Standards for this Module

At the end of this module, students will be able to...

- **E1: Systems as matrices.** Translate back and forth between a system of linear equations and the corresponding augmented matrix.

- **E2: Row reduction.** Put a matrix in reduced row echelon form

- **E3: Solving Linear Systems.** Solve a system of linear equations.

- **E4: Homogeneous Systems.** Find a basis for the solution set of a homogeneous linear system.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.

- Find the unique solution to a two-variable system of linear equations by back-substitution.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-systems-topic/cc-8th-systems-graphica a/systems-of-equations-with-graphing

- https://www.khanacademy.org/math/algebra/systems-of-linear-equations/solving-systems-of-equations- v/practice-using-substitution-for-systems

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Which of these graphs represents the following system of linear equations?

$$x + 2y = 4$$
$$2x - 3y = 1$$

(a)   (b)   (c)   (d)
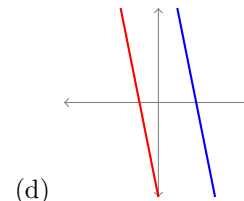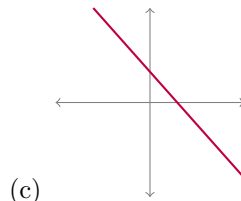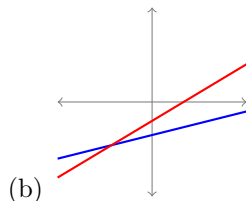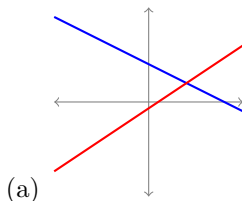
2) How many solutions are there for the system of linear equations represented by the following graph?

(a) Zero   (b) One   (c) Two   (d) Infinitely-many

3) How many solutions are there for the system of linear equations represented by the following graph? (This graph represents two completely overlapping lines.)
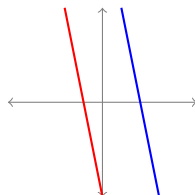
(a) Zero   (b) One   (c) Two   (d) Infinitely-many

4) How many solutions are there for the system of linear equations represented by the following graph? (This graph represents two parallel lines.)

(a) Zero   (b) One   (c) Two   (d) Infinitely-many

5) Solve the following system of linear equations.

$$y = 2x + 5$$
$$y = -x + 2$$

(a) $x = -1$ and $y = 3$

(b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

(c) There are no solutions.

(d) There are infinitely-many solutions.

6) Solve the following system of linear equations.

$$x + 2y = 4$$
$$2x - 3y = 1$$

(a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c) There are no solutions.

(d) There are infinitely-many solutions.

Application Activities - Day 1

Definition. A **linear equation** is an equation of the variables x_i of the form

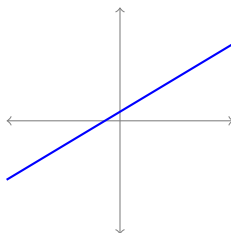$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1 s_1 + a_2 s_2 + \cdots + a_n s_n = b.$$

Observation. The linear equation $3x - 5y = -2$ may be graphed as a line in the xy plane.

The linear equation $x + 2y - z = 4$ may be graphed as a plane in xyz space.

Remark. In previous classes you likely assumed $x = x_1$, $y = x_2$, and $z = x_3$. However, since this course often deals with equations of four or more variables, we will almost always write our variables as x_i.

Definition. A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{array}{cccccc}
a_{11}x_1 + & a_{12}x_2 + & \ldots + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 + & a_{22}x_2 + & \ldots + & a_{2n}x_n & = & b_2 \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{m1}x_1 + & a_{m2}x_2 + & \ldots + & a_{mn}x_n & = & b_m
\end{array}$$

A **solution**

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1} s_1 + a_{i2} s_2 + \cdots + a_{in} s_n = b_i$$

for $1 \leq i \leq m$ (that is, the solution satisfies all equations in the system).

Remark. When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:	Verbose standard form:	Concise standard form:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

$$\begin{aligned} x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - x_2 + x_3 &= -2 \end{aligned}$$

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ - x_2 + x_3 &= -2 \end{aligned}$$

Definition. A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

Fact. All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

Activity. *(5 min)* Consider the following graphs representing linear systems of two variables. Label each graph with **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.
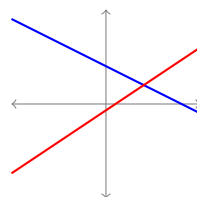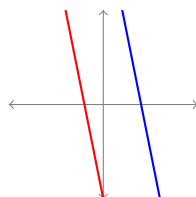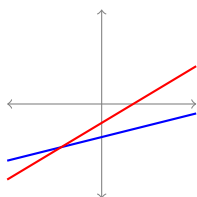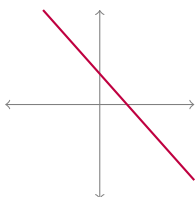
Activity. *(10 min)* All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system by solving for x_1 in the first equation, substituting the resulting expression into the second equation, and then simplifying.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity. *(10 min)* Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part X: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

Part X: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to describe *all* solutions (the **solution set**) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$ for the linear system in terms of a.

Remark. The solution set of a consistent linear system with infinitely many solutions may be described by setting each certain variable equal to an arbitrary parameter, and expressing the other variables in terms of those parameters. (Later we will learn how to do this methodically.)

Activity. *(10 min)* Consider the following linear system.

$$\begin{array}{rcrcr} x_1 + 2x_2 & & - & x_4 & = & 3 \\ & & x_3 & + 4x_4 & = & -2 \end{array}$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix} + \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation. Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but won't cut it for equations with more variables.

Definition. A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{array}{rcrcrcrcl} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Definition. Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

Activity. *(10 min)* Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a conzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

(Instructor Note:) This activity could be ran as a card sort.

Application Activities - Day 2

Definition. The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.

2. Multiply a row by a conzero constant.

3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity. *(15 min)* Show that the following two linear systems:

$$
\begin{aligned}
3x_1 - 2x_2 + 13x_3 &= 6 \\
2x_1 - 2x_2 + 10x_3 &= 2 \\
-1x_1 + 3x_2 - 6x_3 &= 11
\end{aligned}
\qquad\qquad
\begin{aligned}
x_1 - x_2 + 5x_3 &= 1 \\
x_2 - 2x_3 &= 3 \\
x_3 &= 2
\end{aligned}
$$

are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

1. Swap R_1 (first row) and R_2 (second row).

2. Multiply R_2 by $\frac{1}{2}$.

3. Add R_1 to R_3.

4. Add $-3R_1$ to R_2.

5. Add $-2R_2$ to R_3.

6. Multiply R_3 by $\frac{1}{3}$.

Definition. The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix.

Activity. *(10 min)* Reproduce the steps that manipulated the matrix

$$
\left[\begin{array}{ccc|c}
3 & -2 & 13 & 6 \\
2 & -2 & 10 & 2 \\
-1 & 3 & -6 & 11
\end{array}\right]
\sim
\left[\begin{array}{ccc|c}
1 & -1 & 5 & 1 \\
0 & 1 & -2 & 3 \\
0 & 0 & 1 & 2
\end{array}\right]
$$

into row echelon form by using the following algorithm.

1. Identify the top cell of the first non-zero column as your **pivot position**; you will ignore anything in the matrix that is above or left of your current pivot position.

2. If the pivot position contains a 0, swap its row with a lower row that does not contain a 0 in its column.

3. Divide the pivot row by the term in pivot po-

sition to change the pivot term to 1. (If convenient, you can first swap the pivot row with a lower row to make this division easier.)

4. Add multiples of the pivot row to all lower rows so that all terms below pivot position become 0.

5. Move your pivot position down and right one step.

6. If all terms in and below pivot position are zero, move your pivot position right. Repeat this step as needed.

7. If the matrix is not yet in row echelon form, return to Step 2.

Definition. A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0.

Activity. *(10 min)* Show that the following two linear systems:

$$
\begin{aligned}
x_1 - x_2 + 5x_3 &= 1 \\
x_2 - 2x_3 &= 3 \\
x_3 &= 2
\end{aligned}
\qquad\qquad
\begin{aligned}
x_1 \qquad\quad &= -2 \\
x_2 \quad\ &= 7 \\
x_3 &= 2
\end{aligned}
$$

are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

1. Add $2R_3$ to R_2.

2. Add $-5R_3$ to R_1.

3. Add R_2 to R_1.

Then write the solution to the linear system.

Remark. We may verify that $(x_1, x_2, x_3) = (-2, 7, 2)$ is a solution to the original linear system

$$
\begin{aligned}
3x_1 - 2x_2 + 13x_3 &= 6 \\
2x_1 - 2x_2 + 10x_3 &= 2 \\
-1x_1 + 3x_2 - 6x_3 &= 11
\end{aligned}
$$

by plugging the solution into each equation.

Application Activities - Day 3

Fact. Every augmented matrix A reduces to a unique reduced row echelon form matrix. This matrix is denoted as $\mathrm{RREF}(A)$.

Definition. The following algorithm that reduces A to $\mathrm{RREF}(A)$ is known as **Gauss-Jordan elimination**.

1. Identify the top cell of the first non-zero column as your pivot position; you will ignore anything in the matrix that is above or left of your current pivot position.

2. If the pivot position contains a 0, swap its row with a lower row that does not contain a 0 in its column.

3. Divide the pivot row by the term in pivot position to change the pivot term to 1. (If convenient, you can first swap the pivot row with a lower row to make this division easier.)

4. Add multiples of the pivot row to all lower rows

   so that all terms below pivot position become 0.

5. Move your pivot position down and right one step.

6. If all terms in and below pivot position are zero, move your pivot position right. Repeat this step as needed.

7. If the matrix is not yet in row echelon form, return to Step 2.

8. Finally, add multiples of lower rows into higher rows, zeroing out all terms above leading terms.

Activity. *(15 min)* Find $\mathrm{RREF}(A)$ where

$$
A = \left[\begin{array}{cccc|c}
-1 & 1 & -3 & 2 & 0 \\
2 & -1 & 5 & 3 & -11 \\
3 & 2 & 4 & 1 & 1 \\
0 & 1 & -1 & 1 & 1
\end{array}\right].
$$

Definition. The columns of $\mathrm{RREF}(A)$ without a leading term represent **free variables** of the linear system modeled by A that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by A.

Activity. *(10 min)* Given the linear system and its equivalent augmented matrices

$$
\begin{aligned}
-x_1 + x_2 - 3x_3 + 2x_4 &= 0 \\
2x_1 - x_2 + 5x_3 + 3x_4 &= -11 \\
3x_1 + 2x_2 + 4x_3 + x_4 &= 1 \\
x_2 - x_3 + x_4 &= 1
\end{aligned}
\qquad
\left[\begin{array}{cccc|c}
-1 & 1 & -3 & 2 & 0 \\
2 & -1 & 5 & 3 & -11 \\
3 & 2 & 4 & 1 & 1 \\
0 & 1 & -1 & 1 & 1
\end{array}\right]
\sim
\left[\begin{array}{cccc|c}
1 & 0 & 2 & 0 & -1 \\
0 & 1 & -1 & 0 & 3 \\
0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

describe the solution set $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$ to the linear system by setting the free variable $x_3 = a$, and then expressing each of the bounded variables x_1, x_2, x_4 equal to an expression in terms of a.

Remark. It's not necessary to completely find $\mathrm{RREF}(A)$ to deduce that a linear system is inconsistent.

Activity. *(10 min)* Find a contradiction in the inconsistent linear system

$$
\begin{aligned}
2x_1 - 3x_2 &= 17 \\
x_1 + 2x_2 &= -2 \\
-x_1 - x_2 &= 1
\end{aligned}
$$

by considering the following equivalent augmented matrices:

$$
\left[\begin{array}{cc|c}
2 & -3 & 17 \\
1 & 2 & -2 \\
-1 & -1 & 1
\end{array}\right]
\sim
\left[\begin{array}{cc|c}
1 & 2 & -2 \\
0 & 1 & 3 \\
0 & 0 & 2
\end{array}\right].
$$

Remark. In Module TODO, we will

Definition. A **homogeneous system** is a linear system satisfying $b_i = 0$, that is, it is a linear system of the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}
$$

Activity. *(5 min)* Show that all homogeneous systems are consistent by finding a quickly verifiable solution for

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}
$$

Fact. Because the zero vector is always a solution, the solution set to any homogeneous system with infinitely-many solutions may be generated by multiplying the parameters representing the free variables by a minimal set of Euclidean vectors, and adding these up. For example:

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

Definition. A minimal set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Activity. *(10 min)* Find a basis for the solution set of the following homogeneous linear system.

$$\begin{aligned} x_1 + 2x_2 \quad\;\; - \;\; x_4 &= 0 \\ x_3 + 4x_4 &= 0 \\ 2x_1 + 4x_2 + x_3 + 2x_4 &= 0 \end{aligned}$$

Standards for this Module

At the end of this module, students will be able to...

- **V1: Vector Spaces.** Determine if a set with given operations forms a vector space.

- **V2: Linear Combinations.** Determine if a vector can be written as a linear combination of a given set of vectors.

- **V3: Spanning Sets.** Determine if a set of vectors spans a vector space.

- **V4: Subspaces.** Determine if a subset of a vector space is a subset or not.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.

- Add complex numbers and multiply complex numbers by scalars.

- Add polynomials and multiply polynomials by scalars.

- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- https://www.khanacademy.org/math/precalculus/vectors-precalc/vector-addition-subtraction/
  v/adding-and-subtracting-vectors

- https://www.khanacademy.org/math/precalculus/vectors-precalc/combined-vector-operations/
  v/combined-vector-operations-example

- https://www.khanacademy.org/math/precalculus/imaginary-and-complex-numbers/adding-and-subtracting-
  v/adding-complex-numbers

- https://www.khanacademy.org/math/algebra/introduction-to-polynomial-expressions/adding-and-subtrac
  v/adding-and-subtracting-polynomials-1

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Simplify the following vector expression.

$$2 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

(a) $\begin{bmatrix} 0 \\ 4 \\ -7 \end{bmatrix}$ (b) $\begin{bmatrix} 6 \\ -8 \\ -3 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ (d) $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

2) Simplify the complex number expression $-4(3 - 2i) + 2(5 + i)$.

(a) $3 - 7i$ (b) $4 + i$ (c) $-2 + 10i$ (d) $-1 - 5i$

3) Simplify $3f(x) - 2g(x)$ where $f(x) = 7 - x^2$ and $g(x) = 2x^3 + x - 1$.

(a) $x^3 + 4x - 5$ (b) $-4x^3 - 3x^2 - 2x + 23$ (c) $3x^3 + 5x^2 - 3x + 17$ (d) $-x^3 + 19x^2 - 4$

4) TODO some problems based on module E.

Application Activities - Day 1

Activity. *(25 min)* Consider each of the following vector properties. Label each property with $\mathbb{R}^1$, $\mathbb{R}^2$, and/or $\mathbb{R}^3$ if that property holds for Euclidean vectors/scalars $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of that dimension.

1. **Addition associativity.**

   $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

2. **Addition commutivity.**

   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

3. **Addition identity.**

   There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

4. **Addition inverse.**

   There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

5. **Addition midpoint uniqueness.**

   There exists a unique $\mathbf{m}$ where the distance from $\mathbf{u}$ to $\mathbf{m}$ equals the distance from $\mathbf{m}$ to $\mathbf{v}$.

6. **Scalar multiplication associativity.**

   $a(b\mathbf{v}) = (ab)\mathbf{v}$.

7. **Scalar multiplication identity.**

   $1\mathbf{v} = \mathbf{v}$.

8. **Scalar multiplication relativity.**

   There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.

9. **Scalar distribution.**

   $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

10. **Vector distribution.**

    $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

11. **Orthogonality.**

    There exists a non-zero vector $\mathbf{n}$ such that $\mathbf{n}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

12. **Bidimensionality.**

    $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ for some value of a, b.

Definition. A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Addition associativity.**

  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **Addition commutivity.**

  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **Addition identity.**

  There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

- **Addition inverse.**

  There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

- **Scalar multiplication associativity.**

  $a(b\mathbf{v}) = (ab)\mathbf{v}$.

- **Scalar multiplication identity.**

  $1\mathbf{v} = \mathbf{v}$.

- **Scalar distribution.**

  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

- **Vector distribution.**

  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Definition. The most important examples of vector spaces are the **Euclidean vector spaces** $\mathbb{R}^n$, but there are other examples as well.

Activity. *(25 min)* Consider the following vector space that models motion along the curve $y = e^x$. Let $V = \{(x, y) : y = e^x\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 b_2)$, and $c(a, b) = (ca, b^c)$.

Part X: Verify that $3((1, e) + (-2, \frac{1}{e^2})) = 3(1, e) + 3(-2, \frac{1}{e^2})$.
(Instructor Note:) Draw a visualization of these vectors by sketching "curved" vectors in the xy plane.
Part X: Prove the scalar distribution property for this space: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

Application Activities - Day 2

Remark. The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.

- $\mathbb{R}^\infty$: Sequences of real numbers $(v_1, v_2, \dots)$.

- $\mathbb{R}^{m \times n}$: Matrices of real numbers with m rows and n columns.

- $\mathbb{C}$: Complex numbers.

- $\mathcal{P}^n$: Polynomials of degree n or less.

- $\mathcal{P}$: Polynomials of any degree.

- $C(\mathbb{R})$: Real-valued continuous functions.

Activity. *(15 min)* Let $V = \{(a, b) : a, b$ are real numbers$\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$ and $c(a, b) = (a^c, b + c)$. Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.**

  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **Addition commutivity.**

  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **Addition identity.**

  There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

- **Addition inverse.**

  There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

- **Scalar multiplication associativity.**

  $a(b\mathbf{v}) = (ab)\mathbf{v}$.

- **Scalar multiplication identity.**

  $1\mathbf{v} = \mathbf{v}$.

- **Scalar distribution.**

  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

- **Vector distribution.**

  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Definition. A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$ for any choice of scalar multiples $c_1, c_2, \dots, c_m$.

Definition. The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

Activity. *(10 min)* Consider $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

Part X: Sketch $c\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part X: Sketch a representation of all the vectors given by $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ in the xy plane.

Activity. *(10 min)* Consider span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Part X: Sketch $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in the xy plane for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Part X: Sketch a representation of all the vectors given by span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ in the xy plane.

Activity. *(5 min)* Sketch a representation of all the vectors given by span $\left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ in the xy plane.

Activity. *(10 min)* The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector equation

$x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2.

Part X: Reinterpret this vector equation as a system of linear equations.

Part X: Solve this system. (From now on, feel free to use a calculator to solve linear systems.)

Part X: Given this solution, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

Application Activities - Day 3

Fact. A vector $\mathbf{b}$ belongs to span$\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \ldots \mathbf{v}_n \,|\, \mathbf{b}]$ is consistent.

Remark. To determine if $\mathbf{b}$ belongs to span$\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, find RREF$[\mathbf{v}_1 \ldots \mathbf{v}_n \,|\, \mathbf{b}]$.

Activity. *(5 min)* Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Activity. *(5 min)* Determine if $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Observation. So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space $\mathbb{R}^n$; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

Activity. *(5 min)* We previously checked that $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ does not belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$. Does $f(x) = 3x^2 - 2x + 1$ belong to span$\{x^2 - 3, -x^2 - 3x + 2\}$?

Activity. *(10 min)* Does the matrix $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\}$?

Activity. *(10 min)* Does the complex number $2i$ belong to span$\{-3 + i, 6 - 2i\}$?

Activity. *(10 min)* How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your guess.

Activity. *(5 min)* How many vectors are required to span $\mathbb{R}^3$?

Application Activities - Day 4

Fact. At least n vectors are required to span $\mathbb{R}^n$.

Activity. *(15 min)* Find a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by doing the following.

Part X: Choose simple values for x, y, z such that $\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{bmatrix}$ represents an inconsistent linear equation.

Part X: Use row operations to manipulate $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{bmatrix}$.

Part X: Write a sentence explaining why $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ cannot be in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Fact. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ fails to span all of $\mathbb{R}^n$ exactly when $\mathrm{RREF}[\mathbf{v}_1 \ldots \mathbf{v}_m]$ has a row of zeros.

Activity. *(5 min)* Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Prove that $\mathbb{R}^4 =$ span S.

Activity. *(10 min)* Consider the set of third-degree polynomials

$$S = \left\{ 2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2 \right\}.$$

Prove that $\mathcal{P}^3 \neq$ span S.

Definition. A subset of a vector space is called a **subspace** if it is itself a vector space.

Fact. If S is a subset of a vector space V, then span S is a subspace of V.

Remark. To prove that a subset is a subspace, you need only verify that $c\mathbf{v} + d\mathbf{w}$ belongs to the subset for any choice of vectors $\mathbf{v}, \mathbf{w}$ from the subset and any real scalars c, d.

Activity. *(5 min)* Prove that $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$ is a subspace of the vector space of all degree-two polynomials by showing that $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$ belongs to P.

Activity. *(10 min)* Consider the subset of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.

Part X: Find a linear combination $c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ that does not belong to this subset.

Part X: Use this linear combination to sketch a picture illustrating why this subset is not a subspace.

Fact. Suppose a subset S of V is isomorphic to another vector space W. Then S is a subspace of V.

Activity. *(5 min)* Show that the set of 2×2 matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of $\mathbb{R}^{2\times 2}$ by finding a Euclidean space isomorphic to S.

Standards for this Module

At the end of this module, students will be able to...

- **S1. Linear independence** Determine if a set of Euclidean vectors is linearly dependent or independent.

- **S2. Basis verification** Determine if a set of vectors is a basis of a vector space

- **S3. Basis construction** Construct a basis for the subspace spanned by a given set of vectors.

- **S4. Dimension** I can compute the dimension of a vector space.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.

- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.

- Apply linear combinations and spanning sets **(Standard(s) V2)**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- https://www.khanacademy.org/math/precalculus/vectors-precalc/vector-addition-subtraction/v/adding-and-subtracting-vectors

- https://www.khanacademy.org/math/precalculus/vectors-precalc/combined-vector-operations/v/combined-vector-operations-example

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Simplify the following vector expression. (TODO write a new one)

$$2 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

(a) $\begin{bmatrix} 0 \\ 4 \\ -7 \end{bmatrix}$
(b) $\begin{bmatrix} 6 \\ -8 \\ -3 \end{bmatrix}$
(c) $\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$
(d) $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

2) TODO some problems based on module E.

Application Activities - Day 1

Activity. *(15 min)* In the previous module, we considered

$$
S = \left\{
\begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix},
\begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix},
\begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix},
\begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix},
\begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}
\right\}
$$

and showed that span $S \neq \mathbb{R}^4$. Find two vectors that are in the span of the other three vectors.
(Instructor Note:) Actually, the activity involved the corresponding vectors in $\mathcal{P}^3$.

Definition. We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

Standards for this Module

At the end of this module, students will be able to...

- **A1. Linear maps as matrices** I can write the matrix (with respect to the standard bases) corresponding to a linear transformation between Euclidean spaces.

- **A2. Linear map verification** I can determine if a map between vector spaces is linear or not.

- **A3. Injectivity and Surjectivity** I can determine if a given linear map is injective and/or surjective

- **A4. Kernel and Image** I can compute the kernel and image of a linear map, including finding bases.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing **(Standard(s) E1, E2, E3, E4)**.

- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent **(Standard(s) V5)**.

- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace **(Standard(s) V6, V7)**.

- State the definition of a basis, and determine if a set of vectors is a basis **(Standard(s) V8, V9)**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Review the supporting Standards listed above.

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Which of the following is a solution to the system of linear equations

$$x + 3y - z = 2$$
$$2x + 8y + 3z = -1$$
$$-x - y + 9z = -10$$

(a) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
(b) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$
(d) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

2) Find a basis for the solution set of the following homogeneous system of linear equations

$$x + 2y + -z - w = 0$$
$$-2x - 4y + 3z + 5w = 0$$

(a) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$
(b) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$
(c) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$
(d) None of these are a basis.

3) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent

(b) It does not span and is linearly independent

(c) It spans but it is linearly dependent

(d) It is a basis of $\mathbb{R}^3$.

4) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent

(b) It does not span and is linearly independent

(c) It spans but it is linearly dependent

(d) It is a basis of $\mathbb{R}^3$.

5) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

   (a) It does not span and is linearly dependent
   (b) It does not span and is linearly independent
   (c) It spans but it is linearly dependent
   (d) It is a basis of $\mathbb{R}^3$.

6) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

   (a) It does not span and is linearly dependent
   (b) It does not span and is linearly independent
   (c) It spans but it is linearly dependent
   (d) It is a basis of $\mathbb{R}^3$.

7) Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors ...

8) Suppose you know that every vector in $\mathbb{R}^5$ can be written as a linear combination of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. What can you conclude about n?

   (a) $n \leq 5$
   (b) $n = 5$
   (c) $n \geq 5$
   (d) n could be any positive integer

9) Suppose you know that every vector in $\mathbb{R}^5$ can be written uniquely as a linear combination of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. What can you conclude about n?

   (a) $n \leq 5$
   (b) $n = 5$
   (c) $n \geq 5$
   (d) n could be any positive integer

10) Suppose you know that every vector in $\mathbb{R}^5$ can be written uniquely as a linear combination of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. What can you conclude about the set $\{\vec{v}_1, \ldots, \vec{v}_n\}$?

   (a) It does not span and is linearly dependent
   (b) It does not span and is linearly independent
   (c) It spans but it is linearly dependent
   (d) It is a basis of $\mathbb{R}^3$.

Application Activities - Day 1

Definition. A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$

2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}$, $\vec{v} \in V$.

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.
V is called the **domain** of T and W is called the **co-domain** of T.

Activity. *(0 min)* Determine if each of the following maps are linear transformations

(a) $T_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $T_1 \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \sqrt{a^2 + b^2}$

(b) $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T_2 \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$

(c) $T_3 : \mathcal{P}_d \to \mathcal{P}_{d-1}$ given by $T_3(f(x)) = f'(x)$.

(d) $T_4 : C(\mathbb{R}) \to C(\mathbb{R})$ given by $T_4(f(x)) = f(-x)$

(e) $T_5 : \mathcal{P} \to \mathcal{P}$ given by $T_5(f(x)) = f(x) + x^2$

Activity. *(0 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and

$T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute each of the following:

(a) $T \left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right)$

(b) $T \left(\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right)$

(c) $T \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$

(d) $T \left(\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} \right)$

Activity. *(0 min)* Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation. What is the smallest number of vectors needed to determine T? In other words, what is the smallest number n such that there are $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^4$ and given $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ you can determine $T(\vec{w})$ for *any* $\vec{w} \in \mathbb{R}^2$?

Observation. Fix an ordered basis for V. Since every vector can be written *uniquely* as a linear combination of basis vectors, a linear transformation $T : V \to W$ corresponds exactly to a choice of where each basis vector goes. For convenience, we can thus encode a linear transformation as a matrix, with one column for the image of each basis vector (in order).

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard ordered basis.

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the ordered basis

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Activity. *(0 min)* Let $D : \mathcal{P}_3 \to \mathcal{P}_2$ be the derivative map (recall this is a linear transformation). Write the matrix corresponding to D with respect to the ordered basis $\{1, x, x^2, x^3\}$.

Application Activities - Day 2

Definition. Let $T : V \to W$ be a linear transformation.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

- T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\vec{w} \in W$, there is some $v \in V$ with $T(\vec{v}) = \vec{w}$.

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

Activity. *(0 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

Definition. We also have two important sets called the **kernel** of T and the **image** of T.

$$\ker T = \left\{ \vec{v} \in V \mid T(\vec{v}) = 0 \right\}$$
$$\operatorname{Im} T = \left\{ \vec{w} \in W \mid \text{there is some } v \in V \text{ with } T(\vec{v}) = \vec{w} \right\}$$

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T.

Activity. *(0 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T.

Activity. *(0 min)*
Part X: Describe surjective linear transformations in terms of the image.
Part X: Describe injective linear transformations in terms of the kernel.

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

1) Write a system of equations whose solution set is the kernel.

---

2) Compute RREF(A) and solve the system of equations.

3) Compute the kernel of T

4) Find a basis for the kernel of T

Activity. *(0 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

1) Write a system of equations whose solution set is the kernel.

2) Compute RREF(A) and solve the system of equations.

3) Compute the kernel of T

4) Find a basis for the kernel of T

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

1) Find a set of vectors that span the image of T

2) Find a basis for the image of T.

Activity. *(0 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

1) Find a set of vectors that span the image of T

2) Find a basis for the image of T.

Application Activities - Day 3

Activity. *(0 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). You have cards containing a number of statements about T and A. Sort them into groups of equivalent statements, and post them on your board.
(Instructor Note:) Card sort activity for 10-15 minutes: cards contain the following

(a) T is injective

(b) T is not injective

(c) T is surjective

(d) T is not surjective

(e) The system of linear equations given by the augmented matrix $\left[A \mid \vec{b} \right]$ has a solution for all $\vec{b} \in \mathbb{R}^m$

(f) The system of linear equations given by the augmented matrix $\left[A \mid \vec{b} \right]$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$

(g) The system of linear equations given by the augmented matrix $\left[A \mid \vec{0} \right]$ has a non-trivial solution.

(h) The columns of A span $\mathbb{R}^m$

(i) The columns of A are linearly independent

(j) The columns of A are a basis of $\mathbb{R}^m$

(k) Every column of $\text{RREF}(A)$ is a pivot column

(l) $\text{RREF}(A)$ has a non-pivot column

(m) $\text{RREF}(A)$ has n pivot columns

Activity. *(0 min)* **(Instructor Note:)**Gallery walk Cycle around the room counter-clockwise. If they have two things grouped together that you know are not equivalent, write a reason or counter-example on a sticky note.

Activity. *(0 min)* Come up with as many statements as you can, and add them to the appropriate group.

Standards for this Module

At the end of this module, students will be able to...

- **M1. Matrix multiplication** Multiply matrices.

- **M2. Invertible matrices** Determine if a square matrix is invertible or not.

- **M3. Matrix inverses** Compute the inverse matrix of an invertible matrix.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers

- Solve systems of linear equations **(Standard(s) E.3)**

- Find the matrix corresponding to a linear transformation **(Standard(s) A.1)**

- Determine if a linear transformation is injective and/or surjective **(Standard(s) A.3)**

- Interpret the ideas of injectivity and surjectivity in multiple ways

Readiness Assurance Resources

The following resources will help you prepare for this module.

- https://www.khanacademy.org/math/algebra2/manipulating-functions/funciton-composition/
  v/function-composition

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Let $f(x) = x^2 - 2$ and $g(x) = x^2 + 1$. Compute the composition function $(f \circ g)(x)$.

    (a) $x^2 - 1$

    (b) $x^4 + 2x^2 - 1$

    (c) $x^4 - 4x^2 + 5$

    (d) $x^4 - x^2 - 2$

2) Suppose $f(x)$ and $g(x)$ are real-valued functions satisfying

$$f(2) = 1 \qquad\qquad g(2) = 3$$
$$f(3) = 4 \qquad\qquad g(3) = 5$$
$$f(4) = 3 \qquad\qquad g(4) = 6$$

Compute $(f \circ g)(2)$.

    (a) 2

    (b) 3

    (c) 4

    (d) 5

3) Solve the system of linear equations

$$x + 3y = -2$$
$$2x - 7y = 9$$

    (a) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} -2 \\ 9 \end{bmatrix}$

4) Let a, b, c be fixed real numbers. How many solutions does the system of linear equations below have?

$$x + 2y + 3z = a$$
$$y - z = b$$
$$y + z = c$$

    (a) 0   (b) 1   (c) Infinitely many   (d) It depends on the values of a, b, and c.

5) What is the matrix corresponding to the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) =$

$\begin{bmatrix} x + 2y - z \\ y + 3z \\ x + 7y \end{bmatrix}$?

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 7 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ 1 & 7 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 7 \\ -1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 7 \\ -1 & 3 & 0 \end{bmatrix}$

6) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation with associated matrix $A = \begin{bmatrix} 2 & 3 \\ -1 & -1 \\ 0 & 4 \end{bmatrix}$. Compute $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$.

(a) $\begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$

7) Which of the following is true of the linear transformation T : ?

   (a) T is neither injective nor surjective
   (b) T is injective but not surjective
   (c) T is surjective but not injective
   (d) T is both injective and surjective

8) Which of the following is true of the linear transformation T : ?

   (a) T is neither injective nor surjective
   (b) T is injective but not surjective
   (c) T is surjective but not injective
   (d) T is both injective and surjective

9) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with associated matrix $A \in M_{m,n}(\mathbb{R})$. Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?

   (a) T is injective
   (b) T has a non-trivial kernel
   (c) The columns of A are linearly dependent
   (d) RREF(A) has a non-pivot column

10) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with associated matrix $A \in M_{m,n}(\mathbb{R})$. Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?

   (a) T is surjective
   (b) Im $T = \mathbb{R}^m$
   (c) The columns of A span $\mathbb{R}^m$
   (d) RREF(A) has only pivot columns

Application Activities - Day 1

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given

by the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{4,3}$

(b) $M_{4,2}$

(c) $M_{3,2}$

(d) $M_{2,3}$

(e) $M_{2,4}$

(f) $M_{3,4}$

Activity. *(0 min)* Compute $(S \circ T)(\vec{e}_1)$, $(S \circ T)(\vec{e}_2)$, and $(S \circ T)(\vec{e}_3)$.

Activity. *(0 min)* Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity. *(0 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{2,2}$

(b) $M_{2,3}$

(c) $M_{3,2}$

(d) $M_{3,3}$

Activity. *(0 min)* Compute $(S \circ T)(\vec{e}_1)$ and $(S \circ T)(\vec{e}_2)$

Activity. *(0 min)* Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity. *(0 min)* Let $T : \mathbb{R}^1 \to \mathbb{R}^4$ be given by the matrix $B = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ and $S : \mathbb{R}^4 \to \mathbb{R}^1$ be given by the

matrix $A = \begin{bmatrix} 2 & 3 & 2 & 5 \end{bmatrix}$.
What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{1,1}$

(b) $M_{1,4}$

(c) $M_{4,1}$

(d) $M_{4,4}$

Activity. *(0 min)* Compute $(S \circ T)(\vec{e}_1)$

Activity. *(0 min)* Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity. *(0 min)* Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

1. Compute AX

2. Interpret the system of equations below as a matrix equation

$$3x + y - z = 5$$
$$2x + 4z = -7$$
$$-x + 3y + 5z = 2$$

Application Activities - Day 2

Activity. *(0 min)* Each row operation can be interpreted as a matrix multiplication. Let $A \in M_{4,4}$

1) Find a matrix S_1 such that $S_1 A$ is the result of swapping the second and fourth rows of A.

2) Find a matrix S_2 such that $S_2 A$ is the result of adding 5 times the third row of A to the first.

3) Find a matrix S_3 such that $S_3 A$ is the result of doubling the fourth row of A.

Activity. *(0 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). Consider the following statements about T

(a) T is injective

(b) T is surjective

(c) T is bijective (i.e. both injective and surjective)

(d) $AX = B$ has a solution for all $B \in M_{m,1}$

(e) $AX = B$ has a unique solution for all $B \in M_{m,1}$

(f) $AX = 0$ has a non-trivial solution.

(g) The columns of A span $\mathbb{R}^m$

(h) The columns of A are linearly independent

(i) The columns of A are a basis of $\mathbb{R}^m$

(j) RREF(A) has n pivot columns

(k) RREF(A) has m pivot columns

Sort these statements into groups of equivalent statements.

Activity. *(0 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). If T is injective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Activity. *(0 min)* If T is surjective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Activity. *(0 min)* If T is bijective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Application Activities - Day 3

Activity. *(0 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with matrix $A \in M_{n,n}$.

If T is a bijection, then $AX = B$ has a unique solution for all $B \in \mathbb{R}^n$. Thus we can define a map $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by defining $T^{-1}(B)$ to be this solution. It follows immediately that $T \circ T^{-1}$ is the identity map. The matrix corresponding to T^{-1} is denoted A^{-1}.

1) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_1)$

2) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_2)$

3) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_3)$

4) Compute A^{-1}

A (square) matrix is called *invertible* if it corresponds to an invertible linear transformation.

1) Find the inverse of the matrix $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$

2) Find the inverse of the matrix $\begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$

Standards for this Module

At the end of this module, students will be able to...

- **G1. Determinants** Compute the determinant of a square matrix.

- **G2. Eigenvalues** Find the eigenvalues of a square matrix, along with their algebraic multiplicities.

- **G3. Eigenvectors** Find the eigenspace of a square matrix associated to a given eigenvalue.

- **G4. Geometric multiplicity** Compute the geometric multiplicity of an eigenvalue of a square matrix.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.

- Find the matrix corresponding to a linear transformation of Euclidean spaces (**Standard(s) A1**).

- Recall and use the definition of a linear transformation (**Standard(s) A2**).

- Find all roots of quadratic polynomials (including complex ones), and be able to use the rational root theorem to find all rational roots of a higher degree polynomial.

- Interpret the statement "A is an invertible matrix" in many equivalent ways in different contexts.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Finding the area of a parallelogram: `https://www.khanacademy.org/math/basic-geo/basic-geo-area-and-perime` `parallelogram-area/a/area-of-parallelogram`

- Factoring quadratics: `https://www.khanacademy.org/math/algebra2/polynomial-functions/factoring-polynom` `v/factoring-polynomials-1`

- Finding complex roots of quadratics: `https://www.khanacademy.org/math/algebra2/polynomial-functions/` `quadratic-equations-with-complex-numbers/v/complex-roots-from-the-quadratic-formula`

- Finding all roots of polynomials: `https://www.khanacademy.org/math/algebra2/polynomial-functions/` `finding-zeros-of-polynomials/v/finding-roots-or-zeros-of-polynomial-1`

- The Rational Root Theorem: `https://artofproblemsolving.com/wiki/index.php?title=Rational_` `Root_Theorem`

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Find the area of the parallelogram with vertices $(0, 0)$, $(4, 0)$, $(5, 2)$, and $(1, 2)$.

    (a) 5

    (b) 6

    (c) 7

    (d) 8
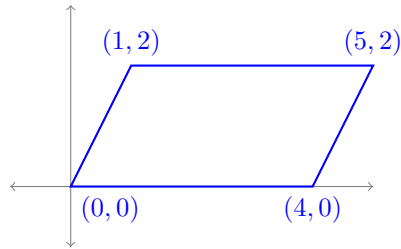
2) Find the area of the parallelogram with vertices $(0, 0)$, $(12, 5)$, $(14, 8)$, and $(2, 3)$.

    (a) 13

    (b) 26

    (c) 39

    (d) 52

3) The parallelogram ABCD has area 6. If AE is $\frac{3}{2}$ the length of AB, what is the area of the parallelogram AEFD?

    (a) 9

    (b) 12

    (c) 15

    (d) 18

4) The parallelogram ABCD has area 6. If AF is one third as long as AD, what is the area of the parallelogram ABEF?

(a) 1

(b) 2

(c) 3

(d) 4

5) Let $T : \mathbb{R}^2 \to \mathbb{R}$ be a linear transformation. Which of the following is equal to $T\left(\begin{bmatrix} a+b \\ a+b \end{bmatrix}\right)$?

(a) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$

(b) $2T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$

(c) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} b \\ a \end{bmatrix}\right)$

(d) $T\left(\begin{bmatrix} a \\ a \end{bmatrix}\right) + T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} b \\ a \end{bmatrix}\right) + T\left(\begin{bmatrix} b \\ b \end{bmatrix}\right)$

6) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with associated matrix $A \in M_n(\mathbb{R})$. Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?

(a) A is not an invertible matrix

(b) T has a non-trivial kernel

(c) $\det(A) \neq 0$

(d) $A\vec{x} = \vec{b}$ has multiple solutions for all $\vec{b} \in \mathbb{R}^n$.

7) What is the matrix corresponding to the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y - z \\ y + z \\ x + 7z \end{bmatrix}$?

(a) $\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 7 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 7 \\ -1 & 0 & 0 \end{bmatrix}$

8) Which of the following conditions imply that the quadratic polynomial $ax^2 + bx + c$ has no real roots?

(a) $a < 0$

(b) $b^2 - 4ac < 0$

(c) $ac - b^2 < 0$

(d) $ab + c^2 < 0$

9) Which of the following is a root of the polynomial $x^2 - 4x + 13$?

(a) $1 + 2i$ (b) $2 - 3i$ (c) $3 + 4i$ (d) $4 - 5i$

10) How many roots does the polynomial $x^4 + 3x^3 + x^2 - 3x - 2$ have?

(a) 1   (b) 2   (c) 3   (d) 4

Application Activities - Days 1-2

Activity. *(0 min)* Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

We can summarize the transformation of the unit square into this rectangle by measuring the following:

(a) How did the area change?

(b) How was th x-axis stretched?

(c) How was the y-axis stretched?

Activity. *(0 min)* Consider the following linear transformations $A_i : \mathbb{R}^2 \to \mathbb{R}^2$.

- $A_2 = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

- $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

- $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

For each linear transformation, do the following:

(a) Draw a graph showing the image of the unit square.

(b) Compute how much the area was stretched out.

(c) Determine which axes (or lines) were preserved; how were they stretched out?

Activity. *(0 min)* Our goal is to define a function det $: M_n \to \mathbb{R}$ that takes a square matrix (linear transformation $\mathbb{R}^n \to \mathbb{R}^n$) and returns its area stretching factor. This function is called the **determinant**. What properties should this function have?

Match the four pictures to the following four expressions

$$\det(\vec{e}_1, \vec{e}_2) \qquad \det(\vec{v}, \vec{v}) \qquad \det(c\vec{v}, \vec{w}) \qquad \det(\vec{u} + \vec{v}, \vec{w})$$

Activity. *(0 min)* What can you conclude about each of the following?

1. $\det(\vec{e}_1, \vec{e}_2)$

2. $\det(\vec{v}, \vec{v})$

3. $\det(c\vec{v}, \vec{w})$

4. $\det(\vec{u} + \vec{v}, \vec{w})$

Definition. To summarize, we have 3 properties (stated here over $\mathbb{R}^n$)

P1: $\det(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n) = 1$

P2: If $\vec{v}_i = \vec{v}_j$ for some $i \neq j$, then $\det(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = 0$.

P3: The determinant is linear in each column.

These three properties uniquely define the **determinant**, as we shall see.

Observation. Note that if $\vec{v}, \vec{w} \in \mathbb{R}^2$ and $A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$ we will write either $\det(A)$ or $\det(\vec{v}, \vec{w})$ as is convenient.

Activity. *(0 min)*

How are $\det(\vec{v}, \vec{w})$ and $\det(\vec{w}, \vec{v})$ related?

(a) $\det(\vec{v}, \vec{w}) = \det(\vec{w}, \vec{v})$

(b) $\det(\vec{v}, \vec{w}) = -\det(\vec{w}, \vec{v})$

(c) They are unrelated

(d) They are related, but not by either (a) or (b).

Observation. Note that this implies that the determinant is actually a *signed* area (volume)!

Activity. *(0 min)*

How are $\det(\vec{v} + \vec{w}, \vec{w})$ and $\det(\vec{v}, \vec{w})$ related?

(a) $\det(\vec{v} + \vec{w}, \vec{w}) = \det(\vec{v}, \vec{w})$

(b) $\det(\vec{v} + \vec{w}, \vec{w}) = -\det(\vec{v}, \vec{w})$

(c) They are unrelated

(d) They are related, but not by either (a) or (b).

Observation. Note that we now understand the effect of any column operation on the determinant.

Activity. *(0 min)* How are $\det(A)$ and $\det(A^T)$ related?

(a) $\det(A) = \det(A^T)$

(b) $\det(A) = -\det(A^T)$

(c) $\det(A) = \frac{1}{\det(A^T)}$

(d) They are unrelated

Observation. Thus, row operations behave like column operations. So we can use row reduction to compute determinants.

Activity. *(0 min)* Compute $\det \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$.

Activity. *(0 min)* Compute $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Activity. *(0 min)* Which of the following is the same as $\det \begin{bmatrix} 3 & -2 & 0 \\ 5 & -1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$?

(a) $\det \begin{bmatrix} 3 & -2 \\ 5 & -1 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$

(c) $\det \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}$

(d) None of these

Activity. *(0 min)* Which of the following is the same as $\det \begin{bmatrix} 3 & 0 & 7 \\ 5 & 1 & 2 \\ -2 & 0 & 6 \end{bmatrix}$?

(a) $\det \begin{bmatrix} 3 & 7 \\ 5 & 2 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix}$

(c) $\det \begin{bmatrix} 5 & 2 \\ -2 & 6 \end{bmatrix}$

(d) None of these

Activity. *(0 min)* Compute $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$

Activity. *(0 min)* Using the fact that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 5 \\ 0 & 3 & 3 \end{bmatrix}$.

Activity. *(0 min)* Compute $\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$

Application Activities - Day 3

Activity. *(0 min)* Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$
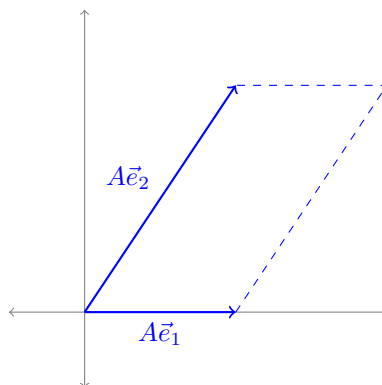
Observe

$$A\vec{e_1} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\vec{e_1}$$

Is there another vector $\vec{v} \in \mathbb{R}^2$ such that $A\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$?

Definition. Let $A \in M_n(\mathbb{R})$. An **eigenvector** is a vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x}$ is parallel to $\vec{x}$; in other words, $A\vec{x} = \lambda\vec{x}$ for some scalar λ, which is called an **eigenvalue**

Observation. Observe that $A\vec{x} = \lambda\vec{x}$ is equivalent to $(A - \lambda I)\vec{x} = 0$.

- To find eigenvalues, we need to find values of λ such that $A - \lambda I$ has a nontrivial kernel; equivalently, $A - \lambda I$ is not invertible, which is equivalent to $\det(A - \lambda I) = 0$. $\det(A - \lambda I)$ is called the **characteristic polynomial**.

- Once an eigenvalue is found, the eigenvectors form a subspace called the **eigenspace**, which is simply the kernel of $A - \lambda I$. Each eigenvalue will have an associated eigenspace.

Activity. *(0 min)* Find the eigenvalues for the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Activity. *(0 min)* Compute the eigenspace associated to the eigenvalue 3.

Activity. *(0 min)* Find all the eigenvalues and associated eigenspaces for the matrix $\begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$.

Application Activities - Day 4

Activity. *(0 min)* If $A \in M_4$, what is the largest number of eigenvalues A can have?

Activity. *(0 min)* 2 is an eigenvalue of each of the matrices $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}$.

Compute the eigenspace associated to 2 for both A and B.

Definition. • The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

  • The **geometric multiplicity** of an eigenvalue is the dimension of the eigenspace.

Activity. *(0 min)* How are the algebraic and geometric multiplicities related?

(a) The algebraic multiplicity is always at least as big as than the geometric multiplicity.

(b) The geometric multiplicity is always at least as big as the algebraic multiplicity.

(c) Sometimes the algebraic multiplicity is larger and sometimes the geometric multiplicity is larger.

Activity. *(0 min)* Find the eigenvalues, along with both their algebraic and geometric multiplicities, for the matrix $\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$

Activity. *(0 min)* Find the eigenvalues of the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Activity. *(0 min)* Describe what this linear transformation is doing geometrically; draw a picture.

Activity. *(0 min)* Fix a real number θ and find the eigenvalues of the matrix $A_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

What are the eigenvalues?

Activity. *(0 min)* D raw pictures and describe the geometric actions of the maps $A_{\frac{\pi}{4}}$, $A_{\frac{\pi}{2}}$, and A_π.

Activity. *(0 min)* For how many values of θ does the rotation matrix A_θ have real eigenvalues?

(a) 0

(b) 1

(c) 2

(d) 3

(e) An infinite number