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Date:	

SEMIFINAL   Math 237 – Linear Algebra
Version 3   Fall 2017
Work any problems you wish on the provided answer sheets; take care to label the standard for each response. Show all work and justify all of your answers. Answers without work or sufficient reasoning will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array}\right]$$

Solution:

$$x_1 + 4x_3 = 1$$
$$x_2 - x_3 = 7$$
$$x_1 - x_2 + 3x_3 = -1$$

☐

E2. Find RREF A, where

$$A = \left[\begin{array}{cccc|c} 2 & 2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 4 & 5 \\ 3 & 3 & -1 & -2 & 1 \end{array}\right]$$

Solution:

$$\text{RREF } A = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

☐

E3. Find the solution set for the following system of linear equations.

$$2x_1 - 2x_2 + 6x_3 - x_4 = -1$$
$$3x_1 + 6x_3 + x_4 = 5$$
$$-4x_1 + x_2 - 9x_3 + 2x_4 = -7$$

Solution: Let $A = \left[\begin{array}{cccc|c} 2 & -2 & 6 & -1 & -1 \\ 3 & 0 & 6 & 1 & 5 \\ -4 & 1 & -9 & 2 & -7 \end{array}\right]$, so RREF $A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$. It follows that the

solution set is given by $\begin{bmatrix} 2 - 2a \\ 3 + a \\ a \\ -1 \end{bmatrix}$ for all real numbers a.

☐

E4. Find a basis for the solution set to the homogeneous system of equations

$$4x_1 + 4x_2 + 3x_3 - 6x_4 = 0$$
$$-2x_3 - 4x_4 = 0$$
$$2x_1 + 2x_2 + x_3 - 4x_4 = 0$$

Solution: Let $A = \begin{bmatrix} 4 & 4 & 3 & -6 & | & 0 \\ 0 & 0 & -2 & -4 & | & 0 \\ 2 & 2 & 1 & -4 & | & 0 \end{bmatrix}$, so RREF $A = \begin{bmatrix} 1 & 1 & 0 & -3 & | & 0 \\ 0 & 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$. It follows that the basis for

the solution set is given by $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

$\square$

V1. Let V be the set of all points on the parabola $y = x^2$ with the operations, for any $(x_1, y_1), (x_2, y_2) \in V$, $c \in \mathbb{R}$,

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 - x_2, y_1 + y_2 - 2x_1x_2)$$
$$c \odot (x_1, y_1) = (cx_1, c^2 y_1)$$

(a) Show that scalar multiplication **distributes scalars** over vector addition:
$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$.

(b) Determine if V is a vector space or not. Justify your answer.

Solution:

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1 - x_2, y_1 + y_2 - 2x_1x_2) = (c(x_1 - x_2), c^2(y_1 + y_2 - 2x_1x_2))$$

$$c \odot (x_1, y_1) \oplus c \odot (x_2, y_2) = (cx_1, c^2 y_1) \oplus (cx_2, c^2 y_2) = (cx_1 - cx_2, c^2 y_1 + c^2 y_2 - 2(cx_1)(cx_2))$$

Not a vector space as addition is not commutative.

$\square$

V2. Determine if $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ belongs to the span of the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution: Since

$$\text{RREF}\left(\begin{bmatrix} 1 & 2 & 0 & | & 3 \\ 2 & 4 & 0 & | & -2 \\ -3 & -6 & 0 & | & 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

contains the contradiction $0 = 1$, $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ is not a linear combination of the three vectors.

$\square$

V3. Determine if the vectors $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ span $\mathbb{R}^4$.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -\dfrac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, the vectors do not span $\mathbb{R}^4$.

$\square$

V4. Let W be the set of all $\mathbb{R}^3$ vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $x + y + z = 0$ (this forms a plane). Determine if W is a subspace of $\mathbb{R}^3$.

Solution: Yes, because $z = -x - y$ and $a\begin{bmatrix} x_1 \\ y_1 \\ -x_1 - y_1 \end{bmatrix} + b\begin{bmatrix} x_2 \\ y_2 \\ -x_2 - y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \\ -(ax_1 + bx_2) - (ay_1 + by_2) \end{bmatrix}$.

Alternately, yes because W is isomorphic to $\mathbb{R}^2$.

$\square$

S1. Determine if the set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}$ is linearly dependent or linearly independent

Solution:

$$\text{RREF}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\dfrac{1}{2} \\ 0 & 1 & \dfrac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, the set is linearly dependent.

$\square$

S2. Determine if the set $\left\{ 2x^2 - x + 3, 2x^2 + 2, -x^2 + 4x + 1 \right\}$ is a basis of $\mathcal{P}^2$.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 2 & 2 & -1 \\ -1 & 0 & 4 \\ 3 & 2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis.

$\square$

S3. Let $W = \text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -8 \\ -1 \end{bmatrix} \right\}$. Find a basis for this vector space.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & -8 \\ 1 & 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis of W.

S4. Let $W = \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\} \right)$. Find the dimension of W.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has two pivot columns, so W has dimension 2.

□

A1. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - 5x_3 \end{bmatrix}.$$

Write the matrix for T with respect to the standard bases of $\mathbb{R}^4$ and $\mathbb{R}^2$.

Solution:

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & -5 & 0 \end{bmatrix}$$

□

A2. Determine if the map $T : \mathcal{P}^6 \to \mathcal{P}^6$ given by $T(f) = f(x) - f(0)$ is a linear transformation or not.

A3. Determine if the following linear maps are injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 3x + 2y \\ x - y \\ x + 4y \end{bmatrix}$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y + z \\ 2y + 3z \\ x - y - 2z \end{bmatrix}$

Solution:

(a)

$$\text{RREF}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\dfrac{1}{2} \\ 0 & 1 & \dfrac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, T is not injective. Since there is a zero row, T is not surjective.

(b)

$$\text{RREF}\left(\begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since all columns are pivot columns, S is injective. Since there is a zero row, S is not surjective.

□

A4. Let $T : \mathbb{R}^{2 \times 2} \to \mathbb{R}^3$ be the linear map given by $T\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = \begin{bmatrix} 8x - 3y - z + 4w \\ y + 3z - 4w \\ -7x + 3y + 2z - 5w \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 8 & -3 & -1 & 4 \\ 0 & 1 & 3 & -4 \\ -7 & 3 & 2 & -5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -1 & -3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for the kernel.

□

M1. Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \qquad\qquad C = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution: AC is the only one that can be computed, and

$$AC = \begin{bmatrix} 9 & -2 & 14 \\ 1 & 0 & 2 \end{bmatrix}$$

□

M2. Determine if the matrix $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix}$ is invertible.

Solution:

$$\text{RREF}\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since it is not row equivalent to the identity matrix, it is not invertible.

□

M3. Compute the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution:

$$\text{RREF}(A|I) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -11 & 37 \\ 0 & 1 & 0 & 0 & 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 1 & 2 & -11 & 37 \\ 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\square$

G1. Compute the determinant of the matrix $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$.

Solution:

$$\det \begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix} + (-2)\det \begin{bmatrix} 3 & 0 & 4 \\ 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = -1(-4) + (-2)(20) = -36$$

$\square$

G2. Compute the eigenvalues, along with their algebraic multiplicities, of the matrix $\begin{bmatrix} 9 & -3 & 2 \\ 23 & -8 & 5 \\ -1 & 0 & 0 \end{bmatrix}$.

Solution: 1 (with algebraic multiplicy 2), and -1 (with algebraic multiplicity 1).

$\square$

G3. Find the eigenspace associated to the eigenvalue 2 in the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 11 & -6 & 1 & -1 \\ -9 & 5 & 1 & 3 \end{bmatrix}$.

Solution: The eigenspace is spanned by $\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

$\square$

G4. Compute the geometric multiplicity of the eigenvalue 2 in the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 11 & -6 & 1 & -1 \\ -9 & 5 & 1 & 3 \end{bmatrix}$.

Solution: The eigenspace is spanned by $\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, so the geometric multiplicity is 2.

$\square$

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