Linear Algebra

University of South Alabama

Fall 2017

Module E: Solving Systems of Linear Equations

At the end of this module, students will be able to...

- **E1: Systems as matrices.** Translate back and forth between a system of linear equations and the corresponding augmented matrix.
- **E2: Row reduction.** Put a matrix in reduced row echelon form
- **E3: Solving Linear Systems.** Solve a system of linear equations.
- **E4: Homogeneous Systems.** Find a basis for the solution set of a homogeneous linear system.

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.

The following resources will help you prepare for this module.

- `https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-systems-topic/cc-8th-systems-graphically/a/systems-of-equations-with-graphing`
- `https://www.khanacademy.org/math/algebra/systems-of-linear-equations/solving-systems-of-equations-with-substitution/v/practice-using-substitution-for-systems`

Application Activities - Module E Part 1 - Class Day 3

Definition 3.1

A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

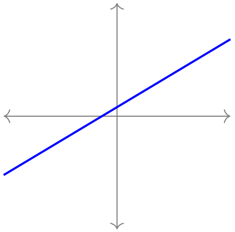
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

Observation 3.2

The linear equation $3x - 5y = -2$ may be graphed as a line in the xy plane.



The linear equation $x + 2y - z = 4$ may be graphed as a plane in xyz space.

Remark 3.3

In previous classes you likely assumed $x = x_1$, $y = x_2$, and $z = x_3$. However, since this course often deals with equations of four or more variables, we will almost always write our variables as x_i.

Definition 3.4

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

A **solution**

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i$$

for $1 \leq i \leq m$ (that is, the solution satisfies all equations in the system).

Remark 3.5

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Concise standard form:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & 3 \\ 3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & -2 \end{array}$$

Definition 3.6

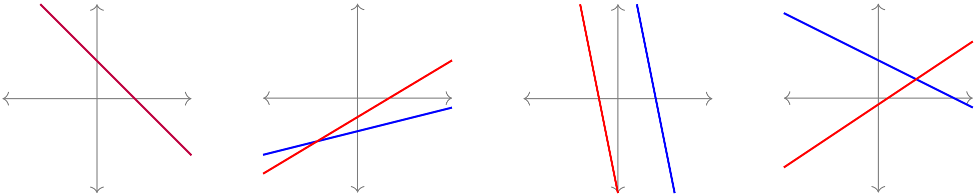
A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

Fact 3.7

All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

Activity 3.8

Consider the following graphs representing linear systems of two variables. Label each graph with **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.



Activity 3.9

All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity 3.10

Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity 3.10

Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

Activity 3.10

Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to describe *all* solutions (the **solution set**) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$ for the linear system in terms of a.

Activity 3.11

Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 \quad - \; x_4 &= \;\; 3 \\ x_3 + 4x_4 &= -2 \end{aligned}$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix} + a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation 3.12

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.

Remark 3.13

The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition 3.14

A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Definition 3.15

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.
For example, both of these systems have a single solution: $(x_1, x_2) = (1, 1)$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \qquad\qquad \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity 3.16

Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a nonzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

Application Activities - Module E Part 2 - Class Day 4

Definition 4.1

The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a nonzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity 4.2

Consider the following two linear systems.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned} \qquad\qquad \begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Activity 4.2

Consider the following two linear systems.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned} \qquad\qquad \begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Part 1: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

1. Swap R_1 (first row) and R_2 (second row).
2. Multiply R_2 by $\frac{1}{2}$.
3. Add R_1 to R_3.
4. Add $-3R_1$ to R_2.
5. Add $-2R_2$ to R_3.
6. Multiply R_3 by $\frac{1}{3}$.

Activity 4.2

Consider the following two linear systems.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Part 1: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

1. Swap R_1 (first row) and R_2 (second row).
2. Multiply R_2 by $\frac{1}{2}$.
3. Add R_1 to R_3.
4. Add $-3R_1$ to R_2.
5. Add $-2R_2$ to R_3.
6. Multiply R_3 by $\frac{1}{3}$.

Part 2: Which linear system would you rather solve?

Definition 4.3

The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix. Examples:

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Activity 4.4

Find your own sequence of row operations to manipulate the matrix

$$\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array}\right]$$

into row echelon form. (Note that row echelon form is not unique.)

The most efficient way to do this is by circling **pivot positions** in your matrix:

1. Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
2. Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
3. Repeat these two steps as often as possible.

Activity 4.5

Solve this simplifed linear system:

$$\begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Observation 4.6

The consise standard form of the solution to this linear system corresponds to a simplified row echelon form matrix:

$$\begin{alignedat}{4} x_1 & & & & &= -2 \\ & & x_2 & & &= 7 \\ & & & & x_3 &= 2 \end{alignedat} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Definition 4.7

A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0. Examples:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Activity 4.8

Show that the following two linear systems:

$$\begin{alignedat}{4} x_1 &- x_2 &+ 5x_3 &= 1 \\ & x_2 &- 2x_3 &= 3 \\ & & x_3 &= 2 \end{alignedat} \qquad\qquad \begin{alignedat}{4} x_1 & & &= -2 \\ & x_2 & &= 7 \\ & & x_3 &= 2 \end{alignedat}$$

are equivalent by converting the first system to an augmented matrix, and then zeroing out all terms above pivot positions (the leading terms).

Remark 4.9

We may verify that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$ is a solution to the original linear system

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

by plugging the solution into each equation.

Fact 4.10

Every augmented matrix A reduces to a unique reduced row echelon form matrix. This matrix is denoted as $\text{RREF}(A)$.

Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array}\right]$$

Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array}\right]$$

Part 1: Find RREF(A).

Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array}\right]$$

Part 1: Find RREF(A).

Part 2: How many solutions does the corresponding linear system have?

Application Activities - Module E Part 3 - Class Day 5

Definition 5.1

An algorithm that reduces A to RREF(A) is called **Gauss-Jordan elimination**. For example:

1. Circle the cell that (a) is in the top-most row without a pivot position and (b) is in the left-most column with a nonzero term either in that position or below it. This position (not the number inside) is called a **pivot**.
2. Change the pivot's value to 1 by using row operations involving only the pivot row and rows below it.
3. Add or subtract multiples of the pivot row to zero out above and below the pivot.
4. Return to Step 1 and repeat as needed until the matrix is in row reduced echelon form.

Observation 5.2

Here is an example of applying Gauss-Jordan elimination to a matrix:

$$\left[\begin{array}{cccc|c} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ -1 & 1 & 3 & -1 & -3 \\ 2 & -2 & -6 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & 2 & -4 & -1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 2 & -4 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 1 & 4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 1 & 4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Definition 5.3

The columns of RREF(A) without a leading term represent **free variables** of the linear system modeled by A that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by A.

Example 5.4

Here, x_3 is the free variable set equal to a since its column lacks a pivot, and the other bounded variables are put in terms of a.

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 &= 1 \end{aligned} \qquad \begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 - 2x_3 &= 1 \\ x_4 &= 3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x_1 &= 1 + 5a \\ x_2 &= 1 + 2a \\ x_3 &= a \\ x_4 &= 3 \end{aligned}$$

$$\Downarrow \qquad\qquad \Uparrow$$

$$\left[\begin{array}{cccc|c} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

So the solution set is $\left\{ \begin{bmatrix} 1 + 5a \\ 1 + 2a \\ a \\ 3 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

Activity 5.5

Solve the system of linear equations, circling the pivot positions in your augmented matrices as you work.

$$\begin{aligned}
-x_1 + x_2 - 3x_3 + 2x_4 &= 0 \\
2x_1 - x_2 + 5x_3 + 3x_4 &= -11 \\
3x_1 + 2x_2 + 4x_3 + x_4 &= 1 \\
x_2 - x_3 + x_4 &= 1
\end{aligned}$$

Remember to find the solution set of the system by setting the free variable (the column without a pivot position) equal to a, and then express each of the other bounded variables equal to an expression in terms of a.

Remark 5.6

From now on, unless specified, there's no need to show your work in finding RREF(A), so you may use a calculator to speed up your work.

Activity 5.7
Solve the linear system

$$\begin{aligned} 2x_1 - 3x_2 &= 17 \\ x_1 + 2x_2 &= -2 \\ -x_1 - x_2 &= 1 \end{aligned}$$

Activity 5.8

Show that all linear systems of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

are consistent by finding a quickly verifiable solution.

Definition 5.9

A **homogeneous system** is a linear system satisfying $b_i = 0$, that is, it is a linear system of the form

$$\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n &= 0 \\
a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n &= 0 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\ \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}$$

Fact 5.10

Because the zero vector is always a solution, the solution set to any homogeneous system with infinitely-many solutions may be generated by multiplying the parameters representing the free variables by a minimal set of Euclidean vectors, and adding these up. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Definition 5.11

A minimal set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Activity 5.12

Find a basis for the solution set of the following homogeneous linear system.

$$\begin{aligned} x_1 + 2x_2 \phantom{{}+x_3} - x_4 &= 0 \\ x_3 + 4x_4 &= 0 \\ 2x_1 + 4x_2 + x_3 + 2x_4 &= 0 \end{aligned}$$

Module V: Vector Spaces

At the end of this module, students will be able to...

- **V1: Vector Spaces.** Determine if a set with given operations forms a vector space.
- **V2: Linear Combinations.** Determine if a vector can be written as a linear combination of a given set of vectors.
- **V3: Spanning Sets.** Determine if a set of vectors spans a vector space.
- **V4: Subspaces.** Determine if a subset of a vector space is a subset or not.

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.

The following resources will help you prepare for this module.

- `https://www.khanacademy.org/math/precalculus/vectors-precalc/vector-addition-subtraction/v/adding-and-subtracting-vectors`
- `https://www.khanacademy.org/math/precalculus/vectors-precalc/combined-vector-operations/v/combined-vector-operations-example`
- `https://www.khanacademy.org/math/precalculus/imaginary-and-complex-numbers/adding-and-subtracting-complex-numbers/v/adding-complex-numbers`
- `https://www.khanacademy.org/math/algebra/introduction-to-polynomial-expressions/adding-and-subtracting-polynomials/v/adding-and-subtracting-polynomials-1`

Application Activities - Module V Part 1 - Class Day 7

Activity 7.1

Consider each of the following vector properties. Label each property with $\mathbb{R}^1$, $\mathbb{R}^2$, and/or $\mathbb{R}^3$ if that property holds for Euclidean vectors/scalars $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of that dimension.

1. **Addition associativity.**
   $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
2. **Addition commutivity.**
   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. **Addition identity.**
   There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
4. **Addition inverse.**
   There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
5. **Addition midpoint uniqueness.**
   There exists a unique $\mathbf{m}$ where the distance from $\mathbf{u}$ to $\mathbf{m}$ equals the distance from $\mathbf{m}$ to $\mathbf{v}$.
6. **Scalar multiplication associativity.**
   $a(b\mathbf{v}) = (ab)\mathbf{v}$.
7. **Scalar multiplication identity.**
   $1\mathbf{v} = \mathbf{v}$.
8. **Scalar multiplication relativity.**
   There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.
9. **Scalar distribution.**
   $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
10. **Vector distribution.**
    $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
11. **Orthogonality.**
    There exists a non-zero vector $\mathbf{n}$ such that $\mathbf{n}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
12. **Bidimensionality.**
    $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ for some value of a, b.

Definition 7.2

A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Addition associativity.** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition commutivity.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Addition identity.** There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- **Addition inverse.** There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.** $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **Scalar multiplication identity.** $1\mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Vector distribution.** $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Definition 7.3

The most important examples of vector spaces are the **Euclidean vector spaces** $\mathbb{R}^n$, but there are other examples as well.

Activity 7.4

Consider the following set that models motion along the curve $y = e^x$. Let $V = \{(x, y) : y = e^x\}$. Let vector addition be defined by $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and let scalar multiplication be defined by $c \odot (x, y) = (cx, y^c)$.

Activity 7.4

Consider the following set that models motion along the curve $y = e^x$. Let $V = \{(x, y) : y = e^x\}$. Let vector addition be defined by $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and let scalar multiplication be defined by $c \odot (x, y) = (cx, y^c)$.

Part 1: Which of the vector space properties are satisfied by V paired with these operations?

- **Addition associativity.** $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$.
- **Addition commutivity.** $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$.
- **Addition identity.** There exists some $\mathbf{0}$ where $\mathbf{v} \oplus \mathbf{0} = \mathbf{v}$.
- **Addition inverse.** There exists some $-\mathbf{v}$ where $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.** $a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}$.
- **Scalar multiplication identity.** $1 \odot \mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$.
- **Vector distribution.** $(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$.

Activity 7.4

Consider the following set that models motion along the curve $y = e^x$. Let $V = \{(x, y) : y = e^x\}$. Let vector addition be defined by $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and let scalar multiplication be defined by $c \odot (x, y) = (cx, y^c)$.

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- **Addition inverse.** There exists some $-\mathbf{v}$ where $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.** $a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}$.
- **Scalar multiplication identity.** $1 \odot \mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$.
- **Vector distribution.** $(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$.

Part 2: Is V a vector space?

Application Activities - Module V Part 2 - Class Day 8

Remark 8.1

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{R}^\infty$: Sequences of real numbers $(v_1, v_2, \dots)$.
- $\mathbb{R}^{m\times n}$: Matrices of real numbers with m rows and n columns.
- $\mathbb{C}$: Complex numbers.
- $\mathcal{P}^n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity 8.2

Let $V = \{(a, b) : a, b \text{ are real numbers}\}$, where $(a_1, b_1) \oplus (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$ and $c \odot (a, b) = (a^c, b + c)$. Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.** $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$.
- **Addition commutivity.** $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$.
- **Addition identity.** There exists some $\mathbf{0}$ where $\mathbf{v} \oplus \mathbf{0} = \mathbf{v}$.
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- **Scalar multiplication identity.** $1 \odot \mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$.
- **Vector distribution.** $(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$.

Definition 8.3

A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ for any choice of scalar multiples $c_1, c_2, \dots, c_m$.

For example, we say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Definition 8.4

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

Activity 8.5

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Activity 8.5

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch $c\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Activity 8.5

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch $c\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part 2: Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ in the xy plane.

Activity 8.6

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Activity 8.6

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane: $1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}$, $0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$, $2\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}$, $2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$.

Activity 8.6

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane: $1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}$, $0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$, $2\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}$, $2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$.

Part 2: Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity 8.7

Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix},\begin{bmatrix}-2\\3\end{bmatrix}\right\}$ in the xy plane.

Activity 8.8

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}$ holds for some scalars x_1, x_2.

Activity 8.8

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}$ holds for some scalars x_1, x_2.

Part 1: Reinterpret this vector equation as a system of linear equations.

Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (Remember, you should use a calculator to help find RREF.)

Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (Remember, you should use a calculator to help find RREF.)

Part 3: Given this solution, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

Application Activities - Module V Part 3 - Class Day 9

Fact 9.1

A vector $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \ \ldots \ \mathbf{v}_n \,|\, \mathbf{b}]$ is consistent.

Remark 9.2

To determine if $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, find $\operatorname{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \,|\, \mathbf{b}]$.

Activity 9.3

Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Activity 9.4

Determine if $\begin{bmatrix}-1\\-9\\0\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ by row-reducing an appropriate matrix.

Observation 9.5

So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space $\mathbb{R}^n$; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

Activity 9.6

We previously checked that $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ does not belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$.

Does $f(x) = 3x^2 - 2x + 1$ belong to span$\{x^2 - 3, -x^2 - 3x + 2\}$?

Activity 9.7

Does the matrix $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\}$?

Activity 9.8

Does the complex number $2i$ belong to span$\{-3+i, 6-2i\}$?

Activity 9.9

How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the *xy* plane to support your answer.

(a) 1

(b) 2

(c) 3

(d) 4

(e) Infinitely Many

Activity 9.10

How many vectors are required to span $\mathbb{R}^3$?

(a) 1

(b) 2

(c) 3

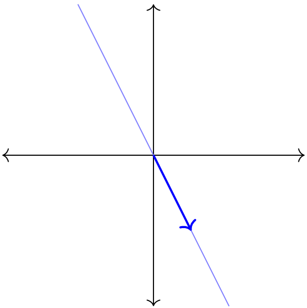
(d) 4

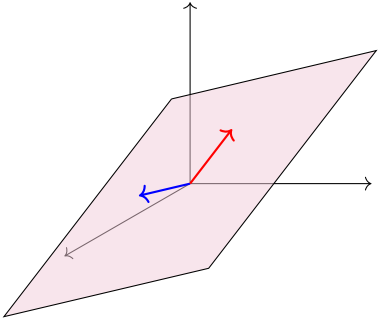
(e) Infinitely Many

Application Activities - Module V Part 4 - Class Day 10

Fact 10.1

At least n vectors are required to span $\mathbb{R}^n$.





Activity 10.2

Choose a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by ensuring $\left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$. (Why does this work?)

Fact 10.3

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ fails to span all of $\mathbb{R}^n$ exactly when RREF$[\mathbf{v}_1 \, \dots \, \mathbf{v}_m]$ has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Activity 10.4

Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does $\mathbb{R}^4 = \operatorname{span} S$?

Activity 10.5

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2$$

Does $\mathcal{P}^3 = \operatorname{span} S$?

Definition 10.6

A subset of a vector space is called a **subspace** if it is itself a vector space.

Fact 10.7

If S is a subset of a vector space V, then $\operatorname{span} S$ is a subspace of V.

Remark 10.8

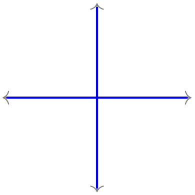
To prove that a subset is a subspace, you need only verify that $c\mathbf{v} + d\mathbf{w}$ belongs to the subset for any choice of vectors $\mathbf{v}, \mathbf{w}$ from the subset and any real scalars c, d.

Activity 10.9

Prove that $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$ is a subspace of the vector space of all degree-two polynomials by showing that $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$ belongs to P.

Activity 10.10

Consider the subset of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.



Find a linear combination $c\mathbf{v} + d\mathbf{w}$ that does not belong to this subset.

Fact 10.11

Suppose a subset S of V is isomorphic to another vector space W. Then S is a subspace of V.

Activity 10.12

Show that the set of 2×2 matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of $\mathbb{R}^{2\times 2}$ by identifying a Euclidean space isomorphic to S.

Module S: Structure of vector spaces

At the end of this module, students will be able to...

- **S1. Linear independence** Determine if a set of Euclidean vectors is linearly dependent or independent.
- **S2. Basis verification** Determine if a set of vectors is a basis of a vector space
- **S3. Basis construction** Construct a basis for the subspace spanned by a given set of vectors.
- **S4. Dimension** I can compute the dimension of a vector space.

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.
- Apply linear combinations and spanning sets **(Standard(s) V2,V3)**.

The following resources will help you prepare for this module.

- https://www.khanacademy.org/math/precalculus/vectors-precalc/vector-addition-subtraction/v/adding-and-subtracting-vectors
- https://www.khanacademy.org/math/precalculus/vectors-precalc/combined-vector-operations/v/combined-vector-operations-example

Application Activities - Module S Part 1 - Class Day 12

Activity 12.1

In the previous module, we considered

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

and showed that span $S \neq \mathbb{R}^4$. Find two vectors from this set that are linear combinations of the other three vectors.

Definition 12.2

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

Activity 12.3

Suppose $3\mathbf{v}_1 - 5\mathbf{v}_2 = \mathbf{v}_3$, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Is the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ consistent with one solution, consistent with infinitely many solutions, or inconsistent?

Fact 12.4

The set $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ is consistent with infinitely many solutions.

Activity 12.5

Find

$$\mathrm{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

Fact 12.6

A set of Euclidean vectors $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ has a column without a pivot position.

Activity 12.7

Is the set of Euclidean vectors $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$ linearly dependent or linearly independent?

Activity 12.8

Is the set of polynomials $\{x^3+1, x^2+2, 4-7x, 2x^3+x\}$ linearly dependent or linearly independent?

Application Activities - Module S Part 2 - Class Day 13

Activity 13.1

Last time we saw that $\left\{x^3+1, x^2+2, 4-7x, 2x^3+x\right\}$ is linearly independent. Show that it spans $\mathcal{P}^3$.

Definition 13.2

A **basis** is a linearly independent set that spans a vector space.

Observation 13.3

A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

Activity 13.4

Which of the following sets are bases for $\mathbb{R}^4$?

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}3\\13\\7\\16\end{bmatrix}, \begin{bmatrix}-1\\10\\7\\14\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix}, \begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}5\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\3\end{bmatrix}, \begin{bmatrix}4\\5\\1\\3\end{bmatrix}\right\}$$

Activity 13.5

If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$, that means RREF$[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$ doesn't have a column without a pivot position, and doesn't have a row of zeros. What is RREF$[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$?

Fact 13.6

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for $\mathbb{R}^n$ if and only if $m = n$ and

$$\mathrm{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, a basis for $\mathbb{R}^n$ must have exactly n vectors and its square matrix must row-reduce to the **identity matrix** containing all zeros except for a downward diagonal of ones.

Activity 13.7

Consider the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$.

Activity 13.7

Consider the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$.

Part 1: Use RREF $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$ to identify which vector may be removed to make the set linearly independent.

Activity 13.7

Consider the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$.

Part 1: Use RREF $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$ to identify which vector may be removed to make the set linearly independent.

Part 2: Find a basis for span $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$.

Application Activities - Module S Part 3 - Class Day 14

Fact 14.1

To compute a basis for the subspace $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, simply remove the vectors corresponding to the non-pivot columns of $\operatorname{RREF}[\mathbf{v}_1 \ \ldots \ \mathbf{v}_m]$.

Activity 14.2

Find all subsets of $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$ that are a basis for span S by changing the order of the vectors in S.

Activity 14.3

Assume $\mathbf{w}_1 \neq \mathbf{w}_2$ are distinct vectors in V, which has a basis containing a single vector: $\{\mathbf{v}\}$. Could $\{\mathbf{w}_1, \mathbf{w}_2\}$ be a basis?

Fact 14.4

All bases for a vector space are the same size.

Definition 14.5

The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

Activity 14.6

Find the dimension of each subspace of $\mathbb{R}^4$.

$$\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}3\\13\\7\\16\end{bmatrix},\begin{bmatrix}-1\\10\\7\\14\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix},\begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}5\\3\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\0\\3\end{bmatrix},\begin{bmatrix}4\\5\\1\\3\end{bmatrix}\right\}$$

Activity 14.7

What is the dimension of the vector space of 7th-degree (or less) polynomials $\mathcal{P}^7$?

a) 6

b) 7

c) 8

d) infinite

Activity 14.8

What is the dimension of the vector space of all polynomials $\mathcal{P}$?

a) 6 b) 7 c) 8 d) infinite

Observation 14.9

Several interesting vector spaces are infinite-dimensional:

- The space of polynomials $\mathcal{P}$ (consider the set $\{1, x, x^2, x^3, \dots\}$).
- The space of continuous functions $C(\mathbb{R})$ (which contains all polynomials, in addition to other functions like $e^x = 1 + x + x^2/2 + x^3/3 + \dots$).
- The space of real number sequences $\mathbb{R}^\infty$ (consider the set $\{(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, \dots), \dots\}$).

Fact 14.10

Every vector space with finite dimension, that is, every vector space with a basis of the form $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is isomorphic to a Euclidean space $\mathbb{R}^n$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Module A: Algebraic properties of linear maps

At the end of this module, students will be able to...

- **A1. Linear maps as matrices** I can write the standard matrix corresponding to a linear transformation between Euclidean spaces.
- **A2. Linear map verification** I can determine if a map between vector spaces is linear or not.
- **A3. Injectivity and Surjectivity** I can determine if a given linear map is injective and/or surjective
- **A4. Kernel and Image** I can compute the kernel and image of a linear map, including finding bases.

Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing **(Standard(s) E1, E2, E3, E4)**.
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace **(Standard(s) V3)**.
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent **(Standard(s) S1)**.
- State the definition of a basis, and determine if a set of vectors is a basis **(Standard(s) S2)**.

The following resources will help you prepare for this module.

- Review the supporting Standards listed above.

Application Activities - Module A Part 1 - Class Day 17

Definition 17.1

A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

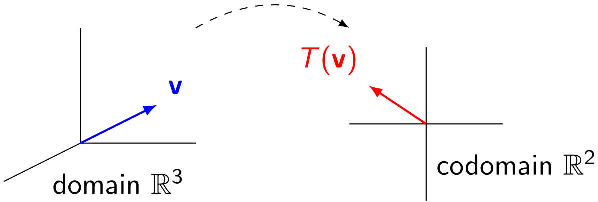
1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}$, $\mathbf{v} \in V$.

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.

Definition 17.2

Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.

Linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$



Example 17.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$$

Example 17.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - z_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - z_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

Example 17.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - z_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - z_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx - cz \\ cy \end{bmatrix} \text{ and } cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - z \\ y \end{bmatrix} = \begin{bmatrix} cx - cz \\ cy \end{bmatrix}$$

Therefore T is a linear transformation.

Activity 17.4

Determine if each of the following maps are linear transformations

Activity 17.4

Determine if each of the following maps are linear transformations

Part 1: $T_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \sqrt{x^2 + y^2}$.

Activity 17.4

Determine if each of the following maps are linear transformations

Part 1: $T_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \sqrt{x^2 + y^2}$.

Part 2: $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$

Activity 17.4

Determine if each of the following maps are linear transformations

Part 1: $T_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \sqrt{x^2 + y^2}$.

Part 2: $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$

Part 3: $T_3 : \mathcal{P}^d \to \mathcal{P}^{d-1}$ given by $T_3(f(x)) = f'(x)$.

Activity 17.4

Determine if each of the following maps are linear transformations

Part 1: $T_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \sqrt{x^2 + y^2}$.

Part 2: $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$

Part 3: $T_3 : \mathcal{P}^d \to \mathcal{P}^{d-1}$ given by $T_3(f(x)) = f'(x)$.

Part 4: $T_4 : \mathcal{P} \to \mathcal{P}$ given by $T_4(f(x)) = f(x) + x^2$

Activity 17.5

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity 17.6

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}0\\0\\-2\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity 17.7

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity 17.8

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}-2\\0\\-3\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity 17.9

Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation. How many facts of the form $T(\mathbf{v}_i) = \mathbf{w}_i$ do you need to know in order to be able to compute $T(\mathbf{v})$ for *any* $\mathbf{v} \in \mathbb{R}^4$?

(a) 2

(b) 3

(c) 4

(d) 5

(e) You need infinitely many

(In this situation, we say that the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ **determine** T.)

Fact 17.10

Consider any basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for V. Since every vector can be written *uniquely* as a linear combination of basis vectors, every linear transformation $T : V \to W$ is determined by those basis vectors.

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = x_1 T(\mathbf{b}_1) + \cdots + x_n T(\mathbf{b}_n)$$

Definition 17.11

The **standard basis** of $\mathbb{R}^n$ is the (ordered) basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Since linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by the values of each $T(\mathbf{e}_i)$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$.

Example 17.12

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\begin{bmatrix}3 & -1 & 5\\2 & 4 & 0\end{bmatrix}.$$

Activity 17.13

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard basis.

Activity 17.14

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Activity 17.15

Let $D : \mathcal{P}^3 \to \mathcal{P}^2$ be the derivative map $D(f(x)) = f'(x)$. (Earlier we showed this is a linear transformation.)

Activity 17.15

Let $D : \mathcal{P}^3 \to \mathcal{P}^2$ be the derivative map $D(f(x)) = f'(x)$. (Earlier we showed this is a linear transformation.)

Part 1: Write down an equivalent linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ by converting $\{1, x, x^2, x^3\}$ and $\{D(1), D(x), D(x^2), D(x^3)\}$ into appropriate vectors in $\mathbb{R}^4$ and $\mathbb{R}^3$.

Activity 17.15

Let $D : \mathcal{P}^3 \to \mathcal{P}^2$ be the derivative map $D(f(x)) = f'(x)$. (Earlier we showed this is a linear transformation.)

Part 1: Write down an equivalent linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ by converting $\{1, x, x^2, x^3\}$ and $\{D(1), D(x), D(x^2), D(x^3)\}$ into appropriate vectors in $\mathbb{R}^4$ and $\mathbb{R}^3$.

Part 2: Write the standard matrix corresponding to T.

Application Activities - Module A Part 2 - Class Day 18

Definition 18.1

Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\mathbf{v}) \neq T(\mathbf{w})$ whenever $\mathbf{v} \neq \mathbf{w}$.

Activity 18.2

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is T injective?

Activity 18.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Is T injective?

Definition 18.4

Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.

Activity 18.5

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is T surjective?

Activity 18.6

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Is T surjective?

Definition 18.7

Let $T : V \to W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{ \mathbf{v} \in V \;\middle|\; T(\mathbf{v}) = \mathbf{0} \right\}$$

Activity 18.8

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the kernel of T.

Activity 18.9

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the kernel of T.

Activity 18.10

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Activity 18.10

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Write a system of equations whose solution set is the kernel.

Activity 18.10

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Write a system of equations whose solution set is the kernel.

Part 2: Use RREF(A) to solve the system of equations and find the kernel of T.

Activity 18.10

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Write a system of equations whose solution set is the kernel.

Part 2: Use RREF(A) to solve the system of equations and find the kernel of T.

Part 3: Find a basis for the kernel of T.

Definition 18.11

Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{ \mathbf{w} \in W \;\middle|\; \text{there is some } v \in V \text{ with } T(\mathbf{v}) = \mathbf{w} \right\}$$

Activity 18.12

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the image of T.

Activity 18.13

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the image of T.

Activity 18.14

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Activity 18.14

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Find a convenient set of vectors $S \subseteq \mathbb{R}^2$ such that $\operatorname{span} S = \operatorname{Im} T$.

Activity 18.14

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Find a convenient set of vectors $S \subseteq \mathbb{R}^2$ such that span $S = \operatorname{Im} T$.

Part 2: Find a convenient basis for the image of T.

Observation 18.15

Let $T : V \to W$ be a linear transformation with corresponding matrix A.

- If A is a matrix corresponding to T, the kernel is the solution set of the homogeneous system with coefficients given by A.
- If A is a matrix corresponding to T, the image is the span of the columns of A.

Application Activities - Module A Part 3 - Class Day 19

Observation 19.1

Let $T : V \to W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.
- The **kernel** of T is the set of all things that are mapped to $\mathbf{0}$. It is a subspace of V.
- The **image** of T is the set of all things in W that are mapped to by something in V. It is a subspace of W.

Activity 19.2

Let $T : V \to W$ be a linear transformation where $\ker T = \{\mathbf{0}\}$. Can you answer either of the following questions about T?

(a) Is T injective?

(b) Is T surjective?

(Hint: If $T(\mathbf{v}) = T(\mathbf{w})$, then what is $T(\mathbf{v} - \mathbf{w})$?)

Fact 19.3

A linear transformation T is injective **if and only if** $\ker T = \{\mathbf{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

Activity 19.4

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation where $\operatorname{Im} T = \operatorname{span}\left\{\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\-1\end{bmatrix}\right\}$.

Can you answer either of the following questions about T?

(a) Is T injective?

(b) Is T surjective?

Fact 19.5

A linear transformation $T : V \to W$ is surjective **if and only if** $\operatorname{Im} T = W$. Put another way, a surjective linear transformation may be recognized by its same codomain and image.

Activity 19.6

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of equivalent statements.

(a) T is injective
(b) T is surjective
(c) The kernel of T is trivial.
(d) The columns of A span $\mathbb{R}^m$
(e) The columns of A are linearly independent
(f) Every column of RREF(A) has a pivot.
(g) Every row of RREF(A) has a pivot.
(h) The image of T equals its codomain.
(i) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$
(j) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$ has exactly one solution.

Definition 19.7

If $T : V \to W$ is both injective and surjective, it is called **bijective**.

Activity 19.8

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

(a) The columns of A form a basis for $\mathbb{R}^m$

(b) RREF(A) is the identity matrix.

(c) The system of linear equations given by the augmented matrix $\left[A \,\middle|\, \mathbf{b}\right]$ has exactly one solution for all $\mathbf{b} \in \mathbb{R}^m$.

Activity 19.9

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity 19.10

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity 19.11

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity 19.12

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Module M: Understanding Matrices Algebraically

At the end of this module, students will be able to...

- **M1. Matrix multiplication** Multiply matrices.
- **M2. Invertible matrices** Determine if a square matrix is invertible or not.
- **M3. Matrix inverses** Compute the inverse matrix of an invertible matrix.

Before beginning this module, each student should be able to...

- Compose functions of real numbers
- Solve systems of linear equations **(Standard(s) E3)**
- Find the matrix corresponding to a linear transformation **(Standard(s) A1)**
- Determine if a linear transformation is injective and/or surjective **(Standard(s) A3)**
- Interpret the ideas of injectivity and surjectivity in multiple ways

The following resources will help you prepare for this module.

- `https://www.khanacademy.org/math/algebra2/manipulating-functions/funciton-composition/v/function-composition`

Application Activities - Module M Part 1 - Class Day 21

Activity 21.1

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity 21.2

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity 21.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by thestandard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

The standard matrix of $S \circ T$ will lie in which matrix space?

(a) 4×3 matrices

(b) 4×2 matrices

(c) 3×2 matrices

(d) 2×3 matrices

(e) 2×4 matrices

(f) 3×4 matrices

Activity 21.4

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Activity 21.4

Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be given by the standard matrix $B=\begin{bmatrix}2 & 1 & -3\\ 5 & -3 & 4\end{bmatrix}$ and $S:\mathbb{R}^2\to\mathbb{R}^4$ be given by the standard matrix $A=\begin{bmatrix}1 & 2\\ 0 & 1\\ 3 & 5\\ -1 & -2\end{bmatrix}$.

Part 1: Compute $(S\circ T)(\mathbf{e}_1)$

Activity 21.4

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute $(S \circ T)(\mathbf{e}_1)$
Part 2: Compute $(S \circ T)(\mathbf{e}_2)$

Activity 21.4

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute $(S \circ T)(\mathbf{e}_1)$
Part 2: Compute $(S \circ T)(\mathbf{e}_2)$
Part 3: Compute $(S \circ T)(\mathbf{e}_3)$.

Activity 21.4

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute $(S \circ T)(\mathbf{e}_1)$
Part 2: Compute $(S \circ T)(\mathbf{e}_2)$
Part 3: Compute $(S \circ T)(\mathbf{e}_3)$.
Part 4: Find the standard matrix of $S \circ T$.

Activity 21.5

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity 21.6

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity 21.7

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

The standard matrix of $S \circ T$ will lie in which matrix space?

(a) 2×2 matrices

(b) 2×3 matrices

(c) 3×2 matrices

(d) 3×3 matrices

Activity 21.8

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

Find the standard matrix of $S \circ T$.

Activity 21.9

Let $T : \mathbb{R}^1 \to \mathbb{R}^4$ be given by the matrix $B = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ and $S : \mathbb{R}^4 \to \mathbb{R}^1$ be given by the matrix $A = \begin{bmatrix} 2 & 3 & 2 & 5 \end{bmatrix}$.

Find the standard matrix of $S \circ T$.

Definition 21.10
We define the product of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix (denoted AB) of the composition map of the two corresponding linear functions.

Fact 21.11

If AB is defined, BA need not be defined, and if it is defined, it is in general different from AB.

Activity 21.12

Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Compute AB.

Activity 21.13

Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Compute AX

Observation 21.14

Consider the system of equations

$$\begin{aligned} 3x + y - z &= 5 \\ 2x + 4z &= -7 \\ -x + 3y + 5z &= 2 \end{aligned}$$

We can interpret this as a **matrix equation** $AX = B$ where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$$

For this reason, we will swap out the use of Euclidean vectors $\mathbf{x} \in \mathbb{R}^n$ and $n \times 1$ matrices X whenever it is convenient.

Application Activities - Module M Part 2 - Class Day 22

Activity 22.1

Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix I such that $IA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Definition 22.2

The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

Fact 22.3

For any square matrix A, $IA = AI = A$:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 7 & -1\\0 & 3 & 2\\1 & 1 & -1\end{bmatrix} = \begin{bmatrix}2 & 7 & -1\\0 & 3 & 2\\1 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 7 & -1\\0 & 3 & 2\\1 & 1 & -1\end{bmatrix}$$

Activity 22.4

Each row operation can be interpreted as a type of matrix multiplication.

Activity 22.4

Each row operation can be interpreted as a type of matrix multiplication.

Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Activity 22.4

Each row operation can be interpreted as a type of matrix multiplication.

Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Activity 22.4

Each row operation can be interpreted as a type of matrix multiplication.

Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5 & 7+5 & -1-5 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Fact 22.5

If R is the result of applying a row operation to I, then RA is the result of applying the same row operation to A.

This means that for any matrix A, we can find a series of matrices $R_1, \dots, R_k$ corresponding to the row operations such that

$$R_1 R_2 \cdots R_k A = \text{RREF}(A).$$

That is, row reduction can be thought of as the result of matrix multiplication.

Activity 22.6

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into groups of statements about T.

(a) T is injective (i.e. one-to-one)
(b) T is surjective (i.e. onto)
(c) T is bijective (i.e. both injective and surjective)
(d) $AX = B$ has a solution for all $m \times 1$ matrices B
(e) $AX = B$ has a unique solution for all $m \times 1$ matrices B
(f) $AX = 0$ has a unique solution.
(g) The columns of A span $\mathbb{R}^m$
(h) The columns of A are linearly independent
(i) The columns of A are a basis of $\mathbb{R}^m$
(j) Every column of RREF(A) has a pivot
(k) Every row of RREF(A) has a pivot
(l) $m = n$ and RREF$(A) = I$

Activity 22.7

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix A. If T is injective, which of the following cannot be true?

(a) A has strictly more columns than rows

(b) A has the same number of rows as columns (i.e. A is square)

(c) A has strictly more rows than columns

Activity 22.8

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix A. If T is surjective, which of the following cannot be true?

(a) A has strictly more columns than rows

(b) A has the same number of rows as columns (i.e. A is square)

(c) A has strictly more rows than columns

Activity 22.9

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix A. If T is bijective, which of the following cannot be true?

(a) A has strictly more columns than rows

(b) A has the same number of rows as columns (i.e. A is square)

(c) A has strictly more rows than columns

Application Activities - Module M Part 3 - Class Day 23

Definition 23.1

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with standard matrix A.

- If T is a bijection and B is any $\mathbb{R}^n$ vector, then $T(X) = AX = B$ has a unique solution X.
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by setting $T^{-1}(B) = X$ to be this unique solution.
- Let A^{-1} be the standard matrix for T^{-1}. We call A^{-1} the **inverse matrix** of A, so we also say that A is **invertible**.

Activity 23.2

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$.

It can be shown that T is bijective and has the inverse map
$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Activity 23.2

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$.

It can be shown that T is bijective and has the inverse map

$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

Activity 23.2

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$.

It can be shown that T is bijective and has the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

Part 2: If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, what must $A^{-1}A$ be?

Observation 23.3

$T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation T. Therefore $A^{-1}A = AA^{-1} = I$ is the identity matrix for any invertible matrix A.

Activity 23.4

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Activity 23.4

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.

Activity 23.4

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.
Part 2: Solve $T(X) = \mathbf{e}_2$ to find $T^{-1}(\mathbf{e}_2)$.

Activity 23.4

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.
Part 2: Solve $T(X) = \mathbf{e}_2$ to find $T^{-1}(\mathbf{e}_2)$.
Part 3: Solve $T(X) = \mathbf{e}_3$ to find $T^{-1}(\mathbf{e}_3)$.

Activity 23.4

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.
Part 2: Solve $T(X) = \mathbf{e}_2$ to find $T^{-1}(\mathbf{e}_2)$.
Part 3: Solve $T(X) = \mathbf{e}_3$ to find $T^{-1}(\mathbf{e}_3)$.
Part 4: Compute A^{-1}, the standard matrix for T^{-1}.

Observation 23.5

We could have solved these three systems simultaneously by row reducing the matrix $[A \mid I]$ at once.

$$A = \left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right]$$

Activity 23.6

Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A \mid I]$.

Activity 23.7

Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

Observation 23.8

A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $\mathrm{RREF}(A) = I_n$.

Module G: Geometry of Linear Maps

At the end of this module, students will be able to...

- **G1. Determinants** Compute the determinant of a square matrix.
- **G2. Eigenvalues** Find the eigenvalues of a square matrix, along with their algebraic multiplicities.
- **G3. Eigenvectors** Find the eigenspace of a square matrix associated to a given eigenvalue.
- **G4. Geometric multiplicity** Compute the geometric multiplicity of an eigenvalue of a square matrix.

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **(Standard(s) A1)**.
- Recall and use the definition of a linear transformation **(Standard(s) A2)**.
- Find all roots of quadratic polynomials (including complex ones), and be able to use the rational root theorem to find all rational roots of a higher degree polynomial.
- Interpret the statement "A is an invertible matrix" in many equivalent ways in different contexts.

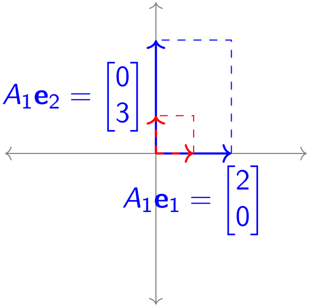
The following resources will help you prepare for this module.

- Finding the area of a parallelogram: `https://www.khanacademy.org/math/basic-geo/basic-geo-area-and-perimeter/parallelogram-area/a/area-of-parallelogram`
- Factoring quadratics: `https://www.khanacademy.org/math/algebra2/polynomial-functions/factoring-polynomials-quadratic-forms-alg2/v/factoring-polynomials-1`
- Finding complex roots of quadratics: `https://www.khanacademy.org/math/algebra2/polynomial-functions/quadratic-equations-with-complex-numbers/v/complex-roots-from-the-quadratic-formula`
- Finding all roots of polynomials: `https://www.khanacademy.org/math/algebra2/polynomial-functions/finding-zeros-of-polynomials/v/finding-roots-or-zeros-of-polynomial-1`
- The Rational Root Theorem: `https://artofproblemsolving.com/wiki/index.php?title=Rational_Root_Theorem`

Application Activities - Module G Part 1 - Class Day 25

Activity 25.1

The image below illustrates how the linear transformation $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.

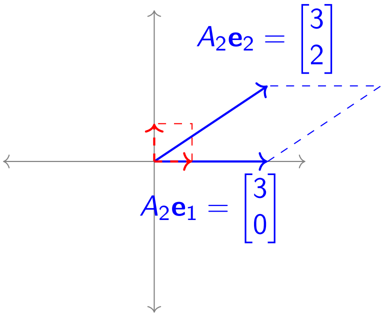


(a) What is the area of the transformed unit square?

(b) Find two vectors that were stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.

Activity 25.2

The image below illustrates how the linear transformation $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A_2 = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$. transforms the unit square.

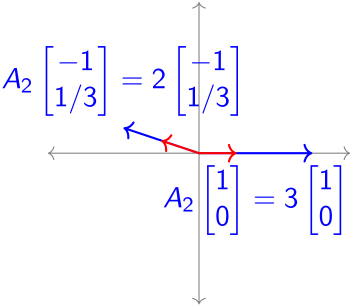


(a) What is the area of the transformed unit square?

(b) Find at least one vector that was stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.

Observation 25.3

It's possible to find two non-parallel vectors that are stretched by the transformation given by A_2:



The process for finding such vectors will be covered later in this module.

Activity 25.4

Consider the linear transformation given by the standard matrix $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(a) Sketch the transformation of the unit square (the parallelogram given by the columns of the standard matrix).

(b) Compute the area of the transformed unit square.

Activity 25.5

Consider the linear transformation given by the standard matrix $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Sketch the transformation of the unit square.

(b) Compute the area of the transformed unit square.

Activity 25.6

Consider the linear transformation given by the standard matrix $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

(a) Sketch the transformation of the unit square.

(b) Compute the area of the transformed unit square.

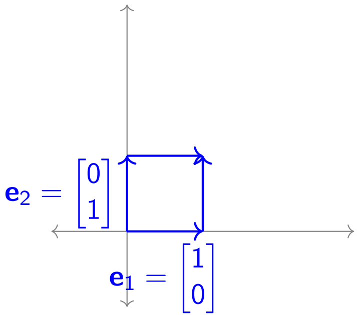
Remark 25.7

The area of the transformed unit square measures the factor by which all areas are transformed by a linear transformation.

We will define the **determinant** of a square matrix A, or $\det(A)$ for short, to be this factor. But what properties must this function satisfy?

Activity 25.8

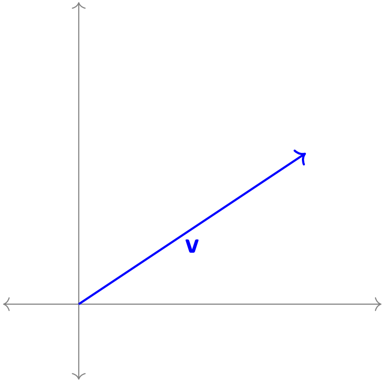
The transformation of the unit square by the standard matrix $[\mathbf{e}_1 \ \ \mathbf{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. What is $\det([\mathbf{e}_1 \ \ \mathbf{e}_2]) = \det(I)$, that is, by what factor has the area of the unit square been scaled?



a) 0
b) 1
c) 2
d) Cannot be determined

Activity 25.9

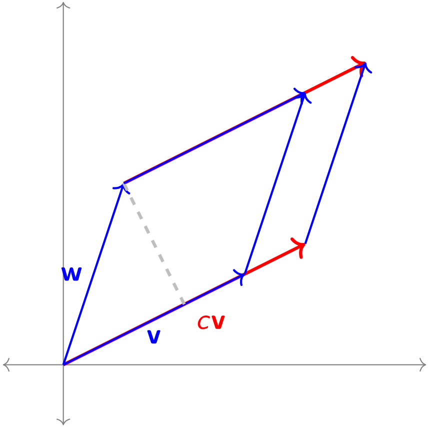
The transformation of the unit square by the standard matrix $[\mathbf{v} \ \ \mathbf{v}]$ is illustrated below: both $T(\mathbf{e}_1) = T(\mathbf{e}_2) = \mathbf{v}$. What is $\det([\mathbf{v} \ \ \mathbf{v}])$, that is, by what factor has area been scaled?



a) 0
b) 1
c) 2
d) Cannot be determined

Activity 25.10

The transformations of the unit square by the standard matrices $[\mathbf{v}\ \ \mathbf{w}]$ and $[c\mathbf{v}\ \ \mathbf{w}]$ are illustrated below. How are $\det([\mathbf{v}\ \ \mathbf{w}])$ and $\det([c\mathbf{v}\ \ \mathbf{w}])$ related?



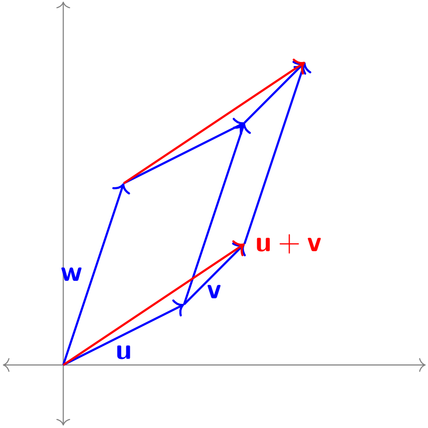
a) $\det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

b) $c + \det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

c) $c\det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

Activity 25.11

The transformations of unit squares by the standard matrices $[\mathbf{u}\ \ \mathbf{w}]$, $[\mathbf{v}\ \ \mathbf{w}]$ and $[\mathbf{u}+\mathbf{v}\ \ \mathbf{w}]$ are illustrated below. How is $\det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$ related to $\det([\mathbf{u}\ \ \mathbf{w}])$ and $\det([\mathbf{v}\ \ \mathbf{w}])$?



a) $\det([\mathbf{u}\ \ \mathbf{w}]) = \det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

b) $\det([\mathbf{u}\ \ \mathbf{w}]) + \det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

c) $\det([\mathbf{u}\ \ \mathbf{w}])\det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

Definition 25.12

The **determinant** is the unique function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfying the following three properties:

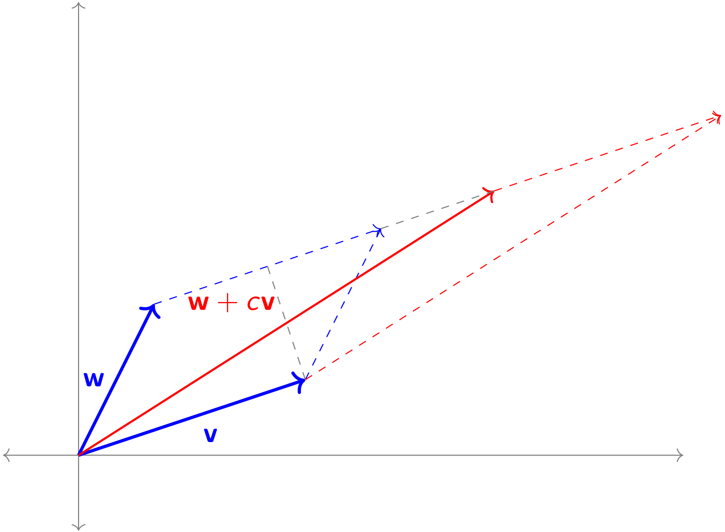
P1: $\det(I) = 1$

P2: $\det([\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \cdots \ \ \mathbf{v}_n]) = 0$ whenever two columns of the matrix are identical.

P3: $\det[\cdots \ \ c\mathbf{v} + d\mathbf{w} \ \ \cdots] = c\det[\cdots \ \ \mathbf{v} \ \ \cdots] + d\det[\cdots \ \ \mathbf{w} \ \ \cdots]$, assuming all other columns are equal.

Observation 25.13

What happens if we had a multiple of one column to another?



The base of both parallelograms is $\mathbf{v}$, while the height has not changed. Thus

$$\det([\mathbf{v} \quad \mathbf{w} + c\mathbf{v}]) = \det([\mathbf{v} \quad \mathbf{w}])$$

Observation 25.14

Swapping columns can be obtained from a sequence of adding column multiples.

$$\begin{aligned}
\det([\mathbf{v} \quad \mathbf{w}]) &= \det([\mathbf{v}+\mathbf{w} \quad \mathbf{w}]) \\
&= \det([\mathbf{v}+\mathbf{w} \quad \mathbf{w}-(\mathbf{v}+\mathbf{w})]) \\
&= \det([\mathbf{v}+\mathbf{w} \quad -\mathbf{v}]) \\
&= \det([\mathbf{v}+\mathbf{w}-\mathbf{v} \quad -\mathbf{v}]) \\
&= \det([\mathbf{w} \quad -\mathbf{v}]) \\
&= -\det([\mathbf{w} \quad \mathbf{v}])
\end{aligned}$$

So swapping two columns results in a negation of the determinant. Therefore, determinants represent a *signed* area, since they are not always positive.

Fact 25.15

We've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c\det([\cdots \quad \mathbf{v} \quad \cdots]) = \det([\cdots \quad c\mathbf{v} \quad \cdots])$$

(b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \quad \mathbf{v} \quad \cdots \quad \mathbf{w} \quad \cdots]) = -\det([\cdots \quad \mathbf{w} \quad \cdots \quad \mathbf{v} \quad \cdots])$$

(c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \quad \mathbf{v} \quad \cdots \quad \mathbf{w} \quad \cdots]) = \det([\cdots \quad \mathbf{v}+c\mathbf{w} \quad \cdots \quad \mathbf{w} \quad \cdots])$$

Activity 25.16

The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. How must the transformation given by the standard matrix AB scale areas?

(a) 1

(b) 7

(c) 12

(d) Cannot be determined

Fact 25.17

Since the transformation given by the standard matrix AB is obtained by applying the transformations given by A and B, it follows that

$$\det(AB) = \det(A)\det(B)$$

Application Activities - Module G Part 2 - Class Day 26

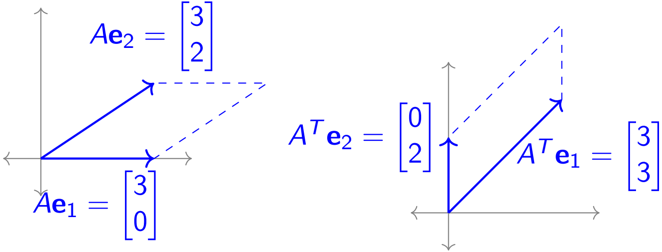
Definition 26.1

The **transpose** of a matrix is given by rewriting its columns as rows and vice versa:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Fact 26.2

It is possible to prove that the determinant of a matrix and its transpose are the same. For example, let $A = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$, so $A^T = \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}$; both matrices scale the unit square by 6, even though the parallelograms are not congruent.



Fact 26.3

We previously figured out that column operations can be used to simplify determinants; since $\det(A) = \det(A^T)$, we can also use row operations:

1. Multiplying rows by scalars: $\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$
2. Swapping two rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$
3. Adding multiples of rows to other rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$

Activity 26.4

Compute the determinant of $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ by row reducing it to a nicer matrix.

For example, $\det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

Fact 26.5

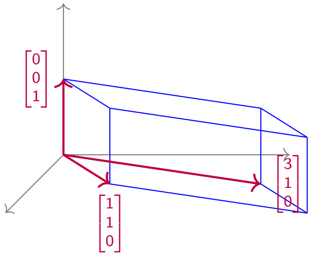
This same process allows us to prove a more convenient formula:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

In higher dimensions, the formulas become unreasonable. For example, the formula for 4×4 matrices has 24 terms!

Activity 26.6

The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.



This volume is equal to which of the following areas?

(a) $\det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ (c) $\det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$ (d) $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Fact 26.7

If column i of a matrix is $\mathbf{e}_i$, then both column and row i may be removed without changing the value of the determinant. For example, the second column of the following matrix is $\mathbf{e}_2$, so:

$$\det \begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Therefore the same holds for the transpose:

$$\det \begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Geometrically, this is the fact that if the height is 1, the base $\times$ height formula reduces to the area/volume/etc. of the $n-1$ dimensional base.

Activity 26.8

Compute $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$.

Activity 26.9

Compute $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$.

(a) -1 (b) 0 (c) 1

Activity 26.10

Compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix}$

Hint: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(a) 3 (b) 6 (c) 9 (d) 12

Activity 26.11

Compute $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$.

Observation 26.12

$$\det\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} = (-1)(0)\det\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} + (1)(3)\det\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} +$$

$$(-1)(2)\det\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} + (1)(0)\det\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$$

$$= 3\det\begin{bmatrix} 2 & 5 & 0 \\ 1 & 0 & 3 \\ -1 & 2 & 2 \end{bmatrix} + (-1)(2)\det\begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ -1 & -1 & 2 \end{bmatrix}$$

This technique is called **Laplace expansion** or **cofactor expansion**.

Activity 26.13

Compute $\det \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 1 & 2 & 0 & 3 \\ -1 & -3 & 2 & -2 \end{bmatrix}$.

Application Activities - Module G Part 3 - Class Day 27

Activity 27.1

An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Compute $\det(M)$ and $\det(M^{-1})$.

Activity 27.2

Suppose the matrix M is invertible, so there exists M^{-1} with $MM^{-1} = I$. It follows that $\det(M)\det(M^{-1}) = \det(I)$.

What is the only number that $\det(M)$ cannot equal?

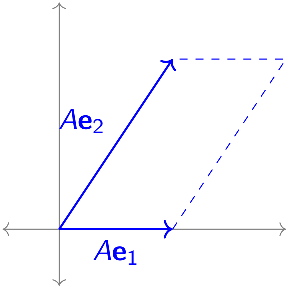
(a) -1 (b) 0 (c) 1 (d) $\frac{1}{\det(M^{-1})}$

Fact 27.3

- For every invertible matrix M, $\det(M^{-1}) = \frac{1}{\det(M)}$.
- Furthermore, a square matrix M is invertible if and only if $\det(M) \neq 0$.

Observation 27.4

Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



It is easy to see geometrically that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily verified by computation) that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Definition 27.5

Let $A \in \mathbb{R}^{n\times n}$. An **eigenvector** is a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x}$ is parallel to $\mathbf{x}$. In other words, $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

We call this λ an **eigenvalue** of A.

Observation 27.6

Since $\lambda\mathbf{x} = \lambda(I\mathbf{x})$, we can find the eigenvalues and eigenvectors satisfying $A\mathbf{x} = \lambda\mathbf{x}$ by inspecting $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

- Since we already know that $(A - \lambda I)\mathbf{0} = \mathbf{0}$ for any value of λ, we are more interested in finding values of λ such that $A - \lambda I$ has a nontrivial kernel.
- Thus RREF$(A - \lambda I)$ must have a non-pivot column, and therefore $A - \lambda I$ cannot be invertible.
- Since $A - \lambda I$ cannot be invertible, our eigenvalues must satisfy $\det(A - \lambda I) = 0$.

Definition 27.7

Computing $\det(A - \lambda I)$ results in the **characteristic polynomial** of A.

For example, when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

Thus the characteristic polynomial of A is

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2$$

Activity 27.8

Compute $\det(A - \lambda I)$ to find the characteristic polynomial of $A = \begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$.

Activity 27.9

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Activity 27.9

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Activity 27.9

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Find the roots of the characteristic polynomial to determine the eigenvalues of A.

Activity 27.9

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Find the roots of the characteristic polynomial to determine the eigenvalues of A.

Part 3: Compute the kernel of the transformation given by

$$A - 2I = \begin{bmatrix} 2-2 & 2 \\ 0 & 3-2 \end{bmatrix}$$

to determine all the eigenvectors associated to the eigenvalue 2.

Activity 27.9

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Find the roots of the characteristic polynomial to determine the eigenvalues of A.

Part 3: Compute the kernel of the transformation given by

$$A - 2I = \begin{bmatrix} 2-2 & 2 \\ 0 & 3-2 \end{bmatrix}$$

to determine all the eigenvectors associated to the eigenvalue 2.

Part 4: Compute the kernel of the transformation given by $A - 3I$ to determine all the eigenvectors associated to the eigenvalue 3.

Definition 27.10

The kernel of the transformation given by $A - \lambda I$ contains all the eigenvectors associated with λ. Since kernel is a subspace of $\mathbb{R}^n$, we call this kernel the **eigenspace** associated with the eigenvalue λ.

Activity 27.11

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

Activity 27.11

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Activity 27.11

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Find the roots of the characteristic polynomial $(3 - \lambda)(\lambda^2 - 4\lambda - 12)$ to determine the eigenvalues of A.

Activity 27.11

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Find the roots of the characteristic polynomial $(3 - \lambda)(\lambda^2 - 4\lambda - 12)$ to determine the eigenvalues of A.

Part 3: Compute the kernels of $A - \lambda I$ for each eigenvalue $\lambda \in \{-2, 3, 6\}$ to determine the respective eigenspaces.

Observation 27.12

Recall that if a is a root of the polynomial $p(\lambda)$, the **multiplicity** of a is the largest number k such that $p(\lambda) = q(\lambda)(\lambda - a)^k$ for some polynomial $q(\lambda)$.

For this reason, the **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Example 27.13

If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & 3 \\ 0 & 0 & -1 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda - 3)^2(\lambda + 1)$.

The eigenvalues are 3 (with algebraic multiplicity 2) and -1 (with algebraic multiplicity 1).

Application Activities - Module G Part 4 - Class Day 28

Observation 28.1

Recall from last class:

- To find the eigenvalues of a matrix A, we need to find values of λ such that $A - \lambda I$ has a nontrivial kernel. Equivalently, we want values where $A - \lambda I$ is not invertible, so we want to know the values of λ where $\det(A - \lambda I) = 0$.
- $\det(A - \lambda I)$ is a polynomial with variable λ, called the **characteristic polynomial** of A. Thus the roots of the characteristic polynomial of A are exactly the eigenvalues of A.
- Once an eigenvalue λ is found, the **eigenspace** containing all **eigenvectors** $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is given by $\ker(A - \lambda I)$.

Activity 28.2

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Activity 28.2

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Part 1: Compute the eigenvalues of A.

Activity 28.2

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Part 1: Compute the eigenvalues of A.

Part 2: Sketch a picture of the transformation of the unit square. What about this picture reveals that A has no real eigenvectors?

Activity 28.3

If A is a 4×4 matrix, what is the largest number of eigenvalues A can have?

(a) 3

(b) 4

(c) 5

(d) 6

(e) It can have infinitely many

Observation 28.4

An $n \times n$ matrix may have between 0 and n real-valued eigenvalues. But the Fundamental Theorem of Algebra implies that if complex eigenvalues are included, then every $n \times n$ matrix has exactly n eigenvalues (counting algebraic multiplicites).

Activity 28.5

The matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$ has characteristic polynomial $-\lambda(\lambda-2)^2$.

Find the dimension of the eigenspace of A associated to the eigenvalue 2 (the dimension of the kernel of $A - 2I$).

Activity 28.6

The matrix $B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}$ has characteristic polynomial $-\lambda(\lambda - 2)^2$.

Find the dimension of the eigenspace of B associated to the eigenvalue 2 (the dimension of the kernel of $B - 2I$).

Observation 28.7

In the first example, the (2 dimensional) plane spanned by $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}4\\-2\\0\end{bmatrix}$ was preserved. In the second example, only the (one dimensional) line spanned by $\begin{bmatrix}1\\0\\1\end{bmatrix}$ is preserved.

Definition 28.8

While the **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial, the **geometric multiplicity** of an eigenvalue is the dimension of its eigenspace.

Fact 28.9

As we've seen, the geometric multiplicity may be different than its algebraic multiplicity, but it cannot exceed it.

This fact is explored deeper and explained in Math 316, Linear Algebra II

Activity 28.10

Consider the 4×4 matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

Activity 28.10

Consider the 4×4 matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

Part 1: Use technology (e.g. Wolfram Alpha) to find its characteristic polynomial.

Activity 28.10

Consider the 4×4 matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

Part 1: Use technology (e.g. Wolfram Alpha) to find its characteristic polynomial.

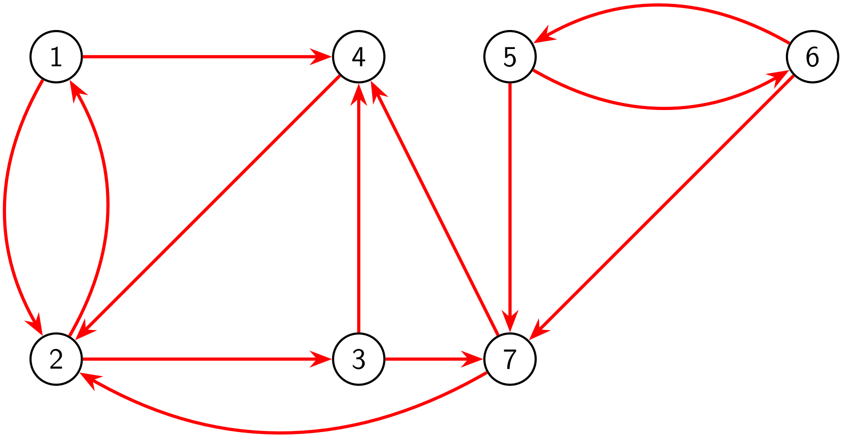
Part 2: Find the algebraic and geometric multiplicities for both eigenvalues.

Application Activities - Module G Part 5 - Class Day 29

Activity 29.1

A $700,000,000,000 Problem:

In the picture below, each circle represents a webpage, and each arrow represents a link from one page to another.



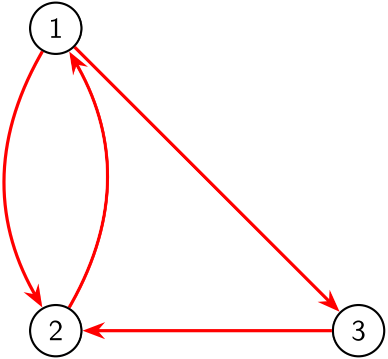
Based on how these pages link to each other, write a list of the 7 webpages in order from most imporant to least important.

Observation 29.2
The $700,000,000,000 Idea:
Links are endorsements.

1. A webpage is important if it is linked to (endorsed) by important pages.
2. A webpage distributes its importance equally among all the pages it links to (endorses).

Example 29.3

Consider this small network with only three pages. Let x_1, x_2, x_3 be the importance of the three pages respectively.

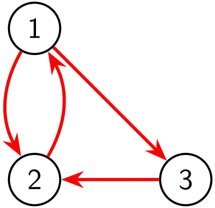


1. x_1 splits its endorsement in half between x_2 and x_3
2. x_2 sends all of its endorsement to x_1
3. x_3 sends all of its endorsement to x_2.

This corresponds to the **page rank system**

$$\begin{aligned} x_1 &= x_2 \\ x_2 &= \frac{1}{2}x_1 + x_3 \\ x_3 &= \frac{1}{2}x_1 \end{aligned}$$

Example 29.4



$$\begin{aligned} x_1 &= x_2 \\ x_2 &= \frac{1}{2}x_1 + x_3 \\ x_3 &= \frac{1}{2}x_1 \end{aligned}$$

We can summarize the left hand side of the system in the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and the right hand side in following **page rank matrix**.

$$A = \begin{bmatrix} 0\% & 100\% & 0\% \\ 50\% & 0\% & 100\% \\ 50\% & 0\% & 0\% \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Thus, the page rank system can be summarized as $A\mathbf{x} = \mathbf{x}$.

Activity 29.5

A **page rank vector** for a page rank matrix A is a vector $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{x}$.

Thus, the $700,000,000,000 problem is what kind of problem?

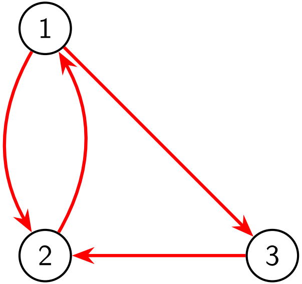
(a) A determinant problem

(b) An eigenvalue problem

(c) An eigenvector problem

Activity 29.6

Find a page rank vector (an eigenvector associated to the eigenvalue 1) for the following network's page rank matrix A.

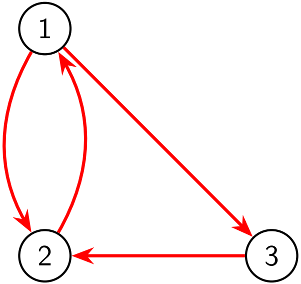


$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Observation 29.7

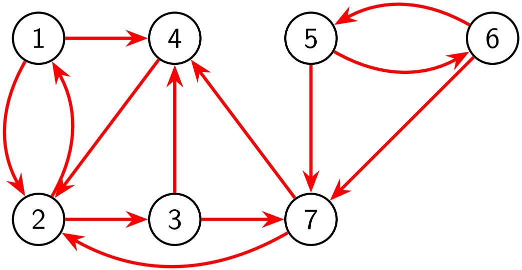
Row-reducing $A - I = \begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ yields the basic eigenvector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Therefore, we may conclude that pages 1 and 2 are equally important, and both pages are twice as important as page 3.



Activity 29.8

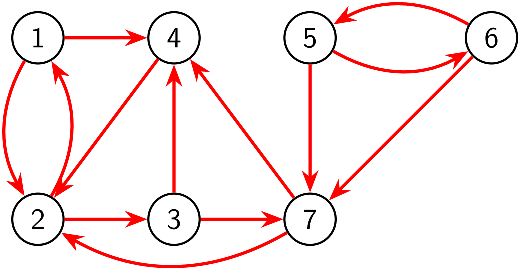
Compute the 7×7 page rank matrix for the following network.



For example, since website 1 distributes its endorsement equally between 2 and 4, the first column is $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity 29.9

Find a page rank vector for the transition matrix.

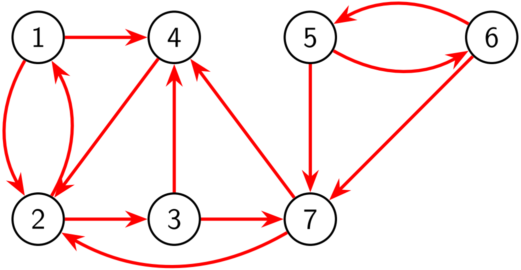


$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Which webpage is most important?

Observation 29.10

Since a steady state vector for the network is given by $\mathbf{x}$, it's reasonable to consider page 2 as the most important page.



$$\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$