

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Linear Algebra

University of South Alabama

Fall 2017

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Module E: Solving Systems of Linear Equations

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

At the end of this module, students will be able to...

- **E1: Systems as matrices.** Translate back and forth between a system of linear equations and the corresponding augmented matrix.
- **E2: Row reduction.** Put a matrix in reduced row echelon form
- **E3: Solving Linear Systems.** Solve a system of linear equations.
- **E4: Homogeneous Systems.** Find a basis for the solution set of a homogeneous linear system.

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

Before beginning this module, each student should be able to...

- Determine if a system to a two-variable system of linear equations will have zero, one, or infinitely-many solutions by graphing.
- Find the unique solution to a two-variable system of linear equations by back-substitution.

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/cc-eighth-grade-math/cc-8th-systems-topic/cc-8th-systems-graphically/a/systems-of-equations-with-graphing>
- <https://www.khanacademy.org/math/algebra/systems-of-linear-equations/solving-systems-of-equations-with-substitution/v/practice-using-substitution-for-systems>

Module E

**Part 1 (Day 3)**

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

# Application Activities - Module E Part 1 - Class Day 3

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 3.1

A **linear equation** is an equation of the variables  $x_i$  of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A **solution** for a linear equation is expressed in terms of the Euclidean vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

and must satisfy

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Observation 3.2**

The linear equation  $3x - 5y = -2$  may be graphed as a line in the  $xy$  plane.



The linear equation  $x + 2y - z = 4$  may be graphed as a plane in  $xyz$  space.



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Remark 3.3

In previous classes you likely assumed  $x = x_1$ ,  $y = x_2$ , and  $z = x_3$ . However, since this course often deals with equations of four or more variables, we will almost always write our variables as  $x_i$ .

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 3.4

A **system of linear equations** (or a **linear system** for short) is a collection of one or more linear equations.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

## A solution

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for a linear system satisfies

$$a_{i1}s_1 + a_{i2}s_2 + \dots + a_{in}s_n = b_i$$

for  $1 \leq i \leq m$  (that is, the solution satisfies all equations in the system).

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

Remark 3.5

When variables in a large linear system are missing, we prefer to write the system in one of the following standard forms:

Original linear system:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Verbose standard form:

$$\begin{aligned}1x_1 + 0x_2 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\0x_1 - 1x_2 + 1x_3 &= -2\end{aligned}$$

Concise standard form:

$$\begin{aligned}x_1 + 3x_3 &= 3 \\3x_1 - 2x_2 + 4x_3 &= 0 \\-x_2 + x_3 &= -2\end{aligned}$$

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

# Definition 3.6

A linear system is **consistent** if there exists a solution for the system. Otherwise it is **inconsistent**.

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Fact 3.7

All linear systems are either **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

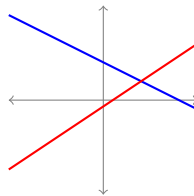
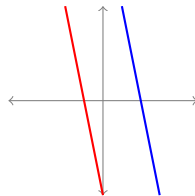
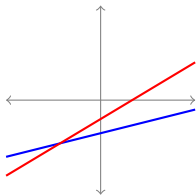
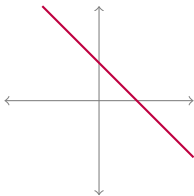
Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 3.8

Consider the following graphs representing linear systems of two variables. Label each graph with **consistent with one solution**, **consistent with infinitely-many solutions**, or **inconsistent**.



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 3.9

All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system.

$$-x_1 + 2x_2 = 5$$

$$2x_1 - 4x_2 = 6$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

*Part 1:* Find three different solutions  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ ,  $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ ,  $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$  for this system.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 3.10

Consider the following consistent linear system.

$$-x_1 + 2x_2 = -3$$

$$2x_1 - 4x_2 = 6$$

*Part 1:* Find three different solutions  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ ,  $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ ,  $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$  for this system.

*Part 2:* Let  $x_2 = a$  where  $a$  is an arbitrary real number, then find an expression for  $x_1$  in terms of  $a$ . Use this to describe *all* solutions (the **solution set**)  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ a \end{bmatrix}$  for the linear system in terms of  $a$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 3.11

Consider the following linear system.

$$x_1 + 2x_2 - x_4 = 3$$

$$x_3 + 4x_4 = -2$$

Describe the solution set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ 0 \\ t_3 \\ 0 \end{bmatrix} + a \begin{bmatrix} ? \\ 1 \\ ? \\ 0 \end{bmatrix} + b \begin{bmatrix} ? \\ 0 \\ ? \\ 1 \end{bmatrix}$$

to the linear system by setting  $x_2 = a$  and  $x_4 = b$ , and then solving for  $x_1$  and  $x_3$ .

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Observation 3.12

Solving linear systems of two variables by graphing or substitution is reasonable for two-variable systems, but these simple techniques won't cut it for equations with more than two variables or more than two equations.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

Remark 3.13

The only important information in a linear system are its coefficients and constants.

Original linear system:	Verbose standard form:	Coefficients/constants:
$x_1 + 3x_3 = 3$	$1x_1 + 0x_2 + 3x_3 = 3$	$1 \quad 0 \quad 3 \mid 3$
$3x_1 - 2x_2 + 4x_3 = 0$	$3x_1 - 2x_2 + 4x_3 = 0$	$3 \quad -2 \quad 4 \mid 0$
$-x_2 + x_3 = -2$	$0x_1 - 1x_2 + 1x_3 = -2$	$0 \quad -1 \quad 1 \mid -2$

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

Definition 3.14

A system of  $m$  linear equations with  $n$  variables is often represented by writing its coefficients and constants in an **augmented matrix**.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 3.15

Two systems of linear equations (and their corresponding augmented matrices) are said to be **equivalent** if they have the same solution set.

For example, both of these systems have a single solution:  $(x_1, x_2) = (1, 1)$ .

$$3x_1 - 2x_2 = 1$$

$$x_1 + 4x_2 = 5$$

$$3x_1 - 2x_2 = 1$$

$$4x_1 + 2x_2 = 6$$

Therefore these augmented matrices are equivalent:

$$\left[ \begin{array}{cc|c} 3 & -2 & 1 \\ 1 & 4 & 5 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} 3 & -2 & 1 \\ 4 & 2 & 6 \end{array} \right]$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 3.16

Following are six procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that would change the solution set of the corresponding linear system as **invalid**.

- a) Swap two rows.
- b) Swap two columns.
- c) Add a constant to every term in a row.
- d) Multiply a row by a nonzero constant.
- e) Add a constant multiple of one row to another row.
- f) Replace a column with zeros.



Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

◀ ◻ ▶ ◀ ◻ ▶ ◀ ≡ ▶ ◀ ≡ ▶ ≡ ↺ 🔍 ↻

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 4.1

The following **row operations** produce equivalent augmented matrices:

- 1 Swap two rows.
- 2 Multiply a row by a nonzero constant.
- 3 Add a constant multiple of one row to another row.

Whenever two matrices  $A, B$  are equivalent (so whenever we do any of these operations), we write  $A \sim B$ .

## Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

### Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

### Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

### Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

### Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

### Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

### Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$x_1 - x_2 + 5x_3 = 1$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$x_2 - 2x_3 = 3$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_3 = 2$$

*Part 1:* Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

- 1 Swap  $R_1$  (first row) and  $R_2$  (second row).
- 2 Multiply  $R_2$  by  $\frac{1}{2}$ .

- 3 Add  $R_1$  to  $R_3$ .
- 4 Add  $-3R_1$  to  $R_2$ .
- 5 Add  $-2R_2$  to  $R_3$ .
- 6 Multiply  $R_3$  by  $\frac{1}{3}$ .

### Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

### Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

### Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

### Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

### Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

### Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 4.2

Consider the following two linear systems.

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$x_1 - x_2 + 5x_3 = 1$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$x_2 - 2x_3 = 3$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

$$x_3 = 2$$

*Part 1:* Show these are equivalent by converting the first system to an augmented matrix, and then performing the following row operations to obtain an augmented matrix equivalent to the second system.

❶ Swap  $R_1$  (first row) and  $R_2$  (second row).

❷ Multiply  $R_2$  by  $\frac{1}{2}$ .

❸ Add  $R_1$  to  $R_3$ .

❹ Add  $-3R_1$  to  $R_2$ .

❺ Add  $-2R_2$  to  $R_3$ .

❻ Multiply  $R_3$  by  $\frac{1}{3}$ .

*Part 2:* Which linear system would you rather solve?

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 4.3

The **leading term** of a matrix row is its first nonzero term. A matrix is in **row echelon form** if all leading terms are 1, the leading term of every row is farther right than every leading term on a higher row, and all zero rows are at the bottom of the matrix. Examples:

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 4.4

Find your own sequence of row operations to manipulate the matrix

$$\left[ \begin{array}{ccc|c} 3 & -2 & 13 & 6 \\ 2 & -2 & 10 & 2 \\ -1 & 3 & -6 & 11 \end{array} \right]$$

into row echelon form. (Note that row echelon form is not unique.)

The most efficient way to do this is by circling **pivot positions** in your matrix:

- 1 Circle the top-left-most cell that (a) is below any existing pivot positions and (b) has a nonzero term either in that position or below it.
- 2 Ignoring any rows above this pivot position, use row operations to change the value of your pivot position to 1, and the terms below it to 0.
- 3 Repeat these two steps as often as possible.

## Module E

Part 1 (Day 3)

**Part 2 (Day 4)**

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 4.5

Solve this simplified linear system:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Observation 4.6

The consise standard form of the solution to this linear system corresponds to a simplified row echelon form matrix:

$$\begin{array}{rcl} x_1 & = & -2 \\ x_2 & = & 7 \\ x_3 & = & 2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Definition 4.7**

A matrix is in **reduced row echelon form** if it is in row echelon form and all terms above leading terms are 0. Examples:

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 4.8

Show that the following two linear systems:

$$x_1 - x_2 + 5x_3 = 1$$

$$x_2 - 2x_3 = 3$$

$$x_3 = 2$$

$$x_1 = -2$$

$$x_2 = 7$$

$$x_3 = 2$$

are equivalent by converting the first system to an augmented matrix, and then zeroing out all terms above pivot positions (the leading terms).

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Remark 4.9**

We may verify that  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$  is a solution to the original linear system

$$3x_1 - 2x_2 + 13x_3 = 6$$

$$2x_1 - 2x_2 + 10x_3 = 2$$

$$-1x_1 + 3x_2 - 6x_3 = 11$$

by plugging the solution into each equation.

Module E

Part 1 (Day 3)

**Part 2 (Day 4)**

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Fact 4.10

Every augmented matrix  $A$  reduces to a unique reduced row echelon form matrix. This matrix is denoted as  $\text{RREF}(A)$ .

## Module E

Part 1 (Day 3)

**Part 2 (Day 4)**

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 4.11

Consider the following matrix.

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 4.11**

Consider the following matrix.

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

*Part 1:* Find  $\text{RREF}(A)$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 4.11**

Consider the following matrix.

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 8 & 0 \end{array} \right]$$

*Part 1:* Find  $\text{RREF}(A)$ .

*Part 2:* How many solutions does the corresponding linear system have?



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)**

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Application Activities - Module E Part 3 - Class Day 5

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 5.1

An algorithm that reduces  $A$  to  $\text{RREF}(A)$  is called **Gauss-Jordan elimination**. For example:

- 1 Circle the cell that (a) is in the top-most row without a pivot position and (b) is in the left-most column with a nonzero term either in that position or below it. This position (not the number inside) is called a **pivot**.
- 2 Change the pivot's value to 1 by using row operations involving only the pivot row and rows below it.
- 3 Add or subtract multiples of the pivot row to zero out above and below the pivot.
- 4 Return to Step 1 and repeat as needed until the matrix is in row reduced echelon form.

**Observation 5.2**

Here is an example of applying Gauss-Jordan elimination to a matrix:

$$\begin{aligned}
 & \left[ \begin{array}{cccc|c} \textcircled{2} & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array} \right] \sim \left[ \begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ -1 & 1 & 3 & -1 & -3 \\ 2 & -2 & -6 & 1 & 3 \end{array} \right] \sim \left[ \begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ 0 & \textcircled{-1} & 2 & 0 & -1 \\ 0 & 2 & -4 & -1 & -1 \end{array} \right] \\
 & \sim \left[ \begin{array}{cccc|c} \textcircled{1} & -2 & -1 & 1 & 2 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 2 & -4 & -1 & -1 \end{array} \right] \sim \left[ \begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 1 & 4 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{-1} & -3 \end{array} \right] \\
 & \sim \left[ \begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 1 & 4 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right] \sim \left[ \begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 0 & 1 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right]
 \end{aligned}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 5.3

The columns of  $\text{RREF}(A)$  without a leading term represent **free variables** of the linear system modeled by  $A$  that may be set equal to arbitrary parameters. The other **bounded variables** can then be expressed in terms of those parameters to describe the solution set to the linear system modeled by  $A$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Example 5.4**

Here,  $x_3$  is the free variable set equal to  $a$  since its column lacks a pivot, and the other bounded variables are put in terms of  $a$ .

$$2x_1 - 2x_2 - 6x_3 + x_4 = 3$$

$$-x_1 + x_2 + 3x_3 - x_4 = -3$$

$$x_1 - 2x_2 - x_3 + x_4 = 1$$

$$x_1 - 5x_3 = 1$$

$$x_2 - 2x_3 = 1$$

$$x_4 = 3$$

$$x_1 = 1 + 5a$$

$$x_2 = 1 + 2a$$

$$x_3 = a$$

$$x_4 = 3$$

 $\Rightarrow$ 

$$\begin{array}{c} \Downarrow \\ \left[ \begin{array}{cccc|c} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{array} \right] \sim \left[ \begin{array}{cccc|c} \textcircled{1} & 0 & -5 & 0 & 1 \\ 0 & \textcircled{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 3 \end{array} \right] \Uparrow \end{array}$$

So the solution set is  $\left\{ \begin{bmatrix} 1 + 5a \\ 1 + 2a \\ a \\ 3 \end{bmatrix} \mid a \in \mathbb{R} \right\}.$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 5.5

Solve the system of linear equations, circling the pivot positions in your augmented matrices as you work.

$$-x_1 + x_2 - 3x_3 + 2x_4 = 0$$

$$2x_1 - x_2 + 5x_3 + 3x_4 = -11$$

$$3x_1 + 2x_2 + 4x_3 + x_4 = 1$$

$$x_2 - x_3 + x_4 = 1$$

Remember to find the solution set of the system by setting the free variable (the column without a pivot position) equal to  $a$ , and then express each of the other bounded variables equal to an expression in terms of  $a$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

**Part 3 (Day 5)**

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Remark 5.6**

From now on, unless specified, there's no need to show your work in finding  $\text{RREF}(A)$ , so you may use a calculator to speed up your work.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 5.7**

Solve the linear system

$$2x_1 - 3x_2 = 17$$

$$x_1 + 2x_2 = -2$$

$$-x_1 - x_2 = 1$$



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Activity 5.8

Show that all linear systems of the form

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & = & 0 \end{array}$$

are consistent by finding a quickly verifiable solution.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Definition 5.9

A **homogeneous system** is a linear system satisfying  $b_i = 0$ , that is, it is a linear system of the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Fact 5.10**

Because the zero vector is always a solution, the solution set to any homogeneous system with infinitely-many solutions may be generated by multiplying the parameters representing the free variables by a minimal set of Euclidean vectors, and adding these up. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

Definition 5.11

A minimal set of Euclidean vectors generating the solution set to a homogeneous system is called a **basis** for the solution set of the homogeneous system. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 5.12

Find a basis for the solution set of the following homogeneous linear system.

$$x_1 + 2x_2 \quad - \quad x_4 = 0$$

$$x_3 + 4x_4 = 0$$

$$2x_1 + 4x_2 + x_3 + 2x_4 = 0$$

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Module V: Vector Spaces

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

At the end of this module, students will be able to...

- **V1: Vector Spaces.** Determine if a set with given operations forms a vector space.
- **V2: Linear Combinations.** Determine if a vector can be written as a linear combination of a given set of vectors.
- **V3: Spanning Sets.** Determine if a set of vectors spans a vector space.
- **V4: Subspaces.** Determine if a subset of a vector space is a subset or not.

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Add complex numbers and multiply complex numbers by scalars.
- Add polynomials and multiply polynomials by scalars.
- Perform basic manipulations of augmented matrices and linear systems  
**(Standard(s) E1,E2,E3).**



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/prec calculus/vectors-prec calc/vector-addition-subtraction/v/adding-and-subtracting-vectors>
- <https://www.khanacademy.org/math/prec calculus/vectors-prec calc/combined-vector-operations/v/combined-vector-operations-example>
- <https://www.khanacademy.org/math/prec calculus/imaginary-and-complex-numbers/v/adding-and-subtracting-complex-numbers/v/adding-complex-numbers>
- <https://www.khanacademy.org/math/algebra/introduction-to-polynomial-expressions/v/adding-and-subtracting-polynomials/v/adding-and-subtracting-polynomials-1>

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)**
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Application Activities - Module V Part 1 - Class Day 7

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 7.1

Consider each of the following vector properties. Label each property with  $\mathbb{R}^1$ ,  $\mathbb{R}^2$ , and/or  $\mathbb{R}^3$  if that property holds for Euclidean vectors/scalars  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  of that dimension.

**1 Addition associativity.**

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$$

**2 Addition commutivity.**

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

**3 Addition identity.**

There exists some  $\mathbf{0}$  where  $\mathbf{v} + \mathbf{0} = \mathbf{v}$ .

**4 Addition inverse.**

There exists some  $-\mathbf{v}$  where  
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$

**5 Addition midpoint uniqueness.**

There exists a unique  $\mathbf{m}$  where the  
 distance from  $\mathbf{u}$  to  $\mathbf{m}$  equals the  
 distance from  $\mathbf{m}$  to  $\mathbf{v}$ .

**6 Scalar multiplication associativity.**

$$a(b\mathbf{v}) = (ab)\mathbf{v}.$$

**7 Scalar multiplication identity.**

$$1\mathbf{v} = \mathbf{v}.$$

**8 Scalar multiplication relativity.**

There exists some scalar  $c$  where either  
 $c\mathbf{v} = \mathbf{w}$  or  $c\mathbf{w} = \mathbf{v}$ .

**9 Scalar distribution.**

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

**10 Vector distribution.**

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$$

**11 Orthogonality.**

There exists a non-zero vector  $\mathbf{n}$  such  
 that  $\mathbf{n}$  is orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$ .

**12 Bidimensionality.**

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} \text{ for some value of } a, b.$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 7.2

A **vector space**  $V$  is any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following properties. Let  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  belong to  $V$ , and let  $a, b$  be scalar numbers.

- **Addition associativity.**  
 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.$
- **Addition commutativity.**  
 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$
- **Addition identity.**  
There exists some  $\mathbf{0}$  where  
 $\mathbf{v} + \mathbf{0} = \mathbf{v}.$
- **Addition inverse.**  
There exists some  $-\mathbf{v}$  where  
 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$
- **Scalar multiplication associativity.**  
 $a(b\mathbf{v}) = (ab)\mathbf{v}.$
- **Scalar multiplication identity.**  
 $1\mathbf{v} = \mathbf{v}.$
- **Scalar distribution.**  
 $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$
- **Vector distribution.**  
 $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 7.3

The most important examples of vector spaces are the **Euclidean vector spaces**  $\mathbb{R}^n$ , but there are other examples as well.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 7.4

Consider the following set that models motion along the curve  $y = e^x$ . Let

$V = \{(x, y) : y = e^x\}$ . Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$ , and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 7.4

Consider the following set that models motion along the curve  $y = e^x$ . Let

$V = \{(x, y) : y = e^x\}$ . Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$ , and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$ .

*Part 1:* Which of the vector space properties are satisfied by  $V$  paired with these operations?

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some  $\mathbf{0}$  where

$$\mathbf{v} \oplus \mathbf{0} = \mathbf{v}.$$

- **Addition inverse.**

There exists some  $-\mathbf{v}$  where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 7.4

Consider the following set that models motion along the curve  $y = e^x$ . Let

$V = \{(x, y) : y = e^x\}$ . Let vector addition be defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2)$ , and let scalar multiplication be defined by

$c \odot (x, y) = (cx, y^c)$ .

*Part 1:* Which of the vector space properties are satisfied by  $V$  paired with these operations?

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some  $\mathbf{0}$  where

$$\mathbf{v} \oplus \mathbf{0} = \mathbf{v}.$$

- **Addition inverse.**

There exists some  $-\mathbf{v}$  where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

*Part 2:* Is  $V$  a vector space?



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)**
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Application Activities - Module V Part 2 - Class Day 8

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Remark 8.1

The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$ : Euclidean vectors with  $n$  components.
- $\mathbb{R}^\infty$ : Sequences of real numbers  $(v_1, v_2, \dots)$ .
- $\mathbb{R}^{m \times n}$ : Matrices of real numbers with  $m$  rows and  $n$  columns.
- $\mathbb{C}$ : Complex numbers.
- $\mathcal{P}^n$ : Polynomials of degree  $n$  or less.
- $\mathcal{P}$ : Polynomials of any degree.
- $C(\mathbb{R})$ : Real-valued continuous functions.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 8.2

Let  $V = \{(a, b) : a, b \text{ are real numbers}\}$ , where

$(a_1, b_1) \oplus (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$  and  $c \odot (a, b) = (a^c, b + c)$ .

Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

- **Addition associativity.**

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$$

- **Addition commutativity.**

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$$

- **Addition identity.**

There exists some  $\mathbf{0}$  where

$$\mathbf{v} \oplus \mathbf{0} = \mathbf{v}.$$

- **Addition inverse.**

There exists some  $-\mathbf{v}$  where

$$\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}.$$

- **Scalar multiplication associativity.**

$$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$$

- **Scalar multiplication identity.**

$$1 \odot \mathbf{v} = \mathbf{v}.$$

- **Scalar distribution.**

$$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$$

- **Vector distribution.**

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 8.3

A **linear combination** of a set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$  is given by  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$  for any choice of scalar multiples  $c_1, c_2, \dots, c_m$ .

For example, we say  $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 8.4

The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
**Part 2 (Day 8)**  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

# Activity 8.5

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 8.5**

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

*Part 1:* Sketch  $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  in the  $xy$  plane for  $c = 1, 3, 0, -2$ .

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
**Part 2 (Day 8)**  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

**Activity 8.5**

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ .

*Part 1:* Sketch  $c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  in the  $xy$  plane for  $c = 1, 3, 0, -2$ .

*Part 2:* Sketch a representation of all the vectors given by  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$  in the  $xy$  plane.



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 8.6

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 8.6**

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

*Part 1:* Sketch the following linear combinations in the  $xy$  plane:  $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix},$

$$0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 8.6**

Consider  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ .

*Part 1:* Sketch the following linear combinations in the  $xy$  plane:  $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix},$

$$0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

*Part 2:* Sketch a representation of all the vectors given by  $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$  in the  $xy$  plane.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 8.7

Sketch a representation of all the vectors given by  $\text{span} \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$  in the  $xy$  plane.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 8.8**

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 8.8**

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector

equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 8.8**

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector

equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

*Part 2:* Solve this system. (Remember, you should use a calculator to help find RREF.)

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

**Part 2 (Day 8)**

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 8.8**

The vector  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  exactly when the vector

equation  $x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  holds for some scalars  $x_1, x_2$ .

*Part 1:* Reinterpret this vector equation as a system of linear equations.

*Part 2:* Solve this system. (Remember, you should use a calculator to help find RREF.)

*Part 3:* Given this solution, does  $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$  belong to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ ?



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)**
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Application Activities - Module V Part 3 - Class Day 9

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Fact 9.1**

A vector  $\mathbf{b}$  belongs to  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  if and only if the linear system corresponding to  $[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$  is consistent.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)**
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Remark 9.2

To determine if **b** belongs to  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ , find  $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n \mid \mathbf{b}]$ .

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)**
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

### Activity 9.3

Determine if  $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  by row-reducing an appropriate matrix.

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)**
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

Activity 9.4

Determine if  $\begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}$  belongs to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$  by row-reducing an appropriate matrix.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Observation 9.5

So far we've only discussed linear combinations of Euclidean vectors. Fortunately, many vector spaces of interest can be reinterpreted as an **isomorphic** Euclidean space  $\mathbb{R}^n$ ; that is, a Euclidean space that mirrors the behavior of the vector space exactly.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 9.6

We previously checked that  $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$  does not belong to  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ .

Does  $f(x) = 3x^2 - 2x + 1$  belong to  $\text{span}\{x^2 - 3, -x^2 - 3x + 2\}$ ?

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 9.7**

Does the matrix  $\begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix}$  belong to  $\text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\}$ ?



Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
**Part 3 (Day 9)**  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 9.8

Does the complex number  $2i$  belong to  $\text{span}\{-3 + i, 6 - 2i\}$ ?

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

**Part 3 (Day 9)**

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 9.9

How many vectors are required to span  $\mathbb{R}^2$ ? Sketch a drawing in the  $xy$  plane to support your answer.

(a) 1

(b) 2

(c) 3

(d) 4

(e) Infinitely Many

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
**Part 3 (Day 9)**  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 9.10

How many vectors are required to span  $\mathbb{R}^3$ ?

- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) Infinitely Many

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
**Part 4 (Day 10)**

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

# Application Activities - Module V Part 4 - Class Day 10

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

**Part 4 (Day 10)**

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Fact 10.1**At least  $n$  vectors are required to span  $\mathbb{R}^n$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 10.2**

Choose a vector  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$  in  $\mathbb{R}^3$  that is not in  $\text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$  by ensuring

$$\left[ \begin{array}{cc|c} 1 & -2 & a \\ -1 & 0 & b \\ 0 & 1 & c \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]. \text{ (Why does this work?)}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

**Fact 10.3**

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$  fails to span all of  $\mathbb{R}^n$  exactly when  $\text{RREF}[\mathbf{v}_1 \dots \mathbf{v}_m]$  has a row of zeros:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & | & a \\ -1 & 0 & | & b \\ 0 & 1 & | & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix}$$

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 10.4**

Consider the set of vectors  $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$ . Does

 $\mathbb{R}^4 = \text{span } S?$



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 10.5

Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2\}$$

Does  $\mathcal{P}^3 = \text{span } S$ ?

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Definition 10.6

A subset of a vector space is called a **subspace** if it is itself a vector space.

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
**Part 4 (Day 10)**

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Fact 10.7

If  $S$  is a subset of a vector space  $V$ , then  $\text{span } S$  is a subspace of  $V$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Remark 10.8**

To prove that a subset is a subspace, you need only verify that  $c\mathbf{v} + d\mathbf{w}$  belongs to the subset for any choice of vectors  $\mathbf{v}, \mathbf{w}$  from the subset and any real scalars  $c, d$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 10.9**

Prove that  $P = \{ax^2 + b : a, b \text{ are both real numbers}\}$  is a subspace of the vector space of all degree-two polynomials by showing that  $c(a_1x^2 + b_1) + d(a_2x^2 + b_2)$  belongs to  $P$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

**Part 4 (Day 10)**

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 10.10**

Consider the subset of  $\mathbb{R}^2$  where at least one coordinate of each vector is 0.



Find a linear combination  $c\mathbf{v} + d\mathbf{w}$  that does not belong to this subset.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Fact 10.11**

Suppose a subset  $S$  of  $V$  is isomorphic to another vector space  $W$ . Then  $S$  is a subspace of  $V$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 10.12**

Show that the set of  $2 \times 2$  matrices

$$S = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : a, b \text{ are real numbers} \right\}$$

is a subspace of  $\mathbb{R}^{2 \times 2}$  by identifying a Euclidean space isomorphic to  $S$ .



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Module S: Structure of vector spaces

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

At the end of this module, students will be able to...

- **S1. Linear independence** Determine if a set of Euclidean vectors is linearly dependent or independent.
- **S2. Basis verification** Determine if a set of vectors is a basis of a vector space
- **S3. Basis construction** Construct a basis for the subspace spanned by a given set of vectors.
- **S4. Dimension** I can compute the dimension of a vector space.

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

Before beginning this module, each student should be able to...

- Add Euclidean vectors and multiply Euclidean vectors by scalars.
- Perform basic manipulations of augmented matrices and linear systems **(Standard(s) E1,E2,E3)**.
- Apply linear combinations and spanning sets **(Standard(s) V2,V3)**.

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/prec calculus/vectors-prec calc/vector-addition-subtraction/v/adding-and-subtracting-vectors>
- <https://www.khanacademy.org/math/prec calculus/vectors-prec calc/combined-vector-operations/v/combined-vector-operations-example>

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

**Part 1 (Day 12)**  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

# Application Activities - Module S Part 1 - Class Day 12

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

**Part 1 (Day 12)**

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 12.1

In the previous module, we considered

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

and showed that  $\text{span } S \neq \mathbb{R}^4$ . Find two vectors from this set that are linear combinations of the other three vectors.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 12.2

We say that a set of vectors is **linearly dependent** if one vector in the set belongs to the span of the others. Otherwise, we say the set is **linearly independent**.

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 12.3

Suppose  $3\mathbf{v}_1 - 5\mathbf{v}_2 = \mathbf{v}_3$ , so the set  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent. Is the vector equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$  consistent with one solution, consistent with infinitely many solutions, or inconsistent?



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Fact 12.4**

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly dependent if and only if  $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$  is consistent with infinitely many solutions.

## Activity 12.5

Find

$$\text{RREF} \left[ \begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 2 & 0 \end{array} \right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly dependent.

### Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

### Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

### Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

### Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

### Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

### Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Fact 12.6**

A set of Euclidean vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly dependent if and only if  $\text{RREF} \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$  has a column without a pivot position.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

**Part 1 (Day 12)**

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 12.7**

Is the set of Euclidean vectors  $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$  linearly dependent or linearly independent?

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 12.8

Is the set of polynomials  $\{x^3 + 1, x^2 + 2, 4 - 7x, 2x^3 + x\}$  linearly dependent or linearly independent?

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Application Activities - Module S Part 2 - Class Day 13

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
**Part 2 (Day 13)**  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 13.1

Last time we saw that  $\{x^3 + 1, x^2 + 2, 4 - 7x, 2x^3 + x\}$  is linearly independent. Show that it spans  $\mathcal{P}^3$ .

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Definition 13.2

A **basis** is a linearly independent set that spans a vector space.



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)**
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Observation 13.3

A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

# Activity 13.4

Which of the following sets are bases for  $\mathbb{R}^4$ ?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$
$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\}$$
$$\left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

**Part 2 (Day 13)**

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 13.5

If  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is a basis for  $\mathbb{R}^4$ , that means  $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$  doesn't have a column without a pivot position, and doesn't have a row of zeros. What is  $\text{RREF}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$ ?

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

Fact 13.6

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$  is a basis for  $\mathbb{R}^n$  if and only if  $m = n$  and

$$\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

That is, a basis for  $\mathbb{R}^n$  must have exactly  $n$  vectors and its square matrix must row-reduce to the **identity matrix** containing all zeros except for a downward diagonal of ones.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

**Part 2 (Day 13)**

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 13.7

Consider the set  $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

**Part 2 (Day 13)**

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 13.7**

Consider the set  $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

*Part 1:* Use RREF  $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$  to identify which vector may be removed to make the set linearly independent.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

**Part 2 (Day 13)**

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 13.7**

Consider the set  $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

*Part 1:* Use RREF  $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$  to identify which vector may be removed to make the set linearly independent.

*Part 2:* Find a basis for  $\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}.$

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)**

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Application Activities - Module S Part 3 - Class Day 14



## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
**Part 3 (Day 14)**

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Fact 14.1

To compute a basis for the subspace  $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ , simply remove the vectors corresponding to the non-pivot columns of  $\text{RREF}[\mathbf{v}_1 \ \dots \ \mathbf{v}_m]$ .

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Activity 14.2

Find all subsets of  $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$  that are a basis for  $\text{span } S$  by changing the order of the vectors in  $S$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

**Part 3 (Day 14)**

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 14.3

Assume  $\mathbf{w}_1 \neq \mathbf{w}_2$  are distinct vectors in  $V$ , which has a basis containing a single vector:  $\{\mathbf{v}\}$ . Could  $\{\mathbf{w}_1, \mathbf{w}_2\}$  be a basis?

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)**

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

**Fact 14.4**  
All bases for a vector space are the same size.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

**Part 3 (Day 14)**

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 14.5

The **dimension** of a vector space is given by the cardinality/size of any basis for the vector space.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 14.6**Find the dimension of each subspace of  $\mathbb{R}^4$ .

a)  $\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  <sup>c)</sup>  $\text{span} \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$  <sup>e)</sup>  $\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\},$

b)  $\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$  <sup>d)</sup>  $\text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
**Part 3 (Day 14)**

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 14.7

What is the dimension of the vector space of 7th-degree (or less) polynomials  $\mathcal{P}^7$ ?

a) 6

b) 7

c) 8

d) infinite

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
**Part 3 (Day 14)**

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 14.8

What is the dimension of the vector space of all polynomials  $\mathcal{P}$ ?

a) 6

b) 7

c) 8

d) infinite



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Observation 14.9

Several interesting vector spaces are infinite-dimensional:

- The space of polynomials  $\mathcal{P}$  (consider the set  $\{1, x, x^2, x^3, \dots\}$ ).
- The space of continuous functions  $C(\mathbb{R})$  (which contains all polynomials, in addition to other functions like  $e^x = 1 + x + x^2/2 + x^3/3 + \dots$ ).
- The space of real number sequences  $\mathbb{R}^\infty$  (consider the set  $\{(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, \dots), \dots\}$ ).

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

**Part 3 (Day 14)**

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Fact 14.10**

Every vector space with finite dimension, that is, every vector space with a basis of the form  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is isomorphic to a Euclidean space  $\mathbb{R}^n$ :

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

◀ ◻ ▶ ◀ ◻ ▶ ◀ ≡ ▶ ◀ ≡ ▶ ≡ 🔍 ↺

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

At the end of this module, students will be able to...

- **A1. Linear maps as matrices** I can write the matrix (with respect to the standard bases) corresponding to a linear transformation between Euclidean spaces.
- **A2. Linear map verification** I can determine if a map between vector spaces is linear or not.
- **A3. Injectivity and Surjectivity** I can determine if a given linear map is injective and/or surjective
- **A4. Kernel and Image** I can compute the kernel and image of a linear map, including finding bases.

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing (**Standard(s) E1, E2, E3, E4**).
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace (**Standard(s) V3**).
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent (**Standard(s) S1**).
- State the definition of a basis, and determine if a set of vectors is a basis (**Standard(s) S2**).

### Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

### Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

### Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

### Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

### Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

### Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

The following resources will help you prepare for this module.

- Review the supporting Standards listed above.

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

## Application Activities - Module A Part 1 - Class Day 17

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 17.1

A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if  $V$  and  $W$  are vector spaces, a map  $T : V \rightarrow W$  is called a linear transformation if

- 1  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$  for any  $\vec{v}, \vec{w} \in V$
- 2  $T(c\vec{v}) = cT(\vec{v})$  for any  $c \in \mathbb{R}, \vec{v} \in V$ .

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.

$V$  is called the **domain** of  $T$  and  $W$  is called the **co-domain** of  $T$ .



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 17.2**

Determine if each of the following maps are linear transformations

(a)  $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $T_1 \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \sqrt{a^2 + b^2}$

(b)  $T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by  $T_2 \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$

(c)  $T_3 : \mathcal{P}_d \rightarrow \mathcal{P}_{d-1}$  given by  $T_3(f(x)) = f'(x)$ .

(d)  $T_4 : C(\mathbb{R}) \rightarrow C(\mathbb{R})$  given by  $T_4(f(x)) = f(-x)$

(e)  $T_5 : \mathcal{P} \rightarrow \mathcal{P}$  given by  $T_5(f(x)) = f(x) + x^2$

# Activity 17.3

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear transformation, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and  $T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute each of the following:

(a)  $T \left( \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right)$

(b)  $T \left( \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right)$

(c)  $T \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$

(d)  $T \left( \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} \right)$

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

**Part 1 (Day 17)**  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

**Part 1 (Day 17)**

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 17.4**

Suppose  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  is a linear transformation. What is the smallest number of vectors needed to determine  $T$ ? In other words, what is the smallest number  $n$  such that there are  $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^4$  and given  $T(\vec{v}_1), \dots, T(\vec{v}_n)$  you can determine  $T(\vec{w})$  for *any*  $\vec{w} \in \mathbb{R}^2$ ?

(a) 2

(b) 3

(c) 4

(d) 5

(e) You need infinitely many

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

**Part 1 (Day 17)**

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Observation 17.5

Fix an ordered basis for  $V$ . Since every vector can be written *uniquely* as a linear combination of basis vectors, a linear transformation  $T : V \rightarrow W$  corresponds exactly to a choice of where each basis vector goes. For convenience, we can thus encode a linear transformation as a matrix, with one column for the image of each basis vector (in order).

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Example 17.6**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be a linear transformation with

$$T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Then the matrix corresponding to  $T$  with respect to the standard bases is

$$\begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix}.$$

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 17.7

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to standard basis.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

**Part 1 (Day 17)**

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 17.8**

Let  $D : \mathcal{P}^3 \rightarrow \mathcal{P}^2$  be the derivative map (recall this is a linear transformation). Write the matrix corresponding to  $D$  with respect to the ordered basis  $\{1, x, x^2, x^3\}$  for  $\mathcal{P}^3$  and  $\{1, x, x^2\}$  for  $\mathcal{P}^2$ .

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
**Part 2 (Day 18)**  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

# Application Activities - Module A Part 2 - Class Day 18



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 18.1

Let  $T : V \rightarrow W$  be a linear transformation.

- $T$  is called **injective** or **one-to-one** if  $T$  does not map two distinct values to the same place. More precisely,  $T$  is injective if  $T(\vec{v}) \neq T(\vec{w})$  whenever  $\vec{v} \neq \vec{w}$ .
- $T$  is called **surjective** or **onto** if every element of  $W$  is mapped to by an element of  $V$ . More precisely, for every  $\vec{w} \in W$ , there is some  $v \in V$  with  $T(\vec{v}) = \vec{w}$ .

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
**Part 2 (Day 18)**  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 18.2

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Determine if  $T$  is injective, surjective, both, or neither.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

**Part 2 (Day 18)**

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 18.3**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ . Determine if  $T$  is injective, surjective, both, or neither.

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
**Part 2 (Day 18)**  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Definition 18.4

We also have two important sets called the **kernel** of  $T$  and the **image** of  $T$ .

$$\ker T = \{ \vec{v} \in V \mid T(\vec{v}) = 0 \}$$

$$\operatorname{Im} T = \{ \vec{w} \in W \mid \text{there is some } v \in V \text{ with } T(\vec{v}) = \vec{w} \}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

**Part 2 (Day 18)**

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 18.5

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$  (for the standard basis). Find the kernel and image of  $T$ .

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
**Part 2 (Day 18)**  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 18.6

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$  (for the standard basis). Find the kernel and image of  $T$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

**Part 2 (Day 18)**

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 18.7**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix} \text{ (for the standard basis).}$$

- 1) Write a system of equations whose solution set is the kernel.
- 2) Compute  $\text{RREF}(A)$  and solve the system of equations.
- 3) Compute the kernel of  $T$
- 4) Find a basis for the kernel of  $T$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

**Part 2 (Day 18)**

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 18.8

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix} \text{ (for the standard basis).}$$

- 1) Find a set of vectors that span the image of  $T$
- 2) Find a basis for the image of  $T$ .



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)**

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Application Activities - Module A Part 3 - Class Day 19

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Part 1 (Day 17)  
Part 2 (Day 18)  
**Part 3 (Day 19)**

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

## Activity 19.1

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)**

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

**Activity 19.1**  
*Part 1:* Describe surjective linear transformations in terms of the image.

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
**Part 3 (Day 19)**

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 19.1

*Part 1:* Describe surjective linear transformations in terms of the image.

*Part 2:* Describe injective linear transformations in terms of the kernel.

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
**Part 3 (Day 19)**

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 19.2

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with matrix  $A$  (for the standard basis). You have cards containing a number of statements about  $T$  and  $A$ . Sort them into groups of equivalent statements, and post them on your board.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

**Part 3 (Day 19)**

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 19.3

Cycle around the room counter-clockwise. If they have two things grouped together that you know are not equivalent, write a reason or counter-example on a sticky note.

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

◀ ◻ ▶ ◀ ◻ ▶ ◀ ≡ ▶ ◀ ≡ ▶ ≡ ↺ 🔍 ↻

#### Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

#### Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

#### Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

#### Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

#### Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

#### Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

At the end of this module, students will be able to...

- **M1. Matrix multiplication** Multiply matrices.
- **M2. Invertible matrices** Determine if a square matrix is invertible or not.
- **M3. Matrix inverses** Compute the inverse matrix of an invertible matrix.



Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

Before beginning this module, each student should be able to...

- Compose functions of real numbers
- Solve systems of linear equations (**Standard(s) E3**)
- Find the matrix corresponding to a linear transformation (**Standard(s) A1**)
- Determine if a linear transformation is injective and/or surjective (**Standard(s) A3**)
- Interpret the ideas of injectivity and surjectivity in multiple ways

#### Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

#### Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

#### Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

#### Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

#### Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

#### Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

The following resources will help you prepare for this module.

- <https://www.khanacademy.org/math/algebra2/manipulating-functions/function-composition/v/function-composition>

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)**
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)
- Part 4 (Day 28)

# Application Activities - Module M Part 1 - Class Day 21

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 21.1**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be

given by the matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

What is the domain of the composition map  $S \circ T$ ?

(a)  $\mathbb{R}$

(b)  $\mathbb{R}^2$

(c)  $\mathbb{R}^3$

(d)  $\mathbb{R}^4$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 21.2**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be

given by the matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

What is the codomain of the composition map  $S \circ T$ ?

(a)  $\mathbb{R}$

(b)  $\mathbb{R}^2$

(c)  $\mathbb{R}^3$

(d)  $\mathbb{R}^4$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 21.3**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be

given by the matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

The matrix corresponding to  $S \circ T$  will lie in which matrix space?

(a)  $4 \times 3$  matrices

(b)  $4 \times 2$  matrices

(c)  $3 \times 2$  matrices

(d)  $2 \times 3$  matrices

(e)  $2 \times 4$  matrices

(f)  $3 \times 4$  matrices

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 21.4

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be

given by the matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

Compute  $(S \circ T)(\vec{e}_1)$ ,  $(S \circ T)(\vec{e}_2)$ , and  $(S \circ T)(\vec{e}_3)$ .

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 21.5

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by the matrix  $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$  and  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be

given by the matrix  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$ .

Find the matrix corresponding to  $S \circ T$  with respect to the standard bases.



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 21.6**

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$  and  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given

by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ .

What is the domain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
- (b)  $\mathbb{R}^2$
- (c)  $\mathbb{R}^3$
- (d)  $\mathbb{R}^4$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 21.7**

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$  and  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given

by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ .

What is the codomain of the composition map  $S \circ T$ ?

- (a)  $\mathbb{R}$
- (b)  $\mathbb{R}^2$
- (c)  $\mathbb{R}^3$
- (d)  $\mathbb{R}^4$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 21.8**

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$  and  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given

by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ .

The matrix corresponding to  $S \circ T$  will lie in which matrix space?

- (a)  $2 \times 2$  matrices
- (b)  $2 \times 3$  matrices
- (c)  $3 \times 2$  matrices
- (d)  $3 \times 3$  matrices

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 21.9**

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by the matrix  $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$  and  $S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given

by the matrix  $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ .

Find the matrix corresponding to  $S \circ T$  with respect to the standard bases.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 21.10**

Let  $T : \mathbb{R}^1 \rightarrow \mathbb{R}^4$  be given by the matrix  $B = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}$  and  $S : \mathbb{R}^4 \rightarrow \mathbb{R}^1$  be given by the matrix  $A = \begin{bmatrix} 2 & 3 & 2 & 5 \end{bmatrix}$ .

Find the matrix corresponding to  $S \circ T$  with respect to the standard bases.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

**Part 1 (Day 21)**

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Definition 21.11**

We define the product of a  $m \times n$  matrix  $A$  and a  $n \times k$  matrix  $B$  to be the  $m \times k$  matrix  $AB$  corresponding to the composition map of the two corresponding linear functions.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

**Part 1 (Day 21)**

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Fact 21.12**

If  $AB$  is defined,  $BA$  need not be defined, and if it is defined, it is in general different from  $AB$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 21.13**

Let  $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$  and  $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- 1 Compute  $AX$
- 2 Interpret the system of equations below as a matrix equation

$$3x + y - z = 5$$

$$2x + 4z = -7$$

$$-x + 3y + 5z = 2$$



Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

Module M

Part 1 (Day 21)

**Part 2 (Day 22)**

Part 3 (Day 23)

Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

# Application Activities - Module M Part 2 - Class Day 22

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
**Part 2 (Day 22)**  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 22.1

Find a  $4 \times 4$  matrix  $I$  such that, for any  $4 \times n$  matrix  $A$ ,  $IA = A$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

**Part 2 (Day 22)**

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Definition 22.2**

The identity matrix  $I_n$  (sometimes written as just  $I$  if  $n$  is understood) is the  $n \times n$  matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 22.3

Each row operation can be interpreted as a matrix multiplication. Let  $A$  be a  $4 \times 4$  matrix.

- 1) Find a matrix  $S_1$  such that  $S_1A$  is the result of swapping the second and fourth rows of  $A$ .
- 2) Find a matrix  $S_2$  such that  $S_2A$  is the result of adding 5 times the third row of  $A$  to the first.
- 3) Find a matrix  $S_3$  such that  $S_3A$  is the result of doubling the fourth row of  $A$ .

*Hint: Tweak the identity matrix slightly.*

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 22.4

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with matrix  $A$  (for the standard basis). Consider the following statements about  $T$

- (a)  $T$  is injective
- (b)  $T$  is surjective
- (c)  $T$  is bijective (i.e. both injective and surjective)
- (d)  $AX = B$  has a solution for all  $m \times 1$  matrices  $B$
- (e)  $AX = B$  has a unique solution for all  $m \times 1$  matrices  $B$
- (f)  $AX = 0$  has a non-trivial solution.
- (g) The columns of  $A$  span  $\mathbb{R}^m$
- (h) The columns of  $A$  are linearly independent
- (i) The columns of  $A$  are a basis of  $\mathbb{R}^m$
- (j)  $\text{RREF}(A)$  has  $n$  pivot columns
- (k)  $\text{RREF}(A)$  has  $m$  pivot columns

Sort these statements into groups of equivalent statements.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

**Part 2 (Day 22)**

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 22.5

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with matrix  $A$ . If  $T$  is injective, which of the following must be true?

- (a)  $A$  has strictly more columns than rows
- (b)  $A$  has at least as many columns as rows
- (c)  $A$  is square
- (d)  $A$  has at least as many rows as columns
- (e)  $A$  has strictly more rows than columns

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
**Part 2 (Day 22)**  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 22.6

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with matrix  $A$ . If  $T$  is surjective, which of the following must be true?

- (a)  $A$  has strictly more columns than rows
- (b)  $A$  has at least as many columns as rows
- (c)  $A$  is square
- (d)  $A$  has at least as many rows as columns
- (e)  $A$  has strictly more rows than columns

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

**Part 2 (Day 22)**

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 22.7**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with matrix  $A$ . If  $T$  is bijective, which of the following must be true?

- (a)  $A$  has strictly more columns than rows
- (b)  $A$  has at least as many columns as rows
- (c)  $A$  is square
- (d)  $A$  has at least as many rows as columns
- (e)  $A$  has strictly more rows than columns



Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
**Part 3 (Day 23)**

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

# Application Activities - Module M Part 3 - Class Day 23

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 23.1

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear map with matrix  $A$ .

If  $T$  is a bijection, then  $AX = B$  has a unique solution for all  $B \in \mathbb{R}^n$ . Thus we can define a map  $T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by defining  $T^{-1}(B)$  to be this solution. It follows immediately that  $T \circ T^{-1}$  is the identity map. The matrix corresponding to  $T^{-1}$  is denoted  $A^{-1}$ , and  $A$  is called an **invertible matrix**.

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
**Part 3 (Day 23)**

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 23.2

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$ .

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
**Part 3 (Day 23)**

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 23.2

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$ .

*Part 1:* Solve  $AX = \vec{e}_1$  to determine  $T^{-1}(\vec{e}_1)$

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
**Part 3 (Day 23)**

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 23.2

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$ .

*Part 1:* Solve  $AX = \vec{e}_1$  to determine  $T^{-1}(\vec{e}_1)$

*Part 2:* Solve  $AX = \vec{e}_2$  to determine  $T^{-1}(\vec{e}_2)$

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
**Part 3 (Day 23)**

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 23.2

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$ .

*Part 1:* Solve  $AX = \vec{e}_1$  to determine  $T^{-1}(\vec{e}_1)$

*Part 2:* Solve  $AX = \vec{e}_2$  to determine  $T^{-1}(\vec{e}_2)$

*Part 3:* Solve  $AX = \vec{e}_3$  to determine  $T^{-1}(\vec{e}_3)$

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 23.2

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$ .

*Part 1:* Solve  $AX = \vec{e}_1$  to determine  $T^{-1}(\vec{e}_1)$

*Part 2:* Solve  $AX = \vec{e}_2$  to determine  $T^{-1}(\vec{e}_2)$

*Part 3:* Solve  $AX = \vec{e}_3$  to determine  $T^{-1}(\vec{e}_3)$

*Part 4:* Compute  $A^{-1}$

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
**Part 3 (Day 23)**

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 23.3

Find the inverse of the matrix  $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$



## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
**Part 3 (Day 23)**

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 23.4

Determine if the matrix  $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$  is invertible or not.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Module G: Geometry of Linear Maps

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

At the end of this module, students will be able to...

- **G1. Determinants** Compute the determinant of a square matrix.
- **G2. Eigenvalues** Find the eigenvalues of a square matrix, along with their algebraic multiplicities.
- **G3. Eigenvectors** Find the eigenspace of a square matrix associated to a given eigenvalue.
- **G4. Geometric multiplicity** Compute the geometric multiplicity of an eigenvalue of a square matrix.

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **(Standard(s) A1)**.
- Recall and use the definition of a linear transformation **(Standard(s) A2)**.
- Find all roots of quadratic polynomials (including complex ones), and be able to use the rational root theorem to find all rational roots of a higher degree polynomial.
- Interpret the statement “ $A$  is an invertible matrix” in many equivalent ways in different contexts.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

The following resources will help you prepare for this module.

- Finding the area of a parallelogram: <https://www.khanacademy.org/math/basic-geo/basic-geo-area-and-perimeter/parallelogram-area/a/area-of-parallelogram>
- Factoring quadratics: <https://www.khanacademy.org/math/algebra2/polynomial-functions/factoring-polynomials-quadratic-forms-alg2/v/factoring-polynomials-1>
- Finding complex roots of quadratics: <https://www.khanacademy.org/math/algebra2/polynomial-functions/quadratic-equations-with-complex-numbers/v/complex-roots-from-the-quadratic-formula>
- Finding all roots of polynomials: <https://www.khanacademy.org/math/algebra2/polynomial-functions/finding-zeros-of-polynomials/v/finding-roots-or-zeros-of-polynomial-1>
- The Rational Root Theorem: [https://artofproblemsolving.com/wiki/index.php?title=Rational\\_Root\\_Theorem](https://artofproblemsolving.com/wiki/index.php?title=Rational_Root_Theorem)

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

**Part 1 (Day 25)**  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

# Application Activities - Module G Part 1 - Class Day 25

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

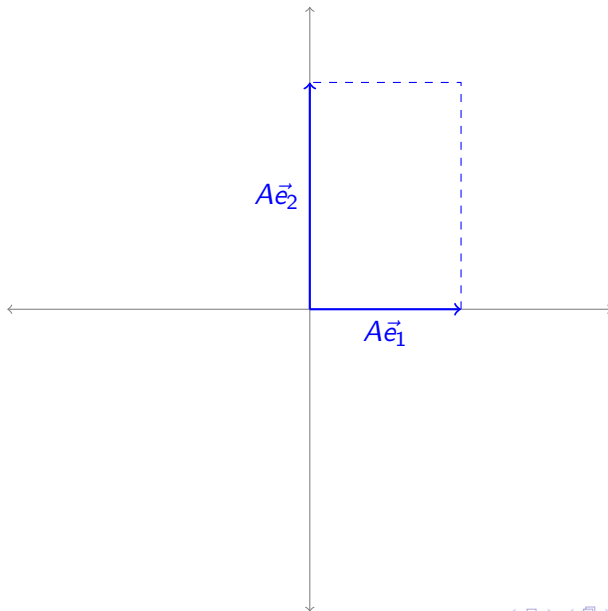
Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

**Part 1 (Day 25)**  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

# Activity 25.1

Consider the linear transformation  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the matrix  $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 25.2

Consider the following linear transformations  $A_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .

- $A_2 = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

- $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

- $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

For each linear transformation, do the following:

- (a) Draw a graph showing the image of the unit square.
- (b) Compute how much the area was stretched out.
- (c) Determine which axes (or lines) were preserved; how were they stretched out?



## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 25.3

Our goal is to define a function  $\det$  that takes a square matrix (linear transformation  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ ) and returns its area stretching factor. This function is called the **determinant**.

What properties should this function have?

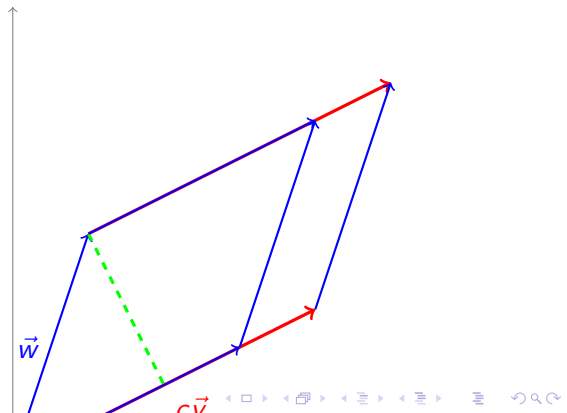
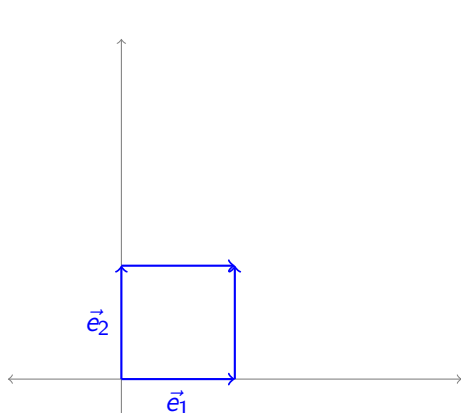
Match the four pictures to the following four expressions

$$\det(\vec{e}_1, \vec{e}_2)$$

$$\det(\vec{v}, \vec{v})$$

$$\det(c\vec{v}, \vec{w})$$

$$\det(\vec{u} + \vec{v}, \vec{w})$$



## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

**Part 1 (Day 25)**  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 25.4

What can you conclude about each of the following?

- 1  $\det(\vec{e}_1, \vec{e}_2)$
- 2  $\det(\vec{v}, \vec{v})$
- 3  $\det(c\vec{v}, \vec{w})$
- 4  $\det(\vec{u} + \vec{v}, \vec{w})$

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Definition 25.5

To summarize, we have 3 properties (stated here over  $\mathbb{R}^n$ )

**P1:**  $\det(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n) = 1$

**P2:** If  $\vec{v}_i = \vec{v}_j$  for some  $i \neq j$ , then  $\det(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = 0$ .

**P3:** The determinant is linear in each column.

These three properties uniquely define the **determinant**, as we shall see.

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

**Part 1 (Day 25)**

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

**Observation 25.6**

Note that if  $\vec{v}, \vec{w} \in \mathbb{R}^2$  and  $A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$  we will write either  $\det(A)$  or  $\det(\vec{v}, \vec{w})$  as is convenient.

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

**Part 1 (Day 25)**  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

**Activity 25.7**

How are  $\det(\vec{v}, \vec{w})$  and  $\det(\vec{w}, \vec{v})$  related?

- (a)  $\det(\vec{v}, \vec{w}) = \det(\vec{w}, \vec{v})$
- (b)  $\det(\vec{v}, \vec{w}) = -\det(\vec{w}, \vec{v})$
- (c) They are unrelated
- (d) They are related, but not by either (a) or (b).

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

**Part 1 (Day 25)**  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Observation 25.8

Note that this implies that the determinant is actually a *signed* area (volume)!

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

**Activity 25.9**

How are  $\det(\vec{v} + \vec{w}, \vec{w})$  and  $\det(\vec{v}, \vec{w})$  related?

- (a)  $\det(\vec{v} + \vec{w}, \vec{w}) = \det(\vec{v}, \vec{w})$
- (b)  $\det(\vec{v} + \vec{w}, \vec{w}) = -\det(\vec{v}, \vec{w})$
- (c) They are unrelated
- (d) They are related, but not by either (a) or (b).

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

**Part 1 (Day 25)**  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Observation 25.10

Note that we now understand the effect of any column operation on the determinant.



Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
**Part 2 (Day 26)**  
Part 3 (Day 27)  
Part 4 (Day 28)

# Application Activities - Module G Part 2 - Class Day 26

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
**Part 2 (Day 26)**  
Part 3 (Day 27)  
Part 4 (Day 28)

## Fact 26.1

By a geometric argument, one can show that the determinant of a matrix and its transpose are the same. Thus, row operations behave like column operations. In particular, we can use row reduction to compute determinants.

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
**Part 2 (Day 26)**  
Part 3 (Day 27)  
Part 4 (Day 28)

**Fact 26.2**

Row operations change the determinant in the following way

- 1 Elementary row operations (adding a multiple of one row to another) do not change the determinant.
- 2 Diagonal operations (multiplying a row by a scalar) multiplies the determinant by the same amount.
- 3 Swapping two rows multiplies the determinant by  $-1$ .

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)**
- Part 3 (Day 27)
- Part 4 (Day 28)

**Activity 26.3**

Compute  $\det \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

**Part 2 (Day 26)**

Part 3 (Day 27)

Part 4 (Day 28)

**Activity 26.4**

Which of the following is the same as  $\det \begin{bmatrix} 3 & -2 & 0 \\ 5 & -1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$ ?

(a)  $\det \begin{bmatrix} 3 & -2 \\ 5 & -1 \end{bmatrix}$

(b)  $\det \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$

(c)  $\det \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}$

(d) None of these

*Hint: Draw a picture*

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

**Part 2 (Day 26)**

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 26.5

Which of the following is the same as  $\det \begin{bmatrix} 3 & 0 & 7 \\ 5 & 1 & 2 \\ -2 & 0 & 6 \end{bmatrix}$ ?

(a)  $\det \begin{bmatrix} 3 & 7 \\ 5 & 2 \end{bmatrix}$

(b)  $\det \begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix}$

(c)  $\det \begin{bmatrix} 5 & 2 \\ -2 & 6 \end{bmatrix}$

(d) None of these

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)**
- Part 3 (Day 27)
- Part 4 (Day 28)

Activity 26.6

Compute  $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$

Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)**
- Part 3 (Day 27)
- Part 4 (Day 28)

Activity 26.7

Using the fact that  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , compute  $\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 5 \\ 0 & 3 & 3 \end{bmatrix}$ .



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)**
- Part 3 (Day 27)
- Part 4 (Day 28)

Activity 26.8

Compute 
$$\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$$

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

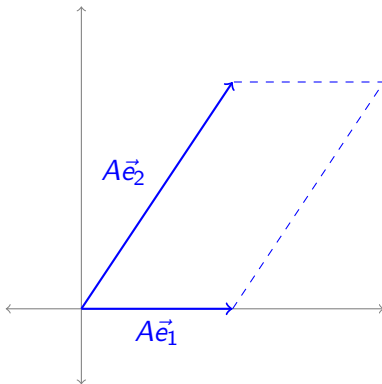
Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
**Part 3 (Day 27)**  
Part 4 (Day 28)

# Application Activities - Module G Part 3 - Class Day 27

## Activity 27.1

Consider the linear transformation  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by the matrix  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



Observe

$$A\vec{e}_1 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\vec{e}_1$$

Is there another vector  $\vec{v} \in \mathbb{R}^2$  such that  $A\vec{v} = \lambda\vec{v}$  for some  $\lambda \in \mathbb{R}$ ?

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

**Part 3 (Day 27)**

Part 4 (Day 28)

## Definition 27.2

Let  $A \in M_n(\mathbb{R})$ . An **eigenvector** is a vector  $\vec{x} \in \mathbb{R}^n$  such that  $A\vec{x}$  is parallel to  $\vec{x}$ ; in other words,  $A\vec{x} = \lambda\vec{x}$  for some scalar  $\lambda$ , which is called an **eigenvalue**

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

**Part 3 (Day 27)**

Part 4 (Day 28)

## Observation 27.3

Observe that  $A\vec{x} = \lambda\vec{x}$  is equivalent to  $(A - \lambda I)\vec{x} = 0$ .

- To find eigenvalues, we need to find values of  $\lambda$  such that  $A - \lambda I$  has a nontrivial kernel; equivalently,  $A - \lambda I$  is not invertible, which is equivalent to  $\det(A - \lambda I) = 0$ .  $\det(A - \lambda I)$  is called the **characteristic polynomial**.
- Once an eigenvalue is found, the eigenvectors form a subspace called the **eigenspace**, which is simply the kernel of  $A - \lambda I$ . Each eigenvalue will have an associated eigenspace.

Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
**Part 3 (Day 27)**  
Part 4 (Day 28)

## Activity 27.4

Let  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
**Part 3 (Day 27)**  
Part 4 (Day 28)

**Activity 27.4**

Let  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .

*Part 1:* Find the eigenvalues of  $A$ .

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
**Part 3 (Day 27)**  
Part 4 (Day 28)

**Activity 27.4**

Let  $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$ .

*Part 1:* Find the eigenvalues of  $A$ .

*Part 2:* Find the eigenspace associated to the eigenvalue 3.



Module E

- Part 1 (Day 3)
- Part 2 (Day 4)
- Part 3 (Day 5)

Module V

- Part 1 (Day 7)
- Part 2 (Day 8)
- Part 3 (Day 9)
- Part 4 (Day 10)

Module S

- Part 1 (Day 12)
- Part 2 (Day 13)
- Part 3 (Day 14)

Module A

- Part 1 (Day 17)
- Part 2 (Day 18)
- Part 3 (Day 19)

Module M

- Part 1 (Day 21)
- Part 2 (Day 22)
- Part 3 (Day 23)

Module G

- Part 1 (Day 25)
- Part 2 (Day 26)
- Part 3 (Day 27)**
- Part 4 (Day 28)

# Activity 27.5

Find all the eigenvalues and associated eigenspaces for the matrix

$$\begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}.$$

Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

**Part 4 (Day 28)**

# Application Activities - Module G Part 4 - Class Day 28

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
**Part 4 (Day 28)**

## Activity 28.1

If  $A$  is a  $4 \times 4$  matrix, what is the largest number of eigenvalues  $A$  can have?

- (a) 3
- (b) 4
- (c) 5
- (d) 6
- (e) It can have infinitely many

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 28.2

2 is an eigenvalue of each of the matrices  $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$  and

$$B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}.$$

Compute the eigenspace associated to 2 for both  $A$  and  $B$ .

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Definition 28.3

- The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.
- The **geometric multiplicity** of an eigenvalue is the dimension of the eigenspace.

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 28.4

How are the algebraic and geometric multiplicities related?

- (a) The algebraic multiplicity is always at least as big as than the geometric multiplicity.
- (b) The geometric multiplicity is always at least as big as the algebraic multiplicity.
- (c) Sometimes the algebraic multiplicity is larger and sometimes the geometric multiplicity is larger.

## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
Part 4 (Day 28)

## Activity 28.5

Find the eigenvalues, along with both their algebraic and geometric multiplicities,

for the matrix 
$$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

**Part 4 (Day 28)**

## Activity 28.6

$$\text{Let } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$



## Module E

Part 1 (Day 3)  
Part 2 (Day 4)  
Part 3 (Day 5)

## Module V

Part 1 (Day 7)  
Part 2 (Day 8)  
Part 3 (Day 9)  
Part 4 (Day 10)

## Module S

Part 1 (Day 12)  
Part 2 (Day 13)  
Part 3 (Day 14)

## Module A

Part 1 (Day 17)  
Part 2 (Day 18)  
Part 3 (Day 19)

## Module M

Part 1 (Day 21)  
Part 2 (Day 22)  
Part 3 (Day 23)

## Module G

Part 1 (Day 25)  
Part 2 (Day 26)  
Part 3 (Day 27)  
**Part 4 (Day 28)**

## Activity 28.6

Let  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

*Part 1:* Find the eigenvalues of  $A$

## Module E

Part 1 (Day 3)

Part 2 (Day 4)

Part 3 (Day 5)

## Module V

Part 1 (Day 7)

Part 2 (Day 8)

Part 3 (Day 9)

Part 4 (Day 10)

## Module S

Part 1 (Day 12)

Part 2 (Day 13)

Part 3 (Day 14)

## Module A

Part 1 (Day 17)

Part 2 (Day 18)

Part 3 (Day 19)

## Module M

Part 1 (Day 21)

Part 2 (Day 22)

Part 3 (Day 23)

## Module G

Part 1 (Day 25)

Part 2 (Day 26)

Part 3 (Day 27)

Part 4 (Day 28)

## Activity 28.6

Let  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .

*Part 1:* Find the eigenvalues of  $A$

*Part 2:* Describe what this linear transformation is doing geometrically; draw a picture.