Linear Algebra

University of South Alabama

Fall 2017

Module E: Solving Systems of Linear Equations

At the end of this module, students will be able to...

- **E1: Systems as matrices.** Translate back and forth between a system of linear equations and the corresponding augmented matrix.
- **E2: Row reduction.** Put a matrix in reduced row echelon form
- **E3: Solving Linear Systems.** Solve a system of linear equations.
- **E4: Homogeneous Systems.** Find a basis for the solution set of a homogeneous linear system.

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.

The following resources will help you prepare for this module.

- `https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-systems-topic/cc-8th-systems-graphically/a/systems-of-equations-with-graphing`
- `https://www.khanacademy.org/math/algebra/systems-of-linear-equations/solving-systems-of-equations-with-substitution/v/practice-using-substitution-for-systems`

Application Activities - Module E Part 1 - Class Day 3

Definition 3.1

A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

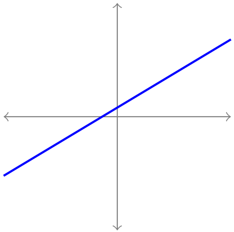
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

Observation 3.2

The linear equation $3x - 5y = -2$ may be graphed as a line in the xy plane.



The linear equation $x + 2y - z = 4$ may be graphed as a plane in xyz space.

Remark 3.3

In previous classes you likely assumed $x = x_1$, $y = x_2$, and $z = x_3$. However, since this course often deals with equations of four or more variables, we will almost always write our variables as x_i.

Definition 3.4

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n \ &= \ b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n \ &= \ b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \quad &\quad\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ &= b_m \end{aligned}$$

A **solution**

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i$$

for $1 \leq i \leq m$ (that is, the solution satisfies all equations in the system).

Remark 3.5

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - x_2 + x_3 &= -2 \end{aligned}$$

Concise standard form:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & 3 \\ 3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & -2 \end{array}$$

Definition 3.6

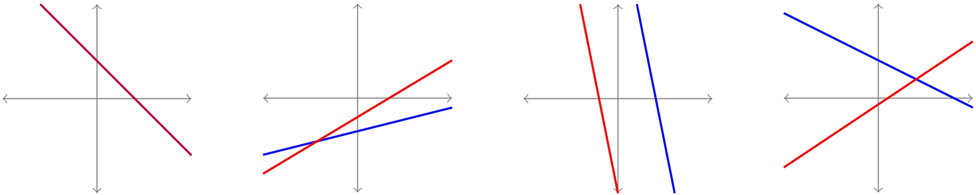
A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

Fact 3.7

All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

Activity 3.8

Consider the following graphs representing linear systems of two variables. Label each graph with **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.



Activity 3.9

All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity 3.10

Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity 3.10

Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

Activity 3.10

Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to describe *all* solutions (the **solution set**) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$ for the linear system in terms of a.

Activity 3.11

Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 \quad - \quad x_4 &= \ \ 3 \\ x_3 + 4x_4 &= -2 \end{aligned}$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix} + a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation 3.12

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.

Remark 3.13

The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - x_2 + x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & 1 & 1 & -2 \end{array}$$

Definition 3.14

A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \dots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \dots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \dots + & a_{mn}x_n & = & b_m \end{array}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Definition 3.15

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution: $(x_1, x_2) = (1, 1)$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned} \qquad \begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \qquad \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity 3.16

Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a conzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

Application Activities - Module E Part 2 - Class Day 4

Definition 4.1

The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a conzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity 4.2

Consider the following two linear systems.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Activity 4.2

Consider the following two linear systems.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Part 1: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

1. Swap R_1 (first row) and R_2 (second row).
2. Multiply R_2 by $\frac{1}{2}$.
3. Add R_1 to R_3.
4. Add $-3R_1$ to R_2.
5. Add $-2R_2$ to R_3.
6. Multiply R_3 by $\frac{1}{3}$.

Activity 4.2

Consider the following two linear systems.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Part 1: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

1. Swap R_1 (first row) and R_2 (second row).
2. Multiply R_2 by $\frac{1}{2}$.
3. Add R_1 to R_3.
4. Add $-3R_1$ to R_2.
5. Add $-2R_2$ to R_3.
6. Multiply R_3 by $\frac{1}{3}$.

Part 2: What is the common solution to these linear systems?

Definition 4.3

The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix. Examples:

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Activity 4.4

Find your own sequence of row operations to manipulate the matrix

$$\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array}\right]$$

into row echelon form. (Note that row echelon form is not unique.)

The most efficient way to do this is by circling **pivot positions** in your matrix:

1. Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
2. Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
3. Repeat these two steps as often as possible.

Activity 4.5

Solve this simplifed linear system:

$$\begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Observation 4.6

The consise standard form of the solution to this linear system corresponds to a simplified row echelon form matrix:

$$\begin{alignedat}{4} x_1 & & & &= -2 \\ & x_2 & & &= 7 \\ & & x_3 & &= 2 \end{alignedat} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Definition 4.7

A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0. Examples:

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & -2\\ 0 & 1 & 0 & 7\\ 0 & 0 & 1 & 2\end{array}\right] \qquad \left[\begin{array}{ccc|c}1 & 0 & -2 & 0\\ 0 & 1 & 3 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \qquad \left[\begin{array}{ccc|c}1 & 3 & 0 & -2\\ 0 & 0 & 1 & 7\\ 0 & 0 & 0 & 0\end{array}\right]$$

Activity 4.8

Show that the following two linear systems:

$$\begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned} \qquad\qquad \begin{aligned} x_1 \quad\quad &= -2 \\ x_2 \quad &= 7 \\ x_3 &= 2 \end{aligned}$$

are equivalent by converting the first system to an augmented matrix, and then zeroing out all terms above pivot positions (the leading terms).

Remark 4.9

We may verify that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$ is a solution to the original linear system

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

by plugging the solution into each equation.

Fact 4.10

Every augmented matrix A reduces to a unique reduced row echelon form matrix. This matrix is denoted as $\mathrm{RREF}(A)$.

Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array}\right]$$

Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array}\right]$$

Part 1: Find RREF(A).

Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array}\right]$$

Part 1: Find RREF(A).

Part 2: How many solutions does the corresponding linear system have?

Application Activities - Module E Part 3 - Class Day 5

Definition 5.1

An algorithm that reduces A to $\mathrm{RREF}(A)$ is called **Gauss-Jordan elimination**. For example:

1. Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
2. Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
3. Repeat these two steps as often as possible.
4. Finally, zero out any terms above pivot positions.

Observation 5.2

Here is an example of applying Gauss-Jordan elimination to a matrix:

$$\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -2 & 10 & 2 \\ 3 & -2 & 13 & 6 \\ -1 & 3 & -6 & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 3 & -2 & 13 & 6 \\ -1 & 3 & -6 & 11 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 2 & -1 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & -9 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Activity 5.3

Find RREF(A) where

$$A = \left[\begin{array}{cccc|c} -1 & 1 & -3 & 2 & 0 \\ 2 & -1 & 5 & 3 & -11 \\ 3 & 2 & 4 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

Definition 5.4

The columns of RREF(A) without a leading term represent **free variables** of the linear system modeled by A that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by A.

Activity 5.5

Given the linear system and its equivalent row-reduced matrix

$$\begin{aligned} -x_1 + x_2 - 3x_3 + 2x_4 &= 0 \\ 2x_1 - x_2 + 5x_3 + 3x_4 &= -11 \\ 3x_1 + 2x_2 + 4x_3 + x_4 &= 1 \\ x_2 - x_3 + x_4 &= 1 \end{aligned} \qquad \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

circle the pivot positions and describe the solution set $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} + a \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$ by setting the free variable (the column without a pivot position) equal to a, and expressing each of the other bounded variables equal to an expression in terms of a.

Remark 5.6

It's not necessary to completely find $\mathrm{RREF}(A)$ to deduce that a linear system is inconsistent.

Activity 5.7

Find a contradiction in the inconsistent linear system

$$\begin{aligned} 2x_1 - 3x_2 &= 17 \\ x_1 + 2x_2 &= -2 \\ -x_1 - x_2 &= 1 \end{aligned}$$

by considering the following equivalent augmented matrices:

$$\left[\begin{array}{cc|c} 2 & -3 & 17 \\ 1 & 2 & -2 \\ -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{array}\right].$$

Activity 5.8

Show that all linear systems of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

are consistent by finding a quickly verifiable solution.

Definition 5.9

A **homogeneous system** is a linear system satisfying $b_i = 0$, that is, it is a linear system of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0\\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0\\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}$$

Fact 5.10

Because the zero vector is always a solution, the solution set to any homogeneous system with infinitely-many solutions may be generated by multiplying the parameters representing the free variables by a minimal set of Euclidean vectors, and adding these up. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Definition 5.11

A minimal set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Activity 5.12

Find a basis for the solution set of the following homogeneous linear system.

$$\begin{alignedat}{5} x_1 &+ 2x_2 & & &-& x_4 &&= 0 \\ & & & x_3 &+& 4x_4 &&= 0 \\ 2x_1 &+ 4x_2 &+& x_3 &+& 2x_4 &&= 0 \end{alignedat}$$

Module V: Vector Spaces

At the end of this module, students will be able to...

- **V1: Vector Spaces.** Determine if a set with given operations forms a vector space.
- **V2: Linear Combinations.** Determine if a vector can be written as a linear combination of a given set of vectors.
- **V3: Spanning Sets.** Determine if a set of vectors spans a vector space.
- **V4: Subspaces.** Determine if a subset of a vector space is a subset or not.

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.

The following resources will help you prepare for this module.

- https://www.khanacademy.org/math/precalculus/vectors-precalc/vector-addition-subtraction/v/adding-and-subtracting-vectors
- https://www.khanacademy.org/math/precalculus/vectors-precalc/combined-vector-operations/v/combined-vector-operations-example
- https://www.khanacademy.org/math/precalculus/imaginary-and-complex-numbers/adding-and-subtracting-complex-numbers/v/adding-complex-numbers
- https://www.khanacademy.org/math/algebra/introduction-to-polynomial-expressions/adding-and-subtracting-polynomials/v/adding-and-subtracting-polynomials-1

Application Activities - Module V Part 1 - Class Day 7

Activity 7.1

Consider each of the following vector properties. Label each property with $\mathbb{R}^1$, $\mathbb{R}^2$, and/or $\mathbb{R}^3$ if that property holds for Euclidean vectors/scalars $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of that dimension.

1. **Addition associativity.**
   $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
2. **Addition commutivity.**
   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. **Addition identity.**
   There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
4. **Addition inverse.**
   There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
5. **Addition midpoint uniqueness.**
   There exists a unique $\mathbf{m}$ where the distance from $\mathbf{u}$ to $\mathbf{m}$ equals the distance from $\mathbf{m}$ to $\mathbf{v}$.
6. **Scalar multiplication associativity.**
7. **Scalar multiplication identity.**
   $1\mathbf{v} = \mathbf{v}$.
8. **Scalar multiplication relativity.**
   There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.
9. **Scalar distribution.**
   $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
10. **Vector distribution.**
    $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
11. **Orthogonality.**
    There exists a non-zero vector $\mathbf{n}$ such that $\mathbf{n}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
12. **Bidimensionality.**

Definition 7.2

A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Addition associativity.** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition commutivity.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Addition identity.** There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- **Addition inverse.** There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.** $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **Scalar multiplication identity.** $1\mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Vector distribution.** $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Definition 7.3

The most important examples of vector spaces are the **Euclidean vector spaces** $\mathbb{R}^n$, but there are other examples as well.

Activity 7.4

Consider the following vector space that models motion along the curve $y = e^x$. Let $V = \{(x, y) : y = e^x\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 b_2)$, and $c(a, b) = (ca, b^c)$.

Activity 7.4

Consider the following vector space that models motion along the curve $y = e^x$. Let $V = \{(x, y) : y = e^x\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 b_2)$, and $c(a, b) = (ca, b^c)$.

Part 1: Verify that $3((1, e) + (-2, \frac{1}{e^2})) = 3(1, e) + 3(-2, \frac{1}{e^2})$.

Activity 7.4

Consider the following vector space that models motion along the curve $y = e^x$. Let $V = \{(x, y) : y = e^x\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 b_2)$, and $c(a, b) = (ca, b^c)$.

Part 1: Verify that $3((1, e) + (-2, \frac{1}{e^2})) = 3(1, e) + 3(-2, \frac{1}{e^2})$.

Part 2: Prove the scalar distribution property for this space: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

Application Activities - Module V Part 2 - Class Day 8

Remark 8.1

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{R}^\infty$: Sequences of real numbers $(v_1, v_2, \ldots)$.
- $\mathbb{R}^{m\times n}$: Matrices of real numbers with m rows and n columns.
- $\mathbb{C}$: Complex numbers.
- $\mathcal{P}^n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity 8.2

Let $V = \{(a, b) : a, b \text{ are real numbers}\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$ and $c(a, b) = (a^c, b + c)$. Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition commutivity.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Addition identity.** There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- **Addition inverse.** There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.** $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **Scalar multiplication identity.** $1\mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Vector distribution.** $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Definition 8.3

A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ for any choice of scalar multiples $c_1, c_2, \dots, c_m$.

Definition 8.4

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

Activity 8.5

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Activity 8.5

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch $c\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Activity 8.5

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch $c\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part 2: Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ in the xy plane.

Activity 8.6

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Activity 8.6

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch $c_1\begin{bmatrix}1\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\1\end{bmatrix}$ in the xy plane for $\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}$.

Activity 8.6

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch $c_1\begin{bmatrix}1\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\1\end{bmatrix}$ in the xy plane for $\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}$.

Part 2: Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity 8.7

Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}$ in the xy plane.

Activity 8.8

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ holds for some scalars x_1, x_2.

Activity 8.8

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}$ holds for some scalars x_1, x_2.

Part 1: Reinterpret this vector equation as a system of linear equations.

Activity 8.8

The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ holds for some scalars x_1, x_2.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (From now on, feel free to use a calculator to solve linear systems.)

Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (From now on, feel free to use a calculator to solve linear systems.)

Part 3: Given this solution, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

Application Activities - Module V Part 3 - Class Day 9

Fact 9.1

A vector $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \,|\, \mathbf{b}]$ is consistent.

Remark 9.2

To determine if $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, find $\operatorname{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \,|\, \mathbf{b}]$.

Activity 9.3

Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Activity 9.4

Determine if $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Observation 9.5

So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space $\mathbb{R}^n$; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

Activity 9.6

We previously checked that $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ does not belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$.

Does $f(x) = 3x^2 - 2x + 1$ belong to span$\{x^2 - 3, -x^2 - 3x + 2\}$?

Activity 9.7

Does the matrix $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\}$?

Activity 9.8

Does the complex number $2i$ belong to $\operatorname{span}\{-3+i, 6-2i\}$?

Activity 9.9

How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the *xy* plane to support your guess.

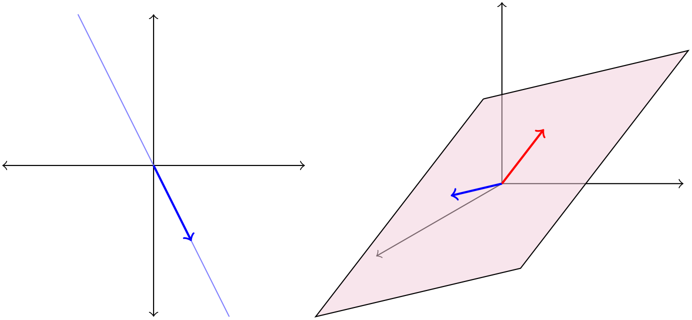
Activity 9.10

How many vectors are required to span $\mathbb{R}^3$?

Application Activities - Module V Part 4 - Class Day 10

Fact 10.1

At least n vectors are required to span $\mathbb{R}^n$.



Activity 10.2

Find a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by doing the following.

Activity 10.2

Find a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by doing the following.

Part 1: Choose simple values for x, y, z such that $\left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{array}\right]$ represents an inconsistent linear equation.

Activity 10.2

Find a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by doing the following.

Part 1: Choose simple values for x, y, z such that $\left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{array}\right]$ represents an inconsistent linear equation.

Part 2: Use row operations to manipulate $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right]$.

Activity 10.2

Find a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by doing the following.

Part 1: Choose simple values for x, y, z such that $\left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{array}\right]$ represents an inconsistent linear equation.

Part 2: Use row operations to manipulate $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right]$.

Part 3: Write a sentence explaining why $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ cannot be in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Fact 10.3

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ fails to span all of $\mathbb{R}^n$ exactly when $\mathrm{RREF}[\mathbf{v}_1 \ \ldots \ \mathbf{v}_m]$ has a row of zeros.

Activity 10.4

Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Prove that $\mathbb{R}^4 = \operatorname{span} S$.

Activity 10.5

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2$$

Prove that $\mathcal{P}^3 \neq \operatorname{span} S$.

Definition 10.6

A subset of a vector space is called a **subspace** if it is itself a vector space.

Fact 10.7

If S is a subset of a vector space V, then $\operatorname{span} S$ is a subspace of V.

Remark 10.8

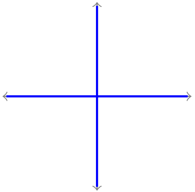
To prove that a subset is a subspace, you need only verify that $c\mathbf{v} + d\mathbf{w}$ belongs to the subset for any choice of vectors $\mathbf{v}, \mathbf{w}$ from the subset and any real scalars c, d.

Activity 10.9

Prove that $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$ is a subspace of the vector space of all degree-two polynomials by showing that $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$ belongs to P.

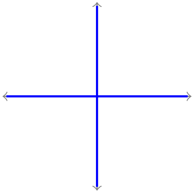
Activity 10.10

Consider the subset of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.



Activity 10.10

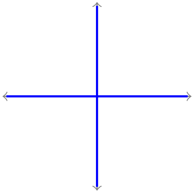
Consider the subset of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.



Part 1: Find a linear combination $c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ that does not belong to this subset.

Activity 10.10

Consider the subset of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.



Part 1: Find a linear combination $c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ that does not belong to this subset.

Part 2: Use this linear combination to sketch a picture illustrating why this subset is not a subspace.

Fact 10.11

Suppose a subset S of V is isomorphic to another vector space W. Then S is a subspace of V.

Activity 10.12

Show that the set of 2×2 matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of $\mathbb{R}^{2\times 2}$ by finding a Euclidean space isomorphic to S.

Module S: Structure of vector spaces

At the end of this module, students will be able to...

- **S1. Linear independence** Determine if a set of Euclidean vectors is linearly dependent or independent.
- **S2. Basis verification** Determine if a set of vectors is a basis of a vector space
- **S3. Basis construction** Construct a basis for the subspace spanned by a given set of vectors.
- **S4. Dimension** I can compute the dimension of a vector space.

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.
- Apply linear combinations and spanning sets **(Standard(s) V2,V3)**.

The following resources will help you prepare for this module.

- `https://www.khanacademy.org/math/precalculus/vectors-precalc/vector-addition-subtraction/v/adding-and-subtracting-vectors`
- `https://www.khanacademy.org/math/precalculus/vectors-precalc/combined-vector-operations/v/combined-vector-operations-example`

Application Activities - Module S Part 1 - Class Day 12

Activity 12.1

In the previous module, we considered

$$S=\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}3\\13\\7\\16\end{bmatrix},\begin{bmatrix}-1\\10\\7\\14\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\}$$

and showed that span $S \neq \mathbb{R}^4$. Find two vectors that are in the span of the other three vectors.

Definition 12.2

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

Activity 12.3

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Is the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ consistent with one solution, consistent with infinitely many solutions, or inconsistent?

Fact 12.4

The set $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ is consistent with infinitely many solutions.

Activity 12.5

Find

$$\mathrm{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array}\right]$$

and circle the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

Fact 12.6

A set of Euclidean vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix}\mathbf{v}_1 & \ldots & \mathbf{v}_n\end{bmatrix}$ has a column without a pivot position.

Activity 12.7

TODO (compute RREF and label each set of vectors as linearly independent/dependent)

Application Activities - Module S Part 2 - Class Day 13

Activity 13.1

(take basis shown to be linearly independent in previous day, and show that it spans)

Definition 13.2

A **basis** is a linearly independent set that spans a vector space.

Observation 13.3

A basis may be thought of as building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

Activity 13.4

(given four sets of general vectors, identify which are bases and which aren't)

Activity 13.5

If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$, that means $\text{RREF}[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$ doesn't have a column without a pivot position, and doesn't have a row of zeros. What is $\text{RREF}[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$?

Fact 13.6

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for $\mathbb{R}^n$ if and only if $m = n$ and

$$\mathrm{RREF}[\mathbf{v}_1 \; \dots \; \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Activity 13.7

(given four sets of IR^5 vectors, identify which are bases and which aren't)

Activity 13.8

How can {u,v,u+v} (but with numbers) be changed to make it linearly independent?

Application Activities - Module S Part 3 - Class Day 14

Activity 14.1
(discover that the redundant vectors are non-pivot columns)

Fact 14.2

To compute a basis for the subspace $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, simply remove the vectors corresponding to the non-pivot columns of $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_m]$.

Activity 14.3
(find ALL the bases for span S that are subsets of S)

Fact 14.4

All bases for a vector space are the same size.

Activity 14.5

Prove that if $\{\mathbf{v}\}$ is a basis for V, then $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly dependent (assuming $\mathbf{w}_1 \neq \mathbf{w}_2$).

Fact 14.6

All bases for a vector space are the same size.

Definition 14.7

The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

Activity 14.8

Reduce a bunch of spans to bases to find their dimension.

Activity 14.9

What is the dimension of the vector space of 7th-degree polynomials $\mathcal{P}^7$?

Activity 14.10

What is the dimension of the vector space of polynomials $\mathcal{P}$?

Observation 14.11

Several interesting vector spaces are infinite-dimensional:

- The space of polynomials $\mathcal{P}$
- The space of real number sequences $\mathbb{R}^\infty$
- The space of continuous functions $C(\mathbb{R})$

Fact 14.12

Every vector space with dimension $n < \infty$ is isomorphic to $\mathbb{R}^n$.

Module A: Algebraic properties of linear maps

At the end of this module, students will be able to...

- **A1. Linear maps as matrices** I can write the matrix (with respect to the standard bases) corresponding to a linear transformation between Euclidean spaces.
- **A2. Linear map verification** I can determine if a map between vector spaces is linear or not.
- **A3. Injectivity and Surjectivity** I can determine if a given linear map is injective and/or surjective
- **A4. Kernel and Image** I can compute the kernel and image of a linear map, including finding bases.

Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing **(Standard(s) E1, E2, E3, E4)**.
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace **(Standard(s) V3)**.
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent **(Standard(s) S1)**.
- State the definition of a basis, and determine if a set of vectors is a basis **(Standard(s) S2)**.

The following resources will help you prepare for this module.

- Review the supporting Standards listed above.

Application Activities - Module A Part 1 - Class Day 17

Definition 17.1

A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}$, $\vec{v} \in V$.

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.

V is called the **domain** of T and W is called the **co-domain** of T.

Activity 17.2

Determine if each of the following maps are linear transformations

(a) $T_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $T_1\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \sqrt{a^2 + b^2}$

(b) $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$

(c) $T_3 : \mathcal{P}_d \to \mathcal{P}_{d-1}$ given by $T_3(f(x)) = f'(x)$.

(d) $T_4 : C(\mathbb{R}) \to C(\mathbb{R})$ given by $T_4(f(x)) = f(-x)$

(e) $T_5 : \mathcal{P} \to \mathcal{P}$ given by $T_5(f(x)) = f(x) + x^2$

Activity 17.3

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute each of the following:

(a) $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$

(b) $T\left(\begin{bmatrix}0\\0\\-2\end{bmatrix}\right)$

(c) $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$

(d) $T\left(\begin{bmatrix}-2\\0\\5\end{bmatrix}\right)$

Activity 17.4

Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation. What is the smallest number of vectors needed to determine T? In other words, what is the smallest number n such that there are $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^4$ and given $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ you can determine $T(\vec{w})$ for *any* $\vec{w} \in \mathbb{R}^2$?

Observation 17.5

Fix an ordered basis for V. Since every vector can be written *uniquely* as a linear combination of basis vectors, a linear transformation $T : V \to W$ corresponds exactly to a choice of where each basis vector goes. For convenience, we can thus encode a linear transformation as a matrix, with one column for the image of each basis vector (in order).

Activity 17.6

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard ordered basis.

Activity 17.7

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the ordered basis

$$\left\{\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$$

Activity 17.8

Let $D : \mathcal{P}_3 \to \mathcal{P}_2$ be the derivative map (recall this is a linear transformation). Write the matrix corresponding to D with respect to the ordered basis $\{1, x, x^2, x^3\}$.

Application Activities - Module A Part 2 - Class Day 18

Definition 18.1

Let $T : V \to W$ be a linear transformation.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.
- T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\vec{w} \in W$, there is some $v \in V$ with $T(\vec{v}) = \vec{w}$.

Activity 18.2

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

Activity 18.3

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

Definition 18.4

We also have two important sets called the **kernel** of T and the **image** of T.

$$\ker T = \left\{ \vec{v} \in V \;\middle|\; T(\vec{v}) = 0 \right\}$$

$$\mathrm{Im}\ T = \left\{ \vec{w} \in W \;\middle|\; \text{there is some } v \in V \text{ with } T(\vec{v}) = \vec{w} \right\}$$

Activity 18.5

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T.

Activity 18.6

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T.

Activity 18.7

Activity 18.7

Part 1: Describe surjective linear transformations in terms of the image.

Activity 18.7

Part 1: Describe surjective linear transformations in terms of the image.

Part 2: Describe injective linear transformations in terms of the kernel.

Activity 18.8

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

1) Write a system of equations whose solution set is the kernel.
2) Compute RREF(A) and solve the system of equations.
3) Compute the kernel of T
4) Find a basis for the kernel of T

Activity 18.9

Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

1) Write a system of equations whose solution set is the kernel.
2) Compute RREF(A) and solve the system of equations.
3) Compute the kernel of T
4) Find a basis for the kernel of T

Activity 18.10

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

1) Find a set of vectors that span the image of T

2) Find a basis for the image of T.

Activity 18.11

Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

1) Find a set of vectors that span the image of T
2) Find a basis for the image of T.

Application Activities - Module A Part 3 - Class Day 19

Activity 19.1

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). You have cards containing a number of statements about T and A. Sort them into groups of equivalent statements, and post them on your board.

Activity 19.2

Cycle around the room counter-clockwise. If they have two things grouped together that you know are not equivalent, write a reason or counter-example on a sticky note.

Activity 19.3

Come up with as many statements as you can, and add them to the appropriate group.

Module M: Understanding Matrices Algebraically

At the end of this module, students will be able to...

- **M1. Matrix multiplication** Multiply matrices.
- **M2. Invertible matrices** Determine if a square matrix is invertible or not.
- **M3. Matrix inverses** Compute the inverse matrix of an invertible matrix.

Before beginning this module, each student should be able to...

- Compose functions of real numbers
- Solve systems of linear equations **(Standard(s) E3)**
- Find the matrix corresponding to a linear transformation **(Standard(s) A1)**
- Determine if a linear transformation is injective and/or surjective **(Standard(s) A3)**
- Interpret the ideas of injectivity and surjectivity in multiple ways

The following resources will help you prepare for this module.

- `https://www.khanacademy.org/math/algebra2/manipulating-functions/funciton-composition/v/function-composition`

Application Activities - Module M Part 1 - Class Day 21

Activity 21.1

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity 21.2

What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity 21.3

The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{4,3}$

(b) $M_{4,2}$

(c) $M_{3,2}$

(d) $M_{2,3}$

(e) $M_{2,4}$

(f) $M_{3,4}$

Activity 21.4

Compute $(S \circ T)(\vec{e}_1)$, $(S \circ T)(\vec{e}_2)$, and $(S \circ T)(\vec{e}_3)$.

Activity 21.5

Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity 21.6

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity 21.7

What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity 21.8

The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{2,2}$

(b) $M_{2,3}$

(c) $M_{3,2}$

(d) $M_{3,3}$

Activity 21.9

Compute $(S \circ T)(\vec{e}_1)$ and $(S \circ T)(\vec{e}_2)$

Activity 21.10

Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity 21.11

Let $T : \mathbb{R}^1 \to \mathbb{R}^4$ be given by the matrix $B = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ and $S : \mathbb{R}^4 \to \mathbb{R}^1$ be given by the matrix $A = \begin{bmatrix} 2 & 3 & 2 & 5 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity 21.12

What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity 21.13

The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{1,1}$

(b) $M_{1,4}$

(c) $M_{4,1}$

(d) $M_{4,4}$

Activity 21.14

Compute $(S \circ T)(\vec{e}_1)$

Activity 21.15

Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity 21.16

Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

1. Compute AX
2. Interpret the system of equations below as a matrix equation

$$\begin{aligned} 3x + y - z &= 5 \\ 2x + 4z &= -7 \\ -x + 3y + 5z &= 2 \end{aligned}$$

Application Activities - Module M Part 2 - Class Day 22

Activity 22.1

Each row operation can be interpreted as a matrix multiplication. Let $A \in M_{4,4}$

1) Find a matrix S_1 such that $S_1 A$ is the result of swapping the second and fourth rows of A.
2) Find a matrix S_2 such that $S_2 A$ is the result of adding 5 times the third row of A to the first.
3) Find a matrix S_3 such that $S_3 A$ is the result of doubling the fourth row of A.

Activity 22.2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). Consider the following statements about T

(a) T is injective

(b) T is surjective

(c) T is bijective (i.e. both injective and surjective)

(d) $AX = B$ has a solution for all $B \in M_{m,1}$

(e) $AX = B$ has a unique solution for all $B \in M_{m,1}$

(f) $AX = 0$ has a non-trivial solution.

(g) The columns of A span $\mathbb{R}^m$

(h) The columns of A are linearly independent

(i) The columns of A are a basis of $\mathbb{R}^m$

(j) RREF(A) has n pivot columns

(k) RREF(A) has m pivot columns

Sort these statements into groups of equivalent statements.

Activity 22.3

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). If T is injective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Activity 22.4

If T is surjective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Activity 22.5

If T is bijective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Application Activities - Module M Part 3 - Class Day 23

Activity 23.1

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with matrix $A \in M_{n,n}$.
If T is a bijection, then $AX = B$ has a unique solution for all $B \in \mathbb{R}^n$. Thus we can define a map $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by defining $T^{-1}(B)$ to be this solution. It follows immediately that $T \circ T^{-1}$ is the identity map. The matrix corresponding to T^{-1} is denoted A^{-1}.

1) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_1)$
2) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_2)$
3) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_3)$
4) Compute A^{-1}

A (square) matrix is called *invertible* if it corresponds to an invertible linear transformation.

1) Find the inverse of the matrix $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$

2) Find the inverse of the matrix $\begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$

Module G: Geometry of Linear Maps

At the end of this module, students will be able to...

- **G1. Determinants** Compute the determinant of a square matrix.
- **G2. Eigenvalues** Find the eigenvalues of a square matrix, along with their algebraic multiplicities.
- **G3. Eigenvectors** Find the eigenspace of a square matrix associated to a given eigenvalue.
- **G4. Geometric multiplicity** Compute the geometric multiplicity of an eigenvalue of a square matrix.

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **(Standard(s) A1)**.
- Recall and use the definition of a linear transformation **(Standard(s) A2)**.
- Find all roots of quadratic polynomials (including complex ones), and be able to use the rational root theorem to find all rational roots of a higher degree polynomial.
- Interpret the statement "A is an invertible matrix" in many equivalent ways in different contexts.

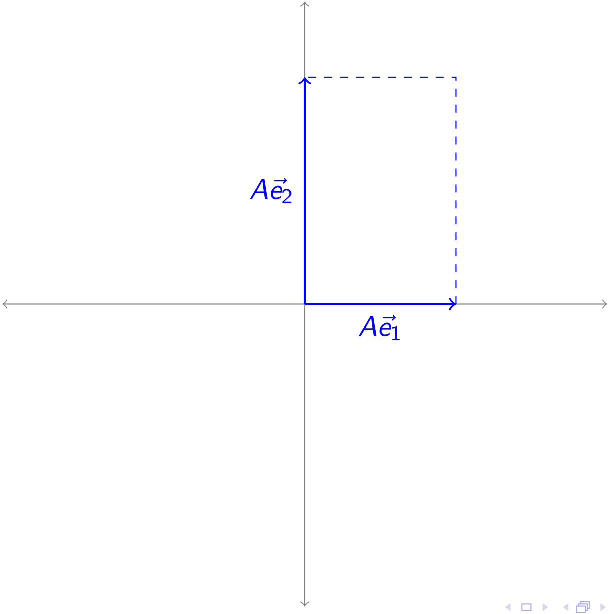
The following resources will help you prepare for this module.

- Finding the area of a parallelogram: `https://www.khanacademy.org/math/basic-geo/basic-geo-area-and-perimeter/parallelogram-area/a/area-of-parallelogram`
- Factoring quadratics: `https://www.khanacademy.org/math/algebra2/polynomial-functions/factoring-polynomials-quadratic-forms-alg2/v/factoring-polynomials-1`
- Finding complex roots of quadratics: `https://www.khanacademy.org/math/algebra2/polynomial-functions/quadratic-equations-with-complex-numbers/v/complex-roots-from-the-quadratic-formula`
- Finding all roots of polynomials: `https://www.khanacademy.org/math/algebra2/polynomial-functions/finding-zeros-of-polynomials/v/finding-roots-or-zeros-of-polynomial-1`
- The Rational Root Theorem: `https://artofproblemsolving.com/wiki/index.php?title=Rational_Root_Theorem`

Application Activities - Module G Part 1 - Class Day 25

Activity 25.1

Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



Activity 25.2

Consider the following linear transformations $A_i : \mathbb{R}^2 \to \mathbb{R}^2$.

- $A_2 = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$
- $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
- $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

For each linear transformation, do the following:

(a) Draw a graph showing the image of the unit square.

(b) Compute how much the area was stretched out.

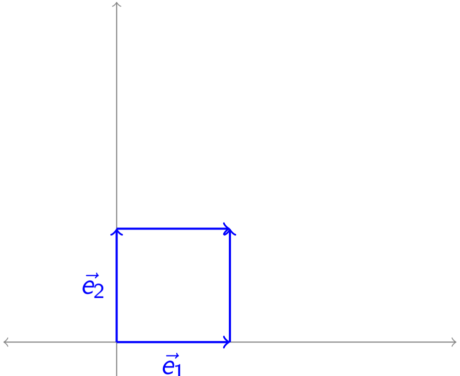
(c) Determine which axes (or lines) were preserved; how were they stretched out?

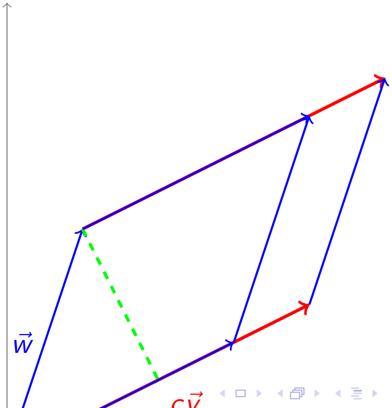
Activity 25.3

Our goal is to define a function $\det : M_n \to \mathbb{R}$ that takes a square matrix (linear transformation $\mathbb{R}^n \to \mathbb{R}^n$) and returns its area stretching factor. This function is called the **determinant**.

What properties should this function have?
Match the four pictures to the following four expressions

$$\det(\vec{e}_1, \vec{e}_2) \qquad \det(\vec{v}, \vec{v}) \qquad \det(c\vec{v}, \vec{w}) \qquad \det(\vec{u} + \vec{v}, \vec{w})$$





Activity 25.4

What can you conclude about each of the following?

1. $\det(\vec{e}_1, \vec{e}_2)$
2. $\det(\vec{v}, \vec{v})$
3. $\det(c\vec{v}, \vec{w})$
4. $\det(\vec{u} + \vec{v}, \vec{w})$

Definition 25.5

To summarize, we have 3 properties (stated here over $\mathbb{R}^n$)

P1: $\det(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n) = 1$

P2: If $\vec{v}_i = \vec{v}_j$ for some $i \neq j$, then $\det(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = 0$.

P3: The determinant is linear in each column.

These three properties uniquely define the **determinant**, as we shall see.

Observation 25.6

Note that if $\vec{v}, \vec{w} \in \mathbb{R}^2$ and $A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$ we will write either $\det(A)$ or $\det(\vec{v}, \vec{w})$ as is convenient.

Activity 25.7

How are $\det(\vec{v}, \vec{w})$ and $\det(\vec{w}, \vec{v})$ related?

(a) $\det(\vec{v}, \vec{w}) = \det(\vec{w}, \vec{v})$

(b) $\det(\vec{v}, \vec{w}) = -\det(\vec{w}, \vec{v})$

(c) They are unrelated

(d) They are related, but not by either (a) or (b).

Observation 25.8

Note that this implies that the determinant is actually a *signed* area (volume)!

Activity 25.9

How are $\det(\vec{v}+\vec{w},\vec{w})$ and $\det(\vec{v},\vec{w})$ related?

(a) $\det(\vec{v}+\vec{w},\vec{w}) = \det(\vec{v},\vec{w})$

(b) $\det(\vec{v}+\vec{w},\vec{w}) = -\det(\vec{v},\vec{w})$

(c) They are unrelated

(d) They are related, but not by either (a) or (b).

Observation 25.10

Note that we now understand the effect of any column operation on the determinant.

Application Activities - Module G Part 2 - Class Day 26

Activity 26.1

How are $\det(A)$ and $\det(A^T)$ related?

(a) $\det(A) = \det(A^T)$

(b) $\det(A) = -\det(A^T)$

(c) $\det(A) = \frac{1}{\det(A^T)}$

(d) They are unrelated

Observation 26.2

Thus, row operations behave like column operations. So we can use row reduction to compute determinants.

Activity 26.3

Compute $\det \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$.

Activity 26.4

Compute $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Activity 26.5

Which of the following is the same as $\det \begin{bmatrix} 3 & -2 & 0 \\ 5 & -1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$?

(a) $\det \begin{bmatrix} 3 & -2 \\ 5 & -1 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$

(c) $\det \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}$

(d) None of these

Activity 26.6

Which of the following is the same as $\det \begin{bmatrix} 3 & 0 & 7 \\ 5 & 1 & 2 \\ -2 & 0 & 6 \end{bmatrix}$?

(a) $\det \begin{bmatrix} 3 & 7 \\ 5 & 2 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix}$

(c) $\det \begin{bmatrix} 5 & 2 \\ -2 & 6 \end{bmatrix}$

(d) None of these

Activity 26.7

Compute $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$

Activity 26.8

Using the fact that $\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}$, compute $\det \begin{bmatrix}1 & 2 & 3\\1 & -2 & 5\\0 & 3 & 3\end{bmatrix}$.

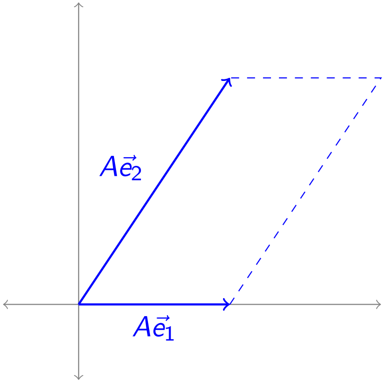
Activity 26.9

Compute $\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$

Application Activities - Module G Part 3 - Class Day 27

Activity 27.1

Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



Observe

$$A\vec{e}_1 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\vec{e}_1$$

Is there another vector $\vec{v} \in \mathbb{R}^2$ such that $A\vec{v} = \lambda \vec{v}$ for some $\lambda \in \mathbb{R}$?

Definition 27.2

Let $A \in M_n(\mathbb{R})$. An **eigenvector** is a vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x}$ is parallel to $\vec{x}$; in other words, $A\vec{x} = \lambda\vec{x}$ for some scalar λ, which is called an **eigenvalue**

Observation 27.3

Observe that $A\vec{x} = \lambda\vec{x}$ is equivalent to $(A - \lambda I)\vec{x} = 0$.

- To find eigenvalues, we need to find values of λ such that $A - \lambda I$ has a nontrivial kernel; equivalently, $A - \lambda I$ is not invertible, which is equivalent to $\det(A - \lambda I) = 0$. $\det(A - \lambda I)$ is called the **characteristic polynomial**.
- Once an eigenvalue is found, the eigenvectors form a subspace called the **eigenspace**, which is simply the kernel of $A - \lambda I$. Each eigenvalue will have an associated eigenspace.

Activity 27.4

Find the eigenvalues for the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Activity 27.5

Compute the eigenspace associated to the eigenvalue 3.

Activity 27.6

Find all the eigenvalues and associated eigenspaces for the matrix $\begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$.

Application Activities - Module G Part 4 - Class Day 28

Activity 28.1

If $A \in M_4$, what is the largest number of eigenvalues A can have?

Activity 28.2

2 is an eigenvalue of each of the matrices $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}$.

Compute the eigenspace associated to 2 for both A and B.

Definition 28.3

- The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.
- The **geometric multiplicity** of an eigenvalue is the dimension of the eigenspace.

Activity 28.4

How are the algebraic and geometric multiplicities related?

(a) The algebraic multiplicity is always at least as big as than the geometric multiplicity.

(b) The geometric multiplicity is always at least as big as the algebraic multiplicity.

(c) Sometimes the algebraic multiplicity is larger and sometimes the geometric multiplicity is larger.

Activity 28.5

Find the eigenvalues, along with both their algebraic and geometric multiplicities, for the matrix $\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$

Activity 28.6

Find the eigenvalues of the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Activity 28.7

Describe what this linear transformation is doing geometrically; draw a picture.

Activity 28.8

Fix a real number θ and find the eigenvalues of the matrix $A_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. What are the eigenvalues?

Activity 28.9

D raw pictures and describe the geometric actions of the maps $A_{\frac{\pi}{4}}$, $A_{\frac{\pi}{2}}$, and A_{π}.

Activity 28.10

For how many values of θ does the rotation matrix A_θ have real eigenvalues?

(a) 0

(b) 1

(c) 2

(d) 3

(e) An infinite number