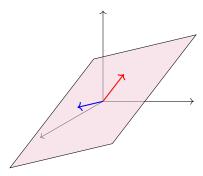
Section V.4

Definition V.35 A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space \mathbb{R}^3 .



Fact V.36 Any subset S of a vector space V that contains the additive identity $\overline{\mathbf{0}}$ satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a subspace, we need to check that addition and multiplication still make sense using only vectors from S. So we need to check two things:

- The set is closed under addition: for any $\vec{x}, \vec{y} \in S$, the sum $\vec{x} + \vec{y}$ is also in S.
- The set is closed under scalar multiplication: for any $\vec{x} \in S$ and scalar $c \in \mathbb{R}$, the product $c\vec{x}$ is also in S.

Activity V.37 (~15 min) Let
$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + 2y + z = 0 \right\}$$
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.

Part 1: Let $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S , so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\vec{\mathbf{v}} + \vec{\mathbf{w}} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Part 2: Let $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\vec{\mathbf{v}} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ also belongs to S for any $c \in \mathbb{R}$ by verifying an appropriate equation.

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$$
 also belongs to S by verifying that $(x+a) + 2(y+b) + (z+c) = 0$.

Part 2: Let
$$\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$$
, so $x + 2y + z = 0$. Show that $c\vec{\mathbf{v}} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ also belongs to S for any $c \in \mathbb{R}$ by

verifying an appropriate equation.

Part 3: Is S is a subspace of \mathbb{R}^3 ?

Activity V.38 (~10 min) Let
$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + 2y + z = 4 \right\}$$
. Choose a vector $\vec{\mathbf{v}} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = ?$, and show that $c\vec{\mathbf{v}}$ isn't in S . Is S a subspace of \mathbb{R}^3 ?

Remark V.39 Since 0 is a scalar and $0\vec{\mathbf{v}} = \vec{\mathbf{z}}$ for any vector $\vec{\mathbf{v}}$, a nonempty set that is closed under scalar multiplication must contain the zero vector $\vec{\mathbf{z}}$ for that vector space.

Put another way, you can check any of the following to show that a nonempty subset W isn't a subspace:

- Show that $\vec{\mathbf{0}} \notin W$.
- Find $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in W$ such that $\vec{\mathbf{u}} + \vec{\mathbf{v}} \notin W$.
- Find $c \in \mathbb{R}, \vec{\mathbf{v}} \in W$ such that $c\vec{\mathbf{v}} \notin W$.

If you cannot do any of these, then W can be proven to be a subspace by doing the following:

- Prove that $\vec{\mathbf{u}} + \vec{\mathbf{v}} \in W$ whenever $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in W$.
- Prove that $c\vec{\mathbf{v}} \in W$ whenever $c \in \mathbb{R}, \vec{\mathbf{v}} \in W$.

Activity V.40 (~ 20 min) Consider these subsets of \mathbb{R}^4 :

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| z = xy \right\}$$

- Part 1: Show R isn't a subspace by showing that $\vec{0} \notin R$.
- Part 2: Show S isn't a subspace by finding two vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in S$ such that $\vec{\mathbf{u}} + \vec{\mathbf{v}} \notin S$.
- Part 3: Show T isn't a subspace by finding a vector $\vec{\mathbf{v}} \in T$ such that $2\vec{\mathbf{v}} \notin T$.

Activity V.41 ($\sim 5 \text{ min}$) Let W be a subspace of a vector space V. How are span W and W related?

- (a) span W is bigger than W
- (b) span W is the same as W
- (c) span W is smaller than W

Fact V.42 If S is any subset of a vector space V, then since span S collects all possible linear combinations, span S is automatically a subspace of V.

In fact, span S is always the smallest subspace of V that contains all the vectors in S.