Module A: Algebraic properties of linear maps

How can we understand linear maps algebraically?

At the end of this module, students will be able to...

A1. Linear map verification. ... determine if a map between vector spaces of polynomials is linear or not.

A2. Linear maps and matrices. ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.

A3. Injectivity and surjectivity. ... determine if a given linear map is injective and/or surjective.

A4. Kernel and Image. ... compute a basis for the kernel and a basis for the image of a linear map.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Module A Section 1

Definition A.1.1

A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

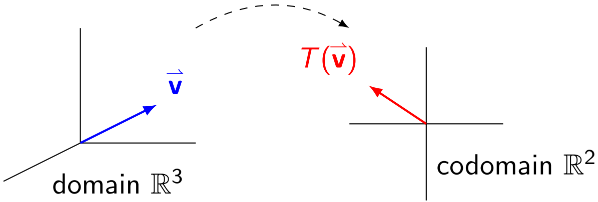
1. $T(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = T(\vec{\mathbf{v}}) + T(\vec{\mathbf{w}})$ for any $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in V$.
2. $T(c\vec{\mathbf{v}}) = cT(\vec{\mathbf{v}})$ for any $c \in \mathbb{R}, \vec{\mathbf{v}} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition A.1.2

Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.

Linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$



Example A.1.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right) = \begin{bmatrix} (x+u)-(z+w) \\ 3(y+v) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x+u)-(z+w) \\ 3(y+v) \end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \text{ and } cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore T is a linear transformation.

Example A.1.4

Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact A.1.5

A map between Euclidean spaces $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of $\mathbb{R}^n$.

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2, $y + 3$, and $y - 2^x$ are not linear combinations (even though $x + y$ is):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

Activity A.1.6 *(∼5 min)*

Recall the following rules from calculus, where $D : \mathcal{P} \to \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f.

$$D(f + g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

a) $\mathcal{P}$ is not a vector space

b) D is a linear map

c) D is not a linear map

Activity A.1.7 *(∼10 min)*
Let the polynomial maps $S : \mathcal{P}^4 \to \mathcal{P}^3$ and $T : \mathcal{P}^4 \to \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4 + x)$, $S(x^4) + S(x)$, $T(x^4 + x)$, and $T(x^4) + T(x)$. Which of these maps is definitely not linear?

Fact A.1.8

If $L : V \to W$ is linear, then $L(\vec{\mathbf{z}}) = L(0\vec{\mathbf{v}}) = 0L(\vec{\mathbf{v}}) = \vec{\mathbf{z}}$ where $\vec{\mathbf{z}}$ is the additive identity of the vector spaces V, W.

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Activity A.1.9 *(∼15 min)*
Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Activity A.1.9 *(∼15 min)*
Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.

Activity A.1.9 *(∼15 min)*
Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.
Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f. Is S linear?

Activity A.1.10 *(∼20 min)*

Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Activity A.1.10 *(∼20 min)*

Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Activity A.1.10 *(∼20 min)*

Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Part 2: Prove that T is linear by verifying that $T(f(x)+g(x)) = T(f(x)) + T(g(x))$ and $T(cf(x)) = cT(f(x))$.

Observation A.1.11

Note that S in the previous activity is not linear, even though $S(0) = (0)^2 = 0$. So showing $S(0) = 0$ isn't enough to prove a map is linear.

This is a similar situation to proving a subset is a subspace: if the subset doesn't contain $\vec{\mathbf{z}}$, then the subset isn't a subspace. But if the subset contains $\vec{\mathbf{z}}$, you cannot conclude anything.

Module A Section 2

Remark A.2.1

Recall that a linear map $T : V \to W$ satisfies

1. $T(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = T(\vec{\mathbf{v}}) + T(\vec{\mathbf{w}})$ for any $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in V$.
2. $T(c\vec{\mathbf{v}}) = cT(\vec{\mathbf{v}})$ for any $c \in \mathbb{R}, \vec{\mathbf{v}} \in V$.

In other words, a map is linear when vecor space operations can be applied before or after the transformation without affecting the result.

Activity A.2.2 *(∼5 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity A.2.3 *(∼3 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.2.4 *(∼2 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}-2\\0\\-3\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.2.5 *(∼5 min)*

Suppose $T:\mathbb{R}^3\to\mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-3\\2\end{bmatrix}$. Do you have enough information to compute $T(\vec{\mathbf{v}})$ for *any* $\vec{\mathbf{v}}\in\mathbb{R}^3$?

(a) Yes.

(b) No, exactly one more piece of information is needed.

(c) No, an infinite amount of information would be necessary to compute the transformation of infinitely-many vectors.

Fact A.2.6

Consider any basis $\{\vec{\mathbf{b}}_1, \dots, \vec{\mathbf{b}}_n\}$ for V. Since every vector $\vec{\mathbf{v}}$ can be written *uniquely* as a linear combination of basis vectors, $x_1\vec{\mathbf{b}}_1 + \cdots + x_n\vec{\mathbf{b}}_n$, we may compute $T(\vec{\mathbf{v}})$ as follows:

$$T(\vec{\mathbf{v}}) = T(x_1\vec{\mathbf{b}}_1 + \cdots + x_n\vec{\mathbf{b}}_n) = x_1 T(\vec{\mathbf{b}}_1) + \cdots + x_n T(\vec{\mathbf{b}}_n).$$

Therefore any linear transformation $T : V \to W$ can be defined by just describing the values of $T(\vec{\mathbf{b}}_i)$.

Put another way, the images of the basis vectors **determine** the transformation T.

Definition A.2.7

Since linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by the standard basis $\{\vec{\mathbf{e}}_1, \dots, \vec{\mathbf{e}}_n\}$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\vec{\mathbf{e}}_1) \cdots T(\vec{\mathbf{e}}_n)]$.

For example, let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T\left(\vec{\mathbf{e}}_1\right) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\vec{\mathbf{e}}_2\right) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\vec{\mathbf{e}}_3\right) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\begin{bmatrix}T(\vec{\mathbf{e}}_1) & T(\vec{\mathbf{e}}_2) & T(\vec{\mathbf{e}}_3)\end{bmatrix} = \begin{bmatrix}3 & -1 & 5\\2 & 4 & 0\end{bmatrix}.$$

Activity A.2.8 *(∼3 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\vec{\mathbf{e}}_1\right) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad T\left(\vec{\mathbf{e}}_2\right) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \qquad T\left(\vec{\mathbf{e}}_3\right) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \qquad T\left(\vec{\mathbf{e}}_4\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix $[T(\vec{\mathbf{e}}_1) \cdots T(\vec{\mathbf{e}}_n)]$ for T.

Activity A.2.9 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Find the standard matrix for T.

Fact A.2.10

Because every linear map $T:\mathbb{R}^m \to \mathbb{R}^n$ has a linear combination of the variables in each component, and thus $T(\vec{\mathbf{e}}_i)$ yields exactly the coefficients of x_i, the standard matrix for T is simply an ordered list of the coefficients of the x_i:

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax+by+cz+dw \\ ex+fy+gz+hw \end{bmatrix} \qquad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Activity A.2.11 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Activity A.2.12 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

Fact A.2.13

To quickly compute $T(\vec{\mathbf{v}})$ from its standard matrix A, compute the **dot product** (defined in Calculus 3) of each matrix row with the vector. For example, if T has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$$

then for $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we will write

$$T(\vec{\mathbf{v}}) = A\vec{\mathbf{v}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 2y + 3z \\ 0x + 1y - 2z \\ 2x - 1y + 0z \end{bmatrix}$$

and for $\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ we will write

$$T(\vec{\mathbf{v}}) = A\vec{\mathbf{v}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(0) + 3(-2) \\ 0(3) + 1(0) - 2(-2) \\ 2(3) - 1(0) + 0(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}.$$

Activity A.2.14 *(∼15 min)*

Compute the following linear transformations of vectors given their standard matrices.

$$T_1\left(\begin{bmatrix}1\\2\end{bmatrix}\right) \text{ for the standard matrix } A_1 = \begin{bmatrix}4 & 3\\0 & -1\\1 & 1\\3 & 0\end{bmatrix}$$

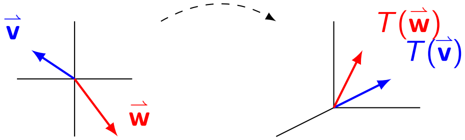
$$T_2\left(\begin{bmatrix}1\\1\\0\\-3\end{bmatrix}\right) \text{ for the standard matrix } A_2 = \begin{bmatrix}4 & 3 & 0 & -1\\1 & 1 & 3 & 0\end{bmatrix}$$

$$T_3\left(\begin{bmatrix}0\\-2\\0\end{bmatrix}\right) \text{ for the standard matrix } A_3 = \begin{bmatrix}4 & 3 & 0\\0 & -1 & 3\\5 & 1 & 1\\3 & 0 & 0\end{bmatrix}$$

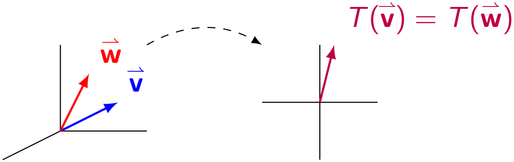
Module A Section 3

Definition A.3.1

Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place. More precisely, T is injective if $T(\vec{\mathbf{v}}) \neq T(\vec{\mathbf{w}})$ whenever $\vec{\mathbf{v}} \neq \vec{\mathbf{w}}$.



injective



not injective

Activity A.3.2 *(∼3 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Show that T is not injective by finding two different vectors $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^3$ such that $T(\vec{\mathbf{v}}) = T(\vec{\mathbf{w}})$.

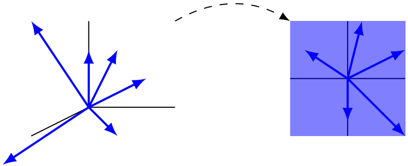
Activity A.3.3 *(~2 min)*
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

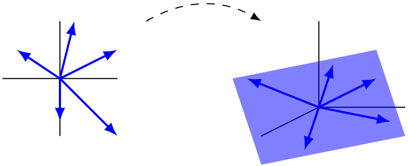
Is T injective? If not, find two different vectors $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^3$ such that $T(\vec{\mathbf{v}}) = T(\vec{\mathbf{w}})$.

Definition A.3.4

Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\vec{\mathbf{w}} \in W$, there is some $\vec{\mathbf{v}} \in V$ with $T(\vec{\mathbf{v}}) = \vec{\mathbf{w}}$.



surjective



not surjective

Activity A.3.5 *(∼3 min)*
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Show that T is not surjective by finding a vector in $\mathbb{R}^3$ that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal.

Activity A.3.6 *(∼2 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective? If not, find a vector in $\mathbb{R}^2$ that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal.

Observation A.3.7

As we will see, it's no coincidence that the RREF of the injective map's standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has all pivot columns. Similarly, the RREF of the surjective map's standard matrix

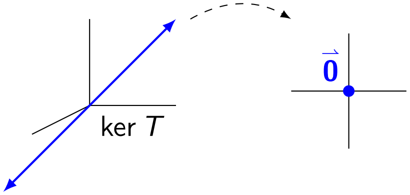
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has a pivot in each row.

Definition A.3.8

Let $T : V \to W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{ \vec{\mathbf{v}} \in V \;\middle|\; T(\vec{\mathbf{v}}) = \vec{\mathbf{z}} \right\}$$



Activity A.3.9 *(∼5 min)*
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\ker T$, the set of all vectors that transform into $\vec{\mathbf{0}}$?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2$

Activity A.3.10 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\ker T$, the set of all vectors that transform into $\vec{\mathbf{0}}$?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle|\, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \middle|\, a \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3$

Activity A.3.11 *(∼10 min)*

Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A=\begin{bmatrix}3 & 4 & -1\\ 1 & 2 & 1\end{bmatrix}.$$

Activity A.3.11 *(∼10 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

Activity A.3.11 *(~10 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

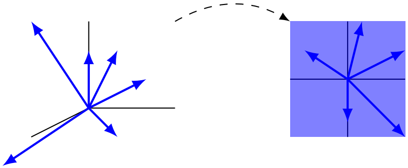
Part 1: Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.
Part 2: Use RREF(A) to solve this homogeneous system of equations and find a basis for the kernel of T.

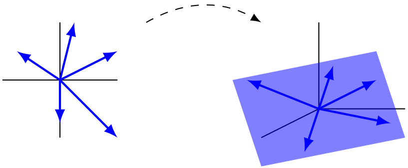
Definition A.3.12

Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{ \vec{\mathbf{w}} \in W \;\middle|\; \text{there is some } \vec{\mathbf{v}} \in V \text{ with } T(\vec{\mathbf{v}}) = \vec{\mathbf{w}} \right\}$$

In the examples below, the left example's image is all of $\mathbb{R}^2$, but the right example's image is a planar subspace of $\mathbb{R}^3$.





Activity A.3.13 *(∼5 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^2$ vectors?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3$

Activity A.3.14 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^3$ vectors?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2$

Activity A.3.15 *(∼5 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} T(\vec{\mathbf{e}}_1) & T(\vec{\mathbf{e}}_2) & T(\vec{\mathbf{e}}_3) & T(\vec{\mathbf{e}}_4) \end{bmatrix}.$$

Since $T(\vec{\mathbf{v}}) = T(x_1\vec{\mathbf{e}}_1 + x_2\vec{\mathbf{e}}_2 + x_3\vec{\mathbf{e}}_3 + x_4\vec{\mathbf{e}}_4)$, the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

a) spans $\operatorname{Im} T$

b) is a linearly independent subset of $\operatorname{Im} T$

c) is a basis for $\operatorname{Im} T$

Observation A.3.16

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}.$$

Since the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ spans $\operatorname{Im} T$, we can obtain a basis for $\operatorname{Im} T$ by finding RREF $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and only using the vectors corresponding to pivot columns:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Fact A.3.17

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

- The kernel of T is the solution set of the homogeneous system given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{\mathbf{0}} \end{array}\right]$. Use the coefficients of its free variables to get a basis for the kernel.
- The image of T is the span of the columns of A. Remove the vectors creating non-pivot columns in RREF A to get a basis for the image.

Activity A.3.18 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T.

Module A Section 4

Observation A.4.1

Let $T : V \to W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T always maps distinct vectors to different places.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.
- The **kernel** of T is the set of all vectors in V that are mapped to $\vec{\mathbf{z}} \in W$. It is a subspace of V.
- The **image** of T is the set of all vectors in W that are mapped to by something in V. It is a subspace of W.

Activity A.4.2 *(∼5 min)*

Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

(a) T is injective

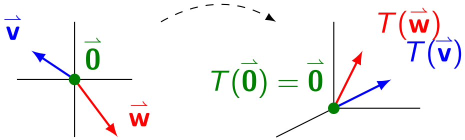
(b) T is not injective

(c) T is surjective

(d) T is not surjective

Fact A.4.3

A linear transformation T is injective **if and only if** $\ker T = \{\vec{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.



Activity A.4.4 *(∼5 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^5$ be a linear transformation where Im T is spanned by four vectors. What can you conclude?

(a) T is injective

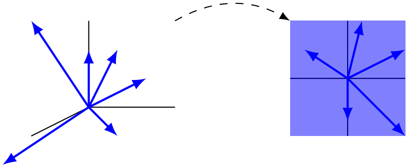
(b) T is not injective

(c) T is surjective

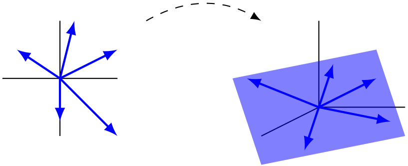
(d) T is not surjective

Fact A.4.5

A linear transformation $T : V \to W$ is surjective **if and only if** $\text{Im } T = W$. Put another way, a surjective linear transformation may be recognized by its identical codomain and image.



surjective, $\text{Im } T = \mathbb{R}^2$



not surjective, $\text{Im } T \neq \mathbb{R}^3$

Activity A.4.6 *(∼15 min)*

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of *equivalent* statements: one group that means T is **injective**, and one group that means T is **surjective**.

(a) The kernel of T is trivial: $\ker T = \{\vec{\mathbf{0}}\}$.

(b) The columns of A span $\mathbb{R}^m$.

(c) The columns of A are linearly independent.

(d) Every column of $\mathrm{RREF}(A)$ has a pivot.

(e) Every row of $\mathrm{RREF}(A)$ has a pivot.

(f) The image of T equals its codomain, i.e. $\mathrm{Im}\, T = \mathbb{R}^m$.

(g) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{\mathbf{b}} \end{array}\right]$ has a solution for all $\vec{\mathbf{b}} \in \mathbb{R}^m$.

(h) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{\mathbf{0}} \end{array}\right]$ has exactly one solution.

Observation A.4.7

The easiest way to show that the linear map with standard matrix A is injective is to show that $\mathrm{RREF}(A)$ has a pivot in each column.

The easiest way to show that the linear map with standard matrix A is surjective is to show that $\mathrm{RREF}(A)$ has a pivot in each row.

Activity A.4.8 *(~3 min)*

What can you immediately conclude (i.e. without computing a RREF) about the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ -3 & 3 \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

d) Its standard matrix has more rows than columns, so T is surjective.

Activity A.4.9 *(∼2 min)*

What can you immediately conclude (i.e. without computing a RREF) about the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 4 \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

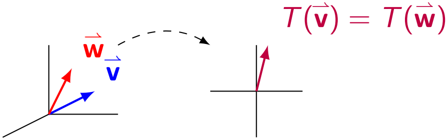
d) Its standard matrix has more rows than columns, so T is surjective.

Fact A.4.10

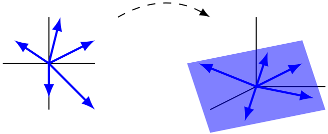
The following are true for any linear map $T : V \to W$:

- If $\dim(V) > \dim(W)$, then T is not injective.
- If $\dim(V) < \dim(W)$, then T is not surjective.

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase the dimension of its image.



not injective, $3 > 2$



not surjective, $2 < 3$

But dimension arguments **cannot** be used to prove a map **is** injective or surjective.

Activity A.4.11 *(∼5 min)*
Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).

Activity A.4.11 *($\sim$5 min)*

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).

Part 1: How many pivot columns must A have?

Activity A.4.11 *(∼5 min)*
Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).
Part 1: How many pivot columns must A have?
Part 2: How many pivot rows must A have?

Activity A.4.11 *(∼5 min)*

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).

Part 1: How many pivot columns must A have?

Part 2: How many pivot rows must A have?

Part 3: What can you conclude about m and n?

Activity A.4.12 *(∼5 min)*

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

(a) The columns of A form a basis for $\mathbb{R}^n$

(b) RREF(A) is the identity matrix.

(c) The system of linear equations given by the augmented matrix $\left[A \,\middle|\, \vec{\mathbf{b}}\right]$ has exactly one solution for each $\vec{\mathbf{b}} \in \mathbb{R}^n$.

Observation A.4.13

The easiest way to show that the linear map with standard matrix A is bijective is to show that $\mathrm{RREF}(A)$ is the identity matrix.

Activity A.4.14 *(∼3 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.15 *(~3 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.16 *(∼3 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x-y \\ x+3y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.17 *(∼3 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.