

## Section A.1

**Definition A.1.1** A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if  $V$  and  $W$  are vector spaces, a map  $T : V \rightarrow W$  is called a linear transformation if

1.  $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$  for any  $\mathbf{v}, \mathbf{w} \in V$ .
2.  $T(c\mathbf{v}) = cT(\mathbf{v})$  for any  $c \in \mathbb{R}, \mathbf{v} \in V$ .

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

**Definition A.1.2** Given a linear transformation  $T : V \rightarrow W$ ,  $V$  is called the **domain** of  $T$  and  $W$  is called the **co-domain** of  $T$ .



**Example A.1.3** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that  $T$  is linear, we must verify...

$$\begin{aligned} T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) &= T \left( \begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix} \right) = \begin{bmatrix} (x+u) - (z+w) \\ 3(y+v) \end{bmatrix} \\ T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) + T \left( \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) &= \begin{bmatrix} x-z \\ 3y \end{bmatrix} + \begin{bmatrix} u-w \\ 3v \end{bmatrix} = \begin{bmatrix} (x+u) - (z+w) \\ 3(y+v) \end{bmatrix} \end{aligned}$$

And also...

$$T \left( c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T \left( \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \quad \text{and} \quad cT \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = c \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore  $T$  is a linear transformation.

**Example A.1.4** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

To show that  $T$  is not linear, we only need to find one counterexample.

$$T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = T \left( \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + T \left( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different,  $T$  is not a linear transformation.

**Fact A.1.5** A map between Euclidean spaces  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear exactly when every component of the output is a linear combination of the variables of  $\mathbb{R}^n$ .

For example, the following map is definitely linear because  $x - z$  and  $3y$  are linear combinations of  $x, y, z$ :

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because  $x^2$ ,  $y + 3$ , and  $y - 2^x$  are not linear combinations (even though  $x + y$  is):

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

**Activity A.1.6** (*~5 min*) Recall the following rules from calculus, where  $D : \mathcal{P} \rightarrow \mathcal{P}$  is the derivative map defined by  $D(f(x)) = f'(x)$  for each polynomial  $f$ .

$$D(f + g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

- a)  $\mathcal{P}$  is not a vector space
- b)  $D$  is a linear map
- c)  $D$  is not a linear map

**Activity A.1.7** (*~10 min*) Let the polynomial maps  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  and  $T : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  be defined by

$$S(f(x)) = 2f'(x) - f''(x) \quad T(f(x)) = f'(x) + x^3$$

Compute  $S(x^4 + x)$ ,  $S(x^4) + S(x)$ ,  $T(x^4 + x)$ , and  $T(x^4) + T(x)$ . Which of these maps is definitely not linear?

**Fact A.1.8** If  $L : V \rightarrow W$  is linear, then  $L(\mathbf{z}) = L(0\mathbf{v}) = 0L(\mathbf{v}) = \mathbf{z}$  where  $\mathbf{z}$  is the additive identity of the vector spaces  $V, W$ .

Put another way, an easy way to prove that a map like  $T(f(x)) = f'(x) + x^3$  can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

**Activity A.1.9** (*~15 min*) Continue to consider  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

*Part 1:* Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to  $S(f(x)) + S(g(x))$  for all polynomials  $f, g$ .

*Part 2:* Verify that  $S(cf(x))$  is equal to  $cS(f(x))$  for all real numbers  $c$  and polynomials  $f$ . Is  $S$  linear?

**Activity A.1.10** (*~20 min*) Let the polynomial maps  $S : \mathcal{P} \rightarrow \mathcal{P}$  and  $T : \mathcal{P} \rightarrow \mathcal{P}$  be defined by

$$S(f(x)) = (f(x))^2 \quad T(f(x)) = 3xf(x^2)$$

*Part 1:* Show that  $S(x + 1) \neq S(x) + S(1)$  to verify that  $S$  is not linear.

*Part 2:* Prove that  $T$  is linear by verifying that  $T(f(x) + g(x)) = T(f(x)) + T(g(x))$  and  $T(cf(x)) = cT(f(x))$ .

**Observation A.1.11** Note that  $S$  in the previous activity is not linear, even though  $S(0) = (0)^2 = 0$ . So showing  $S(0) = 0$  isn't enough to prove a map is linear.

This is a similar situation to proving a subset is a subspace: if the subset doesn't contain  $\mathbf{z}$ , then the subset isn't a subspace. But if the subset contains  $\mathbf{z}$ , you cannot conclude anything.