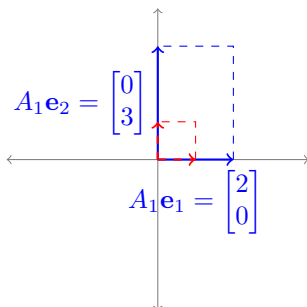


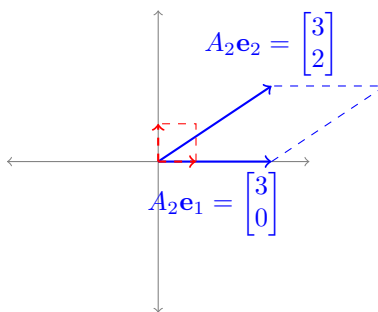
Application Activities - Module G Part 1 - Class Day 25

Activity 25.1 The image below illustrates how the linear transformation $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the standard matrix $A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.



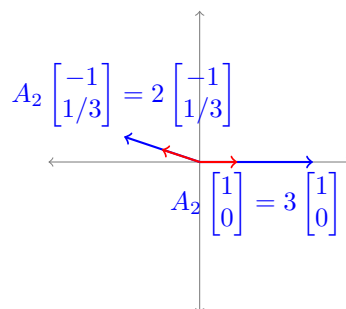
- What is the area of the transformed unit square?
 - Find two vectors that were stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.
-

Activity 25.2 The image below illustrates how the linear transformation $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the standard matrix $A_2 = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$ transforms the unit square.



- What is the area of the transformed unit square?
 - Find at least one vector that was stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.
-

Observation 25.3 It's possible to find two non-parallel vectors that are stretched by the transformation given by A_2 :



The process for finding such vectors will be covered later in this module.

Activity 25.4 Consider the linear transformation given by the standard matrix $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

- Sketch the transformation of the unit square (the parallelogram given by the columns of the standard matrix).
 - Compute the area of the transformed unit square.
-

Activity 25.5 Consider the linear transformation given by the standard matrix $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

- Sketch the transformation of the unit square.
 - Compute the area of the transformed unit square.
-

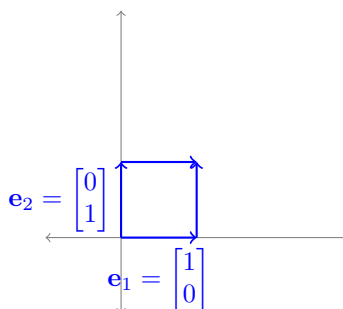
Activity 25.6 Consider the linear transformation given by the standard matrix $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

- Sketch the transformation of the unit square.
 - Compute the area of the transformed unit square.
-

Remark 25.7 The area of the transformed unit square measures the factor by which all areas are transformed by a linear transformation.

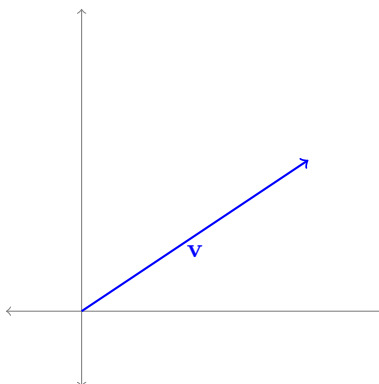
We will define the **determinant** of a square matrix A , or $\det(A)$ for short, to be this factor. But what properties must this function satisfy?

Activity 25.8 The transformation of the unit square by the standard matrix $[\mathbf{e}_1 \ \mathbf{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ is illustrated below. What is $\det([\mathbf{e}_1 \ \mathbf{e}_2]) = \det(I)$, that is, by what factor has the area of the unit square been scaled?



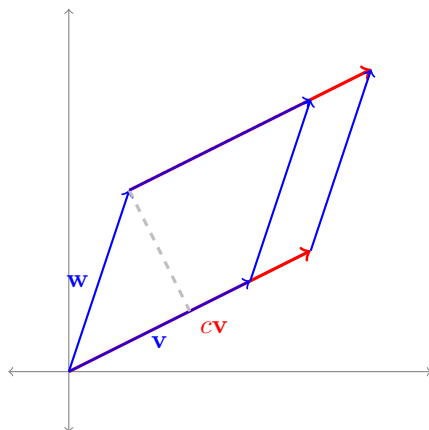
- a) 0
 - b) 1
 - c) 2
 - d) Cannot be determined
-

Activity 25.9 The transformation of the unit square by the standard matrix $[\mathbf{v} \ \mathbf{v}]$ is illustrated below: both $T(\mathbf{e}_1) = T(\mathbf{e}_2) = \mathbf{v}$. What is $\det([\mathbf{v} \ \mathbf{v}])$, that is, by what factor has area been scaled?



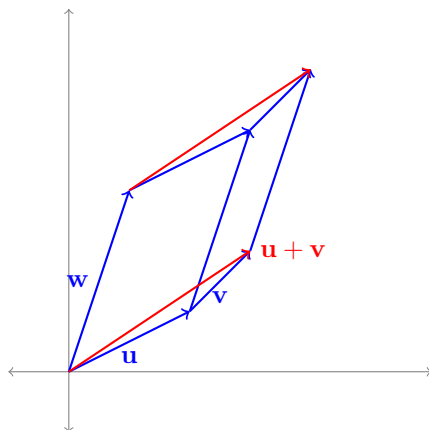
- a) 0
 - b) 1
 - c) 2
 - d) Cannot be determined
-

Activity 25.10 The transformations of the unit square by the standard matrices $[\mathbf{v} \ \mathbf{w}]$ and $[c\mathbf{v} \ \mathbf{w}]$ are illustrated below. How are $\det([\mathbf{v} \ \mathbf{w}])$ and $\det([c\mathbf{v} \ \mathbf{w}])$ related?



- a) $\det([v \ w]) = \det([cv \ w])$
- b) $c + \det([v \ w]) = \det([cv \ w])$
- c) $c \det([v \ w]) = \det([cv \ w])$

Activity 25.11 The transformations of unit squares by the standard matrices $[u \ w]$, $[v \ w]$ and $[u + v \ w]$ are illustrated below. How is $\det([u + v \ w])$ related to $\det([u \ w])$ and $\det([v \ w])$?



- a) $\det([u \ w]) = \det([v \ w]) = \det([u + v \ w])$
- b) $\det([u \ w]) + \det([v \ w]) = \det([u + v \ w])$
- c) $\det([u \ w]) \det([v \ w]) = \det([u + v \ w])$

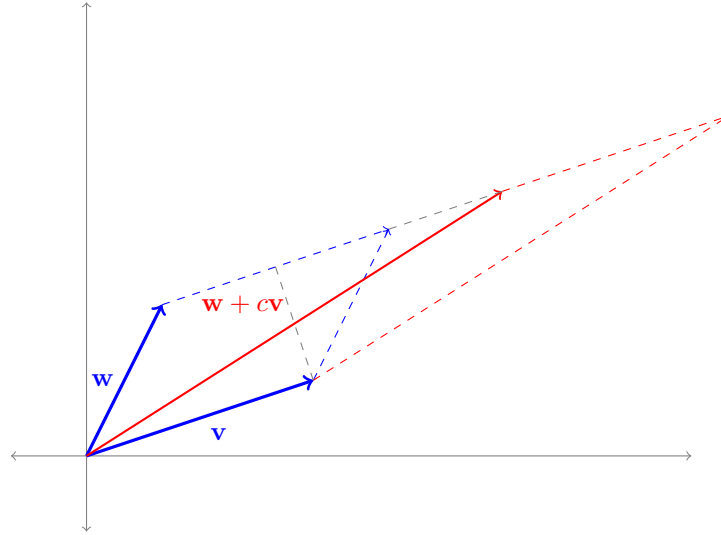
Definition 25.12 The **determinant** is the unique function $\det : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ satisfying the following three properties:

P1: $\det(I) = 1$

P2: $\det([\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]) = 0$ whenever two columns of the matrix are identical.

P3: $\det[\cdots \ c\mathbf{v} + d\mathbf{w} \ \cdots] = c \det[\cdots \ \mathbf{v} \ \cdots] + d \det[\cdots \ \mathbf{w} \ \cdots]$, assuming all other columns are equal.

Observation 25.13 What happens if we had a multiple of one column to another?



The base of both parallelograms is \mathbf{v} , while the height has not changed. Thus

$$\det([\mathbf{v} \ \mathbf{w} + c\mathbf{v}]) = \det([\mathbf{v} \ \mathbf{w}])$$

Observation 25.14 Swapping columns can be obtained from a sequence of adding column multiples.

$$\begin{aligned} \det([\mathbf{v} \ \mathbf{w}]) &= \det([\mathbf{v} + \mathbf{w} \ \mathbf{w}]) \\ &= \det([\mathbf{v} + \mathbf{w} \ \mathbf{w} - (\mathbf{v} + \mathbf{w})]) \\ &= \det([\mathbf{v} + \mathbf{w} \ -\mathbf{v}]) \\ &= \det([\mathbf{v} + \mathbf{w} - \mathbf{v} \ -\mathbf{v}]) \\ &= \det([\mathbf{w} \ -\mathbf{v}]) \\ &= -\det([\mathbf{w} \ \mathbf{v}]) \end{aligned}$$

So swapping two columns results in a negation of the determinant. Therefore, determinants represent a *signed* area, since they are not always positive.

Fact 25.15 We've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c \det([\cdots \ \mathbf{v} \ \cdots]) = \det([\cdots \ c\mathbf{v} \ \cdots])$$

(b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \ \mathbf{v} \ \cdots \ \mathbf{w} \ \cdots]) = -\det([\cdots \ \mathbf{w} \ \cdots \ \mathbf{v} \ \cdots])$$

(c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \quad \mathbf{v} \quad \cdots \quad \mathbf{w} \quad \cdots]) = \det([\cdots \quad \mathbf{v} + c\mathbf{w} \quad \cdots \quad \mathbf{w} \quad \cdots])$$

Activity 25.16 The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. How must the transformation given by the standard matrix AB scale areas?

- (a) 1
- (b) 7
- (c) 12
- (d) Cannot be determined

Fact 25.17 Since the transformation given by the standard matrix AB is obtained by applying the transformations given by A and B , it follows that

$$\det(AB) = \det(A) \det(B)$$