Computational Tools and Techniques for Numerical Macro-Financial Modeling

Victor V. Zhorin

MFM/BFI

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Numerical Building Blocks

Spectral approximation technology (chebfun):

numerical computation in Chebyshev functions

- piece-wise smooth functions
- breakpoints detection
- rootfinding
- functions with singularities
- fast adaptive quadratures
- continuous QR, SVD, least-squares
- linear operators
- solution of linear and non-linear ODE
- Fréchet derivatives via automatic differentiation
- PDEs in one space variable plus time

Stochastic processes:

(quazi) Monte-Carlo simulations, Polynomial Expansion (gPC), finite-differences (FD)

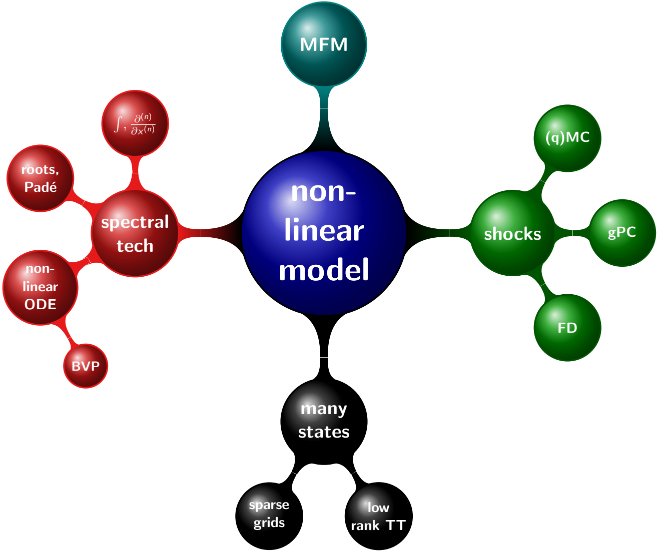
- non-linear IRF
- Borovička-Hansen-Sc[heinkman shock-exposure and shock-price elasticities
- Malliavin derivatives

Many states:

Dimensionality Curse Cure

- low-rank tensor decomposition
- sparse Smolyak grids

Numerical Building Blocks (cont.)



Horse race: methods and models

models

He-Krishnamurthy, "Intermediary Asset Pricing"

Klimenko-Pfeil-Rochet-DeNicolo, "Aggregate Bank Capital and Credit Dynamics"

Brunnermeier-Sannikov, "A Macroeconomic Model with a Financial Sector"

Basak-Cuoco, "An Equilibrium Model with Restricted Stock Market Participation"

Di Tella, "Uncertainty Shocks and Balance Sheet Recessions"

methods

spectral technology vs discrete grids

Monte-Carlo simulations (MC) vs Polynomial Expansion (gPC) vs finite-differences SPDE (FD)

Smolyak sparse grids vs tensor decomposition

criteria

elegance: clean primitives, libraries, ease of mathematical concepts expression in code

speed: feasibility, ready prototypes, LEGO blocks; precision: numerical tests of speed vs stability trade-offs

common metrics: shock exposure elasticity, asset pricing implications

Given Chebyshev interpolation nodes z_k, $k = 1, \ldots, m$ on $[-1, 1]$

$$z_k = -cos\left(\frac{2k-1}{2m}\pi\right)$$

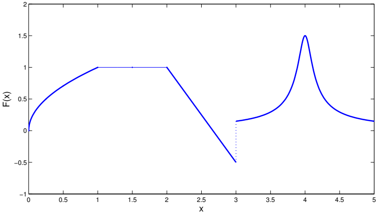
and Chebychev coefficients a_i, $i = 0, \ldots, n$ computed on Chebychev nodes we can approximate

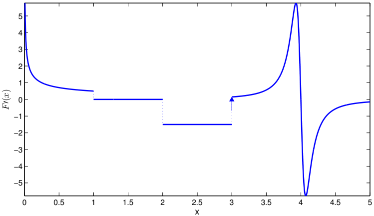
$$\mathbb{F}(x) \approx \sum_{i=0}^{n} a_i T_i(x)\,;\, a_i = \frac{2}{\pi}\int_{-1}^{1} \frac{\mathbb{F}(x) T_i(x)}{\sqrt{1-x^2}}\, dx$$

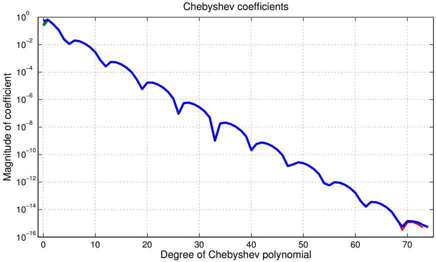
"Approximation Theory and Approximation Practice", by Lloyd N. Trefethen (chebfun.org)
"Chebyshev and Fourier Spectral Methods", by John P. Boyd
Chebyshev interpolation nodes and degree of polynomials have to be adaptive during continuous Newton updates.

> (i) When in doubt, use Chebyshev polynomials unless the solution is spatially periodic, in which case an ordinary Fourier series is better.
>
> (ii) Unless youre sure another set of basis functions is better, use Chebyshev polynomials.
>
> (iii) Unless youre really, really sure that another set of basis functions is better, use Chebyshev polynomials

Adaptive Chebyshev functional approximation: examples (chebfun)







shock-exposure and shock-price elasticity first kind

Multiplicative functional M_t:
1) consumption C_t for consumption shock-exposure
2) stochastic discount factor S_t for consumption shock-price elasticity

$$d \log M_t = \beta(X_t)dt + \alpha(X_t)dB_t$$

$$\epsilon(x, t) = \sigma(x)\frac{\partial}{\partial x} \log \mathbb{E}[M_t | X_0 = x] + \alpha(x)$$

Define

$$\phi(x, t) = \mathbb{E}\left[\frac{M_t}{M_0} | X_0 = x\right]$$

Then solve PDE

$$\frac{\partial \phi(x, t)}{\partial t} = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\phi(x, t) + (\mu(x) + \sigma(x)\alpha(x))\frac{\partial}{\partial x}\phi(x, t) + \left(\beta(x) + \frac{1}{2}|\alpha(x)^2|\right)\phi(x, t)$$

s.t. initial boundary condition $\phi(x, 0) = 1$

shock-exposure and shock-price elasticity second kind

Multiplicative functional M_t:
1) consumption C_t for consumption shock-exposure
2) stochastic discount factor S_t for consumption shock-price elasticity

$$d \log M_t = \beta(X_t)dt + \alpha(X_t)dB_t$$

The elasticity of second kind is

$$\epsilon_2(X_t) = \frac{\mathbb{E}(\mathfrak{D}_t \; M_t)|\mathcal{F}_0}{\mathbb{E}M_t|\mathcal{F}_0}$$

Use specification for M_t to get

$$\epsilon_2(X_t) = \frac{\mathbb{E}(\; M_t\eta(X_t)\alpha(X_t))|\mathcal{F}_0}{\mathbb{E}M_t|\mathcal{F}_0}$$

Then solve PDE twice

$$\frac{\partial\phi(x,t)}{\partial t} = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\phi(x,t) + (\mu(x) + \sigma(x)\alpha(x))\frac{\partial}{\partial x}\phi(x,t) + \left(\beta(x) + \frac{1}{2}|\alpha(x)^2|\right)\phi(x,t)$$

s.t. initial boundary condition $\phi(x,0) = 1$ to get a solution $\phi_1(x,t)$ and initial boundary condition $\phi(x,0) = \alpha(x)$ to get a solution $\phi_2(x,t)$ Then, second type elasticity is

$$\epsilon_2(x,t) = \frac{\phi_2(x,t)}{\phi_1(x,t)}$$

"Intermediary Asset Pricing": Zhiguo He and Arvind Krishnamurthy, AER, 2013

Find $\frac{P_t}{D_t} = F(y),\ \forall y \in \left[0, \frac{1+l}{\rho}\right]$, by solving

$$
\begin{aligned}
& F''(y)\,\theta_s(y) G(y)^2 \frac{(\theta_b(y)\sigma)^2}{2} \frac{G(y)}{\theta_s(y) F(y)} \left(\frac{1+l+\rho y(\gamma-1)}{1+l-\rho y+\rho\gamma G(y)\,\theta_b(y)} \right) \\
= \quad & \rho + g(\gamma-1) - \frac{1}{F(y)} + \frac{\gamma(1-\gamma)\,\sigma^2}{2} \left(1 + \frac{\rho G(y)\,\theta_b(y)}{1+l-\rho y}\right) \frac{y - G(y)\,\theta_b(y)}{\theta_s(y) F(y)} \left[\frac{1+l-\rho y-\rho G(y)\,\theta_b(y)}{1+l-\rho y+\rho\gamma G(y)\,\theta_b(y)} \right] \\
& - \left(\frac{(1+l-\rho y)(G(y)-1)}{\theta_s(y) F(y)} + \gamma\rho G(y) \right) \frac{\theta_s(y) + l + \theta_b(y)(g(\gamma-1)+\rho) - \rho y}{1+l-\rho y+\rho\gamma G(y)\,\theta_b(y)}.
\end{aligned}
$$

where

$$G(y) = \frac{1}{1-\theta_s(y) F'(y)},$$

and

	if $y \in (0, y^c)$	if $y \in \left(y^c, \frac{1+l}{\rho}\right)$
$\theta_s(y) =$	$\frac{(1-\lambda)y}{F(y)-\lambda y}$	$\frac{m}{1+m}$
$\theta_b(y) =$	$\lambda y \frac{F(y)-y}{F(y)-\lambda y}$	$y - \frac{m}{1+m} F(y)$

.

The endogenous threshold y^c is determined by $y^c = \frac{m}{1-\lambda+m} F(y^c)$. Boundary conditions on $y = 0$ and $y = \frac{1+l}{r}$.

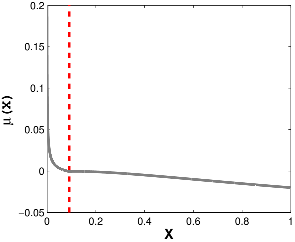
$$F(0) = \frac{1+F'(0)\,l}{\rho + g(\gamma-1) + \frac{\gamma(1-\gamma)\sigma^2}{2} - \frac{l\gamma\rho}{1+l}};\ F\left(\frac{1+l}{\rho}\right) = \frac{1+l}{\rho};\ F'\left(\frac{1+l}{\rho}\right) = 1.$$

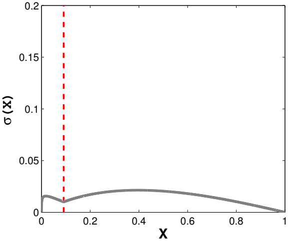
He-Krishnamurthy, BVP solution

$$dX_t = \mu_y(X_t)dt + \sigma_y(X_t)dB_t, X_t = \frac{w_t}{P_t};\ \sigma_y(X_t) = -\frac{\theta_b}{1-\theta_s F'(y)}\sigma; X_t = \frac{F(y)-y}{F(y)}$$

$$d\log C_t = \beta(X_t)dt + \alpha(X_t)dB_t; d\log S_t = -\gamma \log C_t$$

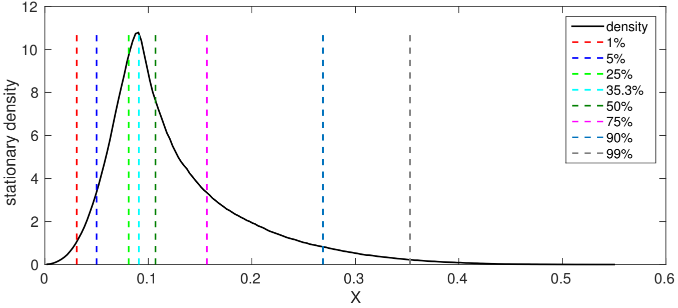
$$\beta(x) = \mu(x)\xi(x) + \frac{1}{2}\sigma(x)^2\frac{\partial\xi}{\partial x} + g_d - \frac{\sigma_d^2}{2}; \alpha(x) = \sigma(x)\xi(x) + \sigma_d; \xi(x) = -\rho\frac{p'(x)(1-x)-p(x)}{1+l-\rho(1-x)p(x)}$$





example, He-Krishnamurthy

stationary density distribution, percentiles



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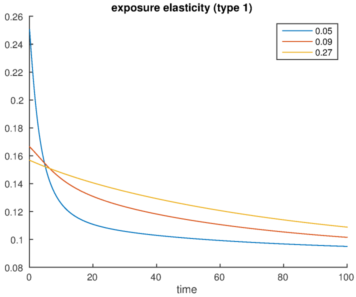
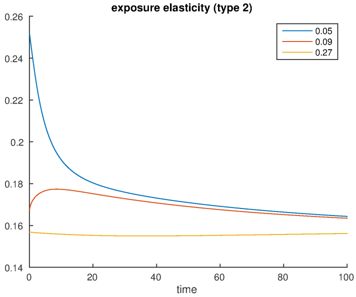
He-Krushnamurthy: shock exposure, specialist C

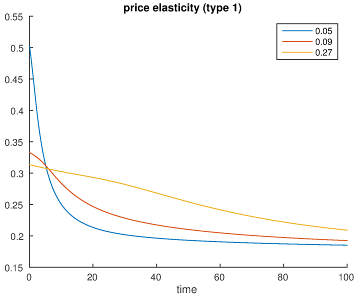


(a) type 1 elasticity



(b) type 2 elasticity

He-Krushnamurthy: price exposure, specialist C



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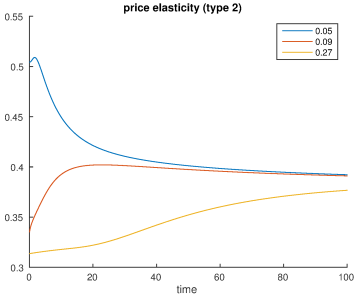


(b) type 2 elasticity

Brunnermeier-Sannikov example

$$\frac{dX_t}{X_t} = \mu(X_t)dt + \sigma(X_t)dB_t - d\zeta_t, X_t = \frac{N_t}{q_t K_t};$$

$X(t)$: expert share of wealth
Consumption is aggregate output net of aggregate investment

$$C_t^a = [a\psi(X_t) + \underline{a}(1 - \psi(X_t))]K_t$$

$$d \log C_t = \beta(X_t)dt + \alpha(X_t)dB_t;$$

where

$$\alpha(X) = \sigma; \beta(X) = \Phi(X) - \delta - \sigma^2/2$$

For log-utility

$$S_t/S_0 = \mathbf{e}^{-\rho t} C_0/C_t$$

Brunnermeier-Sannikov: shock exposure, aggregate C

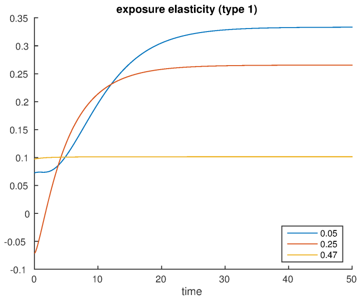
for logarithmic utility consumption of both households and experts are myopic, proportional to their wealth N_t

$$d\zeta_t = \rho dt; d\log C_t = d\log K_t$$

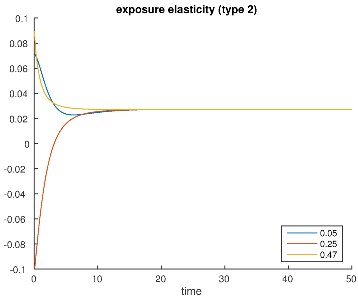
$$d\log K_t = \left(\Phi(i_t) - \delta\psi_t - \underline{\delta}(1-\psi_t) - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t$$

for $\delta = \underline{\delta}$

$$\alpha(X) = \sigma; \beta(X) = \Phi(X) - \delta - \frac{1}{2}\sigma^2$$



(a) type 1 elasticity



(b) type 2 elasticity

Brunnermeier-Sannikov: price exposure, aggregate C

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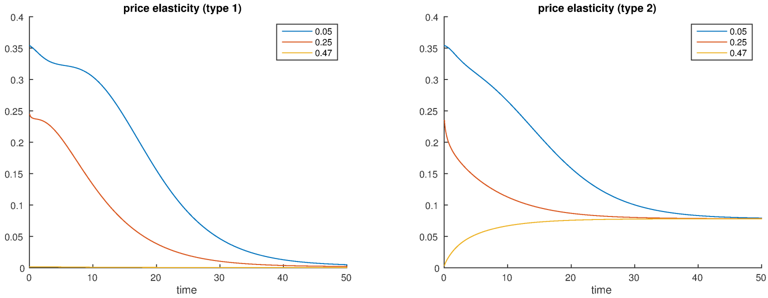


(a) type 1 elasticity

(b) type 2 elasticity

Model Settings, Klimenko, Pfeil, Rochet, DeNicolo (KPRD)

State - equity E
Multiplicative functional $M = R(E)$

$$dE_t = \mu(E_t)dt + \sigma(E_t)dB_t,\, p + r \leq R_t \leq R_{max},$$

where

$$\mu(E) = Er + L(R(E))(R(E) - r - p);\, \sigma(E) = L(R(E))\sigma_0.$$

$$\int_0^{E_{max}} \frac{R(s) - p - r}{\sigma_0^2 L(R(s))} ds = \ln(1+\gamma);\, u(E) = \exp\left(\int_E^{E_{max}} \frac{R(s) - p - r}{\sigma_0^2 L(R(s))} ds\right)$$

with

$$L(R) = \left(\frac{\overline{R} - R}{\overline{R} - p}\right)^{\beta}$$

where u(E) is market-to-book value

$$d\log R = \frac{R(E)'}{R(E)} dE = \psi(E)dE \rightarrow d\log R_t = \beta(E_t)dt + \alpha(E_t)dB_t$$

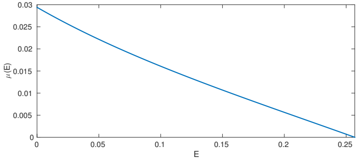
where $\beta(E) = \mu(E)\psi(E) + \frac{1}{2}\sigma(E)^2 \frac{\partial \psi(E)}{\partial E}$, $\alpha(E) = \sigma(E)\psi(E)$; s.t. Neumann b.c.: $\left.\frac{\partial \phi_t(E)}{\partial x}\right|_{E_{min,max}} = 0$

$$R'(E) = -\frac{1}{\sigma_0^2} \frac{2(\rho - r)\sigma_0^2 + [R(E) - p - r]^2 + 2rE[R(E) - p - r]L(R(E))^{-1}}{L(R(E)) - L'(R(E))[R(E) - p - r]},\, R(E_{max}) = p + r$$

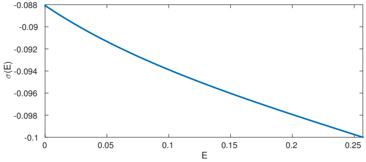
State Space (E) drift $\mu(E)$ and volatility $\sigma(E)$, functional M: $\alpha(E), \beta(E)$

Klimenko, Pfeil, Rochet, DeNicolo (KPRD), revised

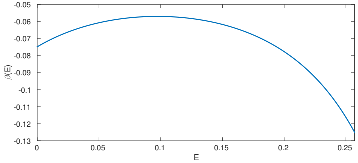
(a) $\mu(E)$



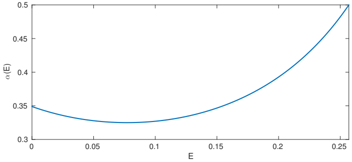
(b) $\sigma(E)$



(a) $\beta(E)$

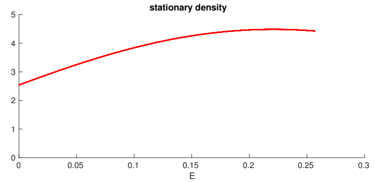


(b) $\alpha(E)$

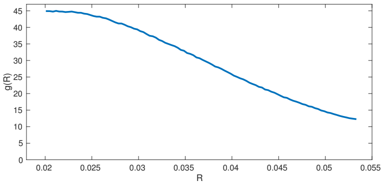


stationary density in E and R, Klimenko, Pfeil, Rochet, DeNicolo (KPRD)

in E space

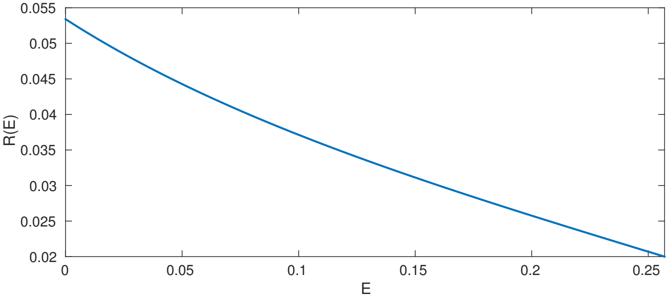


in R space (identical to a graph in KPRD paper)

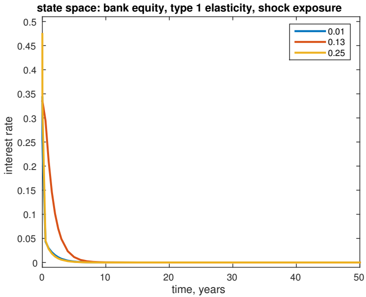


R=R(E), Klimenko, Pfeil, Rochet, DeNicolo (KPRD)

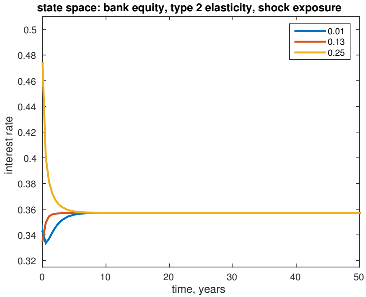
R=R(E)



Shock-exposure elasticity for $R = R(E)$ (KPRD)



(a) type 1 elasticity



(b) type 2 elasticity

SDF and price elasticity, Klimenko, Pfeil, Rochet, DeNicolo (KPRD)

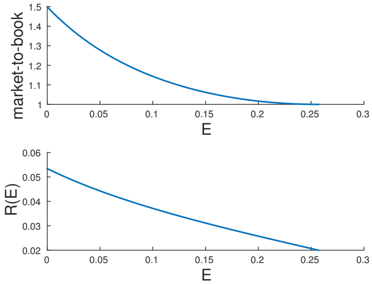
State - equity E
Multiplicative functional for SDF $M = u(E)$

$$u(E) = \exp\left(\int_E^{E_{max}} \frac{R(s) - p - r}{\sigma_0^2 L(R(s))} ds\right)$$

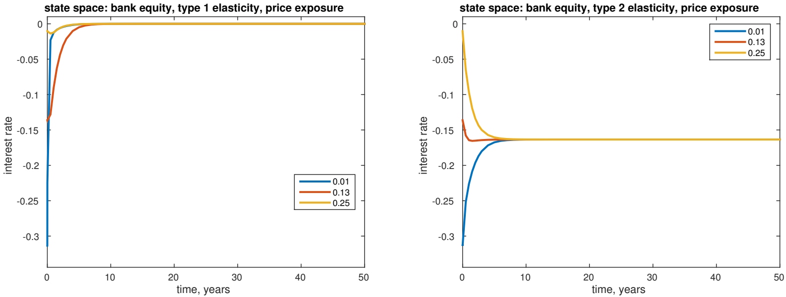
with

$$L(R) = \left(\frac{\overline{R} - R}{\overline{R} - p}\right)^{\beta}$$

where u(E) is market-to-book value



Price elasticity for $R = R(E)$ (KPRD)



(a) type 1 elasticity

(b) type 2 elasticity

Malliavin Derivative

and Generalized Polynomial Chaos Expansion (gPC)

Fast technique for high-precision numerical analysis of stochastic non-linear systems

Malliavin calculus to integrate and differentiate processes that are expressed in generalized Polynomial Chaos (gPC)

Generating function

$$\exp\left(sx - \frac{s^2}{2}\right) = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(x); \, H_n(x) \text{ are orthogonal Hermite polynomials}$$

Let B_t be Brownian motion, $M_t = \exp\left(sB_t - \frac{s^2 t}{2}\right)$

Then (Ito)

$$dM_t = s \sum_{n=0}^{\infty} \frac{s^n}{n!} x_n(t) dB_t, \; x_n(t) = t^{n/2} H_n(B_t/\sqrt{t})$$

$$dx_n(t) = n x_{n-1} dB_t; \; x_n(t) = n! \int_0^t dB(t_{n-1}) \int_0^{t_n - 1} dB(t_{n-2}) \dots \int_0^{t2} dB(t_1)$$

$$u(t, B_t) \approx \sum_{i=0}^{p} u_i(t) H_i(B_t), \text{ Cameron and Martin for Gaussian random variables, Xiu-Karniadakis for generalized}$$

$$u_0(x, t) = \mathbb{E}[u(x, t, \xi) H_0] = \mathbb{E}[u(x, t, \xi)]$$

from SDE to ODEs

Let $\hat{H}_n\xi := H_n(\xi)/\sqrt{n!}$

Then the Malliavin derivative operator (annihilation operator) with respect to random variable ξ is: $\mathbb{D}_\xi(H_n(\xi)) := \sqrt{n}H_{n-1}(\xi)$

Malliavin divergence operator (creation operator, Skorokhod integral) with respect to ξ is: $\delta_\xi(H_n(\xi)) := \sqrt{n+1}H_{n+1}(\xi)$

extended to nonlinear functionals of random variables expressed with PC decomposition

Ornstein-Uhlenbeck operator $L := \delta \circ \mathbb{D}$ (and its semigroup) - the Hermite polynomials are eigenvectors

from SDE to ODEs (cont)

Loan rate $R_t = R(E_t)$:

$$dR_t = \mu(R_t)dt + \sigma(R_t)dB_t, \, p \leq R_t \leq R_{max},$$

$$R_t = r_0 + \int_0^t \mu(R_\tau)d\tau + \int_0^t \sigma(R_\tau)dB_\tau$$

$$R_t \approx \sum_{i=0}^{p} r_i(t)H_i(\xi); \, B(t,\xi) \approx \sum_{i=1}^{n} b_i(t)H_i(\xi), \, \xi \sim N(0,1), \, \{H_i\} - \text{Hermite polynomials}: \ \text{KLE}$$

Integration by parts: $\mathbb{E}\left[F \int_0^t R_\tau \, dB_\tau\right] = \mathbb{E}\left[\int_0^t \mathbb{D}_\xi F R_\tau \, d\tau\right]$

$$\dot{r}_i(t) \approx \langle \mu(R_t) \rangle_i + \sum_{j=1}^{n} \sqrt{j} b_j(t) \langle \sigma(R_t) \rangle_{j-1}, \, i = 0, \ldots, p$$

Non-linear IRF: $F_t = \mathbb{D}_0 R_t$ from Borovicka-Hansen-Scheinkman

Stochastic Galerkin and Polynomial Chaos Expansion

for Stochastic PDE

Spectral expansion in stochastic variable(s): $\xi \in \Omega$:

$$u(x, t, \xi) = \sum_{i=0}^{\infty} u_i(x, t)\psi_i(\xi)$$

where $\psi_i(\xi)$ are orthogonal polynomials (Hermite, Legendre, Chebyshev)
standard approximations (spectral or finite elements) in space and polynomial (also spectral or pseudo-spectral) approximation in the probability domain
Babuska, Ivo, Fabio Nobile, and Raul Tempone. "A stochastic collocation method for elliptic partial differential equations with random input data." SIAM Journal on Numerical Analysis 45.3 (2007): 1005-1034.

$$u(x, t, \xi) \approx \sum_{i=0}^{p} u_i(x, t)\psi_i(\xi)$$

substitute into PDE and do a Galerkin projection by multiplying with $\psi_k(\xi)$

$$\frac{\partial u}{\partial t} = a(\xi)\frac{\partial^2 u}{\partial x^2}; \quad \sum_{i=0}^{p} \frac{\partial u_i(x, t)}{\partial t}\langle\psi_i\psi_k\rangle + \sum_{i=0}^{p} \frac{\partial^2 u_i}{\partial x^2} \sum_{j=0}^{p\sigma} a_j\langle\psi_j\psi_i\psi_k\rangle, k = 0, \ldots, p$$

Monte-Carlo/gPC comparison (sglib)

gPC expansion for $\ln N(\mu, \sigma^2)$ with $\mu = 0.03$, $\sigma = 0.85$, comparison with exact value and MC (1,000 and 100,000 samples)

p= 0	exact	gPC	MC(1,000)	MC(100,000)
mean:	1.47883	1.47863	1.42357	1.48603
var:	2.31722	1.57518	1.85956	2.38018

p= 1	exact	gPC	MC(1,000)	MC(100,000)
mean:	1.47883	1.47883	1.42357	1.48603
var:	2.31722	2.15079	1.85956	2.38018

p= 2	exact	gPC	MC(1,000)	MC(100,000)
mean:	1.47883	1.47883	1.42357	1.48603
var:	2.31722	2.28832	1.85956	2.38018

p= 3	exact	gPC	MC(1,000)	MC(100,000)
mean:	1.47883	1.47883	1.42357	1.48603
var:	2.31722	2.31315	1.85956	2.38018

p= 4	exact	gPC	MC(1,000)	MC(100,000)
mean:	1.47883	1.47883	1.42357	1.48603
var:	2.31722	2.31674	1.85956	2.38018

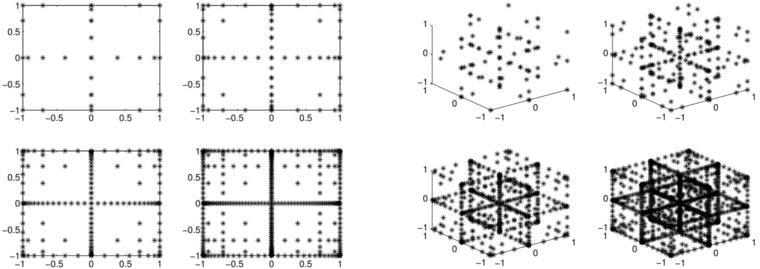
p= 5	exact	gPC	MC(1,000)	MC(100,000)
mean:	1.47883	1.47883	1.42357	1.48603
var:	2.31722	2.31717	1.85956	2.38018

p= 6	exact	gPC	MC(1,000)	MC(100,000)
mean:	1.47883	1.47883	1.42357	1.48603
var:	2.31722	2.31722	1.85956	2.38018

p= 7	exact	gPC	MC(1,000)	MC(100,000)
mean:	1.47883	1.47883	1.42357	1.48603
var:	2.31722	2.31722	1.85956	2.38018

Extension to many dimensions:

Adaptive sparse (Smolyak) grids (Bocola, 2015), tensor approximation



Tensor-train decomposition (TT Toolbox)

$$A(i_1, \ldots, i_d) = G_1(i_1)G_2(i_2)\ldots G_d(i_d),$$

where $G_k(i_k)$ is $r_{k-1} \times r_k$, $r_0 = r_d = 1$.

- basic linear algebra operations in $\mathcal{O}(dnr^{\alpha})$
- rank reduction in $\mathcal{O}(dnr^3)$ operations
- tensor can be recovered exactly by sampling