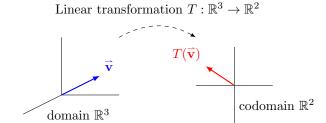
Section A.1

Definition A.1 A linear transformation (also known as a linear map) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T: V \to W$ is called a linear transformation if

- 1. $T(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = T(\vec{\mathbf{v}}) + T(\vec{\mathbf{w}})$ for any $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in V$.
- 2. $T(c\vec{\mathbf{v}}) = cT(\vec{\mathbf{v}})$ for any $c \in \mathbb{R}, \vec{\mathbf{v}} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition A.2 Given a linear transformation $T: V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.



Example A.3 Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right) = \begin{bmatrix} (x+u) - (z+w) \\ 3(y+v) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x-z \\ 3y \end{bmatrix} + \begin{bmatrix} u-w \\ 3v \end{bmatrix} = \begin{bmatrix} (x+u) - (z+w) \\ 3(y+v) \end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = T\left(\begin{bmatrix}cx\\cy\\cz\end{bmatrix}\right) = \begin{bmatrix}cx - cz\\3cy\end{bmatrix} \text{ and } cT\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = c\begin{bmatrix}x - z\\3y\end{bmatrix} = \begin{bmatrix}cx - cz\\3cy\end{bmatrix}$$

Therefore T is a linear transformation.

Example A.4 Let $T: \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix}\right) = T\left(\begin{bmatrix}2\\4\end{bmatrix}\right) = \begin{bmatrix}6\\4\\7\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}1\\0\\4\\-1\end{bmatrix} + \begin{bmatrix}5\\4\\6\\-5\end{bmatrix} = \begin{bmatrix}6\\4\\10\\-6\end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact A.5 A map between Euclidean spaces $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of \mathbb{R}^n .

For example, the following map is definitely linear because x-z and 3y are linear combinations of x,y,z:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2 , y + 3, and $y - 2^x$ are not linear combinations (even though x + y is):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

Activity A.6 (~5 min) Recall the following rules from calculus, where $D: \mathcal{P} \to \mathcal{P}$ is the derivative map defined by D(f(x)) = f'(x) for each polynomial f.

$$D(f+g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

- a) \mathcal{P} is not a vector space
- b) D is a linear map
- c) D is not a linear map

Activity A.7 (~10 min) Let the polynomial maps $S: \mathcal{P}^4 \to \mathcal{P}^3$ and $T: \mathcal{P}^4 \to \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x)$$
 $T(f(x)) = f'(x) + x^3$

Compute $S(x^4+x)$, $S(x^4)+S(x)$, $T(x^4+x)$, and $T(x^4)+T(x)$. Which of these maps is definitely not linear?

Fact A.8 If $L: V \to W$ is linear, then $L(\vec{\mathbf{z}}) = L(0\vec{\mathbf{v}}) = 0$ $L(\vec{\mathbf{v}}) = \vec{\mathbf{z}}$ where $\vec{\mathbf{z}}$ is the additive identity of the vector spaces V, W.

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Observation A.9 Showing $L: V \to W$ is not a linear transformation can be done by finding an example for any one of the following.

- Show $L(\vec{z}) \neq \vec{z}$ (where \vec{z} is the additive identity of L and W).
- Find $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in V$ such that $L(\vec{\mathbf{v}} + \vec{\mathbf{w}}) \neq L(\vec{\mathbf{v}}) + L(\vec{\mathbf{w}})$.
- Find $\vec{\mathbf{v}} \in V$ and $c \in \mathbb{R}$ such that $L(c\vec{\mathbf{v}}) \neq cL(\vec{\mathbf{v}})$.

Otherwise, L can be shown to be linear by proving the following in general.

- For all $\vec{\mathbf{v}}$, $\vec{\mathbf{w}} \in V$, $L(\vec{\mathbf{v}} + \vec{\mathbf{w}}) \neq L(\vec{\mathbf{v}}) + L(\vec{\mathbf{w}})$.
- For all $\vec{\mathbf{v}} \in V$ and $c \in \mathbb{R}$, $L(c\vec{\mathbf{v}}) \neq cL(\vec{\mathbf{v}})$.

Note the similarities between this process and showing that a subset of a vector space is/isn't a subspace.

Activity A.10 (~15 min) Continue to consider $S: \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to S(f(x)) + S(g(x)) for all polynomials f, g.

Part 2: Verify that S(cf(x)) is equal to cS(f(x)) for all real numbers c and polynomials f.

Part 3: Is S linear?

Activity A.11 (~20 min) Let the polynomial maps $S: \mathcal{P} \to \mathcal{P}$ and $T: \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2$$
 $T(f(x)) = 3xf(x^2)$

Part 1: Note that S(0) = 0 and T(0) = 0. So instead, show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Part 2: Prove that T is linear by verifying that T(f(x)+g(x))=T(f(x))+T(g(x)) and T(cf(x))=cT(f(x)).