

Section E.0

Definition E.1 A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector that can be plugged into the equation).

Remark E.2 In previous classes you likely used the variables x, y, z in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as x_i , and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

Definition E.3 A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \mid \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

Remark E.4 When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{array}{rcl} x_1 + 3x_3 & = & 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ -x_2 + x_3 & = & -2 \end{array}$$

Verbose standard form:

$$\begin{array}{rcl} 1x_1 + 0x_2 + 3x_3 & = & 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ 0x_1 - 1x_2 + 1x_3 & = & -2 \end{array}$$

Concise standard form:

$$\begin{array}{rcl} x_1 & + & 3x_3 = 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ -x_2 + x_3 & = & -2 \end{array}$$

Remark E.5 It will often be convenient to think of a system of equations as a vector equation. By applying vector operations and equating components, it is straightforward to see that the vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

is equivalent to the system of equations

$$\begin{array}{rcrcrcrcrcl} x_1 & & & & & + 3x_3 & = & 3 \\ 3x_1 & - 2x_2 & + 4x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & -2 \end{array}$$

Definition E.6 A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**.

Fact E.7 All linear systems are one of the following:

- **Consistent with one solution:** its solution set contains a single vector, e.g. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$
- **Consistent with infinitely-many solutions:** its solution set contains infinitely many vectors, e.g. $\left\{ \begin{bmatrix} 1 \\ 2 - 3a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$
- **Inconsistent:** its solution set is the empty set $\{\} = \emptyset$

Activity E.8 (*~10 min*) All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is \emptyset .

$$\begin{array}{rcrcrcrcrcl} -x_1 & + & 2x_2 & = & 5 \\ 2x_1 & - & 4x_2 & = & 6 \end{array}$$

Activity E.9 (*~10 min*) Consider the following consistent linear system.

$$\begin{array}{rcrcrcrcrcl} -x_1 & + & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 6 \end{array}$$

Part 1: Find three different solutions for this system.

Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a . Use this to write the solution set $\left\{ \begin{bmatrix} ? \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$ for the linear system.

Activity E.10 (*~10 min*) Consider the following linear system.

$$\begin{array}{rcl} x_1 + 2x_2 & - & x_4 = 3 \\ & & x_3 + 4x_4 = -2 \end{array}$$

Describe the solution set

$$\left\{ \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3 .