Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing **(Standard(s) E1, E2, E3, E4)**.
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent **(Standard(s) V5)**.
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace **(Standard(s) V6, V7)**.
- State the definition of a basis, and determine if a set of vectors is a basis **(Standard(s) V8, V9)**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- TODO

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Which of the following is a solution to the system of linear equations

$$\begin{aligned} x + 3y - z &= 2 \\ 2x + 8y + 3z &= -1 \\ -x - y + 9z &= -10 \end{aligned}$$

(a) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

2) Find a basis for the solution set of the following homogeneous system of linear equations

$$\begin{aligned} x + 2y + -z - w &= 0 \\ -2x - 4y + 3z + 5w &= 0 \end{aligned}$$

(a) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$ (d) None of these are a basis.

3) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

4) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

5) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent

(c) It spans but it is linearly dependent

(d) It is a basis of $\mathbb{R}^3$.

6) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent

(b) It does not span and is linearly independent

(c) It spans but it is linearly dependent

(d) It is a basis of $\mathbb{R}^3$.

7) Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors ...

8) Suppose you know that every vector in $\mathbb{R}^5$ can be written as a linear combination of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. What can you conclude about n?

(a) $n \leq 5$

(b) $n = 5$

(c) $n \geq 5$

(d) n could be any positive integer

9) Suppose you know that every vector in $\mathbb{R}^5$ can be written uniquely as a linear combination of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. What can you conclude about n?

(a) $n \leq 5$

(b) $n = 5$

(c) $n \geq 5$

(d) n could be any positive integer

10) Suppose you know that every vector in $\mathbb{R}^5$ can be written uniquely as a linear combination of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. What can you conclude about the set $\{\vec{v}_1, \ldots, \vec{v}_n\}$?

(a) It does not span and is linearly dependent

(b) It does not span and is linearly independent

(c) It spans but it is linearly dependent

(d) It is a basis of $\mathbb{R}^3$.

Application Activities - Day 1

Definition. A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}$, $\vec{v} \in V$.

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.
V is called the **domain** of T and W is called the **co-domain** of T.

Activity. Determine if each of the following maps are linear transformations

(a) $T_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $T_1\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \sqrt{a^2 + b^2}$

(b) $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$

(c) $T_3 : \mathcal{P}_d \to \mathcal{P}_{d-1}$ given by $T_3(f(x)) = f'(x)$.

(d) $T_4 : C(\mathbb{R}) \to C(\mathbb{R})$ given by $T_4(f(x)) = f(-x)$

(e) $T_5 : \mathcal{P} \to \mathcal{P}$ given by $T_5(f(x)) = f(x) + x^2$

Activity. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute each of the following:

(a) $T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$

(b) $T\left(\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}\right)$

(c) $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

(d) $T\left(\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}\right)$

Activity. Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation. What is the smallest number of vectors needed to determine T? In other words, what is the smallest number n such that there are $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^4$ and given $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ you can determine $T(\vec{w})$ for *any* $\vec{w} \in \mathbb{R}^2$?

Observation. Fix an ordered basis for V. Since every vector can be written *uniquely* as a linear combination of basis vectors, a linear transformation $T : V \to W$ corresponds exactly to a choice of where each basis vector goes. For convenience, we can thus encode a linear transformation as a matrix, with one column for the image of each basis vector (in order).

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard ordered basis.

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the ordered basis

$$\left\{\begin{bmatrix}2 & 1 & 1\end{bmatrix}, \begin{bmatrix}-1 & -1 & 3\end{bmatrix}, \begin{bmatrix}0 & 1 & 2\end{bmatrix}\right\}$$

Activity. Let $D : \mathcal{P}_3 \to \mathcal{P}_2$ be the derivative map (recall this is a linear transformation). Write the matrix corresponding to D with respect to the ordered basis $\{1, x, x^2, x^3\}$.

Application Activities - Day 2

Definition. Let $T : V \to W$ be a linear transformation.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.
- T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\vec{w} \in W$, there is some $v \in V$ with $T(\vec{v}) = \vec{w}$.

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

Activity. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

Definition. We also have two important sets called the **kernel** of T and the **image** of T.

$$\begin{aligned} \ker T &= \left\{ \vec{v} \in V \;\middle|\; T(\vec{v}) = 0 \right\} \\ \operatorname{Im} T &= \left\{ \vec{w} \in W \;\middle|\; \text{there is some } v \in V \text{ with } T(\vec{v}) = \vec{w} \right\} \end{aligned}$$

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T.

Activity. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T.

Activity. Describe surjective linear transformations in terms of the image.

Activity. Describe injective linear transformations in terms of the kernel.

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

1) Write a system of equations whose solution set is the kernel.
2) Compute $\mathrm{RREF}(A)$ and solve the system of equations.
3) Compute the kernel of T
4) Find a basis for the kernel of T

Activity. Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

1) Write a system of equations whose solution set is the kernel.
2) Compute RREF(A) and solve the system of equations.
3) Compute the kernel of T
4) Find a basis for the kernel of T

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

1) Find a set of vectors that span the image of T
2) Find a basis for the image of T.

Activity. Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

1) Find a set of vectors that span the image of T
2) Find a basis for the image of T.

Application Activities - Day 3

Activity. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). Consider the following statements about T

(a) T is injective

(b) T is not injective

(c) T is surjective

(d) T is not surjective

(e) The system of linear equations given by the augmented matrix $\left[A \mid \vec{b}\right]$ has a solution for all $\vec{b} \in \mathbb{R}^m$

(f) The system of linear equations given by the augmented matrix $\left[A \mid \vec{b}\right]$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$

(g) The system of linear equations given by the augmented matrix $\left[A \mid \vec{0}\right]$ has a non-trivial solution.

(h) The columns of A span $\mathbb{R}^m$

(i) The columns of A are linearly independent

(j) The columns of A are a basis of $\mathbb{R}^m$

(k) Every column of $\mathrm{RREF}(A)$ is a pivot column

(l) $\mathrm{RREF}(A)$ has a non-pivot column

(m) $\mathrm{RREF}(A)$ has n pivot columns

Sort these statements into groups of equivalent statements.

Activity. Gallery walk–switch boards with a different team. If they have two things grouped together that you know are not equivalent, write a reason or counter-example on a sticky note.

Activity. Update your team's groupings based on feedback.

Activity. Repeat?

Activity. Can you add any statements to any groups?
