

## Section V.0

**Observation V.1** Several properties of the real numbers, such as commutivity:

$$x + y = y + x$$

also hold for Euclidean vectors with multiple components:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

**Activity V.2** ( $\sim 20$  min) Consider each of the following properties of the real numbers  $\mathbb{R}^1$ . Label each property as **valid** if the property also holds for two-dimensional Euclidean vectors  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$  and scalars  $a, b \in \mathbb{R}$ , and **invalid** if it does not.

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|---|---|
| 1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ .  | 6. $a(b\vec{v}) = (ab)\vec{v}$ .  |
| 2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ .  | 7. $1\vec{v} = \vec{v}$ .   |
| 3. There exists some $\vec{z}$ where $\vec{v} + \vec{z} = \vec{v}$ .  | 8. If $\vec{u} \neq \vec{0}$ , then there exists some scalar $c$ such that $c\vec{u} = \vec{v}$ . |
| 4. There exists some $-\vec{v}$ where $\vec{v} + (-\vec{v}) = \vec{z}$ .  | 9. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ .   |
| 5. If $\vec{u} \neq \vec{v}$ , then $\frac{1}{2}(\vec{u} + \vec{v})$ is the only vector equally distant from both $\vec{u}$ and $\vec{v}$ . | 10. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$ .  |

**Definition V.3** A **vector space**  $V$  is any collection of mathematical objects with associated addition  $\oplus$  and scalar multiplication  $\odot$  operations that satisfy the following properties. Let  $\vec{u}, \vec{v}, \vec{w}$  belong to  $V$ , and let  $a, b$  be scalar numbers.

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| • <b>Addition is associative:</b> $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$ . | • <b>Scalar multiplication is associative:</b> $a \odot (b \odot \vec{v}) = (ab) \odot \vec{v}$ .                                     |
| • <b>Addition is commutative:</b> $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$ .                                   | • <b>Scalar multiplication identity exists:</b> $1 \odot \vec{v} = \vec{v}$ .   |
| • <b>Additive identity exists:</b> There exists some $\vec{z}$ where $\vec{v} \oplus \vec{z} = \vec{v}$ .               | • <b>Scalar mult. distributes over vector addition:</b> $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$ . |
| • <b>Additive inverses exist:</b> There exists some $-\vec{v}$ where $\vec{v} \oplus (-\vec{v}) = \vec{z}$ .            | • <b>Scalar mult. distributes over scalar addition:</b> $(a + b) \odot \vec{v} = a\vec{v} \oplus b\vec{v}$ .                          |

**Observation V.4**

Every **Euclidean vector space**

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mid x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$$

satisfies all eight requirements for the usual definitions of addition and scalar multiplication, but we will also study other types of vector spaces.

**Observation V.5** The space of  $m \times n$  **matrices**

$$M_{m,n} = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \mid a_{11}, \dots, a_{mn} \in \mathbb{R} \right\}$$

satisfies all eight requirements for component-wise addition and scalar multiplication.