Module E: Solving Systems of Linear Equations

How can we solve systems of linear equations?

At the end of this module, students will be able to...

E1. **Systems as matrices.** ... translate back and forth between a system of linear equations and the corresponding augmented matrix.

E2. **Row reduction.** ... put a matrix in reduced row echelon form.

E3. **Systems of linear equations.** ... compute the solution set for a system of linear equations.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.
- Describe sets using set-builder notation, and check if an element is a member of a set described by set-builder notation.

The following resources will help you prepare for this module.

- Systems of linear equations (Khan Academy): `http://bit.ly/2l21etm`
- Solving linear systems with substitution (Khan Academy): `http://bit.ly/1SlMpix`
- Set builder notation: `https://youtu.be/xnfUZ-NTsCE`

Module E Section 0

Definition E.0.1

A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector that can be plugged into the equation).

Remark E.0.2

In previous classes you likely used the variables x, y, z in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as x_i, and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

Definition E.0.3

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n \ &= \ b_1 \\
a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n \ &= \ b_2 \\
\vdots \qquad \quad \vdots \qquad \qquad \quad \vdots \ \ &\quad \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ &= b_m
\end{aligned}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \middle| \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

Remark E.0.4

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Concise standard form:

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & 3 \\ 3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & -2 \end{array}$$

Definition E.0.5

A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**.

Fact E.0.6

All linear systems are one of the following:

- **Consistent with one solution**: its solution set contains a single vector, e.g. $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$
- **Consistent with infinitely-many solutions**: its solution set contains infinitely many vectors, e.g. $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$
- **Inconsistent**: its solution set is the empty set $\{\}=\emptyset$

Activity E.0.7 *($\sim$10 min)*

All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is $\emptyset$.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity E.0.8 *(∼10 min)*

Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity E.0.8 *(∼10 min)*
Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions for this system.

Activity E.0.8 *(∼10 min)*
Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions for this system.
Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{ \begin{bmatrix} ? \\ a \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}$ for the linear system.

Activity E.0.9 *(∼10 min)*

Consider the following linear system.

$$\begin{alignedat}{5} x_1 &+ 2x_2 & & &-& x_4 &=& 3 \\ & & & x_3 &+& 4x_4 &=& -2 \end{alignedat}$$

Describe the solution set

$$\left\{ \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation E.0.10

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{aligned} -2x_1 - 4x_2 + x_3 - 4x_4 &= -8 \\ x_1 + 2x_2 + 2x_3 + 12x_4 &= -1 \\ x_1 + 2x_2 + x_3 + 8x_4 &= 1 \end{aligned}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.

Module E Section 1

Remark E.1.1

The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition E.1.2

A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Example E.1.3

The corresopnding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

Definition E.1.4

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \qquad\qquad \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity E.1.5 *(∼10 min)*

Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.
b) Swap two columns.
c) Add a constant to every term in a row.
d) Multiply a row by a nonzero constant.
e) Add a constant multiple of one row to another row.
f) Replace a column with zeros.
g) Replace a row with zeros.

Definition E.1.6

The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a nonzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity E.1.7 *(∼10 min)*
Consider the following (equivalent) linear systems.

(A)

$$\begin{alignedat}{4} -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ x_1 &- 2x_2 &+ 2x_3 &= 7 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(C)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ & & -2x_3 &= -6 \end{alignedat}$$

(E)

$$\begin{alignedat}{4} x_1 &- 2x_2 & &= 1 \\ & & x_3 &= 3 \\ & & 0 &= 0 \end{alignedat}$$

(B)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(D)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & x_3 &= 3 \\ & & -2x_3 &= -6 \end{alignedat}$$

(F)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

Activity E.1.7 *(~10 min)*

Consider the following (equivalent) linear systems.

(A)

$$\begin{aligned} -2x_1+4x_2-2x_3 &= -8\\ x_1-2x_2+2x_3 &= 7\\ 3x_1-6x_2+4x_3 &= 15 \end{aligned}$$

(C)

$$\begin{aligned} x_1-2x_2+2x_3 &= 7\\ 2x_3 &= 6\\ -2x_3 &= -6 \end{aligned}$$

(E)

$$\begin{aligned} x_1-2x_2 \qquad &= 1\\ x_3 &= 3\\ 0 &= 0 \end{aligned}$$

(B)

$$\begin{aligned} x_1-2x_2+2x_3 &= 7\\ -2x_1+4x_2-2x_3 &= -8\\ 3x_1-6x_2+4x_3 &= 15 \end{aligned}$$

(D)

$$\begin{aligned} x_1-2x_2+2x_3 &= 7\\ x_3 &= 3\\ -2x_3 &= -6 \end{aligned}$$

(F)

$$\begin{aligned} x_1-2x_2+2x_3 &= 7\\ 2x_3 &= 6\\ 3x_1-6x_2+4x_3 &= 15 \end{aligned}$$

Part 1: Find a solution to one of these systems.

Activity E.1.7 *(~10 min)*
Consider the following (equivalent) linear systems.

(A)

$$\begin{alignedat}{4} -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ x_1 &- 2x_2 &+ 2x_3 &= 7 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(C)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ & & -2x_3 &= -6 \end{alignedat}$$

(E)

$$\begin{alignedat}{4} x_1 &- 2x_2 & &= 1 \\ & & x_3 &= 3 \\ & & 0 &= 0 \end{alignedat}$$

(B)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ -2x_1 &+ 4x_2 &- 2x_3 &= -8 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

(D)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & x_3 &= 3 \\ & & -2x_3 &= -6 \end{alignedat}$$

(F)

$$\begin{alignedat}{4} x_1 &- 2x_2 &+ 2x_3 &= 7 \\ & & 2x_3 &= 6 \\ 3x_1 &- 6x_2 &+ 4x_3 &= 15 \end{alignedat}$$

Part 1: Find a solution to one of these systems.
Part 2: Rank the six linear systems from most complicated to simplest.

Activity E.1.8 *(∼5 min)*

We can rewrite the previous in terms of equivalences of augmented matrices

$$\left[\begin{array}{ccc|c} -2 & 4 & -2 & -8 \\ 1 & -2 & 2 & 7 \\ 3 & -6 & 4 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ -2 & 4 & -2 & -8 \\ 3 & -6 & 4 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 3 & -6 & 4 & 15 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

Activity E.1.9 (~10 min)

A matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

Circle the leading terms in each example, and label it as RREF or not RREF.

(A)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(B)

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(C)

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(D)

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(E)

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(F)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Remark E.1.10

It is important to understand the **Gauss-Jordan elimination** algorithm that converts a matrix into reduced row echelon form.

A video outlining how to perform the Gauss-Jordan Elimination algorithm by hand is available at `https://youtu.be/Cq0Nxk2dhhU`. Practicing several exercises outside of class using this method is recommended.

In the next section, we will learn to use technology to perform this operation for us, as will be expected when applying row-reduced matrices to solve other problems.

Module E Section 2

Activity E.2.1 *(∼10 min)*

Free browser-based technologies for mathematical computation are available online.

- Go to `http://cocalc.com` and create an account.
- Create a project titled “Linear Algebra Team X” with your appropriate team number. Add all team members as collaborators.
- Open the project and click on “New”
- Give it an appropriate name such as “Class E.2 workbook”. Make a new Jupyter notebook.
- Click on “Kernel” and make sure “Octave” is selected.
- Type `A=[1 3 4 ; 2 5 7]` and press `Shift+Enter` to store the matrix $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{bmatrix}$ in the variable A.
- Type `rref(A)` and press `Shift+Enter` to compute the reduced row echelon form of A.

Remark E.2.2

If you need to find the reduced row echelon form of a matrix during class, you are encouraged to use CoCalc's Octave interpreter.

You can change a cell from "Code" to "Markdown" or "Raw" to put comments around your calculations such as Activity numbers.

Activity E.2.3 *(~10 min)*
Consider the system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

Convert this to an augmented matrix and use CoCalc to compute its reduced row echelon form. Write these on your whiteboard, and use them to write a simpler yet equivalent linear system of equations. Then find its solution set.

Activity E.2.4 *(∼10 min)*
Consider our system of equations from above.

$$\begin{alignedat}{4}
3x_1 &{}-{}& 2x_2 &{}+{}& 13x_3 &{}={}& 6 \\
2x_1 &{}-{}& 2x_2 &{}+{}& 10x_3 &{}={}& 2 \\
-x_1 &&&{}-{}& 3x_3 &{}={}& 1
\end{alignedat}$$

Convert this to an augmented matrix and use CoCalc to compute its reduced row echelon form. Write these on your whiteboard, and use them to write a simpler yet equivalent linear system of equations. Then find its solution set.

Activity E.2.5 *(∼10 min)*

Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Activity E.2.5 *(~10 min)*
Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Part 1: Find its corresponding augmented matrix A and use CoCalc to find RREF(A).

Activity E.2.5 *(∼10 min)*
Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Part 1: Find its corresponding augmented matrix A and use CoCalc to find RREF(A).
Part 2: How many solutions does the corresponding linear system have?

Activity E.2.6 *(∼10 min)*
Consider the simple linear system equivalent to the system from the previous problem:

$$\begin{alignedat}{2} x_1 + 2x_2 & & &= 4 \\ & & x_3 &= -1 \end{alignedat}$$

Activity E.2.6 *(∼10 min)*

Consider the simple linear system equivalent to the system from the previous problem:

$$\begin{aligned} x_1 + 2x_2 \quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

Activity E.2.6 *(∼10 min)*

Consider the simple linear system equivalent to the system from the previous problem:

$$\begin{aligned} x_1 + 2x_2 \quad &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

Part 2: Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$.

Activity E.2.6 *(~10 min)*

Consider the simple linear system equivalent to the system from the previous problem:

$$\begin{alignedat}{3} x_1 + 2x_2 & & &= 4 \\ & & x_3 &= -1 \end{alignedat}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

Part 2: Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$.

Part 3: Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$ caused this?

Definition E.2.7

Recall that the pivots of a matrix in RREF form are the leading 1s in each non-zero row.

The pivot columns in an augmented matrix correspond to the **bound variables** in the system of equations (x_1, x_3 below). The remaining variables are called **free variables** (x_2 below).

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

To efficiently solve a system in RREF form, we may assign letters to free variables and solve for the bound variables.

Activity E.2.8 *(∼10 min)*

Find the solution set for the system

$$\begin{alignedat}{6}
2x_1 &- 2x_2 &- 6x_3 &+ x_4 &- x_5 &= 3 \\
-x_1 &+ x_2 &+ 3x_3 &- x_4 &+ 2x_5 &= -3 \\
x_1 &- 2x_2 &- x_3 &+ x_4 &+ x_5 &= 2
\end{alignedat}$$

by row-reducing its augmented matrix, and then assigning letters to the free variables (given by non-pivot columns) and solving for the bound variables (given by pivot columns) in the corresponding linear system.

Observation E.2.9

The solution set to the system

$$\begin{alignedat}{6} 2x_1 &- 2x_2 &- 6x_3 &+ x_4 &- x_5 &= 3 \\ -x_1 &+ x_2 &+ 3x_3 &- x_4 &+ 2x_5 &= -3 \\ x_1 &- 2x_2 &- x_3 &+ x_4 &+ x_5 &= 2 \end{alignedat}$$

may be written as

$$\left\{ \begin{bmatrix} 1+5a+2b \\ 1+2a+3b \\ a \\ 3+3b \\ b \end{bmatrix} \middle| \; a, b \in \mathbb{R} \right\}.$$

Remark E.2.10

Don't forget to correctly express the solution set of a linear system, using set-builder notation for consistent systems with infintely many solutions.

- **Consistent with one solution**: e.g. $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$
- **Consistent with infinitely-many solutions**: e.g. $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$
- **Inconsistent**: $\emptyset$