

Application Activities - Module S Part 2 - Class Day 13

Activity 13.1 Last time we saw that $\{x^3 + 1, x^2 + 2, 4 - 7x, 2x^3 + x\}$ is linearly independent. Show that it spans \mathcal{P}^3 .

Definition 13.2 A **basis** is a linearly independent set that spans a vector space.

Observation 13.3 A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

Activity 13.4 Which of the following sets are bases for \mathbb{R}^4 ?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Activity 13.5 If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for \mathbb{R}^4 , that means $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$ doesn't have a column without a pivot position, and doesn't have a row of zeros. What is $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$?

Fact 13.6 The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for \mathbb{R}^n if and only if $m = n$ and $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] =$

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, a basis for \mathbb{R}^n must have exactly n vectors and its square matrix must row-reduce to the **identity matrix** containing all zeros except for a downward diagonal of ones.

Activity 13.7 Consider the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$.

Part 1: Use $\text{RREF} \begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$ to identify which vector may be removed to make the set linearly independent.

Part 2: Find a basis for $\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$.
