Section V.2

Remark V.2.1 Recall these definitions from last class:

- A **linear combination** of vectors is given by adding scalar multiples of those vectors, such as:

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

- The **span** of a set of vectors is the collection of all linear combinations of that set, such as:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \middle|\, a, b \in \mathbb{R}\right\}$$

Activity V.2.2 *(∼15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.
Part 2: Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.
Part 3: Given this solution set, does $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belong to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$?

Fact V.2.3 A vector $\vec{\mathbf{b}}$ belongs to $\operatorname{span}\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n\}$ if and only if the linear system corresponding to $[\vec{\mathbf{v}}_1 \, \dots \, \vec{\mathbf{v}}_n \,|\, \vec{\mathbf{b}}]$ is consistent.

Put another way, $\vec{\mathbf{b}}$ belongs to $\operatorname{span}\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_n\}$ exactly when $\operatorname{RREF}[\vec{\mathbf{v}}_1 \, \dots \, \vec{\mathbf{v}}_n \,|\, \vec{\mathbf{b}}]$ doesn't have a row $[0 \cdots 0 \,|\, 1]$ representing the contradiction $0 = 1$.

Activity V.2.4 *(∼10 min)* Determine if $\begin{bmatrix}3\\-2\\1\\5\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\\2\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\\2\end{bmatrix}\right\}$ by row-reducing an appropriate matrix.

Activity V.2.5 *(∼5 min)* Determine if $\begin{bmatrix}-1\\-9\\0\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ by row-reducing an appropriate matrix.

Activity V.2.6 *(∼10 min)* Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?
Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in $\mathbb{R}^4$. (Hint: What four numbers must you know to write a $\mathcal{P}^3$ polynomial?)
Part 2: Solve this equivalent exercise, and use its solution to answer the original question.

Activity V.2.7 *(∼5 min)* Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to $\operatorname{span}\left\{\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\right\}$?

Activity V.2.8 *(∼5 min)* Does the complex number $2i$ belong to $\operatorname{span}\{-3 + i, 6 - 2i\}$?