

## Module A: Algebraic properties of linear maps

# How can we understand linear maps algebraically?

## Module A

Section A.1

Section A.2

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Section A.4

At the end of this module, students will be able to...

- A1. Linear map verification.** ... determine if a map between vector spaces of polynomials is linear or not.
- A2. Linear maps and matrices.** ... translate back and forth between a linear transformation of Euclidean spaces and its standard matrix, and perform related computations.
- A3. Injectivity and surjectivity.** ... determine if a given linear map is injective and/or surjective.
- A4. Kernel and Image.** ... compute a basis for the kernel and a basis for the image of a linear map.

## Readiness Assurance Outcomes

Before beginning this module, each student should be able to...

- State the definition of a spanning set, and determine if a set of Euclidean vectors spans  $\mathbb{R}^n$  **V4**.
- State the definition of linear independence, and determine if a set of Euclidean vectors is linearly dependent or independent **S1**.
- State the definition of a basis, and determine if a set of Euclidean vectors is a basis **S2,S3**.
- Find a basis of the solution space to a homogeneous system of linear equations **S6**.

# Module A Section 1

## Definition A.1.1

A **linear transformation** (also known as a **linear map**) is a map between vector spaces that preserves the vector space operations. More precisely, if  $V$  and  $W$  are vector spaces, a map  $T : V \rightarrow W$  is called a linear transformation if

- ①  $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$  for any  $\mathbf{v}, \mathbf{w} \in V$ .
- ②  $T(c\mathbf{v}) = cT(\mathbf{v})$  for any  $c \in \mathbb{R}, \mathbf{v} \in V$ .

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

## Definition A.1.2

Given a linear transformation  $T : V \rightarrow W$ ,  $V$  is called the **domain** of  $T$  and  $W$  is called the **co-domain** of  $T$ .



**Example A.1.3**

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that  $T$  is linear, we must verify...

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = T \left( \begin{bmatrix} x + u \\ y + v \\ z + w \end{bmatrix} \right) = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) + T \left( \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

And also...

$$T \left( c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T \left( \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \quad \text{and} \quad cT \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = c \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore  $T$  is a linear transformation.



**Example A.1.4**

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

To show that  $T$  is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different,  $T$  is not a linear transformation.

**Fact A.1.5**

A map between Euclidean spaces  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear exactly when every component of the output is a linear combination of the variables of  $\mathbb{R}^n$ .

For example, the following map is definitely linear because  $x - z$  and  $3y$  are linear combinations of  $x, y, z$ :

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because  $x^2$ ,  $y + 3$ , and  $y - 2^x$  are not linear combinations (even though  $x + y$  is):

$$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

**Activity A.1.6** ( $\sim 5$  min)

Recall the following rules from calculus, where  $D : \mathcal{P} \rightarrow \mathcal{P}$  is the derivative map defined by  $D(f(x)) = f'(x)$  for each polynomial  $f$ .

$$D(f + g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

- a)  $\mathcal{P}$  is not a vector space
- b)  $D$  is a linear map
- c)  $D$  is not a linear map

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**Activity A.1.7** (*~10 min*)

Let the polynomial maps  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  and  $T : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute  $S(x^4 + x)$ ,  $S(x^4) + S(x)$ ,  $T(x^4 + x)$ , and  $T(x^4) + T(x)$ . Which of these maps is definitely not linear?

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**Fact A.1.8**

If  $L : V \rightarrow W$  is linear, then  $L(\mathbf{z}) = L(0\mathbf{v}) = 0L(\mathbf{v}) = \mathbf{z}$  where  $\mathbf{z}$  is the additive identity of the vector spaces  $V, W$ .

Put another way, an easy way to prove that a map like  $T(f(x)) = f'(x) + x^3$  can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

**Activity A.1.9** (*~15 min*)

Continue to consider  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

**Activity A.1.9** (*~15 min*)

Continue to consider  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

*Part 1:* Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to  $S(f(x)) + S(g(x))$  for all polynomials  $f, g$ .

**Activity A.1.9** ( $\sim 15$  min)

Continue to consider  $S : \mathcal{P}^4 \rightarrow \mathcal{P}^3$  defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

*Part 1:* Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to  $S(f(x)) + S(g(x))$  for all polynomials  $f, g$ .

*Part 2:* Verify that  $S(cf(x))$  is equal to  $cS(f(x))$  for all real numbers  $c$  and polynomials  $f$ . Is  $S$  linear?



**Activity A.1.10** (*~20 min*)

Let the polynomial maps  $S : \mathcal{P} \rightarrow \mathcal{P}$  and  $T : \mathcal{P} \rightarrow \mathcal{P}$  be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

**Activity A.1.10** (*~20 min*)

Let the polynomial maps  $S : \mathcal{P} \rightarrow \mathcal{P}$  and  $T : \mathcal{P} \rightarrow \mathcal{P}$  be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

*Part 1:* Show that  $S(x+1) \neq S(x) + S(1)$  to verify that  $S$  is not linear.

**Activity A.1.10** (*~20 min*)

Let the polynomial maps  $S : \mathcal{P} \rightarrow \mathcal{P}$  and  $T : \mathcal{P} \rightarrow \mathcal{P}$  be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

*Part 1:* Show that  $S(x+1) \neq S(x) + S(1)$  to verify that  $S$  is not linear.

*Part 2:* Prove that  $T$  is linear by verifying that

$$T(f(x) + g(x)) = T(f(x)) + T(g(x)) \text{ and } T(cf(x)) = cT(f(x)).$$

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**Observation A.1.11**

Note that  $S$  in the previous activity is not linear, even though  $S(0) = (0)^2 = 0$ . So showing  $S(0) = 0$  isn't enough to prove a map is linear.

This is a similar situation to proving a subset is a subspace: if the subset doesn't contain  $\mathbf{z}$ , then the subset isn't a subspace. But if the subset contains  $\mathbf{z}$ , you cannot conclude anything.

## Module A Section 2

## Remark A.2.1

Recall that a linear map  $T : V \rightarrow W$  satisfies

- ①  $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$  for any  $\mathbf{v}, \mathbf{w} \in V$ .
- ②  $T(c\mathbf{v}) = cT(\mathbf{v})$  for any  $c \in \mathbb{R}, \mathbf{v} \in V$ .

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

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**Activity A.2.2** (*~5 min*)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and

$T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b)  $\begin{bmatrix} -9 \\ 6 \end{bmatrix}$

(c)  $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

(d)  $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

**Activity A.2.3** ( $\sim 3$  min)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and

$T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c)  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d)  $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$



**Activity A.2.4** ( $\sim 2$  min)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and

$T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Compute  $T \left( \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} \right)$ .

(a)  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(c)  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(d)  $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

**Activity A.2.5** ( $\sim 5$  min)

Suppose  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map, and you know  $T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and

$T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ . Do you have enough information to compute  $T(\mathbf{v})$  for *any*  $\mathbf{v} \in \mathbb{R}^3$ ?

- (a) Yes.
- (b) No, exactly one more piece of information is needed.
- (c) No, an infinite amount of information would be necessary to compute the transformation of infinitely-many vectors.

### Fact A.2.6

Consider any basis  $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  for  $V$ . Since every vector  $\mathbf{v}$  can be written *uniquely* as a linear combination of basis vectors,  $x_1\mathbf{b}_1 + \dots + x_n\mathbf{b}_n$ , we may compute  $T(\mathbf{v})$  as follows:

$$T(\mathbf{v}) = T(x_1\mathbf{b}_1 + \dots + x_n\mathbf{b}_n) = x_1T(\mathbf{b}_1) + \dots + x_nT(\mathbf{b}_n).$$

Therefore any linear transformation  $T : V \rightarrow W$  can be defined by just describing the values of  $T(\mathbf{b}_i)$ .

Put another way, the images of the basis vectors **determine** the transformation  $T$ .

## Definition A.2.7

Since linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is determined by the standard basis  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ , it's convenient to store this information in the  $m \times n$  **standard matrix**  $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$ .

For example, let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear map determined by the following values for  $T$  applied to the standard basis of  $\mathbb{R}^3$ .

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad T(\mathbf{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Then the standard matrix corresponding to  $T$  is

$$[T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)] = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix}.$$

**Activity A.2.8** ( $\sim 3$  min)

Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be the linear transformation given by

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \quad T(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \quad T(\mathbf{e}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Write the standard matrix  $[T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$  for  $T$ .

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**Activity A.2.9** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 3z \\ 2x - y - 4z \end{bmatrix}$$

Find the standard matrix for  $T$ .

**Fact A.2.10**

Because every linear map  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  has a linear combination of the variables in each component, and thus  $T(\mathbf{e}_i)$  yields exactly the coefficients of  $x_i$ , the standard matrix for  $T$  is simply an ordered list of the coefficients of the  $x_i$ :

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} ax + by + cz + dw \\ ex + fy + gz + hw \end{bmatrix} \quad A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}$$

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**Activity A.2.11** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute  $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ .



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**Activity A.2.12** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 2 \\ 0 & -2 & 1 \end{bmatrix}.$$

Compute  $T \left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$ .

**Fact A.2.13**

To quickly compute  $T(\mathbf{v})$  from its standard matrix  $A$ , compute the **dot product** (defined in Calculus 3) of each matrix row with the vector. For example, if  $T$  has the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$$

then for  $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 2y + 3z \\ 0x + 1y - 2z \\ 2x - 1y + 0z \end{bmatrix}$$

and for  $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$  we will write

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(0) + 3(-2) \\ 0(3) + 1(0) - 2(-2) \\ 2(3) - 1(0) + 0(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 6 \end{bmatrix}.$$

**Activity A.2.14** (*~15 min*)

Compute the following linear transformations of vectors given their standard matrices.

$$T_1 \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \text{ for the standard matrix } A_1 = \begin{bmatrix} 4 & 3 \\ 0 & -1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$$

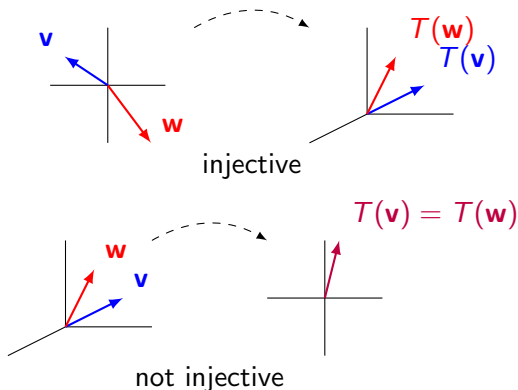
$$T_2 \left( \begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix} \right) \text{ for the standard matrix } A_2 = \begin{bmatrix} 4 & 3 & 0 & -1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

$$T_3 \left( \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right) \text{ for the standard matrix } A_3 = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & 3 \\ 5 & 1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

## Module A Section 3

## Definition A.3.1

Let  $T : V \rightarrow W$  be a linear transformation.  $T$  is called **injective** or **one-to-one** if  $T$  does not map two distinct vectors to the same place. More precisely,  $T$  is injective if  $T(\mathbf{v}) \neq T(\mathbf{w})$  whenever  $\mathbf{v} \neq \mathbf{w}$ .



**Activity A.3.2** ( $\sim 3$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Show that  $T$  is not injective by finding two different vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$  such that  $T(\mathbf{v}) = T(\mathbf{w})$ .

**Activity A.3.3** ( $\sim 2$  min)

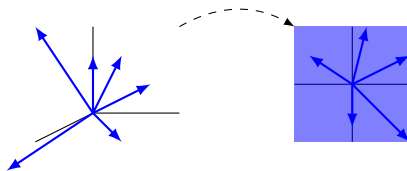
Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

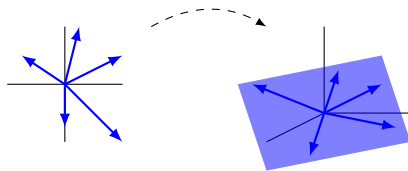
Is  $T$  injective? If not, find two different vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$  such that  $T(\mathbf{v}) = T(\mathbf{w})$ .

### Definition A.3.4

Let  $T : V \rightarrow W$  be a linear transformation.  $T$  is called **surjective** or **onto** if every element of  $W$  is mapped to by an element of  $V$ . More precisely, for every  $\mathbf{w} \in W$ , there is some  $\mathbf{v} \in V$  with  $T(\mathbf{v}) = \mathbf{w}$ .



surjective



not surjective



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**Activity A.3.5** (*~3 min*)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Show that  $T$  is not surjective by finding a vector in  $\mathbb{R}^3$  that  $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$  can never equal.

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**Activity A.3.6** ( $\sim 2$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is  $T$  surjective? If not, find a vector in  $\mathbb{R}^2$  that  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$  can never equal.

**Observation A.3.7**

As we will see, it's no coincidence that the RREF of the injective map's standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has all pivot columns. Similarly, the RREF of the surjective map's standard matrix

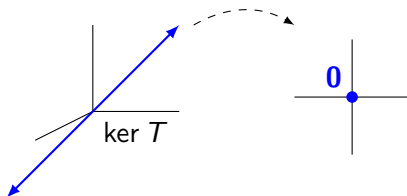
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has all pivot (zero) rows.

### Definition A.3.8

Let  $T : V \rightarrow W$  be a linear transformation. The **kernel** of  $T$  is an important subspace of  $V$  defined by

$$\ker T = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{z}\}$$



**Activity A.3.9** (*~5 min*)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of  $\mathbb{R}^2$  describes  $\ker T$ , the set of all vectors that transform into  $\mathbf{0}$ ?

a)  $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

b)  $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c)  $\mathbb{R}^2$

**Activity A.3.10** ( $\sim 5$  min)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of  $\mathbb{R}^3$  describes  $\ker T$ , the set of all vectors that transform into  $\mathbf{0}$ ?

a)  $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

b)  $\left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} \mid a \in \mathbb{R} \right\}$

c)  $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d)  $\mathbb{R}^3$

**Activity A.3.11** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

**Activity A.3.11** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Set  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  to find a linear system of equations whose solution set is the kernel.



**Activity A.3.11** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

*Part 1:* Set  $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} ? + ? + ? \\ ? + ? + ? \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  to find a linear system of equations

whose solution set is the kernel.

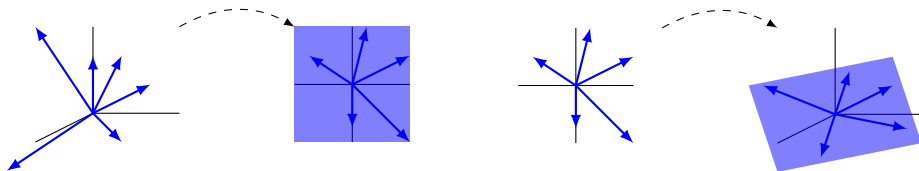
*Part 2:* Use  $\text{RREF}(A)$  to solve this homogeneous system of equations and find a basis for the kernel of  $T$ .

**Definition A.3.12**

Let  $T : V \rightarrow W$  be a linear transformation. The **image** of  $T$  is an important subspace of  $W$  defined by

$$\text{Im } T = \{ \mathbf{w} \in W \mid \text{there is some } \mathbf{v} \in V \text{ with } T(\mathbf{v}) = \mathbf{w} \}$$

In the examples below, the left example's image is all of  $\mathbb{R}^2$ , but the right example's image is a planar subspace of  $\mathbb{R}^3$ .



**Activity A.3.13** ( $\sim 5$  min)

Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Which of these subspaces of  $\mathbb{R}^3$  describes  $\text{Im } T$ , the set of all vectors that are the result of using  $T$  to transform  $\mathbb{R}^2$  vectors?

a)  $\left\{ \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

b)  $\left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$

c)  $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d)  $\mathbb{R}^3$

**Activity A.3.14** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{with standard matrix } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of these subspaces of  $\mathbb{R}^2$  describes  $\text{Im } T$ , the set of all vectors that are the result of using  $T$  to transform  $\mathbb{R}^3$  vectors?

a)  $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \mid a \in \mathbb{R} \right\}$

b)  $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

c)  $\mathbb{R}^2$

**Activity A.3.15** ( $\sim 5$  min)

Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix} = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3) \quad T(\mathbf{e}_4)].$$

Since  $T(\mathbf{v}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4)$ , the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

- a) spans  $\text{Im } T$
- b) is a linearly independent subset of  $\text{Im } T$
- c) is a basis for  $\text{Im } T$

**Observation A.3.16**

Let  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 \\ -1 & 1 & 0 & 2 \\ 2 & 1 & 3 & -1 \end{bmatrix}.$$

Since the set  $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$  spans  $\text{Im } T$ , we can obtain a basis for

$\text{Im } T$  by finding RREF  $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  and only using the vectors

corresponding to pivot columns:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

**Fact A.3.17**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation with standard matrix  $A$ .

- The kernel of  $T$  is the solution set of the homogeneous system given by the augmented matrix  $[A \mid \mathbf{0}]$ . Use the coefficients of its free variables to get a basis for the kernel.
- The image of  $T$  is the span of the columns of  $A$ . Remove the vectors creating non-pivot columns in RREF  $A$  to get a basis for the image.

**Activity A.3.18** (*~10 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be the linear transformation given by the standard matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Find a basis for the kernel and a basis for the image of  $T$ .



## Module A Section 4

## Observation A.4.1

Let  $T : V \rightarrow W$ . We have previously defined the following terms.

- $T$  is called **injective** or **one-to-one** if  $T$  does not map two distinct vectors to the same place.
- $T$  is called **surjective** or **onto** if every element of  $W$  is mapped to by some element of  $V$ .
- The **kernel** of  $T$  is the set of all vectors in  $V$  that are mapped to  $\mathbf{z} \in W$ . It is a subspace of  $V$ .
- The **image** of  $T$  is the set of all vectors in  $W$  that are mapped to by something in  $V$ . It is a subspace of  $W$ .

**Activity A.4.2** ( $\sim 5$  min)

Let  $T : V \rightarrow W$  be a linear transformation where  $\ker T$  contains multiple vectors. What can you conclude?

- (a)  $T$  is injective
- (b)  $T$  is not injective
- (c)  $T$  is surjective
- (d)  $T$  is not surjective

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Section A.1

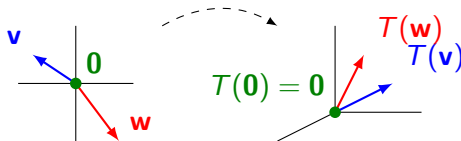
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**Fact A.4.3**

A linear transformation  $T$  is injective **if and only if**  $\ker T = \{\mathbf{0}\}$ . Put another way, an injective linear transformation may be recognized by its **trivial** kernel.



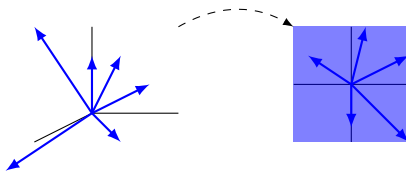
**Activity A.4.4** ( $\sim 5$  min)

Let  $T : \mathbb{R}^5 \rightarrow \mathbb{R}^5$  be a linear transformation where  $\text{Im } T$  is spanned less than five vectors. What can you conclude?

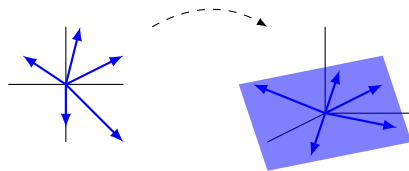
- (a)  $T$  is injective
- (b)  $T$  is not injective
- (c)  $T$  is surjective
- (d)  $T$  is not surjective

**Fact A.4.5**

A linear transformation  $T : V \rightarrow W$  is surjective **if and only if**  $\text{Im } T = W$ . Put another way, a surjective linear transformation may be recognized by its identical codomain and image.



surjective,  $\text{Im } T = \mathbb{R}^2$



not surjective,  $\text{Im } T \neq \mathbb{R}^3$

**Activity A.4.6** ( $\sim 15$  min)

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map with standard matrix  $A$ . Sort the following claims into two groups of *equivalent* statements: one group that means  $T$  is **injective**, and one group that means  $T$  is **surjective**.

- (a) The kernel of  $T$  is trivial:  
 $\ker T = \{\mathbf{0}\}$ .
- (b) The columns of  $A$  span  $\mathbb{R}^m$ .
- (c) The columns of  $A$  are linearly independent.
- (d) Every column of  $\text{RREF}(A)$  has a pivot.
- (e) Every row of  $\text{RREF}(A)$  has a pivot.
- (f) The image of  $T$  equals its codomain:  $\text{Im } T = \mathbb{R}^m$ .
- (g) The system of linear equations given by the augmented matrix  $[A \mid \mathbf{b}]$  has a solution for all  $\mathbf{b} \in \mathbb{R}^m$ .
- (h) The system of linear equations given by the augmented matrix  $[A \mid \mathbf{0}]$  has exactly one solution.

## Observation A.4.7

The easiest way to show that the linear map with standard matrix  $A$  is injective is to show that  $\text{RREF}(A)$  has all pivot columns.

The easiest way to show that the linear map with standard matrix  $A$  is surjective is to show that  $\text{RREF}(A)$  has all pivot rows.



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**Activity A.4.8** ( $\sim 3$  min)

What can you immediately conclude about the linear map  $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ ?

- a) Its standard matrix has more columns than rows, so  $T$  is not injective.
- b) Its standard matrix has more rows than columns, so  $T$  is not surjective.

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**Activity A.4.9** ( $\sim 2$  min)

What can you immediately conclude about the linear map  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^7$ ?

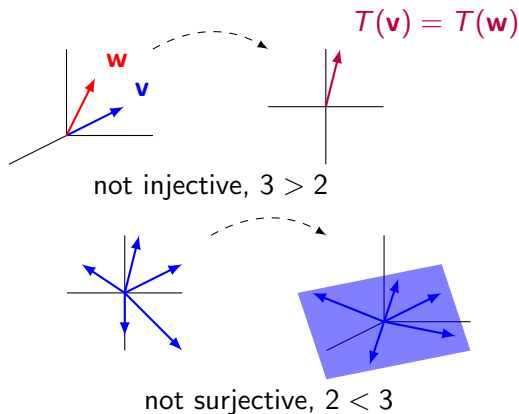
- a) Its standard matrix has more columns than rows, so  $T$  is not injective.
- b) Its standard matrix has more rows than columns, so  $T$  is not surjective.

**Fact A.4.10**

The following are true for any linear map  $T : V \rightarrow W$ :

- If  $\dim(V) > \dim(W)$ , then  $T$  is not injective.
- If  $\dim(V) < \dim(W)$ , then  $T$  is not surjective.

Basically, a linear transformation cannot reduce dimension without collapsing vectors into each other, and a linear transformation cannot increase the dimension of its image.



But these results do **not** reverse. For example,  $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$  might not be

## Definition A.4.11

If  $T : V \rightarrow W$  is both injective and surjective, it is called **bijjective**.

**Activity A.4.12** (*~5 min*)

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a bijective linear map with standard matrix  $A$ . Label each of the following as true or false.

- (a) The columns of  $A$  form a basis for  $\mathbb{R}^m$
- (b)  $\text{RREF}(A)$  is the identity matrix.
- (c) The system of linear equations given by the augmented matrix  $[A \mid \mathbf{b}]$  has exactly one solution for all  $\mathbf{b} \in \mathbb{R}^m$ .

### Observation A.4.13

The easiest way to show that the linear map with standard matrix  $A$  is bijective is to show that  $\text{RREF}(A)$  is the identity matrix.

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**Activity A.4.14** (*~5 min*)

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by the standard matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ 6 & 2 & 1 \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is bijective.

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**Activity A.4.15** (*~5 min*)Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be given by

$$T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2x + y - z \\ 4x + y + z \\ 6x + 2y \end{bmatrix}.$$

Which of the following must be true?

- (a)  $T$  is neither injective nor surjective
- (b)  $T$  is injective but not surjective
- (c)  $T$  is surjective but not injective
- (d)  $T$  is bijective.