Section V.0

Observation V.0.1 Several properites of the real numbers (the **one-dimensional Euclidean vectors** $\mathbb{R}^1$) also work for **two-dimensional Euclidean vectors** ($\mathbb{R}^2$) and **three-dimensional Euclidean vectors** ($\mathbb{R}^3$), such as $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ and $1\mathbf{v}=\mathbf{v}$:

$$x+y=y+x \qquad \begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}y_1\\y_2\end{bmatrix}+\begin{bmatrix}x_1\\x_2\end{bmatrix} \qquad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}+\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$1x=x \qquad 1\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix} \qquad 1\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

Activity V.0.2 *(∼20 min)* Consider each of the following properties of vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ in $\mathbb{R}^1$ (real numbers). Label each property as "VALID" if it also holds for Euclidean vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ in $\mathbb{R}^2$, and "INVALID" if it does not.

1. **Addition associativity.**
   $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$.
2. **Addition commutivity.**
   $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. **Addition identity.**
   There exists some $\mathbf{z}$ where $\mathbf{v}+\mathbf{z}=\mathbf{v}$.
4. **Addition inverse.**
   There exists some $-\mathbf{v}$ where $\mathbf{v}+(-\mathbf{v})=\mathbf{z}$.
5. **Addition midpoint uniqueness.**
   If $\mathbf{u}\neq\mathbf{v}$, then $\frac{1}{2}(\mathbf{u}+\mathbf{v})$ is the only vector equally distant from $\mathbf{u}$ and $\mathbf{v}$.
6. **Scalar multiplication associativity.**
   $a(b\mathbf{v})=(ab)\mathbf{v}$.
7. **Scalar multiplication identity.**
   $1\mathbf{v}=\mathbf{v}$.
8. **Scalar multiplication relativity.**
   If $\mathbf{u}\neq\mathbf{z}$, there exists a scalar c satisfying $c\mathbf{u}=\mathbf{v}$.
9. **Scalar distribution.**
   $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$.
10. **Vector distribution.**
    $(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$.

Definition V.0.3 A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Addition associativity.**
  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition commutivity.**
  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Addition identity.**
  There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.
- **Addition inverse.**
  There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.
- **Scalar multiplication associativity.**
  $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **Scalar multiplication identity.**
  $1\mathbf{v} = \mathbf{v}$.
- **Scalar distribution.**
  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Vector distribution.**
  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Any **Euclidean vector space** $\mathbb{R}^n$ satisfies all eight requirements regardless of the value of n, but we will also study other types of vector spaces.