Linear Algebra

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At the end of this module, students will be able to...

- **E1: Systems as matrices.** Translate back and forth between a system of linear equations and the corresponding augmented matrix.
- **E2: Row reduction.** Put a matrix in reduced row echelon form
- **E3: Solving Linear Systems.** Solve a system of linear equations.
- **E4: Homogeneous Systems.** Find a basis for the solution set of a homogeneous linear system.

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.

The following resources will help you prepare for this module.

- `https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-systems-topic/cc-8th-systems-graphically/a/systems-of-equations-with-graphing`
- `https://www.khanacademy.org/math/algebra/systems-of-linear-equations/solving-systems-of-equations-with-substitution/v/practice-using-substitution-for-systems`

Definition

A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

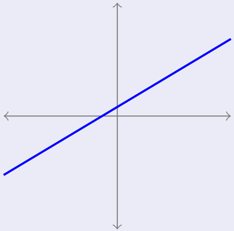
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

Observation

The linear equation $3x - 5y = -2$ may be graphed as a line in the xy plane.



The linear equation $x + 2y - z = 4$ may be graphed as a plane in xyz space.

Remark

In previous classes you likely assumed $x = x_1$, $y = x_2$, and $z = x_3$. However, since this course often deals with equations of four or more variables, we will almost always write our variables as x_i.

Definition

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n \ &= \ b_1 \\
a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n \ &= \ b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ &= b_m
\end{aligned}$$

A **solution**

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i$$

for $1 \leq i \leq m$ (that is, the solution satisfies all equations in the system).

Remark

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{alignedat}{4} & & x_1 &+3x_3 &&= &&3 \\ 3x_1 &-2x_2 &&+4x_3 &&= &&0 \\ & &-x_2 &+x_3 &&= &&-2 \end{alignedat}$$

Verbose standard form:

$$\begin{alignedat}{4} x_1 &+0x_2 &&+3x_3 &&= &&3 \\ 3x_1 &-2x_2 &&+4x_3 &&= &&0 \\ 0x_1 &-x_2 &&+x_3 &&= &&-2 \end{alignedat}$$

Concise standard form:

$$\begin{alignedat}{4} x_1 & && +3x_3 &&= &&3 \\ 3x_1 &-2x_2 &&+4x_3 &&= &&0 \\ &-x_2 &&+x_3 &&= &&-2 \end{alignedat}$$

Definition

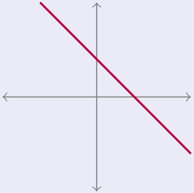
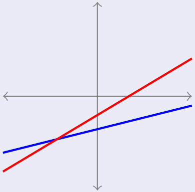
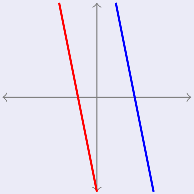
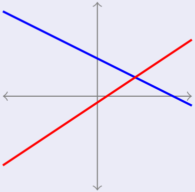
A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

Fact

All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

Activity

Consider the following graphs representing linear systems of two variables. Label each graph with **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

Activity

All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity

Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part X: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

Part X: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to describe *all* solutions (the **solution set**) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$ for the linear system in terms of a.

Activity

Consider the following linear system.

$$\begin{alignedat}{5} x_1 &+ 2x_2 & & &- &x_4 &= &\ 3 \\ & & & x_3 &+ &4x_4 &= &-2 \end{alignedat}$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix} + a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.

Remark

The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - x_2 + x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & 1 & 1 & -2 \end{array}$$

Definition

A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Definition

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

Activity

Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a conzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

Definition

The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a conzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity

Consider the following two linear systems.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Part X: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

1. Swap R_1 (first row) and R_2 (second row).
2. Multiply R_2 by $\frac{1}{2}$.
3. Add R_1 to R_3.
4. Add $-3R_1$ to R_2.
5. Add $-2R_2$ to R_3.
6. Multiply R_3 by $\frac{1}{3}$.

Part X: What is the common solution to these linear systems?

Definition

The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix. Examples:

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Activity

Find your own sequence of row operations to manipulate the matrix

$$\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array}\right]$$

into row echelon form. (Note that row echelon form is not unique.)

The most efficient way to do this is by circling **pivot positions** in your matrix:

1. Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
2. Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
3. Repeat these two steps as often as possible.

Activity

Solve this simplifed linear system:

$$\begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

Observation

The consise standard form of the solution to this linear system corresponds to a simplified row echelon form matrix:

$$\begin{alignedat}{4} x_1 & & & &&= -2 \\ & x_2 & & &&= 7 \\ & & & x_3 &&= 2 \end{alignedat} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Definition

A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0. Examples:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Activity

Show that the following two linear systems:

$$
\begin{aligned}
x_1 - x_2 + 5x_3 &= 1 \\
x_2 - 2x_3 &= 3 \\
x_3 &= 2
\end{aligned}
\qquad\qquad
\begin{aligned}
x_1 &= -2 \\
x_2 &= 7 \\
x_3 &= 2
\end{aligned}
$$

are equivalent by converting the first system to an augmented matrix, and then zeroing out all terms above pivot positions (the leading terms).

Remark

We may verify that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$ is a solution to the original linear system

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

by plugging the solution into each equation.

Fact

Every augmented matrix A reduces to a unique reduced row echelon form matrix. This matrix is denoted as $\mathrm{RREF}(A)$.

Activity

Consider the matrix

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array}\right]$$

.

Part X: Find RREF(A).

Part X: How many solutions does the corresponding linear system have?

Definition

An algorithm that reduces A to $\mathrm{RREF}(A)$ is called **Gauss-Jordan elimination**. For example:

1. Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
2. Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
3. Repeat these two steps as often as possible.
4. Finally, zero out any terms above pivot positions.

Observation

Here is an example of applying Gauss-Jordan elimination to a matrix:

$$\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -2 & 10 & 2 \\ 3 & -2 & 13 & 6 \\ -1 & 3 & -6 & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 3 & -2 & 13 & 6 \\ -1 & 3 & -6 & 11 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 2 & -1 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & -9 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Activity

Find RREF(A) where

$$A = \left[\begin{array}{cccc|c} -1 & 1 & -3 & 2 & 0 \\ 2 & -1 & 5 & 3 & -11 \\ 3 & 2 & 4 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

Definition

The columns of $\mathrm{RREF}(A)$ without a leading term represent **free variables** of the linear system modeled by A that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by A.

Activity

Given the linear system and its equivalent row-reduced matrix

$$\begin{aligned} -x_1 + x_2 - 3x_3 + 2x_4 &= 0 \\ 2x_1 - x_2 + 5x_3 + 3x_4 &= -11 \\ 3x_1 + 2x_2 + 4x_3 + x_4 &= 1 \\ x_2 - x_3 + x_4 &= 1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

circle the pivot positions and describe the solution set $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} + a \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$ by setting the free variable (the column without a pivot position) equal to a, and expressing each of the other bounded variables equal to an expression in terms of a.

Remark

It's not necessary to completely find $\mathrm{RREF}(A)$ to deduce that a linear system is inconsistent.

Activity

Find a contradiction in the inconsistent linear system

$$\begin{aligned} 2x_1 - 3x_2 &= 17 \\ x_1 + 2x_2 &= -2 \\ -x_1 - x_2 &= 1 \end{aligned}$$

by considering the following equivalent augmented matrices:

$$\left[\begin{array}{cc|c} 2 & -3 & 17 \\ 1 & 2 & -2 \\ -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{array}\right].$$

Activity

Show that all linear systems of the form

$$\begin{aligned} a_{11}x_1 + \; a_{12}x_2 + \ldots + \; a_{1n}x_n &= 0 \\ a_{21}x_1 + \; a_{22}x_2 + \ldots + \; a_{2n}x_n &= 0 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \;\; &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

are consistent by finding a quickly verifiable solution.

Definition

A **homogeneous system** is a linear system satisfying $b_i = 0$, that is, it is a linear system of the form

$$\begin{aligned}
a_{11}x_1 + \; a_{12}x_2 + \ldots + \; a_{1n}x_n &= 0 \\
a_{21}x_1 + \; a_{22}x_2 + \ldots + \; a_{2n}x_n &= 0 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\;\; \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}$$

Fact

Because the zero vector is always a solution, the solution set to any homogeneous system with infinitely-many solutions may be generated by multiplying the parameters representing the free variables by a minimal set of Euclidean vectors, and adding these up. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Definition

A minimal set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Activity

Find a basis for the solution set of the following homogeneous linear system.

$$\begin{alignedat}{5} x_1 &{}+{}& 2x_2 &{}{}& &{}-{}& x_4 &= 0 \\ & & & & x_3 &{}+{}& 4x_4 &= 0 \\ 2x_1 &{}+{}& 4x_2 &{}+{}& x_3 &{}+{}& 2x_4 &= 0 \end{alignedat}$$

At the end of this module, students will be able to...

- **V1: Vector Spaces.** Determine if a set with given operations forms a vector space.
- **V2: Linear Combinations.** Determine if a vector can be written as a linear combination of a given set of vectors.
- **V3: Spanning Sets.** Determine if a set of vectors spans a vector space.
- **V4: Subspaces.** Determine if a subset of a vector space is a subset or not.

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.

The following resources will help you prepare for this module.

- https://www.khanacademy.org/math/precalculus/vectors-precalc/vector-addition-subtraction/v/adding-and-subtracting-vectors
- https://www.khanacademy.org/math/precalculus/vectors-precalc/combined-vector-operations/v/combined-vector-operations-example
- https://www.khanacademy.org/math/precalculus/imaginary-and-complex-numbers/adding-and-subtracting-complex-numbers/v/adding-complex-numbers
- https://www.khanacademy.org/math/algebra/introduction-to-polynomial-expressions/adding-and-subtracting-polynomials/v/adding-and-subtracting-polynomials-1

Activity

Consider each of the following vector properties. Label each property with $\mathbb{R}^1$, $\mathbb{R}^2$, and/or $\mathbb{R}^3$ if that property holds for Euclidean vectors/scalars $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of that dimension.

1. **Addition associativity.** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
2. **Addition commutivity.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. **Addition identity.** There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
4. **Addition inverse.** There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
5. **Addition midpoint uniqueness.** There exists a unique $\mathbf{m}$ where the distance from $\mathbf{u}$ to $\mathbf{m}$ equals the distance from $\mathbf{m}$ to $\mathbf{v}$.
6. **Scalar multiplication**
7. **Scalar multiplication identity.** $1\mathbf{v} = \mathbf{v}$.
8. **Scalar multiplication relativity.** There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.
9. **Scalar distribution.** $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
10. **Vector distribution.** $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
11. **Orthogonality.** There exists a non-zero vector $\mathbf{n}$ such that $\mathbf{n}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Definition

A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Addition associativity.** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition commutivity.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Addition identity.** There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- **Addition inverse.** There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.** $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **Scalar multiplication identity.** $1\mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Vector distribution.** $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Definition

The most important examples of vector spaces are the **Euclidean vector spaces** $\mathbb{R}^n$, but there are other examples as well.

Activity

Consider the following vector space that models motion along the curve $y = e^x$. Let $V = \{(x, y) : y = e^x\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 b_2)$, and $c(a, b) = (ca, b^c)$.

Part X: Verify that $3((1, e) + (-2, \frac{1}{e^2})) = 3(1, e) + 3(-2, \frac{1}{e^2})$.

Part X: Prove the scalar distribution property for this space: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

Remark

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{R}^\infty$: Sequences of real numbers $(v_1, v_2, \dots)$.
- $\mathbb{R}^{m\times n}$: Matrices of real numbers with m rows and n columns.
- $\mathbb{C}$: Complex numbers.
- $\mathcal{P}^n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity

Let $V = \{(a, b) : a, b \text{ are real numbers}\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$ and $c(a, b) = (a^c, b + c)$. Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition commutivity.** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Addition identity.** There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- **Addition inverse.** There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.** $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **Scalar multiplication identity.** $1\mathbf{v} = \mathbf{v}$.
- **Scalar distribution.** $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Vector distribution.** $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Definition

A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ for any choice of scalar multiples $c_1, c_2, \dots, c_m$.

Definition

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

Activity

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part X: Sketch $c\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part X: Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ in the xy plane.

Activity

Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part X: Sketch $c_1\begin{bmatrix}1\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\1\end{bmatrix}$ in the xy plane for $\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}$.

Part X: Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity

Sketch a representation of all the vectors given by $\operatorname{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix},\begin{bmatrix}-2\\3\end{bmatrix}\right\}$ in the xy plane.

Activity

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2.

Part X: Reinterpret this vector equation as a system of linear equations.

Part X: Solve this system. (From now on, feel free to use a calculator to solve linear systems.)

Part X: Given this solution, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

Fact

A vector $\mathbf{b}$ belongs to $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \,|\, \mathbf{b}]$ is consistent.

Remark

To determine if $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, find $\operatorname{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \,|\, \mathbf{b}]$.

Activity

Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Activity

Determine if $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Observation

So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space $\mathbb{R}^n$; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

Activity

We previously checked that $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ does not belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$.

Does $f(x) = 3x^2 - 2x + 1$ belong to span$\{x^2 - 3, -x^2 - 3x + 2\}$?

Activity

Does the matrix $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\}$?

Activity

Does the complex number $2i$ belong to $\text{span}\{-3+i, 6-2i\}$?

Activity

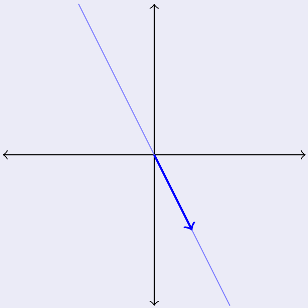
How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your guess.

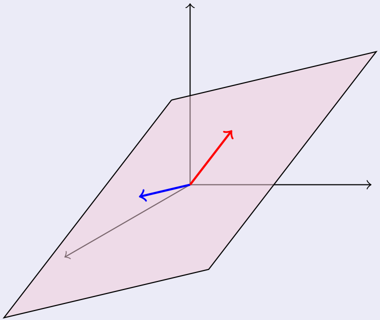
Activity

How many vectors are required to span $\mathbb{R}^3$?

Fact

At least n vectors are required to span $\mathbb{R}^n$.





Activity

Find a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by doing the following.

Part X: Choose simple values for x, y, z such that $\left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{array}\right]$ represents an inconsistent linear equation.

Part X: Use row operations to manipulate $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right]$.

Part X: Write a sentence explaining why $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ cannot be in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Fact

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ fails to span all of $\mathbb{R}^n$ exactly when $\mathrm{RREF}[\mathbf{v}_1 \; \dots \; \mathbf{v}_m]$ has a row of zeros.

Activity

Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Prove that $\mathbb{R}^4 = \operatorname{span} S$.

Activity

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2$$

Prove that $\mathcal{P}^3 \neq \operatorname{span} S$.

Definition

A subset of a vector space is called a **subspace** if it is itself a vector space.

Fact

If S is a subset of a vector space V, then $\operatorname{span} S$ is a subspace of V.

Remark

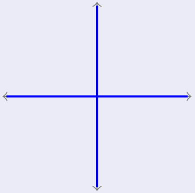
To prove that a subset is a subspace, you need only verify that $c\mathbf{v} + d\mathbf{w}$ belongs to the subset for any choice of vectors $\mathbf{v}, \mathbf{w}$ from the subset and any real scalars c, d.

Activity

Prove that $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$ is a subspace of the vector space of all degree-two polynomials by showing that $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$ belongs to P.

Activity

Consider the subset of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.



Part X: Find a linear combination $c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ that does not belong to this subset.

Part X: Use this linear combination to sketch a picture illustrating why this subset is not a subspace.

Fact

Suppose a subset S of V is isomorphic to another vector space W. Then S is a subspace of V.

Activity

Show that the set of 2×2 matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of $\mathbb{R}^{2\times 2}$ by finding a Euclidean space isomorphic to S.

At the end of this module, students will be able to...

- **S1. Linear independence** Determine if a set of Euclidean vectors is linearly dependent or independent.
- **S2. Basis verification** Determine if a set of vectors is a basis of a vector space
- **S3. Basis construction** Construct a basis for the subspace spanned by a given set of vectors.
- **S4. Dimension** I can compute the dimension of a vector space.

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.
- Apply linear combinations and spanning sets **(Standard(s) V2,V3)**.

The following resources will help you prepare for this module.

- https://www.khanacademy.org/math/precalculus/vectors-precalc/vector-addition-subtraction/v/adding-and-subtracting-vectors
- https://www.khanacademy.org/math/precalculus/vectors-precalc/combined-vector-operations/v/combined-vector-operations-example

Activity

In the previous module, we considered

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

and showed that span $S \neq \mathbb{R}^4$. Find two vectors that are in the span of the other three vectors.

Definition

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

Activity

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Is the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ consistent with one solution, consistent with infinitely many solutions, or inconsistent?

Fact

The set $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ is consistent with infinitely many solutions.

Activity

Find

$$\operatorname{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array}\right]$$

and circle the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

Fact

A set of Euclidean vectors $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ has a column without a pivot position.

Activity

TODO (compute RREF and label each set of vectors as linearly independent/dependent)

Activity

(take basis shown to be linearly independent in previous day, and show that it spans)

Definition

A **basis** is a linearly independent set that spans a vector space.

Observation

A basis may be thought of as building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

Activity

(given four sets of general vectors, identify which are bases and which aren't)

Activity

If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$, that means RREF$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$ doesn't have a column without a pivot position, and doesn't have a row of zeros. What is RREF$[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$?

Fact

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for $\mathbb{R}^n$ if and only if $m = n$ and

$$\mathrm{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Activity

(given four sets of IR^5 vectors, identify which are bases and which aren't)

Activity

How can $\{u, v, u+v\}$ (but with numbers) be changed to make it linearly independent?

Activity

(discover that the redundant vectors are non-pivot columns)

Fact

To compute a basis for the subspace $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, simply remove the vectors corresponding to the non-pivot columns of $\mathrm{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$.

Activity

(find ALL the bases for span S that are subsets of S)

Fact

All bases for a vector space are the same size.

Activity

Prove that if $\{\mathbf{v}\}$ is a basis for V, then $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly dependent (assuming $\mathbf{w}_1 \neq \mathbf{w}_2$).

Fact

All bases for a vector space are the same size.

Definition

The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

Activity

Reduce a bunch of spans to bases to find their dimension.

Activity

What is the dimension of the vector space of 7th-degree polynomials $\mathcal{P}^7$?

Activity

What is the dimension of the vector space of polynomials $\mathcal{P}$?

Observation

Several interesting vector spaces are infinite-dimensional:

- The space of polynomials $\mathcal{P}$
- The space of real number sequences $\mathbb{R}^\infty$
- The space of continuous functions $C(\mathbb{R})$

Fact

Every vector space with dimension $n < \infty$ is isomorphic to $\mathbb{R}^n$.