Module E

Standard E1

E1.1 Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x + 3y - 4z &= 5 \\ 3x + 9y + z &= 0 \\ x - z &= 1 \end{aligned}$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{array}\right]$$

$\square$

E1.2 Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x_1 + 4x_3 &= 1 \\ x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= -1 \end{aligned}$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array}\right]$$

$\square$

E1.3 Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x_1 + 4x_3 &= 1 \\ x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_4 &= -1 \end{aligned}$$

Solution.

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & -1 & 0 & 7 \\ 1 & -1 & 0 & 3 & -1 \end{array}\right]$$

$\square$

E1.4 Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x_1 + 3x_2 - 4x_3 + x_4 &= 5 \\ 3x_1 + 9x_2 + x_3 - 7x_4 &= 0 \\ x_1 - x_3 + x_4 &= 1 \end{aligned}$$

Solution.

$$\left[\begin{array}{cccc|c} 1 & 3 & -4 & 1 & 5 \\ 3 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{array}\right]$$

$\square$

E1.5 Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{cccc|c} 3 & -1 & 0 & 1 & 5 \\ -1 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 0 & -3 \end{array}\right]$$

Solution.

$$\begin{aligned} 3x_1 - x_2 + x_4 &= 5 \\ -x_1 + 9x_2 + x_3 - 7x_4 &= 0 \\ x_1 - x_3 &= -3 \end{aligned}$$

□

E1.6 Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 4 & 1 & -7 \\ 1 & 2 & -1 & 0 \end{array}\right]$$

Solution.

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 4x_2 + x_3 &= -7 \\ x_1 + 2x_2 - x_3 &= 0 \end{aligned}$$

□

E1.7 Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} -4 & -1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{array}\right]$$

Solution.

$$\begin{aligned} -4x_1 - x_2 + 3x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 0 \\ -x_1 + 4x_2 + x_3 &= 4 \end{aligned}$$

□

E1.8 Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array}\right]$$

Solution.

$$\begin{aligned} x_1 + 4x_3 &= 1 \\ x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= -1 \end{aligned}$$

□

Standard E2

E2.1 Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

□

E2.2 Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

E2.3 Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

E2.4 Find the reduced row echelon form of the matrix below.

$$\begin{bmatrix} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{12} & 1 \\ 0 & 1 & 0 & \frac{3}{4} & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{bmatrix}$$

□

E2.5 Find RREF A, where

$$A = \begin{bmatrix} 3 & -2 & 1 & 8 & -5 \\ 2 & 2 & 0 & 6 & -2 \\ -1 & 1 & 1 & -4 & 6 \end{bmatrix}$$

Solution.

$$\operatorname{RREF} A = \begin{bmatrix} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \end{bmatrix}$$

□

E2.6 Find RREF A, where

$$A = \begin{bmatrix} 2 & -7 & 4 \\ 1 & -3 & 2 \\ 3 & 0 & 3 \end{bmatrix}$$

Solution.

$$\operatorname{RREF} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

E2.7 Find RREF A, where

$$A = \begin{bmatrix} 2 & -1 & 5 & 4 \\ -1 & 0 & -2 & -1 \\ 1 & 3 & -1 & -5 \end{bmatrix}$$

Solution.

$$\operatorname{RREF} A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

□

E2.8 Find RREF A, where

$$A = \begin{bmatrix} 2 & 2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 4 & 5 \\ 3 & 3 & -1 & -2 & 1 \end{bmatrix}$$

Solution.

$$\operatorname{RREF} A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

□

Standard E3

E3.1 Find the solution set for the following system of linear equations.

$$\begin{aligned} x + 3y - 4z &= 5 \\ 3x + 9y + z &= 2 \end{aligned}$$

Solution.

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 2 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

So the solution set is

$$\left\{ \begin{bmatrix} 1 - 3c \\ c \\ -1 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

□

E3.2 Find the solution set for the following system of linear equations.

$$\begin{aligned} -3x + y &= 2 \\ -8x + 2y - z &= 6 \\ 2y + 3z &= -2 \end{aligned}$$

Solution.

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the solution set is

$$\left\{ \begin{bmatrix} -1 - \frac{c}{2} \\ -1 - \frac{3c}{2} \\ c \end{bmatrix} \middle| \, c \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} c - 1 \\ 3c - 1 \\ -2c \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

□

E3.3 Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x + y - z + w &= 5 \\ 3x - y - 2w &= 0 \\ -x + 5z + 3w &= -1 \end{aligned}$$

Solution.

$$\mathrm{RREF}\left(\left[\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array}\right]\right) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{4} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{array}\right]$$

So the solution set is

$$\left\{ \begin{bmatrix} 1 + a \\ 3 - 21a \\ -7a \\ 12a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$$

□

E3.4 Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x + y - z + w &= 5 \\ 3x - y - 2w &= 0 \\ -x + 5z + 3w &= -1 \end{aligned}$$

Solution.

$$\operatorname{RREF}\left(\left[\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array}\right]\right) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{4} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{array}\right]$$

So the solution set is

$$\left\{ \begin{bmatrix} 1+a \\ 3-21a \\ -7a \\ 12a \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

□

E3.5 Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x_1 - 2x_2 + 6x_3 - x_4 &= -1 \\ 3x_1 + 6x_3 + x_4 &= 5 \\ -4x_1 + x_2 - 9x_3 + 2x_4 &= -7 \end{aligned}$$

Solution. Let $A = \left[\begin{array}{cccc|c} 2 & -2 & 6 & -1 & -1 \\ 3 & 0 & 6 & 1 & 5 \\ -4 & 1 & -9 & 2 & -7 \end{array}\right]$, so $\operatorname{RREF} A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$. It follows that the solution set is given by $\left\{ \begin{bmatrix} 2-2a \\ 3+a \\ a \\ -1 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$. □

E3.6 Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x_1 + 3x_2 - 5x_3 + 14x_4 &= 8 \\ x_1 + x_2 - x_3 + 5x_4 &= 3 \end{aligned}$$

Solution. Let $A = \left[\begin{array}{cccc|c} 2 & 3 & -5 & 14 & 8 \\ 1 & 1 & -1 & 5 & 3 \end{array}\right]$, so $\operatorname{RREF} A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -3 & 4 & 2 \end{array}\right]$. It follows that the solution set is given by $\left\{ \begin{bmatrix} 1-2a-b \\ 2+3a-4b \\ a \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$. □

E3.7 Find the solution set for the following system of linear equations.

$$\begin{aligned} 4x_1 + 4x_2 + 3x_3 - 6x_4 &= 5 \\ -2x_3 - 4x_4 &= 3 \\ 2x_1 + 2x_2 + x_3 - 4x_4 &= -1 \end{aligned}$$

Solution. Let $A = \left[\begin{array}{cccc|c} 4 & 4 & 3 & -6 & 5 \\ 0 & 0 & -2 & -4 & 3 \\ 2 & 2 & 1 & -4 & -1 \end{array}\right]$, so $\operatorname{RREF} A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$. It follows that the system is inconsistent with no solutions (since the bottom row implies the contradiction $0 = 1$), so its solution set is $\emptyset$. □

E3.8 Find the solution set for the following system of linear equations.

$$\begin{aligned} 3x + 2y + z &= 7 \\ x + y + z &= 1 \\ -2x + 3z &= -11 \end{aligned}$$

Solution. Let $A = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 7 \\ 1 & 1 & 1 & 1 \\ -2 & 0 & 3 & 11 \end{array}\right]$, so RREF $A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right]$. It follows that solution set is $\left\{ \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} \right\}$ $\square$

Module V

Standard V1

V1.1 Let V be the set of all real numbers together with the operations $\oplus$ and $\odot$ defined by, for any $x, y \in V$ and $c \in \mathbb{R}$,

$$\begin{aligned} x \oplus y &= x + y \\ c \odot x &= cx - 3(c-1) \end{aligned}$$

(a) Show that scalar multiplication is associative: $a \odot (b \odot x) = (ab) \odot x$ for all scalars $a, b \in \mathbb{R}$ and $x \in V$.

(b) Explain why V nonetheless isn't a vector space.

Solution. V is not a vector space because scalar multiplication does not distribute over scalar addition. □

V1.2 Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, x_2 + y_2 + 2x_1 y_1) \\ c \odot (x_1, x_2) &= (cx_1, cx_2) \end{aligned}$$

(a) Show that the vector addition $\oplus$ is associative:
$(x_1, x_2) \oplus ((y_1, y_2) \oplus (z_1, z_2)) = ((x_1, x_2) \oplus (y_1, y_2)) \oplus (z_1, z_2)$ for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in V$.

(b) Explain why V nonetheless isn't a vector space.

Solution. V is not a vector space because scalar multiplication does not distribute over vector addition. □

V1.3 Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1 - 1, x_2 + y_2 - 1) \\ c \odot (x_1, x_2) &= (cx_1, cx_2) \end{aligned}$$

(a) Show that this vector space has an additive identity element: there exists $\vec{z} \in V$ satisfying $(x, y) \oplus \vec{z} = (x, y)$ for every $(x, y) \in V$.

(b) Explain why V nonetheless isn't a vector space.

Solution. V is not a vector space because scalar multiplication does not distribute over vector addition. □

V1.4 Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ c \odot (x_1, x_2) &= (0, cx_2) \end{aligned}$$

(a) Show that scalar multiplication distributes over scalar addition:
$(c + d) \odot (x_1, x_2) = c \odot (x_1, x_2) \oplus d \odot (x_1, x_2)$ for every $c, d \in \mathbb{R}$ and $(x_1, x_2) \in V$.

(b) Explain why V nonetheless isn't a vector space.

Solution. V is not a vector space because there is no multiplicative identity element. □

V1.5 Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ c \odot (x_1, x_2) &= (c^2 x_1, c^3 x_2) \end{aligned}$$

(a) Show that scalar multiplication distributes over vector addition:
$c \odot ((x_1, x_2) \oplus (y_1, y_2)) = c \odot (x_1, x_2) \oplus c \odot (y_1, y_2)$ for all $c \in \mathbb{R}$ and $(x_1, x_2), (y_1, y_2) \in V$.

(b) Explain why V nonetheless isn't a vector space.

Solution. V is not a vector space because scalar multiplication does not distribute over scalar addition. $\square$

V1.6 Let V be the set of all real numbers with the operations, for any $x, y \in V$, $c \in \mathbb{R}$,

$$x \oplus y = \sqrt{x^2 + y^2}$$
$$c \odot x = cx$$

(a) Show that the vector addition $\oplus$ is associative: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ for all $x, y, z \in V$.

(b) Explain why V nonetheless isn't a vector space.

Solution. V is not a vector space because scalar multiplication does not distribute over vector addition. $\square$

V1.7 Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 y_2)$$
$$c \odot (x_1, x_2) = (cx_1, cx_2)$$

(a) Show that there is an additive identity element: there exists an element $\vec{z} \in V$ such that $(x_1, x_2) \oplus \vec{z} = (x_1, x_2)$ for any $(x_1, x_2) \in V$.

(b) Explain why V nonetheless isn't a vector space.

Solution. V is not a vector space because additive inverses do not always exist; for instance, $(0, 0)$ does not have an additive inverse. $\square$

Standard V2

V2.1 Determine if $\begin{bmatrix}0\\-1\\2\\6\end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix}3\\-1\\-1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}$.

Solution. $\begin{bmatrix}0\\-1\\2\\6\end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix}3\\-1\\-1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}$, namely

$$\begin{bmatrix}0\\-1\\2\\6\end{bmatrix} = \begin{bmatrix}3\\-1\\-1\\0\end{bmatrix} + 3\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}.$$

□

V2.2 Determine if $\begin{bmatrix}0\\1\\-2\\1\end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix}5\\2\\-3\\2\end{bmatrix}$, $\begin{bmatrix}3\\1\\1\\0\end{bmatrix}$, and $\begin{bmatrix}8\\3\\5\\-1\end{bmatrix}$.

Solution. $\begin{bmatrix}0\\1\\-2\\1\end{bmatrix}$ is not a linear combination of the three other vectors. □

V2.3 Determine if $\begin{bmatrix}0\\0\\2\end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix}-1\\-9\\15\end{bmatrix}$ and $\begin{bmatrix}1\\5\\-5\end{bmatrix}$.

Solution. $\begin{bmatrix}0\\0\\2\end{bmatrix}$ cannot be written as a linear combination of the vectors $\begin{bmatrix}-1\\-9\\15\end{bmatrix}$ and $\begin{bmatrix}1\\5\\-5\end{bmatrix}$. □

V2.4 Determine if $\begin{bmatrix}1\\4\\3\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}2\\3\\-1\end{bmatrix}$, $\begin{bmatrix}1\\-1\\0\end{bmatrix}$, and $\begin{bmatrix}-3\\-2\\5\end{bmatrix}$.

Solution. $\begin{bmatrix}1\\4\\3\end{bmatrix}$ is a linear combination of the three vectors. □

V2.5 Determine if $\begin{bmatrix}1\\4\\3\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}3\\0\\-1\end{bmatrix}$, $\begin{bmatrix}1\\-1\\4\end{bmatrix}$, and $\begin{bmatrix}5\\1\\-6\end{bmatrix}$.

Solution. $\begin{bmatrix}1\\4\\3\end{bmatrix}$ is not a linear combination of the three vectors. □

V2.6 Determine if $\begin{bmatrix}0\\-1\\6\\-7\end{bmatrix}$ belongs to the span of the set $\left\{\begin{bmatrix}2\\0\\-1\\5\end{bmatrix}, \begin{bmatrix}4\\-1\\4\\3\end{bmatrix}\right\}$.

Solution. $\begin{bmatrix}0\\-1\\6\\-7\end{bmatrix}$ belongs to the span of the other two vectors. □

V2.7 Determine if $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ belongs to the span of the set $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \\ 3 \end{bmatrix} \right\}$.

Solution. $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ does not belong to the span of the other two vectors. $\square$

V2.8 Determine if $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ belongs to the span of the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution. $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ is not in the span of the other three vectors. $\square$

Standard V3

V3.1 Determine if the vectors $\begin{bmatrix}-3\\1\\1\end{bmatrix}$, $\begin{bmatrix}5\\-1\\-2\end{bmatrix}$, $\begin{bmatrix}2\\0\\-1\end{bmatrix}$, and $\begin{bmatrix}0\\2\\-1\end{bmatrix}$ span $\mathbb{R}^3$

Solution.

$$\text{RREF}\left(\begin{bmatrix}-3 & 5 & 2 & 0\\ 1 & -1 & 0 & 2\\ 1 & -2 & -1 & -1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & 5\\ 0 & 1 & 1 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Since the resulting matrix has a zero row, the vectors do not span $\mathbb{R}^3$. □

V3.2 Determine if the vectors $\begin{bmatrix}1\\1\\2\\1\end{bmatrix}$, $\begin{bmatrix}3\\3\\6\\3\end{bmatrix}$, $\begin{bmatrix}3\\-1\\3\\-2\end{bmatrix}$, and $\begin{bmatrix}7\\-1\\8\\-3\end{bmatrix}$ span $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}1 & 3 & 3 & 7\\ 1 & 3 & -1 & -1\\ 2 & 6 & 3 & 8\\ 1 & 3 & -2 & -3\end{bmatrix}\right) = \begin{bmatrix}1 & 3 & 0 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Since there are zero rows, they do not span. □

V3.3 Determine if the vectors $\begin{bmatrix}8\\21\\-7\end{bmatrix}$, $\begin{bmatrix}-3\\-8\\3\end{bmatrix}$, $\begin{bmatrix}-1\\-3\\2\end{bmatrix}$, and $\begin{bmatrix}4\\11\\-5\end{bmatrix}$ span $\mathbb{R}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}8 & -3 & -1 & 4\\ 21 & -8 & -3 & 11\\ -7 & 3 & 2 & -5\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & -1\\ 0 & 1 & 3 & -4\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Since there is a zero row, they do not span $\mathbb{R}^3$. □

V3.4 Determine if the vectors $\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}$, $\begin{bmatrix}3\\1\\3\\6\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\2\\0\\1\end{bmatrix}$ span $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}2 & 3 & 0 & 1\\ 0 & 1 & 0 & 2\\ -2 & 3 & 1 & 0\\ 0 & 6 & 1 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0 & -\frac{5}{2}\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -11\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Since there is a zero row, the vectors do not span $\mathbb{R}^4$. □

V3.5 Determine if the vectors $\begin{bmatrix}1\\0\\2\\1\end{bmatrix}$, $\begin{bmatrix}3\\1\\0\\-3\end{bmatrix}$, $\begin{bmatrix}0\\3\\0\\-2\end{bmatrix}$, and $\begin{bmatrix}-1\\1\\-1\\-1\end{bmatrix}$ span $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix}1 & 3 & 0 & -1\\ 0 & 1 & 3 & 1\\ 2 & 0 & 0 & -1\\ 1 & -3 & -2 & -1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Since every row contains a pivot, the vectors span $\mathbb{R}^4$. □

V3.6 Does $\operatorname{span}\left\{\begin{bmatrix}2\\-1\\4\end{bmatrix},\begin{bmatrix}3\\12\\-9\end{bmatrix},\begin{bmatrix}1\\4\\-3\end{bmatrix},\begin{bmatrix}-4\\2\\-8\end{bmatrix}\right\}=\mathbb{R}^3$?

Solution. Since

$$\text{RREF}\begin{bmatrix}2 & 3 & 1 & -4\\-1 & 12 & 4 & 2\\4 & -9 & -3 & -8\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & -2\\0 & 1 & 1/3 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

has a zero row, the vectors fail to span $\mathbb{R}^3$. □

V3.7 Does $\operatorname{span}\left\{\begin{bmatrix}2\\-1\\4\end{bmatrix},\begin{bmatrix}3\\12\\-9\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}-4\\2\\-8\end{bmatrix}\right\}=\mathbb{R}^3$?

Solution. Since

$$\text{RREF}\begin{bmatrix}2 & 3 & 1 & -4\\-1 & 12 & 2 & 2\\4 & -9 & 3 & -8\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & -2\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$$

lacks a zero row, the vectors span $\mathbb{R}^3$. □

V3.8 Does $\operatorname{span}\left\{\begin{bmatrix}2\\-1\\4\\2\\1\end{bmatrix},\begin{bmatrix}-1\\3\\5\\2\\0\end{bmatrix},\begin{bmatrix}1\\0\\5\\1\\-3\end{bmatrix}\right\}=\mathbb{R}^5$?

Solution. Since there are only three vectors, they cannot span $\mathbb{R}^5$. (Or, since RREF must contain a zero row, so they cannot span $\mathbb{R}^5$.) □

Standard V4

V4.1 Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x, y \text{ are integers} \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x = y \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.
Solution. W is not a subspace. U is a subspace. □

V4.2 Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix} \middle| \, x, y, z \in \mathbb{R} \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ 1 \\ z \end{bmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^4$, and that one of the sets is not.
Solution. U is not a subspace. W is a subspace. □

V4.3 Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y + z = 1 \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y + z = 0 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.
Solution. W is not a subspace. U is a subspace. □

V4.4 Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y = 3z \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y = 3, z \in \mathbb{R} \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.
Solution. U is not a subspace. W is a subspace. □

V4.5 Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x + y = 0 \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, xy = 0 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.
Solution. U is not a subspace. W is a subspace. □

V4.6 Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x = y \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, |x| = |y| \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.
Solution. U is not a subspace. W is a subspace. □

V4.7 Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, y = 2x \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, y = x^2 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.
Solution. U is not a subspace. W is a subspace. $\square$

V4.8 Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, z = 2xy \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, z = 2x + y \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.
Solution. W is not a subspace. U is a subspace. $\square$

Standard V5

V5.1 Determine if set of vectors $\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}3\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\-2\end{bmatrix}\right\}$ is linearly dependent or linearly independent.

Solution. The vectors are linearly independent. □

V5.2 Determine if the set of vectors $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\3\\-2\end{bmatrix}\right\}$ is linearly dependent or linearly independent

Solution. The set is linearly dependent. □

V5.3 Determine if the set of vectors $\left\{\begin{bmatrix}-3\\8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}\right\}$ is linearly dependent or linearly independent

Solution. The set is linearly independent. □

V5.4 Determine if the set of vectors $\left\{\begin{bmatrix}-3\\-8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}\right\}$ is linearly dependent or linearly independent

Solution. The set is linearly dependent. □

V5.5 Determine if the set of vectors $\left\{\begin{bmatrix}3\\-1\\0\\4\end{bmatrix}, \begin{bmatrix}1\\2\\-2\\1\end{bmatrix}, \begin{bmatrix}3\\-8\\6\\5\end{bmatrix}\right\}$ is linearly dependent or linearly independent.

Solution. The set is linearly dependent. □

V5.6 Determine if the set of vectors $\left\{\begin{bmatrix}3\\-1\\1\\4\end{bmatrix}, \begin{bmatrix}1\\2\\-2\\1\end{bmatrix}, \begin{bmatrix}3\\-8\\6\\5\end{bmatrix}\right\}$ is linearly dependent or linearly independent.

Solution. The set is linearly independent. □

V5.7 Determine if the set of vectors $\left\{\begin{bmatrix}2\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}-1\\-3\\1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\\0\\0\end{bmatrix}\right\}$ is linearly dependent or linearly independent.

Solution. The set is linearly dependent. □

V5.8 Determine if the set of vectors $\left\{\begin{bmatrix}2\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}-1\\-3\\1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\3\end{bmatrix}\right\}$ is linearly dependent or linearly independent.

Solution. The set is linearly dependent. □

Standard V6

V6.1 Determine if the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

V6.2 Determine if the set $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis. □

V6.3 Determine if the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 4 \\ 2 & 2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

V6.4 Determine if the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 5 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 3 & 2 & 1 & -1 \\ -1 & 0 & -1 & 3 \\ 2 & 2 & 0 & 0 \\ 3 & 4 & -1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

V6.5 Determine if the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

V6.6 Determine if the set $\left\{ \begin{bmatrix} 8 \\ 21 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 8 & -3 & -1 \\ 21 & -8 & -3 \\ -7 & 3 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. $\square$

V6.7 Determine if the set $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Solution. Since

$$\text{RREF}\begin{bmatrix} 2 & 3 & -4 \\ -1 & 12 & 2 \\ 4 & -9 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. $\square$

V6.8 Determine if the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since this is the identity matrix, this set is a basis. $\square$

Standard V7

V7.1 Let $W = \operatorname{span}\left(\left\{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\3\\6\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\3\\-2\end{bmatrix}, \begin{bmatrix}7\\-1\\8\\-3\end{bmatrix}\right\}\right)$. Find a basis for W.

Solution. $\left\{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\3\\-2\end{bmatrix}\right\}$ is a basis for W. □

V7.2 Let $W = \operatorname{span}\left(\left\{\begin{bmatrix}-3\\-8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}\right\}\right)$. Find a basis for W.

Solution. $\left\{\begin{bmatrix}-3\\-8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right\}$ is a basis for W. □

V7.3 Let $W = \operatorname{span}\left(\left\{\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\3\\6\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\1\end{bmatrix}\right\}\right)$. Find a basis of W.

Solution. $\left\{\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\3\\6\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$ is a basis of W. □

V7.4 Let $W = \operatorname{span}\left(\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\0\\1\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\4\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}\right)$. Find a basis of W.

Solution. $\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}$ is a basis for W. □

V7.5 Let $W = \operatorname{span}\left\{\begin{bmatrix}2\\0\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\-8\\-1\end{bmatrix}\right\}$. Find a basis for this vector space.

Solution. $\left\{\begin{bmatrix}2\\0\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\-1\\1\end{bmatrix}\right\}$ is a basis of W. □

V7.6 Let $W = \operatorname{span}\left\{\begin{bmatrix}2\\-1\\4\end{bmatrix}, \begin{bmatrix}3\\12\\-9\end{bmatrix}, \begin{bmatrix}1\\4\\-3\end{bmatrix}, \begin{bmatrix}-4\\2\\-8\end{bmatrix}\right\}$. Find a basis for this vector space.

Solution. $\left\{\begin{bmatrix}2\\-1\\4\end{bmatrix}, \begin{bmatrix}3\\12\\-9\end{bmatrix}\right\}$ is a basis of W. □

V7.7 Let $W = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}$. Find a basis for this vector space.

Solution. $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}$ is a basis of W. $\square$

Standard V8

V8.1 Let $W = \text{span}\left(\left\{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\3\\6\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\3\\-2\end{bmatrix}, \begin{bmatrix}7\\-1\\8\\-3\end{bmatrix}\right\}\right)$. Find the dimension of W.

Solution. W has dimension 2. □

V8.2 Let $W = \text{span}\left(\left\{\begin{bmatrix}-3\\-8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}\right\}\right)$. Compute the dimension of W.

Solution. $\dim W = 2$. □

V8.3 Let $W = \text{span}\left(\left\{\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\3\\6\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\1\end{bmatrix}\right\}\right)$. Compute the dimension of W.

Solution. $\dim(W) = 3$. □

V8.4 Let $W = \text{span}\left(\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\0\\1\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\4\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}\right)$. Compute the dimension of W.

Solution. $\dim(W) = 3$. □

V8.5 Let $W = \text{span}\left\{\begin{bmatrix}2\\0\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\-8\\-1\end{bmatrix}\right\}$. Find the dimension of W.

Solution. The dimension of W is 2. □

V8.6 Let $W = \text{span}\left\{\begin{bmatrix}2\\-1\\4\end{bmatrix}, \begin{bmatrix}3\\12\\-9\end{bmatrix}, \begin{bmatrix}1\\4\\-3\end{bmatrix}, \begin{bmatrix}-4\\2\\-8\end{bmatrix}\right\}$. Find the dimension of W.

Solution. The dimension of W is 2. □

V8.7 Let $W = \text{span}\left\{\begin{bmatrix}2\\-1\\4\\2\end{bmatrix}, \begin{bmatrix}-4\\2\\-8\\-4\end{bmatrix}, \begin{bmatrix}3\\12\\-9\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\\2\end{bmatrix}\right\}$. Find the dimension of W.

Solution. The dimension of W is 3. □

Standard V9

V9.1 Find a basis for the subspace

$$W = \operatorname{span}\left\{x^2+x, x^2+2x-1, x^2+3x-2\right\}$$

of $\mathcal{P}^2$.
Solution. $\left\{x^2+x, x^2+2x-1\right\}$ □

V9.2 Find a basis for the subspace

$$W = \operatorname{span}\left\{-3x^3-8x^2, x^3+2x^2+2, -x^2+3\right\}$$

of $\mathcal{P}^2$.
Solution. $\left\{-3x^3-8x^2, x^3+2x^2+2\right\}$ □

V9.3 Find a basis for the subspace

$$W = \operatorname{span}\left\{\begin{bmatrix}1 & -3\\ 2 & 2\end{bmatrix}, \begin{bmatrix}-1 & 4\\ -1 & 1\end{bmatrix}, \begin{bmatrix}0 & 3\\ 3 & 9\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 2 & 1\end{bmatrix}\right\}$$

of $M_{2,2}$.
Solution.

$$\left\{\begin{bmatrix}1 & -3\\ 2 & 2\end{bmatrix}, \begin{bmatrix}-1 & 4\\ -1 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 2 & 1\end{bmatrix}\right\}$$

□

V9.4 Find a basis for the subspace

$$W = \operatorname{span}\left\{x^3-3x^2+2x+2, -x^3+4x^2-x+1, 3x^2+3x+9, -x^3+2x+1\right\}$$

of $\mathcal{P}^3$.
Solution.

$$\left\{x^3-3x^2+2x+2, -x^3+4x^2-x+1, -x^3+2x+1\right\}$$

□

V9.5 Find a basis for the subspace

$$W = \operatorname{span}\left\{x^3-x, x^2+x+1, x^3-x^2+2, 2x^2-1\right\}$$

of $\mathcal{P}^3$.
Solution.

$$\left\{x^3-x, x^2+x+1, x^3-x^2+2\right\}$$

□

V9.6 Let W be the subspace of $\mathcal{P}^3$ given by

$$W = \operatorname{span}\left(\left\{x^3+x^2+2x+1, 3x^3+3x^2+6x+3, 3x^3-x^2+3x-2, 7x^3-x^2+8x-3\right\}\right).$$

Find a basis for W.
Solution. A basis is $\left\{x^3+x^2+2x+1, 3x^3-x^2+3x-2\right\}$. □

V9.7 Let $W = \operatorname{span}\left\{\begin{bmatrix}2 & 0\\ -2 & 0\end{bmatrix}, \begin{bmatrix}3 & 1\\ 3 & 6\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\right\}$. Find a basis for this vector space.
Solution. $\left\{\begin{bmatrix}2 & 0\\ -2 & 0\end{bmatrix}, \begin{bmatrix}3 & 1\\ 3 & 6\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\right\}$ is a basis of W. □

V9.8 Let W be the subspace of $\mathcal{P}^2$ given by $W = \operatorname{span}\left(\left\{-3x^2-8x, x^2+2x+2, -x+3\right\}\right)$. Find a basis for W.
Solution. $\left\{-3x^2-8x, x^2+2x+2\right\}$ is a basis for W. □

Standard V10

V10.1 Find a basis for the solution space of the homogeneous system of equations

$$\begin{aligned} x+3y+3z+7w &= 0 \\ x+3y-z-w &= 0 \\ 2x+6y+3z+8w &= 0 \\ x+3y-2z-3w &= 0 \end{aligned}$$

Solution. A basis for the solution space is

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

□

V10.2 Find a basis for the solution space of the homogeneous system of equations

$$\begin{aligned} x+2y+3z+w &= 0 \\ 3x-y+z+w &= 0 \\ 2x-3y-2z &= 0 \\ -x+2z+5w &= 0 \end{aligned}$$

Solution. A basis for the solution space is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$. □

V10.3 Find a basis for the solution space of the homogeneous system of equations

$$\begin{aligned} x+2y+3z+w &= 0 \\ 3x-y+z+w &= 0 \\ 2x-3y-2z &= 0 \end{aligned}$$

Solution. A basis for the solution space is $\left\{ \begin{bmatrix} -\frac{5}{7} \\ -\frac{8}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{7} \\ -\frac{2}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$, or $\left\{ \begin{bmatrix} 5 \\ 8 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ -7 \end{bmatrix} \right\}$. □

V10.4 Find a basis for the solution space to the system of equations

$$\begin{aligned} x+2y-3z &= 0 \\ 2x+y-4z &= 0 \\ 3y-2z &= 0 \\ x-y-z &= 0 \end{aligned}$$

Solution. A basis is $\left\{ \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \right\}$. □

V10.5 Find a basis for the solution space to the homogeneous system of equations

$$\begin{aligned} 2x_1+3x_2-5x_3+14x_4 &= 0 \\ x_1+x_2-x_3+5x_4 &= 0 \end{aligned}$$

Solution. A basis for the solution space is $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

V10.6 Find a basis for the solution space to the homogeneous system of equations

$$\begin{aligned} 4x_1 + 4x_2 + 3x_3 - 6x_4 &= 0 \\ -2x_3 - 4x_4 &= 0 \\ 2x_1 + 2x_2 + x_3 - 4x_4 &= 0 \end{aligned}$$

Solution. A basis for the solution space is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$. □

V10.7 Find a basis for the solution space to the homogeneous system of equations given by

$$\begin{aligned} 3x + 2y + z &= 0 \\ x + y + z &= 0 \end{aligned}$$

Solution. A basis for the solution space is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$. □

V10.8 Find a basis for the solution space to the homogeneous system of equations given by

$$\begin{aligned} 2x_1 - 2x_2 + 6x_3 - x_4 &= 0 \\ 3x_1 + 6x_3 + x_4 &= 0 \\ -4x_1 + x_2 - 9x_3 + 2x_4 &= 0 \end{aligned}$$

Solution. A basis for the solution space is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. □

Module A

Standard A1

A1.1 Consider the following maps of polynomials $S : \mathcal{P}^6 \to \mathcal{P}^6$ and $T : \mathcal{P}^6 \to \mathcal{P}^6$ defined by

$$S(f(x)) = f(x) + 3 \text{ and } T(f(x)) = f(x) + f(3).$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. T is linear, S is not. □

A1.2 Consider the following maps of polynomials $S : \mathcal{P}^4 \to \mathcal{P}^5$ and $T : \mathcal{P}^4 \to \mathcal{P}^5$ defined by

$$S(f(x)) = xf(x) - f(1) \text{ and } T(f(x)) = xf(x) - x.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

A1.3 Consider the following maps of polynomials $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ defined by

$$S(f(x)) = f'(x) - f''(x) \text{ and } T(f(x)) = f(x) - (f(x))^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

A1.4 Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^4$ and $T : \mathcal{P}^2 \to \mathcal{P}^4$ defined by

$$S(f(x)) = x^2 f(x) \text{ and } T(f(x)) = (f(x))^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

A1.5 Consider the following maps of polynomials $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ defined by

$$S(f(x)) = (f(x))^2 + 1 \text{ and } T(f(x)) = (x^2 + 1)f(x).$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. T is linear, S is not. □

A1.6 Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^2$ and $T : \mathcal{P}^2 \to \mathcal{P}^2$ defined by

$$S(ax^2 + bx + c) = cx^2 + bx + a \text{ and } T(ax^2 + bx + c) = a^2x^2 + b^2x + c^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

A1.7 Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^1$ and $T : \mathcal{P}^2 \to \mathcal{P}^1$ defined by

$$S(ax^2 + bx + c) = 2ax + b \text{ and } T(ax^2 + bx + c) = a^2x + b.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

A1.8 Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^3$ and $T : \mathcal{P}^2 \to \mathcal{P}^3$ defined by

$$S(ax^2 + bx + c) = ax^3 + bx^2 + cx \text{ and } T(ax^2 + bx + c) = abcx^3.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

Standard A2

A2.1 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \\ 7x \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \\ 7 & 0 & 0 \end{bmatrix}$

(b) $T\left(\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 15 \\ 11 \\ -14 \end{bmatrix}$

□

A2.2 Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3 + 3x_1 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} -1 \\ 7 \\ 3 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \end{bmatrix}$

□

A2.3 Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 + 3x_3 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} 0 & 1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} -12 \end{bmatrix}$

$\square$

A2.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \\ 0 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} -5 \\ -13 \\ -1 \\ 0 \end{bmatrix}$

$\square$

A2.5 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} 2x + y - z \\ y + 3z \\ 2x - y + 5z \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} -2 \\ -2 \\ -10 \\ 0 \end{bmatrix}$

$\square$

A2.6 Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation given by the requirements

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 3 \end{bmatrix} \text{ and } T(\vec{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} 2x + y \\ y \\ 2x - y \\ 3x + 3y \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}$

$\square$

A2.7 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 7x + 2y + 3z \\ 0 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 7 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 6 \\ 13 \\ -5 \\ 0 \end{bmatrix}$

$\square$

A2.8 Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \\ 7 & 0 & 0 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} 3x+y \\ -8x+2y-z \\ 2y+3z \\ 7x \end{bmatrix}$

(b) $\begin{bmatrix} -5 \\ 15 \\ 11 \\ -14 \end{bmatrix}$

$\square$

A2.9 Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & -5 & 0 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} x+3z \\ 3y-5z \end{bmatrix}$

(b) $\begin{bmatrix} -2 \\ 14 \end{bmatrix}$

$\square$

A2.10 Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation given by the following

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} \qquad T(\vec{e}_2) = \begin{bmatrix} 0 \\ 3 \\ -5 \\ 0 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} x \\ 3y \\ 3x-5y \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ -9 \\ 21 \\ 0 \end{bmatrix}$

□

A2.11 Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - x_3 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} -2 \\ 10 \end{bmatrix}$

□

A2.12 Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - 5x_3 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & -5 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} -2 \\ 14 \end{bmatrix}$

$\square$

A2.13 Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 0 & 1 & 3 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} y + 3z \end{bmatrix}$

(b) $\begin{bmatrix} 0 \end{bmatrix}$

$\square$

A2.14 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by the following

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} 3y \\ x \\ 3x \end{bmatrix}$

(b) $\begin{bmatrix} 9 \\ 1 \\ 3 \end{bmatrix}$

$\square$

A2.15 Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}\right)$

Solution.

(a) $\begin{bmatrix} 3x + z \end{bmatrix}$

(b) $\begin{bmatrix} -3 \end{bmatrix}$

$\square$

Standard A3

A3.1 Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+3y+3z+7w \\ x+3y-z-w \\ 2x+6y+3z+8w \\ x+3y-2z-3w \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of T.
Solution. A basis for the kernel is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

and a basis for the image is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}$$

□

A3.2 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x+y \\ -8x+2y-z \\ 2y+3z \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of T.
Solution. A basis for the image is

$$\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

and a basis of the kernel is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}.$$

□

A3.3 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map given by $T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 8x-3y-z+4w \\ y+3z-4w \\ -7x+3y+2z-5w \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution. $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel. □

A3.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 8x-3y-z \\ y+3z \\ -7x+3y+2z \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution. $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel. □

A3.5 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by the standard matrix $\begin{bmatrix} 1 & 2 & 5 \\ 0 & -2 & -4 \\ -3 & 1 & -1 \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel. □

A3.6 Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear map given by the standard matrix $\begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & -2 & -4 & 2 \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel. □

A3.7 Let $T : \mathbb{R}^2 \to \mathbb{R}^5$ be the linear map given by the standard matrix $\begin{bmatrix} 3 & 1 \\ 0 & 0 \\ -6 & -2 \\ 1 & \frac{1}{3} \\ 9 & 3 \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution. $\left\{ \begin{bmatrix} 3 \\ 0 \\ -6 \\ 1 \\ 9 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel. □

A3.8 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map given by the standard matrix $\begin{bmatrix} 2 & 1 & 0 & 3 \\ -4 & -2 & 0 & 5 \\ -2 & -1 & 0 & 0 \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution. $\left\{ \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the kernel. □

Standard A4

A4.1 Determine if the following linear maps are injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x+2y \\ x-y \\ x+4y \end{bmatrix}$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ 2y+3z \\ x-y-2z \end{bmatrix}$

Solution.

(a) S is injective but not surjective.

(b) T is neither injective nor surjective.

$\square$

A4.2 Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ 2y+3z \\ x-y-2z \end{bmatrix}$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x+2y \\ x-y \\ x+4y \end{bmatrix}$

Solution.

(a) S is neither injective nor surjective.

(b) T is injective but not surjective.

$\square$

A4.3 Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the standard matrix $\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 3 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix}$

Solution.

(a) S is both injective and surjective.

(b) T is neither injective nor surjective.

$\square$

A4.4 Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the standard matrix $\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 4 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix}$

Solution.

(a) S is both injective and surjective.

(b) T is not injective but T is surjective.

$\square$

A4.5 Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^4$ given by the standard matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the standard matrix $\begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 11 & -1 & 5 \end{bmatrix}$

Solution.

(a) S is injective but not surjective.

(b) T is neither injective nor surjective.

$\square$

A4.6 Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^4$ given by the standard matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the standard matrix $\begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 7 & -1 & 5 \end{bmatrix}$

Solution.

(a) S is injective but not surjective.

(b) T is not injective, but T is surjective.

$\square$

A4.7 Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^3$ where $S(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $S(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ where $T(\vec{e}_1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution.

(a) S is injective but not surjective.

(b) T is not injective, but it is surjective.

$\square$

A4.8 Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^4 \to \mathbb{R}^3$ where $S(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $S(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $S(\vec{e}_3) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, and $S(\vec{e}_4) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$,

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(\vec{e}_1) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Solution.

(a) S is not injective, but it is surjective.

(b) T is neither injective nor surjective.

$\square$

Module M

Standard M1

M1.1 Let

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 0 & 7 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.
Solution. CA is the only one that can be computed, and

$$CA = \begin{bmatrix} 3 & 9 & 11 & 1 \\ 0 & 0 & 7 & 2 \\ -2 & -6 & -5 & 0 \end{bmatrix}$$

□

M1.2 Let

$$A = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 3 & 7 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.
Solution. CA is the only one that can be computed, and

$$CA = \begin{bmatrix} 2 & 6 & 11 & 1 \\ 1 & 3 & 7 & 2 \\ -1 & -3 & -5 & 0 \end{bmatrix}$$

□

M1.3 Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & -1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.
Solution. AC is the only one that can be computed, and

$$AC = \begin{bmatrix} 3 & -5 & 14 \\ 1 & -1 & 2 \end{bmatrix}$$

□

M1.4 Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.
Solution. AC is the only one that can be computed, and

$$AC = \begin{bmatrix} 9 & -2 & 14 \\ 1 & 0 & 2 \end{bmatrix}$$

$\square$

M1.5 Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. AB is the only ones that can be computed, and

$$AB = \begin{bmatrix} -3 & -5 & 6 & 14 \\ 0 & 0 & 7 & 35 \end{bmatrix}$$

$\square$

M1.6 Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. BC is the only one that can be computed, and

$$BC = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

$\square$

M1.7 Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. CB is the only one that can be computed, and

$$CB = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

$\square$

M1.8 Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & -1 & 3 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. CB is the only one that can be computed, and

$$CB = \begin{bmatrix} 3 & 3 & -5 & 7 \\ 4 & -4 & 12 & -12 \\ 7 & 2 & 0 & 3 \end{bmatrix}$$

$\square$

Standard M2

M2.1 Determine if the matrix $\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}$ is invertible.

Solution. Not invertible. ☐

M2.2 Determine if the matrix $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$ is invertible.

Solution. Not invertible. ☐

M2.3 Determine if the matrix $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & 0 \\ -1 & -1 & 5 \end{bmatrix}$ is invertible.

Solution. Not invertible. ☐

M2.4 Determine if the matrix $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$ is invertible.

Solution. Invertible. ☐

M2.5 Determine if the matrix $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & -2 & 0 & 3 \end{bmatrix}$ is invertible.

Solution. Invertible. ☐

M2.6 Determine if the matrix $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is invertible.

Solution. Invertible. ☐

M2.7 Determine if the matrix $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix}$ is invertible.

Solution. Not invertible. ☐

M2.8 Determine if the matrix $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 0 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix}$ is invertible.

Solution. Invertible. ☐

Standard M3

M3.1 Show how to find the inverse of the matrix $\begin{bmatrix} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$.

Solution. The inverse is $\begin{bmatrix} 1 & 2 & -5 & 12 \\ 1 & 1 & -4 & -9 \\ -4 & -7 & 20 & 47 \\ -1 & 0 & 3 & 7 \end{bmatrix}$. □

M3.2 Compute the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution. The inverse is $\begin{bmatrix} 1 & 2 & -11 & 37 \\ 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ □

M3.3 Show how to find the inverse of the matrix $\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 4 & -3 \\ 8 & -13 & 10 \\ 13 & -24 & 18 \end{bmatrix}$$

□

M3.4 Show how to find the inverse of the matrix $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & \frac{3}{2} & -\frac{3}{2} \\ 1 & -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

□

M3.5 Show how to find the inverse of the matrix $\begin{bmatrix} 4 & -1 & -8 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Solution. The inverse is $\begin{bmatrix} 1 & -4 & 5 \\ -5 & 24 & -28 \\ 1 & -5 & 6 \end{bmatrix}$. □

M3.6 Show how to find the inverse of the matrix $\begin{bmatrix} 1 & -4 & 5 \\ -5 & 24 & -28 \\ 1 & -5 & 6 \end{bmatrix}$.

Solution. The inverse is $\begin{bmatrix} 4 & -1 & -8 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$. □

M3.7 Show how to find the inverse of the matrix $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$.

Solution. The inverse is $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$. ☐

M3.8 Show how to find the inverse of the matrix $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$.

Solution. The inverse is $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$. ☐

M3.9 Show how to find the inverse of the matrix $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$.

Solution. The inverse is $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$. ☐

Module G

Standard G1

G1.1

(a) Find 3×3 matrices S and T whose left multiplication represents the row operations $R_1 + 5R_3 \to R_1$ and $R_2 \leftrightarrow R_3$, respectively.

(b) If $A \in M_{3,3}$ is a matrix with $\det A = 4$, find the determinant of STA.

Solution.

1. $S = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

2. $\det(STA) = -4$.

□

G1.2

(a) Find 3×3 matrices S and T whose left multiplication represents the row operations $R_2 - 4R_3 \to R_2$ and $R_1 \leftrightarrow R_3$, respectively.

(b) If $A \in M_{3,3}$ is a matrix with $\det A = 7$, find the determinant of STA.

Solution.

1. $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

2. $\det(STA) = -7$.

□

G1.3

(a) Find 3×3 matrices S and T whose left multiplication represents the row operations $3R_2 \to R_2$ and $R_3 - 2R_1 \to R_3$, respectively.

(b) If $A \in M_{3,3}$ is a matrix with $\det A = 7$, find the determinant of STA.

Solution.

1. $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

2. $\det(STA) = 21$.

□

G1.4

(a) Find 3×3 matrices S and T whose left multiplication represents the row operations $3R_2 \to R_2$ and $4R_3 \to R_3$, respectively.

(b) If $A \in M_{3,3}$ is a matrix with $\det A = -2$, find the determinant of STA.

Solution.

1. $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

2. $\det(STA) = -24$.

$\square$

G1.5

(a) Find 3×3 matrices S and T whose left multiplication represents the row operations $-8R_1 \to R_1$ and $R_2 + R_1 \to R_2$, respectively.

(b) If $A \in M_{3,3}$ is a matrix with $\det A = -2$, find the determinant of STA.

Solution.

1. $S = \begin{bmatrix} -8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. $\det(STA) = 16$.

$\square$

G1.6

(a) Find 3×3 matrices S and T whose left multiplication represents the row operations $5R_2 \to R_2$ and $R_2 \leftrightarrow R_3$, respectively.

(b) If $A \in M_{3,3}$ is a matrix with $\det A = -3$, find the determinant of STA.

Solution.

1. $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

2. $\det(STA) = 15$.

$\square$

Standard G2

G2.1 Compute the determinant of the matrix $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$.

Solution.

$$\det \begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} = -36$$

□

G2.2 Compute the determinant of the matrix $\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix}$.

Solution.

$$\det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} = -4$$

□

G2.3 Compute the determinant of the matrix $\begin{bmatrix} 2 & 3 & 0 & 1 \\ -1 & 3 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 1 & -1 & 3 & 5 \end{bmatrix}$.

Solution. -60. □

G2.4 Compute the determinant of the matrix $\begin{bmatrix} 3 & -1 & 0 & 7 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 3 & 0 & 0 & 1 \end{bmatrix}$.

Solution. 14. □

G2.5 Compute the determinant of the matrix $\begin{bmatrix} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$.

Solution. -1. □

G2.6 Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 0 & -4 & 0 \\ -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution. 15. □

G2.7 Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ -2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution. -15. $\square$

G2.8 Compute the determinant of the matrix

$$\begin{bmatrix} 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 0 \end{bmatrix}.$$

Solution. 55. $\square$

G2.9 Compute the determinant of the matrix

$$\begin{bmatrix} 0 & -4 & 1 & 1 \\ -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 0 & -4 & 0 \end{bmatrix}.$$

Solution. -55. $\square$

Standard G3

G3.1 Find the eigenvalues of the matrix $\begin{bmatrix} 5 & 1 \\ -24 & -6 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$, yielding the eigenvalues -3 and 2. □

G3.2 Find the eigenvalues of the matrix $\begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 6\lambda + 5 = (\lambda + 1)(\lambda + 5)$, yielding the eigenvalues -1 and -5. □

G3.3 Find the eigenvalues of the matrix $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$, yielding the eigenvalues 1 and 5. □

G3.4 Find the eigenvalues of the matrix $\begin{bmatrix} -2 & 5 \\ 5 & -2 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 4\lambda - 21 = (\lambda + 7)(\lambda - 3)$, yielding the eigenvalues -7 and 3. □

G3.5 Find the eigenvalues of the matrix $\begin{bmatrix} 10 & -8 \\ 4 & -2 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - 8\lambda + 12 = (\lambda - 6)(\lambda - 2)$, yielding the eigenvalues 6 and 2. □

G3.6 Find the eigenvalues of the matrix $\begin{bmatrix} 6 & -4 \\ 11 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2)$, yielding the eigenvalues -5 and 2. □

G3.7 Find the eigenvalues of the matrix $\begin{bmatrix} -6 & -11 \\ 4 & 9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$, yielding the eigenvalues 5 and -2. □

G3.8 Find the eigenvalues of the matrix $\begin{bmatrix} 1 & -5 \\ 4 & -8 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 7\lambda + 12 = (\lambda + 3)(\lambda + 4)$, yielding the eigenvalues -3 and -4. □

G3.9 Find the eigenvalues of the matrix $\begin{bmatrix} -2 & 2 \\ 15 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 11\lambda - 12 = (\lambda + 12)(\lambda - 1)$, yielding the eigenvalues -12 and 1. □

G3.10 Find the eigenvalues of the matrix $\begin{bmatrix} 10 & 2 \\ -39 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3)$, yielding the eigenvalues -3 and 4. □

G3.11 Find the eigenvalues of the matrix $\begin{bmatrix} 8 & 2 \\ -33 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$, yielding the eigenvalues -3 and 2. □

Standard G4

G4.1 Find a basis of the eigenspace associated to the eigenvalue -1 for the matrix $\begin{bmatrix} 4 & -2 & -1 & 1 \\ 15 & -7 & -3 & 1 \\ -5 & 2 & 0 & 1 \\ 10 & -4 & -2 & 0 \end{bmatrix}$.

Solution. A basis for the eigenspace is $\left\{ \begin{bmatrix} \frac{2}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. □

G4.2 Find a basis of the eigenspace associated to the eigenvalue 1 for the matrix $A = \begin{bmatrix} -3 & 1 & 0 & 0 \\ -8 & 2 & 0 & -1 \\ 0 & 2 & 1 & 2 \\ 4 & 1 & 0 & 3 \end{bmatrix}$.

Solution. A basis for the eigenspace is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4.3 Find a basis of the eigenspace associated to the eigenvalue 1 for the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 8 & -3 & -1 \\ 2 & 21 & -8 & -3 \\ 3 & -7 & 3 & 2 \end{bmatrix}$.

Solution. A basis for the eigenspace is $\left\{ \begin{bmatrix} 0 \\ \frac{3}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4.4 Find a basis of the eigenspace associated to the eigenvalue 1 for the matrix
$A = \begin{bmatrix} 9 & -3 & -5 & 2 \\ 19 & -6 & -12 & 5 \\ 1 & 1 & -1 & 3 \\ -11 & 4 & 7 & -2 \end{bmatrix}$.

Solution. A basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4.5 Find a basis of the eigenspace associated to the eigenvalue 2 for the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 11 & -6 & 1 & -1 \\ -9 & 5 & 1 & 3 \end{bmatrix}$.

Solution. A basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4.6 Find a basis of the eigenspace associated to the eigenvalue 2 for the matrix $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$.

Solution. A basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. □

G4.7 Find a basis of the eigenspace associated to the eigenvalue 2 for the matrix
$A = \begin{bmatrix} 0 & -2 & -1 & 0 \\ -4 & -2 & -2 & 0 \\ 14 & 12 & 10 & 2 \\ -13 & -10 & -8 & -1 \end{bmatrix}$.

Solution. A basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4.8 Find a basis of the eigenspace associated to the eigenvalue 3 for the matrix
$A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ -4 & -1 & -2 & 0 \\ 14 & 12 & 11 & 2 \\ -14 & -10 & -9 & -1 \end{bmatrix}$.

Solution. A basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4.9 Find a basis of the eigenspace associated to the eigenvalue -2 for the matrix $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 2 & -3 & 3 \\ 1 & 8 & -9 & 6 \\ 1 & 8 & -7 & 4 \end{bmatrix}$.

Solution. A basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ \frac{1}{4} \\ 1 \\ 1 \end{bmatrix} \right\}$. □

G4.10 Find a basis of the eigenspace associated to the eigenvalue 2 for the matrix $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 6 & 2 & -2 & 0 \\ -9 & 0 & 5 & 0 \\ 15 & 0 & -5 & 2 \end{bmatrix}$.

Solution. A basis for the eigenspace is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. □