

Fall 2017

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Module E: Solving Systems of Linear Equations

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At the end of this module, students will be able to...

- **E1: Systems as matrices.** Translate back and forth between a system of linear equations and the corresponding augmented matrix.
- **E2: Row reduction.** Put a matrix in reduced row echelon form
- **E3: Solving Linear Systems.** Solve a system of linear equations.
- **E4: Homogeneous Systems.** Find a basis for the solution set of a homogeneous linear system.

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Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.

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The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-systems-topic/cc-8th-systems-graphically/a/systems-of-equations-with-graphing>
- <https://www.khanacademy.org/math/algebra/systems-of-linear-equations/solving-systems-of-equations-with-substitution/v/practice-using-substitution-for-systems>

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Application Activities - Module E Part 1 - Class Day 3

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Definition 3.1

A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

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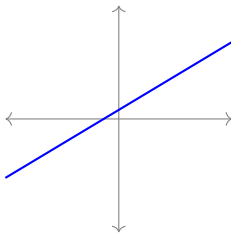
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Observation 3.2

The linear equation $3x - 5y = -2$ may be graphed as a line in the xy plane.



The linear equation $x + 2y - z = 4$ may be graphed as a plane in xyz space.

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Remark 3.3

In previous classes you likely assumed $x = x_1$, $y = x_2$, and $z = x_3$. However, since this course often deals with equations of four or more variables, we will almost always write our variables as x_i .

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Definition 3.4

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

A solution

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \dots + a_{in}s_n = b_i$$

for $1 \leq i \leq m$ (that is, the solution satisfies all equations in the system).

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Remark 3.5

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Verbose standard form:

$$\begin{aligned}1x_1 + 0x_2 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\0x_1 - 1x_2 + 1x_3 &= -2\end{aligned}$$

Concise standard form:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

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Definition 3.6

A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

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Fact 3.7

All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

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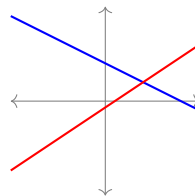
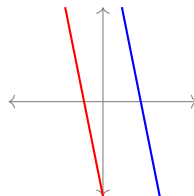
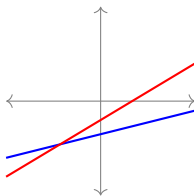
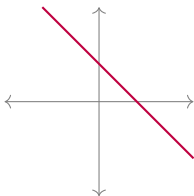
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Activity 3.8

Consider the following graphs representing linear systems of two variables. Label each graph with **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.



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Activity 3.9

All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system.

$$-x_1 + 2x_2 = 5$$

$$2x_1 - 4x_2 = 6$$

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Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

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Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

Part 1: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$, $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$, $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

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Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

Part 1: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$, $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$, $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a . Use this to describe *all* solutions (the **solution set**) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$ for the linear system in terms of a .

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Activity 3.11

Consider the following linear system.

$$x_1 + 2x_2 - x_4 = 3$$

$$x_3 + 4x_4 = -2$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix} + a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3 .

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Observation 3.12

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.

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Remark 3.13

The only important information in a linear system are its coefficients and constants.

Original linear system:	Verbose standard form:	Coefficients/constants:
$x_1 + 3x_3 = 3$	$1x_1 + 0x_2 + 3x_3 = 3$	$1 \quad 0 \quad 3 \mid 3$
$3x_1 - 2x_2 + 4x_3 = 0$	$3x_1 - 2x_2 + 4x_3 = 0$	$3 \quad -2 \quad 4 \mid 0$
$-x_2 + x_3 = -2$	$0x_1 - 1x_2 + 1x_3 = -2$	$0 \quad -1 \quad 1 \mid -2$

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Definition 3.14

A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

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Definition 3.15

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution: $(x_1, x_2) = (1, 1)$.

$$3x_1 - 2x_2 = 1$$

$$x_1 + 4x_2 = 5$$

$$3x_1 - 2x_2 = 1$$

$$4x_1 + 2x_2 = 6$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array} \right]$$

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Activity 3.16

Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

- a) Swap two rows.
- b) Swap two columns.
- c) Add a constant to every term in a row.
- d) Multiply a row by a nonzero constant.
- e) Add a constant multiple of one row to another row.
- f) Replace a column with zeros.

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Application Activities - Module E Part 2 - Class Day 4

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Definition 4.1

The following **row operations** produce equivalent augmented matrices:

- 1 Swap two rows.
- 2 Multiply a row by a nonzero constant.
- 3 Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

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Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$x_1 - x_2 + 5x_3 = 1$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$x_2 - 2x_3 = 3$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_3 = 2$$

Part 1: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

- 1 Swap R_1 (first row) and R_2 (second row).
- 2 Multiply R_2 by $\frac{1}{2}$.

- 3 Add R_1 to R_3 .
- 4 Add $-3R_1$ to R_2 .
- 5 Add $-2R_2$ to R_3 .
- 6 Multiply R_3 by $\frac{1}{3}$.

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Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$x_1 - x_2 + 5x_3 = 1$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$x_2 - 2x_3 = 3$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_3 = 2$$

Part 1: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

① Swap R_1 (first row) and R_2 (second row).

② Multiply R_2 by $\frac{1}{2}$.

③ Add R_1 to R_3 .

④ Add $-3R_1$ to R_2 .

⑤ Add $-2R_2$ to R_3 .

⑥ Multiply R_3 by $\frac{1}{3}$.

Part 2: Which linear system would you rather solve?

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Definition 4.3

The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix. Examples:

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

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Activity 4.4

Find your own sequence of row operations to manipulate the matrix

$$\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array} \right]$$

into row echelon form. (Note that row echelon form is not unique.)

The most efficient way to do this is by circling **pivot positions** in your matrix:

- 1 Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
- 2 Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
- 3 Repeat these two steps as often as possible.

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Activity 4.5

Solve this simplified linear system:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

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Observation 4.6

The consise standard form of the solution to this linear system corresponds to a simplified row echelon form matrix:

$$\begin{array}{rcl} x_1 & = & -2 \\ x_2 & = & 7 \\ x_3 & = & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

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Definition 4.7

A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0. Examples:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

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Activity 4.8

Show that the following two linear systems:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

$$x_1 = -2$$

$$x_2 = 7$$

$$x_3 = 2$$

are equivalent by converting the first system to an augmented matrix, and then zeroing out all terms above pivot positions (the leading terms).

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Remark 4.9

We may verify that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$ is a solution to the original linear system

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

by plugging the solution into each equation.

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Fact 4.10

Every augmented matrix A reduces to a unique reduced row echelon form matrix. This matrix is denoted as $\text{RREF}(A)$.

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Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

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Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

Part 1: Find $\text{RREF}(A)$.

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Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

Part 1: Find $\text{RREF}(A)$.

Part 2: How many solutions does the corresponding linear system have?

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Definition 5.1

An algorithm that reduces A to $\text{RREF}(A)$ is called **Gauss-Jordan elimination**. For example:

- 1 Circle the cell that (a) is in the top-most row without a pivot position and (b) is in the left-most column with a nonzero term either in that position or below it. This position (not the number inside) is called a **pivot**.
- 2 Change the pivot's value to 1 by using row operations involving only the pivot row and rows below it.
- 3 Add or subtract multiples of the pivot row to zero out above and below the pivot.
- 4 Return to Step 1 and repeat as needed until the matrix is in row reduced echelon form.

Observation 5.2

Here is an example of applying Gauss-Jordan elimination to a matrix:

$$\begin{aligned}
& \left[\begin{array}{cccc|c} \textcircled{2} & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array} \right] \sim \left[\begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ -1 & 1 & 3 & -1 & -3 \\ 2 & -2 & -6 & 1 & 3 \end{array} \right] \sim \left[\begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ 0 & \textcircled{-1} & 2 & 0 & -1 \\ 0 & 2 & -4 & -1 & -1 \end{array} \right] \\
& \sim \left[\begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 2 & -4 & -1 & -1 \end{array} \right] \sim \left[\begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 1 & 4 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{-1} & -3 \end{array} \right] \\
& \sim \left[\begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 1 & 4 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right] \sim \left[\begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 0 & 1 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right]
\end{aligned}$$

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Definition 5.3

The columns of $\text{RREF}(A)$ without a leading term represent **free variables** of the linear system modeled by A that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by A .

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Example 5.4

Here, x_3 is the free variable set equal to a since its column lacks a pivot, and the other bounded variables are put in terms of a .

$$2x_1 - 2x_2 - 6x_3 + x_4 = 3$$

$$-x_1 + x_2 + 3x_3 - x_4 = -3$$

$$x_1 - 2x_2 - x_3 + x_4 = 1$$

$$x_1 - 5x_3 = 1$$

$$x_2 - 2x_3 = 1$$

$$x_4 = 3$$

$$x_1 = 1 + 5a$$

$$x_2 = 1 + 2a$$

$$x_3 = a$$

$$x_4 = 3$$

 \Rightarrow

$$\begin{array}{c} \Downarrow \\ \left[\begin{array}{cccc|c} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array} \right] \sim \begin{array}{c} \Uparrow \\ \left[\begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 0 & 1 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right] \end{array}$$

So the solution set is $\left\{ \begin{bmatrix} 1 + 5a \\ 1 + 2a \\ a \\ 3 \end{bmatrix} \mid a \in \mathbb{R} \right\}.$

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Activity 5.5

Solve the system of linear equations, circling the pivot positions in your augmented matrices as you work.

$$-x_1 + x_2 - 3x_3 + 2x_4 = 0$$

$$2x_1 - x_2 + 5x_3 + 3x_4 = -11$$

$$3x_1 + 2x_2 + 4x_3 + x_4 = 1$$

$$x_2 - x_3 + x_4 = 1$$

Remember to find the solution set of the system by setting the free variable (the column without a pivot position) equal to a , and then express each of the other bounded variables equal to an expression in terms of a .

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Remark 5.6

From now on, unless specified, there's no need to show your work in finding $\text{RREF}(A)$, so you may use a calculator to speed up your work.

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Activity 5.7

Solve the linear system

$$2x_1 - 3x_2 = 17$$

$$x_1 + 2x_2 = -2$$

$$-x_1 - x_2 = 1$$

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Activity 5.8

Show that all linear systems of the form

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & 0 \end{array}$$

are consistent by finding a quickly verifiable solution.

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Definition 5.9

A **homogeneous system** is a linear system satisfying $b_i = 0$, that is, it is a linear system of the form

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & 0 \end{array}$$

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Fact 5.10

Because the zero vector is always a solution, the solution set to any homogeneous system with infinitely-many solutions may be generated by multiplying the parameters representing the free variables by a minimal set of Euclidean vectors, and adding these up. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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Definition 5.11

A minimal set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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Activity 5.12

Find a basis for the solution set of the following homogeneous linear system.

$$x_1 + 2x_2 - x_4 = 0$$

$$x_3 + 4x_4 = 0$$

$$2x_1 + 4x_2 + x_3 + 2x_4 = 0$$

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Module V: Vector Spaces

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At the end of this module, students will be able to...

- **V1: Vector Spaces.** Determine if a set with given operations forms a vector space.
- **V2: Linear Combinations.** Determine if a vector can be written as a linear combination of a given set of vectors.
- **V3: Spanning Sets.** Determine if a set of vectors spans a vector space.
- **V4: Subspaces.** Determine if a subset of a vector space is a subset or not.

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Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems
(Standard(s) E1,E2,E3).

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The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/precalculus/vectors-precalc/vector-addition-subtraction/v/adding-and-subtracting-vectors>
- <https://www.khanacademy.org/math/precalculus/vectors-precalc/combined-vector-operations/v/combined-vector-operations-example>
- <https://www.khanacademy.org/math/precalculus/imaginary-and-complex-numbers/v/adding-and-subtracting-complex-numbers/v/adding-complex-numbers>
- <https://www.khanacademy.org/math/algebra/introduction-to-polynomial-expressions/v/adding-and-subtracting-polynomials/v/adding-and-subtracting-polynomials-1>

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Activity 7.1

Consider each of the following vector properties. Label each property with \mathbb{R}^1 , \mathbb{R}^2 , and/or \mathbb{R}^3 if that property holds for Euclidean vectors/scalars \mathbf{u} , \mathbf{v} , \mathbf{w} of that dimension.

1 Addition associativity.

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

2 Addition commutivity.

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

3 Addition identity.

There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

4 Addition inverse.

There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

5 Addition midpoint uniqueness.

There exists a unique \mathbf{m} where the
distance from \mathbf{u} to \mathbf{m} equals the
distance from \mathbf{m} to \mathbf{v} .

6 Scalar multiplication associativity.

$$a(b\mathbf{v}) = (ab)\mathbf{v}.$$

7 Scalar multiplication identity.

$$1\mathbf{v} = \mathbf{v}.$$

8 Scalar multiplication relativity.

There exists some scalar c where either
 $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.

9 Scalar distribution.

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

10 Vector distribution.

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

11 Orthogonality.

There exists a non-zero vector \mathbf{n} such
that \mathbf{n} is orthogonal to both \mathbf{u} and \mathbf{v} .

12 Bidimensionality.

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} \text{ for some value of } a, b.$$

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Definition 7.2

A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V , and let a, b be scalar numbers.

- **Addition associativity.**
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition commutativity.**
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Addition identity.**
There exists some $\mathbf{0}$ where
 $\mathbf{v} + \mathbf{0} = \mathbf{v}.$
- **Addition inverse.**
There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$
- **Scalar multiplication associativity.**
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **Scalar multiplication identity.**
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar distribution.**
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Vector distribution.**
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

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Definition 7.3

The most important examples of vector spaces are the **Euclidean vector spaces** \mathbb{R}^n , but there are other examples as well.

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Activity 7.4

Consider the following set that models motion along the curve $y = e^x$. Let

$V = \{(x, y) : y = e^x\}$. Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$.

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Activity 7.4

Consider the following set that models motion along the curve $y = e^x$. Let

$V = \{(x, y) : y = e^x\}$. Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$.

Part 1: Which of the vector space properties are satisfied by V paired with these operations?

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some $\mathbf{0}$ where

$$\mathbf{v} \oplus \mathbf{0} = \mathbf{v}.$$

- **Addition inverse.**

There exists some $-\mathbf{v}$ where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

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Activity 7.4

Consider the following set that models motion along the curve $y = e^x$. Let

$V = \{(x, y) : y = e^x\}$. Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$.

Part 1: Which of the vector space properties are satisfied by V paired with these operations?

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some $\mathbf{0}$ where

$$\mathbf{v} \oplus \mathbf{0} = \mathbf{v}.$$

- **Addition inverse.**

There exists some $-\mathbf{v}$ where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

Part 2: Is V a vector space?

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Application Activities - Module V Part 2 - Class Day 8

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Remark 8.1

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- \mathbb{R}^n : Euclidean vectors with n components.
- \mathbb{R}^∞ : Sequences of real numbers (v_1, v_2, \dots) .
- $\mathbb{R}^{m \times n}$: Matrices of real numbers with m rows and n columns.
- \mathbb{C} : Complex numbers.
- \mathcal{P}^n : Polynomials of degree n or less.
- \mathcal{P} : Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

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Activity 8.2

Let $V = \{(a, b) : a, b \text{ are real numbers}\}$, where

$(a_1, b_1) \oplus (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$ and $c \odot (a, b) = (a^c, b + c)$.

Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some $\mathbf{0}$ where

$$\mathbf{v} \oplus \mathbf{0} = \mathbf{v}.$$

- **Addition inverse.**

There exists some $-\mathbf{v}$ where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

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Definition 8.3

A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$ for any choice of scalar multiples c_1, c_2, \dots, c_m .

For example, we say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

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Definition 8.4

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

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Activity 8.5
Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

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Activity 8.5

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Part 1: Sketch $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

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Module G

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Activity 8.5

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Part 1: Sketch $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part 2: Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ in the xy plane.

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Activity 8.6

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

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Module G

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Activity 8.6

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Part 1: Sketch the following linear combinations in the xy plane: $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix},$

$$0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

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Activity 8.6

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Part 1: Sketch the following linear combinations in the xy plane: $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix},$

$$0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Part 2: Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ in the xy plane.

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Activity 8.7

Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ in the xy plane.

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

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Module G

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (Remember, you should use a calculator to help find RREF.)

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (Remember, you should use a calculator to help find RREF.)

Part 3: Given this solution, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

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Application Activities - Module V Part 3 - Class Day 9

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Fact 9.1

A vector \mathbf{b} belongs to $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$ is consistent.

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Remark 9.2

To determine if \mathbf{b} belongs to $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, find $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$.

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Activity 9.3

Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

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Activity 9.4

Determine if $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

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Observation 9.5

So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space \mathbb{R}^n ; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

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Activity 9.6

We previously checked that $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ does not belong to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$.

Does $f(x) = 3x^2 - 2x + 1$ belong to $\text{span}\{x^2 - 3, -x^2 - 3x + 2\}$?

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Activity 9.7

Does the matrix $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$ belong to $\text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\}$?

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Activity 9.8

Does the complex number $2i$ belong to $\text{span}\{-3 + i, 6 - 2i\}$?

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Activity 9.9

How many vectors are required to span \mathbb{R}^2 ? Sketch a drawing in the xy plane to support your answer.

(a) 1

(b) 2

(c) 3

(d) 4

(e) Infinitely Many

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Activity 9.10

How many vectors are required to span \mathbb{R}^3 ?

(a) 1

(b) 2

(c) 3

(d) 4

(e) Infinitely Many

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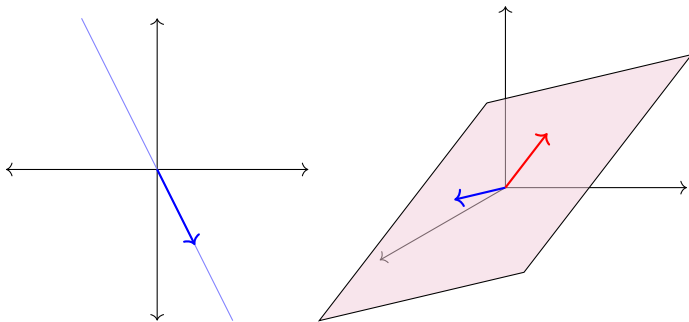
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Fact 10.1At least n vectors are required to span \mathbb{R}^n .

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Activity 10.2

Choose a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in \mathbb{R}^3 that is not in $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by ensuring

$$\left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]. \text{ (Why does this work?)}$$

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Fact 10.3

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ fails to span all of \mathbb{R}^n exactly when $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_m]$ has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & \big| & a \\ -1 & 0 & \big| & b \\ 0 & 1 & \big| & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \big| & 0 \\ 0 & 1 & \big| & 0 \\ 0 & 0 & \big| & 1 \end{bmatrix}$$

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Module E

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Activity 10.4

Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does

$\mathbb{R}^4 = \text{span } S$?

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Activity 10.5

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2\}$$

Does $\mathcal{P}^3 = \text{span } S$?

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Definition 10.6

A subset of a vector space is called a **subspace** if it is itself a vector space.

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Fact 10.7

If S is a subset of a vector space V , then $\text{span } S$ is a subspace of V .

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Remark 10.8

To prove that a subset is a subspace, you need only verify that $c\mathbf{v} + d\mathbf{w}$ belongs to the subset for any choice of vectors \mathbf{v}, \mathbf{w} from the subset and any real scalars c, d .

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Activity 10.9

Prove that $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$ is a subspace of the vector space of all degree-two polynomials by showing that $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$ belongs to P .

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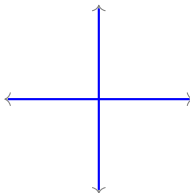
Part 3 (Day 27)

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Activity 10.10

Consider the subset of \mathbb{R}^2 where at least one coordinate of each vector is 0.



Find a linear combination $c\mathbf{v} + d\mathbf{w}$ that does not belong to this subset.

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Fact 10.11

Suppose a subset S of V is isomorphic to another vector space W . Then S is a subspace of V .

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Activity 10.12

Show that the set of 2×2 matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of $\mathbb{R}^{2 \times 2}$ by identifying a Euclidean space isomorphic to S .

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Module S: Structure of vector spaces

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At the end of this module, students will be able to...

- **S1. Linear independence** Determine if a set of Euclidean vectors is linearly dependent or independent.
- **S2. Basis verification** Determine if a set of vectors is a basis of a vector space
- **S3. Basis construction** Construct a basis for the subspace spanned by a given set of vectors.
- **S4. Dimension** I can compute the dimension of a vector space.

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Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.
- Apply linear combinations and spanning sets **(Standard(s) V2,V3)**.

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The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/prec calculus/vectors-prec calc/vector-addition-subtraction/v/adding-and-subtracting-vectors>
- <https://www.khanacademy.org/math/prec calculus/vectors-prec calc/combined-vector-operations/v/combined-vector-operations-example>

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Activity 12.1

In the previous module, we considered

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

and showed that $\text{span } S \neq \mathbb{R}^4$. Find two vectors from this set that are linear combinations of the other three vectors.

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Definition 12.2

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

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Activity 12.3

Suppose $3\mathbf{v}_1 - 5\mathbf{v}_2 = \mathbf{v}_3$, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Is the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ consistent with one solution, consistent with infinitely many solutions, or inconsistent?

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Fact 12.4

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$ is consistent with infinitely many solutions.

Activity 12.5

Find

$$\text{RREF} \left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array} \right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

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Fact 12.6

A set of Euclidean vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if
RREF $\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ has a column without a pivot position.

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Activity 12.7

Is the set of Euclidean vectors $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$ linearly dependent or linearly independent?

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Activity 12.8

Is the set of polynomials $\{x^3 + 1, x^2 + 2, 4 - 7x, 2x^3 + x\}$ linearly dependent or linearly independent?

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Activity 13.1

Last time we saw that $\{x^3 + 1, x^2 + 2, 4 - 7x, 2x^3 + x\}$ is linearly independent. Show that it spans \mathcal{P}^3 .

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Definition 13.2

A **basis** is a linearly independent set that spans a vector space.

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Observation 13.3

A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

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Activity 13.4

Which of the following sets are bases for \mathbb{R}^4 ?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$
$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$
$$\left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

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Activity 13.5

If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for \mathbb{R}^4 , that means $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$ doesn't have a column without a pivot position, and doesn't have a row of zeros. What is $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$?

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Fact 13.6

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for \mathbb{R}^n if and only if $m = n$ and

$$\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, a basis for \mathbb{R}^n must have exactly n vectors and its square matrix must row-reduce to the **identity matrix** containing all zeros except for a downward diagonal of ones.

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Activity 13.7

Consider the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

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Activity 13.7

Consider the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

Part 1: Use RREF $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$ to identify which vector may be removed to make the set linearly independent.

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Activity 13.7

Consider the set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

Part 1: Use RREF $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$ to identify which vector may be removed to make the set linearly independent.

Part 2: Find a basis for $\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

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Fact 14.1

To compute a basis for the subspace $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, simply remove the vectors corresponding to the non-pivot columns of $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$.

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Activity 14.2

Find all subsets of $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$ that are a basis for $\text{span } S$ by changing the order of the vectors in S .

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Activity 14.3

Assume $\mathbf{w}_1 \neq \mathbf{w}_2$ are distinct vectors in V , which has a basis containing a single vector: $\{\mathbf{v}\}$. Could $\{\mathbf{w}_1, \mathbf{w}_2\}$ be a basis?

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Fact 14.4
All bases for a vector space are the same size.

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Definition 14.5

The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

Activity 14.6Find the dimension of each subspace of \mathbb{R}^4 .

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$$\text{span} \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

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Activity 14.7

What is the dimension of the vector space of 7th-degree (or less) polynomials \mathcal{P}^7 ?

a) 6

b) 7

c) 8

d) infinite

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Activity 14.8

What is the dimension of the vector space of all polynomials \mathcal{P} ?

a) 6

b) 7

c) 8

d) infinite

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Observation 14.9

Several interesting vector spaces are infinite-dimensional:

- The space of polynomials \mathcal{P} (consider the set $\{1, x, x^2, x^3, \dots\}$).
- The space of continuous functions $C(\mathbb{R})$ (which contains all polynomials, in addition to other functions like $e^x = 1 + x + x^2/2 + x^3/3 + \dots$).
- The space of real number sequences \mathbb{R}^∞ (consider the set $\{(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, \dots), \dots\}$).

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Fact 14.10

Every vector space with finite dimension, that is, every vector space with a basis of the form $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is isomorphic to a Euclidean space \mathbb{R}^n :

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

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At the end of this module, students will be able to...

- **A1. Linear maps as matrices** I can write the standard matrix corresponding to a linear transformation between Euclidean spaces.
- **A2. Linear map verification** I can determine if a map between vector spaces is linear or not.
- **A3. Injectivity and Surjectivity** I can determine if a given linear map is injective and/or surjective
- **A4. Kernel and Image** I can compute the kernel and image of a linear map, including finding bases.

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Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing (**Standard(s) E1, E2, E3, E4**).
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace (**Standard(s) V3**).
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent (**Standard(s) S1**).
- State the definition of a basis, and determine if a set of vectors is a basis (**Standard(s) S2**).

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The following resources will help you prepare for this module.

- Review the supporting Standards listed above.

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Definition 17.1

A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \rightarrow W$ is called a linear transformation if

- ① $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$
- ② $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.

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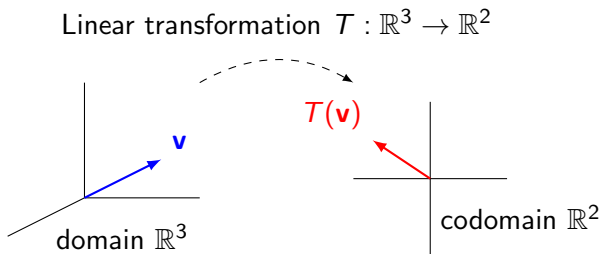
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Definition 17.2

Given a linear transformation $T : V \rightarrow W$, V is called the **domain** of T and W is called the **co-domain** of T .



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Example 17.3

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$$

Example 17.3

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = T \left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \right) = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

$$T \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right) + T \left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 - z_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - z_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

Example 17.3

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = T \left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \right) = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

$$T \left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \right) + T \left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 - z_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 - z_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2) - (z_1 + z_2) \\ (y_1 + y_2) \end{bmatrix}$$

And also...

$$T \left(c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T \left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \right) = \begin{bmatrix} cx - cz \\ cy \end{bmatrix} \quad \text{and} \quad cT \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = c \begin{bmatrix} x - z \\ y \end{bmatrix} = \begin{bmatrix} cx - cz \\ cy \end{bmatrix}$$

Therefore T is a linear transformation.

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Activity 17.4

Determine if each of the following maps are linear transformations

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Activity 17.4

Determine if each of the following maps are linear transformations

Part 1: $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $T_1 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \sqrt{x^2 + y^2}$.

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Activity 17.4

Determine if each of the following maps are linear transformations

Part 1: $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $T_1 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \sqrt{x^2 + y^2}$.

Part 2: $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_2 \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$

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Activity 17.4

Determine if each of the following maps are linear transformations

Part 1: $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $T_1 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \sqrt{x^2 + y^2}$.

Part 2: $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_2 \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$

Part 3: $T_3 : \mathcal{P}^d \rightarrow \mathcal{P}^{d-1}$ given by $T_3(f(x)) = f'(x)$.

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Activity 17.4

Determine if each of the following maps are linear transformations

Part 1: $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $T_1 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \sqrt{x^2 + y^2}$.

Part 2: $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $T_2 \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$

Part 3: $T_3 : \mathcal{P}^d \rightarrow \mathcal{P}^{d-1}$ given by $T_3(f(x)) = f'(x)$.

Part 4: $T_4 : \mathcal{P} \rightarrow \mathcal{P}$ given by $T_4(f(x)) = f(x) + x^2$

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Activity 17.5

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear transformation, and you know $T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and $T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute $T \left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right)$.

(a) $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(d) $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

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Activity 17.6

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear transformation, and you know $T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and $T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute $T \left(\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right)$.

(a) $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(d) $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

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Activity 17.7

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear transformation, and you know $T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and $T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute $T \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$.

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

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Activity 17.8

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear transformation, and you know $T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and $T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute $T \left(\begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} \right)$.

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

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Activity 17.9

Suppose $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is a linear transformation. How many facts of the form $T(\mathbf{v}_i) = \mathbf{w}_i$ do you need to know in order to be able to compute $T(\mathbf{v})$ for *any* $\mathbf{v} \in \mathbb{R}^4$?

(a) 2

(b) 3

(c) 4

(d) 5

(e) You need infinitely many

(In this situation, we say that the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ **determine** T .)

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Fact 17.10

Consider any basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for V . Since every vector can be written *uniquely* as a linear combination of basis vectors, every linear transformation $T : V \rightarrow W$ is determined by those basis vectors.

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = x_1T(\mathbf{b}_1) + \cdots + x_nT(\mathbf{b}_n)$$

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Definition 17.11

The **standard basis** of \mathbb{R}^n is the (ordered) basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \dots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Since linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is determined by the values of each $T(\mathbf{e}_i)$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\mathbf{e}_1) \ \dots \ T(\mathbf{e}_n)]$.

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Example 17.12

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation determined by the following values for T applied to the standard basis of \mathbb{R}^3 .

$$T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix}.$$

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Activity 17.13

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard basis.

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Activity 17.14

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \end{bmatrix}.$$

Compute $T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$.

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Activity 17.15

Let $D : \mathcal{P}^3 \rightarrow \mathcal{P}^2$ be the derivative map $D(f(x)) = f'(x)$. (Earlier we showed this is a linear transformation.)

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Activity 17.15

Let $D : \mathcal{P}^3 \rightarrow \mathcal{P}^2$ be the derivative map $D(f(x)) = f'(x)$. (Earlier we showed this is a linear transformation.)

Part 1: Write down an equivalent linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ by converting $\{1, x, x^2, x^3\}$ and $\{D(1), D(x), D(x^2), D(x^3)\}$ into appropriate vectors in \mathbb{R}^4 and \mathbb{R}^3 .

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Activity 17.15

Let $D : \mathcal{P}^3 \rightarrow \mathcal{P}^2$ be the derivative map $D(f(x)) = f'(x)$. (Earlier we showed this is a linear transformation.)

Part 1: Write down an equivalent linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ by converting $\{1, x, x^2, x^3\}$ and $\{D(1), D(x), D(x^2), D(x^3)\}$ into appropriate vectors in \mathbb{R}^4 and \mathbb{R}^3 .

Part 2: Write the standard matrix corresponding to T .

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Definition 18.1

Let $T : V \rightarrow W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\mathbf{v}) \neq T(\mathbf{w})$ whenever $\mathbf{v} \neq \mathbf{w}$.

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Activity 18.2

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is T injective?

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Activity 18.3

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Is T injective?

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Definition 18.4

Let $T : V \rightarrow W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V . More precisely, for every $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.

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Activity 18.5

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is T surjective?

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Activity 18.6

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Is T surjective?

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Definition 18.7

Let $T : V \rightarrow W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

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Activity 18.8

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the kernel of T .

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Activity 18.9

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the kernel of T .

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Activity 18.10

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

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Activity 18.10

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Write a system of equations whose solution set is the kernel.

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Activity 18.10

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Write a system of equations whose solution set is the kernel.

Part 2: Use $\text{RREF}(A)$ to solve the system of equations and find the kernel of T .

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Activity 18.10

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Write a system of equations whose solution set is the kernel.

Part 2: Use RREF(A) to solve the system of equations and find the kernel of T .

Part 3: Find a basis for the kernel of T .

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Definition 18.11

Let $T : V \rightarrow W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\text{Im } T = \{ \mathbf{w} \in W \mid \text{there is some } v \in V \text{ with } T(v) = \mathbf{w} \}$$

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Activity 18.12

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the image of T .

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Activity 18.13

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the image of T .

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Activity 18.14

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

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Activity 18.14

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Find a convenient set of vectors $S \subseteq \mathbb{R}^2$ such that $\text{span } S = \text{Im } T$.

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Activity 18.14

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Find a convenient set of vectors $S \subseteq \mathbb{R}^2$ such that $\text{span } S = \text{Im } T$.

Part 2: Find a convenient basis for the image of T .

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Observation 18.15

Let $T : V \rightarrow W$ be a linear transformation with corresponding matrix A .

- If A is a matrix corresponding to T , the kernel is the solution set of the homogeneous system with coefficients given by A .
- If A is a matrix corresponding to T , the image is the span of the columns of A .

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Observation 19.1

Let $T : V \rightarrow W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V .
- The **kernel** of T is the set of all things that are mapped to **0**. It is a subspace of V .
- The **image** of T is the set of all things in W that are mapped to by something in V . It is a subspace of W .

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Activity 19.2

Let $T : V \rightarrow W$ be a linear transformation where $\ker T = \{\mathbf{0}\}$. Can you answer either of the following questions about T ?

(a) Is T injective?

(b) Is T surjective?

(Hint: If $T(\mathbf{v}) = T(\mathbf{w})$, then what is $T(\mathbf{v} - \mathbf{w})$?)

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Fact 19.3

A linear transformation T is injective **if and only if** $\ker T = \{\mathbf{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

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Activity 19.4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation where $\text{Im } T = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$.

Can you answer either of the following questions about T ?

(a) Is T injective?

(b) Is T surjective?

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Fact 19.5

A linear transformation $T : V \rightarrow W$ is surjective **if and only if** $\text{Im } T = W$. Put another way, a surjective linear transformation may be recognized by its same codomain and image.

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Activity 19.6

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with standard matrix A . Sort the following claims into two groups of equivalent statements.

- (a) T is injective
- (b) T is surjective
- (c) The kernel of T is trivial.
- (d) The columns of A span \mathbb{R}^m
- (e) The columns of A are linearly independent
- (f) Every column of $\text{RREF}(A)$ has a pivot.
- (g) Every row of $\text{RREF}(A)$ has a pivot.
- (h) The image of T equals its codomain.
- (i) The system of linear equations given by the augmented matrix $[A \mid \mathbf{b}]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$
- (j) The system of linear equations given by the augmented matrix $[A \mid \mathbf{0}]$ has exactly one solution.

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Definition 19.7

If $T : V \rightarrow W$ is both injective and surjective, it is called **bijjective**.

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Activity 19.8

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a bijective linear map with standard matrix A . Label each of the following as true or false.

- (a) The columns of A form a basis for \mathbb{R}^m
- (b) $\text{RREF}(A)$ is the identity matrix.
- (c) The system of linear equations given by the augmented matrix $[A \mid \mathbf{b}]$ has exactly one solution for all $\mathbf{b} \in \mathbb{R}^m$.

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Activity 19.9Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by

$$T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

- (a) T is neither injective nor surjective
- (b) T is injective but not surjective
- (c) T is surjective but not injective
- (d) T is bijective.

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Activity 19.10Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

- (a) T is neither injective nor surjective
- (b) T is injective but not surjective
- (c) T is surjective but not injective
- (d) T is bijective.

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Activity 19.11Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y + z \end{bmatrix}.$$

Which of the following must be true?

- (a) T is neither injective nor surjective
- (b) T is injective but not surjective
- (c) T is surjective but not injective
- (d) T is bijective.

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Activity 19.12Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

- (a) T is neither injective nor surjective
- (b) T is injective but not surjective
- (c) T is surjective but not injective
- (d) T is bijective.

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Module M: Understanding Matrices Algebraically

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At the end of this module, students will be able to...

- **M1. Matrix multiplication** Multiply matrices.
- **M2. Invertible matrices** Determine if a square matrix is invertible or not.
- **M3. Matrix inverses** Compute the inverse matrix of an invertible matrix.

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Before beginning this module, each student should be able to...

- Compose functions of real numbers
- Solve systems of linear equations (**Standard(s) E3**)
- Find the matrix corresponding to a linear transformation (**Standard(s) A1**)
- Determine if a linear transformation is injective and/or surjective (**Standard(s) A3**)
- Interpret the ideas of injectivity and surjectivity in multiple ways

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The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/algebra2/manipulating-functions/function-composition/v/function-composition>

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Application Activities - Module M Part 1 - Class Day 21

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Activity 21.1

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

- (a) \mathbb{R}
- (b) \mathbb{R}^2
- (c) \mathbb{R}^3
- (d) \mathbb{R}^4

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Activity 21.2

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the codomain of the composition map $S \circ T$?

(a) \mathbb{R}

(b) \mathbb{R}^2

(c) \mathbb{R}^3

(d) \mathbb{R}^4

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Module S

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Part 3 (Day 23)

Module G

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Part 4 (Day 28)

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Activity 21.3

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

The standard matrix of $S \circ T$ will lie in which matrix space?

- (a) 4×3 matrices
- (b) 4×2 matrices
- (c) 3×2 matrices
- (d) 2×3 matrices
- (e) 2×4 matrices
- (f) 3×4 matrices

Module E

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Part 3 (Day 23)

Module G

Part 1 (Day 25)

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Activity 21.4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

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Part 5 (Day 29)

Activity 21.4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute $(S \circ T)(\mathbf{e}_1)$

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Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

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Activity 21.4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute $(S \circ T)(\mathbf{e}_1)$

Part 2: Compute $(S \circ T)(\mathbf{e}_2)$

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Part 2 (Day 4)

Part 3 (Day 5)

Module V

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Part 2 (Day 8)

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Activity 21.4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute $(S \circ T)(\mathbf{e}_1)$

Part 2: Compute $(S \circ T)(\mathbf{e}_2)$

Part 3: Compute $(S \circ T)(\mathbf{e}_3)$.

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Part 3 (Day 9)

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Activity 21.4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be given by the standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute $(S \circ T)(\mathbf{e}_1)$

Part 2: Compute $(S \circ T)(\mathbf{e}_2)$

Part 3: Compute $(S \circ T)(\mathbf{e}_3)$.

Part 4: Find the standard matrix of $S \circ T$.

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Activity 21.5

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given

by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

- (a) \mathbb{R}
- (b) \mathbb{R}^2
- (c) \mathbb{R}^3
- (d) \mathbb{R}^4

Module E

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Activity 21.6

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given

by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

What is the codomain of the composition map $S \circ T$?

- (a) \mathbb{R}
- (b) \mathbb{R}^2
- (c) \mathbb{R}^3
- (d) \mathbb{R}^4

Module E

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Part 2 (Day 4)

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Activity 21.7

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given

by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

The standard matrix of $S \circ T$ will lie in which matrix space?

- (a) 2×2 matrices
- (b) 2×3 matrices
- (c) 3×2 matrices
- (d) 3×3 matrices

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Activity 21.8

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given

by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

Find the standard matrix of $S \circ T$.

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Activity 21.9

Let $T : \mathbb{R}^1 \rightarrow \mathbb{R}^4$ be given by the matrix $B = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ and $S : \mathbb{R}^4 \rightarrow \mathbb{R}^1$ be given by

the matrix $A = \begin{bmatrix} 2 & 3 & 2 & 5 \end{bmatrix}$.

Find the standard matrix of $S \circ T$.

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Definition 21.10

We define the product of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix (denoted AB) of the composition map of the two corresponding linear functions.

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Fact 21.11

If AB is defined, BA need not be defined, and if it is defined, it is in general different from AB .

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Activity 21.12

Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Compute AB .

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Activity 21.13

Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Compute AX

Observation 21.14

Consider the system of equations

$$3x + y - z = 5$$

$$2x + 4z = -7$$

$$-x + 3y + 5z = 2$$

We can interpret this as a **matrix equation** $AX = B$ where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$$

For this reason, we will swap out the use of Euclidean vectors $\mathbf{x} \in \mathbb{R}^n$ and $n \times 1$ matrices X whenever it is convenient.

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Application Activities - Module M Part 2 - Class Day 22

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Activity 22.1

Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix I such that $IA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

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Definition 22.2

The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

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Fact 22.3

For any square matrix A , $IA = AI = A$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

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Activity 22.4

Each row operation can be interpreted as a type of matrix multiplication.

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Activity 22.4

Each row operation can be interpreted as a type of matrix multiplication.

Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

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Activity 22.4

Each row operation can be interpreted as a type of matrix multiplication.

Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

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Activity 22.4

Each row operation can be interpreted as a type of matrix multiplication.

Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A :

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5 & 7+5 & -1-5 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

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Fact 22.5

If R is the result of applying a row operation to I , then RA is the result of applying the same row operation to A .

This means that for any matrix A , we can find a series of matrices R_1, \dots, R_k corresponding to the row operations such that

$$R_1 R_2 \cdots R_k A = \text{RREF}(A).$$

That is, row reduction can be thought of as the result of matrix multiplication.

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Activity 22.6

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with standard matrix A . Sort the following items into groups of statements about T .

- (a) T is injective (i.e. one-to-one)
- (b) T is surjective (i.e. onto)
- (c) T is bijective (i.e. both injective and surjective)
- (d) $AX = B$ has a solution for all $m \times 1$ matrices B
- (e) $AX = B$ has a unique solution for all $m \times 1$ matrices B
- (f) $AX = 0$ has a unique solution.
- (g) The columns of A span \mathbb{R}^m
- (h) The columns of A are linearly independent
- (i) The columns of A are a basis of \mathbb{R}^m
- (j) Every column of $\text{RREF}(A)$ has a pivot
- (k) Every row of $\text{RREF}(A)$ has a pivot
- (l) $m = n$ and $\text{RREF}(A) = I$

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Activity 22.7

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with matrix A . If T is injective, which of the following cannot be true?

- (a) A has strictly more columns than rows
- (b) A has the same number of rows as columns (i.e. A is square)
- (c) A has strictly more rows than columns

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Activity 22.8

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with matrix A . If T is surjective, which of the following cannot be true?

- (a) A has strictly more columns than rows
- (b) A has the same number of rows as columns (i.e. A is square)
- (c) A has strictly more rows than columns

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Activity 22.9

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with matrix A . If T is bijective, which of the following cannot be true?

- (a) A has strictly more columns than rows
- (b) A has the same number of rows as columns (i.e. A is square)
- (c) A has strictly more rows than columns

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Definition 23.1

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map with standard matrix A .

- If T is a bijection and B is any \mathbb{R}^n vector, then $T(X) = AX = B$ has a unique solution X .
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by setting $T^{-1}(B) = X$ to be this unique solution.
- Let A^{-1} be the standard matrix for T^{-1} . We call A^{-1} the **inverse matrix** of A , so we also say that A is **invertible**.

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Activity 23.2

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the bijective linear map defined by $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$.

It can be shown that T is bijective and has the inverse map

$$T^{-1} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}.$$

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Activity 23.2

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the bijective linear map defined by $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$.

It can be shown that T is bijective and has the inverse map

$$T^{-1} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}.$$

Part 1: Compute $(T^{-1} \circ T) \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$.

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Activity 23.2

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the bijective linear map defined by $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$.

It can be shown that T is bijective and has the inverse map

$$T^{-1} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}.$$

Part 1: Compute $(T^{-1} \circ T) \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$.

Part 2: If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1} , what must $A^{-1}A$ be?

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Observation 23.3

$T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation T .
Therefore $A^{-1}A = AA^{-1} = I$ is the identity matrix for any invertible matrix A .

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Activity 23.4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

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Activity 23.4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.

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Activity 23.4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.

Part 2: Solve $T(X) = \mathbf{e}_2$ to find $T^{-1}(\mathbf{e}_2)$.

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Activity 23.4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.

Part 2: Solve $T(X) = \mathbf{e}_2$ to find $T^{-1}(\mathbf{e}_2)$.

Part 3: Solve $T(X) = \mathbf{e}_3$ to find $T^{-1}(\mathbf{e}_3)$.

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Activity 23.4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.

Part 2: Solve $T(X) = \mathbf{e}_2$ to find $T^{-1}(\mathbf{e}_2)$.

Part 3: Solve $T(X) = \mathbf{e}_3$ to find $T^{-1}(\mathbf{e}_3)$.

Part 4: Compute A^{-1} , the standard matrix for T^{-1} .

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Observation 23.5

We could have solved these three systems simultaneously by row reducing the matrix $[A | I]$ at once.

$$A = \left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array} \right]$$

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Activity 23.6

Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A \mid I]$.

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Activity 23.7

Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

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Observation 23.8

A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $\text{RREF}(A) = I_n$.

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Module G: Geometry of Linear Maps

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At the end of this module, students will be able to...

- **G1. Determinants** Compute the determinant of a square matrix.
- **G2. Eigenvalues** Find the eigenvalues of a square matrix, along with their algebraic multiplicities.
- **G3. Eigenvectors** Find the eigenspace of a square matrix associated to a given eigenvalue.
- **G4. Geometric multiplicity** Compute the geometric multiplicity of an eigenvalue of a square matrix.

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Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **(Standard(s) A1)**.
- Recall and use the definition of a linear transformation **(Standard(s) A2)**.
- Find all roots of quadratic polynomials (including complex ones), and be able to use the rational root theorem to find all rational roots of a higher degree polynomial.
- Interpret the statement “ A is an invertible matrix” in many equivalent ways in different contexts.

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The following resources will help you prepare for this module.

- Finding the area of a parallelogram: <https://www.khanacademy.org/math/basic-geo/basic-geo-area-and-perimeter/parallelogram-area/a/area-of-parallelogram>
- Factoring quadratics: <https://www.khanacademy.org/math/algebra2/polynomial-functions/factoring-polynomials-quadratic-forms-alg2/v/factoring-polynomials-1>
- Finding complex roots of quadratics: <https://www.khanacademy.org/math/algebra2/polynomial-functions/quadratic-equations-with-complex-numbers/v/complex-roots-from-the-quadratic-formula>
- Finding all roots of polynomials: <https://www.khanacademy.org/math/algebra2/polynomial-functions/finding-zeros-of-polynomials/v/finding-roots-or-zeros-of-polynomial-1>
- The Rational Root Theorem: https://artofproblemsolving.com/wiki/index.php?title=Rational_Root_Theorem

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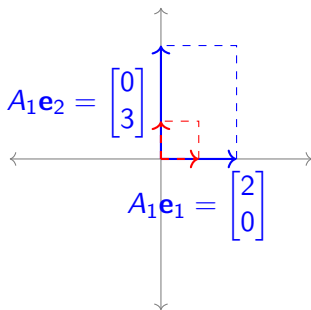
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Activity 25.1

The image below illustrates how the linear transformation $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the standard matrix $A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.



- What is the area of the transformed unit square?
- Find two vectors that were stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.

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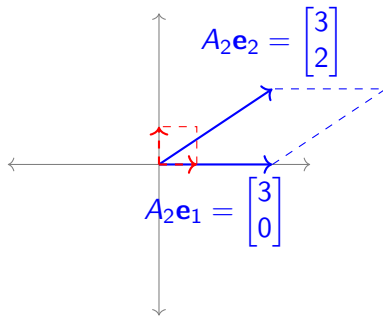
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Activity 25.2

The image below illustrates how the linear transformation $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the standard matrix $A_2 = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$ transforms the unit square.



- (a) What is the area of the transformed unit square?
- (b) Find at least one vector that was stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.

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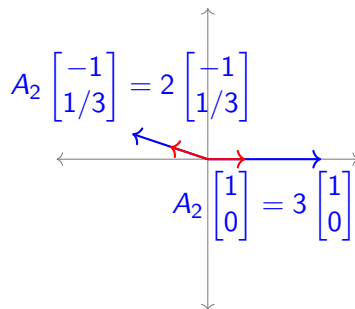
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Observation 25.3

It's possible to find two non-parallel vectors that are stretched by the transformation given by A_2 :



The process for finding such vectors will be covered later in this module.

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Activity 25.4

Consider the linear transformation given by the standard matrix $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

- Sketch the transformation of the unit square (the parallelogram given by the columns of the standard matrix).
- Compute the area of the transformed unit square.

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Activity 25.5

Consider the linear transformation given by the standard matrix $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

- (a) Sketch the transformation of the unit square.
- (b) Compute the area of the transformed unit square.

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Activity 25.6

Consider the linear transformation given by the standard matrix $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

- (a) Sketch the transformation of the unit square.
- (b) Compute the area of the transformed unit square.

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Remark 25.7

The area of the transformed unit square measures the factor by which all areas are transformed by a linear transformation.

We will define the **determinant** of a square matrix A , or $\det(A)$ for short, to be this factor. But what properties must this function satisfy?

Activity 25.8

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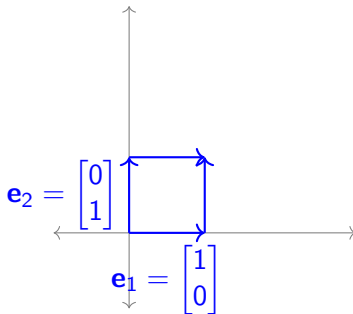
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The transformation of the unit square by the standard matrix $[\mathbf{e}_1 \ \mathbf{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. What is $\det([\mathbf{e}_1 \ \mathbf{e}_2]) = \det(I)$, that is, by what factor has the area of the unit square been scaled?



- a) 0
- b) 1
- c) 2
- d) Cannot be determined

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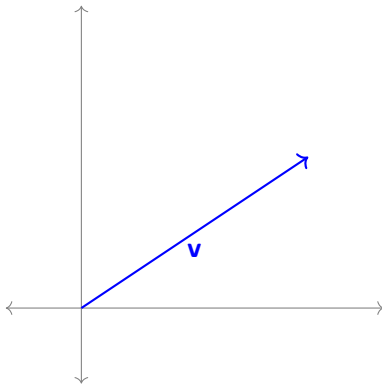
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Activity 25.9

The transformation of the unit square by the standard matrix $[\mathbf{v} \ \mathbf{v}]$ is illustrated below: both $T(\mathbf{e}_1) = T(\mathbf{e}_2) = \mathbf{v}$. What is $\det([\mathbf{v} \ \mathbf{v}])$, that is, by what factor has area been scaled?



- a) 0
- b) 1
- c) 2
- d) Cannot be determined

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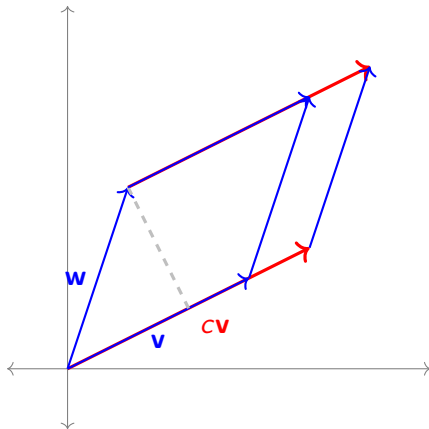
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Activity 25.10

The transformations of the unit square by the standard matrices $[\mathbf{v} \ \mathbf{w}]$ and $[c\mathbf{v} \ \mathbf{w}]$ are illustrated below. How are $\det([\mathbf{v} \ \mathbf{w}])$ and $\det([c\mathbf{v} \ \mathbf{w}])$ related?



- a) $\det([\mathbf{v} \ \mathbf{w}]) = \det([c\mathbf{v} \ \mathbf{w}])$
- b) $c + \det([\mathbf{v} \ \mathbf{w}]) = \det([c\mathbf{v} \ \mathbf{w}])$
- c) $c \det([\mathbf{v} \ \mathbf{w}]) = \det([c\mathbf{v} \ \mathbf{w}])$

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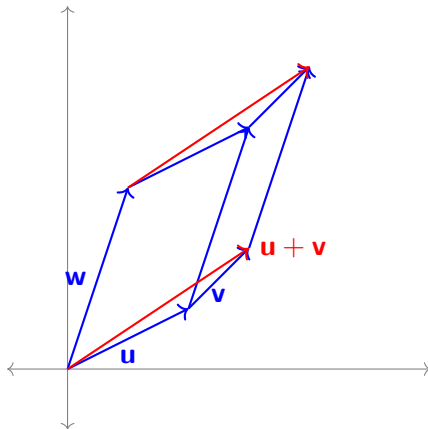
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Part 5 (Day 29)**Activity 25.11**

The transformations of unit squares by the standard matrices $[\mathbf{u} \ \mathbf{w}]$, $[\mathbf{v} \ \mathbf{w}]$ and $[\mathbf{u} + \mathbf{v} \ \mathbf{w}]$ are illustrated below. How is $\det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$ related to $\det([\mathbf{u} \ \mathbf{w}])$ and $\det([\mathbf{v} \ \mathbf{w}])$?



- $\det([\mathbf{u} \ \mathbf{w}]) = \det([\mathbf{v} \ \mathbf{w}]) = \det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$
- $\det([\mathbf{u} \ \mathbf{w}]) + \det([\mathbf{v} \ \mathbf{w}]) = \det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$
- $\det([\mathbf{u} \ \mathbf{w}]) \det([\mathbf{v} \ \mathbf{w}]) = \det([\mathbf{u} + \mathbf{v} \ \mathbf{w}])$

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Definition 25.12

The **determinant** is the unique function $\det : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ satisfying the following three properties:

P1: $\det(I) = 1$

P2: $\det([\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]) = 0$ whenever two columns of the matrix are identical.

P3: $\det[\cdots \ c\mathbf{v} + d\mathbf{w} \ \cdots] = c \det[\cdots \ \mathbf{v} \ \cdots] + d \det[\cdots \ \mathbf{w} \ \cdots]$, assuming all other columns are equal.

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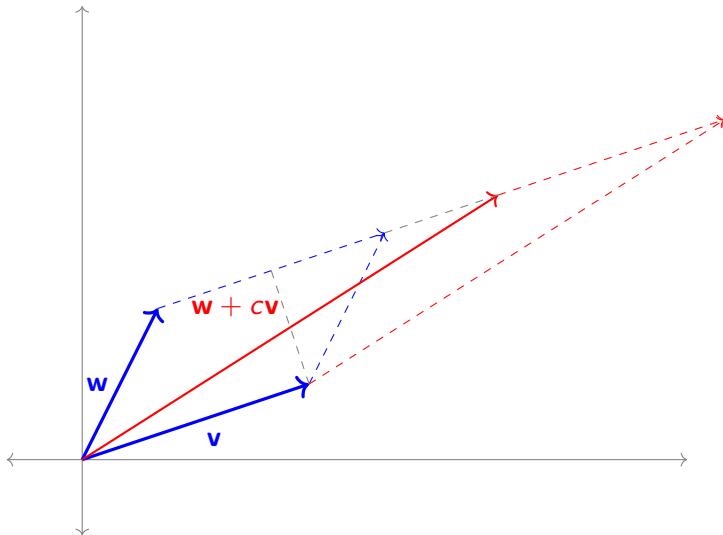
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Observation 25.13

What happens if we had a multiple of one column to another?



The base of both parallelograms is \mathbf{v} , while the height has not changed. Thus

$$\det([\mathbf{v} \quad \mathbf{w} + c\mathbf{v}]) = \det([\mathbf{v} \quad \mathbf{w}])$$

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Observation 25.14

Swapping columns can be obtained from a sequence of adding column multiples.

$$\begin{aligned}
 \det([\mathbf{v} \quad \mathbf{w}]) &= \det([\mathbf{v} + \mathbf{w} \quad \mathbf{w}]) \\
 &= \det([\mathbf{v} + \mathbf{w} \quad \mathbf{w} - (\mathbf{v} + \mathbf{w})]) \\
 &= \det([\mathbf{v} + \mathbf{w} \quad -\mathbf{v}]) \\
 &= \det([\mathbf{v} + \mathbf{w} - \mathbf{v} \quad -\mathbf{v}]) \\
 &= \det([\mathbf{w} \quad -\mathbf{v}]) \\
 &= -\det([\mathbf{w} \quad \mathbf{v}])
 \end{aligned}$$

So swapping two columns results in a negation of the determinant. Therefore, determinants represent a *signed* area, since they are not always positive.

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Fact 25.15

We've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c \det([\cdots \mathbf{v} \cdots]) = \det([\cdots c\mathbf{v} \cdots])$$

(b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \mathbf{v} \cdots \mathbf{w} \cdots]) = -\det([\cdots \mathbf{w} \cdots \mathbf{v} \cdots])$$

(c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \mathbf{v} \cdots \mathbf{w} \cdots]) = \det([\cdots \mathbf{v} + c\mathbf{w} \cdots \mathbf{w} \cdots])$$

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Activity 25.16

The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. How must the transformation given by the standard matrix AB scale areas?

- (a) 1
- (b) 7
- (c) 12
- (d) Cannot be determined

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Fact 25.17

Since the transformation given by the standard matrix AB is obtained by applying the transformations given by A and B , it follows that

$$\det(AB) = \det(A) \det(B)$$

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Definition 26.1

The **transpose** of a matrix is given by rewriting its columns as rows and vice versa:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

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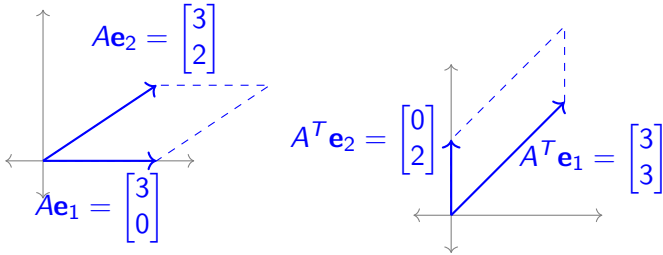
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Fact 26.2

It is possible to prove that the determinant of a matrix and its transpose are the same. For example, let $A = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$, so $A^T = \begin{bmatrix} 3 & 0 \\ 3 & 2 \end{bmatrix}$; both matrices scale the unit square by 6, even though the parallelograms are not congruent.



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Fact 26.3

We previously figured out that column operations can be used to simplify determinants; since $\det(A) = \det(A^T)$, we can also use row operations:

① Multiplying rows by scalars: $\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$

② Swapping two rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$

③ Adding multiples of rows to other rows: $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$

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Activity 26.4

Compute the determinant of $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$ by row reducing it to a nicer matrix.

For example, $\det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

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Fact 26.5

This same process allows us to prove a more convenient formula:

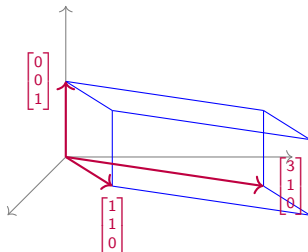
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

In higher dimensions, the formulas become unreasonable. For example, the formula for 4×4 matrices has 24 terms!

Activity 26.6

The following image illustrates the transformation of the unit cube by the matrix

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



This volume is equal to which of the following areas?

(a) $\det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$

(d) $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

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Fact 26.7

If column i of a matrix is \mathbf{e}_i , then both column and row i may be removed without changing the value of the determinant. For example, the second column of the following matrix is \mathbf{e}_2 , so:

$$\det \begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Therefore the same holds for the transpose:

$$\det \begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Geometrically, this is the fact that if the height is 1, the base \times height formula reduces to the area/volume/etc. of the $n - 1$ dimensional base.

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Activity 26.8

Compute $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$.

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Activity 26.9

Compute $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$.

(a) -1

(b) 0

(c) 1

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Activity 26.10

Compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -5 \\ 0 & 3 & 3 \end{bmatrix}$

Hint: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$

(a) 3

(b) 6

(c) 9

(d) 12

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Activity 26.11

Compute $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$.

Observation 26.12

$$\begin{aligned}
 \det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} &= (-1)(0) \det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} + (1)(3) \det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} + \\
 &(-1)(2) \det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} + (1)(0) \det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix} \\
 &= 3 \det \begin{bmatrix} 2 & 5 & 0 \\ 1 & 0 & 3 \\ -1 & 2 & 2 \end{bmatrix} + (-1)(2) \det \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ -1 & -1 & 2 \end{bmatrix}
 \end{aligned}$$

This technique is called **Laplace expansion** or **cofactor expansion**.

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Activity 26.13

Compute $\det \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 1 & 2 & 0 & 3 \\ -1 & -3 & 2 & -2 \end{bmatrix}$.

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Application Activities - Module G Part 3 - Class Day 27

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Activity 27.1

An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Compute $\det(M)$ and $\det(M^{-1})$.

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Activity 27.2

Suppose the matrix M is invertible, so there exists M^{-1} with $MM^{-1} = I$. It follows that $\det(M)\det(M^{-1}) = \det(I)$.

What is the only number that $\det(M)$ cannot equal?

(a) -1

(b) 0

(c) 1

(d) $\frac{1}{\det(M^{-1})}$

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Fact 27.3

For every invertible matrix M , $\det(M^{-1}) = \frac{1}{\det(M)}$.

Furthermore, a square matrix M is invertible if and only if $\det(M) \neq 0$.

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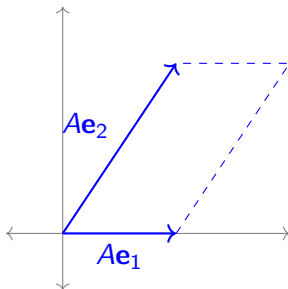
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Observation 27.4

Consider the linear transformation $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



It is easy to see geometrically that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily verified by computation) that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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Definition 27.5

Let $A \in \mathbb{R}^{n \times n}$. An **eigenvector** is a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x}$ is parallel to \mathbf{x} .

In other words, $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ . We call this λ an **eigenvalue** of A .

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Observation 27.6

Since $\lambda \mathbf{x} = \lambda(I\mathbf{x})$, we can find the eigenvalues and eigenvectors satisfying $A\mathbf{x} = \lambda\mathbf{x}$ by inspecting $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

- Since we already know that $(A - \lambda I)\mathbf{0} = \mathbf{0}$ for any value of λ , we are more interested in finding values of λ such that $A - \lambda I$ has a nontrivial kernel.
- Thus $\text{RREF}(A - \lambda I)$ must have a non-pivot column, and therefore $A - \lambda I$ cannot be invertible.
- Since $A - \lambda I$ cannot be invertible, our eigenvalues must satisfy $\det(A - \lambda I) = 0$.

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Definition 27.7

Computing $\det(A - \lambda I)$ results in the **characteristic polynomial** of A .

For example, when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix}$$

Thus the characteristic polynomial of A is

$$\det \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) - 6 = \lambda^2 - 5\lambda - 2$$

Activity 27.8

Complete the following computation of the characteristic polynomial $A - \lambda I$ for

$$A = \begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}.$$

$$\begin{aligned} \begin{bmatrix} 6 - \lambda & -2 & 1 \\ 17 & -5 - \lambda & 5 \\ -4 & 2 & 1 - \lambda \end{bmatrix} &= (6 - \lambda) \det \begin{bmatrix} ? & ? & ? \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} - 2 \det \begin{bmatrix} ? & ? & ? \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} + \det \begin{bmatrix} ? & ? & ? \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \\ &= (6 - \lambda) \det \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} + 2 \det \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} - \det \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \\ &= (6 - \lambda) \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} + 2 \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} - \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \\ &= (6 - \lambda)((-5 - \lambda)(1 - \lambda) - 10) + 2(17(1 - \lambda) + 20) - (-4(-5 - \lambda) - 34) \end{aligned}$$

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Activity 27.9

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

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Activity 27.9Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det \begin{bmatrix} 2 - \lambda & 2 \\ 0 & 3 - \lambda \end{bmatrix}$ to determine the characteristic polynomial of A .

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Activity 27.9

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det \begin{bmatrix} 2 - \lambda & 2 \\ 0 & 3 - \lambda \end{bmatrix}$ to determine the characteristic polynomial of A .

Part 2: Find the roots of the characteristic polynomial to determine the eigenvalues of A .

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Activity 27.9

Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det \begin{bmatrix} 2 - \lambda & 2 \\ 0 & 3 - \lambda \end{bmatrix}$ to determine the characteristic polynomial of A .

Part 2: Find the roots of the characteristic polynomial to determine the eigenvalues of A .

Part 3: Compute the kernel of the transformation given by

$$A - 2I = \begin{bmatrix} 2 - 2 & 2 \\ 0 & 3 - 2 \end{bmatrix}$$

to determine all the eigenvectors associated to the eigenvalue 2.

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Activity 27.9Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det \begin{bmatrix} 2 - \lambda & 2 \\ 0 & 3 - \lambda \end{bmatrix}$ to determine the characteristic polynomial of A .

Part 2: Find the roots of the characteristic polynomial to determine the eigenvalues of A .

Part 3: Compute the kernel of the transformation given by

$$A - 2I = \begin{bmatrix} 2 - 2 & 2 \\ 0 & 3 - 2 \end{bmatrix}$$

to determine all the eigenvectors associated to the eigenvalue 2.

Part 4: Compute the kernel of the transformation given by $A - 3I$ to determine all the eigenvectors associated to the eigenvalue 3.

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Definition 27.10

The kernel of the transformation given by $A - \lambda I$ contains all the eigenvectors associated with λ . Since kernel is a subspace of \mathbb{R}^n , we call this kernel the **eigenspace** associated with the eigenvalue λ .

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Activity 27.11

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

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Activity 27.11

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A .

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Activity 27.11

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A .

Part 2: Find the roots of the characteristic polynomial $(3 - \lambda)(\lambda^2 - 4\lambda - 12)$ to determine the eigenvalues of A .

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Activity 27.11

Find all the eigenvalues and associated eigenspaces for the matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}.$$

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A .

Part 2: Find the roots of the characteristic polynomial $(3 - \lambda)(\lambda^2 - 4\lambda - 12)$ to determine the eigenvalues of A .

Part 3: Compute the kernels of $A - \lambda I$ for each eigenvalue $\lambda \in \{-2, 3, 6\}$ to determine the respective eigenspaces.

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Observation 27.12

Recall that a is a root of the polynomial $p(\lambda)$ if the polynomial may be factored into $p(\lambda) = q(\lambda)(\lambda - a)^k$ for some maximal positive integer k , since $p(a) = q(a)(a - a)^k = 0$.

This k is called the **algebraic multiplicity** of the root.

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Application Activities - Module G Part 4 - Class Day 28

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Observation 28.1

Recall from last class:

- To find the eigenvalues of a matrix A , we need to find values of λ such that $A - \lambda I$ has a nontrivial kernel. Equivalently, we want values where $A - \lambda I$ is not invertible, so we want to know the values of λ where $\det(A - \lambda I) = 0$.
- $\det(A - \lambda I)$ is a polynomial with variable λ , called the **characteristic polynomial** of A . Thus the roots of the characteristic polynomial of A are exactly the eigenvalues of A .
- Once an eigenvalue λ is found, the **eigenspace** containing all **eigenvectors** \mathbf{x} satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is given by $\ker(A - \lambda I)$.

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- Part 4 (Day 28)**
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Activity 28.2

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

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Module S

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Module G

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Activity 28.2

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Part 1: Compute the eigenvalues of A .

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Activity 28.2

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Part 1: Compute the eigenvalues of A .

Part 2: Sketch a picture of the transformation of the unit square. What about this picture reveals that A has no eigenvectors?

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Activity 28.3

If A is a 4×4 matrix, what is the largest number of eigenvalues A can have?

- (a) 3
- (b) 4
- (c) 5
- (d) 6
- (e) It can have infinitely many

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Module S

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Observation 28.4

An $n \times n$ matrix may have anywhere between 0 and n real-valued eigenvalues. But if complex eigenvalues are included, then every $n \times n$ matrix has n eigenvalues (counting algebraic multiplicities).

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Activity 28.5

The matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$ has characteristic polynomial $-\lambda(\lambda - 2)^2$.

Find the dimension of the eigenspace of A associated to the eigenvalue 2 (the dimension of the kernel of $A - 2I$).

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Activity 28.6

The matrix $B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}$ has characteristic polynomial $-\lambda(\lambda - 2)^2$.

Find the dimension of the eigenspace of B associated to the eigenvalue 2 (the dimension of the kernel of $B - 2I$).

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Definition 28.7

While the **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial, the **geometric multiplicity** of an eigenvalue is the dimension of its eigenspace.

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Fact 28.8

As we've seen, the geometric multiplicity may be different than its algebraic multiplicity, but it cannot exceed it.

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Module S

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Module A

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Module G

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Activity 28.9

Consider the 4×4 matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

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Module A

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Part 3 (Day 19)

Module M

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Part 2 (Day 22)

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Activity 28.9

Consider the 4×4 matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

Part 1: Use technology (e.g. Wolfram Alpha) to find its characteristic polynomial.

Module E

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Module S

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Activity 28.9

Consider the 4×4 matrix

$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

Part 1: Use technology (e.g. Wolfram Alpha) to find its characteristic polynomial.

Part 2: Find the algebraic and geometric multiplicities for both eigenvalues.

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Application Activities - Module G Part 5 - Class Day 29

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Module V

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Part 4 (Day 10)

Module S

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Part 2 (Day 13)

Part 3 (Day 14)

Module A

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Module G

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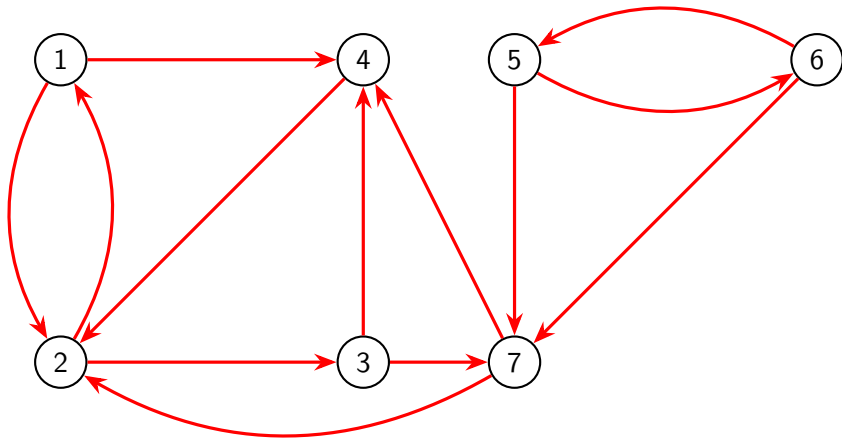
Part 4 (Day 28)

Part 5 (Day 29)

Activity 29.1

A \$700,000,000,000 problem:

In the picture below, each circle represents a webpage, and an arrow is a link to a different webpage.



Rank the 7 webpages from most important to least important.

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Module A

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Module G

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Observation 29.2

The \$700,000,000,000 idea:

- A webpage is important if important webpages link to it.
- A webpage should divide its importance equally among the pages that link to it.

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Part 3 (Day 5)

Module V

Part 1 (Day 7)
Part 2 (Day 8)
Part 3 (Day 9)
Part 4 (Day 10)

Module S

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Module A

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Part 2 (Day 18)
Part 3 (Day 19)

Module M

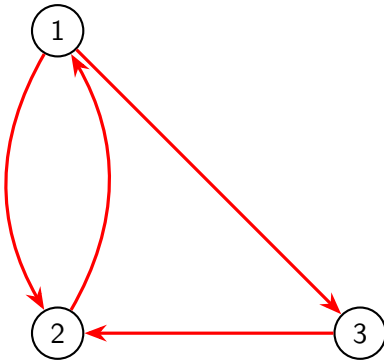
Part 1 (Day 21)
Part 2 (Day 22)
Part 3 (Day 23)

Module G

Part 1 (Day 25)
Part 2 (Day 26)
Part 3 (Day 27)
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Part 5 (Day 29)

Observation 29.3

A simple example:



Transition matrix:

$$x_1 \leftarrow x_2$$

$$x_2 \leftarrow \frac{1}{2}x_1 + x_3$$

$$x_3 \leftarrow \frac{1}{2}x_1$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

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Module V

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Module S

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Fact 29.4

This is called a **Markov chain model**. We can think about iteratively moving importance from webpages to each other by repeatedly multiplying by A .

The **steady state vector** will be a vector satisfying $A\mathbf{x} = \mathbf{x}$.

Thus, the \$700,000,000,000 problem is an eigenvector problem!

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Module M

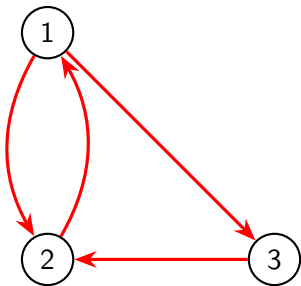
Part 1 (Day 21)
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Part 3 (Day 27)
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Activity 29.5

Find an eigenvector associated to the eigenvalue 1 for our small example



$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

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- Part 1 (Day 3)
- Part 2 (Day 4)
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Module V

- Part 1 (Day 7)
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- Part 4 (Day 10)

Module S

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Module M

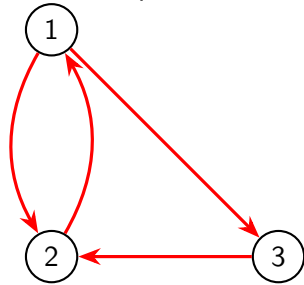
- Part 1 (Day 21)
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- Part 3 (Day 23)

Module G

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- Part 2 (Day 26)
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- Part 4 (Day 28)
- Part 5 (Day 29)

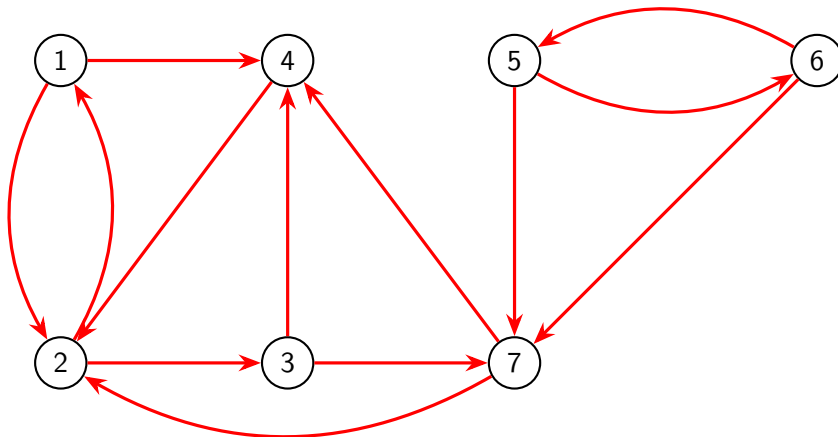
Fact 29.6

The eigenvector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ tells us webpages 1 and 2 are equally important, and both twice as important as webpage 3.



Activity 29.7

Write the 7×7 transition matrix for the website graph below.



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Module V

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Part 2 (Day 8)
Part 3 (Day 9)
Part 4 (Day 10)

Module S

Part 1 (Day 12)
Part 2 (Day 13)
Part 3 (Day 14)

Module A

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Part 2 (Day 18)
Part 3 (Day 19)

Module M

Part 1 (Day 21)
Part 2 (Day 22)
Part 3 (Day 23)

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Part 3 (Day 27)
Part 4 (Day 28)
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Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Part 1 (Day 17)

Part 2 (Day 18)

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Part 3 (Day 23)

Module G

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Part 2 (Day 26)

Part 3 (Day 27)

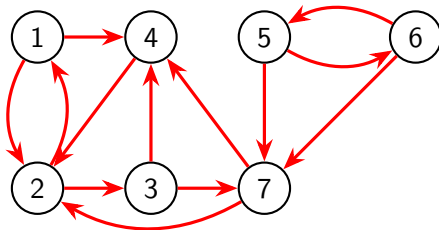
Part 4 (Day 28)

Part 5 (Day 29)

Activity 29.8

Find the **page rank vector**, i.e. an eigenvector associated to 1 for the transition matrix

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$



Which webpage is most important?

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

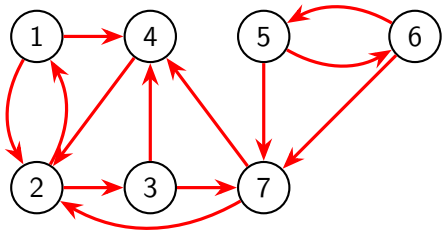
Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
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Observation 29.9



The page rank vector is

$$\begin{bmatrix} 2 \\ 4 \\ 2 \\ 2.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$