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# MIDTERM EXAM

Math 237 – Linear Algebra

## Version 6

Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

<b>Standard E1.</b>	Mark:
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Write a system of linear equations corresponding to the following augmented matrix.

$$\left[ \begin{array}{ccc|c} -4 & -1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{array} \right]$$

**Solution:**

$$-4x_1 - x_2 + 3x_3 = 2$$

$$x_1 + 2x_2 - x_3 = 0$$

$$-x_1 + 4x_2 + x_3 = 4$$

□

<b>Standard E2.</b>	Mark:
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Find RREF  $A$ , where

$$A = \left[ \begin{array}{ccc|c} 2 & -1 & 5 & 4 \\ -1 & 0 & -2 & -1 \\ 1 & 3 & -1 & -5 \end{array} \right]$$

**Solution:**

$$\text{RREF } A = \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

□

<b>Standard E3.</b>	Mark:
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Solve the system of linear equations.

$$2x + y - z + w = 5$$

$$3x - y - 2w = 0$$

$$-x + 5z + 3w = -1$$

**Solution:**

$$\text{RREF} \left( \left[ \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right] \right) = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{12} & 3 \\ 0 & 0 & 1 & \frac{4}{12} & 0 \end{array} \right]$$

So the solutions are

$$\left\{ \left[ \begin{array}{c} 1+a \\ 3-21a \\ -7a \\ 12a \end{array} \right] \mid a \in \mathbb{R} \right\}$$

□

<b>Standard E4.</b>	Mark:
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Find a basis for the solution set of the system of equations

$$x + 2y + 3z + w = 0$$

$$3x - y + z + w = 0$$

$$2x - 3y - 2z = 0$$

**Solution:**

$$\text{RREF} \left( \left[ \begin{array}{cccc|c} 1 & -2 & 3 & 1 & 0 \\ 3 & -1 & 1 & 1 & 0 \\ 2 & -3 & -2 & 0 & 0 \end{array} \right] \right) = \left[ \begin{array}{cccc|c} 1 & 0 & \frac{5}{7} & \frac{3}{7} & 0 \\ 0 & 1 & \frac{8}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Then the solution set is

$$\left\{ \left[ \begin{array}{c} -\frac{5}{7}a - \frac{3}{7}b \\ -\frac{8}{7}a - \frac{2}{7}b \\ a \\ b \end{array} \right] \mid a, b \in \mathbb{R} \right\}$$

So a basis for the solution set is  $\left\{ \left[ \begin{array}{c} -\frac{5}{7} \\ -\frac{8}{7} \\ 1 \\ 0 \end{array} \right], \left[ \begin{array}{c} -\frac{3}{7} \\ -\frac{2}{7} \\ 0 \\ 1 \end{array} \right] \right\}$ , or  $\left\{ \left[ \begin{array}{c} 5 \\ 8 \\ -7 \\ 0 \end{array} \right], \left[ \begin{array}{c} 3 \\ 2 \\ 0 \\ -7 \end{array} \right] \right\}$ .

□

<b>Standard V1.</b>	Mark:
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Let  $V$  be the set of all pairs of real numbers with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$c \odot (x_1, y_1) = (c^2 x_1, c^3 y_1)$$

(a) Show that scalar multiplication **distributes scalars** over vector addition:

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2).$$

(b) Determine if  $V$  is a vector space or not. Justify your answer.

**Solution:** Let  $(x_1, y_1), (x_2, y_2) \in V$  and let  $c \in \mathbb{R}$ .

$$\begin{aligned} c \odot ((x_1, y_1) \oplus (x_2, y_2)) &= c \odot (x_1 + x_2, y_1 + y_2) \\ &= (c^2(x_1 + x_2), c^3(y_1 + y_2)) \\ &= (c^2x_1, c^3y_1) \oplus (c^2x_2, c^3y_2) \\ &= c \odot (x_1, y_1) \oplus c \odot (x_2, y_2) \end{aligned}$$

However,  $V$  is not a vector space, as the other distributive law fails:

$$(c + d) \odot (x_1, y_1) = ((c + d)^2x_1, (c + d)^3y_1) \neq ((c^2 + d^2)x_1, (c^3 + d^3)y_1) = c \odot (x_1, y_1) \oplus d \odot (x_1, y_1).$$

□

<b>Standard V2.</b>	Mark:
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Determine if  $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$  belongs to the span of the set  $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ .

**Solution:** Since

$$\text{RREF} \left( \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & -2 \\ -3 & -6 & 0 & 4 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

contains the contradiction  $0 = 1$ ,  $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$  is not a linear combination of the three vectors.

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<b>Standard V3.</b>	Mark:
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Does  $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$ ?

**Solution:** Since

$$\text{RREF} \left[ \begin{array}{cccc} 2 & 3 & 1 & -4 \\ -1 & 12 & 2 & 2 \\ 4 & -9 & 3 & -8 \end{array} \right] = \left[ \begin{array}{cccc} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

lacks a zero row, the vectors span  $\mathbb{R}^3$ .

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<b>Standard V4.</b>	Mark:
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Let  $W$  be the set of all  $\mathbb{R}^3$  vectors  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  satisfying  $x + y + z = 0$  (this forms a plane). Determine if  $W$  is a subspace of  $\mathbb{R}^3$ .

**Solution:** Yes, because  $z = -x - y$  and  $a \begin{bmatrix} x_1 \\ y_1 \\ -x_1 - y_1 \end{bmatrix} + b \begin{bmatrix} x_2 \\ y_2 \\ -x_2 - y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \\ -(ax_1 + bx_2) - (ay_1 + by_2) \end{bmatrix}$ .  
 Alternately, yes because  $W$  is isomorphic to  $\mathbb{R}^2$ .

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<b>Standard S1.</b>	Mark:
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Determine if the set of vectors  $\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$  is linearly dependent or linearly independent

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

This has a non pivot column, therefore the set is linearly dependent.

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<b>Standard S2.</b>	Mark:
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Determine if the set  $\left\{ \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 9 \end{bmatrix} \right\}$  is a basis of  $M_{2,2}$  or not.

**Solution:**

$$\text{RREF} \begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.

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<b>Standard S3.</b>	Mark:
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Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\} \right)$ . Find a basis for  $W$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis is  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}$ .

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<b>Standard S4.</b>	Mark:
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Let  $W$  be the subspace of  $\mathcal{P}_3$  given by

$W = \text{span}(\{x^3 - x^2 + 3x - 3, 2x^3 + x + 1, 3x^3 - x^2 + 4x - 2, x^3 + x^2 + x - 7\})$ . Compute the dimension of  $W$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so  $\dim(W) = 3$ .

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<b>Additional Notes/Marks</b>	
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