Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Solve a system of linear equations (including finding a basis of the solution space if it is homogeneous) by interpreting as an augmented matrix and row reducing **(Standard(s) E1, E2, E3, E4)**.
- State the definition of linear independence, and determine if a set of vectors is linearly dependent or independent **(Standard(s) V5)**.
- State the definition of a spanning set, and determine if a set of vectors spans a vector space or subspace **(Standard(s) V6, V7)**.
- State the definition of a basis, and determine if a set of vectors is a basis **(Standard(s) V8, V9)**.

Readiness Assurance Resources

The following resources will help you prepare for this module.

- Review the supporting Standards listed above.

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Which of the following is a solution to the system of linear equations

$$\begin{aligned} x + 3y - z &= 2 \\ 2x + 8y + 3z &= -1 \\ -x - y + 9z &= -10 \end{aligned}$$

(a) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

2) Find a basis for the solution set of the following homogeneous system of linear equations

$$\begin{aligned} x + 2y + -z - w &= 0 \\ -2x - 4y + 3z + 5w &= 0 \end{aligned}$$

(a) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$ (d) None of these are a basis.

3) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

4) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

5) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

6) Determine which property applies to the set of vectors

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

7) Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors ...

8) Suppose you know that every vector in $\mathbb{R}^5$ can be written as a linear combination of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. What can you conclude about n?

(a) $n \leq 5$
(b) $n = 5$
(c) $n \geq 5$
(d) n could be any positive integer

9) Suppose you know that every vector in $\mathbb{R}^5$ can be written uniquely as a linear combination of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. What can you conclude about n?

(a) $n \leq 5$
(b) $n = 5$
(c) $n \geq 5$
(d) n could be any positive integer

10) Suppose you know that every vector in $\mathbb{R}^5$ can be written uniquely as a linear combination of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. What can you conclude about the set $\{\vec{v}_1, \ldots, \vec{v}_n\}$?

(a) It does not span and is linearly dependent
(b) It does not span and is linearly independent
(c) It spans but it is linearly dependent
(d) It is a basis of $\mathbb{R}^3$.

Application Activities - Day 1

Definition. A **linear transformation** is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$
2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}$, $\vec{v} \in V$.

In other words, a map is linear if one can do vector space operations before applying the map or after, and obtain the same answer.
V is called the **domain** of T and W is called the **co-domain** of T.

Activity. Determine if each of the following maps are linear transformations

(a) $T_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $T_1\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \sqrt{a^2 + b^2}$

(b) $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y \end{bmatrix}$

(c) $T_3 : \mathcal{P}_d \to \mathcal{P}_{d-1}$ given by $T_3(f(x)) = f'(x)$.

(d) $T_4 : C(\mathbb{R}) \to C(\mathbb{R})$ given by $T_4(f(x)) = f(-x)$

(e) $T_5 : \mathcal{P} \to \mathcal{P}$ given by $T_5(f(x)) = f(x) + x^2$

Activity. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute each of the following:

(a) $T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$

(b) $T\left(\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}\right)$

(c) $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

(d) $T\left(\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}\right)$

Activity. Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation. What is the smallest number of vectors needed to determine T? In other words, what is the smallest number n such that there are $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^4$ and given $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ you can determine $T(\vec{w})$ for *any* $\vec{w} \in \mathbb{R}^2$?

Observation. Fix an ordered basis for V. Since every vector can be written *uniquely* as a linear combination of basis vectors, a linear transformation $T : V \to W$ corresponds exactly to a choice of where each basis vector goes. For convenience, we can thus encode a linear transformation as a matrix, with one column for the image of each basis vector (in order).

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the standard ordered basis.

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Write the matrix corresponding to this linear transformation with respect to the ordered basis

$$\left\{\begin{bmatrix}2 & 1 & 1\end{bmatrix}, \begin{bmatrix}-1 & -1 & 3\end{bmatrix}, \begin{bmatrix}0 & 1 & 2\end{bmatrix}\right\}$$

Activity. Let $D : \mathcal{P}_3 \to \mathcal{P}_2$ be the derivative map (recall this is a linear transformation). Write the matrix corresponding to D with respect to the ordered basis $\{1, x, x^2, x^3\}$.

Application Activities - Day 2

Definition. Let $T : V \to W$ be a linear transformation.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\vec{v}) \neq T(\vec{w})$ whenever $\vec{v} \neq \vec{w}$.
- T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\vec{w} \in W$, there is some $v \in V$ with $T(\vec{v}) = \vec{w}$.

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

Activity. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Determine if T is injective, surjective, both, or neither.

Definition. We also have two important sets called the **kernel** of T and the **image** of T.

$$\ker T = \left\{ \vec{v} \in V \;\middle|\; T(\vec{v}) = 0 \right\}$$
$$\operatorname{Im} T = \left\{ \vec{w} \in W \;\middle|\; \text{there is some } v \in V \text{ with } T(\vec{v}) = \vec{w} \right\}$$

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T.

Activity. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (for the standard basis). Find the kernel and image of T.

Activity. Describe surjective linear transformations in terms of the image.

Activity. Describe injective linear transformations in terms of the kernel.

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

1) Write a system of equations whose solution set is the kernel.

2) Compute RREF(A) and solve the system of equations.
3) Compute the kernel of T
4) Find a basis for the kernel of T

Activity. Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

1) Write a system of equations whose solution set is the kernel.
2) Compute RREF(A) and solve the system of equations.
3) Compute the kernel of T
4) Find a basis for the kernel of T

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ (for the standard basis).

1) Find a set of vectors that span the image of T
2) Find a basis for the image of T.

Activity. Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $B = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 4 \\ 5 & 8 & 9 \end{bmatrix}$ (for the standard basis).

1) Find a set of vectors that span the image of T
2) Find a basis for the image of T.

Application Activities - Day 3

Activity. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). Consider the following statements about T

(a) T is injective

(b) T is not injective

(c) T is surjective

(d) T is not surjective

(e) The system of linear equations given by the augmented matrix $\left[A \,\middle|\, \vec{b}\right]$ has a solution for all $\vec{b} \in \mathbb{R}^m$

(f) The system of linear equations given by the augmented matrix $\left[A \,\middle|\, \vec{b}\right]$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$

(g) The system of linear equations given by the augmented matrix $\left[A \,\middle|\, \vec{0}\right]$ has a non-trivial solution.

(h) The columns of A span $\mathbb{R}^m$

(i) The columns of A are linearly independent

(j) The columns of A are a basis of $\mathbb{R}^m$

(k) Every column of $\mathrm{RREF}(A)$ is a pivot column

(l) $\mathrm{RREF}(A)$ has a non-pivot column

(m) $\mathrm{RREF}(A)$ has n pivot columns

Sort these statements into groups of equivalent statements.

Activity. Gallery walk–switch boards with a different team. If they have two things grouped together that you know are not equivalent, write a reason or counter-example on a sticky note.

Activity. Update your team's groupings based on feedback.

Activity. Repeat?

Activity. Can you add any statements to any groups?

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers
- Solve systems of linear equations **(Standard(s) E.3)**
- Find the matrix corresponding to a linear transformation **(Standard(s) A.1)**
- Determine if a linear transformation is injective and/or surjective **(Standard(s) A.3)**
- Interpret the ideas of injectivity and surjectivity in multiple ways

Readiness Assurance Resources

The following resources will help you prepare for this module.

- `https://www.khanacademy.org/math/algebra2/manipulating-functions/funciton-composition/v/function-composition`

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Let $f(x) = x^2 - 2$ and $g(x) = x^2 + 1$. Compute the composition function $(f \circ g)(x)$.

   (a) $x^2 - 1$

   (b) $x^4 + 2x^2 - 1$

   (c) $x^4 - 4x^2 + 5$

   (d) $x^4 - x^2 - 2$

2) Suppose $f(x)$ and $g(x)$ are real-valued functions satisfying

$$\begin{aligned} f(2) &= 1 & \qquad g(2) &= 3 \\ f(3) &= 4 & \qquad g(3) &= 5 \\ f(4) &= 3 & \qquad g(4) &= 6 \end{aligned}$$

   Compute $(f \circ g)(2)$.

   (a) 2

   (b) 3

   (c) 4

   (d) 5

3) Solve the system of linear equations

$$\begin{aligned} x + 3y &= -2 \\ 2x - 7y &= 9 \end{aligned}$$

   (a) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} -2 \\ 9 \end{bmatrix}$

4) Let a, b, c be fixed real numbers. How many solutions does the system of linear equations below have?

$$\begin{aligned} x + 2y + 3z &= a \\ y - z &= b \\ y + z &= c \end{aligned}$$

   (a) 0 (b) 1 (c) Infinitely many (d) It depends on the values of a, b, and c.

5) What is the matrix corresponding to the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y - z \\ y + 3z \\ x + 7y \end{bmatrix}$?

(a) $\begin{bmatrix}1 & 2 & -1\\0 & 1 & 3\\1 & 7 & 0\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2 & -1\\1 & 3 & 0\\1 & 7 & 0\end{bmatrix}$ (c) $\begin{bmatrix}1 & 1 & 1\\2 & 3 & 7\\-1 & 0 & 0\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0 & 1\\2 & 1 & 7\\-1 & 3 & 0\end{bmatrix}$

6) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation with associated matrix $A = \begin{bmatrix}2 & 3\\-1 & -1\\0 & 4\end{bmatrix}$. Compute $T\left(\begin{bmatrix}2\\-1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}5\\7\\4\end{bmatrix}$ (b) $\begin{bmatrix}1\\-1\\-4\end{bmatrix}$ (c) $\begin{bmatrix}2\\-1\\0\end{bmatrix}$ (d) $\begin{bmatrix}4\\-1\\8\end{bmatrix}$

7) Which of the following is true of the linear transformation $T :$?

(a) T is neither injective nor surjective
(b) T is injective but not surjective
(c) T is surjective but not injective
(d) T is both injective and surjective

8) Which of the following is true of the linear transformation $T :$?

(a) T is neither injective nor surjective
(b) T is injective but not surjective
(c) T is surjective but not injective
(d) T is both injective and surjective

9) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with associated matrix $A \in M_{m,n}(\mathbb{R})$. Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?

(a) T is injective
(b) T has a non-trivial kernel
(c) The columns of A are linearly dependent
(d) RREF(A) has a non-pivot column

10) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with associated matrix $A \in M_{m,n}(\mathbb{R})$. Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?

(a) T is surjective
(b) Im $T = \mathbb{R}^m$
(c) The columns of A span $\mathbb{R}^m$
(d) RREF(A) has only pivot columns

Application Activities - Day 1

Activity. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{4,3}$

(b) $M_{4,2}$

(c) $M_{3,2}$

(d) $M_{2,3}$

(e) $M_{2,4}$

(f) $M_{3,4}$

Activity. Compute $(S \circ T)(\vec{e}_1)$, $(S \circ T)(\vec{e}_2)$, and $(S \circ T)(\vec{e}_3)$.

Activity. Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.
What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{2,2}$

(b) $M_{2,3}$

(c) $M_{3,2}$

(d) $M_{3,3}$

Activity. Compute $(S \circ T)(\vec{e}_1)$ and $(S \circ T)(\vec{e}_2)$

Activity. Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity. Let $T : \mathbb{R}^1 \to \mathbb{R}^4$ be given by the matrix $B = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \end{bmatrix}$ and $S : \mathbb{R}^4 \to \mathbb{R}^1$ be given by the matrix $A = \begin{bmatrix} 2 & 3 & 2 & 5 \end{bmatrix}$.
What is the domain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity. The matrix corresponding to $S \circ T$ will lie in which matrix space?

(a) $M_{1,1}$

(b) $M_{1,4}$

(c) $M_{4,1}$

(d) $M_{4,4}$

Activity. Compute $(S \circ T)(\vec{e}_1)$

Activity. Find the matrix corresponding to $S \circ T$ with respect to the standard bases.

Activity. Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

1. Compute AX
2. Interpret the system of equations below as a matrix equation

$$\begin{aligned} 3x + y - z &= 5 \\ 2x + 4z &= -7 \\ -x + 3y + 5z &= 2 \end{aligned}$$

Application Activities - Day 2

Activity. Each row operation can be interpreted as a matrix multiplication. Let $A \in M_{4,4}$

1) Find a matrix S_1 such that S_1A is the result of swapping the second and fourth rows of A.

2) Find a matrix S_2 such that S_2A is the result of adding 5 times the third row of A to the first.

3) Find a matrix S_3 such that S_3A is the result of doubling the fourth row of A.

Activity. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis). Consider the following statements about T

(a) T is injective

(b) T is surjective

(c) T is bijective (i.e. both injective and surjective)

(d) $AX = B$ has a solution for all $B \in M_{m,1}$

(e) $AX = B$ has a unique solution for all $B \in M_{m,1}$

(f) $AX = 0$ has a non-trivial solution.

(g) The columns of A span $\mathbb{R}^m$

(h) The columns of A are linearly independent

(i) The columns of A are a basis of $\mathbb{R}^m$

(j) RREF(A) has n pivot columns

(k) RREF(A) has m pivot columns

Sort these statements into groups of equivalent statements.

Activity. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A \in M_{m,n}$ (for the standard basis).
If T is injective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Activity. If T is surjective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Activity. If T is bijective, what must be true about how m and n are related?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

Application Activities - Day 3

Activity. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with matrix $A \in M_{n,n}$.
If T is a bijection, then $AX = B$ has a unique solution for all $B \in \mathbb{R}^n$. Thus we can define a map $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by defining $T^{-1}(B)$ to be this solution. It follows immediately that $T \circ T^{-1}$ is the identity map. The matrix corresponding to T^{-1} is denoted A^{-1}.

1) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_1)$
2) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_2)$
3) Solve $AX = \vec{e}_1$ to determine $T^{-1}(\vec{e}_3)$
4) Compute A^{-1}

A (square) matrix is called *invertible* if it corresponds to an invertible linear transformation.

1) Find the inverse of the matrix $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$
2) Find the inverse of the matrix $\begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Calculate the area of a parallelogram.
- Find the matrix corresponding to a linear transformation of Euclidean spaces **(Standard(s) A1)**.
- Recall and use the definition of a linear transformation **(Standard(s) A2)**.
- Find all roots of quadratic polynomials (including complex ones), and be able to use the rational root theorem to find all rational roots of a higher degree polynomial.
- Interpret the statement "A is an invertible matrix" in many equivalent ways in different contexts.

Readiness Assurance Resources

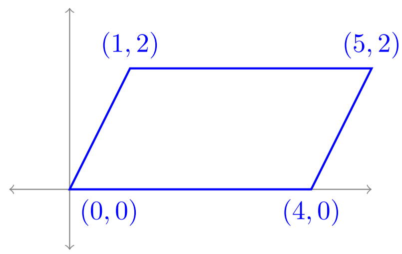
The following resources will help you prepare for this module.

- Finding the area of a parallelogram: `https://www.khanacademy.org/math/basic-geo/basic-geo-area-and-perime parallelogram-area/a/area-of-parallelogram`
- Factoring quadratics: `https://www.khanacademy.org/math/algebra2/polynomial-functions/factoring-polynom v/factoring-polynomials-1`
- Finding complex roots of quadratics: `https://www.khanacademy.org/math/algebra2/polynomial-functions/ quadratic-equations-with-complex-numbers/v/complex-roots-from-the-quadratic-formula`
- Finding all roots of polynomials: `https://www.khanacademy.org/math/algebra2/polynomial-functions/ finding-zeros-of-polynomials/v/finding-roots-or-zeros-of-polynomial-1`
- The Rational Root Theorem: `https://artofproblemsolving.com/wiki/index.php?title=Rational_ Root_Theorem`

Readiness Assurance Test

Choose the most appropriate response for each question.

1) Find the area of the parallelogram with vertices $(0, 0)$, $(4, 0)$, $(5, 2)$, and $(1, 2)$.



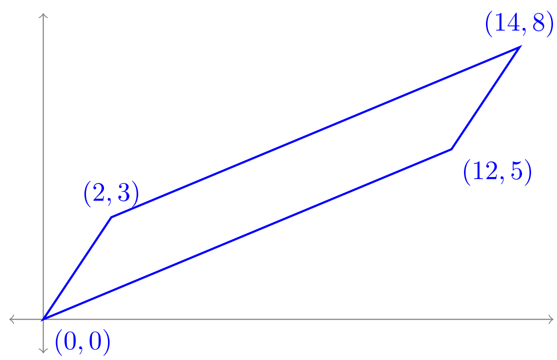
   (a) 5

   (b) 6

   (c) 7

   (d) 8

2) Find the area of the parallelogram with vertices $(0, 0)$, $(12, 5)$, $(14, 8)$, and $(2, 3)$.



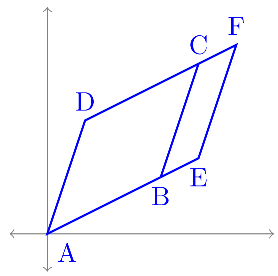
   (a) 13

   (b) 26

   (c) 39

   (d) 52

3) The parallelogram ABCD has area 6. If AE is $\frac{3}{2}$ the length of AB, what is the area of the parallelogram AEFD?



   (a) 9

   (b) 12

   (c) 15

   (d) 18

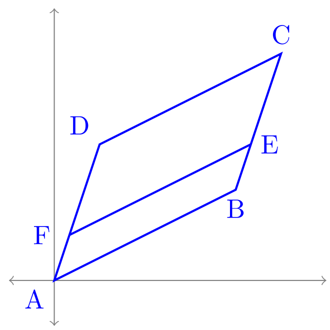
4) The parallelogram ABCD has area 6. If AF is one third as long as AD, what is the area of the parallelogram ABEF?

   (a) 1

   (b) 2

   (c) 3

   (d) 4



5) Let $T : \mathbb{R}^2 \to \mathbb{R}$ be a linear transformation. Which of the following is equal to $T\left(\begin{bmatrix} a+b \\ a+b \end{bmatrix}\right)$?

(a) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$

(b) $2T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$

(c) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} b \\ a \end{bmatrix}\right)$

(d) $T\left(\begin{bmatrix} a \\ a \end{bmatrix}\right) + T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} b \\ a \end{bmatrix}\right) + T\left(\begin{bmatrix} b \\ b \end{bmatrix}\right)$

6) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with associated matrix $A \in M_n(\mathbb{R})$. Three of the four answer choices are equivalent to each other; which one is not equivalent to the other three?

(a) A is not an invertible matrix

(b) T has a non-trivial kernel

(c) $\det(A) \neq 0$

(d) $A\vec{x} = \vec{b}$ has multiple solutions for all $\vec{b} \in \mathbb{R}^n$.

7) What is the matrix corresponding to the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x+2y-z \\ y+z \\ x+7z \end{bmatrix}$?

(a) $\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 7 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 7 \\ -1 & 0 & 0 \end{bmatrix}$

8) Which of the following conditions imply that the quadratic polynomial $ax^2 + bx + c$ has no real roots?

(a) $a < 0$

(b) $b^2 - 4ac < 0$

(c) $ac - b^2 < 0$

(d) $ab + c^2 < 0$

9) Which of the following is a root of the polynomial $x^2 - 4x + 13$?

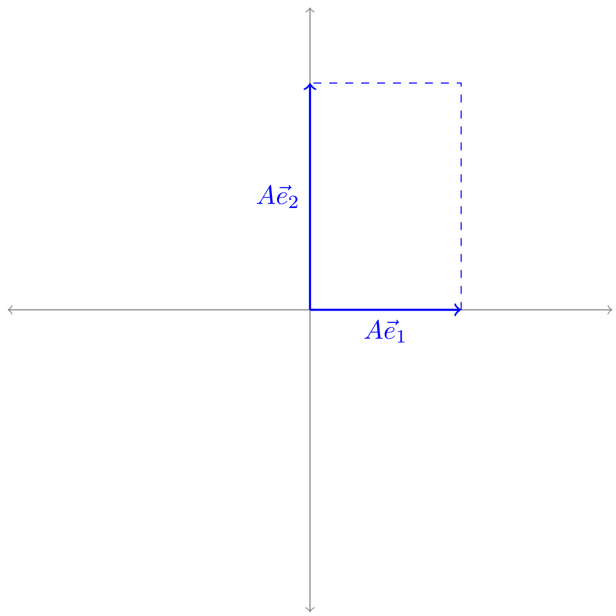
(a) $1+2i$ (b) $2-3i$ (c) $3+4i$ (d) $4-5i$

10) How many roots does the polynomial $x^4+3x^3+x^2-3x-2$ have?

(a) 1 (b) 2 (c) 3 (d) 4

Application Activities - Days 1-2

Activity. Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



We can summarize the transformation of the unit square into this rectangle by measuring the following:

(a) How did the area change?

(b) How was th x-axis stretched?

(c) How was the y-axis stretched?

Activity. Consider the following linear transformations $A_i : \mathbb{R}^2 \to \mathbb{R}^2$.

- $A_2 = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$
- $A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
- $A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

For each linear transformation, do the following:

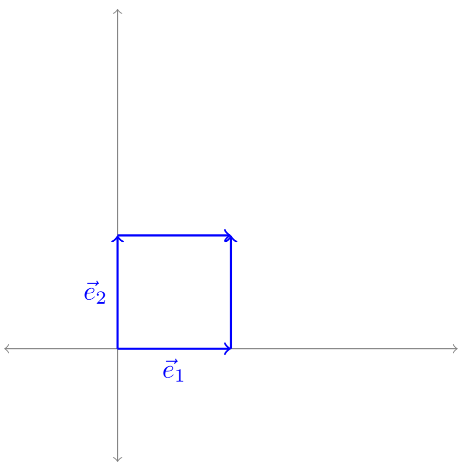
(a) Draw a graph showing the image of the unit square.

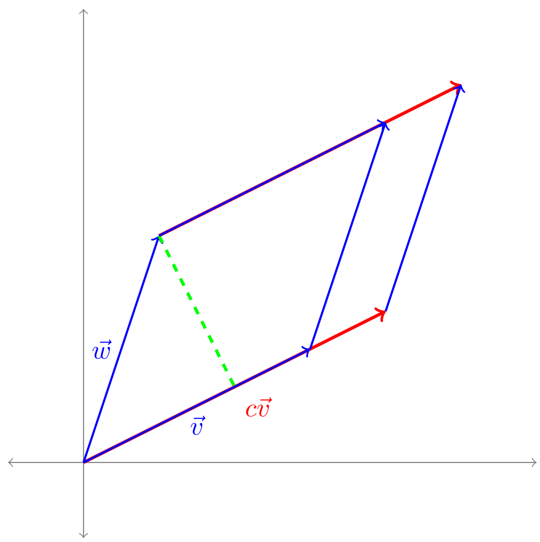
(b) Compute how much the area was stretched out.

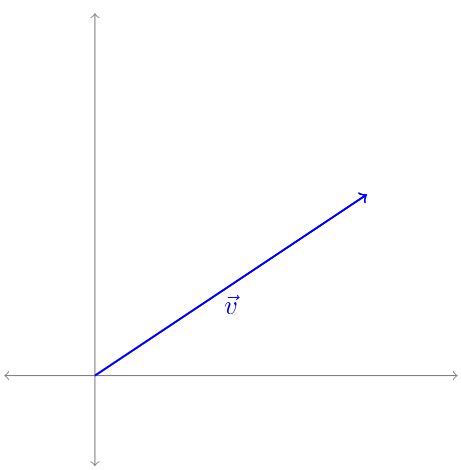
(c) Determine which axes (or lines) were preserved; how were they stretched out?

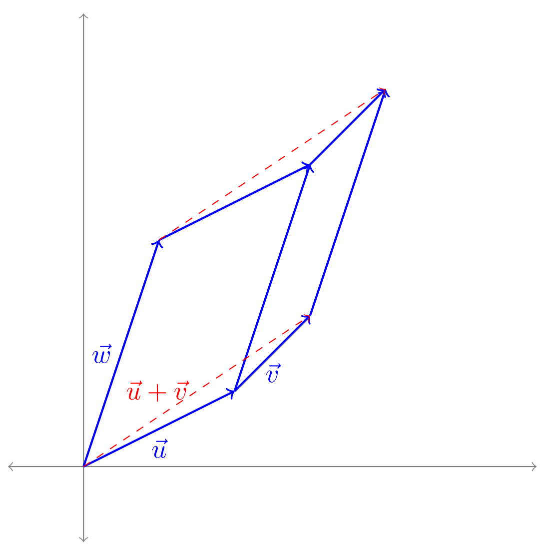
Activity. Our goal is to define a function $\det : M_n \to \mathbb{R}$ that takes a square matrix (linear transformation $\mathbb{R}^n \to \mathbb{R}^n$) and returns its area stretching factor. This function is called the **determinant**.
What properties should this function have?
Match the four pictures to the following four expressions

$\det(\vec{e}_1, \vec{e}_2)$ $\qquad$ $\det(\vec{v}, \vec{v})$ $\qquad$ $\det(c\vec{v}, \vec{w})$ $\qquad$ $\det(\vec{u}+\vec{v}, \vec{w})$









Activity. What can you conclude about each of the following?

1. $\det(\vec{e}_1, \vec{e}_2)$
2. $\det(\vec{v}, \vec{v})$
3. $\det(c\vec{v}, \vec{w})$
4. $\det(\vec{u} + \vec{v}, \vec{w})$

Definition. To summarize, we have 3 properties (stated here over $\mathbb{R}^n$)

P1: $\det(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n) = 1$

P2: If $\vec{v}_i = \vec{v}_j$ for some $i \neq j$, then $\det(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = 0$.

P3: The determinant is linear in each column.

These three properties uniquely define the **determinant**, as we shall see.

Observation. Note that if $\vec{v}, \vec{w} \in \mathbb{R}^2$ and $A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$ we will write either $\det(A)$ or $\det(\vec{v}, \vec{w})$ as is convenient.

Activity.
How are $\det(\vec{v}, \vec{w})$ and $\det(\vec{w}, \vec{v})$ related?

(a) $\det(\vec{v}, \vec{w}) = \det(\vec{w}, \vec{v})$

(b) $\det(\vec{v}, \vec{w}) = -\det(\vec{w}, \vec{v})$

(c) They are unrelated

(d) They are related, but not by either (a) or (b).

Observation. Note that this implies that the determinant is actually a *signed* area (volume)!

Activity.
How are $\det(\vec{v} + \vec{w}, \vec{w})$ and $\det(\vec{v}, \vec{w})$ related?

(a) $\det(\vec{v} + \vec{w}, \vec{w}) = \det(\vec{v}, \vec{w})$

(b) $\det(\vec{v} + \vec{w}, \vec{w}) = -\det(\vec{v}, \vec{w})$

(c) They are unrelated

(d) They are related, but not by either (a) or (b).

Observation. Note that we now understand the effect of any column operation on the determinant.

Activity. How are $\det(A)$ and $\det(A^T)$ related?

(a) $\det(A) = \det(A^T)$

(b) $\det(A) = -\det(A^T)$

(c) $\det(A) = \frac{1}{\det(A^T)}$

(d) They are unrelated

Observation. Thus, row operations behave like column operations. So we can use row reduction to compute determinants.

Activity. Compute $\det \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$.

Activity. Compute $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Activity. Which of the following is the same as $\det \begin{bmatrix} 3 & -2 & 0 \\ 5 & -1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$?

(a) $\det \begin{bmatrix} 3 & -2 \\ 5 & -1 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$

(c) $\det \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}$

(d) None of these

Activity. Which of the following is the same as $\det \begin{bmatrix} 3 & 0 & 7 \\ 5 & 1 & 2 \\ -2 & 0 & 6 \end{bmatrix}$?

(a) $\det \begin{bmatrix} 3 & 7 \\ 5 & 2 \end{bmatrix}$

(b) $\det \begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix}$

(c) $\det \begin{bmatrix} 5 & 2 \\ -2 & 6 \end{bmatrix}$

(d) None of these

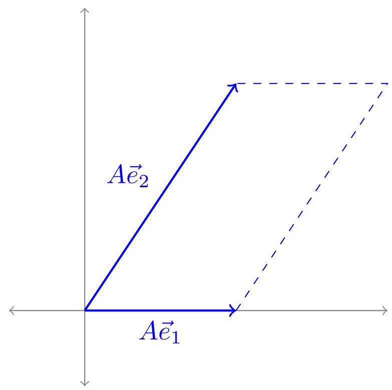
Activity. Compute $\det \begin{bmatrix} 0 & 3 & -2 \\ 1 & 5 & 12 \\ 0 & 2 & -1 \end{bmatrix}$

Activity. Using the fact that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 5 \\ 0 & 3 & 3 \end{bmatrix}$.

Activity. Compute $\begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$

Application Activities - Day 3

Activity. Consider the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$



Observe

$$A\vec{e_1} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\vec{e_1}$$

Is there another vector $\vec{v} \in \mathbb{R}^2$ such that $A\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$?

Definition. Let $A \in M_n(\mathbb{R})$. An **eigenvector** is a vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x}$ is parallel to $\vec{x}$; in other words, $A\vec{x} = \lambda\vec{x}$ for some scalar λ, which is called an **eigenvalue**

Observation. Observe that $A\vec{x} = \lambda\vec{x}$ is equivalent to $(A - \lambda I)\vec{x} = 0$.

- To find eigenvalues, we need to find values of λ such that $A - \lambda I$ has a nontrivial kernel; equivalently, $A - \lambda I$ is not invertible, which is equivalent to $\det(A - \lambda I) = 0$. $\det(A - \lambda I)$ is called the **characteristic polynomial**.

- Once an eigenvalue is found, the eigenvectors form a subspace called the **eigenspace**, which is simply the kernel of $A - \lambda I$. Each eigenvalue will have an associated eigenspace.

Activity. Find the eigenvalues for the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$.

Activity. Compute the eigenspace associated to the eigenvalue 3.

Activity. Find all the eigenvalues and associated eigenspaces for the matrix $\begin{bmatrix} 6 & -2 & 1 \\ 17 & -5 & 5 \\ -4 & 2 & 1 \end{bmatrix}$.

Application Activities - Day 4

Activity. If $A \in M_4$, what is the largest number of eigenvalues A can have?

Activity. 2 is an eigenvalue of each of the matrices $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -9 & 5 \\ -2 & -2 & 2 \\ -7 & -13 & 9 \end{bmatrix}$. Compute the eigenspace associated to 2 for both A and B.

Definition.

- The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.
- The **geometric multiplicity** of an eigenvalue is the dimension of the eigenspace.

Activity. How are the algebraic and geometric multiplicities related?

(a) The algebraic multiplicity is always at least as big as than the geometric multiplicity.

(b) The geometric multiplicity is always at least as big as the algebraic multiplicity.

(c) Sometimes the algebraic multiplicity is larger and sometimes the geometric multiplicity is larger.

Activity. Find the eigenvalues, along with both their algebraic and geometric multiplicities, for the matrix
$\begin{bmatrix} -3 & 1 & 2 & 1 \\ -9 & 5 & -2 & -1 \\ 31 & -17 & 6 & 3 \\ -69 & 39 & -18 & -9 \end{bmatrix}$

Activity. Find the eigenvalues of the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Activity. Describe what this linear transformation is doing geometrically; draw a picture.

Activity. Fix a real number θ and find the eigenvalues of the matrix $A_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. What are the eigenvalues?

Activity. D raw pictures and describe the geometric actions of the maps $A_{\frac{\pi}{4}}$, $A_{\frac{\pi}{2}}$, and A_π.

Activity. For how many values of θ does the rotation matrix A_θ have real eigenvalues?

(a) 0

(b) 1

(c) 2

(d) 3

(e) An infinite number