Sample Assessment Exercises

This document contains one exercise and solution for each standard. The goal is to give you an idea of what the exercises might look like, and what the expectations for a complete solution are.

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & 1 & 1 \\ -1 & -4 & 1 & -7 & 0 \\ 0 & 1 & -1 & 0 & -2 \end{array}\right]$$

Solution:

$$\begin{aligned} 3x_1 + 2x_2 \phantom{{}+x_3} + x_4 &= 1 \\ -x_1 - 4x_2 + x_3 - 7x_4 &= 0 \\ x_2 - x_3 \phantom{{}-7x_4} &= -2 \end{aligned}$$

□

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 2 & -1 & -3 \\ 2 & 4 & -1 & -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 2 & -1 & -3 \\ 2 & 4 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 3 & 1 & 2 \\ 2 & 4 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{13}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{13}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

□

E3. Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x + 4y + z &= 5 \\ x + 2y \phantom{{}+z} &= 3 \end{aligned}$$

Solution:

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} 2 & 4 & 1 & 5 \\ 1 & 2 & 0 & 3 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This corresponds to the system

$$\begin{aligned} x + 2y \phantom{{}+z} &= 3 \\ z &= -1 \end{aligned}$$

Since the y-column is a non-pivot column, it is a free variable, so we let $y = a$; then we have

$$\begin{aligned} x + 2y \phantom{{}+z} &= 3 \\ y \phantom{{}+z} &= a \\ z &= -1 \end{aligned}$$

and thus

$$\begin{aligned} x &= 3 - 2a \\ y &= a \\ z &= -1 \end{aligned}$$

So the solution set is

$$\left\{ \begin{bmatrix} 3-2a \\ a \\ -1 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$$

$\square$

V1. Let V be the set of all polynomials, together with the operations $\oplus$ and $\odot$ defined by the following for all polynomials $f(x), g(x)$ and scalars $c \in \mathbb{R}$:

$$\begin{aligned} f(x) \oplus g(x) &= xf(x) + xg(x) \\ c \odot f(x) &= cf(x) \end{aligned}$$

(a) Show that scalar distribution

$$c \odot (f(x) \oplus g(x)) = c \odot f(x) \oplus c \odot g(x)$$

holds.

(b) Show that addition associativity

$$(f(x) \oplus g(x)) \oplus h(x) = f(x) \oplus (g(x) \oplus h(x))$$

fails.

Solution:

(a) Compute

$$\begin{aligned} c \odot (f(x) \oplus g(x)) &= c \odot (xf(x) + xg(x)) \\ &= c\,(xf(x) + xg(x)) \\ &= cxf(x) + cxg(x) \end{aligned}$$

and

$$\begin{aligned} c \odot f(x) \oplus c \odot g(x) &= (cf(x)) \oplus (cg(x)) \\ &= xcf(x) + xcg(x) \end{aligned}$$

Since these are the same, we have shown that $c \odot (f(x) \oplus g(x)) = c \odot f(x) \oplus c \odot g(x)$ holds.

(b) Suppose $f(x) = 1$, $g(x) = 2$, and $h(x) = 3$. Then

$$\begin{aligned} (f(x) \oplus g(x)) \oplus h(x) &= (x + 2x) \oplus 3 \\ &= 3x \oplus 3 \\ &= 3x^2 + 3x \end{aligned}$$

and

$$\begin{aligned} f(x) \oplus (g(x) \oplus h(x)) &= 1 \oplus (2x + 3x) \\ &= 1 \oplus 5x \\ &= x + 5x^2 \end{aligned}$$

Since $3x^2 + 3x \neq x + 5x^2$, we have shown $(f(x) \oplus g(x)) \oplus h(x) = f(x) \oplus (g(x) \oplus h(x))$ fails.

$\square$

V2. Let V be the set of all non-negative real numbers with the operations $\oplus$ and $\odot$ given by, for all $x, y \in V$ and $c \in \mathbb{R}$,

$$x \oplus y = x + y$$
$$c \odot x = |c|x$$

List the 8 defining properties of a vector space, and label each as "TRUE" or "FALSE" as they apply to V. Based on these, conclude whether V is a vector space or not.

Solution:

1) Addition associativity: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all $x, y, z \in V$. **TRUE**

2) Addition commutivity: $x \oplus y = y \oplus x$ for all $x, y \in V$. **TRUE**

3) Addition identity: there exists an element $z \in V$ such that for all $x \in V$, $x \oplus z = x$. **TRUE**

4) Addition inverses: for every $x \in V$ there is an element $-x \in V$ such that $x \oplus (-x) = z$. **FALSE**

5) Scalar multiplication associativity: for each $c, d \in \mathbb{R}$ and $x \in V$, $c \odot (d \odot x) = (cd) \odot x$. **TRUE**

6) Scalar multiplication identity: for all $x \in V$, $1 \odot x = x$. **TRUE**

7) Scalar distribution: for all $x, y \in V$ and $c \in \mathbb{R}$, $c \odot (x \oplus y) = c \odot x \oplus c \odot y$. **TRUE**

8) Vector distribution: for all $x \in V$ and $c, d \in \mathbb{R}$, $(c + d) \odot x = c \odot x \oplus d \odot x$ **FALSE**

Since at least one property fails, V is not a vector space.

$\square$

V3. Determine if $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Solution:
We compute

$$\text{RREF} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 2 & 1 & -1 \\ 1 & -1 & -1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Since this corresponds to an inconsistent system of equations, $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is **not** a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

$\square$

V4. Determine if the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ span $\mathbb{R}^3$.

Solution:
We compute

$$\text{RREF} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since the last row lacks a pivot, the vectors **do not span** $\mathbb{R}^3$.

$\square$

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y = 3z \right\}$$

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y = 3z + 2 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.

Solution: Let $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in W$, so we know $x_1 + y_1 = 3z_1$ and $x_2 + y_2 = 3z_2$. Consider

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

Since

$$(x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) = 3z_1 + 3z_2 = 3(z_1 + z_2)$$

we see that W is closed under vector addition. Now consider

$$c \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}.$$

Since

$$cx_1 + cx_2 = c(x_1 + x_2) = c(3z_1) = 3(cz_1)$$

we see that W is closed under scalar multiplication. Therefore W is a subspace of $\mathbb{R}^3$.

However, note that $\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ are vectors in U since $0 + 5 = 3(1) + 2$ and $1 + 5 = 3(1) + 2$. But

$$\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix}$$

does not belong to U since $1 + 9 \neq 3(2) + 2$. Since U is not closed under vector addition, U is not a subspace of $\mathbb{R}^3$.

$\square$

S1. Determine if the vectors $\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 5 \\ 1 \\ 5 \end{bmatrix}$ are linearly dependent or linearly independent.

Solution: Compute

$$\text{RREF} \begin{bmatrix} 3 & -1 & 0 & 2 \\ 2 & 1 & 1 & 5 \\ 1 & 2 & -1 & 1 \\ 0 & 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the fourth column is not a pivot column, the vectors are linearly dependent.

$\square$

S2. Determine if the set

$$\left\{\begin{bmatrix}3\\2\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\2\\3\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix},\begin{bmatrix}2\\5\\1\\5\end{bmatrix}\right\}$$

is a basis of $\mathbb{R}^4$ or not.

Solution: Compute

$$\text{RREF}\begin{bmatrix}3&-1&0&2\\2&1&1&5\\1&2&-1&1\\0&3&1&5\end{bmatrix}=\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&2\\0&0&0&0\end{bmatrix}.$$

Since the fourth column is not a pivot column, the vectors are linearly dependent and thus not a basis. (Alternate solution: since the fourth row not a pivot row, the vectors do not span $\mathbb{R}^4$ and thus are not a basis.)

$\square$

S3. Find a basis for W, the subspace of $\mathbb{R}^4$ given by

$$W=\text{span}\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix},\begin{bmatrix}1\\0\\1\\-2\end{bmatrix},\begin{bmatrix}3\\-6\\-1\\2\end{bmatrix},\begin{bmatrix}1\\6\\1\\-1\end{bmatrix},\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}.$$

Solution: Observe that

$$\text{RREF}\begin{bmatrix}1&1&3&1&2\\-3&0&-6&6&3\\-1&1&-1&1&0\\2&-2&2&-1&1\end{bmatrix}=\begin{bmatrix}1&0&2&0&1\\0&1&1&0&0\\0&0&0&1&1\\0&0&0&0&0\end{bmatrix}$$

has pivot columns in the first, second, and fourth columns, and therefore

$$\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix},\begin{bmatrix}1\\0\\1\\-2\end{bmatrix},\begin{bmatrix}1\\6\\1\\-1\end{bmatrix}\right\}$$

is a basis of W.

$\square$

S4. Find the dimension of W, the subspace of $\mathbb{R}^4$ given by

$$W=\text{span}\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix},\begin{bmatrix}1\\0\\1\\-2\end{bmatrix},\begin{bmatrix}3\\-6\\-1\\2\end{bmatrix},\begin{bmatrix}1\\6\\1\\-1\end{bmatrix},\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}.$$

Solution: Observe that

$$\text{RREF}\begin{bmatrix}1&1&3&1&2\\-3&0&-6&6&3\\-1&1&-1&1&0\\2&-2&2&-1&1\end{bmatrix}=\begin{bmatrix}1&0&2&0&1\\0&1&1&0&0\\0&0&0&1&1\\0&0&0&0&0\end{bmatrix}$$

has three pivot columns, and therefore $\dim W=3$.

$\square$

S5. Determine if the polynomials $3x^3+2x^2+x$, $-x^3+x^2+2x+3$, x^2-x+1, and $2x^3+5x^2+x+5$ are linearly dependent or linearly independent.

Solution: This question is equivalent to asking if the Euclidean vectors

$$\begin{bmatrix}3\\2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\5\\1\\5\end{bmatrix}$$

are linearly dependent or linearly independent.
Compute

$$\text{RREF}\begin{bmatrix}3 & -1 & 0 & 2\\ 2 & 1 & 1 & 5\\ 1 & 2 & -1 & 1\\ 0 & 3 & 1 & 5\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Since the fourth column is not a pivot column, the Euclidean vectors (and therefore the polynomials) are linearly dependent.

$\square$

S6. Find a basis for the solution set of the homogeneous system of equations

$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 + 2x_5 &= 0\\ -3x_1 - 6x_3 + 6x_4 + 3x_5 &= 0\\ -x_1 + x_2 - x_3 + x_4 &= 0\\ 2x_1 - 2x_2 + 2x_3 - x_4 + x_5 &= 0. \end{aligned}$$

Solution: Observe that

$$\text{RREF}\left[\begin{array}{ccccc|c}1 & 1 & 3 & 1 & 2 & 0\\ -3 & 0 & -6 & 6 & 3 & 0\\ -1 & 1 & -1 & 1 & 0 & 0\\ 2 & -2 & 2 & -1 & 1 & 0\end{array}\right] = \left[\begin{array}{ccccc|c}1 & 0 & 2 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right]$$

Letting $x_3=a$ and $x_5=b$ (since those correspond to the non-pivot columns), this is equivalent to the system

$$\begin{aligned} x_1 + 2x_3 + x_5 &= 0\\ x_2 + x_3 &= 0\\ x_3 &= a\\ x_4 + x_5 &= 0\\ x_5 &= b \end{aligned}$$

Thus, the solution set is

$$\left\{ \begin{bmatrix}-2a-b\\ -a\\ a\\ -b\\ b\end{bmatrix} \middle| \, a,b\in\mathbb{R}\right\}.$$

Since we can write

$$\begin{bmatrix} -2a-b \\ -a \\ a \\ -b \\ b \end{bmatrix} = a \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

a basis for the solution space is

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$\square$