

Name:
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**SEMIFINAL**

Math 237 – Linear Algebra

**Version 3**

Fall 2017

Work any problems you wish on the provided answer sheets; take care to label the standard for each response. Show all work and justify all of your answers. Answers without work or sufficient reasoning will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

**E1.** Write an augmented matrix corresponding to the following system of linear equations.

$$x_1 + 4x_3 = 1$$

$$x_2 - x_3 = 7$$

$$x_1 - x_2 + 3x_4 = -1$$

**Solution:**

$$\left[ \begin{array}{cccc|c} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & -1 & 0 & 7 \\ 1 & -1 & 0 & 3 & -1 \end{array} \right]$$

□

**E2.** Put the following matrix in reduced row echelon form.

$$\left[ \begin{array}{cccc} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{array} \right]$$

**Solution:**

$$\begin{aligned} \left[ \begin{array}{cccc} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{array} \right] &\sim \left[ \begin{array}{cccc} 1 & -1 & 0 & 2 \\ -3 & 5 & 2 & 0 \\ 1 & -2 & -1 & -1 \end{array} \right] \sim \left[ \begin{array}{cccc} 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & 6 \\ 0 & -1 & -1 & -3 \end{array} \right] \sim \left[ \begin{array}{cccc} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{array} \right] \\ &\sim \left[ \begin{array}{cccc} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

□

**E3.** Solve the system of linear equations.

$$2x + y - z + w = 5$$

$$3x - y - 2w = 0$$

$$-x + 5z + 3w = -1$$

**Solution:**

$$\text{RREF} \left( \left[ \begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right] \right) = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{12} & 3 \\ 0 & 0 & 1 & \frac{4}{12} & 0 \end{array} \right]$$

So the solutions are

$$\left\{ \left[ \begin{array}{c} 1+a \\ 3-21a \\ -7a \\ 12a \end{array} \right] \mid a \in \mathbb{R} \right\}$$

□

**E4.** Find a basis for the solution set of the system of equations

$$x + 2y + 3z + w = 0$$

$$3x - y + z + w = 0$$

$$2x - 3y - 2z = 0$$

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 1 & -2 & 3 & 1 \\ 3 & -1 & 1 & 1 \\ 2 & -3 & -2 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & \frac{5}{7} & \frac{3}{7} \\ 0 & 1 & \frac{8}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution set is

$$\left\{ \begin{bmatrix} -\frac{5}{7}a - \frac{3}{7}b \\ \frac{8}{7}a - \frac{2}{7}b \\ a \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

So a basis for the solution set is  $\left\{ \begin{bmatrix} -\frac{5}{7} \\ \frac{8}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{7} \\ -\frac{2}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$ , or  $\left\{ \begin{bmatrix} 5 \\ 8 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ -7 \end{bmatrix} \right\}$ .

□

**V1.** Let  $V$  be the set of all pairs of real numbers with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$c \odot (x_1, y_1) = (c^2 x_1, c^3 y_1)$$

(a) Show that scalar multiplication **distributes scalars** over vector addition:

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2).$$

(b) Determine if  $V$  is a vector space or not. Justify your answer.

**Solution:** Let  $(x_1, y_1), (x_2, y_2) \in V$  and let  $c \in \mathbb{R}$ .

$$\begin{aligned} c \odot ((x_1, y_1) \oplus (x_2, y_2)) &= c \odot (x_1 + x_2, y_1 + y_2) \\ &= (c^2(x_1 + x_2), c^3(y_1 + y_2)) \\ &= (c^2 x_1, c^3 y_1) \oplus (c^2 x_2, c^3 y_2) \\ &= c \odot (x_1, y_1) \oplus c \odot (x_2, y_2) \end{aligned}$$

However,  $V$  is not a vector space, as the other distributive law fails:

$$(c + d) \odot (x_1, y_1) = ((c + d)^2 x_1, (c + d)^3 y_1) \neq ((c^2 + d^2) x_1, (c^3 + d^3) y_1) = c \odot (x_1, y_1) \oplus d \odot (x_1, y_1).$$

□

**V2.** Determine if  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ , and  $\begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$ .

**Solution:**

$$\text{RREF} \left( \left[ \begin{array}{ccc|c} 3 & 1 & 5 & 1 \\ 0 & -1 & 1 & 4 \\ -1 & 4 & -6 & 3 \end{array} \right] \right) = \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

So  $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$  is not a linear combination of the three vectors.

□

**V3.** Determine if the vectors  $\begin{bmatrix} 8 \\ 21 \\ -7 \end{bmatrix}$ ,  $\begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ , and  $\begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}$  span  $\mathbb{R}^3$ .

**Solution:**

$$\text{RREF} \left( \left[ \begin{array}{cccc} 8 & -3 & -1 & 4 \\ 21 & -8 & -3 & 11 \\ -7 & 3 & 2 & -5 \end{array} \right] \right) = \left[ \begin{array}{cccc} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Since the rank is less than 3, they do not span  $\mathbb{R}^3$ .

□

**V4.** Determine if  $\left\{ \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$  a subspace of  $\mathbb{R}^4$ .

**Solution:** It is closed under addition and scalar multiplication, so it is a subspace. Alternatively, it is the image of the linear transformation from  $\mathbb{R}^3 \rightarrow \mathbb{R}^4$  given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix}.$$

□

**S1.** Determine if the set of vectors  $\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$  is linearly dependent or linearly independent

**Solution:**

$$\text{RREF} \left( \left[ \begin{array}{ccc} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{array} \right] \right) = \left[ \begin{array}{ccc} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{array} \right]$$

This has a non pivot column, therefore the set is linearly dependent.

□

**S2.** Determine if the set  $\left\{ \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ -1 & 8 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix} \right\}$  is a basis of  $\mathbb{R}^{2 \times 2}$ .

**Solution:**

$$\text{RREF} \left( \left[ \begin{array}{cccc} 3 & 2 & 1 & -1 \\ -1 & 0 & 4 & 3 \\ 2 & 2 & -1 & 0 \\ 3 & 4 & 8 & 4 \end{array} \right] \right) = \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Since the resulting matrix is the identity matrix, it is a basis.

□

**S3.** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\} \right)$ . Find a basis for  $W$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis is  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}$ .

□

**S4.** Let  $W = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix} \right\} \right)$ . Compute the dimension of  $W$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so  $\dim(W) = 3$ .

□

**A1.** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the linear transformation given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = [x_3 + 3x_1].$$

Write the matrix for  $T$  with respect to the standard bases of  $\mathbb{R}^3$  and  $\mathbb{R}$ .

**Solution:**

$$\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$$

□

**A2.** Determine if  $D : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$  given by  $D \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a - 3c$  is a linear transformation or not.

**A3.** Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a)  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  where  $S(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$  and  $S(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ .

(b)  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  where  $T(\vec{e}_1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ ,  $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , and  $T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

**Solution:**

(a)  $\text{RREF} \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ . The map is injective since every column has a pivot, but is not surjective because there is a row without a pivot.

(b)  $\text{RREF} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}$ . The map is not injective since there is a column without a pivot, but it is surjective because every row has a pivot.

□

**A4.** Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be the linear map given by  $T \left( \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \right) = \begin{bmatrix} 8x - 3y - z + 4w \\ y + 3z - 4w \\ -7x + 3y + 2z - 5w \end{bmatrix}$ . Compute a basis for the kernel and a basis for the image of  $T$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 8 & -3 & -1 & 4 \\ 0 & 1 & 3 & -4 \\ -7 & 3 & 2 & -5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus  $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$  is a basis for the image, and  $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$  is a basis for the kernel.

□

**M1.** Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

Exactly one of the six products  $AB$ ,  $AC$ ,  $BA$ ,  $BC$ ,  $CA$ ,  $CB$  can be computed. Determine which one, and compute it.

**Solution:**  $CB$  is the only one that can be computed, and

$$CB = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

□

**M2.** Determine if the matrix  $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$  is invertible.

**Solution:**

$$\text{RREF} \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since it is not equivalent to the identity matrix, it is not invertible.

□

**M3.** Find the inverse of the matrix  $\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}$ .

**Solution:**

$$\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 4 & -3 \\ 8 & -13 & 10 \\ 13 & -24 & 18 \end{bmatrix}$$

□

**G1.** Compute the determinant of the matrix  $\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix}$ .

**Solution:**

$$\begin{aligned} \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} &= 2 \det \begin{bmatrix} 3 & 0 & -1 \\ 1 & 3 & 1 \\ -3 & -2 & -1 \end{bmatrix} - (-1) \det \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 1 & -3 & -2 \end{bmatrix} \\ &= 2 \left( 3 \det \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} + (-1) \det \begin{bmatrix} 1 & 3 \\ -3 & -2 \end{bmatrix} \right) + \left( 1 \det \begin{bmatrix} 1 & 3 \\ -3 & -2 \end{bmatrix} - 3 \det \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} \right) \\ &= 2(3(-1) + (-1)(7)) + ((1)(7) - 3(-3)) \\ &= 2(-10) + 16 \\ &= -4 \end{aligned}$$

□

**G2.** Compute the eigenvalues, along with their algebraic multiplicities, of the matrix  $\begin{bmatrix} 9 & -3 & 2 \\ 19 & -6 & 5 \\ -11 & 4 & -2 \end{bmatrix}$ .

**Solution:** 1 with algebraic multiplicity 2, and -1 with algebraic multiplicity 1.

□

**G3.** Find the eigenspace associated to the eigenvalue 2 in the matrix  $A = \begin{bmatrix} 0 & -2 & -1 & 0 \\ -4 & -2 & -2 & 0 \\ 14 & 12 & 10 & 2 \\ -13 & -10 & -8 & -1 \end{bmatrix}$ .

**Solution:** The eigenspace is spanned by  $\begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ .

□

**G4.** Compute the geometric multiplicity of the eigenvalue 3 in the matrix  $A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ -4 & -1 & -2 & 0 \\ 14 & 12 & 11 & 2 \\ -14 & -10 & -9 & -1 \end{bmatrix}$ .

**Solution:** The eigenspace is spanned by  $\begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ , so the geometric multiplicity is 2.

□

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