

Section V.1

Remark V.1.1 Last time, we defined a **vector space** V to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V , and all scalars (i.e. real numbers) a, b .

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| <ul style="list-style-type: none"> • Addition is associative.
$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$ • Addition is commutative.
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$ • Additive identity exists.
There exists some \mathbf{z} where $\mathbf{v} + \mathbf{z} = \mathbf{v}.$ • Additive inverses exist.
There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}.$ | <ul style="list-style-type: none"> • Scalar multiplication is associative.
$a(b\mathbf{v}) = (ab)\mathbf{v}.$ • 1 is a scalar multiplicative identity.
$1\mathbf{v} = \mathbf{v}.$ • Scalar multiplication distributes over vector addition.
$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$ • Scalar multiplication distributes over scalar addition.
$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$ |
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Remark V.1.2 The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- \mathbb{R}^n : Euclidean vectors with n components.
- \mathbb{R}^∞ : Sequences of real numbers (v_1, v_2, \dots) .
- $M_{m,n}$: Matrices of real numbers with m rows and n columns.
- \mathbb{C} : Complex numbers.
- \mathcal{P}^n : Polynomials of degree n or less.
- \mathcal{P} : Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity V.1.3 (~ 20 min) Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \quad c \odot (x, y) = (cx, y^c)$$

Part 1: Show that V satisfies the vector distributive property

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$$

by letting $\mathbf{v} = (x, y)$ and showing both sides simplify to the same expression.

Part 2: Show that V contains an additive identity element by choosing $\mathbf{z} = (?, ?)$ such that $\mathbf{v} \oplus \mathbf{z} = (x, y) \oplus (?, ?) = \mathbf{v}$ for any $\mathbf{v} = (x, y) \in V$.

Remark V.1.4 It turns out $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \quad c \odot (x, y) = (cx, y^c)$$

satisfies all eight properties.

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some \mathbf{z} where $\mathbf{v} \oplus \mathbf{z} = \mathbf{v}$.

- **Addition inverse.**

There exists some $-\mathbf{v}$ where $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{z}$.

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

Thus, V is a vector space.

Activity V.1.5 (~ 15 min) Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \quad c \odot (x, y) = (x^c, y + c - 1).$$

Part 1: Show that the scalar multiplication identity holds by simplifying $1 \odot (x, y)$ to (x, y) .

Part 2: Show that the addition identity property fails by showing that $(0, -1) \oplus \mathbf{z} \neq (0, -1)$ no matter how $\mathbf{z} = (z_1, z_2)$ is chosen.

Part 3: Can V be a vector space?

Definition V.1.6 A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$ for any choice of scalar multiples c_1, c_2, \dots, c_m .

For example, we can say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Definition V.1.7 The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m \mid c_i \in \mathbb{R}\}.$$

For example:

$$\text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\} = \left\{a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R}\right\}$$

Activity V.1.8 (~ 10 min) Consider $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

Part 1: Sketch $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $-2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane.

Part 2: Sketch a representation of all the vectors belonging to $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \left\{a \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid a \in \mathbb{R}\right\}$ in the xy plane.

Activity V.1.9 (~ 10 min) Consider $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$\begin{array}{ccc} 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ -2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & \end{array}$$

Part 2: Sketch a representation of all the vectors belonging to $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ in the xy plane.

Activity V.1.10 (~ 5 min) Sketch a representation of all the vectors belonging to $\text{span}\left\{\begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right\}$ in the xy plane.