# Application Activities - Module G Part 2 - Class Day 26

Definition 26.1 The **transpose** of a matrix is given by rewriting its columns as rows and vice versa:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Fact 26.2 It is possible to prove that the determinant of a matrix and its transpose are the same. For example, let $A = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$, so $A^T = \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}$; both matrices scale the unit square by 6.
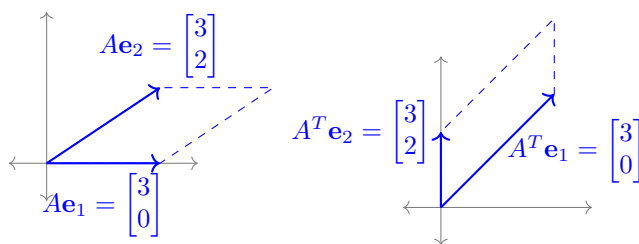
Fact 26.3 We previously figured out that column operations can be used to simplify determinants; since $\det(A) = \det(A^T)$, we can also use row operations:

1. Multiplying rows by scalars: $\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$

2. Swapping two rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$

3. Adding multiples of rows to other rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$

Activity 26.4 Complete the following determinant computation:

$$\det \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} = ? \det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$= ? \det \begin{bmatrix} 1 & 3/2 \\ 4 & 5 \end{bmatrix}$$

$$= ? \det \begin{bmatrix} 1 & 3/2 \\ 0 & -1 \end{bmatrix}$$

$$= ? \det \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$$

$$= ? \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= ?$$

Fact 26.5 This same process allows us to prove a more convenient formula:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \det \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix}$$

$$= a \det \begin{bmatrix} 1 & b/a \\ 0 & d - bc/a \end{bmatrix}$$

$$= a(d - bc/a) \det \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$$

$$= (ad - bc) \det \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$$

$$= (ad - bc) \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= ad - bc$$

Activity 26.6 The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
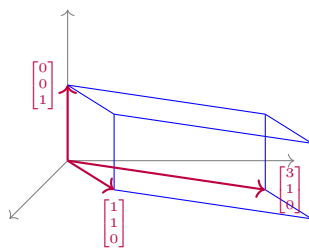
This volume is equal to which of the following areas?

(a) $\det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ (c) $\det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$ (d) $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Fact 26.7 If column i of a matrix is $\mathbf{e}_i$, then both column and row i may be removed without changing the value of the determinant. For example, the second column of the following matrix is $\mathbf{e}_2$, so:

$$\det \begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Therefore the same holds for the transpose:

$$\det \begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Activity 26.8 Complete the following computation of $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$:

$$\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix} = ? \det \begin{bmatrix} 1 & 5 & 12 \\ 0 & 3 & -2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$= ? \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$$= ?$$

Activity 26.9 Complete the following computation of $\det \begin{bmatrix} 2 & 2 & 3 \\ 1 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix}$:

$$\det \begin{bmatrix} 2 & 2 & 3 \\ 1 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix} = ? \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix} + ? \det \begin{bmatrix} 0 & 2 & 3 \\ 1 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix}$$

$$= ? \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} + ? \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$$= ?$$

Activity 26.10 Complete the following computation of $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$:

$$\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} = \det \begin{bmatrix} 2 & 3 & ? & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & ? & 3 \\ -1 & -1 & ? & 2 \end{bmatrix}$$

$$= \det \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

$$= \ldots$$

Observation 26.11 To reduce the dimension of an arbitrary determinant, one may always use linearity to split up a chosen row/column, as seen for the top row in this example:

$$\det \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} = 2\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} + 3\det \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} + 5\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= 2\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} - 3\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} - 5\det \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= 2\det \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} - 3\det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - 5\det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= 2(2) - 3(1) - 5(1) = -4$$

Observation 26.12 Note that choosing rows/columns containing zeros can save some writing:

$$\det \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} = 2\det \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 3 & 5 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= 2\det \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix} - \det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 3 & 5 \end{bmatrix}$$

$$= 2\det \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} - \det \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$$

$$= 2(2) - (8) = -4$$

Observation 26.13 And using row/column operations can save even more work:

$$\det \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 2 & 3 & 5 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 5 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}$$

$$= -(5 - 1) = -4$$