

## Section E.1

**Definition E.1.1** A **linear equation** is an equation of the variables  $x_i$  of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

**Remark E.1.2** In previous classes you likely assumed  $x = x_1$ ,  $y = x_2$ , and  $z = x_3$ . However, since this course often deals with equations of four or more variables, we will almost always write our variables as  $x_i$ .

**Definition E.1.3** A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

A **solution**

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i$$

for  $1 \leq i \leq m$  (that is, the solution satisfies all equations in the system).

**Remark E.1.4** When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{array}{rcl} x_1 + 3x_3 & = & 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ -x_2 + x_3 & = & -2 \end{array}$$

Verbose standard form:

$$\begin{array}{rcl} 1x_1 + 0x_2 + 3x_3 & = & 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ 0x_1 - 1x_2 + 1x_3 & = & -2 \end{array}$$

Concise standard form:

$$\begin{array}{rcl} x_1 & + & 3x_3 = 3 \\ 3x_1 - 2x_2 + 4x_3 & = & 0 \\ -x_2 + x_3 & = & -2 \end{array}$$

**Definition E.1.5** A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

**Fact E.1.6** All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

**Activity E.1.7** (*~10 min*) All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

**Activity E.1.8** (*~10 min*) Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

*Part 1:* Find three different solutions for this system.

*Part 2:* Let  $x_2 = a$  where  $a$  is an arbitrary real number, then find an expression for  $x_1$  in terms of  $a$ . Use this to write *all* solutions (the **solution set**)  $\left\{ \begin{bmatrix} ? \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$  for the linear system in terms of  $a$ .

**Activity E.1.9** (*~10 min*) Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 - x_4 &= 3 \\ x_3 + 4x_4 &= -2 \end{aligned}$$

Describe the solution set

$$\left\{ \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

to the linear system by setting  $x_2 = a$  and  $x_4 = b$ , and then solving for  $x_1$  and  $x_3$ .

**Observation E.1.10** Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.