Module I: Introduction

Remark I.0.1 This brief module gives an overview for the course.

Section I.1

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Remark I.1.1 What is Linear Algebra?
Linear algebra is the study of **linear maps**.

- In Calculus, you learn how to approximate any function by a linear function.
- In Linear Algebra, we learn about how linear maps behave.
- Combining the two, we can approximate how any function behaves.

Remark I.1.2 What is Linear Algebra good for?

- Linear algebra is used throughout several fields in higher mathematics.
- In computer graphics, linear algebra is used to help represent 3D objects in a 2D grid of pixels.
- Linear algebra is used to approximate differential equation solutions in a vast number of engineering applications (e.g. fluid flows, vibrations, heat transfer) whose solutions are very difficult (or impossible) to find precisely.
- Google's search engine is based on its Page Rank algorithm, which ranks websites by computing an eigenvector of a matrix.

Remark I.1.3 What will I learn in this class?
By the end of this class, you will be able to:

- Solve systems of linear equations. (Module E)
- Identify vector spaces and their properties. (Module V)
- Analyze the structure of vector spaces and sets of vectors. (Module S)
- Use and apply the algebraic properties of linear transformations. (Module A)
- Perform fundamental operations in the algebra of matrices. (Module M)
- Use and apply the geometric properties of linear transformations. (Module G)

Module E: Solving Systems of Linear Equations

Standards for this Module

How can we solve systems of linear equations? At the end of this module, students will be able to...

E1. Systems as matrices. ... translate back and forth between a system of linear equations and the corresponding augmented matrix.

E2. Row reduction. ... put a matrix in reduced row echelon form.

E3. Systems of linear equations. ... compute the solution set for a system of linear equations.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Systems of linear equations (Khan Academy): `http://bit.ly/2l21etm`
- Solving linear systems with substitution (Khan Academy): `http://bit.ly/1SlMpix`

Section E.1

Definition E.1.1 A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

Remark E.1.2 In previous classes you likely assumed $x = x_1$, $y = x_2$, and $z = x_3$. However, since this course often deals with equations of four or more variables, we will almost always write our variables as x_i.

Definition E.1.3 A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{array}{rcrcrcrcl} a_{11}x_1 & + & a_{12}x_2 & + \ldots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \ldots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \ldots + & a_{mn}x_n & = & b_m \end{array}$$

A **solution**

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i$$

for $1 \leq i \leq m$ (that is, the solution satisfies all equations in the system).

Remark E.1.4 When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{array}{rcr} x_1 + 3x_3 & = & 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ -x_2 + x_3 & = & -2 \end{array}$$

Verbose standard form:

$$\begin{array}{rcr} 1x_1 + 0x_2 + 3x_3 & = & 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ 0x_1 - 1x_2 + 1x_3 & = & -2 \end{array}$$

Concise standard form:

$$\begin{array}{rrrcr} x_1 & & + 3x_3 & = & 3 \\ 3x_1 & - 2x_2 & + 4x_3 & = & 0 \\ & - x_2 & + x_3 & = & -2 \end{array}$$

Definition E.1.5 A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

Fact E.1.6 All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

Activity E.1.7 *(∼10 min)* All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity E.1.8 *(∼10 min)* Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions for this system.
Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write *all* solutions (the **solution set**) $\left\{ \begin{bmatrix} ? \\ a \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}$ for the linear system in terms of a.

Activity E.1.9 *(∼10 min)* Consider the following linear system.

$$\begin{alignedat}{4} x_1 + 2x_2 & & & {}-{} & x_4 &= {}& 3 \\ & & x_3 & {}+{} & 4x_4 &= {}& -2 \end{alignedat}$$

Describe the solution set

$$\left\{ \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation E.1.10 Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.

Section E.2

Remark E.2.1 The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition E.2.2 A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Definition E.2.3 Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \qquad\qquad \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity E.2.4 *(∼10 min)* Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a nonzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

(Instructor Note:) This activity could be ran as a card sort. Allow 5 additional minutes for intra team discussion.

Definition E.2.5 The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a nonzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity E.2.6 *(∼10 min)* Consider the following linear systems.

(A) $$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

(C) $$\begin{aligned} x_1 - 3x_2 + 6x_3 &= -11 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ 3x_1 - 2x_2 + 13x_3 &= 6 \end{aligned}$$

(E) $$\begin{aligned} x_1 - 3x_2 + 6x_3 &= -11 \\ 4x_2 - 2x_3 &= 24 \\ 7x_2 - 5x_3 &= 39 \end{aligned}$$

(B) $$\begin{aligned} x_1 \quad + 9x_3 &= 16 \\ x_2 + x_3 &= 9 \\ -12x_3 &= -24 \end{aligned}$$

(D) $$\begin{aligned} x_1 - 3x_2 + 6x_3 &= -11 \\ x_2 + x_3 &= 9 \\ 7x_2 - 5x_3 &= 39 \end{aligned}$$

(F) $$\begin{aligned} x_1 \quad + 9x_3 &= 16 \\ x_2 + x_3 &= 9 \\ x_3 &= 2 \end{aligned}$$

Part 1: Which system can be obtained from System (A) in the fewest number of row operations?
Part 2: Rank the six linear systems from easiest to solve to hardest to solve.

Activity E.2.7 *(∼10 min)* Consider the following augmented matrices.

(A) $\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array}\right]$ (C) $\left[\begin{array}{ccc|c} 1 & -3 & 6 & -11 \\ 2 & -2 & 10 & 2 \\ 3 & -2 & 13 & 6 \end{array}\right]$ (E) $\left[\begin{array}{ccc|c} 1 & -3 & 6 & -11 \\ 0 & 4 & -2 & 24 \\ 0 & 7 & -5 & 39 \end{array}\right]$

(B) $\left[\begin{array}{ccc|c} 1 & 0 & 9 & 16 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & -12 & -24 \end{array}\right]$ (D) $\left[\begin{array}{ccc|c} 1 & -3 & 6 & -11 \\ 0 & 1 & 1 & 9 \\ 0 & 7 & -5 & 39 \end{array}\right]$ (F) $\left[\begin{array}{ccc|c} 1 & 0 & 9 & 16 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 2 \end{array}\right]$

Part 1: Rank the six matrices from farthest from a reduced row echelon form (RREF) matrix to closest to a RREF matrix.
Part 2: These matrices are all **row equivalent** and represent equivalent linear systems. Write down one of these linear systems and solve it.

Remark E.2.8 It is important to understand the **Gauss-Jordan elimination** algorithm that converts a matrix into reduced row echelon form, but in practice we don't do thsi by hand; we use technology to do this for us.

Activity E.2.9 *(∼10 min)*

- Go to `http://www.cocalc.com` and create an account.
- Create a project titled "Linear Algebra Team X" with your appropriate team number. Add all team members as collaborators.
- Open the project and click on "New"
- Give it an appropriate name such as "Class 4 workbook". Make a new Jupyter notebook.
- Click on "Kernel" and make sure "Octave" is selected.
- Type `A=[1 3 4 ; 2 5 7]` to store the matrix $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{bmatrix}$ in the variable A; hold shift when you press enter.
- Type `rref(A)` to compute the reduced row echelon form of A.

Remark E.2.10 If you need to find the reduced row echelon form of a matrix during class, you should feel free to use CoCalc/Octave.

You can change a cell from "Code" to "Markdown" or "Raw" to put comments around your calculations such as Activity numbers.

Activity E.2.11 *(~8 min)* Consider our system of equations from above.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

Convert this to an augmented matrix, use CoCalc to compute the reduced row echelon form, and convert back to a simpler system of equations to solve this system. Write your solution on your whiteboard.

Activity E.2.12 *(~7 min)* Consider our system of equations from above.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 \qquad\quad - 3x_3 &= 1 \end{aligned}$$

Convert this to an augmented matrix, use CoCalc to compute the reduced row echelon form, and convert back to a simpler system of equations to solve this system. Write your solution on your whiteboard.

Section E.3

Fact E.3.1 Every augmented matrix A reduces to a unique reduced row echelon form matrix. This matrix is denoted as $\mathrm{RREF}(A)$.

Activity E.3.2 *(∼10 min)* Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array}\right]$$

Part 1: Find $\mathrm{RREF}(A)$ (Use CoCalc).
Part 2: How many solutions does the corresponding linear system have?

Activity E.3.3 *(∼10 min)* Consider the (simpler) system from the previous problem:

$$\begin{aligned} x_1 + 2x_2 \quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

Part 2: Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$

Part 3: Which of these was easier? What features of the RREF matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

cause this?

Definition E.3.4 If a matrix is in reduced row echelon form, a **pivot** is an entry satisfying

1. It is 1
2. Everything else in the same row but to the left of it is zero
3. Everything else in the same column is zero.

For example, the pivots are circled in

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Activity E.3.5 *(∼5 min)* Circle the pivots in each matrix below.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Definition E.3.6 The pivots in a matrix correspond to **bound variables** in the system of equations. The remaining variables are called **free variables**.
To efficiently solve a system in RREF form, assign letters to free variables and solve for the bound variables.

Activity E.3.7 *(∼10 min)* Find the solution set for the system

$$\begin{alignedat}{6} 2x_1 &- 2x_2 &- 6x_3 &+ x_4 &- x_5 &= 3 \\ -x_1 &+ x_2 &+ 3x_3 &- x_4 &+ 2x_5 &= -3 \\ x_1 &- 2x_2 &- x_3 &+ x_4 &+ x_5 &= 2 \end{alignedat}$$

by assigning letters to the free variables and solving for the bounded variables.

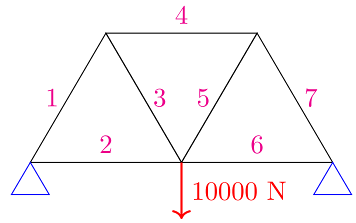
Observation E.3.8 The solution set to the system

$$\begin{alignedat}{6} 2x_1 &- 2x_2 &- 6x_3 &+ x_4 &- x_5 &= 3 \\ -x_1 &+ x_2 &+ 3x_3 &- x_4 &+ 2x_5 &= -3 \\ x_1 &- 2x_2 &- x_3 &+ x_4 &+ x_5 &= 2 \end{alignedat}$$

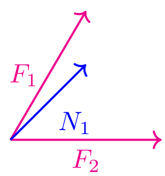
is

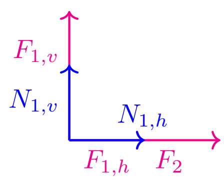
$$\left\{ \begin{bmatrix} 1+5a+2b \\ 1+2a+3b \\ a \\ 3+3b \\ b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

Observation E.3.9 Consider the truss pictured below with two fixed anchor points and a 10000 N load (assume all triangles are equilateral).



The horizontal and vertical forces must balance at each of the five intersecting nodes. For example, at the bottom left node





Apply basic trig:

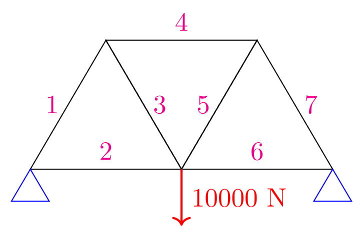
$$F_{1,v} = F_1 \sin(60^\circ)$$
$$F_{1,h} = F_1 \cos(60^\circ)$$

thus

$$F_1 \sin(60^\circ) + N_{1,v} = 0$$
$$F_1 \cos(60^\circ) + N_{1,h} + F_2 = 0$$

We adhere to the convention that a compression force on a strut is positive, while a negative force represents tension.

Activity E.3.10 *(~10 min)* Consider the truss pictured below with two fixed anchor points and a 10000 N load (assume all triangles are equilateral).

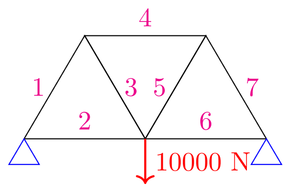


From the bottom left node we obtained 2 equations in the four variables

- F_1 (compression force on strut one)
- $N_{1,v}$ and $N_{1,h}$ (horizontal and vertical components of the normal force from the left anchor)
- F_2 (compression force on strut 2).

Part 1: Determine how many total equations there will be after accounting for all of the nodes, and and list all of the variables. You do not need to actually determine all of the equations.

Activity E.3.11 *(~10 min)*



The resulting system is

$$\begin{array}{rcl}
N_{1,v} + (\sin(60^\circ))F_1 & = & 0 \\
N_{1,h} + (\cos(60^\circ))F_1 + F_2 & = & 0 \\
-(\sin(60^\circ))F_1 - (\sin(60^\circ)F_3 & = & 0 \\
-(\cos(60^\circ))F_1 + (\cos(60^\circ)F_3 + F_4 & = & 0 \\
(\sin(60^\circ)F_3 + (\sin(60^\circ))F_5 & = & 10000 \\
-F_2 - (\cos(60^\circ)F_3 + (\cos(60^\circ))F_5 + F_6 & = & 0 \\
-(\sin(60^\circ))F_5 - (\sin(60^\circ))F_7 & = & 0 \\
-F_4 - (\cos(60^\circ))F_5 + (\cos(60^\circ))F_7 & = & 0 \\
N_{2,v} + (\sin(60^\circ))F_7 & = & 0 \\
N_{2,h} - F_6 - (\cos(60^\circ))F_7 & = & 0
\end{array}$$

Solve this system to determine which struts are compressed and which are in tension.

Observation E.3.12 The determined part of the solution is

$$N_{1,v} = N_{2,v} = 5000$$
$$F_1 = F_4 = F_7 = -5882.4$$
$$F_3 = F_5 = 5882.4$$

So struts 1,4,7 are in tension, while struts 3 and 5 are compressed.
The forces on struts 2 and 6 (and the horizontal normal forces) are not strictly determined in this setting.

Module V: Vector Spaces

Standards for this Module

What is a vector space? At the end of this module, students will be able to...

V1. Vector property verification. ... show why an example satisfies a given vector space property, but does not satisfy another given property.

V2. Vector space identification. ... list the eight defining properties of a vector space, infer which of these properties a given example satisfies, and thus determine if the example is a vector space.

V3. Linear combinations. ... determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors.

V4. Spanning sets. ... determine if a set of Euclidean vectors spans $\mathbb{R}^n$.

V5. Subspaces. ... determine if a subset of $\mathbb{R}^n$ is a subspace or not.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Section V.1

Activity V.1.1 *($\sim$20 min)* Consider each of the following vector properties. Label each property with $\mathbb{R}^1$, $\mathbb{R}^2$, and/or $\mathbb{R}^3$ if that property holds for Euclidean vectors/scalars $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of that dimension.

1. **Addition associativity.**

   $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

2. **Addition commutivity.**

   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

3. **Addition identity.**

   There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

4. **Addition inverse.**

   There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

5. **Addition midpoint uniqueness.**

   There exists a unique $\mathbf{m}$ where the distance from $\mathbf{u}$ to $\mathbf{m}$ equals the distance from $\mathbf{m}$ to $\mathbf{v}$.

6. **Scalar multiplication associativity.**

   $a(b\mathbf{v}) = (ab)\mathbf{v}$.

7. **Scalar multiplication identity.**

   $1\mathbf{v} = \mathbf{v}$.

8. **Scalar multiplication relativity.**

   There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.

9. **Scalar distribution.**

   $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

10. **Vector distribution.**

    $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

11. **Orthogonality.**

    There exists a non-zero vector $\mathbf{n}$ such that $\mathbf{n}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

12. **Bidimensionality.**

    $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ for some value of a, b.

Definition V.1.2 A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Addition associativity.**

  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **Addition commutivity.**

  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **Addition identity.**

  There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

- **Addition inverse.**

  There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

- **Scalar multiplication associativity.**

  $a(b\mathbf{v}) = (ab)\mathbf{v}$.

- **Scalar multiplication identity.**

  $1\mathbf{v} = \mathbf{v}$.

- **Scalar distribution.**

  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

- **Vector distribution.**

  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Definition V.1.3 The most important examples of vector spaces are the **Euclidean vector spaces** $\mathbb{R}^n$, but there are other examples as well.

Activity V.1.4 *(∼25 min)* Consider the following set that models motion along the curve $y = e^x$. Let $V = \{(x, y) : y = e^x\}$. Let vector addition be defined by $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and let scalar multiplication be defined by $c \odot (x, y) = (cx, y^c)$.

Part 1: Which of the vector space properties are satisfied by V paired with these operations?

- **Addition associativity.**
  $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$.
- **Addition commutivity.**
  $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$.
- **Addition identity.**
  There exists some $\mathbf{0}$ where $\mathbf{v} \oplus \mathbf{0} = \mathbf{v}$.
- **Addition inverse.**
  There exists some $-\mathbf{v}$ where $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.**
  $a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}$.
- **Scalar multiplication identity.**
  $1 \odot \mathbf{v} = \mathbf{v}$.
- **Scalar distribution.**
  $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$.
- **Vector distribution.**
  $(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$.

Part 2: Is V a vector space?

Section V.2

Remark V.2.1 The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{R}^\infty$: Sequences of real numbers $(v_1, v_2, \dots)$.
- $\mathbb{R}^{m\times n}$: Matrices of real numbers with m rows and n columns.
- $\mathbb{C}$: Complex numbers.
- $\mathcal{P}^n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity V.2.2 *(∼10 min)* Let $V = \{(a, b) : a, b \text{ are real numbers}\}$, where $(a_1, b_1) \oplus (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$ and $c \odot (a, b) = (a^c, b + c)$. Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.**
  $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$.
- **Addition commutivity.**
  $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$.
- **Addition identity.**
  There exists some $\mathbf{0}$ where $\mathbf{v} \oplus \mathbf{0} = \mathbf{v}$.
- **Addition inverse.**
  There exists some $-\mathbf{v}$ where $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.**
  $a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}$.
- **Scalar multiplication identity.**
  $1 \odot \mathbf{v} = \mathbf{v}$.
- **Scalar distribution.**
  $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$.
- **Vector distribution.**
  $(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$.

Definition V.2.3 A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ for any choice of scalar multiples $c_1, c_2, \dots, c_m$.

For example, we say $\begin{bmatrix}3\\0\\5\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\2\\1\end{bmatrix}$ since

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

Definition V.2.4 The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

Activity V.2.5 *(~10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch $c\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part 2: Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ in the xy plane.

Activity V.2.6 *(~10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane: $1\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix}$, $0\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix}$, $2\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix}$, $2\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix}$.

Part 2: Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity V.2.7 *(~5 min)* Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix}\right\}$ in the xy plane.

Activity V.2.8 *(~15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}$ holds for some scalars x_1, x_2.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (Remember, you should use a calculator to help find RREF.)

Part 3: Given this solution, does $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belong to $\operatorname{span}\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$?

Section V.3

Fact V.3.1 A vector $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \, \dots \, \mathbf{v}_n \,|\, \mathbf{b}]$ is consistent.

Remark V.3.2 To determine if $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, find $\operatorname{RREF}[\mathbf{v}_1 \, \dots \, \mathbf{v}_n \,|\, \mathbf{b}]$.

Activity V.3.3 *(∼5 min)* Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}\right\}$ by row-reducing an appropriate matrix.

Activity V.3.4 *(∼5 min)* Determine if $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$ belongs to $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}\right\}$ by row-reducing an appropriate matrix.

Observation V.3.5 So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space $\mathbb{R}^n$; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

Activity V.3.6 *(∼5 min)* We previously checked that $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ does not belong to $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}\right\}$. Does $f(x) = 3x^2 - 2x + 1$ belong to $\operatorname{span}\{x^2 - 3, -x^2 - 3x + 2\}$?

Activity V.3.7 *(∼10 min)* Does the matrix $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$ belong to $\operatorname{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\right\}$?

Activity V.3.8 *(∼10 min)* Does the complex number $2i$ belong to $\operatorname{span}\{-3 + i, 6 - 2i\}$?

Activity V.3.9 *(∼10 min)* How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your answer.

(a) 1

(b) 2

(c) 3

(d) 4

(e) Infinitely Many

Activity V.3.10 *(∼5 min)* How many vectors are required to span $\mathbb{R}^3$?

(a) 1

(b) 2

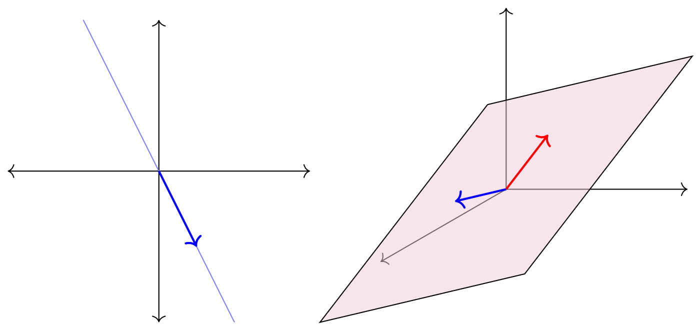
(c) 3

(d) 4

(e) Infinitely Many

Section V.4

Fact V.4.1 At least n vectors are required to span $\mathbb{R}^n$.



Activity V.4.2 *(∼10 min)* Choose a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by ensuring $\left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$. (Why does this work?)

Fact V.4.3 The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ fails to span all of $\mathbb{R}^n$ exactly when $\mathrm{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$ has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Activity V.4.4 *(∼5 min)* Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does $\mathbb{R}^4 = \operatorname{span} S$?

Activity V.4.5 *(∼10 min)* Consider the set of third-degree polynomials

$$S = \left\{ 2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2 \right\}.$$

Does $\mathcal{P}^3 = \operatorname{span} S$?

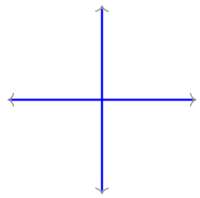
Definition V.4.6 A subset of a vector space is called a **subspace** if it is itself a vector space.

Fact V.4.7 If S is a subset of a vector space V, then $\operatorname{span} S$ is a subspace of V.

Remark V.4.8 To prove that a subset is a subspace, you need only verify that $c\mathbf{v} + d\mathbf{w}$ belongs to the subset for any choice of vectors $\mathbf{v}, \mathbf{w}$ from the subset and any real scalars c, d.

Activity V.4.9 *(~5 min)* Prove that $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$ is a subspace of the vector space of all degree-two polynomials by showing that $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$ belongs to P.

Activity V.4.10 *(~10 min)* Consider the subset of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.



Find a linear combination $c\mathbf{v} + d\mathbf{w}$ that does not belong to this subset. **(Instructor Note:)** Use this linear combination to sketch a picture illustrating why this subset is not a subspace.

Fact V.4.11 Suppose a subset S of V is isomorphic to another vector space W. Then S is a subspace of V.

Activity V.4.12 *(~5 min)* Show that the set of 2×2 matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of $\mathbb{R}^{2\times 2}$ by identifying a Euclidean space isomorphic to S.

Module S: Structure of vector spaces

Standards for this Module

What structure do vector spaces have? At the end of this module, students will be able to...

S1. Linear independence. ... determine if a set of Euclidean vectors is linearly dependent or independent.

S2. Basis verification. ... determine if a set of Euclidean vectors is a basis of $\mathbb{R}^n$.

S3. Basis computation. ... compute a basis for the subspace spanned by a given set of Euclidean vectors.

S4. Dimension. ... compute the dimension of a subspace of $\mathbb{R}^n$.

S5. Abstract vector spaces. ... solve exercises related to standards V3-S4 when posed in terms of polynomials or matrices.

S6. Basis of solution space. ... find a basis for the solution set of a homogeneous system of equations.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V2,V3**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Section S.1

Activity S.1.1 *(~15 min)* In the previous module, we considered

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

and showed that $\operatorname{span} S \neq \mathbb{R}^4$. Find two vectors from this set that are linear combinations of the other three vectors.

(Instructor Note:) Actually, the activity involved the corresponding vectors in $\mathcal{P}^3$.

Definition S.1.2 We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

Activity S.1.3 *(~10 min)* Suppose $3\mathbf{v}_1 - 5\mathbf{v}_2 = \mathbf{v}_3$, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Is the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ consistent with one solution, consistent with infinitely many solutions, or inconsistent?

Fact S.1.4 The set $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ is consistent with infinitely many solutions.

Activity S.1.5 *(~10 min)* Find

$$\operatorname{RREF} \left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

Fact S.1.6 A set of Euclidean vectors $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ is linearly dependent if and only if $\operatorname{RREF} \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ has a column without a pivot position.

Activity S.1.7 *(~5 min)* Is the set of Euclidean vectors $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$ linearly dependent or linearly independent?

Activity S.1.8 *(∼10 min)* Is the set of polynomials $\left\{x^3+1, x^2+2, 4-7x, 2x^3+x\right\}$ linearly dependent or linearly independent?

Section S.2

Activity S.2.1 *(∼10 min)* Last time we saw that $\left\{x^3+1, x^2+2, 4-7x, 2x^3+x\right\}$ is linearly independent. Show that it spans $\mathcal{P}^3$.

Definition S.2.2 A **basis** is a linearly independent set that spans a vector space.

Observation S.2.3 A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

Activity S.2.4 *(∼15 min)* Which of the following sets are bases for $\mathbb{R}^4$?

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}3\\13\\7\\16\end{bmatrix},\begin{bmatrix}-1\\10\\7\\14\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix},\begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}5\\3\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\0\\3\end{bmatrix},\begin{bmatrix}4\\5\\1\\3\end{bmatrix}\right\}$$

Activity S.2.5 *(∼10 min)* If $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$, that means $\mathrm{RREF}[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$ doesn't have a column without a pivot position, and doesn't have a row of zeros. What is $\mathrm{RREF}[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$?

Fact S.2.6 The set $\{\mathbf{v}_1,\dots,\mathbf{v}_m\}$ is a basis for $\mathbb{R}^n$ if and only if $m=n$ and $\mathrm{RREF}[\mathbf{v}_1\,\dots\,\mathbf{v}_n]=\begin{bmatrix}1&0&\dots&0\\0&1&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&1\end{bmatrix}$.

That is, a basis for $\mathbb{R}^n$ must have exactly n vectors and its square matrix must row-reduce to the **identity matrix** containing all zeros except for a downward diagonal of ones.

Activity S.2.7 *(∼10 min)* Consider the set $\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$.

Part 1: Use RREF $\begin{bmatrix}2&2&2&1\\3&0&-3&5\\0&1&2&-1\\1&-1&-3&0\end{bmatrix}$ to identify which vector may be removed to make the set linearly independent.

Part 2: Find a basis for span $\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$.

Section S.3

Fact S.3.1 To compute a basis for the subspace $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, simply remove the vectors corresponding to the non-pivot columns of $\operatorname{RREF}[\mathbf{v}_1 \; \ldots \; \mathbf{v}_m]$.

Activity S.3.2 *(~10 min)* Find all subsets of $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$ that are a basis for $\operatorname{span} S$ by changing the order of the vectors in S.

Activity S.3.3 *(~10 min)* Assume $\mathbf{w}_1 \neq \mathbf{w}_2$ are distinct vectors in V, which has a basis containing a single vector: $\{\mathbf{v}\}$. Could $\{\mathbf{w}_1, \mathbf{w}_2\}$ be a basis?

Fact S.3.4 All bases for a vector space are the same size.

Definition S.3.5 The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

Activity S.3.6 *(~15 min)* Find the dimension of each subspace of $\mathbb{R}^4$.

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} \qquad \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$$\operatorname{span}\left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Activity S.3.7 *(~5 min)* What is the dimension of the vector space of 7th-degree (or less) polynomials $\mathcal{P}^7$?

a) 6   b) 7   c) 8   d) infinite

Activity S.3.8 *(~5 min)* What is the dimension of the vector space of all polynomials $\mathcal{P}$?

a) 6   b) 7   c) 8   d) infinite

Observation S.3.9 Several interesting vector spaces are infinite-dimensional:

- The space of polynomials $\mathcal{P}$ (consider the set $\{1, x, x^2, x^3, \dots\}$).
- The space of continuous functions $C(\mathbb{R})$ (which contains all polynomials, in addition to other functions like $e^x = 1 + x + x^2/2 + x^3/3 + \ldots$).
- The space of real number sequences $\mathbb{R}^\infty$ (consider the set $\{(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, \dots), \dots\}$).

Fact S.3.10 Every vector space with finite dimension, that is, every vector space with a basis of the form $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is isomorphic to a Euclidean space $\mathbb{R}^n$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Module A: Algebraic properties of linear maps

Standards for this Module

How can we understand linear maps algebraically? At the end of this module, students will be able to...

A1. Linear maps and matrices. ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.

A2. Linear map verification. ... determine if a map between vector spaces of polynomials is linear or not.

A3. Injectivity and surjectivity. ... determine if a given linear map is injective and/or surjective.

A4. Kernel and Image. ... compute a basis for the kernel and a basis for the image of a linear map.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing **E1, E2, E3, E4**.
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace **V3**.
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of vectors is a basis **S2**.

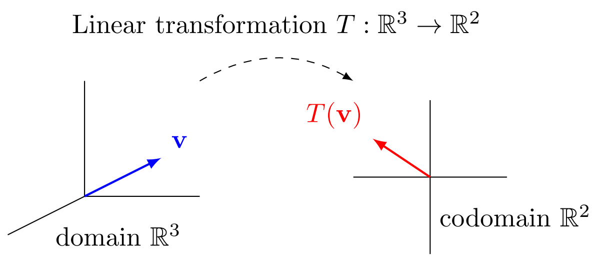
Section A.1

Definition A.1.1 A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}$, $\mathbf{v} \in V$.

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.

Definition A.1.2 Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.



Example A.1.3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - z_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - z_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx - cz \\ cy \end{bmatrix} \text{ and } cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - z \\ y \end{bmatrix} = \begin{bmatrix} cx - cz \\ cy \end{bmatrix}$$

Therefore T is a linear transformation.

Activity A.1.4 *(∼15 min)* Determine if each of the following maps are linear transformations
Part 1: $T_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \sqrt{x^2+y^2}$.
Part 2: $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$
Part 3: $T_3 : \mathcal{P}^d \to \mathcal{P}^{d-1}$ given by $T_3(f(x)) = f'(x)$.
Part 4: $T_4 : \mathcal{P} \to \mathcal{P}$ given by $T_4(f(x)) = f(x) + x^2$

Activity A.1.5 *(∼5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute $T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$.

(a) $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(d) $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

Activity A.1.6 *(∼3 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute $T\left(\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}\right)$.

(a) $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(d) $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

Activity A.1.7 *(∼5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.1.8 *(∼2 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}-2\\0\\-3\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.1.9 *(∼5 min)* Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation. How many facts of the form $T(\mathbf{v}_i) = \mathbf{w}_i$ do you need to know in order to be able to compute $T(\mathbf{v})$ for *any* $\mathbf{v} \in \mathbb{R}^4$?

(a) 2

(b) 3

(c) 4

(d) 5

(e) You need infinitely many

(In this situation, we say that the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ **determine** T.)

Fact A.1.10 Consider any basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for V. Since every vector can be written *uniquely* as a linear combination of basis vectors, every linear transformation $T : V \to W$ is determined by those basis vectors.

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = x_1T(\mathbf{b}_1) + \cdots + x_nT(\mathbf{b}_n)$$

Definition A.1.11 The **standard basis** of $\mathbb{R}^n$ is the (ordered) basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Since linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by the values of each $T(\mathbf{e}_i)$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$.

Example A.1.12 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix}.$$

Activity A.1.13 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard basis.

Activity A.1.14 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Activity A.1.15 *(~10 min)* Let $D : \mathcal{P}^3 \to \mathcal{P}^2$ be the derivative map $D(f(x)) = f'(x)$. (Earlier we showed this is a linear transformation.)
Part 1: Write down an equivalent linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ by converting $\{1, x, x^2, x^3\}$ and $\{D(1), D(x), D(x^2), D(x^3)\}$ into appropriate vectors in $\mathbb{R}^4$ and $\mathbb{R}^3$.
Part 2: Write the standard matrix corresponding to T.

Section A.2

Definition A.2.1 Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\mathbf{v}) \neq T(\mathbf{w})$ whenever $\mathbf{v} \neq \mathbf{w}$.

Activity A.2.2 *(∼5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is T injective?

Activity A.2.3 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Is T injective?

Definition A.2.4 Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.

Activity A.2.5 *(∼5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is T surjective?

Activity A.2.6 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
Is T surjective?

Definition A.2.7 Let $T : V \to W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{\mathbf{v} \in V \;\middle|\; T(\mathbf{v}) = \mathbf{0}\right\}$$

Activity A.2.8 *(∼5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the kernel of T.

Activity A.2.9 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the kernel of T.

Activity A.2.10 *(∼10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.
Part 1: Write a system of equations whose solution set is the kernel.
Part 2: Use $\text{RREF}(A)$ to solve the system of equations and find the kernel of T.
Part 3: Find a basis for the kernel of T.

Definition A.2.11 Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{\mathbf{w} \in W \;\middle|\; \text{there is some } v \in V \text{ with } T(\mathbf{v}) = \mathbf{w}\right\}$$

Activity A.2.12 *(∼5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the image of T.

Activity A.2.13 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the image of T.

Activity A.2.14 *(∼10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.
Part 1: Find a convenient set of vectors $S \subseteq \mathbb{R}^2$ such that $\operatorname{span} S = \operatorname{Im} T$.
Part 2: Find a convenient basis for the image of T.

Observation A.2.15 Let $T : V \to W$ be a linear transformation with corresponding matrix A.

- If A is a matrix corresponding to T, the kernel is the solution set of the homogeneous system with coefficients given by A.
- If A is a matrix corresponding to T, the image is the span of the columns of A.

Section A.3

Observation A.3.1 Let $T : V \to W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.
- The **kernel** of T is the set of all things that are mapped to $\mathbf{0}$. It is a subspace of V.
- The **image** of T is the set of all things in W that are mapped to by something in V. It is a subspace of W.

Activity A.3.2 *(∼5 min)* Let $T : V \to W$ be a linear transformation where $\ker T = \{\mathbf{0}\}$. Can you answer either of the following questions about T?

(a) Is T injective?

(b) Is T surjective?

(Hint: If $T(\mathbf{v}) = T(\mathbf{w})$, then what is $T(\mathbf{v} - \mathbf{w})$?)

Fact A.3.3 A linear transformation T is injective **if and only if** $\ker T = \{\mathbf{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

Activity A.3.4 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation where $\operatorname{Im} T = \operatorname{span}\left\{\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\-1\end{bmatrix}\right\}$. Can you answer either of the following questions about T?

(a) Is T injective?

(b) Is T surjective?

Fact A.3.5 A linear transformation $T : V \to W$ is surjective **if and only if** $\operatorname{Im} T = W$. Put another way, a surjective linear transformation may be recognized by its same codomain and image.

Activity A.3.6 *(∼15 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of equivalent statements.

(a) T is injective

(b) T is surjective

(c) The kernel of T is trivial.

(d) The columns of A span $\mathbb{R}^m$

(e) The columns of A are linearly independent

(f) Every column of RREF(A) has a pivot.

(g) Every row of RREF(A) has a pivot.

(h) The image of T equals its codomain.

(i) The system of linear equations given by the augmented matrix $\left[A \mid \mathbf{b}\right]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$

(j) The system of linear equations given by the augmented matrix $\left[A \mid \mathbf{0}\right]$ has exactly one solution.

(Instructor Note:) This activity may be ran as a card sort.

Definition A.3.7 If $T : V \to W$ is both injective and surjective, it is called **bijective**.

Activity A.3.8 *(∼5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

(a) The columns of A form a basis for $\mathbb{R}^m$

(b) RREF(A) is the identity matrix.

(c) The system of linear equations given by the augmented matrix $\left[A \mid \mathbf{b}\right]$ has exactly one solution for all $\mathbf{b} \in \mathbb{R}^m$.

Activity A.3.9 *(∼10 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.3.10 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.3.11 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.3.12 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Module M: Understanding Matrices Algebraically

Standards for this Module

What algebraic structure do matrices have? At the end of this module, students will be able to...

M1. Matrix Multiplication. ... multiply matrices.

M2. Invertible Matrices. ... determine if a square matrix is invertible or not.

M3. Matrix inverses. ... compute the inverse matrix of an invertible matrix.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers
- Solve systems of linear equations **E3**
- Find the matrix corresponding to a linear transformation **A1**
- Determine if a linear transformation is injective and/or surjective **A3**
- Interpret the ideas of injectivity and surjectivity in multiple ways

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Function composition (Khan Academy): `http://bit.ly/2wkz7f3`

Section M.1

Activity M.1.1 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity M.1.2 *(~2 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity M.1.3 *(~2 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by thestandard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

The standard matrix of $S \circ T$ will lie in which matrix space?

(a) 4×3 matrices

(b) 4×2 matrices

(c) 3×2 matrices

(d) 2×3 matrices

(e) 2×4 matrices

(f) 3×4 matrices

Activity M.1.4 *(~15 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute $(S \circ T)(\mathbf{e}_1)$
Part 2: Compute $(S \circ T)(\mathbf{e}_2)$
Part 3: Compute $(S \circ T)(\mathbf{e}_3)$.
Part 4: Find the standard matrix of $S \circ T$.

Activity M.1.5 *(~2 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity M.1.6 *(~2 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.
What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity M.1.7 *(~2 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.
The standard matrix of $S \circ T$ will lie in which matrix space?

(a) 2×2 matrices

(b) 2×3 matrices

(c) 3×2 matrices

(d) 3×3 matrices

Activity M.1.8 *(~10 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.
Find the standard matrix of $S \circ T$.

Activity M.1.9 *(~5 min)* Let $T : \mathbb{R}^1 \to \mathbb{R}^4$ be given by the matrix $B = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ and $S : \mathbb{R}^4 \to \mathbb{R}^1$ be given by the matrix $A = \begin{bmatrix} 2 & 3 & 2 & 5 \end{bmatrix}$.
Find the standard matrix of $S \circ T$.

Definition M.1.10 We define the product of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix (denoted AB) of the composition map of the two corresponding linear functions.

Fact M.1.11 If AB is defined, BA need not be defined, and if it is defined, it is in general different from AB.

Activity M.1.12 *(~10 min)* Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Compute AB.

Activity M.1.13 *(~5 min)* Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Compute AX

Observation M.1.14 Consider the system of equations

$$\begin{aligned} 3x + y - z &= 5 \\ 2x + 4z &= -7 \\ -x + 3y + 5z &= 2 \end{aligned}$$

We can interpret this as a **matrix equation** $AX = B$ where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$$

For this reason, we will swap out the use of Euclidean vectors $\mathbf{x} \in \mathbb{R}^n$ and $n \times 1$ matrices X whenever it is convenient.

Section M.2

Activity M.2.1 *(∼5 min)* Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix I such that $IA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Definition M.2.2 The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

Fact M.2.3 For any square matrix A, $IA = AI = A$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Activity M.2.4 *(∼15 min)* Each row operation can be interpreted as a type of matrix multiplication.
Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5 & 7+5 & -1-5 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Fact M.2.5 If R is the result of applying a row operation to I, then RA is the result of applying the same row operation to A.
This means that for any matrix A, we can find a series of matrices $R_1, \ldots, R_k$ corresponding to the row operations such that

$$R_1 R_2 \cdots R_k A = \mathrm{RREF}(A).$$

That is, row reduction can be thought of as the result of matrix multiplication.

Activity M.2.6 *(∼15 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into groups of statements about T.

(a) T is injective (i.e. one-to-one)

(b) T is surjective (i.e. onto)

(c) T is bijective (i.e. both injective and surjective)

(d) $AX = B$ has a solution for all $m \times 1$ matrices B

(e) $AX = B$ has a unique solution for all $m \times 1$ matrices B

(f) $AX = 0$ has a unique solution.

(g) The columns of A span $\mathbb{R}^m$

(h) The columns of A are linearly independent

(i) The columns of A are a basis of $\mathbb{R}^m$

(j) Every column of $\mathrm{RREF}(A)$ has a pivot

(k) Every row of $\mathrm{RREF}(A)$ has a pivot

(l) $m = n$ and $\mathrm{RREF}(A) = I$

Activity M.2.7 *(∼5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix A. If T is injective, which of the following cannot be true?

(a) A has strictly more columns than rows

(b) A has the same number of rows as columns (i.e. A is square)

(c) A has strictly more rows than columns

Activity M.2.8 *(∼5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix A. If T is surjective, which of the following cannot be true?

(a) A has strictly more columns than rows

(b) A has the same number of rows as columns (i.e. A is square)

(c) A has strictly more rows than columns

Activity M.2.9 *(∼5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix A. If T is bijective, which of the following cannot be true?

(a) A has strictly more columns than rows

(b) A has the same number of rows as columns (i.e. A is square)

(c) A has strictly more rows than columns

Section M.3

Definition M.3.1 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with standard matrix A.

- If T is a bijection and B is any $\mathbb{R}^n$ vector, then $T(X) = AX = B$ has a unique solution X.
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by setting $T^{-1}(B) = X$ to be this unique solution.
- Let A^{-1} be the standard matrix for T^{-1}. We call A^{-1} the **inverse matrix** of A, so we also say that A is **invertible**.

Activity M.3.2 *(∼10 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$. It can be shown that T is bijective and has the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

Part 2: If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, what must $A^{-1}A$ be?

Observation M.3.3 $T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation T. Therefore $A^{-1}A = AA^{-1} = I$ is the identity matrix for any invertible matrix A.

Activity M.3.4 *(∼20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.
Part 2: Solve $T(X) = \mathbf{e}_2$ to find $T^{-1}(\mathbf{e}_2)$.
Part 3: Solve $T(X) = \mathbf{e}_3$ to find $T^{-1}(\mathbf{e}_3)$.
Part 4: Compute A^{-1}, the standard matrix for T^{-1}.

Observation M.3.5 We could have solved these three systems simultaneously by row reducing the matrix $[A \,|\, I]$ at once.

$$A = \left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right]$$

Activity M.3.6 *(∼10 min)* Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A \,|\, I]$.

Activity M.3.7 *(∼10 min)* Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

Observation M.3.8 A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $\mathrm{RREF}(A) = I_n$.

Module G: Geometry of Linear Maps

Standards for this Module

How can we understand linear maps geometrically? At the end of this module, students will be able to...

G1. Row operations. ... represent a row operation as matrix multiplication, and compute how the operation affects the determinant.

G2. Determinants. ... compute the determinant of a square matrix.

G3. Eigenvalues. ... find the eigenvalues of a 2×2 matrix.

G4. Eigenvectors. ... find a basis for the eigenspace of a square matrix associated with a given eigenvalue.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **A1**.
- Recall and use the definition of a linear transformation **A2**.
- Find all roots of quadratic polynomials (including complex ones), and be able to use the rational root theorem to find all rational roots of a higher degree polynomial.
- Interpret the statement "A is an invertible matrix" in many equivalent ways in different contexts.

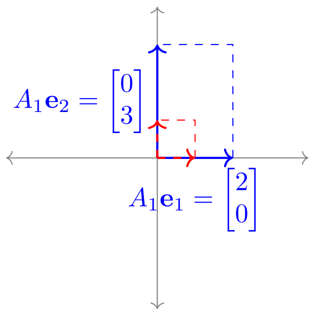
Readiness Assurance Resources

The following resources will help you prepare for this module.

- Finding the area of a parallelogram (Khan Academy): `http://bit.ly/2BO5iWx`
- Factoring quadratics (Khan Academy): `http://bit.ly/1XjfbV2`
- Finding complex roots of quadratics (Khan Academy): `http://bit.ly/1HH3yAA`

Section G.1

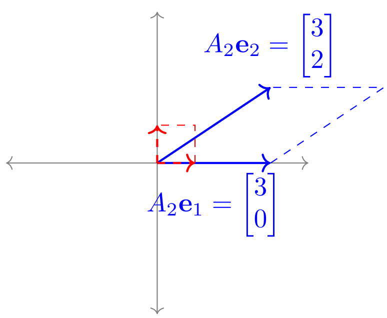
Activity G.1.1 *(∼5 min)* The image below illustrates how the linear transformation $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.



(a) What is the area of the transformed unit square?

(b) Find two vectors that were stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.

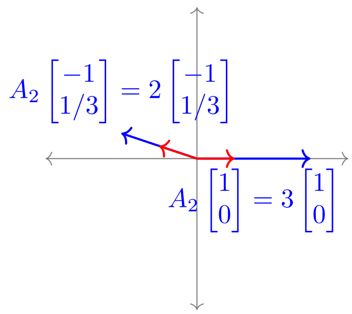
Activity G.1.2 *(∼5 min)* The image below illustrates how the linear transformation $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A_2 = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$. transforms the unit square.



(a) What is the area of the transformed unit square?

(b) Find at least one vector that was stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.

Observation G.1.3 It's possible to find two non-parallel vectors that are stretched by the transformation given by A_2:



The process for finding such vectors will be covered later in this module.

Activity G.1.4 *(~5 min)* Consider the linear transformation given by the standard matrix $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(a) Sketch the transformation of the unit square (the parallelogram given by the columns of the standard matrix).

(b) Compute the area of the transformed unit square.

Activity G.1.5 *(~5 min)* Consider the linear transformation given by the standard matrix $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Sketch the transformation of the unit square.

(b) Compute the area of the transformed unit square.

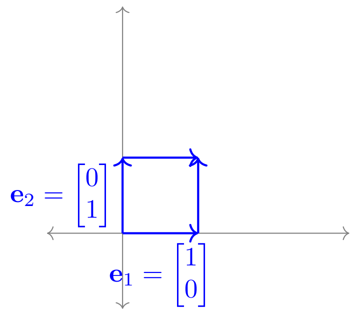
Activity G.1.6 *(~5 min)* Consider the linear transformation given by the standard matrix $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

(a) Sketch the transformation of the unit square.

(b) Compute the area of the transformed unit square.

Remark G.1.7 The area of the transformed unit square measures the factor by which all areas are transformed by a linear transformation.
We will define the **determinant** of a square matrix A, or $\det(A)$ for short, to be this factor. But what properties must this function satisfy?

Activity G.1.8 *(∼2 min)* The transformation of the unit square by the standard matrix $[\mathbf{e}_1\ \ \mathbf{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. What is $\det([\mathbf{e}_1\ \ \mathbf{e}_2]) = \det(I)$, that is, by what factor has the area of the unit square been scaled?



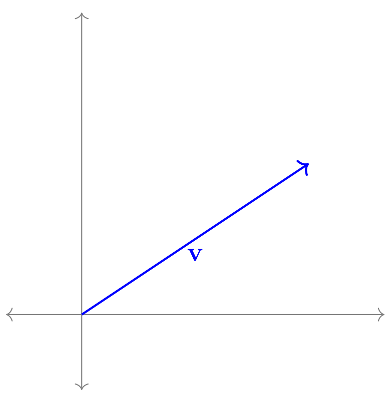
a) 0

b) 1

c) 2

d) Cannot be determined

Activity G.1.9 *(∼2 min)* The transformation of the unit square by the standard matrix $[\mathbf{v}\ \ \mathbf{v}]$ is illustrated below: both $T(\mathbf{e}_1) = T(\mathbf{e}_2) = \mathbf{v}$. What is $\det([\mathbf{v}\ \ \mathbf{v}])$, that is, by what factor has area been scaled?



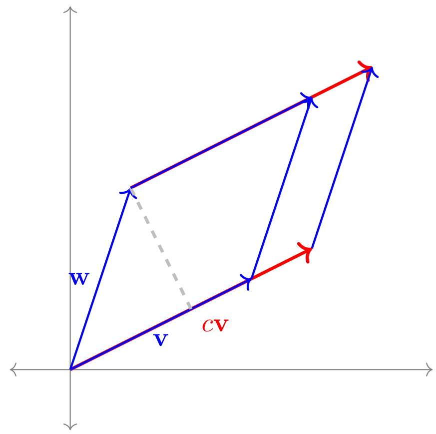
a) 0

b) 1

c) 2

d) Cannot be determined

Activity G.1.10 *(∼5 min)* The transformations of the unit square by the standard matrices $[\mathbf{v}\ \ \mathbf{w}]$ and $[c\mathbf{v}\ \ \mathbf{w}]$ are illustrated below. How are $\det([\mathbf{v}\ \ \mathbf{w}])$ and $\det([c\mathbf{v}\ \ \mathbf{w}])$ related?

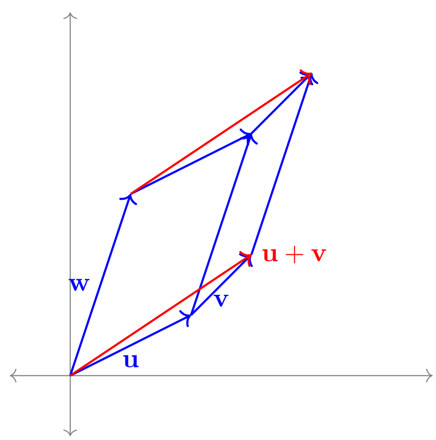


a) $\det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

b) $c + \det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

c) $c\det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

Activity G.1.11 *(∼5 min)* The transformations of unit squares by the standard matrices $[\mathbf{u}\ \ \mathbf{w}]$, $[\mathbf{v}\ \ \mathbf{w}]$ and $[\mathbf{u}+\mathbf{v}\ \ \mathbf{w}]$ are illustrated below. How is $\det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$ related to $\det([\mathbf{u}\ \ \mathbf{w}])$ and $\det([\mathbf{v}\ \ \mathbf{w}])$?



a) $\det([\mathbf{u}\ \ \mathbf{w}]) = \det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

b) $\det([\mathbf{u}\ \ \mathbf{w}]) + \det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

c) $\det([\mathbf{u}\ \ \mathbf{w}])\det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

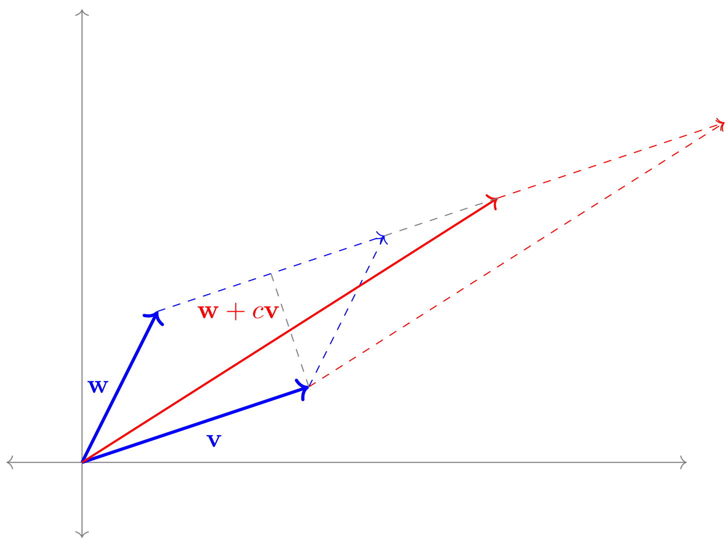
Definition G.1.12 The **determinant** is the unique function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfying the following three properties:

P1: $\det(I) = 1$

P2: $\det([\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \cdots \ \ \mathbf{v}_n]) = 0$ whenever two columns of the matrix are identical.

P3: $\det[\cdots \ \ c\mathbf{v} + d\mathbf{w} \ \ \cdots] = c\det[\cdots \ \ \mathbf{v} \ \ \cdots] + d\det[\cdots \ \ \mathbf{w} \ \ \cdots]$, assuming all other columns are equal.

Observation G.1.13 What happens if we had a multiple of one column to another?



The base of both parallelograms is $\mathbf{v}$, while the height has not changed. Thus

$$\det([\mathbf{v} \quad \mathbf{w} + c\mathbf{v}]) = \det([\mathbf{v} \quad \mathbf{w}])$$

Observation G.1.14 Swapping columns can be obtained from a sequence of adding column multiples.

$$\begin{aligned}
\det([\mathbf{v} \quad \mathbf{w}]) &= \det([\mathbf{v}+\mathbf{w} \quad \mathbf{w}]) \\
&= \det([\mathbf{v}+\mathbf{w} \quad \mathbf{w} - (\mathbf{v}+\mathbf{w})]) \\
&= \det([\mathbf{v}+\mathbf{w} \quad -\mathbf{v}]) \\
&= \det([\mathbf{v}+\mathbf{w}-\mathbf{v} \quad -\mathbf{v}]) \\
&= \det([\mathbf{w} \quad -\mathbf{v}]) \\
&= -\det([\mathbf{w} \quad \mathbf{v}])
\end{aligned}$$

So swapping two columns results in a negation of the determinant. Therefore, determinants represent a *signed* area, since they are not always positive.

Fact G.1.15 We've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c\det([\cdots \ \mathbf{v} \ \cdots]) = \det([\cdots \ c\mathbf{v} \ \cdots])$$

(b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \ \mathbf{v} \ \cdots \ \ \mathbf{w} \ \cdots]) = -\det([\cdots \ \mathbf{w} \ \cdots \ \ \mathbf{v} \ \cdots])$$

(c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \ \mathbf{v} \ \cdots \ \ \mathbf{w} \ \cdots]) = \det([\cdots \ \mathbf{v}+c\mathbf{w} \ \cdots \ \ \mathbf{w} \ \cdots])$$

Activity G.1.16 *(∼5 min)* The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. How must the transformation given by the standard matrix AB scale areas?

(a) 1

(b) 7

(c) 12

(d) Cannot be determined

Fact G.1.17 Since the transformation given by the standard matrix AB is obtained by applying the transformations given by A and B, it follows that

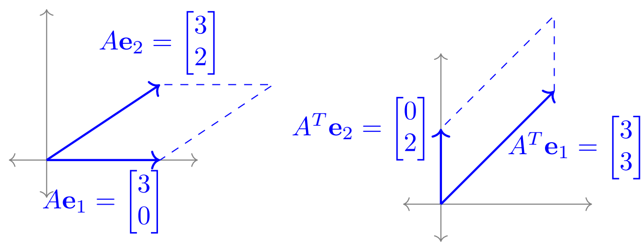
$$\det(AB) = \det(A)\det(B)$$

Section G.2

Definition G.2.1 The **transpose** of a matrix is given by rewriting its columns as rows and vice versa:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Fact G.2.2 It is possible to prove that the determinant of a matrix and its transpose are the same. For example, let $A = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$, so $A^T = \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}$; both matrices scale the unit square by 6, even though the parallelograms are not congruent.



Fact G.2.3 We previously figured out that column operations can be used to simplify determinants; since $\det(A) = \det(A^T)$, we can also use row operations:

1. Multiplying rows by scalars: $\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$
2. Swapping two rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$
3. Adding multiples of rows to other rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$

Activity G.2.4 *(∼10 min)* Compute the determinant of $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ by row reducing it to a nicer matrix.

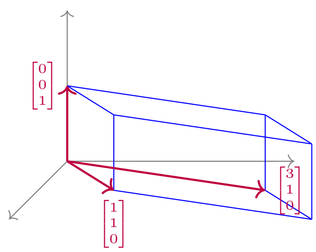
For example, $\det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

Fact G.2.5 This same process allows us to prove a more convenient formula:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

In higher dimensions, the formulas become unreasonable. For example, the formula for 4×4 matrices has 24 terms!

Activity G.2.6 *(∼5 min)* The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.



This volume is equal to which of the following areas?

(a) $\det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ (c) $\det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$ (d) $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Fact G.2.7 If column i of a matrix is $\mathbf{e}_i$, then both column and row i may be removed without changing the value of the determinant. For example, the second column of the following matrix is $\mathbf{e}_2$, so:

$$\det \begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Therefore the same holds for the transpose:

$$\det \begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Geometrically, this is the fact that if the height is 1, the base $\times$ height formula reduces to the area/volume/etc. of the $n-1$ dimensional base.

Activity G.2.8 *(∼5 min)* Compute $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$.

Activity G.2.9 *(∼5 min)* Compute $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$.

(a) -1

(b) 0

(c) 1

Activity G.2.10 *(∼10 min)* Compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix}$

Hint: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(a) 3

(b) 6

(c) 9

(d) 12

Activity G.2.11 *(∼15 min)* Compute $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$.

Observation G.2.12

$$\det\begin{bmatrix}2&3&5&0\\0&3&2&0\\1&2&0&3\\-1&-1&2&2\end{bmatrix}=(-1)(0)\det\begin{bmatrix}2&3&5&0\\0&3&2&0\\1&2&0&3\\-1&-1&2&2\end{bmatrix}+(1)(3)\det\begin{bmatrix}2&3&5&0\\0&3&2&0\\1&2&0&3\\-1&-1&2&2\end{bmatrix}+$$

$$(-1)(2)\det\begin{bmatrix}2&3&5&0\\0&3&2&0\\1&2&0&3\\-1&-1&2&2\end{bmatrix}+(1)(0)\det\begin{bmatrix}2&3&5&0\\0&3&2&0\\1&2&0&3\\-1&-1&2&2\end{bmatrix}$$

$$=3\det\begin{bmatrix}2&5&0\\1&0&3\\-1&2&2\end{bmatrix}+(-1)(2)\det\begin{bmatrix}2&3&0\\1&2&3\\-1&-1&2\end{bmatrix}$$

This technique is called **Laplace expansion** or **cofactor expansion**.

Activity G.2.13 *(∼10 min)* Compute $\det\begin{bmatrix}1&2&1&0\\0&3&2&-1\\1&2&0&3\\-1&-3&2&-2\end{bmatrix}$.

Section G.3

Activity G.3.1 *(∼5 min)* An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Compute $\det(M)$ and $\det(M^{-1})$.

Activity G.3.2 *(∼5 min)* Suppose the matrix M is invertible, so there exists M^{-1} with $MM^{-1} = I$. It follows that $\det(M)\det(M^{-1}) = \det(I)$.

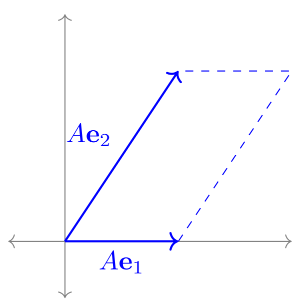
What is the only number that $\det(M)$ cannot equal?

(a) -1 (b) 0 (c) 1 (d) $\frac{1}{\det(M^{-1})}$

Fact G.3.3

- For every invertible matrix M, $\det(M^{-1}) = \frac{1}{\det(M)}$.
- Furthermore, a square matrix M is invertible if and only if $\det(M) \neq 0$.

Observation G.3.4 Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



It is easy to see geometrically that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily verified by computation) that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Definition G.3.5 Let $A \in \mathbb{R}^{n\times n}$. An **eigenvector** is a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x}$ is parallel to $\mathbf{x}$.
In other words, $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.

We call this λ an **eigenvalue** of A.

Observation G.3.6 Since $\lambda\mathbf{x} = \lambda(I\mathbf{x})$, we can find the eigenvalues and eigenvectors satisfying $A\mathbf{x} = \lambda\mathbf{x}$ by inspecting $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

- Since we already know that $(A - \lambda I)\mathbf{0} = \mathbf{0}$ for any value of λ, we are more interested in finding values of λ such that $A - \lambda I$ has a nontrivial kernel.
- Thus $\mathrm{RREF}(A - \lambda I)$ must have a non-pivot column, and therefore $A - \lambda I$ cannot be invertible.
- Since $A - \lambda I$ cannot be invertible, our eigenvalues must satisfy $\det(A - \lambda I) = 0$.

Definition G.3.7 Computing $\det(A - \lambda I)$ results in the **characteristic polynomial** of A.

For example, when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

Thus the characteristic polynomial of A is

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2$$

Activity G.3.8 *(~15 min)* Compute $\det(A - \lambda I)$ to find the characteristic polynomial of $A = \begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$.

Activity G.3.9 *(~15 min)* Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.
Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.
Part 2: Find the roots of the characteristic polynomial to determine the eigenvalues of A.
Part 3: Compute the kernel of the transformation given by

$$A - 2I = \begin{bmatrix} 2-2 & 2 \\ 0 & 3-2 \end{bmatrix}$$

to determine all the eigenvectors associated to the eigenvalue 2.
Part 4: Compute the kernel of the transformation given by $A - 3I$ to determine all the eigenvectors associated to the eigenvalue 3.

Definition G.3.10 The kernel of the transformation given by $A-\lambda I$ contains all the eigenvectors associated with λ. Since kernel is a subspace of $\mathbb{R}^n$, we call this kernel the **eigenspace** associated with the eigenvalue λ.

Activity G.3.11 *(~15 min)* Find all the eigenvalues and associated eigenspaces for the matrix $A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}$.
Part 1: Compute $\det(A-\lambda I)$ to determine the characteristic polynomial of A.
Part 2: Find the roots of the characteristic polynomial $(3-\lambda)(\lambda^2-4\lambda-12)$ to determine the eigenvalues of A.
Part 3: Compute the kernels of $A-\lambda I$ for each eigenvalue $\lambda \in \{-2, 3, 6\}$ to determine the respective eigenspaces.

Observation G.3.12 Recall that if a is a root of the polynomial $p(\lambda)$, the **multiplicity** of a is the largest number k such that $p(\lambda) = q(\lambda)(\lambda-a)^k$ for some polynomial $q(\lambda)$.

For this reason, the **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Example G.3.13 If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & 3 \\ 0 & 0 & -1 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda-3)^2(\lambda+1)$.

The eigenvalues are 3 (with algebraic multiplicity 2) and -1 (with algebraic multiplicity 1).

Section G.4

Observation G.4.1 Recall from last class:

- To find the eigenvalues of a matrix A, we need to find values of λ such that $A - \lambda I$ has a nontrivial kernel. Equivalently, we want values where $A - \lambda I$ is not invertible, so we want to know the values of λ where $\det(A - \lambda I) = 0$.
- $\det(A - \lambda I)$ is a polynomial with variable λ, called the **characteristic polynomial** of A. Thus the roots of the characteristic polynomial of A are exactly the eigenvalues of A.
- Once an eigenvalue λ is found, the **eigenspace** containing all **eigenvectors** $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is given by $\ker(A - \lambda I)$.

Activity G.4.2 *(∼5 min)* Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
Part 1: Compute the eigenvalues of A.
Part 2: Sketch a picture of the transformation of the unit square. What about this picture reveals that A has no real eigenvectors?

Activity G.4.3 *(∼5 min)* If A is a 4×4 matrix, what is the largest number of eigenvalues A can have?

(a) 3

(b) 4

(c) 5

(d) 6

(e) It can have infinitely many

Observation G.4.4 An $n \times n$ matrix may have between 0 and n real-valued eigenvalues. But the Fundamental Theorem of Algebra implies that if complex eigenvalues are included, then every $n \times n$ matrix has exactly n eigenvalues (counting algebraic multiplicites).

Activity G.4.5 *(∼5 min)* The matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$ has characteristic polynomial $-\lambda(\lambda - 2)^2$.

Find the dimension of the eigenspace of A associated to the eigenvalue 2 (the dimension of the kernel of $A - 2I$).

Activity G.4.6 *(∼5 min)* The matrix $B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}$ has characteristic polynomial $-\lambda(\lambda - 2)^2$.

Find the dimension of the eigenspace of B associated to the eigenvalue 2 (the dimension of the kernel of $B - 2I$).

Observation G.4.7 In the first example, the (2 dimensional) plane spanned by $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}4\\-2\\0\end{bmatrix}$ was preserved. In the second example, only the (one dimensional) line spanned by $\begin{bmatrix}1\\0\\1\end{bmatrix}$ is preserved.

Definition G.4.8 While the **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial, the **geometric multiplicity** of an eigenvalue is the dimension of its eigenspace.

Fact G.4.9 As we've seen, the geometric multiplicity may be different than its algebraic multiplicity, but it cannot exceed it.
This fact is explored deeper and explained in Math 316, Linear Algebra II

Activity G.4.10 *(∼20 min)* Consider the 4×4 matrix

$$\begin{bmatrix}-3 & 1 & 2 & 1\\-9 & 5 & -2 & -1\\31 & -17 & 6 & 3\\-69 & 39 & -18 & -9\end{bmatrix}$$

Part 1: Use technology (e.g. Wolfram Alpha) to find its characteristic polynomial.
Part 2: Find the algebraic and geometric multiplicities for both eigenvalues.