Интегралы и дифференциальные уравнения

Лисид Лаконский

November 2023

Содержание

1 Практическое занятие — 02.11.2023

1.1 Дифференциальные уравнения

1.1.1 С разделяющимися переменными

1. $y' = f(x, y)$
   $\frac{\mathrm{d}y}{\mathrm{d}x} = f_1(x) * f_2(y)$
   $\frac{\mathrm{d}y}{f_2(y)} = f_1(x)\,\mathrm{d}x$
   $\int \frac{\mathrm{d}y}{f_2(y)} = \int f_1(x)\,\mathrm{d}x$
   $\Phi_2(y) = \Phi_1(x) + C$

2. $p(x, y)\,\mathrm{d}x + q(x, y)\,\mathrm{d}y = 0$
   $p_1(x)p_2(y)\,\mathrm{d}x + q_1(x)q_2(y)\,\mathrm{d}y = 0$
   $\int \frac{p_1(x)}{q_1(x)}\,\mathrm{d}x = -\int \frac{q_2(y)}{p_2(y)}\,\mathrm{d}y$
   $F_1(x) = F_2(y) + C$

Пример №1 $y' = \frac{x}{y}$
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$
$y\,\mathrm{d}y = x\,\mathrm{d}x$
$\int y\,\mathrm{d}y = \int x\,\mathrm{d}x$
$\frac{y^2}{2} = \frac{x^2}{2} + \frac{C}{2}$
$y^2 = x^2 + C$

Пример №2 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}$
$\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{x}$
$\ln|y| = \ln|x| + C$
$y = Cx$

Пример №3 $(1+y^2)x\,\mathrm{d}x + (1+x^2)\,\mathrm{d}y = 0$
$\int \frac{x\,\mathrm{d}x}{1+x^2} = -\int \frac{\mathrm{d}y}{1+y^2}$
$\frac{1}{2}\ln|x^2+1| = C - \operatorname{arctg} y$

Пример №4 $y' = (4x+y+1)^2$, $y' = f(ax+by+c)$
$z = 4x + y + 1$
$y = z - 4x - 1$, $y' = z' - 4$
$z' - 4 = z^2$, $\frac{\mathrm{d}z}{\mathrm{d}x} = z^2 + 4$, $\int \frac{\mathrm{d}z}{z^2+2^2} = \int \mathrm{d}x$
$\frac{1}{2}\operatorname{arctg}\frac{z}{2} = x + C$
$\frac{1}{2}\operatorname{arctg}\frac{4x+y+1}{2} = x + C$

Пример №3910 $y' + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$
$y' = \sin\frac{x-y}{2} - \sin\frac{x+y}{2}$
$y' = 2\sin\frac{x-y-(x+y)}{2*2}\cos\frac{x-y+x+y}{2*2}$
$y' = 2\sin(\frac{-y}{4})\cos(\frac{x}{2})$
$y' = -2\sin\frac{y}{4}\cos\frac{x}{2}$
$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\sin\frac{y}{4}\cos\frac{x}{2}$
$\mathrm{d}y = -2\sin\frac{y}{4}\cos\frac{x}{2}\,\mathrm{d}x$
$\frac{\mathrm{d}y}{2\sin\frac{y}{2}} = -\cos\frac{x}{2}\,\mathrm{d}x$
$\frac{\mathrm{d}\frac{1}{2}y}{\sin\frac{y}{2}} = -\cos\frac{x}{2}\,\mathrm{d}x$
$t = \frac{1}{2}y, t' = \frac{1}{2}\frac{y^2}{2} = \frac{y^2}{4}\,\mathrm{d}t$
$\int \frac{\mathrm{d}t}{\sin t} = -2\int \cos\frac{x}{2}\,\mathrm{d}\frac{x}{2}$
$\ln\operatorname{tg}\frac{t}{2} = -2\sin\frac{x}{2} + C$
$\ln\operatorname{tg}\frac{y}{4} = -2\sin\frac{x}{2} + C$

Пример №3901 $(xy^2+x)\,\mathrm{d}x + (y - x^2y)\,\mathrm{d}y = 0$
$(y - x^2y)\,\mathrm{d}y = (-xy^2 - x)\,\mathrm{d}x$
$(1-x)y\,\mathrm{d}y = x(-y^2-1)\,\mathrm{d}x$
$\int \frac{y\,\mathrm{d}y}{y^2+1} = -\int \frac{x\,\mathrm{d}x}{x^2-1}$
$\ln|y^2+1| = \ln|x^2-1| + C$

Пример №3902 $xyy' = 1 - x^2$
$\frac{xy\,\mathrm{d}y}{\mathrm{d}x} = 1 - x^2$
$xy\,\mathrm{d}y = (1-x^2)\,\mathrm{d}x$
$y\,\mathrm{d}y = \frac{1-x^2}{x}\,\mathrm{d}x$
$\int y\,\mathrm{d}y = \int \frac{1-x^2}{x}\,\mathrm{d}x$
$\frac{y^2}{2} = \int(\frac{1}{x} - x)\,\mathrm{d}x$
$\frac{y^2}{2} = \ln|x| - \frac{x^2}{2} + C$
$y^2 = 2\ln x - x^2 + C$

Пример №3903 $yy' = \frac{1-2x}{y}$
$\frac{y\,\mathrm{d}y}{\mathrm{d}x} = \frac{1-2x}{y}$
$y\,\mathrm{d}y = \frac{1-2x}{y}\,\mathrm{d}x$
$y^2\,\mathrm{d}y = (1-2x)\,\mathrm{d}x$
$\int y^2\,\mathrm{d}y = \int(1-2x)\,\mathrm{d}x$
$\frac{y^3}{3} = x - \frac{2x^2}{2} + C$

Пример №3907 $\sqrt{1-y^2}\,\mathrm{d}x + y\sqrt{1-x^2}\,\mathrm{d}y = 0$
$\sqrt{1-y^2}\,\mathrm{d}x = -y\sqrt{1-x^2}\,\mathrm{d}y$
$\frac{\sqrt{1-y^2}}{\mathrm{d}y} = -\frac{y\sqrt{1-x^2}}{\mathrm{d}x}$
$\frac{\sqrt{1-y^2}}{y\,\mathrm{d}y} = -\frac{\sqrt{1-x^2}}{\mathrm{d}x}$
$\frac{y\,\mathrm{d}y}{\sqrt{1-y^2}} = -\frac{\mathrm{d}x}{\sqrt{1-x^2}}$
Пусть $t = 1 - y^2$, $t' = -2y\,\mathrm{d}y$
$-\frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} = -\arcsin x + C$
$\frac{1}{2}\frac{\sqrt{t}}{1/2} = \arcsin x + C$
$\sqrt{1-y^2} = \arcsin x + C$

Пример №3913 $y' \sin x = y \ln y;\ y\Big|_{x=\frac{\pi}{2}} = e$

$\frac{dy}{dx}\sin x = y\ln y$
$\frac{\sin x}{dx} = \frac{y\ln y}{dy}$
$\frac{dx}{\sin x} = \frac{dy}{y\ln y}$
$\int\frac{dx}{\sin x} = \int\frac{dy}{y\ln y}$
$\int\frac{d(\ln y)}{\ln y} = \int\frac{dx}{\sin x}$
$\ln|\ln|y|| = \ln|\operatorname{tg}\frac{x}{2}| + \ln C$
$\ln|y| = c * \operatorname{tg}\frac{x}{2}$
$1 = c * 1 \implies c = 1$
$\ln y = \operatorname{tg}\frac{x}{2}$

Пример №3915 $\sin y\cos x\,dy = \cos y\sin x\,dx;\ y\Big|_{x=0} = \frac{\pi}{4}$

$\frac{\sin y}{\cos y}\cos x\,dy = \sin x\,dx$
$\frac{\sin y}{\cos y}\,dy = \frac{\sin x}{\cos x}\,dx$
$\int\frac{\sin y}{\cos y}\,dy = \int\frac{\sin x}{\cos x}\,dx$
$-\int\frac{d(\cos y)}{\cos y} = -\int\frac{d(\cos x)}{\cos x}$
$\ln|\cos y| = \ln|\cos x| + \ln C$
$\cos y = c\cos x$
$\cos\frac{\pi}{4} = c\cos 0$
$\cos\frac{\pi}{4} = c$
$c = \frac{\sqrt{2}}{2}$
$\cos y = \frac{\sqrt{2}\cos x}{2}$

1.1.2 С однородными переменными

$f(x,y),\ f(\lambda x,\lambda y) = \lambda^m f(x,y)$
$f(\lambda x,\lambda y) = f(x,y)$
$y' = f(\frac{y}{x})$
$t = \frac{y}{x}$
$y = t * x$
$y'_x = t' * x + t * 1$

Пример №1 $y' = \frac{x}{y} + \frac{y}{x}$

$t' * x + t = \frac{1}{t} + t$
$\frac{dt}{dx}x = \frac{1}{t}$
$\int t\,dt = \int\frac{dx}{x}$
$\frac{t^2}{2} = \ln|x| + \ln|C|$
$t^2 = \ln x^2 + \ln K$
$\frac{y^2}{x62} = \ln Kx^2$
$y^2 = x^2\ln Kx^2$

Пример №2 $\frac{dx}{dy} = \frac{x}{y} + \frac{x^2}{y^2}$

$\frac{x}{y} = t$
$x = yt$
$x'_y = t + y\frac{dt}{dy}$

Пример №3932 $y' = 3x - 2y + 5$

$z = 3x - 2y + 5$
$y = \frac{3x}{2} + \frac{5}{2} - \frac{z}{2}$
$y' = \frac{1}{2}(3 - z')$
$\frac{1}{2}(3 - z') = z$
$3 - z' = 2z$
$\frac{dz}{dx} = 3 - 2z$
$\frac{dz}{3-2z} = dx$
$\int\frac{dz}{3-2z} = \int dx$
$x = -\frac{1}{2}\ln|3 - 2z| + C$
$x = -\frac{1}{2}\ln|3 - 6x + 4y - 10| + C$

Пример №3934 $y' = \frac{y^2}{x^2} - 2$

$t = \frac{y}{x},\ t^2 = \frac{y^2}{x^2}$
$y' = t^2 - 2$
$t' * x + t = t^2 - 2$
$\frac{dt}{dx} * x = t^2 - t - 2$
$\frac{dt}{t^2-t-2} = \frac{dx}{x}$
$\int\frac{dt}{t^2-t-2} = \int\frac{dx}{x}$
$t^2 - t - 2 = 0$
$t_1 = \frac{1+3}{2} = 2, t_2 = -1$
$\int\frac{dt}{(t-2)(t+1)} = \int\frac{dx}{x}$
$\frac{1}{(t-2)(t+1)} = \frac{a}{t-2} + \frac{b}{t+1}$
$\frac{1}{(t-2)(t+1)} = \frac{at+a+bt-2b}{(t-2)(t+1)}$
$a = \frac{1}{3}, b = -\frac{1}{3}$
$\int\frac{\frac{1}{3}\,dt}{t-2} - \int\frac{\frac{1}{3}}{t+1} = \int\frac{dx}{x}$
$\ln\sqrt[3]{\frac{t-2}{t+1}} = \ln Cx$

Пример №3935 $y' = \frac{x+y}{x-y}$

$y' = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$

$t'x + t = \frac{1+t}{1-t}$

$\frac{x\,\mathrm{d}t}{\mathrm{d}x} = \frac{1+t}{1-t} - t$

$\frac{x\,\mathrm{d}t}{\mathrm{d}x} = \frac{1+t^2}{1-t}$

$\frac{x}{\mathrm{d}x} = \frac{1+t^2}{(1-t)\,\mathrm{d}t}$

$\frac{\mathrm{d}x}{x} = \frac{(1-t)\,\mathrm{d}t}{1+t^2}$

$\int \frac{(1-t)\,\mathrm{d}t}{1+t^2} = \int \frac{1\,\mathrm{d}t}{1+t^2} - \int \frac{t\,\mathrm{d}t}{1+t^2} = \arctan t - \frac{\ln|1+t^2|}{2} = \arctan \frac{y}{x} - \frac{\ln|1+\frac{y^2}{x^2}|}{2}$

$\arctan \frac{y}{x} - \frac{\ln|1+\frac{y^2}{x^2}|}{2} = \ln x + \ln C$

Пример №3941 $y' = e^{\frac{y}{x}} + \frac{y}{x}$

$t'x = e^t$

$\frac{x\,\mathrm{d}t}{\mathrm{d}x} = e^t$

$\int \frac{\mathrm{d}x}{x} = \int \frac{\mathrm{d}t}{e^t}$

$-\frac{1}{e^{\frac{y}{x}}} = \ln|x| + C$

Пример №3942 $xy' = y \ln \frac{y}{x}$

$y' = \frac{y}{x} \ln \frac{y}{x}$

$t'x + t = t \ln t$

$t'x = t \ln t - t$

$\frac{x\,\mathrm{d}t}{\mathrm{d}x} = t \ln t - t$

$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}t}{t(\ln t - 1)}$

$\int \frac{d(\ln t - 1)}{\ln t - 1} = \int \frac{\mathrm{d}x}{x}$

$\ln \ln(t-1) = \ln Cx$

$\ln t - 1 = Cx$

$\ln \frac{y}{x} = Cx + 1$

1.1.3 Приводимые к однородным

$y' = f(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2})$

1. Если $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, то вводим z, например, $z = a_1x + b_1y + c_1$
2. Если $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, то

   $x = u + \alpha$, $y = v + \beta$

   $\begin{cases} a_1\alpha + b_1\beta + c_1 = 0 \\ a_2\alpha + b_2\beta + c_2 = 0 \end{cases}$

   Потом подставляем одно в одно место, другое в другое. В результате не должно остаться свободных членов.

Пример №1 $y' = \frac{2x+y-1}{6x+3y+2}$

$z = 2x + y - 1$

$3z = 6x + 3y - 3$

$6x + 3y + 2 = 3z + 5$

$y = z - 2x + 1$, $y' = z' - 2$

$z' - 2 = \frac{z}{3z+5}$

$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{z}{3z+5} + 2$

$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{7z+10}{3z+5}$

$\int \frac{3z+5}{7z+10}\,\mathrm{d}z = \int \mathrm{d}x$

$\frac{3}{7} \int \frac{z+\frac{5}{3}}{z+\frac{10}{7}}\,\mathrm{d}z = \frac{3}{7} \int \frac{(z+\frac{10}{7})+\frac{5}{21}}{(z+\frac{10}{7})}\,\mathrm{d}z = \frac{3}{7}[\int \mathrm{d}z + \frac{5}{21} \int \frac{\mathrm{d}(z+\frac{10}{7})}{\frac{10}{7}}]$

$\frac{3}{7}[z + \frac{5}{21} \ln|z + \frac{10}{7}|] = x + C$

Пример №2 $(x+y-1)^2\,\mathrm{d}y = 2(y+2)^2\,\mathrm{d}x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2(\frac{y+2}{x+y-1})^2$

$x = u + \alpha$, $y = v + \beta$

$\begin{cases} \beta + 2 = 0 \\ \alpha + \beta - 1 = 0 \end{cases}$, $\beta = -2$, $\alpha = 3$

$x = u + 3$, $y = v - 2$, $\mathrm{d}x = \mathrm{d}u$, $\mathrm{d}y = \mathrm{d}v$

$\frac{\mathrm{d}v}{\mathrm{d}u} = 2(\frac{v-2+2}{u+3+v-2-1})^2$, $\frac{\mathrm{d}v}{\mathrm{d}u} = 2(\frac{v}{u+v})^2$

$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{1}{2}(\frac{u+v}{v})^2$, $\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{1}{2}(\frac{u}{v}+1)^2$

$t = \frac{u}{v}$, $u = v * t$, $\frac{\mathrm{d}u}{\mathrm{d}v} = t + v\frac{\mathrm{d}t}{\mathrm{d}v}$, $t + v\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{1}{2}(t+1)^2 = \frac{t^2}{2} + t + \frac{1}{2}$, $\frac{t^2+1}{2}$, $\int \frac{\mathrm{d}t}{t^2+1} = \frac{1}{2} \int \frac{\mathrm{d}v}{v}$

$\operatorname{arctg} t = \frac{1}{2} \ln|v| + \ln C = \ln c\sqrt{v} \iff \operatorname{arctg} \frac{u}{v} = \ln C\sqrt{v} \iff \operatorname{arctg} \frac{x-3}{y+2} = \ln C\sqrt{y+2}$