Module A: Algebraic properties of linear maps

How can we understand linear maps algebraically?

At the end of this module, students will be able to...

A1 **Linear map verification.** ... determine if a map between vector spaces of polynomials is linear or not.

A2 **Linear maps and matrices.** ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.

A3 **Kernel and Image.** ... compute a basis for the kernel and a basis for the image of a linear map.

A4 **Injectivity and surjectivity.** ... determine if a given linear map is injective and/or surjective.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V3**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **V5**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **V6,V7**.
- Find a basis of the solution space to a homogeneous system of linear equations **V10**.

Module A Section 1

Definition A.1

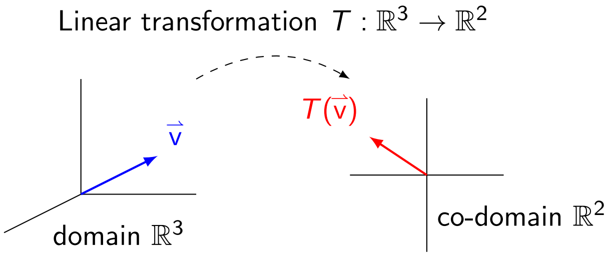
A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$.
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}, \vec{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition A.2

Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.



Example A.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right) = \begin{bmatrix} (x+u)-(z+w) \\ 3(y+v) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x+u)-(z+w) \\ 3(y+v) \end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \text{ and } cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore T is a linear transformation.

Example A.4

Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact A.5

A map between Euclidean spaces $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of $\mathbb{R}^n$.

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2, $y + 3$, and $y - 2^x$ are not linear combinations (even though $x + y$ is):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

Activity A.6 *(∼5 min)* Recall the following rules from calculus, where $D : \mathcal{P} \to \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f.

$$D(f+g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

a) $\mathcal{P}$ is not a vector space
b) D is a linear map
c) D is not a linear map

Activity A.7 *(∼10 min)* Let the polynomial maps $S : \mathcal{P}^4 \to \mathcal{P}^3$ and $T : \mathcal{P}^4 \to \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad\qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4 + x)$, $S(x^4) + S(x)$, $T(x^4 + x)$, and $T(x^4) + T(x)$. Which of these maps is definitely not linear?

Fact A.8

If $L : V \to W$ is linear, then $L(\vec{z}) = L(0\vec{v}) = 0L(\vec{v}) = \vec{z}$ where $\vec{z}$ is the additive identity of the vector spaces V, W.

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Observation A.9

Showing $L : V \to W$ is not a linear transformation can be done by finding an example for any one of the following.

- Show $L(\vec{z}) \neq \vec{z}$ (where $\vec{z}$ is the additive identity of L and W).
- Find $\vec{v}, \vec{w} \in V$ such that $L(\vec{v} + \vec{w}) \neq L(\vec{v}) + L(\vec{w})$.
- Find $\vec{v} \in V$ and $c \in \mathbb{R}$ such that $L(c\vec{v}) \neq cL(\vec{v})$.

Otherwise, L can be shown to be linear by proving the following in general.

- For all $\vec{v}, \vec{w} \in V$, $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$.
- For all $\vec{v} \in V$ and $c \in \mathbb{R}$, $L(c\vec{v}) = cL(\vec{v})$.

Note the similarities between this process and showing that a subset of a vector space is/isn't a subspace.

Activity A.10 *(∼15 min)* Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Activity A.10 *(∼15 min)* Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.

Activity A.10 *(~15 min)* Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.
Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f.

Activity A.10 *($\sim$15 min)* Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.
Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f.
Part 3: Is S linear?

Activity A.11 *(∼20 min)* Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Activity A.11 *(~20 min)* Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Note that $S(0) = 0$ and $T(0) = 0$. So instead, show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Activity A.11 *(∼20 min)* Let the polynomial maps $S:\mathcal{P}\to\mathcal{P}$ and $T:\mathcal{P}\to\mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad\qquad T(f(x)) = 3xf(x^2)$$

Part 1: Note that $S(0)=0$ and $T(0)=0$. So instead, show that $S(x+1)\neq S(x)+S(1)$ to verify that S is not linear.
Part 2: Prove that T is linear by verifying that $T(f(x)+g(x)) = T(f(x))+T(g(x))$ and $T(cf(x)) = cT(f(x))$.

Module A Section 2

Remark A.12

Recall that a linear map $T : V \to W$ satisfies

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$.
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}, \vec{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Activity A.13 *(~5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

a. $\begin{bmatrix}6\\3\end{bmatrix}$

b. $\begin{bmatrix}-9\\6\end{bmatrix}$

c. $\begin{bmatrix}-4\\-2\end{bmatrix}$

d. $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity A.14 *(∼5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

a. $\begin{bmatrix}2\\1\end{bmatrix}$

b. $\begin{bmatrix}3\\-1\end{bmatrix}$

c. $\begin{bmatrix}-1\\3\end{bmatrix}$

d. $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.15 *(∼5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}-2\\0\\-3\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.16 *(~5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix} \text{ and } T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}.$$

What piece of information would help you compute $T\left(\begin{bmatrix}0\\4\\-1\end{bmatrix}\right)$?

a. The value of $T\left(\begin{bmatrix}0\\-4\\0\end{bmatrix}\right)$.

b. The value of $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$.

c. The value of $T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$.

d. Any of the above.

Fact A.17

Consider any basis $\{\vec{b}_1, \ldots, \vec{b}_n\}$ for V. Since every vector $\vec{v}$ can be written as a linear combination of basis vectors, $x_1\vec{b}_1 + \cdots + x_n\vec{b}_n$, we may compute $T(\vec{v})$ as follows:

$$T(\vec{v}) = T(x_1\vec{b}_1 + \cdots + x_n\vec{b}_n) = x_1 T(\vec{b}_1) + \cdots + x_n T(\vec{b}_n).$$

Therefore any linear transformation $T : V \to W$ can be defined by just describing the values of $T(\vec{b}_i)$.

Put another way, the images of the basis vectors **determine** the transformation T.

Definition A.18

Since linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$ is determined by the standard basis $\{\vec{e}_1,\dots,\vec{e}_n\}$, it's convenient to store this information in the $m\times n$ **standard matrix** $[T(\vec{e}_1)\ \cdots\ T(\vec{e}_n)]$.

For example, let $T:\mathbb{R}^3\to\mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T\left(\vec{e}_1\right)=T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}3\\2\end{bmatrix}\qquad T\left(\vec{e}_2\right)=T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\4\end{bmatrix}\qquad T\left(\vec{e}_3\right)=T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}5\\0\end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\left[T(\vec{e}_1)\quad T(\vec{e}_2)\quad T(\vec{e}_3)\right]=\begin{bmatrix}3 & -1 & 5\\2 & 4 & 0\end{bmatrix}.$$

Activity A.19 *(∼3 min)* Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \quad T(\vec{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \quad T(\vec{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \quad T(\vec{e}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix $[T(\vec{e}_1) \cdots T(\vec{e}_n)]$ for T.

Activity A.20 *($\sim$5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Activity A.20 *($\sim$5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Part 1: Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

Activity A.20 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Part 1: Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.
Part 2: Find the standard matrix for T.

Fact A.21

Because every linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ has a linear combination of the variables in each component, and thus $T(\vec{e}_i)$ yields exactly the coefficients of x_i, the standard matrix for T is simply an ordered list of the coefficients of the x_i:

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \qquad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Activity A.22 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Activity A.22 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Part 1: Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Activity A.22 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Part 1: Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Part 2: Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

Fact A.23

To quickly compute $T(\vec{v})$ from its standard matrix A, multiply and add the entries of each row of A with the vector $\vec{v}$. For example, if T has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$$

then for $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we will write

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 2y + 3z \\ 0x + 1y - 2z \\ 2x - 1y + 0z \end{bmatrix}$$

and for $\vec{v} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ we will write

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(0) + 3(-2) \\ 0(3) + 1(0) - 2(-2) \\ 2(3) - 1(0) + 0(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}.$$

Activity A.24 *(∼15 min)* Compute the following linear transformations of vectors given their standard matrices.

$$T_1\left(\begin{bmatrix}1\\2\end{bmatrix}\right) \text{ for the standard matrix } A_1 = \begin{bmatrix}4 & 3\\0 & -1\\1 & 1\\3 & 0\end{bmatrix}$$

$$T_2\left(\begin{bmatrix}1\\1\\0\\-3\end{bmatrix}\right) \text{ for the standard matrix } A_2 = \begin{bmatrix}4 & 3 & 0 & -1\\1 & 1 & 3 & 0\end{bmatrix}$$

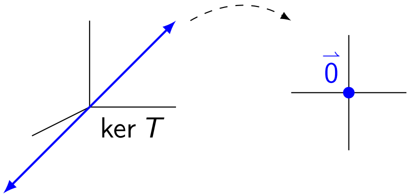
$$T_3\left(\begin{bmatrix}0\\-2\\0\end{bmatrix}\right) \text{ for the standard matrix } A_3 = \begin{bmatrix}4 & 3 & 0\\0 & -1 & 3\\5 & 1 & 1\\3 & 0 & 0\end{bmatrix}$$

Module A Section 3

Definition A.25

Let $T : V \to W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{\vec{v} \in V \;\middle|\; T(\vec{v}) = \vec{z}\right\}$$



Activity A.26 *(~5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\ker T$, the set of all vectors that transform into $\vec{0}$?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x, y \in \mathbb{R} \right\}$

Activity A.27 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\ker T$, the set of all vectors that transform into $\vec{0}$?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$

Activity A.28 *(∼10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Activity A.28 *(∼10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

Activity A.28 *(~10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

Part 2: Use $\mathrm{RREF}(A)$ to solve this homogeneous system of equations and find a basis for the kernel of T.

Activity A.29 *(∼10 min)* Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 2x + 4y + 2z - 4w \\ -2x - 4y + z + w \\ 3x + 6y - z - 4w \end{bmatrix}.$$

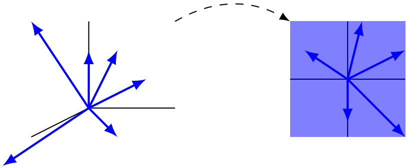
Find a basis for the kernel of T.

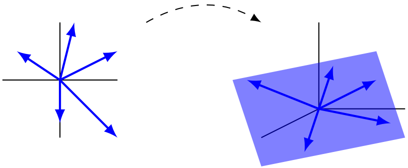
Definition A.30

Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{\vec{w} \in W \;\middle|\; \text{there is some } \vec{v} \in V \text{ with } T(\vec{v}) = \vec{w}\right\}$$

In the examples below, the left example's image is all of $\mathbb{R}^2$, but the right example's image is a planar subspace of $\mathbb{R}^3$.





Activity A.31 *(∼5 min)* Let $T:\mathbb{R}^2\to\mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x\\y\\0\end{bmatrix}\qquad\text{with standard matrix }\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^2$ vectors?

a) $\left\{\begin{bmatrix}0\\0\\a\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$

b) $\left\{\begin{bmatrix}a\\b\\0\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$

c) $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$

d) $\mathbb{R}^3=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\middle|\, x,y,z\in\mathbb{R}\right\}$

Activity A.32 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^3$ vectors?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x, y \in \mathbb{R} \right\}$

Activity A.33 *(∼5 min)* Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) & T(\vec{e}_4) \end{bmatrix}.$$

Since $T(\vec{v}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4)$, the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

a) spans $\operatorname{Im} T$

b) is a linearly independent subset of $\operatorname{Im} T$

c) is a basis for $\operatorname{Im} T$

Observation A.34

Let $T:\mathbb{R}^4\to\mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A=\begin{bmatrix}3 & 4 & 7 & 1\\ -1 & 1 & 0 & 2\\ 2 & 1 & 3 & -1\end{bmatrix}.$$

Since the set $\left\{\begin{bmatrix}3\\-1\\2\end{bmatrix},\begin{bmatrix}4\\1\\1\end{bmatrix},\begin{bmatrix}7\\0\\3\end{bmatrix},\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$ spans $\operatorname{Im} T$, we can obtain a basis for $\operatorname{Im} T$ by finding $\operatorname{RREF} A=\begin{bmatrix}1 & 0 & 1 & -1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$ and only using the vectors corresponding to pivot columns:

$$\left\{\begin{bmatrix}3\\-1\\2\end{bmatrix},\begin{bmatrix}4\\1\\1\end{bmatrix}\right\}$$

Fact A.35

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

- The kernel of T is the solution set of the homogeneous system given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$. Use the coefficients of its free variables to get a basis for the kernel.
- The image of T is the span of the columns of A. Remove the vectors creating non-pivot columns in RREF A to get a basis for the image.

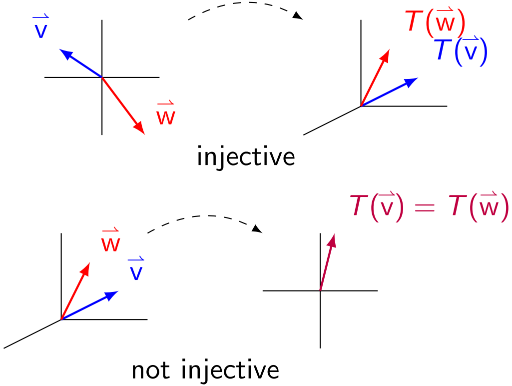
Activity A.36 *(∼10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T.

Definition A.37

Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.



Activity A.38 *(∼3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T injective?

a) Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.

b) Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

c) No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$

d) No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$

Activity A.39 *(∼2 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T injective?

a) Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.

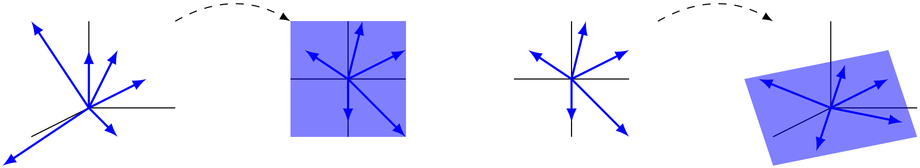
b) Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.

c) No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$

d) No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$

Definition A.40

Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\vec{w} \in W$, there is some $\vec{v} \in V$ with $T(\vec{v}) = \vec{w}$.



surjective

not surjective

Activity A.41 *(∼3 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T surjective?

a) Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, there exists $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T(\vec{v}) = \vec{w}$.

b) No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

c) No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity A.42 *(~2 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective?

a) Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} x \\ y \\ 42 \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

b) Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{v}) = \vec{w}$.

c) No, because $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Module A Section 4

Observation A.43

As we will see, it's no coincidence that the RREF of the injective map's standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has all pivot columns. Similarly, the RREF of the surjective map's standard matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has a pivot in each row.

Observation A.44

Let $T : V \to W$. We have previously defined the following terms.

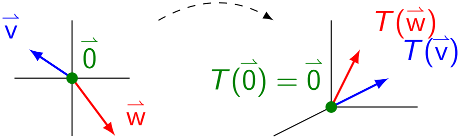
- The **kernel** of T is the set of all vectors in V that are mapped to $\vec{z} \in W$. It is a subspace of V.
- The **image** of T is the set of all vectors in W that are mapped to by something in V. It is a subspace of W.
- T is called **injective** or **one-to-one** if T always maps distinct vectors to different places.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.

Activity A.45 *(∼5 min)* Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

a. T is injective
b. T is not injective
c. T is surjective
d. T is not surjective

Fact A.46

A linear transformation T is injective **if and only if** $\ker T = \{\vec{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

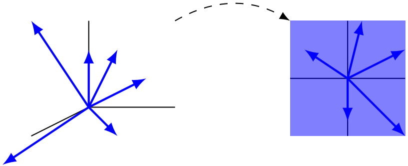


Activity A.47 *(~5 min)* Let $T : V \to \mathbb{R}^5$ be a linear transformation where $\operatorname{Im} T$ is spanned by four vectors. What can you conclude?

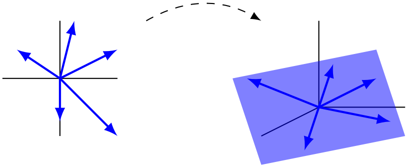
a. T is injective
b. T is not injective
c. T is surjective
d. T is not surjective

Fact A.48

A linear transformation $T : V \to W$ is surjective **if and only if** $\operatorname{Im} T = W$. Put another way, a surjective linear transformation may be recognized by its identical codomain and image.



surjective, $\operatorname{Im} T = \mathbb{R}^2$



not surjective, $\operatorname{Im} T \neq \mathbb{R}^3$

Activity A.49 *(~15 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of *equivalent* statements: one group that means T is **injective**, and one group that means T is **surjective**.

(a) The kernel of T is trivial, i.e. $\ker T = \{\vec{0}\}$.

(b) The columns of A span $\mathbb{R}^m$.

(c) The columns of A are linearly independent.

(d) Every column of RREF(A) has a pivot.

(e) Every row of RREF(A) has a pivot.

(f) The image of T equals its codomain, i.e. $\operatorname{Im} T = \mathbb{R}^m$.

(g) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has a solution for all $\vec{b} \in \mathbb{R}^m$.

(h) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$ has exactly one solution.

Observation A.50

The easiest way to show that the linear map with standard matrix A is injective is to show that $\mathrm{RREF}(A)$ has a pivot in each column.

The easiest way to show that the linear map with standard matrix A is surjective is to show that $\mathrm{RREF}(A)$ has a pivot in each row.

Activity A.51 *(∼3 min)* What can you conclude about the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

d) Its standard matrix has more rows than columns, so T is surjective.

Activity A.52 *($\sim$2 min)* What can you conclude about the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

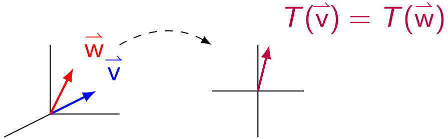
d) Its standard matrix has more rows than columns, so T is surjective.

Fact A.53

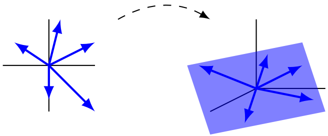
The following are true for any linear map $T : V \to W$:

- If $\dim(V) > \dim(W)$, then T is not injective.
- If $\dim(V) < \dim(W)$, then T is not surjective.

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase dimension from its domain to its image.



not injective, $3 > 2$



not surjective, $2 < 3$

But dimension arguments **cannot** be used to prove a map **is** injective or surjective.

Activity A.54 *(~5 min)* Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijective**).

Activity A.54 *(∼5 min)* Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijective**).

Part 1: How many pivot rows must RREF A have?

Activity A.54 *($\sim$5 min)* Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijective**).

Part 1: How many pivot rows must RREF A have?

Part 2: How many pivot columns must RREF A have?

Activity A.54 *(∼5 min)* Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijective**).

Part 1: How many pivot rows must RREF A have?

Part 2: How many pivot columns must RREF A have?

Part 3: What is RREF A?

Activity A.55 *(~5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

a. RREF(A) is the identity matrix.

b. The columns of A form a basis for $\mathbb{R}^n$

c. The system of linear equations given by the augmented matrix $\left[A \,\middle|\, \vec{b}\right]$ has exactly one solution for each $\vec{b} \in \mathbb{R}^n$.

Observation A.56

The easiest way to show that the linear map with standard matrix A is bijective is to show that $\mathrm{RREF}(A)$ is the identity matrix.

Activity A.57 *(∼3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

- (a) T is neither injective nor surjective
- (b) T is injective but not surjective
- (c) T is surjective but not injective
- (d) T is bijective.

Activity A.58 *(~3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

a. T is neither injective nor surjective
b. T is injective but not surjective
c. T is surjective but not injective
d. T is bijective.

Activity A.59 *(∼3 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x-y \\ x+3y \end{bmatrix}.$$

Which of the following must be true?

a. T is neither injective nor surjective
b. T is injective but not surjective
c. T is surjective but not injective
d. T is bijective.