

## Section E.0

**Definition E.0.1** A **linear equation** is an equation of the variables  $x_i$  of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector that can be plugged into the equation).

**Remark E.0.2** In previous classes you likely used the variables  $x, y, z$  in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as  $x_i$ , and assume  $x = x_1, y = x_2, z = x_3, w = x_4$  when convenient.

**Definition E.0.3** A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \mid \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

**Remark E.0.4** When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{array}{rcl} x_1 + 3x_3 & = & 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ -x_2 + x_3 & = & -2 \end{array}$$

Verbose standard form:

$$\begin{array}{rcl} 1x_1 + 0x_2 + 3x_3 & = & 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ 0x_1 - 1x_2 + 1x_3 & = & -2 \end{array}$$

Concise standard form:

$$\begin{array}{rcl} x_1 & + & 3x_3 = 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ -x_2 + x_3 & = & -2 \end{array}$$

**Definition E.0.5** A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**.

**Fact E.0.6** All linear systems are one of the following:

- **Consistent with one solution:** its solution set contains a single vector, e.g.  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$
- **Consistent with infinitely-many solutions:** its solution set contains infinitely many vectors, e.g.  $\left\{ \begin{bmatrix} 1 \\ 2 - 3a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$
- **Inconsistent:** its solution set is the empty set  $\{\} = \emptyset$

**Activity E.0.7** (*~10 min*) All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is  $\emptyset$ .

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

**Activity E.0.8** (*~10 min*) Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

*Part 1:* Find three different solutions for this system.

*Part 2:* Let  $x_2 = a$  where  $a$  is an arbitrary real number, then find an expression for  $x_1$  in terms of  $a$ . Use this to write the solution set  $\left\{ \begin{bmatrix} ? \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$  for the linear system.

**Activity E.0.9** (*~10 min*) Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 - x_4 &= 3 \\ x_3 + 4x_4 &= -2 \end{aligned}$$

Describe the solution set

$$\left\{ \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

to the linear system by setting  $x_2 = a$  and  $x_4 = b$ , and then solving for  $x_1$  and  $x_3$ .

**Observation E.0.10** Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{aligned} -2x_1 - 4x_2 + x_3 - 4x_4 &= -8 \\ x_1 + 2x_2 + 2x_3 + 12x_4 &= -1 \\ x_1 + 2x_2 + x_3 + 8x_4 &= 1 \end{aligned}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.