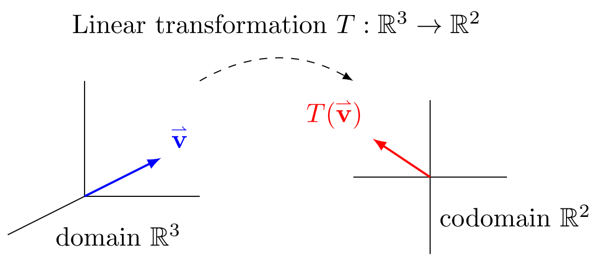
Section A.1

Definition A.1.1 A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = T(\vec{\mathbf{v}}) + T(\vec{\mathbf{w}})$ for any $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in V$.
2. $T(c\vec{\mathbf{v}}) = cT(\vec{\mathbf{v}})$ for any $c \in \mathbb{R}, \vec{\mathbf{v}} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition A.1.2 Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.



Example A.1.3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right) = \begin{bmatrix} (x+u) - (z+w) \\ 3(y+v) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x+u) - (z+w) \\ 3(y+v) \end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \text{ and } cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore T is a linear transformation.

Example A.1.4 Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact A.1.5 A map between Euclidean spaces $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of $\mathbb{R}^n$.

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x+0y-1z \\ 0x+3y+0z \end{bmatrix}$$

But this map is not linear because x^2, $y+3$, and $y-2^x$ are not linear combinations (even though $x+y$ is):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

Activity A.1.6 *(∼5 min)* Recall the following rules from calculus, where $D : \mathcal{P} \to \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f.

$$D(f + g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

a) $\mathcal{P}$ is not a vector space

b) D is a linear map

c) D is not a linear map

Activity A.1.7 *(∼10 min)* Let the polynomial maps $S : \mathcal{P}^4 \to \mathcal{P}^3$ and $T : \mathcal{P}^4 \to \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4+x)$, $S(x^4)+S(x)$, $T(x^4+x)$, and $T(x^4)+T(x)$. Which of these maps is definitely not linear?

Fact A.1.8 If $L : V \to W$ is linear, then $L(\vec{\mathbf{z}}) = L(0\vec{\mathbf{v}}) = 0L(\vec{\mathbf{v}}) = \vec{\mathbf{z}}$ where $\vec{\mathbf{z}}$ is the additive identity of the vector spaces V, W.

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Observation A.1.9 Showing $L : V \to W$ is not a linear transformation can be done by finding an example for any one of the following.

- Show $L(\vec{\mathbf{z}}) \neq \vec{\mathbf{z}}$ (where $\vec{\mathbf{z}}$ is the additive identity of L and W).
- Find $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in V$ such that $L(\vec{\mathbf{v}} + \vec{\mathbf{w}}) \neq L(\vec{\mathbf{v}}) + L(\vec{\mathbf{w}})$.
- Find $\vec{\mathbf{v}} \in V$ and $c \in \mathbb{R}$ such that $L(c\vec{\mathbf{v}}) \neq cL(\vec{\mathbf{v}})$.

Otherwise, L can be shown to be linear by proving the following in general.

- For all $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in V$, $L(\vec{\mathbf{v}} + \vec{\mathbf{w}}) \neq L(\vec{\mathbf{v}}) + L(\vec{\mathbf{w}})$.
- For all $\vec{\mathbf{v}} \in V$ and $c \in \mathbb{R}$, $L(c\vec{\mathbf{v}}) \neq cL(\vec{\mathbf{v}})$.

Note the similarities between this process and showing that a subset of a vector space is/isn't a subspace.

Activity A.1.10 *($\sim$15 min)* Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.
Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f.
Part 3: Is S linear?

Activity A.1.11 *($\sim$20 min)* Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Note that $S(0) = 0$ and $T(0) = 0$. So instead, show that $S(x + 1) \neq S(x) + S(1)$ to verify that S is not linear.
Part 2: Prove that T is linear by verifying that $T(f(x)+g(x)) = T(f(x))+T(g(x))$ and $T(cf(x)) = cT(f(x))$.