Application Activities - Module E Part 2 - Class Day 4

Definition 4.1 The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a conzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity 4.2 Consider the following two linear systems.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Part 1: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

1. Swap R_1 (first row) and R_2 (second row).
2. Multiply R_2 by $\frac{1}{2}$.
3. Add R_1 to R_3.
4. Add $-3R_1$ to R_2.
5. Add $-2R_2$ to R_3.
6. Multiply R_3 by $\frac{1}{3}$.

Part 2: What is the common solution to these linear systems?

Definition 4.3 The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix. Examples:

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Activity 4.4 Find your own sequence of row operations to manipulate the matrix

$$\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array}\right]$$

into row echelon form. (Note that row echelon form is not unique.)
The most efficient way to do this is by circling **pivot positions** in your matrix:

1. Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.

2. Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
3. Repeat these two steps as often as possible.

Activity 4.5 Solve this simplifed linear system:

$$\begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Observation 4.6 The consise standard form of the solution to this linear system corresponds to a simplified row echelon form matrix:

$$\begin{aligned} x_1 \qquad\qquad &= -2 \\ x_2 \qquad &= 7 \\ x_3 &= 2 \end{aligned} \qquad\qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Definition 4.7 A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0. Examples:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Activity 4.8 Show that the following two linear systems:

$$\begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned} \qquad\qquad \begin{aligned} x_1 \qquad\qquad &= -2 \\ x_2 \qquad &= 7 \\ x_3 &= 2 \end{aligned}$$

are equivalent by converting the first system to an augmented matrix, and then zeroing out all terms above pivot positions (the leading terms).

Remark 4.9 We may verify that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$ is a solution to the original linear system

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

by plugging the solution into each equation.

Fact 4.10 Every augmented matrix A reduces to a unique reduced row echelon form matrix. This matrix is denoted as $\mathrm{RREF}(A)$.

Activity 4.11 Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array}\right]$$

Part 1: Find $\mathrm{RREF}(A)$.
Part 2: How many solutions does the corresponding linear system have?