

Application Activities - Module A Part 3 - Class Day 19

Observation 19.1 Let $T : V \rightarrow W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V .
- The **kernel** of T is the set of all things that are mapped to $\mathbf{0}$. It is a subspace of V .
- The **image** of T is the set of all things in W that are mapped to by something in V . It is a subspace of W .

Activity 19.2 Let $T : V \rightarrow W$ be a linear transformation where $\ker T = \{\mathbf{0}\}$. Can you answer either of the following questions about T ?

(a) Is T injective?

(b) Is T surjective?

(Hint: If $T(\mathbf{v}) = T(\mathbf{w})$, then what is $T(\mathbf{v} - \mathbf{w})$?)

Fact 19.3 A linear transformation T is injective **if and only if** $\ker T = \{\mathbf{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

Activity 19.4 Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation where $\text{Im } T = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$. Can you answer either of the following questions about T ?

(a) Is T injective?

(b) Is T surjective?

Fact 19.5 A linear transformation $T : V \rightarrow W$ is surjective **if and only if** $\text{Im } T = W$. Put another way, a surjective linear transformation may be recognized by its same codomain and image.

Activity 19.6 Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with standard matrix A . Sort the following claims into two groups of equivalent statements.

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|---------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------|
| (a) T is injective | (g) Every row of $\text{RREF}(A)$ has a pivot. |
| (b) T is surjective | (h) The image of T equals its codomain. |
| (c) The kernel of T is trivial. | (i) The system of linear equations given by the augmented matrix $[A \mid \mathbf{b}]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$ |
| (d) The columns of A span \mathbb{R}^m | (j) The system of linear equations given by the augmented matrix $[A \mid \mathbf{0}]$ has exactly one solution. |
| (e) The columns of A are linearly independent | |
| (f) Every column of $\text{RREF}(A)$ has a pivot. | |
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Definition 19.7 If $T : V \rightarrow W$ is both injective and surjective, it is called **bijjective**.

Activity 19.8 Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a bijective linear map with standard matrix A . Label each of the following as true or false.

- (a) The columns of A form a basis for \mathbb{R}^m
 - (b) $\text{RREF}(A)$ is the identity matrix.
 - (c) The system of linear equations given by the augmented matrix $[A \mid \mathbf{b}]$ has exactly one solution for all $\mathbf{b} \in \mathbb{R}^m$.
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Activity 19.9 Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

- (a) T is neither injective nor surjective
 - (b) T is injective but not surjective
 - (c) T is surjective but not injective
 - (d) T is bijective.
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Activity 19.10 Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

- (a) T is neither injective nor surjective
 - (b) T is injective but not surjective
 - (c) T is surjective but not injective
 - (d) T is bijective.
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Activity 19.11 Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y + z \end{bmatrix}.$$

Which of the following must be true?

- (a) T is neither injective nor surjective
 - (b) T is injective but not surjective
 - (c) T is surjective but not injective
 - (d) T is bijective.
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Activity 19.12 Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

- (a) T is neither injective nor surjective
 - (b) T is injective but not surjective
 - (c) T is surjective but not injective
 - (d) T is bijective.
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