

Name: \_\_\_\_\_

**SEMIFINAL**

Math 237 – Linear Algebra

**Version 3**

Fall 2017

**Choose up to 6 problems to work.** Work each problem on one of the attached pages; write the standard in the upper left corner. Show all work and justify all of your answers. Answers without work or sufficient reasoning will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

**E1.** Write an augmented matrix corresponding to the following system of linear equations.

$$x + 3y - 4z = 5$$

$$3x + 9y + z = 0$$

$$x - z = 1$$

**Solution:**

$$\left[ \begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{array} \right]$$

□

**E2.** Find RREF  $A$ , where

$$A = \left[ \begin{array}{cccc|c} 3 & -2 & 1 & 8 & -5 \\ 2 & 2 & 0 & 6 & -2 \\ -1 & 1 & 1 & -4 & 6 \end{array} \right]$$

**Solution:**

$$\text{RREF } A = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \end{array} \right]$$

□

**E3.** Solve the system of linear equations.

$$2x + y - z + w = 5$$

$$3x - y - 2w = 0$$

$$-x + 5z + 3w = -1$$

**Solution:**

$$\text{RREF} \left( \left[ \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right] \right) = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{4}{7} & 3 \\ 0 & 0 & 1 & \frac{4}{12} & 0 \end{array} \right]$$

So the solutions are

$$\left\{ \left[ \begin{array}{c} 1+a \\ 3-21a \\ -7a \\ 12a \end{array} \right] \mid a \in \mathbb{R} \right\}$$

□

**E4.** Find a basis for the solution set to the homogeneous system of equations

$$4x_1 + 4x_2 + 3x_3 - 6x_4 = 0$$

$$-2x_3 - 4x_4 = 0$$

$$2x_1 + 2x_2 + x_3 - 4x_4 = 0$$

**Solution:** Let  $A = \left[ \begin{array}{cccc|c} 4 & 4 & 3 & -6 & 0 \\ 0 & 0 & -2 & -4 & 0 \\ 2 & 2 & 1 & -4 & 0 \end{array} \right]$ , so RREF  $A = \left[ \begin{array}{cccc|c} 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$ . It follows that the basis for the solution set is given by  $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ .

□

**V1.** Let  $V$  be the set of all points on the line  $x + y = 2$  with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 - 1, y_1 + y_2 - 1)$$

$$c \odot (x_1, y_1) = (cx_1 - (c - 1), cy_1 - (c - 1))$$

(a) Show that this vector space has an **additive identity** element  $\mathbf{0}$  satisfying  $(x, y) \oplus \mathbf{0} = (x, y)$ .

(b) Determine if  $V$  is a vector space or not. Justify your answer.

**Solution:** Let  $(x_1, y_1) \in V$ ; then  $(x_1, y_1) \oplus (1, 1) = (x_1, y_1)$ , so  $(1, 1)$  is an additive identity element.

Now we will show the other seven properties. Let  $(x_1, y_1), (x_2, y_2) \in V$ , and let  $c, d \in \mathbb{R}$ .

1) Since real addition is associative,  $\oplus$  is associative.

2) Since real addition is commutative,  $\oplus$  is commutative.

3) The additive identity is  $(1, 1)$ .

4)  $(x_1, y_1) \oplus (2 - x_1, 2 - y_1) = (1, 1)$ , so  $(2 - x_1, 2 - y_1)$  is the additive inverse of  $(x_1, y_1)$ .

5)

$$\begin{aligned} c \odot (d \odot (x_1, y_1)) &= c \odot (dx_1 - (d - 1), dy_1 - (d - 1)) \\ &= (c(dx_1 - (d - 1)) - (c - 1), c(dy_1 - (d - 1)) - (c - 1)) \\ &= (cdx_1 - cd + c - (c - 1), cdy_1 - cd + c - (c - 1)) \\ &= (cdx_1 - (cd - 1), cdy_1 - (cd - 1)) \\ &= (cd) \odot (x_1, y_1) \end{aligned}$$

6)  $1 \odot (x_1, y_1) = (x_1 - (1 - 1), y_1 - (1 - 1)) = (x_1, y_1)$

7)

$$\begin{aligned} c \odot ((x_1, y_1) \oplus (x_2, y_2)) &= c \odot (x_1 + y_1 - 1, x_2 + y_2 - 1) \\ &= (c(x_1 + y_1 - 1) - (c - 1), c(x_2 + y_2 - 1) - (c - 1)) \\ &= (cx_1 + cy_1 - 2c + 1, cx_2 + cy_2 - 2c + 1) \\ &= (cx_1 - (c - 1), cy_1 - (c - 1)) \oplus (cx_2 - (c - 1), cy_2 - (c - 1)) \\ &= c \odot (x_1, y_1) \oplus c \odot (x_2, y_2) \end{aligned}$$

8)

$$\begin{aligned}(c+d) \odot (x_1, y_1) &= ((c+d)x_1 - (c+d-1), (c+d)y_1 - (c+d-1)) \\ &= (cx_1 - (c-1), cy_1 - (c-1)) \oplus (dx_1 - (d-1), dy_1 - (d-1)) \\ &= c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)\end{aligned}$$

Therefore  $V$  is a vector space.

□

**V2.** Determine if  $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$  belongs to the span of the set  $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ .

**Solution:** Since

$$\text{RREF} \left( \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & -2 \\ -3 & -6 & 0 & 4 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

contains the contradiction  $0 = 1$ ,  $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$  is not a linear combination of the three vectors.

□

**V3.** Does  $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$ ?

**Solution:** Since

$$\text{RREF} \left[ \begin{array}{cccc} 2 & 3 & 1 & -4 \\ -1 & 12 & 2 & 2 \\ 4 & -9 & 3 & -8 \end{array} \right] = \left[ \begin{array}{cccc} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

lacks a zero row, the vectors span  $\mathbb{R}^3$ .

□

**V4.** Let  $W$  be the set of all complex numbers  $a + bi$  satisfying  $a = 2b$ . Determine if  $W$  is a subspace of  $\mathbb{C}$ .

**Solution:** Yes, because  $c(2b_1 + b_1i) + d(2b_2 + b_2i) = 2(cb_1 + db_2) + (cb_1 + db_2)i$  belongs to  $W$ . Alternately, yes because  $W$  is isomorphic to  $\mathbb{R}$ .

□

**S1.** Determine if the set of vectors  $\left\{ \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$  is linearly dependent or linearly independent

**Solution:**

$$\text{RREF} \left( \left[ \begin{array}{ccc} -3 & 1 & 0 \\ 8 & 2 & -1 \\ 0 & 2 & 3 \end{array} \right] \right) = \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

Every column is a pivot column, therefore the set is linearly independent.

□

**S2.** Determine if the set  $\left\{ \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 9 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^{2 \times 2}$  or not.

**Solution:**

$$\text{RREF} \begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.

□

**S3.** Let  $W$  be the subspace of  $\mathcal{P}^3$  given by  $W = \text{span}(\{x^3 + x^2 + 2x + 1, 3x^3 + 3x^2 + 6x + 3, 3x^3 - x^2 + 3x - 2, 7x^3 - x^2 + 8x - 3\})$ . Find a basis for  $W$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis is  $\{x^3 + x^2 + 2x + 1, 3x^3 - x^2 + 3x - 2\}$ .

□

**S4.** Let  $W = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -8 \\ -1 \end{bmatrix} \right\}$ . Find the dimension of  $W$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & -8 \\ 1 & 1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since it has two pivot columns, its dimension is 2.

□

**A1.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \\ 0 \end{bmatrix}.$$

Write the matrix for  $T$  with respect to the standard bases of  $\mathbb{R}^3$  and  $\mathbb{R}^4$ .

**Solution:**

$$\begin{bmatrix} 3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

□

**A2.** Determine if  $D : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$  given by  $D \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a - 3c$  is a linear transformation or not.

**A3.** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ .

(b)  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  given by the standard matrix  $\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 4 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix}$

**Solution:**

(a)  $\text{RREF} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Since each column is a pivot column,  $S$  is injective. Since there is no zero row,  $S$  is surjective.

(b) Since  $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$ ,  $T$  is not injective.

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 4 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since there are no zero rows,  $T$  is surjective.

□

**A4.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of  $T$ .

**Solution:** Let  $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$ , and compute  $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$ . Then a basis for the image is

its columns,

$$\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

And the kernel is the solution set of  $AX = 0$ , so a basis would be

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}$$

□

**M1.** Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

**Solution:**  $BC$  is the only one that can be computed, and

$$BC = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

□

**M2.** Determine if the matrix  $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$  is invertible.

**Solution:** This matrix is row equivalent to the identity matrix, so it is invertible.

□

**M3.** Find the inverse of the matrix  $\begin{bmatrix} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$ .

**Solution:**

$$\text{RREF} \left( \left[ \begin{array}{cccc|cccc} 8 & 5 & 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 5 & -3 & 1 & -2 & 0 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \right) = \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -5 & 12 \\ 0 & 1 & 0 & 0 & 1 & 1 & -4 & -9 \\ 0 & 0 & 1 & 0 & -4 & -7 & 20 & 47 \\ 0 & 0 & 0 & 1 & -1 & 0 & 3 & 7 \end{array} \right]$$

So the inverse is  $\begin{bmatrix} 1 & 2 & -5 & 12 \\ 1 & 1 & -4 & -9 \\ -4 & -7 & 20 & 47 \\ -1 & 0 & 3 & 7 \end{bmatrix}$ .

□

**G1.** Compute the determinant of the matrix

$$\begin{bmatrix} 0 & -4 & 1 & 1 \\ -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 0 & -4 & 0 \end{bmatrix}.$$

**Solution:**  $-55$ .

□

**G2.** Compute the eigenvalues, along with their algebraic multiplicities, of the matrix  $\begin{bmatrix} 9 & -3 & 2 \\ 19 & -6 & 5 \\ -11 & 4 & -2 \end{bmatrix}$ .

**Solution:** 1 with algebraic multiplicity 2, and -1 with algebraic multiplicity 1.

□

**G3.** Compute the eigenspace associated to the eigenvalue 2 in the matrix  $\begin{bmatrix} -1 & 1 & 0 \\ -9 & 5 & 0 \\ 15 & -5 & 2 \end{bmatrix}$ .

**Solution:** The eigenspace is the solution space of the system  $(B - 2I)X = 0$ .

$$\text{RREF}(B - 2I) = \text{RREF} \left( \begin{bmatrix} -3 & 1 & 0 \\ -9 & 3 & 0 \\ 15 & -5 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the system simplifies to  $x - \frac{y}{3} = 0$ , or  $3x = y$ . Thus the eigenspace is

$$E_2 = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$$

□

**G4.** Compute the geometric multiplicity of the eigenvalue  $-1$  in the matrix  $\begin{bmatrix} 4 & -2 & -1 \\ 15 & -7 & -3 \\ -5 & 2 & 0 \end{bmatrix}$ .

**Solution:**

$$\text{RREF}(A + I) = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the geometric multiplicity is 2.

□

**Standard:** \_\_\_\_\_





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