

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing (**Standard(s) E1, E2, E3, E4**).
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent (**Standard(s) V5**).
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace (**Standard(s) V6, V7**).
- State the definition of a basis, and determine if a set of vectors is a basis (**Standard(s) V8, V9**).

Readiness Assurance Resources

The following resources will help you prepare for this module.

- TODO

Readiness Assurance Test

- 1) Which of the following is a solution to the system of linear equations

$$\begin{aligned}x + 3y - z &= 2 \\ 2x + 8y + 3z &= -1 \\ -x - y + 9z &= -10\end{aligned}$$

- (a) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

- 2) Find a basis for the solution set of the following homogeneous system of linear equations

$$\begin{aligned}x + 2y + -z - w &= 0 \\ -2x - 4y + 3z + 5w &= 0\end{aligned}$$

- (a) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$ (d) None of these are a basis.

- 3) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

- (a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of \mathbb{R}^3 .

- 4) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

- (a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of \mathbb{R}^3 .

- 5) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

- (a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent

(d) It is a basis of \mathbb{R}^3 .

6) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent

(b) It does not span and is linearly independent

(c) It spans but it is linearly dependent

(d) It is a basis of \mathbb{R}^3 .

7) Find a basis for the subspace of \mathbb{R}^4 spanned by the vectors ...

8) Suppose you know that every vector in \mathbb{R}^5 can be written as a linear combination of the vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$. What can you conclude about n ?

(a) $n \leq 5$

(b) $n = 5$

(c) $n \geq 5$

(d) n could be any positive integer

9) Suppose you know that every vector in \mathbb{R}^5 can be written uniquely as a linear combination of the vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$. What can you conclude about n ?

(a) $n \leq 5$

(b) $n = 5$

(c) $n \geq 5$

(d) n could be any positive integer

10) Suppose you know that every vector in \mathbb{R}^5 can be written uniquely as a linear combination of the vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$. What can you conclude about the set $\{\vec{v}_1, \dots, \vec{v}_n\}$?

(a) It does not span and is linearly dependent

(b) It does not span and is linearly independent

(c) It spans but it is linearly dependent

(d) It is a basis of \mathbb{R}^3 .

Day 1

Application Activity 1. A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \rightarrow W$ is called a linear transformation if

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}$, $\vec{v} \in V$.

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.

V is called the **domain** of T and W is called the **co-domain** of T .

1) Determine if each of the following maps are linear transformations

- (a) $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $T_1 \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \sqrt{a^2 + b^2}$
- (b) $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by $T_2 \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$
- (c) $T_3 : \mathcal{P}_d \rightarrow \mathcal{P}_{d-1}$ given by $T_3(f(x)) = f'(x)$.
- (d) $T_4 : C(\mathbb{R}) \rightarrow C(\mathbb{R})$ given by $T_4(f(x)) = f(-x)$
- (e) $T_5 : \mathcal{P} \rightarrow \mathcal{P}$ given by $T_5(f(x)) = f(x) + x^2$

2) Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear transformation, and you know $T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

Compute each of the following:

- (a) $T \left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right)$
- (b) $T \left(\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right)$
- (c) $T \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$
- (d) $T \left(\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} \right)$

3) Suppose $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is a linear transformation. What is the smallest number of vectors needed to determine T ? In other words, what is the smallest number n such that there are $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^4$ and given $T(\vec{v}_1), \dots, T(\vec{v}_n)$ you can determine $T(\vec{w})$ for **any** $\vec{w} \in \mathbb{R}^4$?

Fix an ordered basis for V . Since every vector can be written *uniquely* as a linear combination of basis vectors, a linear transformation $T : V \rightarrow W$ corresponds exactly to a choice of where each basis vector goes. For convenience, we can thus encode a linear transformation as a matrix, with one column for the image of each basis vector (in order).

1) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation with

$$T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad T \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard ordered basis.

- 2) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the ordered basis $\left\{\begin{bmatrix} 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}\right\}$.

- 3) Let $D : \mathcal{P}_3 \rightarrow \mathcal{P}_2$ be the derivative map (recall this is a linear transformation). Write the matrix corresponding to D with respect to the ordered basis $\{1, x, x^2, x^3\}$.

Day 2

Application Activity 2. Let $T : V \rightarrow W$ be a linear transformation.

- T is called *injective* or *one-to-one* if T does not map two distinct values to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.
- T is called *surjective* or *onto* if every element of W is mapped to by an element of V . More precisely, for every $\vec{w} \in W$, there is some $v \in V$ with $T(\vec{v}) = \vec{w}$.

- 1) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

- 2) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

We also have two important sets called the *kernel* of T and the *image* of T .

$$\ker T = \{\vec{v} \in V \mid T(\vec{v}) = 0\}$$

$$\text{Im } T = \{\vec{w} \in W \mid \text{there is some } v \in V \text{ with } T(\vec{v}) = \vec{w}\}$$

- 1) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T .

- 2) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T .

- Describe surjective linear transformations in terms of the image.
- Describe injective linear transformations in terms of the kernel.

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

- Write a system of equations whose solution set is the kernel.
- Compute $\text{RREF}(A)$ and solve the system of equations.
- Compute the kernel of T
- Find a basis for the kernel of T

Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

- 1) Write a system of equations whose solution set is the kernel.
- 2) Compute $\text{RREF}(A)$ and solve the system of equations.
- 3) Compute the kernel of T
- 4) Find a basis for the kernel of T

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

- 1) Find a set of vectors that span the image of T
- 2) Find a basis for the image of T .

Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

- 1) Find a set of vectors that span the image of T
- 2) Find a basis for the image of T .

Day 3

Application Activity 3. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). Consider the following statements about T

- (a) T is injective
- (b) T is not injective
- (c) T is surjective
- (d) T is not surjective
- (e) The system of linear equations given by the augmented matrix $[A \mid \vec{b}]$ has a solution for all $\vec{b} \in \mathbb{R}^m$
- (f) The system of linear equations given by the augmented matrix $[A \mid \vec{b}]$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$
- (g) The system of linear equations given by the augmented matrix $[A \mid \vec{0}]$ has a non-trivial solution.
- (h) The columns of A span \mathbb{R}^m
- (i) The columns of A are linearly independent
- (j) The columns of A are a basis of \mathbb{R}^m
- (k) Every column of $\text{RREF}(A)$ is a pivot column
- (l) $\text{RREF}(A)$ has a non-pivot column
- (m) $\text{RREF}(A)$ has n pivot columns

- 1) Sort these statements into groups of equivalent statements.
- 2) Gallery walk-switch boards with a different team. If they have two things grouped together that you know are not equivalent, write a reason or counter-example on a sticky note.
- 3) Update your team's groupings based on feedback.
- 4) Repeat?
- 5) Can you add any statements to any groups?