

## Section V.4

**Definition V.4.1** A subset of a vector space is called a **subspace** if it is a vector space on its own.

For example, the span of these two vectors forms a planar subspace inside of the larger vector space  $\mathbb{R}^3$ .



**Fact V.4.2** Any nonempty subset  $S$  of a vector space  $V$  satisfies the eight vector space properties automatically, since it is a collection of known vectors.

However, to verify that it's a **subspace**, we need to check that addition and multiplication still make sense using only vectors from  $S$ . So we need to check two things:

- The set is **closed under addition**: for any  $\vec{x}, \vec{y} \in S$ , the sum  $\vec{x} + \vec{y}$  is also in  $S$ .
- The set is **closed under scalar multiplication**: for any  $\vec{x} \in S$  and scalar  $c \in \mathbb{R}$ , the product  $c\vec{x}$  is also in  $S$ .

**Activity V.4.3** ( $\sim 15$  min) Let  $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 0 \right\}$ .

*Part 1:* Let  $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  be vectors in  $S$ , so  $x + 2y + z = 0$  and  $a + 2b + c = 0$ . Show that

$\vec{v} + \vec{w} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$  also belongs to  $S$  by verifying that  $(x+a) + 2(y+b) + (z+c) = 0$ .

*Part 2:* Let  $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$ , so  $x + 2y + z = 0$ . Show that  $c\vec{v} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$  also belongs to  $S$  for any  $c \in \mathbb{R}$  by verifying an appropriate equation.

*Part 3:* Is  $S$  a subspace of  $\mathbb{R}^3$ ?

**Activity V.4.4** ( $\sim 10$  min) Let  $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + 2y + z = 4 \right\}$ . Choose a vector  $\vec{v} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$  in  $S$  and a real number  $c = ?$ , and show that  $c\vec{v}$  isn't in  $S$ . Is  $S$  a subspace of  $\mathbb{R}^3$ ?

**Remark V.4.5** Since 0 is a scalar and  $0\vec{v} = \vec{z}$  for any vector  $\vec{v}$ , a nonempty set that is closed under scalar multiplication must contain the zero vector  $\vec{z}$  for that vector space.

Put another way, you can check any of the following to show that a nonempty subset  $W$  isn't a subspace:

- Show that  $\vec{0} \notin W$ .
- Find  $\vec{u}, \vec{v} \in W$  such that  $\vec{u} + \vec{v} \notin W$ .
- Find  $c \in \mathbb{R}, \vec{v} \in W$  such that  $c\vec{v} \notin W$ .

If you cannot do any of these, then  $W$  can be proven to be a subspace by doing the following:

- Prove that  $\vec{u} + \vec{v} \in W$  whenever  $\vec{u}, \vec{v} \in W$ .
- Prove that  $c\vec{v} \in W$  whenever  $c \in \mathbb{R}, \vec{v} \in W$ .

**Activity V.4.6** ( $\sim 20$  min) Consider these subsets of  $\mathbb{R}^4$ :

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = z + 1 \right\} \quad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| y = |z| \right\} \quad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| z = xy \right\}$$

*Part 1:* Show  $R$  isn't a subspace by showing that  $\vec{0} \notin R$ .

*Part 2:* Show  $S$  isn't a subspace by finding two vectors  $\vec{u}, \vec{v} \in S$  such that  $\vec{u} + \vec{v} \notin S$ .

*Part 3:* Show  $T$  isn't a subspace by finding a vector  $\vec{v} \in T$  such that  $2\vec{v} \notin T$ .

**Activity V.4.7** ( $\sim 5$  min) Let  $W$  be a subspace of a vector space  $V$ . How are  $\text{span } W$  and  $W$  related?

- (a)  $\text{span } W$  is bigger than  $W$
- (b)  $\text{span } W$  is the same as  $W$
- (c)  $\text{span } W$  is smaller than  $W$

**Fact V.4.8** If  $S$  is any subset of a vector space  $V$ , then since  $\text{span } S$  collects all possible linear combinations,  $\text{span } S$  is automatically a subspace of  $V$ .

In fact,  $\text{span } S$  is always the smallest subspace of  $V$  that contains all the vectors in  $S$ .