Section V.0

Observation V.1 Several properties of the real numbers, such as commutivity:

$$x + y = y + x$$

also hold for Euclidean vectors with multiple components:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Activity V.2 *(~20 min)* Consider each of the following properties of the real numbers $\mathbb{R}^1$. Label each property as **valid** if the property also holds for two-dimensional Euclidean vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^2$ and scalars $a, b \in \mathbb{R}$, and **invalid** if it does not.

1. $\vec{\mathbf{u}} + (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = (\vec{\mathbf{u}} + \vec{\mathbf{v}}) + \vec{\mathbf{w}}$.
2. $\vec{\mathbf{u}} + \vec{\mathbf{v}} = \vec{\mathbf{v}} + \vec{\mathbf{u}}$.
3. There exists some $\vec{\mathbf{z}}$ where $\vec{\mathbf{v}} + \vec{\mathbf{z}} = \vec{\mathbf{v}}$.
4. There exists some $-\vec{\mathbf{v}}$ where $\vec{\mathbf{v}} + (-\vec{\mathbf{v}}) = \vec{\mathbf{z}}$.
5. If $\vec{\mathbf{u}} \neq \vec{\mathbf{v}}$, then $\frac{1}{2}(\vec{\mathbf{u}}+\vec{\mathbf{v}})$ is the only vector equally distant from both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$
6. $a(b\vec{\mathbf{v}}) = (ab)\vec{\mathbf{v}}$.
7. $1\vec{\mathbf{v}} = \vec{\mathbf{v}}$.
8. If $\vec{\mathbf{u}} \neq \vec{\mathbf{0}}$, then there exists some scalar c such that $c\vec{\mathbf{u}} = \vec{\mathbf{v}}$.
9. $a(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = a\vec{\mathbf{u}} + a\vec{\mathbf{v}}$.
10. $(a + b)\vec{\mathbf{v}} = a\vec{\mathbf{v}} + b\vec{\mathbf{v}}$.

Definition V.3 A **vector space** V is any collection of mathematical objects with associated addition $\oplus$ and scalar multiplication $\odot$ operations that satisfy the following properties. Let $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ belong to V, and let a, b be scalar numbers.

- **Addition is associative:** $\vec{\mathbf{u}} \oplus (\vec{\mathbf{v}} \oplus \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \oplus \vec{\mathbf{v}}) \oplus \vec{\mathbf{w}}$.
- **Addition is commutative:** $\vec{\mathbf{u}} \oplus \vec{\mathbf{v}} = \vec{\mathbf{v}} \oplus \vec{\mathbf{u}}$.
- **Additive identity exists:** There exists some $\vec{\mathbf{z}}$ where $\vec{\mathbf{v}} \oplus \vec{\mathbf{z}} = \vec{\mathbf{v}}$.
- **Additive inverses exist:** There exists some $-\vec{\mathbf{v}}$ where $\vec{\mathbf{v}} \oplus (-\vec{\mathbf{v}}) = \vec{\mathbf{z}}$.
- **Scalar multiplication is associative:** $a \odot (b \odot \vec{\mathbf{v}}) = (ab) \odot \vec{\mathbf{v}}$.
- **Scalar multiplication identity exists:** $1 \odot \vec{\mathbf{v}} = \vec{\mathbf{v}}$.
- **Scalar mult. distributes over vector addition:** $a \odot (\vec{\mathbf{u}} \oplus \vec{\mathbf{v}}) = a \odot \vec{\mathbf{u}} \oplus a \odot \vec{\mathbf{v}}$.
- **Scalar mult. distributes over scalar addition:** $(a + b) \odot \vec{\mathbf{v}} = a \odot \vec{\mathbf{v}} \oplus b \odot \vec{\mathbf{v}}$.

Observation V.4
Every **Euclidean vector space**

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \, x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}$$

satisfies all eight requirements for the usual definitions of addition and scalar multiplication, but we will also study other types of vector spaces.

Observation V.5 The space of $m \times n$**matrices**

$$M_{m,n} = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \middle| \, a_{11}, \ldots, a_{mn} \in \mathbb{R} \right\}$$

satisfies all eight requirements for component-wise addition and scalar multiplication.