

## Application Activities - Module A Part 3 - Class Day 19

**Observation 19.1** Let  $T : V \rightarrow W$ . We have previously defined the following terms.

- $T$  is called **injective** or **one-to-one** if  $T$  does not map two distinct values to the same place.
- $T$  is called **surjective** or **onto** if every element of  $W$  is mapped to by some element of  $V$ .
- The **kernel** of  $T$  is the set of all things that are mapped to  $\mathbf{0}$ . It is a subspace of  $V$ .
- The **image** of  $T$  is the set of all things in  $W$  that are mapped to by something in  $V$ . It is a subspace of  $W$ .

**Activity 19.2** Let  $T : V \rightarrow W$  be a linear transformation where  $\ker T = \{\mathbf{0}\}$ . Can you answer either of the following questions about  $T$ ?

(a) Is  $T$  injective?

(b) Is  $T$  surjective?

(Hint: If  $T(\mathbf{v}) = T(\mathbf{w})$ , then what is  $T(\mathbf{v} - \mathbf{w})$ ?)

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**Fact 19.3** A linear transformation  $T$  is injective **if and only if**  $\ker T = \{\mathbf{0}\}$ . Put another way, an injective linear transformation may be recognized by its **trivial** kernel.

**Activity 19.4** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear transformation where  $\text{Im } T = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right\}$ . Can you answer either of the following questions about  $T$ ?

(a) Is  $T$  injective?

(b) Is  $T$  surjective?

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**Fact 19.5** A linear transformation  $T : V \rightarrow W$  is surjective **if and only if**  $\text{Im } T = W$ . Put another way, a surjective linear transformation may be recognized by its same codomain and image.

**Activity 19.6** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with standard matrix  $A$ . Sort the following claims into two groups of equivalent statements.

- |                                                   |                                                                                                                                             |
|---------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------|
| (a) $T$ is injective                              | (g) Every row of $\text{RREF}(A)$ has a pivot.                                                                                              |
| (b) $T$ is surjective                             | (h) The image of $T$ equals its codomain.                                                                                                   |
| (c) The kernel of $T$ is trivial.                 | (i) The system of linear equations given by the augmented matrix $[A \mid \mathbf{b}]$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$ |
| (d) The columns of $A$ span $\mathbb{R}^m$        | (j) The system of linear equations given by the augmented matrix $[A \mid \mathbf{0}]$ has exactly one solution.                            |
| (e) The columns of $A$ are linearly independent   |                                                                                                                                             |
| (f) Every column of $\text{RREF}(A)$ has a pivot. |                                                                                                                                             |
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**Definition 19.7** If  $T : V \rightarrow W$  is both injective and surjective, it is called **bijjective**.

**Activity 19.8** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a bijective linear map with standard matrix  $A$ . Label each of the following as true or false.

- (a) The columns of  $A$  form a basis for  $\mathbb{R}^m$
  - (b)  $\text{RREF}(A)$  is the identity matrix.
  - (c) The system of linear equations given by the augmented matrix  $[A \mid \mathbf{b}]$  has exactly one solution for all  $\mathbf{b} \in \mathbb{R}^m$ .
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**Activity 19.9** Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 2x + 3y \\ x - y \\ x + 3y \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
  - (b)  $T$  is injective but not surjective
  - (c)  $T$  is surjective but not injective
  - (d)  $T$  is bijective.
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**Activity 19.10** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
  - (b)  $T$  is injective but not surjective
  - (c)  $T$  is surjective but not injective
  - (d)  $T$  is bijective.
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**Activity 19.11** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y + z \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
  - (b)  $T$  is injective but not surjective
  - (c)  $T$  is surjective but not injective
  - (d)  $T$  is bijective.
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**Activity 19.12** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
  - (b)  $T$  is injective but not surjective
  - (c)  $T$  is surjective but not injective
  - (d)  $T$  is bijective.
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