Section E.1

Remark E.1.1 The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{rrr|r} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition E.1.2 A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Definition E.1.3 Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned}$$

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity E.1.4 *(∼10 min)* Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a nonzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

g) Replace a row with zeros.

Definition E.1.5 The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a nonzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity E.1.6 *(∼10 min)* Consider the following (equivalent) linear systems.

(A)

$$\begin{alignedat}{4} -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ x_1 &- 2x_2 &+ 2x_3 &= 7 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(C)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ & & -2x_3 &= -6 \end{alignedat}$$

(E)

$$\begin{alignedat}{3} x_1 &- 2x_2 & &= 1 \\ & & x_3 &= 3 \\ & & 0 &= 0 \end{alignedat}$$

(B)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(D)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & x_3 &= 3 \\ & & -2x_3 &= -6 \end{alignedat}$$

(F)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

Part 1: Rank the six linear systems from easiest to solve to hardest to solve.
Part 2: Determine the row operation used in each step transforming the hardest system into the easiest.

Observation E.1.7 We can rewrite the previous in terms of augmented matrices

$$\left[\begin{array}{ccc|c} -2 & 4 & -2 & -8 \\ 1 & -2 & 2 & 7 \\ 3 & -6 & 4 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ -2 & 4 & -2 & -8 \\ 3 & -6 & 4 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 3 & -6 & 4 & 15 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We simplify our system by

1. Simplifying one column at a time, moving left to right;
2. Once we have a 1 in a **pivot position**, we zero out above and below.
3. We can always follow this procedure to put a matrix in **reduced row echelon form**, or **RREF**.

Remark E.1.8 It is important to understand the **Gauss-Jordan elimination** algorithm that converts a matrix into reduced row echelon form, but in practice we don't do this by hand; we use technology to do this for us.

Activity E.1.9 *(∼10 min)* A matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term of each nonzero row is a 1 (these will be called **pivots**)
2. Each column containing a pivot is zero except for the pivot
3. All rows of zeroes are at the bottom of the matrix.

Determine which of the following matrices are in RREF:

(A)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(B)
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(C)
$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(D)
$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(E)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(F)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Remark E.1.10 Before the next class, work at least five practice problems putting matrices into reduced row echelon form. You can view a video example at TODO