

Section S.3

Observation S.3.1 In the previous section, we learned that computing a basis for the subspace $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, is as simple as removing the vectors corresponding to the non-pivot columns of $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_m]$.

For example, since

$$\text{RREF} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} \textcircled{1} & 0 & 1 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \right\}$ has $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$ as a basis.

Activity S.3.2 (~ 10 min) Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Part 1: Find a basis for $\text{span } S$.

Part 2: Find a basis for $\text{span } T$.

Observation S.3.3 Even though we found different bases for them, $\text{span } S$ and $\text{span } T$ are exactly the same subspace of \mathbb{R}^4 , since

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} = T$$

Fact S.3.4 Any non-trivial vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size.

For example,

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

are all valid bases for \mathbb{R}^3 , and they all contain three vectors.

Definition S.3.5 The **dimension** of a vector space is equal to the size of any basis for the vector space.

As you'd expect, \mathbb{R}^n has dimension n . For example, \mathbb{R}^3 has dimension 3 because any basis for \mathbb{R}^3 such as

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

contains exactly three vectors.

Activity S.3.6 (*~10 min*) Find the dimension of each subspace of \mathbb{R}^4 by finding RREF for each corresponding matrix.

$$\begin{aligned} \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} & \quad \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} \\ \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\} & \quad \text{span} \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\} \end{aligned}$$

Fact S.3.7 Every vector space with finite dimension, that is, every vector space V with a basis of the form $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is said to be **isomorphic** to a Euclidean space \mathbb{R}^n , since there exists a natural correspondence between vectors in V and vectors in \mathbb{R}^n :

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Observation S.3.8 We've already been taking advantage of the previous fact by converting polynomials and matrices into Euclidean vectors. Since \mathcal{P}^3 and $M_{2,2}$ are both four-dimensional:

$$4x^3 + 0x^2 - 1x + 5 \leftrightarrow \begin{bmatrix} 4 \\ 0 \\ -1 \\ 5 \end{bmatrix} \leftrightarrow \begin{bmatrix} 4 & 0 \\ -1 & 5 \end{bmatrix}$$

Observation S.3.9 The space of polynomials \mathcal{P} (of *any* degree) has the basis $\{1, x, x^2, x^3, \dots\}$, so it is a natural example of an infinite-dimensional vector space.

Since \mathcal{P} and other infinite-dimensional spaces cannot be treated as an isomorphic finite-dimensional Euclidean space \mathbb{R}^n , vectors in such spaces cannot be studied by converting them into Euclidean vectors. Fortunately, most of the examples we will be interested in for this course will be finite-dimensional.

Definition S.3.10 A **homogeneous** system of linear equations is one of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

This system is equivalent to the vector equation:

$$x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array} \right]$$

Activity S.3.11 (~ 5 min) Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are solutions to $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$ so is

$$\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}, \text{ since}$$

$$a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n = \mathbf{0} \text{ and } b_1\mathbf{v}_1 + \dots + b_n\mathbf{v}_n = \mathbf{0}$$

implies

$$(a_1 + b_1)\mathbf{v}_1 + \dots + (a_n + b_n)\mathbf{v}_n = \mathbf{0}.$$

Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution. Thus the solution set of a homogeneous system is...

- a) A basis for \mathbb{R}^n . b) A subspace of \mathbb{R}^n . c) The empty set.

Activity S.3.12 (*~10 min*) Consider the homogeneous system of equations

$$\begin{aligned}x_1 + 2x_2 + x_4 &= 0 \\2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\3x_1 + 6x_2 - x_3 - x_4 &= 0\end{aligned}$$

Part 1: Find its solution set (a subspace of \mathbb{R}^4).

Part 2: Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

Fact S.3.13 The coefficients of the free variables in the solution set of a linear system always yield linearly independent vectors.

Thus if

$$\left\{ a \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

is the solution space for a homogeneous system, then

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

is a basis for the solution space.

Activity S.3.14 (*~10 min*) Consider the homogeneous system of equations

$$\begin{aligned}x_1 - 3x_2 + 2x_3 &= 0 \\2x_1 - 6x_2 + 4x_3 + 3x_4 &= 0 \\-2x_1 + 6x_2 - 4x_3 - 4x_4 &= 0\end{aligned}$$

Find a basis for its solution space.

Activity S.3.15 (*~5 min*) Suppose W is a subspace of \mathcal{P}^8 , and you know that it contains a **linearly independent** set of 3 vectors. What can you conclude about W ?

- (a) The dimension of W is at most 3.
- (b) The dimension of W is exactly 3.
- (c) The dimension of W is at least 3.

Activity S.3.16 (*~5 min*) Suppose W is a subspace of \mathcal{P}^8 , and you know that it contains a **spanning set** of 3 vectors. What can you conclude about W ?

- (a) The dimension of W is at most 3.
- (b) The dimension of W is exactly 3.
- (c) The dimension of W is at least 3.