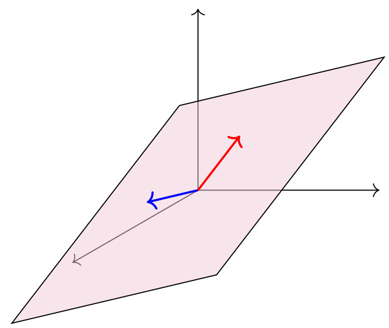
Section V.4

Definition V.4.1 A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space $\mathbb{R}^3$.



Fact V.4.2 Any sub**set** S of a vector space V that contains the additive identity $\vec{\mathbf{0}}$ satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a sub**space**, we need to check that addition and multiplication still make sense using only vectors from S. So we need to check two things:

- The set is **closed under addition**: for any $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in S$, the sum $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ is also in S.
- The set is **closed under scalar multiplication**: for any $\vec{\mathbf{x}} \in S$ and scalar $c \in \mathbb{R}$, the product $c\vec{\mathbf{x}}$ is also in S.

Activity V.4.3 *(~15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Part 1: Let $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\vec{\mathbf{v}} + \vec{\mathbf{w}} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Part 2: Let $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\vec{\mathbf{v}} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ also belongs to S for any $c \in \mathbb{R}$ by verifying an appropriate equation.

Part 3: Is S is a subspace of $\mathbb{R}^3$?

Activity V.4.4 *(~10 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 4 \right\}$. Choose a vector $\vec{\mathbf{v}} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = \,?$, and show that $c\vec{\mathbf{v}}$ isn't in S. Is S a subspace of $\mathbb{R}^3$?

Remark V.4.5 Since 0 is a scalar and $0\vec{\mathbf{v}} = \vec{\mathbf{z}}$ for any vector $\vec{\mathbf{v}}$, a nonempty set that is closed under scalar multiplication must contain the zero vector $\vec{\mathbf{z}}$ for that vector space.

Put another way, you can check any of the following to show that a nonempty subset W isn't a subspace:

- Show that $\vec{\mathbf{0}} \notin W$.
- Find $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in W$ such that $\vec{\mathbf{u}} + \vec{\mathbf{v}} \notin W$.
- Find $c \in \mathbb{R}, \vec{\mathbf{v}} \in W$ such that $c\vec{\mathbf{v}} \notin W$.

If you cannot do any of these, then W can be proven to be a subspace by doing the following:

- Prove that $\vec{\mathbf{u}} + \vec{\mathbf{v}} \in W$ whenever $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in W$.
- Prove that $c\vec{\mathbf{v}} \in W$ whenever $c \in \mathbb{R}, \vec{\mathbf{v}} \in W$.

Activity V.4.6 *(∼20 min)* Consider these subsets of $\mathbb{R}^4$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, z = xy \right\}$$

Part 1: Show R isn't a subspace by showing that $\vec{\mathbf{0}} \notin R$.
Part 2: Show S isn't a subspace by finding two vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in S$ such that $\vec{\mathbf{u}} + \vec{\mathbf{v}} \notin S$.
Part 3: Show T isn't a subspace by finding a vector $\vec{\mathbf{v}} \in T$ such that $2\vec{\mathbf{v}} \notin T$.

Activity V.4.7 *(∼5 min)* Let W be a subspace of a vector space V. How are $\operatorname{span} W$ and W related?

(a) $\operatorname{span} W$ is bigger than W

(b) $\operatorname{span} W$ is the same as W

(c) $\operatorname{span} W$ is smaller than W

Fact V.4.8 If S is any subset of a vector space V, then since $\operatorname{span} S$ collects all possible linear combinations, $\operatorname{span} S$ is automatically a subspace of V.

In fact, $\operatorname{span} S$ is always the smallest subspace of V that contains all the vectors in S.