Module P: Applications of Linear Algebra

Module P Section 1

Definition P.1.1

In chemistry, we learn that when the two substances

- Hydrochloric acid HCl (formed from 1 H and 1 Cl atom)
- Sodium Na (formed from 1 Na atom)

react, their atoms rearrange to form the substances

- Salt $NaCl$ (formed from 1 Na and 1 Cl atom)
- Hydrogen gas H_2 (formed from 2 H atoms).

This may be represented by the **chemical equation**

$$?\,HCl + \,?\,Na \to \,?\,NaCl + \,?\,H_2$$

where each ? represents the amount of that substance before/after the reaction.

Activity P.1.2 *($\sim$5 min)*
The **law of conservation of mass** states that the quantity of atoms before and after a chemical reaction must remain the same.

Find positive integers so that both sides of the chemical equation represent the same amount of matter:

$$? \, HCl + \, ? \, Na \rightarrow \, ? \, NaCl + \, ? \, H_2$$

Definition P.1.3
A chemical equation is **balanced** if the given quantities of each substance before and after the reaction are equal and minimal positive integers:

$$2HCl + 2Na \to 2NaCl + H_2$$

Observation P.1.4

For example, the following equation isn't balanced because all the integers may be divided by three:

$$6HCl + 6Na \to 6NaCl + 3H_2$$

Therefore if a chemical equation can be balanced, there is exactly one correct solution.

Activity P.1.5 *(∼15 min)*
Balance the following chemical equations:

$$? \, Fe + \, ? \, Cl_2 \rightarrow \, ? \, FeCl_3$$

$$? \, Ca(OH)_2 + \, ? \, H_3PO_4 \rightarrow \, ? \, Ca_3(PO_4)_2 + \, ? \, H_2O$$

$$? \, K_4Fe(CN)_6 + \, ? \, H_2SO_4 + \, ? \, H_2O \rightarrow \, ? \, K_2SO_4 + \, ? \, FeSO_4 + \, ? \, (NH_4)_2SO_4 + \, ? \, CO$$

(Note that $(NH_4)_2SO_4$ represents 2 N, 8 H, 1 S, and 4 O.)

Observation P.1.6

For the purposes of balancing chemical equations, the set

$$L = \{\mathbf{A} \mid \mathbf{A} \text{ is combination of elements}\}$$

may be treated as a kind of **vector space**. This means that balancing the chemical equation

$$?\,Fe + \,?\,Cl_2 \to \,?\,FeCl_3$$

may be acheived by finding a solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to the vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl}).$$

Activity P.1.7 *(∼5 min)*

To solve the vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl})$$

we are only concerned with the subspace $W = \operatorname{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ of L. Since the element **Fe** cannot be created from the element **Cl** in a chemical reaction and vice versa, the set $\{\mathbf{Cl}, \mathbf{Fe}\}$:

a) spans W, but is linearly dependent.

b) is linearly independent, but does not span W.

c) is a basis for W.

Observation P.1.8

$W = \operatorname{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ is a two-dimensional subspace of L, so as usual we'd rather work with its isomorphic Euclidean space $\mathbb{R}^2$.

Thus we should assign a transformation of bases such as:

$$\mathbf{Cl} \leftrightarrow \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{Fe} \leftrightarrow \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Activity P.1.9 *(∼10 min)*

Rewrite the $W = \operatorname{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl})$$

using the transformation of bases

$$\mathbf{Cl} \leftrightarrow \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{Fe} \leftrightarrow \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and show how it may be simplifed to

$$x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \end{bmatrix} - z \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Activity P.1.10 *(∼10 min)*

Consider the Euclidean vector equation

$$x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \end{bmatrix} - z \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Activity P.1.10 *(∼10 min)*

Consider the Euclidean vector equation

$$x\begin{bmatrix}0\\1\end{bmatrix}+y\begin{bmatrix}2\\0\end{bmatrix}-z\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

Part 1: Find its solution set.

Activity P.1.10 *(∼10 min)*

Consider the Euclidean vector equation

$$x\begin{bmatrix}0\\1\end{bmatrix}+y\begin{bmatrix}2\\0\end{bmatrix}-z\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

Part 1: Find its solution set.

Part 2: Find a vector in the solution space that consists of minimal positive integers.

Activity P.1.10 *(∼10 min)*
Consider the Euclidean vector equation

$$x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \end{bmatrix} - z \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Part 1: Find its solution set.
Part 2: Find a vector in the solution space that consists of minimal positive integers.
Part 3: Balance the chemical equation

$$?\, Fe + \,?\, Cl_2 \rightarrow \,?\, FeCl_3.$$

Activity P.1.11 *(∼10 min)*
Balance the chemical equation

$$?\,Ca(OH)_2 + \,?\,H_3PO_4 \to \,?\,Ca_3(PO_4)_2 + \,?\,H_2O$$

by first converting it into an $\mathbb{R}^4$ vector equation and finding its solution set.