Module P: Applications of Linear Algebra

Module P Section 1

Definition P.1.1

In geology, a **phase** is any physically separable material in the system, such as various minerals or liquids.

A **component** is a chemical compound necessary to make up the phases; for historical reasons thise are usually oxides such as Calcium Oxide (CaO) or Silicone Dioxide $(\mathrm{SiO_2})$.

In a typical problem, a geologist knows how to build each phase from the components, and is interested in determining reactions among the different phases.

Activity P.1.2 (*∼5 min*)

Consider the 3 components $\mathbf{c}_1 = \mathrm{CaO}$, $\mathbf{c}_2 = \mathrm{MgO}$, and $\mathbf{c}_3 = \mathrm{SiO}_2$, and the 5 phases

$$\mathbf{p}_1 = \mathrm{Ca_3MgSi_2O_8} \qquad \mathbf{p}_2 = \mathrm{CaMgSiO_4} \qquad \mathbf{p}_3 = \mathrm{CaSiO_3}$$
$$\mathbf{p}_4 = \mathrm{CaMgSi_2O_6} \qquad \mathbf{p}_5 = \mathrm{Ca_2MgSi_2O_7}$$

Geologists will know

$$\mathbf{p}_1 = 3\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3 \qquad \mathbf{p}_2 = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 \qquad \mathbf{p}_3 = \mathbf{c}_1 + 0\mathbf{c}_2 + \mathbf{c}_3$$
$$\mathbf{p}_4 = \mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3 \qquad \mathbf{p}_5 = 2\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3$$

or more compactly,

$$\mathbf{p}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{p}_5 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Determine if the 5 phases are linearly dependent or linearly independent.

Activity P.1.3 *(∼15 min)*

Recall our five phases:

$$\mathbf{p}_1 = 3\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3 \qquad \mathbf{p}_2 = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 \qquad \mathbf{p}_3 = \mathbf{c}_1 + 0\mathbf{c}_2 + \mathbf{c}_3$$
$$\mathbf{p}_4 = \mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3 \qquad \mathbf{p}_5 = 2\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3$$

or more compactly,

$$\mathbf{p}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{p}_5 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Geologists want to find chemical reactions among the 5 phases; that is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\mathbf{p}_1 + x_2\mathbf{p}_2 + x_3\mathbf{p}_3 + x_4\mathbf{p}_4 + x_5\mathbf{p}_5 = 0.$$

Activity P.1.3 *(∼15 min)*

Recall our five phases:

$$\mathbf{p}_1 = 3\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3 \qquad \mathbf{p}_2 = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 \qquad \mathbf{p}_3 = \mathbf{c}_1 + 0\mathbf{c}_2 + \mathbf{c}_3$$

$$\mathbf{p}_4 = \mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3 \qquad \mathbf{p}_5 = 2\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3$$

or more compactly,

$$\mathbf{p}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{p}_5 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Geologists want to find chemical reactions among the 5 phases; that is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\mathbf{p}_1 + x_2\mathbf{p}_2 + x_3\mathbf{p}_3 + x_4\mathbf{p}_4 + x_5\mathbf{p}_5 = 0.$$

Part 1: Set up a system of equations that gives these chemical equations.

Activity P.1.3 *(∼15 min)*
Recall our five phases:

$$\mathbf{p}_1 = 3\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3 \qquad \mathbf{p}_2 = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 \qquad \mathbf{p}_3 = \mathbf{c}_1 + 0\mathbf{c}_2 + \mathbf{c}_3$$
$$\mathbf{p}_4 = \mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3 \qquad \mathbf{p}_5 = 2\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3$$

or more compactly,

$$\mathbf{p}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{p}_5 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Geologists want to find chemical reactions among the 5 phases; that is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\mathbf{p}_1 + x_2\mathbf{p}_2 + x_3\mathbf{p}_3 + x_4\mathbf{p}_4 + x_5\mathbf{p}_5 = 0.$$

Part 1: Set up a system of equations that gives these chemical equations.
Part 2: Find a basis for the solution set.

Activity P.1.3 *(∼15 min)*

Recall our five phases:

$$\mathbf{p}_1 = 3\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3 \qquad \mathbf{p}_2 = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 \qquad \mathbf{p}_3 = \mathbf{c}_1 + 0\mathbf{c}_2 + \mathbf{c}_3$$
$$\mathbf{p}_4 = \mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3 \qquad \mathbf{p}_5 = 2\mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3$$

or more compactly,

$$\mathbf{p}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{p}_5 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Geologists want to find chemical reactions among the 5 phases; that is, they want to find numbers x_1, x_2, x_3, x_4, x_5 such that

$$x_1\mathbf{p}_1 + x_2\mathbf{p}_2 + x_3\mathbf{p}_3 + x_4\mathbf{p}_4 + x_5\mathbf{p}_5 = 0.$$

Part 1: Set up a system of equations that gives these chemical equations.
Part 2: Find a basis for the solution set.
Part 3: Interpret each basis vector as a chemical equation.

Activity P.1.4 *(∼10 min)*

We found two basis vector $\begin{bmatrix}-1\\2\\2\\-1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\\0\\-1\end{bmatrix}$, corresponding to two chemical equations

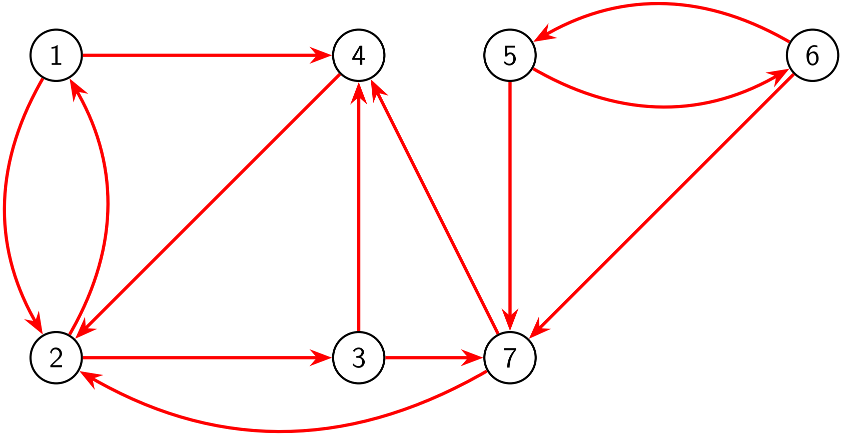
$$\begin{aligned} 2\mathbf{p}_2 + 2\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_4 \quad & 2\mathrm{CaMgSiO_4} + 2\mathrm{CaSiO_3} = \mathrm{Ca_3MgSi_2O_8} + \mathrm{CaMgSi_2O_6} \\ \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{p}_5 \quad & \mathrm{CaMgSiO_4} + \mathrm{CaSiO_3} = \mathrm{Ca_2MgSi_2O_7} \end{aligned}$$

Find a chemical equation among the five phases that does not involve $\mathbf{p}_2 = \mathrm{CaMgSiO_4}$.

Module P Section 2

Activity P.2.1 *(∼10 min)*
A \$700,000,000,000 Problem:

In the picture below, each circle represents a webpage, and each arrow represents a link from one page to another.



Based on how these pages link to each other, write a list of the 7 webpages in order from most imporant to least important.

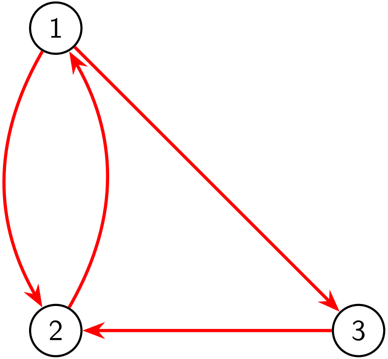
Observation P.2.2
The \$700,000,000,000 Idea:

Links are endorsements.

1. A webpage is important if it is linked to (endorsed) by important pages.
2. A webpage distributes its importance equally among all the pages it links to (endorses).

Example P.2.3

Consider this small network with only three pages. Let x_1, x_2, x_3 be the importance of the three pages respectively.

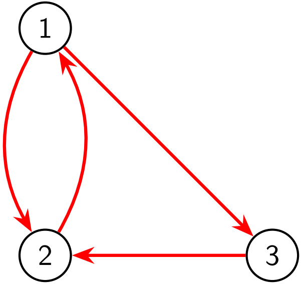


1. x_1 splits its endorsement in half between x_2 and x_3
2. x_2 sends all of its endorsement to x_1
3. x_3 sends all of its endorsement to x_2.

This corresponds to the **page rank system**

$$\begin{aligned} x_2 \quad\quad &= x_1 \\ \frac{1}{2}x_1 \quad + x_3 &= x_2 \\ \frac{1}{2}x_1 \quad\quad &= x_3 \end{aligned}$$

Observation P.2.4



$$
\begin{aligned}
x_2 \quad\quad\quad &= x_1 \\
\frac{1}{2}x_1 \quad + x_3 &= x_2 \\
\frac{1}{2}x_1 \quad\quad\quad &= x_3
\end{aligned}
$$

$$
\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

By writing this linear system in terms of matrix multiplication, we obtain the **page rank matrix** $A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$ and page rank vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Thus, computing the imporance of pages on a network is equivalent to solving the matrix equation $A\mathbf{x} = 1\mathbf{x}$.

Activity P.2.5 *(∼5 min)*

Thus, our \$700,000,000,000 problem is what kind of problem?

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) An antiderivative problem

(b) A bijection problem

(c) A cofactoring problem

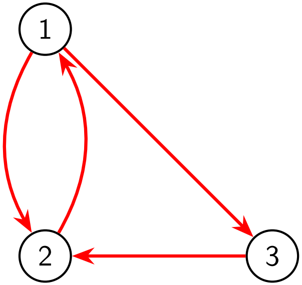
(d) A determinant problem

(e) An eigenvector problem

Activity P.2.6 *(∼10 min)*

Find a page rank vector $\mathbf{x}$ satisfying $A\mathbf{x} = 1\mathbf{x}$ for the following network's page rank matrix A.

That is, find the eigenspace associated with $\lambda = 1$ for the matrix A, and choose a vector from that eigenspace.

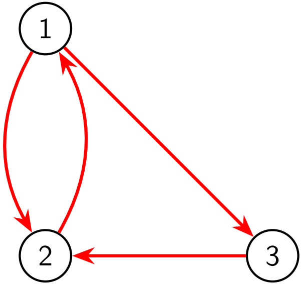


$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Observation P.2.7

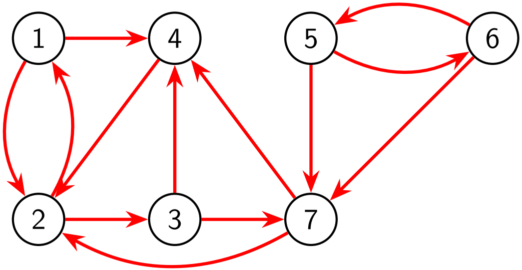
Row-reducing $A - I = \begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ yields the basic eigenvector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Therefore, we may conclude that pages 1 and 2 are equally important, and both pages are twice as important as page 3.



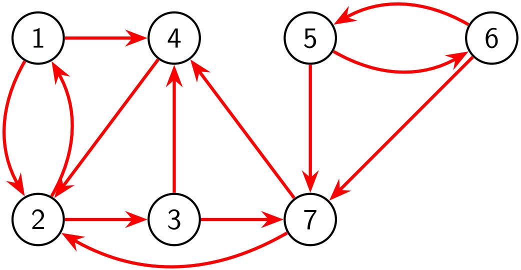
Activity P.2.8 *(∼5 min)*

Compute the 7×7 page rank matrix for the following network.



For example, since website 1 distributes its endorsement equally between 2 and 4, the first column is $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity P.2.9 *(∼10 min)*
Find a page rank vector for the given page rank matrix.

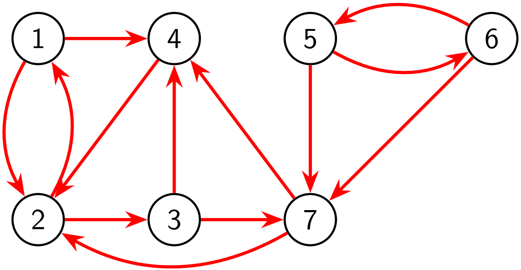


$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Which webpage is most important?

Observation P.2.10

Since a page rank vector for the network is given by $\mathbf{x}$, it's reasonable to consider page 2 as the most important page.



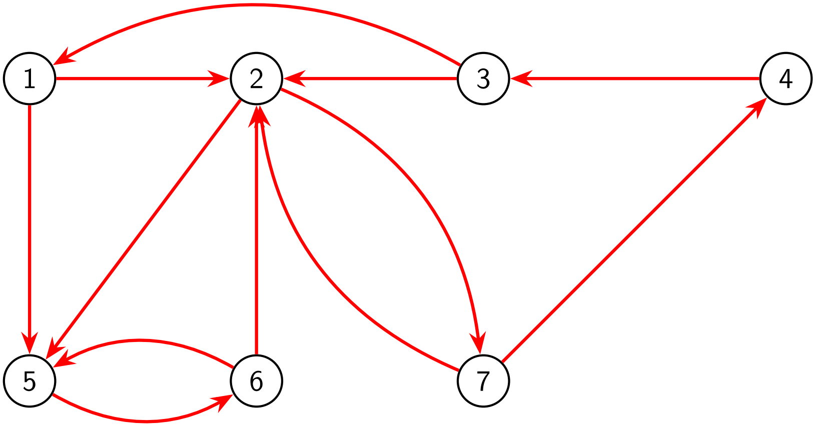
$$\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Based upon this page rank vector, here is a complete ranking of all seven pages from most important to least important:

$$2, 4, 1, 3, 7, 5, 6$$

Activity P.2.11 *(∼10 min)*

Given the following diagram, use a page rank vector to rank the pages 1 through 7 in order from most important to least important.



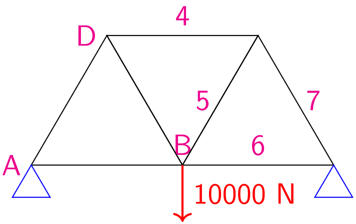
Module P Section 3

Example P.3.1

In engineering, a **truss** is a structure designed from several beams of material called **struts**, assembled to behave as a single object.

Activity P.3.2 *(∼10 min)*

Consider the representation of a simple truss pictured below. All of the seven struts are of equal length, affixed to two anchor points with a 10000N load applied to the center of its base.



Which of the following must hold?

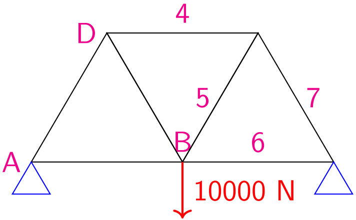
a) All of the struts will experience compression.

b) All of the struts will experience tension.

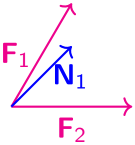
c) Some of the struts will be compressed, and others will be tensioned.

Observation P.3.3

Consider the truss pictured below with two fixed anchor points and a 10000 N load (assume all triangles are equilateral).

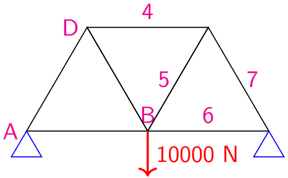


The horizontal and vertical forces must balance at each node. For example, at the bottom left node there are 3 forces acting.

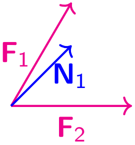


We adhere to the convention that a compression force on a strut is positive, while a negative force represents tension.

Observation P.3.4



We decompose the first node into vertical and horizontal forces:



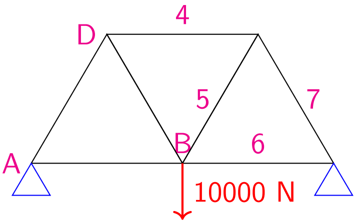
$$\mathbf{F}_1 = F_1 \begin{bmatrix} \cos(60^\circ) \\ \sin(60^\circ) \end{bmatrix}$$

$$\mathbf{N}_1 = \begin{bmatrix} N_{1,h} \\ N_{1,v} \end{bmatrix}$$

$$F_1 \sin(60^\circ) + N_{1,v} = 0$$
$$F_1 \cos(60^\circ) + N_{1,h} + F_2 = 0$$

Activity P.3.5 *(∼10 min)*

Consider the truss pictured below with two fixed anchor points and a 10000 N load (assume all triangles are equilateral).

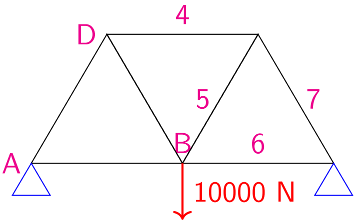


From the bottom left node we obtained 2 equations in the four variables

- F_1 (compression force on strut one)
- $N_{1,v}$ and $N_{1,h}$ (horizontal and vertical components of the normal force from the left anchor)
- F_2 (compression force on strut 2).

Activity P.3.5 *(∼10 min)*

Consider the truss pictured below with two fixed anchor points and a 10000 N load (assume all triangles are equilateral).

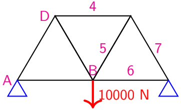


From the bottom left node we obtained 2 equations in the four variables

- F_1 (compression force on strut one)
- $N_{1,v}$ and $N_{1,h}$ (horizontal and vertical components of the normal force from the left anchor)
- F_2 (compression force on strut 2).

Part 1: Determine how many total equations there will be after accounting for all of the nodes, and and list all of the variables. You do not need to actually determine all of the equations.

Activity P.3.6 *(∼10 min)*

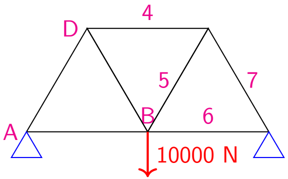


The resulting system is

$$
\begin{aligned}
N_{1,v} + (\sin(60^\circ))F_1 &= 0\\
N_{1,h} + (\cos(60^\circ))F_1 + F_2 &= 0\\
-(\sin(60^\circ))F_1 - (\sin(60^\circ)F_3 &= 0\\
-(\cos(60^\circ))F_1 + (\cos(60^\circ)F_3 + F_4 &= 0\\
(\sin(60^\circ)F_3 + (\sin(60^\circ))F_5 &= 10000\\
-F_2 - (\cos(60^\circ)F_3 + (\cos(60^\circ))F_5 + F_6 &= 0\\
-(\sin(60^\circ))F_5 - (\sin(60^\circ))F_7 &= 0\\
-F_4 - (\cos(60^\circ))F_5 + (\cos(60^\circ))F_7 &= 0\\
N_{2,v} + (\sin(60^\circ))F_7 &= 0\\
N_{2,h} - F_6 - (\cos(60^\circ))F_7 &= 0
\end{aligned}
$$

Solve this system to determine which struts are compressed and which are in tension.

Observation P.3.7



The determined part of the solution is

$$N_{1,v} = N_{2,v} = 5000$$
$$F_1 = F_4 = F_7 = -5882.4$$
$$F_3 = F_5 = 5882.4$$

So struts 1,4,7 are in tension, while struts 3 and 5 are compressed.
The forces on struts 2 and 6 (and the horizontal normal forces) are not strictly determined in this setting.