Section E.1

Observation E.11 Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't usually cut it for equations with more than two variables or more than two equations. For example,

$$\begin{alignedat}{5} -2x_1 &- 4x_2 &+ x_3 &- 4x_4 &= -8 \\ x_1 &+ 2x_2 &+ 2x_3 &+ 12x_4 &= -1 \\ x_1 &+ 2x_2 &+ x_3 &+ 8x_4 &= 1 \end{alignedat}$$

has the exact same solution set as the system in the previous activity, but we'll want to learn new techniques to compute these solutions efficiently.

Remark E.12 The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Verbose standard form:

$$\begin{aligned} 1x_1 + 0x_2 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ 0x_1 - 1x_2 + 1x_3 &= -2 \end{aligned}$$

Coefficients/constants:

$$\begin{array}{rrr|r} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}$$

Definition E.13 A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Example E.14 The corresponding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ 3x_1 - 2x_2 + 4x_3 &= 0 \\ -x_2 + x_3 &= -2 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 3 & -2 & 4 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

Vector equation:

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

Definition E.15 Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ x_1 + 4x_2 &= 5 \end{aligned}$$

$$\begin{aligned} 3x_1 - 2x_2 &= 1 \\ 4x_1 + 2x_2 &= 6 \end{aligned}$$

Therefore these augmented matrices are equivalent, which we denote with $\sim$:

$$\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Activity E.16 *($\sim$10 min)* Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a nonzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

g) Replace a row with zeros.

Definition E.17 The following **row operations** produce equivalent augmented matrices:

1. Swap two rows, for example, $R_1 \leftrightarrow R_2$:
$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}\right]$$

2. Multiply a row by a nonzero constant, for example, $2R_1 \to R_1$:
$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 2(1) & 2(2) & 2(3) \\ 4 & 5 & 6 \end{array}\right]$$

3. Add a constant multiple of one row to another row, for example, $R_2 - 4R_1 \to R_2$:
$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4-4(1) & 5-4(2) & 6-4(3) \end{array}\right]$$

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity E.18 *(∼10 min)* Consider the following (equivalent) linear systems.

(A)
$$\begin{aligned} x+2y+z &= 3 \\ -2x-3y-z &= -5 \\ 2x+5y+4z &= 10 \end{aligned}$$

(C)
$$\begin{aligned} x - z &= 1 \\ y+z &= 1 \\ y+2z &= 4 \end{aligned}$$

(E)
$$\begin{aligned} x - z &= 1 \\ y+z &= 1 \\ z &= 3 \end{aligned}$$

(B)
$$\begin{aligned} 2x+5y+4z &= 10 \\ -2x-3y-z &= -5 \\ x+2y+z &= 3 \end{aligned}$$

(D)
$$\begin{aligned} x+2y+z &= 3 \\ y+z &= 1 \\ 2x+5y+4z &= 10 \end{aligned}$$

(F)
$$\begin{aligned} x+2y+z &= 3 \\ y+z &= 1 \\ y+2z &= 4 \end{aligned}$$

Rank the six linear systems from most complicated to simplest.

Activity E.19 *(∼5 min)* We can rewrite the previous in terms of equivalences of augmented matrices

$$\left[\begin{array}{ccc|c} 2 & 5 & 4 & 10 \\ -2 & -3 & -1 & -5 \\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -2 & -3 & -1 & -5 \\ 2 & 5 & 4 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 5 & 4 & 10 \end{array}\right] \sim$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

Definition E.20 A matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

Every matrix has a unique reduced row echelon form. If A is a matrix, we write $\mathrm{RREF}(A)$ for the reduced row echelon form of that matrix.