

Module M: Understanding Matrices Algebraically

What algebraic structure do matrices have?

Module M

Section M.1

Section M.2

Section M.3

At the end of this module, students will be able to...

M1. Matrix Multiplication. ... multiply matrices.

M2. Invertible Matrices. ... determine if a square matrix is invertible or not.

M3. Matrix inverses. ... compute the inverse matrix of an invertible matrix.

M4. Row operations as multiplication. ... describe the row reduction of a matrix as matrix multiplication.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers.
- Identify the domain and codomain of linear transformations.
- Find the matrix corresponding to a linear transformation and compute the image of a vector given a standard matrix **A2**
- Determine if a linear transformation is injective and/or surjective **A3**
- Interpret the ideas of injectivity and surjectivity in multiple ways.

Module M

Section M.1

Section M.2

Section M.3

The following resources will help you prepare for this module.

- Function composition (Khan Academy): <http://bit.ly/2wkz7f3>
- Domain and codomain: <https://www.youtube.com/watch?v=BQMyeQOLvpg>
- Interpreting injectivity and surjectivity in many ways:
<https://www.youtube.com/watch?v=WpUv72Y6Dl0>

Module M Section 1

Activity M.1.1 (*~5 min*)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

- (a) \mathbb{R}
- (b) \mathbb{R}^2
- (c) \mathbb{R}^3
- (d) \mathbb{R}^4

Activity M.1.2 (~ 3 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the codomain of the composition map $S \circ T$?

- (a) \mathbb{R}
- (b) \mathbb{R}^2
- (c) \mathbb{R}^3
- (d) \mathbb{R}^4

Activity M.1.3 (~ 2 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What size will the standard matrix of $S \circ T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be? (Rows \times Columns)

(a) 4×3

(c) 3×4

(e) 2×4

(b) 4×2

(d) 3×2

(f) 2×3

Activity M.1.4 (*~15 min*)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Activity M.1.4 (~ 15 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Activity M.1.4 (~ 15 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{\mathbf{e}}_2)$.

Activity M.1.4 (~ 15 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{\mathbf{e}}_2)$.

Part 3: Compute $(S \circ T)(\vec{\mathbf{e}}_3)$.

Activity M.1.4 (~ 15 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{\mathbf{e}}_2)$.

Part 3: Compute $(S \circ T)(\vec{\mathbf{e}}_3)$.

Part 4: Write the 4×3 standard matrix of $S \circ T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$.

Definition M.1.5

We define the **product** AB of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For the previous activity, S had a 4×2 matrix and T had a 2×3 matrix, so $S \circ T$ had a 4×3 standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= [(S \circ T)(\vec{e}_1) \quad (S \circ T)(\vec{e}_2) \quad (S \circ T)(\vec{e}_3)] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

Activity M.1.6 (*~15 min*)

Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Activity M.1.6 (~ 15 min)

Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be

given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Part 1: Write the dimensions (rows \times columns) for A , B , AB , and BA .

Activity M.1.6 (*~15 min*)

Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be

given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Part 1: Write the dimensions (rows \times columns) for A , B , AB , and BA .

Part 2: Find the standard matrix AB of $S \circ T$.

Activity M.1.6 (~ 15 min)

Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be

given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Part 1: Write the dimensions (rows \times columns) for A , B , AB , and BA .

Part 2: Find the standard matrix AB of $S \circ T$.

Part 3: Find the standard matrix BA of $T \circ S$.

Activity M.1.7 (~ 10 min)

Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Activity M.1.7 (~ 10 min)

Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Part 1: Label each of these matrices with its number of rows \times columns.

Activity M.1.7 (~ 10 min)

Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Part 1: Label each of these matrices with its number of rows \times columns.

Part 2: Only one of the matrix products AB, AC, BA, BC, CA, CB can actually be computed. Compute it.

Module M Section 2

Remark M.2.1

Recall that the **product** AB of a $m \times n$ matrix A and an $n \times k$ matrix B is the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For example, if S has a $4 \times \textcircled{2}$ matrix A and T has a $\textcircled{2} \times 3$ matrix B , then $S \circ T$ has a 4×3 standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= [(S \circ T)(\vec{e}_1) \quad (S \circ T)(\vec{e}_2) \quad (S \circ T)(\vec{e}_3)] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

Activity M.2.2 (*~15 min*)

Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Activity M.2.2 (*~15 min*)

Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Part 1: Compute the product BA by hand.

Activity M.2.2 (*~15 min*)

Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Part 1: Compute the product BA by hand.

Part 2: Check your work using technology. Using Octave:

- $B = \text{sym}([3 \ -4 \ 0 \ ; \ 2 \ 0 \ -1 \ ; \ 0 \ -3 \ 3])$
- $A = \text{sym}([2 \ 7 \ -1 \ ; \ 0 \ 3 \ 2 \ ; \ 1 \ 1 \ -1])$
- $B*A$

Activity M.2.3 (~ 5 min)

Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix B such that $BA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Check your guess using technology.

Definition M.2.4

The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

Fact M.2.5

For any square matrix A , $IA = AI = A$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Activity M.2.6 (~ 20 min)

Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Activity M.2.6 (~ 20 min)

Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Part 1: Create a matrix that doubles the third row of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Activity M.2.6 (~ 20 min)

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Part 2: Create a matrix that swaps the second and third rows of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Activity M.2.6 (~ 20 min)

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Part 2: Create a matrix that swaps the second and third rows of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 + 5(1) & 7 + 5(1) & -1 + 5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Fact M.2.7

If R is the result of applying a row operation to I , then RA is the result of applying the same row operation to A .

- Scaling a row: $R = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Swapping rows: $R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Adding a row multiple to another row: $R = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Such matrices can be chained together to emulate multiple row operations. In particular,

$$\text{RREF}(A) = R_k \dots R_2 R_1 A$$

for some sequence of matrices R_1, R_2, \dots, R_k .

Activity M.2.8 (~ 10 min)

Consider the two row operations $R_2 \leftrightarrow R_3$ and $R_1 + R_3 \rightarrow R_1$ applied as follows to show $A \sim B$:

$$\begin{aligned} A = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} &\sim \begin{bmatrix} -1 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} \\ &\sim \begin{bmatrix} -1+1 & 4+2 & 5+3 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = B \end{aligned}$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A :

$$B = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} A$$

Check your work using technology.

Module M Section 3

Activity M.3.1 (*~15 min*)

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with standard matrix A . Sort the following items into three groups of statements: a group that means T is **injective**, a group that means T is **surjective**, and a group that means T is **bijjective**.

- (a) $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$
- (b) $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$
- (c) $A\vec{x} = \vec{0}$ has a unique solution.
- (d) The columns of A span \mathbb{R}^m
- (e) The columns of A are linearly independent
- (f) The columns of A are a basis of \mathbb{R}^m
- (g) Every column of $\text{RREF}(A)$ has a pivot
- (h) Every row of $\text{RREF}(A)$ has a pivot
- (i) $m = n$ and $\text{RREF}(A) = I$

Definition M.3.2

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map with standard matrix A .

- If T is a bijection and $\vec{\mathbf{b}}$ is any \mathbb{R}^n vector, then $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution.
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by setting $T^{-1}(\vec{\mathbf{b}})$ to be this unique solution.
- Let A^{-1} be the standard matrix for T^{-1} . We call A^{-1} the **inverse matrix** of A , so we also say that A is **invertible**.

Activity M.3.3 (*~20 min*)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Activity M.3.3 (~ 20 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Part 1: Write an augmented matrix representing the system of equations given by

$T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Activity M.3.3 (~ 20 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

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Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Activity M.3.3 (~ 20 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Part 1: Write an augmented matrix representing the system of equations given by

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Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Part 3: Solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

Activity M.3.3 (~ 20 min)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Part 1: Write an augmented matrix representing the system of equations given by

$$T(\vec{x}) = \vec{e}_1, \text{ that is, } A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \text{ Then solve } T(\vec{x}) = \vec{e}_1 \text{ to find } T^{-1}(\vec{e}_1).$$

Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Part 3: Solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

Part 4: Write A^{-1} , the standard matrix for T^{-1} .

Observation M.3.4

We could have solved these three systems simultaneously by row reducing the matrix $[A \mid I]$ at once.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array} \right]$$

Activity M.3.5 (~ 5 min)

Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A \mid I]$.

Activity M.3.6 (~ 5 min)

Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

Observation M.3.7

An $n \times n$ matrix A is invertible if and only if $\text{RREF}(A) = I_n$.

Activity M.3.8 (~ 10 min)

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the bijective linear map defined by $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$,
with the inverse map $T^{-1} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Activity M.3.8 (~ 10 min)

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the bijective linear map defined by $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$,

with the inverse map $T^{-1} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T) \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$.

Activity M.3.8 (~ 10 min)

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the bijective linear map defined by $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$,

with the inverse map $T^{-1} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T) \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$.

Part 2: If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1} , find the 2×2 matrix

$$A^{-1}A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

Observation M.3.9

$T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation T .
Therefore $A^{-1}A = AA^{-1} = I$ is the identity matrix for any invertible matrix A .