Module A: Algebraic properties of linear maps

How can we understand linear maps algebraically?

At the end of this module, students will be able to...

A1. Linear map verification. ... determine if a map between vector spaces of polynomials is linear or not.

A2. Linear maps and matrices. ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.

A3. Injectivity and surjectivity. ... determine if a given linear map is injective and/or surjective.

A4. Kernel and Image. ... compute a basis for the kernel and a basis for the image of a linear map.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans $\mathbb{R}^n$ **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

Module A Section 1

Definition A.1.1

A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

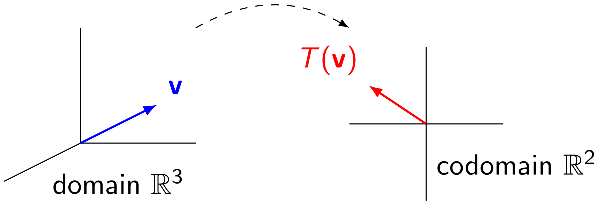
1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Definition A.1.2

Given a linear transformation $T : V \to W$, V is called the **domain** of T and W is called the **co-domain** of T.

Linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$



Example A.1.3

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}\right) = \begin{bmatrix} (x+u)-(z+w) \\ 3(y+v) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x+u)-(z+w) \\ 3(y+v) \end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \text{ and } cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore T is a linear transformation.

Example A.1.4

Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x^2 \\ y+3 \\ y-2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is a linear transformation.

Fact A.1.5

A map between Euclidean spaces $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the variables of $\mathbb{R}^n$.

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2, $y + 3$, and $y - 2^x$ are not linear combinations (even though $x + y$ is):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

Activity A.1.6 *(∼5 min)*

Recall the following rules from calculus, where $D : \mathcal{P} \to \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f.

$$D(f+g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

a) $\mathcal{P}$ is not a vector space

b) D is a linear map

c) D is not a linear map

Activity A.1.7 *(∼10 min)*
Let the polynomial maps $S : \mathcal{P}^4 \to \mathcal{P}^3$ and $T : \mathcal{P}^4 \to \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4 + x)$, $S(x^4) + S(x)$, $T(x^4 + x)$, and $T(x^4) + T(x)$. Which of these maps is definitely not linear?

Fact A.1.8

If $L : V \to W$ is linear, then $L(\mathbf{z}) = L(0\mathbf{v}) = 0L(\mathbf{v}) = \mathbf{z}$ where $\mathbf{z}$ is the additive identity of the vector spaces V, W.

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Activity A.1.9 *(∼15 min)*

Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Activity A.1.9 *(∼15 min)*

Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.

Activity A.1.9 *(∼15 min)*
Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

Part 1: Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.
Part 2: Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f. Is S linear?

Activity A.1.10 *(∼20 min)*

Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Activity A.1.10 *(∼20 min)*
Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.

Activity A.1.10 *(∼20 min)*
Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

Part 1: Show that $S(x+1) \neq S(x) + S(1)$ to verify that S is not linear.
Part 2: Prove that T is linear by verifying that
$T(f(x) + g(x)) = T(f(x)) + T(g(x))$ and $T(cf(x)) = cT(f(x))$.

Observation A.1.11

Note that S in the previous activity is not linear, even though $S(0) = (0)^2 = 0$. So showing $S(0) = 0$ isn't enough to prove a map is linear.

This is a similar situation to proving a subset is a subspace: if the subset doesn't contain $\mathbf{z}$, then the subset isn't a subspace. But if the subset contains $\mathbf{z}$, you cannot conclude anything.

Module A Section 2

Remark A.2.1

Recall that a linear map $T : V \to W$ satisfies

1. $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for any $\mathbf{v}, \mathbf{w} \in V$.
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $c \in \mathbb{R}, \mathbf{v} \in V$.

In other words, a map is linear when vecor space operations can be applied before or after the transformation without affecting the result.

Activity A.2.2 *(∼5 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}3\\0\\0\end{bmatrix}\right)$.

(a) $\begin{bmatrix}6\\3\end{bmatrix}$

(b) $\begin{bmatrix}-9\\6\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-2\end{bmatrix}$

(d) $\begin{bmatrix}6\\-4\end{bmatrix}$

Activity A.2.3 *(∼3 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Compute $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(a) $\begin{bmatrix}2\\1\end{bmatrix}$

(b) $\begin{bmatrix}3\\-1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\3\end{bmatrix}$

(d) $\begin{bmatrix}5\\-8\end{bmatrix}$

Activity A.2.4 *(∼2 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Compute $T\left(\begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix}\right)$.

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

Activity A.2.5 *(∼5 min)*

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map, and you know $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix}$. Do you have enough information to compute $T(\mathbf{v})$ for *any* $\mathbf{v} \in \mathbb{R}^3$?

(a) Yes.

(b) No, exactly one more piece of information is needed.

(c) No, an infinite amount of information would be necessary to compute the transformation of infinitely-many vectors.

Fact A.2.6

Consider any basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for V. Since every vector $\mathbf{v}$ can be written *uniquely* as a linear combination of basis vectors, $x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n$, we may compute $T(\mathbf{v})$ as follows:

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = x_1 T(\mathbf{b}_1) + \cdots + x_n T(\mathbf{b}_n).$$

Therefore any linear transformation $T : V \to W$ can be defined by just describing the values of $T(\mathbf{b}_i)$.

Put another way, the images of the basis vectors **determine** the transformation T.

Definition A.2.7

Since linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, it's convenient to store this information in the $m \times n$ **standard matrix** $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$.

For example, let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map determined by the following values for T applied to the standard basis of $\mathbb{R}^3$.

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix} \qquad T(\mathbf{e}_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\4\end{bmatrix} \qquad T(\mathbf{e}_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

Then the standard matrix corresponding to T is

$$\begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3)\end{bmatrix} = \begin{bmatrix}3 & -1 & 5\\2 & 4 & 0\end{bmatrix}.$$

Activity A.2.8 *(∼3 min)*

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_3) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ for T.

Activity A.2.9 *(∼5 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Find the standard matrix for T.

Fact A.2.10

Because every linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ has a linear combination of the variables in each component, and thus $T(\mathbf{e}_i)$ yields exactly the coefficients of x_i, the standard matrix for T is simply an ordered list of the coefficients of the x_i:

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \qquad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

Activity A.2.11 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Activity A.2.12 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

Fact A.2.13

To quickly compute $T(\mathbf{v})$ from its standard matrix A, compute the **dot product** (defined in Calculus 3) of each matrix row with the vector. For example, if T has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$$

then for $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 2y + 3z \\ 0x + 1y - 2z \\ 2x - 1y + 0z \end{bmatrix}$$

and for $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(0) + 3(-2) \\ 0(3) + 1(0) - 2(-2) \\ 2(3) - 1(0) + 0(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}.$$

Activity A.2.14 *(∼15 min)*

Compute the following linear transformations of vectors given their standard matrices.

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) \text{ for the standard matrix } A=\begin{bmatrix}4&3\\0&-1\\1&1\\3&0\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\\0\\-3\end{bmatrix}\right) \text{ for the standard matrix } A=\begin{bmatrix}4&3&0&-1\\1&1&3&0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\-2\\0\end{bmatrix}\right) \text{ for the standard matrix } A=\begin{bmatrix}4&3&0\\0&-1&3\\5&1&1\\3&0&0\end{bmatrix}$$

Module A Section 3

Definition A.3.1

Let $T : V \to W$ be a linear transformation. T is called **injective** or **one-to-one** if T does not map two distinct values to the same place. More precisely, T is injective if $T(\mathbf{v}) \neq T(\mathbf{w})$ whenever $\mathbf{v} \neq \mathbf{w}$.

Activity A.3.2 *(∼5 min)*
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is T injective?

Activity A.3.3 *(∼5 min)*
Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x\\y\end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$.

Is T injective?

Definition A.3.4

Let $T : V \to W$ be a linear transformation. T is called **surjective** or **onto** if every element of W is mapped to by an element of V. More precisely, for every $\mathbf{w} \in W$, there is some $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$.

Activity A.3.5 *(∼5 min)*
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Is T surjective?

Activity A.3.6 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The standard matrix of T is thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Is T surjective?

Definition A.3.7

Let $T : V \to W$ be a linear transformation. The **kernel** of T is an important subspace of V defined by

$$\ker T = \left\{ \mathbf{v} \in V \;\middle|\; T(\mathbf{v}) = \mathbf{0} \right\}$$

Activity A.3.8 *(∼5 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the kernel of T.

Activity A.3.9 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the kernel of T.

Activity A.3.10 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Activity A.3.10 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Write a system of equations whose solution set is the kernel.

Activity A.3.10 *(∼10 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Write a system of equations whose solution set is the kernel.

Part 2: Use RREF(A) to solve the system of equations and find the kernel of T.

Activity A.3.10 *(∼10 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix
$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.
Part 1: Write a system of equations whose solution set is the kernel.
Part 2: Use RREF(A) to solve the system of equations and find the kernel of T.
Part 3: Find a basis for the kernel of T.

Definition A.3.11

Let $T : V \to W$ be a linear transformation. The **image** of T is an important subspace of W defined by

$$\operatorname{Im} T = \left\{ \mathbf{w} \in W \;\middle|\; \text{there is some } v \in V \text{ with } T(\mathbf{v}) = \mathbf{w} \right\}$$

Activity A.3.12 *(∼5 min)*

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the image of T.

Activity A.3.13 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find the image of T.

Activity A.3.14 *(∼10 min)*

Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be the linear transformation given by the standard matrix $A=\begin{bmatrix}3 & 4 & -1\\ 1 & 2 & 1\end{bmatrix}$.

Activity A.3.14 *(∼10 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Part 1: Find a convenient set of vectors $S \subseteq \mathbb{R}^2$ such that $\operatorname{span} S = \operatorname{Im} T$.

Activity A.3.14 *(∼10 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the standard matrix
$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$
Part 1: Find a convenient set of vectors $S \subseteq \mathbb{R}^2$ such that $\operatorname{span} S = \operatorname{Im} T$.
Part 2: Find a convenient basis for the image of T.

Observation A.3.15

Let $T : V \to W$ be a linear transformation with corresponding matrix A.

- If A is a matrix corresponding to T, the kernel is the solution set of the homogeneous system with coefficients given by A.
- If A is a matrix corresponding to T, the image is the span of the columns of A.

Module A Section 4

Observation A.4.1

Let $T : V \to W$. We have previously defined the following terms.

- T is called **injective** or **one-to-one** if T does not map two distinct values to the same place.
- T is called **surjective** or **onto** if every element of W is mapped to by some element of V.
- The **kernel** of T is the set of all things that are mapped to $\mathbf{0}$. It is a subspace of V.
- The **image** of T is the set of all things in W that are mapped to by something in V. It is a subspace of W.

Activity A.4.2 *(∼5 min)*

Let $T : V \to W$ be a linear transformation where $\ker T = \{\mathbf{0}\}$. Can you answer either of the following questions about T?

(a) Is T injective?

(b) Is T surjective?

(Hint: If $T(\mathbf{v}) = T(\mathbf{w})$, then what is $T(\mathbf{v} - \mathbf{w})$?)

Fact A.4.3

A linear transformation T is injective **if and only if** $\ker T = \{\mathbf{0}\}$. Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

Activity A.4.4 *(~5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation where $\operatorname{Im} T = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$.

Can you answer either of the following questions about T?

(a) Is T injective?

(b) Is T surjective?

Fact A.4.5

A linear transformation $T : V \to W$ is surjective **if and only if** $\operatorname{Im} T = W$. Put another way, a surjective linear transformation may be recognized by its same codomain and image.

Activity A.4.6 *(∼15 min)*

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following claims into two groups of equivalent statements.

(a) T is injective
(b) T is surjective
(c) The kernel of T is trivial.
(d) The columns of A span $\mathbb{R}^m$
(e) The columns of A are linearly independent
(f) Every column of RREF(A) has a pivot.
(g) Every row of RREF(A) has a pivot.
(h) The image of T equals its codomain.
(i) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$
(j) The system of linear equations given by the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$ has exactly one solution.

Definition A.4.7
If $T : V \to W$ is both injective and surjective, it is called **bijective**.

Activity A.4.8 *($\sim$5 min)*

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a bijective linear map with standard matrix A. Label each of the following as true or false.

(a) The columns of A form a basis for $\mathbb{R}^m$

(b) RREF(A) is the identity matrix.

(c) The system of linear equations given by the augmented matrix $\left[A \mid \mathbf{b}\right]$ has exactly one solution for all $\mathbf{b} \in \mathbb{R}^m$.

Activity A.4.9 *(∼10 min)*
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ x-y \\ x+3y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.10 *(∼5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.11 *(∼5 min)*
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y + z \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.

Activity A.4.12 *(~5 min)*

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

(a) T is neither injective nor surjective

(b) T is injective but not surjective

(c) T is surjective but not injective

(d) T is bijective.