Application Activities - Module E Part 3 - Class Day 5

Definition 5.1 An algorithm that reduces A to RREF(A) is called **Gauss-Jordan elimination**. For example:

1. Circle the cell that (a) is in the top-most row without a pivot position and (b) is in the left-most column with a nonzero term either in that position or below it. This position (not the number inside) is called a **pivot**.
2. Change the pivot's value to 1 by using row operations involving only the pivot row and rows below it.
3. Add or subtract multiples of the pivot row to zero out above and below the pivot.
4. Return to Step 1 and repeat as needed until the matrix is in row reduced echelon form.

Observation 5.2 Here is an example of applying Gauss-Jordan elimination to a matrix:

$$\left[\begin{array}{cccc|c} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ -1 & 1 & 3 & -1 & -3 \\ 2 & -2 & -6 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & 2 & -4 & -1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 2 & -4 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 1 & 4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 1 & 4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Definition 5.3 The columns of RREF(A) without a leading term represent **free variables** of the linear system modeled by A that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by A.

Example 5.4 Here, x_3 is the free variable set equal to a since its column lacks a pivot, and the other bounded variables are put in terms of a.

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 &= 1 \end{aligned} \qquad \begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 - 2x_3 &= 1 \\ x_4 &= 3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x_1 &= 1 + 5a \\ x_2 &= 1 + 2a \\ x_3 &= a \\ x_4 &= 3 \end{aligned}$$

$$\Downarrow \qquad\qquad\qquad \Uparrow$$

$$\left[\begin{array}{cccc|c} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

So the solution set is $\left\{ \begin{bmatrix} 1+5a \\ 1+2a \\ a \\ 3 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

Activity 5.5 Solve the system of linear equations, circling the pivot positions in your augmented matrices as you work.

$$\begin{aligned} -x_1 + x_2 - 3x_3 + 2x_4 &= 0 \\ 2x_1 - x_2 + 5x_3 + 3x_4 &= -11 \\ 3x_1 + 2x_2 + 4x_3 + x_4 &= 1 \\ x_2 - x_3 + x_4 &= 1 \end{aligned}$$

Remember to find the solution set of the system by setting the free variable (the column without a pivot position) equal to a, and then express each of the other bounded variables equal to an expression in terms of a.

Remark 5.6 From now on, unless specified, there's no need to show your work in finding RREF(A), so you may use a calculator to speed up your work.

Activity 5.7 Solve the linear system

$$\begin{aligned} 2x_1 - 3x_2 &= 17 \\ x_1 + 2x_2 &= -2 \\ -x_1 - x_2 &= 1 \end{aligned}$$

Activity 5.8 Show that all linear systems of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \;\; &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

are consistent by finding a quickly verifiable solution.

Definition 5.9 A **homogeneous system** is a linear system satisfying $b_i = 0$, that is, it is a linear system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \;\; &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

Fact 5.10 Because the zero vector is always a solution, the solution set to any homogeneous system with infinitely-many solutions may be generated by multiplying the parameters representing the free variables by a minimal set of Euclidean vectors, and adding these up. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Definition 5.11 A minimal set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Activity 5.12 Find a basis for the solution set of the following homogeneous linear system.

$$\begin{aligned} x_1 + 2x_2 \phantom{{}+x_3} - x_4 &= 0 \\ x_3 + 4x_4 &= 0 \\ 2x_1 + 4x_2 + x_3 + 2x_4 &= 0 \end{aligned}$$