Name:
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Date:

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MIDTERM EXAM Math 237 – Linear Algebra

Version 6 Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

Standard E1.	Mark:

Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x + 3y - 4z &= 5 \\ 3x + 9y + z &= 0 \\ x - z &= 1 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{array}\right]$$

$\square$

Standard E2.	Mark:

Put the following matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array}\right]$$

Solution:

$$\left[\begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & -\frac{2}{3} & -1 & \frac{2}{3} \\ 0 & 2 & 3 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 2 & 3 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\square$

Standard E3.	Mark:

Solve the following linear system.

$$
\begin{aligned}
4x_1 + 4x_2 + 3x_3 - 6x_4 &= 5 \\
-2x_3 - 4x_4 &= 3 \\
2x_1 + 2x_2 + x_3 - 4x_4 &= -1
\end{aligned}
$$

Solution: Let $A = \left[\begin{array}{cccc|c} 4 & 4 & 3 & -6 & 5 \\ 0 & 0 & -2 & -4 & 3 \\ 2 & 2 & 1 & -4 & -1 \end{array}\right]$, so RREF $A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$. It follows that the system is inconsistent with no solutions (since the bottom row implies the contradiction $0 = 1$).

□

Standard E4.	Mark:

Find a basis for the solution set to the homogeneous system of equations

$$
\begin{aligned}
4x_1 + 4x_2 + 3x_3 - 6x_4 &= 0 \\
-2x_3 - 4x_4 &= 0 \\
2x_1 + 2x_2 + x_3 - 4x_4 &= 0
\end{aligned}
$$

Solution: Let $A = \left[\begin{array}{cccc|c} 4 & 4 & 3 & -6 & 0 \\ 0 & 0 & -2 & -4 & 0 \\ 2 & 2 & 1 & -4 & 0 \end{array}\right]$, so RREF $A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. It follows that the basis for the solution set is given by $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

□

Standard V1.	Mark:

Let V be the set of all polynomials with the operations, for any $f, g \in V$, $c \in \mathbb{R}$,

$$
\begin{aligned}
f \oplus g &= f' + g' \\
c \odot f &= cf'
\end{aligned}
$$

(here f' denotes the derivative of f).

(a) Show that scalar multiplication **distributes scalars** over vector addition: $c \odot (f \oplus g) = c \odot f \oplus c \odot g$.

(b) Determine if V is a vector space or not. Justify your answer.

Solution: Let $f, g \in \mathcal{P}$, and let $c \in \mathbb{R}$.

$$c \odot (f \oplus g) = c \odot (f' + g') = c(f' + g')' = cf'\,' + cg'\,' = cf' \oplus cg' = c \odot f \oplus c \odot g.$$

However, this is not a vector space, as there is no zero vector. Additionally, $1 \odot f \neq f$ for any nonzero polynomial f.

$\square$

Standard V2.	Mark:

Determine if $\begin{bmatrix} 0 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ belongs to the span of the set $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \\ 3 \end{bmatrix} \right\}$.

Solution: Since

$$\text{RREF} \left(\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & -1 & -1 \\ -1 & 4 & 6 \\ 5 & 3 & -7 \end{array}\right] \right) = \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

does not contain a contradiction, $\begin{bmatrix} 0 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ is a linear combination of the three vectors.

$\square$

Standard V3.	Mark:

Does $\operatorname{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$?

Solution: Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 2 & 2 \\ 4 & -9 & 3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

lacks a zero row, the vectors span $\mathbb{R}^3$.

$\square$

Standard V4.	Mark:

Let W be the set of all $\mathbb{R}^3$ vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $x + y + z = 1$ (this forms a plane). Determine if W is a subspace of $\mathbb{R}^3$.

Solution: No, because $\mathbf{0}$ does not belong to W.

$\square$

Standard S1.	Mark:

Determine if the set of polynomials $\left\{x^2+x, x^2+2x-1, x^2+3x-2\right\}$ is linearly dependent or linearly independent

Solution:

$$\mathrm{RREF}\left(\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 3\\ 1 & -1 & -2\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & \frac{3}{2}\\ 0 & 0 & 0\end{bmatrix}$$

Since there is a nonpivot column, the set is linearly dependent.

□

Standard S2.	Mark:

Determine if the set $\left\{\begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\-1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^4$.

Solution:

$$\mathrm{RREF}\left(\begin{bmatrix}0 & 1 & 1 & 0\\ 1 & -1 & 0 & 2\\ 1 & 0 & -1 & 0\\ 1 & 2 & 0 & -1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.

□

Standard S3.	Mark:

Let $W = \operatorname{span}\left(\left\{\begin{bmatrix}-3\\-8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}\right\}\right)$. Find a basis for W.

Solution: Let $A = \begin{bmatrix}-3 & 1 & 0\\ -8 & 2 & -1\\ 0 & 2 & 3\end{bmatrix}$, and compute $\mathrm{RREF}(A) = \begin{bmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{3}{2}\\ 0 & 0 & 0\end{bmatrix}$. Since the first two columns are pivot columns, $\left\{\begin{bmatrix}-3\\-8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right\}$ is a basis for W.

□

Standard S4.	Mark:

Let W be the subspace of $\mathcal{P}_3$ given by
$W = \text{span}\left(\{x^3 - x^2 + 3x - 3, 2x^3 + x + 1, 3x^3 - x^2 + 4x - 2, x^3 + x^2 + x - 7\}\right)$. Compute the dimension of W.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$.

$\square$

Additional Notes/Marks	