

Name: _____

MIDTERM EXAM

Math 237 – Linear Algebra

Version 4

Fall 2017

Show all work and justify all of your answers. Answers without work or sufficient reasoning will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} -4 & -1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{array} \right]$$

Solution:

$$-4x_1 - x_2 + 3x_3 = 2$$

$$x_1 + 2x_2 - x_3 = 0$$

$$-x_1 + 4x_2 + x_3 = 4$$

□

E2. Find RREF A , where

$$A = \left[\begin{array}{cccc|c} 2 & 2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 4 & 5 \\ 3 & 3 & -1 & -2 & 1 \end{array} \right]$$

Solution:

$$\text{RREF } A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

□

E3. Find the solution set for the following system of linear equations.

$$2x_1 + 3x_2 - 5x_3 + 14x_4 = 8$$

$$x_1 + x_2 - x_3 + 5x_4 = 3$$

Solution: Let $A = \left[\begin{array}{cccc|c} 2 & 3 & -5 & 14 & 8 \\ 1 & 1 & -1 & 5 & 3 \end{array} \right]$, so $\text{RREF } A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -3 & 4 & 2 \end{array} \right]$. It follows that the solution set is given by $\begin{bmatrix} 1 - 2a - b \\ 2 + 3a - 4b \\ a \\ b \end{bmatrix}$ for all real numbers a, b .

□

E4. Find a basis for the solution set to the homogeneous system of equations given by

$$2x_1 - 2x_2 + 6x_3 - x_4 = 0$$

$$3x_1 + 6x_3 + x_4 = 0$$

$$-4x_1 + x_2 - 9x_3 + 2x_4 = 0$$

Solution: Let $A = \left[\begin{array}{cccc|c} 2 & -2 & 6 & -1 & 0 \\ 3 & 0 & 6 & 1 & 0 \\ -4 & 1 & -9 & 2 & 0 \end{array} \right]$, so $\text{RREF } A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$. It follows that the basis for the solution set is given by $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

□

V1. Let V be the set of all points on the line $x + y = 2$ with the operations, for any $(x_1, y_1), (x_2, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, y_1) \oplus (x_2, y_2) &= (x_1 + x_2 - 1, y_1 + y_2 - 1) \\ c \odot (x_1, y_1) &= (cx_1 - (c - 1), cy_1 - (c - 1)) \end{aligned}$$

- (a) Show that this vector space has an **additive identity** element $\mathbf{0}$ satisfying $(x, y) \oplus \mathbf{0} = (x, y)$.
 (b) Determine if V is a vector space or not. Justify your answer.

Solution: Let $(x_1, y_1) \in V$; then $(x_1, y_1) \oplus (1, 1) = (x_1, y_1)$, so $(1, 1)$ is an additive identity element. Now we will show the other seven properties. Let $(x_1, y_1), (x_2, y_2) \in V$, and let $c, d \in \mathbb{R}$.

- 1) Since real addition is associative, \oplus is associative.
- 2) Since real addition is commutative, \oplus is commutative.
- 3) The additive identity is $(1, 1)$.
- 4) $(x_1, y_1) \oplus (2 - x_1, 2 - y_1) = (1, 1)$, so $(2 - x_1, 2 - y_1)$ is the additive inverse of (x_1, y_1) .
- 5)

$$\begin{aligned} c \odot (d \odot (x_1, y_1)) &= c \odot (dx_1 - (d - 1), dy_1 - (d - 1)) \\ &= (c(dx_1 - (d - 1)) - (c - 1), c(dy_1 - (d - 1)) - (c - 1)) \\ &= (cdx_1 - cd + c - (c - 1), cdy_1 - cd + c - (c - 1)) \\ &= (cdx_1 - (cd - 1), cdy_1 - (cd - 1)) \\ &= (cd) \odot (x_1, y_1) \end{aligned}$$

$$6) 1 \odot (x_1, y_1) = (x_1 - (1 - 1), y_1 - (1 - 1)) = (x_1, y_1)$$

7)

$$\begin{aligned} c \odot ((x_1, y_1) \oplus (x_2, y_2)) &= c \odot (x_1 + y_1 - 1, x_2 + y_2 - 1) \\ &= (c(x_1 + y_1 - 1) - (c - 1), c(x_2 + y_2 - 1) - (c - 1)) \\ &= (cx_1 + cx_2 - 2c + 1, cy_1 + cy_2 - 2c + 1) \\ &= (cx_1 - (c - 1), cy_1 - (c - 1)) \oplus (cx_2 - (c - 1), cy_2 - (c - 1)) \\ &= c \odot (x_1, y_1) \oplus c \odot (x_2, y_2) \end{aligned}$$

8)

$$\begin{aligned} (c + d) \odot (x_1, y_1) &= ((c + d)x_1 - (c + d - 1), (c + d)y_1 - (c + d - 1)) \\ &= (cx_1 - (c - 1), cy_1 - (c - 1)) \oplus (dx_1 - (d - 1), dy_1 - (d - 1)) \\ &= c \odot (x_1, y_1) \oplus d \odot (x_1, y_1) \end{aligned}$$

Therefore V is a vector space.

□

V2. Determine if $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix} 5 \\ 2 \\ -3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 8 \\ 3 \\ 5 \\ -1 \end{bmatrix}$.

Solution:

$$\text{RREF} \left(\left[\begin{array}{ccc|c} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{array} \right] \right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

The system has no solution, so $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ is not a linear combination of the three other vectors.

□

V3. Determine if the vectors $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix}$ span \mathbb{R}^4 .

Solution:

$$\text{RREF} \left(\left[\begin{array}{cccc} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{array} \right] \right) = \left[\begin{array}{cccc} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Since there are zero rows, they do not span. Alternatively, by inspection $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, so the set is linearly dependent, so it spans a subspace of dimension at most 3, therefore it does not span \mathbb{R}^4 .

□

V4. Let W be the set of all \mathbb{R}^3 vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $x + y + z = 0$ (this forms a plane). Determine if W is a subspace of \mathbb{R}^3 .

Solution: Yes, because $z = -x - y$ and $a \begin{bmatrix} x_1 \\ y_1 \\ -x_1 - y_1 \end{bmatrix} + b \begin{bmatrix} x_2 \\ y_2 \\ -x_2 - y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \\ -(ax_1 + bx_2) - (ay_1 + by_2) \end{bmatrix}$.

Alternately, yes because W is isomorphic to \mathbb{R}^2 .

□

S1. Determine if the set of polynomials $\{x^2 + x, x^2 + 2x - 1, x^2 + 3x - 2\}$ is linearly dependent or linearly independent

Solution:

$$\text{RREF} \left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, the set is linearly dependent.

□

S2. Determine if the set $\{x^3 - x, x^2 + x + 1, x^3 - x^2 + 2, 2x^2 - 1\}$ is a basis of \mathcal{P}^3 .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis.

□

S3. Let W be the subspace of \mathcal{P}^3 given by $W = \text{span}(\{x^3 + x^2 + 2x + 1, 3x^3 + 3x^2 + 6x + 3, 3x^3 - x^2 + 3x - 2, 7x^3 - x^2 + 8x - 3\})$. Find a basis for W .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis is $\{x^3 + x^2 + 2x + 1, 3x^3 - x^2 + 3x - 2\}$.

□

S4. Let $W = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -8 \\ -1 \end{bmatrix} \right\}$. Find the dimension of W .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & -8 \\ 1 & 1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since it has two pivot columns, its dimension is 2.

□

E1:

E2:

E3:

E4:

V1:

V2:

V3:

V4:

S1:

S2:

S3:

S4: