Sample Assessment Exercises

This document contains one exercise and solution for each standard. The goal is to give you an idea of what the exercises might look like, and what the expectations for a complete solution are.

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & 1 & 1 \\ -1 & -4 & 1 & -7 & 0 \\ 0 & 1 & -1 & 0 & -2 \end{array}\right]$$

Solution:

$$\begin{alignedat}{5} 3x_1 &+ 2x_2 & & &+ x_4 &= 1 \\ -x_1 &- 4x_2 &+ x_3 &&- 7x_4 &= 0 \\ & x_2 &- x_3 && &= -2 \end{alignedat}$$

$\square$

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 2 & -1 & -3 \\ 2 & 4 & -1 & -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 2 & -1 & -3 \\ 2 & 4 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 3 & 1 & 2 \\ 2 & 4 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{13}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{5}{3} & -\frac{13}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$\square$

E3. Find the solution set for the following system of linear equations.

$$\begin{alignedat}{3} 2x &+ 4y &+ z &= 5 \\ x &+ 2y & &= 3 \end{alignedat}$$

Solution:

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} 2 & 4 & 1 & 5 \\ 1 & 2 & 0 & 3 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This corresponds to the system

$$\begin{alignedat}{3} x &+ 2y & &= 3 \\ & & z &= -1 \end{alignedat}$$

Since the y-column is a non-pivot column, it is a free variable, so we let $y = a$; then we have

$$\begin{alignedat}{3} x + 2y & & &= 3 \\ y & & &= a \\ & & z &= -1 \end{alignedat}$$

and thus

$$\begin{aligned} x &= 3 - 2a \\ y &= a \\ z &= -1 \end{aligned}$$

So the solution set is

$$\left\{ \begin{bmatrix} 3-2a \\ a \\ -1 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$$

$\square$

V1. Let V be the set of all polynomials, together with the operations $\oplus$ and $\odot$ defined by the following for all polynomials $f(x), g(x)$ and scalars $c \in \mathbb{R}$:

$$\begin{aligned} f(x) \oplus g(x) &= xf(x) + xg(x) \\ c \odot f(x) &= cf(x) \end{aligned}$$

(a) Show that scalar distribution

$$c \odot (f(x) \oplus g(x)) = c \odot f(x) \oplus c \odot g(x)$$

holds.

(b) Show that addition associativity

$$(f(x) \oplus g(x)) \oplus h(x) = f(x) \oplus (g(x) \oplus h(x))$$

fails.

Solution:

(a) Compute

$$\begin{aligned} c \odot (f(x) \oplus g(x)) &= c \odot (xf(x) + xg(x)) \\ &= c\,(xf(x) + xg(x)) \\ &= cxf(x) + cxg(x) \end{aligned}$$

and

$$\begin{aligned} c \odot f(x) \oplus c \odot g(x) &= (cf(x)) \oplus (cg(x)) \\ &= xcf(x) + xcg(x) \end{aligned}$$

Since these are the same, we have shown that $c \odot (f(x) \oplus g(x)) = c \odot f(x) \oplus c \odot g(x)$ holds.

(b) Suppose $f(x) = 1$, $g(x) = 2$, and $h(x) = 3$. Then

$$\begin{aligned} (f(x) \oplus g(x)) \oplus h(x) &= (x + 2x) \oplus 3 \\ &= 3x \oplus 3 \\ &= 3x^2 + 3x \end{aligned}$$

and

$$\begin{aligned} f(x) \oplus (g(x) \oplus h(x)) &= 1 \oplus (2x + 3x) \\ &= 1 \oplus 5x \\ &= x + 5x^2 \end{aligned}$$

Since $3x^2 + 3x \neq x + 5x^2$, we have shown $(f(x) \oplus g(x)) \oplus h(x) = f(x) \oplus (g(x) \oplus h(x))$ fails.

$\square$

V2. Let V be the set of all non-negative real numbers with the operations $\oplus$ and $\odot$ given by, for all $x, y \in V$ and $c \in \mathbb{R}$,

$$x \oplus y = x + y$$
$$c \odot x = |c|x$$

List the 8 defining properties of a vector space, and label each as "TRUE" or "FALSE" as they apply to V. Based on these, conclude whether V is a vector space or not.

Solution:

1) Addition associativity: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all $x, y, z \in V$. **TRUE**

2) Addition commutivity: $x \oplus y = y \oplus x$ for all $x, y \in V$. **TRUE**

3) Addition identity: there exists an element $z \in V$ such that for all $x \in V$, $x \oplus z = x$. **TRUE**

4) Addition inverses: for every $x \in V$ there is an element $-x \in V$ such that $x \oplus (-x) = z$. **FALSE**

5) Scalar multiplication associativity: for each $c, d \in \mathbb{R}$ and $x \in V$, $c \odot (d \odot x) = (cd) \odot x$. **TRUE**

6) Scalar multiplication identity: for all $x \in V$, $1 \odot x = x$. **TRUE**

7) Scalar distribution: for all $x, y \in V$ and $c \in \mathbb{R}$, $c \odot (x \oplus y) = c \odot x \oplus c \odot y$. **TRUE**

8) Vector distribution: for all $x \in V$ and $c, d \in \mathbb{R}$, $(c + d) \odot x = c \odot x \oplus d \odot x$ **FALSE**

Since at least one property fails, V is not a vector space.

$\square$

V3. Determine if $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Solution:
We compute

$$\text{RREF} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 3 \\ 0 & 2 & 1 & -1 \\ 1 & -1 & -1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Since this corresponds to an inconsistent system of equations, $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is **not** a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

$\square$

V4. Determine if the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ span $\mathbb{R}^3$.

Solution:
We compute

$$\text{RREF} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since the last row lacks a pivot, the vectors **do not span** $\mathbb{R}^3$.

$\square$

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y = 3z \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y = 3z + 2 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.

Solution: Let $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in W$, so we know $x_1 + y_1 = 3z_1$ and $x_2 + y_2 = 3z_2$. Consider

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}.$$

Since

$$(x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) = 3z_1 + 3z_2 = 3(z_1 + z_2)$$

we see that W is closed under vector addition. Now consider

$$c \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}.$$

Since

$$cx_1 + cx_2 = c(x_1 + x_2) = c(3z_1) = 3(cz_1)$$

we see that W is closed under scalar multiplication. Therefore W is a subspace of $\mathbb{R}^3$.

However, note that $\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ are vectors in U since $0 + 5 = 3(1) + 2$ and $1 + 4 = 3(1) + 2$. But

$$\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix}$$

does not belong to U since $1 + 9 \neq 3(2) + 2$. Since U is not closed under vector addition, U is not a subspace of $\mathbb{R}^3$.

$\square$

S1. Determine if the vectors $\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 5 \\ 1 \\ 5 \end{bmatrix}$ are linearly dependent or linearly independent.

Solution: Compute

$$\text{RREF} \begin{bmatrix} 3 & -1 & 0 & 2 \\ 2 & 1 & 1 & 5 \\ 1 & 2 & -1 & 1 \\ 0 & 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the fourth column is not a pivot column, the vectors are linearly dependent.

$\square$

S2. Determine if the set

$$\left\{\begin{bmatrix}3\\2\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\2\\3\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix},\begin{bmatrix}2\\5\\1\\5\end{bmatrix}\right\}$$

is a basis of $\mathbb{R}^4$ or not.

Solution: Compute

$$\text{RREF}\begin{bmatrix}3 & -1 & 0 & 2\\2 & 1 & 1 & 5\\1 & 2 & -1 & 1\\0 & 3 & 1 & 5\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\end{bmatrix}.$$

Since the fourth column is not a pivot column, the vectors are linearly dependent and thus not a basis. (Alternate solution: since the fourth row not a pivot row, the vectors do not span $\mathbb{R}^4$ and thus are not a basis.)

$\square$

S3. Find a basis for W, the subspace of $\mathbb{R}^4$ given by

$$W=\text{span}\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix},\begin{bmatrix}1\\0\\1\\-2\end{bmatrix},\begin{bmatrix}3\\-6\\-1\\2\end{bmatrix},\begin{bmatrix}1\\6\\1\\-1\end{bmatrix},\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}.$$

Solution: Observe that

$$\text{RREF}\begin{bmatrix}1 & 1 & 3 & 1 & 2\\-3 & 0 & -6 & 6 & 3\\-1 & 1 & -1 & 1 & 0\\2 & -2 & 2 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 2 & 0 & 1\\0 & 1 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

has pivot columns in the first, second, and fourth columns, and therefore

$$\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix},\begin{bmatrix}1\\0\\1\\-2\end{bmatrix},\begin{bmatrix}1\\6\\1\\-1\end{bmatrix}\right\}$$

is a basis of W.

$\square$

S4. Find the dimension of W, the subspace of $\mathbb{R}^4$ given by

$$W=\text{span}\left\{\begin{bmatrix}1\\-3\\-1\\2\end{bmatrix},\begin{bmatrix}1\\0\\1\\-2\end{bmatrix},\begin{bmatrix}3\\-6\\-1\\2\end{bmatrix},\begin{bmatrix}1\\6\\1\\-1\end{bmatrix},\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right\}.$$

Solution: Observe that

$$\text{RREF}\begin{bmatrix}1 & 1 & 3 & 1 & 2\\-3 & 0 & -6 & 6 & 3\\-1 & 1 & -1 & 1 & 0\\2 & -2 & 2 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 2 & 0 & 1\\0 & 1 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

has three pivot columns, and therefore $\dim W=3$.

$\square$

S5. Determine if the polynomials $3x^3 + 2x^2 + x$, $-x^3 + x^2 + 2x + 3$, $x^2 - x + 1$, and $2x^3 + 5x^2 + x + 5$ are linearly dependent or linearly independent.

Solution: This question is equivalent to asking if the Euclidean vectors

$$\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \\ 5 \end{bmatrix}$$

are linearly dependent or linearly independent.
Compute

$$\mathrm{RREF} \begin{bmatrix} 3 & -1 & 0 & 2 \\ 2 & 1 & 1 & 5 \\ 1 & 2 & -1 & 1 \\ 0 & 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the fourth column is not a pivot column, the Euclidean vectors (and therefore the polynomials) are linearly dependent.

$\square$

S6. Find a basis for the solution set of the homogeneous system of equations

$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 + 2x_5 &= 0 \\ -3x_1 - 6x_3 + 6x_4 + 3x_5 &= 0 \\ -x_1 + x_2 - x_3 + x_4 &= 0 \\ 2x_1 - 2x_2 + 2x_3 - x_4 + x_5 &= 0. \end{aligned}$$

Solution: Observe that

$$\mathrm{RREF} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 2 & 0 \\ -3 & 0 & -6 & 6 & 3 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -2 & 2 & -1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Letting $x_3 = a$ and $x_5 = b$ (since those correspond to the non-pivot columns), this is equivalent to the system

$$\begin{aligned} x_1 + 2x_3 + x_5 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 &= a \\ x_4 + x_5 &= 0 \\ x_5 &= b \end{aligned}$$

Thus, the solution set is

$$\left\{ \begin{bmatrix} -2a - b \\ -a \\ a \\ -b \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

Since we can write

$$\begin{bmatrix} -2a-b \\ -a \\ a \\ -b \\ b \end{bmatrix} = a\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

a basis for the solution space is

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

□

A1. Determine if $T : M_{2,2} \to \mathbb{R}$ given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 3a - 2b + 4d$ is a linear transformation or not.

Solution:

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) + T\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = 3a_{11} - 2a_{12} + 4a_{22} + 3b_{11} - 2b_{12} + 4b_{22}$$

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = T\left(\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}\right) = 3(a_{11}+b_{11}) - 2(a_{12}+b_{12}) + 4(a_{22}+b_{22})$$

$$cT\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = c\,(3a_{11} - 2a_{12} + 4a_{22}) = 3ca_{11} - 2ca_{12} + 4ca_{22} = T\left(\begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}\right) = T\left(c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)$$

Since T respects both addition and scalar multiplication, it is a linear transformation.

□

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x+y \\ -x+3y-z \\ 7x+y+3z \\ 0 \end{bmatrix}.$$

(a) Write the matrix for T with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^4$.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right)$

Solution:

(a) Since

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1 \\ 7 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 3 & -1 \\ 7 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $T\left(\begin{bmatrix}-2\\0\\3\end{bmatrix}\right)=\begin{bmatrix}0\\-1\\11\\0\end{bmatrix}$

$\square$

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S:\mathbb{R}^2\to\mathbb{R}^2$ given by the standard matrix $\begin{bmatrix}1&1\\-1&0\end{bmatrix}$.

(b) $T:\mathbb{R}^4\to\mathbb{R}^3$ given by the standard matrix $\begin{bmatrix}1&3&2&-3\\2&4&6&-10\\1&6&-1&3\end{bmatrix}$

Solution:

(a) RREF $\begin{bmatrix}1&1\\-1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$. Since each column is a pivot column, S is injective. Since there is no zero row, S is surjective.

(b)

$$\text{RREF}\left(\begin{bmatrix}1&3&2&-3\\2&4&6&-10\\1&6&-1&3\end{bmatrix}\right)=\begin{bmatrix}1&0&5&-9\\0&1&-1&2\\0&0&0&0\end{bmatrix}$$

Since there is a row of zeroes, the image of T does not span $\mathbb{R}^3$, so T is not surjective. Since there are non-pivot columns, T is not injective either. Alternatively, since $\dim\mathbb{R}^4>\dim\mathbb{R}^3$, T is not injective.

$\square$

A4. Let $T:\mathcal{P}^3\to\mathcal{P}^2$ be the linear transformation given by

$$T\left(ax^3+bx^2+cx+d\right)=(a+3b+2c-3d)x^2+(2a+4b+6c-10d)x+(a+6b-c+3d).$$

Compute a basis for the kernel and a basis for the image of T.

Solution: First, write down the standard matrix with respect to the ordered bases $\{x^3,x^2,x,1\}$ and $\{x^2,x,1\}$, namely

$$A=\begin{bmatrix}1&3&2&-3\\2&4&6&-10\\1&6&-1&3\end{bmatrix}.$$

Then we compute

$$\text{RREF}\,(A)=\begin{bmatrix}1&0&5&-9\\0&1&-1&2\\0&0&0&0\end{bmatrix}.$$

The kernel is given by solution set of the corresponding homogeneous system of equations translated back to polynomials i.e.

$$\ker T=\left\{(-5a+9b)x^3+(a-2b)x^2+ax+b\,\middle|\,a,b\in\mathbb{R}\right\}$$

Then a basis for the kernel is

$$\left\{-5x^3+x^2+x,9x^3-2bx^2+b\right\}.$$

A basis for the image is given by the polynomials corresponding to pivot columns, namely

$$\left\{x^2+2x+1,3x^2+4x+6\right\}.$$

$\square$

M1. Let

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 3 \\ 1 & -2 & 5 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution: AC is the only one that can be computed.

$$\begin{aligned}
AC(\mathbf{e}_1) &= A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\
AC(\mathbf{e}_2) &= A\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix} \\
AC(\mathbf{e}_3) &= A\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 5 \end{bmatrix}
\end{aligned}$$

Thus

$$AC = \begin{bmatrix} -3 & 7 & -12 \\ 1 & -2 & 5 \end{bmatrix}.$$

$\square$

M2. Determine if the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 6 \\ 1 & 6 & -1 \end{bmatrix}$ is invertible or not.

Solution: We compute

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 6 \\ 1 & 6 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since this is not the identity matrix, the matrix is not invertible. Alternatively, one might notice that the second row is a multiple of the first row, which means it is similar to a matrix with a row of zeroes, which is not invertible, so it is not invertible either.

$\square$

M3. Compute the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution:

$$\text{RREF}\left(\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -11 & 32 \\ 0 & 1 & 0 & 0 & 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 1 & 2 & -11 & 32 \\ 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\square$

G1. Let $A \in M_{4,4}$.

(a) Write a matrix R such that RA is the result of adding three times the fourth row of A to the second row of A.

(b) How are $\det(RA)$ and $\det(A)$ related?

Solution:

1. $R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. $\det(RA) = \det A$

□

G2. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix}$$

Solution: We first apply a row operation (subtract row three from row 2), and then perform Laplace expansion across the second row. Then, we compute the 3×3 determinants by expanding down the first column and across the first row, respectively.

$$\begin{aligned}
\det A &= \det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ -3 & 1 & 2 & -5 \end{bmatrix} = (-1)\det \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 3 \\ -3 & 1 & -5 \end{bmatrix} + (1)\det \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix} \\
&= (-1)\left((1)\det \begin{bmatrix} 1 & 3 \\ 1 & -5 \end{bmatrix} - (1)\det \begin{bmatrix} 3 & -1 \\ 1 & -5 \end{bmatrix} + (-3)\det \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\right) + \\
&\quad (1)\left((1)\det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - (3)\det \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}\right) \\
&= (-1)\left((1)(-8) - (1)(-14) + (-3)(10)\right) + (1)\left((1)(1) - (3)(5)\right) \\
&= (-1)(-8 + 14 - 30) + (1)(1 - 15) \\
&= 10
\end{aligned}$$

□

G3. Find the eigenvalues of the matrix $\begin{bmatrix} -2 & -2 \\ 10 & 7 \end{bmatrix}$.

Solution: Compute

$$\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & -2 \\ 10 & 7-\lambda \end{bmatrix} = (-2-\lambda)(7-\lambda) + 20 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

The eigenvalues are the roots of the characteristic polynomial, 2 and 3.

□

G4. Find a basis for the eigenspace associated to the eigenvalue 3 in the matrix

$$\begin{bmatrix} -7 & -8 & 2 \\ 8 & 9 & -1 \\ \frac{13}{2} & 5 & 2 \end{bmatrix}.$$

Solution:

$$\mathrm{RREF}(A - 3I) = \mathrm{RREF}\begin{bmatrix} -10 & -8 & 2 \\ 8 & 6 & -1 \\ \frac{13}{2} & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenspace associated to 3 is the kernel of $A - 3I$, namely

$$\left\{ \begin{bmatrix} -a \\ \frac{3}{2}a \\ a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$$

which has a basis of $\left\{ \begin{bmatrix} -1 \\ \frac{3}{2} \\ 1 \end{bmatrix} \right\}$.

$\square$