

## Application Activities - Module M Part 3 - Class Day 23

**Definition 23.1** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear map with standard matrix  $A$ .

- If  $T$  is a bijection and  $B$  is any  $\mathbb{R}^n$  vector, then  $T(X) = AX = B$  has a unique solution  $X$ .
- So we may define an **inverse map**  $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by setting  $T^{-1}(B) = X$  to be this unique solution.
- Let  $A^{-1}$  be the standard matrix for  $T^{-1}$ . We call  $A^{-1}$  the **inverse matrix** of  $A$ , so we also say that  $A$  is **invertible**.

**Activity 23.2** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the bijective linear map defined by  $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$ . It can be shown that  $T$  is bijective and has the inverse map  $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$ .

*Part 1:* Compute  $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$ .

*Part 2:* If  $A$  is the standard matrix for  $T$  and  $A^{-1}$  is the standard matrix for  $T^{-1}$ , what must  $A^{-1}A$  be?

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**Observation 23.3**  $T^{-1} \circ T = T \circ T^{-1}$  is the identity map for any bijective linear transformation  $T$ . Therefore  $A^{-1}A = AA^{-1} = I$  is the identity matrix for any invertible matrix  $A$ .

**Activity 23.4** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the matrix  $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ .

*Part 1:* Solve  $T(X) = \mathbf{e}_1$  to find  $T^{-1}(\mathbf{e}_1)$ .

*Part 2:* Solve  $T(X) = \mathbf{e}_2$  to find  $T^{-1}(\mathbf{e}_2)$ .

*Part 3:* Solve  $T(X) = \mathbf{e}_3$  to find  $T^{-1}(\mathbf{e}_3)$ .

*Part 4:* Compute  $A^{-1}$ , the standard matrix for  $T^{-1}$ .

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**Observation 23.5** We could have solved these three systems simultaneously by row reducing the matrix  $[A \mid I]$  at once.

$$A = \left[ \begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array} \right]$$

**Activity 23.6** Find the inverse  $A^{-1}$  of the matrix  $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$  by row-reducing  $[A \mid I]$ .

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**Activity 23.7** Is the matrix  $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$  invertible? Give a reason for your answer.

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**Observation 23.8** A matrix  $A \in \mathbb{R}^{n \times n}$  is invertible if and only if  $\text{RREF}(A) = I_n$ .

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