

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing (**Standard(s) E1, E2, E3, E4**).
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent (**Standard(s) V5**).
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace (**Standard(s) V6, V7**).
- State the definition of a basis, and determine if a set of vectors is a basis (**Standard(s) V8, V9**).

## Readiness Assurance Resources

The following resources will help you prepare for this module.

- Review the supporting Standards listed above.

## Readiness Assurance Test

Choose the most appropriate response for each question.

- 1) Which of the following is a solution to the system of linear equations

$$\begin{aligned}x + 3y - z &= 2 \\ 2x + 8y + 3z &= -1 \\ -x - y + 9z &= -10\end{aligned}$$

- (a)  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$       (b)  $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$       (c)  $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$       (d)  $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

- 2) Find a basis for the solution set of the following homogeneous system of linear equations

$$\begin{aligned}x + 2y + -z - w &= 0 \\ -2x - 4y + 3z + 5w &= 0\end{aligned}$$

- (a)  $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$       (b)  $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$       (c)  $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$       (d) None of these are a basis.

- 3) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

- (a) It does not span and is linearly dependent  
(b) It does not span and is linearly independent  
(c) It spans but it is linearly dependent  
(d) It is a basis of  $\mathbb{R}^3$ .

- 4) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

- (a) It does not span and is linearly dependent  
(b) It does not span and is linearly independent  
(c) It spans but it is linearly dependent  
(d) It is a basis of  $\mathbb{R}^3$ .

- 5) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

- (a) It does not span and is linearly dependent
- (b) It does not span and is linearly independent
- (c) It spans but it is linearly dependent
- (d) It is a basis of  $\mathbb{R}^3$ .

6) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

- (a) It does not span and is linearly dependent
- (b) It does not span and is linearly independent
- (c) It spans but it is linearly dependent
- (d) It is a basis of  $\mathbb{R}^3$ .

7) Find a basis for the subspace of  $\mathbb{R}^4$  spanned by the vectors ...

8) Suppose you know that every vector in  $\mathbb{R}^5$  can be written as a linear combination of the vectors  $\{\vec{v}_1, \dots, \vec{v}_n\}$ . What can you conclude about  $n$ ?

- (a)  $n \leq 5$
- (b)  $n = 5$
- (c)  $n \geq 5$
- (d)  $n$  could be any positive integer

9) Suppose you know that every vector in  $\mathbb{R}^5$  can be written uniquely as a linear combination of the vectors  $\{\vec{v}_1, \dots, \vec{v}_n\}$ . What can you conclude about  $n$ ?

- (a)  $n \leq 5$
- (b)  $n = 5$
- (c)  $n \geq 5$
- (d)  $n$  could be any positive integer

10) Suppose you know that every vector in  $\mathbb{R}^5$  can be written uniquely as a linear combination of the vectors  $\{\vec{v}_1, \dots, \vec{v}_n\}$ . What can you conclude about the set  $\{\vec{v}_1, \dots, \vec{v}_n\}$ ?

- (a) It does not span and is linearly dependent
- (b) It does not span and is linearly independent
- (c) It spans but it is linearly dependent
- (d) It is a basis of  $\mathbb{R}^3$ .

## Application Activities - Day 1

**Definition.** A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if  $V$  and  $W$  are vector spaces, a map  $T : V \rightarrow W$  is called a linear transformation if

1.  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$  for any  $\vec{v}, \vec{w} \in V$
2.  $T(c\vec{v}) = cT(\vec{v})$  for any  $c \in \mathbb{R}$ ,  $\vec{v} \in V$ .

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.

$V$  is called the **domain** of  $T$  and  $W$  is called the **co-domain** of  $T$ .

**Activity.** Determine if each of the following maps are linear transformations

- (a)  $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $T_1 \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \sqrt{a^2 + b^2}$
- (b)  $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by  $T_2 \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$
- (c)  $T_3 : \mathcal{P}_d \rightarrow \mathcal{P}_{d-1}$  given by  $T_3(f(x)) = f'(x)$ .
- (d)  $T_4 : C(\mathbb{R}) \rightarrow C(\mathbb{R})$  given by  $T_4(f(x)) = f(-x)$
- (e)  $T_5 : \mathcal{P} \rightarrow \mathcal{P}$  given by  $T_5(f(x)) = f(x) + x^2$

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**Activity.** Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear transformation, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute each of the following:

- (a)  $T \left( \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right)$
  - (b)  $T \left( \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right)$
  - (c)  $T \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$
  - (d)  $T \left( \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} \right)$
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**Activity.** Suppose  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  is a linear transformation. What is the smallest number of vectors needed to determine  $T$ ? In other words, what is the smallest number  $n$  such that there are  $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^4$  and given  $T(\vec{v}_1), \dots, T(\vec{v}_n)$  you can determine  $T(\vec{w})$  for *any*  $\vec{w} \in \mathbb{R}^4$ ?

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**Observation.** Fix an ordered basis for  $V$ . Since every vector can be written *uniquely* as a linear combination of basis vectors, a linear transformation  $T : V \rightarrow W$  corresponds exactly to a choice of where each basis vector goes. For convenience, we can thus encode a linear transformation as a matrix, with one column for the image of each basis vector (in order).

**Activity.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be a linear transformation with

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard ordered basis.

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**Activity.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be a linear transformation with

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the ordered basis

$$\left\{ \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \right\}$$


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**Activity.** Let  $D : \mathcal{P}_3 \rightarrow \mathcal{P}_2$  be the derivative map (recall this is a linear transformation). Write the matrix corresponding to  $D$  with respect to the ordered basis  $\{1, x, x^2, x^3\}$ .

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## Application Activities - Day 2

**Definition.** Let  $T : V \rightarrow W$  be a linear transformation.

- $T$  is called **injective** or **one-to-one** if  $T$  does not map two distinct values to the same place. More precisely,  $T$  is injective if  $T(\vec{v}) \neq T(\vec{w})$  whenever  $\vec{v} \neq \vec{w}$ .
- $T$  is called **surjective** or **onto** if every element of  $W$  is mapped to by an element of  $V$ . More precisely, for every  $\vec{w} \in W$ , there is some  $v \in V$  with  $T(\vec{v}) = \vec{w}$ .

**Activity.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Determine if  $T$  is injective, surjective, both, or neither.

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**Activity.** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ . Determine if  $T$  is injective, surjective, both, or neither.

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**Definition.** We also have two important sets called the **kernel** of  $T$  and the **image** of  $T$ .

$$\ker T = \{ \vec{v} \in V \mid T(\vec{v}) = 0 \}$$

$$\text{Im } T = \{ \vec{w} \in W \mid \text{there is some } v \in V \text{ with } T(\vec{v}) = \vec{w} \}$$

**Activity.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$  (for the standard basis). Find the kernel and image of  $T$ .

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**Activity.** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$  (for the standard basis). Find the kernel and image of  $T$ .

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**Activity. Part X:** Describe surjective linear transformations in terms of the image.

**Part X:** Describe injective linear transformations in terms of the kernel.

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**Activity.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the matrix  $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$  (for the standard basis).

- 1) Write a system of equations whose solution set is the kernel.
- 2) Compute  $\text{RREF}(A)$  and solve the system of equations.

- 3) Compute the kernel of  $T$
  - 4) Find a basis for the kernel of  $T$
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**Activity.** Let  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the matrix  $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$  (for the standard basis).

- 1) Write a system of equations whose solution set is the kernel.
  - 2) Compute  $\text{RREF}(A)$  and solve the system of equations.
  - 3) Compute the kernel of  $T$
  - 4) Find a basis for the kernel of  $T$
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**Activity.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the matrix  $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$  (for the standard basis).

- 1) Find a set of vectors that span the image of  $T$
  - 2) Find a basis for the image of  $T$ .
- 

**Activity.** Let  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the matrix  $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$  (for the standard basis).

- 1) Find a set of vectors that span the image of  $T$
  - 2) Find a basis for the image of  $T$ .
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## Application Activities - Day 3

**Activity.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with matrix  $A \in M_{m,n}$  (for the standard basis). Consider the following statements about  $T$

- (a)  $T$  is injective
- (b)  $T$  is not injective
- (c)  $T$  is surjective
- (d)  $T$  is not surjective
- (e) The system of linear equations given by the augmented matrix  $\begin{bmatrix} A & | & \vec{b} \end{bmatrix}$  has a solution for all  $\vec{b} \in \mathbb{R}^m$
- (f) The system of linear equations given by the augmented matrix  $\begin{bmatrix} A & | & \vec{b} \end{bmatrix}$  has a unique solution for all  $\vec{b} \in \mathbb{R}^m$
- (g) The system of linear equations given by the augmented matrix  $\begin{bmatrix} A & | & \vec{0} \end{bmatrix}$  has a non-trivial solution.
- (h) The columns of  $A$  span  $\mathbb{R}^m$
- (i) The columns of  $A$  are linearly independent
- (j) The columns of  $A$  are a basis of  $\mathbb{R}^m$
- (k) Every column of  $\text{RREF}(A)$  is a pivot column
- (l)  $\text{RREF}(A)$  has a non-pivot column
- (m)  $\text{RREF}(A)$  has  $n$  pivot columns

Sort these statements into groups of equivalent statements.

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**Activity.** Gallery walk—switch boards with a different team. If they have two things grouped together that you know are not equivalent, write a reason or counter-example on a sticky note.

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**Activity.** Update your team's groupings based on feedback.

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**Activity.** Repeat?

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**Activity.** Can you add any statements to any groups?

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## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers
- Solve systems of linear equations (**Standard(s) E.3**)
- Find the matrix corresponding to a linear transformation (**Standard(s) A.1**)
- Determine if a linear transformation is injective and/or surjective (**Standard(s) A.3**)
- Interpret the ideas of injectivity and surjectivity in multiple ways

## Readiness Assurance Resources

The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/algebra2/manipulating-functions/function-composition/v/function-composition>

## Readiness Assurance Test

Choose the most appropriate response for each question.

1) Let  $f(x) = x^2 - 2$  and  $g(x) = x^2 + 1$ . Compute the composition function  $(f \circ g)(x)$ .

- (a)  $x^2 - 1$
- (b)  $x^4 + 2x^2 - 1$
- (c)  $x^4 - 4x^2 + 5$
- (d)  $x^4 - x^2 - 2$

2) Suppose  $f(x)$  and  $g(x)$  are real-valued functions satisfying

$$\begin{array}{ll} f(2) = 1 & g(2) = 3 \\ f(3) = 4 & g(3) = 5 \\ f(4) = 3 & g(4) = 6 \end{array}$$

Compute  $(f \circ g)(2)$ .

- (a) 2
- (b) 3
- (c) 4
- (d) 5

3) Solve the system of linear equations

$$\begin{array}{l} x + 3y = -2 \\ 2x - 7y = 9 \end{array}$$

- (a)  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- (b)  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
- (c)  $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
- (d)  $\begin{bmatrix} -2 \\ 9 \end{bmatrix}$

4) Let  $a, b, c$  be fixed real numbers. How many solutions does the system of linear equations below have?

$$\begin{array}{l} x + 2y + 3z = a \\ y - z = b \\ y + z = c \end{array}$$

- (a) 0
- (b) 1
- (c) Infinitely many
- (d) It depends on the values of  $a$ ,  $b$ , and  $c$ .

5) What is the matrix corresponding to the linear transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) =$

$$\begin{bmatrix} x + 2y - z \\ y + 3z \\ x + 7y \end{bmatrix}?$$

(a)  $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 7 & 0 \end{bmatrix}$

(b)  $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ 1 & 7 & 0 \end{bmatrix}$

(c)  $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 7 \\ -1 & 0 & 0 \end{bmatrix}$

(d)  $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 7 \\ -1 & 3 & 0 \end{bmatrix}$

6) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be the linear transformation with associated matrix  $A = \begin{bmatrix} 2 & 3 \\ -1 & -1 \\ 0 & 4 \end{bmatrix}$ . Compute

$T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right).$

(a)  $\begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$

(b)  $\begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$

(c)  $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

(d)  $\begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$

7) Which of the following is true of the linear transformation  $T : ?$

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is both injective and surjective

8) Which of the following is true of the linear transformation  $T : ?$

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is both injective and surjective

9) Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation with associated matrix  $A \in M_{m,n}(\mathbb{R})$ . Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?

- (a)  $T$  is injective
- (b)  $T$  has a non-trivial kernel
- (c) The columns of  $A$  are linearly dependent
- (d)  $\text{RREF}(A)$  has a non-pivot column

10) Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation with associated matrix  $A \in M_{m,n}(\mathbb{R})$ . Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?

- (a)  $T$  is surjective
- (b)  $\text{Im } T = \mathbb{R}^m$
- (c) The columns of  $A$  span  $\mathbb{R}^m$
- (d)  $\text{RREF}(A)$  has only pivot columns

## Application Activities - Day 1

**Activity.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by the matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

What is the domain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
  - (b)  $\mathbb{R}^2$
  - (c)  $\mathbb{R}^3$
  - (d)  $\mathbb{R}^4$
- 

**Activity.** What is the codomain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
  - (b)  $\mathbb{R}^2$
  - (c)  $\mathbb{R}^3$
  - (d)  $\mathbb{R}^4$
- 

**Activity.** The matrix corresponding to  $S \circ T$  will lie in which matrix space?

- (a)  $M_{4,3}$
  - (b)  $M_{4,2}$
  - (c)  $M_{3,2}$
  - (d)  $M_{2,3}$
  - (e)  $M_{2,4}$
  - (f)  $M_{3,4}$
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**Activity.** Compute  $(S \circ T)(\vec{e}_1)$ ,  $(S \circ T)(\vec{e}_2)$ , and  $(S \circ T)(\vec{e}_3)$ .

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**Activity.** Find the matrix corresponding to  $S \circ T$  with respect to the standard bases.

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**Activity.** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$  and  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix

$$A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

What is the domain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
  - (b)  $\mathbb{R}^2$
  - (c)  $\mathbb{R}^3$
  - (d)  $\mathbb{R}^4$
- 

**Activity.** What is the codomain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
  - (b)  $\mathbb{R}^2$
  - (c)  $\mathbb{R}^3$
  - (d)  $\mathbb{R}^4$
- 

**Activity.** The matrix corresponding to  $S \circ T$  will lie in which matrix space?

- (a)  $M_{2,2}$
  - (b)  $M_{2,3}$
  - (c)  $M_{3,2}$
  - (d)  $M_{3,3}$
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**Activity.** Compute  $(S \circ T)(\vec{e}_1)$  and  $(S \circ T)(\vec{e}_2)$

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**Activity.** Find the matrix corresponding to  $S \circ T$  with respect to the standard bases.

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**Activity.** Let  $T : \mathbb{R}^1 \rightarrow \mathbb{R}^4$  be given by the matrix  $B = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}$  and  $S : \mathbb{R}^4 \rightarrow \mathbb{R}^1$  be given by the matrix

$$A = [2 \quad 3 \quad 2 \quad 5].$$

What is the domain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
  - (b)  $\mathbb{R}^2$
  - (c)  $\mathbb{R}^3$
  - (d)  $\mathbb{R}^4$
- 

**Activity.** What is the codomain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
  - (b)  $\mathbb{R}^2$
  - (c)  $\mathbb{R}^3$
  - (d)  $\mathbb{R}^4$
- 

**Activity.** The matrix corresponding to  $S \circ T$  will lie in which matrix space?

- (a)  $M_{1,1}$
  - (b)  $M_{1,4}$
  - (c)  $M_{4,1}$
  - (d)  $M_{4,4}$
- 

**Activity.** Compute  $(S \circ T)(\vec{e}_1)$

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**Activity.** Find the matrix corresponding to  $S \circ T$  with respect to the standard bases.

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**Activity.** Let  $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$  and  $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

1. Compute  $AX$
2. Interpret the system of equations below as a matrix equation

$$\begin{aligned} 3x + y - z &= 5 \\ 2x + 4z &= -7 \\ -x + 3y + 5z &= 2 \end{aligned}$$

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## Application Activities - Day 2

**Activity.** Each row operation can be interpreted as a matrix multiplication. Let  $A \in M_{4,4}$

- 1) Find a matrix  $S_1$  such that  $S_1A$  is the result of swapping the second and fourth rows of  $A$ .
  - 2) Find a matrix  $S_2$  such that  $S_2A$  is the result of adding 5 times the third row of  $A$  to the first.
  - 3) Find a matrix  $S_3$  such that  $S_3A$  is the result of doubling the fourth row of  $A$ .
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**Activity.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with matrix  $A \in M_{m,n}$  (for the standard basis). Consider the following statements about  $T$

- (a)  $T$  is injective
- (b)  $T$  is surjective
- (c)  $T$  is bijective (i.e. both injective and surjective)
- (d)  $AX = B$  has a solution for all  $B \in M_{m,1}$
- (e)  $AX = B$  has a unique solution for all  $B \in M_{m,1}$
- (f)  $AX = 0$  has a non-trivial solution.
- (g) The columns of  $A$  span  $\mathbb{R}^m$
- (h) The columns of  $A$  are linearly independent
- (i) The columns of  $A$  are a basis of  $\mathbb{R}^m$
- (j)  $\text{RREF}(A)$  has  $n$  pivot columns
- (k)  $\text{RREF}(A)$  has  $m$  pivot columns

Sort these statements into groups of equivalent statements.

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**Activity.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with matrix  $A \in M_{m,n}$  (for the standard basis). If  $T$  is injective, what must be true about how  $m$  and  $n$  are related?

- (a)  $m < n$
  - (b)  $m \leq n$
  - (c)  $m = n$
  - (d)  $m \geq n$
  - (e)  $m > n$
- 

**Activity.** If  $T$  is surjective, what must be true about how  $m$  and  $n$  are related?

- (a)  $m < n$
  - (b)  $m \leq n$
  - (c)  $m = n$
  - (d)  $m \geq n$
  - (e)  $m > n$
- 

**Activity.** If  $T$  is bijective, what must be true about how  $m$  and  $n$  are related?

- (a)  $m < n$
  - (b)  $m \leq n$
  - (c)  $m = n$
  - (d)  $m \geq n$
  - (e)  $m > n$
-



## Application Activities - Day 3

**Activity.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear map with matrix  $A \in M_{n,n}$ .

If  $T$  is a bijection, then  $AX = B$  has a unique solution for all  $B \in \mathbb{R}^n$ . Thus we can define a map  $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by defining  $T^{-1}(B)$  to be this solution. It follows immediately that  $T \circ T^{-1}$  is the identity map. The matrix corresponding to  $T^{-1}$  is denoted  $A^{-1}$ .

- 1) Solve  $AX = \vec{e}_1$  to determine  $T^{-1}(\vec{e}_1)$
- 2) Solve  $AX = \vec{e}_2$  to determine  $T^{-1}(\vec{e}_2)$
- 3) Solve  $AX = \vec{e}_3$  to determine  $T^{-1}(\vec{e}_3)$
- 4) Compute  $A^{-1}$

A (square) matrix is called *invertible* if it corresponds to an invertible linear transformation.

- 1) Find the inverse of the matrix  $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$
  - 2) Find the inverse of the matrix  $\begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$
-

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces (**Standard(s) A1**).
- Recall and use the definition of a linear transformation (**Standard(s) A2**).
- Find all roots of quadratic polynomials (including complex ones), and be able to use the rational root theorem to find all rational roots of a higher degree polynomial.
- Interpret the statement “ $A$  is an invertible matrix” in many equivalent ways in different contexts.

## Readiness Assurance Resources

The following resources will help you prepare for this module.

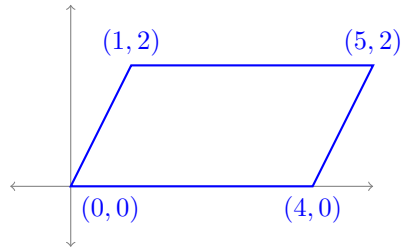
- Finding the area of a parallelogram: <https://www.khanacademy.org/math/basic-geo/basic-geo-area-and-perimeter/parallelogram-area/a/area-of-parallelogram>
- Factoring quadratics: <https://www.khanacademy.org/math/algebra2/polynomial-functions/factoring-polynomials/v/factoring-polynomials-1>
- Finding complex roots of quadratics: <https://www.khanacademy.org/math/algebra2/polynomial-functions/quadratic-equations-with-complex-numbers/v/complex-roots-from-the-quadratic-formula>
- Finding all roots of polynomials: <https://www.khanacademy.org/math/algebra2/polynomial-functions/finding-zeros-of-polynomials/v/finding-roots-or-zeros-of-polynomial-1>
- The Rational Root Theorem: [https://artofproblemsolving.com/wiki/index.php?title=Rational\\_Root\\_Theorem](https://artofproblemsolving.com/wiki/index.php?title=Rational_Root_Theorem)

## Readiness Assurance Test

Choose the most appropriate response for each question.

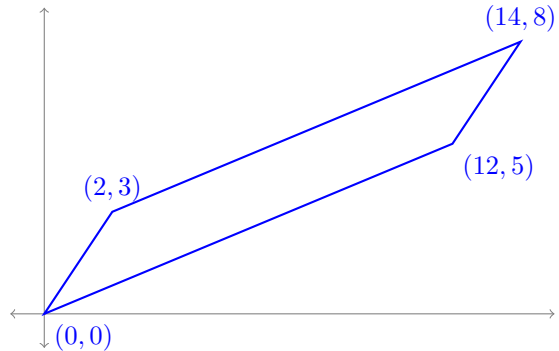
- 1) Find the area of the parallelogram with vertices  $(0, 0)$ ,  $(4, 0)$ ,  $(5, 2)$ , and  $(1, 2)$ .

- (a) 5
- (b) 6
- (c) 7
- (d) 8



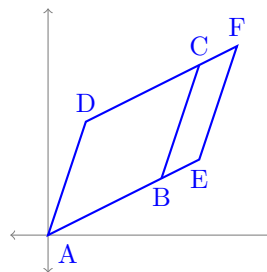
- 2) Find the area of the parallelogram with vertices  $(0, 0)$ ,  $(12, 5)$ ,  $(14, 8)$ , and  $(2, 3)$ .

- (a) 13
- (b) 26
- (c) 39
- (d) 52



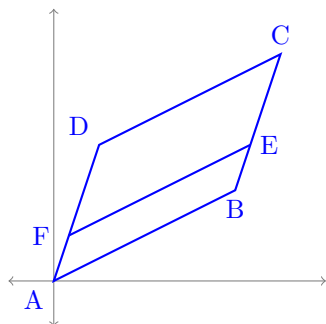
- 3) The parallelogram ABCD has area 6. If AE is  $\frac{3}{2}$  the length of AB, what is the area of the parallelogram AEFD?

- (a) 9
- (b) 12
- (c) 15
- (d) 18



- 4) The parallelogram ABCD has area 6. If AF is one third as long as AD, what is the area of the parallelogram ABEF?

- (a) 1
- (b) 2
- (c) 3
- (d) 4



- 5) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a linear transformation. Which of the following is equal to  $T \left( \begin{bmatrix} a+b \\ a+b \end{bmatrix} \right)$ ?
- (a)  $T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right)$  (c)  $T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) + T \left( \begin{bmatrix} b \\ a \end{bmatrix} \right)$   
 (b)  $2T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right)$  (d)  $T \left( \begin{bmatrix} a \\ a \end{bmatrix} \right) + T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) + T \left( \begin{bmatrix} b \\ a \end{bmatrix} \right) + T \left( \begin{bmatrix} b \\ b \end{bmatrix} \right)$
- 6) Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation with associated matrix  $A \in M_n(\mathbb{R})$ . Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?
- (a)  $A$  is not an invertible matrix  
 (b)  $T$  has a non-trivial kernel  
 (c)  $\det(A) \neq 0$   
 (d)  $A\vec{x} = \vec{b}$  has multiple solutions for all  $\vec{b} \in \mathbb{R}^n$ .
- 7) What is the matrix corresponding to the linear transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 3x + 2y - z \\ y + z \\ x + 7z \end{bmatrix}$ ?
- (a)  $\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 7 \end{bmatrix}$  (b)  $\begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 7 \end{bmatrix}$  (c)  $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 7 & 0 \end{bmatrix}$  (d)  $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 7 \\ -1 & 0 & 0 \end{bmatrix}$
- 8) Which of the following conditions imply that the quadratic polynomial  $ax^2 + bx + c$  has no real roots?
- (a)  $a < 0$   
 (b)  $b^2 - 4ac < 0$   
 (c)  $ac - b^2 < 0$   
 (d)  $ab + c^2 < 0$
- 9) Which of the following is a root of the polynomial  $x^2 - 4x + 13$ ?

(a)  $1 + 2i$

(b)  $2 - 3i$

(c)  $3 + 4i$

(d)  $4 - 5i$

10) How many roots does the polynomial  $x^4 + 3x^3 + x^2 - 3x - 2$  have?

(a) 1

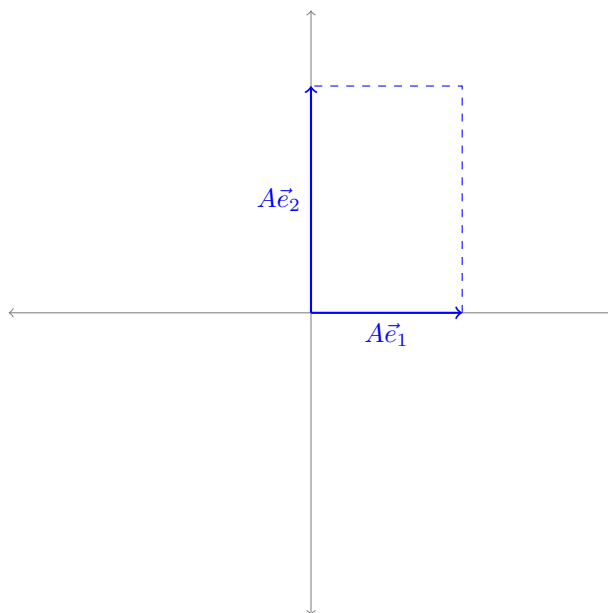
(b) 2

(c) 3

(d) 4

## Application Activities - Days 1-2

**Activity.** Consider the linear transformation  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the matrix  $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



We can summarize the transformation of the unit square into this rectangle by measuring the following:

- How did the area change?
- How was the  $x$ -axis stretched?
- How was the  $y$ -axis stretched?

---

**Activity.** Consider the following linear transformations  $A_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .

- $A_2 = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$
- $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
- $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

For each linear transformation, do the following:

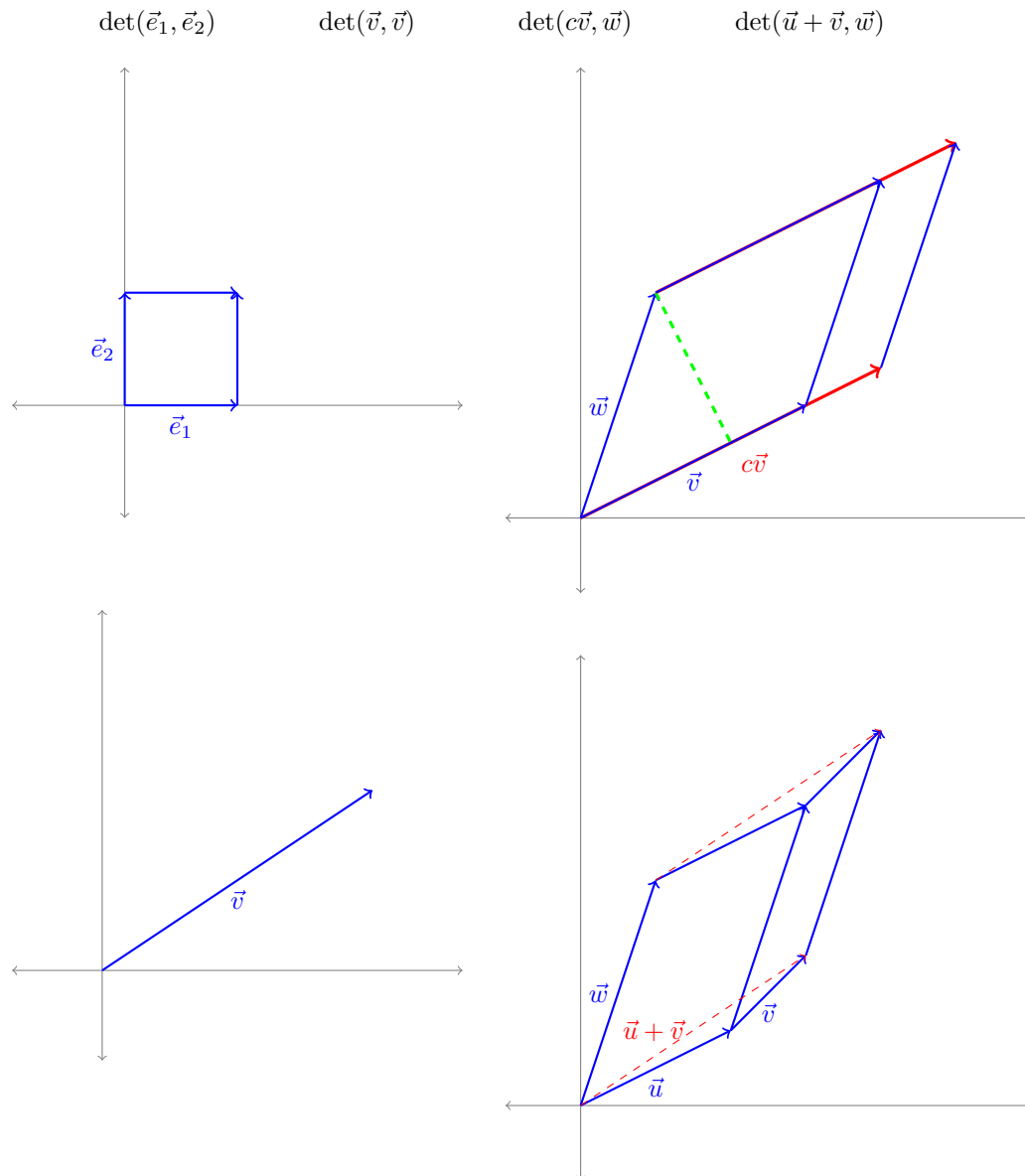
- Draw a graph showing the image of the unit square.

- (b) Compute how much the area was stretched out.
- (c) Determine which axes (or lines) were preserved; how were they stretched out?
- 

**Activity.** Our goal is to define a function  $\det : M_n \rightarrow \mathbb{R}$  that takes a square matrix (linear transformation  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ ) and returns its area stretching factor. This function is called the **determinant**.

What properties should this function have?

Match the four pictures to the following four expressions



**Activity.** What can you conclude about each of the following?

1.  $\det(\vec{e}_1, \vec{e}_2)$
  2.  $\det(\vec{v}, \vec{v})$
  3.  $\det(c\vec{v}, \vec{w})$
  4.  $\det(\vec{u} + \vec{v}, \vec{w})$
- 

**Definition.** To summarize, we have 3 properties (stated here over  $\mathbb{R}^n$ )

P1:  $\det(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n) = 1$

P2: If  $\vec{v}_i = \vec{v}_j$  for some  $i \neq j$ , then  $\det(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = 0$ .

P3: The determinant is linear in each column.

These three properties uniquely define the **determinant**, as we shall see.

**Observation.** Note that if  $\vec{v}, \vec{w} \in \mathbb{R}^2$  and  $A = [\vec{v} \ \vec{w}]$  we will write either  $\det(A)$  or  $\det(\vec{v}, \vec{w})$  as is convenient.

**Activity.**

How are  $\det(\vec{v}, \vec{w})$  and  $\det(\vec{w}, \vec{v})$  related?

- (a)  $\det(\vec{v}, \vec{w}) = \det(\vec{w}, \vec{v})$
  - (b)  $\det(\vec{v}, \vec{w}) = -\det(\vec{w}, \vec{v})$
  - (c) They are unrelated
  - (d) They are related, but not by either (a) or (b).
- 

**Observation.** Note that this implies that the determinant is actually a *signed* area (volume)!

**Activity.**

How are  $\det(\vec{v} + \vec{w}, \vec{w})$  and  $\det(\vec{v}, \vec{w})$  related?

- (a)  $\det(\vec{v} + \vec{w}, \vec{w}) = \det(\vec{v}, \vec{w})$
  - (b)  $\det(\vec{v} + \vec{w}, \vec{w}) = -\det(\vec{v}, \vec{w})$
  - (c) They are unrelated
  - (d) They are related, but not by either (a) or (b).
- 

**Observation.** Note that we now understand the effect of any column operation on the determinant.

**Activity.** How are  $\det(A)$  and  $\det(A^T)$  related?



- (a)  $\det(A) = \det(A^T)$
  - (b)  $\det(A) = -\det(A^T)$
  - (c)  $\det(A) = \frac{1}{\det(A^T)}$
  - (d) They are unrelated
- 

**Observation.** Thus, row operations behave like column operations. So we can use row reduction to compute determinants.

**Activity.** Compute  $\det \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ .

---

**Activity.** Compute  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

---

**Activity.** Which of the following is the same as  $\det \begin{bmatrix} 3 & -2 & 0 \\ 5 & -1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$ ?

- (a)  $\det \begin{bmatrix} 3 & -2 \\ 5 & -1 \end{bmatrix}$
  - (b)  $\det \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$
  - (c)  $\det \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}$
  - (d) None of these
- 

**Activity.** Which of the following is the same as  $\det \begin{bmatrix} 3 & 0 & 7 \\ 5 & 1 & 2 \\ -2 & 0 & 6 \end{bmatrix}$ ?

- (a)  $\det \begin{bmatrix} 3 & 7 \\ 5 & 2 \end{bmatrix}$
  - (b)  $\det \begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix}$
  - (c)  $\det \begin{bmatrix} 5 & 2 \\ -2 & 6 \end{bmatrix}$
  - (d) None of these
-

**Activity.** Compute  $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$

---

**Activity.** Using the fact that  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , compute  $\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 5 \\ 0 & 3 & 3 \end{bmatrix}$ .

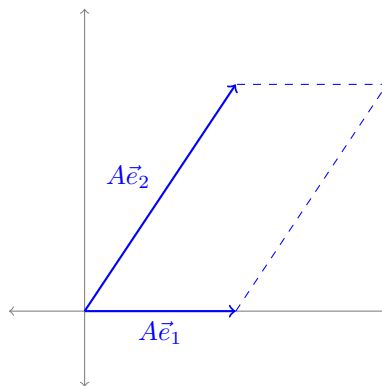
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**Activity.** Compute  $\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$

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## Application Activities - Day 3

**Activity.** Consider the linear transformation  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the matrix  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



Observe

$$A\vec{e}_1 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\vec{e}_1$$

Is there another vector  $\vec{v} \in \mathbb{R}^2$  such that  $A\vec{v} = \lambda\vec{v}$  for some  $\lambda \in \mathbb{R}$ ?

---

**Definition.** Let  $A \in M_n(\mathbb{R})$ . An **eigenvector** is a vector  $\vec{x} \in \mathbb{R}^n$  such that  $A\vec{x}$  is parallel to  $\vec{x}$ ; in other words,  $A\vec{x} = \lambda\vec{x}$  for some scalar  $\lambda$ , which is called an **eigenvalue**

**Observation.** Observe that  $A\vec{x} = \lambda\vec{x}$  is equivalent to  $(A - \lambda I)\vec{x} = 0$ .

- To find eigenvalues, we need to find values of  $\lambda$  such that  $A - \lambda I$  has a nontrivial kernel; equivalently,  $A - \lambda I$  is not invertible, which is equivalent to  $\det(A - \lambda I) = 0$ .  $\det(A - \lambda I)$  is called the **characteristic polynomial**.
- Once an eigenvalue is found, the eigenvectors form a subspace called the **eigenspace**, which is simply the kernel of  $A - \lambda I$ . Each eigenvalue will have an associated eigenspace.

**Activity.** Find the eigenvalues for the matrix  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .

---

**Activity.** Compute the eigenspace associated to the eigenvalue 3.

---

**Activity.** Find all the eigenvalues and associated eigenspaces for the matrix  $\begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$ .

---

## Application Activities - Day 4

**Activity.** If  $A \in M_4$ , what is the largest number of eigenvalues  $A$  can have?

---

**Activity.** 2 is an eigenvalue of each of the matrices  $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}$ .  
Compute the eigenspace associated to 2 for both  $A$  and  $B$ .

---

**Definition.** • The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

• The **geometric multiplicity** of an eigenvalue is the dimension of the eigenspace.

**Activity.** How are the algebraic and geometric multiplicities related?

- (a) The algebraic multiplicity is always at least as big as than the geometric multiplicity.
  - (b) The geometric multiplicity is always at least as big as the algebraic multiplicity.
  - (c) Sometimes the algebraic multiplicity is larger and sometimes the geometric multiplicity is larger.
- 

**Activity.** Find the eigenvalues, along with both their algebraic and geometric multiplicities, for the matrix  $\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$

---

**Activity.** Find the eigenvalues of the matrix  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

---

**Activity.** Describe what this linear transformation is doing geometrically; draw a picture.

---

**Activity.** Fix a real number  $\theta$  and find the eigenvalues of the matrix  $A_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ . What are the eigenvalues?

---

**Activity.** Draw pictures and describe the geometric actions of the maps  $A_{\frac{\pi}{4}}$ ,  $A_{\frac{\pi}{2}}$ , and  $A_\pi$ .

---

**Activity.** For how many values of  $\theta$  does the rotation matrix  $A_\theta$  have real eigenvalues?

- (a) 0
  - (b) 1
  - (c) 2
  - (d) 3
  - (e) An infinite number
-