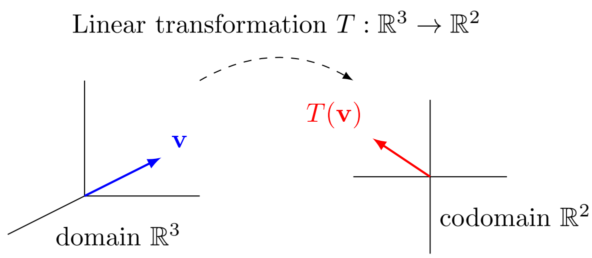
Application Activities - Module A Part 1 - Class Day 17

Definition 17.1 A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}$, $\mathbf{v} \in V$.

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.

Definition 17.2 Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.



Example 17.3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x-z\\y\end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} + \begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}\right) = T\left(\begin{bmatrix}x_1+x_2\\y_1+y_2\\z_1+z_2\end{bmatrix}\right) = \begin{bmatrix}(x_1+x_2)-(z_1+z_2)\\(y_1+y_2)\end{bmatrix}$$

$$T\left(\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}\right) + T\left(\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}\right) = \begin{bmatrix}x_1-z_1\\y_1\end{bmatrix} + \begin{bmatrix}x_2-z_2\\y_2\end{bmatrix} = \begin{bmatrix}(x_1+x_2)-(z_1+z_2)\\(y_1+y_2)\end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = T\left(\begin{bmatrix}cx\\cy\\cz\end{bmatrix}\right) = \begin{bmatrix}cx-cz\\cy\end{bmatrix} \text{ and } cT\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = c\begin{bmatrix}x-z\\y\end{bmatrix} = \begin{bmatrix}cx-cz\\cy\end{bmatrix}$$

Therefore T is a linear transformation.

Activity 17.4 Determine if each of the following maps are linear transformations
Part 1: $T_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $T_1\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \sqrt{x^2+y^2}$.
Part 2: $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_2\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}-x\\-y\\-z\end{bmatrix}$
Part 3: $T_3 : \mathcal{P}^d \to \mathcal{P}^{d-1}$ given by $T_3(f(x)) = f'(x)$.
Part 4: $T_4 : \mathcal{P} \to \mathcal{P}$ given by $T_4(f(x)) = f(x) + x^2$

Activity 17.5 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity 17.6 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}0\\0\\-2\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity 17.7 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity 17.8 Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}-2\\0\\-3\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity 17.9 Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation. How many facts of the form $T(\mathbf{v}_i) = \mathbf{w}_i$ do you need to know in order to be able to compute $T(\mathbf{v})$ for *any* $\mathbf{v} \in \mathbb{R}^4$?

(a) 2

(b) 3

(c) 4

(d) 5

(e) You need infinitely many

(In this situation, we say that the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ **determine** T.)

Fact 17.10 Consider any basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for V. Since every vector can be written *uniquely* as a linear combination of basis vectors, every linear transformation $T : V \to W$ is determined by those basis vectors.

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = x_1T(\mathbf{b}_1) + \cdots + x_nT(\mathbf{b}_n)$$

Definition 17.11 The **standard basis** of $\mathbb{R}^n$ is the (ordered) basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Since linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by the values of each $T(\mathbf{e}_i)$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$.

Example 17.12 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\begin{bmatrix}3 & -1 & 5\\2 & 4 & 0\end{bmatrix}.$$

Activity 17.13 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x + 3z\\2x - y - 4z\end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard basis.

Activity 17.14 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$\begin{bmatrix}3 & -2 & -1\\4 & 5 & 2\end{bmatrix}.$$

Compute $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)$.

Activity 17.15 Let $D : \mathcal{P}^3 \to \mathcal{P}^2$ be the derivative map $D(f(x)) = f'(x)$. (Earlier we showed this is a linear transformation.)
Part 1: Write down an equivalent linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ by converting $\{1, x, x^2, x^3\}$ and $\{D(1), D(x), D(x^2), D(x^3)\}$ into appropriate vectors in $\mathbb{R}^4$ and $\mathbb{R}^3$.
Part 2: Write the standard matrix corresponding to T.
