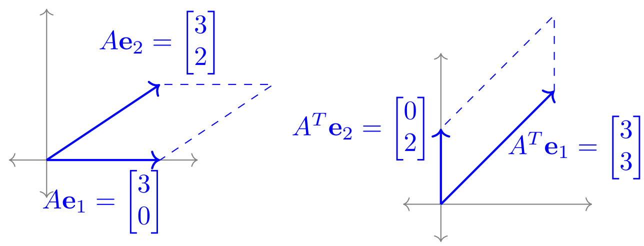
Application Activities - Module G Part 2 - Class Day 26

Definition 26.1 The **transpose** of a matrix is given by rewriting its columns as rows and vice versa:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Fact 26.2 It is possible to prove that the determinant of a matrix and its transpose are the same. For example, let $A = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$, so $A^T = \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}$; both matrices scale the unit square by 6, even though the parallelograms are not congruent.



Fact 26.3 We previously figured out that column operations can be used to simplify determinants; since $\det(A) = \det(A^T)$, we can also use row operations:

1. Multiplying rows by scalars: $\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$
2. Swapping two rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$
3. Adding multiples of rows to other rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$

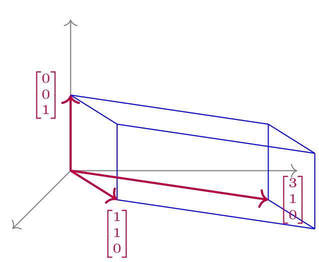
Activity 26.4 Complete the following determinant computation:

$$\begin{aligned}\det\begin{bmatrix}4 & 5\\ 2 & 3\end{bmatrix} &= \ ?\ \det\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}\\ &= \ ?\ \det\begin{bmatrix}1 & 3/2\\ 4 & 5\end{bmatrix}\\ &= \ ?\ \det\begin{bmatrix}1 & 3/2\\ 0 & -1\end{bmatrix}\\ &= \ ?\ \det\begin{bmatrix}1 & 3/2\\ 0 & 1\end{bmatrix}\\ &= \ ?\ \det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\ &= \ ?\end{aligned}$$

Fact 26.5 This same process allows us to prove a more convenient formula:

$$\begin{aligned}\det\begin{bmatrix}a & b\\ c & d\end{bmatrix} &= a\det\begin{bmatrix}1 & b/a\\ c & d\end{bmatrix}\\ &= a\det\begin{bmatrix}1 & b/a\\ 0 & d-bc/a\end{bmatrix}\\ &= a(d-bc/a)\det\begin{bmatrix}1 & b/a\\ 0 & 1\end{bmatrix}\\ &= (ad-bc)\det\begin{bmatrix}1 & b/a\\ 0 & 1\end{bmatrix}\\ &= (ad-bc)\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\ &= ad-bc\end{aligned}$$

Activity 26.6 The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix}3 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$.



This volume is equal to which of the following areas?

(a) $\det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ (c) $\det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$ (d) $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Fact 26.7 If column i of a matrix is $\mathbf{e}_i$, then both column and row i may be removed without changing the value of the determinant. For example, the second column of the following matrix is $\mathbf{e}_2$, so:

$$\det \begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Therefore the same holds for the transpose:

$$\det \begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Activity 26.8 Complete the following computation of $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$:

$$\begin{aligned} \det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix} &= ? \det \begin{bmatrix} 1 & 5 & 12 \\ 0 & 3 & -2 \\ 0 & 2 & -1 \end{bmatrix} \\ &= ? \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \\ &= ? \end{aligned}$$

Activity 26.9 Complete the following computation of $\det \begin{bmatrix} 2 & 2 & 3 \\ 1 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix}$:

$$\begin{aligned} \det \begin{bmatrix} 2 & 2 & 3 \\ 1 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix} &= ? \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix} + ? \det \begin{bmatrix} 0 & 2 & 3 \\ 1 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix} \\ &= ? \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} + ? \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \\ &= ? \end{aligned}$$

Activity 26.10 Complete the following computation of $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$:

$$
\begin{aligned}
\det\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} &= \det\begin{bmatrix} 2 & 3 & ? & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & ? & 3 \\ -1 & -1 & ? & 2 \end{bmatrix} \\
&= \det\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \\
&= \dots
\end{aligned}
$$

Observation 26.11 To reduce the dimension of an arbitrary determinant, one may always use linearity to split up a chosen row/column, as seen for the top row in this example:

$$
\begin{aligned}
\det\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} &= 2\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} + 3\det\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} + 5\det\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \\
&= 2\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} - 3\det\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} - 5\det\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \\
&= 2\det\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} - 3\det\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - 5\det\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \\
&= 2(2) - 3(1) - 5(1) = -4
\end{aligned}
$$

Observation 26.12 Note that choosing rows/columns containing zeros can save some writing:

$$
\begin{aligned}
\det\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} &= 2\det\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & 3 & 5 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \\
&= 2\det\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix} - \det\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 3 & 5 \end{bmatrix} \\
&= 2\det\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} - \det\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} \\
&= 2(2) - (8) = -4
\end{aligned}
$$

Observation 26.13 And using row/column operations can save even more work:

$$
\begin{aligned}
\det \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} &= -\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 2 & 3 & 5 \end{bmatrix} \\
&= -\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 5 \end{bmatrix} \\
&= -\det \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix} \\
&= -(5-1) = -4
\end{aligned}
$$