

Section V.2

Remark V.2.1 Recall these definitions from last class:

- A **linear combination** of vectors is given by adding scalar multiples of those vectors, such as:

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- The **span** of a set of vectors is the collection of all linear combinations of that set, such as:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

Activity V.2.2 (*~15 min*) The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when there exists a solution to the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.

Part 3: Given this solution set, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

Fact V.2.3 A vector $\vec{\mathbf{b}}$ belongs to $\text{span}\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n\}$ if and only if the linear system corresponding to $[\vec{\mathbf{v}}_1 \dots \vec{\mathbf{v}}_n \mid \vec{\mathbf{b}}]$ is consistent.

Put another way, $\vec{\mathbf{b}}$ belongs to $\text{span}\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n\}$ exactly when $\text{RREF}[\vec{\mathbf{v}}_1 \dots \vec{\mathbf{v}}_n \mid \vec{\mathbf{b}}]$ doesn't have a row $[0 \dots 0 \mid 1]$ representing the contradiction $0 = 1$.

Activity V.2.4 (*~10 min*) Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Activity V.2.5 (*~5 min*) Determine if $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Activity V.2.6 (*~10 min*) Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in \mathcal{P}^3 belong to $\text{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in \mathbb{R}^4 . (Hint: What four numbers must you know to write a \mathcal{P}^3 polynomial?)

Part 2: Solve this equivalent exercise, and use its solution to answer the original question.

Activity V.2.7 (*~5 min*) Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to $\text{span}\left\{\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\right\}$?

Activity V.2.8 (*~5 min*) Does the complex number $2i$ belong to $\text{span}\{-3 + i, 6 - 2i\}$?