

## Module E

### Standard E1

**E1.1** Write an augmented matrix corresponding to the following system of linear equations.

$$x + 3y - 4z = 5$$

$$3x + 9y + z = 0$$

$$x - z = 1$$

*Solution.*

$$\left[ \begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{array} \right]$$

□

**E1.2** Write an augmented matrix corresponding to the following system of linear equations.

$$x_1 + 4x_3 = 1$$

$$x_2 - x_3 = 7$$

$$x_1 - x_2 + 3x_3 = -1$$

*Solution.*

$$\left[ \begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array} \right]$$

□

**E1.3** Write an augmented matrix corresponding to the following system of linear equations.

$$x_1 + 4x_3 = 1$$

$$x_2 - x_3 = 7$$

$$x_1 - x_2 + 3x_4 = -1$$

*Solution.*

$$\left[ \begin{array}{cccc|c} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & -1 & 0 & 7 \\ 1 & -1 & 0 & 3 & -1 \end{array} \right]$$

□

**E1.4** Write an augmented matrix corresponding to the following system of linear equations.

$$x_1 + 3x_2 - 4x_3 + x_4 = 5$$

$$3x_1 + 9x_2 + x_3 - 7x_4 = 0$$

$$x_1 - x_3 + x_4 = 1$$

*Solution.*

$$\left[ \begin{array}{cccc|c} 1 & 3 & -4 & 1 & 5 \\ 3 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{array} \right]$$

□

**E1.5** Write a system of linear equations corresponding to the following augmented matrix.

$$\left[ \begin{array}{cccc|c} 3 & -1 & 0 & 1 & 5 \\ -1 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 0 & -3 \end{array} \right]$$

*Solution.*

$$\begin{aligned}3x_1 - x_2 + x_4 &= 5 \\ -x_1 + 9x_2 + x_3 - 7x_4 &= 0 \\ x_1 - x_3 &= -3\end{aligned}$$

□

**E1.6** Write a system of linear equations corresponding to the following augmented matrix.

$$\left[ \begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 4 & 1 & -7 \\ 1 & 2 & -1 & 0 \end{array} \right]$$

*Solution.*

$$\begin{aligned}2x_1 - x_2 &= 1 \\ -x_1 + 4x_2 + x_3 &= -7 \\ x_1 + 2x_2 - x_3 &= 0\end{aligned}$$

□

**E1.7** Write a system of linear equations corresponding to the following augmented matrix.

$$\left[ \begin{array}{ccc|c} -4 & -1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{array} \right]$$

*Solution.*

$$\begin{aligned}-4x_1 - x_2 + 3x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 0 \\ -x_1 + 4x_2 + x_3 &= 4\end{aligned}$$

□

**E1.8** Write a system of linear equations corresponding to the following augmented matrix.

$$\left[ \begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array} \right]$$

*Solution.*

$$\begin{aligned}x_1 + 4x_3 &= 1 \\ x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= -1\end{aligned}$$

□

## Standard E2

**E2.1** Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

*Solution.*

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

□

**E2.2** Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix}$$

*Solution.*

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

**E2.3** Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{bmatrix}$$

*Solution.*

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

**E2.4** Find the reduced row echelon form of the matrix below.

$$\begin{bmatrix} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{bmatrix}$$

*Solution.*

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{12} & 1 \\ 0 & 1 & 0 & \frac{3}{4} & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{bmatrix}$$

□

**E2.5** Find RREF  $A$ , where

$$A = \begin{bmatrix} 3 & -2 & 1 & 8 & -5 \\ 2 & 2 & 0 & 6 & -2 \\ -1 & 1 & 1 & -4 & 6 \end{bmatrix}$$

*Solution.*

$$\text{RREF } A = \begin{bmatrix} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \end{bmatrix}$$

□

**E2.6** Find RREF  $A$ , where

$$A = \begin{bmatrix} 2 & -7 & 4 \\ 1 & -3 & 2 \\ 3 & 0 & 3 \end{bmatrix}$$

*Solution.*

$$\text{RREF } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

**E2.7** Find RREF  $A$ , where

$$A = \begin{bmatrix} 2 & -1 & 5 & 4 \\ -1 & 0 & -2 & -1 \\ 1 & 3 & -1 & -5 \end{bmatrix}$$

*Solution.*

$$\text{RREF } A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

**E2.8** Find RREF  $A$ , where

$$A = \begin{bmatrix} 2 & 2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 4 & 5 \\ 3 & 3 & -1 & -2 & 1 \end{bmatrix}$$

*Solution.*

$$\text{RREF } A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

□

### Standard E3

**E3.1** Find the solution set for the following system of linear equations.

$$x + 3y - 4z = 5$$

$$3x + 9y + z = 2$$

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 2 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

So the solution set is

$$\left\{ \left[ \begin{array}{c} 1 - 3c \\ c \\ -1 \end{array} \right] \mid c \in \mathbb{R} \right\}$$

□

**E3.2** Find the solution set for the following system of linear equations.

$$-3x + y = 2$$

$$-8x + 2y - z = 6$$

$$2y + 3z = -2$$

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

So the solution set is

$$\left\{ \left[ \begin{array}{c} -1 - \frac{c}{2} \\ -1 - \frac{3c}{2} \\ c \end{array} \right] \mid c \in \mathbb{R} \right\} = \left\{ \left[ \begin{array}{c} c - 1 \\ 3c - 1 \\ -2c \end{array} \right] \mid c \in \mathbb{R} \right\}$$

□

**E3.3** Find the solution set for the following system of linear equations.

$$2x + y - z + w = 5$$

$$3x - y - 2w = 0$$

$$-x + 5z + 3w = -1$$

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right] \right) = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{12} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{array} \right]$$

So the solution set is

$$\left\{ \left[ \begin{array}{c} 1 + a \\ 3 - 21a \\ -7a \\ 12a \end{array} \right] \mid a \in \mathbb{R} \right\}$$

□

**E3.4** Find the solution set for the following system of linear equations.

$$2x + y - z + w = 5$$

$$3x - y - 2w = 0$$

$$-x + 5z + 3w = -1$$

*Solution.*

$$\text{RREF} \left( \left[ \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right] \right) = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{4} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{array} \right]$$

So the solution set is

$$\left\{ \left[ \begin{array}{c} 1+a \\ 3-21a \\ -7a \\ 12a \end{array} \right] \mid a \in \mathbb{R} \right\}$$

□

**E3.5** Find the solution set for the following system of linear equations.

$$2x_1 - 2x_2 + 6x_3 - x_4 = -1$$

$$3x_1 + 6x_3 + x_4 = 5$$

$$-4x_1 + x_2 - 9x_3 + 2x_4 = -7$$

*Solution.* Let  $A = \left[ \begin{array}{cccc|c} 2 & -2 & 6 & -1 & -1 \\ 3 & 0 & 6 & 1 & 5 \\ -4 & 1 & -9 & 2 & -7 \end{array} \right]$ , so  $\text{RREF } A = \left[ \begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right]$ . It follows that the

solution set is given by  $\left\{ \left[ \begin{array}{c} 2-2a \\ 3+a \\ a \\ -1 \end{array} \right] \mid a \in \mathbb{R} \right\}$ .

□

**E3.6** Find the solution set for the following system of linear equations.

$$2x_1 + 3x_2 - 5x_3 + 14x_4 = 8$$

$$x_1 + x_2 - x_3 + 5x_4 = 3$$

*Solution.* Let  $A = \left[ \begin{array}{cccc|c} 2 & 3 & -5 & 14 & 8 \\ 1 & 1 & -1 & 5 & 3 \end{array} \right]$ , so  $\text{RREF } A = \left[ \begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -3 & 4 & 2 \end{array} \right]$ . It follows that the solution set

is given by  $\left\{ \left[ \begin{array}{c} 1-2a-b \\ 2+3a-4b \\ a \\ b \end{array} \right] \mid a, b \in \mathbb{R} \right\}$ .

□

**E3.7** Find the solution set for the following system of linear equations.

$$4x_1 + 4x_2 + 3x_3 - 6x_4 = 5$$

$$-2x_3 - 4x_4 = 3$$

$$2x_1 + 2x_2 + x_3 - 4x_4 = -1$$

*Solution.* Let  $A = \left[ \begin{array}{cccc|c} 4 & 4 & 3 & -6 & 5 \\ 0 & 0 & -2 & -4 & 3 \\ 2 & 2 & 1 & -4 & -1 \end{array} \right]$ , so  $\text{RREF } A = \left[ \begin{array}{cccc|c} 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$ . It follows that the system is inconsistent with no solutions (since the bottom row implies the contradiction  $0 = 1$ ), so its solution set is  $\emptyset$ .

□

**E3.8** Find the solution set for the following system of linear equations.

$$3x + 2y + z = 7$$

$$x + y + z = 1$$

$$-2x + 3z = -11$$

*Solution.* Let  $A = \left[ \begin{array}{ccc|c} 3 & 2 & 1 & 7 \\ 1 & 1 & 1 & 1 \\ -2 & 0 & 3 & 11 \end{array} \right]$ , so  $\text{RREF } A = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array} \right]$ . It follows that solution set is

$$\left\{ \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} \right\}$$

□

# Module V

## Standard V1

**V1.1** Let  $V$  be the set of all real numbers together with the operations  $\oplus$  and  $\odot$  defined by, for any  $x, y \in V$  and  $c \in \mathbb{R}$ ,

$$\begin{aligned}x \oplus y &= x + y \\c \odot x &= cx - 3(c - 1)\end{aligned}$$

- (a) Show that scalar multiplication is associative:  $a \odot (b \odot x) = (ab) \odot x$  for all scalars  $a, b \in \mathbb{R}$  and  $x \in V$ .
- (b) Explain why  $V$  nonetheless isn't a vector space.

**V1.2** Let  $V$  be the set of all pairs of real numbers with the operations, for any  $(x_1, x_2), (y_1, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$\begin{aligned}(x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, x_2 + y_2 + 2x_1y_1) \\c \odot (x_1, x_2) &= (cx_1, cx_2)\end{aligned}$$

- (a) Show that the vector addition  $\oplus$  is associative:  
 $(x_1, x_2) \oplus ((y_1, y_2) \oplus (z_1, z_2)) = ((x_1, x_2) \oplus (y_1, y_2)) \oplus (z_1, z_2)$  for all  $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in V$ .
- (b) Explain why  $V$  nonetheless isn't a vector space.

**V1.3** Let  $V$  be the set of all pairs of real numbers with the operations, for any  $(x_1, x_2), (y_1, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$\begin{aligned}(x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1 - 1, x_2 + y_2 - 1) \\c \odot (x_1, x_2) &= (cx_1, cx_2)\end{aligned}$$

- (a) Show that this vector space has an additive identity element: there exists  $\vec{z} \in V$  satisfying  $(x, y) \oplus \vec{z} = (x, y)$  for every  $(x, y) \in V$ .
- (b) Explain why  $V$  nonetheless isn't a vector space.

**V1.4** Let  $V$  be the set of all pairs of real numbers with the operations, for any  $(x_1, x_2), (y_1, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$\begin{aligned}(x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\c \odot (x_1, x_2) &= (0, cx_2)\end{aligned}$$

- (a) Show that scalar multiplication distributes over scalar addition:  
 $(c + d) \odot (x_1, x_2) = c \odot (x_1, x_2) \oplus d \odot (x_1, x_2)$  for every  $c, d \in \mathbb{R}$  and  $(x_1, x_2) \in V$ .
- (b) Explain why  $V$  nonetheless isn't a vector space.

**V1.5** Let  $V$  be the set of all pairs of real numbers with the operations, for any  $(x_1, x_2), (y_1, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$\begin{aligned}(x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\c \odot (x_1, x_2) &= (c^2x_1, c^3x_2)\end{aligned}$$

- (a) Show that scalar multiplication distributes over vector addition:  
 $c \odot ((x_1, x_2) \oplus (y_1, y_2)) = c \odot (x_1, x_2) \oplus c \odot (y_1, y_2)$  for all  $c \in \mathbb{R}$  and  $(x_1, x_2), (y_1, y_2) \in V$ .
- (b) Explain why  $V$  nonetheless isn't a vector space.

**V1.6** Let  $V$  be the set of all real numbers with the operations, for any  $x, y \in V$ ,  $c \in \mathbb{R}$ ,

$$\begin{aligned}x \oplus y &= \sqrt{x^2 + y^2} \\c \odot x &= cx\end{aligned}$$



- (a) Show that the vector addition  $\oplus$  is associative:  $x \oplus (y \oplus z) = (x \oplus y) \oplus z$  for all  $x, y, z \in V$ .
- (b) Explain why  $V$  nonetheless isn't a vector space.

**V1.7** Let  $V$  be the set of all pairs of real numbers with the operations, for any  $(x_1, x_2), (y_1, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$\begin{aligned}(x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, x_2 y_2) \\ c \odot (x_1, x_2) &= (c x_1, c x_2)\end{aligned}$$

- (a) Show that there is an additive identity element: there exists an element  $\vec{z} \in V$  such that  $(x_1, x_2) \oplus \vec{z} = (x_1, x_2)$  for any  $(x_1, x_2) \in V$ .
- (b) Explain why  $V$  nonetheless isn't a vector space.

## Standard V2

**V2.1** Determine if  $\begin{bmatrix} 0 \\ -1 \\ 2 \\ 6 \end{bmatrix}$  can be written as a linear combination of the vectors  $\begin{bmatrix} 3 \\ -1 \\ -1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ .

*Solution.*  $\begin{bmatrix} 0 \\ -1 \\ 2 \\ 6 \end{bmatrix}$  can be written as a linear combination of the vectors  $\begin{bmatrix} 3 \\ -1 \\ -1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ , namely

$$\begin{bmatrix} 0 \\ -1 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

□

**V2.2** Determine if  $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$  can be written as a linear combination of the vectors  $\begin{bmatrix} 5 \\ 2 \\ -3 \\ 2 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ , and  $\begin{bmatrix} 8 \\ 3 \\ 5 \\ -1 \end{bmatrix}$ .

*Solution.*  $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$  is not a linear combination of the three other vectors.

□

**V2.3** Determine if  $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$  can be written as a linear combination of the vectors  $\begin{bmatrix} -1 \\ -9 \\ 15 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 5 \\ -5 \end{bmatrix}$ .

*Solution.*  $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$  cannot be written as a linear combination of the vectors  $\begin{bmatrix} -1 \\ -9 \\ 15 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 5 \\ -5 \end{bmatrix}$ .

□

**V2.4** Determine if  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ , and  $\begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}$ .

*Solution.*  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is a linear combination of the three vectors.

□

**V2.5** Determine if  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ , and  $\begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$ .

*Solution.*  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is not a linear combination of the three vectors.

□

**V2.6** Determine if  $\begin{bmatrix} 0 \\ -1 \\ 6 \\ -7 \end{bmatrix}$  belongs to the span of the set  $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \\ 3 \end{bmatrix} \right\}$ .

*Solution.*  $\begin{bmatrix} 0 \\ -1 \\ 6 \\ -7 \end{bmatrix}$  belongs to the span of the other two vectors.

□

**V2.7** Determine if  $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$  belongs to the span of the set  $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \\ 3 \end{bmatrix} \right\}$ .

*Solution.*  $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$  does not belong to the span of the other two vectors. □

**V2.8** Determine if  $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$  belongs to the span of the set  $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ .

*Solution.*  $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$  is not in the span of the other three vectors. □

### Standard V3

**V3.1** Determine if the vectors  $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ , and  $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$  span  $\mathbb{R}^3$

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix has a zero row, the vectors do not span  $\mathbb{R}^3$ . □

**V3.2** Determine if the vectors  $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}$ , and  $\begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix}$  span  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are zero rows, they do not span. □

**V3.3** Determine if the vectors  $\begin{bmatrix} 8 \\ 21 \\ -7 \end{bmatrix}$ ,  $\begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ , and  $\begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}$  span  $\mathbb{R}^3$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 8 & -3 & -1 & 4 \\ 21 & -8 & -3 & 11 \\ -7 & 3 & 2 & -5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, they do not span  $\mathbb{R}^3$ . □

**V3.4** Determine if the vectors  $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$  span  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, the vectors do not span  $\mathbb{R}^4$ . □

**V3.5** Determine if the vectors  $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 1 \\ 0 \\ -3 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}$ , and  $\begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$  span  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since every row contains a pivot, the vectors span  $\mathbb{R}^4$ . □

**V3.6** Does  $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$ ?

*Solution.* Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 4 & 2 \\ 4 & -9 & -3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has a zero row, the vectors fail to span  $\mathbb{R}^3$ . □

**V3.7** Does  $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$ ?

*Solution.* Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 2 & 2 \\ 4 & -9 & 3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

lacks a zero row, the vectors span  $\mathbb{R}^3$ . □

**V3.8** Does  $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 5 \\ 1 \\ -3 \end{bmatrix} \right\} = \mathbb{R}^5$ ?

*Solution.* Since there are only three vectors, they cannot span  $\mathbb{R}^5$ . (Or, since RREF must contain a zero row, so they cannot span  $\mathbb{R}^5$ .) □

## Standard V4

**V4.1** Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \text{ are integers} \right\} \quad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x = y \right\}$$

Show that one of these sets is a subspace of  $\mathbb{R}^2$ , and that one of the sets is not.

*Solution.*  $W$  is not a subspace.  $U$  is a subspace. □

**V4.2** Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\} \quad U = \left\{ \begin{bmatrix} x \\ y \\ 1 \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

Show that one of these sets is a subspace of  $\mathbb{R}^4$ , and that one of the sets is not.

*Solution.*  $U$  is not a subspace.  $W$  is a subspace. □

**V4.3** Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y + z = 1 \right\} \quad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y + z = 0 \right\}$$

Show that one of these sets is a subspace of  $\mathbb{R}^3$ , and that one of the sets is not.

*Solution.*  $W$  is not a subspace.  $U$  is a subspace. □

**V4.4** Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y = 3z \right\} \quad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y = 3, z \in \mathbb{R} \right\}$$

Show that one of these sets is a subspace of  $\mathbb{R}^3$ , and that one of the sets is not.

*Solution.*  $U$  is not a subspace.  $W$  is a subspace. □

**V4.5** Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x + y = 0 \right\} \quad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid xy = 0 \right\}$$

Show that one of these sets is a subspace of  $\mathbb{R}^2$ , and that one of the sets is not.

*Solution.*  $U$  is not a subspace.  $W$  is a subspace. □

**V4.6** Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x = y \right\} \quad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid |x| = |y| \right\}$$

Show that one of these sets is a subspace of  $\mathbb{R}^2$ , and that one of the sets is not.

*Solution.*  $U$  is not a subspace.  $W$  is a subspace. □

**V4.7** Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid y = 2x \right\} \quad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid y = x^2 \right\}$$

Show that one of these sets is a subspace of  $\mathbb{R}^2$ , and that one of the sets is not.  
*Solution.*  $U$  is not a subspace.  $W$  is a subspace. □

**V4.8** Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid z = 2xy \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid z = 2x + y \right\}$$

Show that one of these sets is a subspace of  $\mathbb{R}^3$ , and that one of the sets is not.  
*Solution.*  $W$  is not a subspace.  $U$  is a subspace. □

## Standard V5

**V5.1** Determine if set of vectors  $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$  is linearly dependent or linearly independent.

*Solution.* The vectors are linearly independent. □

**V5.2** Determine if the set of vectors  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}$  is linearly dependent or linearly independent

*Solution.* The set is linearly dependent. □

**V5.3** Determine if the set of vectors  $\left\{ \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$  is linearly dependent or linearly independent

*Solution.* The set is linearly independent. □

**V5.4** Determine if the set of vectors  $\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$  is linearly dependent or linearly independent

*Solution.* The set is linearly dependent. □

**V5.5** Determine if the set of vectors  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 6 \\ 5 \end{bmatrix} \right\}$  is linearly dependent or linearly independent.

*Solution.* The set is linearly dependent. □

**V5.6** Determine if the set of vectors  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 6 \\ 5 \end{bmatrix} \right\}$  is linearly dependent or linearly

independent.

*Solution.* The set is linearly independent. □

**V5.7** Determine if the set of vectors  $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \end{bmatrix} \right\}$  is linearly dependent or linearly

independent.

*Solution.* The set is linearly dependent. □

**V5.8** Determine if the set of vectors  $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right\}$  is linearly dependent or linearly

independent.

*Solution.* The set is linearly dependent. □



## Standard V6

**V6.1** Determine if the set  $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^3$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

**V6.2** Determine if the set  $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis. □

**V6.3** Determine if the set  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^3$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 4 \\ 2 & 2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

**V6.4** Determine if the set  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 5 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 3 & 2 & 1 & -1 \\ -1 & 0 & -1 & 3 \\ 2 & 2 & 0 & 0 \\ 3 & 4 & -1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

**V6.5** Determine if the set  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\}$  is a basis for  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

**V6.6** Determine if the set  $\left\{ \begin{bmatrix} 8 \\ 21 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  is a basis for  $\mathbb{R}^3$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 8 & -3 & -1 \\ 21 & -8 & -3 \\ -7 & 3 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

**V6.7** Determine if the set  $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\}$  is a basis for  $\mathbb{R}^3$ .

*Solution.* Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & -4 \\ -1 & 12 & 2 \\ 4 & -9 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

**V6.8** Determine if the set  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$  is a basis for  $\mathbb{R}^4$ .

*Solution.*

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since this is the identity matrix, this set is a basis. □

## Standard V7

**V7.1** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\} \right)$ . Find a basis for  $W$ .

*Solution.*  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}$  is a basis for  $W$ . □

**V7.2** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\} \right)$ . Find a basis for  $W$ .

*Solution.*  $\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$  is a basis for  $W$ . □

**V7.3** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$ . Find a basis of  $W$ .

*Solution.*  $\left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$  is a basis of  $W$ . □

**V7.4** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix} \right\} \right)$ . Find a basis of  $W$ .

*Solution.*  $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix} \right\}$  is a basis for  $W$ . □

**V7.5** Let  $W = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -8 \\ -1 \end{bmatrix} \right\}$ . Find a basis for this vector space.

*Solution.*  $\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$  is a basis of  $W$ . □

**V7.6** Let  $W = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\}$ . Find a basis for this vector space.

*Solution.*  $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix} \right\}$  is a basis of  $W$ . □

**V7.7** Let  $W = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}$ . Find a basis for this vector space.

*Solution.*  $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}$  is a basis of  $W$ .

□

## Standard V8

**V8.1** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\} \right)$ . Find the dimension of  $W$ .

*Solution.*  $W$  has dimension 2. □

**V8.2** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\} \right)$ . Compute the dimension of  $W$ .

*Solution.*  $\dim W = 2$ . □

**V8.3** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$ . Compute the dimension of  $W$ .

*Solution.*  $\dim(W) = 3$ . □

**V8.4** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix} \right\} \right)$ . Compute the dimension of  $W$ .

*Solution.*  $\dim(W) = 3$ . □

**V8.5** Let  $W = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -8 \\ -1 \end{bmatrix} \right\}$ . Find the dimension of  $W$ .

*Solution.* The dimension of  $W$  is 2. □

**V8.6** Let  $W = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\}$ . Find the dimension of  $W$ .

*Solution.* The dimension of  $W$  is 2. □

**V8.7** Let  $W = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}$ . Find the dimension of  $W$ .

*Solution.* The dimension of  $W$  is 3. □

## Standard V9

**V9.1** Find a basis for the subspace

$$W = \text{span} \{x^2 + x, x^2 + 2x - 1, x^2 + 3x - 2\}$$

of  $\mathcal{P}^2$ .

*Solution.*  $\{x^2 + x, x^2 + 2x - 1\}$  □

**V9.2** Find a basis for the subspace

$$W = \text{span} \{-3x^3 - 8x^2, x^3 + 2x^2 + 2, -x^2 + 3\}$$

of  $\mathcal{P}^3$ .

*Solution.*  $\{-3x^3 - 8x^2, x^3 + 2x^2 + 2\}$  □

**V9.3** Find a basis for the subspace

$$W = \text{span} \left\{ \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 9 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \right\}$$

of  $M_{2,2}$ .

*Solution.*

$$\left\{ \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \right\}$$

□

**V9.4** Find a basis for the subspace

$$W = \text{span} \{x^3 - 3x^2 + 2x + 2, -x^3 + 4x^2 - x + 1, 3x^2 + 3x + 9, -x^3 + 2x + 1\}$$

of  $\mathcal{P}^3$ .

*Solution.*

$$\{x^3 - 3x^2 + 2x + 2, -x^3 + 4x^2 - x + 1, -x^3 + 2x + 1\}$$

□

**V9.5** Find a basis for the subspace

$$W = \text{span} \{x^3 - x, x^2 + x + 1, x^3 - x^2 + 2, 2x^2 - 1\}$$

of  $\mathcal{P}^3$ .

*Solution.*

$$\{x^3 - x, x^2 + x + 1, x^3 - x^2 + 2\}$$

□

**V9.6** Let  $W$  be the subspace of  $\mathcal{P}^3$  given by

$$W = \text{span} \left( \{x^3 + x^2 + 2x + 1, 3x^3 + 3x^2 + 6x + 3, 3x^3 - x^2 + 3x - 2, 7x^3 - x^2 + 8x - 3\} \right).$$

Find a basis for  $W$ .

*Solution.* A basis is  $\{x^3 + x^2 + 2x + 1, 3x^3 - x^2 + 3x - 2\}$ . □

**V9.7** Let  $W = \text{span} \left\{ \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\}$ . Find a basis for this vector space.

*Solution.*  $\left\{ \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$  is a basis of  $W$ . □

**V9.8** Let  $W$  be the subspace of  $\mathcal{P}^2$  given by  $W = \text{span} \{(-3x^2 - 8x, x^2 + 2x + 2, -x + 3)\}$ . Find a basis for  $W$ .

*Solution.*  $\{-3x^2 - 8x, x^2 + 2x + 2\}$  is a basis for  $W$ . □

## Standard V10

**V10.1** Find a basis for the solution space of the homogeneous system of equations

$$\begin{aligned}x + 3y + 3z + 7w &= 0 \\x + 3y - z - w &= 0 \\2x + 6y + 3z + 8w &= 0 \\x + 3y - 2z - 3w &= 0\end{aligned}$$

*Solution.* A basis for the solution space is

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

□

**V10.2** Find a basis for the solution space of the homogeneous system of equations

$$\begin{aligned}x + 2y + 3z + w &= 0 \\3x - y + z + w &= 0 \\2x - 3y - 2z &= 0 \\-x + 2z + 5w &= 0\end{aligned}$$

*Solution.* A basis for the solution space is  $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$ .

□

**V10.3** Find a basis for the solution space of the homogeneous system of equations

$$\begin{aligned}x + 2y + 3z + w &= 0 \\3x - y + z + w &= 0 \\2x - 3y - 2z &= 0\end{aligned}$$

*Solution.* A basis for the solution space is  $\left\{ \begin{bmatrix} -\frac{5}{7} \\ -\frac{8}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{7} \\ -\frac{2}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$ , or  $\left\{ \begin{bmatrix} 5 \\ 8 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ -7 \end{bmatrix} \right\}$ .

□

**V10.4** Find a basis for the solution space to the system of equations

$$\begin{aligned}x + 2y - 3z &= 0 \\2x + y - 4z &= 0 \\3y - 2z &= 0 \\x - y - z &= 0\end{aligned}$$

*Solution.* A basis is  $\left\{ \begin{bmatrix} 5 \\ 3 \\ 3 \\ 1 \end{bmatrix} \right\}$  or  $\left\{ \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \right\}$ .

□

**V10.5** Find a basis for the solution space to the homogeneous system of equations

$$\begin{aligned}2x_1 + 3x_2 - 5x_3 + 14x_4 &= 0 \\x_1 + x_2 - x_3 + 5x_4 &= 0\end{aligned}$$

*Solution.* A basis for the solution space is  $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}.$  □

**V10.6** Find a basis for the solution space to the homogeneous system of equations

$$4x_1 + 4x_2 + 3x_3 - 6x_4 = 0$$

$$-2x_3 - 4x_4 = 0$$

$$2x_1 + 2x_2 + x_3 - 4x_4 = 0$$

*Solution.* A basis for the solution space is  $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$  □

**V10.7** Find a basis for the solution space to the homogeneous system of equations given by

$$3x + 2y + z = 0$$

$$x + y + z = 0$$

*Solution.* A basis for the solution space is  $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$  □

**V10.8** Find a basis for the solution space to the homogeneous system of equations given by

$$2x_1 - 2x_2 + 6x_3 - x_4 = 0$$

$$3x_1 + 6x_3 + x_4 = 0$$

$$-4x_1 + x_2 - 9x_3 + 2x_4 = 0$$

*Solution.* A basis for the solution space is  $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$  □



## Module A

### Standard A1

**A1.1** Consider the following maps of polynomials  $S : \mathcal{P}^6 \rightarrow \mathcal{P}^6$  and  $T : \mathcal{P}^6 \rightarrow \mathcal{P}^6$  defined by

$$S(f(x)) = f(x) + 3 \text{ and } T(f(x)) = f(x) + f(3).$$

Show that one of these maps is a linear transformation, and that the other map is not.

*Solution.*  $T$  is linear,  $S$  is not. □

**A1.2** Consider the following maps of polynomials  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^5$  and  $T : \mathcal{P}^4 \rightarrow \mathcal{P}^5$  defined by

$$S(f(x)) = xf(x) - f(1) \text{ and } T(f(x)) = xf(x) - x.$$

Show that one of these maps is a linear transformation, and that the other map is not.

*Solution.*  $S$  is linear,  $T$  is not. □

**A1.3** Consider the following maps of polynomials  $S : \mathcal{P} \rightarrow \mathcal{P}$  and  $T : \mathcal{P} \rightarrow \mathcal{P}$  defined by

$$S(f(x)) = f'(x) - f''(x) \text{ and } T(f(x)) = f(x) - (f(x))^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.

*Solution.*  $S$  is linear,  $T$  is not. □

**A1.4** Consider the following maps of polynomials  $S : \mathcal{P}^2 \rightarrow \mathcal{P}^4$  and  $T : \mathcal{P}^2 \rightarrow \mathcal{P}^4$  defined by

$$S(f(x)) = x^2 f(x) \text{ and } T(f(x)) = (f(x))^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.

*Solution.*  $S$  is linear,  $T$  is not. □

**A1.5** Consider the following maps of polynomials  $S : \mathcal{P} \rightarrow \mathcal{P}$  and  $T : \mathcal{P} \rightarrow \mathcal{P}$  defined by

$$S(f(x)) = (f(x))^2 + 1 \text{ and } T(f(x)) = (x^2 + 1)f(x).$$

Show that one of these maps is a linear transformation, and that the other map is not.

*Solution.*  $T$  is linear,  $S$  is not. □

**A1.6** Consider the following maps of polynomials  $S : \mathcal{P}^2 \rightarrow \mathcal{P}^2$  and  $T : \mathcal{P}^2 \rightarrow \mathcal{P}^2$  defined by

$$S(ax^2 + bx + c) = cx^2 + bx + a \text{ and } T(ax^2 + bx + c) = a^2x^2 + b^2x + c^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.

*Solution.*  $S$  is linear,  $T$  is not. □

**A1.7** Consider the following maps of polynomials  $S : \mathcal{P}^2 \rightarrow \mathcal{P}^1$  and  $T : \mathcal{P}^2 \rightarrow \mathcal{P}^1$  defined by

$$S(ax^2 + bx + c) = 2ax + b \text{ and } T(ax^2 + bx + c) = a^2x + b.$$

Show that one of these maps is a linear transformation, and that the other map is not.

*Solution.*  $S$  is linear,  $T$  is not. □

**A1.8** Consider the following maps of polynomials  $S : \mathcal{P}^2 \rightarrow \mathcal{P}^3$  and  $T : \mathcal{P}^2 \rightarrow \mathcal{P}^3$  defined by

$$S(ax^2 + bx + c) = ax^3 + bx^2 + cx \text{ and } T(ax^2 + bx + c) = abcx^3.$$

Show that one of these maps is a linear transformation, and that the other map is not.

*Solution.*  $S$  is linear,  $T$  is not. □

## Standard A2

**A2.1** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \\ 7x \end{bmatrix}.$$

(a) Write the standard matrix for  $T$ .

(b) Compute  $T \left( \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right)$

*Solution.*

(a)  $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \\ 7 & 0 & 0 \end{bmatrix}$

(b)  $T \left( \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ 15 \\ 11 \\ -14 \end{bmatrix}$

□

**A2.2** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the linear transformation given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = [x_3 + 3x_1].$$

(a) Write the standard matrix for  $T$ .

(b) Compute  $T \left( \begin{bmatrix} -1 \\ 7 \\ 3 \end{bmatrix} \right)$

*Solution.*

(a)  $[3 \quad 0 \quad 1]$

(b)  $[0]$

□

**A2.3** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the linear transformation given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = [x_2 + 3x_3].$$

(a) Write the standard matrix for  $T$ .

(b) Compute  $T \left( \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} \right)$

*Solution.*

(a)  $\begin{bmatrix} 0 & 1 & 3 \end{bmatrix}$

(b)  $\begin{bmatrix} -12 \end{bmatrix}$

□

**A2.4** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \\ 0 \end{bmatrix}.$$

(a) Write the standard matrix for  $T$ .

(b) Compute  $T \left( \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right)$

*Solution.*

(a)  $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

(b)  $\begin{bmatrix} -5 \\ -13 \\ -1 \\ 0 \end{bmatrix}$

□

**A2.5** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Compute  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$

(b) Compute  $T \left( \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \right)$

*Solution.*

(a)  $\begin{bmatrix} 2x + y - z \\ y + 3z \\ 2x - y + 5z \\ 0 \end{bmatrix}$

(b)  $\begin{bmatrix} -2 \\ -2 \\ -10 \\ 0 \end{bmatrix}$

□

**A2.6** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be the linear transformation given by the requirements

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 3 \end{bmatrix} \text{ and } T(\vec{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

(a) Compute  $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

(b) Compute  $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

*Solution.*

(a)  $\begin{bmatrix} 2x + y \\ y \\ 2x - y \\ 3x + 3y \end{bmatrix}$

(b)  $\begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}$

□

**A2.7** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 7x + 2y + 3z \\ 0 \end{bmatrix}.$$

(a) Write the standard matrix for  $T$ .

(b) Compute  $T\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right)$

*Solution.*

(a)  $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 7 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

(b)  $\begin{bmatrix} 6 \\ 13 \\ -5 \\ 0 \end{bmatrix}$

□

**A2.8** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \\ 7 & 0 & 0 \end{bmatrix}.$$

(a) Compute  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$

(b) Compute  $T \left( \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right)$

*Solution.*

(a)  $\begin{bmatrix} 3x + y \\ -8x + 2y - z \\ 2y + 3z \\ 7x \end{bmatrix}$

(b)  $\begin{bmatrix} -5 \\ 15 \\ 11 \\ -14 \end{bmatrix}$

□

**A2.9** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & -5 & 0 \end{bmatrix}.$$

(a) Compute  $T \left( \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \right)$

(b) Compute  $T \left( \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix} \right)$

*Solution.*

(a)  $\begin{bmatrix} x + 3z \\ 3y - 5z \end{bmatrix}$

(b)  $\begin{bmatrix} -2 \\ 14 \end{bmatrix}$

□

**A2.10** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be the linear transformation given by the following

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} \quad T(\vec{e}_2) = \begin{bmatrix} 0 \\ 3 \\ -5 \\ 0 \end{bmatrix}.$$

(a) Compute  $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right)$

(b) Compute  $T \left( \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right)$

*Solution.*

$$(a) \begin{bmatrix} x \\ 3y \\ 3x - 5y \\ 0 \end{bmatrix}$$

$$(b) \begin{bmatrix} 2 \\ -9 \\ 21 \\ 0 \end{bmatrix}$$

□

**A2.11** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - x_3 \end{bmatrix}.$$

(a) Write the standard matrix for  $T$ .

$$(b) \text{ Compute } T \left( \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix} \right)$$

*Solution.*

$$(a) \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix}$$

$$(b) \begin{bmatrix} -2 \\ 10 \end{bmatrix}$$

□

**A2.12** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - 5x_3 \end{bmatrix}.$$

(a) Write the standard matrix for  $T$ .

$$(b) \text{ Compute } T \left( \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix} \right)$$

*Solution.*

$$(a) \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & -5 & 0 \end{bmatrix}$$

$$(b) \begin{bmatrix} -2 \\ 14 \end{bmatrix}$$

□

**A2.13** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 0 & 1 & 3 \end{bmatrix}.$$

(a) Compute  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$

(b) Compute  $T \left( \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right)$

*Solution.*

(a)  $[y + 3z]$

(b)  $[0]$

□

**A2.14** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be the linear transformation given by the following

$$T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \quad T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

(a) Compute  $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right)$

(b) Compute  $T \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)$

*Solution.*

(a)  $\begin{bmatrix} 3y \\ x \\ 3x \end{bmatrix}$

(b)  $\begin{bmatrix} 9 \\ 1 \\ 3 \end{bmatrix}$

□

**A2.15** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$$

(a) Compute  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$

(b) Compute  $T \left( \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} \right)$

*Solution.*

(a)  $[3x + z]$

(b)  $[-3]$

□

### Standard A3

**A3.1** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \right) = \begin{bmatrix} x + 3y + 3z + 7w \\ x + 3y - z - w \\ 2x + 6y + 3z + 8w \\ x + 3y - 2z - 3w \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.* A basis for the kernel is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

and a basis for the image is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}$$

□

**A3.2** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.* A basis for the image is

$$\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

and a basis of the kernel is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}.$$

□

**A3.3** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be the linear map given by  $T \left( \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \right) = \begin{bmatrix} 8x - 3y - z + 4w \\ y + 3z - 4w \\ -7x + 3y + 2z - 5w \end{bmatrix}$ . Compute a basis

for the kernel and a basis for the image of  $T$ .

*Solution.*  $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$  is a basis for the kernel. □

**A3.4** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear map given by  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 8x - 3y - z \\ y + 3z \\ -7x + 3y + 2z \end{bmatrix}$ . Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.*  $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$  is a basis for the kernel. □



**A3.5** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear map given by the standard matrix  $\begin{bmatrix} 1 & 2 & 5 \\ 0 & -2 & -4 \\ -3 & 1 & -1 \end{bmatrix}$ . Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.*  $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$  is a basis for the kernel.  $\square$

**A3.6** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  be the linear map given by the standard matrix  $\begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & -2 & -4 & 2 \end{bmatrix}$ . Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.*  $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$  is a basis for the kernel.  $\square$

**A3.7** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^5$  be the linear map given by the standard matrix  $\begin{bmatrix} 3 & 1 \\ 0 & 0 \\ -6 & -2 \\ 1 & \frac{1}{3} \\ 9 & 3 \end{bmatrix}$ . Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.*  $\left\{ \begin{bmatrix} 3 \\ 0 \\ -6 \\ 1 \\ 9 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} \right\}$  is a basis for the kernel.  $\square$

**A3.8** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be the linear map given by the standard matrix  $\begin{bmatrix} 2 & 1 & 0 & 3 \\ -4 & -2 & 0 & 5 \\ -2 & -1 & 0 & 0 \end{bmatrix}$ . Compute a basis for the kernel and a basis for the image of  $T$ .

*Solution.*  $\left\{ \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$  is a basis for the kernel.  $\square$

## Standard A4

**A4.1** Determine if the following linear maps are injective (one-to-one) and/or surjective (onto).

- (a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by  $S \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 3x + 2y \\ x - y \\ x + 4y \end{bmatrix}$
- (b)  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y + z \\ 2y + 3z \\ x - y - 2z \end{bmatrix}$

*Solution.*

- (a)  $S$  is injective but not surjective.
- (b)  $T$  is neither injective nor surjective.

□

**A4.2** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

- (a)  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y + z \\ 2y + 3z \\ x - y - 2z \end{bmatrix}$
- (b)  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by  $S \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 3x + 2y \\ x - y \\ x + 4y \end{bmatrix}$

*Solution.*

- (a)  $S$  is neither injective nor surjective.
- (b)  $T$  is injective but not surjective.

□

**A4.3** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

- (a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ .
- (b)  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  given by the standard matrix  $\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 3 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix}$

*Solution.*

- (a)  $S$  is both injective and surjective.
- (b)  $T$  is neither injective nor surjective.

□

**A4.4** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

- (a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ .

(b)  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  given by the standard matrix  $\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 4 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix}$

*Solution.*

(a)  $S$  is both injective and surjective.

(b)  $T$  is not injective but  $T$  is surjective.

□

**A4.5** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  given by the standard matrix  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$ .

(b)  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  given by the standard matrix  $\begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 11 & -1 & 5 \end{bmatrix}$

*Solution.*

(a)  $S$  is injective but not surjective.

(b)  $T$  is neither injective nor surjective.

□

**A4.6** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  given by the standard matrix  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$ .

(b)  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  given by the standard matrix  $\begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 7 & -1 & 5 \end{bmatrix}$

*Solution.*

(a)  $S$  is injective but not surjective.

(b)  $T$  is not injective, but  $T$  is surjective.

□

**A4.7** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  where  $S(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$  and  $S(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ .

(b)  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  where  $T(\vec{e}_1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ ,  $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , and  $T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

*Solution.*

- (a)  $S$  is injective but not surjective.
- (b)  $T$  is not injective, but it is surjective.

□

**A4.8** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a)  $S : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  where  $S(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ ,  $S(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ,  $S(\vec{e}_3) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ , and  $S(\vec{e}_4) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ ,

(b)  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  where  $T(\vec{e}_1) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ ,  $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ , and  $T(\vec{e}_3) = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ .

*Solution.*

- (a)  $S$  is not injective, but it is surjective.
- (b)  $T$  is neither injective nor surjective.

□

## Module M

### Standard M1

**M1.1** Let

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 0 & 7 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $CA$  is the only one that can be computed, and

$$CA = \begin{bmatrix} 3 & 9 & 11 & 1 \\ 0 & 0 & 7 & 2 \\ -2 & -6 & -5 & 0 \end{bmatrix}$$

□

**M1.2** Let

$$A = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 3 & 7 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $CA$  is the only one that can be computed, and

$$CA = \begin{bmatrix} 2 & 6 & 11 & 1 \\ 1 & 3 & 7 & 2 \\ -1 & -3 & -5 & 0 \end{bmatrix}$$

□

**M1.3** Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $AC$  is the only one that can be computed, and

$$AC = \begin{bmatrix} 3 & -5 & 14 \\ 1 & -1 & 2 \end{bmatrix}$$

□

**M1.4** Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $AC$  is the only one that can be computed, and

$$AC = \begin{bmatrix} 9 & -2 & 14 \\ 1 & 0 & 2 \end{bmatrix}$$

□

**M1.5** Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $AB$  is the only ones that can be computed, and

$$AB = \begin{bmatrix} -3 & -5 & 6 & 14 \\ 0 & 0 & 7 & 35 \end{bmatrix}$$

□

**M1.6** Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $BC$  is the only one that can be computed, and

$$BC = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

□

**M1.7** Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $CB$  is the only one that can be computed, and

$$CB = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

□

**M1.8** Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & -1 & 3 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

*Solution.*  $CB$  is the only one that can be computed, and

$$CB = \begin{bmatrix} 3 & 3 & -5 & 7 \\ 4 & -4 & 12 & -12 \\ 7 & 2 & 0 & 3 \end{bmatrix}$$

□

## Standard M2

**M2.1** Determine if the matrix  $\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}$  is invertible.

*Solution.* Not invertible. □

**M2.2** Determine if the matrix  $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$  is invertible.

*Solution.* Not invertible. □

**M2.3** Determine if the matrix  $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & 0 \\ -1 & -1 & 5 \end{bmatrix}$  is invertible.

*Solution.* Not invertible. □

**M2.4** Determine if the matrix  $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$  is invertible.

*Solution.* Invertible. □

**M2.5** Determine if the matrix  $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & -2 & 0 & 3 \end{bmatrix}$  is invertible.

*Solution.* Invertible. □

**M2.6** Determine if the matrix  $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  is invertible.

*Solution.* Invertible. □

**M2.7** Determine if the matrix  $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix}$  is invertible.

*Solution.* Not invertible. □

**M2.8** Determine if the matrix  $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 0 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix}$  is invertible.

*Solution.* Invertible. □

### Standard M3

**M3.1** Show how to find the inverse of the matrix  $\begin{bmatrix} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$ .

*Solution.* The inverse is  $\begin{bmatrix} 1 & 2 & -5 & 12 \\ 1 & 1 & -4 & -9 \\ -4 & -7 & 20 & 47 \\ -1 & 0 & 3 & 7 \end{bmatrix}$ .

□

**M3.2** Compute the inverse of the matrix  $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ .

*Solution.* The inverse is  $\begin{bmatrix} 1 & 2 & -11 & 37 \\ 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ .

□

**M3.3** Show how to find the inverse of the matrix  $\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}$ .

*Solution.*

$$\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 4 & -3 \\ 8 & -13 & 10 \\ 13 & -24 & 18 \end{bmatrix}$$

□

**M3.4** Show how to find the inverse of the matrix  $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ .

*Solution.*

$$\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & \frac{3}{2} & -\frac{3}{2} \\ 1 & -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

□

**M3.5** Show how to find the inverse of the matrix  $\begin{bmatrix} 4 & -1 & -8 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ .

*Solution.* The inverse is  $\begin{bmatrix} 1 & -4 & 5 \\ -5 & 24 & -28 \\ 1 & -5 & 6 \end{bmatrix}$ .

□

**M3.6** Show how to find the inverse of the matrix  $\begin{bmatrix} 1 & -4 & 5 \\ -5 & 24 & -28 \\ 1 & -5 & 6 \end{bmatrix}$ .

*Solution.* The inverse is  $\begin{bmatrix} 4 & -1 & -8 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ .

□

**M3.7** Show how to find the inverse of the matrix  $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$ .



*Solution.* The inverse is  $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$ .

□

**M3.8** Show how to find the inverse of the matrix  $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$ .

*Solution.* The inverse is  $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$ .

□

**M3.9** Show how to find the inverse of the matrix  $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$ .

*Solution.* The inverse is  $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$ .

□

## Module G

### Standard G1

#### G1.1

- (a) Find  $3 \times 3$  matrices  $S$  and  $T$  whose left multiplication represents the row operations  $R_1 + 5R_3 \rightarrow R_1$  and  $R_2 \leftrightarrow R_3$ , respectively.
- (b) If  $A \in M_{3,3}$  is a matrix with  $\det A = 4$ , find the determinant of  $STA$ .

*Solution.*

1.  $S = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and  $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ .

2.  $\det(STA) = -4$ .

□

#### G1.2

- (a) Find  $3 \times 3$  matrices  $S$  and  $T$  whose left multiplication represents the row operations  $R_2 - 4R_3 \rightarrow R_2$  and  $R_1 \leftrightarrow R_3$ , respectively.
- (b) If  $A \in M_{3,3}$  is a matrix with  $\det A = 7$ , find the determinant of  $STA$ .

*Solution.*

1.  $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$  and  $T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ .

2.  $\det(STA) = -7$ .

□

#### G1.3

- (a) Find  $3 \times 3$  matrices  $S$  and  $T$  whose left multiplication represents the row operations  $3R_2 \rightarrow R_2$  and  $R_3 - 2R_1 \rightarrow R_3$ , respectively.
- (b) If  $A \in M_{3,3}$  is a matrix with  $\det A = 7$ , find the determinant of  $STA$ .

*Solution.*

1.  $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .  $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ .

2.  $\det(STA) = 21$ .

□

#### G1.4

- (a) Find  $3 \times 3$  matrices  $S$  and  $T$  whose left multiplication represents the row operations  $3R_2 \rightarrow R_2$  and  $4R_3 \rightarrow R_3$ , respectively.
- (b) If  $A \in M_{3,3}$  is a matrix with  $\det A = -2$ , find the determinant of  $STA$ .

*Solution.*

$$1. S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

$$2. \det(STA) = -24.$$

□

### G1.5

(a) Find  $3 \times 3$  matrices  $S$  and  $T$  whose left multiplication represents the row operations  $-8R_1 \rightarrow R_1$  and  $R_2 + R_1 \rightarrow R_2$ , respectively.

(b) If  $A \in M_{3,3}$  is a matrix with  $\det A = -2$ , find the determinant of  $STA$ .

*Solution.*

$$1. S = \begin{bmatrix} -8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$2. \det(STA) = 16.$$

□

### G1.6

(a) Find  $3 \times 3$  matrices  $S$  and  $T$  whose left multiplication represents the row operations  $5R_2 \rightarrow R_2$  and  $R_2 \leftrightarrow R_3$ , respectively.

(b) If  $A \in M_{3,3}$  is a matrix with  $\det A = -3$ , find the determinant of  $STA$ .

*Solution.*

$$1. S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$2. \det(STA) = 15.$$

□

## Standard G2

**G2.1** Compute the determinant of the matrix  $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$ .

*Solution.*

$$\det \begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} = -36$$

□

**G2.2** Compute the determinant of the matrix  $\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix}$ .

*Solution.*

$$\det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} = -4$$

□

**G2.3** Compute the determinant of the matrix  $\begin{bmatrix} 2 & 3 & 0 & 1 \\ -1 & 3 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 1 & -1 & 3 & 5 \end{bmatrix}$ .

*Solution.*  $-60$ .

□

**G2.4** Compute the determinant of the matrix  $\begin{bmatrix} 3 & -1 & 0 & 7 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 3 & 0 & 0 & 1 \end{bmatrix}$ .

*Solution.*  $2$ .

□

**G2.5** Compute the determinant of the matrix  $\begin{bmatrix} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$ .

*Solution.*  $-1$ .

□

**G2.6** Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 0 & -4 & 0 \\ -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

*Solution.*  $15$ .

□

**G2.7** Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ -2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

*Solution.*  $-15$ .

□

**G2.8** Compute the determinant of the matrix

$$\begin{bmatrix} 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 0 \end{bmatrix}.$$

*Solution.*  $55$ .

□

**G2.9** Compute the determinant of the matrix

$$\begin{bmatrix} 0 & -4 & 1 & 1 \\ -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 0 & -4 & 0 \end{bmatrix}.$$

*Solution.*  $-55$ .

□

### Standard G3

**G3.1** Find the eigenvalues of the matrix  $\begin{bmatrix} 5 & 1 \\ -24 & -6 \end{bmatrix}$ .

*Solution.* Its characteristic polynomial is  $\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$ , yielding the eigenvalues  $-3$  and  $2$ .  $\square$

**G3.2** Find the eigenvalues of the matrix  $\begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix}$ .

*Solution.* Its characteristic polynomial is  $\lambda^2 + 6\lambda + 5 = (\lambda + 1)(\lambda + 5)$ , yielding the eigenvalues  $-1$  and  $-5$ .  $\square$

**G3.3** Find the eigenvalues of the matrix  $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$ .

*Solution.* Its characteristic polynomial is  $\lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$ , yielding the eigenvalues  $1$  and  $5$ .  $\square$

**G3.4** Find the eigenvalues of the matrix  $\begin{bmatrix} -2 & 5 \\ 5 & -2 \end{bmatrix}$ .

*Solution.* Its characteristic polynomial is  $\lambda^2 + 4\lambda - 21 = (\lambda + 7)(\lambda - 3)$ , yielding the eigenvalues  $-7$  and  $3$ .  $\square$

**G3.5** Find the eigenvalues of the matrix  $\begin{bmatrix} 10 & -8 \\ 4 & -2 \end{bmatrix}$ .

*Solution.* Its characteristic polynomial is  $\lambda^2 - 8\lambda + 12 = (\lambda - 6)(\lambda - 2)$ , yielding the eigenvalues  $6$  and  $2$ .  $\square$

**G3.6** Find the eigenvalues of the matrix  $\begin{bmatrix} 6 & -4 \\ 11 & -9 \end{bmatrix}$ .

*Solution.* Its characteristic polynomial is  $\lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2)$ , yielding the eigenvalues  $-5$  and  $2$ .  $\square$

**G3.7** Find the eigenvalues of the matrix  $\begin{bmatrix} -6 & -11 \\ 4 & 9 \end{bmatrix}$ .

*Solution.* Its characteristic polynomial is  $\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$ , yielding the eigenvalues  $5$  and  $-2$ .  $\square$

**G3.8** Find the eigenvalues of the matrix  $\begin{bmatrix} 1 & -5 \\ 4 & -8 \end{bmatrix}$ .

*Solution.* Its characteristic polynomial is  $\lambda^2 + 7\lambda + 12 = (\lambda + 3)(\lambda + 4)$ , yielding the eigenvalues  $-3$  and  $-4$ .  $\square$

**G3.9** Find the eigenvalues of the matrix  $\begin{bmatrix} -2 & 2 \\ 15 & -9 \end{bmatrix}$ .

*Solution.* Its characteristic polynomial is  $\lambda^2 + 11\lambda - 12 = (\lambda + 12)(\lambda - 1)$ , yielding the eigenvalues  $-12$  and  $1$ .  $\square$

**G3.10** Find the eigenvalues of the matrix  $\begin{bmatrix} 10 & 2 \\ -39 & -9 \end{bmatrix}$ .

*Solution.* Its characteristic polynomial is  $\lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3)$ , yielding the eigenvalues  $-3$  and  $4$ .  $\square$

**G3.11** Find the eigenvalues of the matrix  $\begin{bmatrix} 8 & 2 \\ -33 & -9 \end{bmatrix}$ .

*Solution.* Its characteristic polynomial is  $\lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$ , yielding the eigenvalues  $-3$  and  $2$ .  $\square$

## Standard G4

**G4.1** Find a basis of the eigenspace associated to the eigenvalue  $-1$  for the matrix  $\begin{bmatrix} 4 & -2 & -1 & 1 \\ 15 & -7 & -3 & 1 \\ -5 & 2 & 0 & 1 \\ 10 & -4 & -2 & 0 \end{bmatrix}$ .

*Solution.* A basis for the eigenspace is  $\left\{ \begin{bmatrix} \frac{2}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ . □

**G4.2** Find a basis of the eigenspace associated to the eigenvalue  $1$  for the matrix  $A = \begin{bmatrix} -3 & 1 & 0 & 0 \\ -8 & 2 & 0 & -1 \\ 0 & 2 & 1 & 2 \\ 4 & 1 & 0 & 3 \end{bmatrix}$ .

*Solution.* A basis for the eigenspace is  $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ . □

**G4.3** Find a basis of the eigenspace associated to the eigenvalue  $1$  for the matrix  $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 8 & -3 & -1 \\ 2 & 21 & -8 & -3 \\ 3 & -7 & 3 & 2 \end{bmatrix}$ .

*Solution.* A basis for the eigenspace is  $\left\{ \begin{bmatrix} 0 \\ \frac{3}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$ . □

**G4.4** Find a basis of the eigenspace associated to the eigenvalue  $1$  for the matrix

$$A = \begin{bmatrix} 9 & -3 & -5 & 2 \\ 19 & -6 & -12 & 5 \\ 1 & 1 & -1 & 3 \\ -11 & 4 & 7 & -2 \end{bmatrix}.$$

*Solution.* A basis for the eigenspace is  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ . □

**G4.5** Find a basis of the eigenspace associated to the eigenvalue  $2$  for the matrix  $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 11 & -6 & 1 & -1 \\ -9 & 5 & 1 & 3 \end{bmatrix}$ .

*Solution.* A basis for the eigenspace is  $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ . □

**G4.6** Find a basis of the eigenspace associated to the eigenvalue  $2$  for the matrix  $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$ .

*Solution.* A basis for the eigenspace is  $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ . □

**G4.7** Find a basis of the eigenspace associated to the eigenvalue 2 for the matrix

$$A = \begin{bmatrix} 0 & -2 & -1 & 0 \\ -4 & -2 & -2 & 0 \\ 14 & 12 & 10 & 2 \\ -13 & -10 & -8 & -1 \end{bmatrix}.$$

*Solution.* A basis for the eigenspace is  $\left\{ \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ . □

**G4.8** Find a basis of the eigenspace associated to the eigenvalue 3 for the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ -4 & -1 & -2 & 0 \\ 14 & 12 & 11 & 2 \\ -14 & -10 & -9 & -1 \end{bmatrix}.$$

*Solution.* A basis for the eigenspace is  $\left\{ \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ . □

**G4.9** Find a basis of the eigenspace associated to the eigenvalue  $-2$  for the matrix  $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 2 & -3 & 3 \\ 1 & 8 & -9 & 6 \\ 1 & 8 & -7 & 4 \end{bmatrix}$ .

*Solution.* A basis for the eigenspace is  $\left\{ \begin{bmatrix} -1 \\ \frac{1}{4} \\ 1 \\ 1 \end{bmatrix} \right\}$ . □

**G4.10** Find a basis of the eigenspace associated to the eigenvalue 2 for the matrix  $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 6 & 2 & -2 & 0 \\ -9 & 0 & 5 & 0 \\ 15 & 0 & -5 & 2 \end{bmatrix}$ .

*Solution.* A basis for the eigenspace is  $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ . □