# Application Activities - Module M Part 3 - Class Day 23

**Definition 23.1** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with standard matrix $A$.

- If $T$ is a bijection and $B$ is any $\mathbb{R}^n$ vector, then $T(X) = AX = B$ has a unique solution $X$.
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by setting $T^{-1}(B) = X$ to be this unique solution.
- Let $A^{-1}$ be the standard matrix for $T^{-1}$. We call $A^{-1}$ the **inverse matrix** of $A$, so we also say that $A$ is **invertible**.

**Activity 23.2** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ -3x + 5y \end{bmatrix}$. It can be shown that $T$ is bijective and has the inverse map $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x + 3y \\ 3x + 2y \end{bmatrix}$.

*Part 1:* Compute $(T^{-1} \circ T)\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$.

*Part 2:* If $A$ is the standard matrix for $T$ and $A^{-1}$ is the standard matrix for $T^{-1}$, what must $A^{-1}A$ be?

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**Observation 23.3** $T^{-1} \circ T = T \circ T^{-1}$ is the identity map for any bijective linear transformation $T$. Therefore $A^{-1}A = AA^{-1} = I$ is the identity matrix for any invertible matrix $A$.

**Activity 23.4** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

*Part 1:* Solve $T(X) = \mathbf{e}_1$ to find $T^{-1}(\mathbf{e}_1)$.
*Part 2:* Solve $T(X) = \mathbf{e}_2$ to find $T^{-1}(\mathbf{e}_2)$.
*Part 3:* Solve $T(X) = \mathbf{e}_3$ to find $T^{-1}(\mathbf{e}_3)$.
*Part 4:* Compute $A^{-1}$, the standard matrix for $T^{-1}$.

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**Observation 23.5** We could have solved these three systems simultaneously by row reducing the matrix $[A \,|\, I]$ at once.

$$A = \left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right]$$

**Activity 23.6** Find the inverse $A^{-1}$ of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A \,|\, I]$.

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**Activity 23.7** Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

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**Observation 23.8** A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $\mathrm{RREF}(A) = I_n$.