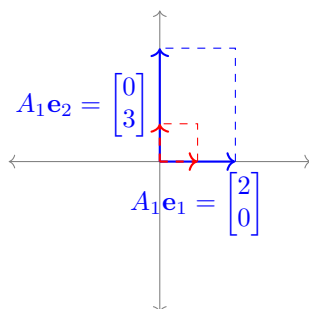


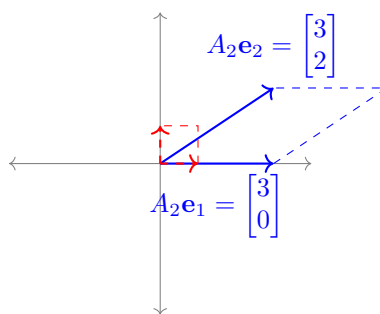
## Application Activities - Module G Part 1 - Class Day 25

**Activity 25.1** The image below illustrates how the linear transformation  $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$  transforms the unit square.



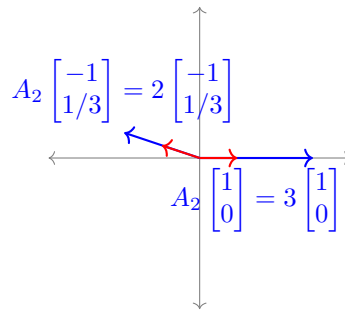
- What is the area of the transformed unit square?
  - Find two vectors that were stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.
- 

**Activity 25.2** The image below illustrates how the linear transformation  $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the standard matrix  $A_2 = \begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix}$  transforms the unit square.



- What is the area of the transformed unit square?
  - Find at least one vector that was stretched/compressed by the transformation (not sheared), and compute how much those vectors were stretched/compressed.
- 

**Observation 25.3** It's possible to find two non-parallel vectors that are stretched by the transformation given by  $A_2$ :



The process for finding such vectors will be covered later in this module.

**Activity 25.4** Consider the linear transformation given by the standard matrix  $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ .

- Sketch the transformation of the unit square (the parallelogram given by the columns of the standard matrix).
  - Compute the area of the transformed unit square.
- 

**Activity 25.5** Consider the linear transformation given by the standard matrix  $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

- Sketch the transformation of the unit square.
  - Compute the area of the transformed unit square.
- 

**Activity 25.6** Consider the linear transformation given by the standard matrix  $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ .

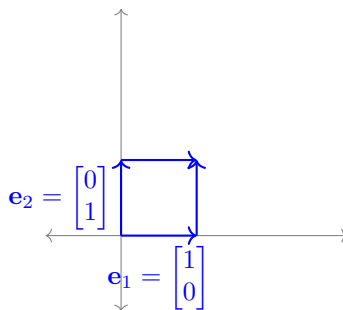
- Sketch the transformation of the unit square.
  - Compute the area of the transformed unit square.
- 

**Remark 25.7** The area of the transformed unit square measures the factor by which all areas are transformed by a linear transformation.

We will define the **determinant** of a square matrix  $A$ , or  $\det(A)$  for short, to be this factor. But what properties must this function satisfy?

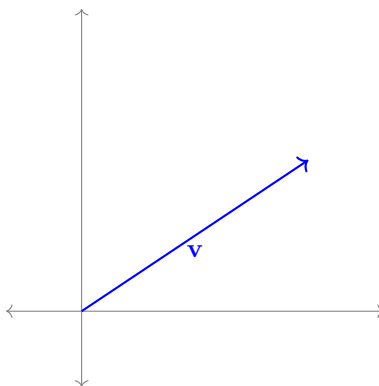
**Activity 25.8** The transformation of the unit square by the standard matrix  $[\mathbf{e}_1 \ \mathbf{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$  is illustrated below. What is  $\det([\mathbf{e}_1 \ \mathbf{e}_2]) = \det(I)$ , that is, by what factor has the area of the unit square been scaled?

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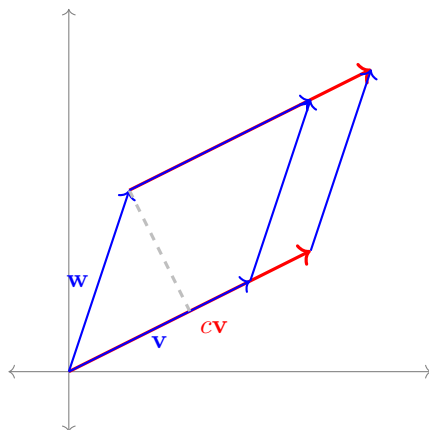
- a) 0
- b) 1
- c) 2
- d) Cannot be determined

**Activity 25.9** The transformation of the unit square by the standard matrix  $[\mathbf{v} \ \mathbf{v}]$  is illustrated below: both  $T(\mathbf{e}_1) = T(\mathbf{e}_2) = \mathbf{v}$ . What is  $\det([\mathbf{v} \ \mathbf{v}])$ , that is, by what factor has area been scaled?



- a) 0
- b) 1
- c) 2
- d) Cannot be determined

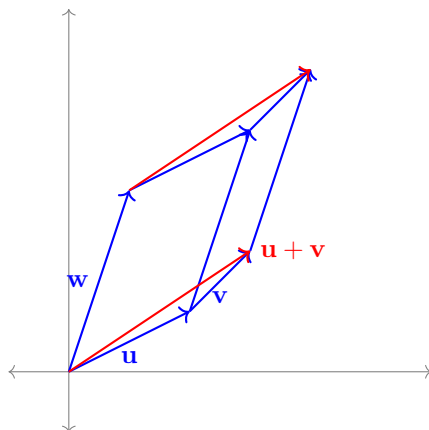
**Activity 25.10** The transformations of the unit square by the standard matrices  $[\mathbf{v} \ \mathbf{w}]$  and  $[c\mathbf{v} \ \mathbf{w}]$  are illustrated below. How are  $\det([\mathbf{v} \ \mathbf{w}])$  and  $\det([c\mathbf{v} \ \mathbf{w}])$  related?



- a)  $\det([v \ w]) = \det([cv \ w])$
- b)  $c + \det([v \ w]) = \det([cv \ w])$
- c)  $c \det([v \ w]) = \det([cv \ w])$

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**Activity 25.11** The transformations of unit squares by the standard matrices  $[u \ w]$ ,  $[v \ w]$  and  $[u + v \ w]$  are illustrated below. How is  $\det([u + v \ w])$  related to  $\det([u \ w])$  and  $\det([v \ w])$ ?



- a)  $\det([u \ w]) = \det([v \ w]) = \det([u + v \ w])$
- b)  $\det([u \ w]) + \det([v \ w]) = \det([u + v \ w])$
- c)  $\det([u \ w]) \det([v \ w]) = \det([u + v \ w])$

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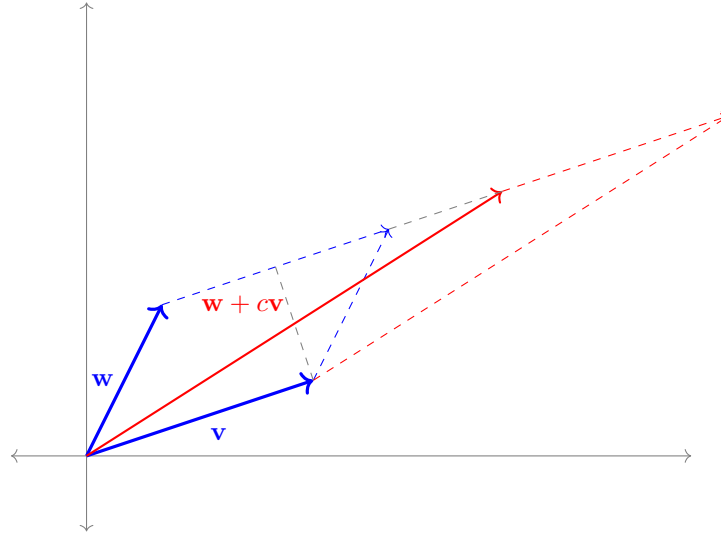
**Definition 25.12** The **determinant** is the unique function  $\det : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$  satisfying the following three properties:

P1:  $\det(I) = 1$

P2:  $\det([\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]) = 0$  whenever two columns of the matrix are identical.

P3:  $\det[\cdots \ c\mathbf{v} + d\mathbf{w} \ \cdots] = c \det[\cdots \ \mathbf{v} \ \cdots] + d \det[\cdots \ \mathbf{w} \ \cdots]$ , assuming all other columns are equal.

**Observation 25.13** What happens if we had a multiple of one column to another?



The base of both parallelograms is  $\mathbf{v}$ , while the height has not changed. Thus

$$\det([\mathbf{v} \ \mathbf{w} + c\mathbf{v}]) = \det([\mathbf{v} \ \mathbf{w}])$$

**Observation 25.14** Swapping columns can be obtained from a sequence of adding column multiples.

$$\begin{aligned} \det([\mathbf{v} \ \mathbf{w}]) &= \det([\mathbf{v} + \mathbf{w} \ \mathbf{w}]) \\ &= \det([\mathbf{v} + \mathbf{w} \ \mathbf{w} - (\mathbf{v} + \mathbf{w})]) \\ &= \det([\mathbf{v} + \mathbf{w} \ -\mathbf{v}]) \\ &= \det([\mathbf{v} + \mathbf{w} - \mathbf{v} \ -\mathbf{v}]) \\ &= \det([\mathbf{w} \ -\mathbf{v}]) \\ &= -\det([\mathbf{w} \ \mathbf{v}]) \end{aligned}$$

So swapping two columns results in a negation of the determinant. Therefore, determinants represent a *signed* area, since they are not always positive.

**Fact 25.15** We've shown that the column versions of the three row-reducing operations a matrix may be used to simplify a determinant:

(a) Multiplying a column by a scalar multiplies the determinant by that scalar:

$$c \det([\cdots \ \mathbf{v} \ \cdots]) = \det([\cdots \ c\mathbf{v} \ \cdots])$$

(b) Swapping two columns changes the sign of the determinant:

$$\det([\cdots \ \mathbf{v} \ \cdots \ \mathbf{w} \ \cdots]) = -\det([\cdots \ \mathbf{w} \ \cdots \ \mathbf{v} \ \cdots])$$

(c) Adding a multiple of a column to another column does not change the determinant:

$$\det([\cdots \quad \mathbf{v} \quad \cdots \quad \mathbf{w} \quad \cdots]) = \det([\cdots \quad \mathbf{v} + c\mathbf{w} \quad \cdots \quad \mathbf{w} \quad \cdots])$$

**Activity 25.16** The transformation given by the standard matrix  $A$  scales areas by 4, and the transformation given by the standard matrix  $B$  scales areas by 3. How must the transformation given by the standard matrix  $AB$  scale areas?

- (a) 1
- (b) 7
- (c) 12
- (d) Cannot be determined

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**Fact 25.17** Since the transformation given by the standard matrix  $AB$  is obtained by applying the transformations given by  $A$  and  $B$ , it follows that

$$\det(AB) = \det(A) \det(B)$$