

## Module G: Geometry of Linear Maps

# How can we understand linear maps geometrically?

## Module G

Section G.1

Section G.2

Section G.3

At the end of this module, students will be able to...

- G1. Row operations.** ... represent a row operation as matrix multiplication, and compute how the operation affects the determinant.
- G2. Determinants.** ... compute the determinant of a square matrix.
- G3. Eigenvalues.** ... find the eigenvalues of a  $2 \times 2$  matrix.
- G4. Eigenvectors.** ... find a basis for the eigenspace of a square matrix associated with a given eigenvalue.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **A1**.
- Recall and use the definition of a linear transformation **A2**.
- Find all roots of quadratic polynomials (including complex ones).
- Interpret the statement “ $A$  is an invertible matrix” in many equivalent ways in different contexts.

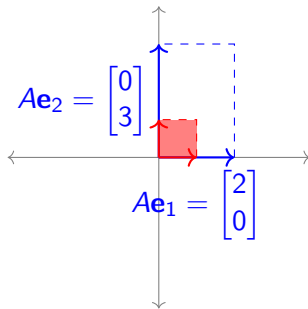
The following resources will help you prepare for this module.

- Finding the area of a parallelogram (Khan Academy):  
<http://bit.ly/2B05iWx>
- Factoring quadratics (Khan Academy): <http://bit.ly/1XjfbV2>
- Factoring quadratics using area models (Youtube):  
<https://youtu.be/Aa-v1EK7DR4>
- Finding complex roots of quadratics (Youtube):  
<https://www.youtube.com/watch?v=2yBhDsNE0wg>

# Module G Section 1

**Activity G.1.1** ( $\sim 5$  min)

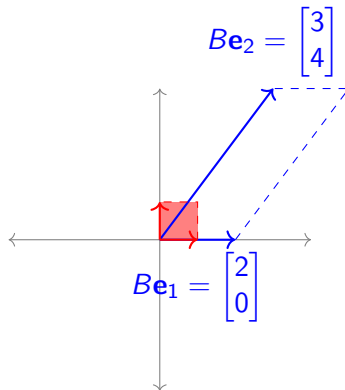
The image below illustrates how the linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$  transforms the unit square.



- (a) What are the lengths of  $Ae_1$  and  $Ae_2$ ?
- (b) What is the area of the transformed unit square?

**Activity G.1.2** ( $\sim 5$  min)

The image below illustrates how the linear transformation  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$  transforms the unit square.



- (a) What are the lengths of  $Be_1$  and  $Be_2$ ?
- (b) What is the area of the transformed unit square?

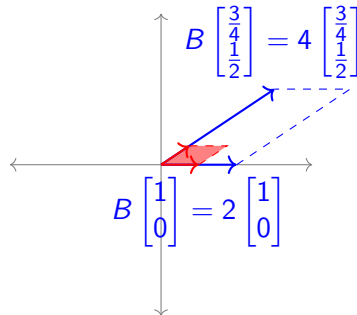


**Observation G.1.3**

It is possible to find two nonparallel vectors that are scaled but not rotated by the linear map given by  $B$ .

$$B\mathbf{e}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\mathbf{e}_1$$

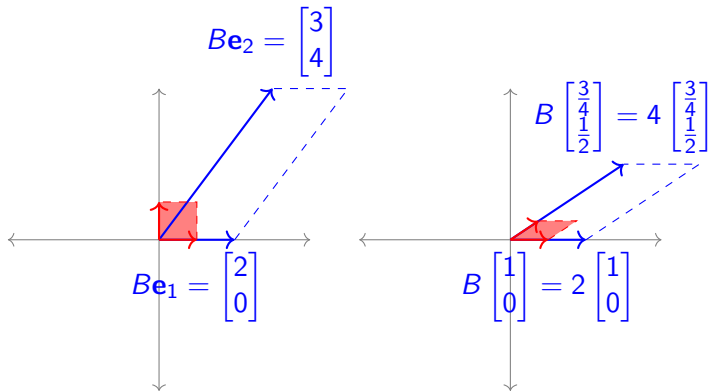
$$B \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$



The process for finding such vectors will be covered later in this module.

## Observation G.1.4

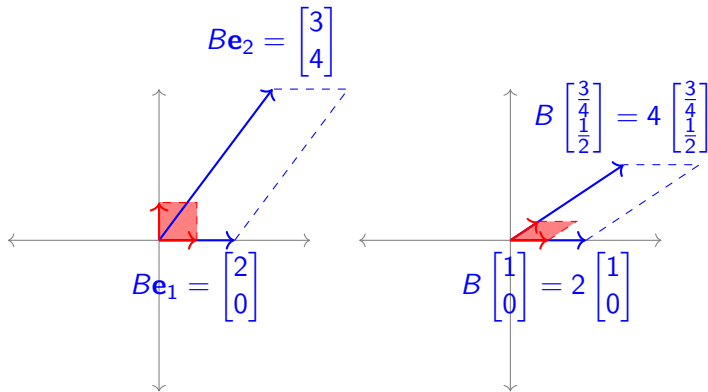
Notice that while a linear map can transform vectors in various ways, linear maps always transform parallelograms into parallelograms, and these areas are always transformed by the same factor: in the case of  $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ , this factor is 8.



Since this change in area is always the same for a given linear map, it will be equal to the value of the transformed unit square (which begins with area 1).

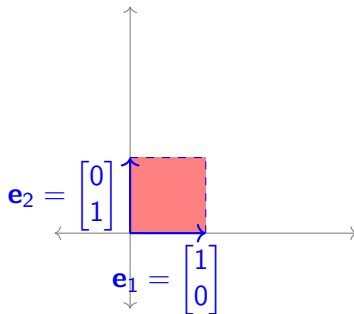
**Remark G.1.5**

We will define the **determinant** of a square matrix  $A$ , or  $\det(A)$  for short, to be the factor by which  $A$  scales areas. In order to figure out how to compute it, we first figure out the properties it must satisfy.



**Activity G.1.6** ( $\sim 2$  min)

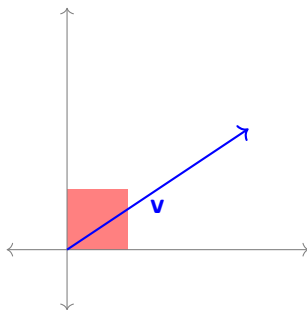
The transformation of the unit square by the standard matrix  $[\mathbf{e}_1 \ \mathbf{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$  is illustrated below. What is  $\det([\mathbf{e}_1 \ \mathbf{e}_2]) = \det(I)$ , the area of the transformed unit square shown here?



- a) 0
- b) 1
- c) 2
- d) Cannot be determined

**Activity G.1.7** ( $\sim 2$  min)

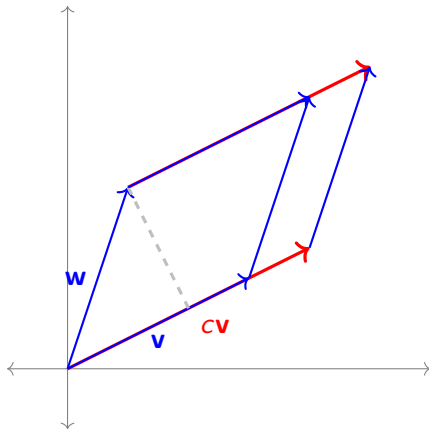
The transformation of the unit square by the standard matrix  $[\mathbf{v} \ \mathbf{v}]$  is illustrated below: both  $T(\mathbf{e}_1) = T(\mathbf{e}_2) = \mathbf{v}$ . What is  $\det([\mathbf{v} \ \mathbf{v}])$ , the area of the transformed unit square shown here?



- a) 0
- b) 1
- c) 2
- d) Cannot be determined

**Activity G.1.8** ( $\sim 5$  min)

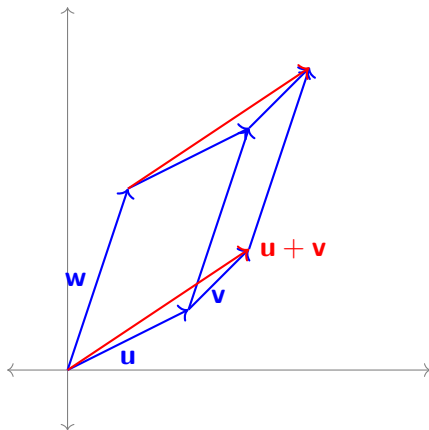
The transformations of the unit square by the standard matrices  $[\mathbf{v} \ \mathbf{w}]$  and  $[c\mathbf{v} \ \mathbf{w}]$  are illustrated below. How are  $\det([\mathbf{v} \ \mathbf{w}])$  and  $\det([c\mathbf{v} \ \mathbf{w}])$  related?



- a)  $\det([\mathbf{v} \ \mathbf{w}]) = \det([c\mathbf{v} \ \mathbf{w}])$
- b)  $c + \det([\mathbf{v} \ \mathbf{w}]) = \det([c\mathbf{v} \ \mathbf{w}])$
- c)  $c \det([\mathbf{v} \ \mathbf{w}]) = \det([c\mathbf{v} \ \mathbf{w}])$

**Activity G.1.9** ( $\sim 5$  min)

The transformations of unit squares by the standard matrices  $[\mathbf{u} \ \mathbf{w}]$ ,  $[\mathbf{v} \ \mathbf{w}]$  and  $[\mathbf{u} + \mathbf{v} \ \mathbf{w}]$  are illustrated below. How is  $\det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$  related to  $\det([\mathbf{u} \ \mathbf{w}])$  and  $\det([\mathbf{v} \ \mathbf{w}])$ ?



- a)  $\det([\mathbf{u} \ \mathbf{w}]) = \det([\mathbf{v} \ \mathbf{w}]) = \det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$
- b)  $\det([\mathbf{u} \ \mathbf{w}]) + \det([\mathbf{v} \ \mathbf{w}]) = \det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$
- c)  $\det([\mathbf{u} \ \mathbf{w}]) \det([\mathbf{v} \ \mathbf{w}]) = \det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$

### Definition G.1.10

The **determinant** is the unique function  $\det : M_{n,n} \rightarrow \mathbb{R}$  satisfying these properties:

P1:  $\det(I) = 1$

P2:  $\det(A) = 0$  whenever two columns of the matrix are identical.

P3:  $\det[\cdots \ c\mathbf{v} \ \cdots] = c \det[\cdots \ \mathbf{v} \ \cdots]$ , assuming no other columns change.

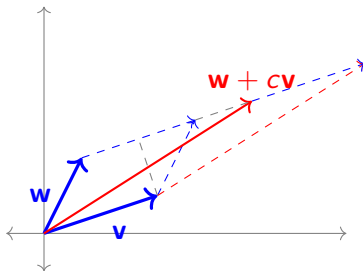
P4:  $\det[\cdots \ \mathbf{v} + \mathbf{w} \ \cdots] = \det[\cdots \ \mathbf{v} \ \cdots] + \det[\cdots \ \mathbf{w} \ \cdots]$ , assuming no other columns change.

Note that these last two properties together can be phrased as “The determinant is linear in each column.”



**Observation G.1.11**

The determinant must also satisfy other properties. Consider  $\det(\begin{bmatrix} \mathbf{v} & \mathbf{w} + c\mathbf{v} \end{bmatrix})$  and  $\det(\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix})$ .



The base of both parallelograms is  $\mathbf{v}$ , while the height has not changed, so the determinant does not change either. This can also be proven using the other properties of the determinant:

$$\begin{aligned}
 \det(\begin{bmatrix} \mathbf{v} + c\mathbf{w} & \mathbf{w} \end{bmatrix}) &= \det(\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}) + \det(\begin{bmatrix} c\mathbf{w} & \mathbf{w} \end{bmatrix}) \\
 &= \det(\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}) + c \det(\begin{bmatrix} \mathbf{w} & \mathbf{w} \end{bmatrix}) \\
 &= \det(\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}) + c \cdot 0 \\
 &= \det(\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix})
 \end{aligned}$$

## Observation G.1.12

Columns may be swapped by adding/subtracting columns from one another, which we've just seen doesn't change the determinant.

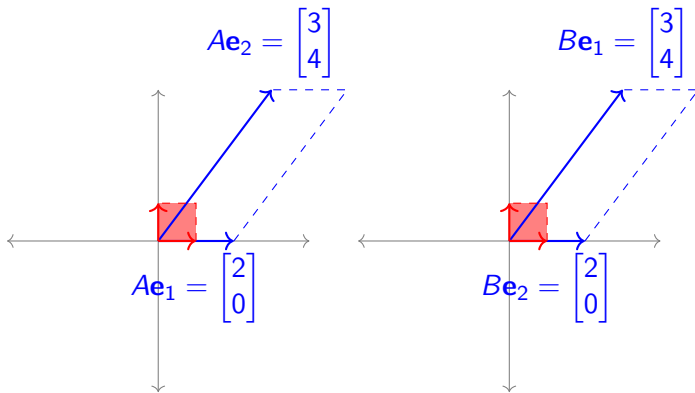
$$\begin{aligned}\det([\mathbf{v} \quad \mathbf{w}]) &= \det([\mathbf{v} + \mathbf{w} \quad \mathbf{w}]) \\ &= \det([\mathbf{v} + \mathbf{w} \quad \mathbf{w} - (\mathbf{v} + \mathbf{w})]) \\ &= \det([\mathbf{v} + \mathbf{w} \quad -\mathbf{v}]) \\ &= \det([\mathbf{v} + \mathbf{w} - \mathbf{v} \quad -\mathbf{v}]) \\ &= \det([\mathbf{w} \quad -\mathbf{v}]) \\ &= -\det([\mathbf{w} \quad \mathbf{v}])\end{aligned}$$

So swapping two columns results in a negation of the determinant. Therefore, determinants represent a *signed* area, since they are not always positive.

**Remark G.1.13**

Swapping columns may be thought of as a reflection, which is represented by a negative determinant. For example, the following matrices transform the unit square into the same parallelogram, but the second matrix reflects its orientation.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}$$



**Fact G.1.14**

To summarize, we've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant in the following way:

- (a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c \det([\cdots \mathbf{v} \cdots]) = \det([\cdots c\mathbf{v} \cdots])$$

- (b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \mathbf{v} \cdots \mathbf{w} \cdots]) = -\det([\cdots \mathbf{w} \cdots \mathbf{v} \cdots])$$

- (c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \mathbf{v} \cdots \mathbf{w} \cdots]) = \det([\cdots \mathbf{v} + c\mathbf{w} \cdots \mathbf{w} \cdots])$$

**Activity G.1.15** ( $\sim 5$  min)

The transformation given by the standard matrix  $A$  scales areas by 4, and the transformation given by the standard matrix  $B$  scales areas by 3. By what factor does the transformation given by the standard matrix  $AB$  scale areas?

- (a) 1
- (b) 7
- (c) 12
- (d) Cannot be determined

**Fact G.1.16**

Since the transformation given by the standard matrix  $AB$  is obtained by applying the transformations given by  $A$  and  $B$ , it follows that

$$\det(AB) = \det(A) \det(B)$$

**Remark G.1.17**

Recall that row operations may be produced by matrix multiplication.

- Multiply the first row of  $A$  by  $c$ :  $\begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$
- Swap the first and second row of  $A$ :  $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$
- Add  $c$  times the third row to the first row of  $A$ :  $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

**Fact G.1.18**

The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row:  $\det \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c$
- Swapping rows:  $\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1$
- Adding a row multiple to another row:  
$$\det \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & c - 1c \\ 0 & 1 & 0 - 0c \\ 0 & 0 & 1 - 0c \end{bmatrix} = \det(I) = 1$$



**Activity G.1.19** ( $\sim 5$  min)

Consider the row operation  $R_1 + 4R_3 \rightarrow R_1$  applied as follows to show  $A \sim B$ :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 + 4(7) & 2 + 4(8) & 3 + 4(9) \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix  $R$  such that  $B = RA$ , by applying the same row operation to

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Find  $\det R$  by comparing with the previous slide.

(c) If  $C \in M_{3,3}$  is a matrix with  $\det(C) = -3$ , find

$$\det(RC) = \det(R) \det(C).$$

**Activity G.1.20** ( $\sim 5$  min)

Consider the row operation  $R_1 \leftrightarrow R_3$  applied as follows to show  $A \sim B$ :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} = B$$

- (a) Find a matrix  $R$  such that  $B = RA$ , by applying the same row operation to  $I$ .
- (b) If  $C \in M_{3,3}$  is a matrix with  $\det(C) = 5$ , find  $\det(RC)$ .

**Activity G.1.21** ( $\sim 5$  min)

Consider the row operation  $3R_2 \rightarrow R_2$  applied as follows to show  $A \sim B$ :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 3(4) & 3(5) & 3(6) \\ 7 & 8 & 9 \end{bmatrix} = B$$

- (a) Find a matrix  $R$  such that  $B = RA$ .
- (b) If  $C \in M_{3,3}$  is a matrix with  $\det(C) = -7$ , find  $\det(RC)$ .

## Module G Section 2

## Remark G.2.1

Recall that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying columns by scalars:

$$\det([\cdots \ c\mathbf{v} \ \cdots]) = c \det([\cdots \ \mathbf{v} \ \cdots])$$

(b) Swapping two columns:

$$\det([\cdots \ \mathbf{v} \ \cdots \ \mathbf{w} \ \cdots]) = -\det([\cdots \ \mathbf{w} \ \cdots \ \mathbf{v} \ \cdots])$$

(c) Adding a multiple of a column to another column:

$$\det([\cdots \ \mathbf{v} \ \cdots \ \mathbf{w} \ \cdots]) = \det([\cdots \ \mathbf{v} + c\mathbf{w} \ \cdots \ \mathbf{w} \ \cdots])$$

## Remark G.2.2

The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row:  $\det \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c$
- Swapping rows:  $\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1$
- Adding a row multiple to another row:  
$$\det \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & c - 1c \\ 0 & 1 & 0 - 0c \\ 0 & 0 & 1 - 0c \end{bmatrix} = \det(I) = 1$$

**Fact G.2.3**

Thus we can also use row operations to simplify determinants:

① Multiplying rows by scalars:  $\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$

② Swapping two rows:  $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$

③ Adding multiples of rows to other rows:  $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$

**Observation G.2.4**

So we may compute the determinant of  $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$  by manipulating its rows/columns to reduce the matrix to  $I$ :

$$\begin{aligned}\det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} &= 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \\ &= 2 \det \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \\ &= -2 \det \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ &= -2 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= -2\end{aligned}$$



## Remark G.2.5

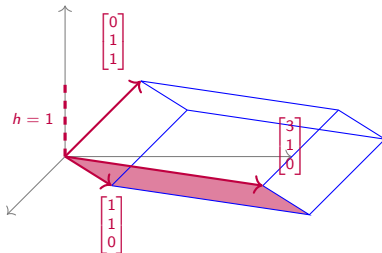
So we see that row reducing all the way into RREF gives us a method of computing determinants!

However, we learned in module E that this can be tedious for large matrices. Thus, we will try to figure out how to turn the determinant of a larger matrix into the determinant of a smaller matrix.

**Activity G.2.6** ( $\sim 5$  min)

The following image illustrates the transformation of the unit cube by the matrix

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$



Recall that for this solid  $V = Bh$ , where  $h$  is the height of the solid and  $B$  is the area of its parallelogram base. So what must its volume be?

(a)  $\det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

(b)  $\det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$

(c)  $\det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$

(d)  $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

**Fact G.2.7**

If row  $i$  contains all zeros except for a 1 on the main (upper-left to lower-right) diagonal, then both column and row  $i$  may be removed without changing the value of the determinant.

$$\det \begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Since row and column operations affect the determinants in the same way, the same technique works for a column of all zeros except for a 1 on the main diagonal.

$$\det \begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

**Activity G.2.8** (*~5 min*)

Remove an appropriate row and column of  $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$  to simplify the determinant to a  $2 \times 2$  determinant.

**Activity G.2.9** (*~5 min*)

Simplify  $\det \begin{bmatrix} 0 & 3 & -2 \\ 2 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$  to a multiple of a  $2 \times 2$  determinant by first doing the following:

- Factor out a 2 from a column.
- Swap rows or columns to put a 1 on the main diagonal.

**Activity G.2.10** ( $\sim 5$  min)

Simplify  $\det \begin{bmatrix} 4 & -2 & 2 \\ 3 & 1 & 4 \\ 1 & -1 & 3 \end{bmatrix}$  to a multiple of a  $2 \times 2$  determinant by first doing the following:

- Use row/column operations to create two zeroes in the same row or column.
- Factor/swap as needed to get a row/column of all zeroes except a 1 on the main diagonal.

**Observation G.2.11**

Using row/column operations, you can introduce zeros and reduce dimension to whittle down the determinant of a large matrix to a determinant of a smaller matrix.

$$\begin{aligned}
 \det \begin{bmatrix} 4 & 3 & 0 & 1 \\ 2 & -2 & 4 & 0 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} &= \det \begin{bmatrix} 4 & 3 & 0 & 1 \\ 6 & -18 & 0 & -20 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} = \det \begin{bmatrix} 4 & 3 & 1 \\ 6 & -18 & -20 \\ 2 & 8 & 3 \end{bmatrix} \\
 &= \dots = -2 \det \begin{bmatrix} 1 & 3 & 4 \\ 0 & 21 & 43 \\ 0 & -1 & -10 \end{bmatrix} = -2 \det \begin{bmatrix} 21 & 43 \\ -1 & -10 \end{bmatrix} \\
 &= \dots = -2 \det \begin{bmatrix} -167 & 21 \\ 0 & 1 \end{bmatrix} = -2 \det[-167] \\
 &= -2(-167) \det(I) = 334
 \end{aligned}$$

**Activity G.2.12** (*~10 min*)

Compute  $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$  by using any combination of row/column operations.



**Observation G.2.13**

Another option is to take advantage of the fact that the determinant is linear in each row or column. This approach is called **Laplace expansion** or **cofactor expansion**.

For example, since  $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ ,

$$\begin{aligned} \det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix} &= 1 \det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 1 & 0 & 0 \end{bmatrix} + 2 \det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 1 & 0 \end{bmatrix} + 4 \det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\ &= -1 \det \begin{bmatrix} 5 & 3 & 2 \\ 5 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 2 & 5 & 3 \\ -1 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix} + 4 \det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\ &= -\det \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} - 2 \det \begin{bmatrix} 2 & 5 \\ -1 & 5 \end{bmatrix} + 4 \det \begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix} \end{aligned}$$

**Observation G.2.14**

Applying Laplace expansion to a  $2 \times 2$  matrix yields a short formula you may have seen:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} + b \det \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} - b \det \begin{bmatrix} 1 & 0 \\ d & c \end{bmatrix} = ad - bc.$$

There are formulas for the determinants of larger matrices, but they can be pretty tedious to use. For example, writing out a formula for a  $4 \times 4$  determinant would require 24 different terms!

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}(a_{22}(a_{33}a_{44} - a_{43}a_{34}) - a_{23}(a_{32}a_{44} - a_{42}a_{34}) + \dots) + \dots$$

So this is why we either use Laplace expansion or row/column operations directly.

**Activity G.2.15** (*~10 min*)

Use Laplace expansion to compute  $\det \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & -3 & 2 & -2 \end{bmatrix}$ .

**Activity G.2.16** ( $\sim 5$  min)

Based on what we've done today, which technique is easier for computing determinants?

- (a) Memorizing formulas.
- (b) Using row/column operations.
- (c) Laplace expansion.
- (d) Some other technique (be prepared to describe it).

**Activity G.2.17** (*~10 min*)

Use your preferred technique to compute  $\det \begin{bmatrix} 4 & -3 & 0 & 0 \\ 1 & -3 & 2 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & -3 & 2 & -2 \end{bmatrix}$ .

## Module G Section 3

**Activity G.3.1** (*~5 min*)

An invertible matrix  $M$  and its inverse  $M^{-1}$  are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Compute  $\det(M)$  and  $\det(M^{-1})$  using the formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

## Fact G.3.2

- For every invertible matrix  $M$ ,

$$\det(M) \det(M^{-1}) = \det(I) = 1$$

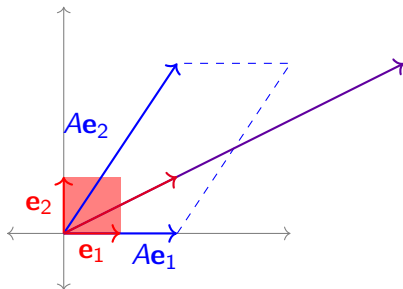
so  $\det(M^{-1}) = \frac{1}{\det(M)}$ .

- Furthermore, a square matrix  $M$  is invertible if and only if  $\det(M) \neq 0$ .



**Observation G.3.3**

Consider the linear transformation  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the matrix  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .



It is easy to see geometrically that

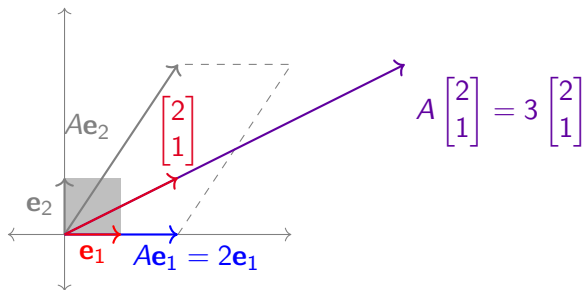
$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily checked once you find it) that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

### Definition G.3.4

Let  $A \in M_{n,n}$ . An **eigenvector** for  $A$  is a vector  $\mathbf{x} \in \mathbb{R}^n$  such that  $A\mathbf{x}$  is parallel to  $\mathbf{x}$ .



In other words,  $A\mathbf{x} = \lambda\mathbf{x}$  for some scalar  $\lambda$ . If  $\mathbf{x} \neq \mathbf{0}$ , then we say  $\mathbf{x}$  is a **nontrivial eigenvector** and we call this  $\lambda$  an **eigenvalue** of  $A$ .

**Activity G.3.5** ( $\sim 5$  min)Finding the eigenvalues  $\lambda$  that satisfy

$$A\mathbf{x} = \lambda\mathbf{x} = \lambda(I\mathbf{x}) = (\lambda I)\mathbf{x}$$

for some nontrivial eigenvector  $\mathbf{x}$  is equivalent to finding nonzero solutions for the matrix equation

$$A\mathbf{x} - (\lambda I)\mathbf{x} = \mathbf{0}.$$

Which of the following must be true for any eigenvalue?

- (a) The kernel of the transformation with standard matrix  $A - \lambda I$  must contain the zero vector, so  $A - \lambda I$  is invertible.
- (b) The kernel of the transformation with standard matrix  $A - \lambda I$  must contain a nonzero vector, so  $A - \lambda I$  is not invertible.
- (c) The image of the transformation with standard matrix  $A - \lambda I$  must contain the zero vector, so  $A - \lambda I$  is invertible.
- (d) The image of the transformation with standard matrix  $A - \lambda I$  must contain a nonzero vector, so  $A - \lambda I$  is invertible.

**Fact G.3.6**

The eigenvalues  $\lambda$  for a matrix  $A$  are the values that make  $A - \lambda I$  non-invertible.

Thus the eigenvalues  $\lambda$  for a matrix  $A$  are the solutions to the equation

$$\det(A - \lambda I) = 0.$$

**Definition G.3.7**

The expression  $\det(A - \lambda I)$  is called **characteristic polynomial** of  $A$ .

For example, when  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix}$$

Thus the characteristic polynomial of  $A$  is

$$\det \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) - (2)(3) = \lambda^2 - 5\lambda - 2$$

and its eigenvalues are the solutions to  $\lambda^2 - 5\lambda - 2 = 0$ .

**Activity G.3.8** ( $\sim 10$  min)

Compute  $\det(A - \lambda I)$  to find the characteristic polynomial of  $A = \begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$ .

**Activity G.3.9** ( $\sim 10$  min)

Let  $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$ .

**Activity G.3.9** ( $\sim 10$  min)

Let  $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$ .

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .



**Activity G.3.9** ( $\sim 10$  min)

Let  $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$ .

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .

*Part 2:* Factor this characteristic polynomial to determine the eigenvalues of  $A$ .

**Activity G.3.10** (*~10 min*)

Find all the eigenvalues for the matrix  $A = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix}$ .

**Activity G.3.11** (*~10 min*)

It's possible to show that  $-2$  is an eigenvalue for  $\begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 9 \\ 3 & 0 & 4 \end{bmatrix}$ .

Compute the kernel of the transformation with standard matrix

$$A - (-2)I = \begin{bmatrix} ? & 4 & -2 \\ 2 & ? & 9 \\ 3 & 0 & ? \end{bmatrix}$$

to find all the eigenvectors  $\mathbf{x}$  such that  $A\mathbf{x} = -2\mathbf{x}$ .

### Definition G.3.12

Since the kernel of a linear map is a subspace of  $\mathbb{R}^n$ , and the kernel obtained from  $A - \lambda I$  contains all the eigenvectors associated with  $\lambda$ , we call this kernel the **eigenspace** of  $A$  associated with  $\lambda$ .

**Activity G.3.13** (*~10 min*)

Find a basis for the eigenspace for the matrix  $\begin{bmatrix} 3 & -6 & 1 \\ -1 & 4 & 2 \\ 3 & -9 & 4 \end{bmatrix}$  associated with the eigenvalue 1.