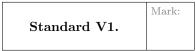
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J#:	Dr. Clontz
Date:	

Math 237 – Linear Algebra Fall 2017

Version 1

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.



Let V be the set of all pairs of real numbers with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V, c \in \mathbb{R}$ ,

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
  
 $c \odot (x_1, y_1) = (0, cy_1)$ 

- (a) Show that scalar multiplication **distributes vectors** over scalar addition:  $(c+d)\odot(x,y)=c\odot(x,y)\oplus d\odot(x,y).$
- (b) Determine if V is a vector space or not. Justify your answer.

**Solution:** Let  $(x_1, y_1) \in V$ , and let  $c, d \in \mathbb{R}$ . Then

$$(c+d)\odot(x_1,y_1)=(0,(c+d)y_1)=(0,cy_1)\oplus(0,dy_1)=c\odot(x_1,y_1)\oplus d\odot(x_1,y_1).$$

However, V is not a vector space, as  $1 \odot (x_1, y_1) = (0, y_1) \neq (x_1, y_1)$ .

Standard V3.

Does span 
$$\left\{ \begin{bmatrix} 2\\-1\\4 \end{bmatrix}, \begin{bmatrix} 3\\12\\-9 \end{bmatrix}, \begin{bmatrix} 1\\2\\3 \end{bmatrix}, \begin{bmatrix} -4\\2\\-8 \end{bmatrix} \right\} = \mathbb{R}^3$$
?

Solution: Since

$$RREF \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 2 & 2 \\ 4 & -9 & 3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

lacks a zero row, the vectors span  $\mathbb{R}^3$ .

Standard V4.

Let W be the set of all complex numbers that are purely real (i.e of the form a + 0i) or purely imaginary (i.e. of the form 0 + bi). Determine if W is a subspace of  $\mathbb{C}$ .

**Solution:** No, because 1 is purely real and i is purely imaginary, but the linear combination 1+i is neither.

Standard S2.

Mark:

Determine if the set  $\{x^3 - 3x^2 + 2x + 2, -x^3 + 4x^2 - x + 1, -x^3 + 2x + 1, 3x^2 + 3x + 9\}$  is a basis of  $\mathcal{P}^3$  or not.

Solution:

RREF 
$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

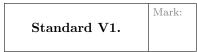
Since this is not the identity matrix, the set is not a basis.

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### Version 2

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.



Let V be the set of all polynomials with the operations, for any  $f, g \in V$ ,  $c \in \mathbb{R}$ ,

$$f \oplus g = f' + g'$$
$$c \odot f = cf'$$

(here f' denotes the derivative of f).

- (a) Show that scalar multiplication **distributes scalars** over vector addition:  $c \odot (f \oplus g) = c \odot f \oplus c \odot g$ .
- (b) Determine if V is a vector space or not. Justify your answer.

**Solution:** Let  $f, g \in \mathcal{P}$ , and let  $c \in \mathbb{R}$ .

$$c \odot (f \oplus g) = c \odot (f' + g') = c(f' + g')' = cf'' + cg'' = cf' \oplus cg' = c \odot f \oplus c \odot g.$$

However, this is not a vector space, as there is no zero vector. Additionally,  $1 \odot f \neq f$  for any nonzero polynomial f.

Standard V3.

Mark:  $\begin{bmatrix} 8 \\ 21 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}, \text{ and } \begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix} \text{ span } \mathbb{R}^3.$ 

Solution:

RREF 
$$\begin{pmatrix} \begin{bmatrix} 8 & -3 & -1 & 4 \\ 21 & -8 & -3 & 11 \\ -7 & 3 & 2 & -5 \end{bmatrix} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the rank is less than 3, they do not span  $\mathbb{R}^3$ .

Standard V4.

Let W be the set of all complex numbers that are purely real (i.e of the form a + 0i) or purely imaginary (i.e. of the form 0 + bi). Determine if W is a subspace of  $\mathbb{C}$ .

**Solution:** No, because 1 is purely real and i is purely imaginary, but the linear combination 1+i is neither.

Standard S2.

Mark:

Determine if the set  $\left\{ \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 9 \end{bmatrix} \right\}$  is a basis of  $M_{2,2}$  or not.

Solution:

RREF 
$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.

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Version 3

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

	Mark:
Standard V1.	

Let V be the set of all points on the line x + y = 2 with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V, c \in \mathbb{R}$ ,

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 - 1, y_1 + y_2 - 1)$$
  
 $c \odot (x_1, y_1) = (cx_1 - (c - 1), cy_1 - (c - 1))$ 

- (a) Show that this vector space has an additive identity element 0 satisfying  $(x,y) \oplus 0 = (x,y)$ .
- (b) Determine if V is a vector space or not. Justify your answer.

**Solution:** Let  $(x_1, y_1) \in V$ ; then  $(x_1, y_1) \oplus (1, 1) = (x_1, y_1)$ , so (1, 1) is an additive identity element. Now we will show the other seven properties. Let  $(x_1, y_1), (x_2, y_2) \in V$ , and let  $c, d \in \mathbb{R}$ .

- 1) Since real addition is associative,  $\oplus$  is associative.
- 2) Since real addition is commutative,  $\oplus$  is commutative.
- 3) The additive identity is (1,1).
- 4)  $(x_1, y_1) \oplus (2 x_1, 2 y_1) = (1, 1)$ , so  $(2 x_1, 2 y_1)$  is the additive inverse of  $(x_1, y_1)$ .

5)

$$\begin{split} c\odot(d\odot(x_1,y_1)) &= c\odot(dx_1-(d-1),dy_1-(d-1))\\ &= (c(dx_1-(d-1))-(c-1),c(dy_1-(d-1)))\\ &= (cdx_1-cd+c-(c-1),cdy_1-cd+c-(c-1))\\ &= (cdx_1-(cd-1),cdy_1-(cd-1))\\ &= (cd)\odot(x_1,y_1) \end{split}$$

6) 
$$1 \odot (x_1, y_1) = (x_1 - (1 - 1), y_1 - (1 - 1)) = (x_1, y_1)$$

7)

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1 + y_1 - 1, x_2 + y_2 - 1)$$

$$= (c(x_1 + y_1 - 1) - (c - 1), c(x_2 + y_2 - 1) - (c - 1))$$

$$= (cx_1 + cx_2 - 2c + 1, cy_1 + cy_2 - 2c + 1)$$

$$= (cx_1 - (c - 1), cy_1 - (c - 1)) \oplus (cx_2 - (c - 1), cy_2 - (c - 1))$$

$$= c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

$$(c+d) \odot (x_1, y_1) = ((c+d)x_1 - (c+d-1), (c+d)y_1 - (c+d-1))$$
  
=  $(cx_1 - (c-1), cy_1 - (c-1)) \oplus (dx_1 - (d-1), dy_1 - (d-1))$   
=  $c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$ 

Therefore V is a vector space.

Standard V3.	Mark:					
Determine if the vectors	$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} \\ \\ - \end{bmatrix}$	$\begin{bmatrix} 3 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}$	, and	$\begin{bmatrix} -1\\1\\-1\\-1\end{bmatrix}$	span $\mathbb{R}^4$ .

Solution:

$$RREF \left( \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since every row contains a pivot, the vectors span  $\mathbb{R}^4$ .

# Standard V4.

Let W be the set of all  $\mathbb{R}^3$  vectors  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  satisfying x + y + z = 0 (this forms a plane). Determine if W is a subspace of  $\mathbb{R}^3$ .

Determine if the set  $\left\{\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 9 \end{bmatrix}\right\}$  is a basis of  $M_{2,2}$  or not.

Solution:

RREF 
$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.

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Version 4

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

Let V be the set of all real numbers with the operations, for any  $x, y \in V$ ,  $c \in \mathbb{R}$ ,

$$x \oplus y = \sqrt{x^2 + y^2}$$
$$c \odot x = cx$$

- (a) Show that the vector addition  $\oplus$  is associative:  $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ .
- (b) Determine if V is a vector space or not. Justify your answer.

**Solution:** Let  $x, y, z \in \mathbb{R}$ . Then

$$(x \oplus y) \oplus z = \sqrt{x^2 + y^2} \oplus z$$

$$= \sqrt{(\sqrt{x^2 + y^2})^2 + z^2}$$

$$= \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{x^2 + (\sqrt{y^2 + z^2})^2}$$

$$= x \oplus \sqrt{y^2 + z^2}$$

$$= x \oplus (y \oplus z)$$

However, this is not a vector space, as there is no zero vector.

Determine if the vectors 
$$\begin{bmatrix} 1\\0\\2\\1 \end{bmatrix}$$
,  $\begin{bmatrix} 3\\1\\0\\-3 \end{bmatrix}$ ,  $\begin{bmatrix} 0\\3\\0\\-2 \end{bmatrix}$ , and  $\begin{bmatrix} -1\\1\\-1\\-1 \end{bmatrix}$  span  $\mathbb{R}^4$ .

Solution:

$$RREF \left( \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since every row contains a pivot, the vectors span  $\mathbb{R}^4$ .

Standard V4.	Mark:

Determine if the set of all lattice points, i.e.  $\{(x,y) \mid x \text{ and } y \text{ are integers}\}$  is a subspace of  $\mathbb{R}^2$ .

Solution: This set is closed under addition, but not under scalar multiplication so it is not a subspace.

Standard S2.

Mark:

Determine if the set  $\{x^3 - 3x^2 + 2x + 2, -x^3 + 4x^2 - x + 1, -x^3 + 2x + 1, 3x^2 + 3x + 9\}$  is a basis of  $\mathcal{P}^3$  or not.

Solution:

RREF 
$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

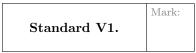
Since this is not the identity matrix, the set is not a basis.

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Version 5

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.



Let V be the set of all polynomials with the operations, for any  $f, g \in V$ ,  $c \in \mathbb{R}$ ,

$$f \oplus g = f' + g'$$
$$c \odot f = cf'$$

(here f' denotes the derivative of f).

- (a) Show that scalar multiplication **distributes scalars** over vector addition:  $c \odot (f \oplus g) = c \odot f \oplus c \odot g$ .
- (b) Determine if V is a vector space or not. Justify your answer.

**Solution:** Let  $f, g \in \mathcal{P}$ , and let  $c \in \mathbb{R}$ .

$$c \odot (f \oplus g) = c \odot (f' + g') = c(f' + g')' = cf'' + cg'' = cf' \oplus cg' = c \odot f \oplus c \odot g.$$

However, this is not a vector space, as there is no zero vector. Additionally,  $1 \odot f \neq f$  for any nonzero polynomial f.

Does span 
$$\left\{ \begin{bmatrix} 2\\-1\\4\\2\\1 \end{bmatrix}, \begin{bmatrix} -1\\3\\5\\2\\0 \end{bmatrix}, \begin{bmatrix} 1\\0\\5\\1\\-3 \end{bmatrix} \right\} = \mathbb{R}^5?$$

**Solution:** Since there are only three vectors, they cannot span  $\mathbb{R}^5$ .

Standard V4.

Let W be the set of all complex numbers a + bi satisfying a = 2b. Determine if W is a subspace of  $\mathbb{C}$ .

**Solution:** Yes, because  $c(2b_1 + b_1i) + d(2b_2 + b_2i) = 2(cb_1 + db_2) + (cb_1 + db_2)i$  belongs to W. Alternately, yes because W is isomorphic to  $\mathbb{R}$ .

Standard S2.  $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  is a basis of  $\mathbb{R}^2$ 

Solution:

$$RREF \left( \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.

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### Version 6

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

Standard V1.	Mark:

Let V be the set of all pairs of real numbers with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
  
 $c \odot (x_1, y_1) = (0, cy_1)$ 

- (a) Show that scalar multiplication **distributes vectors** over scalar addition:  $(c+d)\odot(x,y)=c\odot(x,y)\oplus d\odot(x,y).$
- (b) Determine if V is a vector space or not. Justify your answer.

**Solution:** Let  $(x_1, y_1) \in V$ , and let  $c, d \in \mathbb{R}$ . Then

$$(c+d)\odot(x_1,y_1)=(0,(c+d)y_1)=(0,cy_1)\oplus(0,dy_1)=c\odot(x_1,y_1)\oplus d\odot(x_1,y_1).$$

However, V is not a vector space, as  $1 \odot (x_1, y_1) = (0, y_1) \neq (x_1, y_1)$ .

Determine if the vectors  $\begin{bmatrix} 1\\1\\2\\1 \end{bmatrix}$ ,  $\begin{bmatrix} 3\\3\\6\\3 \end{bmatrix}$ ,  $\begin{bmatrix} 3\\-1\\3\\-2 \end{bmatrix}$ , and  $\begin{bmatrix} 7\\-1\\8\\-3 \end{bmatrix}$  span  $\mathbb{R}^4$ .

Solution:

$$RREF \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are zero rows, they do not span. Alternatively, by inspection  $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ , so the set is linearly

dependent, so it spans a subspace of dimension at most 3, therefore it does not span  $\mathbb{R}^4$ .

Standard V4.	Mark:
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Let W be the set of all complex numbers that are purely real (i.e of the form a + 0i) or purely imaginary (i.e. of the form 0 + bi). Determine if W is a subspace of  $\mathbb{C}$ .

**Solution:** No, because 1 is purely real and i is purely imaginary, but the linear combination 1+i is neither.

Standard S2.

Determine if the set  $\{x^3 - 3x^2 + 2x + 2, -x^3 + 4x^2 - x + 1, -x^3 + 2x + 1, 3x^2 + 3x + 9\}$  is a basis of  $\mathcal{P}^3$  or not.

Solution:

$$RREF \begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.