Section V.1

Remark V.1.1 Previously, we defined a **vector space** V to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ in V, and all scalars (i.e. real numbers) a, b.

- **Addition is associative:** $\vec{\mathbf{u}} \oplus (\vec{\mathbf{v}} \oplus \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \oplus \vec{\mathbf{v}}) \oplus \vec{\mathbf{w}}$.
- **Addition is commutative:** $\vec{\mathbf{u}} \oplus \vec{\mathbf{v}} = \vec{\mathbf{v}} \oplus \vec{\mathbf{u}}$.
- **Additive identity exists:** There exists some $\vec{\mathbf{z}}$ where $\vec{\mathbf{v}} \oplus \vec{\mathbf{z}} = \vec{\mathbf{v}}$.
- **Additive inverses exist:** There exists some $-\vec{\mathbf{v}}$ where $\vec{\mathbf{v}} \oplus (-\vec{\mathbf{v}}) = \vec{\mathbf{z}}$.
- **Scalar multiplication is associative:** $a \odot (b \odot \vec{\mathbf{v}}) = (ab) \odot \vec{\mathbf{v}}$.
- **Scalar multiplication identity exists:** $1 \odot \vec{\mathbf{v}} = \vec{\mathbf{v}}$.
- **Scalar mult. distributes over vector addition:** $a \odot (\vec{\mathbf{u}} \oplus \vec{\mathbf{v}}) = a \odot \vec{\mathbf{u}} \oplus a \odot \vec{\mathbf{v}}$.
- **Scalar mult. distributes over scalar addition:** $(a + b) \odot \vec{\mathbf{v}} = a\vec{\mathbf{v}} \oplus b\vec{\mathbf{v}}$.

Remark V.1.2 Every Euclidean space $\mathbb{R}^n$ is a vector space, but there are other examples of vector spaces as well.

For example, consider the set $\mathbb{C}$ of complex numbers with the usual defintions of addition and scalar multiplication, and let $\vec{\mathbf{u}} = a + b\mathbf{i}$, $\vec{\mathbf{v}} = c + d\mathbf{i}$, and $\vec{\mathbf{w}} = e + f\mathbf{i}$. Then

$$\begin{aligned}
\vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}}) &= (a + b\mathbf{i}) + ((c + d\mathbf{i}) + (e + f\mathbf{i})) \\
&= a + b + c + d\mathbf{i} + e\mathbf{i} + f\mathbf{i} \\
&= ((a + b\mathbf{i}) + (c + d\mathbf{i})) + (e + f\mathbf{i}) \\
&= (\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}}
\end{aligned}$$

All eight properties can be verified in this way.

Remark V.1.3 The following sets are just a few examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{C}$: Complex numbers.
- $M_{m,n}$: Matrices of real numbers with m rows and n columns.
- $\mathcal{P}^n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity V.1.4 *(~20 min)* Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

Part 1: Show that V satisfies the distribution property

$$(a + b) \odot \vec{\mathbf{v}} = (a \odot \vec{\mathbf{v}}) \oplus (b \odot \vec{\mathbf{v}})$$

by substituting $\vec{\mathbf{v}} = (x, y)$ and showing both sides simplify to the same expression.
Part 2: Show that V contains an additive identity element satisfying

$$(x, y) \oplus \vec{\mathbf{z}} = (x, y)$$

for all $(x, y) \in V$ by choosing appropriate values for $\vec{\mathbf{z}} = (\,?\,, \,?\,)$.

Remark V.1.5 It turns out $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

satisifes all eight properties.

- **Addition is associative:** $\vec{\mathbf{u}} \oplus (\vec{\mathbf{v}} \oplus \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \oplus \vec{\mathbf{v}}) \oplus \vec{\mathbf{w}}$.
- **Addition is commutative:** $\vec{\mathbf{u}} \oplus \vec{\mathbf{v}} = \vec{\mathbf{v}} \oplus \vec{\mathbf{u}}$.
- **Additive identity exists:** There exists some $\vec{\mathbf{z}}$ where $\vec{\mathbf{v}} \oplus \vec{\mathbf{z}} = \vec{\mathbf{v}}$.
- **Additive inverses exist:** There exists some $-\vec{\mathbf{v}}$ where $\vec{\mathbf{v}} \oplus (-\vec{\mathbf{v}}) = \vec{\mathbf{z}}$.
- **Scalar multiplication is associative:** $a \odot (b \odot \vec{\mathbf{v}}) = (ab) \odot \vec{\mathbf{v}}$.
- **Scalar multiplication identity exists:** $1 \odot \vec{\mathbf{v}} = \vec{\mathbf{v}}$.
- **Scalar mult. distributes over vector addition:** $a \odot (\vec{\mathbf{u}} \oplus \vec{\mathbf{v}}) = a \odot \vec{\mathbf{u}} \oplus a \odot \vec{\mathbf{v}}$.
- **Scalar mult. distributes over scalar addition:** $(a + b) \odot \vec{\mathbf{v}} = a\vec{\mathbf{v}} \oplus b\vec{\mathbf{v}}$.

Thus, V is a vector space.

Activity V.1.6 *(~15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

Part 1: Show that 1 is the scalar multiplication identity element by simplifying $1 \odot (x, y)$ to (x, y).
Part 2: Show that V does not have an additive identity element by showing that $(0, -1) \oplus \vec{\mathbf{z}} \neq (0, -1)$ no matter how $\vec{\mathbf{z}} = (z_1, z_2)$ is chosen.
Part 3: Is V a vector space?

Definition V.1.7 A **linear combination** of a set of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_m\}$ is given by $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_m\vec{\mathbf{v}}_m$ for any choice of scalar multiples $c_1, c_2, \ldots, c_m$.

For example, we can say $\begin{bmatrix}3\\0\\5\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\2\\1\end{bmatrix}$ since

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

Definition V.1.8 The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_m\} = \{c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \cdots + c_m\vec{\mathbf{v}}_m \,|\, c_i \in \mathbb{R}\}.$$

For example:

$$\text{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$$

Activity V.1.9 *(∼10 min)* Consider $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch
$1\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$, $3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$, $0\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$, and $-2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-2\\-4\end{bmatrix}$
in the xy plane.

Part 2: Sketch a representation of all the vectors belonging to $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\2\end{bmatrix} \,\middle|\, a \in \mathbb{R}\right\}$ in the xy plane.

Activity V.1.10 *(∼10 min)* Consider $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix} \qquad 0\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix} \qquad 1\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix} \qquad -1\begin{bmatrix}1\\2\end{bmatrix} + -2\begin{bmatrix}-1\\1\end{bmatrix}$$

Part 2: Sketch a representation of all the vectors belonging to $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity V.1.11 *(∼5 min)* Sketch a representation of all the vectors belonging to span $\left\{\begin{bmatrix}6\\-4\end{bmatrix}, \begin{bmatrix}-3\\2\end{bmatrix}\right\}$ in the xy plane.