

Application Activities - Module V Part 2 - Class Day 8

Remark 8.1 The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- \mathbb{R}^n : Euclidean vectors with n components.
- \mathbb{R}^∞ : Sequences of real numbers (v_1, v_2, \dots) .
- $\mathbb{R}^{m \times n}$: Matrices of real numbers with m rows and n columns.
- \mathbb{C} : Complex numbers.
- \mathcal{P}^n : Polynomials of degree n or less.
- \mathcal{P} : Polynomials of any degree.
- $C(\mathbb{R})$: Real-valued continuous functions.

Activity 8.2 Let $V = \{(a, b) : a, b \text{ are real numbers}\}$, where $(a_1, b_1) \oplus (a_2, b_2) = (a_1 + b_1 + a_2 + b_2, b_1^2 + b_2^2)$ and $c \odot (a, b) = (a^c, b + c)$. Show that this is not a vector space by finding a counterexample that does not satisfy one of the vector space properties.

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| • Addition associativity.
$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.$ | • Scalar multiplication associativity.
$a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.$ |
| • Addition commutativity.
$\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.$ | • Scalar multiplication identity.
$1 \odot \mathbf{v} = \mathbf{v}.$ |
| • Addition identity.
There exists some $\mathbf{0}$ where $\mathbf{v} \oplus \mathbf{0} = \mathbf{v}.$ | • Scalar distribution.
$a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).$ |
| • Addition inverse.
There exists some $-\mathbf{v}$ where $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}.$ | • Vector distribution.
$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).$ |

Definition 8.3 A **linear combination** of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$ for any choice of scalar multiples c_1, c_2, \dots, c_m .

For example, we say $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Definition 8.4 The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m : c_i \text{ is a real number}\}$$

Activity 8.5 Consider $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

Part 1: Sketch $c\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the xy plane for $c = 1, 3, 0, -2$.

Part 2: Sketch a representation of all the vectors given by $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ in the xy plane.

Activity 8.6 Consider $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

Part 1: Sketch the following linear combinations in the xy plane: $1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $0\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Part 2: Sketch a representation of all the vectors given by $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ in the xy plane.

Activity 8.7 Sketch a representation of all the vectors given by $\text{span}\left\{\begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}$ in the xy plane.

Activity 8.8 The vector $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belongs to $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}\right\}$ exactly when the vector equation $x_1\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ holds for some scalars x_1, x_2 .

Part 1: Reinterpret this vector equation as a system of linear equations.

Part 2: Solve this system. (Remember, you should use a calculator to help find RREF.)

Part 3: Given this solution, does $\begin{bmatrix} -1 \\ -6 \\ 1 \end{bmatrix}$ belong to $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}\right\}$?
