Name: ______________________________

SEMIFINAL Math 237 – Linear Algebra
Version 6 Fall 2017

Choose up to 6 problems to work. Work each problem on one of the attached pages; write the standard in the lower left corner. Show all work and justify all of your answers. Answers without work or sufficient reasoning will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

E1. Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x_1 + 4x_3 &= 1 \\ x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= -1 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array}\right]$$

□

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ -3 & 5 & 2 & 0 \\ 1 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & 6 \\ 0 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

E3. Solve the system of linear equations.

$$\begin{aligned} 2x + y - z + w &= 5 \\ 3x - y - 2w &= 0 \\ -x + 5z + 3w &= -1 \end{aligned}$$

Solution:

$$\mathrm{RREF}\left(\left[\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array}\right]\right) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{4} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{array}\right]$$

So the solutions are

$$\left\{ \begin{bmatrix} 1 + a \\ 3 - 21a \\ -7a \\ 12a \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

$\square$

E4. Find a basis for the solution set of the system of equations

$$\begin{aligned} x + 2y + 3z + w &= 0 \\ 3x - y + z + w &= 0 \\ 2x - 3y - 2z &= 0 \end{aligned}$$

Solution:

$$\text{RREF}\left(\begin{bmatrix} 1 & -2 & 3 & 1 \\ 3 & -1 & 1 & 1 \\ 2 & -3 & -2 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & \frac{5}{7} & \frac{3}{7} \\ 0 & 1 & \frac{8}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution set is

$$\left\{ \begin{bmatrix} -\frac{5}{7}a - \frac{3}{7}b \\ -\frac{8}{7}a - \frac{2}{7}b \\ a \\ b \end{bmatrix} \middle| \ a, b \in \mathbb{R} \right\}$$

So a basis for the solution set is $\left\{ \begin{bmatrix} -\frac{5}{7} \\ -\frac{8}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{7} \\ -\frac{2}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$, or $\left\{ \begin{bmatrix} 5 \\ 8 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ -7 \end{bmatrix} \right\}$.

$\square$

V1. Let V be the set of all real numbers together with the operations $\oplus$ and $\odot$ defined by, for any $x, y \in V$ and $c \in \mathbb{R}$,

$$\begin{aligned} x \oplus y &= x + y - 3 \\ c \odot x &= cx - 3(c - 1) \end{aligned}$$

(a) Show that **scalar multiplication** is **associative**: $a \odot (b \odot x) = (ab) \odot x$.

(b) Determine if V is a vector space or not. Justify your answer

Solution: Let $x, y \in V$, $c, d \in \mathbb{R}$. To show associativity:

$$\begin{aligned} c \odot (d \odot x) &= c \odot (dx - 3(d - 1)) \\ &= c\,(dx - 3(d - 1)) - 3(c - 1) \\ &= cdx - 3(cd - 1) \\ &= (cd) \odot x \end{aligned}$$

We verify the remaining 7 properties to see that V is a vector space.

1) Real addition is associative, so $\oplus$ is associative.

2) $x \oplus 3 = x + 3 - 3 = x$, so 3 is the additive identity.

3) $x \oplus (6 - x) = x + (6 - x) - 3 = 3$, so $6 - x$ is the additive inverse of x.

4) Real addition is commutative, so $\oplus$ is commutative.

5) Associativity shown above

6) $1 \odot x = x - 3(1-1) = x$

7)

$$\begin{aligned} c \odot (x \oplus y) &= c \odot (x + y - 3) \\ &= c(x+y-3) - 3(c-1) \\ &= cx - 3(c-1) + cy - 3(c-1) - 3 \\ &= (c \odot x) \oplus (c \odot y) \end{aligned}$$

8)

$$\begin{aligned} (c+d) \odot x &= (c+d)x - 3(c+d-1) \\ &= cx - 3(c-1) + dx - 3(c-1) - 3 \\ &= (c \odot x) \oplus (d \odot x) \end{aligned}$$

Therefore V is a vector space.

$\square$

V2. Determine if $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ belongs to the span of the set $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \\ 3 \end{bmatrix} \right\}$.

Solution: Since

$$\text{RREF}\left(\left[\begin{array}{cc|c} 2 & 4 & 4 \\ 0 & -1 & -1 \\ -1 & 4 & 6 \\ 5 & 3 & -7 \end{array}\right]\right) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

contains the contradiction $0 = 1$, $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ is not a linear combination of the three vectors.

$\square$

V3. Determine if the vectors $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ span $\mathbb{R}^4$.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, the vectors do not span $\mathbb{R}^4$.

$\square$

V4. Determine if $\left\{ \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^4$.

Solution: It is closed under addition and scalar multiplication, so it is a subspace. Alternatively, it is the image of the linear transformation from $\mathbb{R}^3 \to \mathbb{R}^4$ given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix}.$$

□

S1. Determine if the set of matrices $\left\{\begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -8 \\ 6 & 5 \end{bmatrix}\right\}$ is linearly dependent or linearly independent.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 3 & 1 & 3 \\ -1 & 2 & -8 \\ 0 & -2 & 6 \\ 4 & 1 & 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced row echelon form has a nonpivot column, the vectors are linearly dependent.

□

S2. Determine if the set $\left\{x^2 + x - 1, 3x^2 - x + 1, 2x^2 - 2\right\}$ is a basis of $\mathcal{P}^2$.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis.

□

S3. Let $W = \text{span}\left\{\begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -8 \\ -1 \end{bmatrix}\right\}$. Find a basis for this vector space.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & -8 \\ 1 & 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{\begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right\}$ is a basis of W.

□

S4. Let W be the subspace of $\mathbb{R}^{2\times 2}$ given by $W = \text{span}\left(\left\{\begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right\}\right)$. Compute the dimension of W.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$.

$\square$

A1. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 7x + 2y + 3z \\ 0 \end{bmatrix}.$$

Write the matrix for T with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^4$.

Solution:

$$\begin{bmatrix} 3 & 1 & 0 \\ -8 & 2 & -1 \\ 7 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\square$

A2. Determine if the map $T : \mathcal{P}^6 \to \mathcal{P}^7$ given by $T(f) = xf(x) - f(1)$ is a linear transformation or not.

A3. Determine if the following linear maps are injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y \\ x - y \\ x + 4y \end{bmatrix}$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ 2y + 3z \\ x - y - 2z \end{bmatrix}$

Solution:

(a)

$$\text{RREF}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, T is not injective. Since there is a zero row, T is not surjective.

(b)

$$\text{RREF}\left(\begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 1 & 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since all columns are pivot columns, S is injective. Since there is a zero row, S is not surjective.

$\square$

A4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 8x - 3y - z \\ y + 3z \\ -7x + 3y + 2z \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 8 & -3 & -1 \\ 0 & 1 & 3 \\ -7 & 3 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel.

$\square$

M1. Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & -1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution: AC is the only one that can be computed, and

$$AC = \begin{bmatrix} 3 & -5 & 11 \\ 1 & -1 & 2 \end{bmatrix}$$

$\square$

M2. Determine if the matrix $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$ is invertible.

Solution:

$$\text{RREF} \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since it is not equivalent to the identity matrix, it is not invertible.

$\square$

M3. Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$.

Solution: $\left[\begin{array}{ccc|ccc} 2 & -1 & -3 & 1 & 0 & 0 \\ -14 & 9 & 24 & 0 & 1 & 0 \\ 3 & -2 & -5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3 \\ 0 & 1 & 0 & 2 & -1 & -6 \\ 0 & 0 & 1 & 1 & 1 & 4 \end{array}\right]$. Thus the inverse is $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$.

$\square$

G1. Compute the determinant of the matrix $\begin{bmatrix} 3 & -1 & 0 & 7 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution: 2

$\square$

G2. Compute the eigenvalues, along with their algebraic multiplicities, of the matrix $\begin{bmatrix} 9 & -3 & 2 \\ 23 & -8 & 5 \\ -1 & 0 & 0 \end{bmatrix}$.

Solution: 1 with algebraic multiplicy 2, and -1 with algebraic multiplicity 1.

$\square$

G3. Compute the eigenspace associated to the eigenvalue 2 in the matrix $\begin{bmatrix} -1 & 1 & 0 \\ -9 & 5 & 0 \\ 15 & -5 & 2 \end{bmatrix}$.

Solution: The eigenspace is the solution space of the system $(B-2I)X=0$.

$$\text{RREF}(B-2I) = \text{RREF}\left(\begin{bmatrix} -3 & 1 & 0 \\ -9 & 3 & 0 \\ 15 & -5 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the system simplifies to $x - \dfrac{y}{3} = 0$, or $3x = y$. Thus the eigenspace is

$$E_2 = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

□

G4. Compute the geometric multiplicity of the eigenvalue -1 in the matrix $\begin{bmatrix} 4 & -2 & -1 \\ 15 & -7 & -3 \\ -5 & 2 & 0 \end{bmatrix}$.

Solution:

$$\text{RREF}\,(A+I) = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the geometric multiplicity is 2.

□

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