

Section V.4

Definition V.4.1 A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space \mathbb{R}^3 .



Fact V.4.2 Any subset S of a vector space V that contains the additive identity $\vec{0}$ satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a **subspace**, we need to check that addition and multiplication still make sense using only vectors from S . So we need to check two things:

- The set is **closed under addition**: for any $\vec{x}, \vec{y} \in S$, the sum $\vec{x} + \vec{y}$ is also in S .
- The set is **closed under scalar multiplication**: for any $\vec{x} \in S$ and scalar $c \in \mathbb{R}$, the product $c\vec{x}$ is also in S .

Activity V.4.3 (~ 15 min) Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}$.

Part 1: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S , so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that

$\vec{v} + \vec{w} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$ also belongs to S by verifying that $(x+a) + 2(y+b) + (z+c) = 0$.

Part 2: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\vec{v} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ also belongs to S for any $c \in \mathbb{R}$ by verifying an appropriate equation.

Part 3: Is S a subspace of \mathbb{R}^3 ?

Activity V.4.4 (~ 10 min) Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 4 \right\}$. Choose a vector $\vec{v} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = ?$, and show that $c\vec{v}$ isn't in S . Is S a subspace of \mathbb{R}^3 ?

Remark V.4.5 Since 0 is a scalar and $0\vec{v} = \vec{z}$ for any vector \vec{v} , a nonempty set that is closed under scalar multiplication must contain the zero vector \vec{z} for that vector space.

Put another way, you can check any of the following to show that a nonempty subset W isn't a subspace:

- Show that $\vec{0} \notin W$.
- Find $\vec{u}, \vec{v} \in W$ such that $\vec{u} + \vec{v} \notin W$.
- Find $c \in \mathbb{R}, \vec{v} \in W$ such that $c\vec{v} \notin W$.

If you cannot do any of these, then W can be proven to be a subspace by doing the following:

- Prove that $\vec{u} + \vec{v} \in W$ whenever $\vec{u}, \vec{v} \in W$.
- Prove that $c\vec{v} \in W$ whenever $c \in \mathbb{R}, \vec{v} \in W$.

Activity V.4.6 (~ 20 min) Consider these subsets of \mathbb{R}^4 :

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = z + 1 \right\} \quad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = |z| \right\} \quad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| z = xy \right\}$$

Part 1: Show R isn't a subspace by showing that $\vec{0} \notin R$.

Part 2: Show S isn't a subspace by finding two vectors $\vec{u}, \vec{v} \in S$ such that $\vec{u} + \vec{v} \notin S$.

Part 3: Show T isn't a subspace by finding a vector $\vec{v} \in T$ such that $2\vec{v} \notin T$.

Activity V.4.7 (~ 5 min) Let W be a subspace of a vector space V . How are $\text{span } W$ and W related?

- (a) $\text{span } W$ is bigger than W
- (b) $\text{span } W$ is the same as W
- (c) $\text{span } W$ is smaller than W

Fact V.4.8 If S is any subset of a vector space V , then since $\text{span } S$ collects all possible linear combinations, $\text{span } S$ is automatically a subspace of V .

In fact, $\text{span } S$ is always the smallest subspace of V that contains all the vectors in S .