

## Module A: Algebraic properties of linear maps

# How can we understand linear maps algebraically?

## Module A

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At the end of this module, students will be able to...

- A1. Linear map verification.** ... determine if a map between vector spaces of polynomials is linear or not.
- A2. Linear maps and matrices.** ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.
- A3. Injectivity and surjectivity.** ... determine if a given linear map is injective and/or surjective.
- A4. Kernel and Image.** ... compute a basis for the kernel and a basis for the image of a linear map.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans  $\mathbb{R}^n$  **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

# Module A Section 1

## Definition A.1.1

A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if  $V$  and  $W$  are vector spaces, a map  $T : V \rightarrow W$  is called a linear transformation if

- ①  $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$  for any  $\mathbf{v}, \mathbf{w} \in V$ .
- ②  $T(c\mathbf{v}) = cT(\mathbf{v})$  for any  $c \in \mathbb{R}, \mathbf{v} \in V$ .

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

## Definition A.1.2

Given a linear transformation  $T : V \rightarrow W$ ,  $V$  is called the **domain** of  $T$  and  $W$  is called the **co-domain** of  $T$ .



**Example A.1.3**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that  $T$  is linear, we must verify...

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = T \left( \begin{bmatrix} x + u \\ y + v \\ z + w \end{bmatrix} \right) = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) + T \left( \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

And also...

$$T \left( c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T \left( \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \quad \text{and} \quad cT \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = c \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore  $T$  is a linear transformation.



**Example A.1.4**

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2x \end{bmatrix}$$

To show that  $T$  is not linear, we only need to find one counterexample.

$$T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = T \left( \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + T \left( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ 0 \end{bmatrix}$$

Since the resulting vectors are different,  $T$  is a linear transformation.

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**Activity A.1.5** (*~5 min*)

Show that  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  defined by

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2x \end{bmatrix}$$

is not linear by showing that  $2T \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \neq T \left( 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ .

**Fact A.1.6**

A map between Euclidean spaces  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear exactly when every component of the output is a linear combination of the variables of  $\mathbb{R}^n$ .

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**Example A.1.7**

You can quickly identify

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

as linear because  $x - z$  and  $3y$  are linear combinations of  $x, y, z$ .

But

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2x \end{bmatrix}$$

cannot be linear because  $x^2$  and  $y + 3$  are not linear combinations of  $x, y$  (even though  $x + y$  and  $y - 2x$  are).

**Activity A.1.8** ( $\sim 3$  min)

Recall the following rules from calculus, where  $D : \mathcal{P} \rightarrow \mathcal{P}$  is the derivative map defined by  $D(f) = \frac{df}{dx}$  for each polynomial  $f$ .

$$D(f + g) = \frac{df}{dx} + \frac{dg}{dx}$$

$$D(cf) = c \frac{df}{dx}$$

What can we conclude from these rules?

- a)  $\mathcal{P}$  is not a vector space
- b)  $D$  is a linear map
- c)  $D$  is not a linear map

**Activity A.1.9** (*~12 min*)

Consider the following two polynomial maps.

$$S : \mathcal{P}^4 \rightarrow \mathcal{P}^3 \text{ given by } S(f(x)) = 2f'(x) - f''(x)$$

$$T : \mathcal{P}^2 \rightarrow \mathcal{P}^2 \text{ given by } T(f(x)) = f'(x) + x^2$$

**Activity A.1.9** (*~12 min*)

Consider the following two polynomial maps.

$$S : \mathcal{P}^4 \rightarrow \mathcal{P}^3 \text{ given by } S(f(x)) = 2f'(x) - f''(x)$$

$$T : \mathcal{P}^2 \rightarrow \mathcal{P}^2 \text{ given by } T(f(x)) = f'(x) + x^2$$

*Part 1:* Compare  $S(x^2 + x)$  with  $S(x^2) + S(x)$ , and compare  $T(x^2 + x)$  with  $T(x^2) + T(x)$ . Which of these maps is definitely not linear?

**Activity A.1.9** ( $\sim 12$  min)

Consider the following two polynomial maps.

$$S : \mathcal{P}^4 \rightarrow \mathcal{P}^3 \text{ given by } S(f(x)) = 2f'(x) - f''(x)$$

$$T : \mathcal{P}^2 \rightarrow \mathcal{P}^2 \text{ given by } T(f(x)) = f'(x) + x^2$$

*Part 1:* Compare  $S(x^2 + x)$  with  $S(x^2) + S(x)$ , and compare  $T(x^2 + x)$  with  $T(x^2) + T(x)$ . Which of these maps is definitely not linear?

*Part 2:* Verify that  $S(f + g) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$  is equal to  $S(f) + S(g)$  for all polynomials  $f, g$ .



**Activity A.1.9** ( $\sim 12$  min)

Consider the following two polynomial maps.

$$S : \mathcal{P}^4 \rightarrow \mathcal{P}^3 \text{ given by } S(f(x)) = 2f'(x) - f''(x)$$

$$T : \mathcal{P}^2 \rightarrow \mathcal{P}^2 \text{ given by } T(f(x)) = f'(x) + x^2$$

*Part 1:* Compare  $S(x^2 + x)$  with  $S(x^2) + S(x)$ , and compare  $T(x^2 + x)$  with  $T(x^2) + T(x)$ . Which of these maps is definitely not linear?

*Part 2:* Verify that  $S(f + g) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$  is equal to  $S(f) + S(g)$  for all polynomials  $f, g$ .

*Part 3:* Verify that  $S(cf) = cS(f)$  for all real numbers  $c$  and polynomials  $f$ . Is  $S$  linear?

## Module A Section 2

## Remark A.2.1

Recall that a linear map  $T : V \rightarrow W$  satisfies

- ①  $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$  for any  $\mathbf{v}, \mathbf{w} \in V$ .
- ②  $T(c\mathbf{v}) = cT(\mathbf{v})$  for any  $c \in \mathbb{R}, \mathbf{v} \in V$ .

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

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**Activity A.2.2** ( $\sim 5$  min)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and

$$T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \text{ Compute } T \left( \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right).$$

(a)  $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b)  $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(c)  $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(d)  $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

**Activity A.2.3** ( $\sim 3$  min)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and

$T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c)  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d)  $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

**Activity A.2.4** ( $\sim 2$  min)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and

$T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c)  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d)  $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

**Activity A.2.5** ( $\sim 5$  min)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and

$T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Do you have enough information to compute  $T(\mathbf{v})$  for *any*  $\mathbf{v} \in \mathbb{R}^3$ ?

- (a) Yes.
- (b) No, exactly one more piece of information is needed.
- (c) No, an infinite amount of information would be necessary to compute the transformation of infinitely-many vectors.

## Fact A.2.6

Consider any basis  $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  for  $V$ . Since every vector  $\mathbf{v}$  can be written *uniquely* as a linear combination of basis vectors,  $x_1\mathbf{b}_1 + \dots + x_n\mathbf{b}_n$ , we conclude that

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \dots + x_n\mathbf{b}_n) = x_1T(\mathbf{b}_1) + \dots + x_nT(\mathbf{b}_n).$$

Therefore any linear transformation  $T : V \rightarrow W$  can be defined by just describing the values of  $T(\mathbf{b}_i)$ .

Put another way, the basis vectors **determine** the transformation  $T$ .



## Definition A.2.7

Since linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is determined by the standard basis  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ , it's convenient to store this information in the  $m \times n$  **standard matrix**  $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ .

**Example A.2.8**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear map determined by the following values for  $T$  applied to the standard basis of  $\mathbb{R}^3$ .

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$T(\mathbf{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Then the standard matrix corresponding to  $T$  is

$$[T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)] = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix}.$$

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**Activity A.2.9** (*~5 min*)

TODO Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard basis.

**Activity A.2.10** ( $\sim 5$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \end{bmatrix}.$$

Compute  $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ .

**Activity A.2.11** (*~10 min*)

Let  $D : \mathcal{P}^3 \rightarrow \mathcal{P}^2$  be the derivative map  $D(f(x)) = f'(x)$ . (Earlier we showed this is a linear transformation.)

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**Activity A.2.11** (*~10 min*)

Let  $D : \mathcal{P}^3 \rightarrow \mathcal{P}^2$  be the derivative map  $D(f(x)) = f'(x)$ . (Earlier we showed this is a linear transformation.)

*Part 1:* Write down an equivalent linear transformation  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  by converting  $\{1, x, x^2, x^3\}$  and  $\{D(1), D(x), D(x^2), D(x^3)\}$  into appropriate vectors in  $\mathbb{R}^4$  and  $\mathbb{R}^3$ .

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**Activity A.2.11** ( $\sim 10$  min)

Let  $D : \mathcal{P}^3 \rightarrow \mathcal{P}^2$  be the derivative map  $D(f(x)) = f'(x)$ . (Earlier we showed this is a linear transformation.)

*Part 1:* Write down an equivalent linear transformation  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  by converting  $\{1, x, x^2, x^3\}$  and  $\{D(1), D(x), D(x^2), D(x^3)\}$  into appropriate vectors in  $\mathbb{R}^4$  and  $\mathbb{R}^3$ .

*Part 2:* Write the standard matrix corresponding to  $T$ .

## Module A Section 3



### Definition A.3.1

Let  $T : V \rightarrow W$  be a linear transformation.  $T$  is called **injective** or **one-to-one** if  $T$  does not map two distinct values to the same place. More precisely,  $T$  is injective if  $T(\mathbf{v}) \neq T(\mathbf{w})$  whenever  $\mathbf{v} \neq \mathbf{w}$ .

**Activity A.3.2** ( $\sim 5$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of  $T$  is thus  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

Is  $T$  injective?

**Activity A.3.3** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of  $T$  is thus  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ .

Is  $T$  injective?

### Definition A.3.4

Let  $T : V \rightarrow W$  be a linear transformation.  $T$  is called **surjective** or **onto** if every element of  $W$  is mapped to by an element of  $V$ . More precisely, for every  $\mathbf{w} \in W$ , there is some  $\mathbf{v} \in V$  with  $T(\mathbf{v}) = \mathbf{w}$ .

**Activity A.3.5** ( $\sim 5$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of  $T$  is thus  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

Is  $T$  surjective?

**Activity A.3.6** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of  $T$  is thus  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ .

Is  $T$  surjective?

## Definition A.3.7

Let  $T : V \rightarrow W$  be a linear transformation. The **kernel** of  $T$  is an important subspace of  $V$  defined by

$$\ker T = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

**Activity A.3.8** ( $\sim 5$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the standard matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Find the kernel of  $T$ .



**Activity A.3.9** ( $\sim 5$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ . Find the kernel of  $T$ .

**Activity A.3.10** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

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**Activity A.3.10** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Write a system of equations whose solution set is the kernel.

**Activity A.3.10** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Write a system of equations whose solution set is the kernel.

*Part 2:* Use  $\text{RREF}(A)$  to solve the system of equations and find the kernel of  $T$ .

**Activity A.3.10** ( $\sim 10$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Write a system of equations whose solution set is the kernel.

*Part 2:* Use  $\text{RREF}(A)$  to solve the system of equations and find the kernel of  $T$ .

*Part 3:* Find a basis for the kernel of  $T$ .

### Definition A.3.11

Let  $T : V \rightarrow W$  be a linear transformation. The **image** of  $T$  is an important subspace of  $W$  defined by

$$\text{Im } T = \{ \mathbf{w} \in W \mid \text{there is some } v \in V \text{ with } T(v) = \mathbf{w} \}$$

**Activity A.3.12** (*~5 min*)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the standard matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Find the image of  $T$ .

**Activity A.3.13** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the standard matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ . Find the image of  $T$ .



**Activity A.3.14** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

**Activity A.3.14** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Find a convenient set of vectors  $S \subseteq \mathbb{R}^2$  such that  $\text{span } S = \text{Im } T$ .

**Activity A.3.14** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Find a convenient set of vectors  $S \subseteq \mathbb{R}^2$  such that  $\text{span } S = \text{Im } T$ .

*Part 2:* Find a convenient basis for the image of  $T$ .

### Observation A.3.15

Let  $T : V \rightarrow W$  be a linear transformation with corresponding matrix  $A$ .

- If  $A$  is a matrix corresponding to  $T$ , the kernel is the solution set of the homogeneous system with coefficients given by  $A$ .
- If  $A$  is a matrix corresponding to  $T$ , the image is the span of the columns of  $A$ .

## Module A Section 4

## Observation A.4.1

Let  $T : V \rightarrow W$ . We have previously defined the following terms.

- $T$  is called **injective** or **one-to-one** if  $T$  does not map two distinct values to the same place.
- $T$  is called **surjective** or **onto** if every element of  $W$  is mapped to by some element of  $V$ .
- The **kernel** of  $T$  is the set of all things that are mapped to **0**. It is a subspace of  $V$ .
- The **image** of  $T$  is the set of all things in  $W$  that are mapped to by something in  $V$ . It is a subspace of  $W$ .

**Activity A.4.2** ( $\sim 5$  min)

Let  $T : V \rightarrow W$  be a linear transformation where  $\ker T = \{\mathbf{0}\}$ . Can you answer either of the following questions about  $T$ ?

(a) Is  $T$  injective?

(b) Is  $T$  surjective?

(Hint: If  $T(\mathbf{v}) = T(\mathbf{w})$ , then what is  $T(\mathbf{v} - \mathbf{w})$ ?)

### Fact A.4.3

A linear transformation  $T$  is injective **if and only if**  $\ker T = \{\mathbf{0}\}$ . Put another way, an injective linear transformation may be recognized by its **trivial** kernel.



**Activity A.4.4** ( $\sim 5$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear transformation where  $\text{Im } T = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$ .

Can you answer either of the following questions about  $T$ ?

- (a) Is  $T$  injective?
- (b) Is  $T$  surjective?

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**Fact A.4.5**

A linear transformation  $T : V \rightarrow W$  is surjective **if and only if**  $\text{Im } T = W$ . Put another way, a surjective linear transformation may be recognized by its same codomain and image.

**Activity A.4.6** ( $\sim 15$  min)

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with standard matrix  $A$ . Sort the following claims into two groups of equivalent statements.

- (a)  $T$  is injective
- (b)  $T$  is surjective
- (c) The kernel of  $T$  is trivial.
- (d) The columns of  $A$  span  $\mathbb{R}^m$
- (e) The columns of  $A$  are linearly independent
- (f) Every column of  $\text{RREF}(A)$  has a pivot.
- (g) Every row of  $\text{RREF}(A)$  has a pivot.
- (h) The image of  $T$  equals its codomain.
- (i) The system of linear equations given by the augmented matrix  $[A \mid \mathbf{b}]$  has a solution for all  $\mathbf{b} \in \mathbb{R}^m$
- (j) The system of linear equations given by the augmented matrix  $[A \mid \mathbf{0}]$  has exactly one solution.

## Definition A.4.7

If  $T : V \rightarrow W$  is both injective and surjective, it is called **bijjective**.

**Activity A.4.8** ( $\sim 5$  min)

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a bijective linear map with standard matrix  $A$ . Label each of the following as true or false.

- (a) The columns of  $A$  form a basis for  $\mathbb{R}^m$
- (b)  $\text{RREF}(A)$  is the identity matrix.
- (c) The system of linear equations given by the augmented matrix  $[A \mid \mathbf{b}]$  has exactly one solution for all  $\mathbf{b} \in \mathbb{R}^m$ .

## Module A

Section A.1

Section A.2

Section A.3

Section A.4

**Activity A.4.9** (*~10 min*)Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is bijective.

**Activity A.4.10** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is bijective.

**Activity A.4.11** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y + z \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is bijective.



**Activity A.4.12** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is bijective.