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Module A: Algebraic properties of linear maps

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How can we understand linear maps algebraically?

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At the end of this module, students will be able to...

A1. **Linear map verification.** ... determine if a map between vector spaces of polynomials is linear or not.

A2. **Linear maps and matrices.** ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.

A3. **Injectivity and surjectivity.** ... determine if a given linear map is injective and/or surjective.

A4. **Kernel and Image.** ... compute a basis for the kernel and a basis for the image of a linear map.

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Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

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Module A Section 1

Definition A.1.1

A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

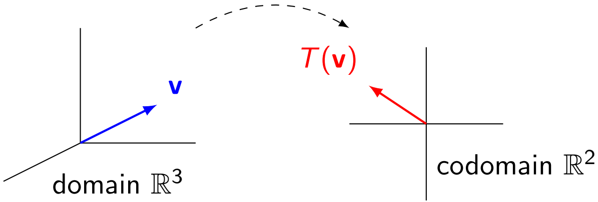
1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition A.1.2

Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.

Linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$



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Example A.1.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right) = \begin{bmatrix} (x+u)-(z+w) \\ 3(y+v) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x+u)-(z+w) \\ 3(y+v) \end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \text{ and } cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore T is a linear transformation.

Example A.1.4

Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ 0 \end{bmatrix}$$

Since the resulting vectors are different, T is a linear transformation.

Activity A.1.5 *(∼5 min)*
Show that $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2x \end{bmatrix}$$

is not linear by showing that $2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \neq T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

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Fact A.1.6

A map between Euclidean spaces $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of $\mathbb{R}^n$.

Example A.1.7
You can quickly identify

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

as linear because $x - z$ and $3y$ are linear combinations of x, y, z.
But

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2x \end{bmatrix}$$

cannot be linear because x^2 and $y + 3$ are not linear combinations of x, y (even though $x + y$ and $y - 2x$ are).

Activity A.1.8 *(∼3 min)*

Recall the following rules from calculus, where $D : \mathcal{P} \to \mathcal{P}$ is the derivative map defined by $D(f) = \frac{df}{dx}$ for each polynomial f.

$$D(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

$$D(cf) = c\frac{df}{dx}$$

What can we conclude from these rules?

a) $\mathcal{P}$ is not a vector space

b) D is a linear map

c) D is not a linear map

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Activity A.1.9 *(∼12 min)*
Consider the following two polynomial maps.

$$S : \mathcal{P}^4 \to \mathcal{P}^3 \text{ given by } S(f(x)) = 2f'(x) - f''(x)$$

$$T : \mathcal{P}^2 \to \mathcal{P}^2 \text{ given by } T(f(x)) = f'(x) + x^2$$

Activity A.1.9 *(∼12 min)*
Consider the following two polynomial maps.

$$S : \mathcal{P}^4 \to \mathcal{P}^3 \text{ given by } S(f(x)) = 2f'(x) - f''(x)$$

$$T : \mathcal{P}^2 \to \mathcal{P}^2 \text{ given by } T(f(x)) = f'(x) + x^2$$

Part 1: Compare $S(x^2 + x)$ with $S(x^2) + S(x)$, and compare $T(x^2 + x)$ with $T(x^2) + T(x)$. Which of these maps is definitely not linear?

Activity A.1.9 *(∼12 min)*
Consider the following two polynomial maps.

$$S : \mathcal{P}^4 \to \mathcal{P}^3 \text{ given by } S(f(x)) = 2f'(x) - f''(x)$$

$$T : \mathcal{P}^2 \to \mathcal{P}^2 \text{ given by } T(f(x)) = f'(x) + x^2$$

Part 1: Compare $S(x^2 + x)$ with $S(x^2) + S(x)$, and compare $T(x^2 + x)$ with $T(x^2) + T(x)$. Which of these maps is definitely not linear?
Part 2: Verify that $S(f + g) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$ is equal to $S(f) + S(g)$ for all polynomials f, g.

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Activity A.1.9 *($\sim$12 min)*
Consider the following two polynomial maps.

$$S : \mathcal{P}^4 \to \mathcal{P}^3 \text{ given by } S(f(x)) = 2f'(x) - f''(x)$$

$$T : \mathcal{P}^2 \to \mathcal{P}^2 \text{ given by } T(f(x)) = f'(x) + x^2$$

Part 1: Compare $S(x^2 + x)$ with $S(x^2) + S(x)$, and compare $T(x^2 + x)$ with $T(x^2) + T(x)$. Which of these maps is definitely not linear?
Part 2: Verify that $S(f + g) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$ is equal to $S(f) + S(g)$ for all polynomials f, g.
Part 3: Verify that $S(cf) = cS(f)$ for all real numbers c and polynomials f. Is S linear?

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Module A Section 2

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Remark A.2.1

Recall that a linear map $T : V \to W$ satisfies

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vecor space operations can be applied before or after the transformation without affecting the result.

Activity A.2.2 *(∼5 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity A.2.3 *(∼3 min)*

Suppose $T:\mathbb{R}^3\to\mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.2.4 *(∼2 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}-2\\0\\-3\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

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Activity A.2.5 *(∼5 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Do you have enough information to compute $T(\mathbf{v})$ for *any* $\mathbf{v} \in \mathbb{R}^3$?

(a) Yes.

(b) No, exactly one more piece of information is needed.

(c) No, an infinite amount of information would be necessary to compute the transformation of infinitely-many vectors.

Fact A.2.6

Consider any basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for V. Since every vector $\mathbf{v}$ can be written *uniquely* as a linear combination of basis vectors, $x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n$, we conclude that

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = x_1 T(\mathbf{b}_1) + \cdots + x_n T(\mathbf{b}_n).$$

Therefore any linear transformation $T : V \to W$ can be defined by just describing the values of $T(\mathbf{b}_i)$.

Put another way, the basis vectors **determine** the transformation T.

Definition A.2.7

Since linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$.

Example A.2.8

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T(\mathbf{e}_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix}$$

$$T(\mathbf{e}_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3)\end{bmatrix} = \begin{bmatrix}3 & -1 & 5\\2 & 4 & 0\end{bmatrix}.$$

Activity A.2.9 *(∼5 min)*
TODO Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard basis.

Activity A.2.10 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Activity A.2.11 *(∼10 min)*
Let $D:\mathcal{P}^3\to\mathcal{P}^2$ be the derivative map $D(f(x))=f'(x)$. (Earlier we showed this is a linear transformation.)

Activity A.2.11 *(∼10 min)*
Let $D : \mathcal{P}^3 \to \mathcal{P}^2$ be the derivative map $D(f(x)) = f'(x)$. (Earlier we showed this is a linear transformation.)
Part 1: Write down an equivalent linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ by converting $\{1, x, x^2, x^3\}$ and $\{D(1), D(x), D(x^2), D(x^3)\}$ into appropriate vectors in $\mathbb{R}^4$ and $\mathbb{R}^3$.

Activity A.2.11 *($\sim$10 min)*
Let $D : \mathcal{P}^3 \to \mathcal{P}^2$ be the derivative map $D(f(x)) = f'(x)$. (Earlier we showed this is a linear transformation.)
Part 1: Write down an equivalent linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ by converting $\{1, x, x^2, x^3\}$ and $\{D(1), D(x), D(x^2), D(x^3)\}$ into appropriate vectors in $\mathbb{R}^4$ and $\mathbb{R}^3$.
Part 2: Write the standard matrix corresponding to T.

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Definition A.3.1

Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\mathbf{v}) \neq T(\mathbf{w})$ whenever $\mathbf{v} \neq \mathbf{w}$.

Activity A.3.2 *(∼5 min)*
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is T injective?

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Activity A.3.3 *(∼5 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Is T injective?

Definition A.3.4

Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.

Activity A.3.5 *(∼5 min)*
Let $T:\mathbb{R}^2\to\mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x\\y\\0\end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$.

Is T surjective?

Activity A.3.6 *(∼5 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Is T surjective?

Definition A.3.7

Let $T : V \to W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{\mathbf{v} \in V \;\middle|\; T(\mathbf{v}) = \mathbf{0}\right\}$$

Activity A.3.8 *(~5 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the kernel of T.

Activity A.3.9 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the kernel of T.

Activity A.3.10 *(∼10 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

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Activity A.3.10 *(~10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Write a system of equations whose solution set is the kernel.

Activity A.3.10 *($\sim$10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Write a system of equations whose solution set is the kernel.

Part 2: Use RREF(A) to solve the system of equations and find the kernel of T.

Activity A.3.10 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix
$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Write a system of equations whose solution set is the kernel.

Part 2: Use RREF(A) to solve the system of equations and find the kernel of T.

Part 3: Find a basis for the kernel of T.

Definition A.3.11

Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{ \mathbf{w} \in W \;\middle|\; \text{there is some } v \in V \text{ with } T(\mathbf{v}) = \mathbf{w} \right\}$$

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Activity A.3.12 *(∼5 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the image of T.

Activity A.3.13 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the image of T.

Activity A.3.14 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Activity A.3.14 *(∼10 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Find a convenient set of vectors $S \subseteq \mathbb{R}^2$ such that $\operatorname{span} S = \operatorname{Im} T$.

Activity A.3.14 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Find a convenient set of vectors $S \subseteq \mathbb{R}^2$ such that $\operatorname{span} S = \operatorname{Im} T$.

Part 2: Find a convenient basis for the image of T.

Observation A.3.15

Let $T : V \to W$ be a linear transformation with corresponding matrix A.

- If A is a matrix corresponding to T, the kernel is the solution set of the homogeneous system with coefficients given by A.
- If A is a matrix corresponding to T, the image is the span of the columns of A.

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Module A Section 4

Observation A.4.1

Let $T : V \to W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.
- The **kernel** of T is the set of all things that are mapped to $\mathbf{0}$. It is a subspace of V.
- The **image** of T is the set of all things in W that are mapped to by something in V. It is a subspace of W.

Activity A.4.2 *(∼5 min)*

Let $T : V \to W$ be a linear transformation where $\ker T = \{\mathbf{0}\}$. Can you answer either of the following questions about T?

(a) Is T injective?

(b) Is T surjective?

(Hint: If $T(\mathbf{v}) = T(\mathbf{w})$, then what is $T(\mathbf{v} - \mathbf{w})$?)

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Fact A.4.3

A linear transformation T is injective **if and only if** $\ker T = \{\mathbf{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

Activity A.4.4 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation where $\operatorname{Im} T = \operatorname{span}\left\{\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\-1\end{bmatrix}\right\}$.

Can you answer either of the following questions about T?

(a) Is T injective?

(b) Is T surjective?

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Fact A.4.5

A linear transformation $T : V \to W$ is surjective **if and only if** $\operatorname{Im} T = W$. Put another way, a surjective linear transformation may be recognized by its same codomain and image.

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Activity A.4.6 *(∼15 min)*

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of equivalent statements.

(a) T is injective

(b) T is surjective

(c) The kernel of T is trivial.

(d) The columns of A span $\mathbb{R}^m$

(e) The columns of A are linearly independent

(f) Every column of $\mathrm{RREF}(A)$ has a pivot.

(g) Every row of $\mathrm{RREF}(A)$ has a pivot.

(h) The image of T equals its codomain.

(i) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$

(j) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$ has exactly one solution.

Definition A.4.7
If $T : V \to W$ is both injective and surjective, it is called **bijective**.

Activity A.4.8 *(∼5 min)*

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

(a) The columns of A form a basis for $\mathbb{R}^m$

(b) RREF(A) is the identity matrix.

(c) The system of linear equations given by the augmented matrix $\left[A \mid \mathbf{b}\right]$ has exactly one solution for all $\mathbf{b} \in \mathbb{R}^m$.

Activity A.4.9 *(∼10 min)*
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.10 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.11 *($\sim$5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.12 *(∼5 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+y-z \\ 4x+y+z \\ 6x+2y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective
(b) T is injective but not surjective
(c) T is surjective but not injective
(d) T is bijective.