Module E: Solving Systems of Linear Equations

Activity E.8 *(∼10 min)* All inconsistent linear systems contain a logical **contradiction**. Find a contradiction in this system to show that its solution set is $\emptyset$.

$$\begin{aligned} -x_1 + 2x_2 &= 5 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Activity E.9 *(∼10 min)* Consider the following consistent linear system.

$$\begin{aligned} -x_1 + 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Part 1: Find three different solutions for this system. *Part 2:* Let $x_2 = a$ where a is an arbitrary real number, then find an expression for x_1 in terms of a. Use this to write the solution set $\left\{ \begin{bmatrix} ? \\ a \end{bmatrix} \middle|\, a \in \mathbb{R} \right\}$ for the linear system.

Activity E.10 *(∼10 min)* Consider the following linear system.

$$\begin{alignedat}{5} x_1 &+ 2x_2 & & & &- & x_4 &= & 3 \\ & & & & x_3 &+ & 4x_4 &= & -2 \end{alignedat}$$

Describe the solution set

$$\left\{ \begin{bmatrix} ? \\ a \\ ? \\ b \end{bmatrix} \middle|\, a, b \in \mathbb{R} \right\}$$

to the linear system by setting $x_2 = a$ and $x_4 = b$, and then solving for x_1 and x_3.

Activity E.16 *(∼10 min)* Following are seven procedures used to manipulate an augmented matrix. Label the procedures that would result in an equivalent augmented matrix as **valid**, and label the procedures that might change the solution set of the corresponding linear system as **invalid**.

a) Swap two rows.

b) Swap two columns.

c) Add a constant to every term in a row.

d) Multiply a row by a nonzero constant.

e) Add a constant multiple of one row to another row.

f) Replace a column with zeros.

g) Replace a row with zeros.

Activity E.18 *(∼10 min)* Consider the following (equivalent) linear systems.

(A)
$$\begin{aligned} x+2y+\ z &= 3 \\ -x-\ y+\ z &= 1 \\ 2x+5y+3z &= 7 \end{aligned}$$

(B)
$$\begin{aligned} 2x+5y+3z &= 7 \\ -x-\ y+\ z &= 1 \\ x+2y+\ z &= 3 \end{aligned}$$

(C)
$$\begin{aligned} x\ -\ z &= 1 \\ y+\ z &= 1 \\ y+2z &= 4 \end{aligned}$$

(D)
$$\begin{aligned} x+2y+\ z &= 3 \\ y+\ z &= 1 \\ 2x+5y+3z &= 7 \end{aligned}$$

(E)
$$\begin{aligned} x\ -z &= 1 \\ y+z &= 1 \\ z &= 3 \end{aligned}$$

(F)
$$\begin{aligned} x+2y+\ z &= 3 \\ y+\ z &= 1 \\ y+2z &= 4 \end{aligned}$$

Rank the six linear systems from most complicated to simplest.

Activity E.19 *(∼5 min)* We can rewrite the previous in terms of equivalences of augmented matrices

$$\left[\begin{array}{ccc|c} 2 & 5 & 13 & 7 \\ -1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -1 & -1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & 5 & 1 & 3 \end{array}\right] \sim$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Determine the row operation(s) necessary in each step to transform the most complicated system's augmented matrix into the simplest.

Activity E.21 *(~15 min)* Recall that a matrix is in **reduced row echelon form** (**RREF**) if

1. The leading term (first nonzero term) of each nonzero row is a 1. Call these terms **pivots**.
2. Each pivot is to the right of every higher pivot.
3. Each term above or below a pivot is zero.
4. All rows of zeroes are at the bottom of the matrix.

(A)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(B)
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(C)
$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(D)
$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(E)
$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(F)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

For each matrix, circle the leading terms, and label it as RREF or not RREF. For the ones not in RREF, find their RREF.

Activity E.23 *(~8 min)* Consider the matrix

$$\begin{bmatrix} 2 & 6 & -1 & 6 \\ 1 & 3 & -1 & 2 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the first move in converting to RREF?

(a) Add row 3 to row 2 ($R_2 + R_3 \to R_2$)

(b) Add row 2 to row 3 ($R_3 + R_2 \to R_3$)

(c) Swap row 1 to row 2 ($R_1 \leftrightarrow R_2$)

(d) Add -2 row 2 to row 1 ($R_1 - 2R_2 \to R_1$)

Activity E.24 *(~7 min)* Consider the matrix

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 6 & -1 & 6 \\ -1 & -3 & 2 & 0 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

(a) Add row 1 to row 3 ($R_3 + R_1 \to R_3$)

(b) Add -2 row 1 to row 2 ($R_2 - 2R_1 \to R_2$)

(c) Add 2 row 2 to row 3 ($R_3 + 2R_2 \to R_3$)

(d) Add 2 row 3 to row 2 ($R_2 + 2R_3 \to R_2$)

Activity E.25 *(~5 min)* Consider the matrix

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Which row operation is the best choice for the next move in converting to RREF?

(a) Add row 1 to row 2 ($R_2 + R_1 \to R_2$)

(b) Add -1 row 3 to row 2 ($R_2 - R_3 \to R_2$)

(c) Add -1 row 2 to row 3 ($R_3 - R_2 \to R_3$)

(d) Add row 2 to row 1 ($R_1 + R_2 \to R_1$)

Activity E.26 *(~10 min)* Consider the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -1 & 1 \end{bmatrix}.$$

Part 1: Perform three row operations to produce a matrix closer to RREF. *Part 2:* Finish putting it in RREF.

Activity E.27 *(~10 min)* Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 2 & 3 \\ -2 & 1 & 6 & 1 \\ -1 & -3 & -4 & 1 \end{bmatrix}.$$

Compute RREF(A).

Activity E.28 *(∼10 min)* Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 & -4 \\ -2 & -4 & 1 & 1 \\ 3 & 6 & -1 & -4 \end{bmatrix}.$$

Compute RREF(A).

Activity E.30 *(∼10 min)* Free browser-based technologies for mathematical computation are available online.

- Go to `https://octave-online.net`.
- Type `A=sym([1 3 4 ; 2 5 7])` and press `Enter` to store the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 7 \end{bmatrix}$ in the variable A.
  - The symbolic function `sym` is used to calculate precise answers rather than floating-point approximations.
  - The vertical bar in an augmented matrix does not affect row operations, so the RREF of $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 5 & 7 \end{array}\right]$ may be computed in the same way.
- Type `rref(A)` and press `Enter` to compute the reduced row echelon form of A.

Activity E.32 *(∼10 min)* Consider the system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - \ \ 6x_3 &= 11 \end{aligned}$$

Part 1: Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\mathrm{RREF}\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

Part 2: Use the RREF matrix to write a linear system equivalent to the original system. Then find its solution set.

Activity E.33 *(∼10 min)* Consider the vector equation

$$x_1 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 13 \\ 10 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

Part 1: Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\mathrm{RREF} \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right] = \left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

Part 2: Use the RREF matrix to write a linear system equivalent to the original system. Then find its solution set.

Activity E.34 *(∼10 min)* Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

Part 1: Find its corresponding augmented matrix A and use technology to find RREF(A). *Part 2:* How many solutions do these linear systems have?

Activity E.35 *(∼10 min)* Consider the simple linear system equivalent to the system from the previous activity:

$$\begin{aligned} x_1 + 2x_2 \qquad &= 4 \\ x_3 &= -1 \end{aligned}$$

Part 1: Let $x_1 = a$ and write the solution set in the form $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \middle|\, a \in \mathbb{R} \right\}$. *Part 2:* Let $x_2 = b$ and write the solution set in the form $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \middle|\, b \in \mathbb{R} \right\}$. *Part 3:* Which of these was easier? What features of the RREF matrix $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$ caused this?

Activity E.37 *(∼10 min)* Find the solution set for the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - x_5 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 + x_5 &= 2 \end{aligned}$$

by row-reducing its augmented matrix, and then assigning letters to the free variables (given by non-pivot columns) and solving for the bound variables (given by pivot columns) in the corresponding linear system.

Module V: Vector Spaces

Activity V.2 *(~20 min)* Consider each of the following properties of the real numbers $\mathbb{R}^1$. Label each property as **valid** if the property also holds for two-dimensional Euclidean vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^2$ and scalars $a, b \in \mathbb{R}$, and **invalid** if it does not.

1. $\vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}}$.
2. $\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$.
3. There exists some $\vec{\mathbf{z}}$ where $\vec{\mathbf{v}} + \vec{\mathbf{z}} = \vec{\mathbf{v}}$.
4. There exists some $-\vec{\mathbf{v}}$ where $\vec{\mathbf{v}} + (-\vec{\mathbf{v}}) = \vec{\mathbf{z}}$.
5. If $\vec{\mathbf{u}} \neq \vec{\mathbf{v}}$, then $\frac{1}{2}(\vec{\mathbf{u}}+\vec{\mathbf{v}})$ is the only vector equally distant from both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$
6. $a(b\vec{\mathbf{v}}) = (ab)\vec{\mathbf{v}}$.
7. $1\vec{\mathbf{v}} = \vec{\mathbf{v}}$.
8. If $\vec{\mathbf{u}} \neq \vec{\mathbf{0}}$, then there exists some scalar c such that $c\vec{\mathbf{u}} = \vec{\mathbf{v}}$.
9. $a(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = a\vec{\mathbf{u}} + a\vec{\mathbf{v}}$.
10. $(a + b)\vec{\mathbf{v}} = a\vec{\mathbf{v}} + b\vec{\mathbf{v}}$.

Activity V.9 *(~20 min)* Consider the set $V = \{(x, y) \mid y = e^x\}$ with operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 y_2) \qquad c \odot (x_1, y_1) = (cx_1, y_1^c)$$

Part 1: Show that V satisfies the distributive property

$$(a + b) \odot (x_1, y_1) = (a \odot (x_1, y_1)) \oplus (b \odot (x_1, y_1))$$

by simplifying both sides and verifying they are the same expression. *Part 2:* Show that V contains an additive identity element satisfying

$$(x_1, y_1) \oplus \vec{\mathbf{z}} = (x_1, y_1)$$

for all $(x_1, y_1) \in V$ by choosing appropriate values for $\vec{\mathbf{z}} = (\,?\,, \,?\,)$.

Activity V.11 *(~15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_1 + x_2 + y_2, x_1^2 + x_2^2) \qquad c \odot (x_1, y_1) = (x_1^c, y_1 + c - 1).$$

Part 1: Show that 1 is the scalar multiplication identity element by simplifying $1 \odot (x, y)$ to (x, y).
Part 2: Show that V does not have an additive identity element by showing that $(0, -1) \oplus \vec{\mathbf{z}} \neq (0, -1)$ no matter how $\vec{\mathbf{z}} = (z, w)$ is chosen.
Part 3: Is V a vector space?

Activity V.12 *(~15 min)* Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$ have operations defined by

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + 3y_2) \qquad c \odot (x_1, y_1) = (cx_1, cy_1).$$

Part 1: Show that scalar multiplication distributes over vector addition, i.e.

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2)$$

for **all** $c \in \mathbb{R}$, $(x_1, y_1), (x_2, y_2) \in V$.
Part 2: Show that vector addition is not associative, i.e.

$$(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) \neq ((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3)$$

for **some** vectors $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$.
Part 3: Is V a vector space?

Activity V.15 *(~10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$. *Part 1:* Sketch

$1\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$, $3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$, $0\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$, and $-2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-2\\-4\end{bmatrix}$

in the xy plane. *Part 2:* Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\2\end{bmatrix} \,\middle|\, a \in \mathbb{R}\right\}$ in the xy plane.

Activity V.16 *(~10 min)* Consider $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$. *Part 1:* Sketch the following linear combinations in the xy plane.

$$1\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}-1\\1\end{bmatrix} \qquad 0\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix} \qquad 1\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix}$$

$$-2\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix} \qquad -1\begin{bmatrix}1\\2\end{bmatrix} + -2\begin{bmatrix}-1\\1\end{bmatrix}$$

Part 2: Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$ in the xy plane.

Activity V.17 *(~5 min)* Sketch a representation of all the vectors belonging to $\operatorname{span}\left\{\begin{bmatrix}6\\-4\end{bmatrix}, \begin{bmatrix}-3\\2\end{bmatrix}\right\}$ in the xy plane.

Activity V.19 *(∼15 min)* The vector $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ exactly when there exists a solution to the vector equation $x_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+x_2\begin{bmatrix}-1\\-3\\2\end{bmatrix}=\begin{bmatrix}-1\\-6\\1\end{bmatrix}$.
Part 1: Reinterpret this vector equation as a system of linear equations.
Part 2: Find its solution set, using technology to find RREF of its corresponding augmented matrix.
Part 3: Given this solution set, does $\begin{bmatrix}-1\\-6\\1\end{bmatrix}$ belong to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$?

Activity V.22 *(∼10 min)* Determine if $\begin{bmatrix}3\\-2\\1\\5\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\\2\end{bmatrix},\begin{bmatrix}-1\\-3\\2\\2\end{bmatrix}\right\}$ by solving an appropriate vector equation.

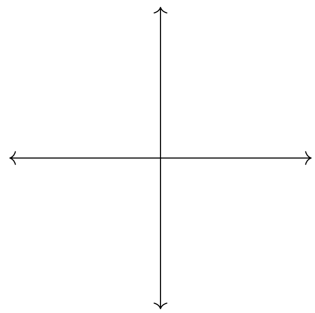
Activity V.23 *(∼5 min)* Determine if $\begin{bmatrix}-1\\-9\\0\end{bmatrix}$ belongs to span $\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}-1\\-3\\2\end{bmatrix}\right\}$ by solving an appropriate vector equation.

Activity V.24 *(∼10 min)* Does the third-degree polynomial $3y^3-2y^2+y+5$ in $\mathcal{P}^3$ belong to span$\{y^3-3y+2,-y^3-3y^2+2y+2\}$? *Part 1:* Reinterpret this question as a question about the solution(s) of a polynomial equation. *Part 2:* Answer this equivalent question, and use its solution to answer the original question.

Activity V.25 *(∼5 min)* Does the polynomial x^2+x+1 belong to span$\{x^2-x,x+1,x^2-1\}$?

Activity V.26 *(∼5 min)* Does the matrix $\begin{bmatrix}3&-2\\1&5\end{bmatrix}$ belong to span $\left\{\begin{bmatrix}1&0\\-3&2\end{bmatrix},\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\right\}$? *Part 1:* Reinterpret this question as a question about the solution(s) of a matrix equation. *Part 2:* Answer this equivalent question, and use its solution to answer the original question.

Activity V.28 *(~5 min)* How many vectors are required to span $\mathbb{R}^2$? Sketch a drawing in the xy plane to support your answer.



(a) 1

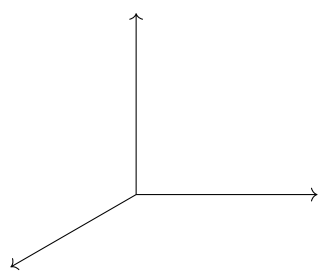
(b) 2

(c) 3

(d) 4

(e) Infinitely Many

Activity V.29 *(~5 min)* How many vectors are required to span $\mathbb{R}^3$?



(a) 1

(b) 2

(c) 3

(d) 4

(e) Infinitely Many

Activity V.31 *(∼15 min)* Choose any vector $\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in $\mathbb{R}^3$ that is not in span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ by using technology to verify that RREF $\left[\begin{array}{cc|c} 1 & -2 & ? \\ -1 & 0 & ? \\ 0 & 1 & ? \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$. (Why does this work?)

Activity V.33 *(∼5 min)* Consider the set of vectors $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix} \right\}$. Does $\mathbb{R}^4 = \operatorname{span} S$? *Part 1:* Rewrite this as a question about the solutions to a vector equation. *Part 2:* Answer your new question, and use this to answer the original question.

Activity V.34 *(∼10 min)* Consider the set of third-degree polynomials

$$S = \{2x^3 + 3x^2 - 1, 2x^3 + 3, 3x^3 + 13x^2 + 7x + 16, \\ -x^3 + 10x^2 + 7x + 14, 4x^3 + 3x^2 + 2\}.$$

Does $\mathcal{P}^3 = \operatorname{span} S$? *Part 1:* Rewrite this as a question about the solutions to a polynomial equation. *Part 2:* Answer your new question, and use this to answer the original question.

Activity V.35 *(∼5 min)* Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Does $M_{2,2} = \operatorname{span} S$? *Part 1:* Rewrite this as a question about the solutions to a matrix equation. *Part 2:* Answer your new question, and use this to answer the original question.

Activity V.36 *(∼5 min)* Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3 \in \mathbb{R}^7$ be three vectors, and suppose $\vec{\mathbf{w}}$ is another vector with $\vec{\mathbf{w}} \in \operatorname{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$. What can you conclude about $\operatorname{span}\{\vec{\mathbf{w}}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$?

(a) $\operatorname{span}\{\vec{\mathbf{w}}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is larger than $\operatorname{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

(b) $\operatorname{span}\{\vec{\mathbf{w}}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} = \operatorname{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

(c) $\operatorname{span}\{\vec{\mathbf{w}}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is smaller than $\operatorname{span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

Activity V.39 *(∼15 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, x + 2y + z = 0 \right\}$.

Part 1: Let $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be vectors in S, so $x + 2y + z = 0$ and $a + 2b + c = 0$. Show that $\vec{\mathbf{v}} + \vec{\mathbf{w}} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$ also belongs to S by verifying that $(x + a) + 2(y + b) + (z + c) = 0$. *Part 2:* Let $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S$, so $x + 2y + z = 0$. Show that $c\vec{\mathbf{v}} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ also belongs to S for any $c \in \mathbb{R}$ by verifying an appropriate equation. *Part 3:* Is S is a subspace of $\mathbb{R}^3$?

Activity V.40 *(∼10 min)* Let $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, x + 2y + z = 4 \right\}$. Choose a vector $\vec{\mathbf{v}} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$ in S and a real number $c = \ ?$, and show that $c\vec{\mathbf{v}}$ isn't in S. Is S a subspace of $\mathbb{R}^3$?

Activity V.42 *(∼20 min)* Consider these subsets of $\mathbb{R}^3$:

$$R = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, y = z + 1 \right\} \qquad S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, y = |z| \right\} \qquad T = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle|\, z = xy \right\}$$

Part 1: Show R isn't a subspace by showing that $\vec{\mathbf{0}} \notin R$. *Part 2:* Show S isn't a subspace by finding two vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in S$ such that $\vec{\mathbf{u}} + \vec{\mathbf{v}} \notin S$. *Part 3:* Show T isn't a subspace by finding a vector $\vec{\mathbf{v}} \in T$ such that $2\vec{\mathbf{v}} \notin T$.

Activity V.43 *(∼5 min)* Let W be a subspace of a vector space V. How are $\operatorname{span} W$ and W related?

(a) $\operatorname{span} W$ is bigger than W

(b) $\operatorname{span} W$ is the same as W

(c) $\operatorname{span} W$ is smaller than W

Activity V.45 *(∼10 min)* Consider the two sets

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -11 \end{bmatrix} \right\}$$

Which of the following is true?

(A) $\operatorname{span} S$ is bigger than $\operatorname{span} T$.

(B) $\operatorname{span} S$ and $\operatorname{span} T$ are the same size.

(C) $\operatorname{span} S$ is smaller than $\operatorname{span} T$.

Activity V.47 *(∼10 min)* Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ be vectors in $\mathbb{R}^n$. Suppose $3\vec{\mathbf{v}}_1 - 5\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_3$, so the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is linearly dependent. Which of the following is true of the vector equation $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + x_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$?

(A) It is consistent with one solution

(B) It is consistent with infinitely many solutions

(C) It is inconsistent.

Activity V.49 *(∼10 min)* Find

$$\operatorname{RREF} \left[\begin{array}{ccccc|c} 2 & 2 & 3 & -1 & 4 & 0 \\ 3 & 0 & 13 & 10 & 3 & 0 \\ 0 & 0 & 7 & 7 & 0 & 0 \\ -1 & 3 & 16 & 14 & 1 & 0 \end{array}\right]$$

and mark the part of the matrix that demonstrates that

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent (the part that shows its linear system has infinitely many solutions).

Activity V.52 *(∼5 min)* Is the set of Euclidean vectors $\left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 10 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 2 \\ 1 \end{bmatrix} \right\}$ linearly dependent or linearly independent? *Part 1:* Reinterpret this question as an appropriate question about solutions to a vector equation. *Part 2:* Use the solution to this question to answer the original question.

Activity V.53 *(∼10 min)* Is the set of polynomials $\left\{x^3+1, x^2+2x, x^2+7x+4\right\}$ linearly dependent or linearly independent? *Part 1:* Reinterpret this question as an appropriate question about solutions to a polynomial equation. *Part 2:* Use the solution to this question to answer the original question.

Activity V.54 *(∼5 min)* What is the largest number of $\mathbb{R}^4$ vectors that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity V.55 *(∼5 min)* What is the largest number of

$$\mathcal{P}^4 = \left\{ax^4+bx^3+cx^2+dx+e \,\middle|\, a,b,c,d,e \in \mathbb{R}\right\}$$

vectors that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity V.56 *(∼5 min)* What is the largest number of

$$\mathcal{P} = \{f(x) \,|\, f(x) \text{ is any polynomial}\}$$

vectors that can form a linearly independent set?

(a) 3

(b) 4

(c) 5

(d) You can have infinitely many vectors and still be linearly independent.

Activity V.59 *(∼15 min)* Label each of the sets A, B, C, D, E as

- SPANS $\mathbb{R}^4$ or DOES NOT SPAN $\mathbb{R}^4$
- LINEARLY INDEPENDENT or LINEARLY DEPENDENT
- BASIS FOR $\mathbb{R}^4$ or NOT A BASIS FOR $\mathbb{R}^4$

by finding RREF for their corresponding matrices.

$$A = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix} \right\}$$

$$C = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}3\\13\\7\\16\end{bmatrix}, \begin{bmatrix}-1\\10\\7\\14\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix} \right\} \qquad D = \left\{ \begin{bmatrix}2\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\3\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\3\end{bmatrix}, \begin{bmatrix}3\\6\\1\\5\end{bmatrix} \right\}$$

$$E = \left\{ \begin{bmatrix}5\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\3\end{bmatrix}, \begin{bmatrix}4\\5\\1\\3\end{bmatrix} \right\}$$

Activity V.60 *(∼10 min)* If $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$ is a basis for $\mathbb{R}^4$, that means $\mathrm{RREF}[\vec{\mathbf{v}}_1\, \vec{\mathbf{v}}_2\, \vec{\mathbf{v}}_3\, \vec{\mathbf{v}}_4]$ doesn't have a non-pivot column, and doesn't have a row of zeros. What is $\mathrm{RREF}[\vec{\mathbf{v}}_1\, \vec{\mathbf{v}}_2\, \vec{\mathbf{v}}_3\, \vec{\mathbf{v}}_4]$?

$$\mathrm{RREF}[\vec{\mathbf{v}}_1\, \vec{\mathbf{v}}_2\, \vec{\mathbf{v}}_3\, \vec{\mathbf{v}}_4] = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Activity V.63 *(∼10 min)* Consider the subspace of $\mathbb{R}^4$ given by $W = \operatorname{span}\left\{ \begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\5\\-1\\0\end{bmatrix} \right\}$.

Part 1: Mark the part of $\mathrm{RREF}\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$ that shows that W's spanning set is linearly dependent.

Part 2: Find a basis for W by removing a vector from its spanning set to make it linearly independent.

Activity V.65 *(∼10 min)* Let W be the subspace of $\mathbb{R}^4$ given by

$$W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Find a basis for W.

Activity V.66 *(∼10 min)* Let W be the subspace of $\mathcal{P}^3$ given by

$$W = \operatorname{span}\left\{x^3 + 3x^2 + x - 1, 2x^3 - x^2 + x + 2, 4x^3 + 5x^2 + 3x, 3x^3 + 2x^2 + 2x + 1\right\}$$

Find a basis for W.

Activity V.67 *(∼10 min)* Let W be the subspace of $M_{2,2}$ given by

$$W = \operatorname{span}\left\{ \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \right\}.$$

Find a basis for W.

Activity V.69 *(∼10 min)* Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\} \qquad \text{and} \qquad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Part 1: Find a basis for $\operatorname{span} S$. *Part 2:* Find a basis for $\operatorname{span} T$.

Activity V.73 *(∼10 min)* Find the dimension of each subspace of $\mathbb{R}^4$ by finding RREF for each corresponding matrix.

$$\operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} \quad \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$$\operatorname{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\} \quad \operatorname{span}\left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Activity V.76 *(~5 min)* Suppose W is a subspace of $\mathcal{P}^8$, and you know that the set $\{x^3+x, x^2+1, x^4-x\}$ is a linearly independent subset of W. What can you conclude about W?

(a) The dimension of W is at most 3.

(b) The dimension of W is exactly 3.

(c) The dimension of W is at least 3.

Activity V.77 *(~5 min)* Suppose W is a subspace of $\mathcal{P}^8$, and you know that W is spanned by the six vectors

$$\{x^4-x, x^3+x, x^3+x+1, x^4+2x, x^3, 2x+1\}.$$

What can you conclude about W?

(a) The dimension of W is at most 6.

(b) The dimension of W is exactly 6.

(c) The dimension of W is at least 6.

Activity V.80 *(~5 min)* Note that if $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are solutions to $x_1\vec{\mathbf{v}}_1+\cdots+x_n\vec{\mathbf{v}}_n = \vec{\mathbf{0}}$ so is $\begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix}$, since

$$a_1\vec{\mathbf{v}}_1+\cdots+a_n\vec{\mathbf{v}}_n = \vec{\mathbf{0}} \text{ and } b_1\vec{\mathbf{v}}_1+\cdots+b_n\vec{\mathbf{v}}_n = \vec{\mathbf{0}}$$

implies

$$(a_1+b_1)\vec{\mathbf{v}}_1+\cdots+(a_n+b_n)\vec{\mathbf{v}}_n = \vec{\mathbf{0}}.$$

Similarly, if $c \in \mathbb{R}$, $\begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$ is a solution. Thus the solution set of a homogeneous system is...

a) A basis for $\mathbb{R}^n$. b) A subspace of $\mathbb{R}^n$. c) The empty set.

Activity V.81 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 + 2x_2 \quad\quad + \; x_4 &= 0 \\ 2x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ 3x_1 + 6x_2 - x_3 - \; x_4 &= 0 \end{aligned}$$

Part 1: Find its solution set (a subspace of $\mathbb{R}^4$). *Part 2:* Rewrite this solution space in the form

$$\left\{ a \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} + b \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

Part 3: Rewrite this solution space in the form

$$\text{span} \left\{ \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix} \right\}.$$

Activity V.83 *(∼10 min)* Consider the homogeneous system of equations

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 - 4x_4 &= 0 \\ -2x_1 - 4x_2 + \; x_3 + \; x_4 &= 0 \\ 3x_1 + 6x_2 - \; x_3 - 4x_4 &= 0 \end{aligned}$$

Find a basis for its solution space.

Activity V.84 *(∼10 min)* Consider the homogeneous vector equation

$$x_1 \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find a basis for its solution space.

Activity V.85 *(∼5 min)* Consider the homogeneous system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ 2x_1 + 6x_2 + 4x_3 &= 0 \\ x_1 + 6x_2 - 4x_3 &= 0 \end{aligned}$$

Find a basis for its solution space.

Module A: Algebraic Properties of Linear Maps

Activity A.6 *($\sim$5 min)* Recall the following rules from calculus, where $D : \mathcal{P} \to \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f.

$$D(f+g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

a) $\mathcal{P}$ is not a vector space

b) D is a linear map

c) D is not a linear map

Activity A.7 *($\sim$10 min)* Let the polynomial maps $S : \mathcal{P}^4 \to \mathcal{P}^3$ and $T : \mathcal{P}^4 \to \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4+x)$, $S(x^4)+S(x)$, $T(x^4+x)$, and $T(x^4)+T(x)$. Which of these maps is definitely not linear?

Activity A.10 *($\sim$15 min)* Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g. *Part 2:* Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f. *Part 3:* Is S linear?

Activity A.11 *($\sim$20 min)* Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Note that $S(0) = 0$ and $T(0) = 0$. So instead, show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear. *Part 2:* Prove that T is linear by verifying that $T(f(x) + g(x)) = T(f(x)) + T(g(x))$ and $T(cf(x)) = cT(f(x))$.

Activity A.13 *(~5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity A.14 *(~5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.15 *(~5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}-2\\0\\-3\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.16 *(∼5 min)* Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$.

What piece of information would help you compute $T\left(\begin{bmatrix}0\\4\\-1\end{bmatrix}\right)$?

(a) The value of $T\left(\begin{bmatrix}0\\-4\\0\end{bmatrix}\right)$.

(b) The value of $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$.

(c) The value of $T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$.

(d) Any of the above.

Activity A.19 *(∼3 min)* Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T(\vec{\mathbf{e}}_1) = \begin{bmatrix}0\\3\\-2\end{bmatrix} \qquad T(\vec{\mathbf{e}}_2) = \begin{bmatrix}-3\\0\\1\end{bmatrix} \qquad T(\vec{\mathbf{e}}_3) = \begin{bmatrix}4\\-2\\1\end{bmatrix} \qquad T(\vec{\mathbf{e}}_4) = \begin{bmatrix}2\\0\\0\end{bmatrix}$$

Write the standard matrix $[T(\vec{\mathbf{e}}_1) \cdots T(\vec{\mathbf{e}}_n)]$ for T.

Activity A.20 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x+3z\\2x-y-4z\end{bmatrix}$$

Part 1: Compute $T(\vec{\mathbf{e}}_1)$, $T(\vec{\mathbf{e}}_2)$, and $T(\vec{\mathbf{e}}_3)$. *Part 2:* Find the standard matrix for T.

Activity A.22 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix}3 & -2 & -1\\4 & 5 & 2\\0 & -2 & 1\end{bmatrix}.$$

Part 1: Compute $T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$. *Part 2:* Compute $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)$.

Activity A.23 *(~15 min)* Compute the following linear transformations of vectors given their standard matrices.

$$T_1\left(\begin{bmatrix}1\\2\end{bmatrix}\right) \text{ for the standard matrix } A_1 = \begin{bmatrix}4 & 3\\0 & -1\\1 & 1\\3 & 0\end{bmatrix}$$

$$T_2\left(\begin{bmatrix}1\\1\\0\\-3\end{bmatrix}\right) \text{ for the standard matrix } A_2 = \begin{bmatrix}4 & 3 & 0 & -1\\1 & 1 & 3 & 0\end{bmatrix}$$

$$T_3\left(\begin{bmatrix}0\\-2\\0\end{bmatrix}\right) \text{ for the standard matrix } A_3 = \begin{bmatrix}4 & 3 & 0\\0 & -1 & 3\\5 & 1 & 1\\3 & 0 & 0\end{bmatrix}$$

Activity A.25 *(~5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x\\y\\0\end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\ker T$, the set of all vectors that transform into $\vec{\mathbf{0}}$?

a) $\left\{\begin{bmatrix}a\\a\end{bmatrix} \,\middle|\, a \in \mathbb{R}\right\}$

b) $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

c) $\mathbb{R}^2 = \left\{\begin{bmatrix}x\\y\end{bmatrix} \,\middle|\, x, y \in \mathbb{R}\right\}$

Activity A.26 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\ker T$, the set of all vectors that transform into $\vec{\mathbf{0}}$?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$

Activity A.27 *(∼10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x + 4y - z \\ x + 2y + z \end{bmatrix}\right)$$

Part 1: Set $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find a linear system of equations whose solution set is the kernel. *Part 2:* Use $\text{RREF}(A)$ to solve this homogeneous system of equations and find a basis for the kernel of T.

Activity A.28 *(∼10 min)* Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 2x + 4y + 2z - 4w \\ -2x - 4y + z + w \\ 3x + 6y - z - 4w \end{bmatrix}.$$

Find a basis for the kernel of T.

Activity A.30 *(∼5 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^3$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^2$ vectors?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$

Activity A.31 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of $\mathbb{R}^2$ describes $\operatorname{Im} T$, the set of all vectors that are the result of using T to transform $\mathbb{R}^3$ vectors?

a) $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

b) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c) $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}$

Activity A.32 *(~5 min)* Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} T(\vec{\mathbf{e}}_1) & T(\vec{\mathbf{e}}_2) & T(\vec{\mathbf{e}}_3) & T(\vec{\mathbf{e}}_4) \end{bmatrix}.$$

Since $T(\vec{\mathbf{v}}) = T(x_1\vec{\mathbf{e}}_1 + x_2\vec{\mathbf{e}}_2 + x_3\vec{\mathbf{e}}_3 + x_4\vec{\mathbf{e}}_4)$, the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

a) spans $\operatorname{Im} T$

b) is a linearly independent subset of $\operatorname{Im} T$

c) is a basis for $\operatorname{Im} T$

Activity A.35 *(~10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of T.

Activity A.36 *(~5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the kernel of T?

(a) The number of pivot columns

(b) The number of non-pivot columns

(c) The number of pivot rows

(d) The number of non-pivot rows

Activity A.37 *(~5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Which of the following is equal to the dimension of the image of T?

(a) The number of pivot columns

(b) The number of non-pivot columns

(c) The number of pivot rows

(d) The number of non-pivot rows

Activity A.39 *(~10 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Verify that the rank-nullity theorem holds for T.

Activity A.41 *(~3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T injective?

a) Yes, because $T(\vec{\mathbf{v}}) = T(\vec{\mathbf{w}})$ whenever $\vec{\mathbf{v}} = \vec{\mathbf{w}}$.

b) Yes, because $T(\vec{\mathbf{v}}) \neq T(\vec{\mathbf{w}})$ whenever $\vec{\mathbf{v}} \neq \vec{\mathbf{w}}$.

c) No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$

d) No, because $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right)$

Activity A.42 *(~2 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T injective?

a) Yes, because $T(\vec{\mathbf{v}}) = T(\vec{\mathbf{w}})$ whenever $\vec{\mathbf{v}} = \vec{\mathbf{w}}$.

b) Yes, because $T(\vec{\mathbf{v}}) \neq T(\vec{\mathbf{w}})$ whenever $\vec{\mathbf{v}} \neq \vec{\mathbf{w}}$.

c) No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$

d) No, because $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$

Activity A.44 *(∼3 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Is T surjective?

a) Yes, because for every $\vec{\mathbf{w}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, there exists $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T(\vec{\mathbf{v}}) = \vec{\mathbf{w}}$.

b) No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

c) No, because $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Activity A.45 *(∼2 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is T surjective?

a) Yes, because for every $\vec{\mathbf{w}} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ 42 \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{\mathbf{v}}) = \vec{\mathbf{w}}$.

b) Yes, because for every $\vec{\mathbf{w}} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, there exists $\vec{\mathbf{v}} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3$ such that $T(\vec{\mathbf{v}}) = \vec{\mathbf{w}}$.

c) No, because $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ can never equal $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Activity A.48 *(∼5 min)* Let $T : V \to W$ be a linear transformation where $\ker T$ contains multiple vectors. What can you conclude?

(a) T is injective

(b) T is not injective

(c) T is surjective

(d) T is not surjective

Activity A.50 *(∼5 min)* Let $T : V \to \mathbb{R}^5$ be a linear transformation where $\operatorname{Im} T$ is spanned by four vectors. What can you conclude?

(a) T is injective

(b) T is not injective

(c) T is surjective

(d) T is not surjective

Activity A.52 *(∼15 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of *equivalent* statements: one group that means T is **injective**, and one group that means T is **surjective**.

(a) The kernel of T is trivial, i.e. $\ker T = \{\vec{\mathbf{0}}\}$.

(b) The columns of A span $\mathbb{R}^m$.

(c) The columns of A are linearly independent.

(d) Every column of $\mathrm{RREF}(A)$ has a pivot.

(e) Every row of $\mathrm{RREF}(A)$ has a pivot.

(f) The image of T equals its codomain, i.e. $\operatorname{Im} T = \mathbb{R}^m$.

(g) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{\mathbf{b}} \end{array}\right]$ has a solution for all $\vec{\mathbf{b}} \in \mathbb{R}^m$.

(h) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{\mathbf{0}} \end{array}\right]$ has exactly one solution.

Activity A.54 *(∼3 min)* What can you conclude about the linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

d) Its standard matrix has more rows than columns, so T is surjective.

Activity A.55 *(∼2 min)* What can you conclude about the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$?

a) Its standard matrix has more columns than rows, so T is not injective.

b) Its standard matrix has more columns than rows, so T is injective.

c) Its standard matrix has more rows than columns, so T is not surjective.

d) Its standard matrix has more rows than columns, so T is surjective.

Activity A.57 *(∼5 min)* Suppose $T : \mathbb{R}^n \to \mathbb{R}^4$ with standard matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ a_{41} & a_{42} & \cdots & a_{4n} \end{bmatrix}$ is both injective and surjective (we call such maps **bijective**). *Part 1:* How many pivot rows must RREF A have? *Part 2:* How many pivot columns must RREF A have? *Part 3:* What is RREF A?

Activity A.58 *(∼5 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

(a) RREF(A) is the identity matrix.

(b) The columns of A form a basis for $\mathbb{R}^n$

(c) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \vec{\mathbf{b}} \end{array}\right]$ has exactly one solution for each $\vec{\mathbf{b}} \in \mathbb{R}^n$.

Activity A.60 *(~3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.61 *(~3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.62 *(~3 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.63 *(∼3 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Module M: Understanding Matrices Algebraically

Activity M.2 *(~5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What are the domain and codomain of the composition map $S \circ T$?

(a) The domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$

(b) The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$

(c) The domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^4$

(d) The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^4$

(e) The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^3$

(f) The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^2$

Activity M.3 *(~2 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What size will the standard matrix of $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$ be? (Rows $\times$ Columns)

(a) 4×3

(b) 4×2

(c) 3×4

(d) 3×2

(e) 2×4

(f) 2×3

Activity M.4 *(~15 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{\mathbf{e}}_1) = S(T(\vec{\mathbf{e}}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{\mathbf{e}}_2)$. *Part 3:* Compute $(S \circ T)(\vec{\mathbf{e}}_3)$. *Part 4:* Write the 4×3 standard matrix of $S \circ T : \mathbb{R}^3 \to \mathbb{R}^4$.

Activity M.6 *(~15 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Part 1: Write the dimensions (rows $\times$ columns) for A, B, AB, and BA. *Part 2:* Find the standard matrix AB of $S \circ T$. *Part 3:* FInd the standard matrix BA of $T \circ S$.

Activity M.7 *(~10 min)* Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Part 1: Find the domain and codomain of each of the three linear maps corresponding to A, B), and C. *Part 2:* Only one of the matrix products AB, AC, BA, BC, CA, CB can actually be computed. Compute it.

Activity M.9 *(~15 min)* Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. *Part 1:* Compute the product BA by hand. *Part 2:* Check your work using technology. Using Octave:

- `B = sym([3 -4 0 ; 2 0 -1 ; 0 -3 3])`
- `A = sym([2 7 -1 ; 0 3 2 ; 1 1 -1])`
- `B*A`

Activity M.10 *(∼5 min)* Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix B such that $BA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Check your guess using technology.

Activity M.13 *(∼20 min)* Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication. *Part 1:* Create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5(1) & 7+5(1) & -1+5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Activity M.15 *(∼10 min)* Consider the two row operations $R_2 \leftrightarrow R_3$ and $R_1 + R_2 \to R_1$ applied as follows to show $A \sim B$:

$$\begin{aligned} A = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} &\sim \begin{bmatrix} -1 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} \\ &\sim \begin{bmatrix} -1+1 & 4+2 & 5+3 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = B \end{aligned}$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A:

$$B = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} A$$

Check your work using technology.

Activity M.16 *(∼15 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into three groups of statements: a group that means T is **injective**, a group that means T is **surjective**, and a group that means T is **bijective**.

(a) $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution for all $\vec{\mathbf{b}} \in \mathbb{R}^m$

(b) $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for all $\vec{\mathbf{b}} \in \mathbb{R}^m$

(c) $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a unique solution.

(d) The columns of A span $\mathbb{R}^m$

(e) The columns of A are linearly independent

(f) The columns of A are a basis of $\mathbb{R}^m$

(g) Every column of $\text{RREF}(A)$ has a pivot

(h) Every row of $\text{RREF}(A)$ has a pivot

(i) $m = n$ and $\text{RREF}(A) = I$

Activity M.17 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 4 \\ 1 & 1 & 3 \end{bmatrix}$.

Write an augmented matrix representing the system of equations given by $T(\vec{\mathbf{x}}) = \vec{\mathbf{0}}$, that is, $A\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{0}}$ to find the kernel of T.

Activity M.19 *(∼20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$. *Part 1:* Write an augmented matrix representing the system of equations given by $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_1$, that is, $A\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_1$ to find $T^{-1}(\vec{\mathbf{e}}_1)$. *Part 2:* Solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_2$ to find $T^{-1}(\vec{\mathbf{e}}_2)$. *Part 3:* Solve $T(\vec{\mathbf{x}}) = \vec{\mathbf{e}}_3$ to find $T^{-1}(\vec{\mathbf{e}}_3)$. *Part 4:* Write A^{-1}, the standard matrix for T^{-1}.

Activity M.21 *(∼5 min)* Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A\,|\,I]$.

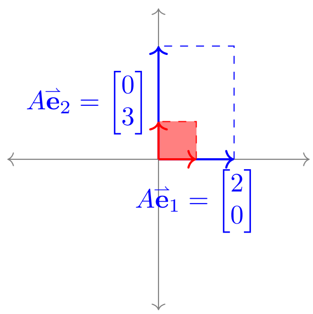
Activity M.22 *(∼5 min)* Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

Activity M.24 *(~10 min)* Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$. *Part 1:* Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$. *Part 2:* If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, find the 2×2 matrix

$$A^{-1}A = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}.$$

Module G: Geometry of Linear Maps

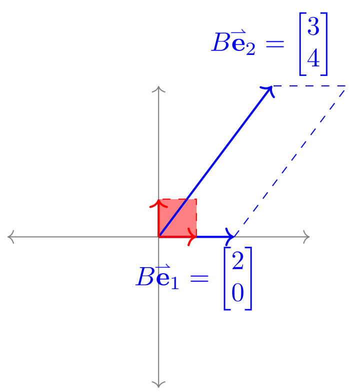
Activity G.1 *(~5 min)* The image below illustrates how the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ transforms the unit square.



(a) What are the lengths of $A\vec{\mathbf{e}}_1$ and $A\vec{\mathbf{e}}_2$?

(b) What is the area of the transformed unit square?

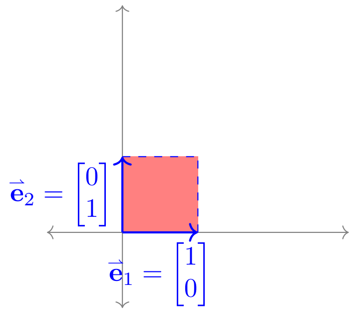
Activity G.2 *(~5 min)* The image below illustrates how the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by the standard matrix $B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$. transforms the unit square.



(a) What are the lengths of $B\vec{\mathbf{e}}_1$ and $B\vec{\mathbf{e}}_2$?

(b) What is the area of the transformed unit square?

Activity G.6 *(∼2 min)* The transformation of the unit square by the standard matrix $[\vec{\mathbf{e}}_1\ \vec{\mathbf{e}}_2] = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I$ is illustrated below. What is $\det([\vec{\mathbf{e}}_1\ \vec{\mathbf{e}}_2]) = \det(I)$, the area of the transformed unit square shown here?



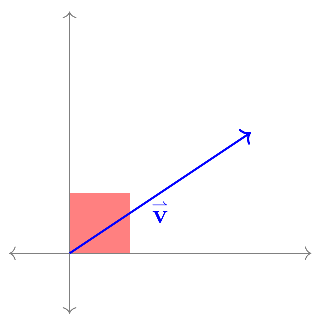
a) 0

b) 1

c) 2

d) 4

Activity G.7 *(∼2 min)* The transformation of the unit square by the standard matrix $[\vec{\mathbf{v}}\ \vec{\mathbf{v}}]$ is illustrated below: both $T(\vec{\mathbf{e}}_1) = T(\vec{\mathbf{e}}_2) = \vec{\mathbf{v}}$. What is $\det([\vec{\mathbf{v}}\ \vec{\mathbf{v}}])$, the area of the transformed unit square shown here?



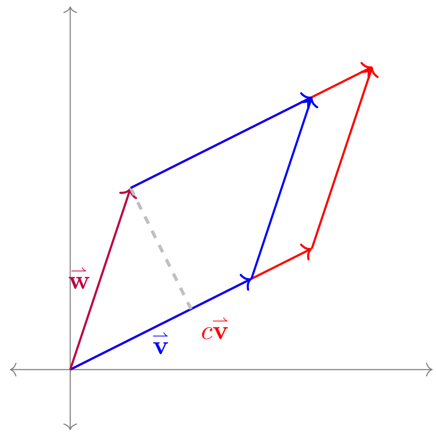
a) 0

b) 1

c) 2

d) 4

Activity G.8 *(∼5 min)* The transformations of the unit square by the standard matrices $[\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]$ and $[c\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]$ are illustrated below. Describe the value of $\det([c\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$.



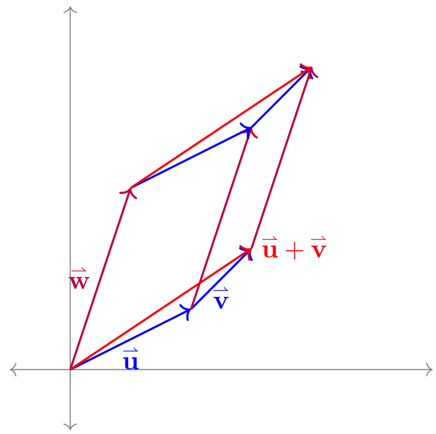
a) $\det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$

b) $\det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]) + c\det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]$

c) $c\det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$

d) Cannot be determined from this information.

Activity G.9 *(~5 min)* The transformations of unit squares by the standard matrices $[\vec{\mathbf{u}}\ \ \vec{\mathbf{w}}]$, $[\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]$ and $[\vec{\mathbf{u}}+\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]$ are illustrated below. Describe the value of $\det([\vec{\mathbf{u}}+\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$.



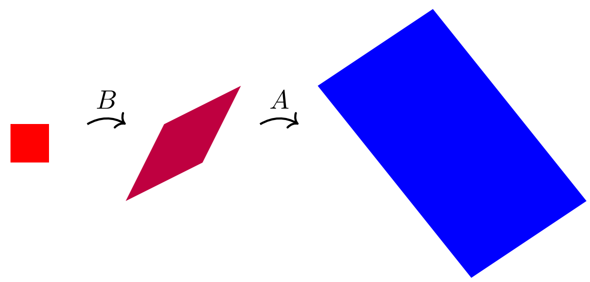
a) $\det([\vec{\mathbf{u}}\ \ \vec{\mathbf{w}}]) = \det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$

b) $\det([\vec{\mathbf{u}}\ \ \vec{\mathbf{w}}]) + \det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$

c) $\det([\vec{\mathbf{u}}\ \ \vec{\mathbf{w}}])\det([\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}])$

d) Cannot be determined from this information.

Activity G.15 *(~5 min)* The transformation given by the standard matrix A scales areas by 4, and the transformation given by the standard matrix B scales areas by 3. By what factor does the transformation given by the standard matrix AB scale areas?



(a) 1

(b) 7

(c) 12

(d) Cannot be determined

Activity G.19 *(~5 min)* Consider the row operation $R_1 + 4R_3 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1+4(9) & 2+4(10) & 3+4(11) & 4+4(12) \\ 5 & 6 & 6 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) Find $\det R$ by comparing with the previous slide.

(c) If $C \in M_{3,3}$ is a matrix with $\det(C) = -3$, find

$$\det(RC) = \det(R)\det(C).$$

Activity G.20 *(~5 min)* Consider the row operation $R_1 \leftrightarrow R_3$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$, by applying the same row operation to I.

(b) If $C \in M_{3,3}$ is a matrix with $\det(C) = 5$, find $\det(RC)$.

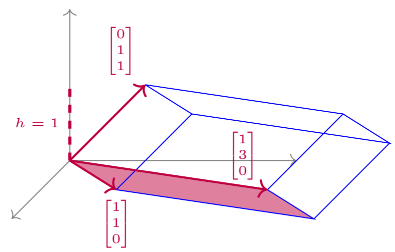
Activity G.21 *(~5 min)* Consider the row operation $3R_2 \to R_2$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3(5) & 3(6) & 3(7) & 3(8) \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = B$$

(a) Find a matrix R such that $B = RA$.

(b) If $C \in M_{3,3}$ is a matrix with $\det(C) = -7$, find $\det(RC)$.

Activity G.27 *(~5 min)* The following image illustrates the transformation of the unit cube by the matrix $\begin{bmatrix}1 & 1 & 0\\ 1 & 3 & 1\\ 0 & 0 & 1\end{bmatrix}$.



Recall that for this solid $V = Bh$, where h is the height of the solid and B is the area of its parallelogram base. So what must its volume be?

(a) $\det\begin{bmatrix}1 & 1\\ 1 & 3\end{bmatrix}$

(b) $\det\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}$

(c) $\det\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

(d) $\det\begin{bmatrix}1 & 3\\ 0 & 0\end{bmatrix}$

Activity G.29 *(~5 min)* Remove an appropriate row and column of $\det\begin{bmatrix}1 & 0 & 0\\ 1 & 5 & 12\\ 3 & 2 & -1\end{bmatrix}$ to simplify the determinant to a 2×2 determinant.

Activity G.30 *(~5 min)* Simplify $\det\begin{bmatrix}0 & 3 & -2\\ 2 & 5 & 12\\ 0 & 2 & -1\end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

- Factor out a 2 from a column.
- Swap rows or columns to put a 1 on the main diagonal.

Activity G.31 *(~5 min)* Simplify $\det\begin{bmatrix}4 & -2 & 2\\ 3 & 1 & 4\\ 1 & -1 & 3\end{bmatrix}$ to a multiple of a 2×2 determinant by first doing the following:

- Use row/column operations to create two zeroes in the same row or column.
- Factor/swap as needed to get a row/column of all zeroes except a 1 on the main diagonal.

Activity G.33 *(∼10 min)* Rewrite

$$\det \begin{bmatrix} 2 & 1 & -2 & 1 \\ 3 & 0 & 1 & 4 \\ -2 & 2 & 3 & 0 \\ -2 & 0 & -3 & -3 \end{bmatrix}$$

as a multiple of a determinant of a 3×3 matrix.

Activity G.34 *(∼20 min)* Compute $\det \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ -1 & -1 & 2 & 2 \end{bmatrix}$ by using any combination of row/column operations.

Activity G.37 *(∼5 min)* Based on what we've done today, which technique is easier for computing determinants?

(a) Memorizing formulas.

(b) Using row/column operations.

(c) Laplace expansion.

(d) Some other technique (be prepared to describe it).

Activity G.38 *(∼10 min)* Use your preferred technique to compute $\det \begin{bmatrix} 4 & -3 & 0 & 0 \\ 1 & -3 & 2 & -1 \\ 3 & 2 & 0 & 3 \\ 0 & -3 & 2 & -2 \end{bmatrix}$.

Activity G.39 *(∼5 min)* An invertible matrix M and its inverse M^{-1} are given below:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Which of the following is equal to $\det(M)\det(M^{-1})$?

a) -1

b) 0

c) 1

d) 4

Activity G.43 *(∼5 min)* Finding the eigenvalues λ that satisfy

$$A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}} = \lambda(I\vec{\mathbf{x}}) = (\lambda I)\vec{\mathbf{x}}$$

for some nontrivial eigenvector $\vec{\mathbf{x}}$ is equivalent to finding nonzero solutions for the matrix equation

$$(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}.$$

Which of the following must be true for any eigenvalue?

(a) The **kernel** of the transformation with standard matrix $A-\lambda I$ must contain **the zero vector**, so $A-\lambda I$ is **invertible**.

(b) The **kernel** of the transformation with standard matrix $A-\lambda I$ must contain **a non-zero vector**, so $A-\lambda I$ is **not invertible**.

(c) The **image** of the transformation with standard matrix $A-\lambda I$ must contain **the zero vector**, so $A-\lambda I$ is **invertible**.

(d) The **image** of the transformation with standard matrix $A-\lambda I$ must contain **a non-zero vector**, so $A-\lambda I$ is **not invertible**.

Activity G.46 *(∼10 min)* Let $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$. *Part 1:* Compute $\det(A - \lambda I)$ to determine the characteristic polynomial of A. *Part 2:* Set this characteristic polynomial equal to zero and factor to determine the eigenvalues of A.

Activity G.47 *(∼5 min)* Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 \\ 2 & -4 \end{bmatrix}$.

Activity G.48 *(∼5 min)* Find all the eigenvalues for the matrix $A = \begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}$.

Activity G.49 *(∼10 min)* Find all the eigenvalues for the matrix $A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

Activity G.50 *(∼10 min)* It's possible to show that -2 is an eigenvalue for $\begin{bmatrix} -1 & 4 & -2 \\ 2 & -7 & 9 \\ 3 & 0 & 4 \end{bmatrix}$.

Compute the kernel of the transformation with standard matrix

$$A - (-2)I = \begin{bmatrix} ? & 4 & -2 \\ 2 & ? & 9 \\ 3 & 0 & ? \end{bmatrix}$$

to find all the eigenvectors $\vec{\mathbf{x}}$ such that $A\vec{\mathbf{x}} = -2\vec{\mathbf{x}}$.

Activity G.52 *(∼10 min)* Find a basis for the eigenspace for the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ associated with the eigenvalue 3.

Activity G.53 *(∼10 min)* Find a basis for the eigenspace for the matrix $\begin{bmatrix} 5 & -2 & 0 & 4 \\ 6 & -2 & 1 & 5 \\ -2 & 1 & 2 & -3 \\ 4 & 5 & -3 & 6 \end{bmatrix}$ associated with the eigenvalue 1.

Activity G.54 *(∼10 min)* Find a basis for the eigenspace for the matrix $\begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ associated with the eigenvalue 2.