

## Section V.1

**Remark V.1.1** Last time, we defined a **vector space**  $V$  to be any collection of mathematical objects with associated addition and scalar multiplication operations that satisfy the following eight properties for all  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in  $V$ , and all scalars (i.e. real numbers)  $a, b$ .

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| <ul style="list-style-type: none"> <li>• <b>Addition is associative.</b><br/><math>\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.</math></li> <li>• <b>Addition is commutative.</b><br/><math>\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.</math></li> <li>• <b>Additive identity exists.</b><br/>There exists some <math>\mathbf{z}</math> where <math>\mathbf{v} + \mathbf{z} = \mathbf{v}.</math></li> <li>• <b>Additive inverses exist.</b><br/>There exists some <math>-\mathbf{v}</math> where <math>\mathbf{v} + (-\mathbf{v}) = \mathbf{z}.</math></li> </ul> | <ul style="list-style-type: none"> <li>• <b>Scalar multiplication is associative.</b><br/><math>a(b\mathbf{v}) = (ab)\mathbf{v}.</math></li> <li>• <b>1 is a scalar multiplicative identity.</b><br/><math>1\mathbf{v} = \mathbf{v}.</math></li> <li>• <b>Scalar multiplication distributes over vector addition.</b><br/><math>a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.</math></li> <li>• <b>Scalar multiplication distributes over scalar addition.</b><br/><math>(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}.</math></li> </ul> |
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**Remark V.1.2** The following sets are examples of vector spaces, with the usual/natural operations for addition and scalar multiplication.

- $\mathbb{R}^n$ : Euclidean vectors with  $n$  components.
- $\mathbb{R}^\infty$ : Sequences of real numbers  $(v_1, v_2, \dots)$ .
- $M_{m,n}$ : Matrices of real numbers with  $m$  rows and  $n$  columns.
- $\mathbb{C}$ : Complex numbers.
- $\mathcal{P}^n$ : Polynomials of degree  $n$  or less.
- $\mathcal{P}$ : Polynomials of any degree.
- $C(\mathbb{R})$ : Real-valued continuous functions.

**Activity V.1.3** ( $\sim 20$  min) Consider the set  $V = \{(x, y) \mid y = e^x\}$  with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \quad c \odot (x, y) = (cx, y^c)$$

*Part 1:* Show that  $V$  satisfies the vector distributive property

$$(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v})$$

by letting  $\mathbf{v} = (x, y)$  and showing both sides simplify to the same expression.

*Part 2:* Show that  $V$  contains an additive identity element by choosing  $\mathbf{z} = (?, ?)$  such that  $\mathbf{v} \oplus \mathbf{z} = (x, y) \oplus (?, ?) = \mathbf{v}$  for any  $\mathbf{v} = (x, y) \in V$ .

**Remark V.1.4** It turns out  $V = \{(x, y) \mid y = e^x\}$  with operations defined by

$$(x, y) \oplus (z, w) = (x + z, yw) \quad c \odot (x, y) = (cx, y^c)$$

satisfies all eight properties.

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| <ul style="list-style-type: none"> <li>• <b>Addition associativity.</b><br/><math>\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}.</math></li> <li>• <b>Addition commutativity.</b><br/><math>\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}.</math></li> <li>• <b>Addition identity.</b><br/>There exists some <math>\mathbf{z}</math> where <math>\mathbf{v} \oplus \mathbf{z} = \mathbf{v}.</math></li> <li>• <b>Addition inverse.</b><br/>There exists some <math>-\mathbf{v}</math> where <math>\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{z}.</math></li> </ul> | <ul style="list-style-type: none"> <li>• <b>Scalar multiplication associativity.</b><br/><math>a \odot (b \odot \mathbf{v}) = (ab) \odot \mathbf{v}.</math></li> <li>• <b>Scalar multiplication identity.</b><br/><math>1 \odot \mathbf{v} = \mathbf{v}.</math></li> <li>• <b>Scalar distribution.</b><br/><math>a \odot (\mathbf{u} \oplus \mathbf{v}) = (a \odot \mathbf{u}) \oplus (a \odot \mathbf{v}).</math></li> <li>• <b>Vector distribution.</b><br/><math>(a + b) \odot \mathbf{v} = (a \odot \mathbf{v}) \oplus (b \odot \mathbf{v}).</math></li> </ul> |
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Thus,  $V$  is a vector space.

**Activity V.1.5** ( $\sim 15$  min) Let  $V = \{(x, y) \mid x, y \in \mathbb{R}\}$  have operations defined by

$$(x, y) \oplus (z, w) = (x + y + z + w, x^2 + z^2) \quad c \odot (x, y) = (x^c, y + c - 1).$$

*Part 1:* Show that the scalar multiplication identity holds by simplifying  $1 \odot (x, y)$  to  $(x, y)$ .

*Part 2:* Show that the addition identity property fails by showing that  $(0, -1) \oplus \mathbf{z} \neq (0, -1)$  no matter how  $\mathbf{z} = (z_1, z_2)$  is chosen.

*Part 3:* Can  $V$  be a vector space?

**Definition V.1.6** A **linear combination** of a set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$  is given by  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$  for any choice of scalar multiples  $c_1, c_2, \dots, c_m$ .

For example, we can say  $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$  is a linear combination of the vectors  $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  since

$$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

**Definition V.1.7** The **span** of a set of vectors is the collection of all linear combinations of that set:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m \mid c_i \in \mathbb{R}\}.$$

For example:

$$\text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\} = \left\{a \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R}\right\}$$

**Activity V.1.8** ( $\sim 10$  min) Consider  $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ .

*Part 1:* Sketch  $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , and  $-2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  in the  $xy$  plane.

*Part 2:* Sketch a representation of all the vectors belonging to  $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \left\{a \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid a \in \mathbb{R}\right\}$  in the  $xy$  plane.

**Activity V.1.9** ( $\sim 10$  min) Consider  $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ .

*Part 1:* Sketch the following linear combinations in the  $xy$  plane.

$$\begin{array}{ccc} 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ -2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & -1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} & \end{array}$$

*Part 2:* Sketch a representation of all the vectors belonging to  $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$  in the  $xy$  plane.

**Activity V.1.10** ( $\sim 5$  min) Sketch a representation of all the vectors belonging to  $\text{span}\left\{\begin{bmatrix} 6 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right\}$  in the  $xy$  plane.