Module G: Geometry of Linear Maps

How can we understand linear maps geometrically?

At the end of this module, students will be able to...

G1. **Row operations.** ... describe how a row operation affects the determinant of a matrix.

G2. **Determinants.** ... compute the determinant of a 4×4 matrix.

G3. **Eigenvalues.** ... find the eigenvalues of a 2×2 matrix.

G4. **Eigenvectors.** ... find a basis for the eigenspace of a 4×4 matrix associated with a given eigenvalue.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **A1**.
- Recall and use the definition of a linear transformation **A2**.
- Find all roots of quadratic polynomials (including complex ones).
- Interpret the statement "A is an invertible matrix" in many equivalent ways in different contexts.

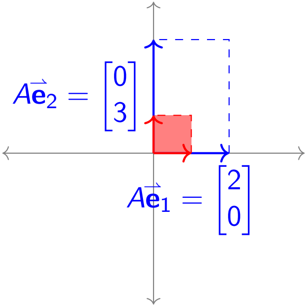
The following resources will help you prepare for this module.

- Finding the area of a parallelogram (Khan Academy): `http://bit.ly/2BO5iWx`
- Factoring quadratics (Khan Academy): `http://bit.ly/1XjfbV2`
- Factoring quadratics using area models (Youtube): `https://youtu.be/Aa-v1EK7DR4`
- Finding complex roots of quadratics (Youtube): `https://www.youtube.com/watch?v=2yBhDsNE0wg`

Module G Section 1

Activity G.1.1 *(~5 min)*

The image below illustrates how the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.

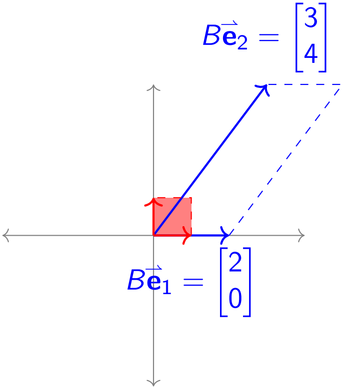


(a) What are the lengths of $A\vec{\mathbf{e}}_1$ and $A\vec{\mathbf{e}}_2$?

(b) What is the area of the transformed unit square?

Activity G.1.2 (*∼5 min*)

The image below illustrates how the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$. transforms the unit square.



(a) What are the lengths of $B\vec{\mathbf{e}}_1$ and $B\vec{\mathbf{e}}_2$?

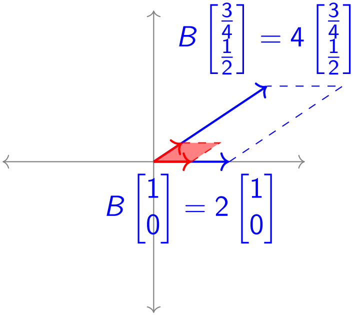
(b) What is the area of the transformed unit square?

Observation G.1.3

It is possible to find two nonparallel vectors that are scaled but not rotated by the linear map given by B.

$$B\vec{\mathbf{e}}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\vec{\mathbf{e}}_1$$

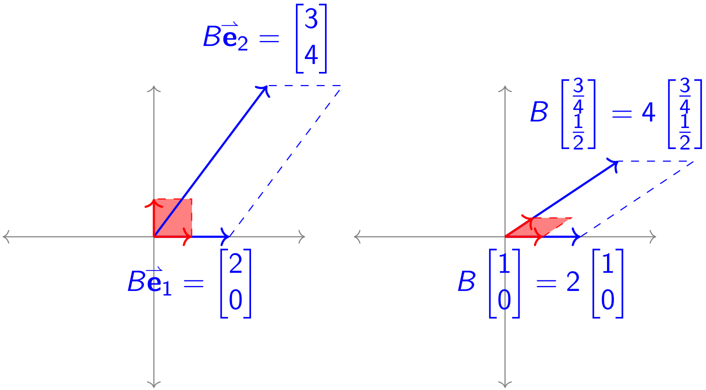
$$B \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$



The process for finding such vectors will be covered later in this module.

Observation G.1.4

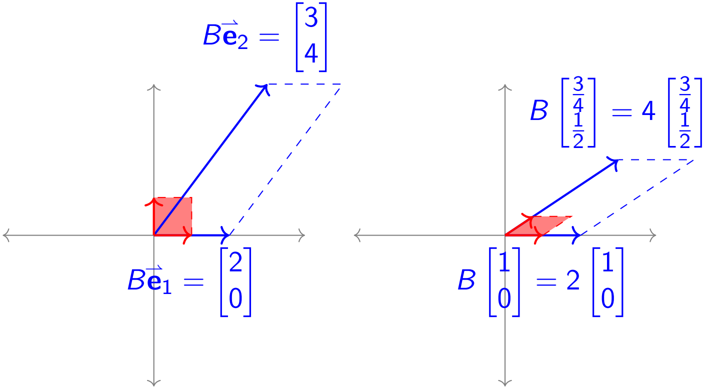
Notice that while a linear map can transform vectors in various ways, linear maps always transform parallelograms into parallelograms, and these areas are always transformed by the same factor: in the case of $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, this factor is 8.



Since this change in area is always the same for a given linear map, it will be equal to the value of the transformed unit square (which begins with area 1).

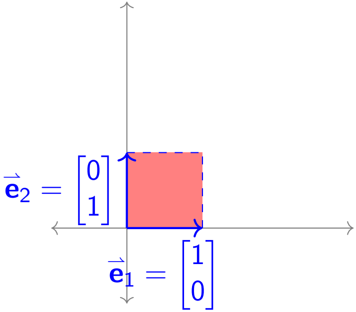
Remark G.1.5

We will define the **determinant** of a square matrix A, or $\det(A)$ for short, to be the factor by which A scales areas. In order to figure out how to compute it, we first figure out the properties it must satisfy.



Activity G.1.6 *(~2 min)*

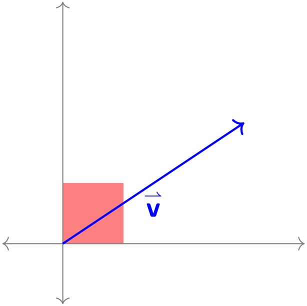
The transformation of the unit square by the standard matrix $[\vec{\mathbf{e}}_1 \ \vec{\mathbf{e}}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. What is $\det([\vec{\mathbf{e}}_1 \ \vec{\mathbf{e}}_2]) = \det(I)$, the area of the transformed unit square shown here?



a) 0
b) 1
c) 2
d) Cannot be determined

Activity G.1.7 *(∼2 min)*

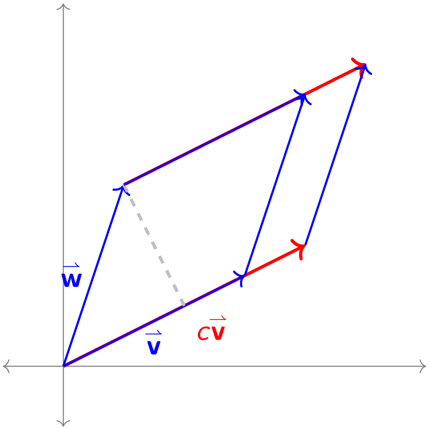
The transformation of the unit square by the standard matrix $[\vec{\mathbf{v}}\ \ \vec{\mathbf{v}}]$ is illustrated below: both $T(\vec{\mathbf{e}}_1) = T(\vec{\mathbf{e}}_2) = \vec{\mathbf{v}}$. What is $\det([\vec{\mathbf{v}}\ \ \vec{\mathbf{v}}])$, the area of the transformed unit square shown here?



a) 0
b) 1
c) 2
d) Cannot be determined

Activity G.1.8 *(∼5 min)*

The transformations of the unit square by the standard matrices $[\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]$ and $[c\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]$ are illustrated below. How are $\det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$ and $\det([c\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$ related?



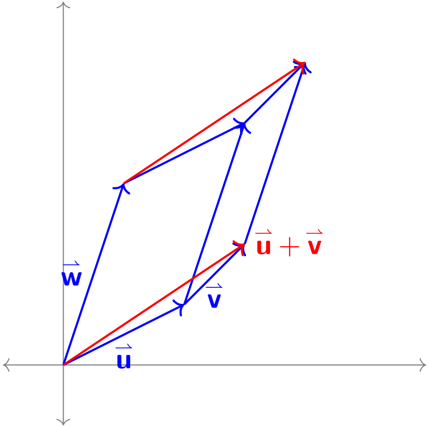
a) $\det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]) = \det([c\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$

b) $c + \det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]) = \det([c\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$

c) $c\det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]) = \det([c\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$

Activity G.1.9 *(∼5 min)*

The transformations of unit squares by the standard matrices $[\vec{\mathbf{u}}\ \ \vec{\mathbf{w}}]$, $[\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]$ and $[\vec{\mathbf{u}}+\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]$ are illustrated below. How is $\det([\vec{\mathbf{u}}+\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$ related to $\det([\vec{\mathbf{u}}\ \ \vec{\mathbf{w}}])$ and $\det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$?



a) $\det([\vec{\mathbf{u}}\ \ \vec{\mathbf{w}}]) = \det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]) = \det([\vec{\mathbf{u}}+\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$

b) $\det([\vec{\mathbf{u}}\ \ \vec{\mathbf{w}}]) + \det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]) = \det([\vec{\mathbf{u}}+\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$

c) $\det([\vec{\mathbf{u}}\ \ \vec{\mathbf{w}}])\det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]) = \det([\vec{\mathbf{u}}+\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$

Definition G.1.10

The **determinant** is the unique function $\det : M_{n,n} \to \mathbb{R}$ satisfying these properties:

P1: $\det(I) = 1$

P2: $\det(A) = 0$ whenever two columns of the matrix are identical.

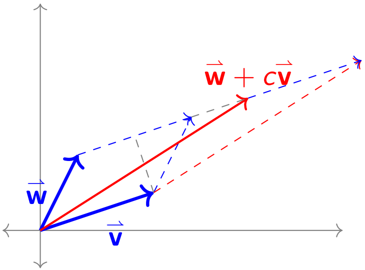
P3: $\det[\cdots \quad c\vec{\mathbf{v}} \quad \cdots] = c\det[\cdots \quad \vec{\mathbf{v}} \quad \cdots]$, assuming no other columns change.

P4: $\det[\cdots \quad \vec{\mathbf{v}} + \vec{\mathbf{w}} \quad \cdots] = \det[\cdots \quad \vec{\mathbf{v}} \quad \cdots] + \det[\cdots \quad \vec{\mathbf{w}} \quad \cdots]$, assuming no other columns change.

Note that these last two properties together can be phrased as "The determinant is linear in each column."

Observation G.1.11

The determinant must also satisfy other properties. Consider $\det([\vec{v} \quad \vec{w} + c\vec{v}])$ and $\det([\vec{v} \quad \vec{w}])$.



The base of both parallelograms is $\vec{v}$, while the height has not changed, so the determinant does not change either. This can also be proven using the other properties of the determinant:

$$\begin{aligned}
\det([\vec{v} + c\vec{w} \quad \vec{w}]) &= \det([\vec{v} \quad \vec{w}]) + \det([c\vec{w} \quad \vec{w}]) \\
&= \det([\vec{v} \quad \vec{w}]) + c\det([\vec{w} \quad \vec{w}]) \\
&= \det([\vec{v} \quad \vec{w}]) + c \cdot 0 \\
&= \det([\vec{v} \quad \vec{w}])
\end{aligned}$$

Observation G.1.12

Columns may be swapped by adding/subtracting columns from one another, which we've just seen doesn't change the determinant.

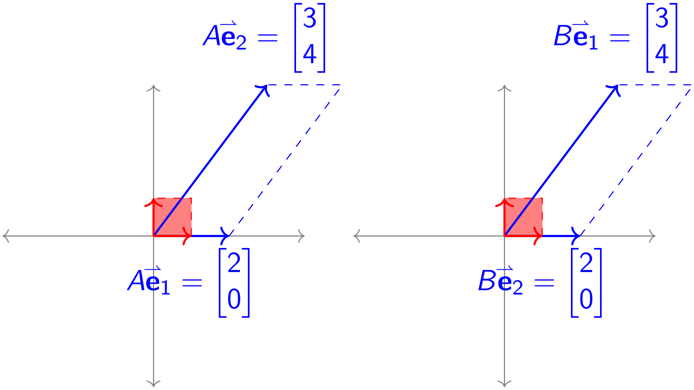
$$\begin{aligned}
\det([\vec{\mathbf{v}} \quad \vec{\mathbf{w}}]) &= \det([\vec{\mathbf{v}}+\vec{\mathbf{w}} \quad \vec{\mathbf{w}}]) \\
&= \det([\vec{\mathbf{v}}+\vec{\mathbf{w}} \quad \vec{\mathbf{w}}-(\vec{\mathbf{v}}+\vec{\mathbf{w}})]) \\
&= \det([\vec{\mathbf{v}}+\vec{\mathbf{w}} \quad -\vec{\mathbf{v}}]) \\
&= \det([\vec{\mathbf{v}}+\vec{\mathbf{w}}-\vec{\mathbf{v}} \quad -\vec{\mathbf{v}}]) \\
&= \det([\vec{\mathbf{w}} \quad -\vec{\mathbf{v}}]) \\
&= -\det([\vec{\mathbf{w}} \quad \vec{\mathbf{v}}])
\end{aligned}$$

So swapping two columns results in a negation of the determinant. Therefore, determinants represent a *signed* area, since they are not always positive.

Remark G.1.13

Swapping columns may be thought of as a reflection, which is represented by a negative determinant. For example, the following matrices transform the unit square into the same parallelogram, but the second matrix reflects its orientation.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}$$



Fact G.1.14

To summarize, we've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant in the following way:

(a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c\det([\cdots \quad \vec{\mathbf{v}} \quad \cdots]) = \det([\cdots \quad c\vec{\mathbf{v}} \quad \cdots])$$

(b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \quad \vec{\mathbf{v}} \quad \cdots \quad \vec{\mathbf{w}} \quad \cdots]) = -\det([\cdots \quad \vec{\mathbf{w}} \quad \cdots \quad \vec{\mathbf{v}} \quad \cdots])$$

(c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \quad \vec{\mathbf{v}} \quad \cdots \quad \vec{\mathbf{w}} \quad \cdots]) = \det([\cdots \quad \vec{\mathbf{v}}+c\vec{\mathbf{w}} \quad \cdots \quad \vec{\mathbf{w}} \quad \cdots])$$

Activity G.1.15 *($\sim$5 min)*

The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. By what factor does the transformation given by the standard matrix AB scale areas?

(a) 1

(b) 7

(c) 12

(d) Cannot be determined

Fact G.1.16

Since the transformation given by the standard matrix AB is obtained by applying the transformations given by A and B, it follows that

$$\det(AB) = \det(A)\det(B)$$

Remark G.1.17

Recall that row operations may be produced by matrix multiplication.

- Multiply the first row of A by c: $\begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$
- Swap the first and second row of A: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$
- Add c times the third row to the first row of A: $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Fact G.1.18

The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row: $\det \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c$
- Swapping rows: $\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1$
- Adding a row multiple to another row:
$$\det \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & c - 1c \\ 0 & 1 & 0 - 0c \\ 0 & 0 & 1 - 0c \end{bmatrix} = \det(I) = 1$$

Activity G.1.19 *(∼5 min)*

Consider the row operation $R_1 + 4R_3 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1+4(7) & 2+4(8) & 3+4(9) \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Find $\det R$ by comparing with the previous slide.

(c) If $C \in M_{3,3}$ is a matrix with $\det(C) = -3$, find

$$\det(RC) = \det(R)\det(C).$$

Activity G.1.20 *($\sim$5 min)*

Consider the row operation $R_1 \leftrightarrow R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to I.

(b) If $C \in M_{3,3}$ is a matrix with $\det(C) = 5$, find $\det(RC)$.

Activity G.1.21 *(∼5 min)*

Consider the row operation $3R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 3(4) & 3(5) & 3(6) \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det(C) = -7$, find $\det(RC)$.

Module G Section 2

Remark G.2.1

Recall that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying columns by scalars:

$$\det([\cdots \quad c\vec{\mathbf{v}} \quad \cdots]) = c\det([\cdots \quad \vec{\mathbf{v}} \quad \cdots])$$

(b) Swapping two columns:

$$\det([\cdots \quad \vec{\mathbf{v}} \quad \cdots \quad \vec{\mathbf{w}} \quad \cdots]) = -\det([\cdots \quad \vec{\mathbf{w}} \quad \cdots \quad \vec{\mathbf{v}} \quad \cdots])$$

(c) Adding a multiple of a column to another column:

$$\det([\cdots \quad \vec{\mathbf{v}} \quad \cdots \quad \vec{\mathbf{w}} \quad \cdots]) = \det([\cdots \quad \vec{\mathbf{v}} + c\vec{\mathbf{w}} \quad \cdots \quad \vec{\mathbf{w}} \quad \cdots])$$

Remark G.2.2

The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row: $\det \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c$
- Swapping rows: $\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1$
- Adding a row multiple to another row:

  $\det \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & c - 1c \\ 0 & 1 & 0 - 0c \\ 0 & 0 & 1 - 0c \end{bmatrix} = \det(I) = 1$

Fact G.2.3

Thus we can also use row operations to simplify determinants:

1. Multiplying rows by scalars: $\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$
2. Swapping two rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = - \det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$
3. Adding multiples of rows to other rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$

Observation G.2.4

So we may compute the determinant of $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ by manipulating its rows/columns to reduce the matrix to I:

$$\begin{aligned} \det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} &= 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \\ &= 2 \det \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \\ &= -2 \det \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ &= -2 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= -2 \end{aligned}$$

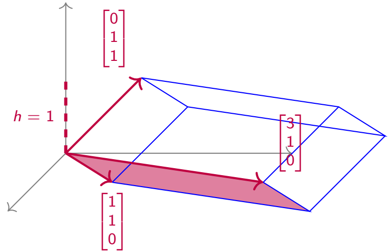
Remark G.2.5

So we see that row reducing all the way into RREF gives us a method of computing determinants!

However, we learned in module E that this can be tedious for large matrices. Thus, we will try to figure out how to turn the determinant of a larger matrix into the determinant of a smaller matrix.

Activity G.2.6 *(~5 min)*

The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.



Recall that for this solid $V = Bh$, where h is the height of the solid and B is the area of its parallelogram base. So what must its volume be?

(a) $\det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ (c) $\det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$ (d) $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Fact G.2.7

If row i contains all zeros except for a 1 on the main (upper-left to lower-right) diagonal, then both column and row i may be removed without changing the value of the determinant.

$$\det \begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Since row and column operations affect the determinants in the same way, the same technique works for a column of all zeros except for a 1 on the main diagonal.

$$\det \begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Activity G.2.8 *(∼5 min)*

Remove an appropriate row and column of $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$ to simplify the determinant to a 2×2 determinant.

Activity G.2.9 *(∼5 min)*

Simplify $\det \begin{bmatrix} 0 & 3 & -2 \\ 2 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

- Factor out a 2 from a column.
- Swap rows or columns to put a 1 on the main diagonal.

Activity G.2.10 *(∼5 min)*

Simplify $\det\begin{bmatrix} 4 & -2 & 2 \\ 3 & 1 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

- Use row/column operations to create two zeroes in the same row or column.
- Factor/swap as needed to get a row/column of all zeroes except a 1 on the main diagonal.

Observation G.2.11

Using row/column operations, you can introduce zeros and reduce dimension to whittle down the determinant of a large matrix to a determinant of a smaller matrix.

$$\det \begin{bmatrix} 4 & 3 & 0 & 1 \\ 2 & -2 & 4 & 0 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} = \det \begin{bmatrix} 4 & 3 & 0 & 1 \\ 6 & -18 & 0 & -20 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} = \det \begin{bmatrix} 4 & 3 & 1 \\ 6 & -18 & -20 \\ 2 & 8 & 3 \end{bmatrix}$$

$$= \cdots = -2 \det \begin{bmatrix} 1 & 3 & 4 \\ 0 & 21 & 43 \\ 0 & -1 & -10 \end{bmatrix} = -2 \det \begin{bmatrix} 21 & 43 \\ -1 & -10 \end{bmatrix}$$

$$= \cdots = -2 \det \begin{bmatrix} -167 & 21 \\ 0 & 1 \end{bmatrix} = -2 \det [-167]$$

$$= -2(-167) \det(I) = 334$$

Activity G.2.12 *(∼10 min)*

Compute $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$ by using any combination of row/column operations.

Observation G.2.13

Another option is to take advantage of the fact that the determinant is linear in each row or column. This approach is called **Laplace expansion** or **cofactor expansion**.

For example, since $\begin{bmatrix}1 & 2 & 4\end{bmatrix} = 1\begin{bmatrix}1 & 0 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 1 & 0\end{bmatrix} + 4\begin{bmatrix}0 & 0 & 1\end{bmatrix}$,

$$\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 1 & 2 & 4\end{bmatrix} = 1\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 1 & 0 & 0\end{bmatrix} + 2\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 0 & 1 & 0\end{bmatrix} + 4\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 0 & 0 & 1\end{bmatrix}$$

$$= -1\det\begin{bmatrix}5 & 3 & 2\\ 5 & 3 & -1\\ 0 & 0 & 1\end{bmatrix} - 2\det\begin{bmatrix}2 & 5 & 3\\ -1 & 5 & 3\\ 0 & 0 & 1\end{bmatrix} + 4\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 0 & 0 & 1\end{bmatrix}$$

$$= -\det\begin{bmatrix}5 & 3\\ 5 & 3\end{bmatrix} - 2\det\begin{bmatrix}2 & 5\\ -1 & 5\end{bmatrix} + 4\det\begin{bmatrix}2 & 3\\ -1 & 3\end{bmatrix}$$

Observation G.2.14

Applying Laplace expansion to a 2×2 matrix yields a short formula you may have seen:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} + b \det \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} - b \det \begin{bmatrix} 1 & 0 \\ d & c \end{bmatrix} = ad - bc.$$

There are formulas for the determinants of larger matrices, but they can be pretty tedious to use. For example, writing out a formula for a 4×4 determinant would require 24 different terms!

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}(a_{22}(a_{33}a_{44} - a_{43}a_{34}) - a_{23}(a_{32}a_{44} - a_{42}a_{34}) + \dots) + \dots$$

So this is why we either use Laplace expansion or row/column operations directly.

Activity G.2.15 *(∼10 min)*

Use Laplace expansion to compute $\det \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & -3 & 2 & -2 \end{bmatrix}$.

Activity G.2.16 *($\sim$5 min)*

Based on what we've done today, which technique is easier for computing determinants?

(a) Memorizing formulas.

(b) Using row/column operations.

(c) Laplace expansion.

(d) Some other technique (be prepared to describe it).

Activity G.2.17 *(∼10 min)*

Use your preferred technique to compute $\det \begin{bmatrix} 4 & -3 & 0 & 0 \\ 1 & -3 & 2 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & -3 & 2 & -2 \end{bmatrix}$.

Module G Section 3

Activity G.3.1 *(∼5 min)*

An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Compute $\det(M)$ and $\det(M^{-1})$ using the formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Fact G.3.2

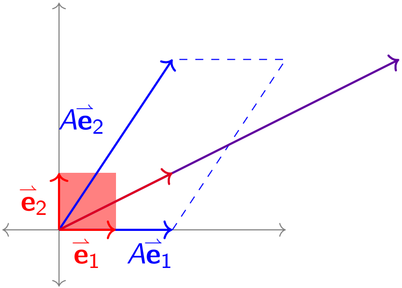
- For every invertible matrix M,

$$\det(M)\det(M^{-1}) = \det(I) = 1$$

  so $\det(M^{-1}) = \frac{1}{\det(M)}$.
- Furthermore, a square matrix M is invertible if and only if $\det(M) \neq 0$.

Observation G.3.3

Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.



It is easy to see geometrically that

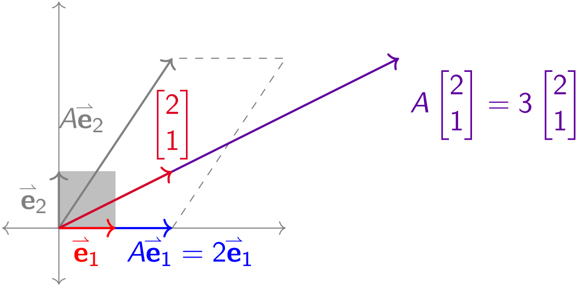
$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily checked once you find it) that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Definition G.3.4

Let $A \in M_{n,n}$. An **eigenvector** for A is a vector $\vec{\mathbf{x}} \in \mathbb{R}^n$ such that $A\vec{\mathbf{x}}$ is parallel to $\vec{\mathbf{x}}$.



In other words, $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ for some scalar λ. If $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$, then we say $\vec{\mathbf{x}}$ is a **nontrivial eigenvector** and we call this λ an **eigenvalue** of A.

Activity G.3.5 *(~5 min)*
Finding the eigenvalues λ that satisfy

$$A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}} = \lambda(I\vec{\mathbf{x}}) = (\lambda I)\vec{\mathbf{x}}$$

for some nontrivial eigenvector $\vec{\mathbf{x}}$ is equivalent to finding nonzero solutions for the matrix equation

$$A\vec{\mathbf{x}} - (\lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}.$$

Which of the following must be true for any eigenvalue?

(a) The kernel of the transformation with standard matrix $A - \lambda I$ must contain the zero vector, so $A - \lambda I$ is invertible.

(b) The kernel of the transformation with standard matrix $A - \lambda I$ must contain a nonzero vector, so $A - \lambda I$ is not invertible.

(c) The image of the transformation with standard matrix $A - \lambda I$ must contain the zero vector, so $A - \lambda I$ is invertible.

(d) The image of the transformation with standard matrix $A - \lambda I$ must contain a nonzero vector, so $A - \lambda I$ is invertible.

Fact G.3.6
The eigenvalues λ for a matrix A are the values that make $A - \lambda I$ non-invertible.

Thus the eigenvalues λ for a matrix A are the solutions to the equation

$$\det(A - \lambda I) = 0.$$

Definition G.3.7
The expression $\det(A - \lambda I)$ is called **characteristic polynomial** of A.

For example, when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

Thus the characteristic polynomial of A is

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(3) = \lambda^2 - 5\lambda - 2$$

and its eigenvalues are the solutions to $\lambda^2 - 5\lambda - 2 = 0$.

Activity G.3.8 *(∼10 min)*

Compute $\det(A - \lambda I)$ to find the characteristic polynomial of $A = \begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$.

Activity G.3.9 *(∼10 min)*

Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$.

Activity G.3.9 *(∼10 min)*

Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Activity G.3.9 *(∼10 min)*

Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.

Part 2: Factor this characteristic polynomial to determine the eigenvalues of A.

Activity G.3.10 *(∼10 min)*

Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix}$.

Activity G.3.11 *(∼10 min)*

It's possible to show that -2 is an eigenvalue for $\begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 9 \\ 3 & 0 & 4 \end{bmatrix}$.

Compute the kernel of the transformation with standard matrix

$$A - (-2)I = \begin{bmatrix} ? & 4 & -2 \\ 2 & ? & 9 \\ 3 & 0 & ? \end{bmatrix}$$

to find all the eigenvectors $\vec{\mathbf{x}}$ such that $A\vec{\mathbf{x}} = -2\vec{\mathbf{x}}$.

Definition G.3.12

Since the kernel of a linear map is a subspace of $\mathbb{R}^n$, and the kernel obtained from $A - \lambda I$ contains all the eigenvectors associated with λ, we call this kernel the **eigenspace** of A associated with λ.

Activity G.3.13 *(∼10 min)*

Find a basis for the eigenspace for the matrix $\begin{bmatrix} 3 & -6 & 1 \\ -1 & 4 & 2 \\ 3 & -9 & 4 \end{bmatrix}$ associated with the eigenvalue 1.