Standards for this Module

At the end of this module, each student should be able to...

- **E1: Augmented Matrices.** Translate back and forth between a system of linear equations and the corresponding augmented matrix.
- **E2: Row Reduction.** Find the equivalent augmented matrix in reduced row echelon form for a given augmented matrix.
- **E3: Solving Linear Systems.** Solve a system of linear equations.
- **E4: Homogeneous Systems.** Find a basis for the solution set of a homogeneous linear system.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.

Readiness Assurance Resources

The following resources will help you prepare for this module.

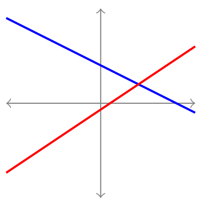
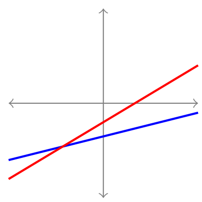
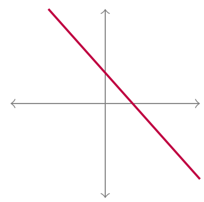
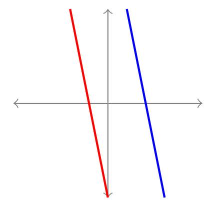
- `https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-systems-topic/cc-8th-systems-graphica a/systems-of-equations-with-graphing`
- `https://www.khanacademy.org/math/algebra/systems-of-linear-equations/solving-systems-of-equations- v/practice-using-substitution-for-systems`

Readiness Assurance Test

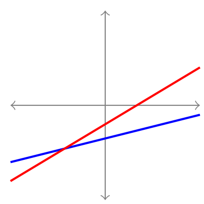
Choose the most appropriate response for each question.

1) Which of these graphs represents the following system of linear equations?

$$x + 2y = 4$$
$$2x - 3y = 1$$

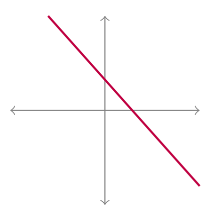
(a) 
(b) 
(c) 
(d) 

2) How many solutions are there for the system of linear equations represented by the following graph?



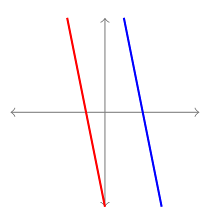
(a) Zero (b) One (c) Two (d) Infinitely-many

3) How many solutions are there for the system of linear equations represented by the following graph? (This graph represents two completely overlapping lines.)



(a) Zero (b) One (c) Two (d) Infinitely-many

4) How many solutions are there for the system of linear equations represented by the following graph? (This graph represents two parallel lines.)



(a) Zero (b) One (c) Two (d) Infinitely-many

5) Solve the following system of linear equations.

$$y = 2x + 5$$
$$y = -x + 2$$

(a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ (c) There are no solutions. (d) There are infinitely-many solutions.

6) Solve the following system of linear equations.

$$x + 2y = 4$$
$$2x - 3y = 1$$

(a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (c) There are no solutions. (d) There are infinitely-many solutions.

Application Activities - Day 1

Definition. A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

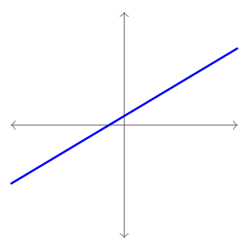
A **solution** for a linear equation is expressed in terms of the Euclidean vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

Observation. The linear equation $3x - 5y = -2$ may be graphed as a line in the xy plane.



The linear equation $x + 2y - z = 4$ may be graphed as a plane in xyz space.

Remark. In previous classes you likely assumed $x = x_1$, $y = x_2$, and $z = x_3$. However, since this course often deals with equations of four or more variables, we will almost always write our variables as x_i.

Definition. A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{aligned} a_{11}x_1 + \; & a_{12}x_2 + \ldots + \; & a_{1n}x_n \; &= \; b_1 \\ a_{21}x_1 + \; & a_{22}x_2 + \ldots + \; & a_{2n}x_n \; &= \; b_2 \\ \vdots \quad & \quad \vdots & \vdots \quad & \quad \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + \ldots + & a_{mn}x_n \; &= b_m \end{aligned}$$

A **solution**

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i$$

for $1 \leq i \leq m$ (that is, the solution satisfies all equations in the system).

Remark. When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - x_2 + x_3 &= -2 \end{aligned}$$

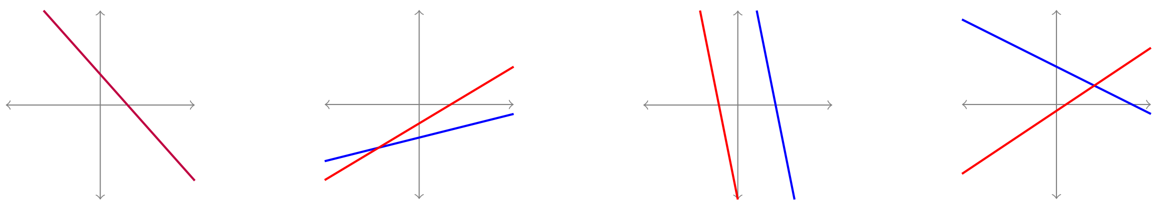
Concise standard form:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & 3 \\ 3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & -2 \end{array}$$

Definition. A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

Fact. All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

Activity. *(5 min)* Consider the following graphs representing linear systems of two variables. Label each graph with **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.



Activity. *(10 min)* All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system by solving for x_1 in the first equation, substituting the resulting expression into the second equation, and then simplifying.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity. *(10 min)* Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part X: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

Part X: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to describe *all* solutions (the **solution set**) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$ for the linear system in terms of a.

Remark. The solution set to a consistent linear system with infinitely many solutions may be described by setting certain variables equal to arbitrary parameters, and expressing the other variables in terms of those parameters.

Activity. *(10 min)* Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 \quad - \; x_4 &= 3 \\ x_3 + 4x_4 &= -2 \end{aligned}$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix} + \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation. Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but won't cut it for equations with more variables. Linear Algebra provides us several tools to make this process more efficient.

Definition. A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + \; a_{12}x_2 + \ldots + \; a_{1n}x_n &= b_1 \\ a_{21}x_1 + \; a_{22}x_2 + \ldots + \; a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \;\; &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \qquad \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Definition. Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they share exactly the same solutions.

Activity. *(10 min)* Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a conzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

(Instructor Note:) This activity could be ran as a card sort.

Application Activities - Day 2

Definition. The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a conzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity. *(15 min)* Show that the following two linear systems:

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned} \qquad\qquad \begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

1. Swap R_1 (first row) and R_2 (second row).
2. Multiply R_2 by $\frac{1}{2}$.
3. Add R_1 to R_3.
4. Add $-3R_1$ to R_2.
5. Add $-2R_2$ to R_3.
6. Multiply R_3 by $\frac{1}{3}$.

Definition. The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix.

Activity. *(10 min)* Reproduce the steps that manipulated the matrix

$$\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

into row echelon form by using the following algorithm.

1. Identify the top cell of the first non-zero column as your **pivot position**; you will ignore anything in the matrix that is above or left of your current pivot position.
2. If the pivot position contains a 0, swap its row with a lower row that does not contain a 0 in its column.
3. Divide the pivot row by the term in pivot position to change the pivot term to 1. (If convenient, you can first swap the pivot row with a lower row to make this division easier.)
4. Add multiples of the pivot row to all lower rows so that all terms below pivot position become 0.
5. Move your pivot position down and right one step.

6. If all terms in and below pivot position are zero, move your pivot position right. Repeat this step as needed.
7. If the matrix is not yet in row echelon form, return to Step 2.

Definition. A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0.

Activity. *(10 min)* Show that the following two linear systems:

$$\begin{aligned} x_1 - x_2 + 5x_3 &= 1 \\ x_2 - 2x_3 &= 3 \\ x_3 &= 2 \end{aligned} \qquad\qquad \begin{aligned} x_1 \quad\quad &= -2 \\ x_2 \quad &= 7 \\ x_3 &= 2 \end{aligned}$$

are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

1. Add $2R_3$ to R_2.
2. Add $-5R_3$ to R_1.
3. Add R_2 to R_1.

Then write the solution to the linear system.

Activity. *(5 min)* Verify that $(x_1, x_2, x_3) = (-2, 7, 2)$ is a solution to the linear system

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -1x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

by plugging the solution into each equation.

Activity. *(10 min)* Reproduce the steps that manipulated the matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

into reduced row echelon form by adding multiples of lower rows into higher rows, zeroing out all terms above leading terms.

Application Activities - Day 3

Fact. Every augmented matrix A reduces to a unique reduced row echelon form matrix. This matrix is denoted as $\mathrm{RREF}(A)$.

Definition. The following algorithm that reduces A to $\mathrm{RREF}(A)$ is known as **Gauss-Jordan elimination**.

1. Identify the top cell of the first non-zero column as your pivot position; you will ignore anything in the matrix that is above or left of your current pivot position.
2. If the pivot position contains a 0, swap its row with a lower row that does not contain a 0 in its column.
3. Divide the pivot row by the term in pivot position to change the pivot term to 1. (If convenient, you can first swap the pivot row with a lower row to make this division easier.)
4. Add multiples of the pivot row to all lower rows so that all terms below pivot position become 0.
5. Move your pivot position down and right one step.
6. If all terms in and below pivot position are zero, move your pivot position right. Repeat this step as needed.
7. If the matrix is not yet in row echelon form, return to Step 2.
8. Finally, add multiples of lower rows into higher rows, zeroing out all terms above leading terms.

Activity. *(15 min)* Find $\mathrm{RREF}(A)$ where

$$A = \left[\begin{array}{cccc|c} -1 & 1 & -3 & 2 & 0 \\ 2 & -1 & 5 & 3 & -11 \\ 3 & 2 & 4 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right].$$

Definition. The columns of $\mathrm{RREF}(A)$ without a leading term represent **free variables** of the linear system modeled by A that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by A.

Activity. *(10 min)* Given the linear system and its equivalent augmented matrices

$$\begin{aligned} -x_1 + x_2 - 3x_3 + 2x_4 &= 0 \\ 2x_1 - x_2 + 5x_3 + 3x_4 &= -11 \\ 3x_1 + 2x_2 + 4x_3 + x_4 &= 1 \\ x_2 - x_3 + x_4 &= 1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} -1 & 1 & -3 & 2 & 0 \\ 2 & -1 & 5 & 3 & -11 \\ 3 & 2 & 4 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

describe the solution set $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$ to the linear system by setting the free variable $x_3 = a$, and then expressing each of the bounded variables x_1, x_2, x_4 equal to an expression in terms of a.

Remark. It's not necessary to completely find RREF(A) to deduce that a linear system is inconsistent.

Activity. *(10 min)* Find the contradiction in the inconsistent linear system

$$\begin{aligned} 2x_1 - 3x_2 &= 17 \\ x_1 + 2x_2 &= -2 \\ -x_1 - x_2 &= 1 \end{aligned}$$

by considering the following equivalent augmented matrices:

$$\left[\begin{array}{cc|c} 2 & -3 & 17 \\ 1 & 2 & -2 \\ -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{array}\right].$$

Definition. A **homogeneous system** is a linear system satisfying $b_i = 0$, that is, it is a linear system of the form

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n \ &= 0 \\ a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n \ &= 0 \\ \vdots \qquad \quad \vdots \qquad \qquad \quad \vdots \ \ &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ &= 0 \end{aligned}$$

Activity. *(5 min)* Prove that all homogeneous systems are consistent. (Hint: Find an obvious solution $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ that satisfies every homogeneous system.)

Fact. The solution set to any homogeneous system with infinitely-many solutions is generated by multiplying the parameters representing the free variables by Euclidean vectors, and adding these up. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Definition. A set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Activity. *(10 min)* Find a basis for the solution set of the following homogeneous linear system.

$$\begin{aligned} x_1 + 2x_2 \phantom{{}+x_3} - x_4 &= 0 \\ x_3 + 4x_4 &= 0 \\ 2x_1 + 4x_2 + x_3 + 2x_4 &= 0 \end{aligned}$$

Standards for this Module

At the end of this module, each student should be able to...

- **V1: Vector Spaces.** Determine if a set with given operations forms a vector space.
- **V2: Linear Combinations.** Determine if a vector can be written as a linear combination of a given set of vectors.
- **V3: Subspaces.** Determine if a subset of a vector space is a subspace.
- **V4: Isomorphisms.** Solve an abstract vector space problem by reinterpreting it in terms of an isomorphic Euclidean space.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- https://www.khanacademy.org/math/precalculus/vectors-precalc/vector-addition-subtraction/v/adding-and-subtracting-vectors
- https://www.khanacademy.org/math/precalculus/vectors-precalc/combined-vector-operations/v/combined-vector-operations-example
- https://www.khanacademy.org/math/precalculus/imaginary-and-complex-numbers/adding-and-subtracting-
v/adding-complex-numbers
- https://www.khanacademy.org/math/algebra/introduction-to-polynomial-expressions/adding-and-subtrac
v/adding-and-subtracting-polynomials-1

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Simplify the following vector expression.

$$2\begin{bmatrix}3\\-1\\0\end{bmatrix} - 3\begin{bmatrix}0\\2\\1\end{bmatrix}$$

(a) $\begin{bmatrix}0\\4\\-7\end{bmatrix}$ (b) $\begin{bmatrix}6\\-8\\-3\end{bmatrix}$ (c) $\begin{bmatrix}3\\2\\-5\end{bmatrix}$ (d) $\begin{bmatrix}-2\\0\\1\end{bmatrix}$

2) Simplify the complex number expression $-4(3-2i)+2(5+i)$.

(a) $3-7i$ (b) $4+i$ (c) $-2+10i$ (d) $-1-5i$

3) Simplify $3f(x)-2g(x)$ where $f(x)=7-x^2$ and $g(x)=2x^3+x-1$.

(a) x^3+4x-5 (b) $-4x^3-3x^2-2x+23$ (c) $3x^3+5x^2-3x+17$ (d) $-x^3+19x^2-4$

Application Activities - Day 1

Activity. *(15 min)* Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. Rewrite the following proofs about two-dimensional Euclidean vector addition, filling in the missing information.

1. Euclidean vector addition is **well-defined**.

   Proof: $\mathbf{u} + \mathbf{v} = ?$. Since this is a Euclidean vector, addition is well-defined. □

2. Euclidean vector addition is **associative**.

   Proof: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = ?$ and $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = ?$. Since both expressions are equal, addition is associative. □

3. Euclidean vector addition is **commutative**.

   Proof: $\mathbf{u} + \mathbf{v} = ?$ and $\mathbf{v} + \mathbf{u} = ?$. Since both expressions are equal, addition is commutative. □

4. There exists an **additive identity** for Euclidean vector addition.

   Proof: Let $\mathbf{0} = ?$. Since $\mathbf{v} + \mathbf{0} = ? = \mathbf{v}$, $\mathbf{0}$ is an identity. □

5. Every Euclidean vector has an **additive inverse**.

   Proof: Let $-\mathbf{v} = ?$. Since $\mathbf{v} + (-\mathbf{v}) = ? = \mathbf{0}$, $-\mathbf{v}$ is an inverse for $\mathbf{v}$. □

Activity. *(15 min)* Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and let a, b be scalar numbers. Rewrite the following proofs about scalar multiplication of two-dimensional Euclidean vectors, filling in the missing information.

1. Scalar multiplication of Euclidean vectors is **well-defined**.

   Proof: $a\mathbf{v} = ?$. Since this is a Euclidean vector, scalar multiplication is well-defined. □

2. Scalar multiplication of Euclidean vectors is **compatible**.

   Proof: $a(b\mathbf{v}) = ?$ and $(ab)\mathbf{v} = ?$. Since both expressions are equal, scalar multiplication is compatible. □

3. Scalar multiplication of Euclidean vectors **distributes scalars**.

   Proof: $a(\mathbf{u} + \mathbf{v}) = ?$ and $a\mathbf{u} + a\mathbf{v} = ?$. Since both expressions are equal, scalars distribute. □

4. Scalar multiplication of Euclidean vectors **distributes vectors**.

   Proof: $(a + b)\mathbf{v} = ?$ and $a\mathbf{v} + b\mathbf{v} = ?$. Since both expressions are equal, vectors distribute. □

5. The scalar 1 is a **multiplicative identity** for the scalar multiplication of Euclidean vectors.

   Proof: Since $1\mathbf{v} = ? = \mathbf{v}$, 1 is an identity. □

Definition. A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Well-defined addition.**
  $\mathbf{v} + \mathbf{w}$ belongs to V.
- **Associative addition.**
  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Commutative addition.**
  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Additive identity.**
  There exists some $\mathbf{0}$ in V where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- **Additive identity.**
  There exists some $-\mathbf{v}$ in V where $\mathbf{v}+(-\mathbf{v}) = \mathbf{0}$.
- **Well-defined scalar multipication.**
  $a\mathbf{v}$ belongs to V.
- **Compatible scalar multipication.**
  $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **Scalar distribution.**
  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Vector distribution.**
  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
- **Scalar multiplication identity.**
  $1\mathbf{v} = \mathbf{v}$.

Definition. The most important examples of vector spaces are the **Euclidean vector spaces** $\mathbb{R}^n$, but there are other examples as well.

Activity. *(15 min)* Assuming the usual definitions of addition and scalar multiplication, label each of the following sets as a **valid** or **non-valid** vector space. If the set is not a valid vector space, give an example of a vector space property that fails to hold.

1. $\left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} : v_i \text{ is an integer} \right\}$
2. $\{a + bi : a, b \text{ are real numbers}\}$
3. $\left\{x^2 + bx + c : b, c \text{ are positive}\right\}$

Application Activities - Day 2

Remark. The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{R}^\infty$: Sequences of real numbers $(v_1, v_2, \dots)$.
- $\mathbb{R}^{m\times n}$: Matrices of real numbers with m rows and n columns.
- $\mathbb{C}$: Complex numbers.
- $\mathbb{P}^n$: Polynomials of degree n or less.
- $\mathbb{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity. *(10 min)* Prove that $\mathbb{P}^2$ satisfies the *commutative addition* property of vector spaces, by showing that if $f(x) = a_1x^2 + b_1x + c_1$ and $g(x) = a_2x^2 + b_2x + c_2$, then $f(x) + g(x) = g(x) + f(x)$.

Definition. A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ for any choice of scalar multiples $c_1, c_2, \dots, c_m$.

Definition. The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

Activity. *(10 min)* Consider $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part X: Sketch $c\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part X: Sketch a representation of all the vectors given by $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ in the xy plane.

Activity. *(10 min)* Consider $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part X: Sketch $c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix}$ in the xy plane for $\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$.

Part X: Sketch a representation of all the vectors given by $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity. *(5 min)* Sketch a representation of all the vectors given by $\text{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix}\right\}$ in the xy plane.

Activity. *(10 min)* Consider the following linear system.

$$\begin{alignedat}{3} x_1 &- {}& x_2 &= -1 \\ &- {}& 3x_2 &= -6 \\ -3x_1 &+ {}& 2x_2 &= 1 \end{alignedat}$$

Part X: Solve this system by using a calculator to find

$$\mathrm{RREF}\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & -3 & -6 \\ -3 & 2 & 1 \end{array}\right]$$

Part X: Given this solution, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

Fact. A vector $\mathbf{b}$ belongs to $\mathrm{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \dots \mathbf{v}_n \,|\, \mathbf{b}]$ is consistent.

Remark. To determine if $\mathbf{b}$ belongs to $\mathrm{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, find $\mathrm{RREF}[\mathbf{v}_1 \dots \mathbf{v}_n \,|\, \mathbf{b}]$.

Activity. *(5 min)* Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Application Activities - Day 3

Observation. So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space $\mathbb{R}^n$; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

Activity. *(5 min)* We previously checked that $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ does not belong to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$. Does $f(x) = 3x^2 - 2x + 1$ belong to $\operatorname{span}\{x^2 - 3, -x^2 - 3x + 2\}$?

Activity. *(10 min)* Does the matrix $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$ belong to span $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\}$?

Activity. *(10 min)* Does the complex number $2i$ belong to $\operatorname{span}\{-3 + i, 6 - 2i\}$?

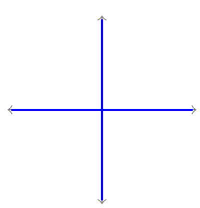
Definition. A subset of a vector space is called a **subspace** if it contains all of its own linear combinations. Every subspace is itself a vector space.

Fact. If S is a subset of a vector space V, then $\operatorname{span} S$ is a subspace of V.

Remark. To prove that a subset is a subspace, you only need to check that $c\mathbf{v} + d\mathbf{w}$ belongs to the subset for any choice of vectors $\mathbf{v}, \mathbf{w}$ from the subset and any real scalars c, d.

Activity. *(5 min)* Prove that $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$ is a subspace of the vector space of all degree-two polynomials by showing that $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$ belongs to P.

Activity. *(10 min)* Consider the subset of $\mathbb{R}^2$ where at least one coordinate on each vector is 0.



Part X: Sketch a picture that demonstrates why this is not a subspace of $\mathbb{R}^2$.

Part X: Define an explicit example of a linear combination $c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ that does not belong to this subset.

Fact. Suppose a subset S of V is isomorphic to another vector space W. Then S is a subspace of V.

Activity. *(10 min)* Show that the upper triangular matrices

$$U^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \text{ are real numbers} \right\}$$

form a subspace of $\mathbb{R}^{2\times 2}$ by finding a Euclidean space isomorphic to $U^{2\times 2}$.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing **(Standard(s) E1, E2, E3, E4)**.
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent **(Standard(s) V5)**.
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace **(Standard(s) V6, V7)**.
- State the definition of a basis, and determine if a set of vectors is a basis **(Standard(s) V8, V9)**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Review the supporting Standards listed above.

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Which of the following is a solution to the system of linear equations

$$\begin{aligned} x + 3y - z &= 2 \\ 2x + 8y + 3z &= -1 \\ -x - y + 9z &= -10 \end{aligned}$$

(a) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

2) Find a basis for the solution set of the following homogeneous system of linear equations

$$\begin{aligned} x + 2y + -z - w &= 0 \\ -2x - 4y + 3z + 5w &= 0 \end{aligned}$$

(a) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$ (d) None of these are a basis.

3) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

4) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

5) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

6) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

7) Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors ...

8) Suppose you know that every vector in $\mathbb{R}^5$ can be written as a linear combination of the vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$. What can you conclude about n?

(a) $n \leq 5$
(b) $n = 5$
(c) $n \geq 5$
(d) n could be any positive integer

9) Suppose you know that every vector in $\mathbb{R}^5$ can be written uniquely as a linear combination of the vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$. What can you conclude about n?

(a) $n \leq 5$
(b) $n = 5$
(c) $n \geq 5$
(d) n could be any positive integer

10) Suppose you know that every vector in $\mathbb{R}^5$ can be written uniquely as a linear combination of the vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$. What can you conclude about the set $\{\vec{v}_1, \dots, \vec{v}_n\}$?

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

Application Activities - Day 1

Definition. A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}$, $\vec{v} \in V$.

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.
V is called the **domain** of T and W is called the **co-domain** of T.

Activity. *(0 min)* Determine if each of the following maps are linear transformations

(a) $T_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $T_1\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \sqrt{a^2 + b^2}$

(b) $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$

(c) $T_3 : \mathcal{P}_d \to \mathcal{P}_{d-1}$ given by $T_3(f(x)) = f'(x)$.

(d) $T_4 : C(\mathbb{R}) \to C(\mathbb{R})$ given by $T_4(f(x)) = f(-x)$

(e) $T_5 : \mathcal{P} \to \mathcal{P}$ given by $T_5(f(x)) = f(x) + x^2$

Activity. *(0 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute each of the following:

(a) $T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$

(b) $T\left(\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}\right)$

(c) $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

(d) $T\left(\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}\right)$

Activity. *(0 min)* Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation. What is the smallest number of vectors needed to determine T? In other words, what is the smallest number n such that there are $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^4$ and given $T(\vec{v}_1), \dots, T(\vec{v}_n)$ you can determine $T(\vec{w})$ for *any* $\vec{w} \in \mathbb{R}^2$?

Observation. Fix an ordered basis for V. Since every vector can be written *uniquely* as a linear combination of basis vectors, a linear transformation $T : V \to W$ corresponds exactly to a choice of where each basis vector goes. For convenience, we can thus encode a linear transformation as a matrix, with one column for the image of each basis vector (in order).

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard ordered basis.

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the ordered basis

$$\left\{\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$$

Activity. *(0 min)* Let $D : \mathcal{P}_3 \to \mathcal{P}_2$ be the derivative map (recall this is a linear transformation). Write the matrix corresponding to D with respect to the ordered basis $\{1, x, x^2, x^3\}$.

Application Activities - Day 2

Definition. Let $T : V \to W$ be a linear transformation.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.
- T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\vec{w} \in W$, there is some $v \in V$ with $T(\vec{v}) = \vec{w}$.

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

Activity. *(0 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

Definition. We also have two important sets called the **kernel** of T and the **image** of T.

$$\begin{aligned} \ker T &= \left\{ \vec{v} \in V \;\middle|\; T(\vec{v}) = 0 \right\} \\ \text{Im } T &= \left\{ \vec{w} \in W \;\middle|\; \text{there is some } v \in V \text{ with } T(\vec{v}) = \vec{w} \right\} \end{aligned}$$

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T.

Activity. *(0 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T.

Activity. *(0 min)*
Part X: Describe surjective linear transformations in terms of the image.
Part X: Describe injective linear transformations in terms of the kernel.

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

1) Write a system of equations whose solution set is the kernel.

2) Compute RREF(A) and solve the system of equations.

3) Compute the kernel of T

4) Find a basis for the kernel of T

Activity. *(0 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

1) Write a system of equations whose solution set is the kernel.

2) Compute RREF(A) and solve the system of equations.

3) Compute the kernel of T

4) Find a basis for the kernel of T

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

1) Find a set of vectors that span the image of T

2) Find a basis for the image of T.

Activity. *(0 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

1) Find a set of vectors that span the image of T

2) Find a basis for the image of T.

Application Activities - Day 3

Activity. *(0 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). You have cards containing a number of statements about T and A. Sort them into groups of equivalent statements, and post them on your board.
(Instructor Note:) Card sort activity for 10-15 minutes: cards contain the following

(a) T is injective

(b) T is not injective

(c) T is surjective

(d) T is not surjective

(e) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has a solution for all $\vec{b} \in \mathbb{R}^m$

(f) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$

(g) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{0} \end{array}\right]$ has a non-trivial solution.

(h) The columns of A span $\mathbb{R}^m$

(i) The columns of A are linearly independent

(j) The columns of A are a basis of $\mathbb{R}^m$

(k) Every column of RREF(A) is a pivot column

(l) RREF(A) has a non-pivot column

(m) RREF(A) has n pivot columns

Activity. *(0 min)* **(Instructor Note:)**Gallery walk Cycle around the room counter-clockwise. If they have two things grouped together that you know are not equivalent, write a reason or counter-example on a sticky note.

Activity. *(0 min)* Come up with as many statements as you can, and add them to the appropriate group.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers
- Solve systems of linear equations **(Standard(s) E.3)**
- Find the matrix corresponding to a linear transformation **(Standard(s) A.1)**
- Determine if a linear transformation is injective and/or surjective **(Standard(s) A.3)**
- Interpret the ideas of injectivity and surjectivity in multiple ways

Readiness Assurance Resources

The following resources will help you prepare for this module.

- `https://www.khanacademy.org/math/algebra2/manipulating-functions/funciton-composition/v/function-composition`

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Let $f(x) = x^2 - 2$ and $g(x) = x^2 + 1$. Compute the composition function $(f \circ g)(x)$.

   (a) $x^2 - 1$
   (b) $x^4 + 2x^2 - 1$
   (c) $x^4 - 4x^2 + 5$
   (d) $x^4 - x^2 - 2$

2) Suppose $f(x)$ and $g(x)$ are real-valued functions satisfying

$$\begin{aligned} f(2) &= 1 \qquad & g(2) &= 3 \\ f(3) &= 4 \qquad & g(3) &= 5 \\ f(4) &= 3 \qquad & g(4) &= 6 \end{aligned}$$

   Compute $(f \circ g)(2)$.

   (a) 2
   (b) 3
   (c) 4
   (d) 5

3) Solve the system of linear equations

$$\begin{aligned} x + 3y &= -2 \\ 2x - 7y &= 9 \end{aligned}$$

   (a) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} -2 \\ 9 \end{bmatrix}$

4) Let a, b, c be fixed real numbers. How many solutions does the system of linear equations below have?

$$\begin{aligned} x + 2y + 3z &= a \\ y - z &= b \\ y + z &= c \end{aligned}$$

   (a) 0 (b) 1 (c) Infinitely many (d) It depends on the values of a, b, and c.

5) What is the matrix corresponding to the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y - z \\ y + 3z \\ x + 7y \end{bmatrix}$?

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 7 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ 1 & 7 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 7 \\ -1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 7 \\ -1 & 3 & 0 \end{bmatrix}$

6) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation with associated matrix $A = \begin{bmatrix} 2 & 3 \\ -1 & -1 \\ 0 & 4 \end{bmatrix}$. Compute $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$.

(a) $\begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$

7) Which of the following is true of the linear transformation $T :$?

(a) T is neither injective nor surjective
(b) T is injective but not surjective
(c) T is surjective but not injective
(d) T is both injective and surjective

8) Which of the following is true of the linear transformation $T :$?

(a) T is neither injective nor surjective
(b) T is injective but not surjective
(c) T is surjective but not injective
(d) T is both injective and surjective

9) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with associated matrix $A \in M_{m,n}(\mathbb{R})$. Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?

(a) T is injective
(b) T has a non-trivial kernel
(c) The columns of A are linearly dependent
(d) RREF(A) has a non-pivot column

10) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with associated matrix $A \in M_{m,n}(\mathbb{R})$. Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?

(a) T is surjective
(b) $\text{Im } T = \mathbb{R}^m$
(c) The columns of A span $\mathbb{R}^m$
(d) RREF(A) has only pivot columns

Application Activities - Day 1

Activity. *(0 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{4,3}$

(b) $M_{4,2}$

(c) $M_{3,2}$

(d) $M_{2,3}$

(e) $M_{2,4}$

(f) $M_{3,4}$

Activity. *(0 min)* Compute $(S \circ T)(\vec{e}_1)$, $(S \circ T)(\vec{e}_2)$, and $(S \circ T)(\vec{e}_3)$.

Activity. *(0 min)* Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity. *(0 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.
What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{2,2}$

(b) $M_{2,3}$

(c) $M_{3,2}$

(d) $M_{3,3}$

Activity. *(0 min)* Compute $(S \circ T)(\vec{e}_1)$ and $(S \circ T)(\vec{e}_2)$

Activity. *(0 min)* Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity. *(0 min)* Let $T : \mathbb{R}^1 \to \mathbb{R}^4$ be given by the matrix $B = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ and $S : \mathbb{R}^4 \to \mathbb{R}^1$ be given by the matrix $A = \begin{bmatrix} 2 & 3 & 2 & 5 \end{bmatrix}$.
What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. *(0 min)* The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{1,1}$

(b) $M_{1,4}$

(c) $M_{4,1}$

(d) $M_{4,4}$

Activity. *(0 min)* Compute $(S \circ T)(\vec{e}_1)$

Activity. *(0 min)* Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity. *(0 min)* Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

1. Compute AX

2. Interpret the system of equations below as a matrix equation

$$\begin{aligned} 3x + y - z &= 5 \\ 2x + 4z &= -7 \\ -x + 3y + 5z &= 2 \end{aligned}$$

Application Activities - Day 2

Activity. *(0 min)* Each row operation can be interpreted as a matrix multiplication. Let $A \in M_{4,4}$

1) Find a matrix S_1 such that S_1A is the result of swapping the second and fourth rows of A.

2) Find a matrix S_2 such that S_2A is the result of adding 5 times the third row of A to the first.

3) Find a matrix S_3 such that S_3A is the result of doubling the fourth row of A.

Activity. *(0 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). Consider the following statements about T

(a) T is injective

(b) T is surjective

(c) T is bijective (i.e. both injective and surjective)

(d) $AX = B$ has a solution for all $B \in M_{m,1}$

(e) $AX = B$ has a unique solution for all $B \in M_{m,1}$

(f) $AX = 0$ has a non-trivial solution.

(g) The columns of A span $\mathbb{R}^m$

(h) The columns of A are linearly independent

(i) The columns of A are a basis of $\mathbb{R}^m$

(j) RREF(A) has n pivot columns

(k) RREF(A) has m pivot columns

Sort these statements into groups of equivalent statements.

Activity. *(0 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). If T is injective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Activity. *(0 min)* If T is surjective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Activity. *(0 min)* If T is bijective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Application Activities - Day 3

Activity. *(0 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with matrix $A \in M_{n,n}$.
If T is a bijection, then $AX = B$ has a unique solution for all $B \in \mathbb{R}^n$. Thus we can define a map $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by defining $T^{-1}(B)$ to be this solution. It follows immediately that $T \circ T^{-1}$ is the identity map. The matrix corresponding to T^{-1} is denoted A^{-1}.

1) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_1)$

2) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_2)$

3) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_3)$

4) Compute A^{-1}

A (square) matrix is called *invertible* if it corresponds to an invertible linear transformation.

1) Find the inverse of the matrix $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$

2) Find the inverse of the matrix $\begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **(Standard(s) A1)**.
- Recall and use the definition of a linear transformation **(Standard(s) A2)**.
- Find all roots of quadratic polynomials (including complex ones), and be able to use the rational root theorem to find all rational roots of a higher degree polynomial.
- Interpret the statement "A is an invertible matrix" in many equivalent ways in different contexts.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Finding the area of a parallelogram: `https://www.khanacademy.org/math/basic-geo/basic-geo-area-and-perime parallelogram-area/a/area-of-parallelogram`
- Factoring quadratics: `https://www.khanacademy.org/math/algebra2/polynomial-functions/factoring-polynom v/factoring-polynomials-1`
- Finding complex roots of quadratics: `https://www.khanacademy.org/math/algebra2/polynomial-functions/quadratic-equations-with-complex-numbers/v/complex-roots-from-the-quadratic-formula`
- Finding all roots of polynomials: `https://www.khanacademy.org/math/algebra2/polynomial-functions/finding-zeros-of-polynomials/v/finding-roots-or-zeros-of-polynomial-1`
- The Rational Root Theorem: `https://artofproblemsolving.com/wiki/index.php?title=Rational_Root_Theorem`

Readiness Assurance Test

Choose the most appropriate response for each question.

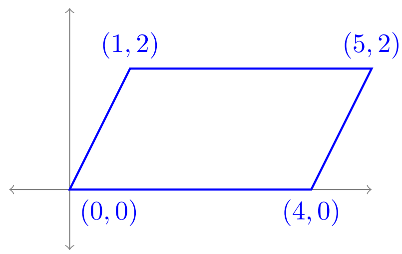
1) Find the area of the parallelogram with vertices $(0, 0)$, $(4, 0)$, $(5, 2)$, and $(1, 2)$.

(a) 5

(b) 6

(c) 7

(d) 8



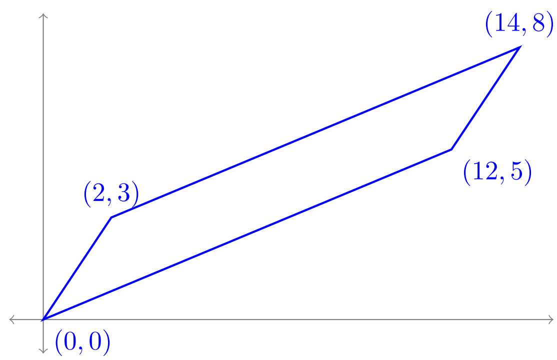
2) Find the area of the parallelogram with vertices $(0, 0)$, $(12, 5)$, $(14, 8)$, and $(2, 3)$.

(a) 13

(b) 26

(c) 39

(d) 52



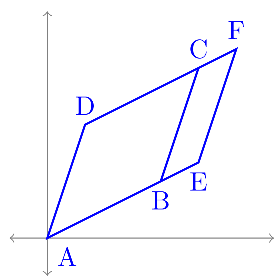
3) The parallelogram ABCD has area 6. If AE is $\frac{3}{2}$ the length of AB, what is the area of the parallelogram AEFD?

(a) 9

(b) 12

(c) 15

(d) 18



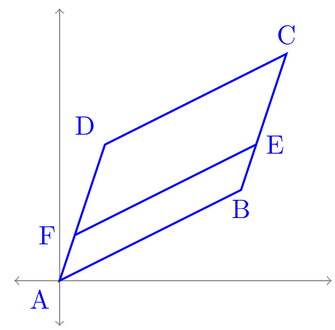
4) The parallelogram ABCD has area 6. If AF is one third as long as AD, what is the area of the parallelogram ABEF?

(a) 1

(b) 2

(c) 3

(d) 4



5) Let $T : \mathbb{R}^2 \to \mathbb{R}$ be a linear transformation. Which of the following is equal to $T\left(\begin{bmatrix} a+b \\ a+b \end{bmatrix}\right)$?

(a) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$

(b) $2T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$

(c) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} b \\ a \end{bmatrix}\right)$

(d) $T\left(\begin{bmatrix} a \\ a \end{bmatrix}\right) + T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} b \\ a \end{bmatrix}\right) + T\left(\begin{bmatrix} b \\ b \end{bmatrix}\right)$

6) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with associated matrix $A \in M_n(\mathbb{R})$. Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?

(a) A is not an invertible matrix

(b) T has a non-trivial kernel

(c) $\det(A) \neq 0$

(d) $A\vec{x} = \vec{b}$ has multiple solutions for all $\vec{b} \in \mathbb{R}^n$.

7) What is the matrix corresponding to the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x+2y-z \\ y+z \\ x+7z \end{bmatrix}$?

(a) $\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 7 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 7 \\ -1 & 0 & 0 \end{bmatrix}$

8) Which of the following conditions imply that the quadratic polynomial $ax^2 + bx + c$ has no real roots?

(a) $a < 0$

(b) $b^2 - 4ac < 0$

(c) $ac - b^2 < 0$

(d) $ab + c^2 < 0$

9) Which of the following is a root of the polynomial $x^2 - 4x + 13$?

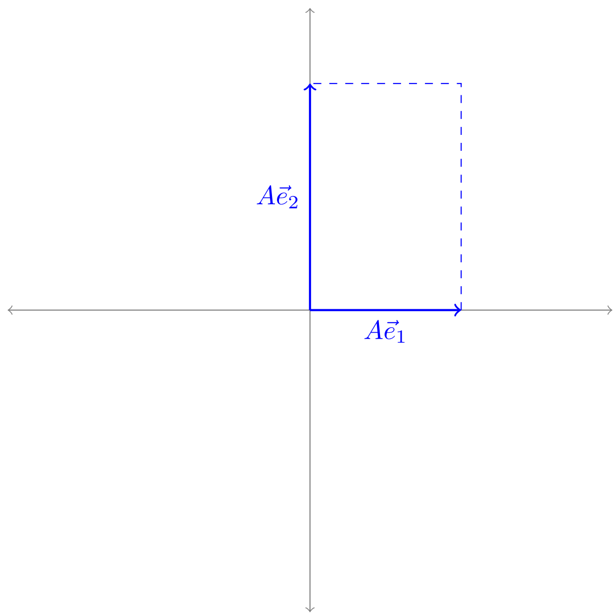
(a) $1 + 2i$ (b) $2 - 3i$ (c) $3 + 4i$ (d) $4 - 5i$

10) How many roots does the polynomial $x^4 + 3x^3 + x^2 - 3x - 2$ have?

(a) 1 (b) 2 (c) 3 (d) 4

Application Activities - Days 1-2

Activity. *(0 min)* Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



We can summarize the transformation of the unit square into this rectangle by measuring the following:

(a) How did the area change?

(b) How was th x-axis stretched?

(c) How was the y-axis stretched?

Activity. *(0 min)* Consider the following linear transformations $A_i : \mathbb{R}^2 \to \mathbb{R}^2$.

- $A_2 = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$
- $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
- $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

For each linear transformation, do the following:

(a) Draw a graph showing the image of the unit square.

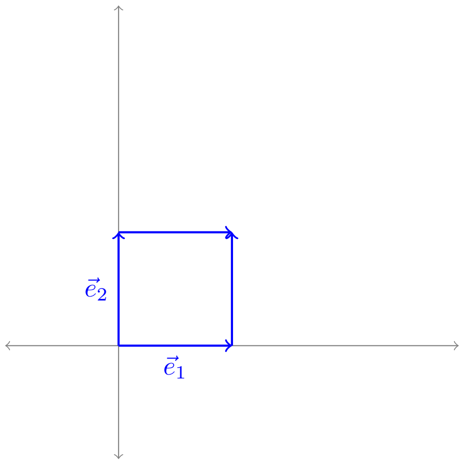
(b) Compute how much the area was stretched out.

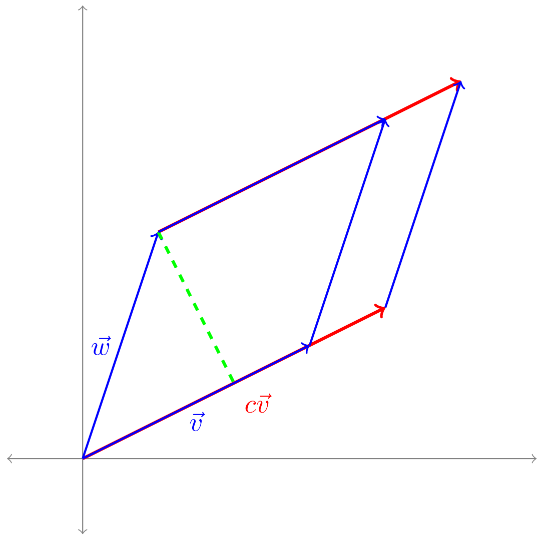
(c) Determine which axes (or lines) were preserved; how were they stretched out?

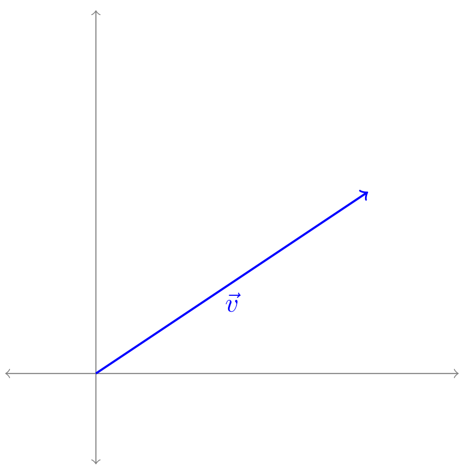
Activity. *(0 min)* Our goal is to define a function $\det : M_n \to \mathbb{R}$ that takes a square matrix (linear transformation $\mathbb{R}^n \to \mathbb{R}^n$) and returns its area stretching factor. This function is called the **determinant**. What properties should this function have?

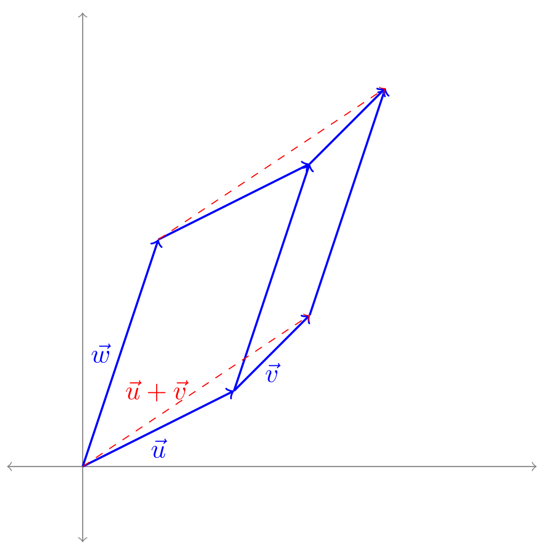
Match the four pictures to the following four expressions

$\det(\vec{e}_1, \vec{e}_2)$ $\det(\vec{v}, \vec{v})$ $\det(c\vec{v}, \vec{w})$ $\det(\vec{u}+\vec{v}, \vec{w})$









Activity. *(0 min)* What can you conclude about each of the following?

1. $\det(\vec{e}_1, \vec{e}_2)$
2. $\det(\vec{v}, \vec{v})$
3. $\det(c\vec{v}, \vec{w})$

4. $\det(\vec{u}+\vec{v},\vec{w})$

Definition. To summarize, we have 3 properties (stated here over $\mathbb{R}^n$)

P1: $\det(\vec{e}_1,\vec{e}_2,\dots,\vec{e}_n)=1$

P2: If $\vec{v}_i=\vec{v}_j$ for some $i\neq j$, then $\det(\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n)=0$.

P3: The determinant is linear in each column.

These three properties uniquely define the **determinant**, as we shall see.

Observation. Note that if $\vec{v},\vec{w}\in\mathbb{R}^2$ and $A=\begin{bmatrix}\vec{v} & \vec{w}\end{bmatrix}$ we will write either $\det(A)$ or $\det(\vec{v},\vec{w})$ as is convenient.

Activity. *(0 min)*
How are $\det(\vec{v},\vec{w})$ and $\det(\vec{w},\vec{v})$ related?

(a) $\det(\vec{v},\vec{w})=\det(\vec{w},\vec{v})$

(b) $\det(\vec{v},\vec{w})=-\det(\vec{w},\vec{v})$

(c) They are unrelated

(d) They are related, but not by either (a) or (b).

Observation. Note that this implies that the determinant is actually a *signed* area (volume)!

Activity. *(0 min)*
How are $\det(\vec{v}+\vec{w},\vec{w})$ and $\det(\vec{v},\vec{w})$ related?

(a) $\det(\vec{v}+\vec{w},\vec{w})=\det(\vec{v},\vec{w})$

(b) $\det(\vec{v}+\vec{w},\vec{w})=-\det(\vec{v},\vec{w})$

(c) They are unrelated

(d) They are related, but not by either (a) or (b).

Observation. Note that we now understand the effect of any column operation on the determinant.

Activity. *(0 min)* How are $\det(A)$ and $\det(A^T)$ related?

(a) $\det(A)=\det(A^T)$

(b) $\det(A)=-\det(A^T)$

(c) $\det(A)=\frac{1}{\det(A^T)}$

(d) They are unrelated

Observation. Thus, row operations behave like column operations. So we can use row reduction to compute determinants.

Activity. *(0 min)* Compute $\det \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$.

Activity. *(0 min)* Compute $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Activity. *(0 min)* Which of the following is the same as $\det \begin{bmatrix} 3 & -2 & 0 \\ 5 & -1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$?

(a) $\det \begin{bmatrix} 3 & -2 \\ 5 & -1 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$

(c) $\det \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}$

(d) None of these

Activity. *(0 min)* Which of the following is the same as $\det \begin{bmatrix} 3 & 0 & 7 \\ 5 & 1 & 2 \\ -2 & 0 & 6 \end{bmatrix}$?

(a) $\det \begin{bmatrix} 3 & 7 \\ 5 & 2 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix}$

(c) $\det \begin{bmatrix} 5 & 2 \\ -2 & 6 \end{bmatrix}$

(d) None of these

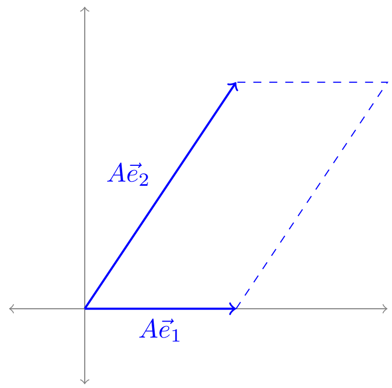
Activity. *(0 min)* Compute $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$

Activity. *(0 min)* Using the fact that $\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}$, compute $\det\begin{bmatrix}1 & 2 & 3\\1 & -2 & 5\\0 & 3 & 3\end{bmatrix}$.

Activity. *(0 min)* Compute $\begin{bmatrix}2 & 3 & 5 & 0\\0 & 1 & -1 & 0\\1 & 2 & 0 & 3\\-1 & -1 & 2 & 2\end{bmatrix}$

Application Activities - Day 3

Activity. *(0 min)* Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



Observe

$$A\vec{e_1} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\vec{e_1}$$

Is there another vector $\vec{v} \in \mathbb{R}^2$ such that $A\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$?

Definition. Let $A \in M_n(\mathbb{R})$. An **eigenvector** is a vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x}$ is parallel to $\vec{x}$; in other words, $A\vec{x} = \lambda\vec{x}$ for some scalar λ, which is called an **eigenvalue**

Observation. Observe that $A\vec{x} = \lambda\vec{x}$ is equivalent to $(A - \lambda I)\vec{x} = 0$.

- To find eigenvalues, we need to find values of λ such that $A - \lambda I$ has a nontrivial kernel; equivalently, $A - \lambda I$ is not invertible, which is equivalent to $\det(A - \lambda I) = 0$. $\det(A - \lambda I)$ is called the **characteristic polynomial**.
- Once an eigenvalue is found, the eigenvectors form a subspace called the **eigenspace**, which is simply the kernel of $A - \lambda I$. Each eigenvalue will have an associated eigenspace.

Activity. *(0 min)* Find the eigenvalues for the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Activity. *(0 min)* Compute the eigenspace associated to the eigenvalue 3.

Activity. *(0 min)* Find all the eigenvalues and associated eigenspaces for the matrix $\begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$.

Application Activities - Day 4

Activity. *(0 min)* If $A \in M_4$, what is the largest number of eigenvalues A can have?

Activity. *(0 min)* 2 is an eigenvalue of each of the matrices $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}$. Compute the eigenspace associated to 2 for both A and B.

Definition.

- The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.
- The **geometric multiplicity** of an eigenvalue is the dimension of the eigenspace.

Activity. *(0 min)* How are the algebraic and geometric multiplicities related?

(a) The algebraic multiplicity is always at least as big as than the geometric multiplicity.

(b) The geometric multiplicity is always at least as big as the algebraic multiplicity.

(c) Sometimes the algebraic multiplicity is larger and sometimes the geometric multiplicity is larger.

Activity. *(0 min)* Find the eigenvalues, along with both their algebraic and geometric multiplicities, for the matrix $\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$

Activity. *(0 min)* Find the eigenvalues of the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Activity. *(0 min)* Describe what this linear transformation is doing geometrically; draw a picture.

Activity. *(0 min)* Fix a real number θ and find the eigenvalues of the matrix $A_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. What are the eigenvalues?

Activity. *(0 min)* D raw pictures and describe the geometric actions of the maps $A_{\frac{\pi}{4}}$, $A_{\frac{\pi}{2}}$, and A_π.

Activity. *(0 min)* For how many values of θ does the rotation matrix A_θ have real eigenvalues?

(a) 0

(b) 1

(c) 2

(d) 3

(e) An infinite number