

# Module G: Geometry of Linear Maps

# How can we understand linear maps geometrically?

## Module G

Section 1

Section 2

Section 3

At the end of this module, students will be able to...

- G1. Row operations and Determinants.** ... describe how a row operation affects the determinant of a matrix.
- G2. Determinants.** ... compute the determinant of a  $4 \times 4$  matrix.
- G3. Eigenvalues.** ... find the eigenvalues of a  $2 \times 2$  matrix.
- G4. Eigenvectors.** ... find a basis for the eigenspace of a  $4 \times 4$  matrix associated with a given eigenvalue.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **A2**.
- Recall and use the definition of a linear transformation **A1**.
- Find all roots of quadratic polynomials (including complex ones).
- Interpret the statement “ $A$  is an invertible matrix” in many equivalent ways in different contexts.

## Module G

Section 1

Section 2

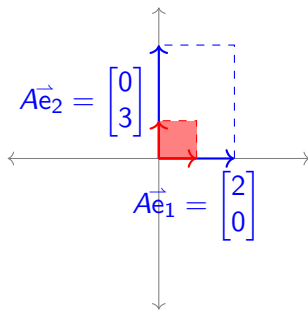
Section 3

The following resources will help you prepare for this module.

- Finding the area of a parallelogram (Khan Academy):  
<http://bit.ly/2B05iWx>
- Factoring quadratics (Khan Academy): <http://bit.ly/1XjfbV2>
- Factoring quadratics using area models (Youtube):  
<https://youtu.be/Aa-v1EK7DR4>
- Finding complex roots of quadratics (Youtube):  
<https://www.youtube.com/watch?v=2yBhDsNE0wg>

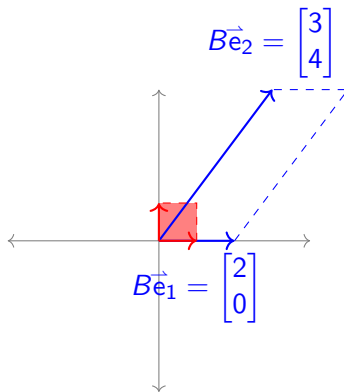
# Module G Section 1

**Activity G.1** ( $\sim 5$  min) The image below illustrates how the linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$  transforms the unit square.



- (a) What are the lengths of  $A\vec{e}_1$  and  $A\vec{e}_2$ ?
- (b) What is the area of the transformed unit square?

**Activity G.2** ( $\sim 5$  min) The image below illustrates how the linear transformation  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$  transforms the unit square.



- (a) What are the lengths of  $B\vec{e}_1$  and  $B\vec{e}_2$ ?
- (b) What is the area of the transformed unit square?

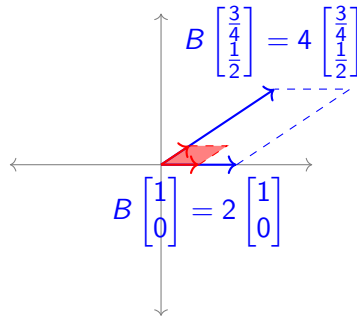


**Observation G.3**

It is possible to find two nonparallel vectors that are scaled but not rotated by the linear map given by  $B$ .

$$B\vec{e}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\vec{e}_1$$

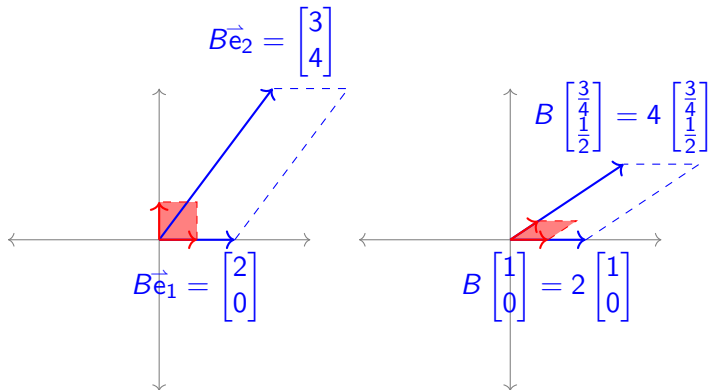
$$B \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$



The process for finding such vectors will be covered later in this module.

## Observation G.4

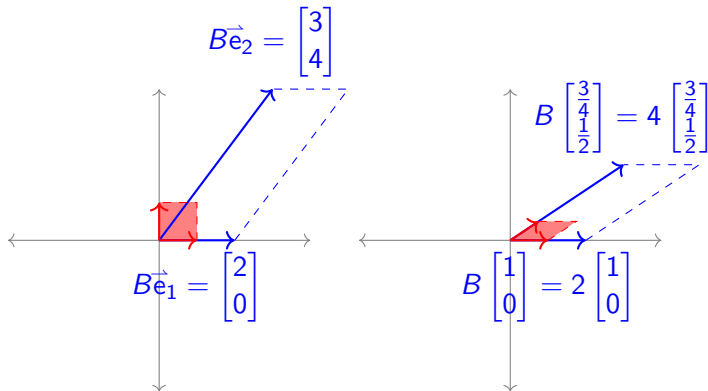
Notice that while a linear map can transform vectors in various ways, linear maps always transform parallelograms into parallelograms, and these areas are always transformed by the same factor: in the case of  $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ , this factor is 8.



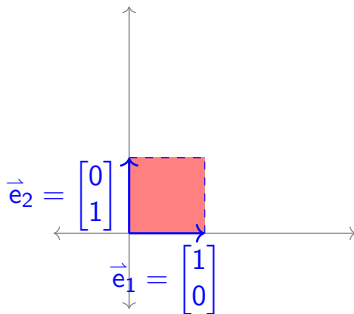
Since this change in area is always the same for a given linear map, it will be equal to the value of the transformed unit square (which begins with area 1).

## Remark G.5

We will define the **determinant** of a square matrix  $A$ , or  $\det(A)$  for short, to be the factor by which  $A$  scales areas. In order to figure out how to compute it, we first figure out the properties it must satisfy.

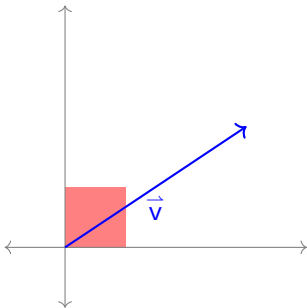


**Activity G.6** ( $\sim 2$  min) The transformation of the unit square by the standard matrix  $[\vec{e}_1 \ \vec{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$  is illustrated below. What is  $\det([\vec{e}_1 \ \vec{e}_2]) = \det(I)$ , the area of the transformed unit square shown here?



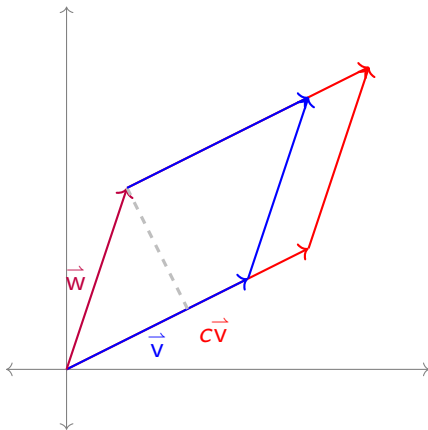
- a) 0
- b) 1
- c) 2
- d) 4

**Activity G.7** ( $\sim 2$  min) The transformation of the unit square by the standard matrix  $[\vec{v} \ \vec{v}]$  is illustrated below: both  $T(\vec{e}_1) = T(\vec{e}_2) = \vec{v}$ . What is  $\det([\vec{v} \ \vec{v}])$ , the area of the transformed unit square shown here?



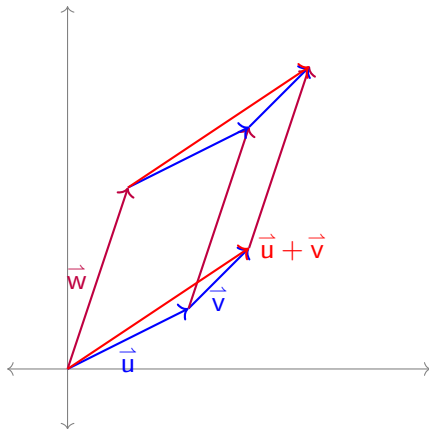
- a) 0
- b) 1
- c) 2
- d) 4

**Activity G.8** ( $\sim 5$  min) The transformations of the unit square by the standard matrices  $[\vec{v} \ \vec{w}]$  and  $[c\vec{v} \ \vec{w}]$  are illustrated below. Describe the value of  $\det([c\vec{v} \ \vec{w}])$ .



- a)  $\det([\vec{v} \ \vec{w}])$
- b)  $\det([\vec{v} \ \vec{w}]) + c \det([\vec{v} \ \vec{w}])$
- c)  $c \det([\vec{v} \ \vec{w}])$
- d) Cannot be determined from this information.

**Activity G.9** ( $\sim 5$  min) The transformations of unit squares by the standard matrices  $[\vec{u} \ \vec{w}]$ ,  $[\vec{v} \ \vec{w}]$  and  $[\vec{u} + \vec{v} \ \vec{w}]$  are illustrated below. Describe the value of  $\det([\vec{u} + \vec{v} \ \vec{w}])$ .



- a)  $\det([\vec{u} \ \vec{w}]) = \det([\vec{v} \ \vec{w}])$
- b)  $\det([\vec{u} \ \vec{w}]) + \det([\vec{v} \ \vec{w}])$
- c)  $\det([\vec{u} \ \vec{w}]) \det([\vec{v} \ \vec{w}])$
- d) Cannot be determined from this information.

## Definition G.10

The **determinant** is the unique function  $\det : M_{n,n} \rightarrow \mathbb{R}$  satisfying these properties:

P1:  $\det(I) = 1$

P2:  $\det(A) = 0$  whenever two columns of the matrix are identical.

P3:  $\det[\cdots \ c\vec{v} \ \cdots] = c \det[\cdots \ \vec{v} \ \cdots]$ , assuming no other columns change.

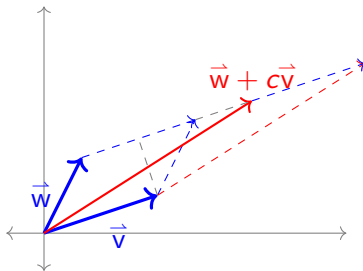
P4:  $\det[\cdots \ \vec{v} + \vec{w} \ \cdots] = \det[\cdots \ \vec{v} \ \cdots] + \det[\cdots \ \vec{w} \ \cdots]$ , assuming no other columns change.

Note that these last two properties together can be phrased as “The determinant is linear in each column.”



**Observation G.11**

The determinant must also satisfy other properties. Consider  $\det(\begin{bmatrix} \vec{v} & \vec{w} + c\vec{v} \end{bmatrix})$  and  $\det(\begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix})$ .



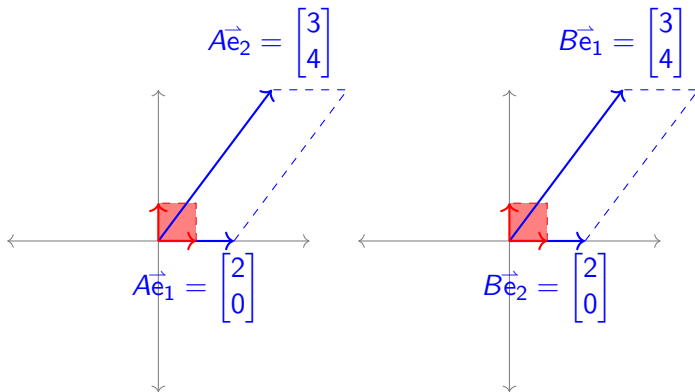
The base of both parallelograms is  $\vec{v}$ , while the height has not changed, so the determinant does not change either. This can also be proven using the other properties of the determinant:

$$\begin{aligned}
 \det(\begin{bmatrix} \vec{v} + c\vec{w} & \vec{w} \end{bmatrix}) &= \det(\begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}) + \det(\begin{bmatrix} c\vec{w} & \vec{w} \end{bmatrix}) \\
 &= \det(\begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}) + c \det(\begin{bmatrix} \vec{w} & \vec{w} \end{bmatrix}) \\
 &= \det(\begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}) + c \cdot 0 \\
 &= \det(\begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix})
 \end{aligned}$$

**Remark G.12**

Swapping columns may be thought of as a reflection, which is represented by a negative determinant. For example, the following matrices transform the unit square into the same parallelogram, but the second matrix reflects its orientation.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \quad \det A = 8 \qquad B = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix} \quad \det B = -8$$



**Observation G.13**

The fact that swapping columns multiplies determinants by a negative may be verified by adding and subtracting columns.

$$\begin{aligned}\det([\vec{v} \quad \vec{w}]) &= \det([\vec{v} + \vec{w} \quad \vec{w}]) \\ &= \det([\vec{v} + \vec{w} \quad \vec{w} - (\vec{v} + \vec{w})]) \\ &= \det([\vec{v} + \vec{w} \quad -\vec{v}]) \\ &= \det([\vec{v} + \vec{w} - \vec{v} \quad -\vec{v}]) \\ &= \det([\vec{w} \quad -\vec{v}]) \\ &= -\det([\vec{w} \quad \vec{v}])\end{aligned}$$

**Fact G.14**

To summarize, we've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant in the following way:

- (a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c \det([\cdots \vec{v} \cdots]) = \det([\cdots c\vec{v} \cdots])$$

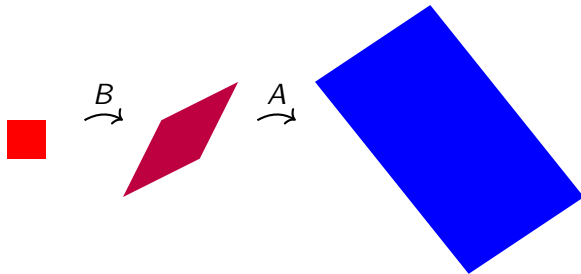
- (b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \vec{v} \cdots \vec{w} \cdots]) = -\det([\cdots \vec{w} \cdots \vec{v} \cdots])$$

- (c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \vec{v} \cdots \vec{w} \cdots]) = \det([\cdots \vec{v} + c\vec{w} \cdots \vec{w} \cdots])$$

**Activity G.15** ( $\sim 5$  min) The transformation given by the standard matrix  $A$  scales areas by 4, and the transformation given by the standard matrix  $B$  scales areas by 3. By what factor does the transformation given by the standard matrix  $AB$  scale areas?



- (a) 1
- (b) 7
- (c) 12
- (d) Cannot be determined

**Fact G.16**

Since the transformation given by the standard matrix  $AB$  is obtained by applying the transformations given by  $A$  and  $B$ , it follows that

$$\det(AB) = \det(A) \det(B) = \det(B) \det(A) = \det(BA)$$

# Module G Section 2

**Remark G.17**

Recall that row operations may be produced by matrix multiplication.

- Multiply the first row of  $A$  by  $c$ : 
$$\begin{bmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

- Swap the first and second row of  $A$ : 
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

- Add  $c$  times the third row to the first row of  $A$ : 
$$\begin{bmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$



**Fact G.18**

The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row:  $\det \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = c \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = c$

- Swapping rows:  $\det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -1$

- Adding a row multiple to another row:

$$\det \begin{bmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & c-1c & 0 \\ 0 & 1 & 0-0c & 0 \\ 0 & 0 & 1-0c & 0 \\ 0 & 0 & 0-0c & 1 \end{bmatrix} = \det(I) = 1$$

**Activity G.19** ( $\sim 5$  min) Consider the row operation  $R_1 + 4R_3 \rightarrow R_1$  applied as follows to show  $A \sim B$ :

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 + 4(9) & 2 + 4(10) & 3 + 4(11) & 4 + 4(12) \\ 5 & 6 & 6 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix  $R$  such that  $B = RA$ , by applying the same row operation to

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Find  $\det R$  by comparing with the previous slide.

(c) If  $C \in M_{3,3}$  is a matrix with  $\det(C) = -3$ , find

$$\det(RC) = \det(R) \det(C).$$

**Activity G.20** ( $\sim 5$  min) Consider the row operation  $R_1 \leftrightarrow R_3$  applied as follows to show  $A \sim B$ :

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

- (a) Find a matrix  $R$  such that  $B = RA$ , by applying the same row operation to  $I$ .
- (b) If  $C \in M_{3,3}$  is a matrix with  $\det(C) = 5$ , find  $\det(RC)$ .

**Activity G.21** ( $\sim 5$  min) Consider the row operation  $3R_2 \rightarrow R_2$  applied as follows to show  $A \sim B$ :

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3(5) & 3(6) & 3(7) & 3(8) \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

- (a) Find a matrix  $R$  such that  $B = RA$ .
- (b) If  $C \in M_{3,3}$  is a matrix with  $\det(C) = -7$ , find  $\det(RC)$ .

**Remark G.22**

Recall that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying columns by scalars:

$$\det([\cdots \ c\vec{v} \ \cdots]) = c \det([\cdots \ \vec{v} \ \cdots])$$

(b) Swapping two columns:

$$\det([\cdots \ \vec{v} \ \cdots \ \vec{w} \ \cdots]) = -\det([\cdots \ \vec{w} \ \cdots \ \vec{v} \ \cdots])$$

(c) Adding a multiple of a column to another column:

$$\det([\cdots \ \vec{v} \ \cdots \ \vec{w} \ \cdots]) = \det([\cdots \ \vec{v} + c\vec{w} \ \cdots \ \vec{w} \ \cdots])$$

**Remark G.23**

The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row: 
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Swapping rows: 
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Adding a row multiple to another row: 
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**Fact G.24**

Thus we can also use row operations to simplify determinants:

1. Multiplying rows by scalars:  $\det \begin{bmatrix} \vdots \\ cR \\ \vdots \end{bmatrix} = c \det \begin{bmatrix} \vdots \\ R \\ \vdots \end{bmatrix}$

2. Swapping two rows:  $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ S \\ \vdots \\ R \\ \vdots \end{bmatrix}$

3. Adding multiples of rows to other rows:  $\det \begin{bmatrix} \vdots \\ R \\ \vdots \\ S \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ R + cS \\ \vdots \\ S \\ \vdots \end{bmatrix}$

**Observation G.25**

So we may compute the determinant of  $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$  by manipulating its rows/columns to reduce the matrix to  $I$ :

$$\begin{aligned}\det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} &= 2 \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \\ &= 2 \det \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \\ &= -2 \det \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ &= -2 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= -2\end{aligned}$$

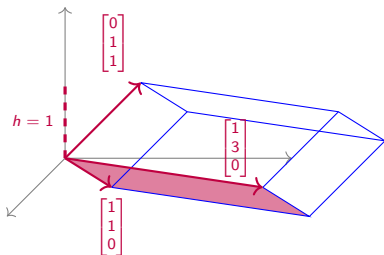


## Remark G.26

So we see that row reducing all the way into RREF gives us a method of computing determinants!

However, we learned in module E that this can be tedious for large matrices. Thus, we will try to figure out how to turn the determinant of a larger matrix into the determinant of a smaller matrix.

**Activity G.27** ( $\sim 5$  min) The following image illustrates the transformation of the unit cube by the matrix  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ .



Recall that for this solid  $V = Bh$ , where  $h$  is the height of the solid and  $B$  is the area of its parallelogram base. So what must its volume be?

(a)  $\det \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

(b)  $\det \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

(c)  $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(d)  $\det \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

**Fact G.28**

If row  $i$  contains all zeros except for a 1 on the main (upper-left to lower-right) diagonal, then both column and row  $i$  may be removed without changing the value of the determinant.

$$\det \begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Since row and column operations affect the determinants in the same way, the same technique works for a column of all zeros except for a 1 on the main diagonal.

$$\det \begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

**Activity G.29** ( $\sim 5$  min) Remove an appropriate row and column of  
 $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$  to simplify the determinant to a  $2 \times 2$  determinant.

**Activity G.30** (*~5 min*) Simplify  $\det \begin{bmatrix} 0 & 3 & -2 \\ 2 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$  to a multiple of a  $2 \times 2$  determinant by first doing the following:

- Factor out a 2 from a column.
- Swap rows or columns to put a 1 on the main diagonal.

**Activity G.31** (*~5 min*) Simplify  $\det \begin{bmatrix} 4 & -2 & 2 \\ 3 & 1 & 4 \\ 1 & -1 & 3 \end{bmatrix}$  to a multiple of a  $2 \times 2$  determinant by first doing the following:

- Use row/column operations to create two zeroes in the same row or column.
- Factor/swap as needed to get a row/column of all zeroes except a 1 on the main diagonal.

**Observation G.32**

Using row/column operations, you can introduce zeros and reduce dimension to whittle down the determinant of a large matrix to a determinant of a smaller matrix.

$$\begin{aligned}
 \det \begin{bmatrix} 4 & 3 & 0 & 1 \\ 2 & -2 & 4 & 0 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} &= \det \begin{bmatrix} 4 & 3 & 0 & 1 \\ 6 & -18 & 0 & -20 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} = \det \begin{bmatrix} 4 & 3 & 1 \\ 6 & -18 & -20 \\ 2 & 8 & 3 \end{bmatrix} \\
 &= \dots = -2 \det \begin{bmatrix} 1 & 3 & 4 \\ 0 & 21 & 43 \\ 0 & -1 & -10 \end{bmatrix} = -2 \det \begin{bmatrix} 21 & 43 \\ -1 & -10 \end{bmatrix} \\
 &= \dots = -2 \det \begin{bmatrix} -167 & 21 \\ 0 & 1 \end{bmatrix} = -2 \det[-167] \\
 &= -2(-167) \det(I) = 334
 \end{aligned}$$

**Activity G.33** (*~10 min*) Rewrite

$$\det \begin{bmatrix} 2 & 1 & -2 & 1 \\ 3 & 0 & 1 & 4 \\ -2 & 2 & 3 & 0 \\ -2 & 0 & -3 & -3 \end{bmatrix}$$

as a multiple of a determinant of a  $3 \times 3$  matrix.



**Activity G.34** (*~20 min*) Compute  $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$  by using any combination of row/column operations.

**Observation G.35**

Another option is to take advantage of the fact that the determinant is linear in each row or column. This approach is called **Laplace expansion** or **cofactor expansion**.

For example, since  $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ ,

$$\begin{aligned}
 \det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix} &= 1 \det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 1 & 0 & 0 \end{bmatrix} + 2 \det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 1 & 0 \end{bmatrix} + 4 \det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= -1 \det \begin{bmatrix} 5 & 3 & 2 \\ 5 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 2 & 5 & 3 \\ -1 & 5 & 3 \\ 0 & 0 & 1 \end{bmatrix} + 4 \det \begin{bmatrix} 2 & 3 & 5 \\ -1 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= -\det \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} - 2 \det \begin{bmatrix} 2 & 5 \\ -1 & 5 \end{bmatrix} + 4 \det \begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix}
 \end{aligned}$$

**Observation G.36**

Applying Laplace expansion to a  $2 \times 2$  matrix yields a short formula you may have seen:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} + b \det \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} = a \det \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} - b \det \begin{bmatrix} 1 & 0 \\ d & c \end{bmatrix} = ad - bc.$$

There are formulas for the determinants of larger matrices, but they can be pretty tedious to use. For example, writing out a formula for a  $4 \times 4$  determinant would require 24 different terms!

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}(a_{22}(a_{33}a_{44} - a_{43}a_{34}) - a_{23}(a_{32}a_{44} - a_{42}a_{34}) + \dots) + \dots$$

So this is why we either use Laplace expansion or row/column operations directly.

**Activity G.37** ( $\sim 5$  min) Based on what we've done today, which technique is easier for computing determinants?

- (a) Memorizing formulas.
- (b) Using row/column operations.
- (c) Laplace expansion.
- (d) Some other technique (be prepared to describe it).

**Activity G.38** ( $\sim 10$  min) Use your preferred technique to compute

$$\det \begin{bmatrix} 4 & -3 & 0 & 0 \\ 1 & -3 & 2 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & -3 & 2 & -2 \end{bmatrix}.$$

# Module G Section 3

**Activity G.39** ( $\sim 5$  min) An invertible matrix  $M$  and its inverse  $M^{-1}$  are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Which of the following is equal to  $\det(M) \det(M^{-1})$ ?

- a)  $-1$
- b)  $0$
- c)  $1$
- d)  $4$

**Fact G.40**

- For every invertible matrix  $M$ ,

$$\det(M) \det(M^{-1}) = \det(I) = 1$$

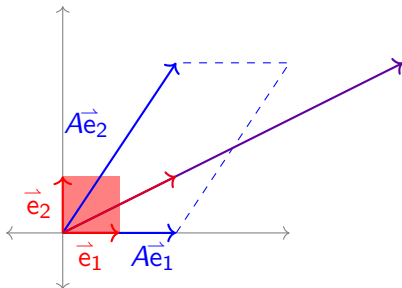
$$\text{so } \det(M^{-1}) = \frac{1}{\det(M)}.$$

- Furthermore, a square matrix  $M$  is invertible if and only if  $\det(M) \neq 0$ .



**Observation G.41**

Consider the linear transformation  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the matrix  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .



It is easy to see geometrically that

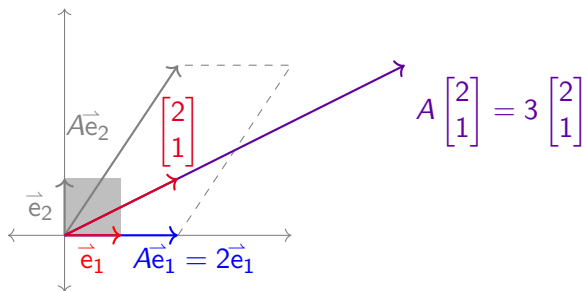
$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily checked once you find it) that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

**Definition G.42**

Let  $A \in M_{n,n}$ . An **eigenvector** for  $A$  is a vector  $\vec{x} \in \mathbb{R}^n$  such that  $A\vec{x}$  is parallel to  $\vec{x}$ .



In other words,  $A\vec{x} = \lambda\vec{x}$  for some scalar  $\lambda$ . If  $\vec{x} \neq \vec{0}$ , then we say  $\vec{x}$  is a **nontrivial eigenvector** and we call this  $\lambda$  an **eigenvalue** of  $A$ .

**Activity G.43** (*~5 min*) Finding the eigenvalues  $\lambda$  that satisfy

$$A\vec{x} = \lambda\vec{x} = \lambda(I\vec{x}) = (\lambda I)\vec{x}$$

for some nontrivial eigenvector  $\vec{x}$  is equivalent to finding nonzero solutions for the matrix equation

$$(A - \lambda I)\vec{x} = \vec{0}.$$

Which of the following must be true for any eigenvalue?

- (a) The **kernel** of the transformation with standard matrix  $A - \lambda I$  must contain **the zero vector**, so  $A - \lambda I$  is **invertible**.
- (b) The **kernel** of the transformation with standard matrix  $A - \lambda I$  must contain a **non-zero vector**, so  $A - \lambda I$  is **not invertible**.
- (c) The **image** of the transformation with standard matrix  $A - \lambda I$  must contain **the zero vector**, so  $A - \lambda I$  is **invertible**.
- (d) The **image** of the transformation with standard matrix  $A - \lambda I$  must contain a **non-zero vector**, so  $A - \lambda I$  is **not invertible**.

**Fact G.44**

The eigenvalues  $\lambda$  for a matrix  $A$  are the values that make  $A - \lambda I$  non-invertible.

Thus the eigenvalues  $\lambda$  for a matrix  $A$  are the solutions to the equation

$$\det(A - \lambda I) = 0.$$

**Definition G.45**

The expression  $\det(A - \lambda I)$  is called **characteristic polynomial** of  $A$ .

For example, when  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix}$$

Thus the characteristic polynomial of  $A$  is

$$\det \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) - (2)(3) = \lambda^2 - 5\lambda - 2$$

and its eigenvalues are the solutions to  $\lambda^2 - 5\lambda - 2 = 0$ .

**Activity G.46** ( $\sim 10$  min) Let  $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$ .

**Activity G.46** ( $\sim 10$  min) Let  $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$ .

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .

**Activity G.46** ( $\sim 10$  min) Let  $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$ .

*Part 1:* Compute  $\det(A - \lambda I)$  to determine the characteristic polynomial of  $A$ .

*Part 2:* Set this characteristic polynomial equal to zero and factor to determine the eigenvalues of  $A$ .



**Activity G.47** (*~5 min*) Find all the eigenvalues for the matrix  $A = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix}$ .

**Activity G.48** (*~5 min*) Find all the eigenvalues for the matrix  $A = \begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}$ .

**Activity G.49** ( $\sim 10$  min) Find all the eigenvalues for the matrix  $A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$ .

**Activity G.50** ( $\sim 10$  min) It's possible to show that  $-2$  is an eigenvalue for

$$\begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 9 \\ 3 & 0 & 4 \end{bmatrix}.$$

Compute the kernel of the transformation with standard matrix

$$A - (-2)I = \begin{bmatrix} ? & 4 & -2 \\ 2 & ? & 9 \\ 3 & 0 & ? \end{bmatrix}$$

to find all the eigenvectors  $\vec{x}$  such that  $A\vec{x} = -2\vec{x}$ .

## Definition G.51

Since the kernel of a linear map is a subspace of  $\mathbb{R}^n$ , and the kernel obtained from  $A - \lambda I$  contains all the eigenvectors associated with  $\lambda$ , we call this kernel the **eigenspace** of  $A$  associated with  $\lambda$ .