Section E.0

Definition E.1 A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector that can be plugged into the equation).

Remark E.2 In previous classes you likely used the variables x, y, z in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as x_i, and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

Definition E.3 A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &=& b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &=& b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots && \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &=& b_m \end{array}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \middle| \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

Remark E.4 When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{array}{rcr} x_1 + 3x_3 &=& 3 \\ 3x_1 - 2x_2 + 4x_3 &=& 0 \\ -x_2 + x_3 &=& -2 \end{array}$$

Verbose standard form:

$$\begin{array}{rcr} 1x_1 + 0x_2 + 3x_3 &=& 3 \\ 3x_1 - 2x_2 + 4x_3 &=& 0 \\ 0x_1 - 1x_2 + 1x_3 &=& -2 \end{array}$$

Concise standard form:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 &=& 3 \\ 3x_1 & - & 2x_2 & + & 4x_3 &=& 0 \\ & - & x_2 & + & x_3 &=& -2 \end{array}$$

Remark E.5 It will often be convenient to think of a system of equations as a vector equation.
By applying vector operations and equating components, it is straightforward to see that the vector equation

$$x_1 \begin{bmatrix}1\\3\\0\end{bmatrix} + x_2 \begin{bmatrix}0\\-2\\-1\end{bmatrix} + x_3 \begin{bmatrix}3\\4\\1\end{bmatrix} = \begin{bmatrix}3\\0\\-2\end{bmatrix}$$

is equivalent to the system of equations

$$\begin{alignedat}{4} x_1 & & &+3x_3 &&= 3 \\ 3x_1 &-2x_2 &&+4x_3 &&= 0 \\ &-\ x_2 &&+\ x_3 &&= -2 \end{alignedat}$$

Definition E.6 A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**.

Fact E.7 All linear systems are one of the following:

- **Consistent with one solution**: its solution set contains a single vector, e.g. $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$
- **Consistent with infinitely-many solutions**: its solution set contains infinitely many vectors, e.g. $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix} \,\middle|\, a \in \mathbb{R}\right\}$
- **Inconsistent**: its solution set is the empty set $\{\} = \emptyset$

Activity E.8 *(∼10 min)* All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is $\emptyset$.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity E.9 *(∼10 min)* Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions for this system.
Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{\begin{bmatrix}?\\a\end{bmatrix} \,\middle|\, a \in \mathbb{R}\right\}$ for the linear system.

Activity E.10 *(~10 min)* Consider the following linear system.

$$\begin{alignedat}{5} x_1 &{}+{}& 2x_2 & & &{}-{}& x_4 &={}& 3 \\ & & & & x_3 &{}+{}& 4x_4 &={}& -2 \end{alignedat}$$

Describe the solution set

$$\left\{ \left[\begin{array}{c} ? \\ a \\ ? \\ b \end{array}\right] \middle|\, a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.