

## Section S.2

**Definition S.2.1** A **basis** is a linearly independent set that spans a vector space.

The **standard basis** of  $\mathbb{R}^n$  is the set  $\{\vec{\mathbf{e}}_1, \dots, \vec{\mathbf{e}}_n\}$  where

$$\vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \dots \quad \vec{\mathbf{e}}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For  $\mathbb{R}^3$ , these are the vectors  $\vec{\mathbf{e}}_1 = \hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\vec{\mathbf{e}}_2 = \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , and  $\vec{\mathbf{e}}_3 = \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ .

**Observation S.2.2** A basis may be thought of as a collection of building blocks for a vector space, since every vector in the space can be expressed as a unique linear combination of basis vectors.

For example, in many calculus courses, vectors in  $\mathbb{R}^3$  are often expressed in their component form

$$(3, -2, 4) = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

or in their standard basic vector form

$$3\vec{\mathbf{e}}_1 - 2\vec{\mathbf{e}}_2 + 4\vec{\mathbf{e}}_3 = 3\hat{i} - 2\hat{j} + 4\hat{k}.$$

Since every vector in  $\mathbb{R}^3$  can be uniquely described as a linear combination of the vectors in  $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3\}$ , this set is indeed a basis.

**Activity S.2.3** ( $\sim 15$  min) Label each of the sets  $A, B, C, D, E$  as

- SPANS  $\mathbb{R}^4$  or DOES NOT SPAN  $\mathbb{R}^4$
- LINEARLY INDEPENDENT or LINEARLY DEPENDENT
- BASIS FOR  $\mathbb{R}^4$  or NOT A BASIS FOR  $\mathbb{R}^4$

by finding RREF for their corresponding matrices.

$$\begin{aligned}
 A &= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} & B &= \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} \\
 C &= \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 7 \\ 16 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 7 \\ 14 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\} & D &= \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 1 \\ 5 \end{bmatrix} \right\} \\
 E &= \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right\}
 \end{aligned}$$

**Activity S.2.4** ( $\sim 10$  min) If  $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$  is a basis for  $\mathbb{R}^4$ , that means  $\text{RREF}[\vec{\mathbf{v}}_1 \ \vec{\mathbf{v}}_2 \ \vec{\mathbf{v}}_3 \ \vec{\mathbf{v}}_4]$  doesn't have a non-pivot column, and doesn't have a row of zeros. What is  $\text{RREF}[\vec{\mathbf{v}}_1 \ \vec{\mathbf{v}}_2 \ \vec{\mathbf{v}}_3 \ \vec{\mathbf{v}}_4]$ ?

$$\text{RREF}[\vec{\mathbf{v}}_1 \ \vec{\mathbf{v}}_2 \ \vec{\mathbf{v}}_3 \ \vec{\mathbf{v}}_4] = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

**Fact S.2.5** The set  $\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_m\}$  is a basis for  $\mathbb{R}^n$  if and only if  $m = n$  and  $\text{RREF}[\vec{\mathbf{v}}_1 \ \dots \ \vec{\mathbf{v}}_n] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$ .

That is, a basis for  $\mathbb{R}^n$  must have exactly  $n$  vectors and its square matrix must row-reduce to the so-called **identity matrix** containing all zeros except for a downward diagonal of ones. (We will learn where the identity matrix gets its name in a later module.)

**Observation S.2.6** Recall that a **subspace** of a vector space is a subset that is itself a vector space.

One easy way to construct a subspace is to take the span of set, but a linearly dependent set contains “redundant” vectors. For example, only two of the three vectors in the following image are needed to span the planar subspace.



**Activity S.2.7** ( $\sim 10$  min) Consider the subspace  $W = \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right\}$  of  $\mathbb{R}^4$ .

*Part 1:* Mark the part of RREF  $\begin{bmatrix} 2 & 2 & 2 & 1 \\ 3 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 1 & -1 & -3 & 0 \end{bmatrix}$  that shows that  $W$ 's spanning set is linearly dependent.

*Part 2:* Find a basis for  $W$  by removing a vector from its spanning set to make it linearly independent.

**Fact S.2.8** Let  $S = \{\vec{v}_1, \dots, \vec{v}_m\}$ . The easiest basis describing  $\text{span } S$  is the set of vectors in  $S$  given by the pivot columns of  $\text{RREF}[\vec{v}_1 \dots \vec{v}_m]$ .

Put another way, to compute a basis for the subspace  $\text{span } S$ , simply remove the vectors corresponding to the non-pivot columns of  $\text{RREF}[\vec{v}_1 \dots \vec{v}_m]$ .

**Activity S.2.9** ( $\sim 10$  min) Let  $W$  be the subspace of  $\mathbb{R}^4$  given by

$$W = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Find a basis for  $W$ .

**Activity S.2.10** ( $\sim 10$  min) Let  $W$  be the subspace of  $\mathcal{P}^3$  given by

$$W = \text{span} \{x^3 + 3x^2 + x - 1, 2x^3 - x^2 + x + 2, 4x^3 + 5x^2 + 3x, 3x^3 + 2x^2 + 2x + 1\}$$

Find a basis for  $W$ .