Linear Algebra

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Module I: Introduction

Remark I.1

This brief module gives an overview for the course.

Remark I.2

What is Linear Algebra?

Linear algebra is the study of **linear maps**.

- In Calculus, you learn how to approximate any function by a linear function.
- In Linear Algebra, we learn about how linear maps behave.
- Combining the two, we can approximate how any function behaves.

Remark I.3

What is Linear Algebra good for?

- Linear algebra is used throughout several fields in higher mathematics.
- In computer graphics, linear algebra is used to help represent 3D objects in a 2D grid of pixels.
- Linear algebra is used to approximate differential equation solutions in a vast number of engineering applications (e.g. fluid flows, vibrations, heat transfer) whose solutions are very difficult (or impossible) to find precisely.
- Google's search engine is based on its Page Rank algorithm, which ranks websites by computing an eigenvector of a matrix.

Remark I.4

What will I learn in this class?

By the end of this class, you will be able to:

- Solve systems of linear equations. (Module E)
- Identify vector spaces and their properties. (Module V)
- Use and apply the algebraic properties of linear transformations. (Module A)
- Perform fundamental operations in the algebra of matrices. (Module M)
- Use and apply the geometric properties of linear transformations. (Module G)

Module E: Solving Systems of Linear Equations

How can we solve systems of linear equations?

At the end of this module, students will be able to...

E1. **Systems as matrices.** ... translate back and forth between a system of linear equations, a vector equation, and the corresponding augmented matrix.

E2. **Row reduction.** ... explain why a matrix isn't in reduced row echelon form, and put a matrix in reduced row echelon form.

E3. **Systems of linear equations.** ... compute the solution set for a system of linear equations or a vector equation.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.
- Describe sets using set-builder notation, and check if an element is a member of a set described by set-builder notation.

The following resources will help you prepare for this module.

- Systems of linear equations (Khan Academy): http://bit.ly/2l21etm
- Solving linear systems with substitution (Khan Academy): http://bit.ly/1SlMpix
- Set builder notation: https://youtu.be/xnfUZ-NTsCE

Definition E.1

A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector that can be plugged into the equation).

Remark E.2

In previous classes you likely used the variables x, y, z in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as x_i, and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

Definition E.3

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n &= b_2 \\ \vdots \qquad \quad \vdots \qquad \qquad \quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \;\middle|\; \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

Remark E.4

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Concise standard form:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & 3 \\ 3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & -2 \end{array}$$

Remark E.5

It will often be convenient to think of a system of equations as a vector equation.

By applying vector operations and equating components, it is straightforward to see that the vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

is equivalent to the system of equations

$$\begin{alignedat}{4} x_1 & & & {}+{} & 3x_3 & {}={} & 3 \\ 3x_1 & {}-{} & 2x_2 & {}+{} & 4x_3 & {}={} & 0 \\ & {}-{} & x_2 & {}+{} & x_3 & {}={} & -2 \end{alignedat}$$

Definition E.6

A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**.

Fact E.7

All linear systems are one of the following:

- **Consistent with one solution**: its solution set contains a single vector, e.g. $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$
- **Consistent with infinitely-many solutions**: its solution set contains infinitely many vectors, e.g. $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix} \middle| a \in \mathbb{R}\right\}$
- **Inconsistent**: its solution set is the empty set $\{\} = \emptyset$

Activity E.8 *(∼10 min)* All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is $\emptyset$.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity E.9 *(∼10 min)* Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity E.9 *(∼10 min)* Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions for this system.

Activity E.9 *(∼10 min)* Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions for this system.
Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{ \begin{bmatrix} ? \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$ for the linear system.

Activity E.10 *(~10 min)* Consider the following linear system.

$$\begin{alignedat}{5} x_1 &+ 2x_2 & & &-& x_4 &=& 3 \\ & & & x_3 &+& 4x_4 &=& -2 \end{alignedat}$$

Describe the solution set

$$\left\{ \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation E.11

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{alignedat}{5}
-2x_1 &\,-\,& 4x_2 &\,+\,& x_3 &\,-\,& 4x_4 &\,=\,& -8 \\
x_1 &\,+\,& 2x_2 &\,+\,& 2x_3 &\,+\,& 12x_4 &\,=\,& -1 \\
x_1 &\,+\,& 2x_2 &\,+\,& x_3 &\,+\,& 8x_4 &\,=\,& 1
\end{alignedat}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.

Remark E.12

The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{alignedat}{4} x_1 & & &+{}& 3x_3 &=& 3 \\ 3x_1 &-{}& 2x_2 &+{}& 4x_3 &=& 0 \\ & & -x_2 &+{}& x_3 &=& -2 \end{alignedat}$$

Verbose standard form:

$$\begin{alignedat}{4} 1x_1 &+{}& 0x_2 &+{}& 3x_3 &=& 3 \\ 3x_1 &-{}& 2x_2 &+{}& 4x_3 &=& 0 \\ 0x_1 &-{}& 1x_2 &+{}& 1x_3 &=& -2 \end{alignedat}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition E.13

A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{alignedat}{5}
a_{11}x_1 &+ a_{12}x_2 &+ \dots + & a_{1n}x_n &= b_1 \\
a_{21}x_1 &+ a_{22}x_2 &+ \dots + & a_{2n}x_n &= b_2 \\
\vdots & \quad \vdots & & \vdots & \vdots \\
a_{m1}x_1 &+ a_{m2}x_2 &+ \dots + & a_{mn}x_n &= b_m
\end{alignedat}
\qquad
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right]$$

Example E.14

The corresponding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

Vector equation:

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

Definition E.15

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent, which we denote with $\sim$:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity E.16 *($\sim$10 min)* Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.
b) Swap two columns.
c) Add a constant to every term in a row.
d) Multiply a row by a nonzero constant.
e) Add a constant multiple of one row to another row.
f) Replace a column with zeros.
g) Replace a row with zeros.

Definition E.17

The following **row operations** produce equivalent augmented matrices:

1. Swap two rows, for example, $R_1 \leftrightarrow R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}\right]$$

2. Multiply a row by a nonzero constant, for example, $2R_1 \to R_1$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 2(1) & 2(2) & 2(3) \\ 4 & 5 & 6 \end{array}\right]$$

3. Add a constant multiple of one row to another row, for example, $R_2 - 4R_1 \to R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4-4(1) & 5-4(2) & 6-4(3) \end{array}\right]$$

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity E.18 *(~10 min)* Consider the following (equivalent) linear systems.

(A)

$$\begin{aligned} x+2y+z &= 3 \\ -x-y+z &= 1 \\ 2x+5y+3z &= 7 \end{aligned}$$

(B)

$$\begin{aligned} 2x+5y+3z &= 7 \\ -x-y+z &= 1 \\ x+2y+z &= 3 \end{aligned}$$

(C)

$$\begin{aligned} x-z &= 1 \\ y+z &= 1 \\ y+2z &= 4 \end{aligned}$$

(D)

$$\begin{aligned} x+2y+z &= 3 \\ y+z &= 1 \\ 2x+5y+3z &= 7 \end{aligned}$$

(E)

$$\begin{aligned} x-z &= 1 \\ y+z &= 1 \\ z &= 3 \end{aligned}$$

(F)

$$\begin{aligned} x+2y+z &= 3 \\ y+z &= 1 \\ y+2z &= 4 \end{aligned}$$

Rank the six linear systems from most complicated to simplest.

Activity E.19 *(∼5 min)* We can rewrite the previous in terms of equivalences of augmented matrices

$$\left[\begin{array}{ccc|c} 2 & 5 & 13 & 7 \\ -1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -1 & -1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

Definition E.20

A matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

Every matrix has a unique reduced row echelon form. If A is a matrix, we write $\mathrm{RREF}(A)$ for the reduced row echelon form of that matrix.

Activity E.21 *(∼15 min)* Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

(A)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(B)
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(C)
$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(D)
$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(E)
$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(F)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

For each matrix, circle the leading terms, and label it as RREF or not RREF. For

Remark E.22

In practice, if we simply need to convert a matrix into reduced row echelon form, we use technology to do so.

However, it is also important to understand the **Gauss-Jordan elimination** algorithm that a computer or calculator uses to convert a matrix (augmented or not) into reduced row echelon form. Understanding this algorithm will help us better understand how to interpret the results in many applications we use it for in Module V.

Activity E.23 *(~8 min)* Consider the matrix

$$\begin{bmatrix} 2 & 6 & -1 & 6 \\ 1 & 3 & -1 & 2 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the first move in converting to RREF?

(a) Add row 3 to row 2 ($R_2 + R_3 \to R_2$)

(b) Add row 2 to row 3 ($R_3 + R_2 \to R_3$)

(c) Swap row 1 to row 2 ($R_1 \leftrightarrow R_2$)

(d) Add -2 row 2 to row 1 ($R_1 - 2R_2 \to R_1$)

Activity E.24 *(~7 min)* Consider the matrix

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 6 & -1 & 6 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

(a) Add row 1 to row 3 ($R_3 + R_1 \to R_3$)

(b) Add -2 row 1 to row 2 ($R_2 - 2R_1 \to R_2$)

(c) Add 2 row 2 to row 3 ($R_3 + 2R_2 \to R_3$)

(d) Add 2 row 3 to row 2 ($R_2 + 2R_3 \to R_2$)

Activity E.25 *(∼5 min)* Consider the matrix

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

(a) Add row 1 to row 2 ($R_2 + R_1 \to R_2$)

(b) Add -1 row 3 to row 2 ($R_2 - R_3 \to R_2$)

(c) Add -1 row 2 to row 3 ($R_3 - R_2 \to R_3$)

(d) Add row 2 to row 1 ($R_1 + R_2 \to R_1$)

Activity E.26 *(∼10 min)* Consider the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -1 & 1 \end{bmatrix}.$$

Activity E.26 *(~10 min)* Consider the matrix

$$\begin{bmatrix}2 & 1 & 0\\ 1 & 0 & 0\\ 3 & -1 & 1\end{bmatrix}.$$

Part 1: Perform three row operations to produce a matrix closer to RREF.

Activity E.26 *(∼10 min)* Consider the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -1 & 1 \end{bmatrix}.$$

Part 1: Perform three row operations to produce a matrix closer to RREF.
Part 2: Finish putting it in RREF.

Activity E.27 *(∼10 min)* Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 2 & 3 \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix}.$$

Compute RREF(A).

Activity E.28 *(∼10 min)* Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{bmatrix}.$$

Compute RREF(A).

Remark E.29

A video example of how to perform the Gauss-Jordan Elimination algorithm by hand is available at `https://youtu.be/Cq0Nxk2dhhU`.

Practicing several exercises on your own using this method is strongly recommended.

Activity E.30 *(∼10 min)* Free browser-based technologies for mathematical computation are available online.

- Go to `https://octave-online.net`.
- Type `A=sym([1 3 4 ; 2 5 7])` and press `Enter` to store the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 7 \end{bmatrix}$ in the variable A.
  - The symbolic function `sym` is used to calculate precise answers rather than floating-point approximations.
  - The vertical bar in an augmented matrix does not affect row operations, so the RREF of $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 5 & 7 \end{array}\right]$ may be computed in the same way.
- Type `rref(A)` and press `Enter` to compute the reduced row echelon form of A.

Remark E.31

We will frequently need to know the reduced row echelon form of matrices during class, so feel free to use Octave-Online.net to compute RREF efficiently.

You may alternatively use the calculator you will use during assessments. Be sure to use fractions mode to compute exact solutions rather than floating-point approximations.

Activity E.32 *(∼10 min)* Consider the system of equations.

$$\begin{alignedat}{4}
3x_1 &- 2x_2 &+ 13x_3 &= 6 \\
2x_1 &- 2x_2 &+ 10x_3 &= 2 \\
-x_1 &+ 3x_2 &- 6x_3 &= 11
\end{alignedat}$$

Activity E.32 *(~10 min)* Consider the system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

Part 1: Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\mathrm{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

Activity E.32 *(~10 min)* Consider the system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

Part 1: Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\mathrm{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

Part 2: Use the RREF matrix to write a linear system equivalent to the original system. Then find its solution set.

Activity E.33 *(~10 min)* Consider the vector equation

$$x_1 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 13 \\ 10 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

Activity E.33 *(~10 min)* Consider the vector equation

$$x_1 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 13 \\ 10 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

Part 1: Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF} \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

Activity E.33 *(∼10 min)* Consider the vector equation

$$x_1 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 13 \\ 10 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

Part 1: Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\mathrm{RREF} \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

Part 2: Use the RREF matrix to write a linear system equivalent to the original system. Then find its solution set.

Activity E.34 *(~10 min)* Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Activity E.34 *(∼10 min)* Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Part 1: Find its corresponding augmented matrix A and use technology to find RREF(A).

Activity E.34 *(∼10 min)* Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Part 1: Find its corresponding augmented matrix A and use technology to find RREF(A).
Part 2: How many solutions do these linear systems have?

Activity E.35 *(∼10 min)* Consider the simple linear system equivalent to the system from the previous activity:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_3 &= -1 \end{aligned}$$

Activity E.35 *(∼10 min)* Consider the simple linear system equivalent to the system from the previous activity:

$$\begin{aligned} x_1 + 2x_2 \quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}$.

Activity E.35 *(~10 min)* Consider the simple linear system equivalent to the system from the previous activity:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}$.

Part 2: Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \,\middle|\, b \in \mathbb{R} \right\}$.

Activity E.35 *(~10 min)* Consider the simple linear system equivalent to the system from the previous activity:

$$\begin{aligned} x_1 + 2x_2 \quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

Part 2: Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$.

Part 3: Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$ caused this?

Definition E.36

Recall that the pivots of a matrix in RREF form are the leading 1s in each non-zero row.

The pivot columns in an augmented matrix correspond to the **bound variables** in the system of equations (x_1, x_3 below). The remaining variables are called **free variables** (x_2 below).

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

To efficiently solve a system in RREF form, assign letters to the free variables, and then solve for the bound variables.

Activity E.37 *(~10 min)* Find the solution set for the system

$$\begin{alignedat}{6}
2x_1 &- 2x_2 &- 6x_3 &+ x_4 &- x_5 &= 3 \\
-x_1 &+ x_2 &+ 3x_3 &- x_4 &+ 2x_5 &= -3 \\
x_1 &- 2x_2 &- x_3 &+ x_4 &+ x_5 &= 2
\end{alignedat}$$

by row-reducing its augmented matrix, and then assigning letters to the free variables (given by non-pivot columns) and solving for the bound variables (given by pivot columns) in the corresponding linear system.

Observation E.38

The solution set to the system

$$\begin{alignedat}{6}
2x_1 &- 2x_2 &- 6x_3 &+ x_4 &- x_5 &= 3\\
-x_1 &+ x_2 &+ 3x_3 &- x_4 &+ 2x_5 &= -3\\
x_1 &- 2x_2 &- x_3 &+ x_4 &+ x_5 &= 2
\end{alignedat}$$

may be written as

$$\left\{ \begin{bmatrix} 1+5a+2b \\ 1+2a+3b \\ a \\ 3+3b \\ b \end{bmatrix} \middle| \, a,b\in\mathbb{R} \right\}.$$

Remark E.39

Don't forget to correctly express the solution set of a linear system, using set-builder notation for consistent systems with infintely many solutions.

- **Consistent with one solution**: e.g. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$
- **Consistent with infinitely-many solutions**: e.g. $\left\{ \begin{bmatrix} 1 \\ 2-3a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$
- **Inconsistent**: $\emptyset$ or $\{\}$

Module V: Vector Spaces

What is a vector space?

At the end of this module, students will be able to...

V1. **Vector spaces.** ... explain why a given set with defined addition and scalar multiplication does satisfy a given vector space property, but nonetheless isn't a vector space.

V2. **Linear combinations.** ... determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors by solving an appropriate vector equation.

V3. **Spanning sets.** ... determine if a set of Euclidean vectors spans $\mathbb{R}^n$ by solving appropriate vector equations.

V4. **Subspaces.** ... determine if a subset of $\mathbb{R}^n$ is a subspace or not.

V5. **Linear independence.** ... determine if a set of Euclidean vectors is linearly dependent or independent by solving an appropriate vector equation.

V6. **Basis verification.** ... explain why a set of Euclidean vectors is or is not a basis of $\mathbb{R}^n$.

V7. **Basis computation.** ... compute a basis for the subspace spanned by a given set of Euclidean vectors, and determine the dimension of the subspace.

V8. **Polynomial and Matrix computation.** ... answer questions about vector spaces of polynomials or matrices.

V9. **Basis of solution space.** ... find a basis for the solution set of a homogeneous system of equations.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Use set builder notation to describe sets of vectors.
- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.

The following resources will help you prepare for this module.

- Set Builder Notation: `https://youtu.be/xnfUZ-NTsCE`
- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Observation V.1

Several properties of the real numbers, such as commutivity:

$$x + y = y + x$$

also hold for Euclidean vectors with multiple components:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Activity V.2 *(~20 min)* Consider each of the following properties of the real numbers $\mathbb{R}^1$. Label each property as **valid** if the property also holds for two-dimensional Euclidean vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ and scalars $a, b \in \mathbb{R}$, and **invalid** if it does not.

1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. There exists some $\vec{z}$ where $\vec{v} + \vec{z} = \vec{v}$.
4. There exists some $-\vec{v}$ where $\vec{v} + (-\vec{v}) = \vec{z}$.
5. If $\vec{u} \neq \vec{v}$, then $\frac{1}{2}(\vec{u} + \vec{v})$ is the only vector equally distant from both $\vec{u}$ and $\vec{v}$
6. $a(b\vec{v}) = (ab)\vec{v}$.
7. $1\vec{v} = \vec{v}$.
8. If $\vec{u} \neq \vec{0}$, then there exists some scalar c such that $c\vec{u} = \vec{v}$.
9. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$.
10. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$.

Definition V.3

A **vector space** V is any collection of mathematical objects with associated addition $\oplus$ and scalar multiplication $\odot$ operations that satisfy the following properties. Let $\vec{u}, \vec{v}, \vec{w}$ belong to V, and let a, b be scalar numbers.

- **Addition is associative:** $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$.
- **Addition is commutative:** $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.
- **Additive identity exists:** There exists some $\vec{z}$ where $\vec{v} \oplus \vec{z} = \vec{v}$.
- **Additive inverses exist:** There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$.
- **Scalar multiplication is associative:** $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$.
- **Scalar multiplication identity exists:** $1 \odot \vec{v} = \vec{v}$.
- **Scalar mult. distributes over vector addition:** $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$.
- **Scalar mult. distributes over scalar addition:** $(a + b) \odot \vec{v} = a \odot \vec{v} \oplus b \odot \vec{v}$.

Observation V.4

Every **Euclidean vector space**

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle|\, x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$$

satisfies all eight requirements for the usual definitions of addition and scalar multiplication, but we will also study other types of vector spaces.

Observation V.5

The space of $m \times n$**matrices**

$$M_{m,n} = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \middle| \, a_{11}, \ldots, a_{mn} \in \mathbb{R} \right\}$$

satisfies all eight requirements for component-wise addition and scalar multiplication.

Remark V.6

Every Euclidean space $\mathbb{R}^n$ is a vector space, but there are other examples of vector spaces as well.

For example, consider the set $\mathbb{C}$ of complex numbers with the usual defintions of addition and scalar multiplication, and let $\vec{\mathrm{u}} = a + b\mathrm{i}$, $\vec{\mathrm{v}} = c + d\mathrm{i}$, and $\vec{\mathrm{w}} = e + f\mathrm{i}$. Then

$$\begin{aligned}
\vec{\mathrm{u}} + (\vec{\mathrm{v}} + \vec{\mathrm{w}}) &= (a + b\mathrm{i}) + ((c + d\mathrm{i}) + (e + f\mathrm{i})) \\
&= (a + b\mathrm{i}) + ((c + e) + (d + f)\mathrm{i}) \\
&= (a + c + e) + (b + d + f)\mathrm{i} \\
&= ((a + c) + (b + d)\mathrm{i}) + (e + f\mathrm{i}) \\
&= (\vec{\mathrm{u}} + \vec{\mathrm{v}}) + \vec{\mathrm{w}}
\end{aligned}$$

All eight properties can be verified in this way.

Remark V.7

The following sets are just a few examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{C}$: Complex numbers.
- $M_{m,n}$: Matrices of real numbers with m rows and n columns.
- $\mathcal{P}^n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Remark V.8

Previously, we defined a **vector space** V to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all $\vec{u}, \vec{v}, \vec{w}$ in V, and all scalars (i.e. real numbers) a, b.

- **Addition is associative:** $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$.
- **Addition is commutative:** $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.
- **Additive identity exists:** There exists some $\vec{z}$ where $\vec{v} \oplus \vec{z} = \vec{v}$.
- **Additive inverses exist:** There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$.
- **Scalar multiplication is associative:** $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$.
- **Scalar multiplication identity exists:** $1 \odot \vec{v} = \vec{v}$.
- **Scalar mult. distributes over vector addition:** $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$.
- **Scalar mult. distributes over scalar addition:** $(a + b) \odot \vec{v} = a \odot \vec{v} \oplus b \odot \vec{v}$.

Activity V.9 *(~20 min)* Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x_1, y_1) = (cx_1, y_1^c)$$

Activity V.9 *(~20 min)* Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x_1, y_1) = (cx_1, y_1^c)$$

Part 1: Show that V satisfies the distributive property

$$(a + b) \odot (x_1, y_1) = (a \odot (x_1, y_1)) \oplus (b \odot (x_1, y_1))$$

by simplifying both sides and verifying they are the same expression.

Activity V.9 *(∼20 min)* Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x_1, y_1) = (cx_1, y_1^c)$$

Part 1: Show that V satisfies the distributive property

$$(a + b) \odot (x_1, y_1) = (a \odot (x_1, y_1)) \oplus (b \odot (x_1, y_1))$$

by simplifying both sides and verifying they are the same expression.

Part 2: Show that V contains an additive identity element satisfying

$$(x_1, y_1) \oplus \vec{z} = (x_1, y_1)$$

for all $(x_1, y_1) \in V$ by choosing appropriate values for $\vec{z} = (\ ?\ ,\ ?\ )$.

Remark V.10

It turns out $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x_1, y_1) = (cx_1, y_1^c)$$

satisifes all eight properties.

- **Addition is associative:** $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$.
- **Addition is commutative:** $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.
- **Additive identity exists:** There exists some $\vec{z}$ where $\vec{v} \oplus \vec{z} = \vec{v}$.
- **Additive inverses exist:** There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$.
- **Scalar multiplication is associative:** $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$.
- **Scalar multiplication identity exists:** $1 \odot \vec{v} = \vec{v}$.
- **Scalar mult. distributes over vector addition:** $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$.
- **Scalar mult. distributes over scalar addition:** $(a + b) \odot \vec{v} = a \odot \vec{v} \oplus b \odot \vec{v}$.

Thus, V is a vector space.

Activity V.11 *(∼15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2) \qquad c \odot (x_1, y_1) = (x_1^c, y_1 + c - 1).$$

Activity V.11 *($\sim$15 min)* Let $V = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2) \qquad c \odot (x_1, y_1) = (x_1^c, y_1 + c - 1).$$

Part 1: Show that 1 is the scalar multiplication identity element by simplifying $1 \odot (x, y)$ to (x, y).

Activity V.11 *(∼15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2) \qquad c \odot (x_1, y_1) = (x_1^c, y_1 + c - 1).$$

Part 1: Show that 1 is the scalar multiplication identity element by simplifying $1 \odot (x, y)$ to (x, y).

Part 2: Show that V does not have an additive identity element by showing that $(0, -1) \oplus \vec{z} \neq (0, -1)$ no matter how $\vec{z} = (z, w)$ is chosen.

Activity V.11 *(~15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2) \qquad c \odot (x_1, y_1) = (x_1^c, y_1 + c - 1).$$

Part 1: Show that 1 is the scalar multiplication identity element by simplifying $1 \odot (x, y)$ to (x, y).

Part 2: Show that V does not have an additive identity element by showing that $(0, -1) \oplus \vec{z} \neq (0, -1)$ no matter how $\vec{z} = (z, w)$ is chosen.

Part 3: Is V a vector space?

Activity V.12 *(∼15 min)* Let $V = \{(x,y) \,|\, x,y \in \mathbb{R}\}$ have operations defined by

$$(x_1,y_1) \oplus (x_2,y_2) = (x_1+x_2, y_1+3y_2) \qquad c \odot (x_1,y_1) = (cx_1, cy_1).$$

Activity V.12 *(∼15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + 3y_2) \qquad c \odot (x_1, y_1) = (cx_1, cy_1).$$

Part 1: Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

for **all** $c \in \mathbb{R}$, $(x_1, y_1), (x_2, y_2) \in V$.

Activity V.12 *(~15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + 3y_2) \qquad c \odot (x_1, y_1) = (cx_1, cy_1).$$

Part 1: Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

for **all** $c \in \mathbb{R}$, $(x_1, y_1), (x_2, y_2) \in V$.
Part 2: Show that vector addition is not associative, i.e.

$$(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) \neq ((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3)$$

for **some** vectors $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$.

Activity V.12 *(~15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + 3y_2) \qquad c \odot (x_1, y_1) = (cx_1, cy_1).$$

Part 1: Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

for **all** $c \in \mathbb{R}$, $(x_1, y_1), (x_2, y_2) \in V$.

Part 2: Show that vector addition is not associative, i.e.

$$(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) \neq ((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3)$$

for **some** vectors $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$.

Part 3: Is V a vector space?

Definition V.13

A **linear combination** of a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is given by $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$ for any choice of scalar multiples $c_1, c_2, \dots, c_m$.

For example, we can say $\begin{bmatrix}3\\0\\5\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\2\\1\end{bmatrix}$ since

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

Definition V.14

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m \,|\, c_i \in \mathbb{R}\}.$$

For example:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$$

Activity V.15 *(∼10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Activity V.15 *(∼10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch

$$1\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},\qquad 3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix},\qquad 0\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\qquad \text{and } -2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-2\\-4\end{bmatrix}$$

in the xy plane.

Activity V.15 *(~10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch

$$1\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},\qquad 3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix},\qquad 0\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\qquad \text{and } -2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-2\\-4\end{bmatrix}$$

in the xy plane.

Part 2: Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}\,\middle|\, a\in\mathbb{R}\right\}$ in the xy plane.

Activity V.16 *(∼10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Activity V.16 *(~10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix} \qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix} \qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix} \qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}$$

Activity V.16 *(∼10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix} \qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix} \qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix} \qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}$$

Part 2: Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity V.17 *(∼5 min)* Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix},\begin{bmatrix}-3\\2\end{bmatrix}\right\}$ in the xy plane.

Remark V.18

Recall these definitions from last class:

- A **linear combination** of vectors is given by adding scalar multiples of those vectors, such as:

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

- The **span** of a set of vectors is the collection of all linear combinations of that set, such as:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \middle|\, a, b \in \mathbb{R}\right\}$$

Activity V.19 *(∼15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation

$$x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}.$$

Activity V.19 *(~15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation

$$x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}.$$

Part 1: Reinterpret this vector equation as a system of linear equations.

Activity V.19 *(~15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation

$$x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}.$$

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Find its solution set, using technology to find RREF of its corresponding augmented matrix.

Activity V.19 *(∼15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation

$$x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}.$$

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Find its solution set, using technology to find RREF of its corresponding augmented matrix.

Part 3: Given this solution set, does $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belong to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$?

Fact V.20

A vector $\vec{b}$ belongs to $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ if and only if the vector equation $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{b}$ is consistent.

Quick Check V.21

The following are all equivalent statements:

- The vector $\vec{b}$ belongs to $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\}$.
- The vector equation $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{b}$ is consistent.
- The linear system corresponding to $[\vec{v}_1 \ \dots \ \vec{v}_n \,|\, \vec{b}]$ is consistent.
- $\operatorname{RREF}[\vec{v}_1 \ \dots \ \vec{v}_n \,|\, \vec{b}]$ doesn't have a row $[0 \cdots 0 \,|\, 1]$ representing the contradiction $0 = 1$.

Activity V.22 *(~10 min)* Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \\ 2 \end{bmatrix} \right\}$ by solving an appropriate vector equation.

Activity V.23 *(∼5 min)* Determine if $\begin{bmatrix}-1\\-9\\0\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ by solving an appropriate vector equation.

Activity V.24 *(∼10 min)* Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Activity V.24 *(∼10 min)* Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Part 1: Reinterpret this question as a question about the solution(s) of a polynomial equation.

Activity V.24 *(∼10 min)* Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to span$\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Part 1: Reinterpret this question as a question about the solution(s) of a polynomial equation.

Part 2: Answer this equivalent question, and use its solution to answer the original question.

Activity V.25 *(∼5 min)* Does the polynomial $x^2 + x + 1$ belong to $\operatorname{span}\{x^2 - x, x + 1, x^2 - 1\}$?

Activity V.26 *(~5 min)* Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to $\operatorname{span}\left\{\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\right\}$?

Activity V.26 *($\sim$5 min)* Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to $\operatorname{span}\left\{\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\right\}$?

Part 1: Reinterpret this question as a question about the solution(s) of a matrix equation.

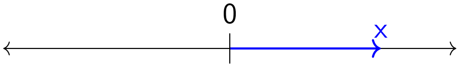
Activity V.26 *(∼5 min)* Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to $\operatorname{span}\left\{\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\right\}$?

Part 1: Reinterpret this question as a question about the solution(s) of a matrix equation.

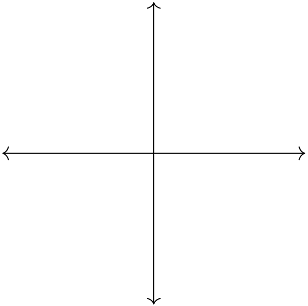
Part 2: Answer this equivalent question, and use its solution to answer the original question.

Observation V.27

Any single non-zero vector/number x in $\mathbb{R}^1$ spans $\mathbb{R}^1$, since $\mathbb{R}^1 = \{cx \mid c \in \mathbb{R}\}$.



Activity V.28 *(∼5 min)* How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your answer.



(a) 1

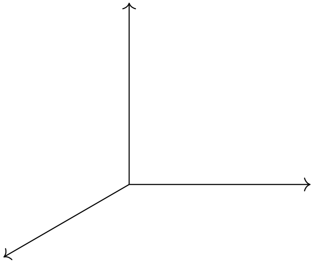
(b) 2

(c) 3

(d) 4

(e) Infinitely Many

Activity V.29 *(∼5 min)* How many vectors are required to span $\mathbb{R}^3$?



(a) 1

(b) 2

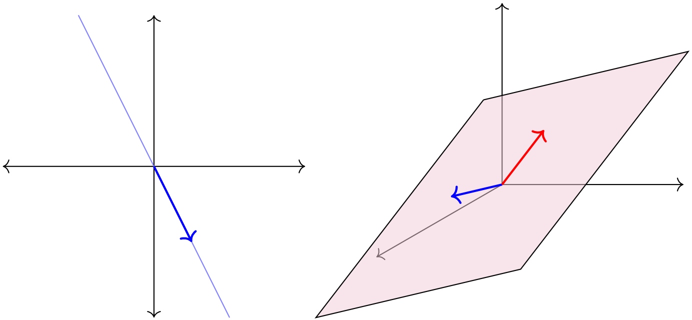
(c) 3

(d) 4

(e) Infinitely Many

Fact V.30

At least n vectors are required to span $\mathbb{R}^n$.



Activity V.31 *(~15 min)* Choose any vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in $\mathbb{R}^3$ that is not in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}$ by using technology to verify that $\operatorname{RREF}\left[\begin{array}{cc|c} 1 & -2 & ? \\ -1 & 0 & ? \\ 0 & 1 & ? \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$. (Why does this work?)

Fact V.32

The set $\{\vec{v}_1, \dots, \vec{v}_m\}$ fails to span all of $\mathbb{R}^n$ exactly when the vector equation

$$x_1\vec{v}_1 + \cdots x_m\vec{v}_m = \vec{w}$$

is inconsistent for **some** vector $\vec{w}$.

Note that this happens exactly when $\mathrm{RREF}[\vec{v}_1 \ \dots \ \vec{v}_m]$ has a non-pivot row of zeros.

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \text{ for some choice of vector } \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Activity V.33 *(∼5 min)* Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does $\mathbb{R}^4 = \operatorname{span} S$?

Activity V.33 *(∼5 min)* Consider the set of vectors

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}.$$ Does $\mathbb{R}^4 = \operatorname{span} S$?

Part 1: Rewrite this as a question about the solutions to a vector equation.

Activity V.33 *(~5 min)* Consider the set of vectors

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}.$$

Does $\mathbb{R}^4 = \operatorname{span} S$?

Part 1: Rewrite this as a question about the solutions to a vector equation.

Part 2: Answer your new question, and use this to answer the original question.

Activity V.34 *(∼10 min)* Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16,$$
$$-x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2\}.$$

Does $\mathcal{P}^3 = \operatorname{span} S$?

Activity V.34 *(∼10 min)* Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, \\ -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2\}.$$

Does $\mathcal{P}^3 = \operatorname{span} S$?

Part 1: Rewrite this as a question about the solutions to a polynomial equation.

Activity V.34 *(∼10 min)* Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16,\\ -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2\}.$$

Does $\mathcal{P}^3 = \operatorname{span} S$?
Part 1: Rewrite this as a question about the solutions to a polynomial equation.
Part 2: Answer your new question, and use this to answer the original question.

Activity V.35 *(~5 min)* Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Does $M_{2,2} = \operatorname{span} S$?

Activity V.35 *(∼5 min)* Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Does $M_{2,2} = \operatorname{span} S$?
Part 1: Rewrite this as a question about the solutions to a matrix equation.

Activity V.35 *(∼5 min)* Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Does $M_{2,2} = \operatorname{span} S$?

Part 1: Rewrite this as a question about the solutions to a matrix equation.

Part 2: Answer your new question, and use this to answer the original question.

Activity V.36 *(~5 min)* Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^7$ be three vectors, and suppose $\vec{w}$ is another vector with $\vec{w} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. What can you conclude about $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

(a) $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is larger than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

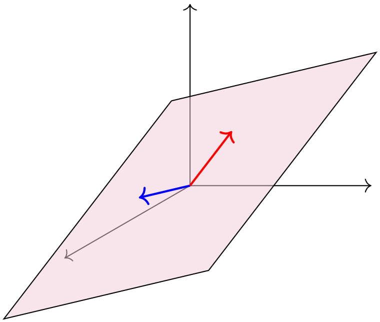
(b) $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(c) $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is smaller than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Definition V.37

A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space $\mathbb{R}^3$.



Fact V.38

Any sub**set** S of a vector space V that contains the additive identity $\vec{0}$ satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a sub**space**, we need to check that addition and multiplication still make sense using only vectors from S. So we need to check two things:

- The set is **closed under addition**: for any $\vec{x}, \vec{y} \in S$, the sum $\vec{x} + \vec{y}$ is also in S.
- The set is **closed under scalar multiplication**: for any $\vec{x} \in S$ and scalar $c \in \mathbb{R}$, the product $c\vec{x}$ is also in S.

Activity V.39 *(∼15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, x + 2y + z = 0 \right\}$.

Activity V.39 *(∼15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Part 1: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\vec{v} + \vec{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Activity V.39 *(∼15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Part 1: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\vec{v} + \vec{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Part 2: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\vec{v} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ also belongs to S for any $c \in \mathbb{R}$ by verifying an appropriate equation.

Activity V.39 *(~15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Part 1: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\vec{v} + \vec{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Part 2: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\vec{v} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ also belongs to S for any $c \in \mathbb{R}$ by verifying an appropriate equation.

Part 3: Is S is a subspace of $\mathbb{R}^3$?

Activity V.40 *(∼10 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 4 \right\}$. Choose a vector $\vec{v} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = ?$, and show that $c\vec{v}$ isn't in S. Is S a subspace of $\mathbb{R}^3$?

Remark V.41

Since 0 is a scalar and $0\vec{v} = \vec{z}$ for any vector $\vec{v}$, a nonempty set that is closed under scalar multiplication must contain the zero vector $\vec{z}$ for that vector space.

Put another way, you can check any of the following to show that a nonempty subset W isn't a subspace:

- Show that $\vec{0} \notin W$.
- Find $\vec{u}, \vec{v} \in W$ such that $\vec{u} + \vec{v} \notin W$.
- Find $c \in \mathbb{R}, \vec{v} \in W$ such that $c\vec{v} \notin W$.

If you cannot do any of these, then W can be proven to be a subspace by doing the following:

- Prove that $\vec{u} + \vec{v} \in W$ whenever $\vec{u}, \vec{v} \in W$.
- Prove that $c\vec{v} \in W$ whenever $c \in \mathbb{R}, \vec{v} \in W$.

Activity V.42 *(∼20 min)* Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, z = xy \right\}$$

Activity V.42 *($\sim$20 min)* Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, z = xy \right\}$$

Part 1: Show R isn't a subspace by showing that $\vec{0} \notin R$.

Activity V.42 *(∼20 min)* Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, z = xy \right\}$$

Part 1: Show R isn't a subspace by showing that $\vec{0} \notin R$.
Part 2: Show S isn't a subspace by finding two vectors $\vec{u}, \vec{v} \in S$ such that $\vec{u} + \vec{v} \notin S$.

Activity V.42 *(∼20 min)* Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, z = xy \right\}$$

Part 1: Show R isn't a subspace by showing that $\vec{0} \notin R$.
Part 2: Show S isn't a subspace by finding two vectors $\vec{u}, \vec{v} \in S$ such that $\vec{u} + \vec{v} \notin S$.
Part 3: Show T isn't a subspace by finding a vector $\vec{v} \in T$ such that $2\vec{v} \notin T$.

Activity V.43 *(∼5 min)* Let W be a subspace of a vector space V. How are span W and W related?

(a) span W is bigger than W

(b) span W is the same as W

(c) span W is smaller than W

Fact V.44

If S is any subset of a vector space V, then since $\operatorname{span} S$ collects all possible linear combinations, $\operatorname{span} S$ is automatically a subspace of V.

In fact, $\operatorname{span} S$ is always the smallest subspace of V that contains all the vectors in S.

Activity V.45 *(∼10 min)* Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

Which of the following is true?

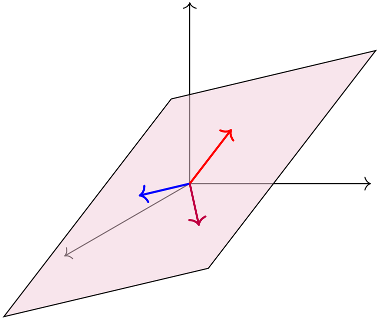
(A) span S is bigger than span T.

(B) span S and span T are the same size.

(C) span S is smaller than span T.

Definition V.46

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.



You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set. In the above image, all three vectors lay on the same planar subspace, but only two vectors are needed to span the plane, so the set is linearly dependent.

Activity V.47 *(~10 min)* Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be vectors in $\mathbb{R}^n$. Suppose $3\vec{v}_1 - 5\vec{v}_2 = \vec{v}_3$, so the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent. Which of the following is true of the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$?

(A) It is consistent with one solution

(B) It is consistent with infinitely many solutions

(C) It is inconsistent.

Fact V.48

For any vector space, the set $\{\vec{v}_1, \ldots \vec{v}_n\}$ is linearly dependent if and only if the vector equation $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$ is consistent with infinitely many solutions.

Activity V.49 *(~10 min)* Find

$$\operatorname{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 1 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Quick Check V.50

A set of Euclidean vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix}\vec{v}_1 & \dots & \vec{v}_n\end{bmatrix}$ has a column without a pivot position.

Observation V.51

Compare the following results:

- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is linearly independent if and only if RREF $\begin{bmatrix}\vec{v}_1 & \dots & \vec{v}_n\end{bmatrix}$ has all pivot columns.
- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ spans $\mathbb{R}^m$ if and only if RREF $\begin{bmatrix}\vec{v}_1 & \dots & \vec{v}_n\end{bmatrix}$ has all pivot rows.

Activity V.52 *(~5 min)* Is the set of Euclidean vectors

$$\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$$

linearly dependent or linearly independent?

Activity V.52 *(~5 min)* Is the set of Euclidean vectors

$$\left\{\begin{bmatrix}-4\\2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}1\\2\\0\\0\\3\end{bmatrix},\begin{bmatrix}1\\10\\10\\2\\6\end{bmatrix},\begin{bmatrix}3\\4\\7\\2\\1\end{bmatrix}\right\}$$ linearly dependent or linearly independent?

Part 1: Reinterpret this question as an appropriate question about solutions to a vector equation.

Activity V.52 *(∼5 min)* Is the set of Euclidean vectors

$$\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$$ linearly dependent or linearly independent?

Part 1: Reinterpret this question as an appropriate question about solutions to a vector equation.

Part 2: Use the solution to this question to answer the original question.

Activity V.53 *(∼10 min)* Is the set of polynomials $\{x^3+1, x^2+2x, x^2+7x+4\}$ linearly dependent or linearly independent?

Activity V.53 *(∼10 min)* Is the set of polynomials $\{x^3+1, x^2+2x, x^2+7x+4\}$ linearly dependent or linearly independent?

Part 1: Reinterpret this question as an appropriate question about solutions to a polynomial equation.

Activity V.53 *(∼10 min)* Is the set of polynomials $\{x^3+1, x^2+2x, x^2+7x+4\}$ linearly dependent or linearly independent?

Part 1: Reinterpret this question as an appropriate question about solutions to a polynomial equation.

Part 2: Use the solution to this question to answer the original question.

Activity V.54 *(∼5 min)* What is the largest number of $\mathbb{R}^4$ vectors that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity V.55 *(∼5 min)* What is the largest number of

$$\mathcal{P}^4 = \left\{ax^4 + bx^3 + cx^2 + dx + e \,\middle|\, a, b, c, d, e \in \mathbb{R}\right\}$$

vectors that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity V.56 *(∼5 min)* What is the largest number of

$$\mathcal{P} = \{f(x) \,|\, f(x) \text{ is any polynomial}\}$$

vectors that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Definition V.57

A **basis** is a linearly independent set that spans a vector space.

The **standard basis** of $\mathbb{R}^n$ is the set $\{\vec{e}_1, \ldots, \vec{e}_n\}$ where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For $\mathbb{R}^3$, these are the vectors $\vec{e}_1 = \hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{e}_3 = \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Observation V.58

A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

For example, in many calculus courses, vectors in $\mathbb{R}^3$ are often expressed in their component form

$$(3, -2, 4) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

or in their standard basic vector form

$$3\vec{e}_1 - 2\vec{e}_2 + 4\vec{e}_3 = 3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}.$$

Since every vector in $\mathbb{R}^3$ can be uniquely described as a linear combination of the vectors in $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, this set is indeed a basis.

Activity V.59 *(~15 min)* Label each of the sets A, B, C, D, E as

- SPANS $\mathbb{R}^4$ or DOES NOT SPAN $\mathbb{R}^4$
- LINEARLY INDEPENDENT or LINEARLY DEPENDENT
- BASIS FOR $\mathbb{R}^4$ or NOT A BASIS FOR $\mathbb{R}^4$

by finding RREF for their corresponding matrices.

$$A = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}3\\13\\7\\16\end{bmatrix}, \begin{bmatrix}-1\\10\\7\\14\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix} \right\} \qquad D = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix}, \begin{bmatrix}3\\6\\1\\5\end{bmatrix} \right\}$$

$$E = \left\{ \begin{bmatrix}5\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\3\end{bmatrix}, \begin{bmatrix}4\\5\\1\\3\end{bmatrix} \right\}$$

Activity V.60 *(~10 min)* If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$, that means $\mathrm{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$ doesn't have a non-pivot column, and doesn't have a row of zeros. What is $\mathrm{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$?

$$\mathrm{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4] = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Quick Check V.61

The set $\{\vec{v}_1, \dots, \vec{v}_m\}$ is a basis for $\mathbb{R}^n$ if and only if $m = n$ and

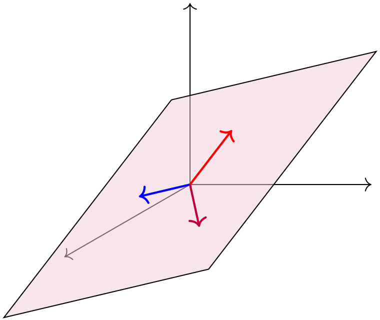
$$\mathrm{RREF}[\vec{v}_1 \ \dots \ \vec{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, a basis for $\mathbb{R}^n$ must have exactly n vectors and its square matrix must row-reduce to the so-called **identity matrix** containing all zeros except for a downward diagonal of ones. (We will learn where the identity matrix gets its name in a later module.)

Observation V.62

Recall that a **subspace** of a vector space is a subset that is itself a vector space.

One easy way to construct a subspace is to take the span of set, but a linearly dependent set contains "redundant" vectors. For example, only two of the three vectors in the following image are needed to span the planar subspace.



Activity V.63 *(∼10 min)* Consider the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}.$$

Activity V.63 *(∼10 min)* Consider the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}.$$

Part 1: Mark the part of RREF $\begin{bmatrix}2 & 2 & 2 & 1\\3 & 0 & -3 & 5\\0 & 1 & 2 & -1\\1 & -1 & -3 & 0\end{bmatrix}$ that shows that W's spanning set is linearly dependent.

Activity V.63 *($\sim$10 min)* Consider the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}.$$

Part 1: Mark the part of RREF $\begin{bmatrix}2 & 2 & 2 & 1\\3 & 0 & -3 & 5\\0 & 1 & 2 & -1\\1 & -1 & -3 & 0\end{bmatrix}$ that shows that W's spanning set is linearly dependent.

Part 2: Find a basis for W by removing a vector from its spanning set to make it linearly independent.

Fact V.64

Let $S = \{\vec{v}_1, \dots, \vec{v}_m\}$. The easiest basis describing span S is the set of vectors in S given by the pivot columns of RREF$[\vec{v}_1 \dots \vec{v}_m]$.

Put another way, to compute a basis for the subspace span S, simply remove the vectors corresponding to the non-pivot columns of RREF$[\vec{v}_1 \dots \vec{v}_m]$. For example, since

$$\mathrm{RREF}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}\right\}$ has $\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}$ as a basis.

Activity V.65 *(∼10 min)* Let W be the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Find a basis for W.

Activity V.66 *(∼10 min)* Let W be the subspace of $\mathcal{P}^3$ given by

$$W = \operatorname{span}\left\{x^3+3x^2+x-1, 2x^3-x^2+x+2, 4x^3+5x^2+3x, 3x^3+2x^2+2x+1\right\}$$

Find a basis for W.

Activity V.67 *(∼10 min)* Let W be the subspace of $M_{2,2}$ given by

$$W = \operatorname{span}\left\{\begin{bmatrix}1 & 3\\ 1 & -1\end{bmatrix}, \begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}, \begin{bmatrix}4 & 5\\ 3 & 0\end{bmatrix}, \begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}\right\}.$$

Find a basis for W.

Observation V.68

In the previous section, we learned that computing a basis for the subspace $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_m\}$, is as simple as removing the vectors corresponding to the non-pivot columns of $\operatorname{RREF}[\vec{v}_1 \ \dots \ \vec{v}_m]$.

For example, since

$$\operatorname{RREF}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}\right\}$ has $\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}$ as a basis.

Activity V.69 *(∼10 min)* Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Activity V.69 *(~10 min)* Let

$$S=\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}\quad\text{and}\quad T=\left\{\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}$$

Part 1: Find a basis for span S.

Activity V.69 *(~10 min)* Let

$$S=\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\} \text{ and } T=\left\{\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}$$

Part 1: Find a basis for span S.
Part 2: Find a basis for span T.

Observation V.70

Even though we found different bases for them, span S and span T are exactly the same subspace of $\mathbb{R}^4$, since

$$S=\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}=\left\{\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}=T$$

Fact V.71

Any non-trivial vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size.

For example,

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

are all valid bases for $\mathbb{R}^3$, and they all contain three vectors.

Definition V.72
The **dimension** of a vector space is equal to the size of any basis for the vector space.

As you'd expect, $\mathbb{R}^n$ has dimension n. For example, $\mathbb{R}^3$ has dimension 3 because any basis for $\mathbb{R}^3$ such as

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

contains exactly three vectors.

Activity V.73 *(~10 min)* Find the dimension of each subspace of $\mathbb{R}^4$ by finding RREF for each corresponding matrix.

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\} \qquad \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}3\\13\\7\\16\end{bmatrix},\begin{bmatrix}-1\\10\\7\\14\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix},\begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\} \qquad \operatorname{span}\left\{\begin{bmatrix}5\\3\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\0\\3\end{bmatrix},\begin{bmatrix}4\\5\\1\\3\end{bmatrix}\right\}$$

Fact V.74

Every vector space with finite dimension, that is, every vector space V with a basis of the form $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is said to be **isomorphic** to a Euclidean space $\mathbb{R}^n$, since there exists a natural correspondance between vectors in V and vectors in $\mathbb{R}^n$:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Observation V.75

We've already been taking advantage of the previous fact by converting polynomials and matrices into Euclidean vectors. Since $\mathcal{P}^3$ and $M_{2,2}$ are both four-dimensional:

$$4x^3 + 0x^2 - 1x + 5 \leftrightarrow \begin{bmatrix} 4 \\ 0 \\ -1 \\ 5 \end{bmatrix} \leftrightarrow \begin{bmatrix} 4 & 0 \\ -1 & 5 \end{bmatrix}$$

Activity V.76 *(∼5 min)* Suppose W is a subspace of $\mathcal{P}^8$, and you know that the set $\{x^3+x, x^2+1, x^4-x\}$ is a linearly independent subset of W. What can you conclude about W?

(a) The dimension of W is at most 3.

(b) The dimension of W is exactly 3.

(c) The dimension of W is at least 3.

Activity V.77 *(∼5 min)* Suppose W is a subspace of $\mathcal{P}^8$, and you know that W is spanned by the six vectors

$$\{x^4 - x, x^3 + x, x^3 + x + 1, x^4 + 2x, x^3, 2x + 1\}.$$

What can you conclude about W?

(a) The dimension of W is at most 6.

(b) The dimension of W is exactly 6.

(c) The dimension of W is at least 6.

Observation V.78

The space of polynomials $\mathcal{P}$ (of *any* degree) has the basis $\{1, x, x^2, x^3, \dots\}$, so it is a natural example of an infinite-dimensional vector space.

Since $\mathcal{P}$ and other infinite-dimensional spaces cannot be treated as an isomorphic finite-dimensional Euclidean space $\mathbb{R}^n$, vectors in such spaces cannot be studied by converting them into Euclidean vectors. Fortunately, most of the examples we will be interested in for this course will be finite-dimensional.

Definition V.79

A **homogeneous** system of linear equations is one of the form:

$$\begin{alignedat}{4} a_{11}x_1 &+ a_{12}x_2 &+ \dots &+ a_{1n}x_n &&= 0 \\ a_{21}x_1 &+ a_{22}x_2 &+ \dots &+ a_{2n}x_n &&= 0 \\ \vdots & \quad \vdots & & \quad \vdots && \vdots \\ a_{m1}x_1 &+ a_{m2}x_2 &+ \dots &+ a_{mn}x_n &&= 0 \end{alignedat}$$

This system is equivalent to the vector equation:

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

Activity V.80 *(∼5 min)* Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are solutions to $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$ so is $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$, since

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0} \text{ and } b_1\vec{v}_1 + \cdots + b_n\vec{v}_n = \vec{0}$$

implies

$$(a_1 + b_1)\vec{v}_1 + \cdots + (a_n + b_n)\vec{v}_n = \vec{0}.$$

Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution. Thus the solution set of a homogeneous system is...

a) A basis for $\mathbb{R}^n$.

b) A subspace of $\mathbb{R}^n$.

c) The empty set.

Activity V.81 *(~10 min)* Consider the homogeneous system of equations

$$\begin{alignedat}{5}
x_1 &{}+{}& 2x_2 &   &   &{}+{}& x_4 &= 0\\
2x_1 &{}+{}& 4x_2 &{}-{}& x_3 &{}-{}& 2x_4 &= 0\\
3x_1 &{}+{}& 6x_2 &{}-{}& x_3 &{}-{}& x_4 &= 0
\end{alignedat}$$

Activity V.81 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{alignedat}{5} x_1 &{}+{}& 2x_2 & & &{}+{}& x_4 &= 0\\ 2x_1 &{}+{}& 4x_2 &{}-{}& x_3 &{}-{}& 2x_4 &= 0\\ 3x_1 &{}+{}& 6x_2 &{}-{}& x_3 &{}-{}& x_4 &= 0 \end{alignedat}$$

Part 1: Find its solution set (a subspace of $\mathbb{R}^4$).

Activity V.81 *(~10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}- x_3} + x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - x_4 &= 0 \end{aligned}$$

Part 1: Find its solution set (a subspace of $\mathbb{R}^4$).

Part 2: Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

Activity V.81 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}- x_3} + x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - x_4 &= 0 \end{aligned}$$

Part 1: Find its solution set (a subspace of $\mathbb{R}^4$).

Part 2: Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle|\, a, b \in \mathbb{R} \right\}.$$

Part 3: Rewrite this solution space in the form

$$\operatorname{span}\left\{ \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \right\}.$$

Fact V.82

The coefficients of the free variables in the solution set of a linear system always yield linearly independent vectors.

Thus if

$$\left\{ a\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + b\begin{bmatrix}-1\\0\\-4\\1\end{bmatrix} \middle|\, a,b\in\mathbb{R} \right\} = \operatorname{span}\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-4\\1\end{bmatrix}\right\}$$

is the solution space for a homogeneous system, then

$$\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-4\\1\end{bmatrix}\right\}$$

is a basis for the solution space.

Activity V.83 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{aligned}
2x_1 + 4x_2 + 2x_3 - 4x_4 &= 0 \\
-2x_1 - 4x_2 + x_3 + x_4 &= 0 \\
3x_1 + 6x_2 - x_3 - 4x_4 &= 0
\end{aligned}$$

Find a basis for its solution space.

Activity V.84 *(∼10 min)* Consider the homogeneous vector equation

$$x_1 \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find a basis for its solution space.

Activity V.85 *(∼5 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ 2x_1 + 6x_2 + 4x_3 &= 0 \\ x_1 + 6x_2 - 4x_3 &= 0 \end{aligned}$$

Find a basis for its solution space.

Observation V.86

The basis of the trivial vector space is the empty set. You can denote this as either $\emptyset$ or $\{\}$.

Thus, if $\vec{0}$ is the only solution of a homogeneous system, the basis of the solution space is $\emptyset$.

Module A: Algebraic Properties of Linear Maps

How can we understand linear maps algebraically?

At the end of this module, students will be able to...

A1. Linear map verification. ... determine if a map between vector spaces of polynomials is linear or not.

A2. Linear maps and matrices. ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.

A3. Kernel and Image. ... compute a basis for the kernel and a basis for the image of a linear map, and verify that the rank-nullity theorem holds for a given linear map.

A4. Injectivity and surjectivity. ... determine if a given linear map is injective and/or surjective.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V3**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **V5**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **V6,V7**.
- Find a basis of the solution space to a homogeneous system of linear equations **V10**.

Definition A.1

A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

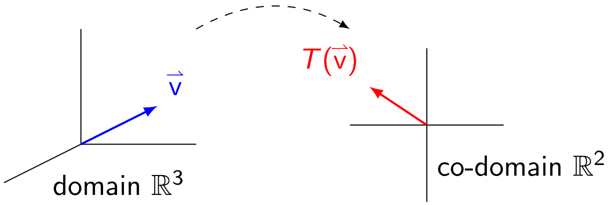
1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$.
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}, \vec{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition A.2

Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.

Linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$



Example A.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right) = \begin{bmatrix} (x+u)-(z+w) \\ 3(y+v) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x-z \\ 3y \end{bmatrix} + \begin{bmatrix} u-w \\ 3v \end{bmatrix} = \begin{bmatrix} (x+u)-(z+w) \\ 3(y+v) \end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx-cz \\ 3cy \end{bmatrix} \text{ and } cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x-z \\ 3y \end{bmatrix} = \begin{bmatrix} cx-cz \\ 3cy \end{bmatrix}$$

Therefore T is a linear transformation.

Example A.4

Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact A.5

A map between Euclidean spaces $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of $\mathbb{R}^n$.

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2, $y + 3$, and $y - 2^x$ are not linear combinations (even though $x + y$ is):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

Activity A.6 *(∼5 min)* Recall the following rules from calculus, where $D : \mathcal{P} \to \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f.

$$D(f+g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

a) $\mathcal{P}$ is not a vector space
b) D is a linear map
c) D is not a linear map

Activity A.7 *(∼10 min)* Let the polynomial maps $S : \mathcal{P}^4 \to \mathcal{P}^3$ and $T : \mathcal{P}^4 \to \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4 + x)$, $S(x^4) + S(x)$, $T(x^4 + x)$, and $T(x^4) + T(x)$. Which of these maps is definitely not linear?

Fact A.8

If $L : V \to W$ is linear, then $L(\vec{z}) = L(0\vec{v}) = 0L(\vec{v}) = \vec{z}$ where $\vec{z}$ is the additive identity of the vector spaces V, W.

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Observation A.9

Showing $L : V \to W$ is not a linear transformation can be done by finding an example for any one of the following.

- Show $L(\vec{z}) \neq \vec{z}$ (where $\vec{z}$ is the additive identity of L and W).
- Find $\vec{v}, \vec{w} \in V$ such that $L(\vec{v} + \vec{w}) \neq L(\vec{v}) + L(\vec{w})$.
- Find $\vec{v} \in V$ and $c \in \mathbb{R}$ such that $L(c\vec{v}) \neq cL(\vec{v})$.

Otherwise, L can be shown to be linear by proving the following in general.

- For all $\vec{v}, \vec{w} \in V$, $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$.
- For all $\vec{v} \in V$ and $c \in \mathbb{R}$, $L(c\vec{v}) = cL(\vec{v})$.

Note the similarities between this process and showing that a subset of a vector space is/isn't a subspace.

Activity A.10 *(∼15 min)* Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Activity A.10 *(∼15 min)* Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.

Activity A.10 *(~15 min)* Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.
Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f.

Activity A.10 *(∼15 min)* Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.
Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f.
Part 3: Is S linear?

Activity A.11 *(∼20 min)* Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad\qquad T(f(x)) = 3xf(x^2)$$

Activity A.11 *(∼20 min)* Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Note that $S(0) = 0$ and $T(0) = 0$. So instead, show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Activity A.11 *(~20 min)* Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad\qquad T(f(x)) = 3xf(x^2)$$

Part 1: Note that $S(0) = 0$ and $T(0) = 0$. So instead, show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.
Part 2: Prove that T is linear by verifying that $T(f(x) + g(x)) = T(f(x)) + T(g(x))$ and $T(cf(x)) = cT(f(x))$.

Remark A.12

Recall that a linear map $T : V \to W$ satisfies

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$.
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}, \vec{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Activity A.13 *(∼5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity A.14 *(~5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.15 *(~5 min)* Suppose $T: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}-2\\0\\-3\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.16 *(~5 min)* Suppose $T:\mathbb{R}^3\to\mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-3\\2\end{bmatrix}$.

What piece of information would help you compute $T\left(\begin{bmatrix}0\\4\\-1\end{bmatrix}\right)$?

(a) The value of $T\left(\begin{bmatrix}0\\-4\\0\end{bmatrix}\right)$.

(b) The value of $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$.

(c) The value of $T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$.

(d) Any of the above.

Fact A.17

Consider any basis $\{\vec{b}_1, \dots, \vec{b}_n\}$ for V. Since every vector $\vec{v}$ can be written as a linear combination of basis vectors, $x_1\vec{b}_1 + \cdots + x_n\vec{b}_n$, we may compute $T(\vec{v})$ as follows:

$$T(\vec{v}) = T(x_1\vec{b}_1 + \cdots + x_n\vec{b}_n) = x_1 T(\vec{b}_1) + \cdots + x_n T(\vec{b}_n).$$

Therefore any linear transformation $T : V \to W$ can be defined by just describing the values of $T(\vec{b}_i)$.

Put another way, the images of the basis vectors **determine** the transformation T.

Definition A.18

Since linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by the standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\vec{e}_1) \cdots T(\vec{e}_n)]$.

For example, let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T\left(\vec{e}_1\right) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\vec{e}_2\right) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\vec{e}_3\right) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\left[T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3)\right] = \begin{bmatrix}3 & -1 & 5\\2 & 4 & 0\end{bmatrix}.$$

Activity A.19 *(~3 min)* Let $T:\mathbb{R}^4\to\mathbb{R}^3$ be the linear transformation given by

$$T\left(\vec{e}_1\right)=\begin{bmatrix}0\\3\\-2\end{bmatrix}\qquad T\left(\vec{e}_2\right)=\begin{bmatrix}-3\\0\\1\end{bmatrix}\qquad T\left(\vec{e}_3\right)=\begin{bmatrix}4\\-2\\1\end{bmatrix}\qquad T\left(\vec{e}_4\right)=\begin{bmatrix}2\\0\\0\end{bmatrix}$$

Write the standard matrix $[\,T(\vec{e}_1)\ \cdots\ T(\vec{e}_n)]$ for T.

Activity A.20 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Activity A.20 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Part 1: Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

Activity A.20 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Part 1: Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.
Part 2: Find the standard matrix for T.

Fact A.21

Because every linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ has a linear combination of the variables in each component, and thus $T(\vec{e}_i)$ yields exactly the coefficients of x_i, the standard matrix for T is simply an ordered list of the coefficients of the x_i:

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \qquad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Activity A.22 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Activity A.22 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Part 1: Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

Activity A.22 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Part 1: Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

Part 2: Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Activity A.23 *(~15 min)* Compute the following linear transformations of vectors given their standard matrices.

$$T_1\left(\begin{bmatrix}1\\2\end{bmatrix}\right) \text{ for the standard matrix } A_1 = \begin{bmatrix}4 & 3\\0 & -1\\1 & 1\\3 & 0\end{bmatrix}$$

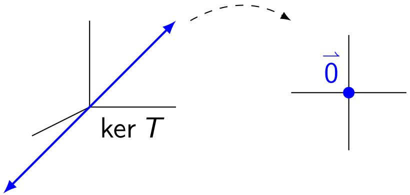
$$T_2\left(\begin{bmatrix}1\\1\\0\\-3\end{bmatrix}\right) \text{ for the standard matrix } A_2 = \begin{bmatrix}4 & 3 & 0 & -1\\1 & 1 & 3 & 0\end{bmatrix}$$

$$T_3\left(\begin{bmatrix}0\\-2\\0\end{bmatrix}\right) \text{ for the standard matrix } A_3 = \begin{bmatrix}4 & 3 & 0\\0 & -1 & 3\\5 & 1 & 1\\3 & 0 & 0\end{bmatrix}$$

Definition A.24

Let $T : V \to W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{ \vec{v} \in V \;\middle|\; T(\vec{v}) = \vec{z} \right\}$$



Activity A.25 *(∼5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\ker T$, the set of all vectors that transform into $\vec{0}$?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x, y \in \mathbb{R} \right\}$

Activity A.26 *(~5 min)* Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x\\y\end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\ker T$, the set of all vectors that transform into $\vec{0}$?

a) $\left\{\begin{bmatrix}0\\0\\a\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$

b) $\left\{\begin{bmatrix}a\\a\\0\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$

c) $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$

d) $\mathbb{R}^3=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\middle|\, x,y,z\in\mathbb{R}\right\}$

Activity A.27 *(∼10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 4y - z \\ x + 2y + z \end{bmatrix}$$

Activity A.27 *(~10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 4y - z \\ x + 2y + z \end{bmatrix}\Bigg)$$

Part 1: Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

Activity A.27 *(∼10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 4y - z \\ x + 2y + z \end{bmatrix}$$

Part 1: Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

Part 2: Use RREF(A) to solve this homogeneous system of equations and find a basis for the kernel of T.

Activity A.28 *(∼10 min)* Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 2x+4y+2z-4w \\ -2x-4y+z+w \\ 3x+6y-z-4w \end{bmatrix}.$$

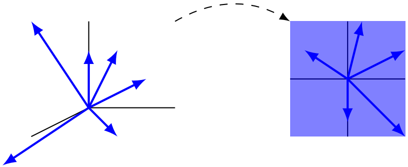
Find a basis for the kernel of T.

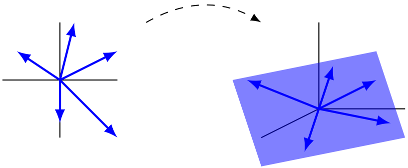
Definition A.29

Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{ \vec{w} \in W \;\middle|\; \text{there is some } \vec{v} \in V \text{ with } T(\vec{v}) = \vec{w} \right\}$$

In the examples below, the left example's image is all of $\mathbb{R}^2$, but the right example's image is a planar subspace of $\mathbb{R}^3$.





Activity A.30 *(∼5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^2$ vectors?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\}$

Activity A.31 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^3$ vectors?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}$

Activity A.32 *(∼5 min)* Let $T:\mathbb{R}^4\to\mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A=\begin{bmatrix}3&4&7&1\\-1&1&0&2\\2&1&3&-1\end{bmatrix}=\begin{bmatrix}T(\vec{e}_1)&T(\vec{e}_2)&T(\vec{e}_3)&T(\vec{e}_4)\end{bmatrix}.$$

Since $T(\vec{v})=T(x_1\vec{e}_1+x_2\vec{e}_2+x_3\vec{e}_3+x_4\vec{e}_4)$, the set of vectors

$$\left\{\begin{bmatrix}3\\-1\\2\end{bmatrix},\begin{bmatrix}4\\1\\1\end{bmatrix},\begin{bmatrix}7\\0\\3\end{bmatrix},\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$$

a) spans Im T
b) is a linearly independent subset of Im T
c) is a basis for Im T

Observation A.33

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}.$$

Since the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ spans $\operatorname{Im} T$, we can obtain a basis for $\operatorname{Im} T$ by finding $\operatorname{RREF} A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and only using the vectors corresponding to pivot columns:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Fact A.34

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

- The kernel of T is the solution set of the homogeneous system given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$. Use the coefficients of its free variables to get a basis for the kernel.
- The image of T is the span of the columns of A. Remove the vectors creating non-pivot columns in RREF A to get a basis for the image.

Activity A.35 *(∼10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T.

Activity A.36 *(∼5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the kernel of T?

(a) The number of pivot columns

(b) The number of non-pivot columns

(c) The number of pivot rows

(d) The number of non-pivot rows

Activity A.37 *(∼5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the image of T?

(a) The number of pivot columns

(b) The number of non-pivot columns

(c) The number of pivot rows

(d) The number of non-pivot rows

Observation A.38

Combining these with the observation that the number of columns is the dimension of the domain of T, we have the **rank-nullity theorem**:

The dimension of the domain of T equals $\dim(\ker T) + \dim(\operatorname{Im} T)$.

The dimension of the image is called the **rank** of T (or A) and the dimension of the kernel is called the **nullity**.

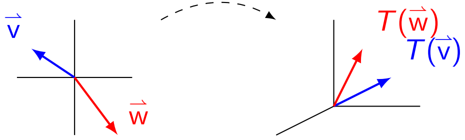
Activity A.39 *(~10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

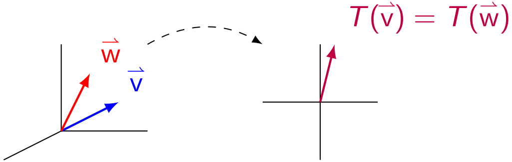
Verify that the rank-nullity theorem holds for T.

Definition A.40

Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.



injective



not injective

Activity A.41 *(~3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T injective?

a) Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.

b) Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

c) No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$

d) No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$

Activity A.42 *(∼2 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T injective?

a) Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.

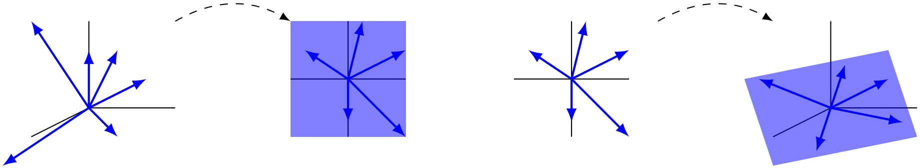
b) Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

c) No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$

d) No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$

Definition A.43

Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\vec{w} \in W$, there is some $\vec{v} \in V$ with $T(\vec{v}) = \vec{w}$.



surjective

not surjective

Activity A.44 *(~3 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x\\y\\0\end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

Is T surjective?

a) Yes, because for every $\vec{w} = \begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3$, there exists $\vec{v} = \begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$ such that $T(\vec{v}) = \vec{w}$.

b) No, because $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$ can never equal $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

c) No, because $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$ can never equal $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Activity A.45 *(~2 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective?

a) Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} x \\ y \\ 42 \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

b) Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

c) No, because $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Observation A.46

As we will see, it's no coincidence that the RREF of the injective map's standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has all pivot columns. Similarly, the RREF of the surjective map's standard matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has a pivot in each row.

Observation A.47

Let $T : V \to W$. We have previously defined the following terms.

- The **kernel** of T is the set of all vectors in V that are mapped to $\vec{z} \in W$. It is a subspace of V.
- The **image** of T is the set of all vectors in W that are mapped to by something in V. It is a subspace of W.
- T is called **injective** or **one-to-one** if T always maps distinct vectors to different places.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.

Activity A.48 *(∼5 min)* Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

(a) T is injective

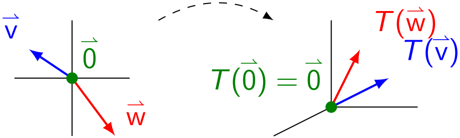
(b) T is not injective

(c) T is surjective

(d) T is not surjective

Fact A.49

A linear transformation T is injective **if and only if** $\ker T = \{\vec{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.



Activity A.50 *(∼5 min)* Let $T : V \to \mathbb{R}^5$ be a linear transformation where $\operatorname{Im} T$ is spanned by four vectors. What can you conclude?

(a) T is injective

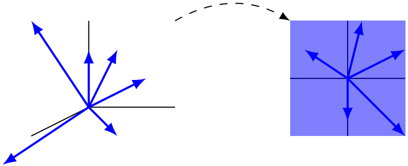
(b) T is not injective

(c) T is surjective

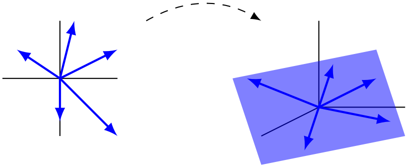
(d) T is not surjective

Fact A.51

A linear transformation $T : V \to W$ is surjective **if and only if** $\operatorname{Im} T = W$. Put another way, a surjective linear transformation may be recognized by its identical codomain and image.



surjective, $\operatorname{Im} T = \mathbb{R}^2$



not surjective, $\operatorname{Im} T \neq \mathbb{R}^3$

Activity A.52 *(~15 min)* Let $T:\mathbb{R}^n\to\mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of *equivalent* statements: one group that means T is **injective**, and one group that means T is **surjective**.

(a) The kernel of T is trivial, i.e. $\ker T=\{\vec{0}\}$.

(b) The columns of A span $\mathbb{R}^m$.

(c) The columns of A are linearly independent.

(d) Every column of RREF(A) has a pivot.

(e) Every row of RREF(A) has a pivot.

(f) The image of T equals its codomain, i.e. $\operatorname{Im} T=\mathbb{R}^m$.

(g) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has a solution for all $\vec{b}\in\mathbb{R}^m$.

(h) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$ has exactly one solution.

Quick Check A.53

The easiest way to determine if the linear map with standard matrix A is injective is to see if $\text{RREF}(A)$ has a pivot in each column.

The easiest way to determine if the linear map with standard matrix A is surjective is to see if $\text{RREF}(A)$ has a pivot in each row.

Activity A.54 *(∼3 min)* What can you conclude about the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

d) Its standard matrix has more rows than columns, so T is surjective.

Activity A.55 *(∼2 min)* What can you conclude about the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

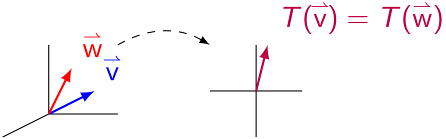
d) Its standard matrix has more rows than columns, so T is surjective.

Fact A.56

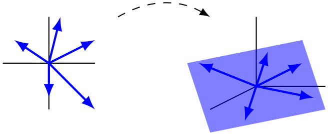
The following are true for any linear map $T : V \to W$:

- If $\dim(V) > \dim(W)$, then T is not injective.
- If $\dim(V) < \dim(W)$, then T is not surjective.

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase dimension from its domain to its image.



not injective, $3 > 2$



not surjective, $2 < 3$

But dimension arguments **cannot** be used to prove a map **is** injective or surjective.

Activity A.57 *(∼5 min)* Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijective**).

Activity A.57 *(~5 min)* Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix
$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijective**).

Part 1: How many pivot rows must RREF A have?

Activity A.57 *(~5 min)* Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijective**).

Part 1: How many pivot rows must RREF A have?

Part 2: How many pivot columns must RREF A have?

Activity A.57 *(∼5 min)* Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijective**).

Part 1: How many pivot rows must RREF A have?

Part 2: How many pivot columns must RREF A have?

Part 3: What is RREF A?

Activity A.58 *(∼5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

(a) RREF(A) is the identity matrix.

(b) The columns of A form a basis for $\mathbb{R}^n$

(c) The system of linear equations given by the augmented matrix $\left[A \mid \vec{b}\right]$ has exactly one solution for each $\vec{b} \in \mathbb{R}^n$.

Observation A.59

The easiest way to show that the linear map with standard matrix A is bijective is to show that $\mathrm{RREF}(A)$ is the identity matrix.

Activity A.60 *(∼3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective
(b) T is injective but not surjective
(c) T is surjective but not injective
(d) T is bijective.

Activity A.61 *(∼3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.62 *(∼3 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x-y \\ x+3y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.63 *(∼3 min)* Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}2x+y-z\\4x+y+z\end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Module M: Understanding Matrices Algebraically

What algebraic structure do matrices have?

At the end of this module, students will be able to...

M1. **Matrix Multiplication.** ... multiply matrices.

M2. **Row operations as matrix multiplication.** ... can express row operations through matrix multiplication.

M3. **Invertible Matrices.** ... determine if a square matrix is invertible or not.

M4. **Matrix inverses.** ... compute the inverse matrix of an invertible matrix.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers.
- Identify the domain and codomain of linear transformations.
- Find the matrix corresponding to a linear transformation and compute the image of a vector given a standard matrix **A2**
- Determine if a linear transformation is injective and/or surjective **A4**
- Interpret the ideas of injectivity and surjectivity in multiple ways.

The following resources will help you prepare for this module.

- Function composition (Khan Academy): `http://bit.ly/2wkz7f3`
- Domain and codomain: `https://www.youtube.com/watch?v=BQMyeQOLvpg`
- Interpreting injectivity and surjectivity in many ways: `https://www.youtube.com/watch?v=WpUv72Y6D10`

Observation M.1

If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^k$ are linear maps, then the composition map $S \circ T$ is a linear map from $\mathbb{R}^n \to \mathbb{R}^k$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T"] \arrow[rr, bend right, "S \circ T"'] & \mathbb{R}^m \arrow[r, "S"] & \mathbb{R}^k \end{tikzcd}$$

Recall that for a vector, $\vec{v} \in \mathbb{R}^n$, the composition is computed as $(S \circ T)(\vec{v}) = S(T(\vec{v}))$.

Activity M.2 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What are the domain and codomain of the composition map $S \circ T$?

(a) The domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$

(b) The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$

(c) The domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^4$

(d) The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^4$

(e) The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^3$

(f) The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^2$

Activity M.3 *(∼2 min)* Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be given by the 2×3 standard matrix $B=\begin{bmatrix}2 & 1 & -3\\ 5 & -3 & 4\end{bmatrix}$ and $S:\mathbb{R}^2\to\mathbb{R}^4$ be given by the 4×2 standard matrix $A=\begin{bmatrix}1 & 2\\ 0 & 1\\ 3 & 5\\ -1 & -2\end{bmatrix}$.

What size will the standard matrix of $S\circ T:\mathbb{R}^3\to\mathbb{R}^4$ be? (Rows $\times$ Columns)

(a) 4×3
(b) 4×2
(c) 3×4
(d) 3×2
(e) 2×4
(f) 2×3

Activity M.4 *(∼15 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Activity M.4 *(~15 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Activity M.4 *(∼15 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{e}_2)$.

Activity M.4 *(~15 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{e}_2)$.

Part 3: Compute $(S \circ T)(\vec{e}_3)$.

Activity M.4 *(∼15 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{e}_2)$.

Part 3: Compute $(S \circ T)(\vec{e}_3)$.

Part 4: Write the 4×3 standard matrix of $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$.

Definition M.5

We define the **product** AB of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For the previous activity, T was a map $\mathbb{R}^3 \to \mathbb{R}^2$, and S was a map $\mathbb{R}^2 \to \mathbb{R}^4$, so $S \circ T$ gave a map $\mathbb{R}^3 \to \mathbb{R}^4$ with a 4×3 standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= \left[(S \circ T)(\vec{e}_1) \quad (S \circ T)(\vec{e}_2) \quad (S \circ T)(\vec{e}_3) \right] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

Activity M.6 *(~15 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Activity M.6 *($\sim$15 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Part 1: Write the dimensions (rows $\times$ columns) for A, B, AB, and BA.

Activity M.6 *(∼15 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Part 1: Write the dimensions (rows $\times$ columns) for A, B, AB, and BA.

Part 2: Find the standard matrix AB of $S \circ T$.

Activity M.6 *(∼15 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Part 1: Write the dimensions (rows $\times$ columns) for A, B, AB, and BA.

Part 2: Find the standard matrix AB of $S \circ T$.

Part 3: FInd the standard matrix BA of $T \circ S$.

Activity M.7 *(∼10 min)* Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Activity M.7 *(∼10 min)* Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Part 1: Find the domain and codomain of each of the three linear maps corresponding to A, B), and C.

Activity M.7 *(∼10 min)* Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Part 1: Find the domain and codomain of each of the three linear maps corresponding to A, B), and C.

Part 2: Only one of the matrix products AB, AC, BA, BC, CA, CB can actually be computed. Compute it.

Remark M.8

Recall that the **product** AB of a $m \times n$ matrix A and an $n \times k$ matrix B is the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For example, if T was a map $\mathbb{R}^3 \to \mathbb{R}^2$, and S was a map $\mathbb{R}^2 \to \mathbb{R}^4$, then $S \circ T$ gave a map $\mathbb{R}^3 \to \mathbb{R}^4$ with a 4×3 standard matrix, such as:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= \left[(S \circ T)(\vec{e}_1) \quad (S \circ T)(\vec{e}_2) \quad (S \circ T)(\vec{e}_3) \right] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

Activity M.9 *(∼15 min)* Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Activity M.9 *(∼15 min)* Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Part 1: Compute the product BA by hand.

Activity M.9 *(~15 min)* Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Part 1: Compute the product BA by hand.

Part 2: Check your work using technology. Using Octave:

- `B = sym([3 -4 0 ; 2 0 -1 ; 0 -3 3])`
- `A = sym([2 7 -1 ; 0 3 2 ; 1 1 -1])`
- `B*A`

Activity M.10 *(∼5 min)* Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix B such that $BA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Check your guess using technology.

Definition M.11

The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

Fact M.12

For any square matrix A, $IA = AI = A$:

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 7 & -1\\ 0 & 3 & 2\\ 1 & 1 & -1\end{bmatrix} = \begin{bmatrix}2 & 7 & -1\\ 0 & 3 & 2\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 7 & -1\\ 0 & 3 & 2\\ 1 & 1 & -1\end{bmatrix}$$

Activity M.13 *($\sim$20 min)* Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Activity M.13 *(∼20 min)* Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Part 1: Create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Activity M.13 *(∼20 min)* Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

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$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Activity M.13 *(∼20 min)* Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Part 1: Create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5(1) & 7+5(1) & -1+5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Fact M.14

If R is the result of applying a row operation to I, then RA is the result of applying the same row operation to A.

- Scaling a row: $R = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Swapping rows: $R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Adding a row multiple to another row: $R = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Such matrices can be chained together to emulate multiple row operations. In particular,

$$\mathrm{RREF}(A) = R_k \dots R_2 R_1 A$$

for some sequence of matrices $R_1, R_2, \dots, R_k$.

Activity M.15 *(∼10 min)* Consider the two row operations $R_2 \leftrightarrow R_3$ and $R_1 + R_2 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1+1 & 4+2 & 5+3 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = B$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A:

$$B = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} A$$

Check your work using technology.

Activity M.16 *(∼15 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into three groups of statements: a group that means T is **injective**, a group that means T is **surjective**, and a group that means T is **bijective**.

(a) $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$

(b) $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$

(c) $A\vec{x} = \vec{0}$ has a unique solution.

(d) The columns of A span $\mathbb{R}^m$

(e) The columns of A are linearly independent

(f) The columns of A are a basis of $\mathbb{R}^m$

(g) Every column of RREF(A) has a pivot

(h) Every row of RREF(A) has a pivot

(i) $m = n$ and RREF(A) $= I$

Activity M.17 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 4 \\ 1 & 1 & 3 \end{bmatrix}$.

Write an augmented matrix representing the system of equations given by $T(\vec{x}) = \vec{0}$, that is, $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{0}$ to find the kernel of T.

Definition M.18

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with standard matrix A.

- If T is a bijection and $\vec{b}$ is any $\mathbb{R}^n$ vector, then $T(\vec{x}) = A\vec{x} = \vec{b}$ has a unique solution.
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by setting $T^{-1}(\vec{b})$ to be this unique solution.
- Let A^{-1} be the standard matrix for T^{-1}. We call A^{-1} the **inverse matrix** of A, so we also say that A is **invertible**.

Activity M.19 *(∼20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Activity M.19 *(~20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Activity M.19 *(∼20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Activity M.19 *(~20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Part 3: Solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

Activity M.19 *(∼20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Part 3: Solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

Part 4: Write A^{-1}, the standard matrix for T^{-1}.

Observation M.20

We could have solved these three systems simultaneously by row reducing the matrix $[A \,|\, I]$ at once.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right]$$

Activity M.21 *(~5 min)* Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A \,|\, I]$.

Activity M.22 *(∼5 min)* Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

Quick Check M.23

An $n \times n$ matrix A is invertible if and only if $\operatorname{RREF}(A) = I_n$.

Activity M.24 *(∼10 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Activity M.24 *(~10 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

Activity M.24 *(∼10 min)* Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}2x-3y\\-3x+5y\end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}5x+3y\\3x+2y\end{bmatrix}$.

Part 1: Compute $(T^{-1}\circ T)\left(\begin{bmatrix}-2\\1\end{bmatrix}\right)$.

Part 2: If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, find the 2×2 matrix

$$A^{-1}A=\begin{bmatrix}? & ?\\ ? & ?\end{bmatrix}.$$

Observation M.25

$T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation T. Therefore $A^{-1}A = AA^{-1} = I$ is the identity matrix for any invertible matrix A.

Module G: Geometry of Linear Maps

How can we understand linear maps geometrically?

At the end of this module, students will be able to...

G1. **Row operations and Determinants.** ... describe how a row operation affects the determinant of a matrix.

G2. **Determinants.** ... compute the determinant of a 4×4 matrix.

G3. **Eigenvalues.** ... find the eigenvalues of a 2×2 matrix.

G4. **Eigenvectors.** ... find a basis for the eigenspace of a 4×4 matrix associated with a given eigenvalue.

Readiness Assurance Outcomes

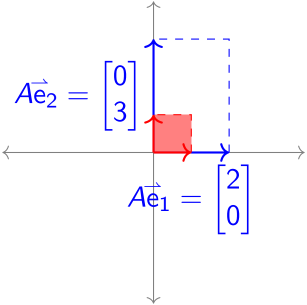
Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **A2**.
- Recall and use the definition of a linear transformation **A1**.
- Find all roots of quadratic polynomials (including complex ones).
- Interpret the statement "A is an invertible matrix" in many equivalent ways in different contexts.

The following resources will help you prepare for this module.

- Finding the area of a parallelogram (Khan Academy): `http://bit.ly/2BO5iWx`
- Factoring quadratics (Khan Academy): `http://bit.ly/1XjfbV2`
- Factoring quadratics using area models (Youtube): `https://youtu.be/Aa-v1EK7DR4`
- Finding complex roots of quadratics (Youtube): `https://www.youtube.com/watch?v=2yBhDsNE0wg`

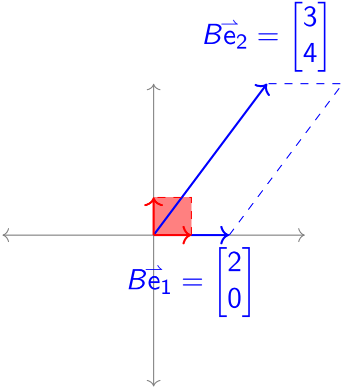
Activity G.1 *(∼5 min)* The image below illustrates how the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.



(a) What are the lengths of $A\vec{e}_1$ and $A\vec{e}_2$?

(b) What is the area of the transformed unit square?

Activity G.2 *(∼5 min)* The image below illustrates how the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$. transforms the unit square.



(a) What are the lengths of $B\vec{e}_1$ and $B\vec{e}_2$?

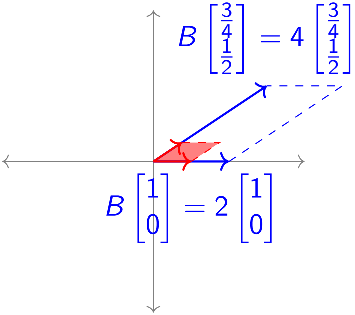
(b) What is the area of the transformed unit square?

Observation G.3

It is possible to find two nonparallel vectors that are scaled but not rotated by the linear map given by B.

$$B\vec{e}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\vec{e}_1$$

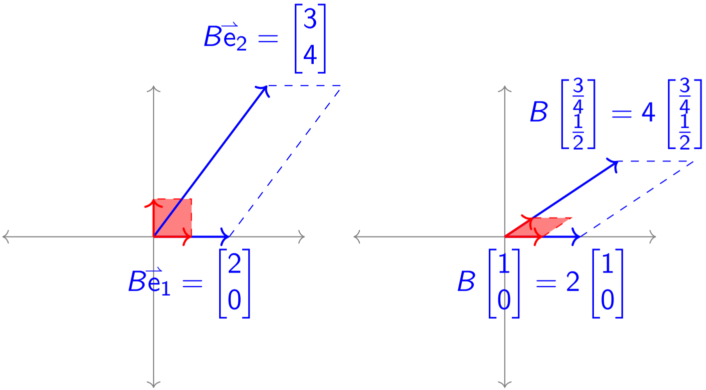
$$B\begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4\begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$



The process for finding such vectors will be covered later in this module.

Observation G.4

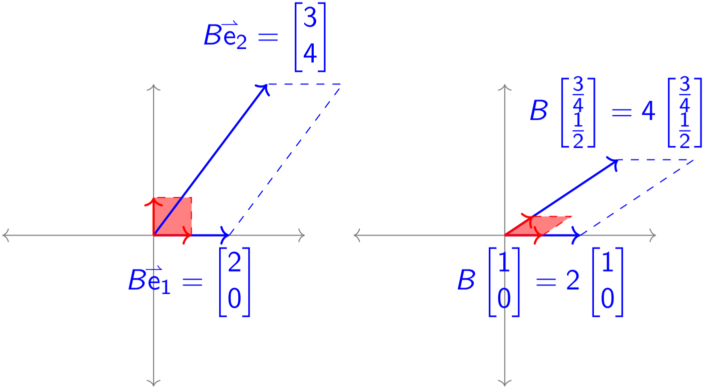
Notice that while a linear map can transform vectors in various ways, linear maps always transform parallelograms into parallelograms, and these areas are always transformed by the same factor: in the case of $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, this factor is 8.



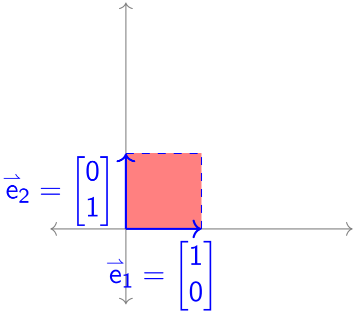
Since this change in area is always the same for a given linear map, it will be equal to the value of the transformed unit square (which begins with area 1).

Remark G.5

We will define the **determinant** of a square matrix A, or $\det(A)$ for short, to be the factor by which A scales areas. In order to figure out how to compute it, we first figure out the properties it must satisfy.

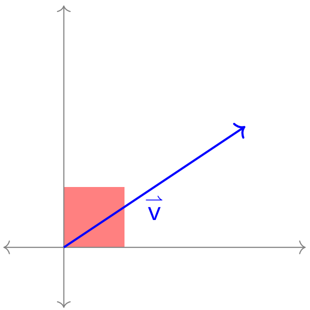


Activity G.6 *(∼2 min)* The transformation of the unit square by the standard matrix $[\vec{e}_1 \ \vec{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. What is $\det([\vec{e}_1 \ \vec{e}_2]) = \det(I)$, the area of the transformed unit square shown here?



a) 0
b) 1
c) 2
d) 4

Activity G.7 *(∼2 min)* The transformation of the unit square by the standard matrix $[\vec{v}\ \ \vec{v}]$ is illustrated below: both $T(\vec{e}_1) = T(\vec{e}_2) = \vec{v}$. What is $\det([\vec{v}\ \ \vec{v}])$, the area of the transformed unit square shown here?



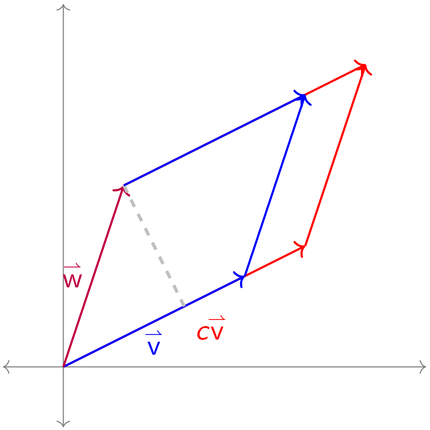
a) 0

b) 1

c) 2

d) 4

Activity G.8 *(∼5 min)* The transformations of the unit square by the standard matrices $[\vec{v}\ \ \vec{w}]$ and $[c\vec{v}\ \ \vec{w}]$ are illustrated below. Describe the value of $\det([c\vec{v}\ \ \vec{w}])$.



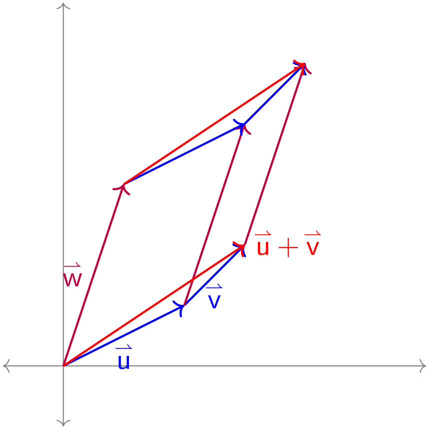
a) $\det([\vec{v}\ \ \vec{w}])$

b) $\det([\vec{v}\ \ \vec{w}]) + c\det([\vec{v}\ \ \vec{w}]$

c) $c\det([\vec{v}\ \ \vec{w}])$

d) Cannot be determined from this information.

Activity G.9 *(~5 min)* The transformations of unit squares by the standard matrices $[\vec{u}\ \ \vec{w}]$, $[\vec{v}\ \ \vec{w}]$ and $[\vec{u}+\vec{v}\ \ \vec{w}]$ are illustrated below. Describe the value of $\det([\vec{u}+\vec{v}\ \ \vec{w}])$.



a) $\det([\vec{u}\ \ \vec{w}]) = \det([\vec{v}\ \ \vec{w}])$

b) $\det([\vec{u}\ \ \vec{w}]) + \det([\vec{v}\ \ \vec{w}])$

c) $\det([\vec{u}\ \ \vec{w}])\det([\vec{v}\ \ \vec{w}])$

d) Cannot be determined from this information.

Definition G.10

The **determinant** is the unique function $\det : M_{n,n} \to \mathbb{R}$ satisfying these properties:

P1: $\det(I) = 1$

P2: $\det(A) = 0$ whenever two columns of the matrix are identical.

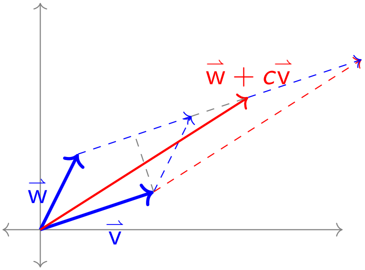
P3: $\det[\cdots \quad c\vec{v} \quad \cdots] = c\det[\cdots \quad \vec{v} \quad \cdots]$, assuming no other columns change.

P4: $\det[\cdots \quad \vec{v} + \vec{w} \quad \cdots] = \det[\cdots \quad \vec{v} \quad \cdots] + \det[\cdots \quad \vec{w} \quad \cdots]$, assuming no other columns change.

Note that these last two properties together can be phrased as "The determinant is linear in each column."

Observation G.11

The determinant must also satisfy other properties. Consider $\det([\vec{v} \quad \vec{w} + c\vec{v}])$ and $\det([\vec{v} \quad \vec{w}])$.



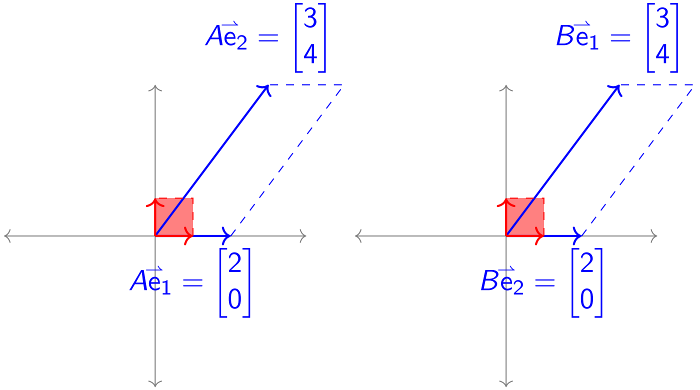
The base of both parallelograms is $\vec{v}$, while the height has not changed, so the determinant does not change either. This can also be proven using the other properties of the determinant:

$$\begin{aligned}
\det([\vec{v} + c\vec{w} \quad \vec{w}]) &= \det([\vec{v} \quad \vec{w}]) + \det([c\vec{w} \quad \vec{w}]) \\
&= \det([\vec{v} \quad \vec{w}]) + c\det([\vec{w} \quad \vec{w}]) \\
&= \det([\vec{v} \quad \vec{w}]) + c \cdot 0 \\
&= \det([\vec{v} \quad \vec{w}])
\end{aligned}$$

Remark G.12

Swapping columns may be thought of as a reflection, which is represented by a negative determinant. For example, the following matrices transform the unit square into the same parallelogram, but the second matrix reflects its orientation.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \quad \det A = 8 \qquad B = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix} \quad \det B = -8$$



Observation G.13

The fact that swapping columns multiplies determinants by a negative may be verified by adding and subtracting columns.

$$\begin{aligned}
\det([\vec{v} \quad \vec{w}]) &= \det([\vec{v}+\vec{w} \quad \vec{w}]) \\
&= \det([\vec{v}+\vec{w} \quad \vec{w}-(\vec{v}+\vec{w})]) \\
&= \det([\vec{v}+\vec{w} \quad -\vec{v}]) \\
&= \det([\vec{v}+\vec{w}-\vec{v} \quad -\vec{v}]) \\
&= \det([\vec{w} \quad -\vec{v}]) \\
&= -\det([\vec{w} \quad \vec{v}])
\end{aligned}$$

Fact G.14

To summarize, we've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant in the following way:

(a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c\det([\cdots \quad \vec{v} \quad \cdots]) = \det([\cdots \quad c\vec{v} \quad \cdots])$$

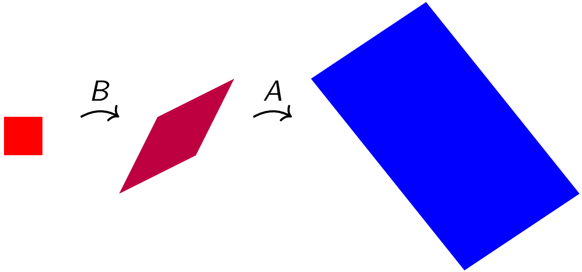
(b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = -\det([\cdots \quad \vec{w} \quad \cdots \quad \vec{v} \quad \cdots])$$

(c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = \det([\cdots \quad \vec{v}+c\vec{w} \quad \cdots \quad \vec{w} \quad \cdots])$$

Activity G.15 *(∼5 min)* The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. By what factor does the transformation given by the standard matrix AB scale areas?



(a) 1
(b) 7
(c) 12
(d) Cannot be determined

Fact G.16

Since the transformation given by the standard matrix AB is obtained by applying the transformations given by A and B, it follows that

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

Remark G.17

Recall that row operations may be produced by matrix multiplication.

- Multiply the first row of A by c: $\begin{bmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$
- Swap the first and second row of A: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$
- Add c times the third row to the first row of A: $\begin{bmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$

Fact G.18

The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row: $\det \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = c \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = c$
- Swapping rows: $\det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -1$
- Adding a row multiple to another row:

$$\det \begin{bmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & c-1c & 0 \\ 0 & 1 & 0-0c & 0 \\ 0 & 0 & 1-0c & 0 \\ 0 & 0 & 0-0c & 1 \end{bmatrix} = \det(I) = 1$$

Activity G.19 *(~5 min)* Consider the row operation $R_1 + 4R_3 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1+4(9) & 2+4(10) & 3+4(11) & 4+4(12) \\ 5 & 6 & 6 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to
$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Find $\det R$ by comparing with the previous slide.

(c) If $C \in M_{3,3}$ is a matrix with $\det(C) = -3$, find
$$\det(RC) = \det(R)\det(C).$$

Activity G.20 *(∼5 min)* Consider the row operation $R_1 \leftrightarrow R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to I.

(b) If $C \in M_{3,3}$ is a matrix with $\det(C) = 5$, find $\det(RC)$.

Activity G.21 *(~5 min)* Consider the row operation $3R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3(5) & 3(6) & 3(7) & 3(8) \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det(C) = -7$, find $\det(RC)$.

Remark G.22

Recall that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying columns by scalars:

$$\det([\cdots \quad c\vec{v} \quad \cdots]) = c\det([\cdots \quad \vec{v} \quad \cdots])$$

(b) Swapping two columns:

$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = -\det([\cdots \quad \vec{w} \quad \cdots \quad \vec{v} \quad \cdots])$$

(c) Adding a multiple of a column to another column:

$$\det([\cdots \quad \vec{v} \quad \cdots \quad \vec{w} \quad \cdots]) = \det([\cdots \quad \vec{v}+c\vec{w} \quad \cdots \quad \vec{w} \quad \cdots])$$

Remark G.23

The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row: $\begin{bmatrix}1&0&0&0\\0&c&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$
- Swapping rows: $\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$
- Adding a row multiple to another row: $\begin{bmatrix}1&0&0&0\\0&1&c&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$

Fact G.24

Thus we can also use row operations to simplify determinants:

1. Multiplying rows by scalars: $\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$
2. Swapping two rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$
3. Adding multiples of rows to other rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$

Observation G.25

So we may compute the determinant of $\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix}$ by manipulating its rows/columns to reduce the matrix to I:

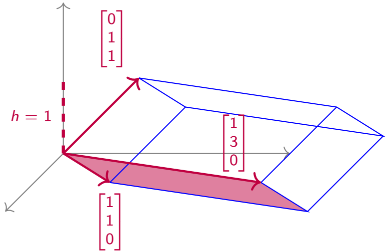
$$\begin{aligned}\det\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix} &= 2\det\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}\\ &= 2\det\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\\ &= -2\det\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}\\ &= -2\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\ &= -2\end{aligned}$$

Remark G.26

So we see that row reducing all the way into RREF gives us a method of computing determinants!

However, we learned in module E that this can be tedious for large matrices. Thus, we will try to figure out how to turn the determinant of a larger matrix into the determinant of a smaller matrix.

Activity G.27 *(~5 min)* The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.



Recall that for this solid $V = Bh$, where h is the height of the solid and B is the area of its parallelogram base. So what must its volume be?

(a) $\det \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

(b) $\det \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

(c) $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(d) $\det \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

Fact G.28

If row i contains all zeros except for a 1 on the main (upper-left to lower-right) diagonal, then both column and row i may be removed without changing the value of the determinant.

$$\det\begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det\begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Since row and column operations affect the determinants in the same way, the same technique works for a column of all zeros except for a 1 on the main diagonal.

$$\det\begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Activity G.29 *(~5 min)* Remove an appropriate row and column of $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$ to simplify the determinant to a 2×2 determinant.

Activity G.30 *(∼5 min)* Simplify $\det \begin{bmatrix} 0 & 3 & -2 \\ 2 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

- Factor out a 2 from a column.
- Swap rows or columns to put a 1 on the main diagonal.

Activity G.31 *(∼5 min)* Simplify $\det \begin{bmatrix} 4 & -2 & 2 \\ 3 & 1 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

- Use row/column operations to create two zeroes in the same row or column.
- Factor/swap as needed to get a row/column of all zeroes except a 1 on the main diagonal.

Observation G.32

Using row/column operations, you can introduce zeros and reduce dimension to whittle down the determinant of a large matrix to a determinant of a smaller matrix.

$$\det\begin{bmatrix} 4 & 3 & 0 & 1 \\ 2 & -2 & 4 & 0 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} = \det\begin{bmatrix} 4 & 3 & 0 & 1 \\ 6 & -18 & 0 & -20 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} = \det\begin{bmatrix} 4 & 3 & 1 \\ 6 & -18 & -20 \\ 2 & 8 & 3 \end{bmatrix}$$

$$= \cdots = -2\det\begin{bmatrix} 1 & 3 & 4 \\ 0 & 21 & 43 \\ 0 & -1 & -10 \end{bmatrix} = -2\det\begin{bmatrix} 21 & 43 \\ -1 & -10 \end{bmatrix}$$

$$= \cdots = -2\det\begin{bmatrix} -167 & 21 \\ 0 & 1 \end{bmatrix} = -2\det[-167]$$

$$= -2(-167)\det(I) = 334$$

Activity G.33 *(∼10 min)* Rewrite

$$\det \begin{bmatrix} 2 & 1 & -2 & 1 \\ 3 & 0 & 1 & 4 \\ -2 & 2 & 3 & 0 \\ -2 & 0 & -3 & -3 \end{bmatrix}$$

as a multiple of a determinant of a 3×3 matrix.

Activity G.34 *(∼20 min)* Compute $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$ by using any combination of row/column operations.

Observation G.35

Another option is to take advantage of the fact that the determinant is linear in each row or column. This approach is called **Laplace expansion** or **cofactor expansion**.

For example, since $\begin{bmatrix}1 & 2 & 4\end{bmatrix} = 1\begin{bmatrix}1 & 0 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 1 & 0\end{bmatrix} + 4\begin{bmatrix}0 & 0 & 1\end{bmatrix}$,

$$\begin{aligned}
\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 1 & 2 & 4\end{bmatrix} &= 1\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 1 & 0 & 0\end{bmatrix} + 2\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 0 & 1 & 0\end{bmatrix} + 4\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 0 & 0 & 1\end{bmatrix}\\
&= -1\det\begin{bmatrix}5 & 3 & 2\\ 5 & 3 & -1\\ 0 & 0 & 1\end{bmatrix} - 2\det\begin{bmatrix}2 & 5 & 3\\ -1 & 5 & 3\\ 0 & 0 & 1\end{bmatrix} + 4\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 0 & 0 & 1\end{bmatrix}\\
&= -\det\begin{bmatrix}5 & 3\\ 5 & 3\end{bmatrix} - 2\det\begin{bmatrix}2 & 5\\ -1 & 5\end{bmatrix} + 4\det\begin{bmatrix}2 & 3\\ -1 & 3\end{bmatrix}
\end{aligned}$$

Observation G.36

Applying Laplace expansion to a 2×2 matrix yields a short formula you may have seen:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} + b \det \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} - b \det \begin{bmatrix} 1 & 0 \\ d & c \end{bmatrix} = ad - bc.$$

There are formulas for the determinants of larger matrices, but they can be pretty tedious to use. For example, writing out a formula for a 4×4 determinant would require 24 different terms!

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}(a_{22}(a_{33}a_{44} - a_{43}a_{34}) - a_{23}(a_{32}a_{44} - a_{42}a_{34}) + \dots) + \dots$$

So this is why we either use Laplace expansion or row/column operations directly.

Activity G.37 *(∼5 min)* Based on what we've done today, which technique is easier for computing determinants?

(a) Memorizing formulas.

(b) Using row/column operations.

(c) Laplace expansion.

(d) Some other technique (be prepared to describe it).

Activity G.38 *(~10 min)* Use your preferred technique to compute

$$\det \begin{bmatrix} 4 & -3 & 0 & 0 \\ 1 & -3 & 2 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & -3 & 2 & -2 \end{bmatrix}.$$

Activity G.39 *(∼5 min)* An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Which of the following is equal to $\det(M)\det(M^{-1})$?

a) -1
b) 0
c) 1
d) 4

Fact G.40

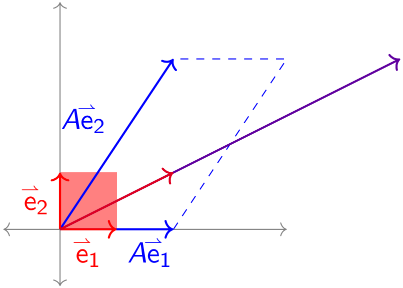
- For every invertible matrix M,

  $$\det(M)\det(M^{-1}) = \det(I) = 1$$

  so $\det(M^{-1}) = \frac{1}{\det(M)}$.
- Furthermore, a square matrix M is invertible if and only if $\det(M) \neq 0$.

Observation G.41

Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.



It is easy to see geometrically that

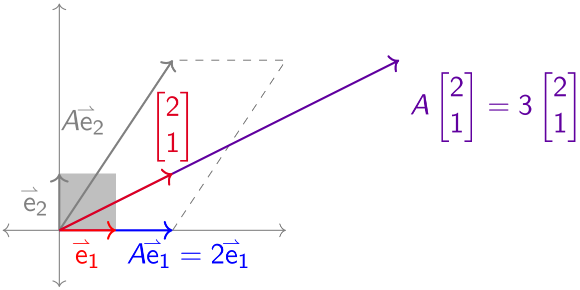
$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily checked once you find it) that

$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Definition G.42

Let $A \in M_{n,n}$. An **eigenvector** for A is a vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x}$ is parallel to $\vec{x}$.



In other words, $A\vec{x} = \lambda\vec{x}$ for some scalar λ. If $\vec{x} \neq \vec{0}$, then we say $\vec{x}$ is a **nontrivial eigenvector** and we call this λ an **eigenvalue** of A.

Activity G.43 *(~5 min)* Finding the eigenvalues λ that satisfy

$$A\vec{x} = \lambda\vec{x} = \lambda(I\vec{x}) = (\lambda I)\vec{x}$$

for some nontrivial eigenvector $\vec{x}$ is equivalent to finding nonzero solutions for the matrix equation

$$(A - \lambda I)\vec{x} = \vec{0}.$$

Which of the following must be true for any eigenvalue?

(a) The **kernel** of the transformation with standard matrix $A - \lambda I$ must contain **the zero vector**, so $A - \lambda I$ is **invertible**.

(b) The **kernel** of the transformation with standard matrix $A - \lambda I$ must contain **a non-zero vector**, so $A - \lambda I$ is **not invertible**.

(c) The **image** of the transformation with standard matrix $A - \lambda I$ must contain **the zero vector**, so $A - \lambda I$ is **invertible**.

(d) The **image** of the transformation with standard matrix $A - \lambda I$ must contain **a non-zero vector**, so $A - \lambda I$ is **not invertible**.

Fact G.44
The eigenvalues λ for a matrix A are the values that make $A - \lambda I$ non-invertible.

Thus the eigenvalues λ for a matrix A are the solutions to the equation

$$\det(A - \lambda I) = 0.$$

Definition G.45

The expression $\det(A - \lambda I)$ is called **characteristic polynomial** of A.

For example, when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix}$$

Thus the characteristic polynomial of A is

$$\det \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) - (2)(3) = \lambda^2 - 5\lambda - 2$$

and its eigenvalues are the solutions to $\lambda^2 - 5\lambda - 2 = 0$.

Activity G.46 *(~10 min)* Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$.

Activity G.46 *(∼10 min)* Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Activity G.46 *(∼10 min)* Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Set this characteristic polynomial equal to zero and factor to determine the eigenvalues of A.

Activity G.47 *(~5 min)* Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix}$.

Activity G.48 *(∼5 min)* Find all the eigenvalues for the matrix $A = \begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}$.

Activity G.49 *(∼10 min)* Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

Activity G.50 *(∼10 min)* It's possible to show that -2 is an eigenvalue for $\begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 9 \\ 3 & 0 & 4 \end{bmatrix}$.

Compute the kernel of the transformation with standard matrix

$$A-(-2)I=\begin{bmatrix} ? & 4 & -2 \\ 2 & ? & 9 \\ 3 & 0 & ? \end{bmatrix}$$

to find all the eigenvectors $\vec{x}$ such that $A\vec{x}=-2\vec{x}$.

Definition G.51

Since the kernel of a linear map is a subspace of $\mathbb{R}^n$, and the kernel obtained from $A - \lambda I$ contains all the eigenvectors associated with λ, we call this kernel the **eigenspace** of A associated with λ.

Activity G.52 *(∼10 min)* Find a basis for the eigenspace for the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ associated with the eigenvalue 3.

Activity G.53 *(∼10 min)* Find a basis for the eigenspace for the matrix $\begin{bmatrix} 5 & -2 & 0 & 4 \\ 6 & -2 & 1 & 5 \\ -2 & 1 & 2 & -3 \\ 4 & 5 & -3 & 6 \end{bmatrix}$ associated with the eigenvalue 1.

Activity G.54 *(∼10 min)* Find a basis for the eigenspace for the matrix $\begin{bmatrix}4 & 3 & 0 & 0\\3 & 3 & 0 & 0\\0 & 0 & 2 & 5\\0 & 0 & 0 & 2\end{bmatrix}$ associated with the eigenvalue 2.

Module P: Applications of Linear Algebra

Module P Section 1

Definition P.1

In geology, a **phase** is any physically separable material in the system, such as various minerals or liquids.

A **component** is a chemical compound necessary to make up the phases; these are usually oxides such as Calcium Oxide (CaO) or Silicone Dioxide $(\mathrm{SiO_2})$.

In a typical application, a geologist knows how to build each phase from the components, and is interested in determining reactions among the different phases.

Observation P.2

Consider the 3 components

$$\vec{c}_1 = \mathrm{CaO} \quad \vec{c}_2 = \mathrm{MgO} \quad \text{and } \vec{c}_3 = \mathrm{SiO_2}$$

and the 5 phases:

$$\vec{p}_1 = \mathrm{Ca_3MgSi_2O_8} \qquad \vec{p}_2 = \mathrm{CaMgSiO_4} \qquad \vec{p}_3 = \mathrm{CaSiO_3}$$
$$\vec{p}_4 = \mathrm{CaMgSi_2O_6} \qquad \vec{p}_5 = \mathrm{Ca_2MgSi_2O_7}$$

Geologists already know (or can easily deduce) that

$$\vec{p}_1 = 3\vec{c}_1 + \vec{c}_2 + 2\vec{c}_3 \qquad \vec{p}_2 = \vec{c}_1 + \vec{c}_2 + \vec{c}_3 \qquad \vec{p}_3 = \vec{c}_1 + 0\vec{c}_2 + \vec{c}_3$$
$$\vec{p}_4 = \vec{c}_1 + \vec{c}_2 + 2\vec{c}_3 \qquad \vec{p}_5 = 2\vec{c}_1 + \vec{c}_2 + 2\vec{c}_3$$

since, for example:

$$\vec{c}_1 + \vec{c}_3 = \mathrm{CaO} + \mathrm{SiO_2} = \mathrm{CaSiO_3} = \vec{p}_3$$

Activity P.3 *(∼5 min)* To study this vector space, each of the three components $\vec{c}_1, \vec{c}_2, \vec{c}_3$ may be considered as the three components of a Euclidean vector.

$$\vec{p}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{p}_5 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Determine if the set of phases is linearly dependent or linearly independent.

Activity P.4 *(~15 min)* Geologists are interested in knowing all the possible chemical reactions among the 5 phases:

$$\vec{p}_1 = \mathrm{Ca_3MgSi_2O_8} = \begin{bmatrix}3\\1\\2\end{bmatrix} \quad \vec{p}_2 = \mathrm{CaMgSiO_4} = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad \vec{p}_3 = \mathrm{CaSiO_3} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\vec{p}_4 = \mathrm{CaMgSi_2O_6} = \begin{bmatrix}1\\1\\2\end{bmatrix} \quad \vec{p}_5 = \mathrm{Ca_2MgSi_2O_7} = \begin{bmatrix}2\\1\\2\end{bmatrix}.$$

That is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\vec{p}_1 + x_2\vec{p}_2 + x_3\vec{p}_3 + x_4\vec{p}_4 + x_5\vec{p}_5 = 0.$$

Activity P.4 *(∼15 min)* Geologists are interested in knowing all the possible chemical reactions among the 5 phases:

$$\vec{p}_1 = \mathrm{Ca_3MgSi_2O_8} = \begin{bmatrix}3\\1\\2\end{bmatrix} \quad \vec{p}_2 = \mathrm{CaMgSiO_4} = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad \vec{p}_3 = \mathrm{CaSiO_3} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\vec{p}_4 = \mathrm{CaMgSi_2O_6} = \begin{bmatrix}1\\1\\2\end{bmatrix} \quad \vec{p}_5 = \mathrm{Ca_2MgSi_2O_7} = \begin{bmatrix}2\\1\\2\end{bmatrix}.$$

That is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\vec{p}_1 + x_2\vec{p}_2 + x_3\vec{p}_3 + x_4\vec{p}_4 + x_5\vec{p}_5 = 0.$$

Part 1: Set up a system of equations equivalent to this vector equation.

Activity P.4 *(~15 min)* Geologists are interested in knowing all the possible chemical reactions among the 5 phases:

$$\vec{p}_1 = \mathrm{Ca_3MgSi_2O_8} = \begin{bmatrix}3\\1\\2\end{bmatrix} \quad \vec{p}_2 = \mathrm{CaMgSiO_4} = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad \vec{p}_3 = \mathrm{CaSiO_3} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\vec{p}_4 = \mathrm{CaMgSi_2O_6} = \begin{bmatrix}1\\1\\2\end{bmatrix} \quad \vec{p}_5 = \mathrm{Ca_2MgSi_2O_7} = \begin{bmatrix}2\\1\\2\end{bmatrix}.$$

That is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\vec{p}_1 + x_2\vec{p}_2 + x_3\vec{p}_3 + x_4\vec{p}_4 + x_5\vec{p}_5 = 0.$$

Part 1: Set up a system of equations equivalent to this vector equation.
Part 2: Find a basis for its solution space.

Activity P.4 *(∼15 min)* Geologists are interested in knowing all the possible chemical reactions among the 5 phases:

$$\vec{p}_1 = \mathrm{Ca_3MgSi_2O_8} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \vec{p}_2 = \mathrm{CaMgSiO_4} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{p}_3 = \mathrm{CaSiO_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{p}_4 = \mathrm{CaMgSi_2O_6} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \vec{p}_5 = \mathrm{Ca_2MgSi_2O_7} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

That is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\vec{p}_1 + x_2\vec{p}_2 + x_3\vec{p}_3 + x_4\vec{p}_4 + x_5\vec{p}_5 = 0.$$

Part 1: Set up a system of equations equivalent to this vector equation.
Part 2: Find a basis for its solution space.
Part 3: Interpret each basis vector as a vector equation and a chemical equation.

Activity P.5 *(∼10 min)* We found two basis vectors $\begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, corresponding to the vector and chemical equations

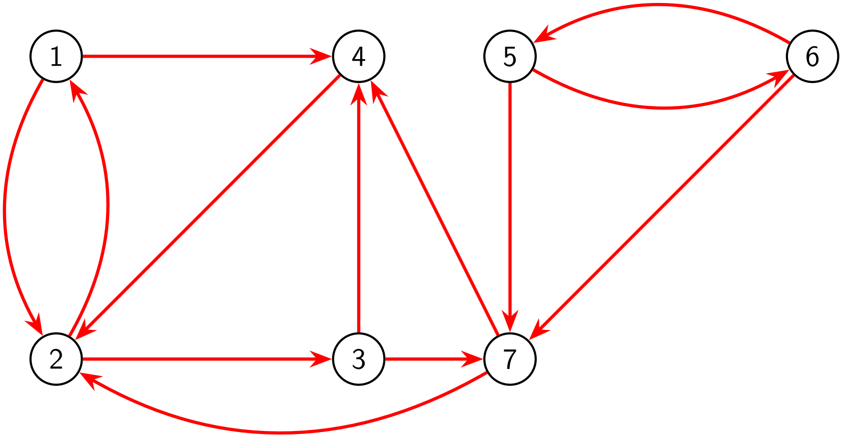
$$\begin{array}{ll} 2\vec{p}_2 + 2\vec{p}_3 = \vec{p}_1 + \vec{p}_4 & 2\mathrm{CaMgSiO_4} + 2\mathrm{CaSiO_3} = \mathrm{Ca_3MgSi_2O_8} + \mathrm{CaMgSi_2O_6} \\ \vec{p}_2 + \vec{p}_3 = \vec{p}_5 & \mathrm{CaMgSiO_4} + \mathrm{CaSiO_3} = \mathrm{Ca_2MgSi_2O_7} \end{array}$$

Combine the basis vectors to produce a chemical equation among the five phases that does not involve $\vec{p}_2 = \mathrm{CaMgSiO_4}$.

Module P Section 2

Activity P.6 *(∼10 min)*
A $700,000,000,000 Problem:

In the picture below, each circle represents a webpage, and each arrow represents a link from one page to another.



Based on how these pages link to each other, write a list of the 7 webpages in order from most imptorant to least important.

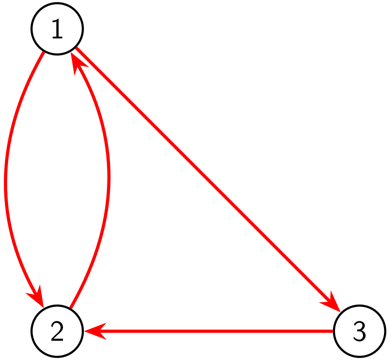
Observation P.7
The $700,000,000,000 Idea:

Links are endorsements.

1. A webpage is important if it is linked to (endorsed) by important pages.
2. A webpage distributes its importance equally among all the pages it links to (endorses).

Example P.8

Consider this small network with only three pages. Let x_1, x_2, x_3 be the importance of the three pages respectively.

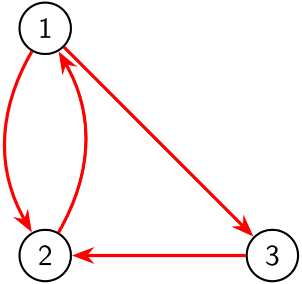


1. x_1 splits its endorsement in half between x_2 and x_3
2. x_2 sends all of its endorsement to x_1
3. x_3 sends all of its endorsement to x_2.

This corresponds to the **page rank system**

$$\begin{alignedat}{3} & x_2 & & = x_1 \\ \frac{1}{2}x_1 & & +x_3 & = x_2 \\ \frac{1}{2}x_1 & & & = x_3 \end{alignedat}$$

Observation P.9



$$\begin{array}{rcrcl} & x_2 & & = & x_1 \\ \frac{1}{2}x_1 & & +x_3 & = & x_2 \\ \frac{1}{2}x_1 & & & = & x_3 \end{array}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

By writing this linear system in terms of matrix multiplication, we obtain the **page rank matrix** $A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$ and page rank vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Thus, computing the importance of pages on a network is equivalent to solving the matrix equation $A\vec{x} = 1\vec{x}$.

Activity P.10 *(∼5 min)* Thus, our \$700,000,000,000 problem is what kind of problem?

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) An antiderivative problem

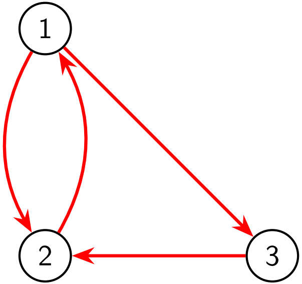
(b) A bijection problem

(c) A cofactoring problem

(d) A determinant problem

(e) An eigenvector problem

Activity P.11 *(∼10 min)* Find a page rank vector $\vec{x}$ satisfying $A\vec{x} = 1\vec{x}$ for the following network's page rank matrix A.
That is, find the eigenspace associated with $\lambda = 1$ for the matrix A, and choose a vector from that eigenspace.

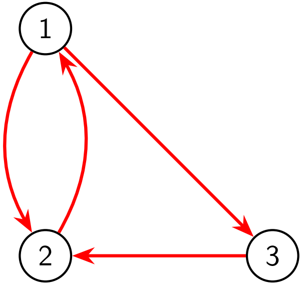


$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

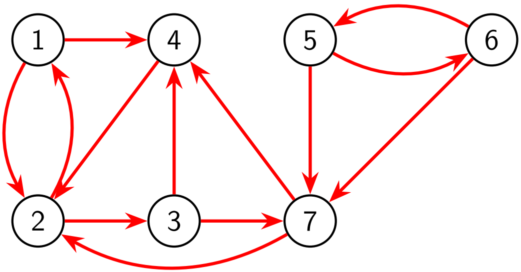
Observation P.12

Row-reducing $A - I = \begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ yields the basic eigenvector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Therefore, we may conclude that pages 1 and 2 are equally important, and both pages are twice as important as page 3.

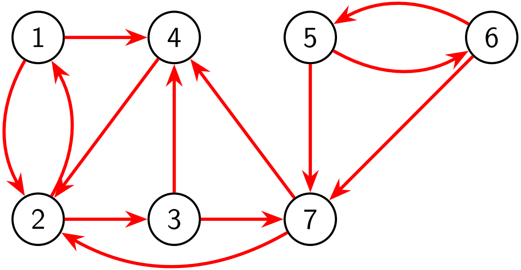


Activity P.13 *(∼5 min)* Compute the 7×7 page rank matrix for the following network.



For example, since website 1 distributes its endorsement equally between 2 and 4, the first column is $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity P.14 *(∼10 min)* Find a page rank vector for the given page rank matrix.

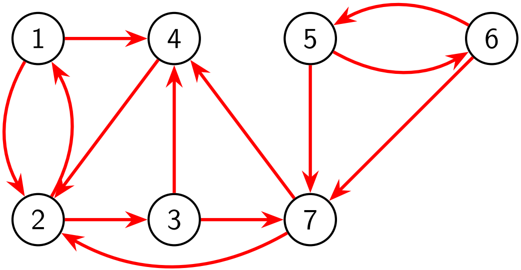


$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Which webpage is most important?

Observation P.15

Since a page rank vector for the network is given by $\vec{x}$, it's reasonable to consider page 2 as the most important page.

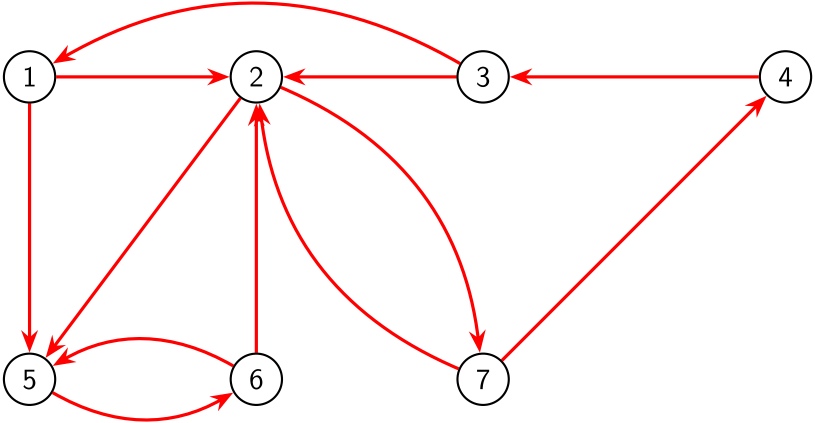


$$\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Based upon this page rank vector, here is a complete ranking of all seven pages from most important to least important:

$$2, 4, 1, 3, 7, 5, 6$$

Activity P.16 *(∼10 min)* Given the following diagram, use a page rank vector to rank the pages 1 through 7 in order from most important to least important.

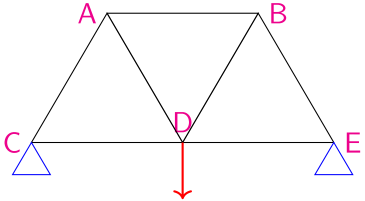


Module P Section 3

Example P.17

In engineering, a **truss** is a structure designed from several beams of material called **struts**, assembled to behave as a single object.

Activity P.18 *(∼5 min)* Consider the representation of a simple truss pictured below. All of the seven struts are of equal length, affixed to two anchor points applying a normal force to nodes C and E, and with a $10000N$ load applied to the node given by D.



Which of the following must hold for the truss to be stable?

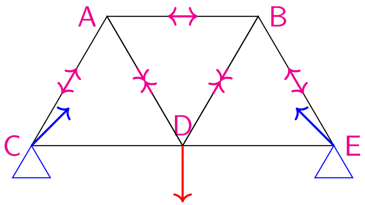
a) All of the struts will experience compression.

b) All of the struts will experience tension.

c) Some of the struts will be compressed, but others will be tensioned.

Observation P.19

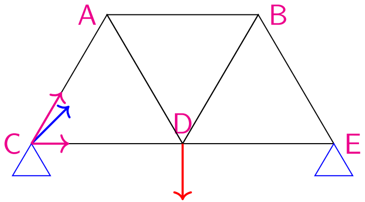
Since the forces must balance at each node for the truss to be stable, some of the struts will be compressed, while others will be tensioned.



By finding vector equations that must hold at each node, we may determine many of the forces at play.

Remark P.20

For example, at the bottom left node there are 3 forces acting.

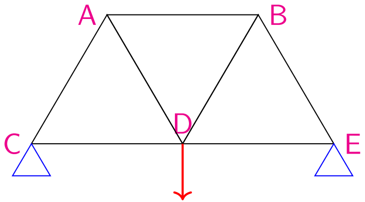


Let $\vec{F}_{CA}$ be the force on C given by the compression/tension of the strut CA, let $\vec{F}_{CD}$ be defined similarly, and let $\vec{N}_C$ be the normal force of the anchor point on C.

For the truss to be stable, we must have

$$\vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}.$$

Activity P.21 *(∼10 min)* Using the conventions of the previous slide, and where $\vec{L}$ represents the load vector on node D, find four more vector equations that must be satisfied for each of the other four nodes of the truss.



$$A : ?$$

$$B : ?$$

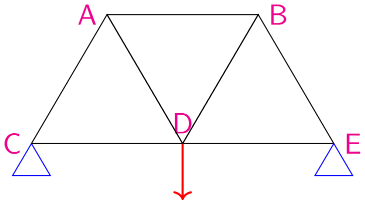
$$C : \vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}$$

$$D : ?$$

$$E : ?$$

Remark P.22

The five vector equations may be written as follows.



$$A : \vec{F}_{AC} + \vec{F}_{AD} + \vec{F}_{AB} = \vec{0}$$

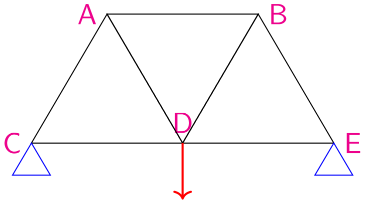
$$B : \vec{F}_{BA} + \vec{F}_{BD} + \vec{F}_{BE} = \vec{0}$$

$$C : \vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_{C} = \vec{0}$$

$$D : \vec{F}_{DC} + \vec{F}_{DA} + \vec{F}_{DB} + \vec{F}_{DE} + \vec{L} = \vec{0}$$

$$E : \vec{F}_{EB} + \vec{F}_{ED} + \vec{N}_{E} = \vec{0}$$

Observation P.23

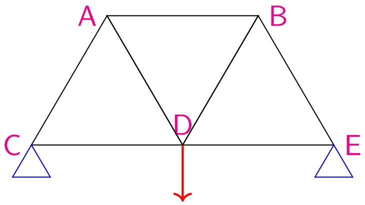


Each vector has a vertical and horizontal component, so it may be treated as a vector in $\mathbb{R}^2$. Note that $\vec{F}_{CA}$ must have the same magnitude (but opposite direction) as $\vec{F}_{AC}$.

$$\vec{F}_{CA} = x \begin{bmatrix} \cos(60^\circ) \\ \sin(60^\circ) \end{bmatrix} = x \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

$$\vec{F}_{AC} = x \begin{bmatrix} \cos(-120^\circ) \\ \sin(-120^\circ) \end{bmatrix} = x \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

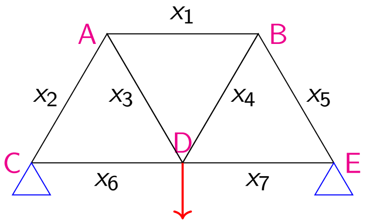
Activity P.24 *(∼5 min)* To write a linear system that models the truss under consideration with constant load 10000 newtons, how many scalar variables will be required?



a) 7: 5 from the nodes, 2 from the anchors
b) 9: 7 from the struts, 2 from the anchors
c) 11: 7 from the struts, 4 from the anchors
d) 12: 7 from the struts, 4 from the anchors, 1 from the load
e) 13: 5 from the nodes, 7 from the struts, 1 from the load

Observation P.25

Since the angles for each strut are known, one variable may be used to represent each.



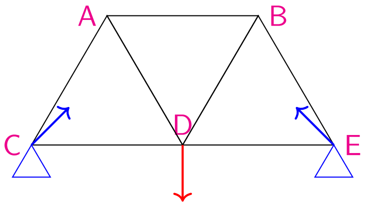
For example:

$$\vec{F}_{AB} = -\vec{F}_{BA} = x_1 \begin{bmatrix} \cos(0) \\ \sin(0) \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{F}_{BE} = -\vec{F}_{EB} = x_5 \begin{bmatrix} \cos(-60^\circ) \\ \sin(-60^\circ) \end{bmatrix} = x_5 \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

Observation P.26

Since the angle of the normal forces for each anchor point are unknown, two variables may be used to represent each.



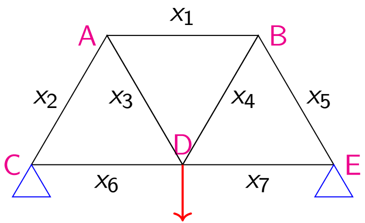
$$\vec{\mathrm{N}}_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \vec{\mathrm{N}}_D = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

The load vector is constant.

$$\vec{\mathrm{L}} = \begin{bmatrix} 0 \\ -10000 \end{bmatrix}$$

Remark P.27

Each of the five vector equations found previously represent two linear equations: one for the horizontal component and one for the vertical.



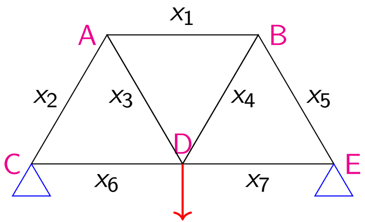
$$C : \vec{F}_{CA} + \vec{F}_{CD} + \vec{N}_C = \vec{0}$$

$$\Leftrightarrow x_2 \begin{bmatrix} \cos(60^\circ) \\ \sin(60^\circ) \end{bmatrix} + x_6 \begin{bmatrix} \cos(0^\circ) \\ \sin(0^\circ) \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using the approximation $\sqrt{3}/2 \approx 0.866$, we have

$$\Leftrightarrow x_2 \begin{bmatrix} 0.5 \\ 0.866 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Activity P.28 *(∼10 min)* Expand the vector equation given below using sine and cosine of appropriate angles, then compute each component (approximating $\sqrt{3}/2 \approx 0.866$).



$$D : \vec{F}_{DA} + \vec{F}_{DB} + \vec{F}_{DC} + \vec{F}_{DE} = -\vec{L}$$

$$\Leftrightarrow x_3 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} + x_4 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} + x_6 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} + x_7 \begin{bmatrix} \cos(\,?\,) \\ \sin(\,?\,) \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$\Leftrightarrow x_3 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_4 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_6 \begin{bmatrix} ? \\ ? \end{bmatrix} + x_7 \begin{bmatrix} ? \\ ? \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

Observation P.29

The full augmented matrix given by the ten equations in this linear system is given below, where the elevent columns correspond to $x_1, \ldots, x_7, y_1, y_2, z_1, z_2$, and the ten rows correspond to the horizontal and vertical components of the forces acting at $A, \ldots, E$.

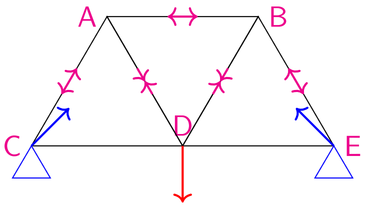
$$\left[\begin{array}{ccccccccccc|c}
1 & -0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.866 & -0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.866 & -0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -0.5 & 0.5 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.866 & 0.866 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 \\
0 & 0 & 0 & 0 & -0.5 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.866 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]$$

Observation P.30

This matrix row-reduces to the following.

$$\sim \left[\begin{array}{ccccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5773.7 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5773.7 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5773.7 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5773.7 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -5773.7 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 2886.8 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 2886.8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5000
\end{array}\right]$$

Observation P.31



Thus we know the truss must satisfy the following conditions.

$$
\begin{aligned}
x_1 = x_2 = x_5 &= -5882.4 \\
x_3 = x_4 &= 5882.4 \\
x_6 = x_7 &= 2886.8 + z_1 \\
y_1 &= -z_1 \\
y_2 = z_2 &= 5000
\end{aligned}
$$

In particular, the negative x_1, x_2, x_5 represent tension (forces pointing into the nodes), and the postive x_3, x_4 represent compression (forces pointing out of the nodes). The vertical normal forces $y_2 + z_2$ counteract the 10000 load.

Module P Section 4

Definition P.32

Cryptography is the practice and study of encoding messages so that only the intended receiver can decode them.

For example, the ROT13 cipher both encodes and decodes messages by shifting each letter thirteen places in the alphabet, cycling from Z back to A. This may be accomplished by converting each letter to a number

$$\mathtt{A} \equiv 1, \mathtt{B} \equiv 2, \dots, \mathtt{Y} \equiv 25, \mathtt{Z} \equiv 0$$

and adding 13 (modulo 26):

$$\mathtt{HELLO} \equiv \begin{bmatrix} 8 \\ 5 \\ 12 \\ 12 \\ 15 \end{bmatrix} \underset{ROT13}{\leftrightarrow} \begin{bmatrix} 21 \\ 18 \\ 25 \\ 25 \\ 2 \end{bmatrix} \equiv \mathtt{URYYB}$$

Activity P.33 *(∼10 min)* Suppose your instructor saw another student passing a note that said

`MFUT DIFBU PO UIF UFTU`

How could the instructor decode this message, taking advantage of the fact that `THE` is one of the most commonly used words in the English language?

Observation P.34

Frequency analysis is a common tool used in breaking **substitution ciphers** that simply substitute letters for other letters. In the message

`MFUT DIFBU PO UIF UFTU`

the common word `THE` is encoded as `UIF`, and the most common letters in the English language `E,T` match the most common letters used in this message: `F,U`.

This suggests the following partial decryption:

`-ET- -HE-T -- THE TE-T`

By considering the context, or the fact that all letters were shifted the same amount, or perhaps by an analysis of other messages sent using the same code, the completed message may be revealed:

`LETS CHEAT ON THE TEST`

Remark P.35

To defeat naive frequency analysis attacks, one method that may be used is to create a rule that converts groups of letters into new groups of letters, rather than converting single letters individually.

So to send the message

$$\texttt{LETS CHEAT ON THE TEST}$$

one might first break it into three-letter pieces.

$$\texttt{LET SCH EAT ONT HET EST}$$

Remark P.36

Each piece then may be converted to a Euclidean vector in $\mathbb{R}^3$, which may be linearly transformed by multiplying by a matrix A with $\det(A) = 1 = \det(A^{-1})$.

For $A = \begin{bmatrix} 3 & -2 & -3 \\ -2 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}$:

$$\mathrm{LET} \equiv \begin{bmatrix} 12 \\ 5 \\ 20 \end{bmatrix} \to \begin{bmatrix} 3 & -2 & -3 \\ -2 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 12 \\ 5 \\ 20 \end{bmatrix} = \begin{bmatrix} -34 \\ -9 \\ 28 \end{bmatrix}$$

Remark P.37

The resulting vector may be converted back into English letters by adding multiples of 26 to each component to obtain numbers between 0 and 25.

$$\begin{bmatrix} -34 \\ -9 \\ 28 \end{bmatrix} \equiv \begin{bmatrix} -34 + 52 \\ -9 + 26 \\ 28 - 26 \end{bmatrix} = \begin{bmatrix} 18 \\ 17 \\ 2 \end{bmatrix} \equiv \text{RPB}$$

Observation P.38

This process may be done all at once by converting the entire message into a matrix:

$$\texttt{LET SCH} \;\ldots \equiv \begin{bmatrix} 12 & 19 & \\ 5 & 3 & \ldots \\ 20 & 8 & \end{bmatrix}$$

$$\to \begin{bmatrix} 3 & -2 & -3 \\ -2 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 12 & 19 & \\ 5 & 3 & \ldots \\ 20 & 8 & \end{bmatrix} = \begin{bmatrix} -34 & 27 & \\ -9 & -29 & \ldots \\ 28 & -3 & \end{bmatrix}$$

$$\equiv \begin{bmatrix} 18 & 1 & \\ 17 & 23 & \ldots \\ 2 & 23 & \end{bmatrix} \equiv \texttt{RQB AVV} \;\ldots$$

Activity P.39 *(∼10 min)* Complete the following encoding of the entire message given below, using the encoding matrix $A = \begin{bmatrix} 3 & -2 & -3 \\ -2 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

$$\texttt{LET SCH EAT ONT HET EST} \equiv \begin{bmatrix} 12 & 19 & \\ 5 & 3 & \dots \\ 20 & 8 & \end{bmatrix}$$

$$\to \begin{bmatrix} 3 & -2 & -3 \\ -2 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 12 & 19 & \\ 5 & 3 & \dots \\ 20 & 8 & \end{bmatrix} = \begin{bmatrix} -34 & 27 & \\ -9 & -29 & \dots \\ 28 & -3 & \end{bmatrix}$$

$$\equiv \begin{bmatrix} 18 & 1 & \\ 17 & 23 & \dots \\ 2 & 23 & \end{bmatrix} \equiv \texttt{RQB AWW ESI ILY FYF UUI}$$

Activity P.40 *(∼10 min)* Reverse this process by using the decoding matrix, $A^{-1} = \begin{bmatrix} 6 & 4 & 9 \\ 4 & 3 & 6 \\ 3 & 2 & 5 \end{bmatrix}$.

$$\texttt{RQB AWW ESI ILY FYF UUI} \equiv \begin{bmatrix} 18 & 1 & \\ 17 & 23 & \dots \\ 2 & 23 & \end{bmatrix}$$

$$\to \begin{bmatrix} 6 & 4 & 9 \\ 4 & 3 & 6 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 18 & 1 & \\ 17 & 23 & \dots \\ 2 & 23 & \end{bmatrix} = \begin{bmatrix} 194 & 305 & \\ 135 & 211 & \dots \\ 98 & 164 & \end{bmatrix}$$

$$\equiv \begin{bmatrix} 12 & 19 & \\ 5 & 3 & \dots \\ 20 & 8 & \end{bmatrix} \equiv \texttt{LET SCH EAT ONT HET EST}$$