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Module A: Algebraic properties of linear maps

How can we understand linear maps algebraically?

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At the end of this module, students will be able to...

- A1. Linear map verification.** ... determine if a map between vector spaces of polynomials is linear or not.
- A2. Linear maps and matrices.** ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.
- A3. Kernel and Image.** ... compute a basis for the kernel and a basis for the image of a linear map, and verify that the rank-nullity theorem holds for a given linear map.
- A4. Injectivity and surjectivity.** ... determine if a given linear map is injective and/or surjective.

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Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans \mathbb{R}^n **V3**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **V5**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **V6,V7**.
- Find a basis of the solution space to a homogeneous system of linear equations **V10**.

Module A Section 1

Definition A.1

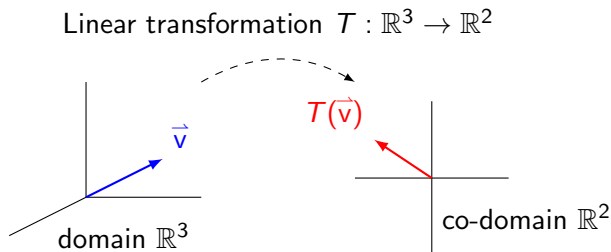
A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \rightarrow W$ is called a linear transformation if

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$.
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}, \vec{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition A.2

Given a linear transformation $T : V \rightarrow W$, V is called the **domain** of T and W is called the **co-domain** of T .



Example A.3

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = T \left(\begin{bmatrix} x + u \\ y + v \\ z + w \end{bmatrix} \right) = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) + T \left(\begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

And also...

$$T \left(c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T \left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \text{ and } cT \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = c \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore T is a linear transformation.

Example A.4

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact A.5

A map between Euclidean spaces $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of \mathbb{R}^n .

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z :

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2 , $y + 3$, and $y - 2^x$ are not linear combinations (even though $x + y$ is):

$$T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

Activity A.6 (*~5 min*) Recall the following rules from calculus, where $D : \mathcal{P} \rightarrow \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f .

$$D(f + g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

- a) \mathcal{P} is not a vector space
- b) D is a linear map
- c) D is not a linear map

Activity A.7 (*~10 min*) Let the polynomial maps $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ and $T : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4 + x)$, $S(x^4) + S(x)$, $T(x^4 + x)$, and $T(x^4) + T(x)$. Which of these maps is definitely not linear?

Fact A.8

If $L : V \rightarrow W$ is linear, then $L(\vec{z}) = L(0\vec{v}) = 0L(\vec{v}) = \vec{z}$ where \vec{z} is the additive identity of the vector spaces V, W .

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Observation A.9

Showing $L : V \rightarrow W$ is not a linear transformation can be done by finding an example for any one of the following.

- Show $L(\vec{z}) \neq \vec{z}$ (where \vec{z} is the additive identity of L and W).
- Find $\vec{v}, \vec{w} \in V$ such that $L(\vec{v} + \vec{w}) \neq L(\vec{v}) + L(\vec{w})$.
- Find $\vec{v} \in V$ and $c \in \mathbb{R}$ such that $L(c\vec{v}) \neq cL(\vec{v})$.

Otherwise, L can be shown to be linear by proving the following in general.

- For all $\vec{v}, \vec{w} \in V$, $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$.
- For all $\vec{v} \in V$ and $c \in \mathbb{R}$, $L(c\vec{v}) = cL(\vec{v})$.

Note the similarities between this process and showing that a subset of a vector space is/isn't a subspace.

Activity A.10 (*~15 min*) Continue to consider $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Activity A.10 (*~15 min*) Continue to consider $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g .

Activity A.10 (*~15 min*) Continue to consider $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g .

Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f .

Activity A.10 (*~15 min*) Continue to consider $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g .

Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f .

Part 3: Is S linear?

Activity A.11 (*~20 min*) Let the polynomial maps $S : \mathcal{P} \rightarrow \mathcal{P}$ and $T : \mathcal{P} \rightarrow \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Activity A.11 (~ 20 min) Let the polynomial maps $S : \mathcal{P} \rightarrow \mathcal{P}$ and $T : \mathcal{P} \rightarrow \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Note that $S(0) = 0$ and $T(0) = 0$. So instead, show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Activity A.11 (~ 20 min) Let the polynomial maps $S : \mathcal{P} \rightarrow \mathcal{P}$ and $T : \mathcal{P} \rightarrow \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Note that $S(0) = 0$ and $T(0) = 0$. So instead, show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Part 2: Prove that T is linear by verifying that $T(f(x) + g(x)) = T(f(x)) + T(g(x))$ and $T(cf(x)) = cT(f(x))$.

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Remark A.12

Recall that a linear map $T : V \rightarrow W$ satisfies

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$.
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}, \vec{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

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Activity A.13 (~ 5 min) Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear map, and you know

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \text{ Compute } T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right).$$

(a) $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(c) $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(d) $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

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Activity A.14 (~ 5 min) Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear map, and you know

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \text{ Compute } T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right).$$

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

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Activity A.15 (~ 5 min) Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear map, and you know

$$T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \text{ Compute } T \left(\begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} \right).$$

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

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Activity A.16 (~ 5 min) Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear map, and you know

$$T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

What piece of information would help you compute $T \left(\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \right)$?

(a) The value of $T \left(\begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} \right)$.

(b) The value of $T \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$.

(c) The value of $T \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.

(d) Any of the above.

Fact A.17

Consider any basis $\{\vec{b}_1, \dots, \vec{b}_n\}$ for V . Since every vector \vec{v} can be written as a linear combination of basis vectors, $x_1\vec{b}_1 + \dots + x_n\vec{b}_n$, we may compute $T(\vec{v})$ as follows:

$$T(\vec{v}) = T(x_1\vec{b}_1 + \dots + x_n\vec{b}_n) = x_1T(\vec{b}_1) + \dots + x_nT(\vec{b}_n).$$

Therefore any linear transformation $T : V \rightarrow W$ can be defined by just describing the values of $T(\vec{b}_i)$.

Put another way, the images of the basis vectors **determine** the transformation T .

Definition A.18

Since linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is determined by the standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\vec{e}_1) \cdots T(\vec{e}_n)]$.

For example, let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of \mathbb{R}^3 .

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad T(\vec{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Then the standard matrix corresponding to T is

$$[T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3)] = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix}.$$

Activity A.19 (~ 3 min) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation given by

$$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \quad T(\vec{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \quad T(\vec{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \quad T(\vec{e}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix $[T(\vec{e}_1) \cdots T(\vec{e}_n)]$ for T .

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Activity A.20 (*~5 min*) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

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Activity A.20 (*~5 min*) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Part 1: Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

Activity A.20 (~ 5 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Part 1: Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

Part 2: Find the standard matrix for T .

Fact A.21

Because every linear map $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ has a linear combination of the variables in each component, and thus $T(\vec{e}_i)$ yields exactly the coefficients of x_i , the standard matrix for T is simply an ordered list of the coefficients of the x_i :

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \quad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Activity A.22 (*~5 min*) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Activity A.22 (~ 5 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Part 1: Compute $T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$.

Activity A.22 (~ 5 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Part 1: Compute $T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$.

Part 2: Compute $T \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$.

Fact A.23

To quickly compute $T(\vec{v})$ from its standard matrix A , multiply and add the entries of each row of A with the vector \vec{v} . For example, if T has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$$

then for $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we will write

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 2y + 3z \\ 0x + 1y - 2z \\ 2x - 1y + 0z \end{bmatrix}$$

and for $\vec{v} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ we will write

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(0) + 3(-2) \\ 0(3) + 1(0) - 2(-2) \\ 2(3) - 1(0) + 0(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}.$$

Activity A.24 (*~15 min*) Compute the following linear transformations of vectors given their standard matrices.

$$T_1 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \text{ for the standard matrix } A_1 = \begin{bmatrix} 4 & 3 \\ 0 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$$

$$T_2 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix} \right) \text{ for the standard matrix } A_2 = \begin{bmatrix} 4 & 3 & 0 & -1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

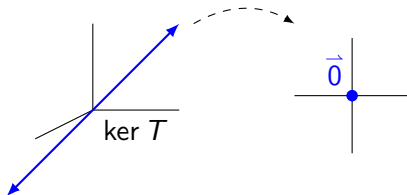
$$T_3 \left(\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right) \text{ for the standard matrix } A_3 = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & 3 \\ 5 & 1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

Module A Section 3

Definition A.25

Let $T : V \rightarrow W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \{\vec{v} \in V \mid T(\vec{v}) = \vec{z}\}$$



Activity A.26 (~ 5 min) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of \mathbb{R}^2 describes $\ker T$, the set of all vectors that transform into $\vec{0}$?

- a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$
- b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
- c) $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Activity A.27 (~ 5 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of \mathbb{R}^3 describes $\ker T$, the set of all vectors that transform into $\vec{0}$?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \mid a \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

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Activity A.28 (*~10 min*) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 3x + 4y - z \\ x + 2y + z \end{bmatrix}$$

Activity A.28 (~ 10 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 3x + 4y - z \\ x + 2y + z \end{bmatrix}$$

Part 1: Set $T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

Activity A.28 (~ 10 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 3x + 4y - z \\ x + 2y + z \end{bmatrix}$$

Part 1: Set $T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.

Part 2: Use $\text{RREF}(A)$ to solve this homogeneous system of equations and find a basis for the kernel of T .

Activity A.29 (*~10 min*) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation given by

$$T \left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \right) = \begin{bmatrix} 2x + 4y + 2z - 4w \\ -2x - 4y + z + w \\ 3x + 6y - z - 4w \end{bmatrix}.$$

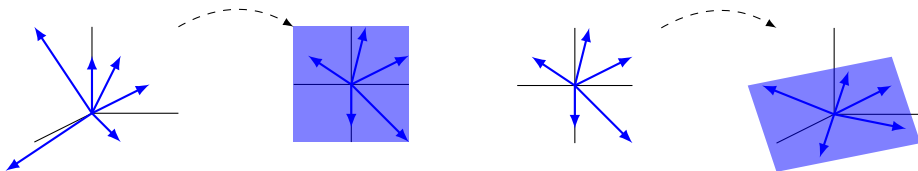
Find a basis for the kernel of T .

Definition A.30

Let $T : V \rightarrow W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\text{Im } T = \{ \vec{w} \in W \mid \text{there is some } \vec{v} \in V \text{ with } T(\vec{v}) = \vec{w} \}$$

In the examples below, the left example's image is all of \mathbb{R}^2 , but the right example's image is a planar subspace of \mathbb{R}^3 .



Activity A.31 (~ 5 min) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of \mathbb{R}^3 describes $\text{Im } T$, the set of all vectors that are the result of using T to transform \mathbb{R}^2 vectors?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

b) $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$

d) $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

Activity A.32 (~ 5 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of \mathbb{R}^2 describes $\text{Im } T$, the set of all vectors that are the result of using T to transform \mathbb{R}^3 vectors?

- a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$
- b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
- c) $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Activity A.33 (~ 5 min) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3) \quad T(\vec{e}_4)] .$$

Since $T(\vec{v}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4)$, the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

- a) spans $\text{Im } T$
- b) is a linearly independent subset of $\text{Im } T$
- c) is a basis for $\text{Im } T$

Observation A.34

Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}.$$

Since the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ spans $\text{Im } T$, we can obtain a basis for

$\text{Im } T$ by finding RREF $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and only using the vectors

corresponding to pivot columns:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Fact A.35

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation with standard matrix A .

- The kernel of T is the solution set of the homogeneous system given by the augmented matrix $\left[A \mid \vec{0} \right]$. Use the coefficients of its free variables to get a basis for the kernel.
- The image of T is the span of the columns of A . Remove the vectors creating non-pivot columns in RREF A to get a basis for the image.

Activity A.36 (~ 10 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T .

Activity A.37 (~ 5 min) Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation with standard matrix A . Which of the following is equal to the dimension of the kernel of T ?

- (a) The number of pivot columns
- (b) The number of non-pivot columns
- (c) The number of pivot rows
- (d) The number of non-pivot rows

Activity A.38 (~ 5 min) Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation with standard matrix A . Which of the following is equal to the dimension of the image of T ?

- (a) The number of pivot columns
- (b) The number of non-pivot columns
- (c) The number of pivot rows
- (d) The number of non-pivot rows

Observation A.39

Combining these with the observation that the number of columns is the dimension of the domain of T , we have the **rank-nullity theorem**:

The dimension of the domain of T equals $\dim(\ker T) + \dim(\operatorname{Im} T)$.

The dimension of the image is called the **rank** of T (or A) and the dimension of the kernel is called the **nullity**.

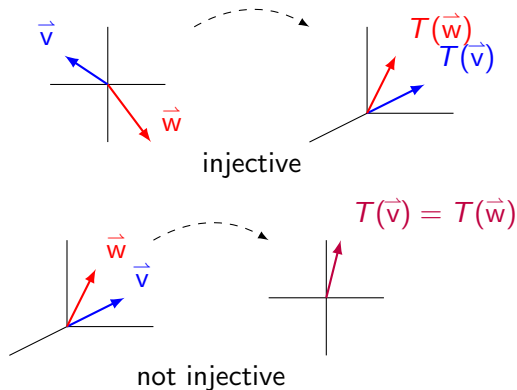
Activity A.40 (~ 10 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Verify that the rank-nullity theorem holds for T .

Definition A.41

Let $T : V \rightarrow W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.



Activity A.42 (~ 3 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T injective?

- a) Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.
- b) Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.
- c) No, because $T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \neq T \left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right)$
- d) No, because $T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = T \left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right)$

Module A Section 4

Activity A.43 (~ 2 min) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

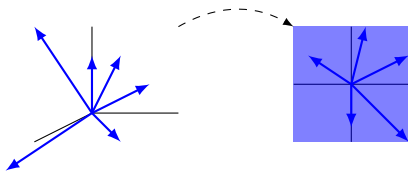
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T injective?

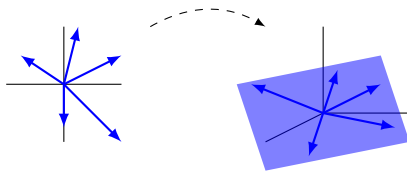
- a) Yes, because $T(\vec{v}) = T(\vec{w})$ whenever $\vec{v} = \vec{w}$.
- b) Yes, because $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.
- c) No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$
- d) No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$

Definition A.44

Let $T : V \rightarrow W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V . More precisely, for every $\vec{w} \in W$, there is some $\vec{v} \in V$ with $T(\vec{v}) = \vec{w}$.



surjective



not surjective

Activity A.45 (~ 3 min) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T surjective?

a) Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, there exists $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T(\vec{v}) = \vec{w}$.

b) No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

c) No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity A.46 (~ 2 min) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective?

a) Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} x \\ y \\ 42 \end{bmatrix} \in \mathbb{R}^3$ such that

$$T(\vec{v}) = \vec{w}.$$

b) Yes, because for every $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3$ such that

$$T(\vec{v}) = \vec{w}.$$

c) No, because $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Observation A.47

As we will see, it's no coincidence that the RREF of the injective map's standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has all pivot columns. Similarly, the RREF of the surjective map's standard matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has a pivot in each row.

Observation A.48

Let $T : V \rightarrow W$. We have previously defined the following terms.

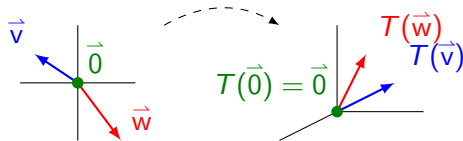
- The **kernel** of T is the set of all vectors in V that are mapped to $\vec{0} \in W$. It is a subspace of V .
- The **image** of T is the set of all vectors in W that are mapped to by something in V . It is a subspace of W .
- T is called **injective** or **one-to-one** if T always maps distinct vectors to different places.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V .

Activity A.49 (~ 5 min) Let $T : V \rightarrow W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

- (a) T is injective
- (b) T is not injective
- (c) T is surjective
- (d) T is not surjective

Fact A.50

A linear transformation T is injective **if and only if** $\ker T = \{\vec{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

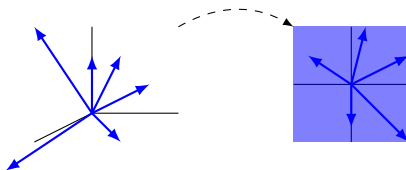


Activity A.51 (~ 5 min) Let $T : V \rightarrow \mathbb{R}^5$ be a linear transformation where $\text{Im } T$ is spanned by four vectors. What can you conclude?

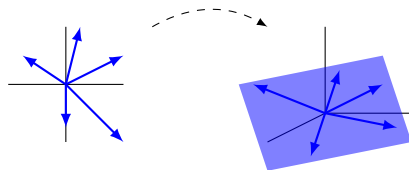
- (a) T is injective
- (b) T is not injective
- (c) T is surjective
- (d) T is not surjective

Fact A.52

A linear transformation $T : V \rightarrow W$ is surjective **if and only if** $\text{Im } T = W$. Put another way, a surjective linear transformation may be recognized by its identical codomain and image.



surjective, $\text{Im } T = \mathbb{R}^2$



not surjective, $\text{Im } T \neq \mathbb{R}^3$

Activity A.53 (~ 15 min) Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with standard matrix A . Sort the following claims into two groups of *equivalent* statements: one group that means T is **injective**, and one group that means T is **surjective**.

- (a) The kernel of T is trivial, i.e. $\ker T = \{\vec{0}\}$.
- (b) The columns of A span \mathbb{R}^m .
- (c) The columns of A are linearly independent.
- (d) Every column of $\text{RREF}(A)$ has a pivot.
- (e) Every row of $\text{RREF}(A)$ has a pivot.
- (f) The image of T equals its codomain, i.e. $\text{Im } T = \mathbb{R}^m$.
- (g) The system of linear equations given by the augmented matrix $\begin{bmatrix} A & | & \vec{b} \end{bmatrix}$ has a solution for all $\vec{b} \in \mathbb{R}^m$.
- (h) The system of linear equations given by the augmented matrix $\begin{bmatrix} A & | & \vec{0} \end{bmatrix}$ has exactly one solution.

Observation A.54

The easiest way to show that the linear map with standard matrix A is injective is to show that $\text{RREF}(A)$ has a pivot in each column.

The easiest way to show that the linear map with standard matrix A is surjective is to show that $\text{RREF}(A)$ has a pivot in each row.

Activity A.55 (~ 3 min) What can you conclude about the linear map

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with standard matrix $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$?

- a) Its standard matrix has more columns than rows, so T is not injective.
- b) Its standard matrix has more columns than rows, so T is injective.
- c) Its standard matrix has more rows than columns, so T is not surjective.
- d) Its standard matrix has more rows than columns, so T is surjective.

Activity A.56 (~ 2 min) What can you conclude about the linear map

$T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with standard matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$?

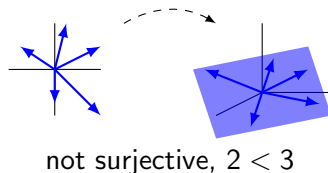
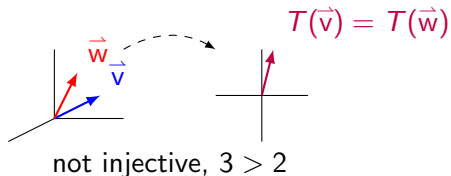
- a) Its standard matrix has more columns than rows, so T is not injective.
- b) Its standard matrix has more columns than rows, so T is injective.
- c) Its standard matrix has more rows than columns, so T is not surjective.
- d) Its standard matrix has more rows than columns, so T is surjective.

Fact A.57

The following are true for any linear map $T : V \rightarrow W$:

- If $\dim(V) > \dim(W)$, then T is not injective.
- If $\dim(V) < \dim(W)$, then T is not surjective.

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase dimension from its domain to its image.



But dimension arguments **cannot** be used to prove a map **is** injective or surjective.

Activity A.58 (~ 5 min) Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^4$ with standard matrix

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijjective**).

Activity A.58 (*~5 min*) Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^4$ with standard matrix

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijjective**).

Part 1: How many pivot rows must RREF A have?

Activity A.58 (~ 5 min) Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^4$ with standard matrix

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijjective**).

Part 1: How many pivot rows must RREF A have?

Part 2: How many pivot columns must RREF A have?

Activity A.58 (~ 5 min) Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^4$ with standard matrix

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijjective**).

Part 1: How many pivot rows must RREF A have?

Part 2: How many pivot columns must RREF A have?

Part 3: What is RREF A ?

Activity A.59 (~ 5 min) Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a bijective linear map with standard matrix A . Label each of the following as true or false.

- (a) $\text{RREF}(A)$ is the identity matrix.
- (b) The columns of A form a basis for \mathbb{R}^n .
- (c) The system of linear equations given by the augmented matrix $\left[A \mid \vec{b} \right]$ has exactly one solution for each $\vec{b} \in \mathbb{R}^n$.