Section E.2

Remark E.2.1 The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{rrr|r} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition E.2.2 A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Definition E.2.3 Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \qquad\qquad \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity E.2.4 *(∼10 min)* Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a nonzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

Definition E.2.5 The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a nonzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity E.2.6 *(∼10 min)* Consider the following linear systems.

(A) $\begin{alignedat}{4} 3x_1 &- 2x_2 &+ 13x_3 &= 6 \\ 2x_1 &- 2x_2 &+ 10x_3 &= 2 \\ -x_1 &+ 3x_2 &- 6x_3 &= 11 \end{alignedat}$

(C) $\begin{alignedat}{4} x_1 &- 3x_2 &+ 6x_3 &= -11 \\ 2x_1 &- 2x_2 &+ 10x_3 &= 2 \\ 3x_1 &- 2x_2 &+ 13x_3 &= 6 \end{alignedat}$

(E) $\begin{alignedat}{4} x_1 &- 3x_2 &+ 6x_3 &= -11 \\ & 4x_2 &- 2x_3 &= 24 \\ & 7x_2 &- 5x_3 &= 39 \end{alignedat}$

(B) $\begin{alignedat}{4} x_1 & &+ 9x_3 &= 16 \\ & x_2 &+ x_3 &= 9 \\ & &- 12x_3 &= -24 \end{alignedat}$

(D) $\begin{alignedat}{4} x_1 &- 3x_2 &+ 6x_3 &= -11 \\ & x_2 &+ x_3 &= 9 \\ & 7x_2 &- 5x_3 &= 39 \end{alignedat}$

(F) $\begin{alignedat}{4} x_1 & &+ 9x_3 &= 16 \\ & x_2 &+ x_3 &= 9 \\ & & x_3 &= 2 \end{alignedat}$

Part 1: Which system can be obtained from System (A) in the fewest number of row operations?
Part 2: Rank the six linear systems from easiest to solve to hardest to solve.

Activity E.2.7 *(∼10 min)* Consider the following augmented matrices.

(A) $\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array}\right]$

(C) $\left[\begin{array}{ccc|c} 1 & -3 & 6 & -11 \\ 2 & -2 & 10 & 2 \\ 3 & -2 & 13 & 6 \end{array}\right]$

(E) $\left[\begin{array}{ccc|c} 1 & -3 & 6 & -11 \\ 0 & 4 & -2 & 24 \\ 0 & 7 & -5 & 39 \end{array}\right]$

(B) $\left[\begin{array}{ccc|c} 1 & 0 & 9 & 16 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & -12 & -24 \end{array}\right]$

(D) $\left[\begin{array}{ccc|c} 1 & -3 & 6 & -11 \\ 0 & 1 & 1 & 9 \\ 0 & 7 & -5 & 39 \end{array}\right]$

(F) $\left[\begin{array}{ccc|c} 1 & 0 & 9 & 16 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 2 \end{array}\right]$

Part 1: Rank the six matrices from farthest from a reduced row echelon form (RREF) matrix to closest to a RREF matrix.
Part 2: These matrices are all **row equivalent** and represent equivalent linear systems. Write down one of these linear systems and solve it.

Remark E.2.8 It is important to understand the **Gauss-Jordan elimination** algorithm that converts a matrix into reduced row echelon form, but in practice we don't do thsi by hand; we use technology to do this for us.

Activity E.2.9 *(~10 min)*

- Go to `http://www.cocalc.com` and create an account.
- Create a project titled "Linear Algebra Team X" with your appropriate team number. Add all team members as collaborators.
- Open the project and click on "New"
- Give it an appropriate name such as "Class 4 workbook". Make a new Jupyter notebook.
- Click on "Kernel" and make sure "Octave" is selected.
- Type `A=[1 3 4 ; 2 5 7]` to store the matrix $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{bmatrix}$ in the variable A; hold shift when you press enter.
- Type `rref(A)` to compute the reduced row echelon form of A.

Remark E.2.10 If you need to find the reduced row echelon form of a matrix during class, you should feel free to use CoCalc/Octave.

You can change a cell from "Code" to "Markdown" or "Raw" to put comments around your calculations such as Activity numbers.

Activity E.2.11 *(~8 min)* Consider our system of equations from above.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

Convert this to an augmented matrix, use CoCalc to compute the reduced row echelon form, and convert back to a simpler system of equations to solve this system. Write your solution on your whiteboard.

Activity E.2.12 *(~7 min)* Consider our system of equations from above.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 \qquad\quad - 3x_3 &= 1 \end{aligned}$$

Convert this to an augmented matrix, use CoCalc to compute the reduced row echelon form, and convert back to a simpler system of equations to solve this system. Write your solution on your whiteboard.