Section E.1

Remark E.11 The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition E.12 A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Example E.13 The corresponding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

Definition E.14 Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned}$$

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent, which we denote with $\sim$:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity E.15 *(∼10 min)* Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a nonzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

g) Replace a row with zeros.

Definition E.16 The following **row operations** produce equivalent augmented matrices:

1. Swap two rows, for example, $R_1 \leftrightarrow R_2$:

$$\left[\begin{array}{cc|c}1 & 2 & 3\\4 & 5 & 6\end{array}\right] \sim \left[\begin{array}{cc|c}4 & 5 & 6\\1 & 2 & 3\end{array}\right]$$

2. Multiply a row by a nonzero constant, for example, $2R_1 \to R_1$:

$$\left[\begin{array}{cc|c}1 & 2 & 3\\4 & 5 & 6\end{array}\right] \sim \left[\begin{array}{cc|c}2(1) & 2(2) & 2(3)\\4 & 5 & 6\end{array}\right]$$

3. Add a constant multiple of one row to another row, for example, $R_2 - 4R_1 \to R_2$:

$$\left[\begin{array}{cc|c}1 & 2 & 3\\4 & 5 & 6\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 2 & 3\\4-4(1) & 5-4(2) & 6-4(3)\end{array}\right]$$

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity E.17 *(∼10 min)* Consider the following (equivalent) linear systems.

(A)
$$\begin{alignedat}{4} x &+ 2y &+ z &= 3\\ -x &- y &+ z &= 1\\ 2x &+ 5y &+ 3z &= 7 \end{alignedat}$$

(C)
$$\begin{alignedat}{3} x & & - z &= 1\\ & y &+ z &= 1\\ & y &+ 2z &= 4 \end{alignedat}$$

(E)
$$\begin{alignedat}{3} x & & - z &= 1\\ & y &+ z &= 1\\ & & z &= 3 \end{alignedat}$$

(B)
$$\begin{alignedat}{4} 2x &+ 5y &+ 3z &= 7\\ -x &- y &+ z &= 1\\ x &+ 2y &+ z &= 3 \end{alignedat}$$

(D)
$$\begin{alignedat}{4} x &+ 2y &+ z &= 3\\ & y &+ z &= 1\\ 2x &+ 5y &+ 3z &= 7 \end{alignedat}$$

(F)
$$\begin{alignedat}{4} x &+ 2y &+ z &= 3\\ & y &+ z &= 1\\ & y &+ 2z &= 4 \end{alignedat}$$

Rank the six linear systems from most complicated to simplest.

Activity E.18 *(∼5 min)* We can rewrite the previous in terms of equivalences of augmented matrices

$$\left[\begin{array}{ccc|c} 2 & 5 & 13 & 7 \\ -1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -1 & -1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array}\right] \sim$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

Definition E.19 A matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

Every matrix has a unique reduced row echelon form. If A is a matrix, we write RREF(A) for the reduced row echelon form of that matrix.

Activity E.20 *(∼15 min)* Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

(A)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(B)
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(C)
$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(D)
$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(E)
$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(F)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

For each matrix, circle the leading terms, and label it as RREF or not RREF. For the ones not in RREF, find their RREF.