

## Section V.4

**Definition V.4.1** A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space  $\mathbb{R}^3$ .



**Fact V.4.2** Any subset  $S$  of a vector space  $V$  satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a **subspace**, we need to check that addition and multiplication still make sense using only vectors from  $S$ . So we need to check two things:

- The set is **closed under addition**: for any  $\mathbf{x}, \mathbf{y} \in S$ , the sum  $\mathbf{x} + \mathbf{y}$  is also in  $S$ .
- The set is **closed under scalar multiplication**: for any  $\mathbf{x} \in S$  and scalar  $c \in \mathbb{R}$ , the product  $c\mathbf{x}$  is also in  $S$ .

**Activity V.4.3** ( $\sim 15$  min) Let  $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}$ .

*Part 1:* Let  $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  be vectors in  $S$ , so  $x + 2y + z = 0$  and  $a + 2b + c = 0$ . Show that

$\mathbf{v} + \mathbf{w} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$  also belongs to  $S$  by verifying that  $(x+a) + 2(y+b) + (z+c) = 0$ .

*Part 2:* Let  $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$ , so  $x + 2y + z = 0$ . Show that  $c\mathbf{v}$  also belongs to  $S$  for any  $c \in \mathbb{R}$ .

*Part 3:* Is  $S$  a subspace of  $\mathbb{R}^3$ ?

**Activity V.4.4** ( $\sim 10$  min) Let  $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 4 \right\}$ . Choose a vector  $\mathbf{v} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$  in  $S$  and a real number  $c = ?$ , and show that  $c\mathbf{v}$  isn't in  $S$ . Is  $S$  a subspace of  $\mathbb{R}^3$ ?

**Remark V.4.5** Since  $0$  is a scalar and  $0\mathbf{v} = \mathbf{z}$  for any vector  $\mathbf{v}$ , a set that is closed under scalar multiplication must contain the zero vector  $\mathbf{z}$  for that vector space.

Put another way, an easy way to check that a subset isn't a subspace is to show it doesn't contain  $\mathbf{0}$ .

**Activity V.4.6** (*~10 min*) Consider these two subsets of  $\mathbb{R}^4$ :

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \mid a, b \text{ are real numbers} \right\} \quad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

*Part 1:* Which set is not a subspace of  $\mathbb{R}^4$ ?

*Part 2:* Is the set of polynomials

$$S = \{ax^3 + bx^2 + (b-1)x + (a-1) \mid a, b \text{ are real numbers}\}$$

a subspace of  $\mathcal{P}^3$ ?

**Activity V.4.7** (*~10 min*) Consider the subset  $A$  of  $\mathbb{R}^2$  where at least one coordinate of each vector is 0.



This set contains  $\mathbf{0}$ , and it's not hard to show that for every  $\mathbf{v}$  in  $A$  and scalar  $c \in \mathbb{R}$ ,  $c\mathbf{v}$  is also in  $A$ . Is  $A$  a subspace of  $\mathbb{R}^2$ ? Why?

**Activity V.4.8** (*~5 min*) Let  $W$  be a subspace of a vector space  $V$ . How are  $\text{span } W$  and  $W$  related?

- (a)  $\text{span } W$  is bigger than  $W$
- (b)  $\text{span } W$  is the same as  $W$
- (c)  $\text{span } W$  is smaller than  $W$

**Fact V.4.9** If  $S$  is any subset of a vector space  $V$ , then since  $\text{span } S$  collects all possible linear combinations,  $\text{span } S$  is automatically a subspace of  $V$ .

In fact,  $\text{span } S$  is always the smallest subspace of  $V$  that contains all the vectors in  $S$ .