

Name: \_\_\_\_\_

# **MASTERY QUIZ DAY 12**

Math 237 – Linear Algebra

## **Version 1**

Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

**V1.** Let  $V$  be the set of all pairs of real numbers with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$c \odot (x_1, y_1) = (c^2 x_1, c^3 y_1)$$

(a) Show that scalar multiplication **distributes scalars** over vector addition:

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2).$$

(b) Determine if  $V$  is a vector space or not. Justify your answer.

**Solution:** Let  $(x_1, y_1), (x_2, y_2) \in V$  and let  $c \in \mathbb{R}$ .

$$\begin{aligned} c \odot ((x_1, y_1) \oplus (x_2, y_2)) &= c \odot (x_1 + x_2, y_1 + y_2) \\ &= (c^2(x_1 + x_2), c^3(y_1 + y_2)) \\ &= (c^2 x_1, c^3 y_1) \oplus (c^2 x_2, c^3 y_2) \\ &= c \odot (x_1, y_1) \oplus c \odot (x_2, y_2) \end{aligned}$$

However,  $V$  is not a vector space, as the other distributive law fails:

$$(c + d) \odot (x_1, y_1) = ((c + d)^2 x_1, (c + d)^3 y_1) \neq ((c^2 + d^2) x_1, (c^3 + d^3) y_1) = c \odot (x_1, y_1) \oplus d \odot (x_1, y_1).$$

□

**V3.** Determine if the vectors  $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}$ , and  $\begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix}$  span  $\mathbb{R}^4$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are zero rows, they do not span. Alternatively, by inspection  $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ , so the set is linearly dependent, so it spans a subspace of dimension at most 3, therefore it does not span  $\mathbb{R}^4$ .

□

**V4.** Let  $W$  be the set of all complex numbers that are purely real (i.e. of the form  $a + 0i$ ) or purely imaginary (i.e. of the form  $0 + bi$ ). Determine if  $W$  is a subspace of  $\mathbb{C}$ .

**Solution:** No, because 1 is purely real and  $i$  is purely imaginary, but the linear combination  $1 + i$  is neither.

□

**V1:**

**V3:**

**V4:**

Name: \_\_\_\_\_

**MASTERY QUIZ DAY 12**

Math 237 – Linear Algebra

**Version 2**

Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

**V1.** Let  $V$  be the set of all polynomials with the operations, for any  $f, g \in V$ ,  $c \in \mathbb{R}$ ,

$$\begin{aligned}f \oplus g &= f' + g' \\c \odot f &= cf'\end{aligned}$$

(here  $f'$  denotes the derivative of  $f$ ).

(a) Show that scalar multiplication **distributes scalars** over vector addition:  $c \odot (f \oplus g) = c \odot f \oplus c \odot g$ .

(b) Determine if  $V$  is a vector space or not. Justify your answer.

**Solution:** Let  $f, g \in \mathcal{P}$ , and let  $c \in \mathbb{R}$ .

$$c \odot (f \oplus g) = c \odot (f' + g') = c(f' + g')' = cf'' + cg'' = cf' \oplus cg' = c \odot f \oplus c \odot g.$$

However, this is not a vector space, as there is no zero vector. Additionally,  $1 \odot f \neq f$  for any nonzero polynomial  $f$ .

□

**V3.** Determine if the vectors  $\begin{bmatrix} 8 \\ 21 \\ -7 \end{bmatrix}$ ,  $\begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ , and  $\begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}$  span  $\mathbb{R}^3$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 8 & -3 & -1 & 4 \\ 21 & -8 & -3 & 11 \\ -7 & 3 & 2 & -5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the rank is less than 3, they do not span  $\mathbb{R}^3$ .

□

**V4.** Let  $W$  be the set of all  $\mathbb{R}^3$  vectors  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  satisfying  $x + y + z = 1$  (this forms a plane). Determine if  $W$  is a subspace of  $\mathbb{R}^3$ .

**Solution:** No, because  $\mathbf{0}$  does not belong to  $W$ .

□

**V1:**

**V3:**

**V4:**

Name: \_\_\_\_\_

## MASTERY QUIZ DAY 12

Math 237 – Linear Algebra

### Version 3

Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

**V1.** Let  $V$  be the set of all real numbers together with the operations  $\oplus$  and  $\odot$  defined by, for any  $x, y \in V$  and  $c \in \mathbb{R}$ ,

$$\begin{aligned}x \oplus y &= x + y - 3 \\c \odot x &= cx - 3(c - 1)\end{aligned}$$

- (a) Show that **scalar multiplication** is **associative**:  $a \odot (b \odot x) = (ab) \odot x$ .
- (b) Determine if  $V$  is a vector space or not. Justify your answer

**Solution:** Let  $x, y \in V$ ,  $c, d \in \mathbb{R}$ . To show associativity:

$$\begin{aligned}c \odot (d \odot x) &= c \odot (dx - 3(d - 1)) \\&= c(dx - 3(d - 1)) - 3(c - 1) \\&= cdx - 3(cd - 1) \\&= (cd) \odot x\end{aligned}$$

We verify the remaining 7 properties to see that  $V$  is a vector space.

- 1) Real addition is associative, so  $\oplus$  is associative.
- 2)  $x \oplus 3 = x + 3 - 3 = x$ , so 3 is the additive identity.
- 3)  $x \oplus (6 - x) = x + (6 - x) - 3 = 3$ , so  $6 - x$  is the additive inverse of  $x$ .
- 4) Real addition is commutative, so  $\oplus$  is commutative.
- 5) Associativity shown above
- 6)  $1 \odot x = x - 3(1 - 1) = x$
- 7)

$$\begin{aligned}c \odot (x \oplus y) &= c \odot (x + y - 3) \\&= c(x + y - 3) - 3(c - 1) \\&= cx - 3(c - 1) + cy - 3(c - 1) - 3 \\&= (c \odot x) \oplus (c \odot y)\end{aligned}$$

8)

$$\begin{aligned}(c + d) \odot x &= (c + d)x - 3(c + d - 1) \\&= cx - 3(c - 1) + dx - 3(d - 1) - 3 \\&= (c \odot x) \oplus (d \odot x)\end{aligned}$$

Therefore  $V$  is a vector space.

□

**V3.** Determine if the vectors  $\begin{bmatrix} 8 \\ 21 \\ -7 \end{bmatrix}$ ,  $\begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ , and  $\begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}$  span  $\mathbb{R}^3$ .

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} 8 & -3 & -1 & 4 \\ 21 & -8 & -3 & 11 \\ -7 & 3 & 2 & -5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the rank is less than 3, they do not span  $\mathbb{R}^3$ .

□

**V4.** Determine if  $\left\{ \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$  a subspace of  $\mathbb{R}^4$ .

**Solution:** It is closed under addition and scalar multiplication, so it is a subspace. Alternatively, it is the image of the linear transformation from  $\mathbb{R}^3 \rightarrow \mathbb{R}^4$  given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix}.$$

□

**V1:**

**V3:**

**V4:**

Name: \_\_\_\_\_

## MASTERY QUIZ DAY 12

Math 237 – Linear Algebra

### Version 4

Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

**V1.** Let  $V$  be the set of all real numbers with the operations, for any  $x, y \in V$ ,  $c \in \mathbb{R}$ ,

$$x \oplus y = \sqrt{x^2 + y^2}$$

$$c \odot x = cx$$

(a) Show that the vector **addition**  $\oplus$  is **associative**:  $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ .

(b) Determine if  $V$  is a vector space or not. Justify your answer.

**Solution:** Let  $x, y, z \in \mathbb{R}$ . Then

$$\begin{aligned}(x \oplus y) \oplus z &= \sqrt{x^2 + y^2} \oplus z \\&= \sqrt{(\sqrt{x^2 + y^2})^2 + z^2} \\&= \sqrt{x^2 + y^2 + z^2} \\&= \sqrt{x^2 + (\sqrt{y^2 + z^2})^2} \\&= x \oplus \sqrt{y^2 + z^2} \\&= x \oplus (y \oplus z)\end{aligned}$$

However, this is not a vector space, as there is no zero vector.

□

**V3.** Does  $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$ ?

**Solution:** Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 4 & 2 \\ 4 & -9 & -3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has a zero row, the vectors fail to span  $\mathbb{R}^3$ .

□

**V4.** Let  $W$  be the set of all complex numbers  $a + bi$  satisfying  $a = 2b$ . Determine if  $W$  is a subspace of  $\mathbb{C}$ .

**Solution:** Yes, because  $c(2b_1 + b_1i) + d(2b_2 + b_2i) = 2(cb_1 + db_2) + (cb_1 + db_2)i$  belongs to  $W$ . Alternately, yes because  $W$  is isomorphic to  $\mathbb{R}$ .

□

**V1:**

**V3:**

**V4:**

Name: \_\_\_\_\_

## MASTERY QUIZ DAY 12

Math 237 – Linear Algebra

### Version 5

Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

**V1.** Let  $V$  be the set of all points on the line  $x + y = 2$  with the operations, for any  $(x_1, y_1), (x_2, y_2) \in V$ ,  $c \in \mathbb{R}$ ,

$$\begin{aligned}(x_1, y_1) \oplus (x_2, y_2) &= (x_1 + x_2 - 1, y_1 + y_2 - 1) \\ c \odot (x_1, y_1) &= (cx_1 - (c - 1), cy_1 - (c - 2))\end{aligned}$$

- (a) Show that this vector space has an **additive identity** element  $\mathbf{0}$  satisfying  $(x, y) \oplus \mathbf{0} = (x, y)$ .  
(b) Determine if  $V$  is a vector space or not. Justify your answer.

**Solution:** Let  $(x_1, y_1) \in V$ ; then  $(x_1, y_1) \oplus (1, 1) = (x_1, y_1)$ , so  $(1, 1)$  is an additive identity element.

Now we will show the other seven properties. Let  $(x_1, y_1), (x_2, y_2) \in V$ , and let  $c, d \in \mathbb{R}$ .

- 1) Since real addition is associative,  $\oplus$  is associative.
- 2) Since real addition is commutative,  $\oplus$  is commutative.
- 3) The additive identity is  $(1, 1)$ .
- 4)  $(x_1, y_1) \oplus (2 - x_1, 2 - y_1) = (1, 1)$ , so  $(2 - x_1, 2 - y_1)$  is the additive inverse of  $(x_1, y_1)$ .
- 5)

$$\begin{aligned}c \odot (d \odot (x_1, y_1)) &= c \odot (dx_1 - (d - 1), dy_1 - (d - 1)) \\ &= (c(dx_1 - (d - 1)) - (c - 1), c(dy_1 - (d - 1))) \\ &= (cdx_1 - cd + c - (c - 1), cdy_1 - cd + c - (c - 1)) \\ &= (cdx_1 - (cd - 1), cdy_1 - (cd - 1)) \\ &= (cd) \odot (x_1, y_1)\end{aligned}$$

- 6)  $1 \odot (x_1, y_1) = (x_1 - (1 - 1), y_1 - (1 - 1)) = (x_1, y_1)$   
7)

$$\begin{aligned}c \odot ((x_1, y_1) \oplus (x_2, y_2)) &= c \odot (x_1 + y_1 - 1, x_2 + y_2 - 1) \\ &= (c(x_1 + y_1 - 1) - (c - 1), c(x_2 + y_2 - 1) - (c - 1)) \\ &= (cx_1 + cx_2 - 2c + 1, cy_1 + cy_2 - 2c + 1) \\ &= (cx_1 - (c - 1), cy_1 - (c - 1)) \oplus (cx_2 - (c - 1), cy_2 - (c - 1)) \\ &= c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)\end{aligned}$$

8)

$$\begin{aligned}(c + d) \odot (x_1, y_1) &= ((c + d)x_1 - (c + d - 1), (c + d)y_1 - (c + d - 1)) \\ &= (cx_1 - (c - 1), cy_1 - (c - 1)) \oplus (dx_1 - (d - 1), dy_1 - (d - 1)) \\ &= c \odot (x_1, y_1) \oplus d \odot (x_1, y_1)\end{aligned}$$

Therefore  $V$  is a vector space.

□

**V3.** Determine if the vectors  $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}$ ,  $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ , and  $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$  span  $\mathbb{R}^3$

**Solution:**

$$\text{RREF} \left( \begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix has only two pivot columns, the vectors do not span  $\mathbb{R}^3$ .

□

**V4.** Let  $W$  be the set of all  $\mathbb{R}^3$  vectors  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  satisfying  $x + y + z = 1$  (this forms a plane). Determine if  $W$  is a subspace of  $\mathbb{R}^3$ .

**Solution:** No, because  $\mathbf{0}$  does not belong to  $W$ .

□

**V1:**

**V3:**

**V4:**

Name: \_\_\_\_\_

## MASTERY QUIZ DAY 12

Math 237 – Linear Algebra

### Version 6

Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

**V1.** Let  $V$  be the set of all real numbers with the operations, for any  $x, y \in V$ ,  $c \in \mathbb{R}$ ,

$$x \oplus y = \sqrt{x^2 + y^2}$$

$$c \odot x = cx$$

(a) Show that the vector **addition**  $\oplus$  is **associative**:  $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ .

(b) Determine if  $V$  is a vector space or not. Justify your answer.

**Solution:** Let  $x, y, z \in \mathbb{R}$ . Then

$$\begin{aligned}(x \oplus y) \oplus z &= \sqrt{x^2 + y^2} \oplus z \\&= \sqrt{(\sqrt{x^2 + y^2})^2 + z^2} \\&= \sqrt{x^2 + y^2 + z^2} \\&= \sqrt{x^2 + (\sqrt{y^2 + z^2})^2} \\&= x \oplus \sqrt{y^2 + z^2} \\&= x \oplus (y \oplus z)\end{aligned}$$

However, this is not a vector space, as there is no zero vector.

□

**V3.** Does  $\text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$ ?

**Solution:** Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 4 & 2 \\ 4 & -9 & -3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has a zero row, the vectors fail to span  $\mathbb{R}^3$ .

□

**V4.** Let  $W$  be the set of all complex numbers that are purely real (i.e. of the form  $a + 0i$ ) or purely imaginary (i.e. of the form  $0 + bi$ ). Determine if  $W$  is a subspace of  $\mathbb{C}$ .

**Solution:** No, because 1 is purely real and  $i$  is purely imaginary, but the linear combination  $1 + i$  is neither.

□

**V1:**

**V3:**

**V4:**