Name: ______________________________

MIDTERM EXAM Math 237 – Linear Algebra
Version 5 Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

E1. Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x + 3y - 4z &= 5 \\ 3x + 9y + z &= 0 \\ x - z &= 1 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{array}\right]$$

□

E2. Find the reduced row echelon form of the matrix below.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array}\right]$$

Solution:

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} -1 & 0 & 5 & 0 & -1 \\ 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & -1 & 15 & -2 & -3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & 0 & 24 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{5}{12} & 1 \\ 0 & 1 & 0 & \frac{3}{4} & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{array}\right]$$

□

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x_1 + 3x_2 - 5x_3 + 14x_4 &= 8 \\ x_1 + x_2 - x_3 + 5x_4 &= 3 \end{aligned}$$

Solution: Let $A = \left[\begin{array}{cccc|c} 2 & 3 & -5 & 14 & 8 \\ 1 & 1 & -1 & 5 & 3 \end{array}\right]$, so RREF $A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -3 & 4 & 2 \end{array}\right]$. It follows that the solution set is given by $\begin{bmatrix} 1-2a-b \\ 2+3a-4b \\ a \\ b \end{bmatrix}$ for all real numbers a, b.

□

E4. Find a basis for the solution set to the homogeneous system of equations given by

$$\begin{aligned} 3x + 2y + z &= 0 \\ x + y + z &= 0 \end{aligned}$$

Solution: Let $A = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$, so RREF $A = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$. It follows that the basis for the solution set is given by $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

$\square$

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, y_1), (x_2, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, y_1) \oplus (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ c \odot (x_1, y_1) &= (0, cy_1) \end{aligned}$$

(a) Show that scalar multiplication **distributes vectors** over scalar addition: $(c + d) \odot (x, y) = c \odot (x, y) \oplus d \odot (x, y)$.

(b) Determine if V is a vector space or not. Justify your answer.

Solution: Let $(x_1, y_1) \in V$, and let $c, d \in \mathbb{R}$. Then

$$(c + d) \odot (x_1, y_1) = (0, (c + d)y_1) = (0, cy_1) \oplus (0, dy_1) = c \odot (x_1, y_1) \oplus d \odot (x_1, y_1).$$

However, V is not a vector space, as $1 \odot (x_1, y_1) = (0, y_1) \neq (x_1, y_1)$.

$\square$

V2. Determine if $\begin{bmatrix} 0 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ belongs to the span of the set $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \\ 3 \end{bmatrix} \right\}$.

Solution: Since

$$\text{RREF} \left(\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & -1 & -1 \\ -1 & 4 & 6 \\ 5 & 3 & -7 \end{array}\right] \right) = \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

does not contain a contradiction, $\begin{bmatrix} 0 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ is a linear combination of the three vectors.

$\square$

V3. Determine if the vectors $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix}$ span $\mathbb{R}^4$.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are zero rows, they do not span. Alternatively, by inspection $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, so the set is linearly dependent, so it spans a subspace of dimension at most 3, therefore it does not span $\mathbb{R}^4$.

$\square$

V4. Let W be the set of all polynomials of the form $ax^3 + bx$. Determine if W is a subspace of $\mathcal{P}^3$.

Solution: Yes because $s(a_1x^3 + b_1x) + t(a_2x^3 + b_2x) = (sa_1 + ta_2)x^3 + (sb_1 + tb_2)x$ also belongs to W. Alternately, yes because W is isomorphic to $\mathbb{R}^2$.

$\square$

S1. Determine if the set of vectors $\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$ is linearly dependent or linearly independent

Solution:

$$\text{RREF}\left(\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

This has a non pivot column, therefore the set is linearly dependent.

$\square$

S2. Determine if the set $\left\{x^2 + x - 1, 3x^2 - x + 1, 2x - 2\right\}$ is a basis of $\mathcal{P}_2$

Solution:

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis.

$\square$

S3. Let W be the subspace of $\mathcal{P}_3$ given by
$W = \text{span}\left(\left\{x^3 + x^2 + 2x + 1, 3x^3 + 3x^2 + 6x + 3, 3x^3 - x^2 + 3x - 2, 7x^3 - x^2 + 8x - 3\right\}\right)$. Find a basis for W.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis is $\left\{x^3 + x^2 + 2x + 1, 3x^3 - x^2 + 3x - 2\right\}$.

$\square$

S4. Let $W = \text{span}\left(\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\0\\1\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\4\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}\right)$. Compute the dimension of W.

Solution:

$$\text{RREF}\left(\begin{bmatrix}1 & 2 & 3 & 1\\-1 & 0 & -1 & 1\\3 & 1 & 4 & 1\\-3 & 1 & -2 & -7\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$.

$\square$

E1: $\boxed{}$

E2: $\boxed{}$

E3: $\boxed{}$

E4: $\boxed{}$

V1: $\boxed{}$

V2: $\boxed{}$

V3: $\boxed{}$

V4: $\boxed{}$

S1: $\boxed{}$

S2: $\boxed{}$

S3: $\boxed{}$

S4: $\boxed{}$