

# Module P: Applications of Linear Algebra

# Module P Section 1

## Definition P.1.1

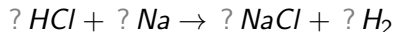
In chemistry, we learn that when the two substances

- Hydrochloric acid  $HCl$  (formed from 1  $H$  and 1  $Cl$  atom)
- Sodium  $Na$  (formed from 1  $Na$  atom)

react, their atoms rearrange to form the substances

- Salt  $NaCl$  (formed from 1  $Na$  and 1  $Cl$  atom)
- Hydrogen gas  $H_2$  (formed from 2  $H$  atoms).

This may be represented by the **chemical equation**

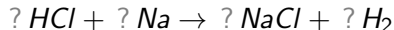


where each  $?$  represents the amount of that substance before/after the reaction.

**Activity P.1.2** (*~5 min*)

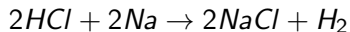
The **law of conservation of mass** states that the quantity of atoms before and after a chemical reaction must remain the same.

Find positive integers so that both sides of the chemical equation represent the same amount of matter:



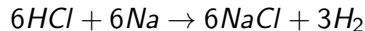
**Definition P.1.3**

A chemical equation is **balanced** if the given quantities of each substance before and after the reaction are equal and minimal positive integers:



**Observation P.1.4**

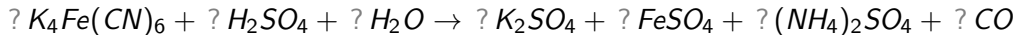
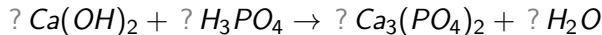
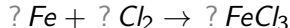
For example, the following equation isn't balanced because all the integers may be divided by three:



Therefore if a chemical equation can be balanced, there is exactly one correct solution.

**Activity P.1.5** (*~15 min*)

Balance the following chemical equations:



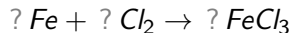
(Note that  $(NH_4)_2SO_4$  represents 2 *N*, 8 *H*, 1 *S*, and 4 *O*.)

## Observation P.1.6

For the purposes of balancing chemical equations, the set

$$L = \{\mathbf{A} \mid \mathbf{A} \text{ is combination of elements}\}$$

may be treated as a kind of **vector space**. This means that balancing the chemical equation



may be achieved by finding a solution  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  to the vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl}).$$



**Activity P.1.7** (*~5 min*)

To solve the vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl})$$

we are only concerned with the subspace  $W = \text{span}\{\mathbf{Cl}, \mathbf{Fe}\}$  of  $L$ . Since the element  $\mathbf{Fe}$  cannot be created from the element  $\mathbf{Cl}$  in a chemical reaction and vice versa, the set  $\{\mathbf{Cl}, \mathbf{Fe}\}$ :

- a) spans  $W$ , but is linearly dependent.
- b) is linearly independent, but does not span  $W$ .
- c) is a basis for  $W$ .

**Observation P.1.8**

$W = \text{span}\{\mathbf{Cl}, \mathbf{Fe}\}$  is a two-dimensional subspace of  $L$ , so as usual we'd rather work with its isomorphic Euclidean space  $\mathbb{R}^2$ .

Thus we should assign a transformation of bases such as:

$$\mathbf{Cl} \leftrightarrow \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{Fe} \leftrightarrow \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**Activity P.1.9** (*~10 min*)

Rewrite the  $W = \text{span}\{\mathbf{Cl}, \mathbf{Fe}\}$  vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl})$$

using the transformation of bases

$$\mathbf{Cl} \leftrightarrow \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{Fe} \leftrightarrow \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and show how it may be simplified to

$$x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \end{bmatrix} - z \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

**Activity P.1.10** (*~10 min*)

Consider the Euclidean vector equation

$$x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \end{bmatrix} - z \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

**Activity P.1.10** ( $\sim 10$  min)

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*Part 1:* Find its solution set.

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*Part 1:* Find its solution set.

*Part 2:* Find a vector in the solution space that consists of minimal positive integers.

**Activity P.1.10** ( $\sim 10$  min)

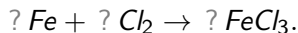
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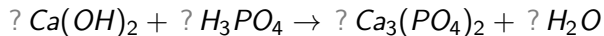
*Part 2:* Find a vector in the solution space that consists of minimal positive integers.

*Part 3:* Balance the chemical equation



**Activity P.1.11** (*~10 min*)

Balance the chemical equation



by first converting it into an  $\mathbb{R}^4$  vector equation and finding its solution set.