

### Section S.3

**Observation S.3.1** In the previous section, we learned that computing a basis for the subspace  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ , is as simple as removing the vectors corresponding to the non-pivot columns of  $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_m]$ .

For example, since

$$\text{RREF} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} \textcircled{1} & 0 & 1 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the subspace  $W = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \right\}$  has  $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$  as a basis.

**Activity S.3.2** ( $\sim 10$  min) Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

*Part 1:* Find a basis for  $\text{span } S$ .

*Part 2:* Find a basis for  $\text{span } T$ .

**Observation S.3.3** Even though we found different bases for them,  $\text{span } S$  and  $\text{span } T$  are exactly the same subspace of  $\mathbb{R}^4$ , since

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} = T$$

**Fact S.3.4** Any non-trivial vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size.

For example,

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

are all valid bases for  $\mathbb{R}^3$ , and they all contain three vectors.

**Definition S.3.5** The **dimension** of a vector space is equal to the size of any basis for the vector space.

As you'd expect,  $\mathbb{R}^n$  has dimension  $n$ . For example,  $\mathbb{R}^3$  has dimension 3 because any basis for  $\mathbb{R}^3$  such as

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

contains exactly three vectors.

**Activity S.3.6** (*~10 min*) Find the dimension of each subspace of  $\mathbb{R}^4$  by finding RREF for each corresponding matrix.

$$\begin{aligned} \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} & \quad \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} \\ \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\} & \quad \text{span} \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\} \end{aligned}$$

**Fact S.3.7** Every vector space with finite dimension, that is, every vector space  $V$  with a basis of the form  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is said to be **isomorphic** to a Euclidean space  $\mathbb{R}^n$ , since there exists a natural correspondence between vectors in  $V$  and vectors in  $\mathbb{R}^n$ :

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

**Observation S.3.8** We've already been taking advantage of the previous fact by converting polynomials and matrices into Euclidean vectors. Since  $\mathcal{P}^3$  and  $M_{2,2}$  are both four-dimensional:

$$4x^3 + 0x^2 - 1x + 5 \leftrightarrow \begin{bmatrix} 4 \\ 0 \\ -1 \\ 5 \end{bmatrix} \leftrightarrow \begin{bmatrix} 4 & 0 \\ -1 & 5 \end{bmatrix}$$

**Observation S.3.9** The space of polynomials  $\mathcal{P}$  (of *any* degree) has the basis  $\{1, x, x^2, x^3, \dots\}$ , so it is a natural example of an infinite-dimensional vector space.

Since  $\mathcal{P}$  and other infinite-dimensional spaces cannot be treated as an isomorphic finite-dimensional Euclidean space  $\mathbb{R}^n$ , vectors in such spaces cannot be studied by converting them into Euclidean vectors. Fortunately, most of the examples we will be interested in for this course will be finite-dimensional.

**Definition S.3.10** A **homogeneous** system of linear equations is one of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

This system is equivalent to the vector equation:

$$x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$$

and the augmented matrix:

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array} \right]$$

**Activity S.3.11** ( $\sim 5$  min) Note that if  $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$  and  $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$  are solutions to  $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$  so is

$$\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}, \text{ since}$$

$$a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n = \mathbf{0} \text{ and } b_1\mathbf{v}_1 + \dots + b_n\mathbf{v}_n = \mathbf{0}$$

implies

$$(a_1 + b_1)\mathbf{v}_1 + \dots + (a_n + b_n)\mathbf{v}_n = \mathbf{0}.$$

Similarly, if  $c \in \mathbb{R}$ ,  $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$  is a solution. Thus the solution set of a homogeneous system is...

- a) A basis for  $\mathbb{R}^n$ .                      b) A subspace of  $\mathbb{R}^n$ .                      c) The empty set.

**Activity S.3.12** ( $\sim 10$  min) Consider the homogeneous system of equations

$$\begin{aligned}x_1 + 2x_2 + x_4 &= 0 \\2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\3x_1 + 6x_2 - x_3 - x_4 &= 0\end{aligned}$$

*Part 1:* Find its solution set (a subspace of  $\mathbb{R}^4$ ).

*Part 2:* Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

**Fact S.3.13** The coefficients of the free variables in the solution set of a linear system always yield linearly independent vectors.

Thus if

$$\left\{ a \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

is the solution space for a homogeneous system, then

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

is a basis for the solution space.

**Activity S.3.14** ( $\sim 10$  min) Consider the homogeneous system of equations

$$\begin{aligned}x_1 - 3x_2 + 2x_3 &= 0 \\2x_1 - 6x_2 + 4x_3 + 3x_4 &= 0 \\-2x_1 + 6x_2 - 4x_3 - 4x_4 &= 0\end{aligned}$$

Find a basis for its solution space.

**Activity S.3.15** ( $\sim 5$  min) Suppose  $W$  is a subspace of  $\mathcal{P}^8$ , and you know that it contains a **linearly independent** set of 3 vectors. What can you conclude about  $W$ ?

- (a) The dimension of  $W$  is at most 3.
- (b) The dimension of  $W$  is exactly 3.
- (c) The dimension of  $W$  is at least 3.

**Activity S.3.16** ( $\sim 5$  min) Suppose  $W$  is a subspace of  $\mathcal{P}^8$ , and you know that it contains a **spanning set** of 3 vectors. What can you conclude about  $W$ ?

- (a) The dimension of  $W$  is at most 3.
- (b) The dimension of  $W$  is exactly 3.
- (c) The dimension of  $W$  is at least 3.