

## Application Activities - Module G Part 3 - Class Day 27

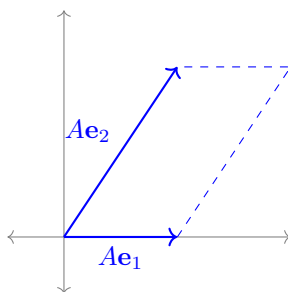
**Activity 27.1** Suppose the matrix  $M$  is invertible, so there exists  $M^{-1}$  with  $MM^{-1} = I$ . It follows that  $\det(M)\det(M^{-1}) = \det(I)$ .

What is the only number that  $\det(M)$  cannot equal?

- (a)  $-1$                       (b)  $0$                       (c)  $1$                       (d)  $\frac{1}{\det(M^{-1})}$

**Fact 27.2** Since  $\det(M^{-1}) = \frac{1}{\det(M)}$  for every invertible matrix  $M$ , a square matrix  $M$  is invertible if and only if  $\det(M) \neq 0$ .

**Observation 27.3** Consider the linear transformation  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the matrix  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



It is easy to see geometrically that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily verified by computation) that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

**Definition 27.4** Let  $A \in \mathbb{R}^{n \times n}$ . An **eigenvector** is a vector  $\mathbf{x} \in \mathbb{R}^n$  such that  $A\mathbf{x}$  is parallel to  $\mathbf{x}$ . In other words,  $A\mathbf{x} = \lambda\mathbf{x}$  for some scalar  $\lambda$ . We call this  $\lambda$  an **eigenvalue** of  $A$ .

**Observation 27.5** Since  $\lambda\mathbf{x} = \lambda(I\mathbf{x})$ , we can find the eigenvalues and eigenvectors satisfying  $A\mathbf{x} = \lambda\mathbf{x}$  by inspecting  $(A - \lambda I)\mathbf{x} = \mathbf{0}$ .

- Since we already know that  $(A - \lambda I)\mathbf{0} = \mathbf{0}$  for any value of  $\lambda$ , we are more interested in finding values of  $\lambda$  such that  $A - \lambda I$  has a nontrivial kernel.
- Thus  $\text{RREF}(A - \lambda I)$  must have a non-pivot column, and therefore  $A - \lambda I$  cannot be invertible.
- Since  $A - \lambda I$  cannot be invertible, our eigenvalues must satisfy  $\det(A - \lambda I) = 0$ .

**Definition 27.6** Computing  $\det(A - \lambda I)$  results in the **characteristic polynomial** of  $A$ .

For example, when  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix}$$

Thus the characteristic polynomial of  $A$  is

$$\det \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) - 6 = \lambda^2 - 5\lambda - 2$$

**Activity 27.7** Complete the following computation of the characteristic polynomial  $A - \lambda I$  for  $A = \begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$ .

$$\begin{aligned} \begin{bmatrix} 6 - \lambda & -2 & 1 \\ 17 & -5 - \lambda & 5 \\ -4 & 2 & 1 - \lambda \end{bmatrix} &= (6 - \lambda) \det \begin{bmatrix} ? & ? & ? \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} - 2 \det \begin{bmatrix} ? & ? & ? \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} + \det \begin{bmatrix} ? & ? & ? \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \\ &= (6 - \lambda) \det \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} + 2 \det \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} - \det \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \\ &= (6 - \lambda) \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} + 2 \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} - \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \\ &= (6 - \lambda)((-5 - \lambda)(1 - \lambda) - 10) + 2(17(1 - \lambda) + 20) - (-4(-5 - \lambda) - 34) \end{aligned}$$

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**Activity 27.8** Let  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .

*Part 1:* Compute  $\det \begin{bmatrix} 2 - \lambda & 2 \\ 0 & 3 - \lambda \end{bmatrix}$  to determine the characteristic polynomial of  $A$ .

*Part 2:* Find the roots of the characteristic polynomial to determine the eigenvalues of  $A$ .

*Part 3:* Compute the kernel of the transformation given by

$$A - 2I = \begin{bmatrix} 2 - 2 & 2 \\ 0 & 3 - 2 \end{bmatrix}$$

to determine all the eigenvectors associated to the eigenvalue 2.

*Part 4:* Compute the kernel of the transformation given by  $A - 3I$  to determine all the eigenvectors associated to the eigenvalue 3.

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**Definition 27.9** The kernel of the transformation given by  $A - \lambda I$  contains all the eigenvectors associated with  $\lambda$ . Since kernel is a subspace of  $\mathbb{R}^n$ , we call this kernel the **eigenspace** associated with the eigenvalue  $\lambda$ .

**Activity 27.10** Find all the eigenvalues and associated eigenspaces for the matrix  $A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}$ .

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .

*Part 2:* Find the roots of the characteristic polynomial  $(3 - \lambda)(\lambda^2 - 4\lambda - 12)$  to determine the eigenvalues of  $A$ .

*Part 3:* Compute the kernels of  $A - \lambda I$  for each eigenvalue  $\lambda \in \{-2, 3, 6\}$  to determine the respective eigenspaces.

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