

Section A.1

Definition A.1.1 A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \rightarrow W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition A.1.2 Given a linear transformation $T : V \rightarrow W$, V is called the **domain** of T and W is called the **co-domain** of T .



Example A.1.3 Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$\begin{aligned} T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) &= T \left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix} \right) = \begin{bmatrix} (x+u) - (z+w) \\ 3(y+v) \end{bmatrix} \\ T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) + T \left(\begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) &= \begin{bmatrix} x-z \\ 3y \end{bmatrix} + \begin{bmatrix} u-w \\ 3v \end{bmatrix} = \begin{bmatrix} (x+u) - (z+w) \\ 3(y+v) \end{bmatrix} \end{aligned}$$

And also...

$$T \left(c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T \left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \quad \text{and} \quad cT \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = c \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore T is a linear transformation.

Example A.1.4 Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact A.1.5 A map between Euclidean spaces $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of \mathbb{R}^n .

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z :

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2 , $y + 3$, and $y - 2^x$ are not linear combinations (even though $x + y$ is):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

Activity A.1.6 (~ 5 min) Recall the following rules from calculus, where $D : \mathcal{P} \rightarrow \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f .

$$D(f + g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

- a) \mathcal{P} is not a vector space
- b) D is a linear map
- c) D is not a linear map

Activity A.1.7 (~ 10 min) Let the polynomial maps $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ and $T : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \quad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4 + x)$, $S(x^4) + S(x)$, $T(x^4 + x)$, and $T(x^4) + T(x)$. Which of these maps is definitely not linear?

Fact A.1.8 If $L : V \rightarrow W$ is linear, then $L(\mathbf{z}) = L(0\mathbf{v}) = 0L(\mathbf{v}) = \mathbf{z}$ where \mathbf{z} is the additive identity of the vector spaces V, W .

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Activity A.1.9 (~ 15 min) Continue to consider $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g .

Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f . Is S linear?

Activity A.1.10 (~ 20 min) Let the polynomial maps $S : \mathcal{P} \rightarrow \mathcal{P}$ and $T : \mathcal{P} \rightarrow \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \quad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x + 1) \neq S(x) + S(1)$ to verify that S is not linear.

Part 2: Prove that T is linear by verifying that $T(f(x) + g(x)) = T(f(x)) + T(g(x))$ and $T(cf(x)) = cT(f(x))$.

Observation A.1.11 Note that S in the previous activity is not linear, even though $S(0) = (0)^2 = 0$. So showing $S(0) = 0$ isn't enough to prove a map is linear.

This is a similar situation to proving a subset is a subspace: if the subset doesn't contain \mathbf{z} , then the subset isn't a subspace. But if the subset contains \mathbf{z} , you cannot conclude anything.