Module E

Standard E1

E1. Write an augmented matrix corresponding to the following system of linear equations.

$$x + 3y - 4z = 5$$
$$3x + 9y + z = 0$$
$$x - z = 1$$

Solution.

$$\begin{bmatrix} 1 & 3 & -4 & 5 \\ 3 & 9 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix}$$

□

E1. Write an augmented matrix corresponding to the following system of linear equations.

$$x_1 + 4x_3 = 1$$
$$x_2 - x_3 = 7$$
$$x_1 - x_2 + 3x_3 = -1$$

Solution.

$$\begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{bmatrix}$$

□

E1. Write an augmented matrix corresponding to the following system of linear equations.

$$x_1 + 4x_3 = 1$$
$$x_2 - x_3 = 7$$
$$x_1 - x_2 + 3x_4 = -1$$

Solution.

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & -1 & 0 & 7 \\ 1 & -1 & 0 & 3 & -1 \end{bmatrix}$$

□

E1. Write an augmented matrix corresponding to the following system of linear equations.

$$x_1 + 3x_2 - 4x_3 + x_4 = 5$$
$$3x_1 + 9x_2 + x_3 - 7x_4 = 0$$
$$x_1 - x_3 + x_4 = 1$$

Solution.

$$\begin{bmatrix} 1 & 3 & -4 & 1 & 5 \\ 3 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{bmatrix}$$

□

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\begin{bmatrix} 3 & -1 & 0 & 1 & 5 \\ -1 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 0 & -3 \end{bmatrix}$$

Solution.

$$3x_1 - x_2 + x_4 = 5$$
$$-x_1 + 9x_2 + x_3 - 7x_4 = 0$$
$$x_1 - x_3 = -3$$

□

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 4 & 1 & -7 \\ 1 & 2 & -1 & 0 \end{array}\right]$$

Solution.

$$2x_1 - x_2 = 1$$
$$-x_1 + 4x_2 + x_3 = -7$$
$$x_1 + 2x_2 - x_3 = 0$$

□

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} -4 & -1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{array}\right]$$

Solution.

$$-4x_1 - x_2 + 3x_3 = 2$$
$$x_1 + 2x_2 - x_3 = 0$$
$$-x_1 + 4x_2 + x_3 = 4$$

□

E1. Write a system of linear equations corresponding to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 1 & -1 & 7 \\ 1 & -1 & 3 & -1 \end{array}\right]$$

Solution.

$$x_1 + 4x_3 = 1$$
$$x_2 - x_3 = 7$$
$$x_1 - x_2 + 3x_3 = -1$$

□

Standard E2

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 \\ 3 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

□

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ -3 & 5 & 2 & 0 \\ 1 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & 6 \\ 0 & -1 & -1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

E2. Put the following matrix in reduced row echelon form.

$$\begin{bmatrix} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} -3 & 1 & 0 & 2 \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ -8 & 2 & -1 & 6 \\ 0 & 2 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & -\frac{2}{3} & -1 & \frac{2}{3} \\ 0 & 2 & 3 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 2 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

E2. Find the reduced row echelon form of the matrix below.

$$\begin{bmatrix} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 5 & 0 & -1 \\ 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 1 \\ 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & -1 & 15 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & 0 & 24 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 9 & 0 & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{12} & 1 \\ 0 & 1 & 0 & \frac{3}{4} & 3 \\ 0 & 0 & 1 & -\frac{1}{12} & 0 \end{bmatrix}$$

□

E2. Find RREF A, where

$$A = \begin{bmatrix} 3 & -2 & 1 & 8 & -5 \\ 2 & 2 & 0 & 6 & -2 \\ -1 & 1 & 1 & -4 & 6 \end{bmatrix}$$

Solution.

$$\mathrm{RREF}\,A = \begin{bmatrix} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \end{bmatrix}$$

□

E2. Find RREF A, where

$$A = \begin{bmatrix} 2 & -7 & 4 \\ 1 & -3 & 2 \\ 3 & 0 & 3 \end{bmatrix}$$

Solution.

$$\mathrm{RREF}\,A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

E2. Find RREF A, where

$$A = \begin{bmatrix} 2 & -1 & 5 & 4 \\ -1 & 0 & -2 & -1 \\ 1 & 3 & -1 & -5 \end{bmatrix}$$

Solution.

$$\mathrm{RREF}\,A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

□

E2. Find RREF A, where

$$A = \begin{bmatrix} 2 & 2 & 1 & 2 & -1 \\ 1 & 1 & 2 & 4 & 5 \\ 3 & 3 & -1 & -2 & 1 \end{bmatrix}$$

Solution.

$$\mathrm{RREF}\,A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

□

Standard E3

E3. Find the solution set for the following system of linear equations.

$$x + 3y - 4z = 5$$
$$3x + 9y + z = 2$$

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & -4 & | & 5 \\ 3 & 9 & 1 & | & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 0 & | & 1 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

So the solution set is

$$\left\{ \begin{bmatrix} 1 - 3c \\ c \\ -1 \end{bmatrix} \,\middle|\, c \in \mathbb{R} \right\}$$

□

E3. Find the solution set for the following system of linear equations.

$$-3x + y = 2$$
$$-8x + 2y - z = 6$$
$$2y + 3z = -2$$

Solution.

$$\text{RREF}\left(\begin{bmatrix} -3 & 1 & 0 & | & 2 \\ -8 & 2 & -1 & | & 6 \\ 0 & 2 & 3 & | & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} & | & -1 \\ 0 & 1 & \frac{3}{2} & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

So the solution set is

$$\left\{ \begin{bmatrix} -1 - \frac{c}{2} \\ -1 - \frac{3c}{2} \\ c \end{bmatrix} \,\middle|\, c \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} c - 1 \\ 3c - 1 \\ -2c \end{bmatrix} \,\middle|\, c \in \mathbb{R} \right\}$$

□

E3. Find the solution set for the following system of linear equations.

$$2x + y - z + w = 5$$
$$3x - y - 2w = 0$$
$$-x + 5z + 3w = -1$$

Solution.

$$\text{RREF}\left(\begin{bmatrix} 2 & 1 & -1 & 0 & | & 5 \\ 3 & -1 & 0 & -2 & | & 0 \\ -1 & 0 & 5 & 0 & | & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{12} & | & 1 \\ 0 & 1 & 0 & \frac{7}{4} & | & 3 \\ 0 & 0 & 1 & \frac{7}{12} & | & 0 \end{bmatrix}$$

So the solution set is

$$\left\{ \begin{bmatrix} 1 + a \\ 3 - 21a \\ -7a \\ 12a \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}$$

□

E3. Find the solution set for the following system of linear equations.

$$2x + y - z + w = 5$$
$$3x - y - 2w = 0$$
$$-x + 5z + 3w = -1$$

Solution.

$$\text{RREF}\left(\begin{bmatrix} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{7}{4} & 3 \\ 0 & 0 & 1 & \frac{7}{12} & 0 \end{bmatrix}$$

So the solution set is

$$\left\{ \begin{bmatrix} 1+a \\ 3-21a \\ -7a \\ 12a \end{bmatrix} \mid a \in \mathbb{R} \right\}$$

$\square$

E3. Find the solution set for the following system of linear equations.

$$2x_1 - 2x_2 + 6x_3 - x_4 = -1$$
$$3x_1 + 6x_3 + x_4 = 5$$
$$-4x_1 + x_2 - 9x_3 + 2x_4 = -7$$

Solution. Let $A = \begin{bmatrix} 2 & -2 & 6 & -1 & -1 \\ 3 & 0 & 6 & 1 & 5 \\ -4 & 1 & -9 & 2 & -7 \end{bmatrix}$, so RREF $A = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$. It follows that the

solution set is given by $\left\{ \begin{bmatrix} 2-2a \\ 3+a \\ a \\ -1 \end{bmatrix} \mid a \in \mathbb{R} \right\}$.

$\square$

E3. Find the solution set for the following system of linear equations.

$$2x_1 + 3x_2 - 5x_3 + 14x_4 = 8$$
$$x_1 + x_2 - x_3 + 5x_4 = 3$$

Solution. Let $A = \begin{bmatrix} 2 & 3 & -5 & 14 & 8 \\ 1 & 1 & -1 & 5 & 3 \end{bmatrix}$, so RREF $A = \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -3 & 4 & 2 \end{bmatrix}$. It follows that the solution set

is given by $\left\{ \begin{bmatrix} 1-2a-b \\ 2+3a-4b \\ a \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$.

$\square$

E3. Find the solution set for the following system of linear equations.

$$4x_1 + 4x_2 + 3x_3 - 6x_4 = 5$$
$$-2x_3 - 4x_4 = 3$$
$$2x_1 + 2x_2 + x_3 - 4x_4 = -1$$

Solution. Let $A = \begin{bmatrix} 4 & 4 & 3 & -6 & 5 \\ 0 & 0 & -2 & -4 & 3 \\ 2 & 2 & 1 & -4 & -1 \end{bmatrix}$, so RREF $A = \begin{bmatrix} 1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. It follows that the system is

inconsistent with no solutions (since the bottom row implies the contradiction $0 = 1$), so its solution set is
$\emptyset$.

$\square$

E3. Find the solution set for the following system of linear equations.

$$3x + 2y + z = 7$$
$$x + y + z = 1$$
$$-2x + 3z = -11$$

Solution. Let $A = \begin{bmatrix} 3 & 2 & 1 & | & 7 \\ 1 & 1 & 1 & | & 1 \\ -2 & 0 & 3 & | & 11 \end{bmatrix}$, so RREF $A = \begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$. It follows that solution set is

$$\left\{ \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} \right\}$$

$\square$

Module V

Standard V1

V1. Let V be the set of all real numbers together with the operations $\oplus$ and $\odot$ defined by, for any $x, y \in V$ and $c \in \mathbb{R}$,

$$x \oplus y = x + y$$
$$c \odot x = cx - 3(c - 1)$$

(a) Show that **scalar multiplication** is **associative**: $a \odot (b \odot x) = (ab) \odot x$ for all scalars $a, b \in \mathbb{R}$ and $x \in V$.

(b) Show that scalar multiplication does not distribute over vector addition, i.e. for some scalar $a \in \mathbb{R}$ and $x, y \in V$, $a \odot (x \oplus y) \neq a \odot x \oplus a \odot y$.

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2 + 2x_1 y_1)$$
$$c \odot (x_1, x_2) = (cx_1, cx_2)$$

(a) Show that the vector **addition** $\oplus$ is **associative**:
$(x_1, x_2) \oplus ((y_1, y_2) \oplus (z_1, z_2)) = ((x_1, x_2) \oplus (y_1, y_2)) \oplus (z_1, z_2)$ for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in V$.

(b) Show that scalar multiplication does not distribute over scalar addition, i.e. for some scalars $a, b \in \mathbb{R}$ and $(x_1, x_2) \in V$, $(a + b) \odot (x_1, x_2) \neq a \odot (x_1, x_2) \oplus b \odot (x_1, x_2)$.

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1 - 1, x_2 + y_2 - 1)$$
$$c \odot (x_1, x_2) = (cx_1, cx_2)$$

(a) Show that this vector space has an **additive identity** element, i.e. an element $\mathbf{z} \in V$ satisfying $(x, y) \oplus \mathbf{z} = (x, y)$ for every $(x, y) \in V$.

(b) Show that scalar multiplication does not distribute over vector addition, i.e. for some $a \in \mathbb{R}$ and $(x_1, x_2), (y_1, y_2) \in V$, $a \odot ((x_1, x_2) \oplus (y_1, y_2)) \neq a \odot (x_1, x_2) \oplus a \odot (y_1, y_2)$.

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$c \odot (x_1, x_2) = (0, cx_2)$$

(a) Show that **scalar multiplication distributes over scalar addition**, i.e. that $(c + d) \odot (x_1, x_2) = c \odot (x_1, x_2) \oplus d \odot (x_1, x_2)$ for every $c, d \in \mathbb{R}$ and $(x_1, x_2) \in V$.

(b) Show that 1 is not a scalar multiplicative identity.

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$c \odot (x_1, x_2) = (c^2 x_1, c^3 x_2)$$

(a) Show that **scalar multiplication distributes over vector addition**, i.e. that $c \odot ((x_1, x_2) \oplus (y_1, y_2)) = c \odot (x_1, x_2) \oplus c \odot (y_1, y_2)$ for all $c \in \mathbb{R}$ and $(x_1, x_2), (y_1, y_2) \in V$.

(b) Show that scalar multiplication does not distribute over scalar addition, i.e. for some $c, d \in \mathbb{R}$ and $(x_1, x_2) \in V$ $(c + d) \odot (x_1, x_2) \neq c \odot (x_1, x_2) \oplus d \odot (x_1, x_2)$.

V1. Let V be the set of all real numbers with the operations, for any $x, y \in V$, $c \in \mathbb{R}$,

$$x \oplus y = \sqrt{x^2 + y^2}$$
$$c \odot x = cx$$

(a) Show that the **vector addition $\oplus$ is associative**, i.e. that $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ for all $x, y, z \in V$.

(b) Show that there is no additive identity element, i.e. there is no element $\mathbf{z} \in V$ such that $x \oplus \mathbf{z} = x$ for all $x \in V$.

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V$, $c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 y_2)$$
$$c \odot (x_1, x_2) = (cx_1, cx_2)$$

(a) Show that there is an **additive identity element**, i.e. an element $\mathbf{z} \in V$ such that $(x_1, x_2) \oplus \vec{z} = (x_1, x_2)$ for any $(x_1, x_2) \in V$.

(b) Show that scalar multiplication does not distribute over vector addition, i.e. for some $a \in \mathbb{R}$ and $(x_1, x_2), (y_1, y_2) \in V$, that $a \odot ((x_1, x_2) \oplus (y_1, y_2)) \neq a \odot (x_1, x_2) \oplus a \odot (y_1, y_2)$.

Standard V2

V2. Let V be the set of all real numbers together with the operations defined by, for any $x, y \in V$ and $c \in \mathbb{R}$,

$$x \oplus y = x + y - 3$$
$$c \odot x = cx - 3(c-1)$$

List the 8 defining properties of a vector space, and label each as "TRUE" or "FALSE" as they apply to V. Based on these, conclude whether V is a vector space or not.

Solution. All properties are TRUE, so V is a vector space. $\qquad\square$

V2. Let $V = \left\{(x_1, x_2) \,\middle|\, x_1^2 = x_2\right\}$ with the operations, for any $(x_1, x_2), (y_1, y_2) \in V, c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2 + 2x_1 y_1)$$
$$c \odot (x_1, x_2) = (cx_1, c^2 x_2)$$

List the 8 defining properties of a vector space, and label each as "TRUE" or "FALSE" as they apply to V. Based on these, conclude whether V is a vector space or not.

Solution. All properties are TRUE, so V is a vector space. $\qquad\square$

V2. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V, c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2 + 2x_1 y_1)$$
$$c \odot (x_1, x_2) = (cx_1, c^2 x_2)$$

List the 8 defining properties of a vector space, and label each as "TRUE" or "FALSE" as they apply to V. Based on these, conclude whether V is a vector space or not.

Solution. Not a vector space as the following properties are FALSE: vector distribution. $\qquad\square$

V2. Let $V = \left\{(x_1, x_2) \,\middle|\, x_1^2 = x_2\right\}$ with the operations, for any $(x_1, x_2), (y_1, y_2) \in V, c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 - y_1, x_2 + y_2 - 2x_1 y_1)$$
$$c \odot (x_1, x_2) = (cx_1, c^2 x_2)$$

List the 8 defining properties of a vector space, and label each as "TRUE" or "FALSE" as they apply to V. Based on these, conclude whether V is a vector space or not.

Solution. Not a vector space as the following properties are FALSE: addition commutivity, addition associativity, vector distribution. $\qquad\square$

V2. Let V be the set of all points on the line $x + y = 2$ with the operations, for any $(x_1, x_2), (y_1, y_2) \in V, c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1 - 1, x_2 + y_2 - 1)$$
$$c \odot (x_1, x_2) = (cx_1 - (c-1), cx_2 - (c-1))$$

List the 8 defining properties of a vector space, and label each as "TRUE" or "FALSE" as they apply to V. Based on these, conclude whether V is a vector space or not.

Solution. All properties are TRUE, so V is a vector space. $\qquad\square$

V2. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V, c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$c \odot (x_1, x_2) = (0, cx_2)$$

List the 8 defining properties of a vector space, and label each as "TRUE" or "FALSE" as they apply to V. Based on these, conclude whether V is a vector space or not.

Solution. Not a vector space as the following properties are FALSE: 1 is not a scalar multiplicative identity.
$\square$

V2. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V, c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$c \odot (x_1, x_2) = (c, cx_2)$$

List the 8 defining properties of a vector space, and label each as "TRUE" or "FALSE" as they apply to V. Based on these, conclude whether V is a vector space or not.

Solution. Not a vector space as the following properties are FALSE: scalar multiplication is not associative, 1 is not a scalar multiplication identity, scalar multiplication does not distribute over vector addition. $\square$

V2. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, x_2), (y_1, y_2) \in V, c \in \mathbb{R}$,

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$c \odot (x_1, x_2) = (c^2 x_1, c^3 x_2)$$

List the 8 defining properties of a vector space, and label each as "TRUE" or "FALSE" as they apply to V. Based on these, conclude whether V is a vector space or not.

Solution. Not a vector space as the following properties are FALSE: scalar multiplication does not distribute over scalar addition. $\square$

Standard V3

V3. Determine if $\begin{bmatrix} 0 \\ -1 \\ 2 \\ 6 \end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix} 3 \\ -1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

Solution.

$$\text{RREF}\left(\left[\begin{array}{cc|c} 3 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{array}\right]\right) = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Since this system has a solution, $\begin{bmatrix} 0 \\ -1 \\ 2 \\ 6 \end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix} 3 \\ -1 \\ -1 \\ 0 \end{bmatrix}$ and

$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, namely

$$\begin{bmatrix} 0 \\ -1 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

$\square$

V3. Determine if $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix} 5 \\ 2 \\ -3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 8 \\ 3 \\ 5 \\ -1 \end{bmatrix}$.

Solution.

$$\text{RREF}\left(\left[\begin{array}{ccc|c} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The system has no solution, so $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ is not a linear combination of the three other vectors. $\square$

V3. Determine if $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ can be written as a linear combination of the vectors $\begin{bmatrix} -1 \\ -9 \\ 15 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 5 \\ -5 \end{bmatrix}$.

Solution.

$$\text{RREF}\left(\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -9 & 5 & 0 \\ 15 & -5 & 2 \end{array}\right]\right) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Since this system has no solution, $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ cannot be written as a linear combination of the vectors $\begin{bmatrix} -1 \\ -9 \\ 15 \end{bmatrix}$ and

$\begin{bmatrix} 1 \\ 5 \\ -5 \end{bmatrix}$.

$\square$

V3. Determine if $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}$.

Solution.

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} 2 & 1 & -3 & 1 \\ 3 & -1 & -2 & 4 \\ -1 & 0 & 5 & 3 \end{array} \right] \right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Since this system has a solution, $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is a linear combination of the three vectors. $\qquad\square$

V3. Determine if $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$.

Solution.

$$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} 3 & 1 & 5 & 1 \\ 0 & -1 & 1 & 4 \\ -1 & 4 & -6 & 3 \end{array} \right] \right) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Since the corresponding system has no solutions, $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ is not a linear combination of the three vectors. $\qquad\square$

V3. Determine if $\begin{bmatrix} 0 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ belongs to the span of the set $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \\ 3 \end{bmatrix} \right\}$.

Solution. Since

$$\mathrm{RREF}\left(\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & -1 & -1 \\ -1 & 4 & 6 \\ 5 & 3 & -7 \end{array} \right] \right) = \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

does not require a contradiction, $\begin{bmatrix} 0 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ is a linear combination of the three vectors. $\qquad\square$

V3. Determine if $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ belongs to the span of the set $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \\ 3 \end{bmatrix} \right\}$.

Solution. Since

$$\mathrm{RREF}\left(\left[\begin{array}{cc|c} 2 & 4 & 4 \\ 0 & -1 & -1 \\ -1 & 4 & 6 \\ 5 & 3 & -7 \end{array} \right] \right) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

requires the contradiction $0 = 1$, $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ is not a linear combination of the three vectors. $\qquad\square$

V3. Determine if $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ belongs to the span of the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution. Since

$$\text{RREF}\left(\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & -2 \\ -3 & -6 & 0 & 4 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

requires the contradiction $0 = 1$, $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ is not a linear combination of the three vectors. $\qquad\square$

Standard V4

V4. Determine if the vectors $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ span $\mathbb{R}^3$

Solution.

$$\text{RREF}\left(\begin{bmatrix} -3 & 5 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -2 & -1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix has a zero row, the vectors do not span $\mathbb{R}^3$. $\qquad\square$

V4. Determine if the vectors $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix}$ span $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are zero rows, they do not span. $\qquad\square$

V4. Determine if the vectors $\begin{bmatrix} 8 \\ 21 \\ -7 \end{bmatrix}$, $\begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}$ span $\mathbb{R}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 8 & -3 & -1 & 4 \\ 21 & -8 & -3 & 11 \\ -7 & 3 & 2 & -5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, they do not span $\mathbb{R}^3$. $\qquad\square$

V4. Determine if the vectors $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ span $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, the vectors do not span $\mathbb{R}^4$. $\qquad\square$

V4. Determine if the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 0 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ span $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since every row contains a pivot, the vectors span $\mathbb{R}^4$. $\qquad\square$

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V4. Does span $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$?

Solution. Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 4 & 2 \\ 4 & -9 & -3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has a zero row, the vectors fail to span $\mathbb{R}^3$. □

V4. Does span $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\} = \mathbb{R}^3$?

Solution. Since

$$\text{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 2 & 2 \\ 4 & -9 & 3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

lacks a zero row, the vectors span $\mathbb{R}^3$. □

V4. Does span $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 5 \\ 1 \\ -3 \end{bmatrix} \right\} = \mathbb{R}^5$?

Solution. Since there are only three vectors, they cannot span $\mathbb{R}^5$. (Or, since RREF must contain a zero row, so they cannot span $\mathbb{R}^5$.) □

Standard V5

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x, y \text{ are integers} \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x = y \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.

Solution. W is not a subspace; for example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ belongs to W while $\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does not.

U is a subspace. ☐

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \\ z \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ 1 \\ z \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^4$, and that one of the sets is not.

Solution. U is not a subspace; for example, it doesn't contain the zero vector.

W is a subspace. ☐

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + y + z = 1 \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + y + z = 0 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.

Solution. W is not a subspace; for example, it doesn't contain the zero vector.

U is a subspace. ☐

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + y = 3z \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x + y = 3, z \in \mathbb{R} \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.

Solution. U is not a subspace; for example, it doesn't contain the zero vector.

W is a subspace. ☐

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x + y = 0 \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| xy = 0 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.

Solution. U is not a subspace; for example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ belong to U while $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ does not.

W is a subspace. ☐

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x = y \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| |x| = |y| \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.

Solution. U is not a subspace; for example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ belong to U while $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ does not.

W is a subspace. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, y = 2x \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, y = x^2 \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^2$, and that one of the sets is not.

Solution. U is not a subspace; for example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ belong to U while $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ does not.

W is a subspace. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

V5. Consider the following two sets of Euclidean vectors.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,\middle|\, z = 2xy \right\} \qquad U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,\middle|\, z = 2x + y \right\}$$

Show that one of these sets is a subspace of $\mathbb{R}^3$, and that one of the sets is not.

Solution. W is not a subspace; for example, $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ belongs to W while $\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ does not.

U is a subspace. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Module S

Standard S1

S1. Determine if the vectors in the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$ are linearly dependent or linearly independent.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since each column is a pivot column, the vectors are linearly independent. □

S1. Determine if the vectors in the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}$ are linearly dependent or linearly independent

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, the set is linearly dependent. □

S1. Determine if the vectors in the set $\left\{ \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$ are linearly dependent or linearly independent

Solution.

$$\text{RREF}\left(\begin{bmatrix} -3 & 1 & 0 \\ 8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every column is a pivot column, therefore the set is linearly independent. □

S1. Determine if the vectors in the set $\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$ are linearly dependent or linearly independent

Solution.

$$\text{RREF}\left(\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

This has a non pivot column, therefore the set is linearly dependent. □

S1. Determine if the vectors in the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 6 \\ 5 \end{bmatrix} \right\}$ are linearly dependent or linearly independent.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 3 & 1 & 3 \\ -1 & 2 & -8 \\ 0 & -2 & 6 \\ 4 & 1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced row echelon form has a nonpivot column, the vectors are linearly dependent. □

S1. Determine if the vectors in the set $\left\{\begin{bmatrix} 3 \\ -1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 6 \\ 5 \end{bmatrix}\right\}$ are linearly dependent or linearly

independent.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 3 & 1 & 3 \\ -1 & 2 & -8 \\ 1 & -2 & 6 \\ 4 & 1 & 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced row echelon form has only pivot columns, the vectors are linearly independent. □

S1. Determine if the vectors in the set $\left\{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \end{bmatrix}\right\}$ are linearly dependent or linearly

independent.

Solution.

$$\mathrm{RREF}\begin{bmatrix} 2 & -1 & 2 \\ 1 & -3 & -4 \\ -1 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the third column is not a pivot column, the set is linearly dependent. □

S1. Determine if the vectors in the set $\left\{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}\right\}$ are linearly dependent or linearly

independent.

Solution.

$$\mathrm{RREF}\begin{bmatrix} 2 & -1 & 2 & 1 \\ 1 & -3 & -4 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the third column is not a pivot column, the set is linearly dependent. □

Standard S2

S2. Determine if the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

S2. Determine if the set $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis. □

S2. Determine if the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 4 \\ 2 & 2 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

S2. Determine if the set $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 5 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 3 & 2 & 1 & -1 \\ -1 & 0 & -1 & 3 \\ 2 & 2 & 0 & 0 \\ 3 & 4 & -1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

S2. Determine if the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

S2. Determine if the set $\left\{ \begin{bmatrix} 8 \\ 21 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

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Solution.

$$\text{RREF}\left(\begin{bmatrix} 8 & -3 & -1 \\ 21 & -8 & -3 \\ -7 & 3 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis.   □

S2. Determine if the set $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Solution. Since

$$\text{RREF}\begin{bmatrix} 2 & 3 & -4 \\ -1 & 12 & 2 \\ 4 & -9 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis.   □

S2. Determine if the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since this is the identity matrix, this set is a basis.   □

Standard S3

S3. Let $W = \text{span}\left(\left\{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\3\\6\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\3\\-2\end{bmatrix}, \begin{bmatrix}7\\-1\\8\\-3\end{bmatrix}\right\}\right)$. Find a basis for W.

Solution.

$$\text{RREF}\left(\begin{bmatrix}1 & 3 & 3 & 7\\1 & 3 & -1 & -1\\2 & 6 & 3 & 8\\1 & 3 & -2 & -3\end{bmatrix}\right) = \begin{bmatrix}1 & 3 & 0 & 1\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

Thus a basis is given by its pivot columns: $\left\{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\3\\-2\end{bmatrix}\right\}$. $\qquad\square$

S3. Let $W = \text{span}\left(\left\{\begin{bmatrix}-3\\-8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}\right\}\right)$. Find a basis for W.

Solution. Let $A = \begin{bmatrix}-3 & 1 & 0\\-8 & 2 & -1\\0 & 2 & 3\end{bmatrix}$, and compute $\text{RREF}(A) = \begin{bmatrix}1 & 0 & \frac{1}{2}\\0 & 1 & \frac{3}{2}\\0 & 0 & 0\end{bmatrix}$. Since the first two columns are

pivot columns, $\left\{\begin{bmatrix}-3\\-8\\0\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right\}$ is a basis for W. $\qquad\square$

S3. Let $W = \text{span}\left(\left\{\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\3\\6\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\1\end{bmatrix}\right\}\right)$. Find a basis of W.

Solution.

$$\text{RREF}\left(\begin{bmatrix}2 & 3 & 0 & 1\\0 & 1 & 0 & 2\\-2 & 3 & 1 & 0\\0 & 6 & 1 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0 & -\frac{5}{2}\\0 & 1 & 0 & 2\\0 & 0 & 1 & -11\\0 & 0 & 0 & 0\end{bmatrix}$$

Then its pivot columns $\left\{\begin{bmatrix}2\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\3\\6\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$ form a basis of W. $\qquad\square$

S3. Let $W = \text{span}\left(\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\0\\1\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\4\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}\right)$. Find a basis of W.

Solution.

$$\text{RREF}\left(\begin{bmatrix}1 & 2 & 3 & 1\\-1 & 0 & -1 & 1\\3 & 1 & 4 & 1\\-3 & 1 & -2 & -7\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

Thus its pivot columns $\left\{\begin{bmatrix}1\\-1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-7\end{bmatrix}\right\}$ form a basis for W. $\qquad\square$

S3. Let $W = \text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -8 \\ -1 \end{bmatrix} \right\}$. Find a basis for this vector space.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & -8 \\ 1 & 1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus its pivot columns $\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ form a basis of W. ☐

S3. Let $W = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\}$. Find a basis for this vector space.

Solution.

$$\text{RREF}\begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 4 & 2 \\ 4 & -9 & -3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus its pivot columns $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix} \right\}$ form a basis of W. ☐

S3. Let $W = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}$. Find a basis for this vector space.

Solution.

$$\text{RREF}\begin{bmatrix} 2 & -4 & 3 & 1 \\ -1 & 2 & 12 & 2 \\ 4 & -8 & -9 & 3 \\ 2 & -4 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus its pivot columns $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}$ form a basis of W. ☐

Standard S4

S4. Let $W = \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 8 \\ -3 \end{bmatrix} \right\}\right)$. Find the dimension of W.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has two pivot columns, so W has dimension 2. $\qquad\square$

S4. Let $W = \text{span}\left(\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}\right)$. Compute the dimension of W.

Solution. Let $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$, and compute $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$. Since there are two pivot

columns, $\dim W = 2$. $\qquad\square$

S4. Let $W = \text{span}\left(\left\{ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}\right)$. Compute the dimension of W.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$. $\qquad\square$

S4. Let $W = \text{span}\left(\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix} \right\}\right)$. Compute the dimension of W.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$. $\qquad\square$

S4. Let $W = \text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -8 \\ -1 \end{bmatrix} \right\}$. Find the dimension of W.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & -8 \\ 1 & 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since it has two pivot columns, its dimension is 2. $\qquad\square$

S4. Let $W = \operatorname{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \end{bmatrix} \right\}$. Find the dimension of W.

Solution.

$$\operatorname{RREF} \begin{bmatrix} 2 & 3 & 1 & -4 \\ -1 & 12 & 4 & 2 \\ 4 & -9 & -3 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since it has two pivot columns, its dimension is 2. □

S4. Let $W = \operatorname{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -8 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}$. Find the dimension of W.

Solution.

$$\operatorname{RREF} \begin{bmatrix} 2 & -4 & 3 & 1 \\ -1 & 2 & 12 & 2 \\ 4 & -8 & -9 & 3 \\ 2 & -4 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since it has three pivot columns, its dimension is 3. □

Standard S5

S5. Determine if the set of polynomials $\{x^2 + x, x^2 + 2x - 1, x^2 + 3x - 2\}$ is linearly dependent or linearly independent.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, the set is linearly dependent. ☐

S5. Determine if the set of polynomials $\{x^2 + x, x^2 + 2x - 1, x^2 + 3x - 2\}$ spans $\mathcal{P}^2$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, the set does not span. ☐

S5. Determine if the set of polynomials $\{-3x^3 - 8x^2, x^3 + 2x^2 + 2, -x^2 + 3\}$ is linearly dependent or linearly independent.

Solution.

$$\text{RREF}\left(\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has a non pivot column, therefore the set is linearly dependent. ☐

S5. Determine if the set of polynomials $\{-3x^3 - 8x^2, x^3 + 2x^2 + 2, -x^2 + 3\}$ spans $\mathcal{P}^3$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, the set does not span (also, $\mathcal{P}^3$ is four-dimensional and only three vectors are given). ☐

S5. Determine if the set of matrices $\left\{\begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -8 \\ 6 & 5 \end{bmatrix}\right\}$ is linearly dependent or linearly independent.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 3 & 1 & 3 \\ -1 & 2 & -8 \\ 0 & -2 & 6 \\ 4 & 1 & 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the reduced row echelon form has a nonpivot column, the vectors are linearly dependent. ☐

S5. Determine if the set $\{x^2 + x - 1, 3x^2 - x + 1, 2x^2 - 2\}$ is a basis of $\mathcal{P}^2$.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. ☐

S5. Determine if the set $\{2x^2 - x + 3, 2x^2 + 2, -x^2 + 4x + 1\}$ is a basis of $\mathcal{P}^2$.

Solution.

$$\operatorname{RREF}\left(\begin{bmatrix} 2 & 2 & -1 \\ -1 & 0 & 4 \\ 3 & 2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

S5. Determine if the set $\left\{\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 9 \end{bmatrix}\right\}$ is a basis of $M_{2,2}$ or not.

Solution.

$$\operatorname{RREF}\begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis. □

S5. Determine if the set $\left\{\begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ -1 & 8 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}\right\}$ is a basis of $M_{2,2}$.

Solution.

$$\operatorname{RREF}\left(\begin{bmatrix} 3 & 2 & 1 & -1 \\ -1 & 0 & 4 & 3 \\ 2 & 2 & -1 & 0 \\ 3 & 4 & 8 & 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis. □

S5. Determine if the set $\{x^3 - 3x^2 + 2x + 2, -x^3 + 4x^2 - x + 1, -x^3 + 2x + 1, 3x^2 + 3x + 9\}$ is a basis of $\mathcal{P}^3$ or not.

Solution.

$$\operatorname{RREF}\begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 4 & 0 & 3 \\ 2 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the identity matrix, the set is not a basis. □

S5. Determine if the set $\{x^3 - x, x^2 + x + 1, x^3 - x^2 + 2, 2x^2 - 1\}$ is a basis of $\mathcal{P}^3$.

Solution.

$$\operatorname{RREF}\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the resulting matrix is not the identity matrix, it is not a basis. □

S5. Let W be the subspace of $\mathcal{P}^3$ given by

$$\left(W = \operatorname{span}\left(\{x^3 + x^2 + 2x + 1, 3x^3 + 3x^2 + 6x + 3, 3x^3 - x^2 + 3x - 2, 7x^3 - x^2 + 8x - 3\}\right).\right.$$

Find a basis for W.

Solution.

$$\operatorname{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis is $\{x^3 + x^2 + 2x + 1, 3x^3 - x^2 + 3x - 2\}$. □

S5. Let $W = \operatorname{span}\left\{\begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right\}$. Find a basis for this vector space.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis of W. □

S5. Let W be the subspace of $\mathcal{P}^2$ given by $W = \mathrm{span}\left(\{-3x^2 - 8x, x^2 + 2x + 2, -x + 3\}\right)$. Find a basis for W.

Solution. Let $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$, and compute $\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$. Since the first two columns are pivot columns, $\{-3x^2 - 8x, x^2 + 2x + 2\}$ is a basis for W. □

S5. Let W be the subspace of $\mathcal{P}^3$ given by

$$W = \mathrm{span}\left(\{x^3 - x^2 + 3x - 3, 2x^3 + x + 1, 3x^3 - x^2 + 4x - 2, x^3 + x^2 + x - 7\}\right).$$

Compute the dimension of W.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$. □

S5. Let W be the subspace of $M_{2,2}$ given by $W = \mathrm{span}\left(\left\{ \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\}\right)$. Compute the dimension of W.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$. □

S5. Let $W = \mathrm{span}\left\{2x^2 - x + 3, 2x^2 + 2, -x^2 + 4x + 1\right\}$. Find the dimension of W.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 2 & 2 & -1 \\ -1 & 0 & 4 \\ 3 & 2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since it has three pivot columns, its dimension is 3. □

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Standard S6

S6. Find a basis for the solution space of the homogeneous system of equations

$$x + 3y + 3z + 7w = 0$$
$$x + 3y - z - w = 0$$
$$2x + 6y + 3z + 8w = 0$$
$$x + 3y - 2z - 3w = 0$$

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution space is

$$\left\{ \begin{bmatrix} -3a - b \\ a \\ -2b \\ b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

So a basis for the solution space is

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

□

S6. Find a basis for the solution space of the homogeneous system of equations

$$x + 2y + 3z + w = 0$$
$$3x - y + z + w = 0$$
$$2x - 3y - 2z = 0$$
$$-x + 2z + 5w = 0$$

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & -2 & 3 & 1 \\ 3 & -1 & 1 & 1 \\ 2 & -3 & -2 & 0 \\ -1 & 0 & 2 & 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution space is

$$\left\{ \begin{bmatrix} a \\ 2a \\ -2a \\ a \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}$$

So a basis for the solution space is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$.

□

S6. Find a basis for the solution space of the homogeneous system of equations

$$x + 2y + 3z + w = 0$$
$$3x - y + z + w = 0$$
$$2x - 3y - 2z = 0$$

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & -2 & 3 & 1 \\ 3 & -1 & 1 & 1 \\ 2 & -3 & -2 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & \frac{5}{7} & \frac{3}{7} \\ 0 & 1 & \frac{8}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the solution space is

$$\left\{ \begin{bmatrix} -\frac{5}{7}a - \frac{3}{7}b \\ -\frac{8}{7}a - \frac{2}{7}b \\ a \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

So a basis for the solution space is $\left\{ \begin{bmatrix} -\frac{5}{7} \\ -\frac{8}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{7} \\ -\frac{2}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$, or $\left\{ \begin{bmatrix} 5 \\ 8 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ -7 \end{bmatrix} \right\}$. $\qquad\square$

S6. Find a basis for the solution space to the system of equations

$$x + 2y - 3z = 0$$
$$2x + y - 4z = 0$$
$$3y - 2z = 0$$
$$x - y - z = 0$$

Solution.

$$\text{RREF}\left(\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -4 \\ 0 & 3 & -2 \\ 1 & -1 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{5}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the solution space is

$$\left\{ \begin{bmatrix} \frac{5}{3}a \\ \frac{2}{3}a \\ a \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

So a basis is $\left\{ \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \right\}$. $\qquad\square$

S6. Find a basis for the solution space to the homogeneous system of equations

$$2x_1 + 3x_2 - 5x_3 + 14x_4 = 0$$
$$x_1 + x_2 - x_3 + 5x_4 = 0$$

Solution. Let $A = \begin{bmatrix} 2 & 3 & -5 & 14 & | & 0 \\ 1 & 1 & -1 & 5 & | & 0 \end{bmatrix}$, so RREF $A = \begin{bmatrix} 1 & 0 & 2 & 1 & | & 0 \\ 0 & 1 & -3 & 4 & | & 0 \end{bmatrix}$. It follows that the basis for the

solution space is given by $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$. $\qquad\square$

S6. Find a basis for the solution space to the homogeneous system of equations

$$4x_1 + 4x_2 + 3x_3 - 6x_4 = 0$$
$$-2x_3 - 4x_4 = 0$$
$$2x_1 + 2x_2 + x_3 - 4x_4 = 0$$

Solution. Let $A = \begin{bmatrix} 4 & 4 & 3 & -6 & | & 0 \\ 0 & 0 & -2 & -4 & | & 0 \\ 2 & 2 & 1 & -4 & | & 0 \end{bmatrix}$, so RREF $A = \begin{bmatrix} 1 & 1 & 0 & -3 & | & 0 \\ 0 & 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$. It follows that the basis for

the solution space is given by $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$. □

S6. Find a basis for the solution space to the homogeneous system of equations given by

$$3x + 2y + z = 0$$
$$x + y + z = 0$$

Solution. Let $A = \begin{bmatrix} 3 & 2 & 1 & | & 0 \\ 1 & 1 & 1 & | & 0 \end{bmatrix}$, so RREF $A = \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$. It follows that the basis for the solution

space is given by $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$. □

S6. Find a basis for the solution space to the homogeneous system of equations given by

$$2x_1 - 2x_2 + 6x_3 - x_4 = 0$$
$$3x_1 + 6x_3 + x_4 = 0$$
$$-4x_1 + x_2 - 9x_3 + 2x_4 = 0$$

Solution. Let $A = \begin{bmatrix} 2 & -2 & 6 & -1 & | & 0 \\ 3 & 0 & 6 & 1 & | & 0 \\ -4 & 1 & -9 & 2 & | & 0 \end{bmatrix}$, so RREF $A = \begin{bmatrix} 1 & 0 & 2 & 0 & | & 0 \\ 0 & 1 & -1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$. It follows that the basis

for the solution space is given by $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. □

Module A

Standard A1

A1. Consider the following maps of polynomials $S : \mathcal{P}^6 \to \mathcal{P}^6$ and $T : \mathcal{P}^6 \to \mathcal{P}^6$ defined by

$$S(f(x)) = f(x) + 3 \text{ and } T(f(x)) = f(x) + f(3).$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. T is linear, S is not. □

A1. Consider the following maps of polynomials $S : \mathcal{P}^4 \to \mathcal{P}^5$ and $T : \mathcal{P}^4 \to \mathcal{P}^5$ defined by

$$S(f(x)) = xf(x) - f(1) \text{ and } T(f(x)) = xf(x) - x.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

A1. Consider the following maps of polynomials $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ defined by

$$S(f(x)) = f'(x) - f''(x) \text{ and } T(f(x)) = f(x) - (f(x))^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

A1. Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^4$ and $T : \mathcal{P}^2 \to \mathcal{P}^4$ defined by

$$S(f(x)) = x^2 f(x) \text{ and } T(f(x)) = (f(x))^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

A1. Consider the following maps of polynomials $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ defined by

$$S(f(x)) = (f(x))^2 + 1 \text{ and } T(f(x)) = (x^2 + 1)f(x).$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. T is linear, S is not. □

A1. Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^2$ and $T : \mathcal{P}^2 \to \mathcal{P}^2$ defined by

$$S(ax^2 + bx + c) = cx^2 + bx + a \text{ and } T(ax^2 + bx + c) = a^2 x^2 + b^2 x + c^2.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

A1. Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^1$ and $T : \mathcal{P}^2 \to \mathcal{P}^1$ defined by

$$S(ax^2 + bx + c) = 2ax + b \text{ and } T(ax^2 + bx + c) = a^2 x + b.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

A1. Consider the following maps of polynomials $S : \mathcal{P}^2 \to \mathcal{P}^3$ and $T : \mathcal{P}^2 \to \mathcal{P}^3$ defined by

$$S(ax^2 + bx + c) = ax^3 + bx^2 + cx \text{ and } T(ax^2 + bx + c) = abcx^3.$$

Show that one of these maps is a linear transformation, and that the other map is not.
Solution. S is linear, T is not. □

Standard A2

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \\ 0 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation given by the requirements

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 3 \end{bmatrix} \text{ and } T(\vec{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 7x + 2y + 3z \\ 0 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \\ 7x \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}\right)$

Solution.

$$\begin{bmatrix} 3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \\ 7 & 0 & 0 \end{bmatrix}$$

□

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \\ 7 & 0 & 0 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & -5 & 0 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}\right)$

Solution. □

A2. Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation given by the following

$$T(\vec{e_1}) = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} \qquad T(\vec{e_2}) = \begin{bmatrix} 0 \\ 3 \\ -5 \\ 0 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$

Solution. □

A2. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - x_3 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - 5x_3 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 + 3x_3 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}\right)$

Solution.

$$\begin{bmatrix} 0 & 1 & 3 \end{bmatrix}$$

□

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 0 & 1 & 3 \end{bmatrix}.$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by the following

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3 + 3x_1 \end{bmatrix}.$$

(a) Write the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix} -1 \\ 7 \\ 3 \end{bmatrix}\right)$

Solution.

$$\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$$

$\square$

A2. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$$

(a) Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

(b) Compute $T\left(\begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}\right)$

Standard A3

A3. Determine if the following linear maps are injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y \\ x - y \\ x + 4y \end{bmatrix}$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ 2y + 3z \\ x - y - 2z \end{bmatrix}$

Solution.

(a)

$$\text{RREF}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, T is not injective. Since there is a zero row, T is not surjective.

(b)

$$\text{RREF}\left(\begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 1 & 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since all columns are pivot columns, S is injective. Since there is a zero row, S is not surjective.

□

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ 2y + 3z \\ x - y - 2z \end{bmatrix}$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y \\ x - y \\ x + 4y \end{bmatrix}$

Solution.

(a)

$$\text{RREF}\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & -1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a nonpivot column, S is not injective. Since there is a zero row, S is not surjective.

(b)

$$\text{RREF}\left(\begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 1 & 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since all columns are pivot columns, T is injective. Since there is a zero row, T is not surjective.

□

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the standard matrix $\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 3 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix}$

Solution.

(a) RREF $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since each column is a pivot column, S is injective. Since there is no zero row, S is surjective.

(b) Since $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$, T is not injective.

$$\text{RREF} \left(\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 3 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -5 & -\frac{5}{2} \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are only two pivot columns, T is not surjective.

□

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the standard matrix $\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 4 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix}$

Solution.

(a) RREF $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since each column is a pivot column, S is injective. Since there is no zero row, S is surjective.

(b) Since $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$, T is not injective.

$$\text{RREF} \left(\begin{bmatrix} 2 & 3 & -1 & -2 \\ 0 & 1 & 3 & 1 \\ 2 & 1 & -7 & -4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since there are no zero rows, T is surjective.

□

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^4$ given by the standard matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the standard matrix $\begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 11 & -1 & 5 \end{bmatrix}$

Solution.

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Since each column is a pivot column, S is injective. Since there a no zero row, S is not surjective.

(b) Since $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$, T is not injective.

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 11 & -1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -\frac{4}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a zero row, T is not surjective.

$\square$

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^4$ given by the standard matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the standard matrix $\begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 7 & -1 & 5 \end{bmatrix}$

Solution.

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Since each column is a pivot column, S is injective. Since there a no zero row, S is not surjective.

(b) Since $\dim \mathbb{R}^4 > \dim \mathbb{R}^3$, T is not injective.

$$\text{RREF}\left(\begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 4 & 7 & -1 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Since there is not a zero row, T is surjective.

$\square$

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^2 \to \mathbb{R}^3$ where $S(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $S(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ where $T(\vec{e}_1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution.

(a) RREF $\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. The map is injective since every column has a pivot, but is not surjective because there is a row without a pivot.

(b) RREF $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}$. The map is not injective since there is a column without a pivot, but it is surjective because every row has a pivot.

$\square$

A3. Determine if each of the following linear transformations is injective (one-to-one) and/or surjective (onto).

(a) $S : \mathbb{R}^4 \to \mathbb{R}^3$ where $S(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $S(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $S(\vec{e}_3) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, and $S(\vec{e}_4) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$,

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where $T(\vec{e}_1) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, and $T(\vec{e}_3) = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Solution.

(a) RREF $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The map is not injective since it has a column without pivot, but it is surjective because every row has a pivot.

(b) RREF $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The map is not injective since there is a column without a pivot, and it is not surjective because there is a row without a pivot.

$\square$

Standard A4

A4. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x + 3y + 3z + 7w \\ x + 3y - z - w \\ 2x + 6y + 3z + 8w \\ x + 3y - 2z - 3w \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of T.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis for the kernel is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

and a basis for the image is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}$$

$\square$

A4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -3x + y \\ -8x + 2y - z \\ 2y + 3z \end{bmatrix}$$

Compute a basis for the kernel and a basis for the image of T.

Solution. Let $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$, and compute $\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$. Then a basis for the image is its columns,

$$\left\{ \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

And the kernel is the solution set of $AX = 0$, so a basis would be

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}$$

$\square$

A4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map given by $T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 8x - 3y - z + 4w \\ y + 3z - 4w \\ -7x + 3y + 2z - 5w \end{bmatrix}$. Compute a basis

for the kernel and a basis for the image of T.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 8 & -3 & -1 & 4 \\ 0 & 1 & 3 & -4 \\ -7 & 3 & 2 & -5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel. $\qquad \square$

A4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 8x - 3y - z \\ y + 3z \\ -7x + 3y + 2z \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 8 & -3 & -1 \\ 0 & 1 & 3 \\ -7 & 3 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 8 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel. $\qquad \square$

A4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by the standard matrix $\begin{bmatrix} 1 & 2 & 5 \\ 0 & -2 & -4 \\ -3 & 1 & -1 \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 1 & 2 & 5 \\ 0 & -2 & -4 \\ -3 & 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel. $\qquad \square$

A4. Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear map given by the standard matrix $\begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & -2 & -4 & 2 \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution.

$$\mathrm{RREF}\left(\begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & -2 & -4 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel. $\qquad \square$

A4. Let $T : \mathbb{R}^2 \to \mathbb{R}^5$ be the linear map given by the standard matrix $\begin{bmatrix} 3 & 1 \\ 0 & 0 \\ -6 & -2 \\ 1 & \frac{1}{3} \\ 9 & 3 \end{bmatrix}$. Compute a basis for the kernel and a basis for the image of T.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 3 & 1 \\ 0 & 0 \\ -6 & -2 \\ 1 & \frac{1}{3} \\ 9 & 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 3 \\ 0 \\ -6 \\ 1 \\ 9 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} \right\}$ is a basis for the kernel. $\square$

A4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map given by the standard matrix $\begin{bmatrix} 2 & 1 & 0 & 3 \\ -4 & -2 & 0 & 5 \\ -2 & -1 & 0 & 0 \end{bmatrix}$. Compute a basis

for the kernel and a basis for the image of T.

Solution.

$$\text{RREF}\left(\begin{bmatrix} 2 & 1 & 0 & 3 \\ -4 & -2 & 0 & 5 \\ -2 & -1 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} \right\}$ is a basis for the image, and $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the kernel. $\square$

Module M

Standard M1

M1. Let

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 0 & 7 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. CA is the only one that can be computed, and

$$CA = \begin{bmatrix} 3 & 9 & 11 & 1 \\ 0 & 0 & 7 & 2 \\ -2 & -6 & -5 & 0 \end{bmatrix}$$

☐

M1. Let

$$A = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 3 & 7 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. CA is the only one that can be computed, and

$$CA = \begin{bmatrix} 2 & 6 & 11 & 1 \\ 1 & 3 & 7 & 2 \\ -1 & -3 & -5 & 0 \end{bmatrix}$$

☐

M1. Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & -1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. AC is the only one that can be computed, and

$$AC = \begin{bmatrix} 3 & -5 & 14 \\ 1 & -1 & 2 \end{bmatrix}$$

☐

M1. Let

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. AC is the only one that can be computed, and

$$AC = \begin{bmatrix} 9 & -2 & 14 \\ 1 & 0 & 2 \end{bmatrix}$$

M1. Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 7 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 0 & 1 & 7 & 7 \\ -1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad\qquad C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. AB is the only ones that can be computed, and

$$AB = \begin{bmatrix} -3 & -5 & 6 & 14 \\ 0 & 0 & 7 & 35 \end{bmatrix}$$

$\square$

M1. Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix} \qquad\qquad C = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. BC is the only one that can be computed, and

$$BC = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

$\square$

M1. Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 1 & -1 & 3 & -3 \\ 2 & 1 & -1 & 2 \end{bmatrix} \qquad\qquad C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. CB is the only one that can be computed, and

$$CB = \begin{bmatrix} 0 & -3 & 7 & -8 \\ 8 & 4 & -4 & 8 \\ 5 & -2 & 8 & -7 \end{bmatrix}$$

$\square$

M1. Let

$$A = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & -1 & 3 & -3 \end{bmatrix} \qquad\qquad C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

Exactly one of the six products AB, AC, BA, BC, CA, CB can be computed. Determine which one, and compute it.

Solution. CB is the only one that can be computed, and

$$CB = \begin{bmatrix} 3 & 3 & -5 & 7 \\ 4 & -4 & 12 & -12 \\ 7 & 2 & 0 & 3 \end{bmatrix}$$

$\square$

Standard M2

M2. Determine if the matrix $\begin{bmatrix} 1 & 3 & 3 & 7 \\ 1 & 3 & -1 & -1 \\ 2 & 6 & 3 & 8 \\ 1 & 3 & -2 & -3 \end{bmatrix}$ is invertible.

Solution. The second column is a multiple of the first, so it is not invertible. □

M2. Determine if the matrix $\begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$ is invertible.

Solution.
$$\text{RREF} \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since it is not equivalent to the identity matrix, it is not invertible. □

M2. Determine if the matrix $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & 0 \\ -1 & -1 & 5 \end{bmatrix}$ is invertible.

Solution.
$$\text{RREF} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 7 & 0 \\ -1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since it is not equivalent to the identity matrix, it is not invertible. □

M2. Determine if the matrix $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$ is invertible.

Solution. This matrix is row equivalent to the identity matrix, so it is invertible. □

M2. Determine if the matrix $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & -2 & 0 & 3 \end{bmatrix}$ is invertible.

Solution.
$$\text{RREF} \begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & -2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is not row equivalent to the identity matrix, so it is not invertible. □

M2. Determine if the matrix $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is invertible.

Solution. It is row equivalent to the identity matrix, so it is invertible. □

M2. Determine if the matrix $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix}$ is invertible.

Solution.

$$\mathrm{RREF} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 3 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since it is not row equivalent to the identity matrix, it is not invertible. □

M2. Determine if the matrix $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 0 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix}$ is invertible.

Solution.

$$\mathrm{RREF} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 0 & 1 \\ 3 & 2 & -1 & 7 \\ 4 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since it is row equivalent to the identity matrix, it is invertible. □

Standard M3

M3. Show how to find the inverse of the matrix $\begin{bmatrix} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$.

Solution.

$$\text{RREF}\left(\left[\begin{array}{cccc|cccc} 8 & 5 & 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 5 & -3 & 1 & -2 & 0 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -5 & 12 \\ 0 & 1 & 0 & 0 & 1 & 1 & -4 & -9 \\ 0 & 0 & 1 & 0 & -4 & -7 & 20 & 47 \\ 0 & 0 & 0 & 1 & -1 & 0 & 3 & 7 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 1 & 2 & -5 & 12 \\ 1 & 1 & -4 & -9 \\ -4 & -7 & 20 & 47 \\ -1 & 0 & 3 & 7 \end{bmatrix}$. $\qquad\square$

M3. Compute the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution.

$$\text{RREF}(A|I) = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -11 & 37 \\ 0 & 1 & 0 & 0 & 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 1 & 2 & -11 & 37 \\ 0 & -1 & 4 & -14 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\qquad\square$

M3. Show how to find the inverse of the matrix $\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 6 & 0 & 1 \\ -14 & 3 & -4 \\ -23 & 4 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 4 & -3 \\ 8 & -13 & 10 \\ 13 & -24 & 18 \end{bmatrix}$$

$\qquad\square$

M3. Show how to find the inverse of the matrix $\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & \frac{3}{2} & -\frac{3}{2} \\ 1 & -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

$\qquad\square$

M3. Show how to find the inverse of the matrix $\begin{bmatrix} 4 & -1 & -8 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Solution. $\begin{bmatrix} 4 & -1 & -8 & | & 1 & 0 & 0 \\ 2 & 1 & 3 & | & 0 & 1 & 0 \\ 1 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & 1 & -4 & 5 \\ 0 & 1 & 0 & | & -5 & 24 & -28 \\ 0 & 0 & 1 & | & 1 & -5 & 6 \end{bmatrix}$. Thus the inverse is $\begin{bmatrix} 1 & -4 & 5 \\ -5 & 24 & -28 \\ 1 & -5 & 6 \end{bmatrix}$. $\square$

M3. Show how to find the inverse of the matrix $\begin{bmatrix} 1 & -4 & 5 \\ -5 & 24 & -28 \\ 1 & -5 & 6 \end{bmatrix}$.

Solution. $\begin{bmatrix} 1 & -4 & 5 & | & 1 & 0 & 0 \\ -5 & 24 & -28 & | & 0 & 1 & 0 \\ 1 & -5 & 6 & | & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & 4 & -1 & -8 \\ 0 & 1 & 0 & | & 2 & 1 & 3 \\ 0 & 0 & 1 & | & 1 & 1 & 4 \end{bmatrix}$. Thus the inverse is $\begin{bmatrix} 4 & -1 & -8 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$. $\square$

M3. Show how to find the inverse of the matrix $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$.

Solution. $\begin{bmatrix} 3 & 1 & 3 & | & 1 & 0 & 0 \\ 2 & -1 & -6 & | & 0 & 1 & 0 \\ 1 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & 2 & -1 & -3 \\ 0 & 1 & 0 & | & -14 & 9 & 24 \\ 0 & 0 & 1 & | & 3 & -2 & -5 \end{bmatrix}$. Thus the inverse is $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$. $\square$

M3. Show how to find the inverse of the matrix $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$.

Solution. $\begin{bmatrix} 3 & 1 & 3 & | & 1 & 0 & 0 \\ 2 & -1 & -6 & | & 0 & 1 & 0 \\ 1 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & 2 & -1 & -3 \\ 0 & 1 & 0 & | & -14 & 9 & 24 \\ 0 & 0 & 1 & | & 3 & -2 & -5 \end{bmatrix}$. Thus the inverse is $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$. $\square$

M3. Show how to find the inverse of the matrix $\begin{bmatrix} 2 & -1 & -3 \\ -14 & 9 & 24 \\ 3 & -2 & -5 \end{bmatrix}$.

Solution. $\begin{bmatrix} 2 & -1 & -3 & | & 1 & 0 & 0 \\ -14 & 9 & 24 & | & 0 & 1 & 0 \\ 3 & -2 & -5 & | & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & 3 & 1 & 3 \\ 0 & 1 & 0 & | & 2 & -1 & -6 \\ 0 & 0 & 1 & | & 1 & 1 & 4 \end{bmatrix}$. Thus the inverse is $\begin{bmatrix} 3 & 1 & 3 \\ 2 & -1 & -6 \\ 1 & 1 & 4 \end{bmatrix}$. $\square$

Module G

Standard G1

G1. Consider the row operation $R_1 + 5R_3 \rightarrow R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1+5(7) & 2+5(8) & 3+5(9) \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = 4$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (1)(4) = 4$.

$\square$

G1. Consider the row operation $R_2 - 4R_3 \rightarrow R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 4-4(7) & 5-4(8) & 6-4(9) \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = 7$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (1)(7) = 7$.

$\square$

G1. Consider the row operation $R_3 - 2R_1 \rightarrow R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7-2(1) & 8-2(1) & 9-2(1) \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = -8$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (1)(-8) = -8$.

G1. Consider the row operation $4R_3 \to R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ (4)7 & (4)8 & (4)9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = -12$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (4)(-12) = -48$.

G1. Consider the row operation $-8R_1 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} (-8)1 & (-8)2 & (-8)3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = -2$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} -8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (-8)(-2) = 16$.

G1. Consider the row operation $5R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ (5)4 & (5)5 & (5)6 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = 3$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (5)(3) = 15$.

G1. Consider the row operation that swaps R_1 and R_2 applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = 3$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (-1)(3) = -3$.

G1. Consider the row operation that swaps R_3 and R_2 applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = -7$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (-1)(-7) = 7$.

G1. Consider the row operation that swaps R_3 and R_1 applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det C = -11$, find the determinant of RC.

Solution.

1. $R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

2. $\det(RC) = \det(R)\det(C) = (-1)(-11) = 11$.

Standard G2

G2. Compute the determinant of the matrix $\begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix}$.

Solution.

$$\det \begin{bmatrix} 3 & -1 & 0 & 4 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix} + (-2)\det \begin{bmatrix} 3 & 0 & 4 \\ 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = -1(-4) + (-2)(20) = -36$$

$\square$

G2. Compute the determinant of the matrix $\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix}$.

Solution.

$$\det \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 2 & 0 & 0 & -1 \\ 1 & -3 & -2 & -1 \end{bmatrix} = 2\det \begin{bmatrix} 3 & 0 & -1 \\ 1 & 3 & 1 \\ -3 & -2 & -1 \end{bmatrix} - (-1)\det \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 1 & -3 & -2 \end{bmatrix}$$

$$= 2\left(3\det \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} + (-1)\det \begin{bmatrix} 1 & 3 \\ -3 & -2 \end{bmatrix} \right) + \left(1\det \begin{bmatrix} 1 & 3 \\ -3 & -2 \end{bmatrix} - 3\det \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} \right)$$

$$= 2\left(3(-1) + (-1)(7) \right) + ((1)(7) - 3(-3))$$

$$= 2(-10) + 16$$

$$= -4$$

$\square$

G2. Compute the determinant of the matrix $\begin{bmatrix} 2 & 3 & 0 & 1 \\ -1 & 3 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 1 & -1 & 3 & 5 \end{bmatrix}$.

Solution. -60. $\square$

G2. Compute the determinant of the matrix $\begin{bmatrix} 3 & -1 & 0 & 7 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution. 2 $\square$

G2. Compute the determinant of the matrix $\begin{bmatrix} 8 & 5 & 3 & 0 \\ 3 & 2 & 1 & 1 \\ 5 & -3 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix}$.

Solution. -1. $\square$

G2. Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 0 & -4 & 0 \\ -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution. 15. □

G2. Compute the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ -2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution. −15. □

G2. Compute the determinant of the matrix

$$\begin{bmatrix} 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ -2 & 3 & -1 & 1 \\ 5 & 0 & -4 & 0 \end{bmatrix}.$$

Solution. 55. □

G2. Compute the determinant of the matrix

$$\begin{bmatrix} 0 & -4 & 1 & 1 \\ -2 & 3 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 5 & 0 & -4 & 0 \end{bmatrix}.$$

Solution. −55. □

Standard G3

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 5 & 1 \\ -24 & -6 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$, yielding the eigenvalues -3 and 2. □

G3. Find the eigenvalues of the matrix $\begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 6\lambda + 5 = (\lambda + 1)(\lambda + 5)$, yielding the eigenvalues -1 and -5. □

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$, yielding the eigenvalues 1 and 5. □

G3. Find the eigenvalues of the matrix $\begin{bmatrix} -2 & 5 \\ 5 & -2 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 4\lambda - 21 = (\lambda + 7)(\lambda - 3)$, yielding the eigenvalues -7 and 3. □

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 10 & -8 \\ 4 & -2 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - 8\lambda + 12 = (\lambda - 6)(\lambda - 2)$, yielding the eigenvalues 6 and 2. □

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 6 & -4 \\ 11 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2)$, yielding the eigenvalues -5 and 2. □

G3. Find the eigenvalues of the matrix $\begin{bmatrix} -6 & -11 \\ 4 & 9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$, yielding the eigenvalues 5 and -2. □

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 1 & -5 \\ 4 & -8 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 7\lambda + 12 = (\lambda + 3)(\lambda + 4)$, yielding the eigenvalues -3 and -4. □

G3. Find the eigenvalues of the matrix $\begin{bmatrix} -2 & 2 \\ 15 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + 11\lambda - 12 = (\lambda + 12)(\lambda - 1)$, yielding the eigenvalues -12 and 1. □

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 10 & 2 \\ -39 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3)$, yielding the eigenvalues -3 and 4. □

G3. Find the eigenvalues of the matrix $\begin{bmatrix} 8 & 2 \\ -33 & -9 \end{bmatrix}$.

Solution. Its characteristic polynomial is $\lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$, yielding the eigenvalues -3 and 2. □

Standard G4

G4. Find a basis of the eigenspace associated to the eigenvalue -1 in the matrix $\begin{bmatrix} 4 & -2 & -1 \\ 15 & -7 & -3 \\ -5 & 2 & 0 \end{bmatrix}$.

Solution.

$$\text{RREF}\,(A+I) = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So a basis for the eigenspace is given by $\left\{ \begin{bmatrix} \frac{2}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{5} \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 1 in the matrix $A = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 2 & -1 \\ 0 & 2 & 3 \end{bmatrix}$

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} -\frac{1}{4} \\ -1 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 1 in the matrix $A = \begin{bmatrix} 8 & -3 & -1 \\ 21 & -8 & -3 \\ -7 & 3 & 2 \end{bmatrix}$

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} \frac{3}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{7} \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue -1 in the matrix $A = \begin{bmatrix} 9 & -3 & 2 \\ 19 & -6 & 5 \\ -11 & 4 & -2 \end{bmatrix}$

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} -\frac{5}{7} \\ -\frac{12}{7} \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 1 in the matrix $A = \begin{bmatrix} 9 & -3 & 2 \\ 19 & -6 & 5 \\ -11 & 4 & -2 \end{bmatrix}$

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 2 in the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 11 & -6 & 1 & -1 \\ -9 & 5 & 1 & 3 \end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 2 in the matrix $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$.

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 2 in the matrix
$$A = \begin{bmatrix} 0 & -2 & -1 & 0 \\ -4 & -2 & -2 & 0 \\ 14 & 12 & 10 & 2 \\ -13 & -10 & -8 & -1 \end{bmatrix}.$$

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 3 in the matrix
$$A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ -4 & -1 & -2 & 0 \\ 14 & 12 & 11 & 2 \\ -14 & -10 & -9 & -1 \end{bmatrix}.$$

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 2 in the matrix $A = \begin{bmatrix} 8 & -3 & 2 \\ 15 & -5 & 5 \\ -3 & 2 & 1 \end{bmatrix}$

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue -2 in the matrix $A = \begin{bmatrix} 8 & -3 & 2 \\ 23 & -9 & 5 \\ -7 & 2 & -3 \end{bmatrix}$

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue -2 in the matrix $A = \begin{bmatrix} 2 & -3 & 2 \\ 8 & -9 & 5 \\ 8 & -7 & 3 \end{bmatrix}$

Solution. A basis for the eigenspace is given by $\left\{ \begin{bmatrix} \frac{1}{4} \\ 1 \\ 1 \end{bmatrix} \right\}$. □

G4. Find a basis of the eigenspace associated to the eigenvalue 2 in the matrix $\begin{bmatrix} -1 & 1 & 0 \\ -9 & 5 & 0 \\ 15 & -5 & 2 \end{bmatrix}$.

Solution.
$$\text{RREF}(B - 2I) = \text{RREF}\left(\begin{bmatrix} -3 & 1 & 0 \\ -9 & 3 & 0 \\ 15 & -5 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace is given by $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. $\qquad\square$