Application Activities - Module M Part 2 - Class Day 22

Activity 22.1 Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix I such that $IA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Definition 22.2 The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

Fact 22.3 For any square matrix A, $IA = AI = A$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Activity 22.4 Each row operation can be interpreted as a type of matrix multiplication.
Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5 & 7+5 & -1-5 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Fact 22.5 If R is the result of applying a row operation to I, then RA is the result of applying the same row operation to A.

This means that for any matrix A, we can find a series of matrices $R_1, \ldots, R_k$ corresponding to the row operations such that

$$R_1 R_2 \cdots R_k A = \mathrm{RREF}(A).$$

That is, row reduction can be thought of as the result of matrix multiplication.

Activity 22.6 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into groups of statements about T.

(a) T is injective (i.e. one-to-one)

(b) T is surjective (i.e. onto)

(c) T is bijective (i.e. both injective and surjective)

(d) $AX = B$ has a solution for all $m \times 1$ matrices B

(e) $AX = B$ has a unique solution for all $m \times 1$ matrices B

(f) $AX = 0$ has a unique solution.

(g) The columns of A span $\mathbb{R}^m$

(h) The columns of A are linearly independent

(i) The columns of A are a basis of $\mathbb{R}^m$

(j) Every column of $\mathrm{RREF}(A)$ has a pivot

(k) Every row of $\mathrm{RREF}(A)$ has a pivot

(l) $m = n$ and $\mathrm{RREF}(A) = I$

Activity 22.7 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix A. If T is injective, which of the following cannot be true?

(a) A has strictly more columns than rows

(b) A has the same number of rows as columns (i.e. A is square)

(c) A has strictly more rows than columns

Activity 22.8 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix A. If T is surjective, which of the following cannot be true?

(a) A has strictly more columns than rows

(b) A has the same number of rows as columns (i.e. A is square)

(c) A has strictly more rows than columns

Activity 22.9 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix A. If T is bijective, which of the following cannot be true?

(a) A has strictly more columns than rows

(b) A has the same number of rows as columns (i.e. A is square)

(c) A has strictly more rows than columns