

## Section E.2

**Remark E.2.1** The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Verbose standard form:

$$\begin{aligned}1x_1 + 0x_2 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\0x_1 - 1x_2 + 1x_3 &= -2\end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c}1 & 0 & 3 & 3 \\3 & -2 & 4 & 0 \\0 & -1 & 1 & -2\end{array}$$

**Definition E.2.2** A system of  $m$  linear equations with  $n$  variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{array}{l}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m\end{array} \qquad \left[ \begin{array}{cccc|c}a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\\vdots & \vdots & \ddots & \vdots & \vdots \\a_{m1} & a_{m2} & \cdots & a_{mn} & b_m\end{array} \right]$$

**Definition E.2.3** Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution:  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

$$\begin{aligned}3x_1 - 2x_2 &= 1 \\x_1 + 4x_2 &= 5\end{aligned}$$

$$\begin{aligned}3x_1 - 2x_2 &= 1 \\4x_1 + 2x_2 &= 6\end{aligned}$$

Therefore these augmented matrices are equivalent:

$$\left[ \begin{array}{cc|c}3 & -2 & 1 \\1 & 4 & 5\end{array} \right]$$

$$\left[ \begin{array}{cc|c}3 & -2 & 1 \\4 & 2 & 6\end{array} \right]$$

**Activity E.2.4** ( $\sim 10$  min) Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

- |   |   |
|---|---|
| a) Swap two rows.                         | d) Multiply a row by a nonzero constant.              |
| b) Swap two columns.                      | e) Add a constant multiple of one row to another row. |
| c) Add a constant to every term in a row. | f) Replace a column with zeros.                       |

**(Instructor Note:)** This activity could be ran as a card sort. Allow 5 additional minutes for intra team discussion.

**Definition E.2.5** The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a nonzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices  $A, B$  are equivalent (so whenever we do any of these operations), we write  $A \sim B$ .

**Activity E.2.6** ( $\sim 10$  min) Consider the following linear systems.

$3x_1 - 2x_2 + 13x_3 = 6$	$x_1 - 3x_2 + 6x_3 = -11$	$x_1 - 3x_2 + 6x_3 = -11$
(A) $2x_1 - 2x_2 + 10x_3 = 2$	(C) $2x_1 - 2x_2 + 10x_3 = 2$	(E) $4x_2 - 2x_3 = 24$
$-x_1 + 3x_2 - 6x_3 = 11$	$3x_1 - 2x_2 + 13x_3 = 6$	$7x_2 - 5x_3 = 39$
$x_1 + 9x_3 = 16$	$x_1 - 3x_2 + 6x_3 = -11$	$x_1 + 9x_3 = 16$
(B) $x_2 + x_3 = 9$	(D) $x_2 + x_3 = 9$	(F) $x_2 + x_3 = 9$
$-12x_3 = -24$	$7x_2 - 5x_3 = 39$	$x_3 = 2$

*Part 1:* Which system can be obtained from System (A) in the fewest number of row operations?

*Part 2:* Rank the six linear systems from easiest to solve to hardest to solve.

**Activity E.2.7** ( $\sim 10$  min) Consider the following augmented matrices.

$$(A) \left[ \begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array} \right]$$

$$(C) \left[ \begin{array}{ccc|c} 1 & -3 & 6 & -11 \\ 2 & -2 & 10 & 2 \\ 3 & -2 & 13 & 6 \end{array} \right]$$

$$(E) \left[ \begin{array}{ccc|c} 1 & -3 & 6 & -11 \\ 0 & 4 & -2 & 24 \\ 0 & 7 & -5 & 39 \end{array} \right]$$

$$(B) \left[ \begin{array}{ccc|c} 1 & 0 & 9 & 16 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & -12 & -24 \end{array} \right]$$

$$(D) \left[ \begin{array}{ccc|c} 1 & -3 & 6 & -11 \\ 0 & 1 & 1 & 9 \\ 0 & 7 & -5 & 39 \end{array} \right]$$

$$(F) \left[ \begin{array}{ccc|c} 1 & 0 & 9 & 16 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

*Part 1:* Rank the six matrices from farthest from a reduced row echelon form (RREF) matrix to closest to a RREF matrix.

*Part 2:* These matrices are all **row equivalent** and represent equivalent linear systems. Write down one of these linear systems and solve it.

**Remark E.2.8** It is important to understand the **Gauss-Jordan elimination** algorithm that converts a matrix into reduced row echelon form, but in practice we don't do this by hand; we use technology to do this for us.

**Activity E.2.9** ( $\sim 10$  min)

- Go to <http://www.cocalc.com> and create an account.
- Create a project titled “Linear Algebra Team X” with your appropriate team number. Add all team members as collaborators.
- Open the project and click on “New”
- Give it an appropriate name such as “Class 4 workbook”. Make a new Jupyter notebook.
- Click on “Kernel” and make sure “Octave” is selected.
- Type `A=[1 3 4 ; 2 5 7]` to store the matrix  $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{bmatrix}$  in the variable  $A$ ; hold shift when you press enter.
- Type `rref(A)` to compute the reduced row echelon form of  $A$ .

**Remark E.2.10** If you need to find the reduced row echelon form of a matrix during class, you should feel free to use CoCalc/Octave.

You can change a cell from “Code” to “Markdown” or “Raw” to put comments around your calculations such as Activity numbers.

**Activity E.2.11** ( $\sim 8$  min) Consider our system of equations from above.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-x_1 + 3x_2 - 6x_3 = 11$$

Convert this to an augmented matrix, use CoCalc to compute the reduced row echelon form, and convert back to a simpler system of equations to solve this system. Write your solution on your whiteboard.

**Activity E.2.12** ( $\sim 7$  min) Consider our system of equations from above.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-x_1 \quad \quad - 3x_3 = 1$$

Convert this to an augmented matrix, use CoCalc to compute the reduced row echelon form, and convert back to a simpler system of equations to solve this system. Write your solution on your whiteboard.