

Linear Algebra

University of South Alabama

Fall 2017

Module E

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- Part 2 (Day 4)
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Module E: Solving Systems of Linear Equations

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At the end of this module, students will be able to...

- **E1: Systems as matrices.** Translate back and forth between a system of linear equations and the corresponding augmented matrix.
- **E2: Row reduction.** Put a matrix in reduced row echelon form
- **E3: Solving Linear Systems.** Solve a system of linear equations.
- **E4: Homogeneous Systems.** Find a basis for the solution set of a homogeneous linear system.

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Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.

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The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-systems-topic/cc-8th-systems-graphically/a/systems-of-equations-with-graphing>
- <https://www.khanacademy.org/math/algebra/systems-of-linear-equations/solving-systems-of-equations-with-substitution/v/practice-using-substitution-for-systems>

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Application Activities - Module E Part 1 - Class Day 3

Definition 3.1

A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

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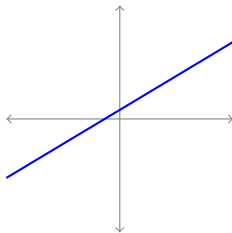
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Observation 3.2

The linear equation $3x - 5y = -2$ may be graphed as a line in the xy plane.



The linear equation $x + 2y - z = 4$ may be graphed as a plane in xyz space.

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Remark 3.3

In previous classes you likely assumed $x = x_1$, $y = x_2$, and $z = x_3$. However, since this course often deals with equations of four or more variables, we will almost always write our variables as x_i .

Definition 3.4

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

A solution

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \dots + a_{in}s_n = b_i$$

for $1 \leq i \leq m$ (that is, the solution satisfies all equations in the system).

Remark 3.5

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2\end{aligned}$$

Verbose standard form:

$$\begin{aligned}x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - x_2 + x_3 &= -2\end{aligned}$$

Concise standard form:

$$\begin{aligned}x_1 \quad \quad + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ \quad - x_2 + x_3 &= -2\end{aligned}$$

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Definition 3.6

A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

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Fact 3.7

All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

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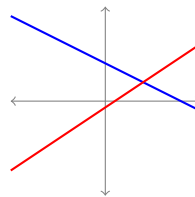
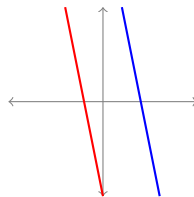
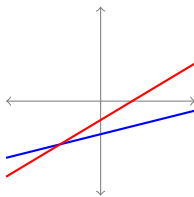
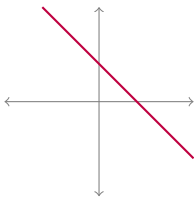
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Activity 3.8

Consider the following graphs representing linear systems of two variables. Label each graph with **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.



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Activity 3.9

All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system.

$$-x_1 + 2x_2 = 5$$

$$2x_1 - 4x_2 = 6$$

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Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

Part 1: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$, $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$, $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

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Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

Part 1: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$, $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$, $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a . Use this to describe *all* solutions (the **solution set**) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$ for the linear system in terms of a .

Activity 3.11

Consider the following linear system.

$$x_1 + 2x_2 - x_4 = 3$$

$$x_3 + 4x_4 = -2$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix} + a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3 .

Observation 3.12

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.

Remark 3.13

The only important information in a linear system are its coefficients and constants.

Original linear system:

$$x_1 + 3x_3 = 3$$

$$3x_1 - 2x_2 + 4x_3 = 0$$

$$-x_2 + x_3 = -2$$

Verbose standard form:

$$x_1 + 0x_2 + 3x_3 = 3$$

$$3x_1 - 2x_2 + 4x_3 = 0$$

$$0x_1 - x_2 + x_3 = -2$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \end{array}$$

$$\begin{array}{ccc|c} 3 & -2 & 4 & 0 \end{array}$$

$$\begin{array}{ccc|c} 0 & 1 & 1 & -2 \end{array}$$

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Definition 3.14

A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

Definition 3.15

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution: $(x_1, x_2) = (1, 1)$.

$$3x_1 - 2x_2 = 1$$

$$x_1 + 4x_2 = 5$$

$$3x_1 - 2x_2 = 1$$

$$4x_1 + 2x_2 = 6$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array} \right]$$

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Activity 3.16

Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

- a) Swap two rows.
- b) Swap two columns.
- c) Add a constant to every term in a row.
- d) Multiply a row by a nonzero constant.
- e) Add a constant multiple of one row to another row.
- f) Replace a column with zeros.

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Definition 4.1

The following **row operations** produce equivalent augmented matrices:

- 1 Swap two rows.
- 2 Multiply a row by a nonzero constant.
- 3 Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

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Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$x_1 - x_2 + 5x_3 = 1$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$x_2 - 2x_3 = 3$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_3 = 2$$

Part 1: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

- 1 Swap R_1 (first row) and R_2 (second row).
- 2 Multiply R_2 by $\frac{1}{2}$.

- 3 Add R_1 to R_3 .
- 4 Add $-3R_1$ to R_2 .
- 5 Add $-2R_2$ to R_3 .
- 6 Multiply R_3 by $\frac{1}{3}$.

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Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$x_1 - x_2 + 5x_3 = 1$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$x_2 - 2x_3 = 3$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_3 = 2$$

Part 1: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

- ① Swap R_1 (first row) and R_2 (second row).
- ② Multiply R_2 by $\frac{1}{2}$.
- ③ Add R_1 to R_3 .
- ④ Add $-3R_1$ to R_2 .
- ⑤ Add $-2R_2$ to R_3 .
- ⑥ Multiply R_3 by $\frac{1}{3}$.

Part 2: What is the common solution to these linear systems?

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Definition 4.3

The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix. Examples:

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Activity 4.4

Find your own sequence of row operations to manipulate the matrix

$$\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array} \right]$$

into row echelon form. (Note that row echelon form is not unique.)

The most efficient way to do this is by circling **pivot positions** in your matrix:

- 1 Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
- 2 Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
- 3 Repeat these two steps as often as possible.

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Activity 4.5

Solve this simplified linear system:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

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Observation 4.6

The concise standard form of the solution to this linear system corresponds to a simplified row echelon form matrix:

$$x_1 = -2$$

$$x_2 = 7$$

$$x_3 = 2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

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Definition 4.7

A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0. Examples:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Activity 4.8

Show that the following two linear systems:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

$$x_1 = -2$$

$$x_2 = 7$$

$$x_3 = 2$$

are equivalent by converting the first system to an augmented matrix, and then zeroing out all terms above pivot positions (the leading terms).

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Remark 4.9

We may verify that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$ is a solution to the original linear system

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

by plugging the solution into each equation.

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Fact 4.10

Every augmented matrix A reduces to a unique reduced row echelon form matrix. This matrix is denoted as $\text{RREF}(A)$.

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Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

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Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

Part 1: Find $\text{RREF}(A)$.

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Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

Part 1: Find $\text{RREF}(A)$.

Part 2: How many solutions does the corresponding linear system have?

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Definition 5.1

An algorithm that reduces A to $\text{RREF}(A)$ is called **Gauss-Jordan elimination**. For example:

- 1 Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
- 2 Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
- 3 Repeat these two steps as often as possible.
- 4 Finally, zero out any terms above pivot positions.

Observation 5.2

Here is an example of applying Gauss-Jordan elimination to a matrix:

$$\begin{aligned}
 & \left[\begin{array}{ccc|c} \textcircled{3} & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array} \right] \sim \left[\begin{array}{ccc|c} \textcircled{2} & -2 & 10 & 2 \\ 3 & -2 & 13 & 6 \\ -1 & 3 & -6 & 11 \end{array} \right] \sim \left[\begin{array}{ccc|c} \textcircled{1} & -1 & 5 & 1 \\ 3 & -2 & 13 & 6 \\ -1 & 3 & -6 & 11 \end{array} \right] \\
 & \sim \left[\begin{array}{ccc|c} \textcircled{1} & -1 & 5 & 1 \\ 0 & \textcircled{1} & -2 & 3 \\ 0 & 2 & -1 & 12 \end{array} \right] \sim \left[\begin{array}{ccc|c} \textcircled{1} & -1 & 5 & 1 \\ 0 & \textcircled{1} & -2 & 3 \\ 0 & 0 & \textcircled{3} & 6 \end{array} \right] \sim \left[\begin{array}{ccc|c} \textcircled{1} & -1 & 5 & 1 \\ 0 & \textcircled{1} & -2 & 3 \\ 0 & 0 & \textcircled{1} & 2 \end{array} \right] \\
 & \sim \left[\begin{array}{ccc|c} \textcircled{1} & -1 & 5 & 1 \\ 0 & \textcircled{1} & -2 & 3 \\ 0 & 0 & \textcircled{1} & 2 \end{array} \right] \sim \left[\begin{array}{ccc|c} \textcircled{1} & -1 & 0 & -9 \\ 0 & \textcircled{1} & 0 & 7 \\ 0 & 0 & \textcircled{1} & 2 \end{array} \right] \sim \left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & -2 \\ 0 & \textcircled{1} & 0 & 7 \\ 0 & 0 & \textcircled{1} & 2 \end{array} \right]
 \end{aligned}$$

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Activity 5.3Find $\text{RREF}(A)$ where

$$A = \left[\begin{array}{cccc|c} -1 & 1 & -3 & 2 & 0 \\ 2 & -1 & 5 & 3 & -11 \\ 3 & 2 & 4 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{array} \right].$$

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Definition 5.4

The columns of $\text{RREF}(A)$ without a leading term represent **free variables** of the linear system modeled by A that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by A .

Activity 5.5

Given the linear system and its equivalent row-reduced matrix

$$-x_1 + x_2 - 3x_3 + 2x_4 = 0$$

$$2x_1 - x_2 + 5x_3 + 3x_4 = -11$$

$$3x_1 + 2x_2 + 4x_3 + x_4 = 1$$

$$x_2 - x_3 + x_4 = 1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

circle the pivot positions and describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} + a \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \text{ by}$$

setting the free variable (the column without a pivot position) equal to a , and expressing each of the other bounded variables equal to an expression in terms of a .

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Remark 5.6

It's not necessary to completely find $\text{RREF}(A)$ to deduce that a linear system is inconsistent.

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Activity 5.7

Find a contradiction in the inconsistent linear system

$$2x_1 - 3x_2 = 17$$

$$x_1 + 2x_2 = -2$$

$$-x_1 - x_2 = 1$$

by considering the following equivalent augmented matrices:

$$\left[\begin{array}{cc|c} 2 & -3 & 17 \\ 1 & 2 & -2 \\ -1 & -1 & 1 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{array} \right].$$

Activity 5.8

Show that all linear systems of the form

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & 0 \end{array}$$

are consistent by finding a quickly verifiable solution.

Definition 5.9

A **homogeneous system** is a linear system satisfying $b_i = 0$, that is, it is a linear system of the form

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & 0 \end{array}$$

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Fact 5.10

Because the zero vector is always a solution, the solution set to any homogeneous system with infinitely-many solutions may be generated by multiplying the parameters representing the free variables by a minimal set of Euclidean vectors, and adding these up. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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Definition 5.11

A minimal set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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Activity 5.12

Find a basis for the solution set of the following homogeneous linear system.

$$x_1 + 2x_2 - x_4 = 0$$

$$x_3 + 4x_4 = 0$$

$$2x_1 + 4x_2 + x_3 + 2x_4 = 0$$

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Module V: Vector Spaces

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At the end of this module, students will be able to...

- **V1: Vector Spaces.** Determine if a set with given operations forms a vector space.
- **V2: Linear Combinations.** Determine if a vector can be written as a linear combination of a given set of vectors.
- **V3: Spanning Sets.** Determine if a set of vectors spans a vector space.
- **V4: Subspaces.** Determine if a subset of a vector space is a subset or not.

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Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems
(Standard(s) E1,E2,E3).

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The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/prec calculus/vectors-prec alc/vector-addition-subtraction/v/adding-and-subtracting-vectors>
- <https://www.khanacademy.org/math/prec calculus/vectors-prec alc/combined-vector-operations/v/combined-vector-operations-example>
- <https://www.khanacademy.org/math/prec calculus/imaginary-and-complex-numbers/v/adding-and-subtracting-complex-numbers/v/adding-complex-numbers>
- <https://www.khanacademy.org/math/algebra/introduction-to-polynomial-expressions/v/adding-and-subtracting-polynomials/v/adding-and-subtracting-polynomials-1>

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Activity 7.1

Consider each of the following vector properties. Label each property with \mathbb{R}^1 , \mathbb{R}^2 , and/or \mathbb{R}^3 if that property holds for Euclidean vectors/scalars \mathbf{u} , \mathbf{v} , \mathbf{w} of that dimension.

1 Addition associativity.

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

2 Addition commutativity.

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

3 Addition identity.

There exists some $\mathbf{0}$ where

$$\mathbf{v} + \mathbf{0} = \mathbf{v}.$$

4 Addition inverse.

There exists some $-\mathbf{v}$ where

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

5 Addition midpoint uniqueness.

There exists a unique \mathbf{m} where the distance from \mathbf{u} to \mathbf{m} equals the distance from \mathbf{m} to \mathbf{v} .

6 Scalar multiplication associativity.

7 Scalar multiplication identity.

$$1\mathbf{v} = \mathbf{v}.$$

8 Scalar multiplication relativity.

There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.

9 Scalar distribution.

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

10 Vector distribution.

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

11 Orthogonality.

There exists a non-zero vector \mathbf{n} such that \mathbf{n} is orthogonal to both \mathbf{u} and \mathbf{v} .

12 Bidimensionality.

Definition 7.2

A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V , and let a, b be scalar numbers.

- **Addition associativity.**
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition commutativity.**
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Addition identity.**
There exists some $\mathbf{0}$ where
 $\mathbf{v} + \mathbf{0} = \mathbf{v}.$
- **Addition inverse.**
There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$
- **Scalar multiplication associativity.**
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **Scalar multiplication identity.**
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar distribution.**
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Vector distribution.**
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

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Definition 7.3

The most important examples of vector spaces are the **Euclidean vector spaces** \mathbb{R}^n , but there are other examples as well.

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Activity 7.4

Consider the following vector space that models motion along the curve $y = e^x$.

Let $V = \{(x, y) : y = e^x\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 b_2)$, and $c(a, b) = (ca, b^c)$.

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Activity 7.4

Consider the following vector space that models motion along the curve $y = e^x$.

Let $V = \{(x, y) : y = e^x\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 b_2)$, and $c(a, b) = (ca, b^c)$.

Part 1: Verify that $3((1, e) + (-2, \frac{1}{e^2})) = 3(1, e) + 3(-2, \frac{1}{e^2})$.

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Activity 7.4

Consider the following vector space that models motion along the curve $y = e^x$.

Let $V = \{(x, y) : y = e^x\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 b_2)$, and $c(a, b) = (ca, b^c)$.

Part 1: Verify that $3((1, e) + (-2, \frac{1}{e^2})) = 3(1, e) + 3(-2, \frac{1}{e^2})$.

Part 2: Prove the scalar distribution property for this space: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

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Application Activities - Module V Part 2 - Class Day 8

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Remark 8.1

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- \mathbb{R}^n : Euclidean vectors with n components.
- \mathbb{R}^∞ : Sequences of real numbers (v_1, v_2, \dots) .
- $\mathbb{R}^{m \times n}$: Matrices of real numbers with m rows and n columns.
- \mathbb{C} : Complex numbers.
- \mathcal{P}^n : Polynomials of degree n or less.
- \mathcal{P} : Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity 8.2

Let $V = \{(a, b) : a, b \text{ are real numbers}\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$ and $c(a, b) = (a^c, b + c)$. Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.**
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition commutativity.**
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Addition identity.**
There exists some $\mathbf{0}$ where
 $\mathbf{v} + \mathbf{0} = \mathbf{v}.$
- **Addition inverse.**
There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$
- **Scalar multiplication associativity.**
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **Scalar multiplication identity.**
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar distribution.**
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Vector distribution.**
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

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Definition 8.3

A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$ for any choice of scalar multiples c_1, c_2, \dots, c_m .

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Definition 8.4

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

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Activity 8.5

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

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Activity 8.5

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Part 1: Sketch $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

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Activity 8.5

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Part 1: Sketch $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part 2: Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ in the xy plane.

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Activity 8.6

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

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Activity 8.6

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Part 1: Sketch $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in the xy plane for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

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Activity 8.6

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Part 1: Sketch $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in the xy plane for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Part 2: Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ in the xy plane.

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Activity 8.7

Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ in the xy plane.

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (From now on, feel free to use a calculator to solve linear systems.)

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (From now on, feel free to use a calculator to solve linear systems.)

Part 3: Given this solution, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

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Fact 9.1

A vector \mathbf{b} belongs to $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$ is consistent.

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Remark 9.2

To determine if \mathbf{b} belongs to $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, find $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$.

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Activity 9.3

Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

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Activity 9.4

Determine if $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

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Observation 9.5

So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space \mathbb{R}^n ; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

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Activity 9.6

We previously checked that $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ does not belong to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$.

Does $f(x) = 3x^2 - 2x + 1$ belong to $\text{span}\{x^2 - 3, -x^2 - 3x + 2\}$?

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Activity 9.7

Does the matrix $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$ belong to $\text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\}$?

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Activity 9.8

Does the complex number $2i$ belong to $\text{span}\{-3 + i, 6 - 2i\}$?

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Activity 9.9

How many vectors are required to span \mathbb{R}^2 ? Sketch a drawing in the xy plane to support your guess.

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Activity 9.10

How many vectors are required to span \mathbb{R}^3 ?

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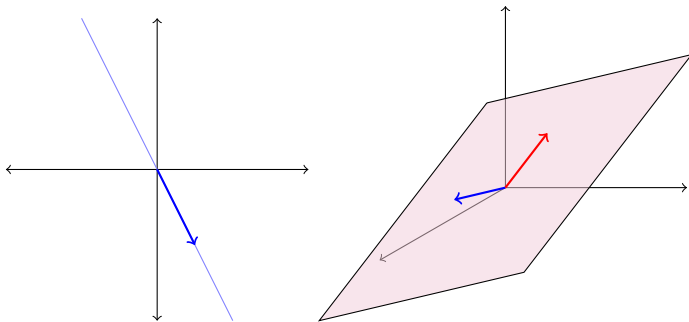
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Fact 10.1At least n vectors are required to span \mathbb{R}^n .

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Activity 10.2

Find a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in \mathbb{R}^3 that is not in $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by doing the following.

Activity 10.2

Find a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in \mathbb{R}^3 that is not in $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by doing the following.

Part 1: Choose simple values for x, y, z such that $\begin{bmatrix} 1 & 0 & | & x \\ 0 & 1 & | & y \\ 0 & 0 & | & z \end{bmatrix}$ represents an inconsistent linear equation.

Activity 10.2

Find a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in \mathbb{R}^3 that is not in $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by doing the following.

Part 1: Choose simple values for x, y, z such that $\begin{bmatrix} 1 & 0 & | & x \\ 0 & 1 & | & y \\ 0 & 0 & | & z \end{bmatrix}$ represents an inconsistent linear equation.

Part 2: Use row operations to manipulate $\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & | & a \\ -1 & 0 & | & b \\ 0 & 1 & | & c \end{bmatrix}$.

Activity 10.2

Find a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in \mathbb{R}^3 that is not in $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by doing the following.

Part 1: Choose simple values for x, y, z such that $\begin{bmatrix} 1 & 0 & | & x \\ 0 & 1 & | & y \\ 0 & 0 & | & z \end{bmatrix}$ represents an inconsistent linear equation.

Part 2: Use row operations to manipulate $\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & | & a \\ -1 & 0 & | & b \\ 0 & 1 & | & c \end{bmatrix}$.

Part 3: Write a sentence explaining why $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ cannot be in $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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Fact 10.3

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ fails to span all of \mathbb{R}^n exactly when $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$ has a row of zeros.

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Activity 10.4

Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Prove that

$$\mathbb{R}^4 = \text{span } S.$$

Activity 10.5

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2$$

Prove that $\mathcal{P}^3 \neq \text{span } S$.

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Definition 10.6

A subset of a vector space is called a **subspace** if it is itself a vector space.

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Fact 10.7

If S is a subset of a vector space V , then $\text{span } S$ is a subspace of V .

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Remark 10.8

To prove that a subset is a subspace, you need only verify that $c\mathbf{v} + d\mathbf{w}$ belongs to the subset for any choice of vectors \mathbf{v}, \mathbf{w} from the subset and any real scalars c, d .

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Activity 10.9

Prove that $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$ is a subspace of the vector space of all degree-two polynomials by showing that $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$ belongs to P .

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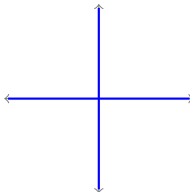
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Activity 10.10

Consider the subset of \mathbb{R}^2 where at least one coordinate of each vector is 0.



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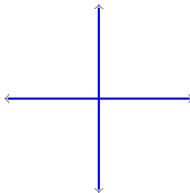
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Activity 10.10

Consider the subset of \mathbb{R}^2 where at least one coordinate of each vector is 0.



Part 1: Find a linear combination $c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ that does not belong to this subset.

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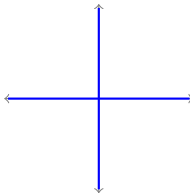
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Activity 10.10

Consider the subset of \mathbb{R}^2 where at least one coordinate of each vector is 0.



Part 1: Find a linear combination $c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ that does not belong to this subset.

Part 2: Use this linear combination to sketch a picture illustrating why this subset is not a subspace.

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Fact 10.11

Suppose a subset S of V is isomorphic to another vector space W . Then S is a subspace of V .

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Activity 10.12

Show that the set of 2×2 matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of $\mathbb{R}^{2 \times 2}$ by finding a Euclidean space isomorphic to S .

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Module S: Structure of vector spaces

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At the end of this module, students will be able to...

- **S1. Linear independence** Determine if a set of Euclidean vectors is linearly dependent or independent.
- **S2. Basis verification** Determine if a set of vectors is a basis of a vector space
- **S3. Basis construction** Construct a basis for the subspace spanned by a given set of vectors.
- **S4. Dimension** I can compute the dimension of a vector space.

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Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.
- Apply linear combinations and spanning sets **(Standard(s) V2,V3)**.

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The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/prec calculus/vectors-prec alc/vector-addition-subtraction/v/adding-and-subtracting-vectors>
- <https://www.khanacademy.org/math/prec calculus/vectors-prec alc/combined-vector-operations/v/combined-vector-operations-example>

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Activity 12.1

In the previous module, we considered

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

and showed that $\text{span } S \neq \mathbb{R}^4$. Find two vectors that are in the span of the other three vectors.

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Definition 12.2

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

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Activity 12.3

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Is the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ consistent with one solution, consistent with infinitely many solutions, or inconsistent?

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Fact 12.4

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$ is consistent with infinitely many solutions.

Activity 12.5

Find

$$\text{RREF} \left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array} \right]$$

and circle the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Fact 12.6

A set of Euclidean vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ has a column without a pivot position.

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Activity 12.7

TODO (compute RREF and label each set of vectors as linearly independent/dependent)

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Application Activities - Module S Part 2 - Class Day 13

(take basis shown to be linearly independent in previous day, and show that it spans)

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

Module M

Module G

Definition 13.2

A **basis** is a linearly independent set that spans a vector space.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

Module M

Module G

Observation 13.3

A basis may be thought of as building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

Module M

Module G

Activity 13.4
(given four sets of general vectors, identify which are bases and which aren't)

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Activity 13.5

If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for \mathbb{R}^4 , that means $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$ doesn't have a column without a pivot position, and doesn't have a row of zeros. What is $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$?

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Fact 13.6

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for \mathbb{R}^n if and only if $m = n$ and

$$\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

Module M

Module G

Activity 13.7
(given four sets of \mathbb{R}^5 vectors, identify which are bases and which aren't)

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

Module M

Module G

Activity 13.8
How can $\{u,v,u+v\}$ (but with numbers) be changed to make it linearly independent?

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Part 1 (Day 12)
Part 2 (Day 13)
Part 3 (Day 14)

Module G

Application Activities - Module S Part 3 - Class Day 14

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)**

Module A

Module M

Module G

Activity 14.1
(discover that the redundant vectors are non-pivot columns)

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Fact 14.2

To compute a basis for the subspace $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, simply remove the vectors corresponding to the non-pivot columns of $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)**

Module A

Module M

Module G

Activity 14.3
(find ALL the bases for $\text{span } S$ that are subsets of S)

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Part 1 (Day 12)
Part 2 (Day 13)
Part 3 (Day 14)

Module M

Module G

All bases for a vector space are the same size.

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Activity 14.5

Prove that if $\{\mathbf{v}\}$ is a basis for V , then $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly dependent (assuming $\mathbf{w}_1 \neq \mathbf{w}_2$).

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Fact 14.6

All bases for a vector space are the same size.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

Module M

Module G

Definition 14.7

The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

Reduce a bunch of spans to bases to find their dimension.

Reduce a bunch of spans to bases to find their dimension.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)**

Module A

Module M

Module G

Activity 14.9

What is the dimension of the vector space of 7th-degree polynomials \mathcal{P}^7 ?

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)**

Module A

Module M

Module G

Activity 14.10

What is the dimension of the vector space of polynomials \mathcal{P} ?

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Observation 14.11

Several interesting vector spaces are infinite-dimensional:

- The space of polynomials \mathcal{P}
- The space of real number sequences \mathbb{R}^∞
- The space of continuous functions $C(\mathbb{R})$

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)**

Module A

Module M

Module G

Fact 14.12

Every vector space with dimension $n < \infty$ is isomorphic to \mathbb{R}^n .

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Module M

Module G

Module A: TODO

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

Module M

Module G

Module M: TODO

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

Module M

Module G

Module G: TODO