

## Section V.1

**Remark V.6** Previously, we defined a **vector space**  $V$  to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all  $\vec{u}, \vec{v}, \vec{w}$  in  $V$ , and all scalars (i.e. real numbers)  $a, b$ .

- |   |   |
|---|---|
| • <b>Addition is associative:</b> $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$ . | • <b>Scalar multiplication is associative:</b> $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$ .                                     |
| • <b>Addition is commutative:</b> $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$ .                                   | • <b>Scalar multiplication identity exists:</b> $1 \odot \vec{v} = \vec{v}$ .   |
| • <b>Additive identity exists:</b> There exists some $\vec{z}$ where $\vec{v} \oplus \vec{z} = \vec{v}$ .               | • <b>Scalar mult. distributes over vector addition:</b> $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$ . |
| • <b>Additive inverses exist:</b> There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$ .            | • <b>Scalar mult. distributes over scalar addition:</b> $(a + b) \odot \vec{v} = a \odot \vec{v} \oplus b \odot \vec{v}$ .            |

**Remark V.7** Every Euclidean space  $\mathbb{R}^n$  is a vector space, but there are other examples of vector spaces as well.

For example, consider the set  $\mathbb{C}$  of complex numbers with the usual definitions of addition and scalar multiplication, and let  $\vec{u} = a + b\mathbf{i}$ ,  $\vec{v} = c + d\mathbf{i}$ , and  $\vec{w} = e + f\mathbf{i}$ . Then

$$\begin{aligned}
 \vec{u} + (\vec{v} + \vec{w}) &= (a + b\mathbf{i}) + ((c + d\mathbf{i}) + (e + f\mathbf{i})) \\
 &= (a + b\mathbf{i}) + ((c + e) + (d + f)\mathbf{i}) \\
 &= (a + c + e) + (b + d + f)\mathbf{i} \\
 &= ((a + c) + (b + d)\mathbf{i}) + (e + f\mathbf{i}) \\
 &= (\vec{u} + \vec{v}) + \vec{w}
 \end{aligned}$$

All eight properties can be verified in this way.

**Remark V.8** The following sets are just a few examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$ : Euclidean vectors with  $n$  components.
- $\mathbb{C}$ : Complex numbers.
- $M_{m,n}$ : Matrices of real numbers with  $m$  rows and  $n$  columns.
- $\mathcal{P}^n$ : Polynomials of degree  $n$  or less.
- $\mathcal{P}$ : Polynomials of any degree.
- $C(\mathbb{R})$ : Real-valued continuous functions.

**Activity V.9** ( $\sim 20$  min) Consider the set  $V = \{(x, y) \mid y = e^x\}$  with operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \quad c \odot (x_1, y_1) = (cx_1, y_1^c)$$

*Part 1:* Show that  $V$  satisfies the distributive property

$$(a + b) \odot (x_1, y_1) = (a \odot (x_1, y_1)) \oplus (b \odot (x_1, y_1))$$

by simplifying both sides and verifying they are the same expression.

*Part 2:* Show that  $V$  contains an additive identity element satisfying

$$(x_1, y_1) \oplus \vec{z} = (x_1, y_1)$$

for all  $(x_1, y_1) \in V$  by choosing appropriate values for  $\vec{z} = (?, ?)$ .

**Remark V.10** It turns out  $V = \{(x, y) \mid y = e^x\}$  with operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \quad c \odot (x_1, y_1) = (cx_1, y_1^c)$$

satisfies all eight properties.

- |   |   |
|---|---|
| • <b>Addition is associative:</b> $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$ . | • <b>Scalar multiplication is associative:</b> $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$ .                                     |
| • <b>Addition is commutative:</b> $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$ .                                   | • <b>Scalar multiplication identity exists:</b> $1 \odot \vec{v} = \vec{v}$ .   |
| • <b>Additive identity exists:</b> There exists some $\vec{z}$ where $\vec{v} \oplus \vec{z} = \vec{v}$ .               | • <b>Scalar mult. distributes over vector addition:</b> $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$ . |
| • <b>Additive inverses exist:</b> There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$ .            | • <b>Scalar mult. distributes over scalar addition:</b> $(a + b) \odot \vec{v} = a \odot \vec{v} \oplus b \odot \vec{v}$ .            |

Thus,  $V$  is a vector space.

**Activity V.11** ( $\sim 15$  min) Let  $V = \{(x, y) \mid x, y \in \mathbb{R}\}$  have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2) \quad c \odot (x_1, y_1) = (x_1^c, y_1 + c - 1).$$

*Part 1:* Show that 1 is the scalar multiplication identity element by simplifying  $1 \odot (x, y)$  to  $(x, y)$ .

*Part 2:* Show that  $V$  does not have an additive identity element by showing that  $(0, -1) \oplus \vec{z} \neq (0, -1)$  no matter how  $\vec{z} = (z, w)$  is chosen.

*Part 3:* Is  $V$  a vector space?

**Activity V.12** (*~15 min*) Let  $V = \{(x, y) \mid x, y \in \mathbb{R}\}$  have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + 3y_2) \quad c \odot (x_1, y_1) = (cx_1, cy_1).$$

*Part 1:* Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

for **all**  $c \in \mathbb{R}$ ,  $(x_1, y_1), (x_2, y_2) \in V$ .

*Part 2:* Show that vector addition is not associative, i.e.

$$(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) \neq ((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3)$$

for **some** vectors  $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$ .

*Part 3:* Is  $V$  a vector space?

**Definition V.13** A **linear combination** of a set of vectors  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$  is given by  $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m$  for any choice of scalar multiples  $c_1, c_2, \dots, c_m$ .

For example, we can say  $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

**Definition V.14** The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m \mid c_i \in \mathbb{R}\}.$$

For example:

$$\text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\} = \left\{a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R}\right\}$$

**Activity V.15** (*~10 min*) Consider  $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ .

*Part 1:* Sketch

$$1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{and } -2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

in the  $xy$  plane.

*Part 2:* Sketch a representation of all the vectors belonging to  $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \left\{a \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid a \in \mathbb{R}\right\}$  in the  $xy$  plane.

**Activity V.16** (*~10 min*) Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

*Part 1:* Sketch the following linear combinations in the  $xy$  plane.

$$\begin{array}{ccc} 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ -2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & \end{array}$$

*Part 2:* Sketch a representation of all the vectors belonging to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$  in the  $xy$  plane.