Application Activities - Module A Part 3 - Class Day 19

Observation 19.1 Let $T : V \to W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.
- The **kernel** of T is the set of all things that are mapped to $\mathbf{0}$. It is a subspace of V.
- The **image** of T is the set of all things in W that are mapped to by something in V. It is a subspace of W.

Activity 19.2 Let $T : V \to W$ be a linear transformation where $\ker T = \{\mathbf{0}\}$. Can you answer either of the following questions about T?

(a) Is T injective?

(b) Is T surjective?

(Hint: If $T(\mathbf{v}) = T(\mathbf{w})$, then what is $T(\mathbf{v} - \mathbf{w})$?)

Fact 19.3 A linear transformation T is injective **if and only if** $\ker T = \{\mathbf{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

Activity 19.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation where $\operatorname{Im} T = \operatorname{span}\left\{\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\-1\end{bmatrix}\right\}$. Can you answer either of the following questions about T?

(a) Is T injective?

(b) Is T surjective?

Fact 19.5 A linear transformation $T : V \to W$ is surjective **if and only if** $\operatorname{Im} T = W$. Put another way, a surjective linear transformation may be recognized by its same codomain and image.

Activity 19.6 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of equivalent statements.

(a) T is injective

(b) T is surjective

(c) The kernel of T is trivial.

(d) The columns of A span $\mathbb{R}^m$

(e) The columns of A are linearly independent

(f) Every column of $\operatorname{RREF}(A)$ has a pivot.

(g) Every row of $\operatorname{RREF}(A)$ has a pivot.

(h) The image of T equals its codomain.

(i) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b}\end{array}\right]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$

(j) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0}\end{array}\right]$ has exactly one solution.

Definition 19.7 If $T : V \to W$ is both injective and surjective, it is called **bijective**.

Activity 19.8 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

(a) The columns of A form a basis for $\mathbb{R}^m$

(b) RREF(A) is the identity matrix.

(c) The system of linear equations given by the augmented matrix $\left[A \mid \mathbf{b}\right]$ has exactly one solution for all $\mathbf{b} \in \mathbb{R}^m$.

Activity 19.9 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity 19.10 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity 19.11 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity 19.12 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.
