Application Activities - Module V Part 2 - Class Day 8

Remark 8.1 The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{R}^\infty$: Sequences of real numbers $(v_1, v_2, \dots)$.
- $\mathbb{R}^{m\times n}$: Matrices of real numbers with m rows and n columns.
- $\mathbb{C}$: Complex numbers.
- $\mathcal{P}^n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity 8.2 Let $V = \{(a, b) : a, b \text{ are real numbers}\}$, where $(a_1, b_1) \oplus (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$ and $c \odot (a, b) = (a^c, b + c)$. Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.**
  $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$.
- **Addition commutivity.**
  $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$.
- **Addition identity.**
  There exists some $\mathbf{0}$ where $\mathbf{v} \oplus \mathbf{0} = \mathbf{v}$.
- **Addition inverse.**
  There exists some $-\mathbf{v}$ where $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.**
  $a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}$.
- **Scalar multiplication identity.**
  $1 \odot \mathbf{v} = \mathbf{v}$.
- **Scalar distribution.**
  $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$.
- **Vector distribution.**
  $(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$.

Definition 8.3 A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ for any choice of scalar multiples $c_1, c_2, \dots, c_m$.

For example, we say $\begin{bmatrix}3\\0\\5\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\2\\1\end{bmatrix}$ since

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

Definition 8.4 The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

Activity 8.5 Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch $c\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part 2: Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ in the xy plane.

Activity 8.6 Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane: $1\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix}$, $0\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix}$, $2\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix}$, $2\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix}$.

Part 2: Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity 8.7 Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix}\right\}$ in the xy plane.

Activity 8.8 The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}$ holds for some scalars x_1, x_2.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (Remember, you should use a calculator to help find RREF.)

Part 3: Given this solution, does $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belong to $\operatorname{span}\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$?
