

## Section E.2

**Activity E.2.1** (*~10 min*) Free browser-based technologies for mathematical computation are available online.

- Go to <https://octave-online.net>.
- Type `A=sym([1 3 4 ; 2 5 7])` and press **Enter** to store the matrix  $\begin{bmatrix} 1 & 3 & 2 \\ 2 & -9 & -1 \end{bmatrix}$  in the variable  $A$ .
  - The symbolic function `sym` is used to calculate precise answers rather than floating-point approximations.
  - The vertical bar in an augmented matrix does not affect row operations, so the RREF of  $\begin{bmatrix} 1 & 3 & 2 \\ 2 & -9 & -1 \end{bmatrix}$  may be computed in the same way.
- Type `rref(A)` and press **Enter** to compute the reduced row echelon form of  $A$ .

**Remark E.2.2** We will frequently need to know the reduced row echelon form of matrices during class, so feel free to use Octave-Online.net to compute RREF efficiently.

You may alternatively use the calculator you will use during assessments. Be sure to use fractions mode to compute exact solutions rather than floating-point approximations.

**Activity E.2.3** (*~10 min*) Consider the system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 + 3x_2 - 6x_3 &= 11 \end{aligned}$$

*Part 1:* Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF} \left[ \begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array} \right] = \left[ \begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array} \right]$$

*Part 2:* Use the RREF matrix to write a linear system equivalent to the original system. Then find its solution set.

**Activity E.2.4** (~10 min) Consider the system of equations.

$$\begin{aligned} 3x_1 - 2x_2 + 13x_3 &= 6 \\ 2x_1 - 2x_2 + 10x_3 &= 2 \\ -x_1 &\quad - 3x_3 = 1 \end{aligned}$$

*Part 1:* Convert this to an augmented matrix and use technology to compute its reduced row echelon form:

$$\text{RREF} \left[ \begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array} \right] = \left[ \begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array} \right]$$

*Part 2:* Use the RREF matrix to write a linear system equivalent to the original system. Then find its solution set.

**Activity E.2.5** (~10 min) Consider the following linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

*Part 1:* Find its corresponding augmented matrix  $A$  and use technology to find  $\text{RREF}(A)$ .

*Part 2:* How many solutions do these linear systems have?

**Activity E.2.6** (~10 min) Consider the simple linear system equivalent to the system from the previous activity:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_3 &= -1 \end{aligned}$$

*Part 1:* Let  $x_1 = a$  and write the solution set in the form  $\left\{ \begin{bmatrix} a \\ ? \\ ? \end{bmatrix} \mid a \in \mathbb{R} \right\}$ .

*Part 2:* Let  $x_2 = b$  and write the solution set in the form  $\left\{ \begin{bmatrix} ? \\ b \\ ? \end{bmatrix} \mid b \in \mathbb{R} \right\}$ .

*Part 3:* Which of these was easier? What features of the RREF matrix  $\left[ \begin{array}{ccc|c} \textcircled{1} & 2 & 0 & 4 \\ 0 & 0 & \textcircled{1} & -1 \end{array} \right]$  caused this?

**Definition E.2.7** Recall that the pivots of a matrix in RREF form are the leading 1s in each non-zero row.

The pivot columns in an augmented matrix correspond to the **bound variables** in the system of equations ( $x_1, x_3$  below). The remaining variables are called **free variables** ( $x_2$  below).

$$\left[ \begin{array}{ccc|c} \textcircled{1} & 2 & 0 & 4 \\ 0 & 0 & \textcircled{1} & -1 \end{array} \right]$$

To efficiently solve a system in RREF form, assign letters to the free variables, and then solve for the bound variables.

**Activity E.2.8** ( $\sim 10$  min) Find the solution set for the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - x_5 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 + x_5 &= 2 \end{aligned}$$

by row-reducing its augmented matrix, and then assigning letters to the free variables (given by non-pivot columns) and solving for the bound variables (given by pivot columns) in the corresponding linear system.

**Observation E.2.9** The solution set to the system

$$\begin{aligned} 2x_1 - 2x_2 - 6x_3 + x_4 - x_5 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 + 2x_5 &= -3 \\ x_1 - 2x_2 - x_3 + x_4 + x_5 &= 2 \end{aligned}$$

may be written as

$$\left\{ \left[ \begin{array}{c} 1 + 5a + 2b \\ 1 + 2a + 3b \\ a \\ 3 + 3b \\ b \end{array} \right] \middle| a, b \in \mathbb{R} \right\}.$$

**Remark E.2.10** Don't forget to correctly express the solution set of a linear system, using set-builder notation for consistent systems with infinitely many solutions.

- **Consistent with one solution:** e.g.  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$
- **Consistent with infinitely-many solutions:** e.g.  $\left\{ \left[ \begin{array}{c} 1 \\ 2 - 3a \\ a \end{array} \right] \middle| a \in \mathbb{R} \right\}$
- **Inconsistent:**  $\emptyset$  or  $\{\}$