Table 5.4. Errors obtained using the preconditioned gradient method (PG), the preconditioned conjugate gradient method (PCG), and the direct method implemented in the MATLAB command \ for the solution of the Hilbert system. For the iterative methods also the number of iterations is reported

		\	PG		PCG	
n	$K(\mathrm{A}_n)$	Error	Error	Iter	Error	Iter
4	1.55e+04	7.72e-13	8.72e-03	995	1.12e-02	3
6	1.50e+07	7.61e-10	3.60e-03	1813	3.88e-03	4
8	1.53e+10	6.38e-07	6.30e-03	1089	7.53e-03	4
10	1.60e+13	5.24e-04	7.98e-03	875	2.21e-03	5
12	1.70e+16	6.27e-01	5.09e-03	1355	3.26e-03	5
14	6.06e+17	4.12e+01	3.91e-03	1379	4.32e-03	5

In this case the error estimate (5.64) still holds, however $K(\mathrm{A})$ is replaced by the more favourable $K(\mathrm{P}^{-1}\mathrm{A})$.
The PCG method is implemented in the MATLAB function `pcg`. `pcg`

Example 5.16 Let us go back to Example 5.9 on the Hilbert matrix and solve the related system (for different values of n) by the preconditioned gradient (PG) and the preconditioned conjugate gradient (PCG) methods, using as preconditioner the diagonal matrix D made of the diagonal entries of the Hilbert matrix. We fix $\mathbf{x}^{(0)}$ to be the null vector and iterate until the relative residual (5.64) is less than 10^{-6}. In Table 5.4 we report the absolute errors (with respect to the exact solution) obtained with PG and PCG methods, as well as the errors obtained using the MATLAB command \. For the latter, the error degenerates when n gets large. On the other hand, we can appreciate the beneficial effect that a suitable iterative method such as the PCG scheme can have on the number of iterations. ■

Remark 5.5 (Non-symmetric systems) The CG method is a special instance of the so-called *Krylov* (or *Lanczos*) *methods* that can be used for the solution of systems which are not necessarily symmetric. Their description is provided, e.g., in [Axe94], [Saa03] and [vdV03].

Some of them share with the CG method the notable property of finite termination, that is, in exact arithmetic they provide the exact solution in a finite number of iterations also for nonsymmetric systems. A remarkable example is the *GMRES* (Generalized Minimum RESidual) *method*, available in MATLAB under the name of `gmres`. `gmres`

Another method, the Bi-CGStab ([vdV03]), is very competitive with GMRES from the efficiency point of view. The MATLAB command is `bicgstab`. `bicgstab`
■

See Exercises 5.15-5.19.