Module M: Understanding Matrices Algebraically

What algebraic structure do matrices have?

At the end of this module, students will be able to...

M1. Matrix Multiplication. ... multiply matrices.

M2. Invertible Matrices. ... determine if a square matrix is invertible or not.

M3. Matrix inverses. ... compute the inverse matrix of an invertible matrix.

M4. Row operations as multiplication. ... describe the row reduction of a matrix as matrix multiplication.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers.
- Identify the domain and codomain of linear transformations.
- Find the matrix corresponding to a linear transformation and compute the image of a vector given a standard matrix **A2**
- Determine if a linear transformation is injective and/or surjective **A3**
- Interpret the ideas of injectivity and surjectivity in multiple ways.

The following resources will help you prepare for this module.

- Function composition (Khan Academy): `http://bit.ly/2wkz7f3`
- Domain and codomain: `https://www.youtube.com/watch?v=BQMyeQOLvpg`
- Interpreting injectivity and surjectivity in many ways: `https://www.youtube.com/watch?v=WpUv72Y6D10`

Module M Section 1

Activity M.1.1 *(∼5 min)*

Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be given by the 2×3 standard matrix $B=\begin{bmatrix}2 & 1 & -3\\ 5 & -3 & 4\end{bmatrix}$ and $S:\mathbb{R}^2\to\mathbb{R}^4$ be given by the 4×2 standard matrix $A=\begin{bmatrix}1 & 2\\ 0 & 1\\ 3 & 5\\ -1 & -2\end{bmatrix}$.

What is the domain of the composition map $S\circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity M.1.2 *(∼3 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What is the codomain of the composition map $S \circ T$?

(a) $\mathbb{R}$

(b) $\mathbb{R}^2$

(c) $\mathbb{R}^3$

(d) $\mathbb{R}^4$

Activity M.1.3 *(∼2 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What size will the standard matrix of $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$ be? (Rows $\times$ Columns)

(a) 4×3

(b) 4×2

(c) 3×4

(d) 3×2

(e) 2×4

(f) 2×3

Activity M.1.4 *(∼15 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Activity M.1.4 *(∼15 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Activity M.1.4 *(∼15 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{\mathbf{e}}_2)$.

Activity M.1.4 *(∼15 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{\mathbf{e}}_2)$.
Part 3: Compute $(S \circ T)(\vec{\mathbf{e}}_3)$.

Activity M.1.4 *(∼15 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{\mathbf{e}}_2)$.
Part 3: Compute $(S \circ T)(\vec{\mathbf{e}}_3)$.
Part 4: Find the 4×3 standard matrix of $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$.

Definition M.1.5

We define the **product** AB of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For the previous activity, S had a 4×2 matrix and T had a 2×3 matrix, so $S \circ T$ had a 4×3 standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= [(S \circ T)(\vec{\mathbf{e}}_1)(S \circ T)(\vec{\mathbf{e}}_2)(S \circ T)(\vec{\mathbf{e}}_3)] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

Activity M.1.6 *(∼10 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

Find the standard matrix AB of $S \circ T$.

Activity M.1.7 *(∼5 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.
Find the standard matrix BA of $T \circ S$.

Activity M.1.8 *(∼10 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be given by the matrix $B = \begin{bmatrix} 3 & 2 & 5 & -4 \\ -1 & -3 & 1 & 2 \end{bmatrix}$ and let $S : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ -4 & 2 \end{bmatrix}$. Compute AB, the standard matrix of the composition $S \circ T$.

Observation M.1.9

Note that an $\mathbb{R}^n$ vector acts exactly the same as an $n \times 1$ matrix, so we will use them interchangablely, as follows.

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix} \qquad X = \vec{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \vec{\mathbf{b}} = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$$

So we may study the linear system

$$\begin{aligned} 3x + y - z &= 5 \\ 2x + 4z &= -7 \\ -x + 3y + 5z &= 2 \end{aligned}$$

as both a vector equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and a matrix equation $AX = B$:

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$$

Module M Section 2

Observation M.2.1

Recall that if $T : \mathbb{R}^n \to \mathbb{R}^k$ is a linear map with standard matrix $B \in M_{k,n}$ and $S : \mathbb{R}^k \to \mathbb{R}^m$ is a linear map with standard matrix $A \in M_{m,k}$, the product matrix $AB \in M_{m,n}$ is defined to be the standard matrix of the composition map

$$S \circ T : \mathbb{R}^n \to \mathbb{R}^m.$$

Activity M.2.2 *(∼5 min)*

Matrix multiplication only makes sense if the first matrix has as many columns as the second matrix has rows. Label each of these matrices with **rows** $\times$ **columns**, and then figure out which of the products AB, AC, BA, BC, CA, CB can actually be computed.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Activity M.2.3 *(∼10 min)*

Let $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Compute the product BA.

Activity M.2.4 *(∼5 min)*

Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix I such that $IA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Definition M.2.5

The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

Fact M.2.6

For any square matrix A, $IA = AI = A$:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 7 & -1\\0 & 3 & 2\\1 & 1 & -1\end{bmatrix} = \begin{bmatrix}2 & 7 & -1\\0 & 3 & 2\\1 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 7 & -1\\0 & 3 & 2\\1 & 1 & -1\end{bmatrix}$$

Activity M.2.7 *(∼20 min)*

Each row operation can be interpreted as a type of matrix multiplication.

Activity M.2.7 *($\sim$20 min)*

Each row operation can be interpreted as a type of matrix multiplication.

Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Activity M.2.7 *(~20 min)*

Each row operation can be interpreted as a type of matrix multiplication.

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$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Activity M.2.7 *(∼20 min)*

Each row operation can be interpreted as a type of matrix multiplication.

Part 1: Tweak the identity matrix slightly to create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5(1) & 7+5(1) & -1+5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Fact M.2.8

If R is the result of applying a row operation to I, then RA is the result of applying the same row operation to A.

This means that for any matrix A, we can find a series of matrices $R_1, \dots, R_k$ corresponding to the row operations such that

$$R_1 R_2 \cdots R_k A = \mathrm{RREF}(A).$$

That is, row reduction can be thought of as the result of matrix multiplication.

Module M Section 3

Activity M.3.1 (*∼15 min*)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into three groups of statements: a group that means T is **injective**, a group that means T is **surjective**, and a group that means T is **bijective**.

(a) $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution for all $\vec{\mathbf{b}} \in \mathbb{R}^m$

(b) $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for all $\vec{\mathbf{b}} \in \mathbb{R}^m$

(c) $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a unique solution.

(d) The columns of A span $\mathbb{R}^m$

(e) The columns of A are linearly independent

(f) The columns of A are a basis of $\mathbb{R}^m$

(g) Every column of RREF(A) has a pivot

(h) Every row of RREF(A) has a pivot

(i) $m = n$ and RREF(A) $= I$

Definition M.3.2

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with standard matrix A.

- If T is a bijection and $\vec{\mathbf{b}}$ is any $\mathbb{R}^n$ vector, then $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution.
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by setting $T^{-1}(\vec{\mathbf{b}})$ to be this unique solution.
- Let A^{-1} be the standard matrix for T^{-1}. We call A^{-1} the **inverse matrix** of A, so we also say that A is **invertible**.

Activity M.3.3 *(∼20 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix
$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Activity M.3.3 *(∼20 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Part 1: Write an augmented matrix representing the system of equations given by

$T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_1$, that is, $A\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Activity M.3.3 *(∼20 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Part 1: Write an augmented matrix representing the system of equations given by

$$T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_1, \text{ that is, } A\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Part 2: Solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_1$ to find $T^{-1}(\vec{\mathbf{e}}_1)$.

Activity M.3.3 *(∼20 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix
$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_1$, that is, $A\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Part 2: Solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_1$ to find $T^{-1}(\vec{\mathbf{e}}_1)$.

Part 3: Solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_2$ to find $T^{-1}(\vec{\mathbf{e}}_2)$.

Activity M.3.3 *(∼20 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_1$, that is, $A\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Part 2: Solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_1$ to find $T^{-1}(\vec{\mathbf{e}}_1)$.

Part 3: Solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_2$ to find $T^{-1}(\vec{\mathbf{e}}_2)$.

Part 4: Solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_3$ to find $T^{-1}(\vec{\mathbf{e}}_3)$.

Activity M.3.3 *(∼20 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}.$$

Part 1: Write an augmented matrix representing the system of equations given by $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_1$, that is, $A\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Part 2: Solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_1$ to find $T^{-1}(\vec{\mathbf{e}}_1)$.

Part 3: Solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_2$ to find $T^{-1}(\vec{\mathbf{e}}_2)$.

Part 4: Solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_3$ to find $T^{-1}(\vec{\mathbf{e}}_3)$.

Part 5: Write A^{-1}, the standard matrix for T^{-1}.

Observation M.3.4

We could have solved these three systems simultaneously by row reducing the matrix $[A \,|\, I]$ at once.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right]$$

Activity M.3.5 *(∼5 min)*

Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A \,|\, I]$.

Activity M.3.6 *(∼5 min)*

Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

Observation M.3.7

An $n \times n$ matrix A is invertible if and only if $\mathrm{RREF}(A) = I_n$.

Activity M.3.8 *(∼10 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Activity M.3.8 *(∼10 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

Activity M.3.8 *(~10 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

Part 1: Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

Part 2: If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, find the 2×2 matrix

$$A^{-1}A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

Observation M.3.9

$T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation T. Therefore $A^{-1}A = AA^{-1} = I$ is the identity matrix for any invertible matrix A.