Computational Tools and Techniques for Numerical Macro-Financial Modeling

Victor V. Zhorin

MFM/BFI

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Numerical Building Blocks

Spectral approximation technology:

numerical computation in Chebyshev functions

- piece-wise smooth functions
- breakpoints detection
- rootfinding
- functions with singularities
- fast adaptive quadratures
- continuous QR, SVD, least-squares
- linear operators
- solution of linear and non-linear ODE
- Fréchet derivatives via automatic differentiation
- PDEs in one space variable plus time

Stochastic processes:

(quazi) Monte-Carlo simulations, Polynomial Expansion (gPC), finite-differences (FD)

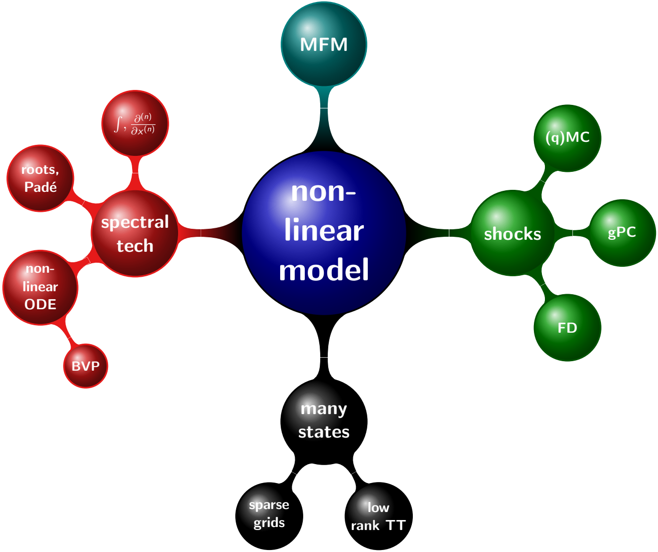
- non-linear IRF
- Borovička-Hansen-Sc[heinkman shock-exposure and shock-price elasticities
- Malliavin derivatives

Many states:

Dimensionality Curse Cure

- low-rank tensor decomposition
- sparse Smolyak grids

Numerical Building Blocks (cont.)



Horse race: methods and models

models

He-Krishnamurthy, "Intermediary Asset Pricing"

Klimenko-Pfeil-Rochet-DeNicolo, "Aggregate Bank Capital and Credit Dynamics"

Brunnermeier-Sannikov, "A Macroeconomic Model with a Financial Sector"

Basak-Cuoco, "An Equilibrium Model with Restricted Stock Market Participation"

Di Tella, "Uncertainty Shocks and Balance Sheet Recessions"

methods

spectral technology vs discrete grids

Monte-Carlo simulations (MC) vs Polynomial Expansion (gPC) vs finite-differences SPDE (FD)

Smolyak sparse grids vs tensor decomposition

criteria

elegance: clean primitives, libraries, ease of mathematical concepts expression in code

speed: feasibility, ready prototypes, LEGO blocks; precision: numerical tests of speed vs stability trade-offs

common metrics: shock exposure elasticity, asset pricing implications

shock-exposure and shock-price elasticity first kind

Multiplicative functional M_t:
1) consumption C_t for consumption shock-exposure
2) stochastic discount factor S_t for consumption shock-price elasticity

$$d \log M_t = \beta(X_t)dt + \alpha(X_t)dB_t$$

$$\epsilon(x,t) = \sigma(x)\frac{\partial}{\partial x}\log \mathbb{E}[M_t|X_0 = x] + \alpha(x)$$

Define

$$\phi(x,t) = \mathbb{E}\left[\frac{M_t}{M_0}|X_0 = x\right]$$

Then solve PDE

$$\frac{\partial \phi(x,t)}{\partial t} = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\phi(x,t) + (\mu(x) + \sigma(x)\alpha(x))\frac{\partial}{\partial x}\phi(x,t) + \left(\beta(x) + \frac{1}{2}|\alpha(x)^2|\right)\phi(x,t)$$

s.t. initial boundary condition $\phi(x,0) = 1$

shock-exposure and shock-price elasticity second kind

Multiplicative functional M_t:
1) consumption C_t for consumption shock-exposure
2) stochastic discount factor S_t for consumption shock-price elasticity

$$d \log M_t = \beta(X_t)dt + \alpha(X_t)dB_t$$

The elasticity of second kind is

$$\epsilon_2(X_t) = \frac{\mathbb{E}(\mathfrak{D}_t\ M_t)|\mathcal{F}_0}{\mathbb{E}M_t|\mathcal{F}_0}$$

Use specification for M_t to get

$$\epsilon_2(X_t) = \frac{\mathbb{E}(\ M_t\eta(X_t)\alpha(X_t))|\mathcal{F}_0}{\mathbb{E}M_t|\mathcal{F}_0}$$

Then solve PDE twice

$$\frac{\partial\phi(x,t)}{\partial t} = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\phi(x,t) + (\mu(x)+\sigma(x)\alpha(x))\frac{\partial}{\partial x}\phi(x,t) + \left(\beta(x) + \frac{1}{2}|\alpha(x)^2|\right)\phi(x,t)$$

s.t. initial boundary condition $\phi(x,0) = 1$ to get a solution $\phi_1(x,t)$ and initial boundary condition $\phi(x,0) = \alpha(x)$ to get a solution $\phi_2(x,t)$ Then, second type elasticity is

$$\epsilon_2(x,t) = \frac{\phi_2(x,t)}{\phi_1(x,t)}$$

Find $\frac{P_t}{D_t} = F(y)$, $\forall y \in \left[0, \frac{1+l}{\rho}\right]$, by solving

$$
\begin{aligned}
& F''(y)\,\theta_s(y)G(y)^2\,\frac{(\theta_b(y)\sigma)^2}{2}\,\frac{G(y)}{\theta_s(y)F(y)}\left(\frac{1+l+\rho y(\gamma-1)}{1+l-\rho y+\rho\gamma G(y)\,\theta_b(y)}\right) \\
= \quad & \rho+g(\gamma-1)-\frac{1}{F(y)}+\frac{\gamma(1-\gamma)\sigma^2}{2}\left(1+\frac{\rho G(y)\,\theta_b(y)}{1+l-\rho y}\right)\frac{y-G(y)\,\theta_b(y)}{\theta_s(y)F(y)}\left[\frac{1+l-\rho y-\rho G(y)\,\theta_b(y)}{1+l-\rho y+\rho\gamma G(y)\,\theta_b(y)}\right] \\
& -\left(\frac{(1+l-\rho y)(G(y)-1)}{\theta_s(y)F(y)}+\gamma\rho G(y)\right)\frac{\theta_s(y)+l+\theta_b(y)(g(\gamma-1)+\rho)-\rho y}{1+l-\rho y+\rho\gamma G(y)\,\theta_b(y)}.
\end{aligned}
$$

where

$$G(y) = \frac{1}{1-\theta_s(y)F'(y)},$$

and

	if $y \in (0, y^c)$	if $y \in \left(y^c, \frac{1+l}{\rho}\right)$
$\theta_s(y) =$	$\frac{(1-\lambda)y}{F(y)-\lambda y}$	$\frac{m}{1+m}$
$\theta_b(y) =$	$\lambda y\frac{F(y)-y}{F(y)-\lambda y}$	$y-\frac{m}{1+m}F(y)$

The endogenous threshold y^c is determined by $y^c = \frac{m}{1-\lambda+m}F(y^c)$. Boundary conditions on $y=0$ and $y=\frac{1+l}{r}$.

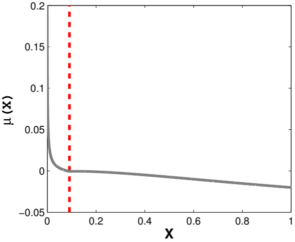
$$F(0) = \frac{1+F'(0)\,l}{\rho+g(\gamma-1)+\frac{\gamma(1-\gamma)\sigma^2}{2}-\frac{l\gamma\rho}{1+l}};\ F\left(\frac{1+l}{\rho}\right) = \frac{1+l}{\rho};\ F'\left(\frac{1+l}{\rho}\right) = 1.$$

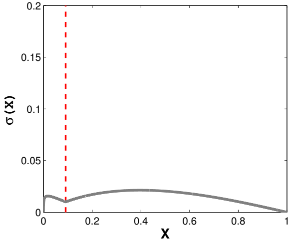
He-Krishnamurthy, BVP solution

$$dX_t = \mu_y(X_t)dt + \sigma_y(X_t)dB_t, X_t = \frac{w_t}{P_t};\ \sigma_y(X_t) = -\frac{\theta_b}{1-\theta_s F'(y)}\sigma; X_t = \frac{F(y)-y}{F(y)}$$

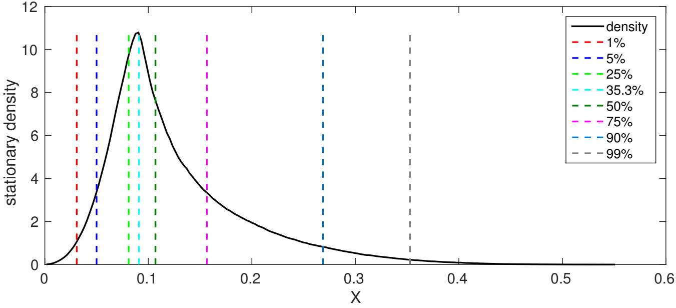
$$d\log C_t = \beta(X_t)dt + \alpha(X_t)dB_t; d\log S_t = -\gamma \log C_t$$

$$\beta(x) = \mu(x)\xi(x) + \frac{1}{2}\sigma(x)^2\frac{\partial\xi}{\partial x} + g_d - \frac{\sigma_d^2}{2}; \alpha(x) = \sigma(x)\xi(x) + \sigma_d; \xi(x) = -\rho\frac{p'(x)(1-x)-p(x)}{1+l-\rho(1-x)p(x)}$$





stationary density distribution, percentiles



*

Given Chebyshev interpolation nodes z_k, $k = 1, \ldots, m$ on $[-1, 1]$

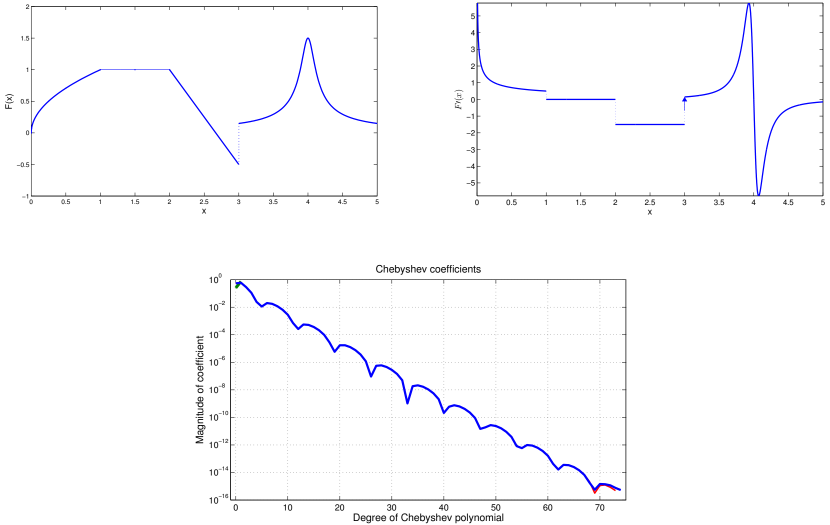
$$z_k = -cos\left(\frac{2k-1}{2m}\pi\right)$$

and Chebychev coefficients a_i, $i = 0, \ldots, n$ computed on Chebychev nodes we can approximate

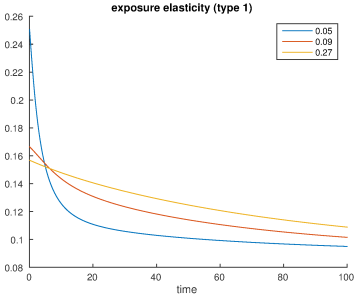
$$\mathbb{F}(x) \approx \sum_{i=0}^{n} a_i T_i(x)\,;\, a_i = \frac{2}{\pi}\int_{-1}^{1}\frac{\mathbb{F}(x)T_i(x)}{\sqrt{1-x^2}}dx$$

"Approximation Theory and Approximation Practice", by Lloyd N. Trefethen (chebfun.org)
"Chebyshev and Fourier Spectral Methods", by John P. Boyd
Chebyshev interpolation nodes and degree of polynomials have to be adaptive during continuous Newton updates.

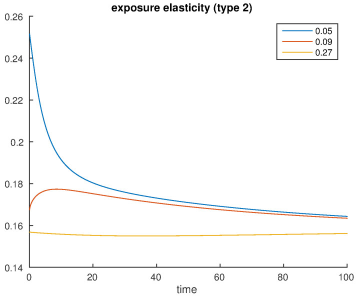
> (i) When in doubt, use Chebyshev polynomials unless the solution is spatially periodic, in which case an ordinary Fourier series is better.
>
> (ii) Unless youre sure another set of basis functions is better, use Chebyshev polynomials.
>
> (iii) Unless youre really, really sure that another set of basis functions is better, use Chebyshev polynomials


F(x)
x
F'(x)
x
Chebyshev coefficients
Magnitude of coefficient
Degree of Chebyshev polynomial

He-Krushnamurthy: shock exposure, specialist C

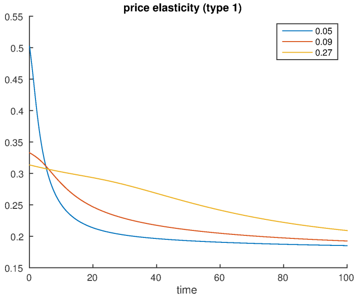


(a) type 1 elasticity

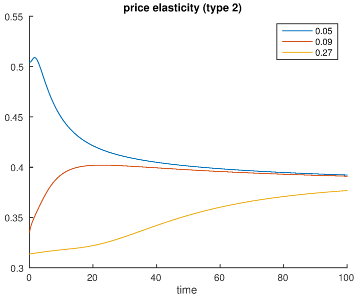


(b) type 2 elasticity

He-Krushnamurthy: price exposure, specialist C



(a) type 1 elasticity



(b) type 2 elasticity

Brunnermeier-Sannikov example

$$\frac{dX_t}{X_t} = \mu(X_t)dt + \sigma(X_t)dB_t - d\zeta_t, X_t = \frac{N_t}{q_t K_t};$$

$X(t)$: expert share of wealth
Consumption is aggregate output net of aggregate investment

$$C_t^a = [a\psi(X_t) + \underline{a}(1 - \psi(X_t))]K_t$$

$$d \log C_t = \beta(X_t)dt + \alpha(X_t)dB_t;$$

where

$$\alpha(X) = \sigma; \beta(X) = \Phi(X) - \delta - \sigma^2/2$$

For log-utility

$$S_t/S_0 = \mathbf{e}^{-\rho t} C_0/C_t$$

Brunnermeier-Sannikov: shock exposure, aggregate C

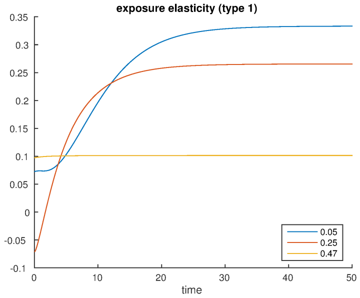
for logarithmic utility consumption of both households and experts are myopic, proportional to their wealth N_t

$$d\zeta_t = \rho dt; d\log C_t = d\log K_t$$

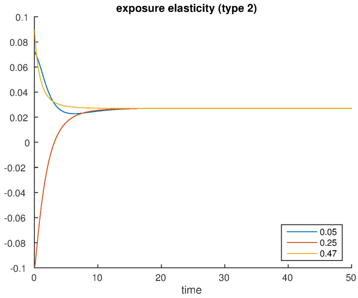
$$d\log K_t = \left(\Phi(i_t) - \delta\psi_t - \underline{\delta}(1-\psi_t) - \frac{1}{2}\sigma^2\right)dt + \sigma dB_t$$

for $\delta = \underline{\delta}$

$$\alpha(X) = \sigma; \beta(X) = \Phi(X) - \delta - \frac{1}{2}\sigma^2$$



(a) type 1 elasticity



(b) type 2 elasticity

Brunnermeier-Sannikov: price exposure, aggregate C

for logarithmic utility consumption of both households and experts are myopic, proportional to their wealth N_t

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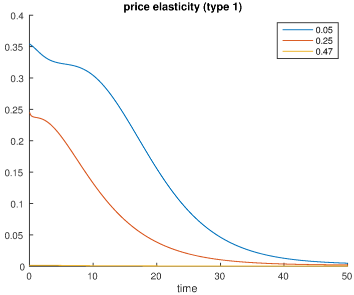
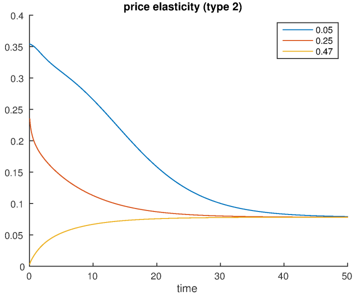


(a) type 1 elasticity



(b) type 2 elasticity

Model Settings, Klimenko, Pfeil, Rochet, DeNicolo (KPRD)

State - equity E
Multiplicative functional $M = R(E)$

$$dE_t = \mu(E_t)dt + \sigma(E_t)dB_t, p + r \leq R_t \leq R_{max},$$

where

$$\mu(E) = Er + L(R(E))(R(E) - r - p); \sigma(E) = L(R(E))\sigma_0.$$

$$\int_0^{E_{max}} \frac{R(s) - p - r}{\sigma_0^2 L(R(s))} ds = \ln(1+\gamma); u(E) = \exp\left(\int_E^{E_{max}} \frac{R(s) - p - r}{\sigma_0^2 L(R(s))} ds\right)$$

with

$$L(R) = \left(\frac{\overline{R} - R}{\overline{R} - p}\right)^{\beta}$$

where u(E) is market-to-book value

$$d \log R = \frac{R(E)'}{R(E)} dE = \psi(E)dE \to d \log R_t = \beta(E_t)dt + \alpha(E_t)dB_t$$

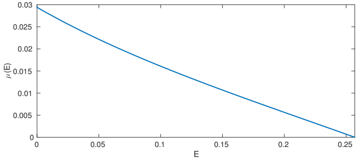
where $\beta(E) = \mu(E)\psi(E) + \frac{1}{2}\sigma(E)^2 \frac{\partial \psi(E)}{\partial E}$, $\alpha(E) = \sigma(E)\psi(E)$; s.t. Neumann b.c.: $\left.\frac{\partial \phi_t(E)}{\partial x}\right|_{E_{min,max}} = 0$

$$R'(E) = -\frac{1}{\sigma_0^2} \frac{2(\rho - r)\sigma_0^2 + [R(E) - p - r]^2 + 2rE[R(E) - p - r]L(R(E))^{-1}}{L(R(E)) - L'(R(E))[R(E) - p - r]}, R(E_{max}) = p + r$$

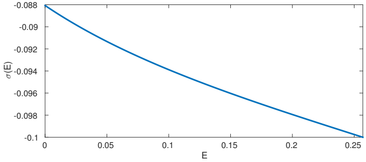
State Space (E) drift $\mu(E)$ and volatility $\sigma(E)$, functional M: $\alpha(E), \beta(E)$

Klimenko, Pfeil, Rochet, DeNicolo (KPRD), revised

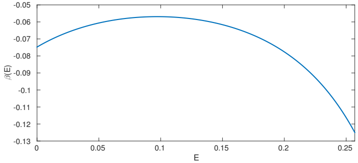
(a) $\mu(E)$



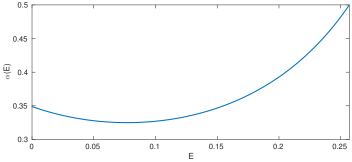
(b) $\sigma(E)$



(a) $\beta(E)$

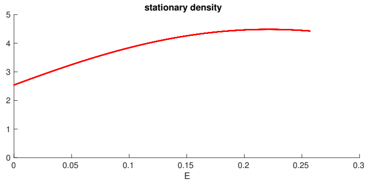


(b) $\alpha(E)$

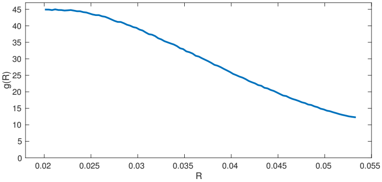


stationary density in E and R, Klimenko, Pfeil, Rochet, DeNicolo (KPRD)

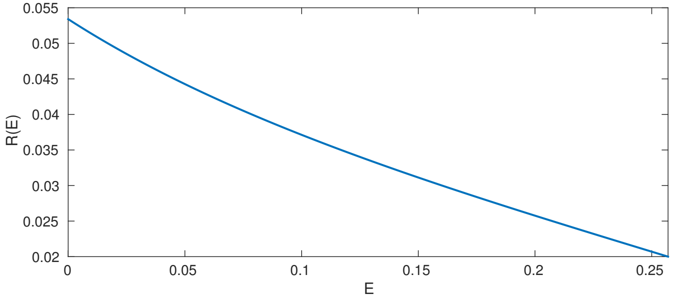
in E space



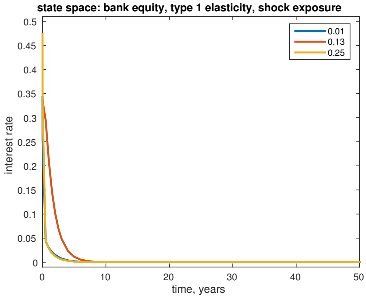
in R space (identical to a graph in KPRD paper)



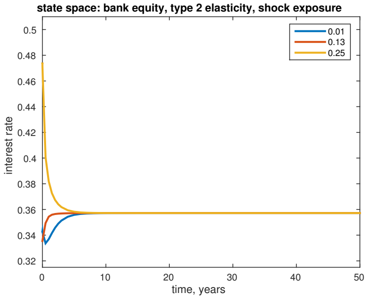
R=R(E)



Shock-exposure elasticity for $R = R(E)$ (KPRD)



(a) type 1 elasticity



(b) type 2 elasticity

SDF and price elasticity, Klimenko, Pfeil, Rochet, DeNicolo (KPRD)

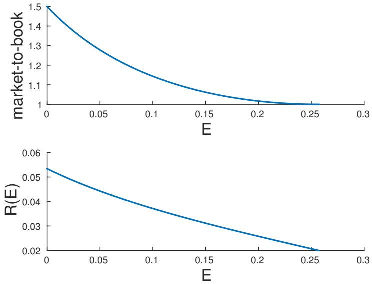
State - equity E
Multiplicative functional for SDF $M = u(E)$

$$u(E) = \exp\left(\int_E^{E_{max}} \frac{R(s) - p - r}{\sigma_0^2 L(R(s))} ds\right)$$

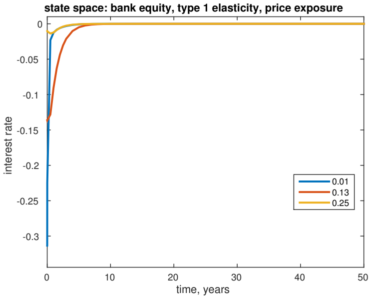
with

$$L(R) = \left(\frac{\overline{R} - R}{\overline{R} - p}\right)^{\beta}$$

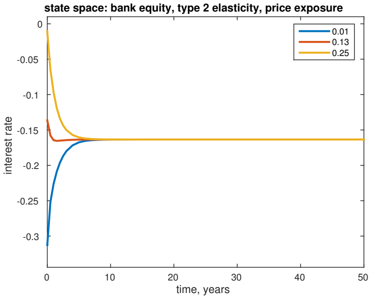
where u(E) is market-to-book value



Price elasticity for $R = R(E)$ (KPRD)



(a) type 1 elasticity



(b) type 2 elasticity