Application Activities - Module G Part 3 - Class Day 27

Activity 27.1 An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

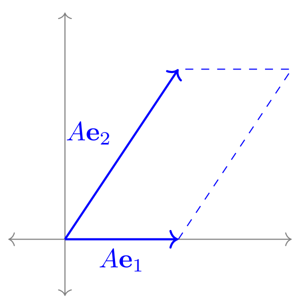
Compute $\det(M)$ and $\det(M^{-1})$.

Activity 27.2 Suppose the matrix M is invertible, so there exists M^{-1} with $MM^{-1} = I$. It follows that $\det(M)\det(M^{-1}) = \det(I)$.
What is the only number that $\det(M)$ cannot equal?

(a) -1 (b) 0 (c) 1 (d) $\frac{1}{\det(M^{-1})}$

Fact 27.3 For every invertible matrix M, $\det(M^{-1}) = \frac{1}{\det(M)}$.
Furthermore, a square matrix M is invertible if and only if $\det(M) \neq 0$.

Observation 27.4 Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



It is easy to see geometrically that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily verified by computation) that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Definition 27.5 Let $A \in \mathbb{R}^{n \times n}$. An **eigenvector** is a vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x}$ is parallel to $\mathbf{x}$.
In other words, $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. We call this λ an **eigenvalue** of A.

Observation 27.6 Since $\lambda\mathbf{x} = \lambda(I\mathbf{x})$, we can find the eigenvalues and eigenvectors satisfying $A\mathbf{x} = \lambda\mathbf{x}$ by inspecting $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

- Since we already know that $(A - \lambda I)\mathbf{0} = \mathbf{0}$ for any value of λ, we are more interested in finding values of λ such that $A - \lambda I$ has a nontrivial kernel.
- Thus $\text{RREF}(A - \lambda I)$ must have a non-pivot column, and therefore $A - \lambda I$ cannot be invertible.
- Since $A - \lambda I$ cannot be invertible, our eigenvalues must satisfy $\det(A - \lambda I) = 0$.

Definition 27.7 Computing $\det(A - \lambda I)$ results in the **characteristic polynomial** of A.

For example, when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

Thus the characteristic polynomial of A is

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2$$

Activity 27.8 Complete the following computation of the characteristic polynomial $A - \lambda I$ for $A = \begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$.

$$\begin{aligned}
\begin{bmatrix} 6-\lambda & -2 & 1 \\ 17 & -5-\lambda & 5 \\ -4 & 2 & 1-\lambda \end{bmatrix} &= (6-\lambda)\det \begin{bmatrix} ? & ? & ? \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} - 2\det \begin{bmatrix} ? & ? & ? \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} + \det \begin{bmatrix} ? & ? & ? \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \\
&= (6-\lambda)\det \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} + 2\det \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} - \det \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \\
&= (6-\lambda)\det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} + 2\det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} - \det \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \\
&= (6-\lambda)((-5-\lambda)(1-\lambda) - 10) + 2(17(1-\lambda) + 20) - (-4(-5-\lambda) - 34)
\end{aligned}$$

Activity 27.9 Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Part 1: Compute $\det \begin{bmatrix} 2-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix}$ to determine the characteristic polynomial of A.

Part 2: Find the roots of the characteristic polynomial to determine the eigenvalues of A.

Part 3: Compute the kernel of the transformation given by

$$A - 2I = \begin{bmatrix} 2-2 & 2 \\ 0 & 3-2 \end{bmatrix}$$

to determine all the eigenvectors associated to the eigenvalue 2.

Part 4: Compute the kernel of the transformation given by $A - 3I$ to determine all the eigenvectors associated to the eigenvalue 3.

Definition 27.10 The kernel of the transformation given by $A - \lambda I$ contains all the eigenvectors associated with λ. Since kernel is a subspace of $\mathbb{R}^n$, we call this kernel the **eigenspace** associated with the eigenvalue λ.

Activity 27.11 Find all the eigenvalues and associated eigenspaces for the matrix $A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix}$.

Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.
Part 2: Find the roots of the characteristic polynomial $(3 - \lambda)(\lambda^2 - 4\lambda - 12)$ to determine the eigenvalues of A.
Part 3: Compute the kernels of $A - \lambda I$ for each eigenvalue $\lambda \in \{-2, 3, 6\}$ to determine the respective eigenspaces.

Observation 27.12 Recall that a is a root of the polynomial $p(\lambda)$ if the polynomial may be factored into $p(\lambda) = q(\lambda)(\lambda - a)^k$ for some maximal positive integer k, since $p(a) = q(a)(a - a)^k = 0$.
This k is called the **algebraic multiplicity** of the root.