Sample Assessment Exercises

This document contains one exercise and solution for each standard. The goal is to give you an idea of what the exercises might look like, and what the expectations for a complete solution are.

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & 1 & 1 \\ -1 & -4 & 1 & -7 & 0 \\ 0 & 1 & -1 & 0 & -2 \end{array}\right]$$

Solution:

$$\begin{alignedat}{5} 3x_1 &+ 2x_2 & & &+ x_4 &= 1 \\ -x_1 &- 4x_2 &+ x_3 &- 7x_4 & &= 0 \\ & x_2 &- x_3 & & &= -2 \end{alignedat}$$

$\square$

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 2 & -1 & -3 \\ 2 & 4 & -1 & -1 \end{bmatrix}$$

Solution:

$$\begin{aligned}
\begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 2 & -1 & -3 \\ 2 & 4 & -1 & -1 \end{bmatrix} &\sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 3 & 1 & 2 \\ 2 & 4 & -1 & -1 \end{bmatrix} && \text{Swap Rows 1 and 2} \\
&\sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} && \text{Add } -2 \text{ Row 1 to Row 3} \\
&\sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 5 \end{bmatrix} && \text{Multiply Row 3 by } \frac{1}{3} \\
&\sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{13}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 5 \end{bmatrix} && \text{Add } -2 \text{ Row 2 to Row 1} \\
&\sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{13}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix} && \text{Add } -\frac{1}{3} \text{ Row 3 to Row 2} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix} && \text{Add } \frac{5}{3} \text{ Row 3 to Row 1}
\end{aligned}$$

$\square$

E3. Find the solution set for the following system of linear equations.

$$\begin{alignedat}{3} 2x &+ 4y &+ z &= 5 \\ x &+ 2y & &= 3 \end{alignedat}$$

Solution: First, note that this system corresponds to the matrix $\left(\left[\begin{array}{ccc|c} 2 & 4 & 1 & 5 \\ 1 & 2 & 0 & 3 \end{array}\right]\right)$. Then we compute (using technology)

$$\text{RREF}\left(\left[\begin{array}{ccc|c} 2 & 4 & 1 & 5 \\ 1 & 2 & 0 & 3 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

This corresponds to the system

$$\begin{aligned} x + 2y \quad &= 3 \\ z &= -1 \end{aligned}$$

Since the y-column is a non-pivot column, it is a free variable, so we let $y = a$; then we have

$$\begin{aligned} x + 2y \quad &= 3 \\ y \quad &= a \\ z &= -1 \end{aligned}$$

and thus

$$\begin{aligned} x &= 3 - 2a \\ y &= a \\ z &= -1 \end{aligned}$$

So the solution set is

$$\left\{ \begin{bmatrix} 3 - 2a \\ a \\ -1 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$$

$\square$

V1. Let V be the set of all pairs of numbers $(x, y) \in \mathbb{R}^2$, together with the operations $\oplus$ and $\odot$ defined by the following for all vectors $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2)$ and scalars $c \in \mathbb{R}$:

$$\begin{aligned} (x_1, x_2) \oplus (y_1, y_2) &= (2x_1 + 2y_1, 2x_2 + 2y_2) \\ c \odot (x_1, x_2) &= (cx_1, c^2 x_2) \end{aligned}$$

(a) Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot ((x_1, x_2) \oplus (y_1, y_2)) = c \odot (x_1, x_2) \oplus c \odot (y_1, y_2)$$

holds.

(b) Explain why V nonetheless is not a vector space.

Solution:

(a) Compute

$$\begin{aligned} c \odot ((x_1, x_2) \oplus (y_1, y_2)) &= c \odot (2x_1 + 2y_1, 2x_2 + 2y_2) \\ &= (c(2x_1 + 2y_1), c^2(2x_2 + 2y_2)) \\ &= (2cx_1 + 2cy_1, 2c^2 x_2 + 2c^2 y_2) \end{aligned}$$

and

$$\begin{aligned} c \odot (x_1, x_2) \oplus c \odot (y_1, y_2) &= (cx_1, c^2 x_2) \oplus (cy_1, c^2 y_2) \\ &= (2cx_1 + 2cy_1, 2c^2 x_2 + 2c^2 y_2). \end{aligned}$$

Since these are the same, we have shown that $c \odot ((x_1, x_2) \oplus (y_1, y_2)) = c \odot (x_1, x_2) \oplus c \odot (y_1, y_2)$.

(b) To show V is not a vector space, we must show that it fails one of the 8 defining properties of vector spaces. We will show that scalar multiplication does not distribute over scalar addition, i.e.

$$(c+d) \odot (x_1, x_2) \neq c \odot (x_1, x_2) \oplus d \odot (x_1, x_2)$$

for all scalars $c, d \in \mathbb{R}$ and vectors $(x_1, x_2) \in \mathbb{R}^2$. First, we compute

$$\begin{aligned} (c+d) \odot (x_1, x_2) &= ((c+d)x_1, (c+d)^2 x_2) \\ &= ((c+d)x_1, (c^2 + 2cd + d^2)x_2). \end{aligned}$$

Then we compute

$$\begin{aligned} c \odot (x_1, x_2) \oplus d \odot (x_1, x_2) &= (cx_1, c^2 x_2) \oplus (dx_1, d^2 x_2) \\ &= (2cx_1 + 2dx_2, 2c^2 x^2 + 2d^2 x_2). \end{aligned}$$

Thus, we see that $(c+d) \odot (x_1, x_2) \neq c \odot (x_1, x_2) \oplus d \odot (x_1, x_2)$, and therefor V is not a vector space.

$\square$

V2. Explain why the vector $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is or is not a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Solution: By definition, the statement

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \text{ is a linear combination of the vectors } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

is equivalent to the statement

$$\text{There exists a solution to the system of equations } x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

This system corresponds to the augmented matrix $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 2 & 1 & -1 \\ 1 & -1 & -1 & 2 \end{array}\right]$. Therefore, we compute

$$\text{RREF} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 2 & 1 & -1 \\ 1 & -1 & -1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Since this corresponds to an inconsistent system of equations, the system of equations

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

has no solution, and therefore $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

$\square$

V3. Explain why the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ span or don't span $\mathbb{R}^3$.

Solution: By definition, the statement

$$\text{The vectors } \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}3\\2\\-1\end{bmatrix}, \text{ and } \begin{bmatrix}1\\1\\-1\end{bmatrix} \text{ span } \mathbb{R}^3$$

is equivalent to the statement

$$\text{For every } \vec{\mathbf{v}} \in \mathbb{R}^3, \text{ the system } x_1 \begin{bmatrix}1\\0\\1\end{bmatrix} + x_2 \begin{bmatrix}3\\2\\-1\end{bmatrix} + x_3 \begin{bmatrix}1\\1\\-1\end{bmatrix} = \vec{\mathbf{v}} \text{ has a solution.}$$

We compute

$$\text{RREF}\begin{bmatrix}1 & 3 & 1\\0 & 2 & 1\\1 & -1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & -\frac{1}{2}\\0 & 1 & \frac{1}{2}\\0 & 0 & 0\end{bmatrix}$$

Since the last row lacks a pivot, there is some vector $\vec{\mathbf{v}} \in \mathbb{R}^3$ that upon augmenting this matrix will produce an inconsistent system. That vector will not be in the span of these three vectos, so the vectors do not span $\mathbb{R}^3$.

$\square$

V4. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix}x\\y\\z\\w\end{bmatrix} \middle| \, x+y = 3z+2w \right\} \qquad U = \left\{ \begin{bmatrix}x\\y\\z\\w\end{bmatrix} \middle| \, x+y = 3z+w^2 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.

Solution:

To show that W is a subspace, let $\vec{\mathbf{v}} = \begin{bmatrix}x_1\\y_1\\z_1\\w_1\end{bmatrix} \in W$ and $\vec{\mathbf{w}} = \begin{bmatrix}x_2\\y_2\\z_2\\w_2\end{bmatrix} \in W$, so we know that $x_1+y_1 = 3z_1+2w_1$ and $x_2+y_2 = 3z_2+2w_2$. Consider

$$\begin{bmatrix}x_1\\y_1\\z_1\\w_1\end{bmatrix} + \begin{bmatrix}x_2\\y_2\\z_2\\w_2\end{bmatrix} = \begin{bmatrix}x_1+x_2\\y_1+y_2\\z_1+z_2\\w_1+w_2\end{bmatrix}.$$

To see if $\vec{\mathbf{v}}+\vec{\mathbf{w}} \in W$, we need to check if $(x_1+x_2)+(y_1+y_2) = 3(z_1+z_2)+2(w_1+w_2)$. We compute

$$\begin{aligned}
(x_1+x_2)+(y_1+y_2) &= (x_1+y_1)+(x_2+y_2) && \text{by regrouping}\\
&= (3z_1+2w_1)+(3z_2+2w_2) && \text{since } \vec{\mathbf{v}}, \vec{\mathbf{w}} \in W\\
&= 3(z_1+z_2)+2(w_1+w_2) && \text{by regrouping.}
\end{aligned}$$

Thus $\vec{\mathbf{v}}+\vec{\mathbf{w}} \in W$, so W is closed under vector addition.
Now consider

$$c\vec{\mathbf{v}} = \begin{bmatrix}cx_1\\cy_1\\cz_1\\cw_1\end{bmatrix}.$$

Similarly, to check that $c\vec{\mathbf{v}} \in W$, we need to check if $cx_1 + cy_1 = 3(cz_1) + 2(cw_1)$, so we compute

$$\begin{aligned} cx_1 + cy_1 &= c(x_1 + y_1) && \text{by factoring} \\ &= c(3z_1 + 2w_1) && \text{since } \vec{\mathbf{v}} \in W \\ &= 3(cz_1) + 2(cw_1) && \text{by regrouping} \end{aligned}$$

and we see that $c\vec{\mathbf{v}} \in W$, so W is closed under scalar multiplication. Therefore W is a subspace of $\mathbb{R}^3$.
Now, to show U is not a subspace, we will show that it is not closed under vector addition. Now let $\vec{\mathbf{v}} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix} \in U$ and $\vec{\mathbf{w}} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{bmatrix} \in U$, so we know that $x_1 + y_1 = 3z_1 + w_1^2$ and $x_2 + y_2 = 3z_2 + w_2^2$. Consider

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{bmatrix}.$$

To see if $\vec{\mathbf{v}} + \vec{\mathbf{w}} \in U$, we need to check if $(x_1 + x_2) + (y_1 + y_2) = 3(z_1 + z_2) + (w_1 + w_2)^2$. We compute

$$\begin{aligned} (x_1 + x_2) + (y_1 + y_2) &= (x_1 + y_1) + (x_2 + y_2) && \text{by regrouping} \\ &= (3z_1 + w_1^2) + (3z_2 + w_2^2) && \text{since } \vec{\mathbf{v}}, \vec{\mathbf{w}} \in W \\ &= 3(z_1 + z_2) + (w_1^2 + w_2^2) && \text{by regrouping} \end{aligned}$$

and thus $\vec{\mathbf{v}} + \vec{\mathbf{w}} \in W$ **only when** $w_1^2 + w_2^2 = (w_1 + w_2)^2$. Since this is not true in general, W is not closed under vector addition, and thus cannot be a subspace.

$\square$

V5. Explain why the vectors $\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 5 \\ 1 \\ 5 \end{bmatrix}$ are linearly dependent or linearly independent.

Solution: The vectors $\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 5 \\ 1 \\ 5 \end{bmatrix}$ are linearly independent precisely when the system of equations

$$x_1 \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 5 \\ 1 \\ 5 \end{bmatrix} = \vec{\mathbf{0}}$$

has a unique solution (namely, $\vec{\mathbf{0}}$).
Converting this system to the corresponding augmented matrix and row reducing, we have

$$\text{RREF} \begin{bmatrix} cccc|c3 & -1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 5 & 0 \\ 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 1 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the fourth column is not a pivot column, the system has (infinitely many) nontrivial solutions. Thus the vectors are linearly dependent.

$\square$

V6. Explain why the vectors

$$\begin{bmatrix}3\\2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\5\\1\\5\end{bmatrix}$$

are or are not a basis of $\mathbb{R}^4$.

Solution: Compute

$$\text{RREF}\begin{bmatrix}3 & -1 & 0 & 2\\2 & 1 & 1 & 5\\1 & 2 & -1 & 1\\0 & 3 & 1 & 5\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\end{bmatrix}.$$

Since the fourth column is not a pivot column, the vectors are linearly dependent and thus not a basis of $\mathbb{R}^4$.
(Alternate solutions: Since the fourth row not a pivot row, the vectors do not span $\mathbb{R}^4$ and thus are not a basis of $\mathbb{R}^4$. Or since the resulting matrix is not the identity matrix, the vectors do not form a basis of $\mathbb{R}^4$.)

$\square$

V7. Find a basis for the subspace

$$W = \text{span}\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\\-2\end{bmatrix}, \begin{bmatrix}3\\-6\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\6\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}.$$

Solution: Observe that

$$\text{RREF}\begin{bmatrix}1 & 1 & 3 & 1 & 2\\-3 & 0 & -6 & 6 & 3\\-1 & 1 & -1 & 1 & 0\\2 & -2 & 2 & -1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 2 & 0 & 1\\0 & 1 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

has pivot columns in the first, second, and fourth columns, and therefore removing the corresponding vectors shows

$$\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\\-2\end{bmatrix}, \begin{bmatrix}1\\6\\1\\-1\end{bmatrix}\right\}$$

is a basis of W.

$\square$

V8. Explain how to find the dimension of

$$W = \text{span}\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\\-2\end{bmatrix}, \begin{bmatrix}3\\-6\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\6\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}.$$

Solution: Observe that

$$\text{RREF}\begin{bmatrix}1 & 1 & 3 & 1 & 2\\-3 & 0 & -6 & 6 & 3\\-1 & 1 & -1 & 1 & 0\\2 & -2 & 2 & -1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 2 & 0 & 1\\0 & 1 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

has three pivot columns, so a basis of W has three elements, and therefore $\dim W = 3$.

$\square$

V9. Find a basis for the subspace of $\mathcal{P}^3$

$$W = \operatorname{span}\left\{3x^3 + 2x^2 + x, -x^3 + x^2 + 2x + 3, x^2 - x + 1, 2x^3 + 5x^2 + x + 5\right\}.$$

Solution: This question is equivalent to finding a basis for the subspace

$$W' = \operatorname{span}\left\{\begin{bmatrix}3\\2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\5\\1\\5\end{bmatrix}\right\}$$

of Euclidean vectors.
Compute

$$\operatorname{RREF}\begin{bmatrix}3 & -1 & 0 & 2\\ 2 & 1 & 1 & 5\\ 1 & 2 & -1 & 1\\ 0 & 3 & 1 & 5\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Since the fourth column is not a pivot column, a basis for W' is given by

$$\left\{\begin{bmatrix}3\\2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\right\}$$

Thus a basis for W is given by

$$\left\{3x^3 + 2x^2 + x, -x^3 + x^2 + 2x + 3, x^2 - x + 1\right\}$$

$\square$

V10. Find a basis for the solution set of the homogeneous system of equations

$$\begin{aligned}
x_1 + x_2 + 3x_3 + x_4 + 2x_5 &= 0\\
-3x_1 \phantom{{}+x_2} - 6x_3 + 6x_4 + 3x_5 &= 0\\
-x_1 + x_2 - x_3 + x_4 \phantom{{}+2x_5} &= 0\\
2x_1 - 2x_2 + 2x_3 - x_4 + x_5 &= 0.
\end{aligned}$$

Solution: Observe that

$$\operatorname{RREF}\left[\begin{array}{ccccc|c}1 & 1 & 3 & 1 & 2 & 0\\ -3 & 0 & -6 & 6 & 3 & 0\\ -1 & 1 & -1 & 1 & 0 & 0\\ 2 & -2 & 2 & -1 & 1 & 0\end{array}\right] = \left[\begin{array}{ccccc|c}1 & 0 & 2 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right]$$

Letting $x_3 = a$ and $x_5 = b$ (since those correspond to the non-pivot columns), this is equivalent to the system

$$\begin{aligned}
x_1 + 2x_3 + x_5 &= 0\\
x_2 + x_3 &= 0\\
x_3 &= a\\
x_4 + x_5 &= 0\\
x_5 &= b
\end{aligned}$$

Thus, the solution set is

$$\left\{ \begin{bmatrix} -2a-b \\ -a \\ a \\ -b \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

Since we can write

$$\begin{bmatrix} -2a-b \\ -a \\ a \\ -b \\ b \end{bmatrix} = a \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

a basis for the solution space is

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

□

A1. Consider the following maps of polynomials $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ defined by

$$S(f(x)) = 3xf(x) \text{ and } T(f(x)) = 3f'(x)f(x).$$

Show that one of these maps is a linear transformation, and that the other map is not.

Solution: To show S is a linear transformation, we must show two things:

$$\begin{aligned} &S\left(f(x) + g(x)\right) = S(f(x)) + s(g(x)) \\ &S(cf(x)) = cS(f(x)) \end{aligned}$$

To show S respects addition, we compute

$$\begin{aligned} S\left(f(x) + g(x)\right) &= 3x\left(f(x) + g(x)\right) && \text{by definition of } S \\ &= 3xf(x) + 3xg(x) && \text{by distributing} \end{aligned}$$

But note that $S(f(x)) = 3xf(x)$ and $S(g(x)) = 3xg(x)$, so we have $S(f(x) + g(x)) = S(f(x)) + S(g(x))$.
For the second part, we compute

$$\begin{aligned} S\left(cf(x)\right) &= 3x\left(cf(x)\right) && \text{by definition of } S \\ &= 3cxf(x) && \text{rewriting the multiplication.} \end{aligned}$$

But note that $cS(f(x)) = c(3xf(x)) = 3cxf(x)$ as well, so we have $S(cf(x)) = cS(f(x))$. Now, since S respects both addition and scalar multiplication, we can conclude S is a linear transformation.
As for T, we compute

$$\begin{aligned} T(f(x) + g(x)) &= 3(f(x) + g(x))'(f(x) + g(x)) && \text{by definition of } T \\ &= 3(f'(x) + g'(x))(f(x) + g(x)) && \text{since the derivative is linear} \\ &= 3f(x)f'(x) + 3f(x)g'(x) + 3f'(x)g(x) + 3g(x)g'(x) && \text{by distributing} \end{aligned}$$

However, note that $T(f(x)) + T(g(x)) = 3f'(x)f(x) + 3g'(x)g(x)$, so we see that $T(f(x) + g(x)) \neq T(f(x)) + T(g(x))$, so T does not respect addition and is therefore not a linear transformation.

□

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x+y \\ -x+3y-z \\ 7x+y+3z \\ 0 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right)$

Solution:

(a) Since

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \\ 7 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 3 & -1 \\ 7 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $T\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -(-2)+(0) \\ -(-2)+3(0)-(3) \\ 7(-2)+(0)+3(3) \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5 \\ 0 \end{bmatrix}$

Alternatively, $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 3 & -1 \\ 7 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1(-2)+1(0)+0(3) \\ -1(-2)+3(0)-1(3) \\ 7(-2)+1(0)+3(3) \\ 0(-2)+0(0)+0(3) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -5 \\ 0 \end{bmatrix}$.

□

A3. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+3y+2z-3w \\ 2x+4y+6z-10w \\ x+6y-z+3w \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of T.

Solution: First, we note the standard matrix

$$A = \begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 4 & 6 & -10 \\ 1 & 6 & -1 & 3 \end{bmatrix}$$

and compute

$$\mathrm{RREF}\,(A) = \begin{bmatrix} 1 & 0 & 5 & -9 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The kernel is given by solution set of the corresponding homogeneous system of equations

$$\ker T = \left\{ \begin{bmatrix} -5a+9b \\ a-2b \\ a \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

so a basis for the kernel is

$$\left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A basis for the image is given by the pivot columns, namely

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \right\}.$$

□

A4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $\begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 4 & 6 & -10 \\ 1 & 6 & -1 & 3 \end{bmatrix}$.

(a) Explain why T is or is not injective.

(b) Explain why T is or is not surjective.

Solution: Compute

$$\text{RREF} \begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 4 & 6 & -10 \\ 1 & 6 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 & -9 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Note that the third and fourth columns are non-pivot columns, which means $\ker T$ contains infinitely many vectors, so T is not injective.

(b) Since the third row lacks a pivot, the image (i.e. the span of the columns) is a 2-dimensional subspace (and thus does not equal $\mathbb{R}^3$), so T is not surjective.

□

M1. Let

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 3 \\ 1 & -2 & 5 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and show how to compute it.

Solution: AC is the only one that can be computed, since A is 2×2 and C is 2×3. Thus AC will be the 2×3 matrix given by

$$AC(\vec{\mathbf{e}}_1) = A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$AC(\vec{\mathbf{e}}_2) = A\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$AC(\vec{\mathbf{e}}_3) = A\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 5 \end{bmatrix}$$

Thus

$$AC = \begin{bmatrix} -3 & 7 & -12 \\ 1 & -2 & 5 \end{bmatrix}.$$

$\square$

M2. Explain why the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 6 \\ 1 & 6 & -1 \end{bmatrix}$ is or is not invertible.

Solution: We compute

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 6 \\ 1 & 6 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since its RREF is not the identity matrix, the linear map is not bijective and thus the matrix is not invertible.

$\square$

M3. Show how to compute the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution:

$$\text{RREF}\left(\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -11 & 32 \\ 0 & 1 & 0 & 0 & 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 1 & 2 & -11 & 32 \\ 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

$\square$

G1.

(a) Find 3×3 matrices S and T whose left multiplication represents the row operations $R_2 - 4R_1 \to R_2$ and $R_3 \leftrightarrow R_2$, respectively.

(b) If $A \in M_{3,3}$ is a matrix with $\det A = 12$, find the determinant of STA.

Solution:

1. $S = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
2. $\det(STA) = \det(S)\det(T)\det(A) = (1)(-1)(12) = -12$.

$\square$

G2. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix}$$

Solution: Here is one possible solution, first applying a single row operation, and then performing Laplace/cofactor expansions to reduce the determinant to a linear combination of 2×2 determinants:

$$\det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} = (-1) \det \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 3 \\ -3 & 1 & -5 \end{bmatrix} + (1) \det \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} &= (-1) \left((1) \det \begin{bmatrix} 1 & 3 \\ 1 & -5 \end{bmatrix} - (1) \det \begin{bmatrix} 3 & -1 \\ 1 & -5 \end{bmatrix} + (-3) \det \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \right) + \\ &\quad (1) \left((1) \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - (3) \det \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \right) \\ &= (-1)\,(-8 + 14 - 30) + (1)\,(1 - 15) \\ &= 10 \end{aligned}$$

Here is another possible solution, using row and column operations to first reduce the determinant to a 3×3 matrix and then applying a formula:

$$\det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ -3 & 1 & 2 & -7 \end{bmatrix}$$

$$\begin{aligned} &= -\det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & -7 \end{bmatrix} = -\det \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 2 \\ -3 & 1 & -7 \end{bmatrix} \\ &= -((-7 - 18 - 1) - (3 + 2 - 21)) \\ &= 10 \end{aligned}$$

□

G3. Find the eigenvalues of the matrix $\begin{bmatrix} -2 & -2 \\ 10 & 7 \end{bmatrix}$.

Solution: Compute the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} -2 - \lambda & -2 \\ 10 & 7 - \lambda \end{bmatrix} = (-2 - \lambda)(7 - \lambda) + 20 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

The eigenvalues are the roots of the characteristic polynomial, namely 2 and 3.

□

G4. Find a basis for the eigenspace associated to the eigenvalue 3 in the matrix

$$\begin{bmatrix} -7 & -8 & 2 \\ 8 & 9 & -1 \\ \frac{13}{2} & 5 & 2 \end{bmatrix}.$$

Solution: The eigenspace associated to 3 is the kernel of $A - 3I$, so we compute

$$\mathrm{RREF}(A - 3I) = \mathrm{RREF} \begin{bmatrix} -7 - 3 & -8 & 2 \\ 8 & 9 - 3 & -1 \\ \frac{13}{2} & 5 & 2 - 3 \end{bmatrix} = \mathrm{RREF} \begin{bmatrix} -10 & -8 & 2 \\ 8 & 6 & -1 \\ \frac{13}{2} & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we see the kernel is

$$\left\{ \begin{bmatrix} -a \\ \frac{3}{2}a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$$

which has a basis of $\left\{ \begin{bmatrix} -1 \\ \frac{3}{2} \\ 1 \end{bmatrix} \right\}$.

$\square$