

Name: \_\_\_\_\_

**SEMIFINAL**

Math 237 – Linear Algebra

**Version 6**

Fall 2017

Work any problems you wish on the provided answer sheets; take care to label the standard for each response. Show all work and justify all of your answers. Answers without work or sufficient reasoning will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

**E1.** Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned}x_1 + 3x_2 - 4x_3 + x_4 &= 5 \\3x_1 + 9x_2 + x_3 - 7x_4 &= 0 \\x_1 - x_3 + x_4 &= 1\end{aligned}$$

**Solution:**

$$\left[ \begin{array}{cccc|c} 1 & 3 & -4 & 1 & 5 \\ 3 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{array} \right]$$

□

**E2.** Find the reduced row echelon form of the matrix below.

$$\left[ \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right]$$

**Solution:**

$$\begin{aligned} & \left[ \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right] \sim \left[ \begin{array}{cccc|c} -1 & 0 & 5 & 0 & -1 \\ 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \end{array} \right] \\ & \sim \left[ \begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & -1 & 15 & -2 & -3 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & 0 & 24 & -2 & 0 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{5}{12} & 1 \\ 0 & 1 & 0 & \frac{3}{4} & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{array} \right] \end{aligned}$$

□

**E3.** Solve the system of linear equations.

$$\begin{aligned}2x + y - z + w &= 5 \\3x - y - 2w &= 0 \\-x + 5z + 3w &= -1\end{aligned}$$

**Solution:**

$$\text{RREF} \left( \left( \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right) \right) = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{12} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{array} \right]$$

So the solutions are

$$\left\{ \left[ \begin{array}{c} 1+a \\ 3-21a \\ -7a \\ 12a \end{array} \right] \mid a \in \mathbb{R} \right\}$$

□

**E4.** Find a basis for the solution set of the system of equations

$$\begin{aligned}x + 3y + 3z + 7w &= 0 \\x + 3y - z - w &= 0 \\2x + 6y + 3z + 8w &= 0 \\x + 3y - 2z - 3w &= 0\end{aligned}$$

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution set is

$$\left\{ \begin{bmatrix} -3a - b \\ a \\ -2b \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

So a basis for the solution set is

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

□

**V1.** Let  $V$  be the set of all points on the parabola  $y = x^2$  with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$\begin{aligned}(x_1, y_1) \oplus (x_2, y_2) &= (x_1 - x_2, y_1 + y_2 - 2x_1x_2) \\c \odot (x_1, y_1) &= (cx_1, c^2y_1)\end{aligned}$$

(a) Show that scalar multiplication **distributes scalars** over vector addition:

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2).$$

(b) Determine if  $V$  is a vector space or not. Justify your answer.

**Solution:**

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1 - x_2, y_1 + y_2 - 2x_1x_2) = (c(x_1 - x_2), c^2(y_1 + y_2 - 2x_1x_2))$$

$$c \odot (x_1, y_1) \oplus c \odot (x_2, y_2) = (cx_1, c^2y_1) \oplus (cx_2, c^2y_2) = (cx_1 - cx_2, c^2y_1 + c^2y_2 - 2(cx_1)(cx_2))$$

Not a vector space as addition is not commutative.

□

**V2.** Determine if  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ , and  $\begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 3 & 1 & 5 & \mid & 1 \\ 0 & -1 & 1 & \mid & 4 \\ -1 & 4 & -6 & \mid & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 2 & \mid & 0 \\ 0 & 1 & -1 & \mid & 0 \\ 0 & 0 & 0 & \mid & 1 \end{bmatrix}$$

So  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is not a linear combination of the three vectors.

□

**V3.** Determine if the vectors  $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 1 \\ 0 \\ -3 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}$ , and  $\begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$  span  $\mathbb{R}^4$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since every row contains a pivot, the vectors span  $\mathbb{R}^4$ .

□

**V4.** Let  $W$  be the set of all polynomials of even degree. Determine if  $W$  is a subspace of the vector space of all polynomials.

**Solution:**  $W$  is closed under scalar multiplication, but not under addition. For example,  $x - x^2$  and  $x^2$  are both in  $W$ , but  $(x - x^2) + (x^2) = x \notin W$ .

□

**S1.** Determine if the set of vectors  $\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$  is linearly dependent or linearly independent

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

This has a non pivot column, therefore the set is linearly dependent.

□

**S2.** Determine if the set  $\left\{ \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 9 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^{2 \times 2}$  or not.

**Solution:**

$$\text{RREF} \begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.

□

**S3.** Let  $W = \text{span} \left( \left( \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right) \right)$ . Find a basis for  $W$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis is  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}.$

□

**S4.** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$ . Compute the dimension of  $W$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so  $\dim(W) = 3$ .

□

**A1.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the linear transformation given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = [x_2 + 3x_3].$$

Write the matrix for  $T$  with respect to the standard bases of  $\mathbb{R}^3$  and  $\mathbb{R}$ .

**Solution:**

$$\begin{bmatrix} 0 & 1 & 3 \end{bmatrix}$$

□

**A2.** Determine if  $D : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$  given by  $D \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$  is a linear transformation or not.

**Solution:**  $D(I) = 1$  but  $D(2I) = 4 \neq 2D(I)$ , so  $D$  is not linear.

□

**A3.** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ .

(b)  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  given by the standard matrix  $\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 4 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix}$

**Solution:**

(a)  $\text{RREF} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Since each column is a pivot column,  $S$  is injective. Since there is no zero row,  $S$  is surjective.

(b) Since  $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$ ,  $T$  is not injective.

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 3 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since there are no zero rows,  $T$  is surjective.

□

**A4.** Let  $T : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^3$  be the linear map given by  $T \left( \begin{bmatrix} a & b \\ x & y \end{bmatrix} \right) = \begin{bmatrix} a+x \\ 0 \\ b+y \end{bmatrix}$ . Compute a basis for the kernel and a basis for the image of  $T$ .

**Solution:** Rewrite as  $T' \left( \begin{bmatrix} a \\ b \\ x \\ y \end{bmatrix} \right) = \begin{bmatrix} a+x \\ 0 \\ b+y \end{bmatrix}$ .

$$\text{RREF} \left( \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Thus  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \right\}$  is a basis for the kernel.

□

**M1.** Let

$$A = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 3 & 7 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

**Solution:**  $CA$  is the only one that can be computed, and

$$CA = \begin{bmatrix} 2 & 6 & 11 & 1 \\ 1 & 3 & 7 & 2 \\ -1 & -3 & -5 & 0 \end{bmatrix}$$

□

**M2.** Determine if the matrix  $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$  is invertible.

**Solution:** This matrix is row equivalent to the identity matrix, so it is invertible.

□

**M3.** Find the inverse of the matrix  $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$ .

**Solution:**  $\left[ \begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 2 & -1 & -6 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -3 \\ 0 & 1 & 0 & -14 & 9 & 24 \\ 0 & 0 & 1 & 3 & -2 & -5 \end{array} \right]$ . Thus the inverse is  $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$ .

□

**G1.** Compute the determinant of the matrix  $\begin{bmatrix} 2 & 3 & 0 & 1 \\ -1 & 3 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 1 & -1 & 3 & 5 \end{bmatrix}$ .

**Solution:**  $-60$ .

□

**G2.** Compute the eigenvalues, along with their algebraic multiplicities, of the matrix  $\begin{bmatrix} 9 & -3 & 2 \\ 19 & -6 & 5 \\ -11 & 4 & -2 \end{bmatrix}$ .

**Solution:** 1 with algebraic multiplicity 2, and -1 with algebraic multiplicity 1.

□

**G3.** Compute the eigenspace of the eigenvalue  $-1$  in the matrix  $\begin{bmatrix} 4 & -2 & -1 \\ 15 & -7 & -3 \\ -5 & 2 & 0 \end{bmatrix}$ .

**Solution:**

$$\text{RREF}(A + I) = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenspace is spanned by  $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$ .

□

**G4.** Compute the geometric multiplicity of the eigenvalue  $-1$  in the matrix  $\begin{bmatrix} 4 & -2 & -1 \\ 15 & -7 & -3 \\ -5 & 2 & 0 \end{bmatrix}$ .

**Solution:**

$$\text{RREF}(A + I) = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the geometric multiplicity is 2.

□

**Standard:** \_\_\_\_\_



**Standard:** \_\_\_\_\_





**Standard:** \_\_\_\_\_



**Standard:** \_\_\_\_\_



**Standard:** \_\_\_\_\_



**Standard:** \_\_\_\_\_

