

Section V.4

Definition V.4.1 A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space \mathbb{R}^3 .



Fact V.4.2 Any subset S of a vector space V satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a **subspace**, we need to check that addition and multiplication still make sense using only vectors from S . So we need to check two things:

- The set is **closed under addition**: for any $\mathbf{x}, \mathbf{y} \in S$, the sum $\mathbf{x} + \mathbf{y}$ is also in S .
- The set is **closed under scalar multiplication**: for any $\mathbf{x} \in S$ and scalar $c \in \mathbb{R}$, the product $c\mathbf{x}$ is also in S .

Activity V.4.3 (~ 15 min) Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}$.

Part 1: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S , so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that

$\mathbf{v} + \mathbf{w} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$ also belongs to S by verifying that $(x+a) + 2(y+b) + (z+c) = 0$.

Part 2: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\mathbf{v}$ also belongs to S for any $c \in \mathbb{R}$.

Part 3: Is S a subspace of \mathbb{R}^3 ?

Activity V.4.4 (~ 10 min) Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 4 \right\}$. Choose a vector $\mathbf{v} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = ?$, and show that $c\mathbf{v}$ isn't in S . Is S a subspace of \mathbb{R}^3 ?

Remark V.4.5 Since 0 is a scalar and $0\mathbf{v} = \mathbf{z}$ for any vector \mathbf{v} , a set that is closed under scalar multiplication must contain the zero vector \mathbf{z} for that vector space.

Put another way, an easy way to check that a subset isn't a subspace is to show it doesn't contain $\mathbf{0}$.

Activity V.4.6 (*~10 min*) Consider these two subsets of \mathbb{R}^4 :

$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \mid a, b \text{ are real numbers} \right\} \quad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \mid a, b \text{ are real numbers} \right\}$$

Part 1: Which set is not a subspace of \mathbb{R}^4 ?

Part 2: Is the set of polynomials

$$S = \{ax^3 + bx^2 + (b-1)x + (a-1) \mid a, b \text{ are real numbers}\}$$

a subspace of \mathcal{P}^3 ?

Activity V.4.7 (*~10 min*) Consider the subset A of \mathbb{R}^2 where at least one coordinate of each vector is 0.



This set contains $\mathbf{0}$, and it's not hard to show that for every \mathbf{v} in A and scalar $c \in \mathbb{R}$, $c\mathbf{v}$ is also in A . Is A a subspace of \mathbb{R}^2 ? Why?

Activity V.4.8 (*~5 min*) Let W be a subspace of a vector space V . How are $\text{span } W$ and W related?

- (a) $\text{span } W$ is bigger than W
- (b) $\text{span } W$ is the same as W
- (c) $\text{span } W$ is smaller than W

Fact V.4.9 If S is any subset of a vector space V , then since $\text{span } S$ collects all possible linear combinations, $\text{span } S$ is automatically a subspace of V .

In fact, $\text{span } S$ is always the smallest subspace of V that contains all the vectors in S .