

## Module S: Structure of vector spaces

# What structure do vector spaces have?

At the end of this module, students will be able to...

- S1. Linear independence.** ... determine if a set of Euclidean vectors is linearly dependent or independent.
- S2. Basis verification.** ... determine if a set of Euclidean vectors is a basis of  $\mathbb{R}^n$ .
- S3. Basis computation.** ... compute a basis for the subspace spanned by a given set of Euclidean vectors.
- S4. Dimension.** ... compute the dimension of a subspace of  $\mathbb{R}^n$ .
- S5. Abstract vector spaces.** ... solve exercises related to standards V3-S4 when posed in terms of polynomials or matrices.
- S6. Basis of solution space.** ... find a basis for the solution set of a homogeneous system of equations.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V3,V4**.

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Academy):  
<http://bit.ly/2y8A0wa>
- Linear combinations of Euclidean vectors (Khan Academy):  
<http://bit.ly/2nK3wne>
- Adding and subtracting complex numbers (Khan Academy):  
<http://bit.ly/1PE3ZMQ>
- Adding and subtracting polynomials (Khan Academy):  
<http://bit.ly/2d5SLGZ>

# Module S Section 1

**Activity S.1.1** ( $\sim 10$  min)

Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

$$T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

Which of the following is true?

- (A)  $\text{span } S$  is bigger than  $\text{span } T$ .
- (B)  $\text{span } S$  and  $\text{span } T$  are the same size.
- (C)  $\text{span } S$  is smaller than  $\text{span } T$ .

## Definition S.1.2

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set.



**Activity S.1.3** ( $\sim 10$  min)

Suppose  $3\mathbf{v}_1 - 5\mathbf{v}_2 = \mathbf{v}_3$ , so the set  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent. Which of the following is true of the vector equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$  ?

- (A) It is consistent with one solution
- (B) It is consistent with infinitely many solutions
- (C) It is inconsistent.

**Fact S.1.4**

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly dependent if and only if  $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$  is consistent with infinitely many solutions.

**Activity S.1.5** (*~10 min*)

Find

$$\text{RREF} \left[ \begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array} \right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

**Fact S.1.6**

A set of Euclidean vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly dependent if and only if  $\text{RREF} \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$  has a column without a pivot position.

**Activity S.1.7** ( $\sim 5$  min)

Is the set of Euclidean vectors  $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$  linearly dependent or linearly independent?

**Activity S.1.8** (*~10 min*)

Is the set of polynomials  $\{x^3 + 1, x^2 + 2x, x^2 + 7x + 4\}$  linearly dependent or linearly independent?

**Activity S.1.9** (*~5 min*)

What is the largest number of vectors in  $\mathbb{R}^4$  that can form a linearly independent set?

- (a) 3
- (b) 4
- (c) 5
- (d) You can have infinitely many vectors and still be linearly independent.

**Activity S.1.10** ( $\sim 5$  min)

What is the largest number of vectors in  $\mathcal{P}^4$  that can form a linearly independent set?

- (a) 3
- (b) 4
- (c) 5
- (d) You can have infinitely many vectors and still be linearly independent.



**Activity S.1.11** ( $\sim 5$  min)

What is the largest number of vectors in  $\mathcal{P}$  that can form a linearly independent set?

- (a) 3
- (b) 4
- (c) 5
- (d) You can have infinitely many vectors and still be linearly independent.

## Module S Section 2

## Definition S.2.1

A **basis** is a linearly independent set that spans a vector space.

## Observation S.2.2

A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

## Definition S.2.3

The **standard basis** of  $\mathbb{R}^n$  is the set  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

 $\dots$ 

$$\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

**Activity S.2.4** ( $\sim 15$  min)

Which of the following sets are bases for  $\mathbb{R}^4$ ?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

**Activity S.2.5** (*~10 min*)

If  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is a basis for  $\mathbb{R}^4$ , that means  $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$  doesn't have a column without a pivot position, and doesn't have a row of zeros. What is  $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$ ?

## Fact S.2.6

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$  is a basis for  $\mathbb{R}^n$  if and only if  $m = n$  and

$$\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, a basis for  $\mathbb{R}^n$  must have exactly  $n$  vectors and its square matrix must row-reduce to the **identity matrix** containing all zeros except for a downward diagonal of ones.



## Observation S.2.7

Recall that a **subspace** of a vector space is a subset that is itself a vector space.

One easy way to construct a subspace is to take the span of set, but if the set is linearly dependent it may contain redundant vectors.

**Activity S.2.8** (*~10 min*)

Consider the set  $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

**Activity S.2.8** ( $\sim 10$  min)

Consider the set  $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

*Part 1:* Use RREF  $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$  to identify which vector may be removed to make the set linearly independent.

**Activity S.2.8** ( $\sim 10$  min)

Consider the set  $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

*Part 1:* Use RREF  $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$  to identify which vector may be removed to make the set linearly independent.

*Part 2:* Let  $W = \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$   $W$  is a subspace of  $\mathbb{R}^4$ ;

find a basis for  $W$ .

## Fact S.2.9

To compute a basis for the subspace  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ , simply remove the vectors corresponding to the non-pivot columns of  $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$ .

**Activity S.2.10** (*~10 min*)

Let  $W$  be the subspace of  $\mathbb{R}^4$  given by

$$W = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Find a basis for  $W$ .

**Activity S.2.11** (*~10 min*)

Let  $W$  be the subspace of  $\mathcal{P}^3$  given by

$$W = \text{span} \{x^3 + 3x^2 + x - 1, 2x^3 - x^2 + x + 2, 4x^3 + 5x^3 + 3x, 3x^3 + 2x^2 + 2x + 1\}$$

Find a basis for  $W$ .

## Module S Section 3



### Observation S.3.1

Recall from last class: to compute a basis for the subspace  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ , simply remove the vectors corresponding to the non-pivot columns of  $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$ .

**Activity S.3.2** (*~10 min*)

Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

**Activity S.3.2** (*~10 min*)

Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

*Part 1:* Find a basis for  $\text{span } S$

**Activity S.3.2** (*~10 min*)

Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

*Part 1:* Find a basis for  $\text{span } S$ *Part 2:* Find a basis for  $\text{span } T$

### Fact S.3.3

A vector space has a lot of bases, but all bases for a given vector space must be the same size.

### Definition S.3.4

The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

**Activity S.3.5** (*~15 min*)Find the dimension of each subspace of  $\mathbb{R}^4$ .

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

## Fact S.3.6

Every vector space with finite dimension, that is, every vector space with a basis of the form  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is **isomorphic** to a Euclidean space  $\mathbb{R}^n$ :

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$



### Observation S.3.7

Several interesting vector spaces are infinite-dimensional:

- The space of polynomials  $\mathcal{P}$  (consider the set  $\{1, x, x^2, x^3, \dots\}$ ).
- The space of continuous functions  $C(\mathbb{R})$  (which contains all polynomials, in addition to other functions like  $e^x$ ).
- The space of real number sequences  $\mathbb{R}^\infty$  (consider the set  $\{(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, \dots), \dots\}$ ).

**Definition S.3.8**

A **homogeneous** system of linear equations is one of the form

$$x_1 \mathbf{v}_1 + \cdots + x_n \mathbf{v}_n = \mathbf{0}.$$

Note that if  $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$  and  $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$  are solutions, so is  $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$  i.e. if

$$a_1 \mathbf{v}_1 + \cdots + a_n \mathbf{v}_n = \mathbf{0}$$

and

$$b_1 \mathbf{v}_1 + \cdots + b_n \mathbf{v}_n = \mathbf{0}$$

then

$$(a_1 + b_1) \mathbf{v}_1 + \cdots + (a_n + b_n) \mathbf{v}_n = \mathbf{0}.$$

Similarly, if  $c \in \mathbb{R}$ ,  $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$  is a solution. Thus the solution set of a homogeneous

system is a subspace.

**Activity S.3.9** (*~10 min*)

Consider the homogeneous system of equations

$$x_1 + 2x_2 \quad + \quad x_4 = 0$$

$$2x_1 + 4x_2 - x_3 - 2x_4 = 0$$

$$3x_1 + 6x_2 - x_3 - x_4 = 0$$

**Activity S.3.9** (*~10 min*)

Consider the homogeneous system of equations

$$x_1 + 2x_2 \quad + \quad x_4 = 0$$

$$2x_1 + 4x_2 - x_3 - 2x_4 = 0$$

$$3x_1 + 6x_2 - x_3 - x_4 = 0$$

*Part 1:* Find the solution set.

**Activity S.3.9** ( $\sim 10$  min)

Consider the homogeneous system of equations

$$x_1 + 2x_2 \quad + \quad x_4 = 0$$

$$2x_1 + 4x_2 - x_3 - 2x_4 = 0$$

$$3x_1 + 6x_2 - x_3 - x_4 = 0$$

*Part 1:* Find the solution set.

*Part 2:* Rewrite the solution set in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

.

**Activity S.3.9** (*~10 min*)

Consider the homogeneous system of equations

$$x_1 + 2x_2 \quad + \quad x_4 = 0$$

$$2x_1 + 4x_2 - x_3 - 2x_4 = 0$$

$$3x_1 + 6x_2 - x_3 - x_4 = 0$$

*Part 1:* Find the solution set.

*Part 2:* Rewrite the solution set in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

.

*Part 3:* Find a basis for the solution set.

**Activity S.3.10** (*~10 min*)

Consider the homogeneous system of equations

$$x_1 - 3x_2 + 2x_3 = 0$$

$$2x_1 - 6x_2 + 4x_3 + 3x_4 = 0$$

$$-2x_1 + 6x_2 - 4x_3 - 4x_4 = 0$$

Find a basis for the solution set.

**Activity S.3.11** ( $\sim 5$  min)

Suppose  $W$  is a subspace of  $\mathcal{P}^8$ , and you know that the set  $\{x^3 + x, x^2 + 1, x^4 - x\}$  is a linearly independent subset of  $W$ . What can you conclude about  $W$ ?

- (a) The dimension of  $W$  is no more than 3
- (b) The dimension of  $W$  is 3
- (c) The dimension of  $W$  is at least 3



**Activity S.3.12** (*~5 min*)

Suppose  $W$  is a subspace of  $\mathcal{P}^8$ , and you know that  $W$  is spanned by the six vectors

$$\{x^4 - x, x^3 + x, x^3 + x + 1, x^4 + 2x, x^3, 2x + 1\}$$

Without doing any calculation, what can you conclude about  $W$ ?

- (a) The dimension of  $W$  is no more than 6
- (b) The dimension of  $W$  is 6
- (c) The dimension of  $W$  is at least 6