Module M: Understanding Matrices Algebraically

What algebraic structure do matrices have?

At the end of this module, students will be able to...

- M1 **Matrix Multiplication.** ... multiply matrices.
- M2 **Row operations as matrix multiplication.** ... can express row operations through matrix multiplication.
- M3 **Invertible Matrices.** ... determine if a square matrix is invertible or not.
- M4 **Matrix inverses.** ... compute the inverse matrix of an invertible matrix.

Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- Compose functions of real numbers.
- Identify the domain and codomain of linear transformations.
- Find the matrix corresponding to a linear transformation and compute the image of a vector given a standard matrix **A2**
- Determine if a linear transformation is injective and/or surjective **A4**
- Interpret the ideas of injectivity and surjectivity in multiple ways.

The following resources will help you prepare for this module.

- Function composition (Khan Academy): `http://bit.ly/2wkz7f3`
- Domain and codomain: `https://www.youtube.com/watch?v=BQMyeQOLvpg`
- Interpreting injectivity and surjectivity in many ways: `https://www.youtube.com/watch?v=WpUv72Y6D10`

Module M Section 1

Observation M.1

If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^k$ are linear maps, then the composition map $S \circ T$ is a linear map from $\mathbb{R}^n \to \mathbb{R}^k$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T"] \arrow[rr, bend right, "S \circ T"'] & \mathbb{R}^m \arrow[r, "S"] & \mathbb{R}^k \end{tikzcd}$$

Recall that for a vector, $\vec{v} \in \mathbb{R}^n$, the composition is computed as $(S \circ T)(\vec{v}) = S(T(\vec{v}))$.

Activity M.2 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

What are the domain and codomain of the composition map $S \circ T$?

a. The domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$
b. The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$
c. The domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^4$
d. The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^4$
e. The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^3$
f. The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^2$

Activity M.3 *(~2 min)* Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be given by the 2×3 standard matrix $B=\begin{bmatrix}2 & 1 & -3\\ 5 & -3 & 4\end{bmatrix}$ and $S:\mathbb{R}^2\to\mathbb{R}^4$ be given by the 4×2 standard matrix $A=\begin{bmatrix}1 & 2\\ 0 & 1\\ 3 & 5\\ -1 & -2\end{bmatrix}$.

What size will the standard matrix of $S\circ T:\mathbb{R}^3\to\mathbb{R}^4$ be? (Rows $\times$ Columns)

a. 4×3
b. 4×2
c. 3×4
d. 3×2
e. 2×4
f. 2×3

Activity M.4 *(∼15 min)* Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be given by the 2×3 standard matrix $B=\begin{bmatrix}2 & 1 & -3\\ 5 & -3 & 4\end{bmatrix}$ and $S:\mathbb{R}^2\to\mathbb{R}^4$ be given by the 4×2 standard matrix $A=\begin{bmatrix}1 & 2\\ 0 & 1\\ 3 & 5\\ -1 & -2\end{bmatrix}$.

Activity M.4 *(~15 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Activity M.4 *(~15 min)* Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S: \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{e}_2)$.

Activity M.4 *(~15 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the 2×3 standard matrix $B = \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the 4×2 standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix}$.

Part 1: Compute

$$(S \circ T)(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Part 2: Compute $(S \circ T)(\vec{e}_2)$.

Part 3: Compute $(S \circ T)(\vec{e}_3)$.

Activity M.4 *(~15 min)* Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be given by the 2×3 standard matrix $B=\begin{bmatrix}2 & 1 & -3\\ 5 & -3 & 4\end{bmatrix}$ and $S:\mathbb{R}^2\to\mathbb{R}^4$ be given by the 4×2 standard matrix $A=\begin{bmatrix}1 & 2\\ 0 & 1\\ 3 & 5\\ -1 & -2\end{bmatrix}$.

Part 1: Compute

$$(S\circ T)(\vec{e}_1)=S(T(\vec{e}_1))=S\left(\begin{bmatrix}2\\5\end{bmatrix}\right)=\begin{bmatrix}?\\?\\?\\?\end{bmatrix}.$$

Part 2: Compute $(S\circ T)(\vec{e}_2)$.

Part 3: Compute $(S\circ T)(\vec{e}_3)$.

Part 4: Write the 4×3 standard matrix of $S\circ T:\mathbb{R}^3\to\mathbb{R}^4$.

Definition M.5

We define the **product** AB of a $m \times n$ matrix A and a $n \times k$ matrix B to be the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For the previous activity, T was a map $\mathbb{R}^3 \to \mathbb{R}^2$, and S was a map $\mathbb{R}^2 \to \mathbb{R}^4$, so $S \circ T$ gave a map $\mathbb{R}^3 \to \mathbb{R}^4$ with a 4×3 standard matrix:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (S \circ T)(\vec{e}_1) & (S \circ T)(\vec{e}_2) & (S \circ T)(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

Activity M.6 *(∼15 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Activity M.6 *(∼15 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Part 1: Write the dimensions (rows $\times$ columns) for A, B, AB, and BA.

Activity M.6 *($\sim$15 min)* Let $S:\mathbb{R}^3\to\mathbb{R}^2$ be given by the matrix $A=\begin{bmatrix}-4 & -2 & 3\\ 0 & 1 & 1\end{bmatrix}$ and $T:\mathbb{R}^2\to\mathbb{R}^3$ be given by the matrix $B=\begin{bmatrix}2 & 3\\ 1 & -1\\ 0 & -1\end{bmatrix}$.

Part 1: Write the dimensions (rows $\times$ columns) for A, B, AB, and BA.

Part 2: Find the standard matrix AB of $S\circ T$.

Activity M.6 *(∼15 min)* Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the matrix $A = \begin{bmatrix} -4 & -2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the matrix $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -1 \end{bmatrix}$.

Part 1: Write the dimensions (rows $\times$ columns) for A, B, AB, and BA.

Part 2: Find the standard matrix AB of $S \circ T$.

Part 3: FInd the standard matrix BA of $T \circ S$.

Activity M.7 *(∼10 min)* Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Activity M.7 *(∼10 min)* Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Part 1: Find the domain and codomain of each of the three linear maps corresponding to A, B), and C.

Activity M.7 *(∼10 min)* Consider the following three matrices.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ -1 & 5 & 7 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 2 \\ 0 & -1 \\ 3 & 1 \\ 4 & 0 \end{bmatrix}$$

Part 1: Find the domain and codomain of each of the three linear maps corresponding to A, B), and C.

Part 2: Only one of the matrix products AB, AC, BA, BC, CA, CB can actually be computed. Compute it.

Module M Section 2

Remark M.8

Recall that the **product** AB of a $m \times n$ matrix A and an $n \times k$ matrix B is the $m \times k$ standard matrix of the composition map of the two corresponding linear functions.

For example, if T was a map $\mathbb{R}^3 \to \mathbb{R}^2$, and S was a map $\mathbb{R}^2 \to \mathbb{R}^4$, then $S \circ T$ gave a map $\mathbb{R}^3 \to \mathbb{R}^4$ with a 4×3 standard matrix, such as:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 5 & -3 & 4 \end{bmatrix}$$

$$= \left[(S \circ T)(\vec{e}_1) \quad (S \circ T)(\vec{e}_2) \quad (S \circ T)(\vec{e}_3)\right] = \begin{bmatrix} 12 & -5 & 5 \\ 5 & -3 & 4 \\ 31 & -12 & 11 \\ -12 & 5 & -5 \end{bmatrix}.$$

Activity M.9 *(∼15 min)* Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Activity M.9 *(∼15 min)* Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Part 1: Compute the product BA by hand.

Activity M.9 *(∼15 min)* Let $B = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

Part 1: Compute the product BA by hand.

Part 2: Check your work using technology. Using Octave:

- `B = sym([3 -4 0 ; 2 0 -1 ; 0 -3 3])`
- `A = sym([2 7 -1 ; 0 3 2 ; 1 1 -1])`
- `B*A`

Activity M.10 *(∼5 min)* Let $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Find a 3×3 matrix B such that $BA = A$, that is,

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Check your guess using technology.

Definition M.11

The identity matrix I_n (or just I when n is obvious from context) is the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It has a 1 on each diagonal element and a 0 in every other position.

Fact M.12

For any square matrix A, $IA = AI = A$:

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 7 & -1\\ 0 & 3 & 2\\ 1 & 1 & -1\end{bmatrix} = \begin{bmatrix}2 & 7 & -1\\ 0 & 3 & 2\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 7 & -1\\ 0 & 3 & 2\\ 1 & 1 & -1\end{bmatrix}$$

Activity M.13 *($\sim$20 min)* Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Activity M.13 *(∼20 min)* Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Part 1: Create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Activity M.13 *(∼20 min)* Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Part 1: Create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Activity M.13 *(∼20 min)* Tweaking the identity matrix slightly allows us to write row operations in terms of matrix multiplication.

Part 1: Create a matrix that doubles the third row of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 2 & 2 & -2 \end{bmatrix}$$

Part 2: Create a matrix that swaps the second and third rows of A:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -1 \\ 1 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix}$$

Part 3: Create a matrix that adds 5 times the third row of A to the first row:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 2 & 7 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2+5(1) & 7+5(1) & -1+5(-1) \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Fact M.14

If R is the result of applying a row operation to I, then RA is the result of applying the same row operation to A.

- Scaling a row: $R = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Swapping rows: $R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Adding a row multiple to another row: $R = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Such matrices can be chained together to emulate multiple row operations. In particular,

$$\mathrm{RREF}(A) = R_k \dots R_2 R_1 A$$

for some sequence of matrices $R_1, R_2, \dots, R_k$.

Activity M.15 *(∼10 min)* Consider the two row operations $R_2 \leftrightarrow R_3$ and $R_1 + R_2 \to R_1$ applied as follows to show $A \sim B$:

$$A = \begin{bmatrix} -1 & 4 & 5 \\ 0 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1+1 & 4+2 & 5+3 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 8 \\ 1 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} = B$$

Express these row operations as matrix multiplication by expressing B as the product of two matrices and A:

$$B = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} A$$

Check your work using technology.

Module M Section 3

Activity M.16 *($\sim$15 min)* Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with standard matrix A. Sort the following items into three groups of statements: a group that means T is **injective**, a group that means T is **surjective**, and a group that means T is **bijective**.

a. $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$
b. $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^m$
c. $A\vec{x} = \vec{0}$ has a unique solution.
d. The columns of A span $\mathbb{R}^m$
e. The columns of A are linearly independent
f. The columns of A are a basis of $\mathbb{R}^m$
g. Every column of RREF(A) has a pivot
h. Every row of RREF(A) has a pivot
i. $m = n$ and RREF$(A) = I$

Activity M.17 *(∼5 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 4 \\ 1 & 1 & 3 \end{bmatrix}$.

Write an augmented matrix representing the system of equations given by $T(\vec{x}) = \vec{0}$, that is, $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{0}$ to find the kernel of T.

Definition M.18

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with standard matrix A.

- If T is a bijection and $\vec{b}$ is any $\mathbb{R}^n$ vector, then $T(\vec{x}) = A\vec{x} = \vec{b}$ has a unique solution.
- So we may define an **inverse map** $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ by setting $T^{-1}(\vec{b})$ to be this unique solution.
- Let A^{-1} be the standard matrix for T^{-1}. We call A^{-1} the **inverse matrix** of A, so we also say that A is **invertible**.

Activity M.19 *(∼20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Activity M.19 *(~20 min)* Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation given by the standard matrix $A=\begin{bmatrix}2 & -1 & -6\\ 2 & 1 & 3\\ 1 & 1 & 4\end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(\vec{x})=\vec{e}_1$, that is, $A\vec{x}=\begin{bmatrix}1\\0\\0\end{bmatrix}$. Then solve $T(\vec{x})=\vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Activity M.19 *(∼20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Activity M.19 *(~20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Part 3: Solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

Activity M.19 *(∼20 min)* Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the standard matrix $A = \begin{bmatrix} 2 & -1 & -6 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Part 1: Write an augmented matrix representing the system of equations given by $T(\vec{x}) = \vec{e}_1$, that is, $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then solve $T(\vec{x}) = \vec{e}_1$ to find $T^{-1}(\vec{e}_1)$.

Part 2: Solve $T(\vec{x}) = \vec{e}_2$ to find $T^{-1}(\vec{e}_2)$.

Part 3: Solve $T(\vec{x}) = \vec{e}_3$ to find $T^{-1}(\vec{e}_3)$.

Part 4: Write A^{-1}, the standard matrix for T^{-1}.

Observation M.20

We could have solved these three systems simultaneously by row reducing the matrix $[A \mid I]$ at once.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & -5 & 14 & -18 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right]$$

Activity M.21 *(∼5 min)* Find the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ by row-reducing $[A \,|\, I]$.

Activity M.22 *(∼5 min)* Is the matrix $\begin{bmatrix} 2 & 3 & 1 \\ -1 & -4 & 2 \\ 0 & -5 & 5 \end{bmatrix}$ invertible? Give a reason for your answer.

Observation M.23

An $n \times n$ matrix A is invertible if and only if $\mathrm{RREF}(A) = I_n$.

Activity M.24 *($\sim$10 min)* Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}2x-3y\\-3x+5y\end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}5x+3y\\3x+2y\end{bmatrix}$.

Activity M.24 *(∼10 min)* Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}2x-3y\\-3x+5y\end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}5x+3y\\3x+2y\end{bmatrix}$.

Part 1: Compute $(T^{-1}\circ T)\left(\begin{bmatrix}-2\\1\end{bmatrix}\right)$.

Activity M.24 *(∼10 min)* Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the bijective linear map defined by $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}2x-3y\\-3x+5y\end{bmatrix}$, with the inverse map $T^{-1}\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}5x+3y\\3x+2y\end{bmatrix}$.

Part 1: Compute $(T^{-1}\circ T)\left(\begin{bmatrix}-2\\1\end{bmatrix}\right)$.

Part 2: If A is the standard matrix for T and A^{-1} is the standard matrix for T^{-1}, find the 2×2 matrix

$$A^{-1}A=\begin{bmatrix}? & ?\\ ? & ?\end{bmatrix}.$$