

Name:
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MIDTERM EXAM

Math 237 – Linear Algebra

Version 5

Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

Standard E1.	Mark:
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Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 4 & 1 & -7 \\ 1 & 2 & -1 & 0 \end{array} \right]$$

Solution:

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 4x_2 + x_3 &= -7 \\ x_1 + 2x_2 - x_3 &= 0 \end{aligned}$$

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Standard E2.	Mark:
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Put the following matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array} \right]$$

Solution:

$$\begin{aligned} \left[\begin{array}{ccc|c} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array} \right] &\sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & -\frac{2}{3} & -1 & \frac{2}{3} \\ 0 & 2 & 3 & -2 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 2 & 3 & -2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

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Standard E3.	Mark:
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Find the solution set for the following system of linear equations.

$$2x_1 + 3x_2 - 5x_3 + 14x_4 = 8$$

$$x_1 + x_2 - x_3 + 5x_4 = 3$$

Solution: Let $A = \begin{bmatrix} 2 & 3 & -5 & 14 & | & 8 \\ 1 & 1 & -1 & 5 & | & 3 \end{bmatrix}$, so $\text{RREF } A = \begin{bmatrix} 1 & 0 & 2 & 1 & | & 1 \\ 0 & 1 & -3 & 4 & | & 2 \end{bmatrix}$. It follows that the solution set is given by $\begin{bmatrix} 1 - 2a - b \\ 2 + 3a - 4b \\ a \\ b \end{bmatrix}$ for all real numbers a, b .

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Standard E4.	Mark:
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Find a basis for the solution set to the homogeneous system of equations given by

$$2x_1 - 2x_2 + 6x_3 - x_4 = 0$$

$$3x_1 + 6x_3 + x_4 = 0$$

$$-4x_1 + x_2 - 9x_3 + 2x_4 = 0$$

Solution: Let $A = \begin{bmatrix} 2 & -2 & 6 & -1 & | & 0 \\ 3 & 0 & 6 & 1 & | & 0 \\ -4 & 1 & -9 & 2 & | & 0 \end{bmatrix}$, so $\text{RREF } A = \begin{bmatrix} 1 & 0 & 2 & 0 & | & 0 \\ 0 & 1 & -1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$. It follows that the basis for the solution set is given by $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

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Standard V1.	Mark:
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Let V be the set of all real numbers with the operations, for any $x, y \in V$, $c \in \mathbb{R}$,

$$x \oplus y = \sqrt{x^2 + y^2}$$

$$c \odot x = cx$$

(a) Show that the vector **addition** \oplus is **associative**: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

(b) Determine if V is a vector space or not. Justify your answer.

Solution: Let $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned}
 (x \oplus y) \oplus z &= \sqrt{x^2 + y^2} \oplus z \\
 &= \sqrt{(\sqrt{x^2 + y^2})^2 + z^2} \\
 &= \sqrt{x^2 + y^2 + z^2} \\
 &= \sqrt{x^2 + (\sqrt{y^2 + z^2})^2} \\
 &= x \oplus \sqrt{y^2 + z^2} \\
 &= x \oplus (y \oplus z)
 \end{aligned}$$

However, this is not a vector space, as there is no zero vector.

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Standard V2.	Mark:
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Determine if $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$.

Solution:

$$\text{RREF} \left(\left[\begin{array}{ccc|c} 3 & 1 & 5 & 1 \\ 0 & -1 & 1 & 4 \\ -1 & 4 & -6 & 3 \end{array} \right] \right) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

So $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is not a linear combination of the three vectors.

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Standard V3.	Mark:
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Determine if the vectors $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix}$ span \mathbb{R}^4 .

Solution:

$$\text{RREF} \left(\left[\begin{array}{cccc} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{array} \right] \right) = \left[\begin{array}{cccc} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Since there are zero rows, they do not span. Alternatively, by inspection $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, so the set is linearly dependent, so it spans a subspace of dimension at most 3, therefore it does not span \mathbb{R}^4 .

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Standard V4.	Mark:
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Determine if the set of all lattice points, i.e. $\{(x, y) \mid x \text{ and } y \text{ are integers}\}$ is a subspace of \mathbb{R}^2 .

Solution: This set is closed under addition, but not under scalar multiplication so it is not a subspace.

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Standard S1.	Mark:
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Determine if the set of matrices $\left\{ \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -8 \\ 6 & 5 \end{bmatrix} \right\}$ is linearly dependent or linearly independent.

Solution:

$$\text{RREF} \left(\begin{bmatrix} 3 & 1 & 3 \\ -1 & 2 & -8 \\ 0 & -2 & 6 \\ 4 & 1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced row echelon form has a nonpivot column, the vectors are linearly dependent.

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Standard S2.	Mark:
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Determine if the set $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis of \mathbb{R}^4 .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis.

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Standard S3.	Mark:
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Let $W = \text{span} \left(\left(\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right) \right)$. Find a basis of W .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of W .

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Standard S4.	Mark:
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Let W be the subspace of \mathcal{P}_3 given by

$W = \text{span}(\{x^3 - x^2 + 3x - 3, 2x^3 + x + 1, 3x^3 - x^2 + 4x - 2, x^3 + x^2 + x - 7\})$. Compute the dimension of W .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$.

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Additional Notes/Marks	
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