrix  $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

- (a) Sketch the transformation of the unit square.
- (b) Compute the area of the transformed unit square.

**Activity G.1.6** ( $\sim 5$  min)

Consider the linear transformation given by the standard matrix  $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ .

- (a) Sketch the transformation of the unit square.
- (b) Compute the area of the transformed unit square.

**Remark G.1.7**

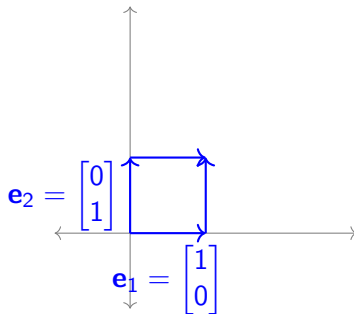
The area of the transformed unit square measures the factor by which all areas are transformed by a linear transformation.

We will define the **determinant** of a square matrix  $A$ , or  $\det(A)$  for short, to be this factor. But what properties must this function satisfy?



**Activity G.1.8** ( $\sim 2$  min)

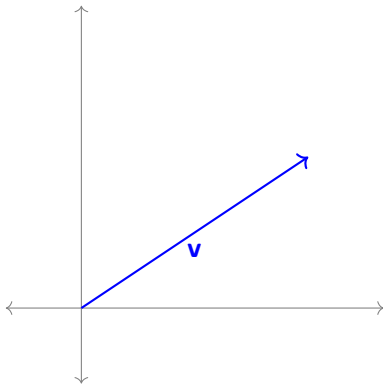
The transformation of the unit square by the standard matrix  $[\mathbf{e}_1 \ \mathbf{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$  is illustrated below. What is  $\det([\mathbf{e}_1 \ \mathbf{e}_2]) = \det(I)$ , that is, by what factor has the area of the unit square been scaled?



- a) 0
- b) 1
- c) 2
- d) Cannot be determined

**Activity G.1.9** ( $\sim 2$  min)

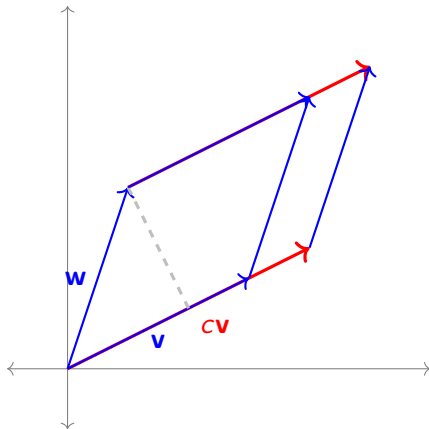
The transformation of the unit square by the standard matrix  $[\mathbf{v} \ \mathbf{v}]$  is illustrated below: both  $T(\mathbf{e}_1) = T(\mathbf{e}_2) = \mathbf{v}$ . What is  $\det([\mathbf{v} \ \mathbf{v}])$ , that is, by what factor has area been scaled?



- a) 0
- b) 1
- c) 2
- d) Cannot be determined

**Activity G.1.10** ( $\sim 5$  min)

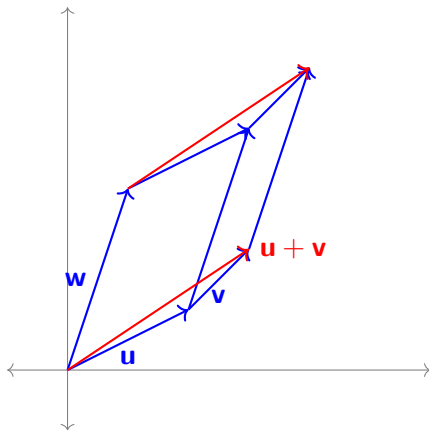
The transformations of the unit square by the standard matrices  $[\mathbf{v} \ \mathbf{w}]$  and  $[c\mathbf{v} \ \mathbf{w}]$  are illustrated below. How are  $\det([\mathbf{v} \ \mathbf{w}])$  and  $\det([c\mathbf{v} \ \mathbf{w}])$  related?



- a)  $\det([\mathbf{v} \ \mathbf{w}]) = \det([c\mathbf{v} \ \mathbf{w}])$
- b)  $c + \det([\mathbf{v} \ \mathbf{w}]) = \det([c\mathbf{v} \ \mathbf{w}])$
- c)  $c \det([\mathbf{v} \ \mathbf{w}]) = \det([c\mathbf{v} \ \mathbf{w}])$

**Activity G.1.11** ( $\sim 5$  min)

The transformations of unit squares by the standard matrices  $[\mathbf{u} \ \mathbf{w}]$ ,  $[\mathbf{v} \ \mathbf{w}]$  and  $[\mathbf{u} + \mathbf{v} \ \mathbf{w}]$  are illustrated below. How is  $\det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$  related to  $\det([\mathbf{u} \ \mathbf{w}])$  and  $\det([\mathbf{v} \ \mathbf{w}])$ ?



- a)  $\det([\mathbf{u} \ \mathbf{w}]) = \det([\mathbf{v} \ \mathbf{w}]) = \det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$
- b)  $\det([\mathbf{u} \ \mathbf{w}]) + \det([\mathbf{v} \ \mathbf{w}]) = \det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$
- c)  $\det([\mathbf{u} \ \mathbf{w}]) \det([\mathbf{v} \ \mathbf{w}]) = \det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$

**Definition G.1.12**

The **determinant** is the unique function  $\det : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$  satisfying the following three properties:

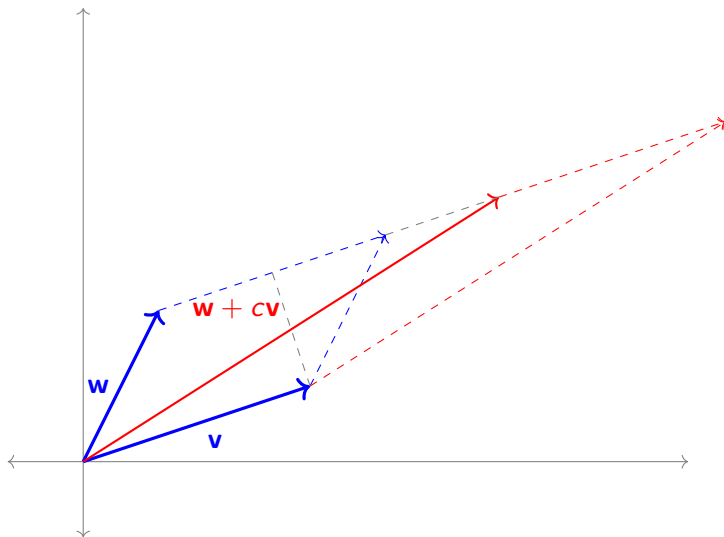
P1:  $\det(I) = 1$

P2:  $\det([\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]) = 0$  whenever two columns of the matrix are identical.

P3:  $\det[\cdots \ c\mathbf{v} + d\mathbf{w} \ \cdots] = c \det[\cdots \ \mathbf{v} \ \cdots] + d \det[\cdots \ \mathbf{w} \ \cdots]$ , assuming all other columns are equal.

**Observation G.1.13**

What happens if we had a multiple of one column to another?



The base of both parallelograms is  $\mathbf{v}$ , while the height has not changed. Thus

$$\det([\mathbf{v} \quad \mathbf{w} + c\mathbf{v}]) = \det([\mathbf{v} \quad \mathbf{w}])$$

**Observation G.1.14**

Swapping columns can be obtained from a sequence of adding column multiples.

$$\begin{aligned}\det([\mathbf{v} \quad \mathbf{w}]) &= \det([\mathbf{v} + \mathbf{w} \quad \mathbf{w}]) \\ &= \det([\mathbf{v} + \mathbf{w} \quad \mathbf{w} - (\mathbf{v} + \mathbf{w})]) \\ &= \det([\mathbf{v} + \mathbf{w} \quad -\mathbf{v}]) \\ &= \det([\mathbf{v} + \mathbf{w} - \mathbf{v} \quad -\mathbf{v}]) \\ &= \det([\mathbf{w} \quad -\mathbf{v}]) \\ &= -\det([\mathbf{w} \quad \mathbf{v}])\end{aligned}$$

So swapping two columns results in a negation of the determinant. Therefore, determinants represent a *signed* area, since they are not always positive.

**Fact G.1.15**

We've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

- (a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c \det([\cdots \mathbf{v} \cdots]) = \det([\cdots c\mathbf{v} \cdots])$$

- (b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \mathbf{v} \cdots \mathbf{w} \cdots]) = -\det([\cdots \mathbf{w} \cdots \mathbf{v} \cdots])$$

- (c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \mathbf{v} \cdots \mathbf{w} \cdots]) = \det([\cdots \mathbf{v} + c\mathbf{w} \cdots \mathbf{w} \cdots])$$



**Activity G.1.16** (*~5 min*)

The transformation given by the standard matrix  $A$  scales areas by 4, and the transformation given by the standard matrix  $B$  scales areas by 3. How must the transformation given by the standard matrix  $AB$  scale areas?

- (a) 1
- (b) 7
- (c) 12
- (d) Cannot be determined

**Fact G.1.17**

Since the transformation given by the standard matrix  $AB$  is obtained by applying the transformations given by  $A$  and  $B$ , it follows that

$$\det(AB) = \det(A) \det(B)$$

## Module G Section 2

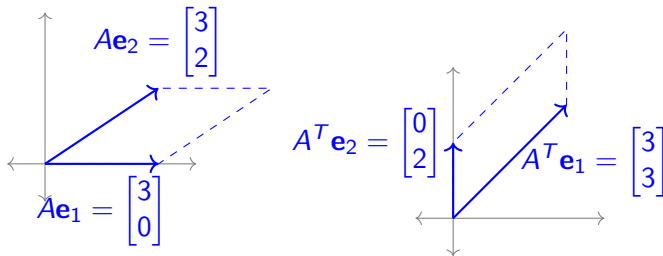
**Definition G.2.1**

The **transpose** of a matrix is given by rewriting its columns as rows and vice versa:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

**Fact G.2.2**

It is possible to prove that the determinant of a matrix and its transpose are the same. For example, let  $A = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$ , so  $A^T = \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}$ ; both matrices scale the unit square by 6, even though the parallelograms are not congruent.



**Fact G.2.3**

We previously figured out that column operations can be used to simplify determinants; since  $\det(A) = \det(A^T)$ , we can also use row operations:

① Multiplying rows by scalars:  $\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$

② Swapping two rows:  $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$

③ Adding multiples of rows to other rows:  $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$

**Activity G.2.4** (*~10 min*)

Compute the determinant of  $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$  by row reducing it to a nicer matrix.

For example,  $\det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ .

**Fact G.2.5**

This same process allows us to prove a more convenient formula:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

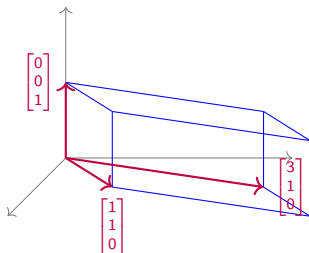
In higher dimensions, the formulas become unreasonable. For example, the formula for  $4 \times 4$  matrices has 24 terms!



**Activity G.2.6** ( $\sim 5$  min)

The following image illustrates the transformation of the unit cube by the matrix

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



This volume is equal to which of the following areas?

(a)  $\det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

(b)  $\det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$

(c)  $\det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$

(d)  $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

**Fact G.2.7**

If column  $i$  of a matrix is  $\mathbf{e}_i$ , then both column and row  $i$  may be removed without changing the value of the determinant. For example, the second column of the following matrix is  $\mathbf{e}_2$ , so:

$$\det \begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Therefore the same holds for the transpose:

$$\det \begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Geometrically, this is the fact that if the height is 1, the base  $\times$  height formula reduces to the area/volume/etc. of the  $n - 1$  dimensional base.

**Activity G.2.8** ( $\sim 5$  min)

Compute  $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$ .

**Activity G.2.9** ( $\sim 5$  min)

Compute  $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$ .

(a)  $-1$ (b)  $0$ (c)  $1$

**Activity G.2.10** ( $\sim 10$  min)

Compute  $\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix}$

*Hint:*  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ .

(a) 3

(b) 6

(c) 9

(d) 12

**Activity G.2.11** (*~15 min*)

Compute  $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$ .

**Observation G.2.12**

$$\begin{aligned}
 \det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} &= (-1)(0) \det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} + (1)(3) \det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} + \\
 &\quad (-1)(2) \det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} + (1)(0) \det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} \\
 &= 3 \det \begin{bmatrix} 2 & 5 & 0 \\ 1 & 0 & 3 \\ -1 & 2 & 2 \end{bmatrix} + (-1)(2) \det \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ -1 & -1 & 2 \end{bmatrix}
 \end{aligned}$$



This technique is called **Laplace expansion** or **cofactor expansion**.

**Activity G.2.13** (*~10 min*)

Compute  $\det \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 1 & 2 & 0 & 3 \\ -1 & -3 & 2 & -2 \end{bmatrix}$ .



## Module G Section 3

**Activity G.3.1** (*~5 min*)

An invertible matrix  $M$  and its inverse  $M^{-1}$  are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Compute  $\det(M)$  and  $\det(M^{-1})$ .

**Activity G.3.2** ( $\sim 5$  min)

Suppose the matrix  $M$  is invertible, so there exists  $M^{-1}$  with  $MM^{-1} = I$ . It follows that  $\det(M)\det(M^{-1}) = \det(I)$ .

What is the only number that  $\det(M)$  cannot equal?

(a)  $-1$

(b)  $0$

(c)  $1$

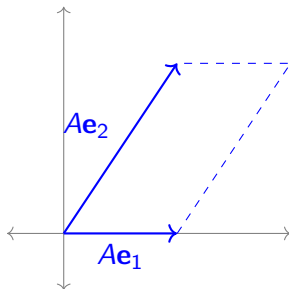
(d)  $\frac{1}{\det(M^{-1})}$

**Fact G.3.3**

- For every invertible matrix  $M$ ,  $\det(M^{-1}) = \frac{1}{\det(M)}$ .
- Furthermore, a square matrix  $M$  is invertible if and only if  $\det(M) \neq 0$ .

**Observation G.3.4**

Consider the linear transformation  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the matrix  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



It is easy to see geometrically that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily verified by computation) that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

**Definition G.3.5**

Let  $A \in \mathbb{R}^{n \times n}$ . An **eigenvector** is a vector  $\mathbf{x} \in \mathbb{R}^n$  such that  $A\mathbf{x}$  is parallel to  $\mathbf{x}$ . In other words,  $A\mathbf{x} = \lambda\mathbf{x}$  for some scalar  $\lambda$ .

We call this  $\lambda$  an **eigenvalue** of  $A$ .

### Observation G.3.6

Since  $\lambda \mathbf{x} = \lambda(I\mathbf{x})$ , we can find the eigenvalues and eigenvectors satisfying  $A\mathbf{x} = \lambda\mathbf{x}$  by inspecting  $(A - \lambda I)\mathbf{x} = \mathbf{0}$ .

- Since we already know that  $(A - \lambda I)\mathbf{0} = \mathbf{0}$  for any value of  $\lambda$ , we are more interested in finding values of  $\lambda$  such that  $A - \lambda I$  has a nontrivial kernel.
- Thus  $\text{RREF}(A - \lambda I)$  must have a non-pivot column, and therefore  $A - \lambda I$  cannot be invertible.
- Since  $A - \lambda I$  cannot be invertible, our eigenvalues must satisfy  $\det(A - \lambda I) = 0$ .

**Definition G.3.7**

Computing  $\det(A - \lambda I)$  results in the **characteristic polynomial** of  $A$ .

For example, when  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix}$$

Thus the characteristic polynomial of  $A$  is

$$\det \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) - 6 = \lambda^2 - 5\lambda - 2$$



**Activity G.3.8** (*~15 min*)

Compute  $\det(A - \lambda I)$  to find the characteristic polynomial of  $A = \begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$ .

**Activity G.3.9** ( $\sim 15$  min)

Let  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .

**Activity G.3.9** ( $\sim 15$  min)

Let  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .

**Activity G.3.9** ( $\sim 15$  min)

Let  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .

*Part 2:* Find the roots of the characteristic polynomial to determine the eigenvalues of  $A$ .

**Activity G.3.9** ( $\sim 15$  min)

Let  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .

*Part 2:* Find the roots of the characteristic polynomial to determine the eigenvalues of  $A$ .

*Part 3:* Compute the kernel of the transformation given by

$$A - 2I = \begin{bmatrix} 2 - 2 & 2 \\ 0 & 3 - 2 \end{bmatrix}$$

to determine all the eigenvectors associated to the eigenvalue 2.

**Activity G.3.9** ( $\sim 15$  min)

Let  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .

*Part 2:* Find the roots of the characteristic polynomial to determine the eigenvalues of  $A$ .

*Part 3:* Compute the kernel of the transformation given by

$$A - 2I = \begin{bmatrix} 2 - 2 & 2 \\ 0 & 3 - 2 \end{bmatrix}$$

to determine all the eigenvectors associated to the eigenvalue 2.

*Part 4:* Compute the kernel of the transformation given by  $A - 3I$  to determine all the eigenvectors associated to the eigenvalue 3.

**Definition G.3.10**

The kernel of the transformation given by  $A - \lambda I$  contains all the eigenvectors associated with  $\lambda$ . Since kernel is a subspace of  $\mathbb{R}^n$ , we call this kernel the **eigenspace** associated with the eigenvalue  $\lambda$ .

**Activity G.3.11** (*~15 min*)

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$



**Activity G.3.11** (*~15 min*)

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .

**Activity G.3.11** (*~15 min*)

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .

*Part 2:* Find the roots of the characteristic polynomial  $(3 - \lambda)(\lambda^2 - 4\lambda - 12)$  to determine the eigenvalues of  $A$ .

**Activity G.3.11** (*~15 min*)

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .

*Part 2:* Find the roots of the characteristic polynomial  $(3 - \lambda)(\lambda^2 - 4\lambda - 12)$  to determine the eigenvalues of  $A$ .

*Part 3:* Compute the kernels of  $A - \lambda I$  for each eigenvalue  $\lambda \in \{-2, 3, 6\}$  to determine the respective eigenspaces.

**Observation G.3.12**

Recall that if  $a$  is a root of the polynomial  $p(\lambda)$ , the **multiplicity** of  $a$  is the largest number  $k$  such that  $p(\lambda) = q(\lambda)(\lambda - a)^k$  for some polynomial  $q(\lambda)$ .

For this reason, the **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

**Example G.3.13**

If  $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & 3 \\ 0 & 0 & -1 \end{bmatrix}$ , the characteristic polynomial is  $p(\lambda) = (\lambda - 3)^2(\lambda + 1)$ .

The eigenvalues are 3 (with algebraic multiplicity 2) and  $-1$  (with algebraic multiplicity 1).

## Module G Section 4

## Observation G.4.1

Recall from last class:

- To find the eigenvalues of a matrix  $A$ , we need to find values of  $\lambda$  such that  $A - \lambda I$  has a nontrivial kernel. Equivalently, we want values where  $A - \lambda I$  is not invertible, so we want to know the values of  $\lambda$  where  $\det(A - \lambda I) = 0$ .
- $\det(A - \lambda I)$  is a polynomial with variable  $\lambda$ , called the **characteristic polynomial** of  $A$ . Thus the roots of the characteristic polynomial of  $A$  are exactly the eigenvalues of  $A$ .
- Once an eigenvalue  $\lambda$  is found, the **eigenspace** containing all **eigenvectors**  $\mathbf{x}$  satisfying  $A\mathbf{x} = \lambda\mathbf{x}$  is given by  $\ker(A - \lambda I)$ .

**Activity G.4.2** ( $\sim 5$  min)

Let  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .



**Activity G.4.2** ( $\sim 5$  min)

Let  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

*Part 1:* Compute the eigenvalues of  $A$ .

**Activity G.4.2** ( $\sim 5$  min)

Let  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

*Part 1:* Compute the eigenvalues of  $A$ .

*Part 2:* Sketch a picture of the transformation of the unit square. What about this picture reveals that  $A$  has no real eigenvectors?

**Activity G.4.3** (*~5 min*)

If  $A$  is a  $4 \times 4$  matrix, what is the largest number of eigenvalues  $A$  can have?

- (a) 3
- (b) 4
- (c) 5
- (d) 6
- (e) It can have infinitely many

**Observation G.4.4**

An  $n \times n$  matrix may have between 0 and  $n$  real-valued eigenvalues. But the Fundamental Theorem of Algebra implies that if complex eigenvalues are included, then every  $n \times n$  matrix has exactly  $n$  eigenvalues (counting algebraic multiplicities).

**Activity G.4.5** (*~5 min*)

The matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$  has characteristic polynomial  $-\lambda(\lambda - 2)^2$ .

Find the dimension of the eigenspace of  $A$  associated to the eigenvalue 2 (the dimension of the kernel of  $A - 2I$ ).

**Activity G.4.6** ( $\sim 5$  min)

The matrix  $B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}$  has characteristic polynomial  $-\lambda(\lambda - 2)^2$ .

Find the dimension of the eigenspace of  $B$  associated to the eigenvalue 2 (the dimension of the kernel of  $B - 2I$ ).

**Observation G.4.7**

In the first example, the (2 dimensional) plane spanned by  $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$  was preserved. In the second example, only the (one dimensional) line spanned by  $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  is preserved.

**Definition G.4.8**

While the **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial, the **geometric multiplicity** of an eigenvalue is the dimension of its eigenspace.



**Fact G.4.9**

As we've seen, the geometric multiplicity may be different than its algebraic multiplicity, but it cannot exceed it.

This fact is explored deeper and explained in Math 316, Linear Algebra II

**Activity G.4.10** (*~20 min*)

Consider the  $4 \times 4$  matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

**Activity G.4.10** (*~20 min*)

Consider the  $4 \times 4$  matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

*Part 1:* Use technology (e.g. Wolfram Alpha) to find its characteristic polynomial.

**Activity G.4.10** ( $\sim 20$  min)

Consider the  $4 \times 4$  matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

*Part 1:* Use technology (e.g. Wolfram Alpha) to find its characteristic polynomial.

*Part 2:* Find the algebraic and geometric multiplicities for both eigenvalues.