

# Module M: Understanding Matrices Algebraically

# What algebraic structure do matrices have?

## Module M

Section M.1

Section M.2

Section M.3

At the end of this module, students will be able to...

**M1. Matrix Multiplication.** ... multiply matrices.

**M2. Invertible Matrices.** ... determine if a square matrix is invertible or not.

**M3. Matrix inverses.** ... compute the inverse matrix of an invertible matrix.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers.
- Identify the domain and codomain of linear transformations.
- Find the matrix corresponding to a linear transformation and compute the image of a vector given a standard matrix **A2**
- Determine if a linear transformation is injective and/or surjective **A3**
- Interpret the ideas of injectivity and surjectivity in multiple ways.

## Module M

Section M.1

Section M.2

Section M.3

The following resources will help you prepare for this module.

- Function composition (Khan Academy): <http://bit.ly/2wkz7f3>
- Domain and codomain: <https://www.youtube.com/watch?v=BQMyeQOLvpg>
- Interpreting injectivity and surjectivity in many ways:  
<https://www.youtube.com/watch?v=WpUv72Y6Dl0>

# Module M Section 1

**Activity M.1.1** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the  $2 \times 3$  standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the  $4 \times 2$  standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

What is the domain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
- (b)  $\mathbb{R}^2$
- (c)  $\mathbb{R}^3$
- (d)  $\mathbb{R}^4$

**Activity M.1.2** ( $\sim 3$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the  $2 \times 3$  standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the  $4 \times 2$  standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

What is the codomain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
- (b)  $\mathbb{R}^2$
- (c)  $\mathbb{R}^3$
- (d)  $\mathbb{R}^4$



**Activity M.1.3** ( $\sim 2$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the  $2 \times 3$  standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the  $4 \times 2$  standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

What size will the standard matrix of  $S \circ T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be? (Rows  $\times$  Columns)

(a)  $4 \times 3$

(c)  $3 \times 4$

(e)  $2 \times 4$

(b)  $4 \times 2$

(d)  $3 \times 2$

(f)  $2 \times 3$

**Activity M.1.4** (*~15 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the  $2 \times 3$  standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the  $4 \times 2$  standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

**Activity M.1.4** ( $\sim 15$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the  $2 \times 3$  standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the  $4 \times 2$  standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

*Part 1:* Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

**Activity M.1.4** ( $\sim 15$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the  $2 \times 3$  standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

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*Part 1:* Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

*Part 2:* Compute  $(S \circ T)(\vec{e}_2)$ .

**Activity M.1.4** ( $\sim 15$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the  $2 \times 3$  standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the  $4 \times 2$  standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

*Part 1:* Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

*Part 2:* Compute  $(S \circ T)(\vec{\mathbf{e}}_2)$ .

*Part 3:* Compute  $(S \circ T)(\vec{\mathbf{e}}_3)$ .

**Activity M.1.4** ( $\sim 15$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the  $2 \times 3$  standard matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the  $4 \times 2$  standard matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

*Part 1:* Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

*Part 2:* Compute  $(S \circ T)(\vec{\mathbf{e}}_2)$ .

*Part 3:* Compute  $(S \circ T)(\vec{\mathbf{e}}_3)$ .

*Part 4:* Write the  $4 \times 3$  standard matrix of  $S \circ T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ .

**Definition M.1.5**

We define the **product**  $AB$  of a  $m \times n$  matrix  $A$  and a  $n \times k$  matrix  $B$  to be the  $m \times k$  standard matrix of the composition map of the two corresponding linear functions.

For the previous activity,  $S$  had a  $4 \times 2$  matrix and  $T$  had a  $2 \times 3$  matrix, so  $S \circ T$  had a  $4 \times 3$  standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= [(S \circ T)(\vec{e}_1) \quad (S \circ T)(\vec{e}_2) \quad (S \circ T)(\vec{e}_3)] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

**Activity M.1.6** (*~15 min*)

Let  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$  and  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be

given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ .



**Activity M.1.6** ( $\sim 15$  min)

Let  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$  and  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be

given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ .

*Part 1:* Write the dimensions (rows  $\times$  columns) for  $A$ ,  $B$ ,  $AB$ , and  $BA$ .

**Activity M.1.6** (*~15 min*)

Let  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$  and  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be

given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ .

*Part 1:* Write the dimensions (rows  $\times$  columns) for  $A$ ,  $B$ ,  $AB$ , and  $BA$ .

*Part 2:* Find the standard matrix  $AB$  of  $S \circ T$ .

**Activity M.1.6** ( $\sim 15$  min)

Let  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$  and  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be

given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ .

*Part 1:* Write the dimensions (rows  $\times$  columns) for  $A$ ,  $B$ ,  $AB$ , and  $BA$ .

*Part 2:* Find the standard matrix  $AB$  of  $S \circ T$ .

*Part 3:* Find the standard matrix  $BA$  of  $T \circ S$ .

**Activity M.1.7** ( $\sim 10$  min)

Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

**Activity M.1.7** ( $\sim 10$  min)

Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

*Part 1:* Label each of these matrices with its number of rows  $\times$  columns.

**Activity M.1.7** ( $\sim 10$  min)

Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

*Part 1:* Label each of these matrices with its number of rows  $\times$  columns.

*Part 2:* Only one of the matrix products  $AB, AC, BA, BC, CA, CB$  can actually be computed. Compute it.

## Module M Section 2

**Remark M.2.1**

Recall that the **product**  $AB$  of a  $m \times n$  matrix  $A$  and an  $n \times k$  matrix  $B$  is the  $m \times k$  standard matrix of the composition map of the two corresponding linear functions.

For example, if  $S$  has a  $4 \times \textcircled{2}$  matrix  $A$  and  $T$  has a  $\textcircled{2} \times 3$  matrix  $B$ , then  $S \circ T$  has a  $4 \times 3$  standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= [(S \circ T)(\vec{e}_1) \quad (S \circ T)(\vec{e}_2) \quad (S \circ T)(\vec{e}_3)] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$



**Activity M.2.2** (*~15 min*)

Let  $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$ , and let  $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ .

**Activity M.2.2** ( $\sim 15$  min)

Let  $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$ , and let  $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ .

*Part 1:* Compute the product  $BA$  by hand.

**Activity M.2.2** ( $\sim 15$  min)

Let  $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$ , and let  $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ .

*Part 1:* Compute the product  $BA$  by hand.

*Part 2:* Check your work using technology. Using Octave:

- $B = \text{sym}([3 \ -4 \ 0 \ ; \ 2 \ 0 \ -1 \ ; \ 0 \ -3 \ 3])$
- $A = \text{sym}([2 \ 7 \ -1 \ ; \ 0 \ 3 \ 2 \ ; \ 1 \ 1 \ -1])$
- $B*A$

**Activity M.2.3** ( $\sim 5$  min)

Let  $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ . Find a  $3 \times 3$  matrix  $B$  such that  $BA = A$ , that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Check your guess using technology.

**Definition M.2.4**

The identity matrix  $I_n$  (or just  $I$  when  $n$  is obvious from context) is the  $n \times n$  matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

**Fact M.2.5**

For any square matrix  $A$ ,  $IA = AI = A$ :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

**Activity M.2.6** ( $\sim 20$  min)

Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

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Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

*Part 1:* Create a matrix that doubles the third row of  $A$ :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$



**Activity M.2.6** ( $\sim 20$  min)

Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

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*Part 2:* Create a matrix that swaps the second and third rows of  $A$ :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

**Activity M.2.6** ( $\sim 20$  min)

Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

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$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

*Part 3:* Create a matrix that adds 5 times the third row of  $A$  to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 + 5(1) & 7 + 5(1) & -1 + 5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

**Fact M.2.7**

If  $R$  is the result of applying a row operation to  $I$ , then  $RA$  is the result of applying the same row operation to  $A$ .

- Scaling a row:  $R = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Swapping rows:  $R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Adding a row multiple to another row:  $R = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Such matrices can be chained together to emulate multiple row operations. In particular,

$$\text{RREF}(A) = R_k \dots R_2 R_1 A$$

for some sequence of matrices  $R_1, R_2, \dots, R_k$ .

**Activity M.2.8** ( $\sim 10$  min)

Consider the two row operations  $R_2 \leftrightarrow R_3$  and  $R_1 + R_3 \rightarrow R_1$  applied as follows to show  $A \sim B$ :

$$\begin{aligned}
 A = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} &\sim \begin{bmatrix} -1 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} \\
 &\sim \begin{bmatrix} -1+1 & 4+2 & 5+3 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = B
 \end{aligned}$$

Express these row operations as matrix multiplication by expressing  $B$  as the product of two matrices and  $A$ :

$$B = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} A$$

Check your work using technology.

## Module M Section 3

**Activity M.3.1** (*~15 min*)

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with standard matrix  $A$ . Sort the following items into three groups of statements: a group that means  $T$  is **injective**, a group that means  $T$  is **surjective**, and a group that means  $T$  is **bijjective**.

- (a)  $A\vec{x} = \vec{b}$  has a solution for all  $\vec{b} \in \mathbb{R}^m$
- (b)  $A\vec{x} = \vec{b}$  has a unique solution for all  $\vec{b} \in \mathbb{R}^m$
- (c)  $A\vec{x} = \vec{0}$  has a unique solution.
- (d) The columns of  $A$  span  $\mathbb{R}^m$
- (e) The columns of  $A$  are linearly independent
- (f) The columns of  $A$  are a basis of  $\mathbb{R}^m$
- (g) Every column of  $\text{RREF}(A)$  has a pivot
- (h) Every row of  $\text{RREF}(A)$  has a pivot
- (i)  $m = n$  and  $\text{RREF}(A) = I$

## Definition M.3.2

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear map with standard matrix  $A$ .

- If  $T$  is a bijection and  $\vec{\mathbf{b}}$  is any  $\mathbb{R}^n$  vector, then  $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}} = \vec{\mathbf{b}}$  has a unique solution.
- So we may define an **inverse map**  $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by setting  $T^{-1}(\vec{\mathbf{b}})$  to be this unique solution.
- Let  $A^{-1}$  be the standard matrix for  $T^{-1}$ . We call  $A^{-1}$  the **inverse matrix** of  $A$ , so we also say that  $A$  is **invertible**.

**Activity M.3.3** ( $\sim 20$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$



**Activity M.3.3** ( $\sim 20$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

*Part 1:* Write an augmented matrix representing the system of equations given by

$T(\vec{x}) = \vec{e}_1$ , that is,  $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ . Then solve  $T(\vec{x}) = \vec{e}_1$  to find  $T^{-1}(\vec{e}_1)$ .

**Activity M.3.3** ( $\sim 20$  min)

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*Part 2:* Solve  $T(\vec{x}) = \vec{e}_2$  to find  $T^{-1}(\vec{e}_2)$ .

**Activity M.3.3** ( $\sim 20$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

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*Part 2:* Solve  $T(\vec{x}) = \vec{e}_2$  to find  $T^{-1}(\vec{e}_2)$ .

*Part 3:* Solve  $T(\vec{x}) = \vec{e}_3$  to find  $T^{-1}(\vec{e}_3)$ .

**Activity M.3.3** ( $\sim 20$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

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*Part 1:* Write an augmented matrix representing the system of equations given by

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*Part 2:* Solve  $T(\vec{x}) = \vec{e}_2$  to find  $T^{-1}(\vec{e}_2)$ .

*Part 3:* Solve  $T(\vec{x}) = \vec{e}_3$  to find  $T^{-1}(\vec{e}_3)$ .

*Part 4:* Write  $A^{-1}$ , the standard matrix for  $T^{-1}$ .

**Observation M.3.4**

We could have solved these three systems simultaneously by row reducing the matrix  $[A \mid I]$  at once.

$$\left[ \begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array} \right]$$

**Activity M.3.5** ( $\sim 5$  min)

Find the inverse  $A^{-1}$  of the matrix  $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$  by row-reducing  $[A \mid I]$ .

**Activity M.3.6** ( $\sim 5$  min)

Is the matrix  $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$  invertible? Give a reason for your answer.

**Observation M.3.7**

An  $n \times n$  matrix  $A$  is invertible if and only if  $\text{RREF}(A) = I_n$ .



**Activity M.3.8** ( $\sim 10$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the bijective linear map defined by  $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$ ,  
with the inverse map  $T^{-1} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$ .

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*Part 1:* Compute  $(T^{-1} \circ T) \left( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$ .

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*Part 1:* Compute  $(T^{-1} \circ T) \left( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$ .

*Part 2:* If  $A$  is the standard matrix for  $T$  and  $A^{-1}$  is the standard matrix for  $T^{-1}$ , find the  $2 \times 2$  matrix

$$A^{-1}A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

**Observation M.3.9**

$T^{-1} \circ T = T \circ T^{-1}$  is the identity map for any bijective linear transformation  $T$ .  
Therefore  $A^{-1}A = AA^{-1} = I$  is the identity matrix for any invertible matrix  $A$ .