

Section V.0

Observation V.1 Several properties of the real numbers, such as commutativity:

$$x + y = y + x$$

also hold for Euclidean vectors with multiple components:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Activity V.2 (~ 20 min) Consider each of the following properties of the real numbers \mathbb{R}^1 . Label each property as **valid** if the property also holds for two-dimensional Euclidean vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ and scalars $a, b \in \mathbb{R}$, and **invalid** if it does not.

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|---|---|
| 1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$. | 6. $a(b\vec{v}) = (ab)\vec{v}$. |
| 2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$. | 7. $1\vec{v} = \vec{v}$. |
| 3. There exists some \vec{z} where $\vec{v} + \vec{z} = \vec{v}$. | 8. If $\vec{u} \neq \vec{0}$, then there exists some scalar c such that $c\vec{u} = \vec{v}$. |
| 4. There exists some $-\vec{v}$ where $\vec{v} + (-\vec{v}) = \vec{z}$. | 9. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$. |
| 5. If $\vec{u} \neq \vec{v}$, then $\frac{1}{2}(\vec{u} + \vec{v})$ is the only vector equally distant from both \vec{u} and \vec{v} . | 10. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$. |

Definition V.3 A **vector space** V is any collection of mathematical objects with associated addition \oplus and scalar multiplication \odot operations that satisfy the following properties. Let $\vec{u}, \vec{v}, \vec{w}$ belong to V , and let a, b be scalar numbers.

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| • Addition is associative: $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$. | • Scalar multiplication is associative: $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$. |
| • Addition is commutative: $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$. | • Scalar multiplication identity exists: $1 \odot \vec{v} = \vec{v}$. |
| • Additive identity exists: There exists some \vec{z} where $\vec{v} \oplus \vec{z} = \vec{v}$. | • Scalar mult. distributes over vector addition: $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$. |
| • Additive inverses exist: There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$. | • Scalar mult. distributes over scalar addition: $(a + b) \odot \vec{v} = a\vec{v} \oplus b\vec{v}$. |

Observation V.4

Every **Euclidean vector space**

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mid x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$$

satisfies all eight requirements for the usual definitions of addition and scalar multiplication, but we will also study other types of vector spaces.

Observation V.5 The space of $m \times n$ **matrices**

$$M_{m,n} = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \mid a_{11}, \dots, a_{mn} \in \mathbb{R} \right\}$$

satisfies all eight requirements for component-wise addition and scalar multiplication.