

Fall 2017

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At the end of this module, students will be able to...

- **E1: Systems as matrices.** Translate back and forth between a system of linear equations and the corresponding augmented matrix.
- **E2: Row reduction.** Put a matrix in reduced row echelon form
- **E3: Solving Linear Systems.** Solve a system of linear equations.
- **E4: Homogeneous Systems.** Find a basis for the solution set of a homogeneous linear system.

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Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.

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The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-systems-topic/cc-8th-systems-graphically/a/systems-of-equations-with-graphing>
- <https://www.khanacademy.org/math/algebra/systems-of-linear-equations/solving-systems-of-equations-with-substitution/v/practice-using-substitution-for-systems>

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Application Activities - Module E Part 1 - Class Day 3

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Definition 3.1

A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

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Observation 3.2

The linear equation $3x - 5y = -2$ may be graphed as a line in the xy plane.



The linear equation $x + 2y - z = 4$ may be graphed as a plane in xyz space.

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Remark 3.3

In previous classes you likely assumed $x = x_1$, $y = x_2$, and $z = x_3$. However, since this course often deals with equations of four or more variables, we will almost always write our variables as x_i .

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Definition 3.4

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

A solution

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \dots + a_{in}s_n = b_i$$

for $1 \leq i \leq m$ (that is, the solution satisfies all equations in the system).

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Remark 3.5

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Verbose standard form:

$$\begin{aligned}1x_1 + 0x_2 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\0x_1 - 1x_2 + 1x_3 &= -2\end{aligned}$$

Concise standard form:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

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Definition 3.6

A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

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Fact 3.7

All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

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Activity 3.8

Consider the following graphs representing linear systems of two variables. Label each graph with **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.



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Activity 3.9

All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system.

$$-x_1 + 2x_2 = 5$$

$$2x_1 - 4x_2 = 6$$

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Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

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Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

Part 1: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$, $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$, $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

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Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

Part 1: Find three different solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$, $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$, $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ for this system.

Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a . Use this to describe *all* solutions (the **solution set**) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$ for the linear system in terms of a .

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Activity 3.11

Consider the following linear system.

$$x_1 + 2x_2 - x_4 = 3$$

$$x_3 + 4x_4 = -2$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix} + a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3 .

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Observation 3.12

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.

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Remark 3.13

The only important information in a linear system are its coefficients and constants.

Original linear system:	Verbose standard form:	Coefficients/constants:
$x_1 + 3x_3 = 3$	$1x_1 + 0x_2 + 3x_3 = 3$	$1 \quad 0 \quad 3 \mid 3$
$3x_1 - 2x_2 + 4x_3 = 0$	$3x_1 - 2x_2 + 4x_3 = 0$	$3 \quad -2 \quad 4 \mid 0$
$-x_2 + x_3 = -2$	$0x_1 - 1x_2 + 1x_3 = -2$	$0 \quad -1 \quad 1 \mid -2$

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Definition 3.14

A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

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Definition 3.15

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution: $(x_1, x_2) = (1, 1)$.

$$3x_1 - 2x_2 = 1$$

$$x_1 + 4x_2 = 5$$

$$3x_1 - 2x_2 = 1$$

$$4x_1 + 2x_2 = 6$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array} \right]$$

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Activity 3.16

Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

- a) Swap two rows.
- b) Swap two columns.
- c) Add a constant to every term in a row.
- d) Multiply a row by a nonzero constant.
- e) Add a constant multiple of one row to another row.
- f) Replace a column with zeros.

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Definition 4.1

The following **row operations** produce equivalent augmented matrices:

- 1 Swap two rows.
- 2 Multiply a row by a nonzero constant.
- 3 Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

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Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$x_1 - x_2 + 5x_3 = 1$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$x_2 - 2x_3 = 3$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_3 = 2$$

Part 1: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

- 1 Swap R_1 (first row) and R_2 (second row).
- 2 Multiply R_2 by $\frac{1}{2}$.

- 3 Add R_1 to R_3 .
- 4 Add $-3R_1$ to R_2 .
- 5 Add $-2R_2$ to R_3 .
- 6 Multiply R_3 by $\frac{1}{3}$.

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Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$x_1 - x_2 + 5x_3 = 1$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$x_2 - 2x_3 = 3$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_3 = 2$$

Part 1: Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

① Swap R_1 (first row) and R_2 (second row).

② Multiply R_2 by $\frac{1}{2}$.

③ Add R_1 to R_3 .

④ Add $-3R_1$ to R_2 .

⑤ Add $-2R_2$ to R_3 .

⑥ Multiply R_3 by $\frac{1}{3}$.

Part 2: Which linear system would you rather solve?

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Definition 4.3

The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix. Examples:

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

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Activity 4.4

Find your own sequence of row operations to manipulate the matrix

$$\left[\begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array} \right]$$

into row echelon form. (Note that row echelon form is not unique.)

The most efficient way to do this is by circling **pivot positions** in your matrix:

- 1 Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
- 2 Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
- 3 Repeat these two steps as often as possible.

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Activity 4.5

Solve this simplified linear system:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

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Observation 4.6

The consise standard form of the solution to this linear system corresponds to a simplified row echelon form matrix:

$$\begin{array}{rcl} x_1 & = & -2 \\ x_2 & = & 7 \\ x_3 & = & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

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Definition 4.7

A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0. Examples:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

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Activity 4.8

Show that the following two linear systems:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

$$x_1 = -2$$

$$x_2 = 7$$

$$x_3 = 2$$

are equivalent by converting the first system to an augmented matrix, and then zeroing out all terms above pivot positions (the leading terms).

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Remark 4.9

We may verify that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$ is a solution to the original linear system

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

by plugging the solution into each equation.

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Fact 4.10

Every augmented matrix A reduces to a unique reduced row echelon form matrix. This matrix is denoted as $\text{RREF}(A)$.

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Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

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Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

Part 1: Find $\text{RREF}(A)$.

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Activity 4.11

Consider the following matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

Part 1: Find $\text{RREF}(A)$.

Part 2: How many solutions does the corresponding linear system have?

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Definition 5.1

An algorithm that reduces A to $\text{RREF}(A)$ is called **Gauss-Jordan elimination**. For example:

- 1 Circle the cell that (a) is in the top-most row without a pivot position and (b) is in the left-most column with a nonzero term either in that position or below it. This position (not the number inside) is called a **pivot**.
- 2 Change the pivot's value to 1 by using row operations involving only the pivot row and rows below it.
- 3 Add or subtract multiples of the pivot row to zero out above and below the pivot.
- 4 Return to Step 1 and repeat as needed until the matrix is in row reduced echelon form.

Observation 5.2

Here is an example of applying Gauss-Jordan elimination to a matrix:

$$\begin{aligned}
& \left[\begin{array}{cccc|c} \textcircled{2} & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array} \right] \sim \left[\begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ -1 & 1 & 3 & -1 & -3 \\ 2 & -2 & -6 & 1 & 3 \end{array} \right] \sim \left[\begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ 0 & \textcircled{-1} & 2 & 0 & -1 \\ 0 & 2 & -4 & -1 & -1 \end{array} \right] \\
& \sim \left[\begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 2 & -4 & -1 & -1 \end{array} \right] \sim \left[\begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 1 & 4 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{-1} & -3 \end{array} \right] \\
& \sim \left[\begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 1 & 4 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right] \sim \left[\begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 0 & 1 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right]
\end{aligned}$$

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Definition 5.3

The columns of $\text{RREF}(A)$ without a leading term represent **free variables** of the linear system modeled by A that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by A .

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Example 5.4

Here, x_3 is the free variable set equal to a since its column lacks a pivot, and the other bounded variables are put in terms of a .

$$2x_1 - 2x_2 - 6x_3 + x_4 = 3$$

$$-x_1 + x_2 + 3x_3 - x_4 = -3$$

$$x_1 - 2x_2 - x_3 + x_4 = 1$$

$$x_1 - 5x_3 = 1$$

$$x_2 - 2x_3 = 1$$

$$x_4 = 3$$

$$x_1 = 1 + 5a$$

$$x_2 = 1 + 2a$$

$$x_3 = a$$

$$x_4 = 3$$

 \Rightarrow

$$\begin{array}{c} \Downarrow \\ \left[\begin{array}{cccc|c} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array} \right] \sim \left[\begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 0 & 1 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right] \Uparrow \end{array}$$

So the solution set is $\left\{ \begin{bmatrix} 1 + 5a \\ 1 + 2a \\ a \\ 3 \end{bmatrix} \mid a \in \mathbb{R} \right\}.$

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Activity 5.5

Solve the system of linear equations, circling the pivot positions in your augmented matrices as you work.

$$-x_1 + x_2 - 3x_3 + 2x_4 = 0$$

$$2x_1 - x_2 + 5x_3 + 3x_4 = -11$$

$$3x_1 + 2x_2 + 4x_3 + x_4 = 1$$

$$x_2 - x_3 + x_4 = 1$$

Remember to find the solution set of the system by setting the free variable (the column without a pivot position) equal to a , and then express each of the other bounded variables equal to an expression in terms of a .

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Remark 5.6

From now on, unless specified, there's no need to show your work in finding $\text{RREF}(A)$, so you may use a calculator to speed up your work.

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Activity 5.7

Solve the linear system

$$2x_1 - 3x_2 = 17$$

$$x_1 + 2x_2 = -2$$

$$-x_1 - x_2 = 1$$

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Activity 5.8

Show that all linear systems of the form

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & 0 \end{array}$$

are consistent by finding a quickly verifiable solution.

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Definition 5.9

A **homogeneous system** is a linear system satisfying $b_i = 0$, that is, it is a linear system of the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

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Fact 5.10

Because the zero vector is always a solution, the solution set to any homogeneous system with infinitely-many solutions may be generated by multiplying the parameters representing the free variables by a minimal set of Euclidean vectors, and adding these up. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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Definition 5.11

A minimal set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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Activity 5.12

Find a basis for the solution set of the following homogeneous linear system.

$$x_1 + 2x_2 - x_4 = 0$$

$$x_3 + 4x_4 = 0$$

$$2x_1 + 4x_2 + x_3 + 2x_4 = 0$$

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Module V: Vector Spaces

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At the end of this module, students will be able to...

- **V1: Vector Spaces.** Determine if a set with given operations forms a vector space.
- **V2: Linear Combinations.** Determine if a vector can be written as a linear combination of a given set of vectors.
- **V3: Spanning Sets.** Determine if a set of vectors spans a vector space.
- **V4: Subspaces.** Determine if a subset of a vector space is a subset or not.

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Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems
(Standard(s) E1,E2,E3).

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The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/prec calculus/vectors-prec alc/vector-addition-subtraction/v/adding-and-subtracting-vectors>
- <https://www.khanacademy.org/math/prec calculus/vectors-prec alc/combined-vector-operations/v/combined-vector-operations-example>
- <https://www.khanacademy.org/math/prec calculus/imaginary-and-complex-numbers/v/adding-and-subtracting-complex-numbers/v/adding-complex-numbers>
- <https://www.khanacademy.org/math/algebra/introduction-to-polynomial-expressions/v/adding-and-subtracting-polynomials/v/adding-and-subtracting-polynomials-1>

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Activity 7.1

Consider each of the following vector properties. Label each property with \mathbb{R}^1 , \mathbb{R}^2 , and/or \mathbb{R}^3 if that property holds for Euclidean vectors/scalars \mathbf{u} , \mathbf{v} , \mathbf{w} of that dimension.

1 Addition associativity.

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

2 Addition commutivity.

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

3 Addition identity.

There exists some $\mathbf{0}$ where $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

4 Addition inverse.

There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$

5 Addition midpoint uniqueness.

There exists a unique \mathbf{m} where the
 distance from \mathbf{u} to \mathbf{m} equals the
 distance from \mathbf{m} to \mathbf{v} .

6 Scalar multiplication associativity.

$$a(b\mathbf{v}) = (ab)\mathbf{v}.$$

7 Scalar multiplication identity.

$$1\mathbf{v} = \mathbf{v}.$$

8 Scalar multiplication relativity.

There exists some scalar c where either
 $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.

9 Scalar distribution.

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

10 Vector distribution.

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

11 Orthogonality.

There exists a non-zero vector \mathbf{n} such
 that \mathbf{n} is orthogonal to both \mathbf{u} and \mathbf{v} .

12 Bidimensionality.

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} \text{ for some value of } a, b.$$

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Definition 7.2

A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V , and let a, b be scalar numbers.

- **Addition associativity.**
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition commutativity.**
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Addition identity.**
There exists some $\mathbf{0}$ where
 $\mathbf{v} + \mathbf{0} = \mathbf{v}.$
- **Addition inverse.**
There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$
- **Scalar multiplication associativity.**
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **Scalar multiplication identity.**
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar distribution.**
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Vector distribution.**
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

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Definition 7.3

The most important examples of vector spaces are the **Euclidean vector spaces** \mathbb{R}^n , but there are other examples as well.

Activity 7.4

Consider the following vector space that models motion along the curve $y = e^x$.

Let $V = \{(x, y) : y = e^x\}$, where $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$, and $c(x, y) = (cx, y^c)$.

Prove that V paired with these operations satisfies all the vector space properties.

- **Addition associativity.**

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

- **Addition identity.**

There exists some $\mathbf{0}$ where

$$\mathbf{v} + \mathbf{0} = \mathbf{v}.$$

- **Addition inverse.**

There exists some $-\mathbf{v}$ where

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

- **Scalar multiplication associativity.**

$$a(b\mathbf{v}) = (ab)\mathbf{v}.$$

- **Scalar multiplication identity.**

$$1\mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

- **Vector distribution.**

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

For example, Addition Commutivity may be proven by showing

$$\mathbf{u} \oplus \mathbf{v} = (x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) = (x_2 + x_1, y_2 y_1) = (x_2, y_2) \oplus (x_1, y_1) = \mathbf{v} \oplus \mathbf{u}$$

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Remark 8.1

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- \mathbb{R}^n : Euclidean vectors with n components.
- \mathbb{R}^∞ : Sequences of real numbers (v_1, v_2, \dots) .
- $\mathbb{R}^{m \times n}$: Matrices of real numbers with m rows and n columns.
- \mathbb{C} : Complex numbers.
- \mathcal{P}^n : Polynomials of degree n or less.
- \mathcal{P} : Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

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Activity 8.2

Let $V = \{(a, b) : a, b \text{ are real numbers}\}$, where $(a_1, b_1) + (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$ and $c(a, b) = (a^c, b + c)$. Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.**
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition commutativity.**
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Addition identity.**
There exists some $\mathbf{0}$ where
 $\mathbf{v} + \mathbf{0} = \mathbf{v}.$
- **Addition inverse.**
There exists some $-\mathbf{v}$ where
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$
- **Scalar multiplication associativity.**
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **Scalar multiplication identity.**
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar distribution.**
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Vector distribution.**
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

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Definition 8.3

A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$ for any choice of scalar multiples c_1, c_2, \dots, c_m .

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Definition 8.4

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

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Activity 8.5

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

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Activity 8.5

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Part 1: Sketch $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

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Activity 8.5

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Part 1: Sketch $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part 2: Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ in the xy plane.

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Activity 8.6

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

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Activity 8.6

Consider span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Part 1: Sketch $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in the xy plane for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

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Activity 8.6

Consider $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Part 1: Sketch $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in the xy plane for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Part 2: Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ in the xy plane.

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Activity 8.7

Sketch a representation of all the vectors given by $\text{span} \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ in the xy plane.

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (Remember, you should use a calculator to help find RREF.)

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Activity 8.8

The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ exactly when the vector

equation $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (Remember, you should use a calculator to help find RREF.)

Part 3: Given this solution, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$?

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Application Activities - Module V Part 3 - Class Day 9

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Fact 9.1

A vector \mathbf{b} belongs to $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$ is consistent.

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Remark 9.2

To determine if \mathbf{b} belongs to $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, find $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$.

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Activity 9.3

Determine if $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

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Activity 9.4

Determine if $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$ belongs to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ by row-reducing an appropriate matrix.

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Observation 9.5

So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space \mathbb{R}^n ; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

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Activity 9.6

We previously checked that $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ does not belong to $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$.

Does $f(x) = 3x^2 - 2x + 1$ belong to $\text{span}\{x^2 - 3, -x^2 - 3x + 2\}$?

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Activity 9.7

Does the matrix $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$ belong to $\text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\}$?

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Activity 9.8

Does the complex number $2i$ belong to $\text{span}\{-3 + i, 6 - 2i\}$?

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Activity 9.9

How many vectors are required to span \mathbb{R}^2 ? Sketch a drawing in the xy plane to support your guess.

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Activity 9.10

How many vectors are required to span \mathbb{R}^3 ?

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Application Activities - Module V Part 4 - Class Day 10

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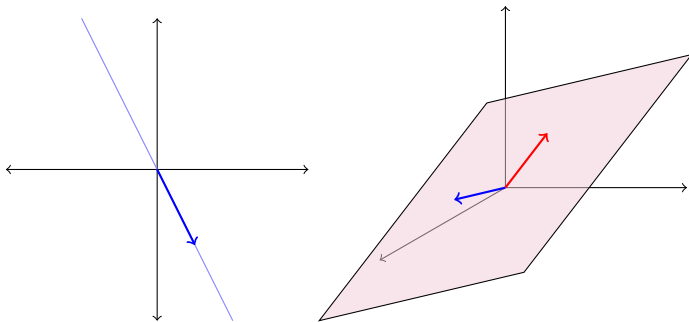
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Fact 10.1At least n vectors are required to span \mathbb{R}^n .

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Activity 10.2

Choose a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in \mathbb{R}^3 that is not in $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by ensuring

$$\left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]. \text{ (Why does this work?)}$$

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Fact 10.3

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ fails to span all of \mathbb{R}^n exactly when $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_m]$ has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & | & a \\ -1 & 0 & | & b \\ 0 & 1 & | & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix}$$

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Activity 10.4

Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does

$\mathbb{R}^4 = \text{span } S$?

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Activity 10.5

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2\}$$

Does $\mathcal{P}^3 = \text{span } S$?

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Definition 10.6

A subset of a vector space is called a **subspace** if it is itself a vector space.

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Fact 10.7

If S is a subset of a vector space V , then $\text{span } S$ is a subspace of V .

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Remark 10.8

To prove that a subset is a subspace, you need only verify that $c\mathbf{v} + d\mathbf{w}$ belongs to the subset for any choice of vectors \mathbf{v}, \mathbf{w} from the subset and any real scalars c, d .

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Activity 10.9

Prove that $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$ is a subspace of the vector space of all degree-two polynomials by showing that $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$ belongs to P .

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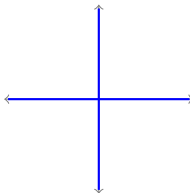
Part 2 (Day 26)

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Activity 10.10

Consider the subset of \mathbb{R}^2 where at least one coordinate of each vector is 0.



Find a linear combination $c\mathbf{v} + d\mathbf{w}$ that does not belong to this subset.

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Fact 10.11

Suppose a subset S of V is isomorphic to another vector space W . Then S is a subspace of V .

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Activity 10.12

Show that the set of 2×2 matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of $\mathbb{R}^{2 \times 2}$ by selecting a Euclidean space isomorphic to S .

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Module S: Structure of vector spaces

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At the end of this module, students will be able to...

- **S1. Linear independence** Determine if a set of Euclidean vectors is linearly dependent or independent.
- **S2. Basis verification** Determine if a set of vectors is a basis of a vector space
- **S3. Basis construction** Construct a basis for the subspace spanned by a given set of vectors.
- **S4. Dimension** I can compute the dimension of a vector space.

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Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.
- Apply linear combinations and spanning sets **(Standard(s) V2,V3)**.

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The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/prec calculus/vectors-prec calc/vector-addition-subtraction/v/adding-and-subtracting-vectors>
- <https://www.khanacademy.org/math/prec calculus/vectors-prec calc/combined-vector-operations/v/combined-vector-operations-example>

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Application Activities - Module S Part 1 - Class Day 12

Module E

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Module S

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Module A

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Activity 12.1

In the previous module, we considered

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

and showed that $\text{span } S \neq \mathbb{R}^4$. Find two vectors that are in the span of the other three vectors.

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Definition 12.2

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

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Activity 12.3

Suppose $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Is the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ consistent with one solution, consistent with infinitely many solutions, or inconsistent?

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Fact 12.4

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$ is consistent with infinitely many solutions.

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Activity 12.5

Find

$$\text{RREF} \left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array} \right]$$

and circle the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

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Fact 12.6

A set of Euclidean vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if RREF $[\mathbf{v}_1 \quad \dots \quad \mathbf{v}_n]$ has a column without a pivot position.

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Activity 12.7

TODO (compute RREF and label each set of vectors as linearly independent/dependent)

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Application Activities - Module S Part 2 - Class Day 13

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Activity 13.1
(take basis shown to be linearly independent in previous day, and show that it spans)

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Module M

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Definition 13.2

A **basis** is a linearly independent set that spans a vector space.

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Module M

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Observation 13.3

A basis may be thought of as building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

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Module S

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Module A

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Module M

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Activity 13.4
(given four sets of general vectors, identify which are bases and which aren't)

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Activity 13.5

If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for \mathbb{R}^4 , that means $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$ doesn't have a column without a pivot position, and doesn't have a row of zeros. What is $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$?

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Fact 13.6

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for \mathbb{R}^n if and only if $m = n$ and

$$\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

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Activity 13.7

(given four sets of \mathbb{R}^5 vectors, identify which are bases and which aren't)

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Module S

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- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

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Module M

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Activity 13.8
How can $\{u,v,u+v\}$ (but with numbers) be changed to make it linearly independent?

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Module S

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Module M

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Application Activities - Module S Part 3 - Class Day 14

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Module V

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Module S

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Module M

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Activity 14.1

(discover that the redundant vectors are non-pivot columns)

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Fact 14.2

To compute a basis for the subspace $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, simply remove the vectors corresponding to the non-pivot columns of $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$.

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Activity 14.3
(find ALL the bases for span S that are subsets of S)

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Module S

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Fact 14.4
All bases for a vector space are the same size.

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Activity 14.5

Prove that if $\{\mathbf{v}\}$ is a basis for V , then $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly dependent (assuming $\mathbf{w}_1 \neq \mathbf{w}_2$).

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Fact 14.6
All bases for a vector space are the same size.

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Definition 14.7

The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

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Activity 14.8

Reduce a bunch of spans to bases to find their dimension.

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Module M

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Activity 14.9

What is the dimension of the vector space of 7th-degree polynomials \mathcal{P}^7 ?

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Activity 14.10

What is the dimension of the vector space of polynomials \mathcal{P} ?

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Observation 14.11

Several interesting vector spaces are infinite-dimensional:

- The space of polynomials \mathcal{P}
- The space of real number sequences \mathbb{R}^∞
- The space of continuous functions $C(\mathbb{R})$

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Fact 14.12
Every vector space with dimension $n < \infty$ is isomorphic to \mathbb{R}^n .

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Module A: Algebraic properties of linear maps

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At the end of this module, students will be able to...

- **A1. Linear maps as matrices** I can write the matrix (with respect to the standard bases) corresponding to a linear transformation between Euclidean spaces.
- **A2. Linear map verification** I can determine if a map between vector spaces is linear or not.
- **A3. Injectivity and Surjectivity** I can determine if a given linear map is injective and/or surjective
- **A4. Kernel and Image** I can compute the kernel and image of a linear map, including finding bases.

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Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing (**Standard(s) E1, E2, E3, E4**).
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace (**Standard(s) V3**).
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent (**Standard(s) S1**).
- State the definition of a basis, and determine if a set of vectors is a basis (**Standard(s) S2**).

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The following resources will help you prepare for this module.

- Review the supporting Standards listed above.

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Application Activities - Module A Part 1 - Class Day 17

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Definition 17.1

A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \rightarrow W$ is called a linear transformation if

- 1 $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$
- 2 $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}, \vec{v} \in V$.

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.

V is called the **domain** of T and W is called the **co-domain** of T .

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Activity 17.2

Determine if each of the following maps are linear transformations

(a) $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $T_1 \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \sqrt{a^2 + b^2}$

(b) $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by $T_2 \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$

(c) $T_3 : \mathcal{P}_d \rightarrow \mathcal{P}_{d-1}$ given by $T_3(f(x)) = f'(x)$.

(d) $T_4 : C(\mathbb{R}) \rightarrow C(\mathbb{R})$ given by $T_4(f(x)) = f(-x)$

(e) $T_5 : \mathcal{P} \rightarrow \mathcal{P}$ given by $T_5(f(x)) = f(x) + x^2$

Activity 17.3

Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear transformation, and you know $T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and $T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute each of the following:

(a) $T \left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right)$

(b) $T \left(\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right)$

(c) $T \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$

(d) $T \left(\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} \right)$

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Activity 17.4

Suppose $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is a linear transformation. What is the smallest number of vectors needed to determine T ? In other words, what is the smallest number n such that there are $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^4$ and given $T(\vec{v}_1), \dots, T(\vec{v}_n)$ you can determine $T(\vec{w})$ for *any* $\vec{w} \in \mathbb{R}^2$?

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Observation 17.5

Fix an ordered basis for V . Since every vector can be written *uniquely* as a linear combination of basis vectors, a linear transformation $T : V \rightarrow W$ corresponds exactly to a choice of where each basis vector goes. For convenience, we can thus encode a linear transformation as a matrix, with one column for the image of each basis vector (in order).

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Activity 17.6

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation with

$$T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard ordered basis.

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Activity 17.7

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation with

$$T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad T \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the ordered basis

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

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Activity 17.8

Let $D : \mathcal{P}_3 \rightarrow \mathcal{P}_2$ be the derivative map (recall this is a linear transformation). Write the matrix corresponding to D with respect to the ordered basis $\{1, x, x^2, x^3\}$.

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Definition 18.1

Let $T : V \rightarrow W$ be a linear transformation.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.
- T is called **surjective** or **onto** if every element of W is mapped to by an element of V . More precisely, for every $\vec{w} \in W$, there is some $v \in V$ with $T(\vec{v}) = \vec{w}$.

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Activity 18.2

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

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Activity 18.3

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

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Definition 18.4

We also have two important sets called the **kernel** of T and the **image** of T .

$$\ker T = \{ \vec{v} \in V \mid T(\vec{v}) = 0 \}$$

$$\operatorname{Im} T = \{ \vec{w} \in W \mid \text{there is some } v \in V \text{ with } T(\vec{v}) = \vec{w} \}$$

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Activity 18.5

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T .

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Activity 18.6

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T .

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Activity 18.7

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Part 1: Describe surjective linear transformations in terms of the image.

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Activity 18.7

Part 1: Describe surjective linear transformations in terms of the image.

Part 2: Describe injective linear transformations in terms of the kernel.

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Activity 18.8

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix} \text{ (for the standard basis).}$$

- 1) Write a system of equations whose solution set is the kernel.
- 2) Compute $\text{RREF}(A)$ and solve the system of equations.
- 3) Compute the kernel of T
- 4) Find a basis for the kernel of T

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Activity 18.9

Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

- 1) Write a system of equations whose solution set is the kernel.
- 2) Compute $\text{RREF}(A)$ and solve the system of equations.
- 3) Compute the kernel of T
- 4) Find a basis for the kernel of T

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Activity 18.10

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix} \text{ (for the standard basis).}$$

- 1) Find a set of vectors that span the image of T
- 2) Find a basis for the image of T .

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Activity 18.11

Let $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$

(for the standard basis).

- 1) Find a set of vectors that span the image of T
- 2) Find a basis for the image of T .

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Application Activities - Module A Part 3 - Class Day 19

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Activity 19.1

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). You have cards containing a number of statements about T and A . Sort them into groups of equivalent statements, and post them on your board.

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Activity 19.2

Cycle around the room counter-clockwise. If they have two things grouped together that you know are not equivalent, write a reason or counter-example on a sticky note.

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Activity 19.3

Come up with as many statements as you can, and add them to the appropriate group.

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Module M: Understanding Matrices Algebraically

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At the end of this module, students will be able to...

- **M1. Matrix multiplication** Multiply matrices.
- **M2. Invertible matrices** Determine if a square matrix is invertible or not.
- **M3. Matrix inverses** Compute the inverse matrix of an invertible matrix.

Module E

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Before beginning this module, each student should be able to...

- Compose functions of real numbers
- Solve systems of linear equations (**Standard(s) E3**)
- Find the matrix corresponding to a linear transformation (**Standard(s) A1**)
- Determine if a linear transformation is injective and/or surjective (**Standard(s) A3**)
- Interpret the ideas of injectivity and surjectivity in multiple ways

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The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/algebra2/manipulating-functions/function-composition/v/function-composition>

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Activity 21.1

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by the matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be

given by the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) \mathbb{R}

(b) \mathbb{R}^2

(c) \mathbb{R}^3

(d) \mathbb{R}^4

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Activity 21.2

What is the codomain of the composition map $S \circ T$?

(a) \mathbb{R}

(b) \mathbb{R}^2

(c) \mathbb{R}^3

(d) \mathbb{R}^4

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Activity 21.3

The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{4,3}$

(b) $M_{4,2}$

(c) $M_{3,2}$

(d) $M_{2,3}$

(e) $M_{2,4}$

(f) $M_{3,4}$

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Activity 21.4

Compute $(S \circ T)(\vec{e}_1)$, $(S \circ T)(\vec{e}_2)$, and $(S \circ T)(\vec{e}_3)$.

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Activity 21.5

Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

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Activity 21.6

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given

by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

- (a) \mathbb{R}
- (b) \mathbb{R}^2
- (c) \mathbb{R}^3
- (d) \mathbb{R}^4

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Activity 21.7

What is the codomain of the composition map $S \circ T$?

- (a) \mathbb{R}
- (b) \mathbb{R}^2
- (c) \mathbb{R}^3
- (d) \mathbb{R}^4

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Activity 21.8

The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{2,2}$

(b) $M_{2,3}$

(c) $M_{3,2}$

(d) $M_{3,3}$

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Activity 21.9

Compute $(S \circ T)(\vec{e}_1)$ and $(S \circ T)(\vec{e}_2)$

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Activity 21.10

Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

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Activity 21.11

Let $T : \mathbb{R}^1 \rightarrow \mathbb{R}^4$ be given by the matrix $B = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ and $S : \mathbb{R}^4 \rightarrow \mathbb{R}^1$ be given by

the matrix $A = \begin{bmatrix} 2 & 3 & 2 & 5 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

- (a) \mathbb{R}
- (b) \mathbb{R}^2
- (c) \mathbb{R}^3
- (d) \mathbb{R}^4

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Activity 21.12

What is the codomain of the composition map $S \circ T$?

(a) \mathbb{R}

(b) \mathbb{R}^2

(c) \mathbb{R}^3

(d) \mathbb{R}^4

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Activity 21.13

The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{1,1}$

(b) $M_{1,4}$

(c) $M_{4,1}$

(d) $M_{4,4}$

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Activity 21.14

Compute $(S \circ T)(\vec{e}_1)$

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Module S

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Activity 21.15
Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

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Activity 21.16

Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- 1 Compute AX
- 2 Interpret the system of equations below as a matrix equation

$$3x + y - z = 5$$

$$2x + 4z = -7$$

$$-x + 3y + 5z = 2$$

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Activity 22.1

Each row operation can be interpreted as a matrix multiplication. Let $A \in M_{4,4}$

- 1) Find a matrix S_1 such that S_1A is the result of swapping the second and fourth rows of A .
- 2) Find a matrix S_2 such that S_2A is the result of adding 5 times the third row of A to the first.
- 3) Find a matrix S_3 such that S_3A is the result of doubling the fourth row of A .

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Activity 22.2

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). Consider the following statements about T

- (a) T is injective
- (b) T is surjective
- (c) T is bijective (i.e. both injective and surjective)
- (d) $AX = B$ has a solution for all $B \in M_{m,1}$
- (e) $AX = B$ has a unique solution for all $B \in M_{m,1}$
- (f) $AX = 0$ has a non-trivial solution.
- (g) The columns of A span \mathbb{R}^m
- (h) The columns of A are linearly independent
- (i) The columns of A are a basis of \mathbb{R}^m
- (j) $\text{RREF}(A)$ has n pivot columns
- (k) $\text{RREF}(A)$ has m pivot columns

Sort these statements into groups of equivalent statements.

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Activity 22.3

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis).

If T is injective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

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Activity 22.4

If T is surjective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

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Activity 22.5

If T is bijective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

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Activity 23.1

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map with matrix $A \in M_{n,n}$.

If T is a bijection, then $AX = B$ has a unique solution for all $B \in \mathbb{R}^n$. Thus we can define a map $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by defining $T^{-1}(B)$ to be this solution. It follows immediately that $T \circ T^{-1}$ is the identity map. The matrix corresponding to T^{-1} is denoted A^{-1} .

- 1) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_1)$
- 2) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_2)$
- 3) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_3)$
- 4) Compute A^{-1}

A (square) matrix is called *invertible* if it corresponds to an invertible linear transformation.

- 1) Find the inverse of the matrix $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$
- 2) Find the inverse of the matrix $\begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$

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At the end of this module, students will be able to...

- **G1. Determinants** Compute the determinant of a square matrix.
- **G2. Eigenvalues** Find the eigenvalues of a square matrix, along with their algebraic multiplicities.
- **G3. Eigenvectors** Find the eigenspace of a square matrix associated to a given eigenvalue.
- **G4. Geometric multiplicity** Compute the geometric multiplicity of an eigenvalue of a square matrix.

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Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **(Standard(s) A1)**.
- Recall and use the definition of a linear transformation **(Standard(s) A2)**.
- Find all roots of quadratic polynomials (including complex ones), and be able to use the rational root theorem to find all rational roots of a higher degree polynomial.
- Interpret the statement “ A is an invertible matrix” in many equivalent ways in different contexts.

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The following resources will help you prepare for this module.

- Finding the area of a parallelogram: <https://www.khanacademy.org/math/basic-geo/basic-geo-area-and-perimeter/parallelogram-area/a/area-of-parallelogram>
- Factoring quadratics: <https://www.khanacademy.org/math/algebra2/polynomial-functions/factoring-polynomials-quadratic-forms-alg2/v/factoring-polynomials-1>
- Finding complex roots of quadratics: <https://www.khanacademy.org/math/algebra2/polynomial-functions/quadratic-equations-with-complex-numbers/v/complex-roots-from-the-quadratic-formula>
- Finding all roots of polynomials: <https://www.khanacademy.org/math/algebra2/polynomial-functions/finding-zeros-of-polynomials/v/finding-roots-or-zeros-of-polynomial-1>
- The Rational Root Theorem: https://artofproblemsolving.com/wiki/index.php?title=Rational_Root_Theorem

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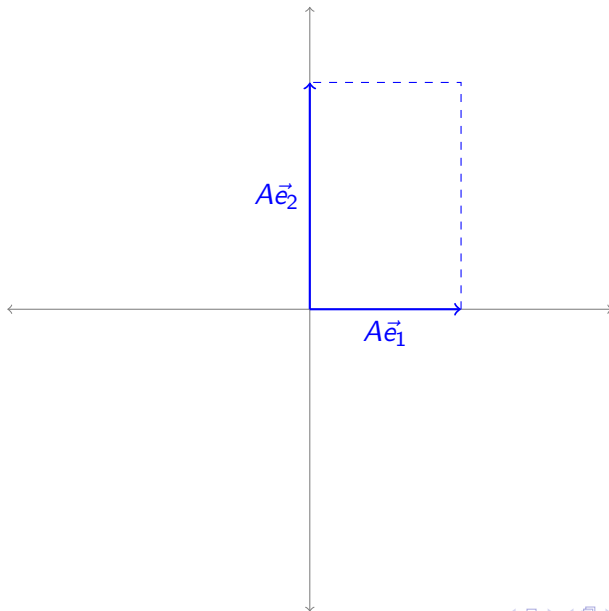
Module G

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Application Activities - Module G Part 1 - Class Day 25

Activity 25.1

Consider the linear transformation $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



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Activity 25.2

Consider the following linear transformations $A_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

- $A_2 = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$
- $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
- $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

For each linear transformation, do the following:

- Draw a graph showing the image of the unit square.
- Compute how much the area was stretched out.
- Determine which axes (or lines) were preserved; how were they stretched out?

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Activity 25.3

Our goal is to define a function $\det : M_n \rightarrow \mathbb{R}$ that takes a square matrix (linear transformation $\mathbb{R}^n \rightarrow \mathbb{R}^n$) and returns its area stretching factor. This function is called the **determinant**.

What properties should this function have?

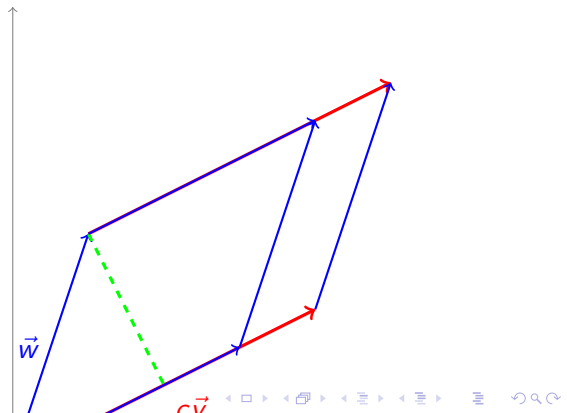
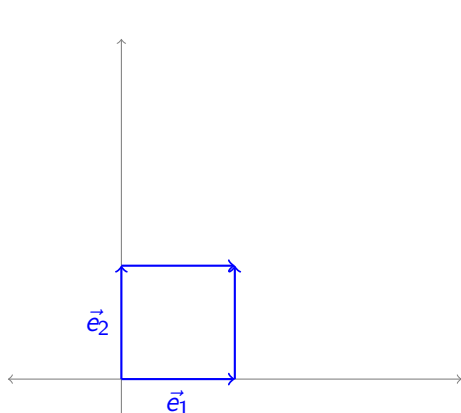
Match the four pictures to the following four expressions

$$\det(\vec{e}_1, \vec{e}_2)$$

$$\det(\vec{v}, \vec{v})$$

$$\det(c\vec{v}, \vec{w})$$

$$\det(\vec{u} + \vec{v}, \vec{w})$$



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Activity 25.4

What can you conclude about each of the following?

① $\det(\vec{e}_1, \vec{e}_2)$

② $\det(\vec{v}, \vec{v})$

③ $\det(c\vec{v}, \vec{w})$

④ $\det(\vec{u} + \vec{v}, \vec{w})$

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Definition 25.5

To summarize, we have 3 properties (stated here over \mathbb{R}^n)

P1: $\det(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n) = 1$

P2: If $\vec{v}_i = \vec{v}_j$ for some $i \neq j$, then $\det(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = 0$.

P3: The determinant is linear in each column.

These three properties uniquely define the **determinant**, as we shall see.

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Observation 25.6

Note that if $\vec{v}, \vec{w} \in \mathbb{R}^2$ and $A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$ we will write either $\det(A)$ or $\det(\vec{v}, \vec{w})$ as is convenient.

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Activity 25.7

How are $\det(\vec{v}, \vec{w})$ and $\det(\vec{w}, \vec{v})$ related?

- (a) $\det(\vec{v}, \vec{w}) = \det(\vec{w}, \vec{v})$
- (b) $\det(\vec{v}, \vec{w}) = -\det(\vec{w}, \vec{v})$
- (c) They are unrelated
- (d) They are related, but not by either (a) or (b).

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Observation 25.8

Note that this implies that the determinant is actually a *signed* area (volume)!

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Activity 25.9

How are $\det(\vec{v} + \vec{w}, \vec{w})$ and $\det(\vec{v}, \vec{w})$ related?

- (a) $\det(\vec{v} + \vec{w}, \vec{w}) = \det(\vec{v}, \vec{w})$
- (b) $\det(\vec{v} + \vec{w}, \vec{w}) = -\det(\vec{v}, \vec{w})$
- (c) They are unrelated
- (d) They are related, but not by either (a) or (b).

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Observation 25.10

Note that we now understand the effect of any column operation on the determinant.

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Activity 26.1

How are $\det(A)$ and $\det(A^T)$ related?

(a) $\det(A) = \det(A^T)$

(b) $\det(A) = -\det(A^T)$

(c) $\det(A) = \frac{1}{\det(A^T)}$

(d) They are unrelated

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Observation 26.2

Thus, row operations behave like column operations. So we can use row reduction to compute determinants.

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Activity 26.3

Compute $\det \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$.

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Activity 26.4

Compute $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

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Activity 26.5

Which of the following is the same as $\det \begin{bmatrix} 3 & -2 & 0 \\ 5 & -1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$?

(a) $\det \begin{bmatrix} 3 & -2 \\ 5 & -1 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$

(c) $\det \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}$

(d) None of these

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Activity 26.6

Which of the following is the same as $\det \begin{bmatrix} 3 & 0 & 7 \\ 5 & 1 & 2 \\ -2 & 0 & 6 \end{bmatrix}$?

(a) $\det \begin{bmatrix} 3 & 7 \\ 5 & 2 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix}$

(c) $\det \begin{bmatrix} 5 & 2 \\ -2 & 6 \end{bmatrix}$

(d) None of these

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Activity 26.7

Compute $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$

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Activity 26.8

Using the fact that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 5 \\ 0 & 3 & 3 \end{bmatrix}$.

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Activity 26.9

Compute
$$\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$$

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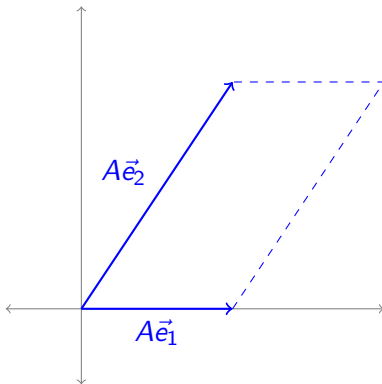
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Application Activities - Module G Part 3 - Class Day 27

Activity 27.1

Consider the linear transformation $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



Observe

$$A\vec{e}_1 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\vec{e}_1$$

Is there another vector $\vec{v} \in \mathbb{R}^2$ such that $A\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$?

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Definition 27.2

Let $A \in M_n(\mathbb{R})$. An **eigenvector** is a vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x}$ is parallel to \vec{x} ; in other words, $A\vec{x} = \lambda\vec{x}$ for some scalar λ , which is called an **eigenvalue**

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Observation 27.3

Observe that $A\vec{x} = \lambda\vec{x}$ is equivalent to $(A - \lambda I)\vec{x} = 0$.

- To find eigenvalues, we need to find values of λ such that $A - \lambda I$ has a nontrivial kernel; equivalently, $A - \lambda I$ is not invertible, which is equivalent to $\det(A - \lambda I) = 0$. $\det(A - \lambda I)$ is called the **characteristic polynomial**.
- Once an eigenvalue is found, the eigenvectors form a subspace called the **eigenspace**, which is simply the kernel of $A - \lambda I$. Each eigenvalue will have an associated eigenspace.

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Activity 27.4

Find the eigenvalues for the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

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Activity 27.5
Compute the eigenspace associated to the eigenvalue 3.

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Activity 27.6

Find all the eigenvalues and associated eigenspaces for the matrix

$$\begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}.$$

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Application Activities - Module G Part 4 - Class Day 28

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Activity 28.1

If $A \in M_4$, what is the largest number of eigenvalues A can have?

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Activity 28.2

2 is an eigenvalue of each of the matrices $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$ and

$$B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}.$$

Compute the eigenspace associated to 2 for both A and B .

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Definition 28.3

- The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.
- The **geometric multiplicity** of an eigenvalue is the dimension of the eigenspace.

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Activity 28.4

How are the algebraic and geometric multiplicities related?

- (a) The algebraic multiplicity is always at least as big as than the geometric multiplicity.
- (b) The geometric multiplicity is always at least as big as the algebraic multiplicity.
- (c) Sometimes the algebraic multiplicity is larger and sometimes the geometric multiplicity is larger.

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Activity 28.5

Find the eigenvalues, along with both their algebraic and geometric multiplicities,

for the matrix
$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

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Activity 28.6

Find the eigenvalues of the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

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Activity 28.7

Describe what this linear transformation is doing geometrically; draw a picture.

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Activity 28.8

Fix a real number θ and find the eigenvalues of the matrix

$$A_{\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}. \text{ What are the eigenvalues?}$$

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Activity 28.9

D raw pictures and describe the geometric actions of the maps $A_{\frac{\pi}{4}}$, $A_{\frac{\pi}{2}}$, and A_{π} .

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Activity 28.10

For how many values of θ does the rotation matrix A_θ have real eigenvalues?

- (a) 0
- (b) 1
- (c) 2
- (d) 3
- (e) An infinite number