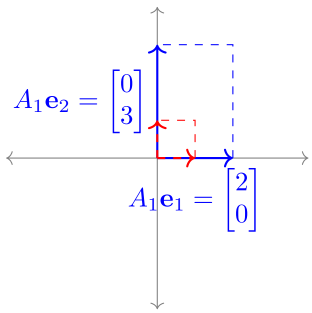
Application Activities - Module G Part 1 - Class Day 25

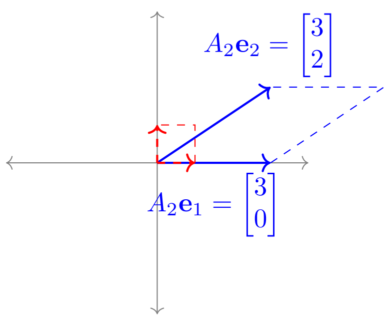
Activity 25.1 The image below illustrates how the linear transformation $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.



(a) What is the area of the transformed unit square?

(b) Find two vectors that were stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.

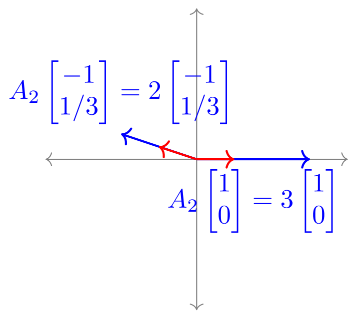
Activity 25.2 The image below illustrates how the linear transformation $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A_2 = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$. transforms the unit square.



(a) What is the area of the transformed unit square?

(b) Find at least one vector that was stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.

Observation 25.3 It's possible to find two non-parallel vectors that are stretched by the transformation given by A_2:



The process for finding such vectors will be covered later in this module.

Activity 25.4 Consider the linear transformation given by the standard matrix $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(a) Sketch the transformation of the unit square (the parallelogram given by the columns of the standard matrix).

(b) Compute the area of the transformed unit square.

Activity 25.5 Consider the linear transformation given by the standard matrix $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Sketch the transformation of the unit square.

(b) Compute the area of the transformed unit square.

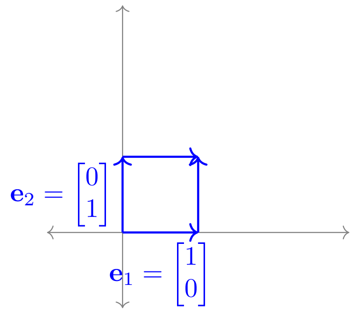
Activity 25.6 Consider the linear transformation given by the standard matrix $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

(a) Sketch the transformation of the unit square.

(b) Compute the area of the transformed unit square.

Remark 25.7 The area of the transformed unit square measures the factor by which all areas are transformed by a linear transformation.
We will define the **determinant** of a square matrix A, or $\det(A)$ for short, to be this factor. But what properties must this function satisfy?

Activity 25.8 The transformation of the unit square by the standard matrix $[\mathbf{e}_1 \ \mathbf{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. What is $\det([\mathbf{e}_1 \ \mathbf{e}_2]) = \det(I)$, that is, by what factor has the area of the unit square been scaled?



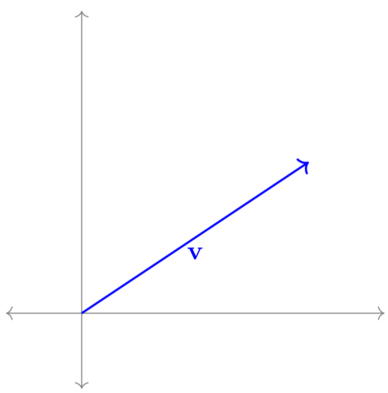
a) 0

b) 1

c) 2

d) Cannot be determined

Activity 25.9 The transformation of the unit square by the standard matrix $[\mathbf{v}\ \ \mathbf{v}]$ is illustrated below: both $T(\mathbf{e}_1) = T(\mathbf{e}_2) = \mathbf{v}$. What is $\det([\mathbf{v}\ \ \mathbf{v}])$, that is, by what factor has area been scaled?



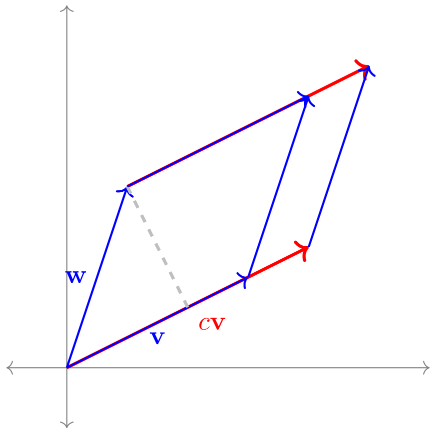
a) 0

b) 1

c) 2

d) Cannot be determined

Activity 25.10 The transformations of the unit square by the standard matrices $[\mathbf{v}\ \ \mathbf{w}]$ and $[c\mathbf{v}\ \ \mathbf{w}]$ are illustrated below. How are $\det([\mathbf{v}\ \ \mathbf{w}])$ and $\det([c\mathbf{v}\ \ \mathbf{w}])$ related?

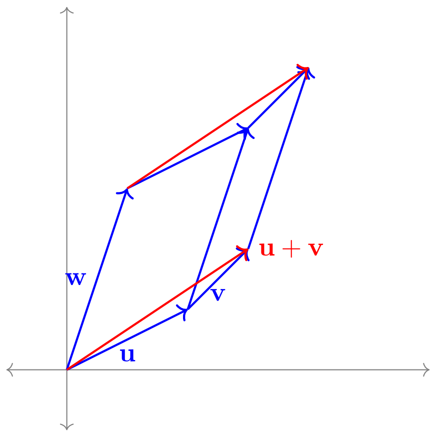


a) $\det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

b) $c + \det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

c) $c\det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

Activity 25.11 The transformations of unit squares by the standard matrices $[\mathbf{u}\ \ \mathbf{w}]$, $[\mathbf{v}\ \ \mathbf{w}]$ and $[\mathbf{u}+\mathbf{v}\ \ \mathbf{w}]$ are illustrated below. How is $\det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$ related to $\det([\mathbf{u}\ \ \mathbf{w}])$ and $\det([\mathbf{v}\ \ \mathbf{w}])$?



a) $\det([\mathbf{u}\ \ \mathbf{w}]) = \det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

b) $\det([\mathbf{u}\ \ \mathbf{w}]) + \det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

c) $\det([\mathbf{u}\ \ \mathbf{w}])\det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

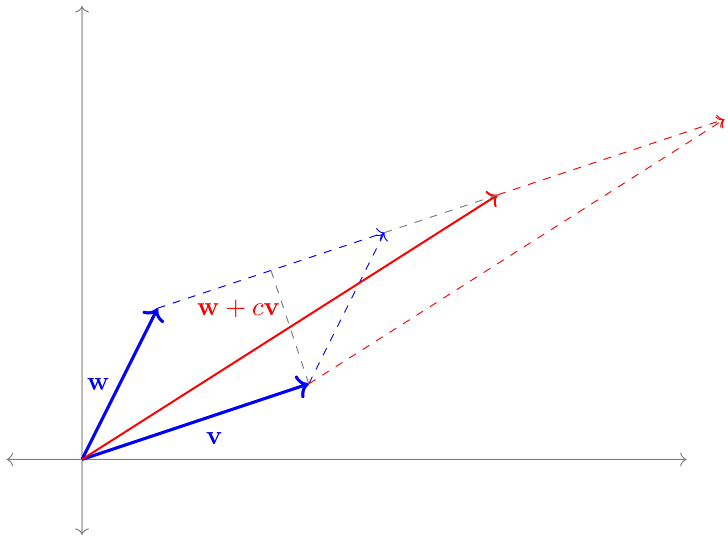
Definition 25.12 The **determinant** is the unique function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfying the following three properties:

P1: $\det(I) = 1$

P2: $\det([\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \cdots \;\; \mathbf{v}_n]) = 0$ whenever two columns of the matrix are identical.

P3: $\det[\cdots \;\; c\mathbf{v} + d\mathbf{w} \;\; \cdots] = c \det[\cdots \;\; \mathbf{v} \;\; \cdots] + d \det[\cdots \;\; \mathbf{w} \;\; \cdots]$, assuming all other columns are equal.

Observation 25.13 What happens if we had a multiple of one column to another?



The base of both parallelograms is $\mathbf{v}$, while the height has not changed. Thus

$$\det([\mathbf{v} \quad \mathbf{w} + c\mathbf{v}]) = \det([\mathbf{v} \quad \mathbf{w}])$$

Observation 25.14 Swapping columns can be obtained from a sequence of adding column multiples.

$$\begin{aligned}
\det([\mathbf{v} \quad \mathbf{w}]) &= \det([\mathbf{v} + \mathbf{w} \quad \mathbf{w}]) \\
&= \det([\mathbf{v} + \mathbf{w} \quad \mathbf{w} - (\mathbf{v} + \mathbf{w})]) \\
&= \det([\mathbf{v} + \mathbf{w} \quad -\mathbf{v}]) \\
&= \det([\mathbf{v} + \mathbf{w} - \mathbf{v} \quad -\mathbf{v}]) \\
&= \det([\mathbf{w} \quad -\mathbf{v}]) \\
&= -\det([\mathbf{w} \quad \mathbf{v}])
\end{aligned}$$

So swapping two columns results in a negation of the determinant. Therefore, determinants represent a *signed* area, since they are not always positive.

Fact 25.15 We've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c \det([\cdots \;\; \mathbf{v} \;\; \cdots]) = \det([\cdots \;\; c\mathbf{v} \;\; \cdots])$$

(b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \;\; \mathbf{v} \;\; \cdots \;\; \mathbf{w} \;\; \cdots]) = -\det([\cdots \;\; \mathbf{w} \;\; \cdots \;\; \mathbf{v} \;\; \cdots])$$

(c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \quad \mathbf{v} \quad \cdots \quad \mathbf{w} \quad \cdots]) = \det([\cdots \quad \mathbf{v}+c\mathbf{w} \quad \cdots \quad \mathbf{w} \quad \cdots])$$

Activity 25.16 The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. How must the transformation given by the standard matrix AB scale areas?

(a) 1

(b) 7

(c) 12

(d) Cannot be determined

Fact 25.17 Since the transformation given by the standard matrix AB is obtained by applying the transformations given by A and B, it follows that

$$\det(AB) = \det(A)\det(B)$$