Section E.2

Activity E.2.1 *(∼10 min)* Free browser-based technologies for mathematical computation are available online.

- Go to `https://octave-online.net`.
- Type `A=sym([1 3 4 ; 2 5 7])` and press `Enter` to store the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 2 & -9 & -1 \end{bmatrix}$ in the variable A.
  - The symbolic function `sym` is used to calculate precise answers rather than floating-point approximations.
  - The vertical bar in an augmented matrix does not affect row operations, so the RREF of $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & -9 & -1 \end{array}\right]$ may be computed in the same way.
- Type `rref(A)` and press `Enter` to compute the reduced row echelon form of A.

Remark E.2.2 We will frequently need to know the reduced row echelon form of matrices during class, so feel free to use Octave-Online.net to compute RREF efficiently.

You may alternatively use the calculator you will use during assessments. Be sure to use fractions mode to compute exact solutions rather than floating-point approximations.

Activity E.2.3 *(∼10 min)* Consider the system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

Part 1: Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

Part 2: Use the RREF matrix to write a linear system equivalent to the original system. Then find its solution set.

Activity E.2.4 *(~10 min)* Consider the system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 \qquad\quad - \ 3x_3 &= 1 \end{aligned}$$

Part 1: Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

Part 2: Use the RREF matrix to write a linear system equivalent to the original system. Then find its solution set.

Activity E.2.5 *(~10 min)* Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Part 1: Find its corresponding augmented matrix A and use technology to find RREF(A).
Part 2: How many solutions do these linear systems have?

Activity E.2.6 *(~10 min)* Consider the simple linear system equivalent to the system from the previous activity:

$$\begin{aligned} x_1 + 2x_2 \qquad &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \left[\begin{array}{c} a \\ ? \\ ? \end{array}\right] \middle| \, a \in \mathbb{R} \right\}$.

Part 2: Let $x_2 = b$ and write the solution set in the form $\left\{ \left[\begin{array}{c} ? \\ b \\ ? \end{array}\right] \middle| \, b \in \mathbb{R} \right\}$.

Part 3: Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$ caused this?

Definition E.2.7 Recall that the pivots of a matrix in RREF form are the leading 1s in each non-zero row.

The pivot columns in an augmented matrix correspond to the **bound variables** in the system of equations (x_1, x_3 below). The remaining variables are called **free variables** (x_2 below).

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

To efficiently solve a system in RREF form, assign letters to the free variables, and then solve for the bound variables.

Activity E.2.8 *(~10 min)* Find the solution set for the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - \ \ x_5 &= \ \ 3 \\ -x_1 + \ \ x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - \ \ x_3 + x_4 + \ \ x_5 &= \ \ 2 \end{aligned}$$

by row-reducing its augmented matrix, and then assigning letters to the free variables (given by non-pivot columns) and solving for the bound variables (given by pivot columns) in the corresponding linear system.

Observation E.2.9 The solution set to the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - \ \ x_5 &= \ \ 3 \\ -x_1 + \ \ x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - \ \ x_3 + x_4 + \ \ x_5 &= \ \ 2 \end{aligned}$$

may be written as

$$\left\{ \begin{bmatrix} 1+5a+2b \\ 1+2a+3b \\ a \\ 3+3b \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

Remark E.2.10 Don't forget to correctly express the solution set of a linear system, using set-builder notation for consistent systems with infintely many solutions.

- **Consistent with one solution**: e.g. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$
- **Consistent with infinitely-many solutions**: e.g. $\left\{ \begin{bmatrix} 1 \\ 2-3a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$
- **Inconsistent**: $\emptyset$ or $\{\}$