Module I: Introduction

Remark I.0.1 This brief module gives an overview for the course.

Section I.0

Remark I.0.1 What is Linear Algebra?
Linear algebra is the study of **linear maps**.

- In Calculus, you learn how to approximate any function by a linear function.
- In Linear Algebra, we learn about how linear maps behave.
- Combining the two, we can approximate how any function behaves.

Remark I.0.2 What is Linear Algebra good for?

- Linear algebra is used throughout several fields in higher mathematics.
- In computer graphics, linear algebra is used to help represent 3D objects in a 2D grid of pixels.
- Linear algebra is used to approximate differential equation solutions in a vast number of engineering applications (e.g. fluid flows, vibrations, heat transfer) whose solutions are very difficult (or impossible) to find precisely.
- Google's search engine is based on its Page Rank algorithm, which ranks websites by computing an eigenvector of a matrix.

Remark I.0.3 What will I learn in this class?
By the end of this class, you will be able to:

- Solve systems of linear equations. (Module E)
- Identify vector spaces and their properties. (Module V)
- Analyze the structure of vector spaces and sets of vectors. (Module S)
- Use and apply the algebraic properties of linear transformations. (Module A)
- Perform fundamental operations in the algebra of matrices. (Module M)
- Use and apply the geometric properties of linear transformations. (Module G)

Module E: Solving Systems of Linear Equations

Standards for this Module

How can we solve systems of linear equations? At the end of this module, students will be able to...

E1. Systems as matrices. ... translate back and forth between a system of linear equations and the corresponding augmented matrix.

E2. Row reduction. ... put a matrix in reduced row echelon form.

E3. Systems of linear equations. ... compute the solution set for a system of linear equations.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.
- Describe sets using set-builder notation, and check if an element is a member of a set described by set-builder notation.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Systems of linear equations (Khan Academy): `http://bit.ly/2l21etm`
- Solving linear systems with substitution (Khan Academy): `http://bit.ly/1SlMpix`
- Set builder notation: `https://youtu.be/xnfUZ-NTsCE`

Section E.0

Definition E.0.1 A **linear equation** is an equation of the variables x_i of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is a Euclidean vector

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

that satisfies

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

(that is, a Euclidean vector that can be plugged into the equation).

Remark E.0.2 In previous classes you likely used the variables x, y, z in equations. However, since this course often deals with equations of four or more variables, we will often write our variables as x_i, and assume $x = x_1, y = x_2, z = x_3, w = x_4$ when convenient.

Definition E.0.3 A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$\begin{alignedat}{5} a_{11}x_1 &+ \; & a_{12}x_2 &+ \ldots + \; & a_{1n}x_n &\; = \; & b_1 \\ a_{21}x_1 &+ \; & a_{22}x_2 &+ \ldots + \; & a_{2n}x_n &\; = \; & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 &+ \; & a_{m2}x_2 &+ \ldots + \; & a_{mn}x_n &\; = \; & b_m \end{alignedat}$$

Its **solution set** is given by

$$\left\{ \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \middle| \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \text{ is a solution to all equations in the system} \right\}.$$

Remark E.0.4 When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Concise standard form:

$$\begin{alignedat}{4} x_1 & & &+ 3x_3 &&= 3 \\ 3x_1 &- 2x_2 & &+ 4x_3 &&= 0 \\ &- \; x_2 & &+ \; x_3 &&= -2 \end{alignedat}$$

Definition E.0.5 A linear system is **consistent** if its solution set is non-empty (that is, there exists a solution for the system). Otherwise it is **inconsistent**.

Fact E.0.6 All linear systems are one of the following:

- **Consistent with one solution**: its solution set contains a single vector, e.g. $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$
- **Consistent with infinitely-many solutions**: its solution set contains infinitely many vectors, e.g. $\left\{\begin{bmatrix}1\\2-3a\\a\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$
- **Inconsistent**: its solution set is the empty set $\{\} = \emptyset$

Activity E.0.7 *(∼10 min)* All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is $\emptyset$.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity E.0.8 *(∼10 min)* Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions for this system.
Part 2: Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{\begin{bmatrix}?\\a\end{bmatrix}\middle|\, a\in\mathbb{R}\right\}$ for the linear system.

Activity E.0.9 *(∼10 min)* Consider the following linear system.

$$\begin{alignedat}{4} x_1 + 2x_2 & & &- x_4 &&= &&3 \\ & &x_3 &+ 4x_4 &&= &-&2 \end{alignedat}$$

Describe the solution set

$$\left\{\begin{bmatrix}?\\a\\?\\b\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Observation E.0.10 Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{aligned}
-2x_1 - 4x_2 + x_3 - 4x_4 &= -8 \\
x_1 + 2x_2 + 2x_3 + 12x_4 &= -1 \\
x_1 + 2x_2 + x_3 + 8x_4 &= 1
\end{aligned}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.

Section E.1

Remark E.1.1 The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition E.1.2 A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Example E.1.3 The corresopnding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

Definition E.1.4 Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned}$$

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity E.1.5 *(~10 min)* Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a nonzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

g) Replace a row with zeros.

(Instructor Note:) This activity could be ran as a card sort. Allow 5 additional minutes for intra team discussion.

Definition E.1.6 The following **row operations** produce equivalent augmented matrices:

1. Swap two rows.
2. Multiply a row by a nonzero constant.
3. Add a constant multiple of one row to another row.

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity E.1.7 *(~10 min)* Consider the following (equivalent) linear systems.

(A)

$$\begin{aligned} -2x_1 + 4x_2 - 2x_3 &= -8 \\ x_1 - 2x_2 + 2x_3 &= 7 \\ 3x_1 - 6x_2 + 4x_3 &= 15 \end{aligned}$$

(B)

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ -2x_1 + 4x_2 - 2x_3 &= -8 \\ 3x_1 - 6x_2 + 4x_3 &= 15 \end{aligned}$$

(C)

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ 2x_3 &= 6 \\ -2x_3 &= -6 \end{aligned}$$

(D)

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ x_3 &= 3 \\ -2x_3 &= -6 \end{aligned}$$

(E)

$$\begin{aligned} x_1 - 2x_2 \quad &= 1 \\ x_3 &= 3 \\ 0 &= 0 \end{aligned}$$

(F)

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 7 \\ 2x_3 &= 6 \\ 3x_1 - 6x_2 + 4x_3 &= 15 \end{aligned}$$

Part 1: Find a solution to one of these systems.
Part 2: Rank the six linear systems from most complicated to simplest.

Activity E.1.8 *(∼5 min)* We can rewrite the previous in terms of equivalences of augmented matrices

$$\left[\begin{array}{ccc|c} -2 & 4 & -2 & -8 \\ 1 & -2 & 2 & 7 \\ 3 & -6 & 4 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ -2 & 4 & -2 & -8 \\ 3 & -6 & 4 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 3 & -6 & 4 & 15 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

Activity E.1.9 *(∼10 min)* A matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

Circle the leading terms in each example, and label it as RREF or not RREF.

(A)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(B)
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(C)
$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(D)
$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(E)
$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(F)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Remark E.1.10 It is important to understand the **Gauss-Jordan elimination** algorithm that converts a matrix into reduced row echelon form.

A video outlining how to perform the Gauss-Jordan Elimination algorithm by hand is available at `https://youtu.be/CqONxk2dhhU`. Practicing several exercises outside of class using this method is recommended.

In the next section, we will learn to use technology to perform this operation for us, as will be expected when applying row-reduced matrices to solve other problems.

Section E.2

Activity E.2.1 *(~10 min)* Free browser-based technologies for mathematical computation are available online.

- Go to `http://cocalc.com` and create an account.
- Create a project titled "Linear Algebra Team X" with your appropriate team number. Add all team members as collaborators.
- Open the project and click on "New"
- Give it an appropriate name such as "Class E.2 workbook". Make a new Jupyter notebook.
- Click on "Kernel" and make sure "Octave" is selected.
- Type `A=[1 3 4 ; 2 5 7]` and press `Shift+Enter` to store the matrix $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{bmatrix}$ in the variable A.
- Type `rref(A)` and press `Shift+Enter` to compute the reduced row echelon form of A.

Remark E.2.2 If you need to find the reduced row echelon form of a matrix during class, you are encouraged to use CoCalc's Octave interpreter.

You can change a cell from "Code" to "Markdown" or "Raw" to put comments around your calculations such as Activity numbers.

Activity E.2.3 *(~10 min)* Consider the system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

Convert this to an augmented matrix and use CoCalc to compute its reduced row echelon form. Write these on your whiteboard, and use them to write a simpler yet equivalent linear system of equations. Then find its solution set.

Activity E.2.4 *(~10 min)* Consider our system of equations from above.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 \qquad\quad - 3x_3 &= 1 \end{aligned}$$

Convert this to an augmented matrix and use CoCalc to compute its reduced row echelon form. Write these on your whiteboard, and use them to write a simpler yet equivalent linear system of equations. Then find its solution set.

Activity E.2.5 *(∼10 min)* Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Part 1: Find its corresponding augmented matrix A and use CoCalc to find RREF(A).
Part 2: How many solutions does the corresponding linear system have?

Activity E.2.6 *(∼10 min)* Consider the simple linear system equivalent to the system from the previous problem:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$.

Part 2: Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}$.

Part 3: Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$ caused this?

Definition E.2.7 Recall that the pivots of a matrix in RREF form are the leading 1s in each non-zero row.

The pivot columns in an augmented matrix correspond to the **bound variables** in the system of equations (x_1, x_3 below). The remaining variables are called **free variables** (x_2 below).

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

To efficiently solve a system in RREF form, we may assign letters to free variables and solve for the bound variables.

Activity E.2.8 *(∼10 min)* Find the solution set for the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - x_5 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 + x_5 &= 2 \end{aligned}$$

by row-reducing its augmented matrix, and then assigning letters to the free variables (given by non-pivot columns) and solving for the bound variables (given by pivot columns) in the corresponding linear system.

Observation E.2.9 The solution set to the system

$$\begin{alignedat}{6} 2x_1 &-{}& 2x_2 &-{}& 6x_3 &+{}& x_4 &-{}& x_5 &={}& 3 \\ -x_1 &+{}& x_2 &+{}& 3x_3 &-{}& x_4 &+{}& 2x_5 &={}& -3 \\ x_1 &-{}& 2x_2 &-{}& x_3 &+{}& x_4 &+{}& x_5 &={}& 2 \end{alignedat}$$

may be written as

$$\left\{ \begin{bmatrix} 1+5a+2b \\ 1+2a+3b \\ a \\ 3+3b \\ b \end{bmatrix} \middle| \, a,b \in \mathbb{R} \right\}.$$

Remark E.2.10 Don't forget to correctly express the solution set of a linear system, using set-builder notation for consistent systems with infintely many solutions.

- **Consistent with one solution**: e.g. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$
- **Consistent with infinitely-many solutions**: e.g. $\left\{ \begin{bmatrix} 1 \\ 2-3a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$
- **Inconsistent**: $\emptyset$

Module V: Vector Spaces

Standards for this Module

What is a vector space? At the end of this module, students will be able to...

V1. Vector property verification. ... show why an example satisfies a given vector space property, but does not satisfy another given property.

V2. Vector space identification. ... list the eight defining properties of a vector space, infer which of these properties a given example satisfies, and thus determine if the example is a vector space.

V3. Linear combinations. ... determine if a Euclidean vector can be written as a linear combination of a given set of Euclidean vectors.

V4. Spanning sets. ... determine if a set of Euclidean vectors spans $\mathbb{R}^n$.

V5. Subspaces. ... determine if a subset of $\mathbb{R}^n$ is a subspace or not.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Section V.0

Activity V.0.1 *(∼20 min)* Consider each of the following vector properties. Label each property with $\mathbb{R}^1$, $\mathbb{R}^2$, and/or $\mathbb{R}^3$ if that property holds for Euclidean vectors/scalars $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of that dimension.

1. **Addition associativity.**

   $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

2. **Addition commutivity.**

   $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

3. **Addition identity.**

   There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.

4. **Addition inverse.**

   There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.

5. **Addition midpoint uniqueness.**

   There exists a unique $\mathbf{m}$ where the distance from $\mathbf{u}$ to $\mathbf{m}$ equals the distance from $\mathbf{m}$ to $\mathbf{v}$.

6. **Scalar multiplication associativity.**

   $a(b\mathbf{v}) = (ab)\mathbf{v}$.

7. **Scalar multiplication identity.**

   $1\mathbf{v} = \mathbf{v}$.

8. **Scalar multiplication relativity.**

   There exists some scalar c where either $c\mathbf{v} = \mathbf{w}$ or $c\mathbf{w} = \mathbf{v}$.

9. **Scalar distribution.**

   $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

10. **Vector distribution.**

    $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

11. **Orthogonality.**

    There exists a non-zero vector $\mathbf{n}$ such that $\mathbf{n}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

12. **Bidimensionality.**

    $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ for some value of a, b.

Definition V.0.2 A **vector space** V is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ belong to V, and let a, b be scalar numbers.

- **Addition is associative.**

  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **Addition is commutative.**

  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **Additive identity exists.**

  There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.

- **Additive inverses exist.**

  There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.

- **Scalar multiplication is associative.**

  $a(b\mathbf{v}) = (ab)\mathbf{v}$.

- 1 **is a scalar multiplicative identity.**

  $1\mathbf{v} = \mathbf{v}$.

- **Scalar multiplication distributes over vector addition.**

  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

- **Scalar multiplication distributes over scalar addition.**

  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Any **Euclidean vector space** $\mathbb{R}^n$ satisfies all eight requirements regardless of the value of n, but we will also study other types of vector spaces.

Section V.1

Remark V.1.1 Last time, we defined a **vector space** V to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, and all scalars (i.e. real numbers) a, b.

- **Addition is associative.**

  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- **Addition is commutative.**

  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- **Additive identity exists.**

  There exists some $\mathbf{z}$ where $\mathbf{v} + \mathbf{z} = \mathbf{v}$.

- **Additive inverses exist.**

  There exists some $-\mathbf{v}$ where $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$.

- **Scalar multiplication is associative.**

  $a(b\mathbf{v}) = (ab)\mathbf{v}$.

- 1 **is a scalar multiplicative identity.**

  $1\mathbf{v} = \mathbf{v}$.

- **Scalar multiplication distributes over vector addition.**

  $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

- **Scalar multiplication distributes over scalar addition.**

  $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Remark V.1.2 The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$: Euclidean vectors with n components.
- $\mathbb{R}^\infty$: Sequences of real numbers $(v_1, v_2, \dots)$.
- $M_{m,n}$: Matrices of real numbers with m rows and n columns.
- $\mathbb{C}$: Complex numbers.
- $\mathcal{P}^n$: Polynomials of degree n or less.
- $\mathcal{P}$: Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity V.1.3 *(∼20 min)* Consider the set $V = \{(x, y) \,|\, y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

Part 1: Show that V satisfies the vector distributive property

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$$

by letting $\mathbf{v} = (x, y)$ and showing both sides simplify to the same expression.
Part 2: Show that V contains an additive identity element by choosing $\mathbf{z} = (\,?\,,\,?\,)$ such that $\mathbf{v} \oplus \mathbf{z} = (x, y) \oplus (\,?\,,\,?\,) = \mathbf{v}$ for any $\mathbf{v} = (x, y) \in V$.

Remark V.1.4 It turns out $V = \{(x, y) \,|\, y = e^x\}$ with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \qquad c \odot (x, y) = (cx, y^c)$$

satisifes all eight properties.

- **Addition associativity.**
  $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$.
- **Addition commutivity.**
  $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$.
- **Addition identity.**
  There exists some $\mathbf{z}$ where $\mathbf{v} \oplus \mathbf{z} = \mathbf{v}$.
- **Addition inverse.**
  There exists some $-\mathbf{v}$ where $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{z}$.
- **Scalar multiplication associativity.**
  $a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}$.
- **Scalar multiplication identity.**
  $1 \odot \mathbf{v} = \mathbf{v}$.
- **Scalar distribution.**
  $a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v})$.
- **Vector distribution.**
  $(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$.

Thus, V is a vector space.

Activity V.1.5 *(∼15 min)* Let $V = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \qquad c \odot (x, y) = (x^c, y + c - 1).$$

Part 1: Show that the scalar multiplication identity holds by simplifying $1 \odot (x, y)$ to (x, y).
Part 2: Show that the addition identity property fails by showing that $(0, -1) \oplus \mathbf{z} \neq (0, -1)$ no matter how $\mathbf{z} = (z_1, z_2)$ is chosen.
Part 3: Can V be a vector space?

Definition V.1.6 A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$ for any choice of scalar multiples $c_1, c_2, \ldots, c_m$.

For example, we can say $\begin{bmatrix}3\\0\\5\end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\2\\1\end{bmatrix}$ since

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

Definition V.1.7 The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m \,|\, c_i \in \mathbb{R}\}\,.$$

For example:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$$

Activity V.1.8 *(~10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$.

Part 1: Sketch $1\begin{bmatrix}1\\2\end{bmatrix}$, $3\begin{bmatrix}1\\2\end{bmatrix}$, $0\begin{bmatrix}1\\2\end{bmatrix}$, and $-2\begin{bmatrix}1\\2\end{bmatrix}$ in the xy plane.

Part 2: Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\2\end{bmatrix} \,\middle|\, a \in \mathbb{R}\right\}$ in the xy plane.

Activity V.1.9 *(~10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix} \qquad 0\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix} \qquad 1\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix} \qquad -1\begin{bmatrix}1\\2\end{bmatrix} + -2\begin{bmatrix}-1\\1\end{bmatrix}$$

Part 2: Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity V.1.10 *(~5 min)* Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix}, \begin{bmatrix}-3\\2\end{bmatrix}\right\}$ in the xy plane.

Section V.2

Remark V.2.1 Recall these definitions from last class:

- A **linear combination** of vectors is given by adding scalar multiples of those vectors, such as:

$$\begin{bmatrix}3\\0\\5\end{bmatrix} = 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

- The **span** of a set of vectors is the collection of all linear combinations of that set, such as:

$$\operatorname{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\-1\\2\end{bmatrix} + b\begin{bmatrix}1\\2\\1\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$$

Activity V.2.2 *(∼15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix} + x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\\1\end{bmatrix}$.

Part 1: Reinterpret this vector equation as a system of linear equations.
Part 2: Find its solution set, using CoCalc.com to find RREF of its corresponding augmented matrix.
Part 3: Given this solution set, does $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belong to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$?

Fact V.2.3 A vector $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ if and only if the linear system corresponding to $[\mathbf{v}_1 \ldots \mathbf{v}_n \,|\, \mathbf{b}]$ is consistent.

Put another way, $\mathbf{b}$ belongs to $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ exactly when $\operatorname{RREF}[\mathbf{v}_1 \ldots \mathbf{v}_n \,|\, \mathbf{b}]$ doesn't have a row $[0 \cdots 0 \,|\, 1]$ representing the contradiction $0 = 1$.

Activity V.2.4 *(∼10 min)* Determine if $\begin{bmatrix}3\\-2\\1\\5\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\\2\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\\2\end{bmatrix}\right\}$ by row-reducing an appropriate matrix.

Activity V.2.5 *(∼5 min)* Determine if $\begin{bmatrix}-1\\-9\\0\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ by row-reducing an appropriate matrix.

Activity V.2.6 *(~10 min)* Does the third-degree polynomial $3y^3 - 2y^2 + y + 5$ in $\mathcal{P}^3$ belong to $\operatorname{span}\{y^3 - 3y + 2, -y^3 - 3y^2 + 2y + 2\}$?
Part 1: Reinterpret this question as an equivalent exercise involving Euclidean vectors in $\mathbb{R}^4$. (Hint: What four numbers must you know to write a $\mathcal{P}^3$ polynomial?)
Part 2: Solve this equivalent exercise, and use its solution to answer the original question.

Activity V.2.7 *(~5 min)* Does the matrix $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ belong to $\operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \right\}$?

Activity V.2.8 *(~5 min)* Does the complex number $2i$ belong to $\operatorname{span}\{-3 + i, 6 - 2i\}$?

Section V.3

Activity V.3.1 *(∼5 min)* How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your answer.

(a) 1

(b) 2

(c) 3

(d) 4

(e) Infinitely Many

Activity V.3.2 *(∼5 min)* How many vectors are required to span $\mathbb{R}^3$?

(a) 1

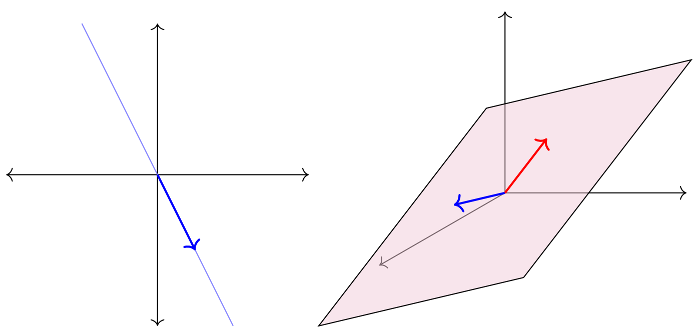
(b) 2

(c) 3

(d) 4

(e) Infinitely Many

Fact V.3.3 At least n vectors are required to span $\mathbb{R}^n$.



Activity V.3.4 *(∼15 min)* Choose a vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by using CoCalc to verify that RREF $\left[\begin{array}{cc|c} 1 & -2 & ? \\ -1 & 0 & ? \\ 0 & 1 & ? \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$. (Why does this work?)

Fact V.3.5 The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ fails to span all of $\mathbb{R}^n$ exactly when $\mathrm{RREF}[\mathbf{v}_1 \, \dots \, \mathbf{v}_m]$ has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \text{for some choice of vector } \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Activity V.3.6 *(∼5 min)* Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does $\mathbb{R}^4 = \operatorname{span} S$?

Activity V.3.7 *(∼10 min)* Consider the set of third-degree polynomials

$$\begin{aligned} S = \{&2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, \\ &- x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2\}. \end{aligned}$$

Does $\mathcal{P}^3 = \operatorname{span} S$? (Hint: first rewrite the question so it is about Euclidean vectors.)

Activity V.3.8 *(∼5 min)* Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Does $M_{2,2} = \operatorname{span} S$?

Activity V.3.9 *(∼5 min)* Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^7$ be three vectors, and suppose $\mathbf{w}$ is another vector with $\mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. What can you conclude about $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is larger than $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

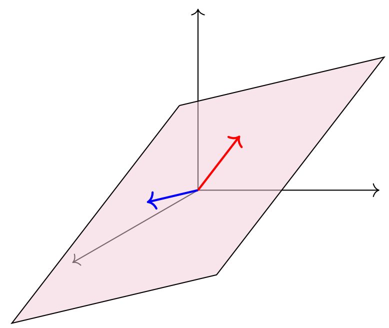
(b) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(c) $\operatorname{span}\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is smaller than $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Section V.4

Definition V.4.1 A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space $\mathbb{R}^3$.



Fact V.4.2 Any sub**set** S of a vector space V satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a sub**space**, we need to check that addition and multiplication still make sense using only vectors from S. So we need to check two things:

- The set is **closed under addition**: for any $\mathbf{x}, \mathbf{y} \in S$, the sum $\mathbf{x}+\mathbf{y}$ is also in S.
- The set is **closed under scalar multiplication**: for any $\mathbf{x} \in S$ and scalar $c \in \mathbb{R}$, the product $c\mathbf{x}$ is also in S.

Activity V.4.3 *(~15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, x+2y+z=0 \right\}$.

Part 1: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x+2y+z=0$ and $a+2b+c=0$. Show that $\mathbf{v}+\mathbf{w} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$ also belongs to S by verifying that $(x+a)+2(y+b)+(z+c)=0$.

Part 2: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x+2y+z=0$. Show that $c\mathbf{v}$ also belongs to S for any $c \in \mathbb{R}$.

Part 3: Is S is a subspace of $\mathbb{R}^3$?

Activity V.4.4 *(~10 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, x+2y+z=4 \right\}$. Choose a vector $\mathbf{v} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = ?$, and show that $c\mathbf{v}$ isn't in S. Is S a subspace of $\mathbb{R}^3$?

Remark V.4.5 Since 0 is a scalar and $0\mathbf{v} = \mathbf{z}$ for any vector $\mathbf{v}$, a set that is closed under scalar multiplication must contain the zero vector $\mathbf{z}$ for that vector space.

Put another way, an easy way to check that a subset isn't a subspace is to show it doesn't contain $\mathbf{0}$.

Activity V.4.6 *(~10 min)* Consider these two subsets of $\mathbb{R}^4$:

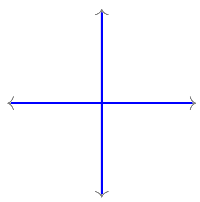
$$S = \left\{ \begin{bmatrix} a \\ b \\ -b \\ -a \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\} \qquad T = \left\{ \begin{bmatrix} a \\ b \\ b-1 \\ a-1 \end{bmatrix} \middle| \, a, b \text{ are real numbers} \right\}$$

Part 1: Which set is not a subspace of $\mathbb{R}^4$?
Part 2: Is the set of polynomials

$$S = \left\{ ax^3 + bx^2 + (b-1)x + (a-1) \,\middle|\, a, b \text{ are real numbers} \right\}$$

a subspace of $\mathcal{P}^3$?

Activity V.4.7 *(~10 min)* Consider the subset A of $\mathbb{R}^2$ where at least one coordinate of each vector is 0.



This set contains $\mathbf{0}$, and it's not hard to show that for every $\mathbf{v}$ in A and scalar $c \in \mathbb{R}$, $c\mathbf{v}$ is also in A. Is A a subspace of $\mathbb{R}^2$? Why?
(Instructor Note:) Sketch the sum of two vectors on different axes to give a geometrical argument.

Activity V.4.8 *(~5 min)* Let W be a subspace of a vector space V. How are $\operatorname{span} W$ and W related?

(a) $\operatorname{span} W$ is bigger than W

(b) $\operatorname{span} W$ is the same as W

(c) $\operatorname{span} W$ is smaller than W

Fact V.4.9 If S is any subset of a vector space V, then since $\operatorname{span} S$ collects all possible linear combinations, $\operatorname{span} S$ is automatically a subspace of V.

In fact, $\operatorname{span} S$ is always the smallest subspace of V that contains all the vectors in S.

Module S: Structure of vector spaces

Standards for this Module

What structure do vector spaces have? At the end of this module, students will be able to...

S1. Linear independence. ... determine if a set of Euclidean vectors is linearly dependent or independent.

S2. Basis verification. ... determine if a set of Euclidean vectors is a basis of $\mathbb{R}^n$.

S3. Basis computation. ... compute a basis for the subspace spanned by a given set of Euclidean vectors.

S4. Dimension. ... compute the dimension of a subspace of $\mathbb{R}^n$.

S5. Abstract vector spaces. ... solve exercises related to standards V3-S4 when posed in terms of polynomials or matrices.

S6. Basis of solution space. ... find a basis for the solution set of a homogeneous system of equations.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **E1,E2,E3**.
- Apply linear combinations and spanning sets **V3,V4**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Adding and subtracting Euclidean vectors (Khan Acaemdy): `http://bit.ly/2y8AOwa`
- Linear combinations of Euclidean vectors (Khan Academy): `http://bit.ly/2nK3wne`
- Adding and subtracting complex numbers (Khan Academy): `http://bit.ly/1PE3ZMQ`
- Adding and subtracting polynomials (Khan Academy): `http://bit.ly/2d5SLGZ`

Section S.1

Activity S.1.1 *(~10 min)* Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad\qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

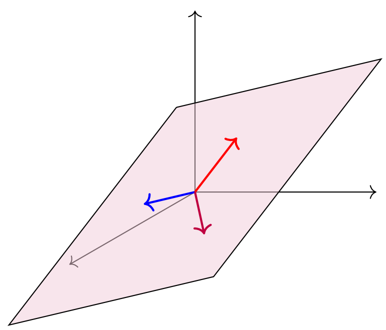
Which of the following is true?

(A) $\operatorname{span} S$ is bigger than $\operatorname{span} T$.

(B) $\operatorname{span} S$ and $\operatorname{span} T$ are the same size.

(C) $\operatorname{span} S$ is smaller than $\operatorname{span} T$.

Definition S.1.2 We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.



You can think of linearly dependent sets as containing a redundant vector, in the sense that you can drop a vector out without reducing the span of the set. In the above image, all three vectors lay on the same planar subspace, but only two vectors are needed to span the plane, so the set is linearly dependent.

Activity S.1.3 *(~10 min)* Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$. Suppose $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$, so the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. Which of the following is true of the vector equation $x\mathbf{u} + y\mathbf{v} + z\mathbf{w} = \mathbf{0}$?

(A) It is consistent with one solution

(B) It is consistent with infinitely many solutions

(C) It is inconsistent.

Fact S.1.4 For any vector space, the set $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ is linearly dependent if and only if $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{z}$ is consistent with infinitely many solutions.

Activity S.1.5 *(∼10 min)* Find

$$\mathrm{RREF}\left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Fact S.1.6 A set of Euclidean vectors $\{\mathbf{v}_1, \dots \mathbf{v}_n\}$ is linearly dependent if and only if RREF $\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ has a column without a pivot position.

Activity S.1.7 *(∼5 min)* Is the set of Euclidean vectors $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$ linearly dependent or linearly independent?

Activity S.1.8 *(∼10 min)* Is the set of polynomials $\{x^3 + 1, x^2 + 2x, x^2 + 7x + 4\}$ linearly dependent or linearly independent?

Activity S.1.9 *(∼5 min)* What is the largest number of vectors in $\mathbb{R}^4$ that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity S.1.10 *(∼5 min)* What is the largest number of vectors in

$$\mathcal{P}^4 = \left\{ax^4 + bx^3 + cx^2 + dx + e \,\middle|\, a, b, c, d, e \in \mathbb{R}\right\}$$

that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity S.1.11 *(∼5 min)* What is the largest number of vectors in

$$\mathcal{P} = \{f(x) \,|\, f(x) \text{ is any polynomial}\}$$

that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Section S.2

Definition S.2.1 A **basis** is a linearly independent set that spans a vector space.

The **standard basis** of $\mathbb{R}^n$ is the set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \cdots \qquad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For $\mathbb{R}^3$, these are the vectors $\mathbf{e}_1 = \hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{e}_3 = \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Observation S.2.2 A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

For example, in many calculus courses, vectors in $\mathbb{R}^3$ are often expressed in their component form

$$(3, -2, 4) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

or in their standard basic vector form

$$3\mathbf{e}_1 - 2\mathbf{e}_2 + 4\mathbf{e}_3 = 3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}.$$

Since every vector in $\mathbb{R}^3$ can be uniquely described as a linear combination of the vectors in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, this set is indeed a basis.

Activity S.2.3 *(~15 min)* Label each of the sets A, B, C, D, E as

- SPANS $\mathbb{R}^4$ or DOES NOT SPAN $\mathbb{R}^4$
- LINEARLY INDEPENDENT or LINEARLY DEPENDENT
- BASIS FOR $\mathbb{R}^4$ or NOT A BASIS FOR $\mathbb{R}^4$

by finding RREF for their corresponding matrices.

$$A = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}3\\13\\7\\16\end{bmatrix}, \begin{bmatrix}-1\\10\\7\\14\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix} \right\} \qquad D = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix}, \begin{bmatrix}3\\6\\1\\5\end{bmatrix} \right\}$$

$$E = \left\{ \begin{bmatrix}5\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\3\end{bmatrix}, \begin{bmatrix}4\\5\\1\\3\end{bmatrix} \right\}$$

Activity S.2.4 *(~10 min)* If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$, that means RREF$[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$ doesn't have a non-pivot column, and doesn't have a row of zeros. What is RREF$[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4]$?

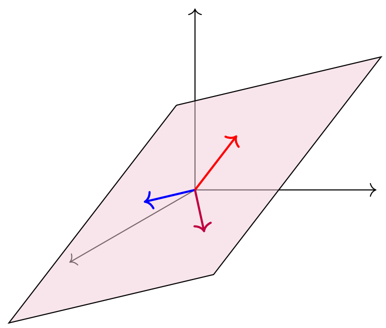
$$\text{RREF}[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3\,\mathbf{v}_4] = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Fact S.2.5 The set $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for $\mathbb{R}^n$ if and only if $m = n$ and $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$.

That is, a basis for $\mathbb{R}^n$ must have exactly n vectors and its square matrix must row-reduce to the so-called **identity matrix** containing all zeros except for a downward diagonal of ones. (We will learn where the identity matrix gets its name in a later module.)

Observation S.2.6 Recall that a **subspace** of a vector space is a subset that is itself a vector space.

One easy way to construct a subspace is to take the span of set, but a linearly dependent set contains "redundant" vectors. For example, only two of the three vectors in the following image are needed to span the planar subspace.



Activity S.2.7 *(~10 min)* Consider the subspace $W = \operatorname{span}\left\{\begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix}\right\}$ of $\mathbb{R}^4$.

Part 1: Mark the part of RREF $\begin{bmatrix}2&2&2&1\\3&0&-3&5\\0&1&2&-1\\1&-1&-3&0\end{bmatrix}$ that shows that W's spanning set is linearly dependent.

Part 2: Find a basis for W by removing a vector from its spanning set to make it linearly independent.

Fact S.2.8 Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$. The easiest basis describing $\operatorname{span} S$ is the set of vectors in S given by the pivot columns of RREF$[\mathbf{v}_1 \dots \mathbf{v}_m]$.

Put another way, to compute a basis for the subspace $\operatorname{span} S$, simply remove the vectors corresponding to the non-pivot columns of RREF$[\mathbf{v}_1 \dots \mathbf{v}_m]$.

Activity S.2.9 *(~10 min)* Let W be the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{\begin{bmatrix}1\\3\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\2\end{bmatrix}, \begin{bmatrix}4\\5\\3\\0\end{bmatrix}, \begin{bmatrix}3\\2\\2\\1\end{bmatrix}\right\}$$

Find a basis for W.

Activity S.2.10 *(~10 min)* Let W be the subspace of $\mathcal{P}^3$ given by

$$W = \operatorname{span}\left\{x^3 + 3x^2 + x - 1, 2x^3 - x^2 + x + 2, 4x^3 + 5x^2 + 3x, 3x^3 + 2x^2 + 2x + 1\right\}$$

Find a basis for W.

Section S.3

Observation S.3.1 In the previous section, we learned that computing a basis for the subspace $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, is as simple as removing the vectors corresponding to the non-pivot columns of $\operatorname{RREF}[\mathbf{v}_1 \, \dots \, \mathbf{v}_m]$.

For example, since

$$\operatorname{RREF}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ -3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the subspace $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}\right\}$ has $\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}$ as a basis.

Activity S.3.2 *(∼10 min)* Let

$$S = \left\{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}\right\} \qquad \text{and} \qquad T = \left\{\begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Part 1: Find a basis for $\operatorname{span} S$.
Part 2: Find a basis for $\operatorname{span} T$.

Observation S.3.3 Even though we found different bases for them, $\operatorname{span} S$ and $\operatorname{span} T$ are exactly the same subspace of $\mathbb{R}^4$, since

$$S = \left\{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right\} = T$$

Fact S.3.4 Any non-trivial vector space has infinitely-many different bases, but all the bases for a given vector space are exactly the same size.

For example,

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \text{ and } \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} \text{ and } \left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}\right\}$$

are all valid bases for $\mathbb{R}^3$, and they all contain three vectors.

Definition S.3.5 The **dimension** of a vector space is equal to the size of any basis for the vector space.

As you'd expect, $\mathbb{R}^n$ has dimension n. For example, $\mathbb{R}^3$ has dimension 3 because any basis for $\mathbb{R}^3$ such as

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \text{ and } \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}2\\-2\\1\end{bmatrix}, \begin{bmatrix}3\\-2\\5\end{bmatrix} \right\}$$

contains exactly three vectors.

Activity S.3.6 *(∼10 min)* Find the dimension of each subspace of $\mathbb{R}^4$ by finding RREF for each corresponding matrix.

$$\text{span}\left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix} \right\} \quad \text{span}\left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}3\\13\\7\\16\end{bmatrix}, \begin{bmatrix}-1\\10\\7\\14\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix} \right\}$$

$$\text{span}\left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix}, \begin{bmatrix}3\\6\\1\\5\end{bmatrix} \right\} \quad \text{span}\left\{ \begin{bmatrix}5\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\3\end{bmatrix}, \begin{bmatrix}4\\5\\1\\3\end{bmatrix} \right\}$$

Fact S.3.7 Every vector space with finite dimension, that is, every vector space V with a basis of the form $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is said to be **isomorphic** to a Euclidean space $\mathbb{R}^n$, since there exists a natural correspondance between vectors in V and vectors in $\mathbb{R}^n$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \leftrightarrow \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$$

Observation S.3.8 We've already been taking advantage of the previous fact by converting polynomials and matrices into Euclidean vectors. Since $\mathcal{P}^3$ and $M_{2,2}$ are both four-dimensional:

$$4x^3 + 0x^2 - 1x + 5 \leftrightarrow \begin{bmatrix}4\\0\\-1\\5\end{bmatrix} \leftrightarrow \begin{bmatrix}4 & 0\\-1 & 5\end{bmatrix}$$

Observation S.3.9 The space of polynomials $\mathcal{P}$ (of *any* degree) has the basis $\{1, x, x^2, x^3, \dots\}$, so it is a natural example of an infinite-dimensional vector space.

Since $\mathcal{P}$ and other infinite-dimensional spaces cannot be treated as an isomorphic finite-dimensional Euclidean space $\mathbb{R}^n$, vectors in such spaces cannot be studied by converting them into Euclidean vectors. Fortunately, most of the examples we will be interested in for this course will be finite-dimensional.

Definition S.3.10 A **homogeneous** system of linear equations is one of the form:

$$\begin{aligned}
a_{11}x_1 + \; a_{12}x_2 + \ldots + \; a_{1n}x_n \; &= 0 \\
a_{21}x_1 + \; a_{22}x_2 + \ldots + \; a_{2n}x_n \; &= 0 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \;\; &\;\;\; \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \; &= 0
\end{aligned}$$

This system is equivalent to the vector equation:

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

and the augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

Activity S.3.11 *(∼5 min)* Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are solutions to $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ so is $\begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$, since

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0} \text{ and } b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n = \mathbf{0}$$

implies

$$(a_1 + b_1)\mathbf{v}_1 + \cdots + (a_n + b_n)\mathbf{v}_n = \mathbf{0}.$$

Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution. Thus the solution set of a homogeneous system is...

a) A basis for $\mathbb{R}^n$.

b) A subspace of $\mathbb{R}^n$.

c) The empty set.

Activity S.3.12 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \phantom{{}-x_3} + x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - x_4 &= 0 \end{aligned}$$

Part 1: Find its solution set (a subspace of $\mathbb{R}^4$).
Part 2: Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

Fact S.3.13 The coefficients of the free variables in the solution set of a linear system always yield linearly independent vectors.

Thus if

$$\left\{ a \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

is the solution space for a homoegeneous system, then

$$\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

is a basis for the solution space.

Activity S.3.14 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 \phantom{{}+3x_4} &= 0 \\ 2x_1 - 6x_2 + 4x_3 + 3x_4 &= 0 \\ -2x_1 + 6x_2 - 4x_3 - 4x_4 &= 0 \end{aligned}$$

Find a basis for its solution space.

Activity S.3.15 *(∼5 min)* Suppose W is a subspace of $\mathcal{P}^8$, and you know that it contains a **linearly independent** set of 3 vectors. What can you conclude about W?

(a) The dimension of W is at most 3.

(b) The dimension of W is exactly 3.

(c) The dimension of W is at least 3.

Activity S.3.16 *(∼5 min)* Suppose W is a subspace of $\mathcal{P}^8$, and you know that it contains a **spanning set** of 3 vectors. What can you conclude about W?

(a) The dimension of W is at most 3.

(b) The dimension of W is exactly 3.

(c) The dimension of W is at least 3.

Module A: Algebraic properties of linear maps

Standards for this Module

How can we understand linear maps algebraically? At the end of this module, students will be able to...

A1. Linear map verification. ... determine if a map between vector spaces of polynomials is linear or not.

A2. Linear maps and matrices. ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.

A3. Injectivity and surjectivity. ... determine if a given linear map is injective and/or surjective.

A4. Kernel and Image. ... compute a basis for the kernel and a basis for the image of a linear map.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

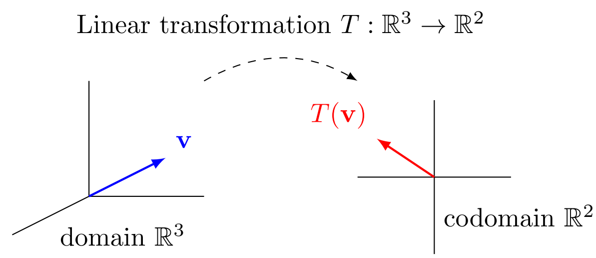
Section A.1

Definition A.1.1 A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition A.1.2 Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.



Example A.1.3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right) = \begin{bmatrix} (x+u) - (z+w) \\ 3(y+v) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x+u) - (z+w) \\ 3(y+v) \end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \text{ and } cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore T is a linear transformation.

Example A.1.4 Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact A.1.5 A map between Euclidean spaces $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of $\mathbb{R}^n$.

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2, $y + 3$, and $y - 2^x$ are not linear combinations (even though $x + y$ is):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

Activity A.1.6 *($\sim$5 min)* Recall the following rules from calculus, where $D : \mathcal{P} \to \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f.

$$D(f+g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

a) $\mathcal{P}$ is not a vector space

b) D is a linear map

c) D is not a linear map

Activity A.1.7 *($\sim$10 min)* Let the polynomial maps $S : \mathcal{P}^4 \to \mathcal{P}^3$ and $T : \mathcal{P}^4 \to \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4+x)$, $S(x^4)+S(x)$, $T(x^4+x)$, and $T(x^4)+T(x)$. Which of these maps is definitely not linear?

Fact A.1.8 If $L : V \to W$ is linear, then $L(\mathbf{z}) = L(0\mathbf{v}) = 0L(\mathbf{v}) = \mathbf{z}$ where $\mathbf{z}$ is the additive identity of the vector spaces V, W.

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Activity A.1.9 *($\sim$15 min)* Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.
Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f. Is S linear?

Activity A.1.10 *($\sim$20 min)* Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.
Part 2: Prove that T is linear by verifying that $T(f(x)+g(x)) = T(f(x))+T(g(x))$ and $T(cf(x)) = cT(f(x))$.

Observation A.1.11 Note that S in the previous activity is not linear, even though $S(0) = (0)^2 = 0$. So showing $S(0) = 0$ isn't enough to prove a map is linear.

This is a similar situation to proving a subset is a subspace: if the subset doesn't contain $\mathbf{z}$, then the subset isn't a subspace. But if the subset contains $\mathbf{z}$, you cannot conclude anything.

Section A.2

Remark A.2.1 Recall that a linear map $T : V \to W$ satisfies

1. $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vecor space operations can be applied before or after the transformation without affecting the result.

Activity A.2.2 *(∼5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity A.2.3 *(∼3 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.2.4 *(~2 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}-2\\0\\-3\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.2.5 *(~5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Do you have enough information to compute $T(\mathbf{v})$ for *any* $\mathbf{v} \in \mathbb{R}^3$?

(a) Yes.

(b) No, exactly one more piece of information is needed.

(c) No, an infinite amount of information would be necessary to compute the transformation of infinitely-many vectors.

Fact A.2.6 Consider any basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for V. Since every vector $\mathbf{v}$ can be written *uniquely* as a linear combination of basis vectors, $x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n$, we may compute $T(\mathbf{v})$ as follows:

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = x_1T(\mathbf{b}_1) + \cdots + x_nT(\mathbf{b}_n).$$

Therefore any linear transformation $T : V \to W$ can be defined by just describing the values of $T(\mathbf{b}_i)$. Put another way, the images of the basis vectors **determine** the transformation T.

Definition A.2.7 Since linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$.

For example, let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T(\mathbf{e}_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T(\mathbf{e}_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3)\end{bmatrix} = \begin{bmatrix}3 & -1 & 5\\2 & 4 & 0\end{bmatrix}.$$

Activity A.2.8 *(~3 min)* Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ for T.

Activity A.2.9 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Find the standard matrix for T.

Fact A.2.10 Because every linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ has a linear combination of the variables in each component, and thus $T(\mathbf{e}_i)$ yields exactly the coefficients of x_i, the standard matrix for T is simply an ordered list of the coefficients of the x_i:

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \qquad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Activity A.2.11 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Activity A.2.12 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

Fact A.2.13 To quickly compute $T(\mathbf{v})$ from its standard matrix A, compute the **dot product** (defined in Calculus 3) of each matrix row with the vector. For example, if T has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$$

then for $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 2y + 3z \\ 0x + 1y - 2z \\ 2x - 1y + 0z \end{bmatrix}$$

and for $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(0) + 3(-2) \\ 0(3) + 1(0) - 2(-2) \\ 2(3) - 1(0) + 0(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}.$$

Activity A.2.14 *(∼15 min)* Compute the following linear transformations of vectors given their standard matrices.

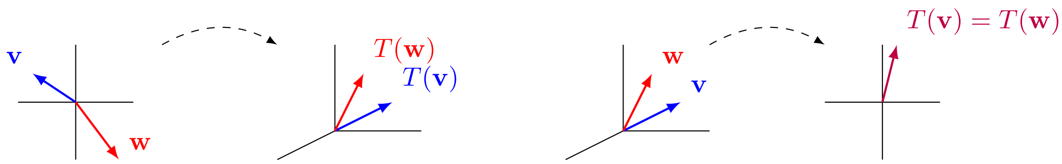
$$T_1\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \text{ for the standard matrix } A_1 = \begin{bmatrix} 4 & 3 \\ 0 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix}\right) \text{ for the standard matrix } A_2 = \begin{bmatrix} 4 & 3 & 0 & -1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

$$T_3\left(\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}\right) \text{ for the standard matrix } A_3 = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & 3 \\ 5 & 1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

Section A.3

Definition A.3.1 Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct vectors to the same place. More precisely, T is injective if $T(\mathbf{v}) \neq T(\mathbf{w})$ whenever $\mathbf{v} \neq \mathbf{w}$.



injective not injective

Activity A.3.2 *(~3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

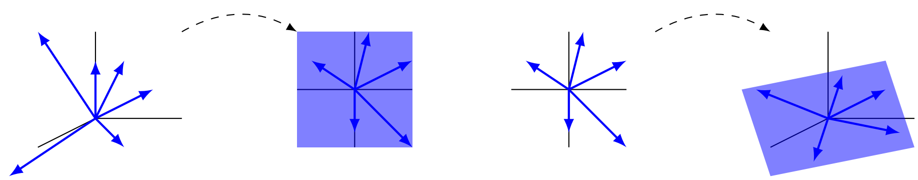
Show that T is not injective by finding two different vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $T(\mathbf{v}) = T(\mathbf{w})$.

Activity A.3.3 *(~2 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T injective? If not, find two different vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $T(\mathbf{v}) = T(\mathbf{w})$.

Definition A.3.4 Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.



surjective not surjective

Activity A.3.5 *(∼3 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Show that T is not surjective by finding a vector in $\mathbb{R}^3$ that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal.

Activity A.3.6 *(∼2 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective? If not, find a vector in $\mathbb{R}^2$ that $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal.

Observation A.3.7 As we will see, it's no coincidence that the RREF of the injective map's standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

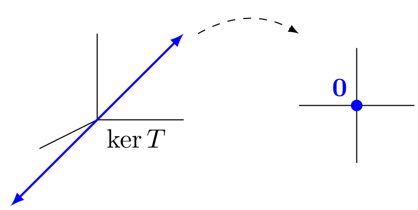
has all pivot columns. Similarly, the RREF of the surjective map's standard matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has a pivot in each row.

Definition A.3.8 Let $T : V \to W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{\mathbf{v} \in V \;\middle|\; T(\mathbf{v}) = \mathbf{z}\right\}$$



Activity A.3.9 *(∼5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\ker T$, the set of all vectors that transform into $\mathbf{0}$?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2$

Activity A.3.10 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\ker T$, the set of all vectors that transform into $\mathbf{0}$?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3$

Activity A.3.11 *(∼10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

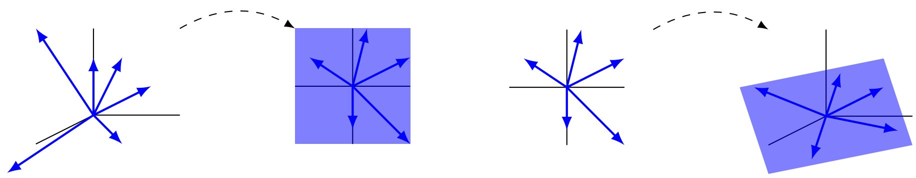
$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Part 1: Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel.
Part 2: Use RREF(A) to solve this homogeneous system of equations and find a basis for the kernel of T.

Definition A.3.12 Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{ \mathbf{w} \in W \;\middle|\; \text{there is some } \mathbf{v} \in V \text{ with } T(\mathbf{v}) = \mathbf{w} \right\}$$

In the examples below, the left example's image is all of $\mathbb{R}^2$, but the right example's image is a planar subspace of $\mathbb{R}^3$.



Activity A.3.13 *(∼5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^2$ vectors?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle|\; a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \middle|\; a, b \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3$

Activity A.3.14 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^3$ vectors?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2$

Activity A.3.15 *(∼5 min)* Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) & T(\mathbf{e}_4) \end{bmatrix}.$$

Since $T(\mathbf{v}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4)$, the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

a) spans $\operatorname{Im} T$

b) is a linearly independent subset of $\operatorname{Im} T$

c) is a basis for $\operatorname{Im} T$

Observation A.3.16 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}.$$

Since the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ spans $\operatorname{Im} T$, we can obtain a basis for $\operatorname{Im} T$ by finding $\operatorname{RREF} A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and only using the vectors corresponding to pivot columns:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Fact A.3.17 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

- The kernel of T is the solution set of the homogeneous system given by the augmented matrix $\left[A \mid \mathbf{0}\right]$. Use the coefficients of its free variables to get a basis for the kernel.
- The image of T is the span of the columns of A. Remove the vectors creating non-pivot columns in $\operatorname{RREF} A$ to get a basis for the image.

Activity A.3.18 *(~10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T.

Section A.4

Observation A.4.1 Let $T : V \to W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T always maps distinct vectors to different places.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.
- The **kernel** of T is the set of all vectors in V that are mapped to $\mathbf{z} \in W$. It is a subspace of V.
- The **image** of T is the set of all vectors in W that are mapped to by something in V. It is a subspace of W.

Activity A.4.2 *(∼5 min)* Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

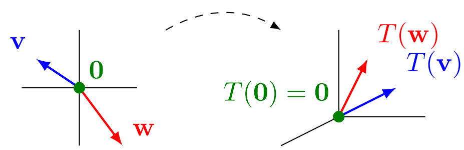
(a) T is injective

(b) T is not injective

(c) T is surjective

(d) T is not surjective

Fact A.4.3 A linear transformation T is injective **if and only if** $\ker T = \{\mathbf{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.



Activity A.4.4 *(∼5 min)* Let $T : \mathbb{R}^4 \to \mathbb{R}^5$ be a linear transformation where $\operatorname{Im} T$ is spanned by four vectors. What can you conclude?

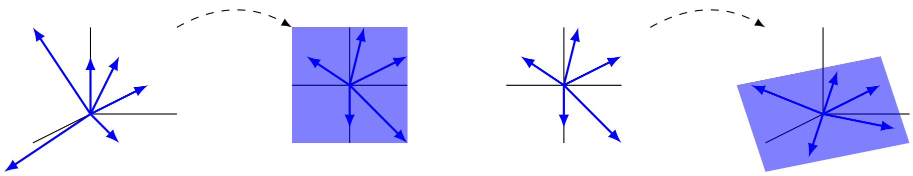
(a) T is injective

(b) T is not injective

(c) T is surjective

(d) T is not surjective

Fact A.4.5 A linear transformation $T : V \to W$ is surjective **if and only if** $\operatorname{Im} T = W$. Put another way, a surjective linear transformation may be recognized by its identical codomain and image.



surjective, $\operatorname{Im} T = \mathbb{R}^2$ not surjective, $\operatorname{Im} T \neq \mathbb{R}^3$

Activity A.4.6 *(∼15 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of *equivalent* statements: one group that means T is **injective**, and one group that means T is **surjective**.

(a) The kernel of T is trivial: $\ker T = \{\mathbf{0}\}$.

(b) The columns of A span $\mathbb{R}^m$.

(c) The columns of A are linearly independent.

(d) Every column of $\operatorname{RREF}(A)$ has a pivot.

(e) Every row of $\operatorname{RREF}(A)$ has a pivot.

(f) The image of T equals its codomain, i.e. $\operatorname{Im} T = \mathbb{R}^m$.

(g) The system of linear equations given by the augmented matrix $\left[A \mid \mathbf{b}\right]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$.

(h) The system of linear equations given by the augmented matrix $\left[A \mid \mathbf{0}\right]$ has exactly one solution.

(Instructor Note:) This activity may be ran as a card sort.

Observation A.4.7 The easiest way to show that the linear map with standard matrix A is injective is to show that $\operatorname{RREF}(A)$ has a pivot in each column.

The easiest way to show that the linear map with standard matrix A is surjective is to show that $\operatorname{RREF}(A)$ has a pivot in each row.

Activity A.4.8 *(∼3 min)* What can you immediately conclude (i.e. without computing a RREF) about the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ -3 & 3 \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

d) Its standard matrix has more rows than columns, so T is surjective.

Activity A.4.9 *(∼2 min)* What can you immediately conclude (i.e. without computing a RREF) about the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 4 \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

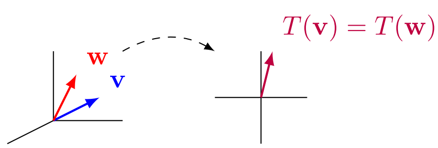
c) Its standard matrix has more rows than columns, so T is not surjective.

d) Its standard matrix has more rows than columns, so T is surjective.

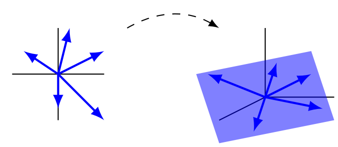
Fact A.4.10 The following are true for any linear map $T : V \to W$:

- If $\dim(V) > \dim(W)$, then T is not injective.
- If $\dim(V) < \dim(W)$, then T is not surjective.

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase the dimension of its image.



not injective, $3 > 2$


not surjective, $2 < 3$

But dimension arguments **cannot** be used to prove a map **is** injective or surjective.

Activity A.4.11 *(∼5 min)* Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A is both injective and surjective (we call such maps **bijective**).
Part 1: How many pivot columns must A have?
Part 2: How many pivot rows must A have?
Part 3: What can you conclude about m and n?

Activity A.4.12 *(∼5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

(a) The columns of A form a basis for $\mathbb{R}^n$

(b) $\mathrm{RREF}(A)$ is the identity matrix.

(c) The system of linear equations given by the augmented matrix $\left[A \mid \mathbf{b}\right]$ has exactly one solution for each $\mathbf{b} \in \mathbb{R}^n$.

Observation A.4.13 The easiest way to show that the linear map with standard matrix A is bijective is to show that $\mathrm{RREF}(A)$ is the identity matrix.

Activity A.4.14 *(∼3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.15 *(∼3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.16 *(∼3 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.17 *(∼3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Module M: Understanding Matrices Algebraically

Standards for this Module

What algebraic structure do matrices have? At the end of this module, students will be able to...

M1. Matrix Multiplication. ... multiply matrices.

M2. Invertible Matrices. ... determine if a square matrix is invertible or not.

M3. Matrix inverses. ... compute the inverse matrix of an invertible matrix.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers.
- Identify the domain and codomain of linear transformations.
- Find the matrix corresponding to a linear transformation and compute the image of a vector given a standard matrix **A2**
- Determine if a linear transformation is injective and/or surjective **A3**
- Interpret the ideas of injectivity and surjectivity in multiple ways.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Function composition (Khan Academy): `http://bit.ly/2wkz7f3`
- Domain and codomain: `https://www.youtube.com/watch?v=BQMyeQOLvpg`
- Interpreting injectivity and surjectivity in many ways: `https://www.youtube.com/watch?v=WpUv72Y6D10`

Section M.1

Activity M.1.1 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity M.1.2 *(∼3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity M.1.3 *(∼2 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What size will the standard matrix of $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$ be? (Rows × Columns)

(a) 4×3

(b) 4×2

(c) 3×4

(d) 3×2

(e) 2×4

(f) 2×3

Activity M.1.4 *(~15 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\mathbf{e}_1) = S(T(\mathbf{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\mathbf{e}_2)$.
Part 3: Compute $(S \circ T)(\mathbf{e}_3)$.
Part 4: Find the 4×3 standard matrix of $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$.

Definition M.1.5 We define the **product** AB of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For the previous activity, S had a 4×2 matrix and T had a 2×3 matrix, so $S \circ T$ had a 4×3 standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= [(S \circ T)(\mathbf{e}_1) (S \circ T)(\mathbf{e}_2) (S \circ T)(\mathbf{e}_3)] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

Activity M.1.6 *(~10 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.
Find the standard matrix AB of $S \circ T$.

Activity M.1.7 *(~5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.
Find the standard matrix BA of $T \circ S$.

Activity M.1.8 *(~10 min)* Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be given by the matrix $B = \begin{bmatrix} 3 & 2 & 5 & -4 \\ -1 & -3 & 1 & 2 \end{bmatrix}$ and let $S : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ -4 & 2 \end{bmatrix}$. Compute AB, the standard matrix of the composition $S \circ T$.

Observation M.1.9 Note that an $\mathbb{R}^n$ vector acts exactly the same as an $n \times 1$ matrix, so we will use them interchangablely, as follows.

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix} \qquad X = \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \mathbf{b} = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$$

So we may study the linear system

$$\begin{aligned} 3x + y - z &= 5 \\ 2x + 4z &= -7 \\ -x + 3y + 5z &= 2 \end{aligned}$$

as both a vector equation $A\mathbf{x} = \mathbf{b}$ and a matrix equation $AX = B$:

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$$

Section M.2

Observation M.2.1 Recall that if $T : \mathbb{R}^n \to \mathbb{R}^k$ is a linear map with standard matrix $B \in M_{k,n}$ and $S : \mathbb{R}^k \to \mathbb{R}^m$ is a linear map with standard matrix $A \in M_{m,k}$, the product matrix $AB \in M_{m,n}$ is defined to be the standard matrix of the composition map

$$S \circ T : \mathbb{R}^n \to \mathbb{R}^m.$$

Activity M.2.2 *(~5 min)* Matrix multiplication only makes sense if the first matrix has as many columns as the second matrix has rows. Label each of these matrices with **rows** $\times$ **columns**, and then figure out which of the products AB, AC, BA, BC, CA, CB can actually be computed.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Activity M.2.3 *(~10 min)* Let $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Compute the product BA.

Activity M.2.4 *(~5 min)* Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix I such that $IA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Definition M.2.5 The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

Fact M.2.6 For any square matrix A, $IA = AI = A$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Activity M.2.7 *(∼20 min)* Each row operation can be interpreted as a type of matrix multiplication.
Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5(1) & 7+5(1) & -1+5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Fact M.2.8 If R is the result of applying a row operation to I, then RA is the result of applying the same row operation to A.
This means that for any matrix A, we can find a series of matrices $R_1, \dots, R_k$ corresponding to the row operations such that

$$R_1 R_2 \cdots R_k A = \mathrm{RREF}(A).$$

That is, row reduction can be thought of as the result of matrix multiplication.

Section M.3

Activity M.3.1 *(∼15 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into three groups of statements: a group that means T is **injective**, a group that means T is **surjective**, and a group that means T is **bijective**.

(a) $AX = B$ has a solution for all $m \times 1$ matrices B

(b) $AX = B$ has a unique solution for all $m \times 1$ matrices B

(c) $AX = 0$ has a unique solution.

(d) The columns of A span $\mathbb{R}^m$

(e) The columns of A are linearly independent

(f) The columns of A are a basis of $\mathbb{R}^m$

(g) Every column of RREF(A) has a pivot

(h) Every row of RREF(A) has a pivot

(i) $m = n$ and RREF$(A) = I$

Definition M.3.2 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with standard matrix A.

- If T is a bijection and B is any $\mathbb{R}^n$ vector, then $T(X) = AX = B$ has a unique solution X.
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by setting $T^{-1}(B) = X$ to be this unique solution.
- Let A^{-1} be the standard matrix for T^{-1}. We call A^{-1} the **inverse matrix** of A, so we also say that A is **invertible**.

Activity M.3.3 *(∼20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(X) = \mathbf{e}_1$ (or in matrix form, $AX = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$).

Part 2: Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.

Part 3: Solve $T(X) = \mathbf{e}_2$ to find $T^{-1}(\mathbf{e}_2)$.

Part 4: Solve $T(X) = \mathbf{e}_3$ to find $T^{-1}(\mathbf{e}_3)$.

Part 5: Compute A^{-1}, the standard matrix for T^{-1}.

Observation M.3.4 We could have solved these three systems simultaneously by row reducing the matrix $[A \,|\, I]$ at once.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right]$$

Activity M.3.5 *(∼5 min)* Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A \,|\, I]$.

Activity M.3.6 *(∼5 min)* Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

Observation M.3.7 An $n \times n$ matrix A is invertible if and only if $\mathrm{RREF}(A) = I_n$.

Activity M.3.8 *(∼10 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

Part 2: If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, find the 2×2 matrix

$$A^{-1}A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

Observation M.3.9 $T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation T. Therefore $A^{-1}A = AA^{-1} = I$ is the identity matrix for any invertible matrix A.

Module G: Geometry of Linear Maps

Standards for this Module

How can we understand linear maps geometrically? At the end of this module, students will be able to...

G1. Row operations. ... represent a row operation as matrix multiplication, and compute how the operation affects the determinant.

G2. Determinants. ... compute the determinant of a square matrix.

G3. Eigenvalues. ... find the eigenvalues of a 2×2 matrix.

G4. Eigenvectors. ... find a basis for the eigenspace of a square matrix associated with a given eigenvalue.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **A1**.
- Recall and use the definition of a linear transformation **A2**.
- Find all roots of quadratic polynomials (including complex ones).
- Interpret the statement "A is an invertible matrix" in many equivalent ways in different contexts.

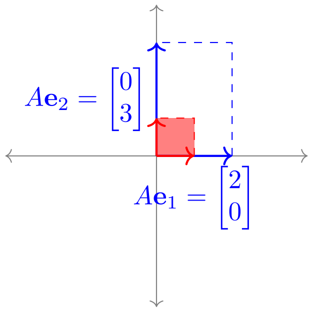
Readiness Assurance Resources

The following resources will help you prepare for this module.

- Finding the area of a parallelogram (Khan Academy): `http://bit.ly/2BO5iWx`
- Factoring quadratics (Khan Academy): `http://bit.ly/1XjfbV2`
- Factoring quadratics using area models (Youtube): `https://youtu.be/Aa-v1EK7DR4`
- Finding complex roots of quadratics (Youtube): `https://www.youtube.com/watch?v=2yBhDsNE0wg`

Section G.1

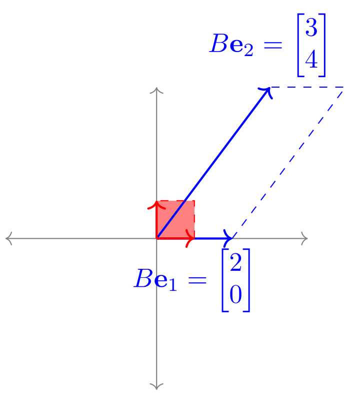
Activity G.1.1 *(~5 min)* The image below illustrates how the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.



(a) What are the lengths of $A\mathbf{e}_1$ and $A\mathbf{e}_2$?

(b) What is the area of the transformed unit square?

Activity G.1.2 *(~5 min)* The image below illustrates how the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$. transforms the unit square.



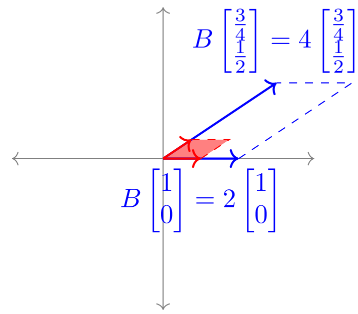
(a) What are the lengths of $B\mathbf{e}_1$ and $B\mathbf{e}_2$?

(b) What is the area of the transformed unit square?

Observation G.1.3 It is possible to find two nonparallel vectors that are scaled but not rotated by the linear map given by B.

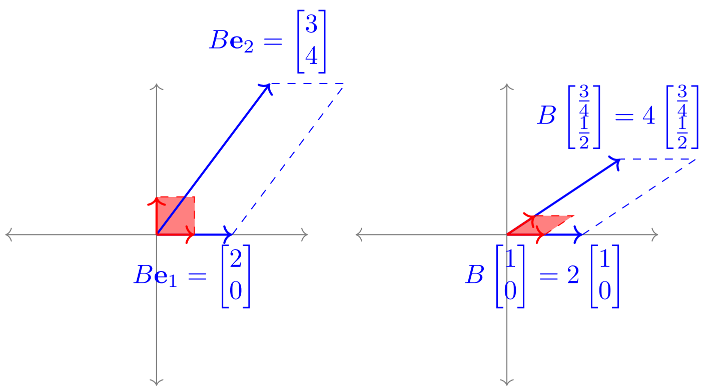
$$B\mathbf{e}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\mathbf{e}_1$$

$$B \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$



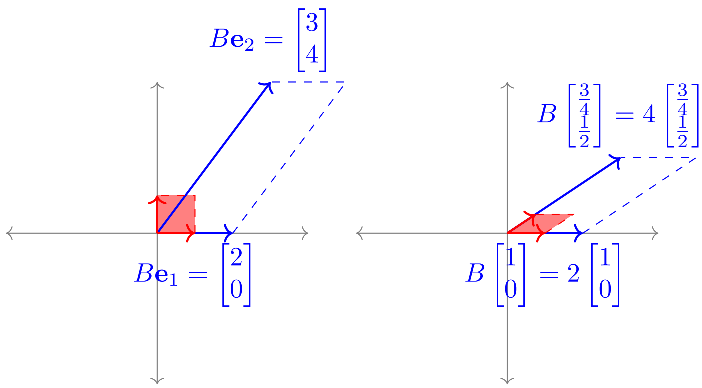
The process for finding such vectors will be covered later in this module.

Observation G.1.4 Notice that while a linear map can transform vectors in various ways, linear maps always transform parallelograms into parallelograms, and these areas are always transformed by the same factor: in the case of $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, this factor is 8.

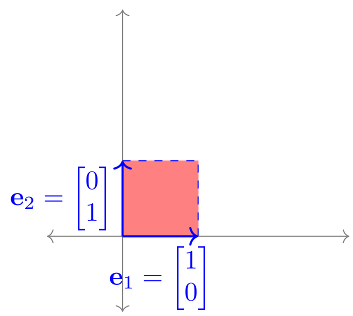


Since this change in area is always the same for a given linear map, it will be equal to the value of the transformed unit square (which begins with area 1).

Remark G.1.5 We will define the **determinant** of a square matrix A, or $\det(A)$ for short, to be the factor by which A scales areas. In order to figure out how to compute it, we first figure out the properties it must satisfy.



Activity G.1.6 *(∼2 min)* The transformation of the unit square by the standard matrix $[\mathbf{e}_1\ \mathbf{e}_2] = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$ is illustrated below. What is $\det([\mathbf{e}_1\ \mathbf{e}_2]) = \det(I)$, the area of the transformed unit square shown here?



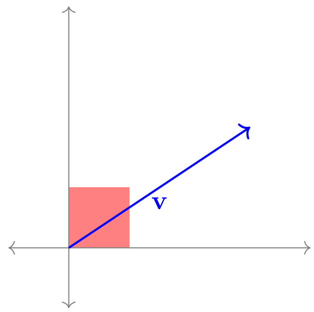
a) 0

b) 1

c) 2

d) Cannot be determined

Activity G.1.7 *(~2 min)* The transformation of the unit square by the standard matrix $[\mathbf{v}\ \ \mathbf{v}]$ is illustrated below: both $T(\mathbf{e}_1) = T(\mathbf{e}_2) = \mathbf{v}$. What is $\det([\mathbf{v}\ \ \mathbf{v}])$, the area of the transformed unit square shown here?



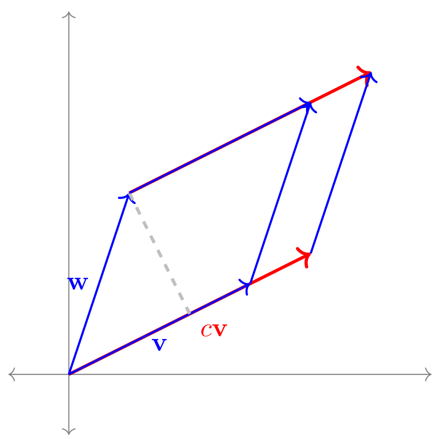
a) 0

b) 1

c) 2

d) Cannot be determined

Activity G.1.8 *(~5 min)* The transformations of the unit square by the standard matrices $[\mathbf{v}\ \ \mathbf{w}]$ and $[c\mathbf{v}\ \ \mathbf{w}]$ are illustrated below. How are $\det([\mathbf{v}\ \ \mathbf{w}])$ and $\det([c\mathbf{v}\ \ \mathbf{w}])$ related?

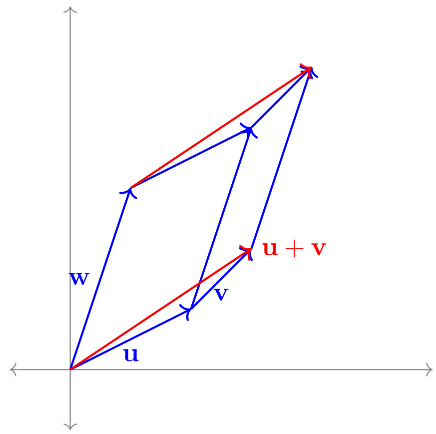


a) $\det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

b) $c + \det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

c) $c\det([\mathbf{v}\ \ \mathbf{w}]) = \det([c\mathbf{v}\ \ \mathbf{w}])$

Activity G.1.9 *(∼5 min)* The transformations of unit squares by the standard matrices $[\mathbf{u}\ \ \mathbf{w}]$, $[\mathbf{v}\ \ \mathbf{w}]$ and $[\mathbf{u}+\mathbf{v}\ \ \mathbf{w}]$ are illustrated below. How is $\det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$ related to $\det([\mathbf{u}\ \ \mathbf{w}])$ and $\det([\mathbf{v}\ \ \mathbf{w}])$?



a) $\det([\mathbf{u}\ \ \mathbf{w}]) = \det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

b) $\det([\mathbf{u}\ \ \mathbf{w}]) + \det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

c) $\det([\mathbf{u}\ \ \mathbf{w}])\det([\mathbf{v}\ \ \mathbf{w}]) = \det([\mathbf{u}+\mathbf{v}\ \ \mathbf{w}])$

Definition G.1.10 The **determinant** is the unique function $\det : M_{n,n} \to \mathbb{R}$ satisfying these properties:

P1: $\det(I) = 1$

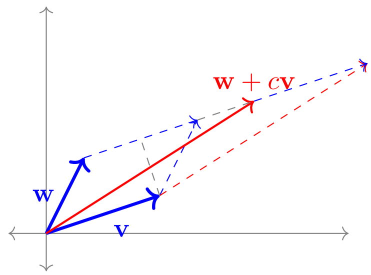
P2: $\det(A) = 0$ whenever two columns of the matrix are identical.

P3: $\det[\cdots\ \ c\mathbf{v}\ \ \cdots] = c\det[\cdots\ \ \mathbf{v}\ \ \cdots]$, assuming no other columns change.

P4: $\det[\cdots\ \ \mathbf{v}+\mathbf{w}\ \ \cdots] = \det[\cdots\ \ \mathbf{v}\ \ \cdots] + \det[\cdots\ \ \mathbf{w}\ \ \cdots]$, assuming no other columns change.

Note that these last two properties together can be phrased as “The determinant is linear in each column.”

Observation G.1.11 The determinant must also satisfy other properties. Consider $\det([\mathbf{v} + c\mathbf{w} \quad \mathbf{w}])$ and $\det([\mathbf{v} \quad \mathbf{w}])$.



The base of both parallelograms is $\mathbf{v}$, while the height has not changed, so the determinant does not change either. This can also be proven using the other properties of the determinant:

$$\begin{aligned}
\det([\mathbf{v} + c\mathbf{w} \quad \mathbf{w}]) &= \det([\mathbf{v} \quad \mathbf{w}]) + \det([c\mathbf{w} \quad \mathbf{w}]) \\
&= \det([\mathbf{v} \quad \mathbf{w}]) + c\det([\mathbf{w} \quad \mathbf{w}]) \\
&= \det([\mathbf{v} \quad \mathbf{w}]) + c \cdot 0 \\
&= \det([\mathbf{v} \quad \mathbf{w}])
\end{aligned}$$

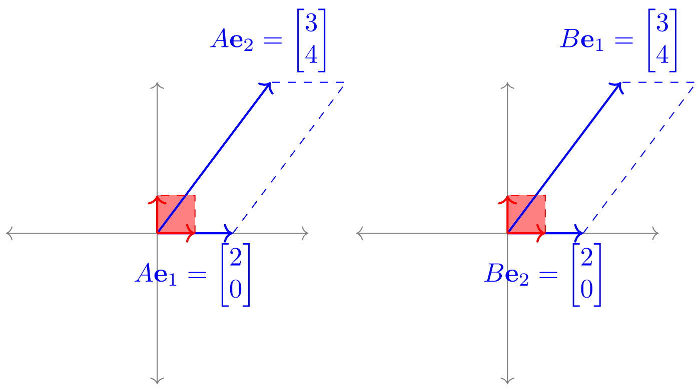
Observation G.1.12 Columns may be swapped by adding/subtracting columns from one another, which we've just seen doesn't change the determinant.

$$\begin{aligned}
\det([\mathbf{v} \quad \mathbf{w}]) &= \det([\mathbf{v} + \mathbf{w} \quad \mathbf{w}]) \\
&= \det([\mathbf{v} + \mathbf{w} \quad \mathbf{w} - (\mathbf{v} + \mathbf{w})]) \\
&= \det([\mathbf{v} + \mathbf{w} \quad -\mathbf{v}]) \\
&= \det([\mathbf{v} + \mathbf{w} - \mathbf{v} \quad -\mathbf{v}]) \\
&= \det([\mathbf{w} \quad -\mathbf{v}]) \\
&= -\det([\mathbf{w} \quad \mathbf{v}])
\end{aligned}$$

So swapping two columns results in a negation of the determinant. Therefore, determinants represent a *signed* area, since they are not always positive.

Remark G.1.13 Swapping columns may be thought of as a reflection, which is represented by a negative determinant. For example, the following matrices transform the unit square into the same parallelogram, but the second matrix reflects its orientation.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}$$



Fact G.1.14 To summarize, we've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant in the following way:

(a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c\det([\cdots \ \mathbf{v} \ \cdots]) = \det([\cdots \ c\mathbf{v} \ \cdots])$$

(b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \ \mathbf{v} \ \cdots \ \mathbf{w} \ \cdots]) = -\det([\cdots \ \mathbf{w} \ \cdots \ \mathbf{v} \ \cdots])$$

(c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \ \mathbf{v} \ \cdots \ \mathbf{w} \ \cdots]) = \det([\cdots \ \mathbf{v}+c\mathbf{w} \ \cdots \ \mathbf{w} \ \cdots])$$

Activity G.1.15 *(∼5 min)* The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. By what factor does the transformation given by the standard matrix AB scale areas?

(a) 1

(b) 7

(c) 12

(d) Cannot be determined

Fact G.1.16 Since the transformation given by the standard matrix AB is obtained by applying the transformations given by A and B, it follows that

$$\det(AB) = \det(A)\det(B)$$

Remark G.1.17 Recall that row operations may be produced by matrix multiplication.

- Multiply the first row of A by c: $\begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$
- Swap the first and second row of A: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$
- Add c times the third row to the first row of A: $\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Fact G.1.18 The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row: $\det \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c$
- Swapping rows: $\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1$
- Adding a row multiple to another row: $\det \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & c - 1c \\ 0 & 1 & 0 - 0c \\ 0 & 0 & 1 - 0c \end{bmatrix} = \det(I) = 1$

Activity G.1.19 *(∼5 min)* Consider the row operation $R_1 + 4R_3 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1+4(7) & 2+4(8) & 3+4(9) \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Find $\det R$ by comparing with the previous slide.

(c) If $C \in M_{3,3}$ is a matrix with $\det(C) = -3$, find

$$\det(RC) = \det(R)\det(C).$$

Activity G.1.20 *(∼5 min)* Consider the row operation $R_1 \leftrightarrow R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to I.

(b) If $C \in M_{3,3}$ is a matrix with $\det(C) = 5$, find $\det(RC)$.

Activity G.1.21 *(∼5 min)* Consider the row operation $3R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 3(4) & 3(5) & 3(6) \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det(C) = -7$, find $\det(RC)$.

Section G.2

Remark G.2.1 Recall that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying columns by scalars:

$$\det([\cdots \; c\mathbf{v} \; \cdots]) = c\det([\cdots \; \mathbf{v} \; \cdots])$$

(b) Swapping two columns:

$$\det([\cdots \; \mathbf{v} \; \cdots \; \mathbf{w} \; \cdots]) = -\det([\cdots \; \mathbf{w} \; \cdots \; \mathbf{v} \; \cdots])$$

(c) Adding a multiple of a column to another column:

$$\det([\cdots \; \mathbf{v} \; \cdots \; \mathbf{w} \; \cdots]) = \det([\cdots \; \mathbf{v}+c\mathbf{w} \; \cdots \; \mathbf{w} \; \cdots])$$

Remark G.2.2 The determinants of row operation matrices may be computed by manipulating columns to reduce each matrix to the identity:

- Scaling a row: $\det \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c$
- Swapping rows: $\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1$
- Adding a row multiple to another row: $\det \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & c-1c \\ 0 & 1 & 0-0c \\ 0 & 0 & 1-0c \end{bmatrix} = \det(I) = 1$

Fact G.2.3 Thus we can also use row operations to simplify determinants:

1. Multiplying rows by scalars: $\det\begin{bmatrix}\vdots\\cR\\\vdots\end{bmatrix} = c\det\begin{bmatrix}\vdots\\R\\\vdots\end{bmatrix}$

2. Swapping two rows: $\det\begin{bmatrix}\vdots\\R\\\vdots\\S\\\vdots\end{bmatrix} = -\det\begin{bmatrix}\vdots\\S\\\vdots\\R\\\vdots\end{bmatrix}$

3. Adding multiples of rows to other rows: $\det\begin{bmatrix}\vdots\\R\\\vdots\\S\\\vdots\end{bmatrix} = \det\begin{bmatrix}\vdots\\R+cS\\\vdots\\S\\\vdots\end{bmatrix}$

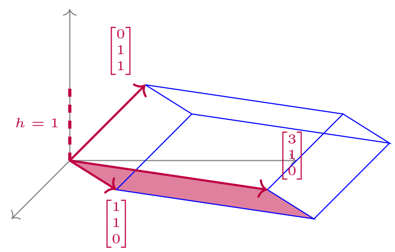
Observation G.2.4 So we may compute the determinant of $\begin{bmatrix}2 & 4\\2 & 3\end{bmatrix}$ by manipulating its rows/columns to reduce the matrix to I:

$$\begin{aligned}
\det\begin{bmatrix}2 & 4\\2 & 3\end{bmatrix} &= 2\det\begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}\\
&= 2\det\begin{bmatrix}1 & 2\\2-2(1) & 3-2(2)\end{bmatrix} = 2\det\begin{bmatrix}1 & 2\\0 & -1\end{bmatrix}\\
&= 2(-1)\det\begin{bmatrix}1 & -2\\0 & +1\end{bmatrix} = -2\det\begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}\\
&= -2\det\begin{bmatrix}1+2(0) & -2+2(1)\\0 & 1\end{bmatrix} = -2\det\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\\
&= -2\det I = -2(1) = -2
\end{aligned}$$

Remark G.2.5 While a formula might make this 2×2 determinant easier, memorizing a formula for 3×3, 4×4, or larger determinants is difficult. So we will start by focusing on how to use row/column operations on 3×3 determinants.

But rather than always turning the original matrix into I, let's figure out how to turn the determinant of a larger matrix into the determinant of a smaller matrix.

Activity G.2.6 *(∼5 min)* The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.



Recall that for this solid $V = Bh$, where h is the height of the solid and B is the area of its parallelogram base. So what must its volume be?

(a) $\det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$

(d) $\det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Fact G.2.7 If row i contains all zeros except for a 1 on the main (upper-left to lower-right) diagonal, then both column and row i may be removed without changing the value of the determinant.

$$\det \begin{bmatrix} 3 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 5 & 0 & 11 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 3 \\ -1 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$$

Since row and column operations affect the determinants in the same way, the same technique works for a column of all zeros except for a 1 on the main diagonal.

$$\det \begin{bmatrix} 3 & 0 & -1 & 5 \\ 2 & 1 & 4 & 0 \\ -1 & 0 & 1 & 11 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 3 & -1 & 5 \\ -1 & 1 & 11 \\ 3 & 0 & 1 \end{bmatrix}$$

Activity G.2.8 *(∼5 min)* Remove an appropriate row and column of $\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 12 \\ 3 & 2 & -1 \end{bmatrix}$ to simplify the determinant to a 2×2 determinant.

Activity G.2.9 *(~5 min)* Simplify $\det \begin{bmatrix} 0 & 3 & -2 \\ 2 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

- Factor out a 2 from a column.
- Swap rows or columns to put a 1 on the main diagonal.

Activity G.2.10 *(~5 min)* Simplify $\det \begin{bmatrix} 4 & -2 & 2 \\ 3 & 1 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

- Use row/column operations to create two zeroes in the same row or column.
- Factor/swap as needed to get a row/column of all zeroes except a 1 on the main diagonal.

Observation G.2.11 Using row/column operations, you can introduce zeros and reduce dimension to whittle down the determinant of a large matrix to a determinant of a smaller matrix.

$$\begin{aligned}
\det \begin{bmatrix} 4 & 3 & 0 & 1 \\ 2 & -2 & 4 & 0 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} &= \det \begin{bmatrix} 4 & 3 & 0 & 1 \\ 6 & -18 & 0 & -20 \\ -1 & 4 & 1 & 5 \\ 2 & 8 & 0 & 3 \end{bmatrix} = \det \begin{bmatrix} 4 & 3 & 1 \\ 6 & -18 & -20 \\ 2 & 8 & 3 \end{bmatrix} \\
&= \cdots = -2 \det \begin{bmatrix} 1 & 3 & 4 \\ 0 & 21 & 43 \\ 0 & -1 & -10 \end{bmatrix} = -2 \det \begin{bmatrix} 21 & 43 \\ -1 & -10 \end{bmatrix} \\
&= \cdots = -2 \det \begin{bmatrix} -167 & 21 \\ 0 & 1 \end{bmatrix} = -2 \det [-167] \\
&= -2(-167) \det(I) = 334
\end{aligned}$$

Activity G.2.12 *(~10 min)* Compute $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$ by using any combination of row/column operations.

Observation G.2.13 Another option is to take advantage of the fact that the determinant is linear in each row or column. This approach is called **Laplace expansion** or **cofactor expansion**.
For example, since $\begin{bmatrix}1 & 2 & 4\end{bmatrix} = 1\begin{bmatrix}1 & 0 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 1 & 0\end{bmatrix} + 4\begin{bmatrix}0 & 0 & 1\end{bmatrix}$,

$$\begin{aligned}
\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 1 & 2 & 4\end{bmatrix} &= 1\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 1 & 0 & 0\end{bmatrix} + 2\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 0 & 1 & 0\end{bmatrix} + 4\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 0 & 0 & 1\end{bmatrix}\\
&= -1\det\begin{bmatrix}5 & 3 & 2\\ 5 & 3 & -1\\ 0 & 0 & 1\end{bmatrix} - 2\det\begin{bmatrix}2 & 5 & 3\\ -1 & 5 & 3\\ 0 & 0 & 1\end{bmatrix} + 4\det\begin{bmatrix}2 & 3 & 5\\ -1 & 3 & 5\\ 0 & 0 & 1\end{bmatrix}\\
&= -\det\begin{bmatrix}5 & 3\\ 5 & 3\end{bmatrix} - 2\det\begin{bmatrix}2 & 5\\ -1 & 5\end{bmatrix} + 4\det\begin{bmatrix}2 & 3\\ -1 & 3\end{bmatrix}
\end{aligned}$$

Observation G.2.14 Applying Laplace expansion to a 2×2 matrix yields a short formula:

$$\det\begin{bmatrix}a & b\\ c & d\end{bmatrix} = a\det\begin{bmatrix}1 & 0\\ c & d\end{bmatrix} + b\det\begin{bmatrix}0 & 1\\ c & d\end{bmatrix} = a\det\begin{bmatrix}1 & 0\\ c & d\end{bmatrix} - b\det\begin{bmatrix}1 & 0\\ d & c\end{bmatrix} = ad - bc.$$

There are formulas for the determinants of larger matrices, but they can be pretty tedious to use. For example, writing out a formula for a 4×4 determinant would require 24 different terms!

$$\det\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix} = a_{11}(a_{22}(a_{33}a_{44} - a_{43}a_{34}) - a_{23}(a_{32}a_{44} - a_{42}a_{34}) + \dots) + \dots$$

So this is why we either use Laplace expansion or row/column operations directly.

Activity G.2.15 *(~10 min)* Use Laplace expansion to compute $\det\begin{bmatrix}2 & 2 & 1 & 0\\ 0 & 3 & 2 & -1\\ 3 & 2 & 0 & 3\\ 0 & -3 & 2 & -2\end{bmatrix}$.

Activity G.2.16 *(~5 min)* Based on what we've done today, which technique is easier for computing determinants?

(a) Memorizing formulas.

(b) Using row/column operations.

(c) Laplace expansion.

(d) Some other technique (be prepared to describe it).

Activity G.2.17 *(∼10 min)* Use your preferred technique to compute $\det \begin{bmatrix} 4 & -3 & 0 & 0 \\ 1 & -3 & 2 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & -3 & 2 & -2 \end{bmatrix}$.

Section G.3

Activity G.3.1 *(∼5 min)* An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Compute $\det(M)$ and $\det(M^{-1})$ using the formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Fact G.3.2

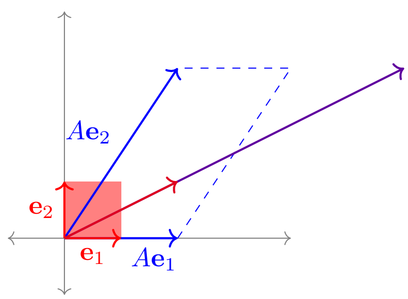
- For every invertible matrix M,

$$\det(M)\det(M^{-1}) = \det(I) = 1$$

  so $\det(M^{-1}) = \frac{1}{\det(M)}$.

- Furthermore, a square matrix M is invertible if and only if $\det(M) \neq 0$.

Observation G.3.3 Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.



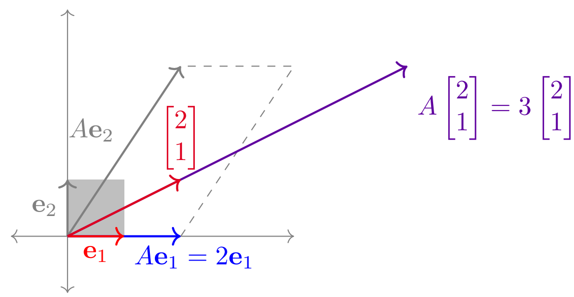
It is easy to see geometrically that

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is less obvious (but easily checked once you find it) that

$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Definition G.3.4 Let $A \in M_{n,n}$. An **eigenvector** for A is a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x}$ is parallel to $\mathbf{x}$.



In other words, $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. If $\mathbf{x} \neq \mathbf{0}$, then we say $\mathbf{x}$ is a **nontrivial eigenvector** and we call this λ an **eigenvalue** of A.

Activity G.3.5 *(~5 min)* Finding the eigenvalues λ that satisfy

$$A\mathbf{x} = \lambda\mathbf{x} = \lambda(I\mathbf{x}) = (\lambda I)\mathbf{x}$$

for some nontrivial eigenvector $\mathbf{x}$ is equivalent to finding nonzero solutions for the matrix equation

$$A\mathbf{x} - (\lambda I)\mathbf{x} = \mathbf{0}.$$

Which of the following must be true for any eigenvalue?

(a) The kernel of the transformation with standard matrix $A - \lambda I$ must contain the zero vector, so $A - \lambda I$ is invertible.

(b) The kernel of the transformation with standard matrix $A - \lambda I$ must contain a nonzero vector, so $A - \lambda I$ is not invertible.

(c) The image of the transformation with standard matrix $A - \lambda I$ must contain the zero vector, so $A - \lambda I$ is invertible.

(d) The image of the transformation with standard matrix $A - \lambda I$ must contain a nonzero vector, so $A - \lambda I$ is invertible.

Fact G.3.6 The eigenvalues λ for a matrix A are the values that make $A - \lambda I$ non-invertible.

Thus the eigenvalues λ for a matrix A are the solutions to the equation

$$\det(A - \lambda I) = 0.$$

Definition G.3.7 The expression $\det(A - \lambda I)$ is called **characteristic polynomial** of A.

For example, when $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we have

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

Thus the characteristic polynomial of A is

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(3) = \lambda^2 - 5\lambda - 2$$

and its eigenvalues are the solutions to $\lambda^2 - 5\lambda - 2 = 0$.

Activity G.3.8 *(~10 min)* Compute $\det(A-\lambda I)$ to find the characteristic polynomial of $A = \begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$.

Activity G.3.9 *(~10 min)* Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$.
Part 1: Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A.
Part 2: Factor this characteristic polynomial to determine the eigenvalues of A.

Activity G.3.10 *(~10 min)* Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix}$.

Activity G.3.11 *(~10 min)* It's possible to show that -2 is an eigenvalue for $\begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 9 \\ 3 & 0 & 4 \end{bmatrix}$.

Compute the kernel of the transformation with standard matrix

$$A - (-2)I = \begin{bmatrix} ? & 4 & -2 \\ 2 & ? & 9 \\ 3 & 0 & ? \end{bmatrix}$$

to find all the eigenvectors $\mathbf{x}$ such that $A\mathbf{x} = -2\mathbf{x}$.

Definition G.3.12 Since the kernel of a linear map is a subspace of $\mathbb{R}^n$, and the kernel obtained from $A-\lambda I$ contains all the eigenvectors associated with λ, we call this kernel the **eigenspace** of A associated with λ.

Activity G.3.13 *(~10 min)* Find a basis for the eigenspace for the matrix $\begin{bmatrix} 3 & -6 & 1 \\ -1 & 4 & 2 \\ 3 & -9 & 4 \end{bmatrix}$ associated with the eigenvalue -1.

Module P: Applications of Linear Algebra

Section P.1

Definition P.1.1 In chemistry, we learn that when the two substances

- Hydrochloric acid HCl (formed from 1 H and 1 Cl atom)
- Sodium Na (formed from 1 Na atom)

react, their atoms rearrange to form the substances

- Salt $NaCl$ (formed from 1 Na and 1 Cl atom)
- Hydrogen gas H_2 (formed from 2 H atoms).

This may be represented by the **chemical equation**

$$? \, HCl + \, ? \, Na \to \, ? \, NaCl + \, ? \, H_2$$

where each $?$ represents the amount of that substance before/after the reaction.

Activity P.1.2 *(~5 min)* The **law of conservation of mass** states that the quantity of atoms before and after a chemical reaction must remain the same.

Find positive integers so that both sides of the chemical equation represent the same amount of matter:

$$? \, HCl + \, ? \, Na \to \, ? \, NaCl + \, ? \, H_2$$

Definition P.1.3 A chemical equation is **balanced** if the given quantities of each substance before and after the reaction are equal and minimal positive integers:

$$2HCl + 2Na \to 2NaCl + H_2$$

Observation P.1.4 For example, the following equation isn't balanced because all the integers may be divided by three:

$$6HCl + 6Na \to 6NaCl + 3H_2$$

Therefore if a chemical equation can be balanced, there is exactly one correct solution.

Activity P.1.5 *(~15 min)* Balance the following chemical equations:

$$? \, Fe + \, ? \, Cl_2 \to \, ? \, FeCl_3$$

$$? \, Ca(OH)_2 + \, ? \, H_3PO_4 \to \, ? \, Ca_3(PO_4)_2 + \, ? \, H_2O$$

$$? \, K_4Fe(CN)_6 + \, ? \, H_2SO_4 + \, ? \, H_2O \to \, ? \, K_2SO_4 + \, ? \, FeSO_4 + \, ? \, (NH_4)_2SO_4 + \, ? \, CO$$

(Note that $(NH_4)_2SO_4$ represents 2 N, 8 H, 1 S, and 4 O.)

Observation P.1.6 For the purposes of balancing chemical equations, the set

$$L = \{\mathbf{A} \mid \mathbf{A} \text{ is combination of elements}\}$$

may be treated as a kind of **vector space**. This means that balancing the chemical equation

$$?\,Fe + \,?\,Cl_2 \to \,?\,FeCl_3$$

may be acheived by finding a solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to the vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl}).$$

Activity P.1.7 *(~5 min)* To solve the vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl})$$

we are only concerned with the subspace $W = \operatorname{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ of L. Since the element **Fe** cannot be created from the element **Cl** in a chemical reaction and vice versa, the set $\{\mathbf{Cl}, \mathbf{Fe}\}$:

a) spans W, but is linearly dependent.

b) is linearly independent, but does not span W.

c) is a basis for W.

Observation P.1.8 $W = \operatorname{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ is a two-dimensional subspace of L, so as usual we'd rather work with its isomorphic Euclidean space $\mathbb{R}^2$.

Thus we should assign a transformation of bases such as:

$$\mathbf{Cl} \leftrightarrow \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{Fe} \leftrightarrow \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Activity P.1.9 *(~10 min)* Rewrite the $W = \operatorname{span}\{\mathbf{Cl}, \mathbf{Fe}\}$ vector equation

$$x\mathbf{Fe} + y(2\mathbf{Cl}) = z(\mathbf{Fe} + 3\mathbf{Cl})$$

using the transformation of bases

$$\mathbf{Cl} \leftrightarrow \mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix} \qquad \mathbf{Fe} \leftrightarrow \mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

and show how it may be simplifed to

$$x\begin{bmatrix}0\\1\end{bmatrix} + y\begin{bmatrix}2\\0\end{bmatrix} - z\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Activity P.1.10 *(~10 min)* Consider the Euclidean vector equation

$$x\begin{bmatrix}0\\1\end{bmatrix} + y\begin{bmatrix}2\\0\end{bmatrix} - z\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Part 1: Find its solution set.
Part 2: Find a vector in the solution space that consists of minimal positive integers.
Part 3: Balance the chemical equation

$$?\,Fe + \,?\,Cl_2 \to \,?\,FeCl_3.$$

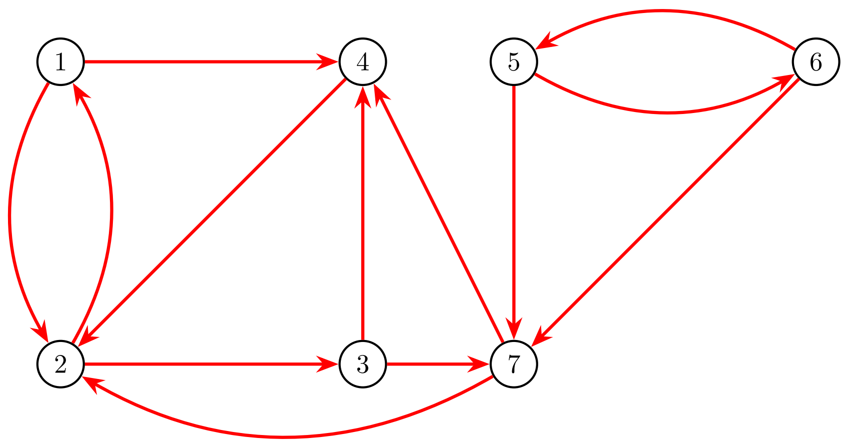
Activity P.1.11 *(~10 min)* Balance the chemical equation

$$?\,Ca(OH)_2 + \,?\,H_3PO_4 \to \,?\,Ca_3(PO_4)_2 + \,?\,H_2O$$

by first converting it into an $\mathbb{R}^4$ vector equation and finding its solution set.

Section P.2

Activity P.2.1 *(∼10 min)*
A \$700,000,000,000 Problem:
In the picture below, each circle represents a webpage, and each arrow represents a link from one page to another.

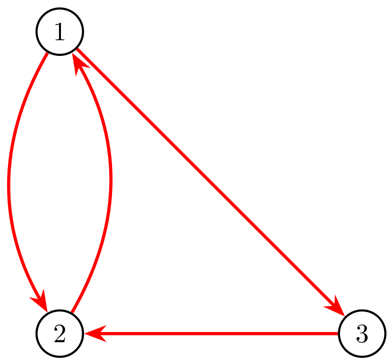


Based on how these pages link to each other, write a list of the 7 webpages in order from most imporant to least important.

Observation P.2.2 The \$700,000,000,000 Idea:
Links are endorsements.

1. A webpage is important if it is linked to (endorsed) by important pages.
2. A webpage distributes its importance equally among all the pages it links to (endorses).

Example P.2.3 Consider this small network with only three pages. Let x_1, x_2, x_3 be the importance of the three pages respectively.

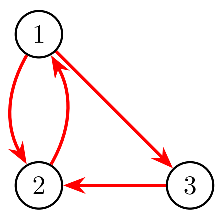


1. x_1 splits its endorsement in half between x_2 and x_3
2. x_2 sends all of its endorsement to x_1
3. x_3 sends all of its endorsement to x_2.

This corresponds to the **page rank system**

$$\begin{aligned} x_2 \qquad &= x_1 \\ \frac{1}{2}x_1 \quad + x_3 &= x_2 \\ \frac{1}{2}x_1 \qquad &= x_3 \end{aligned}$$

Example P.2.4



$$\begin{aligned} x_2 \qquad &= x_1 \\ \frac{1}{2}x_1 \quad + x_3 &= x_2 \\ \frac{1}{2}x_1 \qquad &= x_3 \end{aligned}$$

We can summarize the left hand side of the system by putting its coefficients into a **page rank matrix** $A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$, and store the right hand side of the system as the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.
Thus, computing the imporance of pages on a network is equivalent to solving the matrix equation $A\mathbf{x} = \mathbf{x}$.

Activity P.2.5 *(~5 min)* A **page rank vector** for a page rank matrix A is a vector $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{x}$. This vector describes the relative importance of webpages on the network described by A.
Thus, the $700,000,000,000 problem is what kind of problem?

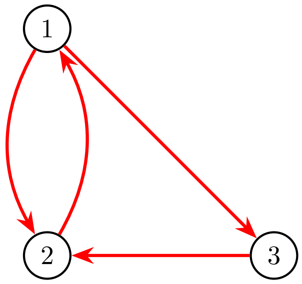
(a) A bijection problem

(b) A calculus problem

(c) A determinant problem

(d) An eigenvector problem

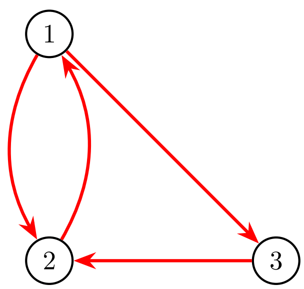
Activity P.2.6 *($\sim$10 min)* Find a page rank vector $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{x}$ (an eigenvector associated to the eigenvalue 1) for the following network's page rank matrix A.



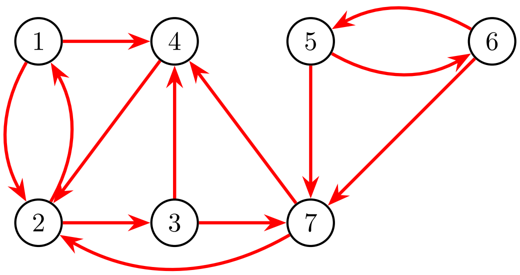
$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Observation P.2.7 Row-reducing $A - I = \begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ yields the basic eigenvector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Therefore, we may conclude that pages 1 and 2 are equally important, and both pages are twice as important as page 3.

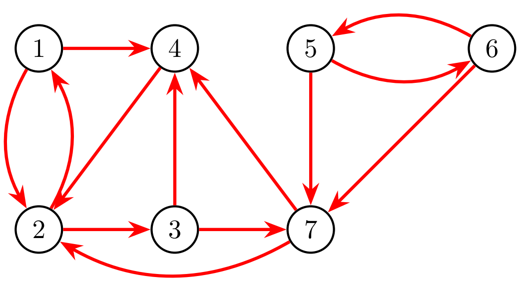


Activity P.2.8 *(∼10 min)* Compute the 7×7 page rank matrix for the following network.



For example, since website 1 distributes its endorsement equally between 2 and 4, the first column is $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

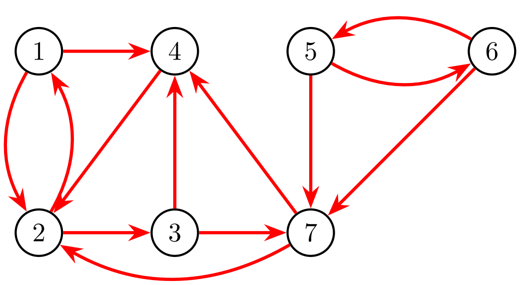
Activity P.2.9 *(∼10 min)* Find a page rank vector for the transition matrix.



$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Which webpage is most important?

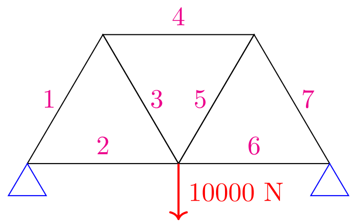
Observation P.2.10 Since a page rank vector for the network is given by $\mathbf{x}$, it's reasonable to consider page 2 as the most important page.



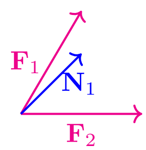
$$\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Section P.3

Observation P.3.1 Consider the truss pictured below with two fixed anchor points and a 10000 N load (assume all triangles are equilateral).

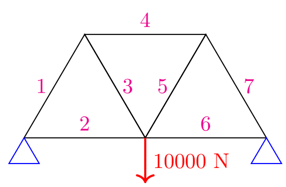


The horizontal and vertical forces must balance at each node. For example, at the bottom left node there are 3 forces acting.

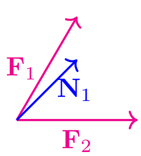


We adhere to the convention that a compression force on a strut is positive, while a negative force represents tension.

Observation P.3.2



We decompose the first node into vertical and horizontal forces:

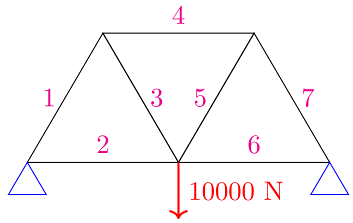


$$\mathbf{F}_1 = F_1 \begin{bmatrix} \cos(60^\circ) \\ \sin(60^\circ) \end{bmatrix}$$

$$\mathbf{N}_1 = \begin{bmatrix} N_{1,h} \\ N_{1,v} \end{bmatrix}$$

$$\begin{aligned} F_1 \sin(60^\circ) + N_{1,v} &= 0 \\ F_1 \cos(60^\circ) + N_{1,h} + F_2 &= 0 \end{aligned}$$

Activity P.3.3 *(~10 min)* Consider the truss pictured below with two fixed anchor points and a 10000 N load (assume all triangles are equilateral).

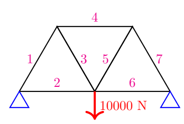


From the bottom left node we obtained 2 equations in the four variables

- F_1 (compression force on strut one)
- $N_{1,v}$ and $N_{1,h}$ (horizontal and vertical components of the normal force from the left anchor)
- F_2 (compression force on strut 2).

Part 1: Determine how many total equations there will be after accounting for all of the nodes, and and list all of the variables. You do not need to actually determine all of the equations.

Activity P.3.4 *(~10 min)*

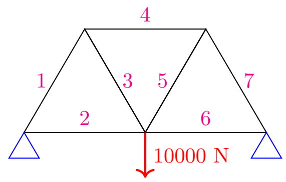


The resulting system is

$$
\begin{aligned}
N_{1,v} + (\sin(60^\circ))F_1 &= 0 \\
N_{1,h} + (\cos(60^\circ))F_1 + F_2 &= 0 \\
-(\sin(60^\circ))F_1 - (\sin(60^\circ)F_3 &= 0 \\
-(\cos(60^\circ))F_1 + (\cos(60^\circ)F_3 + F_4 &= 0 \\
(\sin(60^\circ)F_3 + (\sin(60^\circ))F_5 &= 10000 \\
-F_2 - (\cos(60^\circ)F_3 + (\cos(60^\circ))F_5 + F_6 &= 0 \\
-(\sin(60^\circ))F_5 - (\sin(60^\circ))F_7 &= 0 \\
-F_4 - (\cos(60^\circ))F_5 + (\cos(60^\circ))F_7 &= 0 \\
N_{2,v} + (\sin(60^\circ))F_7 &= 0 \\
N_{2,h} - F_6 - (\cos(60^\circ))F_7 &= 0
\end{aligned}
$$

Solve this system to determine which struts are compressed and which are in tension.

Observation P.3.5



The determined part of the solution is

$$\begin{aligned} N_{1,v} = N_{2,v} &= 5000 \\ F_1 = F_4 = F_7 &= -5882.4 \\ F_3 = F_5 &= 5882.4 \end{aligned}$$

So struts 1,4,7 are in tension, while struts 3 and 5 are compressed.
The forces on struts 2 and 6 (and the horizontal normal forces) are not strictly determined in this setting.