

## Section E.1

**Remark E.1.1** The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Verbose standard form:

$$\begin{aligned}1x_1 + 0x_2 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\0x_1 - 1x_2 + 1x_3 &= -2\end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c}1 & 0 & 3 & 3 \\3 & -2 & 4 & 0 \\0 & -1 & 1 & -2\end{array}$$

**Definition E.1.2** A system of  $m$  linear equations with  $n$  variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m\end{aligned} \qquad \left[ \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

**Example E.1.3** The corresponding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Augmented matrix:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array} \right]$$

**Definition E.1.4** Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set  $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ .

$$\begin{aligned}3x_1 - 2x_2 &= 1 \\x_1 + 4x_2 &= 5\end{aligned} \qquad \begin{aligned}3x_1 - 2x_2 &= 1 \\4x_1 + 2x_2 &= 6\end{aligned}$$

Therefore these augmented matrices are equivalent, which we denote with  $\sim$ :

$$\left[ \begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array} \right] \sim \left[ \begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array} \right]$$

**Activity E.1.5** ( $\sim 10$  min) Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

- |                                           |                                                       |
|-------------------------------------------|-------------------------------------------------------|
| a) Swap two rows.                         | e) Add a constant multiple of one row to another row. |
| b) Swap two columns.                      | f) Replace a column with zeros.                       |
| c) Add a constant to every term in a row. | g) Replace a row with zeros.                          |
| d) Multiply a row by a nonzero constant.  |                                                       |

**Definition E.1.6** The following **row operations** produce equivalent augmented matrices:

1. Swap two rows, for example,  $R_1 \leftrightarrow R_2$ :

$$\left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \sim \left[ \begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array} \right]$$

2. Multiply a row by a nonzero constant, for example,  $2R_1 \rightarrow R_1$ :

$$\left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \sim \left[ \begin{array}{cc|c} 2(1) & 2(2) & 2(3) \\ 4 & 5 & 6 \end{array} \right]$$

3. Add a constant multiple of one row to another row, for example,  $R_2 - 4R_1 \rightarrow R_2$ :

$$\left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 4 - 4(1) & 5 - 4(2) & 6 - 4(3) \end{array} \right]$$

Whenever two matrices  $A, B$  are equivalent (so whenever we do any of these operations), we write  $A \sim B$ .

**Activity E.1.7** ( $\sim 10$  min) Consider the following (equivalent) linear systems.

<p>(A)</p> $\begin{aligned} -2x_1 + 4x_2 - 2x_3 &= -8 \\ x_1 - 2x_2 + 2x_3 &= 7 \\ 3x_1 - 6x_2 + 4x_3 &= 15 \end{aligned}$	<p>(C)</p> $\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ 2x_3 &= 6 \\ -2x_3 &= -6 \end{aligned}$	<p>(E)</p> $\begin{aligned} x_1 - 2x_2 &= 1 \\ x_3 &= 3 \\ 0 &= 0 \end{aligned}$
<p>(B)</p> $\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ -2x_1 + 4x_2 - 2x_3 &= -8 \\ 3x_1 - 6x_2 + 4x_3 &= 15 \end{aligned}$	<p>(D)</p> $\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ x_3 &= 3 \\ -2x_3 &= -6 \end{aligned}$	<p>(F)</p> $\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ 2x_3 &= 6 \\ 3x_1 - 6x_2 + 4x_3 &= 15 \end{aligned}$

*Part 1:* Find a solution to one of these systems.

*Part 2:* Rank the six linear systems from most complicated to simplest.

**Activity E.1.8** (*~5 min*) We can rewrite the previous in terms of equivalences of augmented matrices

$$\begin{aligned} \left[ \begin{array}{ccc|c} -2 & 4 & -2 & -8 \\ 1 & -2 & 2 & 7 \\ 3 & -6 & 4 & 15 \end{array} \right] &\sim \left[ \begin{array}{ccc|c} \textcircled{1} & -2 & 2 & 7 \\ -2 & 4 & -2 & -8 \\ 3 & -6 & 4 & 15 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{1} & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 3 & -6 & 4 & 15 \end{array} \right] \\ &\sim \left[ \begin{array}{ccc|c} \textcircled{1} & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -2 & -6 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{1} & -2 & 2 & 7 \\ 0 & 0 & \textcircled{1} & 3 \\ 0 & 0 & -2 & -6 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 3 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

**Activity E.1.9** (*~10 min*) A matrix is in **reduced row echelon form (RREF)** if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

Circle the leading terms in each example, and label it as RREF or not RREF.

(A) $\left[ \begin{array}{ccc c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$	(C) $\left[ \begin{array}{ccc c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]$	(E) $\left[ \begin{array}{ccc c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$
(B) $\left[ \begin{array}{ccc c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$	(D) $\left[ \begin{array}{ccc c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$	(F) $\left[ \begin{array}{ccc c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array} \right]$

**Remark E.1.10** It is important to understand the **Gauss-Jordan elimination** algorithm that converts a matrix (augmented or not) into reduced row echelon form.

A video outlining how to perform the Gauss-Jordan Elimination algorithm by hand is available at <https://youtu.be/Cq0Nvk2dhhU>. Practicing several exercises outside of class using this method is recommended.

In the next section, we will learn to use technology to perform this operation for us, as will be expected when applying row-reduced matrices to solve other problems.