

Name: _____

MIDTERM EXAM

Math 237 – Linear Algebra

Version 6

Fall 2017

Show all work. Answers without work will not receive credit. You may use a calculator, but you must show all relevant work to receive credit for a standard.

E1. Write an augmented matrix corresponding to the following system of linear equations.

$$\begin{aligned}x_1 + 3x_2 - 4x_3 + x_4 &= 5 \\3x_1 + 9x_2 + x_3 - 7x_4 &= 0 \\x_1 - x_3 + x_4 &= 1\end{aligned}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 3 & -4 & 1 & 5 \\ 3 & 9 & 1 & -7 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{array} \right]$$

□

E2. Find RREF A , where

$$A = \left[\begin{array}{ccc|c} 2 & -1 & 5 & 4 \\ -1 & 0 & -2 & -1 \\ 1 & 3 & -1 & -5 \end{array} \right]$$

Solution:

$$\text{RREF } A = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

□

E3. Solve the system of linear equations.

$$\begin{aligned}2x + y - z + w &= 5 \\3x - y - 2w &= 0 \\-x + 5z + 3w &= -1\end{aligned}$$

Solution:

$$\text{RREF} \left(\left[\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 5 \\ 3 & -1 & 0 & -2 & 0 \\ -1 & 0 & 5 & 0 & -1 \end{array} \right] \right) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{12} & 1 \\ 0 & 1 & 0 & \frac{4}{7} & 3 \\ 0 & 0 & 1 & \frac{1}{12} & 0 \end{array} \right]$$

So the solutions are

$$\left\{ \left[\begin{array}{c} 1+a \\ 3-21a \\ -7a \\ 12a \end{array} \right] \mid a \in \mathbb{R} \right\}$$

□

E4. Find a basis for the solution set to the homogeneous system of equations given by

$$\begin{aligned} 3x + 2y + z &= 0 \\ x + y + z &= 0 \end{aligned}$$

Solution: Let $A = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right]$, so $\text{RREF } A = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$. It follows that the basis for the solution set is given by $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

□

V1. Let V be the set of all pairs of real numbers with the operations, for any $(x_1, y_1), (x_2, y_2) \in V$, $c \in \mathbb{R}$,

$$\begin{aligned} (x_1, y_1) \oplus (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ c \odot (x_1, y_1) &= (0, cy_1) \end{aligned}$$

(a) Show that scalar multiplication **distributes vectors** over scalar addition:
 $(c + d) \odot (x, y) = c \odot (x, y) \oplus d \odot (x, y)$.

(b) Determine if V is a vector space or not. Justify your answer.

Solution: Let $(x_1, y_1) \in V$, and let $c, d \in \mathbb{R}$. Then

$$(c + d) \odot (x_1, y_1) = (0, (c + d)y_1) = (0, cy_1) \oplus (0, dy_1) = c \odot (x_1, y_1) \oplus d \odot (x_1, y_1).$$

However, V is not a vector space, as $1 \odot (x_1, y_1) = (0, y_1) \neq (x_1, y_1)$.

□

V2. Determine if $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ belongs to the span of the set $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \\ 3 \end{bmatrix} \right\}$.

Solution: Since

$$\text{RREF} \left(\left(\begin{array}{cc|c} 2 & 4 & 4 \\ 0 & -1 & -1 \\ -1 & 4 & 6 \\ 5 & 3 & -7 \end{array} \right) \right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right)$$

contains the contradiction $0 = 1$, $\begin{bmatrix} 4 \\ -1 \\ 6 \\ -7 \end{bmatrix}$ is not a linear combination of the three vectors.

□

V3. Determine if the vectors $\begin{bmatrix} 8 \\ 21 \\ -7 \end{bmatrix}$, $\begin{bmatrix} -3 \\ -8 \\ 3 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}$ span \mathbb{R}^3 .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 8 & -3 & -1 & 4 \\ 21 & -8 & -3 & 11 \\ -7 & 3 & 2 & -5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the rank is less than 3, they do not span \mathbb{R}^3 .

□

V4. Let W be the set of all \mathbb{R}^3 vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $x + y + z = 1$ (this forms a plane). Determine if W is a subspace of \mathbb{R}^3 .

Solution: No, because $\mathbf{0}$ does not belong to W .

□

S1. Determine if the vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ are linearly dependent or linearly independent

Solution:

$$\text{RREF} \left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since each column is a pivot column, the vectors are linearly independent.

□

S2. Determine if the set $\{x^2 + x - 1, 3x^2 - x + 1, 2x - 2\}$ is a basis of \mathcal{P}_2

Solution:

$$\text{RREF} \left(\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting matrix is the identity matrix, it is a basis.

□

S3. Let $W = \text{span} \left(\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix} \right\} \right)$. Find a basis of W .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 \\ 3 & 1 & 4 & 1 \\ -3 & 1 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -7 \end{bmatrix} \right\}$ is a basis for W .

□

S4. Let W be the subspace of $M_{2,2}$ given by $W = \text{span} \left(\left\{ \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\} \right)$. Compute the dimension of W .

Solution:

$$\text{RREF} \left(\begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ -2 & 3 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This has 3 pivot columns so $\dim(W) = 3$.

□

E1:	<input type="text"/>	E2:	<input type="text"/>	E3:	<input type="text"/>	E4:	<input type="text"/>	V1:	<input type="text"/>	V2:	<input type="text"/>	V3:
<input type="text"/>	V4:	<input type="text"/>	S1:	<input type="text"/>	S2:	<input type="text"/>	S3:	<input type="text"/>	S4:	<input type="text"/>		