

## Section E.1

**Remark E.11** The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Verbose standard form:

$$\begin{aligned}1x_1 + 0x_2 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\0x_1 - 1x_2 + 1x_3 &= -2\end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c}1 & 0 & 3 & 3 \\3 & -2 & 4 & 0 \\0 & -1 & 1 & -2\end{array}$$

**Definition E.12** A system of  $m$  linear equations with  $n$  variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{array}{cccccc}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & b_1 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & b_2 \\\vdots & & \vdots \\a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & b_m\end{array} \qquad \left[ \begin{array}{cccc|c}a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\\vdots & \vdots & \ddots & \vdots & \vdots \\a_{m1} & a_{m2} & \cdots & a_{mn} & b_m\end{array} \right]$$

**Example E.13** The corresponding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Augmented matrix:

$$\left[ \begin{array}{ccc|c}1 & 0 & 3 & 3 \\3 & -2 & 4 & 0 \\0 & -1 & 1 & -2\end{array} \right]$$

**Definition E.14** Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set  $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ .

$$\begin{array}{rcl}3x_1 - 2x_2 & = & 1 \\x_1 + 4x_2 & = & 5\end{array} \qquad \begin{array}{rcl}3x_1 - 2x_2 & = & 1 \\4x_1 + 2x_2 & = & 6\end{array}$$

Therefore these augmented matrices are equivalent, which we denote with  $\sim$ :

$$\left[ \begin{array}{cc|c}3 & -2 & 1 \\1 & 4 & 5\end{array} \right] \sim \left[ \begin{array}{cc|c}3 & -2 & 1 \\4 & 2 & 6\end{array} \right]$$

**Activity E.15** ( $\sim 10$  min) Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

- |   |   |
|---|---|
| a) Swap two rows.                         | e) Add a constant multiple of one row to another row. |
| b) Swap two columns.                      | f) Replace a column with zeros.                       |
| c) Add a constant to every term in a row. | g) Replace a row with zeros.                          |
| d) Multiply a row by a nonzero constant.  |   |

**Definition E.16** The following **row operations** produce equivalent augmented matrices:

1. Swap two rows, for example,  $R_1 \leftrightarrow R_2$ :

$$\left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \sim \left[ \begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array} \right]$$

2. Multiply a row by a nonzero constant, for example,  $2R_1 \rightarrow R_1$ :

$$\left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \sim \left[ \begin{array}{cc|c} 2(1) & 2(2) & 2(3) \\ 4 & 5 & 6 \end{array} \right]$$

3. Add a constant multiple of one row to another row, for example,  $R_2 - 4R_1 \rightarrow R_2$ :

$$\left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 4 - 4(1) & 5 - 4(2) & 6 - 4(3) \end{array} \right]$$

Whenever two matrices  $A, B$  are equivalent (so whenever we do any of these operations), we write  $A \sim B$ .

**Activity E.17** ( $\sim 10$  min) Consider the following (equivalent) linear systems.

- |  |   |   |
|--|---|---|
| <p>(A)</p> $\begin{aligned} x + 2y + z &= 3 \\ -x - y + z &= 1 \\ 2x + 5y + 3z &= 7 \end{aligned}$ | <p>(C)</p> $\begin{aligned} x - z &= 1 \\ y + z &= 1 \\ y + 2z &= 4 \end{aligned}$            | <p>(E)</p> $\begin{aligned} x - z &= 1 \\ y + z &= 1 \\ z &= 3 \end{aligned}$           |
| <p>(B)</p> $\begin{aligned} 2x + 5y + 3z &= 7 \\ -x - y + z &= 1 \\ x + 2y + z &= 3 \end{aligned}$ | <p>(D)</p> $\begin{aligned} x + 2y + z &= 3 \\ y + z &= 1 \\ 2x + 5y + 3z &= 7 \end{aligned}$ | <p>(F)</p> $\begin{aligned} x + 2y + z &= 3 \\ y + z &= 1 \\ y + 2z &= 4 \end{aligned}$ |

Rank the six linear systems from most complicated to simplest.

**Activity E.18** (*~5 min*) We can rewrite the previous in terms of equivalences of augmented matrices

$$\begin{aligned} \left[ \begin{array}{ccc|c} 2 & 5 & 13 & 7 \\ -1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{array} \right] &\sim \left[ \begin{array}{ccc|c} \textcircled{1} & 2 & 1 & 3 \\ -1 & -1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{1} & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array} \right] \sim \\ \left[ \begin{array}{ccc|c} \textcircled{1} & 2 & 1 & 3 \\ 0 & \textcircled{1} & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array} \right] &\sim \left[ \begin{array}{ccc|c} \textcircled{1} & 0 & -1 & 1 \\ 0 & \textcircled{1} & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{1} & 0 & -1 & 1 \\ 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & \textcircled{1} & 3 \end{array} \right] \end{aligned}$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

**Definition E.19** A matrix is in **reduced row echelon form (RREF)** if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

Every matrix has a unique reduced row echelon form. If  $A$  is a matrix, we write  $\text{RREF}(A)$  for the reduced row echelon form of that matrix.

**Activity E.20** (*~15 min*) Recall that a matrix is in **reduced row echelon form (RREF)** if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

(A) $\left[ \begin{array}{ccc c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$	(C) $\left[ \begin{array}{ccc c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]$	(E) $\left[ \begin{array}{ccc c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$
(B) $\left[ \begin{array}{ccc c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$	(D) $\left[ \begin{array}{ccc c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$	(F) $\left[ \begin{array}{ccc c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array} \right]$

For each matrix, circle the leading terms, and label it as RREF or not RREF. For the ones not in RREF, find their RREF.