

Application Activities - Module M Part 3 - Class Day 23

Definition 23.1 Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map with standard matrix A .

- If T is a bijection and B is any \mathbb{R}^n vector, then $T(X) = AX = B$ has a unique solution X .
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by setting $T^{-1}(B) = X$ to be this unique solution.
- Let A^{-1} be the standard matrix for T^{-1} . We call A^{-1} the **inverse matrix** of A , so we also say that A is **invertible**.

Activity 23.2 Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$. It can be shown that T is bijective and has the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

Part 2: If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1} , what must $A^{-1}A$ be?

Observation 23.3 $T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation T . Therefore $A^{-1}A = AA^{-1} = I$ is the identity matrix for any invertible matrix A .

Activity 23.4 Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.

Part 2: Solve $T(X) = \mathbf{e}_2$ to find $T^{-1}(\mathbf{e}_2)$.

Part 3: Solve $T(X) = \mathbf{e}_3$ to find $T^{-1}(\mathbf{e}_3)$.

Part 4: Compute A^{-1} , the standard matrix for T^{-1} .

Observation 23.5 We could have solved these three systems simultaneously by row reducing the matrix $[A | I]$ at once.

$$A = \left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array} \right]$$

Activity 23.6 Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A | I]$.

Activity 23.7 Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

Observation 23.8 A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $\text{RREF}(A) = I_n$.