Name: ____________________

FINAL EXAM Math 237 – Linear Algebra
Version 6 Fall 2017

Show all work and justify all of your answers. Answers without work or sufficient reasoning will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} -4 & -1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{array}\right]$$

Solution:

$$\begin{aligned} -4x_1 - x_2 + 3x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 0 \\ -x_1 + 4x_2 + x_3 &= 4 \end{aligned}$$

$\square$

E2. Put the following matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array}\right]$$

Solution:

$$\left[\begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & -\frac{2}{3} & -1 & \frac{2}{3} \\ 0 & 2 & 3 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 2 & 3 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\square$

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x_1 + 3x_2 - 5x_3 + 14x_4 &= 8 \\ x_1 + x_2 - x_3 + 5x_4 &= 3 \end{aligned}$$

Solution: Let $A = \left[\begin{array}{cccc|c} 2 & 3 & -5 & 14 & 8 \\ 1 & 1 & -1 & 5 & 3 \end{array}\right]$, so RREF $A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -3 & 4 & 2 \end{array}\right]$. It follows that the solution set is given by $\begin{bmatrix} 1-2a-b \\ 2+3a-4b \\ a \\ b \end{bmatrix}$ for all real numbers a, b.

$\square$

E4. Find a basis for the solution set to the homogeneous system of equations

$$\begin{aligned} 4x_1 + 4x_2 + 3x_3 - 6x_4 &= 0 \\ -2x_3 - 4x_4 &= 0 \\ 2x_1 + 2x_2 + x_3 - 4x_4 &= 0 \end{aligned}$$

Solution: Let $A = \left[\begin{array}{cccc|c} 4 & 4 & 3 & -6 & 0 \\ 0 & 0 & -2 & -4 & 0 \\ 2 & 2 & 1 & -4 & 0 \end{array}\right]$, so RREF $A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. It follows that the basis for the solution set is given by $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

□

V1. Let V be the set of all polynomials with the operations, for any $f, g \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} f \oplus g &= f' + g' \\ c \odot f &= cf' \end{aligned}$$

(here f' denotes the derivative of f).

(a) Show that scalar multiplication **distributes scalars** over vector addition: $c \odot (f \oplus g) = c \odot f \oplus c \odot g$.

(b) Determine if V is a vector space or not. Justify your answer.

Solution: Let $f, g \in \mathcal{P}$, and let $c \in \mathbb{R}$.

$$c \odot (f \oplus g) = c \odot (f' + g') = c(f' + g')' = cf'\,' + cg'\,' = cf' \oplus cg' = c \odot f \oplus c \odot g.$$

However, this is not a vector space, as there is no zero vector. Additionally, $1 \odot f \neq f$ for any nonzero polynomial f.

□

V2. Determine if $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix} 5 \\ 2 \\ -3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 8 \\ 3 \\ 5 \\ -1 \end{bmatrix}$.

Solution:

$$\text{RREF}\left(\left[\begin{array}{ccc|c} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The system has no solution, so $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ is not a linear combination of the three other vectors.

□

V3. Does span $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$?

Solution: Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 4 & 2 \\ 4 & -9 & -3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has a zero row, the vectors fail to span $\mathbb{R}^3$.

$\square$

V4. Let W be the set of all complex numbers $a + bi$ satisfying $a = 2b$. Determine if W is a subspace of $\mathbb{C}$.

Solution: Yes, because $c(2b_1 + b_1 i) + d(2b_2 + b_2 i) = 2(cb_1 + db_2) + (cb_1 + db_2)i$ belongs to W. Alternately, yes because W is isomorphic to $\mathbb{R}$.

$\square$

S1. Determine if the set of vectors $\left\{ \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$ is linearly dependent or linearly independent

Solution:

$$\text{RREF} \left(\begin{bmatrix} -3 & 1 & 0 \\ 8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every column is a pivot column, therefore the set is linearly independent.

$\square$

S2. Determine if the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

Solution:

$$\text{RREF} \left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis.

$\square$

S3. Let W be the subspace of $\mathcal{P}^2$ given by $W = \text{span}\left(\left\{-3x^2 - 8x, x^2 + 2x + 2, -x + 3\right\}\right)$. Find a basis for W.

Solution: Let $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$, and compute $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$. Since the first two columns are pivot columns, $\left\{-3x^2 - 8x, x^2 + 2x + 2\right\}$ is a basis for W.

$\square$

S4. Let $W = \operatorname{span}\left(\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\0\\1\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\4\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}\right)$. Compute the dimension of W.

Solution:

$$\operatorname{RREF}\left(\begin{bmatrix}1 & 2 & 3 & 1\\-1 & 0 & -1 & 1\\3 & 1 & 4 & 1\\-3 & 1 & -2 & -7\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$.

$\square$

A1. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}-3x+y\\-8x+2y-z\\2y+3z\\7x\end{bmatrix}.$$

Write the matrix for T with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^4$.

Solution:

$$\begin{bmatrix}3 & 1 & 0\\-8 & 2 & -1\\0 & 2 & 3\\7 & 0 & 0\end{bmatrix}$$

$\square$

A2. Determine if the map $T : \mathcal{P} \to \mathcal{P}$ given by $T(f) = f' - f''$ is a linear transformation or not.

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x+y+z\\2y+3z\\x-y-2z\end{bmatrix}$

(b) $S : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}3x+2y\\x-y\\x+4y\end{bmatrix}$

Solution:

(a)

$$\operatorname{RREF}\left(\begin{bmatrix}1 & 1 & 1\\0 & 2 & 3\\1 & -1 & -2\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & -\frac{1}{2}\\0 & 1 & \frac{3}{2}\\0 & 0 & 0\end{bmatrix}$$

Since there is a nonpivot column, T is not injective. Since there is a zero row, T is not surjective.

(b)

$$\text{RREF}\left(\begin{bmatrix}3 & 2\\ 1 & -1\\ 1 & 4\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$$

Since all columns are pivot columns, S is injective. Since there is a zero row, S is not surjective.

□

A4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix}x\\ y\\ z\end{bmatrix}\right) = \begin{bmatrix}-3x+y\\ -8x+2y-z\\ 2y+3z\end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of T.

Solution: Let $A = \begin{bmatrix}-3 & 1 & 0\\ -8 & 2 & -1\\ 0 & 2 & 3\end{bmatrix}$, and compute $\text{RREF}(A) = \begin{bmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{3}{2}\\ 0 & 0 & 0\end{bmatrix}$. Then a basis for the image is its columns,

$$\left\{\begin{bmatrix}-3\\ -8\\ 0\end{bmatrix}, \begin{bmatrix}1\\ 2\\ 2\end{bmatrix}\right\}$$

And the kernel is the solution set of $AX = 0$, so a basis would be

$$\left\{\begin{bmatrix}1\\ 3\\ -2\end{bmatrix}\right\}$$

□

M1. Let

$$A = \begin{bmatrix}1 & 3 & -1 & -1\\ 0 & 0 & 7 & 2\end{bmatrix} \qquad B = \begin{bmatrix}0 & 1 & 7 & 7\\ -1 & -2 & 0 & 4\\ 0 & 0 & 1 & 5\end{bmatrix} \qquad C = \begin{bmatrix}3 & 2\\ 0 & 1\\ -2 & -1\end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution: CA is the only one that can be computed, and

$$CA = \begin{bmatrix}3 & 9 & 11 & 1\\ 0 & 0 & 7 & 2\\ -2 & -6 & -5 & 0\end{bmatrix}$$

□

M2. Determine if the matrix $\begin{bmatrix}2 & 1 & 0 & 3\\ 1 & -1 & 0 & 1\\ 3 & 2 & -1 & 7\\ 4 & 1 & 2 & 0\end{bmatrix}$ is invertible.

Solution:

$$\text{RREF}\begin{bmatrix}2 & 1 & 0 & 3\\ 1 & -1 & 0 & 1\\ 3 & 2 & -1 & 7\\ 4 & 1 & 2 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Since it is row equivalent to the identity matrix, it is invertible.

□

M3. Find the inverse of the matrix $\begin{bmatrix}4 & -1 & -8\\ 2 & 1 & 3\\ 1 & 1 & 4\end{bmatrix}$.

Solution: $\left[\begin{array}{ccc|ccc}4 & -1 & -8 & 1 & 0 & 0\\ 2 & 1 & 3 & 0 & 1 & 0\\ 1 & 1 & 4 & 0 & 0 & 1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & -4 & 5\\ 0 & 1 & 0 & -5 & 24 & -28\\ 0 & 0 & 1 & 1 & -5 & 6\end{array}\right]$. Thus the inverse is $\begin{bmatrix}1 & -4 & 5\\ -5 & 24 & -28\\ 1 & -5 & 6\end{bmatrix}$.

□

G1. Compute the determinant of the matrix $\begin{bmatrix}1 & 3 & 0 & -1\\ 0 & 1 & 3 & 1\\ 2 & 0 & 0 & -1\\ 1 & -3 & -2 & -1\end{bmatrix}$.

Solution:

$$\begin{aligned}
\det\begin{bmatrix}1 & 3 & 0 & -1\\ 0 & 1 & 3 & 1\\ 2 & 0 & 0 & -1\\ 1 & -3 & -2 & -1\end{bmatrix} &= 2\det\begin{bmatrix}3 & 0 & -1\\ 1 & 3 & 1\\ -3 & -2 & -1\end{bmatrix} - (-1)\det\begin{bmatrix}1 & 3 & 0\\ 0 & 1 & 3\\ 1 & -3 & -2\end{bmatrix}\\
&= 2\left(3\det\begin{bmatrix}3 & 1\\ -2 & -1\end{bmatrix} + (-1)\det\begin{bmatrix}1 & 3\\ -3 & -2\end{bmatrix}\right) + \left(1\det\begin{bmatrix}1 & 3\\ -3 & -2\end{bmatrix} - 3\begin{bmatrix}0 & 3\\ 1 & -2\end{bmatrix}\right)\\
&= 2\left(3(-1) + (-1)(7)\right) + \left((1)(7) - 3(-3)\right)\\
&= 2(-10) + 16\\
&= -4
\end{aligned}$$

□

G2. Compute the eigenvalues, along with their algebraic multiplicities, of the matrix $\begin{bmatrix}9 & -3 & 2\\ 19 & -6 & 5\\ -11 & 4 & -2\end{bmatrix}$.

Solution: 1 (with algebraic multiplicity 2), and -1 (with algebraic multiplicity 1).

□

G3. Find the eigenspace associated to the eigenvalue -2 in the matrix $A = \begin{bmatrix}2 & -3 & 2\\ 8 & -9 & 5\\ 8 & -7 & 3\end{bmatrix}$

Solution: The eigenspace is spanned by $\begin{bmatrix}\frac{1}{4}\\ 1\\ 1\end{bmatrix}$.

□

G4. Compute the geometric multiplicity of the eigenvalue 1 in the matrix $A = \begin{bmatrix} 8 & -3 & -1 \\ 21 & -8 & -3 \\ -7 & 3 & 2 \end{bmatrix}$

Solution: The eigenspace is spanned by $\begin{bmatrix} \frac{3}{7} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{7} \\ 0 \\ 1 \end{bmatrix}$, so the geometric multiplicity is 2.

□

E1:

E2:

E3:

E4:

V1:

V2:

V3:

V4:

S1:

S2:

S3:

S4:

A1:

A2:

A3:

A4:

M1:

M2:

M3:

G1:

G2:

G3:

G4: