Module V: Vector Spaces

What is a vector space?

At the end of this module, students will be able to...

V1 **Vector spaces.** ... explain why a given set with defined addition and scalar multiplication does satisfy a given vector space property, but nonetheless isn't a vector space.

V2 **Linear combinations.** ... determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors by solving an appropriate vector equation.

V3 **Spanning sets.** ... determine if a set of Euclidean vectors spans $\mathbb{R}^n$ by solving appropriate vector equations.

V4 **Subspaces.** ... determine if a subset of $\mathbb{R}^n$ is a subspace or not.

V5 **Linear independence.** ... determine if a set of Euclidean vectors is linearly dependent or independent by solving an appropriate vector equation.

V6 **Basis verification.** ... explain why a set of Euclidean vectors is or is not a basis of $\mathbb{R}^n$.

V7 **Basis computation.** ... compute a basis for the subspace spanned by a given set of Euclidean vectors, and determine the dimension of the subspace.

V8 **Polynomial and Matrix computation.** ... answer questions about vector spaces of polynomials or matrices.

V9 **Basis of solution space.** ... find a basis for the solution set of a homogeneous system of equations.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Use set builder notation to describe sets of vectors.
- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.

The following resources will help you prepare for this module.

- Set Builder Notation: `https://youtu.be/xnfUZ-NTsCE`
- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Module V Section 0

Observation V.1

Several properties of the real numbers, such as commutivity:

$$x + y = y + x$$

also hold for Euclidean vectors with multiple components:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Activity V.2 *($\sim$20 min)* Consider each of the following properties of the real numbers $\mathbb{R}^1$. Label each property as **valid** if the property also holds for two-dimensional Euclidean vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ and scalars $a, b \in \mathbb{R}$, and **invalid** if it does not.

1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. There exists some $\vec{z}$ where $\vec{v} + \vec{z} = \vec{v}$.
4. There exists some $-\vec{v}$ where $\vec{v} + (-\vec{v}) = \vec{z}$.
5. If $\vec{u} \neq \vec{v}$, then $\frac{1}{2}(\vec{u} + \vec{v})$ is the only vector equally distant from both $\vec{u}$ and $\vec{v}$
6. $a(b\vec{v}) = (ab)\vec{v}$.
7. $1\vec{v} = \vec{v}$.
8. If $\vec{u} \neq \vec{0}$, then there exists some scalar c such that $c\vec{u} = \vec{v}$.
9. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$.
10. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$.

Definition V.3

A **vector space** V is any collection of mathematical objects with associated addition $\oplus$ and scalar multiplication $\odot$ operations that satisfy the following properties. Let $\vec{u}, \vec{v}, \vec{w}$ belong to V, and let a, b be scalar numbers.

- **Addition is associative:** $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$.
- **Addition is commutative:** $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.
- **Additive identity exists:** There exists some $\vec{z}$ where $\vec{v} \oplus \vec{z} = \vec{v}$.
- **Additive inverses exist:** There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$.
- **Scalar multiplication is associative:** $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$.
- **Scalar multiplication identity exists:** $1 \odot \vec{v} = \vec{v}$.
- **Scalar mult. distributes over vector addition:** $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$.
- **Scalar mult. distributes over scalar addition:** $(a + b) \odot \vec{v} = a \odot \vec{v} \oplus b \odot \vec{v}$.

Observation V.4

Every **Euclidean vector space**

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \, x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$$

satisfies all eight requirements for the usual definitions of addition and scalar multiplication, but we will also study other types of vector spaces.

Observation V.5

The space of $m \times n$ **matrices**

$$M_{m,n} = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \middle| \, a_{11}, \dots, a_{mn} \in \mathbb{R} \right\}$$

satisfies all eight requirements for component-wise addition and scalar multiplication.

Module V Section 1

Remark V.6

Previously, we defined a **vector space** V to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all $\vec{u}, \vec{v}, \vec{w}$ in V, and all scalars (i.e. real numbers) a, b.

- **Addition is associative:** $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$.
- **Addition is commutative:** $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.
- **Additive identity exists:** There exists some $\vec{z}$ where $\vec{v} \oplus \vec{z} = \vec{v}$.
- **Additive inverses exist:** There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$.
- **Scalar multiplication is associative:** $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$.
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- **Scalar mult. distributes over vector addition:** $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$.
- **Scalar mult. distributes over scalar addition:** $(a + b) \odot \vec{v} = a \odot \vec{v} \oplus b \odot \vec{v}$.

Remark V.7

Every Euclidean space $\mathbb{R}^n$ is a vector space, but there are other examples of vector spaces as well.

For example, consider the set $\mathbb{C}$ of complex numbers with the usual defintions of addition and scalar multiplication, and let $\vec{\mathrm{u}} = a + b\mathrm{i}$, $\vec{\mathrm{v}} = c + d\mathrm{i}$, and $\vec{\mathrm{w}} = e + f\mathrm{i}$. Then

$$\begin{aligned}\vec{\mathrm{u}} + (\vec{\mathrm{v}} + \vec{\mathrm{w}}) &= (a + b\mathrm{i}) + ((c + d\mathrm{i}) + (e + f\mathrm{i}))\\ &= (a + b\mathrm{i}) + ((c + e) + (d + f)\mathrm{i})\\ &= (a + c + e) + (b + d + f)\mathrm{i}\\ &= ((a + c) + (b + d)\mathrm{i}) + (e + f\mathrm{i})\\ &= (\vec{\mathrm{u}} + \vec{\mathrm{v}}) + \vec{\mathrm{w}}\end{aligned}$$

All eight properties can be verified in this way.

Remark V.8

The following sets are just a few examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{C}$: Complex numbers.
- $M_{m,n}$: Matrices of real numbers with m rows and n columns.
- $\mathcal{P}^n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity V.9 *(∼20 min)* Consider the set $V = \{(x,y) \mid y = e^x\}$ with operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x_1, y_1) = (cx_1, y_1^c)$$

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Part 1: Show that V satisfies the distributive property

$$(a + b) \odot (x_1, y_1) = (a \odot (x_1, y_1)) \oplus (b \odot (x_1, y_1))$$

by simplifying both sides and verifying they are the same expression.

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Part 2: Show that V contains an additive identity element satisfying

$$(x_1, y_1) \oplus \vec{z} = (x_1, y_1)$$

for all $(x_1, y_1) \in V$ by choosing appropriate values for $\vec{z} = (\,?\,,\,?\,)$.

Remark V.10

It turns out $V = \{(x,y) \mid y = e^x\}$ with operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x_1, y_1) = (cx_1, y_1^c)$$

satisifes all eight properties.

- **Addition is associative:** $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$.
- **Addition is commutative:** $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.
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- **Scalar mult. distributes over scalar addition:** $(a + b) \odot \vec{v} = a \odot \vec{v} \oplus b \odot \vec{v}$.

Thus, V is a vector space.

Activity V.11 *(∼15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2) \qquad c \odot (x_1, y_1) = (x_1^c, y_1 + c - 1).$$

Activity V.11 *(∼15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2) \qquad c \odot (x_1, y_1) = (x_1^c, y_1 + c - 1).$$

Part 1: Show that 1 is the scalar multiplication identity element by simplifying $1 \odot (x, y)$ to (x, y).

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Part 1: Show that 1 is the scalar multiplication identity element by simplifying $1 \odot (x, y)$ to (x, y).

Part 2: Show that V does not have an additive identity element by showing that $(0, -1) \oplus \vec{z} \neq (0, -1)$ no matter how $\vec{z} = (z, w)$ is chosen.

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Part 3: Is V a vector space?

Activity V.12 *(∼15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + 3y_2) \qquad c \odot (x_1, y_1) = (cx_1, cy_1).$$

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$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + 3y_2) \qquad c \odot (x_1, y_1) = (cx_1, cy_1).$$

Part 1: Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

for **all** $c \in \mathbb{R}$, $(x_1, y_1), (x_2, y_2) \in V$.

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$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

for **all** $c \in \mathbb{R}$, $(x_1, y_1), (x_2, y_2) \in V$.
Part 2: Show that vector addition is not associative, i.e.

$$(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) \neq ((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3)$$

for **some** vectors $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$.

Activity V.12 *(~15 min)* Let $V = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ have operations defined by

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for **some** vectors $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$.
Part 3: Is V a vector space?

Definition V.13

A **linear combination** of a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ is given by $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$ for any choice of scalar multiples $c_1, c_2, \ldots, c_m$.

For example, we can say $\begin{bmatrix}3\\0\\5\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\2\\1\end{bmatrix}$ since

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

Definition V.14

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m \,|\, c_i \in \mathbb{R}\}.$$

For example:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$$

Activity V.15 *(∼10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

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Part 1: Sketch

$$1\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},\qquad 3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix},\qquad 0\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\qquad \text{and } -2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-2\\-4\end{bmatrix}$$

in the xy plane.

Activity V.15 *(∼10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

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in the xy plane.

Part 2: Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\end{bmatrix}\,\middle|\,a\in\mathbb{R}\right\}$ in the xy plane.

Activity V.16 *(~10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Activity V.16 *(∼10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix}\qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}\qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}\qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}$$

Activity V.16 *(∼10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix} \qquad 0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix} \qquad 1\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix} \qquad -1\begin{bmatrix}1\\2\end{bmatrix}+-2\begin{bmatrix}-1\\1\end{bmatrix}$$

Part 2: Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Module V Section 2

Activity V.17 *(∼5 min)* Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix},\begin{bmatrix}-3\\2\end{bmatrix}\right\}$ in the xy plane.

Remark V.18

Recall these definitions from last class:

- A **linear combination** of vectors is given by adding scalar multiples of those vectors, such as:

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

- The **span** of a set of vectors is the collection of all linear combinations of that set, such as:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \middle| a, b \in \mathbb{R}\right\}$$

Activity V.19 *(~15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation

$$x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}.$$

Activity V.19 *(∼15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation

$$x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}.$$

Part 1: Reinterpret this vector equation as a system of linear equations.

Activity V.19 *(∼15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation

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Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Find its solution set, using technology to find RREF of its corresponding augmented matrix.

Activity V.19 *(∼15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation

$$x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}.$$

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Find its solution set, using technology to find RREF of its corresponding augmented matrix.

Part 3: Given this solution set, does $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belong to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$?

Fact V.20

A vector $\vec{b}$ belongs to $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\}$ if and only if the linear system corresponding to $[\vec{v}_1 \ \dots \ \vec{v}_n \,|\, \vec{b}]$ is consistent.

Put another way, $\vec{b}$ belongs to $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\}$ exactly when $\mathrm{RREF}[\vec{v}_1 \ \dots \ \vec{v}_n \,|\, \vec{b}]$ doesn't have a row $[0 \cdots 0 \,|\, 1]$ representing the contradiction $0 = 1$.

Activity V.21 *(∼10 min)* Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 5 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

Activity V.22 *(~5 min)* Determine if $\begin{bmatrix}-1\\-9\\0\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ by row-reducing an appropriate matrix.

Activity V.23 *(∼10 min)* Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to span$\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Activity V.23 *(∼10 min)* Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in $\mathbb{R}^4$. (Hint: What four numbers must you know to write a $\mathcal{P}^3$ polynomial?)

Activity V.23 *(∼10 min)* Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?

Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in $\mathbb{R}^4$. (Hint: What four numbers must you know to write a $\mathcal{P}^3$ polynomial?)

Part 2: Solve this equivalent exercise, and use its solution to answer the original question.

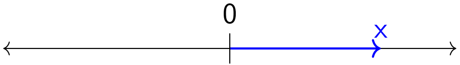
Activity V.24 *(∼5 min)* Does the polynomial $x^2 + x + 1$ belong to $\operatorname{span}\{x^2 - x, x + 1, x^2 - 1\}$?

Module V Section 3

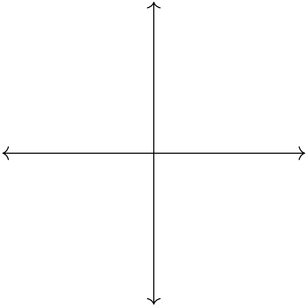
Activity V.25 *(∼5 min)* Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to $\operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \right\}$?

Observation V.26

Any single non-zero vector/number x in $\mathbb{R}^1$ spans $\mathbb{R}^1$, since $\mathbb{R}^1 = \{cx \mid c \in \mathbb{R}\}$.

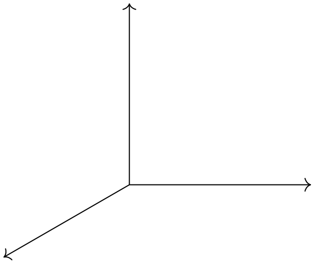


Activity V.27 *(∼5 min)* How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your answer.



- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) Infinitely Many

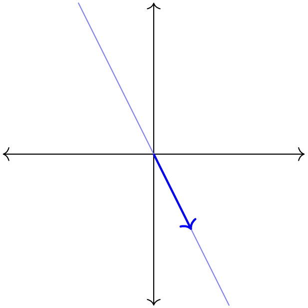
Activity V.28 *(∼5 min)* How many vectors are required to span $\mathbb{R}^3$?

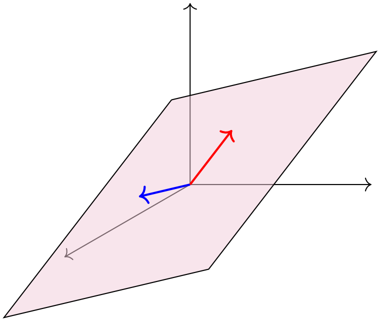


a. 1
b. 2
c. 3
d. 4
e. Infinitely Many

Fact V.29

At least n vectors are required to span $\mathbb{R}^n$.





Activity V.30 *(∼15 min)* Choose any vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in $\mathbb{R}^3$ that is not in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}$ by using technology to verify that $\operatorname{RREF}\left[\begin{array}{cc|c} 1 & -2 & ? \\ -1 & 0 & ? \\ 0 & 1 & ? \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$. (Why does this work?)

Fact V.31

The set $\{\vec{v}_1, \dots, \vec{v}_m\}$ fails to span all of $\mathbb{R}^n$ exactly when $\mathrm{RREF}[\vec{v}_1 \, \dots \, \vec{v}_m]$ has a non-pivot row of zeros.

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \text{ for some choice of vector } \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Activity V.32 *(∼5 min)* Consider the set of vectors

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}.$$ Does $\mathbb{R}^4 = \operatorname{span} S$?

Activity V.33 *(∼10 min)* Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, \\ -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2\}.$$

Does $\mathcal{P}^3 = \operatorname{span} S$? (Hint: first rewrite the question so it is about Euclidean vectors.)

Activity V.34 *(∼5 min)* Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Does $M_{2,2} = \operatorname{span} S$?

Module V Section 4

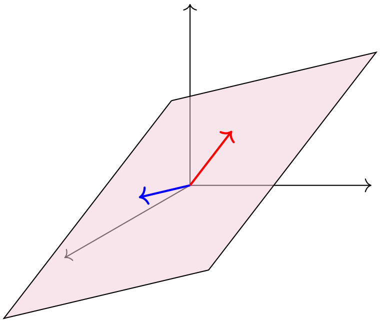
Activity V.35 *(∼5 min)* Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^7$ be three vectors, and suppose $\vec{w}$ is another vector with $\vec{w} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. What can you conclude about $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

a. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is larger than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.
b. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.
c. $\operatorname{span}\{\vec{w}, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is smaller than $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Definition V.36

A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space $\mathbb{R}^3$.



Fact V.37

Any sub**set** S of a vector space V that contains the additive identity $\vec{0}$ satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a sub**space**, we need to check that addition and multiplication still make sense using only vectors from S. So we need to check two things:

- The set is **closed under addition**: for any $\vec{x}, \vec{y} \in S$, the sum $\vec{x} + \vec{y}$ is also in S.
- The set is **closed under scalar multiplication**: for any $\vec{x} \in S$ and scalar $c \in \mathbb{R}$, the product $c\vec{x}$ is also in S.

Activity V.38 *(~15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Activity V.38 *(∼15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, x + 2y + z = 0 \right\}$.

Part 1: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\vec{v} + \vec{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Activity V.38 *(∼15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Part 1: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\vec{v} + \vec{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Part 2: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\vec{v} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ also belongs to S for any $c \in \mathbb{R}$ by verifying an appropriate equation.

Activity V.38 *(~15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + 2y + z = 0 \right\}$.

Part 1: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\vec{v} + \vec{w} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$.

Part 2: Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\vec{v} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ also belongs to S for any $c \in \mathbb{R}$ by verifying an appropriate equation.

Part 3: Is S is a subspace of $\mathbb{R}^3$?

Activity V.39 *(∼10 min)* Let $S=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\,\middle|\,x+2y+z=4\right\}$. Choose a vector $\vec{v}=\begin{bmatrix}?\\?\\?\end{bmatrix}$ in S and a real number $c=\,?$, and show that $c\vec{v}$ isn't in S. Is S a subspace of $\mathbb{R}^3$?

Remark V.40

Since 0 is a scalar and $0\vec{v} = \vec{z}$ for any vector $\vec{v}$, a nonempty set that is closed under scalar multiplication must contain the zero vector $\vec{z}$ for that vector space.

Put another way, you can check any of the following to show that a nonempty subset W isn't a subspace:

- Show that $\vec{0} \not\in W$.
- Find $\vec{u}, \vec{v} \in W$ such that $\vec{u} + \vec{v} \not\in W$.
- Find $c \in \mathbb{R}, \vec{v} \in W$ such that $c\vec{v} \not\in W$.

If you cannot do any of these, then W can be proven to be a subspace by doing the following:

- Prove that $\vec{u} + \vec{v} \in W$ whenever $\vec{u}, \vec{v} \in W$.
- Prove that $c\vec{v} \in W$ whenever $c \in \mathbb{R}, \vec{v} \in W$.

Activity V.41 *(∼20 min)* Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| z = xy \right\}$$

Activity V.41 *(∼20 min)* Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| z = xy \right\}$$

Part 1: Show R isn't a subspace by showing that $\vec{0} \notin R$.

Activity V.41 *(∼20 min)* Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, z = xy \right\}$$

Part 1: Show R isn't a subspace by showing that $\vec{0} \notin R$.
Part 2: Show S isn't a subspace by finding two vectors $\vec{u}, \vec{v} \in S$ such that $\vec{u} + \vec{v} \notin S$.

Activity V.41 *(∼20 min)* Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, z = xy \right\}$$

Part 1: Show R isn't a subspace by showing that $\vec{0} \notin R$.
Part 2: Show S isn't a subspace by finding two vectors $\vec{u}, \vec{v} \in S$ such that $\vec{u} + \vec{v} \notin S$.
Part 3: Show T isn't a subspace by finding a vector $\vec{v} \in T$ such that $2\vec{v} \notin T$.

Activity V.42 *(∼5 min)* Let W be a subspace of a vector space V. How are span W and W related?

a. span W is bigger than W
b. span W is the same as W
c. span W is smaller than W

Module V Section 5

Fact V.43

If S is any subset of a vector space V, then since $\operatorname{span} S$ collects all possible linear combinations, $\operatorname{span} S$ is automatically a subspace of V.

In fact, $\operatorname{span} S$ is always the smallest subspace of V that contains all the vectors in S.

Activity V.44 *(∼10 min)* Consider the two sets

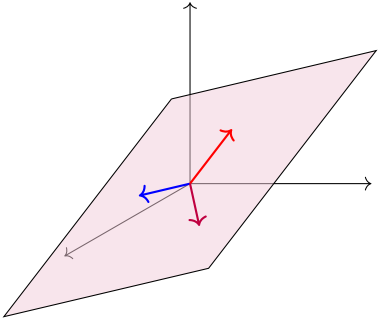
$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

Which of the following is true?

- A. span S is bigger than span T.
- B. span S and span T are the same size.
- C. span S is smaller than span T.

Definition V.45

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.



You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set. In the above image, all three vectors lay on the same planar subspace, but only two vectors are needed to span the plane, so the set is linearly dependent.

Activity V.46 *(∼10 min)* Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$. Suppose $3\vec{u} - 5\vec{v} = \vec{w}$, so the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent. Which of the following is true of the vector equation $x\vec{u} + y\vec{v} + z\vec{w} = \vec{0}$?

- (A) It is consistent with one solution
- (B) It is consistent with infinitely many solutions
- (C) It is inconsistent.

Fact V.47

For any vector space, the set $\{\vec{v}_1, \ldots \vec{v}_n\}$ is linearly dependent if and only if $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{z}$ is consistent with infinitely many solutions.

Activity V.48 *(∼10 min)* Find

$$\text{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Fact V.49

A set of Euclidean vectors $\{\vec{v}_1, \dots \vec{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$ has a column without a pivot position.

Observation V.50

Compare the following results:

- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \ldots \vec{v}_n\}$ is linearly independent if and only if RREF $\begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_n \end{bmatrix}$ has all pivot columns.
- A set of $\mathbb{R}^m$ vectors $\{\vec{v}_1, \ldots \vec{v}_n\}$ spans $\mathbb{R}^m$ if and only if RREF $\begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_n \end{bmatrix}$ has all pivot rows.

Activity V.51 *(∼5 min)* Is the set of Euclidean vectors

$$\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$$

linearly dependent or linearly independent?

Activity V.52 *(∼10 min)* Is the set of polynomials $\{x^3+1, x^2+2x, x^2+7x+4\}$ linearly dependent or linearly independent?

Activity V.53 *(∼5 min)* What is the largest number of $\mathbb{R}^4$ vectors that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity V.54 *(~5 min)* What is the largest number of

$$\mathcal{P}^4 = \left\{ax^4 + bx^3 + cx^2 + dx + e \,\middle|\, a, b, c, d, e \in \mathbb{R}\right\}$$

vectors that can form a linearly independent set?

a. 3
b. 4
c. 5
d. You can have infinitely many vectors and still be linearly independent.

Module V Section 6

Activity V.55 *($\sim$5 min)* What is the largest number of

$$\mathcal{P} = \{f(x) \,|\, f(x) \text{ is any polynomial}\}$$

vectors that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Definition V.56

A **basis** is a linearly independent set that spans a vector space.

The **standard basis** of $\mathbb{R}^n$ is the set $\{\vec{e}_1, \ldots, \vec{e}_n\}$ where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For $\mathbb{R}^3$, these are the vectors $\vec{e}_1 = \hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{e}_3 = \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Observation V.57

A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

For example, in many calculus courses, vectors in $\mathbb{R}^3$ are often expressed in their component form

$$(3, -2, 4) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

or in their standard basic vector form

$$3\vec{e}_1 - 2\vec{e}_2 + 4\vec{e}_3 = 3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}.$$

Since every vector in $\mathbb{R}^3$ can be uniquely described as a linear combination of the vectors in $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, this set is indeed a basis.

Activity V.58 *(~15 min)* Label each of the sets A, B, C, D, E as

- SPANS $\mathbb{R}^4$ or DOES NOT SPAN $\mathbb{R}^4$
- LINEARLY INDEPENDENT or LINEARLY DEPENDENT
- BASIS FOR $\mathbb{R}^4$ or NOT A BASIS FOR $\mathbb{R}^4$

by finding RREF for their corresponding matrices.

$$A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} \qquad D = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$$E = \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Activity V.59 *(~10 min)* If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$, that means $\text{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$ doesn't have a non-pivot column, and doesn't have a row of zeros. What is $\text{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4]$?

$$\text{RREF}[\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,\vec{v}_4] = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Fact V.60

The set $\{\vec{v}_1, \dots, \vec{v}_m\}$ is a basis for $\mathbb{R}^n$ if and only if $m = n$ and

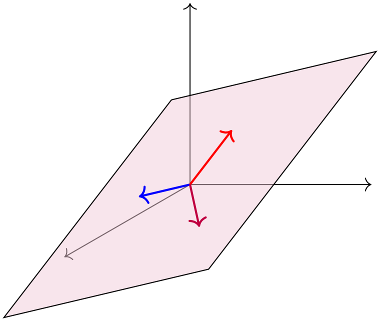
$$\mathrm{RREF}[\vec{v}_1 \ \dots \ \vec{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, a basis for $\mathbb{R}^n$ must have exactly n vectors and its square matrix must row-reduce to the so-called **identity matrix** containing all zeros except for a downward diagonal of ones. (We will learn where the identity matrix gets its name in a later module.)

Observation V.61

Recall that a **subspace** of a vector space is a subset that is itself a vector space.

One easy way to construct a subspace is to take the span of set, but a linearly dependent set contains "redundant" vectors. For example, only two of the three vectors in the following image are needed to span the planar subspace.



Activity V.62 *(∼10 min)* Consider the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}.$$

Activity V.62 *(∼10 min)* Consider the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}.$$

Part 1: Mark the part of RREF $\begin{bmatrix}2 & 2 & 2 & 1\\3 & 0 & -3 & 5\\0 & 1 & 2 & -1\\1 & -1 & -3 & 0\end{bmatrix}$ that shows that W's spanning set is linearly dependent.

Activity V.62 *(∼10 min)* Consider the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Part 1: Mark the part of RREF $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$ that shows that W's spanning set is linearly dependent.

Part 2: Find a basis for W by removing a vector from its spanning set to make it linearly independent.

Fact V.63

Let $S = \{\vec{v}_1, \dots, \vec{v}_m\}$. The easiest basis describing span S is the set of vectors in S given by the pivot columns of $\operatorname{RREF}[\vec{v}_1 \dots \vec{v}_m]$.

Put another way, to compute a basis for the subspace span S, simply remove the vectors corresponding to the non-pivot columns of $\operatorname{RREF}[\vec{v}_1 \dots \vec{v}_m]$. For example, since

$$\operatorname{RREF}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}\right\}$ has $\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}$ as a basis.

Activity V.64 *(∼10 min)* Let W be the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Find a basis for W.

Activity V.65 *(~10 min)* Let W be the subspace of $\mathcal{P}^3$ given by

$$W = \operatorname{span}\left\{x^3 + 3x^2 + x - 1, 2x^3 - x^2 + x + 2, 4x^3 + 5x^2 + 3x, 3x^3 + 2x^2 + 2x + 1\right\}$$

Find a basis for W.

Module V Section 7

Observation V.66

In the previous section, we learned that computing a basis for the subspace $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_m\}$, is as simple as removing the vectors corresponding to the non-pivot columns of $\operatorname{RREF}[\vec{v}_1 \, \dots \, \vec{v}_m]$.

For example, since

$$\operatorname{RREF}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \right\}$ has $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$ as a basis.

Activity V.67 *(∼10 min)* Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Activity V.67 *(~10 min)* Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Part 1: Find a basis for span S.

Activity V.67 *(∼10 min)* Let

$$S=\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix},\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}\quad\text{and}\quad T=\left\{\begin{bmatrix}2\\0\\1\\-1\end{bmatrix},\begin{bmatrix}2\\-3\\2\\-3\end{bmatrix},\begin{bmatrix}1\\5\\-1\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}$$

Part 1: Find a basis for span S.
Part 2: Find a basis for span T.

Observation V.68

Even though we found different bases for them, span S and span T are exactly the same subspace of $\mathbb{R}^4$, since

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} = T$$

Fact V.69

Any non-trivial vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size.

For example,

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

are all valid bases for $\mathbb{R}^3$, and they all contain three vectors.

Definition V.70

The **dimension** of a vector space is equal to the size of any basis for the vector space.

As you'd expect, $\mathbb{R}^n$ has dimension n. For example, $\mathbb{R}^3$ has dimension 3 because any basis for $\mathbb{R}^3$ such as

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$$

contains exactly three vectors.

Activity V.71 *(~10 min)* Find the dimension of each subspace of $\mathbb{R}^4$ by finding RREF for each corresponding matrix.

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix}\right\} \quad \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\0\\3\end{bmatrix},\begin{bmatrix}3\\13\\7\\16\end{bmatrix},\begin{bmatrix}-1\\10\\7\\14\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix}\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\-1\end{bmatrix},\begin{bmatrix}4\\3\\0\\2\end{bmatrix},\begin{bmatrix}-3\\0\\1\\3\end{bmatrix},\begin{bmatrix}3\\6\\1\\5\end{bmatrix}\right\} \quad \operatorname{span}\left\{\begin{bmatrix}5\\3\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\1\\0\\3\end{bmatrix},\begin{bmatrix}4\\5\\1\\3\end{bmatrix}\right\}$$

Fact V.72

Every vector space with finite dimension, that is, every vector space V with a basis of the form $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is said to be **isomorphic** to a Euclidean space $\mathbb{R}^n$, since there exists a natural correspondance between vectors in V and vectors in $\mathbb{R}^n$:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Observation V.73

We've already been taking advantage of the previous fact by converting polynomials and matrices into Euclidean vectors. Since $\mathcal{P}^3$ and $M_{2,2}$ are both four-dimensional:

$$4x^3 + 0x^2 - 1x + 5 \leftrightarrow \begin{bmatrix} 4 \\ 0 \\ -1 \\ 5 \end{bmatrix} \leftrightarrow \begin{bmatrix} 4 & 0 \\ -1 & 5 \end{bmatrix}$$

Activity V.74 *(∼5 min)* Suppose W is a subspace of $\mathcal{P}^8$, and you know that the set $\{x^3 + x, x^2 + 1, x^4 - x\}$ is a linearly independent subset of W. What can you conclude about W?

a. The dimension of W is at most 3.
b. The dimension of W is exactly 3.
c. The dimension of W is at least 3.

Activity V.75 *(∼5 min)* Suppose W is a subspace of $\mathcal{P}^8$, and you know that W is spanned by the six vectors

$$\{x^4 - x, x^3 + x, x^3 + x + 1, x^4 + 2x, x^3, 2x + 1\}.$$

What can you conclude about W?

- (a) The dimension of W is at most 6.
- (b) The dimension of W is exactly 6.
- (c) The dimension of W is at least 6.

Observation V.76

The space of polynomials $\mathcal{P}$ (of *any* degree) has the basis $\{1, x, x^2, x^3, \dots\}$, so it is a natural example of an infinite-dimensional vector space.

Since $\mathcal{P}$ and other infinite-dimensional spaces cannot be treated as an isomorphic finite-dimensional Euclidean space $\mathbb{R}^n$, vectors in such spaces cannot be studied by converting them into Euclidean vectors. Fortunately, most of the examples we will be interested in for this course will be finite-dimensional.

Definition V.77

A **homogeneous** system of linear equations is one of the form:

$$\begin{aligned}
a_{11}x_1 + \ & a_{12}x_2 + \ldots + \ & a_{1n}x_n \ & = 0 \\
a_{21}x_1 + \ & a_{22}x_2 + \ldots + \ & a_{2n}x_n \ & = 0 \\
\vdots \ & \quad \vdots & \vdots & \quad \vdots \\
a_{m1}x_1 + \ & a_{m2}x_2 + \ldots + \ & a_{mn}x_n \ & = 0
\end{aligned}$$

This system is equivalent to the vector equation:

$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & 0 \\
a_{21} & a_{22} & \cdots & a_{2n} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & 0
\end{array}\right]$$

Activity V.78 *(∼5 min)* Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are solutions to $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$ so is $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$, since

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{0} \text{ and } b_1\vec{v}_1 + \cdots + b_n\vec{v}_n = \vec{0}$$

implies

$$(a_1 + b_1)\vec{v}_1 + \cdots + (a_n + b_n)\vec{v}_n = \vec{0}.$$

Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution. Thus the solution set of a homogeneous system is...

a) A basis for $\mathbb{R}^n$.

b) A subspace of $\mathbb{R}^n$.

c) The empty set.

Activity V.79 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}- x_3} + x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - x_4 &= 0 \end{aligned}$$

Activity V.79 *(∼10 min)* Consider the homogeneous system of equations

$$
\begin{aligned}
x_1 + 2x_2 \phantom{{}-x_3} + x_4 &= 0\\
2x_1 + 4x_2 - x_3 - 2x_4 &= 0\\
3x_1 + 6x_2 - x_3 - x_4 &= 0
\end{aligned}
$$

Part 1: Find its solution set (a subspace of $\mathbb{R}^4$).

Activity V.79 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \quad\quad + \quad x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - \quad x_4 &= 0 \end{aligned}$$

Part 1: Find its solution set (a subspace of $\mathbb{R}^4$).
Part 2: Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle|\, a, b \in \mathbb{R} \right\}.$$

Activity V.79 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}-x_3} + x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - x_4 &= 0 \end{aligned}$$

Part 1: Find its solution set (a subspace of $\mathbb{R}^4$).

Part 2: Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}.$$

Part 3: Rewrite this solution space in the form

$$\operatorname{span}\left\{ \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \right\}.$$

Fact V.80

The coefficients of the free variables in the solution set of a linear system always yield linearly independent vectors.

Thus if

$$\left\{ a\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + b\begin{bmatrix}-1\\0\\-4\\1\end{bmatrix} \middle|\, a,b\in\mathbb{R} \right\} = \operatorname{span}\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-4\\1\end{bmatrix}\right\}$$

is the solution space for a homoegeneous system, then

$$\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-4\\1\end{bmatrix}\right\}$$

is a basis for the solution space.