

Section E.2

Activity E.2.1 (*~10 min*) Free browser-based technologies for mathematical computation are available online.

- Go to <https://octave-online.net>.
- Type `A=sym([1 3 4 ; 2 5 7])` and press **Enter** to store the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 2 & -9 & -1 \end{bmatrix}$ in the variable A .
 - The symbolic function `sym` is used to calculate precise answers rather than floating-point approximations.
 - The vertical bar in an augmented matrix does not affect row operations, so the RREF of $\begin{bmatrix} 1 & 3 & 2 \\ 2 & -9 & -1 \end{bmatrix}$ may be computed in the same way.
- Type `rref(A)` and press **Enter** to compute the reduced row echelon form of A .

Remark E.2.2 We will frequently need to know the reduced row echelon form of matrices during class, so feel free to use Octave-Online.net to compute RREF efficiently.

You may alternatively use the calculator you will use during assessments. Be sure to use fractions mode to compute exact solutions rather than floating-point approximations.

Activity E.2.3 (*~10 min*) Consider the system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

Part 1: Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF} \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array} \right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array} \right]$$

Part 2: Use the RREF matrix to write a linear system equivalent to the original system. Then find its solution set.

Activity E.2.4 (~10 min) Consider the system of equations.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-x_1 \quad \quad - \quad 3x_3 = 1$$

Part 1: Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF} \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array} \right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array} \right]$$

Part 2: Use the RREF matrix to write a linear system equivalent to the original system. Then find its solution set.

Activity E.2.5 (~10 min) Consider the following linear system.

$$x_1 + 2x_2 + 3x_3 = 1$$

$$2x_1 + 4x_2 + 8x_3 = 0$$

Part 1: Find its corresponding augmented matrix A and use technology to find $\text{RREF}(A)$.

Part 2: How many solutions do these linear systems have?

Activity E.2.6 (~10 min) Consider the simple linear system equivalent to the system from the previous activity:

$$\begin{array}{rcl} x_1 + 2x_2 & = & 4 \\ x_3 & = & -1 \end{array}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \mid a \in \mathbb{R} \right\}$.

Part 2: Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \mid b \in \mathbb{R} \right\}$.

Part 3: Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 0 & 4 \\ 0 & 0 & \textcircled{1} & -1 \end{array} \right]$ caused this?

Definition E.2.7 Recall that the pivots of a matrix in RREF form are the leading 1s in each non-zero row.

The pivot columns in an augmented matrix correspond to the **bound variables** in the system of equations (x_1, x_3 below). The remaining variables are called **free variables** (x_2 below).

$$\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 0 & 4 \\ 0 & 0 & \textcircled{1} & -1 \end{array} \right]$$

To efficiently solve a system in RREF form, assign letters to the free variables, and then solve for the bound variables.

Activity E.2.8 (~ 10 min) Find the solution set for the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - x_5 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 + x_5 &= 2 \end{aligned}$$

by row-reducing its augmented matrix, and then assigning letters to the free variables (given by non-pivot columns) and solving for the bound variables (given by pivot columns) in the corresponding linear system.

Observation E.2.9 The solution set to the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - x_5 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 + x_5 &= 2 \end{aligned}$$

may be written as

$$\left\{ \left[\begin{array}{c} 1 + 5a + 2b \\ 1 + 2a + 3b \\ a \\ 3 + 3b \\ b \end{array} \right] \middle| a, b \in \mathbb{R} \right\}.$$

Remark E.2.10 Don't forget to correctly express the solution set of a linear system, using set-builder notation for consistent systems with infinitely many solutions.

- **Consistent with one solution:** e.g. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$
- **Consistent with infinitely-many solutions:** e.g. $\left\{ \left[\begin{array}{c} 1 \\ 2 - 3a \\ a \end{array} \right] \middle| a \in \mathbb{R} \right\}$
- **Inconsistent:** \emptyset or $\{\}$