Name:	
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MIDTERM EXAM Math 237 – Linear Algebra
Version 2 Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

Standard E1.	Mark:

Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x_1 + 4x_3 &= 1 \\ x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= -1 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array}\right]$$

□

Standard E2.	Mark:

Find RREF A, where

$$A = \left[\begin{array}{ccc|c} 2 & -1 & 5 & 4 \\ -1 & 0 & -2 & -1 \\ 1 & 3 & -1 & -5 \end{array}\right]$$

Solution:

$$\text{RREF } A = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

□

Standard E3.	Mark:

Solve the following linear system.

$$\begin{aligned} 3x + 2y + z &= 7 \\ x + y + z &= 1 \\ -2x + 3z &= -11 \end{aligned}$$

Solution: Let $A = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 7 \\ 1 & 1 & 1 & 1 \\ -2 & 0 & 3 & 11 \end{array}\right]$, so RREF $A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right]$. It follows that the system has exactly one solution: $\begin{bmatrix} 4 & -2 & -1 \end{bmatrix}$

□

Standard E4.	Mark:

Find a basis for the solution set of the system of equations

$$\begin{aligned} x + 2y + 3z + w &= 0 \\ 3x - y + z + w &= 0 \\ 2x - 3y - 2z &= 0 \end{aligned}$$

Solution:

$$\text{RREF}\left(\begin{bmatrix} 1 & -2 & 3 & 1 \\ 3 & -1 & 1 & 1 \\ 2 & -3 & -2 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & \frac{5}{7} & \frac{3}{7} \\ 0 & 1 & \frac{8}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution set is

$$\left\{ \begin{bmatrix} -\frac{5}{7}a - \frac{3}{7}b \\ -\frac{8}{7}a - \frac{2}{7}b \\ a \\ b \end{bmatrix} \middle|\, a, b \in \mathbb{R} \right\}$$

So a basis for the solution set is $\left\{ \begin{bmatrix} -\frac{5}{7} \\ -\frac{8}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{7} \\ -\frac{2}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$, or $\left\{ \begin{bmatrix} 5 \\ 8 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ -7 \end{bmatrix} \right\}$.

□

Standard V1.	Mark:

Let V be the set of all polynomials with the operations, for any $f, g \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} f \oplus g &= f' + g' \\ c \odot f &= cf' \end{aligned}$$

(here f' denotes the derivative of f).

(a) Show that scalar multiplication **distributes scalars** over vector addition: $c \odot (f \oplus g) = c \odot f \oplus c \odot g$.

(b) Determine if V is a vector space or not. Justify your answer.

Solution: Let $f, g \in \mathcal{P}$, and let $c \in \mathbb{R}$.

$$c \odot (f \oplus g) = c \odot (f' + g') = c(f' + g')' = cf'\,' + cg'\,' = cf' \oplus cg' = c \odot f \oplus c \odot g.$$

However, this is not a vector space, as there is no zero vector. Additionally, $1 \odot f \neq f$ for any nonzero polynomial f.

$\square$

Standard V2.	Mark:

Determine if $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}$.

Solution:

$$\text{RREF}\left(\left[\begin{array}{ccc|c} 2 & 1 & -3 & 1 \\ 3 & -1 & -2 & 4 \\ -1 & 0 & 5 & 3 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Since this system has a solution, $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is a linear combination of the three vectors.

$\square$

Standard V3.	Mark:

Does $\operatorname{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$?

Solution: Since

$$\text{RREF}\begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 2 & 2 \\ 4 & -9 & 3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

lacks a zero row, the vectors span $\mathbb{R}^3$.

$\square$

Standard V4.	Mark:

Let W be the set of all complex numbers $a + bi$ satisfying $a = 2b$. Determine if W is a subspace of $\mathbb{C}$.

Solution: Yes, because $c(2b_1 + b_1 i) + d(2b_2 + b_2 i) = 2(cb_1 + db_2) + (cb_1 + db_2)i$ belongs to W. Alternately, yes because W is isomorphic to $\mathbb{R}$.

$\square$

Standard S1.	Mark:

Determine if the vectors $\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\begin{bmatrix}3\\-1\\1\end{bmatrix}$, and $\begin{bmatrix}2\\0\\-2\end{bmatrix}$ are linearly dependent or linearly independent

Solution:

$$\text{RREF}\left(\begin{bmatrix}1 & 3 & 2\\1 & -1 & 0\\-1 & 1 & -2\end{bmatrix}\right)=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Since each column is a pivot column, the vectors are linearly independent.

□

Standard S2.	Mark:

Determine if the set $\left\{\begin{bmatrix}1 & -3\\2 & 2\end{bmatrix},\begin{bmatrix}-1 & 4\\-1 & 1\end{bmatrix},\begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix},\begin{bmatrix}0 & 3\\3 & 9\end{bmatrix}\right\}$ is a basis of $M_{2,2}$ or not.

Solution:

$$\text{RREF}\begin{bmatrix}1 & -1 & -1 & 0\\-3 & 4 & 0 & 3\\2 & -1 & 2 & 3\\2 & 1 & 1 & 9\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 3\\0 & 1 & 0 & 3\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.

□

Standard S3.	Mark:

Let $W=\text{span}\left(\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix},\begin{bmatrix}2\\0\\1\\1\end{bmatrix},\begin{bmatrix}3\\-1\\4\\-2\end{bmatrix},\begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}\right)$. Find a basis of W.

Solution:

$$\text{RREF}\left(\begin{bmatrix}1 & 2 & 3 & 1\\-1 & 0 & -1 & 1\\3 & 1 & 4 & 1\\-3 & 1 & -2 & -7\end{bmatrix}\right)=\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

Then $\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix},\begin{bmatrix}2\\0\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}$ is a basis for W.

□

Standard S4.	Mark:

Let W be the subspace of $\mathcal{P}_3$ given by
$W = \operatorname{span}\left(\left\{x^3 - x^2 + 3x - 3, 2x^3 + x + 1, 3x^3 - x^2 + 4x - 2, x^3 + x^2 + x - 7\right\}\right)$. Compute the dimension of W.

Solution:

$$\mathrm{RREF}\left(\begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$.

$\square$

Additional Notes/Marks	