Application Activities - Module V Part 1 - Class Day 7

Activity 7.1 Consider each of the following vector properties. Label each property with $\mathbb{R}^1$, $\mathbb{R}^2$, and/or $\mathbb{R}^3$ if that property holds for Euclidean vectors/scalars $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of that dimension.

1. **Addition associativity.**
   $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
2. **Addition commutivity.**
   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. **Addition identity.**
   There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
4. **Addition inverse.**
   There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
5. **Addition midpoint uniqueness.**
   There exists a unique $\mathbf{m}$ where the distance from $\mathbf{u}$ to $\mathbf{m}$ equals the distance from $\mathbf{m}$ to $\mathbf{v}$.
6. **Scalar multiplication associativity.**
   $a(b\mathbf{v}) = (ab)\mathbf{v}$.
7. **Scalar multiplication identity.**
   $1\mathbf{v} = \mathbf{v}$.
8. **Scalar multiplication relativity.**
   There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.
9. **Scalar distribution.**
   $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
10. **Vector distribution.**
    $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
11. **Orthogonality.**
    There exists a non-zero vector $\mathbf{n}$ such that $\mathbf{n}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
12. **Bidimensionality.**
    $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ for some value of a, b.

Definition 7.2 A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Addition associativity.**
  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition commutivity.**
  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Addition identity.**
  There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- **Addition inverse.**
  There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.**
  $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **Scalar multiplication identity.**
  $1\mathbf{v} = \mathbf{v}$.
- **Scalar distribution.**
  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Vector distribution.**
  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Definition 7.3 The most important examples of vector spaces are the **Euclidean vector spaces** $\mathbb{R}^n$, but there are other examples as well.

Activity 7.4 Consider the following vector space that models motion along the curve $y = e^x$. Let $V = \{(x, y) : y = e^x\}$, where $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and $c(x, y) = (cx, y^c)$.
Prove that V paired with these operations satisfies all the vector space properties.

- **Addition associativity.**
  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Addition commutivity.**
  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **Addition identity.**
  There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- **Addition inverse.**
  There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- **Scalar multiplication associativity.**
  $a(b\mathbf{v}) = (ab)\mathbf{v}$.
- **Scalar multiplication identity.**
  $1\mathbf{v} = \mathbf{v}$.
- **Scalar distribution.**
  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
- **Vector distribution.**
  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

For example, Addition Commutivity may be proven by showing

$$\mathbf{u} \oplus \mathbf{v} = (x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) = (x_2 + x_1, y_2 y_1) = (x_2, y_2) + (x_1, y_1) = \mathbf{v} \oplus \mathbf{u}$$