## Section S.3

**Observation S.3.1** In the previous section, we learned that computing a basis for the subspace $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, is as simple as removing the vectors corresponding to the non-pivot columns of $\operatorname{RREF}[\mathbf{v}_1 \ \ldots \ \mathbf{v}_m]$.

For example, since

$$\operatorname{RREF}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}\right\}$ has $\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}$ as a basis.

**Activity S.3.2** *(~10 min)* Let

$$S = \left\{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}\right\} \qquad \text{and} \qquad T = \left\{\begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right\}$$

*Part 1:* Find a basis for $\operatorname{span} S$.
*Part 2:* Find a basis for $\operatorname{span} T$.

**Observation S.3.3** Even though we found different bases for them, $\operatorname{span} S$ and $\operatorname{span} T$ are exactly the same subspace of $\mathbb{R}^4$, since

$$S = \left\{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right\} = T$$

**Fact S.3.4** Any non-trivial vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size.

For example,

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \text{ and } \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} \text{ and } \left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}\right\}$$

are all valid bases for $\mathbb{R}^3$, and they all contain three vectors.

**Definition S.3.5** The **dimension** of a vector space is equal to the size of any basis for the vector space.

As you'd expect, $\mathbb{R}^n$ has dimension $n$. For example, $\mathbb{R}^3$ has dimension 3 because any basis for $\mathbb{R}^3$ such as

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \text{ and } \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}2\\-2\\1\end{bmatrix}, \begin{bmatrix}3\\-2\\5\end{bmatrix} \right\}$$

contains exactly three vectors.

**Activity S.3.6** *(~10 min)* Find the dimension of each subspace of $\mathbb{R}^4$ by finding RREF for each corresponding matrix.

$$\operatorname{span}\left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix} \right\} \quad \operatorname{span}\left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}3\\13\\7\\16\end{bmatrix}, \begin{bmatrix}-1\\10\\7\\14\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix} \right\}$$

$$\operatorname{span}\left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix}, \begin{bmatrix}3\\6\\1\\5\end{bmatrix} \right\} \quad \operatorname{span}\left\{ \begin{bmatrix}5\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\3\end{bmatrix}, \begin{bmatrix}4\\5\\1\\3\end{bmatrix} \right\}$$

**Fact S.3.7** Every vector space with finite dimension, that is, every vector space $V$ with a basis of the form $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is said to be **isomorphic** to a Euclidean space $\mathbb{R}^n$, since there exists a natural correspondance between vectors in $V$ and vectors in $\mathbb{R}^n$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \leftrightarrow \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$$

**Observation S.3.8** We've already been taking advantage of the previous fact by converting polynomials and matrices into Euclidean vectors. Since $\mathcal{P}^3$ and $M_{2,2}$ are both four-dimensional:

$$4x^3 + 0x^2 - 1x + 5 \leftrightarrow \begin{bmatrix}4\\0\\-1\\5\end{bmatrix} \leftrightarrow \begin{bmatrix}4 & 0\\-1 & 5\end{bmatrix}$$

**Observation S.3.9** The space of polynomials $\mathcal{P}$ (of *any* degree) has the basis $\{1, x, x^2, x^3, \dots\}$, so it is a natural example of an infinite-dimensional vector space.

Since $\mathcal{P}$ and other infinite-dimensional spaces cannot be treated as an isomorphic finite-dimensional Euclidean space $\mathbb{R}^n$, vectors in such spaces cannot be studied by converting them into Euclidean vectors. Fortunately, most of the examples we will be interested in for this course will be finite-dimensional.

**Definition S.3.10** A **homogeneous** system of linear equations is one of the form:

$$\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n \ &= 0 \\
a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n \ &= 0 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\ \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ &= 0
\end{aligned}$$

This system is equivalent to the vector equation:

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

**Activity S.3.11** *(~5 min)* Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are solutions to $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ so is $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$, since

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0} \text{ and } b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n = \mathbf{0}$$

implies

$$(a_1 + b_1)\mathbf{v}_1 + \cdots + (a_n + b_n)\mathbf{v}_n = \mathbf{0}.$$

Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution. Thus the solution set of a homogeneous system is...

a) A basis for $\mathbb{R}^n$.

b) A subspace of $\mathbb{R}^n$.

c) The empty set.

**Activity S.3.12** *(~10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}-x_3} + \phantom{2}x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - \phantom{2}x_4 &= 0 \end{aligned}$$

*Part 1:* Find its solution set (a subspace of $\mathbb{R}^4$).
*Part 2:* Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle|\, a, b \in \mathbb{R} \right\}.$$

**Fact S.3.13** The coefficients of the free variables in the solution set of a linear system always yield linearly independent vectors.

Thus if

$$\left\{ a \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \middle|\, a, b \in \mathbb{R} \right\}$$

is the solution space for a homoegeneous system, then

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

is a basis for the solution space.

**Activity S.3.14** *(~10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 \phantom{{}+3x_4} &= 0 \\ 2x_1 - 6x_2 + 4x_3 + 3x_4 &= 0 \\ -2x_1 + 6x_2 - 4x_3 - 4x_4 &= 0 \end{aligned}$$

Find a basis for its solution space.

**Activity S.3.15** *($\sim$5 min)* Suppose $W$ is a subspace of $\mathcal{P}^8$, and you know that it contains a **linearly independent** set of 3 vectors. What can you conclude about $W$?

(a) The dimension of $W$ is at most 3.

(b) The dimension of $W$ is exactly 3.

(c) The dimension of $W$ is at least 3.

**Activity S.3.16** *($\sim$5 min)* Suppose $W$ is a subspace of $\mathcal{P}^8$, and you know that it contains a **spanning set** of 3 vectors. What can you conclude about $W$?

(a) The dimension of $W$ is at most 3.

(b) The dimension of $W$ is exactly 3.

(c) The dimension of $W$ is at least 3.