

Module A: Algebraic properties of linear maps

How can we understand linear maps algebraically?

Module A

Section A.1

Section A.2

Section A.3

Section A.4

At the end of this module, students will be able to...

- A1. Linear map verification.** ... determine if a map between vector spaces of polynomials is linear or not.
- A2. Linear maps and matrices.** ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.
- A3. Injectivity and surjectivity.** ... determine if a given linear map is injective and/or surjective.
- A4. Kernel and Image.** ... compute a basis for the kernel and a basis for the image of a linear map.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans \mathbb{R}^n **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Module A Section 1

Definition A.1.1

A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \rightarrow W$ is called a linear transformation if

- ① $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
- ② $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition A.1.2

Given a linear transformation $T : V \rightarrow W$, V is called the **domain** of T and W is called the **co-domain** of T .



Example A.1.3

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = T \left(\begin{bmatrix} x + u \\ y + v \\ z + w \end{bmatrix} \right) = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) + T \left(\begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

And also...

$$T \left(c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T \left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \quad \text{and} \quad cT \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = c \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore T is a linear transformation.

Example A.1.4

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact A.1.5

A map between Euclidean spaces $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of \mathbb{R}^n .

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z :

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2 , $y + 3$, and $y - 2^x$ are not linear combinations (even though $x + y$ is):

$$T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

Activity A.1.6 (~ 5 min)

Recall the following rules from calculus, where $D : \mathcal{P} \rightarrow \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f .

$$D(f + g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

- a) \mathcal{P} is not a vector space
- b) D is a linear map
- c) D is not a linear map

Module A

Section A.1

Section A.2

Section A.3

Section A.4

Activity A.1.7 (*~10 min*)

Let the polynomial maps $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ and $T : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4 + x)$, $S(x^4) + S(x)$, $T(x^4 + x)$, and $T(x^4) + T(x)$. Which of these maps is definitely not linear?

Fact A.1.8

If $L : V \rightarrow W$ is linear, then $L(\mathbf{z}) = L(0\mathbf{v}) = 0L(\mathbf{v}) = \mathbf{z}$ where \mathbf{z} is the additive identity of the vector spaces V, W .

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Activity A.1.9 (*~15 min*)

Continue to consider $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Activity A.1.9 (~ 15 min)

Continue to consider $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g .

Activity A.1.9 (*~15 min*)

Continue to consider $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g .

Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f . Is S linear?

Activity A.1.10 (*~20 min*)

Let the polynomial maps $S : \mathcal{P} \rightarrow \mathcal{P}$ and $T : \mathcal{P} \rightarrow \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Activity A.1.10 (*~20 min*)

Let the polynomial maps $S : \mathcal{P} \rightarrow \mathcal{P}$ and $T : \mathcal{P} \rightarrow \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Module A

Section A.1

Section A.2

Section A.3

Section A.4

Activity A.1.10 (*~20 min*)

Let the polynomial maps $S : \mathcal{P} \rightarrow \mathcal{P}$ and $T : \mathcal{P} \rightarrow \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Part 2: Prove that T is linear by verifying that

$$T(f(x) + g(x)) = T(f(x)) + T(g(x)) \text{ and } T(cf(x)) = cT(f(x)).$$

Module A

Section A.1

Section A.2

Section A.3

Section A.4

Observation A.1.11

Note that S in the previous activity is not linear, even though $S(0) = (0)^2 = 0$. So showing $S(0) = 0$ isn't enough to prove a map is linear.

This is a similar situation to proving a subset is a subspace: if the subset doesn't contain \mathbf{z} , then the subset isn't a subspace. But if the subset contains \mathbf{z} , you cannot conclude anything.

Module A Section 2

Remark A.2.1

Recall that a linear map $T : V \rightarrow W$ satisfies

- ① $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
- ② $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Module A

Section A.1

Section A.2

Section A.3

Section A.4

Activity A.2.2 (*~5 min*)

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear map, and you know $T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and

$T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute $T \left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right)$.

(a) $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(d) $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

Activity A.2.3 (~ 3 min)

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear map, and you know $T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and

$T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute $T \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$.

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

Activity A.2.4 (~ 2 min)

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear map, and you know $T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and

$T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute $T \left(\begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} \right)$.

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

Activity A.2.5 (~ 5 min)

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear map, and you know $T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and

$T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Do you have enough information to compute $T(\mathbf{v})$ for *any* $\mathbf{v} \in \mathbb{R}^3$?

- (a) Yes.
- (b) No, exactly one more piece of information is needed.
- (c) No, an infinite amount of information would be necessary to compute the transformation of infinitely-many vectors.

Fact A.2.6

Consider any basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for V . Since every vector \mathbf{v} can be written *uniquely* as a linear combination of basis vectors, $x_1\mathbf{b}_1 + \dots + x_n\mathbf{b}_n$, we may compute $T(\mathbf{v})$ as follows:

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \dots + x_n\mathbf{b}_n) = x_1T(\mathbf{b}_1) + \dots + x_nT(\mathbf{b}_n).$$

Therefore any linear transformation $T : V \rightarrow W$ can be defined by just describing the values of $T(\mathbf{b}_i)$.

Put another way, the images of the basis vectors **determine** the transformation T .

Definition A.2.7

Since linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is determined by the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$.

For example, let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of \mathbb{R}^3 .

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad T(\mathbf{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Then the standard matrix corresponding to T is

$$[T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)] = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix}.$$

Activity A.2.8 (~ 3 min)

Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation given by

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \quad T(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \quad T(\mathbf{e}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ for T .

Activity A.2.9 (*~5 min*)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Find the standard matrix for T .

Fact A.2.10

Because every linear map $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ has a linear combination of the variables in each component, and thus $T(\mathbf{e}_i)$ yields exactly the coefficients of x_i , the standard matrix for T is simply an ordered list of the coefficients of the x_i :

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \quad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Module A

Section A.1

Section A.2

Section A.3

Section A.4

Activity A.2.11 (*~5 min*)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Module A

Section A.1

Section A.2

Section A.3

Section A.4

Activity A.2.12 (*~5 min*)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute $T \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$.

Fact A.2.13

To quickly compute $T(\mathbf{v})$ from its standard matrix A , compute the **dot product** (defined in Calculus 3) of each matrix row with the vector. For example, if T has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$$

then for $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 2y + 3z \\ 0x + 1y - 2z \\ 2x - 1y + 0z \end{bmatrix}$$

and for $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(0) + 3(-2) \\ 0(3) + 1(0) - 2(-2) \\ 2(3) - 1(0) + 0(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}.$$

Activity A.2.14 (*~15 min*)

Compute the following linear transformations of vectors given their standard matrices.

$$T_1 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \text{ for the standard matrix } A_1 = \begin{bmatrix} 4 & 3 \\ 0 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$$

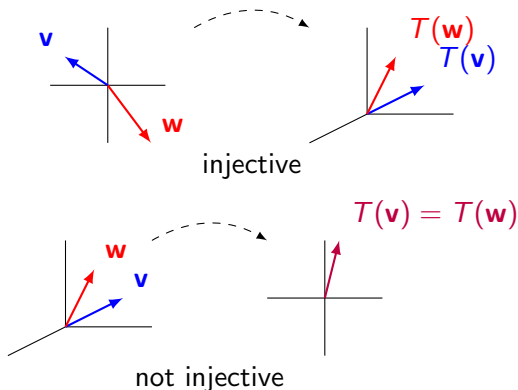
$$T_2 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix} \right) \text{ for the standard matrix } A_2 = \begin{bmatrix} 4 & 3 & 0 & -1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

$$T_3 \left(\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right) \text{ for the standard matrix } A_3 = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & 3 \\ 5 & 1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

Module A Section 3

Definition A.3.1

Let $T : V \rightarrow W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place. More precisely, T is injective if $T(\mathbf{v}) \neq T(\mathbf{w})$ whenever $\mathbf{v} \neq \mathbf{w}$.



Activity A.3.2 (~ 3 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Show that T is not injective by finding two different vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $T(\mathbf{v}) = T(\mathbf{w})$.

Activity A.3.3 (~ 2 min)

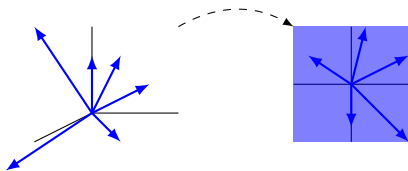
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

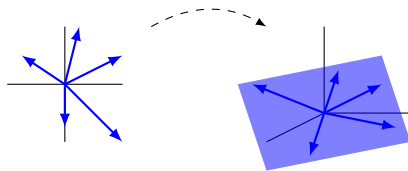
Is T injective? If not, find two different vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $T(\mathbf{v}) = T(\mathbf{w})$.

Definition A.3.4

Let $T : V \rightarrow W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V . More precisely, for every $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.



surjective



not surjective

Module A

Section A.1

Section A.2

Section A.3

Section A.4

Activity A.3.5 (*~3 min*)

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Show that T is not surjective by finding a vector in \mathbb{R}^3 that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal.

Module A

Section A.1

Section A.2

Section A.3

Section A.4

Activity A.3.6 (~ 2 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective? If not, find a vector in \mathbb{R}^2 that $T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$ can never equal.

Observation A.3.7

As we will see, it's no coincidence that the RREF of the injective map's standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has all pivot columns. Similarly, the RREF of the surjective map's standard matrix

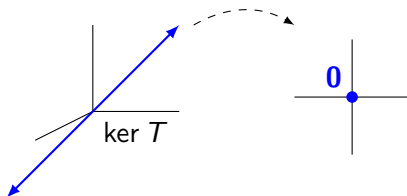
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has a pivot in each row.

Definition A.3.8

Let $T : V \rightarrow W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{z}\}$$



Activity A.3.9 (*~5 min*)

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of \mathbb{R}^2 describes $\ker T$, the set of all vectors that transform into $\mathbf{0}$?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) \mathbb{R}^2

Activity A.3.10 (~ 5 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of \mathbb{R}^3 describes $\ker T$, the set of all vectors that transform into $\mathbf{0}$?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \mid a \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) \mathbb{R}^3

Activity A.3.11 (*~10 min*)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Activity A.3.11 (*~10 min*)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Set $T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

Activity A.3.11 (~ 10 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations

whose solution set is the kernel.

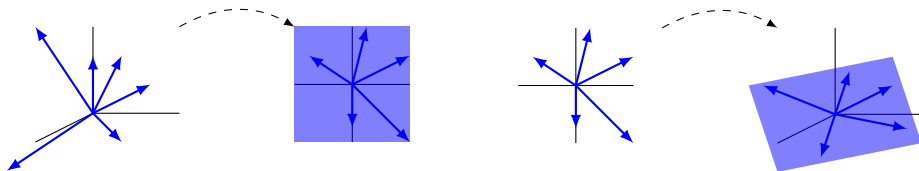
Part 2: Use $\text{RREF}(A)$ to solve this homogeneous system of equations and find a basis for the kernel of T .

Definition A.3.12

Let $T : V \rightarrow W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\text{Im } T = \{ \mathbf{w} \in W \mid \text{there is some } \mathbf{v} \in V \text{ with } T(\mathbf{v}) = \mathbf{w} \}$$

In the examples below, the left example's image is all of \mathbb{R}^2 , but the right example's image is a planar subspace of \mathbb{R}^3 .



Activity A.3.13 (~ 5 min)

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of \mathbb{R}^3 describes $\text{Im } T$, the set of all vectors that are the result of using T to transform \mathbb{R}^2 vectors?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) \mathbb{R}^3

Activity A.3.14 (*~5 min*)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of \mathbb{R}^2 describes $\text{Im } T$, the set of all vectors that are the result of using T to transform \mathbb{R}^3 vectors?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) \mathbb{R}^2

Activity A.3.15 (~ 5 min)

Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3) \quad T(\mathbf{e}_4)].$$

Since $T(\mathbf{v}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4)$, the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

- a) spans $\text{Im } T$
- b) is a linearly independent subset of $\text{Im } T$
- c) is a basis for $\text{Im } T$

Observation A.3.16

Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}.$$

Since the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ spans $\text{Im } T$, we can obtain a basis for

$\text{Im } T$ by finding RREF $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and only using the vectors

corresponding to pivot columns:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Fact A.3.17

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation with standard matrix A .

- The kernel of T is the solution set of the homogeneous system given by the augmented matrix $[A \mid \mathbf{0}]$. Use the coefficients of its free variables to get a basis for the kernel.
- The image of T is the span of the columns of A . Remove the vectors creating non-pivot columns in RREF A to get a basis for the image.

Activity A.3.18 (*~10 min*)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T .

Module A Section 4

Observation A.4.1

Let $T : V \rightarrow W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V .
- The **kernel** of T is the set of all vectors in V that are mapped to $\mathbf{z} \in W$. It is a subspace of V .
- The **image** of T is the set of all vectors in W that are mapped to by something in V . It is a subspace of W .

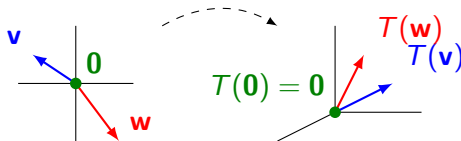
Activity A.4.2 (~ 5 min)

Let $T : V \rightarrow W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

- (a) T is injective
- (b) T is not injective
- (c) T is surjective
- (d) T is not surjective

Fact A.4.3

A linear transformation T is injective **if and only if** $\ker T = \{\mathbf{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.



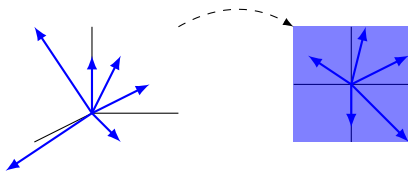
Activity A.4.4 (~ 5 min)

Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ be a linear transformation where $\text{Im } T$ is spanned by four vectors. What can you conclude?

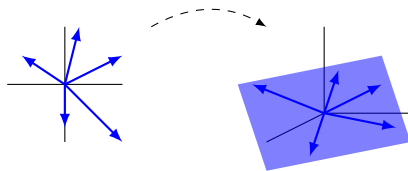
- (a) T is injective
- (b) T is not injective
- (c) T is surjective
- (d) T is not surjective

Fact A.4.5

A linear transformation $T : V \rightarrow W$ is surjective **if and only if** $\text{Im } T = W$. Put another way, a surjective linear transformation may be recognized by its identical codomain and image.



surjective, $\text{Im } T = \mathbb{R}^2$



not surjective, $\text{Im } T \neq \mathbb{R}^3$

Activity A.4.6 (~ 15 min)

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with standard matrix A . Sort the following claims into two groups of *equivalent* statements: one group that means T is **injective**, and one group that means T is **surjective**.

- (a) The kernel of T is trivial:
 $\ker T = \{\mathbf{0}\}$.
- (b) The columns of A span \mathbb{R}^m .
- (c) The columns of A are linearly independent.
- (d) Every column of $\text{RREF}(A)$ has a pivot.
- (e) Every row of $\text{RREF}(A)$ has a pivot.
- (f) The image of T equals its codomain: $\text{Im } T = \mathbb{R}^m$.
- (g) The system of linear equations given by the augmented matrix $[A \mid \mathbf{b}]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$.
- (h) The system of linear equations given by the augmented matrix $[A \mid \mathbf{0}]$ has exactly one solution.

Observation A.4.7

The easiest way to show that the linear map with standard matrix A is injective is to show that $\text{RREF}(A)$ has all pivot columns.

The easiest way to show that the linear map with standard matrix A is surjective is to show that $\text{RREF}(A)$ has all pivot rows.

Activity A.4.8 (~ 3 min)

What can you immediately conclude about the linear map $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$?

- a) Its standard matrix has more columns than rows, so T is not
- b) Its standard matrix has more columns than rows, so T is injective.
- c) Its standard matrix has more rows than columns, so T is not
- d) Its standard matrix has more rows than columns, so T is surjective.

Activity A.4.9 (~ 2 min)

What can you immediately conclude about the linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^7$?

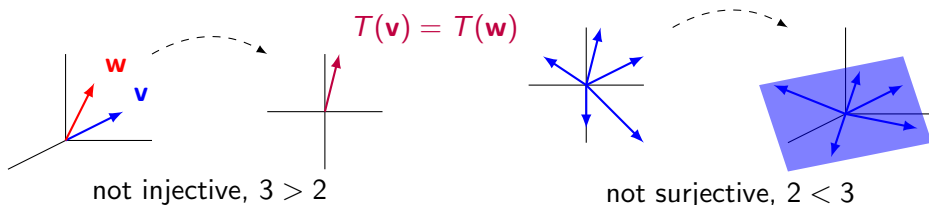
- a) Its standard matrix has more columns than rows, so T is not
- b) Its standard matrix has more columns than rows, so T is injective.
- c) Its standard matrix has more rows than columns, so T is not
- d) Its standard matrix has more rows than columns, so T is surjective.

Fact A.4.10

The following are true for any linear map $T : V \rightarrow W$:

- If $\dim(V) > \dim(W)$, then T is not injective.
- If $\dim(V) < \dim(W)$, then T is not surjective.

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase the dimension of its image.



But these results do **not** reverse. For example, $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ might not be surjective, even though $5 \not< 4$.

Definition A.4.11

If $T : V \rightarrow W$ is both injective and surjective, it is called **bijjective**.

Activity A.4.12 (*~5 min*)

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a bijective linear map with standard matrix A . Label each of the following as true or false.

- (a) The columns of A form a basis for \mathbb{R}^m
- (b) $\text{RREF}(A)$ is the identity matrix.
- (c) The system of linear equations given by the augmented matrix $[A \mid \mathbf{b}]$ has exactly one solution for all $\mathbf{b} \in \mathbb{R}^m$.

Observation A.4.13

The easiest way to show that the linear map with standard matrix A is bijective is to show that $\text{RREF}(A)$ is the identity matrix.

Activity A.4.14 (*~5 min*)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

- (a) T is neither injective nor surjective
- (b) T is injective but not surjective
- (c) T is surjective but not injective
- (d) T is bijective.

Module A

Section A.1

Section A.2

Section A.3

Section A.4

Activity A.4.15 (*~5 min*)Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

- (a) T is neither injective nor surjective
- (b) T is injective but not surjective
- (c) T is surjective but not injective
- (d) T is bijective.