

Интегралы и дифференциальные уравнения

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1 Практическое занятие — 02.11.2023

1.1 Дифференциальные уравнения

1.1.1 С разделяющимися переменными

1. $y' = f(x, y)$

$$\frac{dy}{dx} = f_1(x) * f_2(y)$$

$$\frac{dy}{f_2(y)} = f_1(x) dx$$

$$\int \frac{dy}{f_2(y)} = \int f_1(x) dx$$

$$\Phi_2(y) = \Phi_1(x) + C$$

2. $p(x, y) dx + q(x, y) dy = 0$

$$p_1(x)p_2(y) dx + q_1(x)q_2(y) dy = 0$$

$$\int \frac{p_1(x)}{q_1(x)} dx = - \int \frac{q_2(y)}{p_2(y)} dy$$

$$F_1(x) = F_2(y) + C$$

Пример №1 $y' = \frac{x}{y}$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y dy = x dx$$

$$\int y dy = \int x dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{C}{2}$$

$$y^2 = x^2 + C$$

Пример №2 $\frac{dy}{dx} = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln |y| = \ln |x| + C$$

$$y = Cx$$

Пример №3 $(1 + y^2)x dx + (1 + x^2) dy = 0$

$$\int \frac{x dx}{1+x^2} = - \int \frac{dy}{1+y^2}$$

$$\frac{1}{2} \ln |x^2 + 1| = C - \arctg y$$

Пример №4 $y' = (4x + y + 1)^2$, $y' = f(ax + by + c)$

$$z = 4x + y + 1$$

$$y = z - 4x - 1, y' = z' - 4$$

$$z' - 4 = z^2, \frac{dz}{dx} = z^2 + 4, \int \frac{dz}{z^2+2^2} = \int dx$$

$$\frac{1}{2} \arctg \frac{z}{2} = x + C$$

$$\frac{1}{2} \arctg \frac{4x+y+1}{2} = x + C$$

Пример №3910 $y' + \sin \frac{x+y}{2} = \sin \frac{x-y}{2}$

$$y' = \sin \frac{x-y}{2} - \sin \frac{x+y}{2}$$

$$y' = 2 \sin \frac{x-y-(x+y)}{2*2} \cos \frac{x-y+x+y}{2*2}$$

$$y' = 2 \sin \left(\frac{-y}{2} \right) \cos \left(\frac{x}{2} \right)$$

$$y' = -2 \sin \frac{y}{2} \cos \frac{x}{2}$$

$$\frac{dy}{dx} = -2 \sin \frac{y}{2} \cos \frac{x}{2}$$

$$dy = -2 \sin \frac{y}{2} \cos \frac{x}{2} dx$$

$$\frac{dy}{2 \sin \frac{y}{2}} = - \cos \frac{x}{2} dx$$

$$\frac{d \frac{1}{2} y}{\sin \frac{y}{2}} = - \cos \frac{x}{2} dx$$

$$t = \frac{1}{2} y, t' = \frac{1}{2} \frac{y'}{2} = \frac{y'}{4} dt$$

$$\int \frac{dt}{\sin t} = -2 \int \cos \frac{x}{2} d \frac{x}{2}$$

$$\ln \operatorname{tg} \frac{t}{2} = -2 \sin \frac{x}{2} + C$$

$$\ln \operatorname{tg} \frac{y}{4} = -2 \sin \frac{x}{2} + C$$

Пример №3901 $(xy^2 + x) dx + (y - x^2y) dy = 0$

$$(y - x^2y) dy = (-xy^2 - x) dx$$

$$(1 - x)y dy = x(-y^2 - 1) dx$$

$$\int \frac{y dy}{y^2+1} = - \int \frac{x dx}{x^2-1}$$

$$\ln |y^2 + 1| = \ln |x^2 - 1| + C$$

Пример №3902 $xyy' = 1 - x^2$

$$\frac{xy dy}{dx} = 1 - x^2$$

$$xy dy = (1 - x^2) dx$$

$$y dy = \frac{1-x^2}{x} dx$$

$$\int y dy = \int \frac{1-x^2}{x} dx$$

$$\frac{y^2}{2} = \int \left(\frac{1}{x} - x \right) dx$$

$$\frac{y^2}{2} = \ln |x| - \frac{x^2}{2} + C$$

$$y^2 = 2 \ln x - x^2 + C$$

Пример №3903 $yy' = \frac{1-2x}{y}$

$$\frac{y dy}{dx} = \frac{1-2x}{y}$$

$$y dy = \frac{1-2x}{y} dx$$

$$y^2 dy = (1 - 2x) dx$$

$$\int y^2 dy = \int (1 - 2x) dx$$

$$\frac{y^3}{3} = x - \frac{2x^2}{2} + C$$

Пример №3907 $\sqrt{1-y^2} dx + y\sqrt{1-x^2} dy = 0$

$$\sqrt{1-y^2} dx = -y\sqrt{1-x^2} dy$$

$$\frac{\sqrt{1-y^2}}{dy} = - \frac{y\sqrt{1-x^2}}{dx}$$

$$\frac{\sqrt{1-y^2}}{y dy} = - \frac{\sqrt{1-x^2}}{dx}$$

$$\frac{y dy}{\sqrt{1-y^2}} = - \frac{dx}{\sqrt{1-x^2}}$$

$$\text{Пусть } t = 1 - y^2, t' = -2y dy$$

$$-\frac{1}{2} \int \frac{dt}{\sqrt{t}} = - \arcsin x + C$$

$$\frac{1}{2} \frac{\sqrt{t}}{1/2} = \arcsin x + C$$

$$\sqrt{1-y^2} = \arcsin x + C$$

Пример №3913 $y' \sin x = y \ln y; y \Big|_{x=\frac{\pi}{2}} = e$

$$\begin{aligned}\frac{dy}{dx} \sin x &= y \ln y \\ \frac{\sin x}{dx} &= \frac{y \ln y}{dy} \\ \frac{dx}{\sin x} &= \frac{dy}{y \ln y} \\ \int \frac{dx}{\sin x} &= \int \frac{dy}{y \ln y} \\ \int \frac{d(\ln y)}{\ln y} &= \int \frac{dx}{\sin x} \\ \ln |\ln |y|| &= \ln |\operatorname{tg} \frac{x}{2}| + \ln C \\ \ln |y| &= c * \operatorname{tg} \frac{x}{2} \\ 1 = c * 1 &\implies c = 1 \\ \ln y &= \operatorname{tg} \frac{x}{2}\end{aligned}$$

Пример №3915 $\sin y \cos x \, dy = \cos y \sin x \, dx; y \Big|_{x=0} = \frac{\pi}{4}$

$$\begin{aligned}\frac{\sin y}{\cos y} \cos x \, dy &= \sin x \, dx \\ \frac{\sin y}{\cos y} \, dy &= \frac{\sin x}{\cos x} \, dx \\ \int \frac{\sin y}{\cos y} \, dy &= \int \frac{\sin x}{\cos x} \, dx \\ - \int \frac{d(\cos y)}{\cos y} &= - \int \frac{d(\cos x)}{\cos x} \\ \ln |\cos y| &= \ln |\cos x| + \ln C \\ \cos y &= c \cos x \\ \cos \frac{\pi}{4} &= c \cos 0 \\ \cos \frac{\pi}{4} &= c \\ c &= \frac{\sqrt{2}}{2} \\ \cos y &= \frac{\sqrt{2} \cos x}{2}\end{aligned}$$

1.1.2 С однородными переменными

$$\begin{aligned}f(x, y), f(\lambda x, \lambda y) &= \lambda^m f(x, y) \\ f(\lambda x, \lambda y) &= f(x, y) \\ y' &= f\left(\frac{y}{x}\right) \\ t &= \frac{y}{x} \\ y &= t * x \\ y'_x &= t' * x + t * 1\end{aligned}$$

Пример №1 $y' = \frac{x}{y} + \frac{y}{x}$

$$\begin{aligned}t' * x + t &= \frac{1}{t} + t \\ \frac{dt}{dx} x &= \frac{1}{t} + t \\ \int t \, dt &= \int \frac{dx}{x} \\ \frac{t^2}{2} &= \ln |x| + \ln |C| \\ t^2 &= \ln x^2 + \ln K \\ \frac{y^2}{x^2} &= \ln K x^2 \\ y^2 &= x^2 \ln K x^2\end{aligned}$$

Пример №2 $\frac{dx}{dy} = \frac{x}{y} + \frac{x^2}{y^2}$

$$\begin{aligned}\frac{x}{y} &= t \\ x &= yt \\ x'_y &= t + y \frac{dt}{dy}\end{aligned}$$

Пример №3932 $y' = 3x - 2y + 5$

$$\begin{aligned}z &= 3x - 2y + 5 \\ y &= \frac{3x}{2} + \frac{5}{2} - \frac{z}{2} \\ y' &= \frac{1}{2}(3 - z') \\ \frac{1}{2}(3 - z') &= z \\ 3 - z' &= 2z \\ \frac{dz}{dx} &= 3 - 2z \\ \frac{dz}{3-2z} &= dx \\ \int \frac{dz}{3-2z} &= \int dx \\ x &= -\frac{1}{2} \ln |3 - 2z| + C \\ x &= -\frac{1}{2} \ln |3 - 6x + 4y - 10| + C\end{aligned}$$

Пример №3934 $y' = \frac{y^2}{x^2} - 2$

$$\begin{aligned}t &= \frac{y}{x}, t^2 = \frac{y^2}{x^2} \\ y' &= t^2 - 2 \\ t' * x + t &= t^2 - 2 \\ \frac{dt}{dx} * x &= t^2 - t - 2 \\ \frac{dt}{t^2-t-2} &= \frac{dx}{x} \\ \int \frac{dt}{t^2-t-2} &= \int \frac{dx}{x} \\ t^2 - t - 2 &= 0 \\ t_1 &= \frac{1+3}{2} = 2, t_2 = -1 \\ \int \frac{dt}{(t-2)(t+1)} &= \int \frac{dx}{x} \\ \frac{1}{(t-2)(t+1)} &= \frac{a}{t-2} + \frac{b}{t+1} \\ \frac{1}{(t-2)(t+1)} &= \frac{at+a+bt-2b}{(t-2)(t+1)} \\ a &= \frac{1}{3}, b = -\frac{1}{3} \\ \int \frac{\frac{1}{3} dt}{t-2} - \int \frac{\frac{1}{3} dt}{t+1} &= \int \frac{dx}{x} \\ \ln \sqrt[3]{\frac{t-2}{t+1}} &= \ln Cx\end{aligned}$$

Пример №3935 $y' = \frac{x+y}{x-y}$

$$\begin{aligned} y' &= \frac{1+\frac{y}{x}}{1-\frac{y}{x}} \\ t'x + t &= \frac{1+t}{1-t} \\ \frac{x dt}{dx} &= \frac{1+t}{1-t} - t \\ \frac{x dt}{dx} &= \frac{1+t^2}{1-t} \\ \frac{x}{dx} &= \frac{1+t^2}{(1-t) dt} \\ \frac{dx}{x} &= \frac{(1-t) dt}{1+t^2} \\ \int \frac{(1-t) dt}{1+t^2} &= \int \frac{1 dt}{1+t^2} - \int \frac{t dt}{1+t^2} = \arctan t - \frac{\ln|1+t^2|}{2} = \arctan \frac{y}{x} - \frac{\ln|1+\frac{y^2}{x^2}|}{2} \\ \arctan \frac{y}{x} - \frac{\ln|1+\frac{y^2}{x^2}|}{2} &= \ln x + \ln C \end{aligned}$$

Пример №3941 $y' = e^{\frac{y}{x}} + \frac{y}{x}$

$$\begin{aligned} t'x &= e^t \\ \frac{x dt}{dx} &= e^t \\ \int \frac{dx}{x} &= \int \frac{dt}{e^t} \\ -\frac{1}{e^t} &= \ln|x| + C \end{aligned}$$

Пример №3942 $xy' = y \ln \frac{y}{x}$

$$\begin{aligned} y' &= \frac{y}{x} \ln \frac{y}{x} \\ t'x + t &= t \ln t \\ t'x &= t \ln t - t \\ \frac{x dt}{dx} &= t \ln t - t \\ \frac{dx}{x} &= \frac{dt}{t(\ln t - 1)} \\ \int \frac{d(\ln t - 1)}{\ln t - 1} &= \int \frac{dx}{x} \\ \ln \ln(t - 1) &= \ln Cx \\ \ln t - 1 &= Cx \\ \ln \frac{y}{x} &= Cx + 1 \end{aligned}$$

1.1.3 Приводимые к однородным

$$y' = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$$

1. Если $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, то вводим z , например, $z = a_1x + b_1y + c_1$
2. Если $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, то

$$\begin{aligned} x &= u + \alpha, y = v + \beta \\ \begin{cases} a_1\alpha + b_1\beta + c_1 = 0 \\ a_2\alpha + b_2\beta + c_2 = 0 \end{cases} \end{aligned}$$

Потом подставляем одно в одно место, другое в другое. В результате не должно остаться свободных членов.

Пример №1 $y' = \frac{2x+y-1}{6x+3y+2}$

$$\begin{aligned} z &= 2x + y - 1 \\ 3z &= 6x + 3y - 3 \\ 6x + 3y + 2 &= 3z + 5 \\ y &= z - 2x + 1, y' = z' - 2 \\ z' - 2 &= \frac{z}{3z+5} \\ \frac{dz}{dx} &= \frac{z}{3z+5} + 2 \\ \frac{dz}{dx} &= \frac{7z+10}{3z+5} \\ \int \frac{7z+10}{3z+5} dz &= \int dx \\ \frac{3}{7} \int \frac{z+\frac{5}{3}}{z+\frac{10}{7}} dz &= \frac{3}{7} \int \frac{(z+\frac{10}{7})+\frac{5}{21}}{(z+\frac{10}{7})} dz = \frac{3}{7} [\int dz + \frac{5}{21} \int \frac{d(z+\frac{10}{7})}{z+\frac{10}{7}}] \\ \frac{3}{7} [z + \frac{5}{21} \ln|z + \frac{10}{7}|] &= x + C \end{aligned}$$

Пример №2 $(x+y-1)^2 dy = 2(y+2)^2 dx$

$$\begin{aligned} \frac{dy}{dx} &= 2\left(\frac{y+2}{x+y-1}\right)^2 \\ x &= u + \alpha, y = v + \beta \\ \begin{cases} \beta + 2 = 0 \\ \alpha + \beta - 1 = 0 \end{cases}, \beta &= -2, \alpha = 3 \\ x &= u + 3, y = v - 2, dx = du, dy = dv \\ \frac{dv}{du} &= 2\left(\frac{v-2+2}{u+3+v-2-1}\right)^2, \frac{dv}{du} = 2\left(\frac{v}{u+v}\right)^2 \\ \frac{du}{dv} &= \frac{1}{2}\left(\frac{u+v}{v}\right)^2, \frac{du}{dv} = \frac{1}{2}\left(\frac{u}{v} + 1\right)^2 \\ t &= \frac{u}{v}, u = v * t, \frac{du}{dv} = t + v \frac{dt}{dv}, t + v \frac{dt}{dv} = \frac{1}{2}(t+1)^2 = \frac{t^2}{2} + t + \frac{1}{2}, \\ \frac{t^2+1}{2}, \int \frac{dt}{t^2+1} &= \frac{1}{2} \int \frac{dv}{v} \\ \operatorname{arctg} t &= \frac{1}{2} \ln|v| + \ln C = \ln C\sqrt{v} \iff \operatorname{arctg} \frac{u}{v} = \ln C\sqrt{v} \iff \\ \operatorname{arctg} \frac{x-3}{y+2} &= \ln C\sqrt{y+2} \end{aligned}$$