

## Application Activities - Module V Part 2 - Class Day 8

**Remark 8.1** The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$ : Euclidean vectors with  $n$  components.
- $\mathbb{R}^\infty$ : Sequences of real numbers  $(v_1, v_2, \dots)$ .
- $\mathbb{R}^{m \times n}$ : Matrices of real numbers with  $m$  rows and  $n$  columns.
- $\mathbb{C}$ : Complex numbers.
- $\mathcal{P}^n$ : Polynomials of degree  $n$  or less.
- $\mathcal{P}$ : Polynomials of any degree.
- $C(\mathbb{R})$ : Real-valued continuous functions.

**Activity 8.2** Let  $V = \{(a, b) : a, b \text{ are real numbers}\}$ , where  $(a_1, b_1) \oplus (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$  and  $c \odot (a, b) = (a^c, b + c)$ . Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

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| <ul style="list-style-type: none"> <li>• <b>Addition associativity.</b><br/><math>\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.</math></li> <li>• <b>Addition commutativity.</b><br/><math>\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.</math></li> <li>• <b>Addition identity.</b><br/>There exists some <math>\mathbf{0}</math> where <math>\mathbf{v} \oplus \mathbf{0} = \mathbf{v}.</math></li> <li>• <b>Addition inverse.</b><br/>There exists some <math>-\mathbf{v}</math> where <math>\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}.</math></li> </ul> | <ul style="list-style-type: none"> <li>• <b>Scalar multiplication associativity.</b><br/><math>a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.</math></li> <li>• <b>Scalar multiplication identity.</b><br/><math>1 \odot \mathbf{v} = \mathbf{v}.</math></li> <li>• <b>Scalar distribution.</b><br/><math>a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).</math></li> <li>• <b>Vector distribution.</b><br/><math>(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).</math></li> </ul> |
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**Definition 8.3** A **linear combination** of a set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$  is given by  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$  for any choice of scalar multiples  $c_1, c_2, \dots, c_m$ .

For example, we say  $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

**Definition 8.4** The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

**Activity 8.5** Consider  $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ .

*Part 1:* Sketch  $c\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  in the  $xy$  plane for  $c = 1, 3, 0, -2$ .

*Part 2:* Sketch a representation of all the vectors given by  $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$  in the  $xy$  plane.

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**Activity 8.6** Consider  $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ .

*Part 1:* Sketch the following linear combinations in the  $xy$  plane:  $1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ,  $0\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ,  $2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ,  $2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

*Part 2:* Sketch a representation of all the vectors given by  $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$  in the  $xy$  plane.

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**Activity 8.7** Sketch a representation of all the vectors given by  $\text{span}\left\{\begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}$  in the  $xy$  plane.

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**Activity 8.8** The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}\right\}$  exactly when the vector equation  $x_1\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

*Part 2:* Solve this system. (Remember, you should use a calculator to help find RREF.)

*Part 3:* Given this solution, does  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belong to  $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}\right\}$ ?

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