$f(x) = -(\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$ 是凸函数

$f(x) = (x_1 \cdots x_n)^{\frac{1}{n}}$

$\nabla f(x) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$.

之有 $\frac{\partial f}{\partial x_i} = -\frac{1}{n} \frac{\pi(x_i)^{\frac{1}{n}}}{x_i}$

则 $\nabla f(x) = -(\frac{1}{n}\frac{\pi(x_i)^{\frac{1}{n}}}{x_1}, \cdots\cdots, \frac{1}{n}\frac{\pi(x_n)^{\frac{1}{n}}}{x_n})$.

而后对于 $\nabla^2 f(x)$ 有 $\frac{\partial^2 f}{\partial x_i^2} = \frac{n-1}{n^2} \frac{\pi(x_i)^{\frac{1}{n}}}{x_i^2}$.

$\frac{\partial^2 f}{\partial x_i x_j} = -\frac{1}{n^2} \frac{\pi(x_i)^{\frac{1}{n}}}{x_i x_j}$

则

$$\nabla^2 f(x) = \frac{\pi(x_i)^{\frac{1}{n}}}{n^2} \begin{pmatrix} \frac{n-1}{x_1^2} & -\frac{1}{x_1 x_2} & \cdots\cdots & \frac{-1}{x_1 x_n} \\ \vdots & \frac{n-1}{x_2^2} & & \vdots \\ \vdots & & \ddots & \\ \frac{-1}{x_1 x_n} & \cdots & & \frac{n-1}{x_n^2} \end{pmatrix}$$

为对称矩阵.

即 $\nabla^2 f(x) = \frac{\pi(x_i)^{\frac{1}{n}}}{n^2} (n\,\mathrm{diag}(n/x_1^2, \cdots, n/x_n^2) - r r^T)$

其中 $r = (\frac{1}{x_1} \cdots \frac{1}{x_n})$

若 $\nabla^2 f(x)$ 为半正定矩阵 则对任意向量 u 有. $u^T \nabla f(x) u \geq 0$.

即. $\frac{\pi(x_i)^{\frac{1}{n}}}{n^2} (\sum_{i=1}^{n} \frac{n u_i^2}{x_i^2} - (\sum_{i=1}^{n} \frac{u_i}{x_i})^2)$

由柯西不等式知 后式 ≥ 0 即 $\nabla^2 f(x)$ 半正定. 原函数为

凸函数