

# Linear Algebra

University of South Alabama

Fall 2017

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

Module M

Module G

# Module E: Solving Systems of Linear Equations

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

At the end of this module, students will be able to...

- **E1: Systems as matrices.** Translate back and forth between a system of linear equations and the corresponding augmented matrix.
- **E2: Row reduction.** Put a matrix in reduced row echelon form
- **E3: Solving Linear Systems.** Solve a system of linear equations.
- **E4: Homogeneous Systems.** Find a basis for the solution set of a homogeneous linear system.

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-systems-topic/cc-8th-systems-graphically/a/systems-of-equations-with-graphing>
- <https://www.khanacademy.org/math/algebra/systems-of-linear-equations/solving-systems-of-equations-with-substitution/v/practice-using-substitution-for-systems>

Module E

**Part 1 (Day 3)**

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

# Application Activities - Module E Part 1 - Class Day 3

## Definition 3.1

A **linear equation** is an equation of the variables  $x_i$  of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

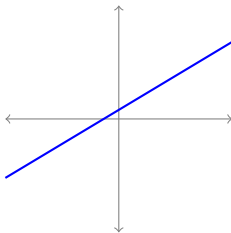
## Module A

## Module M

## Module G

**Observation 3.2**

The linear equation  $3x - 5y = -2$  may be graphed as a line in the  $xy$  plane.



The linear equation  $x + 2y - z = 4$  may be graphed as a plane in  $xyz$  space.



## Module E

## Part 1 (Day 3)

## Part 2 (Day 4)

## Part 3 (Day 5)

## Module V

## Part 1 (Day 7)

## Part 2 (Day 8)

## Part 3 (Day 9)

## Part 4 (Day 10)

## Module S

## Part 1 (Day 12)

## Part 2 (Day 13)

## Part 3 (Day 14)

## Module A

## Module M

## Module G

## Remark 3.3

In previous classes you likely assumed  $x = x_1$ ,  $y = x_2$ , and  $z = x_3$ . However, since this course often deals with equations of four or more variables, we will almost always write our variables as  $x_i$ .

## Definition 3.4

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

## A solution

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \dots + a_{in}s_n = b_i$$

for  $1 \leq i \leq m$  (that is, the solution satisfies all equations in the system).

## Remark 3.5

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Verbose standard form:

$$\begin{aligned}x_1 + 0x_2 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\0x_1 - x_2 + x_3 &= -2\end{aligned}$$

Concise standard form:

$$\begin{aligned}x_1 \quad \quad + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

## Module E

## Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Definition 3.6

A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

# Fact 3.7

All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

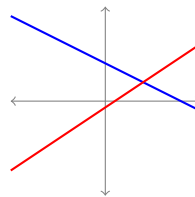
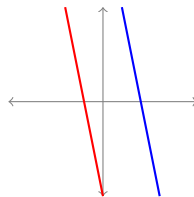
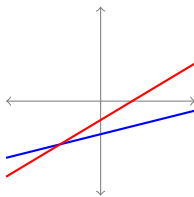
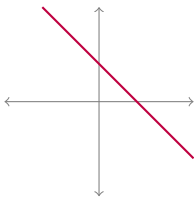
## Module A

## Module M

## Module G

## Activity 3.8

Consider the following graphs representing linear systems of two variables. Label each graph with **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 3.9

All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system.

$$-x_1 + 2x_2 = 5$$

$$2x_1 - 4x_2 = 6$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$



## Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

*Part 1:* Find three different solutions  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ ,  $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ ,  $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$  for this system.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

*Part 1:* Find three different solutions  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ ,  $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ ,  $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$  for this system.

*Part 2:* Let  $x_2 = a$  where  $a$  is an arbitrary real number, then find an expression for  $x_1$  in terms of  $a$ . Use this to describe *all* solutions (the **solution set**)  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$  for the linear system in terms of  $a$ .

## Activity 3.11

Consider the following linear system.

$$x_1 + 2x_2 - x_4 = 3$$

$$x_3 + 4x_4 = -2$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix} + a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$$

to the linear system by setting  $x_2 = a$  and  $x_4 = b$ , and then solving for  $x_1$  and  $x_3$ .

## Observation 3.12

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.

## Remark 3.13

The only important information in a linear system are its coefficients and constants.

Original linear system:

$$x_1 + 3x_3 = 3$$

$$3x_1 - 2x_2 + 4x_3 = 0$$

$$-x_2 + x_3 = -2$$

Verbose standard form:

$$x_1 + 0x_2 + 3x_3 = 3$$

$$3x_1 - 2x_2 + 4x_3 = 0$$

$$0x_1 - x_2 + x_3 = -2$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \end{array}$$

$$\begin{array}{ccc|c} 3 & -2 & 4 & 0 \end{array}$$

$$\begin{array}{ccc|c} 0 & 1 & 1 & -2 \end{array}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Definition 3.14

A system of  $m$  linear equations with  $n$  variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

## Definition 3.15

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution:  $(x_1, x_2) = (1, 1)$ .

$$3x_1 - 2x_2 = 1$$

$$x_1 + 4x_2 = 5$$

$$3x_1 - 2x_2 = 1$$

$$4x_1 + 2x_2 = 6$$

Therefore these augmented matrices are equivalent:

$$\left[ \begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array} \right]$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 3.16

Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

- a) Swap two rows.
- b) Swap two columns.
- c) Add a constant to every term in a row.
- d) Multiply a row by a nonzero constant.
- e) Add a constant multiple of one row to another row.
- f) Replace a column with zeros.



Module E

Part 1 (Day 3)

**Part 2 (Day 4)**

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

# Application Activities - Module E Part 2 - Class Day 4

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Definition 4.1

The following **row operations** produce equivalent augmented matrices:

- 1 Swap two rows.
- 2 Multiply a row by a nonzero constant.
- 3 Add a constant multiple of one row to another row.

Whenever two matrices  $A, B$  are equivalent (so whenever we do any of these operations), we write  $A \sim B$ .

## Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

### Module E

Part 1 (Day 3)

**Part 2 (Day 4)**

Part 3 (Day 5)

### Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

### Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

### Module A

### Module M

### Module G

## Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$x_1 - x_2 + 5x_3 = 1$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$x_2 - 2x_3 = 3$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_3 = 2$$

*Part 1:* Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

- ① Swap  $R_1$  (first row) and  $R_2$  (second row).
- ② Multiply  $R_2$  by  $\frac{1}{2}$ .
- ③ Add  $R_1$  to  $R_3$ .
- ④ Add  $-3R_1$  to  $R_2$ .
- ⑤ Add  $-2R_2$  to  $R_3$ .
- ⑥ Multiply  $R_3$  by  $\frac{1}{3}$ .

## Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$x_1 - x_2 + 5x_3 = 1$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$x_2 - 2x_3 = 3$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_3 = 2$$

*Part 1:* Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

- ① Swap  $R_1$  (first row) and  $R_2$  (second row).
- ② Multiply  $R_2$  by  $\frac{1}{2}$ .
- ③ Add  $R_1$  to  $R_3$ .
- ④ Add  $-3R_1$  to  $R_2$ .
- ⑤ Add  $-2R_2$  to  $R_3$ .
- ⑥ Multiply  $R_3$  by  $\frac{1}{3}$ .

*Part 2:* What is the common solution to these linear systems?

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Definition 4.3

The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix. Examples:

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

## Activity 4.4

Find your own sequence of row operations to manipulate the matrix

$$\left[ \begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array} \right]$$

into row echelon form. (Note that row echelon form is not unique.)

The most efficient way to do this is by circling **pivot positions** in your matrix:

- 1 Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
- 2 Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
- 3 Repeat these two steps as often as possible.

## Module E

Part 1 (Day 3)

**Part 2 (Day 4)**

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 4.5

Solve this simplified linear system:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Observation 4.6**

The concise standard form of the solution to this linear system corresponds to a simplified row echelon form matrix:

$$x_1 = -2$$

$$x_2 = 7$$

$$x_3 = 2$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Definition 4.7**

A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0. Examples:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 4.8

Show that the following two linear systems:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

$$x_1 = -2$$

$$x_2 = 7$$

$$x_3 = 2$$

are equivalent by converting the first system to an augmented matrix, and then zeroing out all terms above pivot positions (the leading terms).

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Remark 4.9**

We may verify that  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$  is a solution to the original linear system

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

by plugging the solution into each equation.

## Module E

Part 1 (Day 3)

**Part 2 (Day 4)**

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Fact 4.10**

Every augmented matrix  $A$  reduces to a unique reduced row echelon form matrix. This matrix is denoted as  $\text{RREF}(A)$ .

## Module E

Part 1 (Day 3)

**Part 2 (Day 4)**

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 4.11**

Consider the following matrix.

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

## Module E

Part 1 (Day 3)

**Part 2 (Day 4)**

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 4.11**

Consider the following matrix.

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

*Part 1:* Find  $\text{RREF}(A)$ .

## Module E

Part 1 (Day 3)

**Part 2 (Day 4)**

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 4.11**

Consider the following matrix.

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

*Part 1:* Find  $\text{RREF}(A)$ .

*Part 2:* How many solutions does the corresponding linear system have?



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)**

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

Module M

Module G

# Application Activities - Module E Part 3 - Class Day 5

## Definition 5.1

An algorithm that reduces  $A$  to  $\text{RREF}(A)$  is called **Gauss-Jordan elimination**. For example:

- 1 Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
- 2 Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
- 3 Repeat these two steps as often as possible.
- 4 Finally, zero out any terms above pivot positions.

**Observation 5.2**

Here is an example of applying Gauss-Jordan elimination to a matrix:

$$\begin{aligned}
 & \left[ \begin{array}{ccc|c} \textcircled{3} & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{2} & -2 & 10 & 2 \\ 3 & -2 & 13 & 6 \\ -1 & 3 & -6 & 11 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{1} & -1 & 5 & 1 \\ 3 & -2 & 13 & 6 \\ -1 & 3 & -6 & 11 \end{array} \right] \\
 & \sim \left[ \begin{array}{ccc|c} \textcircled{1} & -1 & 5 & 1 \\ 0 & \textcircled{1} & -2 & 3 \\ 0 & 2 & -1 & 12 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{1} & -1 & 5 & 1 \\ 0 & \textcircled{1} & -2 & 3 \\ 0 & 0 & \textcircled{3} & 6 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{1} & -1 & 5 & 1 \\ 0 & \textcircled{1} & -2 & 3 \\ 0 & 0 & \textcircled{1} & 2 \end{array} \right] \\
 & \sim \left[ \begin{array}{ccc|c} \textcircled{1} & -1 & 5 & 1 \\ 0 & \textcircled{1} & -2 & 3 \\ 0 & 0 & \textcircled{1} & 2 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{1} & -1 & 0 & -9 \\ 0 & \textcircled{1} & 0 & 7 \\ 0 & 0 & \textcircled{1} & 2 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{1} & 0 & 0 & -2 \\ 0 & \textcircled{1} & 0 & 7 \\ 0 & 0 & \textcircled{1} & 2 \end{array} \right]
 \end{aligned}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 5.3**Find  $\text{RREF}(A)$  where

$$A = \left[ \begin{array}{cccc|c} -1 & 1 & -3 & 2 & 0 \\ 2 & -1 & 5 & 3 & -11 \\ 3 & 2 & 4 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{array} \right].$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Definition 5.4

The columns of  $\text{RREF}(A)$  without a leading term represent **free variables** of the linear system modeled by  $A$  that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by  $A$ .

**Activity 5.5**

Given the linear system and its equivalent row-reduced matrix

$$-x_1 + x_2 - 3x_3 + 2x_4 = 0$$

$$2x_1 - x_2 + 5x_3 + 3x_4 = -11$$

$$3x_1 + 2x_2 + 4x_3 + x_4 = 1$$

$$x_2 - x_3 + x_4 = 1$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

circle the pivot positions and describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} + a \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \text{ by}$$

setting the free variable (the column without a pivot position) equal to  $a$ , and expressing each of the other bounded variables equal to an expression in terms of  $a$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

Module E

Part 1 (Day 3)

Part 2 (Day 4)

**Part 3 (Day 5)**

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

## Remark 5.6

It's not necessary to completely find  $\text{RREF}(A)$  to deduce that a linear system is inconsistent.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 5.7

Find a contradiction in the inconsistent linear system

$$2x_1 - 3x_2 = 17$$

$$x_1 + 2x_2 = -2$$

$$-x_1 - x_2 = 1$$

by considering the following equivalent augmented matrices:

$$\left[ \begin{array}{cc|c} 2 & -3 & 17 \\ 1 & 2 & -2 \\ -1 & -1 & 1 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{array} \right].$$



# Activity 5.8

Show that all linear systems of the form

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & 0 \end{array}$$

are consistent by finding a quickly verifiable solution.

# Definition 5.9

A **homogeneous system** is a linear system satisfying  $b_i = 0$ , that is, it is a linear system of the form

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & 0 \end{array}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Fact 5.10**

Because the zero vector is always a solution, the solution set to any homogeneous system with infinitely-many solutions may be generated by multiplying the parameters representing the free variables by a minimal set of Euclidean vectors, and adding these up. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Definition 5.11**

A minimal set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 5.12

Find a basis for the solution set of the following homogeneous linear system.

$$x_1 + 2x_2 - x_4 = 0$$

$$x_3 + 4x_4 = 0$$

$$2x_1 + 4x_2 + x_3 + 2x_4 = 0$$

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

## Module G

## Module V: Vector Spaces

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

At the end of this module, students will be able to...

- **V1: Vector Spaces.** Determine if a set with given operations forms a vector space.
- **V2: Linear Combinations.** Determine if a vector can be written as a linear combination of a given set of vectors.
- **V3: Spanning Sets.** Determine if a set of vectors spans a vector space.
- **V4: Subspaces.** Determine if a subset of a vector space is a subset or not.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems  
**(Standard(s) E1,E2,E3).**



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/precalculus/vectors-precalc/vector-addition-subtraction/v/adding-and-subtracting-vectors>
- <https://www.khanacademy.org/math/precalculus/vectors-precalc/combined-vector-operations/v/combined-vector-operations-example>
- <https://www.khanacademy.org/math/precalculus/imaginary-and-complex-numbers/v/adding-and-subtracting-complex-numbers/v/adding-complex-numbers>
- <https://www.khanacademy.org/math/algebra/introduction-to-polynomial-expressions/v/adding-and-subtracting-polynomials/v/adding-and-subtracting-polynomials-1>

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

## Module G

◀ ◻ ▶ ◀ ◻ ▶ ◀ ≡ ▶ ◀ ≡ ▶ ≡ ▶ ↺ 🔍 ↻

## Activity 7.1

Consider each of the following vector properties. Label each property with  $\mathbb{R}^1$ ,  $\mathbb{R}^2$ , and/or  $\mathbb{R}^3$  if that property holds for Euclidean vectors/scalars  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  of that dimension.

**1 Addition associativity.**

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

**2 Addition commutativity.**

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

**3 Addition identity.**

There exists some  $\mathbf{0}$  where

$$\mathbf{v} + \mathbf{0} = \mathbf{v}.$$

**4 Addition inverse.**

There exists some  $-\mathbf{v}$  where

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

**5 Addition midpoint uniqueness.**

There exists a unique  $\mathbf{m}$  where the distance from  $\mathbf{u}$  to  $\mathbf{m}$  equals the distance from  $\mathbf{m}$  to  $\mathbf{v}$ .

**6 Scalar multiplication associativity.**

**7 Scalar multiplication identity.**

$$1\mathbf{v} = \mathbf{v}.$$

**8 Scalar multiplication relativity.**

There exists some scalar  $c$  where either  $c\mathbf{v} = \mathbf{w}$  or  $c\mathbf{w} = \mathbf{v}$ .

**9 Scalar distribution.**

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

**10 Vector distribution.**

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

**11 Orthogonality.**

There exists a non-zero vector  $\mathbf{n}$  such that  $\mathbf{n}$  is orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$ .

**12 Bidimensionality.**

## Definition 7.2

A **vector space**  $V$  is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties.

Let  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  belong to  $V$ , and let  $a, b$  be scalar numbers.

- **Addition associativity.**  

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$
- **Addition commutativity.**  

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$
- **Addition identity.**  
 There exists some  $\mathbf{0}$  where  

$$\mathbf{v} + \mathbf{0} = \mathbf{v}.$$
- **Addition inverse.**  
 There exists some  $-\mathbf{v}$  where  

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$
- **Scalar multiplication associativity.**  

$$a(b\mathbf{v}) = (ab)\mathbf{v}.$$
- **Scalar multiplication identity.**  

$$1\mathbf{v} = \mathbf{v}.$$
- **Scalar distribution.**  

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$
- **Vector distribution.**  

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Definition 7.3

The most important examples of vector spaces are the **Euclidean vector spaces**  $\mathbb{R}^n$ , but there are other examples as well.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

**Part 1 (Day 7)**

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 7.4

Consider the following vector space that models motion along the curve  $y = e^x$ .

Let  $V = \{(x, y) : y = e^x\}$ , where  $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 b_2)$ , and  $c(a, b) = (ca, b^c)$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 7.4**

Consider the following vector space that models motion along the curve  $y = e^x$ .

Let  $V = \{(x, y) : y = e^x\}$ , where  $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 b_2)$ , and  $c(a, b) = (ca, b^c)$ .

*Part 1:* Verify that  $3((1, e) + (-2, \frac{1}{e^2})) = 3(1, e) + 3(-2, \frac{1}{e^2})$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 7.4

Consider the following vector space that models motion along the curve  $y = e^x$ .

Let  $V = \{(x, y) : y = e^x\}$ , where  $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 b_2)$ , and  $c(a, b) = (ca, b^c)$ .

*Part 1:* Verify that  $3((1, e) + (-2, \frac{1}{e^2})) = 3(1, e) + 3(-2, \frac{1}{e^2})$ .

*Part 2:* Prove the scalar distribution property for this space:  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ .



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)**
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

Module M

Module G

# Application Activities - Module V Part 2 - Class Day 8

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Remark 8.1

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$ : Euclidean vectors with  $n$  components.
- $\mathbb{R}^\infty$ : Sequences of real numbers  $(v_1, v_2, \dots)$ .
- $\mathbb{R}^{m \times n}$ : Matrices of real numbers with  $m$  rows and  $n$  columns.
- $\mathbb{C}$ : Complex numbers.
- $\mathcal{P}^n$ : Polynomials of degree  $n$  or less.
- $\mathcal{P}$ : Polynomials of any degree.
- $C(\mathbb{R})$ : Real-valued continuous functions.

## Activity 8.2

Let  $V = \{(a, b) : a, b \text{ are real numbers}\}$ , where  $(a_1, b_1) + (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$  and  $c(a, b) = (a^c, b + c)$ . Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.**  
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition commutativity.**  
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Addition identity.**  
There exists some  $\mathbf{0}$  where  
 $\mathbf{v} + \mathbf{0} = \mathbf{v}.$
- **Addition inverse.**  
There exists some  $-\mathbf{v}$  where  
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$
- **Scalar multiplication associativity.**  
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **Scalar multiplication identity.**  
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar distribution.**  
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Vector distribution.**  
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)**
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

Module M

Module G

# Definition 8.3

A **linear combination** of a set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$  is given by  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$  for any choice of scalar multiples  $c_1, c_2, \dots, c_m$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Definition 8.4

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
**Part 2 (Day 8)**  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Module M

Module G

# Activity 8.5

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 8.5**

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

*Part 1:* Sketch  $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  in the  $xy$  plane for  $c = 1, 3, 0, -2$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 8.5**

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

*Part 1:* Sketch  $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  in the  $xy$  plane for  $c = 1, 3, 0, -2$ .

*Part 2:* Sketch a representation of all the vectors given by  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$  in the  $xy$  plane.



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 8.6**

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 8.6**

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

*Part 1:* Sketch  $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  in the  $xy$  plane for  $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 8.6**

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

*Part 1:* Sketch  $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  in the  $xy$  plane for  $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ .

*Part 2:* Sketch a representation of all the vectors given by  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$  in the  $xy$  plane.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 8.7

Sketch a representation of all the vectors given by  $\text{span} \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$  in the  $xy$  plane.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 8.8**

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 8.8**

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 8.8**

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

*Part 2:* Solve this system. (From now on, feel free to use a calculator to solve linear systems.)

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 8.8**

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector

equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

*Part 2:* Solve this system. (From now on, feel free to use a calculator to solve linear systems.)

*Part 3:* Given this solution, does  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belong to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ ?



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)**
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

Module M

Module G

# Application Activities - Module V Part 3 - Class Day 9

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Fact 9.1**

A vector  $\mathbf{b}$  belongs to  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  if and only if the linear system corresponding to  $[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$  is consistent.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Remark 9.2**

To determine if  $\mathbf{b}$  belongs to  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ , find  $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 9.3**

Determine if  $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  by row-reducing an appropriate matrix.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 9.4**

Determine if  $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  by row-reducing an appropriate matrix.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Observation 9.5

So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space  $\mathbb{R}^n$ ; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 9.6**

We previously checked that  $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$  does not belong to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ .

Does  $f(x) = 3x^2 - 2x + 1$  belong to  $\text{span}\{x^2 - 3, -x^2 - 3x + 2\}$ ?

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 9.7**

Does the matrix  $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$  belong to  $\text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\}$ ?



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)**
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

Module M

Module G

Activity 9.8

Does the complex number  $2i$  belong to  $\text{span}\{-3 + i, 6 - 2i\}$ ?

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 9.9

How many vectors are required to span  $\mathbb{R}^2$ ? Sketch a drawing in the  $xy$  plane to support your guess.

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

## Activity 9.10

How many vectors are required to span  $\mathbb{R}^3$ ?

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

**Part 4 (Day 10)**

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

# Application Activities - Module V Part 4 - Class Day 10

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

**Part 4 (Day 10)**

## Module S

Part 1 (Day 12)

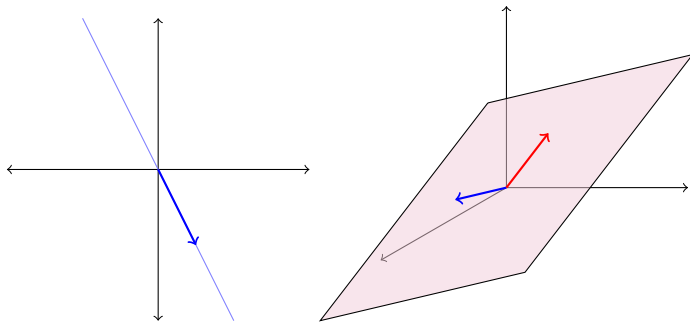
Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Fact 10.1**At least  $n$  vectors are required to span  $\mathbb{R}^n$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

**Part 4 (Day 10)**

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 10.2**

Find a vector  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  in  $\mathbb{R}^3$  that is not in  $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$  by doing the following.

## Activity 10.2

Find a vector  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  in  $\mathbb{R}^3$  that is not in  $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$  by doing the following.

*Part 1:* Choose simple values for  $x, y, z$  such that  $\left[ \begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{array} \right]$  represents an inconsistent linear equation.

## Activity 10.2

Find a vector  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  in  $\mathbb{R}^3$  that is not in  $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$  by doing the following.

*Part 1:* Choose simple values for  $x, y, z$  such that  $\begin{bmatrix} 1 & 0 & | & x \\ 0 & 1 & | & y \\ 0 & 0 & | & z \end{bmatrix}$  represents an inconsistent linear equation.

*Part 2:* Use row operations to manipulate  $\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & | & a \\ -1 & 0 & | & b \\ 0 & 1 & | & c \end{bmatrix}$ .



## Activity 10.2

Find a vector  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  in  $\mathbb{R}^3$  that is not in  $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$  by doing the following.

*Part 1:* Choose simple values for  $x, y, z$  such that  $\begin{bmatrix} 1 & 0 & | & x \\ 0 & 1 & | & y \\ 0 & 0 & | & z \end{bmatrix}$  represents an inconsistent linear equation.

*Part 2:* Use row operations to manipulate  $\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & | & a \\ -1 & 0 & | & b \\ 0 & 1 & | & c \end{bmatrix}$ .

*Part 3:* Write a sentence explaining why  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  cannot be in  $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Fact 10.3**

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$  fails to span all of  $\mathbb{R}^n$  exactly when  $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$  has a row of zeros.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 10.4**

Consider the set of vectors  $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$ . Prove that

$$\mathbb{R}^4 = \text{span } S.$$

## Activity 10.5

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2\}$$

Prove that  $\mathcal{P}^3 \neq \text{span } S$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Definition 10.6

A subset of a vector space is called a **subspace** if it is itself a vector space.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Fact 10.7**

If  $S$  is a subset of a vector space  $V$ , then  $\text{span } S$  is a subspace of  $V$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Remark 10.8**

To prove that a subset is a subspace, you need only verify that  $c\mathbf{v} + d\mathbf{w}$  belongs to the subset for any choice of vectors  $\mathbf{v}, \mathbf{w}$  from the subset and any real scalars  $c, d$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 10.9**

Prove that  $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$  is a subspace of the vector space of all degree-two polynomials by showing that  $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$  belongs to  $P$ .



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

**Part 4 (Day 10)**

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

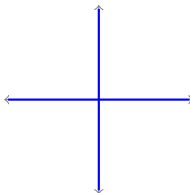
## Module A

## Module M

## Module G

## Activity 10.10

Consider the subset of  $\mathbb{R}^2$  where at least one coordinate of each vector is 0.



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

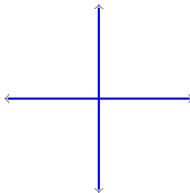
## Module A

## Module M

## Module G

**Activity 10.10**

Consider the subset of  $\mathbb{R}^2$  where at least one coordinate of each vector is 0.



*Part 1:* Find a linear combination  $c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$  that does not belong to this subset.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

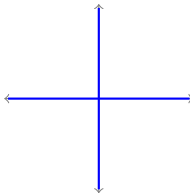
## Module A

## Module M

## Module G

**Activity 10.10**

Consider the subset of  $\mathbb{R}^2$  where at least one coordinate of each vector is 0.



*Part 1:* Find a linear combination  $c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$  that does not belong to this subset.

*Part 2:* Use this linear combination to sketch a picture illustrating why this subset is not a subspace.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Fact 10.11**

Suppose a subset  $S$  of  $V$  is isomorphic to another vector space  $W$ . Then  $S$  is a subspace of  $V$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Activity 10.12**

Show that the set of  $2 \times 2$  matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of  $\mathbb{R}^{2 \times 2}$  by finding a Euclidean space isomorphic to  $S$ .

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Module M

Module G

# Module S: Structure of vector spaces

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

At the end of this module, students will be able to...

- **S1. Linear independence** Determine if a set of Euclidean vectors is linearly dependent or independent.
- **S2. Basis verification** Determine if a set of vectors is a basis of a vector space
- **S3. Basis construction** Construct a basis for the subspace spanned by a given set of vectors.
- **S4. Dimension** I can compute the dimension of a vector space.

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.
- Apply linear combinations and spanning sets **(Standard(s) V2,V3)**.



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/prec calculus/vectors-prec alc/vector-addition-subtraction/v/adding-and-subtracting-vectors>
- <https://www.khanacademy.org/math/prec calculus/vectors-prec alc/combined-vector-operations/v/combined-vector-operations-example>

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

**Part 1 (Day 12)**

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

# Application Activities - Module S Part 1 - Class Day 12

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 12.1

In the previous module, we considered

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

and showed that  $\text{span } S \neq \mathbb{R}^4$ . Find two vectors that are in the span of the other three vectors.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Definition 12.2

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 12.3

Suppose  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ , so the set  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent. Is the vector equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$  consistent with one solution, consistent with infinitely many solutions, or inconsistent?

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Fact 12.4**

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly dependent if and only if  $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$  is consistent with infinitely many solutions.

**Activity 12.5**

Find

$$\text{RREF} \left[ \begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array} \right]$$

and circle the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Fact 12.6**

A set of Euclidean vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly dependent if and only if RREF  $\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$  has a column without a pivot position.



Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

## Activity 12.7

TODO (compute RREF and label each set of vectors as linearly independent/dependent)

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module G

## Application Activities - Module S Part 2 - Class Day 13

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module G

(take basis shown to be linearly independent in previous day, and show that it spans)

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

Module M

Module G

Definition 13.2

A **basis** is a linearly independent set that spans a vector space.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

Module M

Module G

# Observation 13.3

A basis may be thought of as building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

Module M

Module G

**Activity 13.4**  
(given four sets of general vectors, identify which are bases and which aren't)

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 13.5

If  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is a basis for  $\mathbb{R}^4$ , that means  $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$  doesn't have a column without a pivot position, and doesn't have a row of zeros. What is  $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$ ?

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

**Part 2 (Day 13)**

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Fact 13.6**

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$  is a basis for  $\mathbb{R}^n$  if and only if  $m = n$  and

$$\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

Module M

Module G

**Activity 13.7**  
(given four sets of  $\mathbb{R}^5$  vectors, identify which are bases and which aren't)

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

Module M

Module G

**Activity 13.8**  
How can  $\{u,v,u+v\}$  (but with numbers) be changed to make it linearly independent?

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Application Activities - Module S Part 3 - Class Day 14

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)**

Module A

Module M

Module G

# Activity 14.1

(discover that the redundant vectors are non-pivot columns)

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

**Fact 14.2**

To compute a basis for the subspace  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ , simply remove the vectors corresponding to the non-pivot columns of  $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$ .

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)**

Module A

Module M

Module G

**Activity 14.3**  
(find ALL the bases for  $\text{span } S$  that are subsets of  $S$ )

### Fact 14.4

All bases for a vector space are the same size.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 14.5

Prove that if  $\{\mathbf{v}\}$  is a basis for  $V$ , then  $\{\mathbf{w}_1, \mathbf{w}_2\}$  is linearly dependent (assuming  $\mathbf{w}_1 \neq \mathbf{w}_2$ ).



- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Part 1 (Day 12)  
Part 2 (Day 13)  
**Part 3 (Day 14)**

## Module G

All bases for a vector space are the same size.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

Module M

Module G

Definition 14.7

The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Activity 14.8

Reduce a bunch of spans to bases to find their dimension.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)**

Module A

Module M

Module G

# Activity 14.9

What is the dimension of the vector space of 7th-degree polynomials  $\mathcal{P}^7$ ?

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)**

Module A

Module M

Module G

# Activity 14.10

What is the dimension of the vector space of polynomials  $\mathcal{P}$ ?

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Observation 14.11

Several interesting vector spaces are infinite-dimensional:

- The space of polynomials  $\mathcal{P}$
- The space of real number sequences  $\mathbb{R}^\infty$
- The space of continuous functions  $C(\mathbb{R})$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

### Fact 14.12

Every vector space with dimension  $n < \infty$  is isomorphic to  $\mathbb{R}^n$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

## Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

## Module M

## Module G

## Module A: TODO



## Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

## Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

## Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

## Module A

## Module M

## Module G

## Module M: TODO

## Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

## Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

## Module M

## Module G

## Module G: TODO