

Module V

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Module V: Vector Spaces

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What is a vector space?

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At the end of this module, students will be able to...

- V1. Vector property verification.** ... show why an example satisfies a given vector space property, but does not satisfy another given property.
- V2. Vector space identification.** ... list the eight defining properties of a vector space, infer which of these properties a given example satisfies, and thus determine if the example is a vector space.
- V3. Linear combinations.** ... determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors.
- V4. Spanning sets.** ... determine if a set of Euclidean vectors spans \mathbb{R}^n .
- V5. Subspaces.** ... determine if a subset of \mathbb{R}^n is a subspace or not.

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Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems
E1,E2,E3.

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The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Academy):
<http://bit.ly/2y8A0wa>
- Linear combinations of Euclidean vectors (Khan Academy):
<http://bit.ly/2nK3wne>
- Adding and subtracting complex numbers (Khan Academy):
<http://bit.ly/1PE3ZMQ>
- Adding and subtracting polynomials (Khan Academy):
<http://bit.ly/2d5SLGZ>

Module V Section 0

Activity V.0.1 (~ 20 min)

Consider each of the following vector properties. Label each property with \mathbb{R}^1 , \mathbb{R}^2 , and/or \mathbb{R}^3 if that property holds for Euclidean vectors/scalars \mathbf{u} , \mathbf{v} , \mathbf{w} of that dimension.

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1 Addition associativity.

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

2 Addition commutivity.

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

3 Addition identity.

There exists some \mathbf{z} where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.

4 Addition inverse.

There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.

5 Addition midpoint uniqueness.

There exists a unique \mathbf{m} where the distance from \mathbf{u} to \mathbf{m} equals the distance from \mathbf{m} to \mathbf{v} .

6 Scalar multiplication associativity.

$$a(b\mathbf{v}) = (ab)\mathbf{v}.$$

7 Scalar multiplication identity.

$$1\mathbf{v} = \mathbf{v}.$$

8 Scalar multiplication relativity.

There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.

9 Scalar distribution.

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

10 Vector distribution.

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

11 Orthogonality.

There exists a non-zero vector \mathbf{n} such that \mathbf{n} is orthogonal to both \mathbf{u} and \mathbf{v} .

12 Bidimensionality.

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} \text{ for some value of } a, b.$$

Definition V.0.2

A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V , and let a, b be scalar numbers.

- **Addition is associative.**
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition is commutative.**
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Additive identity exists.**
There exists some \mathbf{z} where
 $\mathbf{v} + \mathbf{z} = \mathbf{v}.$
- **Additive inverses exist.**
There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}.$
- **Scalar multiplication is associative.**
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **1 is a scalar multiplicative identity.**
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar multiplication distributes over vector addition.**
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Scalar multiplication distributes over scalar addition.**
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

Any **Euclidean vector space** \mathbb{R}^n satisfies all eight requirements regardless of the value of n , but we will also study other types of vector spaces.

Module V Section 1

Remark V.1.1

Last time, we defined a **vector space** V to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V , and all scalars (i.e. real numbers) a, b .

- **Addition is associative.**
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition is commutative.**
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Additive identity exists.**
There exists some \mathbf{z} where
 $\mathbf{v} + \mathbf{z} = \mathbf{v}.$
- **Additive inverses exist.**
There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}.$
- **Scalar multiplication is associative.**
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **1 is a scalar multiplicative identity.**
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar multiplication distributes over vector addition.**
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Scalar multiplication distributes over scalar addition.**
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

Remark V.1.2

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- \mathbb{R}^n : Euclidean vectors with n components.
- \mathbb{R}^∞ : Sequences of real numbers (v_1, v_2, \dots) .
- $M_{m,n}$: Matrices of real numbers with m rows and n columns.
- \mathbb{C} : Complex numbers.
- \mathcal{P}^n : Polynomials of degree n or less.
- \mathcal{P} : Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

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Activity V.1.3 (~ 20 min)

Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

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Activity V.1.3 (~ 20 min)

Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

Part 1: Show that V satisfies the vector distributive property

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$$

by letting $\mathbf{v} = (x, y)$ and showing both sides simplify to the same expression.

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Activity V.1.3 (~ 20 min)

Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

Part 1: Show that V satisfies the vector distributive property

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$$

by letting $\mathbf{v} = (x, y)$ and showing both sides simplify to the same expression.

Part 2: Show that V contains an additive identity element by choosing $\mathbf{z} = (?, ?)$ such that $\mathbf{v} \oplus \mathbf{z} = (x, y) \oplus (?, ?) = \mathbf{v}$ for any $\mathbf{v} = (x, y) \in V$.

Remark V.1.4

It turns out $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

satisfies all eight properties.

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutivity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some \mathbf{z} where

$$\mathbf{v} \oplus \mathbf{z} = \mathbf{v}.$$

- **Addition inverse.**

There exists some $-\mathbf{v}$ where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{z}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

Thus, V is a vector space.

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Activity V.1.5 (*~15 min*)

Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

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Activity V.1.5 (*~15 min*)

Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

Part 1: Show that the scalar multiplication identity holds by simplifying $1 \odot (x, y)$ to (x, y) .

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Activity V.1.5 (*~15 min*)

Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

Part 1: Show that the scalar multiplication identity holds by simplifying $1 \odot (x, y)$ to (x, y) .

Part 2: Show that the addition identity property fails by showing that $(0, -1) \oplus \mathbf{z} \neq (0, -1)$ no matter how $\mathbf{z} = (z_1, z_2)$ is chosen.

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Activity V.1.5 (*~15 min*)

Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

Part 1: Show that the scalar multiplication identity holds by simplifying $1 \odot (x, y)$ to (x, y) .

Part 2: Show that the addition identity property fails by showing that $(0, -1) \oplus \mathbf{z} \neq (0, -1)$ no matter how $\mathbf{z} = (z_1, z_2)$ is chosen.

Part 3: Can V be a vector space?

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Definition V.1.6

A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$ for any choice of scalar multiples c_1, c_2, \dots, c_m .

For example, we can say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Definition V.1.7

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m \mid c_i \in \mathbb{R}\}.$$

For example:

$$\text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\} = \left\{a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R}\right\}$$

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Activity V.1.8 (~ 10 min)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

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Activity V.1.8 (~ 10 min)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Part 1: Sketch $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $-2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane.

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Activity V.1.8 (~ 10 min)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Part 1: Sketch $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $-2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane.

Part 2: Sketch a representation of all the vectors belonging to $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid a \in \mathbb{R} \right\}$ in the xy plane.

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Activity V.1.9 (*~ 10 min*)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

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Activity V.1.9 (*~10 min*)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$-2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

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Activity V.1.9 (~ 10 min)

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$-2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Part 2: Sketch a representation of all the vectors belonging to $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ in the xy plane.

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Activity V.1.10 (~ 5 min)

Sketch a representation of all the vectors belonging to $\text{span} \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ in the xy plane.

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Remark V.2.1

Recall these definitions from last class:

- A **linear combination** of vectors is given by adding scalar multiples of those vectors, such as:

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- The **span** of a set of vectors is the collection of all linear combinations of that set, such as:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

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Activity V.2.2 (*~15 min*)

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when there exists a solution to the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$.

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Activity V.2.2 (*~15 min*)

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when there exists a solution to the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.

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Activity V.2.2 (*~15 min*)

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when there exists a solution to the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.

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Activity V.2.2 (*~15 min*)

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when there exists a

solution to the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.

Part 3: Given this solution set, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

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Fact V.2.3

A vector \mathbf{b} belongs to $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$ is consistent.

Put another way, \mathbf{b} belongs to $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ exactly when $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$ doesn't have a row $[0 \ \dots \ 0 \mid 1]$ representing the contradiction $0 = 1$.

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Activity V.2.4 (*~10 min*)

Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

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Activity V.2.5 (*~5 min*)

Determine if $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

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Activity V.2.6 (*~10 min*)

Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in \mathcal{P}^3 belong to $\text{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

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Activity V.2.6 (~ 10 min)

Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in \mathcal{P}^3 belong to $\text{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in \mathbb{R}^4 . (Hint: What four numbers must you know to write a \mathcal{P}^3 polynomial?)

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Activity V.2.6 (*~10 min*)

Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in \mathcal{P}^3 belong to $\text{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in \mathbb{R}^4 . (Hint: What four numbers must you know to write a \mathcal{P}^3 polynomial?)

Part 2: Solve this equivalent exercise, and use its solution to answer the original question.

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Activity V.2.7 (*~5 min*)

Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to $\text{span} \left\{ \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \right\}$?

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Activity V.2.8 (~ 5 min)

Does the complex number $2i$ belong to $\text{span}\{-3 + i, 6 - 2i\}$?

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Activity V.3.1 (*~5 min*)

How many vectors are required to span \mathbb{R}^2 ? Sketch a drawing in the xy plane to support your answer.

- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) Infinitely Many

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Activity V.3.2 (*~5 min*)

How many vectors are required to span \mathbb{R}^3 ?

- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) Infinitely Many

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Fact V.3.3At least n vectors are required to span \mathbb{R}^n .

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Activity V.3.4 (*~15 min*)

Choose a vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in \mathbb{R}^3 that is not in $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by using CoCalc

to verify that $\text{RREF} \left[\begin{array}{cc|c} 1 & -2 & ? \\ -1 & 0 & ? \\ 0 & 1 & ? \end{array} \right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$. (Why does this work?)

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Fact V.3.5

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ fails to span all of \mathbb{R}^n exactly when $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_m]$ has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \text{ for some choice of vector } \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

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Activity V.3.6 (*~5 min*)

Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does

$\mathbb{R}^4 = \text{span } S$?

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Activity V.3.7 (*~10 min*)

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, \\ -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2\}.$$

Does $\mathcal{P}^3 = \text{span } S$? (Hint: first rewrite the question so it is about Euclidean vectors.)

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Activity V.3.8 (*~5 min*)

Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Does $M_{2,2} = \text{span } S$?

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Activity V.3.9 (~ 5 min)

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^7$ be three vectors, and suppose \mathbf{w} is another vector with $\mathbf{w} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. What can you conclude about $\text{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

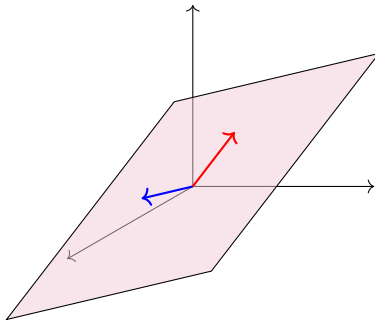
- (a) $\text{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is larger than $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
- (b) $\text{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
- (c) $\text{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is smaller than $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Module V Section 4

Definition V.4.1

A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space \mathbb{R}^3 .



Fact V.4.2

Any **subset** S of a vector space V satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a **subspace**, we need to check that addition and multiplication still make sense using only vectors from S . So we need to check two things:

- The set is **closed under addition**: for any $\mathbf{x}, \mathbf{y} \in S$, the sum $\mathbf{x} + \mathbf{y}$ is also in S .
- The set is **closed under scalar multiplication**: for any $\mathbf{x} \in S$ and scalar $c \in \mathbb{R}$, the product $c\mathbf{x}$ is also in S .

Activity V.4.3 (~ 15 min)

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}.$

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Activity V.4.3 (~ 15 min)

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}.$

Part 1: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S , so $x + 2y + z = 0$ and

$a + 2b + c = 0$. Show that $\mathbf{v} + \mathbf{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that

$$(x + a) + 2(y + b) + (z + c) = 0.$$

Activity V.4.3 (~ 15 min)

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}.$

Part 1: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S , so $x + 2y + z = 0$ and

$a + 2b + c = 0$. Show that $\mathbf{v} + \mathbf{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that

$$(x + a) + 2(y + b) + (z + c) = 0.$$

Part 2: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\mathbf{v}$ also belongs to S for any $c \in \mathbb{R}$.

Activity V.4.3 (~ 15 min)

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}.$

Part 1: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S , so $x + 2y + z = 0$ and

$a + 2b + c = 0$. Show that $\mathbf{v} + \mathbf{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that

$$(x + a) + 2(y + b) + (z + c) = 0.$$

Part 2: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\mathbf{v}$ also belongs to S for

any $c \in \mathbb{R}$.

Part 3: Is S is a subspace of \mathbb{R}^3 ?

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Activity V.4.4 (*~10 min*)

Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 4 \right\}$. Choose a vector $\mathbf{v} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = ?$, and show that $c\mathbf{v}$ isn't in S . Is S a subspace of \mathbb{R}^3 ?

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Remark V.4.5

Since 0 is a scalar and $0\mathbf{v} = \mathbf{z}$ for any vector \mathbf{v} , a set that is closed under scalar multiplication must contain the zero vector \mathbf{z} for that vector space.

Put another way, an easy way to check that a subset isn't a subspace is to show it doesn't contain $\mathbf{0}$.

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Section V.4

Activity V.4.6 (*~10 min*)Consider these two subsets of \mathbb{R}^4 :

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

$$T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

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Activity V.4.6 (*~10 min*)Consider these two subsets of \mathbb{R}^4 :

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

$$T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

Part 1: Which set is not a subspace of \mathbb{R}^4 ?

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Activity V.4.6 (~ 10 min)

Consider these two subsets of \mathbb{R}^4 :

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \mid a, b \text{ are real numbers} \right\} \qquad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

Part 1: Which set is not a subspace of \mathbb{R}^4 ?

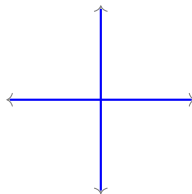
Part 2: Is the set of polynomials

$$S = \{ ax^3 + bx^2 + (b-1)x + (a-1) \mid a, b \text{ are real numbers} \}$$

a subspace of \mathcal{P}^3 ?

Activity V.4.7 (*~10 min*)

Consider the subset A of \mathbb{R}^2 where at least one coordinate of each vector is 0.



This set contains $\mathbf{0}$, and it's not hard to show that for every \mathbf{v} in A and scalar $c \in \mathbb{R}$, $c\mathbf{v}$ is also in A . Is A a subspace of \mathbb{R}^2 ? Why?

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Activity V.4.8 (~ 5 min)

Let W be a subspace of a vector space V . How are $\text{span } W$ and W related?

- (a) $\text{span } W$ is bigger than W
- (b) $\text{span } W$ is the same as W
- (c) $\text{span } W$ is smaller than W

Module V

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Fact V.4.9

If S is any subset of a vector space V , then since $\text{span } S$ collects all possible linear combinations, $\text{span } S$ is automatically a subspace of V .

In fact, $\text{span } S$ is always the smallest subspace of V that contains all the vectors in S .