Application Activities - Module G Part 4 - Class Day 28

Observation 28.1 Recall from last class:

- To find the eigenvalues of a matrix A, we need to find values of λ such that $A - \lambda I$ has a nontrivial kernel. Equivalently, we want values where $A - \lambda I$ is not invertible, so we want to know the values of λ where $\det(A - \lambda I) = 0$.
- $\det(A - \lambda I)$ is a polynomial with variable λ, called the **characteristic polynomial** of A. Thus the roots of the characteristic polynomial of A are exactly the eigenvalues of A.
- Once an eigenvalue λ is found, the **eigenspace** containing all **eigenvectors** $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is given by $\ker(A - \lambda I)$.

Activity 28.2 Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
Part 1: Compute the eigenvalues of A.
Part 2: Sketch a picture of the transformation of the unit square. What about this picture reveals that A has no real eigenvectors?

Activity 28.3 If A is a 4×4 matrix, what is the largest number of eigenvalues A can have?

(a) 3

(b) 4

(c) 5

(d) 6

(e) It can have infinitely many

Observation 28.4 An $n \times n$ matrix may have between 0 and n real-valued eigenvalues. But the Fundamental Theorem of Algebra implies that if complex eigenvalues are included, then every $n \times n$ matrix has exactly n eigenvalues (counting algebraic multiplicites).

Activity 28.5 The matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$ has characteristic polynomial $-\lambda(\lambda - 2)^2$.
Find the dimension of the eigenspace of A associated to the eigenvalue 2 (the dimension of the kernel of $A - 2I$).

Activity 28.6 The matrix $B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}$ has characteristic polynomial $-\lambda(\lambda - 2)^2$.
Find the dimension of the eigenspace of B associated to the eigenvalue 2 (the dimension of the kernel of $B - 2I$).

Observation 28.7 In the first example, the (2 dimensional) plane spanned by $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}4\\-2\\0\end{bmatrix}$ was preserved.

In the second example, only the (one dimensional) line spanned by $\begin{bmatrix}1\\0\\1\end{bmatrix}$ is preserved.

Definition 28.8 While the **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial, the **geometric multiplicity** of an eigenvalue is the dimension of its eigenspace.

Fact 28.9 As we've seen, the geometric multiplicity may be different than its algebraic multiplicity, but it cannot exceed it.
This fact is explored deeper and explained in Math 316, Linear Algebra II

Activity 28.10 Consider the 4×4 matrix

$$\begin{bmatrix}-3 & 1 & 2 & 1\\ -9 & 5 & -2 & -1\\ 31 & -17 & 6 & 3\\ -69 & 39 & -18 & -9\end{bmatrix}$$

Part 1: Use technology (e.g. Wolfram Alpha) to find its characteristic polynomial.
Part 2: Find the algebraic and geometric multiplicities for both eigenvalues.
