Sample Assessment Exercises

This document contains one exercise and solution for each standard. The goal is to give you an idea of what the exercises might look like, and what the expectations for a complete solution are.

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & 1 & 1 \\ -1 & -4 & 1 & -7 & 0 \\ 0 & 1 & -1 & 0 & -2 \end{array}\right]$$

Solution:

$$\begin{alignedat}{5} 3x_1 &+{}& 2x_2 & & &+{}& x_4 &={}& 1 \\ -x_1 &-{}& 4x_2 &+{}& x_3 &-{}& 7x_4 &={}& 0 \\ & & x_2 &-{}& x_3 & & &={}& -2 \end{alignedat}$$

□

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 2 & -1 & -3 \\ 2 & 4 & -1 & -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 2 & -1 & -3 \\ 2 & 4 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 3 & 1 & 2 \\ 2 & 4 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{13}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{13}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

□

E3. Find the solution set for the following system of linear equations.

$$\begin{alignedat}{4} 2x &+{}& 4y &+{}& z &={}& 5 \\ x &+{}& 2y & & &={}& 3 \end{alignedat}$$

Solution:

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} 2 & 4 & 1 & 5 \\ 1 & 2 & 0 & 3 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This corresponds to the system

$$\begin{alignedat}{4} x &+{}& 2y & & &={}& 3 \\ & & & & z &={}& -1 \end{alignedat}$$

Since the y-column is a non-pivot column, it is a free variable, so we let $y = a$; then we have

$$\begin{alignedat}{4} x &+{}& 2y & & &={}& 3 \\ & & y & & &={}& a \\ & & & & z &={}& -1 \end{alignedat}$$

and thus

$$\begin{aligned}
x &= 3 - 2a \\
y &= a \\
z &= -1
\end{aligned}$$

So the solution set is

$$\left\{ \begin{bmatrix} 3-2a \\ a \\ -1 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$$

$\square$

V1. Let V be the set of all polynomials, together with the operations $\oplus$ and $\odot$ defined by, for all polynomials $f(x), g(x)$ and scalars $c \in \mathbb{R}$:

$$\begin{aligned}
f(x) \oplus g(x) &= xf(x) + xg(x) \\
c \odot f(x) &= cf(x)
\end{aligned}$$

(a) Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot (f(x) \oplus g(x)) = c \odot f(x) \oplus c \odot g(x).$$

(b) Show that addition is not associative, i.e. for all polynomials $f(x), g(x), h(x)$,

$$(f(x) \oplus g(x)) \oplus h(x) \neq f(x) \oplus (g(x) \oplus h(x)).$$

Solution:

(a) Compute

$$\begin{aligned}
c \odot (f(x) \oplus g(x)) &= c \odot (xf(x) + xg(x)) \\
&= c\,(xf(x) + xg(x)) \\
&= cxf(x) + cxg(x)
\end{aligned}$$

and

$$\begin{aligned}
c \odot f(x) \oplus c \odot g(x) &= (cf(x)) \oplus (cg(x)) \\
&= xcf(x) + xcg(x)
\end{aligned}$$

Since these are the same, we have shown $c \odot (f(x) \oplus g(x)) = c \odot f(x) \oplus c \odot g(x)$.

(b) Compute

$$\begin{aligned}
(f(x) \oplus g(x)) \oplus h(x) &= (xf(x) + xg(x)) \oplus h(x) \\
&= x\,(xf(x) + xg(x)) + xh(x) \\
&= x^2 f(x) + x^2 g(x) + xh(x)
\end{aligned}$$

and

$$\begin{aligned}
f(x) \oplus (g(x) \oplus h(x)) &= f(x) \oplus (xg(x) + xh(x)) \\
&= xf(x) + x\,(xg(x) + xh(x)) \\
&= xf(x) + x^2 g(x) + x^2 h(x)
\end{aligned}$$

Since $x^2 f(x) + x^2 g(x) + xh(x) \neq xf(x) + x^2 g(x) + x^2 h(x)$, we have shown $(f(x) \oplus g(x)) \oplus h(x) \neq f(x) \oplus (g(x) \oplus h(x))$.

$\square$

L. et V be the set of all non-negative real numbers with the operations $\oplus$ and $\odot$ given by, for all $x, y \in V$ and $c \in \mathbb{R}$,

$$\begin{aligned} x \oplus y &= x + y \\ c \odot x &= |c|x \end{aligned}$$

(a) List the 8 defining properties of a vector space.

(b) Determine which of the 8 hold for V with these operations, and conclude whether V is a vector space or not.

Solution:

(a) The eight properties are

1) Addition is associative, i.e. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all $x, y, z \in V$.
2) Addition is commutative, i.e. $x \oplus y = y \oplus x$ for all $x, y \in V$.
3) There exists a zero element, i.e. an element $0 \in V$ such that for all $x \in V$, $x \oplus 0 = x$.
4) Additive inverses exist, i.e. for every $x \in V$ there is an element $-x \in V$ such that $x \oplus (-x) = 0$
5) Scalar multiplication is associative, i.e. for each $c, d \in \mathbb{R}$ and $x \in V$, $c \odot (d \odot x) = (cd) \odot x$.
6) 1 is the multiplicative identity, i.e. for all $x \in V$, $1 \odot x = x$.
7) Scalar multiplication distributes over vector addition, i.e. for all $x, y \in V$ and $c \in \mathbb{R}$, $c \odot (x \oplus y) = c \odot x \oplus c \odot y$
8) Scalar addition distributes over scalar multiplication, i.e. for all $x \in V$ and $c, d \in \mathbb{R}$, $(c + d) \odot x = c \odot x \oplus d \odot x$

(b) V is not a vector space, as only six properties hold: it does not have additive inverses, and scalar addition does not distribute over scalar multiplication.

$\square$

V3. Determine if $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Solution:
We compute

$$\mathrm{RREF} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 2 & 1 & -1 \\ 1 & -1 & -1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Since this corresponds to an inconsistent system of equations, $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is **not** a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

$\square$

V4. Determine if the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ span $\mathbb{R}^3$.

Solution:

We compute

$$\text{RREF}\begin{bmatrix}1 & 3 & 1\\ 0 & 2 & 1\\ 1 & -1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & \frac{1}{2}\\ 0 & 0 & 0\end{bmatrix}$$

Since the last row lacks a pivot, the vectors **do not span** $\mathbb{R}^3$.

$\square$

V5. Determine if the set

$$W = \left\{ \begin{bmatrix}x\\ y\\ z\end{bmatrix} \middle| \, x + y = 3z + 2 \right\}$$

is a subspace of $\mathbb{R}^3$.

Solution: Let $\begin{bmatrix}x_1\\ y_1\\ z_1\end{bmatrix}, \begin{bmatrix}x_2\\ y_2\\ z_2\end{bmatrix} \in W$ (so we know $x_1 + y_1 = 3z_1 + 2$ and $x_2 + y_2 = 3z_2 + 2$. We compute

$$\begin{bmatrix}x_1\\ y_1\\ z_1\end{bmatrix} + \begin{bmatrix}x_2\\ y_2\\ z_2\end{bmatrix} = \begin{bmatrix}x_1 + x_2\\ y_1 + y_2\\ z_1 + z_2\end{bmatrix}$$

However,

$$(x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) = (3z_1 + 2) + (3z_2 + 2) = 3z_1 + 3z_2 + 4 \neq 3(z_1 + z_2) + 2.$$

Thus, W is not closed under addition, so it is not a subspace.

$\square$

S1. Determine if the vectors $\begin{bmatrix}3\\ 2\\ 1\\ 0\end{bmatrix}$, $\begin{bmatrix}-1\\ 1\\ 2\\ 3\end{bmatrix}$, $\begin{bmatrix}0\\ 1\\ -1\\ 1\end{bmatrix}$, and $\begin{bmatrix}2\\ 5\\ 1\\ 5\end{bmatrix}$ are linearly dependent or linearly independent.

Solution: Compute

$$\text{RREF}\begin{bmatrix}3 & -1 & 0 & 2\\ 2 & 1 & 1 & 5\\ 1 & 2 & -1 & 1\\ 0 & 3 & 1 & 5\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Since the fourth column is not a pivot column, the vectors are linearly dependent.

$\square$

S2. Determine if the set

$$\left\{ \begin{bmatrix}3\\ 2\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}-1\\ 1\\ 2\\ 3\end{bmatrix}, \begin{bmatrix}0\\ 1\\ -1\\ 1\end{bmatrix}, \begin{bmatrix}2\\ 5\\ 1\\ 5\end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^4$ or not.

Solution: Compute

$$\text{RREF}\begin{bmatrix}3 & -1 & 0 & 2\\ 2 & 1 & 1 & 5\\ 1 & 2 & -1 & 1\\ 0 & 3 & 1 & 5\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Since the fourth column is not a pivot column, the vectors are linearly dependent and thus not a basis. Alternatively, since the fourth row is all zeroes, the vectors do not span $\mathbb{R}^4$ and thus are not a basis.

$\square$