Name: ______________________________________

**MIDTERM EXAM** Math 237 – Linear Algebra
**Version 5** Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

**E1.** Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned} x_1 + 3x_2 - 4x_3 + x_4 &= 5 \\ 3x_1 + 9x_2 + x_3 - 7x_4 &= 0 \\ x_1 - x_3 + x_4 &= 1 \end{aligned}$$

**Solution:**

$$\left[\begin{array}{cccc|c} 1 & 3 & -4 & 1 & 5 \\ 3 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{array}\right]$$

□

**E2.** Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ -3 & 5 & 2 & 0 \\ 1 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & 6 \\ 0 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

**E3.** Find the solution set for the following system of linear equations.

$$\begin{aligned} 2x_1 - 2x_2 + 6x_3 - x_4 &= -1 \\ 3x_1 + 6x_3 + x_4 &= 5 \\ -4x_1 + x_2 - 9x_3 + 2x_4 &= -7 \end{aligned}$$

**Solution:** Let $A = \left[\begin{array}{cccc|c} 2 & -2 & 6 & -1 & -1 \\ 3 & 0 & 6 & 1 & 5 \\ -4 & 1 & -9 & 2 & -7 \end{array}\right]$, so RREF $A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$. It follows that the solution set is given by $\begin{bmatrix} 2-2a \\ 3+a \\ a \\ -1 \end{bmatrix}$ for all real numbers $a$.

□

**E4.** Find a basis for the solution set to the homogeneous system of equations given by

$$\begin{aligned}2x_1 - 2x_2 + 6x_3 - x_4 &= 0\\ 3x_1 + 6x_3 + x_4 &= 0\\ -4x_1 + x_2 - 9x_3 + 2x_4 &= 0\end{aligned}$$

**Solution:** Let $A = \left[\begin{array}{cccc|c} 2 & -2 & 6 & -1 & 0\\ 3 & 0 & 6 & 1 & 0\\ -4 & 1 & -9 & 2 & 0\end{array}\right]$, so RREF $A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0\\ 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{array}\right]$. It follows that the basis for the solution set is given by $\left\{\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}\right\}$.

□

**V1.** Let $V$ be the set of all real numbers together with the operations $\oplus$ and $\odot$ defined by, for any $x, y \in V$ and $c \in \mathbb{R}$,

$$\begin{aligned}x \oplus y &= x + y - 3\\ c \odot x &= cx - 3(c-1)\end{aligned}$$

(a) Show that **scalar multiplication** is **associative**: $a \odot (b \odot x) = (ab) \odot x$.

(b) Determine if $V$ is a vector space or not. Justify your answer

**Solution:** Let $x, y \in V$, $c, d \in \mathbb{R}$. To show associativity:

$$\begin{aligned}c \odot (d \odot x) &= c \odot (dx - 3(d-1))\\ &= c\,(dx - 3(d-1)) - 3(c-1)\\ &= cdx - 3(cd-1)\\ &= (cd) \odot x\end{aligned}$$

We verify the remaining 7 properties to see that $V$ is a vector space.

1) Real addition is associative, so $\oplus$ is associative.

2) $x \oplus 3 = x + 3 - 3 = x$, so 3 is the additive identity.

3) $x \oplus (6 - x) = x + (6 - x) - 3 = 3$, so $6 - x$ is the additive inverse of $x$.

4) Real addition is commutative, so $\oplus$ is commutative.

5) Associativity shown above

6) $1 \odot x = x - 3(1-1) = x$

7)

$$\begin{aligned}c \odot (x \oplus y) &= c \odot (x + y - 3)\\ &= c(x + y - 3) - 3(c-1)\\ &= cx - 3(c-1) + cy - 3(c-1) - 3\\ &= (c \odot x) \oplus (c \odot y)\end{aligned}$$

8)

$$\begin{aligned}(c+d)\odot x &= (c+d)x - 3(c+d-1)\\ &= cx - 3(c-1) + dx - 3(c-1) - 3\\ &= (c\odot x)\oplus(d\odot x)\end{aligned}$$

Therefore $V$ is a vector space.

$\square$

**V2.** Determine if $\begin{bmatrix}3\\-2\\4\end{bmatrix}$ belongs to the span of the set $\left\{\begin{bmatrix}1\\2\\-3\end{bmatrix},\begin{bmatrix}2\\4\\-6\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$.

**Solution:** Since

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c}1&2&0&3\\2&4&0&-2\\-3&-6&0&4\end{array}\right]\right)=\left[\begin{array}{ccc|c}1&2&0&0\\0&0&0&1\\0&0&0&0\end{array}\right]$$

contains the contradiction $0=1$, $\begin{bmatrix}3\\-2\\4\end{bmatrix}$ is not a linear combination of the three vectors.

$\square$

**V3.** Determine if the vectors $\begin{bmatrix}-3\\1\\1\end{bmatrix},\begin{bmatrix}5\\-1\\-2\end{bmatrix},\begin{bmatrix}2\\0\\-1\end{bmatrix}$, and $\begin{bmatrix}0\\2\\-1\end{bmatrix}$ span $\mathbb{R}^3$

**Solution:**

$$\mathrm{RREF}\left(\begin{bmatrix}-3&5&2&0\\1&-1&0&2\\1&-2&-1&-1\end{bmatrix}\right)=\begin{bmatrix}1&0&1&5\\0&1&1&3\\0&0&0&0\end{bmatrix}$$

Since the resulting matrix has only two pivot columns, the vectors do not span $\mathbb{R}^3$.

$\square$

**V4.** Let $W$ be the set of all complex numbers $a+bi$ satisfying $a=2b$. Determine if $W$ is a subspace of $\mathbb{C}$.

**Solution:** Yes, because $c(2b_1+b_1i)+d(2b_2+b_2i)=2(cb_1+db_2)+(cb_1+db_2)i$ belongs to $W$. Alternately, yes because $W$ is isomorphic to $\mathbb{R}$.

$\square$

**S1.** Determine if the set of polynomials $\{x^2+x, x^2+2x-1, x^2+3x-2\}$ is linearly dependent or linearly independent

**Solution:**

$$\mathrm{RREF}\left(\begin{bmatrix}1&1&1\\0&2&3\\1&-1&-2\end{bmatrix}\right)=\begin{bmatrix}1&0&-\frac{1}{2}\\0&1&\frac{3}{2}\\0&0&0\end{bmatrix}$$

Since there is a nonpivot column, the set is linearly dependent.

$\square$

**S2.** Determine if the set $\left\{x^2+x-1, 3x^2-x+1, 2x-2\right\}$ is a basis of $\mathcal{P}_2$

**Solution:**

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis.

□

**S3.** Let $W = \text{span}\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix}\right\}\right)$. Find a basis of $W$.

**Solution:**

$$\text{RREF}\left(\begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\left\{\begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix}\right\}$ is a basis for $W$.

□

**S4.** Let $W$ be the subspace of $\mathcal{P}_3$ given by
$W = \text{span}\left(\left\{x^3-x^2+3x-3, 2x^3+x+1, 3x^3-x^2+4x-2, x^3+x^2+x-7\right\}\right)$. Compute the dimension of $W$.

**Solution:**

$$\text{RREF}\left(\begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$.

□

**E1:**

**E2:**

**E3:**

**E4:**

**V1:**

**V2:**

**V3:**

**V4:**

**S1:**

**S2:**

**S3:**

**S4:**