

## Application Activities - Module A Part 1 - Class Day 17

**Definition 17.1** A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if  $V$  and  $W$  are vector spaces, a map  $T : V \rightarrow W$  is called a linear transformation if

1.  $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$  for any  $\mathbf{v}, \mathbf{w} \in V$
2.  $T(c\mathbf{v}) = cT(\mathbf{v})$  for any  $c \in \mathbb{R}, \mathbf{v} \in V$ .

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.

**Definition 17.2** Given a linear transformation  $T : V \rightarrow W$ ,  $V$  is called the **domain** of  $T$  and  $W$  is called the **co-domain** of  $T$ .



**Example 17.3** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$$

To show that  $T$  is linear, we must verify...

$$\begin{aligned} T \left( \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) &= T \left( \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \right) = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix} \\ T \left( \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right) + T \left( \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) &= \begin{bmatrix} x_1 - z_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - z_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix} \end{aligned}$$

And also...

$$T \left( c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T \left( \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \right) = \begin{bmatrix} cx - cz \\ cy \end{bmatrix} \text{ and } cT \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = c \begin{bmatrix} x - z \\ y \end{bmatrix} = \begin{bmatrix} cx - cz \\ cy \end{bmatrix}$$

Therefore  $T$  is a linear transformation.

**Activity 17.4** Determine if each of the following maps are linear transformations

*Part 1:*  $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $T_1 \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \sqrt{x^2 + y^2}$ .

Part 2:  $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T_2 \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$

Part 3:  $T_3 : \mathcal{P}^d \rightarrow \mathcal{P}^{d-1}$  given by  $T_3(f(x)) = f'(x)$ .

Part 4:  $T_4 : \mathcal{P} \rightarrow \mathcal{P}$  given by  $T_4(f(x)) = f(x) + x^2$

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**Activity 17.5** Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear transformation, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and

$T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

(c)  $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(b)  $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(d)  $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$

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**Activity 17.6** Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear transformation, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and

$T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

(c)  $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(b)  $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(d)  $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$

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**Activity 17.7** Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear transformation, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and

$T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c)  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(b)  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(d)  $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

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**Activity 17.8** Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear transformation, and you know  $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T\left(\begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix}\right)$ .

(a)  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c)  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(b)  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(d)  $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

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**Activity 17.9** Suppose  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  is a linear transformation. How many facts of the form  $T(\mathbf{v}_i) = \mathbf{w}_i$  do you need to know in order to be able to compute  $T(\mathbf{v})$  for *any*  $\mathbf{v} \in \mathbb{R}^4$ ?

(a) 2

(b) 3

(c) 4

(d) 5

(e) You need infinitely many

(In this situation, we say that the vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  **determine**  $T$ .)

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**Fact 17.10** Consider any basis  $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  for  $V$ . Since every vector can be written *uniquely* as a linear combination of basis vectors, every linear transformation  $T : V \rightarrow W$  is determined by those basis vectors.

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \dots x_n\mathbf{b}_n) = x_1T(\mathbf{b}_1) + \dots + x_nT(\mathbf{b}_n)$$

**Definition 17.11** The **standard basis** of  $\mathbb{R}^n$  is the (ordered) basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \dots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Since linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is determined by the values of each  $T(\mathbf{e}_i)$ , it's convenient to store this information in the  $m \times n$  **standard matrix**  $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ .

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**Example 17.12** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation determined by the following values for  $T$  applied to the standard basis of  $\mathbb{R}^3$ .

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Then the standard matrix corresponding to  $T$  is

$$\begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix}.$$

**Activity 17.13** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard basis.

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**Activity 17.14** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \end{bmatrix}.$$

Compute  $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ .

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**Activity 17.15** Let  $D : \mathcal{P}^3 \rightarrow \mathcal{P}^2$  be the derivative map  $D(f(x)) = f'(x)$ . (Earlier we showed this is a linear transformation.)

*Part 1:* Write down an equivalent linear transformation  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  by converting  $\{1, x, x^2, x^3\}$  and  $\{D(1), D(x), D(x^2), D(x^3)\}$  into appropriate vectors in  $\mathbb{R}^4$  and  $\mathbb{R}^3$ .

*Part 2:* Write the standard matrix corresponding to  $T$ .

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