

Name: _____

MIDTERM EXAM

Math 237 – Linear Algebra

Version 4

Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{cccc|c} 3 & -1 & 0 & 1 & 5 \\ -1 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 0 & -3 \end{array} \right]$$

Solution:

$$\begin{aligned} 3x_1 - x_2 + x_4 &= 5 \\ -x_1 + 9x_2 + x_3 - 7x_4 &= 0 \\ x_1 - x_3 &= -3 \end{aligned}$$

□

E2. Find RREF A , where

$$A = \left[\begin{array}{cccc|c} 2 & 2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 4 & 5 \\ 3 & 3 & -1 & -2 & 1 \end{array} \right]$$

Solution:

$$\text{RREF } A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

□

E3. Solve the system of linear equations.

$$\begin{aligned} 2x + y - z + w &= 5 \\ 3x - y - 2w &= 0 \\ -x + 5z + 3w &= -1 \end{aligned}$$

Solution:

$$\text{RREF} \left(\left[\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right] \right) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{12} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{array} \right]$$

So the solutions are

$$\left\{ \left[\begin{array}{c} 1+a \\ 3-21a \\ -7a \\ 12a \end{array} \right] \mid a \in \mathbb{R} \right\}$$

□

E4. Find a basis for the solution set of the system of equations

$$x + 2y + 3z + w = 0$$

$$3x - y + z + w = 0$$

$$2x - 3y - 2z = 0$$

Solution:

$$\text{RREF} \left(\begin{bmatrix} 1 & -2 & 3 & 1 \\ 3 & -1 & 1 & 1 \\ 2 & -3 & -2 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & \frac{5}{7} & \frac{3}{7} \\ 0 & 1 & \frac{8}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution set is

$$\left\{ \begin{bmatrix} -\frac{5}{7}a - \frac{3}{7}b \\ -\frac{8}{7}a - \frac{2}{7}b \\ a \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

So a basis for the solution set is $\left\{ \begin{bmatrix} -\frac{5}{7} \\ -\frac{8}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{7} \\ -\frac{2}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$, or $\left\{ \begin{bmatrix} 5 \\ 8 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ -7 \end{bmatrix} \right\}$.

□

V1. Let V be the set of all polynomials with the operations, for any $f, g \in V$, $c \in \mathbb{R}$,

$$f \oplus g = f' + g'$$

$$c \odot f = cf'$$

(here f' denotes the derivative of f).

(a) Show that scalar multiplication **distributes scalars** over vector addition: $c \odot (f \oplus g) = c \odot f \oplus c \odot g$.

(b) Determine if V is a vector space or not. Justify your answer.

Solution: Let $f, g \in \mathcal{P}$, and let $c \in \mathbb{R}$.

$$c \odot (f \oplus g) = c \odot (f' + g') = c(f' + g')' = cf'' + cg'' = cf' \oplus cg' = c \odot f \oplus c \odot g.$$

However, this is not a vector space, as there is no zero vector. Additionally, $1 \odot f \neq f$ for any nonzero polynomial f .

□

V2. Determine if $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$.

Solution:

$$\text{RREF} \left(\left[\begin{array}{ccc|c} 3 & 1 & 5 & 1 \\ 0 & -1 & 1 & 4 \\ -1 & 4 & -6 & 3 \end{array} \right] \right) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

So $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is not a linear combination of the three vectors.

□

V3. Does $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$?

Solution: Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 2 & 2 \\ 4 & -9 & 3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

lacks a zero row, the vectors span \mathbb{R}^3 .

□

V4. Determine if $\left\{ \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$ a subspace of \mathbb{R}^4 .

Solution: It is closed under addition and scalar multiplication, so it is a subspace. Alternatively, it is the image of the linear transformation from $\mathbb{R}^3 \rightarrow \mathbb{R}^4$ given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix}.$$

□

S1. Determine if the set of polynomials $\{-3x^3 - 8x^2, x^3 + 2x^2 + 2, -x^2 + 3\}$ is linearly dependent or linearly independent

Solution:

$$\text{RREF} \left(\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has a non pivot column, therefore the set is linearly dependent.

□

S2. Determine if the set $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis of \mathbb{R}^4 .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.

□

S3. Let $W = \text{span} \left(\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\} \right)$. Find a basis for W .

Solution: Let $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$, and compute $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$. Since the first two columns are

pivot columns, $\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$ is a basis for W .

□

S4. Let $W = \text{span} \left(\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\} \right)$. Compute the dimension of W .

Solution: Let $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$, and compute $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$. Since there are two pivot

columns, $\dim W = 2$.

□

E1:	<input type="text"/>	E2:	<input type="text"/>	E3:	<input type="text"/>	E4:	<input type="text"/>	V1:	<input type="text"/>	V2:	<input type="text"/>	V3:	<input type="text"/>
<input type="text"/>		V4:	<input type="text"/>	S1:	<input type="text"/>	S2:	<input type="text"/>	S3:	<input type="text"/>	S4:	<input type="text"/>	<input type="text"/>	