

Module M

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Module M: Understanding Matrices Algebraically

What algebraic structure do matrices have?

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At the end of this module, students will be able to...

- M1. Matrix Multiplication.** ... multiply matrices.
- M2. Row operations as matrix multiplication.** ... can express row operations through matrix multiplication.
- M3. Invertible Matrices.** ... determine if a square matrix is invertible or not.
- M4. Matrix inverses.** ... compute the inverse matrix of an invertible matrix.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers.
- Identify the domain and codomain of linear transformations.
- Find the matrix corresponding to a linear transformation and compute the image of a vector given a standard matrix **A2**
- Determine if a linear transformation is injective and/or surjective **A4**
- Interpret the ideas of injectivity and surjectivity in multiple ways.

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The following resources will help you prepare for this module.

- Function composition (Khan Academy): <http://bit.ly/2wkz7f3>
- Domain and codomain: <https://www.youtube.com/watch?v=BQMyeQOLvpg>
- Interpreting injectivity and surjectivity in many ways:
<https://www.youtube.com/watch?v=WpUv72Y6D10>

Module M Section 1

Today's goals: Today we will learn about composing linear maps.

Observation M.1

If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $S : \mathbb{R}^m \rightarrow \mathbb{R}^k$ are linear maps, then the composition map $S \circ T$ is a linear map from $\mathbb{R}^n \rightarrow \mathbb{R}^k$.

$$\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{T} & \mathbb{R}^m & \xrightarrow{S} & \mathbb{R}^k \\ & \searrow & & \nearrow & \\ & & S \circ T & & \end{array}$$

Recall that for a vector, $\vec{v} \in \mathbb{R}^n$, the composition is computed as $(S \circ T)(\vec{v}) = S(T(\vec{v}))$.

Activity M.2 (~ 5 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix

$$B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix} \text{ and } S : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \text{ be given by the } 4 \times 2 \text{ standard matrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}.$$

What are the domain and codomain of the composition map $S \circ T$?

- (a) The domain is \mathbb{R}^2 and the codomain is \mathbb{R}^3
- (b) The domain is \mathbb{R}^3 and the codomain is \mathbb{R}^2
- (c) The domain is \mathbb{R}^2 and the codomain is \mathbb{R}^4
- (d) The domain is \mathbb{R}^3 and the codomain is \mathbb{R}^4
- (e) The domain is \mathbb{R}^4 and the codomain is \mathbb{R}^3
- (f) The domain is \mathbb{R}^4 and the codomain is \mathbb{R}^2

Activity M.3 (~ 2 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix

$B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}.$$

What size will the standard matrix of $S \circ T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be? (Rows \times Columns)

(a) 4×3

(c) 3×4

(e) 2×4

(b) 4×2

(d) 3×2

(f) 2×3

Activity M.4 (~ 15 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix

$B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}.$$

Activity M.4 (~ 15 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix

$B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}.$$

Part 1: Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Activity M.4 (~ 15 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix

$B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}.$$

Part 1: Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{e}_2)$.

Activity M.4 (~ 15 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix

$B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}.$$

Part 1: Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{e}_2)$.

Part 3: Compute $(S \circ T)(\vec{e}_3)$.

Activity M.4 (~ 15 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix

$B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}.$$

Part 1: Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{e}_2)$.

Part 3: Compute $(S \circ T)(\vec{e}_3)$.

Part 4: Write the 4×3 standard matrix of $S \circ T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$.

Definition M.5

We define the **product** AB of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For the previous activity, T was a map $\mathbb{R}^3 \rightarrow \mathbb{R}^2$, and S was a map $\mathbb{R}^2 \rightarrow \mathbb{R}^4$, so $S \circ T$ gave a map $\mathbb{R}^3 \rightarrow \mathbb{R}^4$ with a 4×3 standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= [(S \circ T)(\vec{e}_1) \quad (S \circ T)(\vec{e}_2) \quad (S \circ T)(\vec{e}_3)] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

Activity M.6 (*~15 min*) Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix

$$A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \text{ be given by the matrix } B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}.$$

Activity M.6 (~ 15 min) Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix

$$A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \text{ be given by the matrix } B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}.$$

Part 1: Write the dimensions (rows \times columns) for A , B , AB , and BA .

Activity M.6 (~ 15 min) Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix

$$A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \text{ be given by the matrix } B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}.$$

Part 1: Write the dimensions (rows \times columns) for A , B , AB , and BA .

Part 2: Find the standard matrix AB of $S \circ T$.

Activity M.6 (~ 15 min) Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix

$$A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \text{ be given by the matrix } B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}.$$

Part 1: Write the dimensions (rows \times columns) for A , B , AB , and BA .

Part 2: Find the standard matrix AB of $S \circ T$.

Part 3: Find the standard matrix BA of $T \circ S$.

Activity M.7 (~ 10 min) Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Activity M.7 (~ 10 min) Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Part 1: Find the domain and codomain of each of the three linear maps corresponding to A , B , and C .

Activity M.7 (~ 10 min) Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Part 1: Find the domain and codomain of each of the three linear maps corresponding to A , B , and C .

Part 2: Only one of the matrix products AB , AC , BA , BC , CA , CB can actually be computed. Compute it.

Summary: Today we learned that matrix multiplication is simply composing linear maps (Standard M1)

Module M Section 2

Today's goals: Today we will learn how row operations can be interpreted via matrix multiplication (Standard M2). This will be useful when we compute **determinants** in module G.

Remark M.8

Recall that the **product** AB of a $m \times n$ matrix A and an $n \times k$ matrix B is the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For example, if T was a map $\mathbb{R}^3 \rightarrow \mathbb{R}^2$, and S was a map $\mathbb{R}^2 \rightarrow \mathbb{R}^4$, then $S \circ T$ gave a map $\mathbb{R}^3 \rightarrow \mathbb{R}^4$ with a 4×3 standard matrix, such as:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= [(S \circ T)(\vec{e}_1) \quad (S \circ T)(\vec{e}_2) \quad (S \circ T)(\vec{e}_3)] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

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Activity M.9 (~ 15 min) Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

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Activity M.9 (~ 15 min) Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Part 1: Compute the product BA by hand.

Activity M.9 (~ 15 min) Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Part 1: Compute the product BA by hand.

Part 2: Check your work using technology. Using Octave:

- `B = sym([3 -4 0 ; 2 0 -1 ; 0 -3 3])`
- `A = sym([2 7 -1 ; 0 3 2 ; 1 1 -1])`
- `B*A`

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Activity M.10 (~ 5 min) Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix B such that $BA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Check your guess using technology.

Definition M.11

The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

Fact M.12

For any square matrix A , $IA = AI = A$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Activity M.13 (~ 20 min) Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Activity M.13 (~ 20 min) Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Part 1: Create a matrix that doubles the third row of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Activity M.13 (~ 20 min) Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Part 1: Create a matrix that doubles the third row of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

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Activity M.13 (~ 20 min) Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Part 1: Create a matrix that doubles the third row of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 + 5(1) & 7 + 5(1) & -1 + 5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Fact M.14

If R is the result of applying a row operation to I , then RA is the result of applying the same row operation to A .

- Scaling a row: $R = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Swapping rows: $R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Adding a row multiple to another row: $R = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Such matrices can be chained together to emulate multiple row operations. In particular,

$$\text{RREF}(A) = R_k \dots R_2 R_1 A$$

for some sequence of matrices R_1, R_2, \dots, R_k .

Summary: Today we learned how row operations can be interpreted via matrix multiplication (Standard M2). This will be useful when we compute **determinants** in module G.

Module M Section 3

Today's goals: Today we will learn how to determine when a matrix is **invertible**(Standard M3).

Activity M.15 (~ 10 min) Consider the two row operations $R_2 \leftrightarrow R_3$ and $R_1 + R_2 \rightarrow R_1$ applied as follows to show $A \sim B$:

$$\begin{aligned} A = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} &\sim \begin{bmatrix} -1 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} \\ &\sim \begin{bmatrix} -1+1 & 4+2 & 5+3 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = B \end{aligned}$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A :

$$B = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} A$$

Check your work using technology.

Activity M.16 (~ 15 min) Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with standard matrix A . Sort the following items into three groups of statements: a group that means T is **injective**, a group that means T is **surjective**, and a group that means T is **bijjective**.

- (a) $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$
- (b) $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$
- (c) $A\vec{x} = \vec{0}$ has a unique solution.
- (d) The columns of A span \mathbb{R}^m
- (e) The columns of A are linearly independent
- (f) The columns of A are a basis of \mathbb{R}^m
- (g) Every column of $\text{RREF}(A)$ has a pivot
- (h) Every row of $\text{RREF}(A)$ has a pivot
- (i) $m = n$ and $\text{RREF}(A) = I$

Activity M.17 (*~5 min*) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by

the standard matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 4 \\ 1 & 1 & 3 \end{bmatrix}$.

Write an augmented matrix representing the system of equations given by

$T(\vec{x}) = \vec{0}$, that is, $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{0}$ to find the kernel of T .

Definition M.18

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map with standard matrix A .

- If T is a bijection and \vec{b} is any \mathbb{R}^n vector, then $T(\vec{x}) = A\vec{x} = \vec{b}$ has a unique solution.
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by setting $T^{-1}(\vec{b})$ to be this unique solution.
- Let A^{-1} be the standard matrix for T^{-1} . We call A^{-1} the **inverse matrix** of A , so we also say that A is **invertible**.

Summary: Today we learned how to determine when a matrix is **invertible** (Standard M3).

Module M Section 4

Today's goals: Today we will learn how to find the **inverse** of an invertible matrix (Standard M4).

Activity M.19 (~ 20 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

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Activity M.19 (~ 20 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by

the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by

$T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

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Activity M.19 (~ 20 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by

the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by

$T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Activity M.19 (~ 20 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by

the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by

$T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Part 3: Solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

Activity M.19 (~ 20 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by

the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by

$T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Part 3: Solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

Part 4: Write A^{-1} , the standard matrix for T^{-1} .

Observation M.20

We could have solved these three systems simultaneously by row reducing the matrix $[A \mid I]$ at once.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array} \right]$$

Activity M.21 (~ 5 min) Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A \mid I]$.

Activity M.22 (*~5 min*) Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

Quick Check M.23

An $n \times n$ matrix A is invertible if and only if $\text{RREF}(A) = I_n$.

Activity M.24 (*~10 min*) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Activity M.24 (*~10 min*) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the bijective linear map defined by

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Part 1: Compute $(T^{-1} \circ T) \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$.

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Part 1: Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

Part 2: If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1} , find the 2×2 matrix

$$A^{-1}A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

Summary: Today we learned how to find the **inverse** of an invertible matrix (Standard M4).