

Section E.1

Remark E.11 The only important information in a linear system are its coefficients and constants.

Original linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Verbose standard form:

$$\begin{aligned}1x_1 + 0x_2 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\0x_1 - 1x_2 + 1x_3 &= -2\end{aligned}$$

Coefficients/constants:

$$\begin{array}{ccc|c}1 & 0 & 3 & 3 \\3 & -2 & 4 & 0 \\0 & -1 & 1 & -2\end{array}$$

Definition E.12 A system of m linear equations with n variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$\begin{array}{cccccc}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & b_1 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & b_2 \\\vdots & & \vdots \\a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & b_m\end{array} \qquad \left[\begin{array}{cccc|c}a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\\vdots & \vdots & \ddots & \vdots & \vdots \\a_{m1} & a_{m2} & \cdots & a_{mn} & b_m\end{array} \right]$$

Example E.13 The corresponding augmented matrix for this system is obtained by simply writing the coefficients and constants in matrix form.

Linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c}1 & 0 & 3 & 3 \\3 & -2 & 4 & 0 \\0 & -1 & 1 & -2\end{array} \right]$$

Definition E.14 Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems share the same solution set $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

$$\begin{array}{rcl}3x_1 - 2x_2 & = & 1 \\x_1 + 4x_2 & = & 5\end{array} \qquad \begin{array}{rcl}3x_1 - 2x_2 & = & 1 \\4x_1 + 2x_2 & = & 6\end{array}$$

Therefore these augmented matrices are equivalent, which we denote with \sim :

$$\left[\begin{array}{cc|c}3 & -2 & 1 \\1 & 4 & 5\end{array} \right] \sim \left[\begin{array}{cc|c}3 & -2 & 1 \\4 & 2 & 6\end{array} \right]$$

Activity E.15 (~ 10 min) Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

- | | |
|---|---|
| a) Swap two rows. | e) Add a constant multiple of one row to another row. |
| b) Swap two columns. | f) Replace a column with zeros. |
| c) Add a constant to every term in a row. | g) Replace a row with zeros. |
| d) Multiply a row by a nonzero constant. | |

Definition E.16 The following **row operations** produce equivalent augmented matrices:

1. Swap two rows, for example, $R_1 \leftrightarrow R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \sim \left[\begin{array}{cc|c} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array} \right]$$

2. Multiply a row by a nonzero constant, for example, $2R_1 \rightarrow R_1$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \sim \left[\begin{array}{cc|c} 2(1) & 2(2) & 2(3) \\ 4 & 5 & 6 \end{array} \right]$$

3. Add a constant multiple of one row to another row, for example, $R_2 - 4R_1 \rightarrow R_2$:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 - 4(1) & 5 - 4(2) & 6 - 4(3) \end{array} \right]$$

Whenever two matrices A, B are equivalent (so whenever we do any of these operations), we write $A \sim B$.

Activity E.17 (~ 10 min) Consider the following (equivalent) linear systems.

- | | | |
|--------------------|--------------------|------------------|
| (A) | (C) | (E) |
| $x + 2y + z = 3$ | $x - z = 1$ | $x - z = 1$ |
| $-x - y + z = 1$ | $y + z = 1$ | $y + z = 1$ |
| $2x + 5y + 3z = 7$ | $y + 2z = 4$ | $z = 3$ |
| (B) | (D) | (F) |
| $2x + 5y + 3z = 7$ | $x + 2y + z = 3$ | $x + 2y + z = 3$ |
| $-x - y + z = 1$ | $y + z = 1$ | $y + z = 1$ |
| $x + 2y + z = 3$ | $2x + 5y + 3z = 7$ | $y + 2z = 4$ |

Rank the six linear systems from most complicated to simplest.

Activity E.18 (*~5 min*) We can rewrite the previous in terms of equivalences of augmented matrices

$$\begin{aligned} \left[\begin{array}{ccc|c} 2 & 5 & 13 & 7 \\ -1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{array} \right] &\sim \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 1 & 3 \\ -1 & -1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array} \right] \sim \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array} \right] \sim \\ \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 1 & 3 \\ 0 & \textcircled{1} & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array} \right] &\sim \left[\begin{array}{ccc|c} \textcircled{1} & 0 & -1 & 1 \\ 0 & \textcircled{1} & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array} \right] \sim \left[\begin{array}{ccc|c} \textcircled{1} & 0 & -1 & 1 \\ 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & \textcircled{1} & 3 \end{array} \right] \end{aligned}$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

Definition E.19 A matrix is in **reduced row echelon form (RREF)** if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

Every matrix has a unique reduced row echelon form. If A is a matrix, we write $\text{RREF}(A)$ for the reduced row echelon form of that matrix.

Activity E.20 (*~15 min*) Recall that a matrix is in **reduced row echelon form (RREF)** if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

<p>(A) $\left[\begin{array}{ccc c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$</p>	<p>(C) $\left[\begin{array}{ccc c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]$</p>	<p>(E) $\left[\begin{array}{ccc c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$</p>
<p>(B) $\left[\begin{array}{ccc c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$</p>	<p>(D) $\left[\begin{array}{ccc c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$</p>	<p>(F) $\left[\begin{array}{ccc c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array} \right]$</p>

For each matrix, circle the leading terms, and label it as RREF or not RREF. For the ones not in RREF, find their RREF.