

Section V.1

Remark V.1.1 Previously, we defined a **vector space** V to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all $\vec{u}, \vec{v}, \vec{w}$ in V , and all scalars (i.e. real numbers) a, b .

- $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$.
- $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.
- There exists some \vec{z} where $\vec{v} \oplus \vec{z} = \vec{v}$.
- There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$.
- $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$.
- $1 \odot \vec{v} = \vec{v}$.
- $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$.
- $(a + b) \odot \vec{v} = a\vec{v} \oplus b\vec{v}$.

Remark V.1.2 Every Euclidean space \mathbb{R}^n is a vector space, but there are other examples of vector spaces as well.

For example, consider the set \mathbb{C} of complex numbers with the usual definitions of addition and scalar multiplication, and let $\vec{u} = a + b\mathbf{i}$, $\vec{v} = c + d\mathbf{i}$, and $\vec{w} = e + f\mathbf{i}$. Then

$$\begin{aligned}
 \vec{u} + (\vec{v} + \vec{w}) &= (a + b\mathbf{i}) + ((c + d\mathbf{i}) + (e + f\mathbf{i})) \\
 &= a + b + c + d\mathbf{i} + e\mathbf{i} + f\mathbf{i} \\
 &= ((a + b\mathbf{i}) + (c + d\mathbf{i})) + (e + f\mathbf{i}) \\
 &= (\vec{u} + \vec{v}) + \vec{w}
 \end{aligned}$$

All eight properties can be verified in this way.

Remark V.1.3 The following sets are just a few examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- \mathbb{R}^n : Euclidean vectors with n components.
- \mathbb{C} : Complex numbers.
- $M_{m,n}$: Matrices of real numbers with m rows and n columns.
- \mathcal{P}^n : Polynomials of degree n or less.
- \mathcal{P} : Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity V.1.4 (~ 20 min) Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \quad c \odot (x, y) = (cx, y^c)$$

Part 1: Show that V satisfies the distribution property

$$(a + b) \odot \vec{v} = (a \odot \vec{v}) \oplus (b \odot \vec{v})$$

by substituting $\vec{v} = (x, y)$ and showing both sides simplify to the same expression.

Part 2: Show that V contains an additive identity element satisfying

$$(x, y) + \vec{z} = (x, y)$$

for all $(x, y) \in V$ by choosing appropriate values for $\vec{z} = (?, ?)$.

Remark V.1.5 It turns out $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \quad c \odot (x, y) = (cx, y^c)$$

satisfies all eight properties.

- $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$.
- $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.
- There exists some \vec{z} where $\vec{v} \oplus \vec{z} = \vec{v}$.
- There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$.
- $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$.
- $1 \odot \vec{v} = \vec{v}$.
- $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$.
- $(a + b) \odot \vec{v} = a\vec{v} \oplus b\vec{v}$.

Thus, V is a vector space.

Activity V.1.6 (~ 15 min) Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \quad c \odot (x, y) = (x^c, y + c - 1).$$

Part 1: Show that 1 is the scalar multiplication identity satisfying

$$1 \odot (x, y) = (x, y)$$

by simplifying $1 \odot (x, y)$ to (x, y) .

Part 2: Show that there cannot exist an addition identity property satisfying

$$\vec{v} \oplus \vec{z} = \vec{v}$$

for all vectors $\vec{v} \in V$ by showing that $(0, -1) \oplus \vec{z} \neq (0, -1)$ no matter how $\vec{z} = (z_1, z_2)$ is chosen.

Part 3: Is V a vector space?

Definition V.1.7 A **linear combination** of a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is given by $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m$ for any choice of scalar multiples c_1, c_2, \dots, c_m .

For example, we can say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Definition V.1.8 The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m \mid c_i \in \mathbb{R}\}.$$

For example:

$$\text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\} = \left\{a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R}\right\}$$

Activity V.1.9 (*~10 min*) Consider $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

Part 1: Sketch

$$1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{and } -2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

in the xy plane.

Part 2: Sketch a representation of all the vectors belonging to $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \left\{a \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid a \in \mathbb{R}\right\}$ in the xy plane.

Activity V.1.10 (*~10 min*) Consider $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$\begin{array}{ccc} 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ -2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & \end{array}$$

Part 2: Sketch a representation of all the vectors belonging to $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ in the xy plane.

Activity V.1.11 (*~5 min*) Sketch a representation of all the vectors belonging to $\text{span} \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ in the xy plane.