Name:	
J#:	Dr. Clontz
Date:	

MIDTERM EXAM Math 237 – Linear Algebra
Version 1 Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

Standard E1.	Mark:

Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 4 & 1 & -7 \\ 1 & 2 & -1 & 0 \end{array}\right]$$

Solution:

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 4x_2 + x_3 &= -7 \\ x_1 + 2x_2 - x_3 &= 0 \end{aligned}$$

□

Standard E2.	Mark:

Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 \\ 3 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

□

Standard E3.	Mark:

Solve the system of equations

$$\begin{aligned} x + 3y - 4z &= 5 \\ 3x + 9y + z &= 2 \end{aligned}$$

Solution:

$$\text{RREF}\left(\left[\begin{array}{ccc|c} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 2 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

So the solution set is

$$\left\{ \begin{bmatrix} 1-3c \\ c \\ -1 \end{bmatrix} \,\middle|\, c \in \mathbb{R} \right\}$$

□

Standard E4.	Mark:

Find a basis for the solution set to the homogeneous system of equations given by

$$\begin{aligned} 2x_1 - 2x_2 + 6x_3 - x_4 &= 0 \\ 3x_1 + 6x_3 + x_4 &= 0 \\ -4x_1 + x_2 - 9x_3 + 2x_4 &= 0 \end{aligned}$$

Solution: Let $A = \left[\begin{array}{cccc|c} 2 & -2 & 6 & -1 & 0 \\ 3 & 0 & 6 & 1 & 0 \\ -4 & 1 & -9 & 2 & 0 \end{array}\right]$, so $\text{RREF}\, A = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$. It follows that the basis for the solution set is given by $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

□

Standard V1.	Mark:

Let V be the set of all points on the line $x + y = 2$ with the operations, for any $(x_1, y_1), (x_2, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, y_1) \oplus (x_2, y_2) &= (x_1 + x_2 - 1, y_1 + y_2 - 1) \\ c \odot (x_1, y_1) &= (cx_1 - (c-1), cy_1 - (c-1)) \end{aligned}$$

(a) Show that this vector space has an **additive identity** element $\mathbf{0}$ satisfying $(x, y) \oplus \mathbf{0} = (x, y)$.

(b) Determine if V is a vector space or not. Justify your answer.

Solution: Let $(x_1, y_1) \in V$; then $(x_1, y_1) \oplus (1, 1) = (x_1, y_1)$, so $(1, 1)$ is an additive identity element.
Now we will show the other seven properties. Let $(x_1, y_1), (x_2, y_2) \in V$, and let $c, d \in \mathbb{R}$.

1) Since real addition is associative, $\oplus$ is associative.

2) Since real addition is commutative, $\oplus$ is commutative.

3) The additive identity is $(1,1)$.

4) $(x_1,y_1)\oplus(2-x_1,2-y_1)=(1,1)$, so $(2-x_1,2-y_1)$ is the additive inverse of (x_1,y_1).

5)

$$\begin{aligned}
c\odot(d\odot(x_1,y_1)) &= c\odot(dx_1-(d-1),dy_1-(d-1))\\
&= (c\,(dx_1-(d-1))-(c-1),c\,(dy_1-(d-1)))\\
&= (cdx_1-cd+c-(c-1),cdy_1-cd+c-(c-1))\\
&= (cdx_1-(cd-1),cdy_1-(cd-1))\\
&= (cd)\odot(x_1,y_1)
\end{aligned}$$

6) $1\odot(x_1,y_1)=(x_1-(1-1),y_1-(1-1)=(x_1,y_1)$

7)

$$\begin{aligned}
c\odot((x_1,y_1)\oplus(x_2,y_2)) &= c\odot(x_1+y_1-1,x_2+y_2-1)\\
&= (c(x_1+y_1-1)-(c-1),c(x_2+y_2-1)-(c-1))\\
&= (cx_1+cx_2-2c+1,cy_1+cy_2-2c+1)\\
&= (cx_1-(c-1),cy_1-(c-1))\oplus(cx_2-(c-1),cy_2-(c-1))\\
&= c\odot(x_1,y_1)\oplus c\odot(x_2,y_2)
\end{aligned}$$

8)

$$\begin{aligned}
(c+d)\odot(x_1,y_1) &= ((c+d)x_1-(c+d-1),(c+d)y_1-(c+d-1))\\
&= (cx_1-(c-1),cy_1-(c-1))\oplus(dx_1-(d-1),dy_1-(d-1))\\
&= c\odot(x_1,y_1)\oplus c\odot(x_2,y_2)
\end{aligned}$$

Therefore V is a vector space.

$\square$

Standard V2.	Mark:

Determine if $\begin{bmatrix}0\\-1\\6\\-7\end{bmatrix}$ belongs to the span of the set $\left\{\begin{bmatrix}2\\0\\-1\\5\end{bmatrix},\begin{bmatrix}4\\-1\\4\\3\end{bmatrix}\right\}$.

Solution: Since

$$\mathrm{RREF}\left(\left[\begin{array}{cc|c}2&4&0\\0&-1&-1\\-1&4&6\\5&3&-7\end{array}\right]\right)=\left[\begin{array}{cc|c}1&0&-2\\0&1&1\\0&0&0\\0&0&0\end{array}\right]$$

does not contain a contradiction, $\begin{bmatrix}0\\-1\\6\\-7\end{bmatrix}$ is a linear combination of the three vectors.

$\square$

Standard V3.	Mark:

Determine if the vectors $\begin{bmatrix}-3\\1\\1\end{bmatrix}, \begin{bmatrix}5\\-1\\-2\end{bmatrix}, \begin{bmatrix}2\\0\\-1\end{bmatrix}$, and $\begin{bmatrix}0\\2\\-1\end{bmatrix}$ span $\mathbb{R}^3$

Solution:

$$\text{RREF}\left(\begin{bmatrix}-3 & 5 & 2 & 0\\1 & -1 & 0 & 2\\1 & -2 & -1 & -1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & 5\\0 & 1 & 1 & 3\\0 & 0 & 0 & 0\end{bmatrix}$$

Since the resulting matrix has only two pivot columns, the vectors do not span $\mathbb{R}^3$.

□

Standard V4.	Mark:

Let W be the set of all $\mathbb{R}^3$ vectors $\begin{bmatrix}x\\y\\z\end{bmatrix}$ satisfying $x+y+z=0$ (this forms a plane). Determine if W is a subspace of $\mathbb{R}^3$.

Solution: Yes, because $z=-x-y$ and $a\begin{bmatrix}x_1\\y_1\\-x_1-y_1\end{bmatrix}+b\begin{bmatrix}x_2\\y_2\\-x_2-y_2\end{bmatrix}=\begin{bmatrix}ax_1+bx_2\\ay_1+by_2\\-(ax_1+bx_2)-(ay_1+by_2)\end{bmatrix}$.
Alternately, yes because W is isomorphic to $\mathbb{R}^2$.

□

Standard S1.	Mark:

Determine if the set of vectors $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\3\\-2\end{bmatrix}\right\}$ is linearly dependent or linearly independent

Solution:

$$\text{RREF}\left(\begin{bmatrix}1 & 1 & 1\\0 & 2 & 3\\1 & -1 & -2\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & -\frac{1}{2}\\0 & 1 & \frac{3}{2}\\0 & 0 & 0\end{bmatrix}$$

Since there is a nonpivot column, the set is linearly dependent.

□

Standard S2.	Mark:

Determine if the set $\{x^2+x-1, 3x^2-x+1, 2x-2\}$ is a basis of $\mathcal{P}_2$

Solution:

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis.

□

Standard S3.	Mark:

Let $W = \text{span}\left(\left\{\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$. Find a basis of W.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\left\{\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ is a basis of W.

□

Standard S4.	Mark:

Let W be the subspace of $M_{2,2}$ given by $W = \text{span}\left(\left\{\begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right\}\right)$. Compute the dimension of W.

Solution:

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$.

□

Additional Notes/Marks	