Application Activities - Module A Part 2 - Class Day 18

Definition 18.1 Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\mathbf{v}) \neq T(\mathbf{w})$ whenever $\mathbf{v} \neq \mathbf{w}$.

Activity 18.2 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is T injective?

Activity 18.3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Is T injective?

Definition 18.4 Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.

Activity 18.5 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is T surjective?

Activity 18.6 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Is T surjective?

Definition 18.7 Let $T : V \to W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{ \mathbf{v} \in V \;\middle|\; T(\mathbf{v}) = \mathbf{0} \right\}$$

Activity 18.8 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the kernel of T.

Activity 18.9 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the kernel of T.

Activity 18.10 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Write a system of equations whose solution set is the kernel.
Part 2: Use RREF(A) to solve the system of equations and find the kernel of T.
Part 3: Find a basis for the kernel of T.

Definition 18.11 Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{ \mathbf{w} \in W \;\middle|\; \text{there is some } v \in V \text{ with } T(\mathbf{v}) = \mathbf{w} \right\}$$

Activity 18.12 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the image of T.

Activity 18.13 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the image of T.

Activity 18.14 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Find a convenient set of vectors $S \subseteq \mathbb{R}^2$ such that $\operatorname{span} S = \operatorname{Im} T$.
Part 2: Find a convenient basis for the image of T.

Observation 18.15 Let $T : V \to W$ be a linear transformation with corresponding matrix A.

- If A is a matrix corresponding to T, the kernel is the solution set of the homogeneous system with coefficients given by A.
- If A is a matrix corresponding to T, the image is the span of the columns of A.