最优化大作业

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目录

1 Problem 5.6

对于 $q(\boldsymbol{x}) = (10x_1^2 - 18x_1x_2 + 10x_2^2)/2 + 4x_1 - 15x_2 + 13$

1.1 重要参数

$$G = \begin{bmatrix} 10 & -9 \\ -9 & 10 \end{bmatrix}, \qquad \lambda_1 = 19, \lambda_2 = 1, \qquad (\frac{\lambda_1 - \lambda_2}{\lambda_1 + x_2})^2 = 0.81$$

1.2 算法伪代码

Algorithm 1 Steepest-denscent method for problem(5.6)

Given $\boldsymbol{x}^{(0)}$ and G
Set $\boldsymbol{p}^{(0)} = -\boldsymbol{g}^{(0)}, k = 0$
while $\|\boldsymbol{g}^{(k)}\| > \epsilon$ **do**
    Set $\alpha_k = -\dfrac{{\boldsymbol{p}^{(k)}}^T \boldsymbol{g}^{(k)}}{{\boldsymbol{p}^{(k)}}^T \boldsymbol{G} \boldsymbol{p}^{(k)}}$
    Set $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha_k \boldsymbol{p}^{(k)}$
    Set $\boldsymbol{g}^{(k+1)} = g(\boldsymbol{x}^{(k+1)})$
    Set $\boldsymbol{p}^{(k)} = -\boldsymbol{g}^{(k)}$
    k=k+1
end while

1.3 计算结果展示

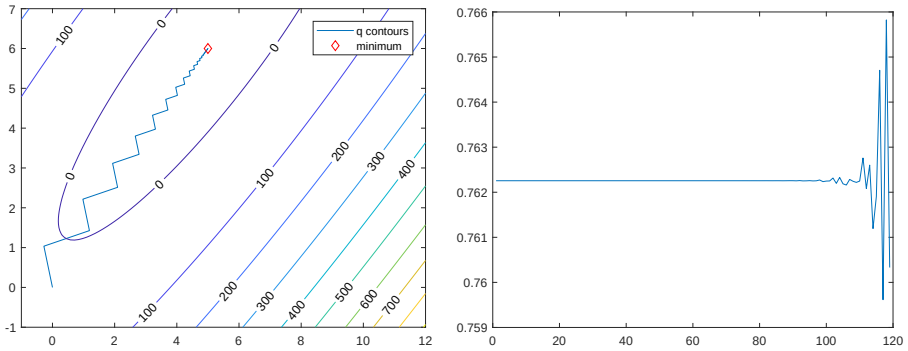


图 1: Steepest-denscent in (0,0)

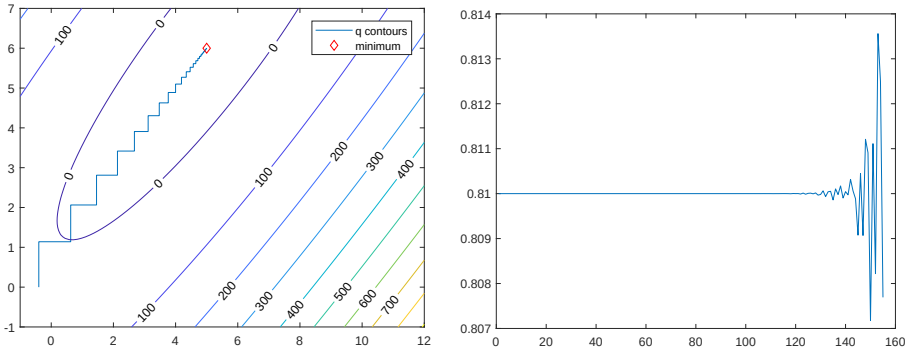


图 2: Steepest-denscent in (-0.4,0)

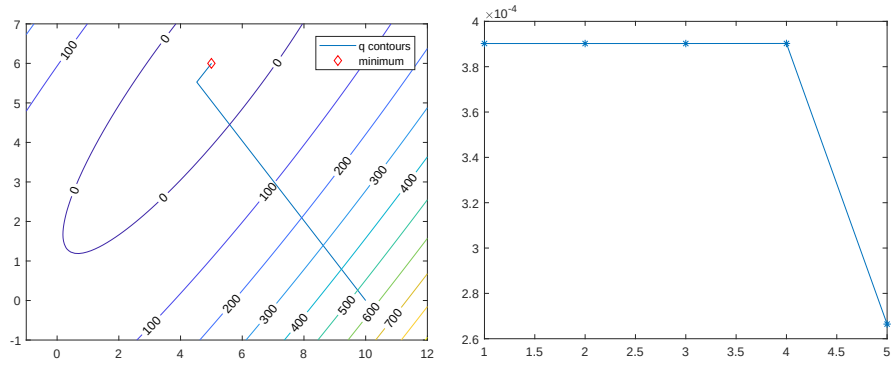


图 3: Steepest-denscent in (10,0)

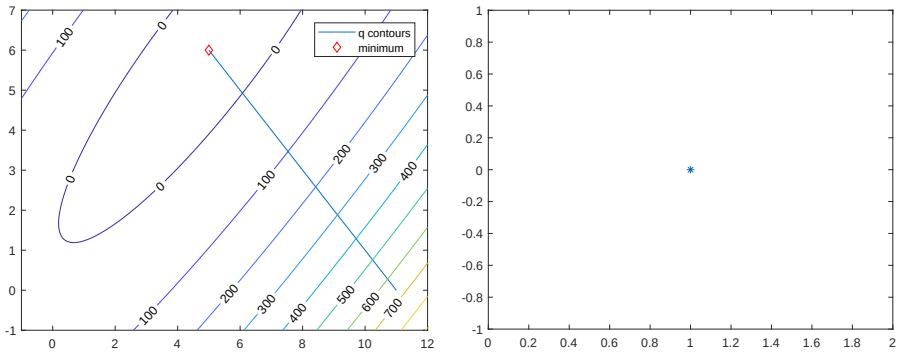


图 4: Steepest-denscent in (11,0)

表 1: 收敛因子比较

起始点	$(0,0)$	$(-0.4,0)$	$(10,0)$	$(11,0)$
收敛因子	0.762255	0.810000	0.000390	0

1.4 分析

由于目标函数为凸函数，故使用梯度下降法从这四个不同的起始点出发都能收敛到全局最优点，然而线性收敛因子却互不相同.

由图像可知：在迭代开始后，函数值的收敛速度稳定在一个值左右，直到接近最优点时，收敛速度开始较大幅度波动。

而且，可以看出，初始点越接近等值线椭圆的狭长端，线性收敛因子越大，在点 $(-0.4, 0)$ 处甚至达到了线性收敛因子的上界 0.81, 而离狭长端越远，收敛因子越小。这是由于梯度下降在构造搜索方向时没有充分利用到函数的二阶导数信息，在面临“峡谷”状的函数时，会反复震荡到“峡谷”的另一端，而不能直接向最优值方向前进。

容易看出，等值线椭圆的长轴端斜率为 $9/10$，且 $(-0.4, 0) = (5, 6) - 0.6 * (9, 10)$，这说明点 $(-0.4, 0)$ 刚好处在等值线椭圆的长轴上，因此也是震荡最剧烈的地方，收敛速度达到了最坏收敛速度。

2 Problem 5.7

2.1 重要参数

$$g(x) = f'(x) = 9 - \frac{4}{x-7}$$
$$G(x) = g'(x) = \frac{4}{(x-7)^2}$$

其迭代公式为:

$$x^{(k+1)} = x^{(k)} - \frac{1}{4}(x^{(k)} - 7)(9x^{(k)} - 67)$$

2.2 算法伪代码

Algorithm 2 Newton method for problem(5.7)

1: Given $x^{(0)}$ and compute $g(x) = f'(x)$
2: Compute $G(x) = g'(x)$
3: Set $g^{(0)} = g(x^{(0)}), k = 0$
4: **while** $|g^{(k)}| > \epsilon$ **do**
5: Set $s^{(k)} = -g^{(k)}/G^{(k)}$
6: Set $x^{(k+1)} = x^{(k)} + s^{(k)}$
7: Set k=k+1
8: **end while**

2.3 计算结果展示

由 MATLAB 函数 fminsearch 求得最优解为:

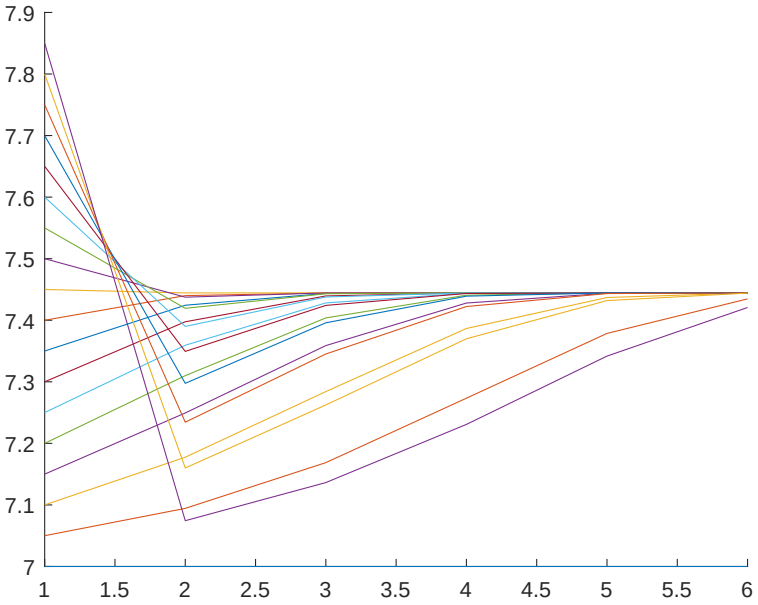
$$x^\star = 7.444421386718750, \quad f(x^\star) = 70.243720870248540$$

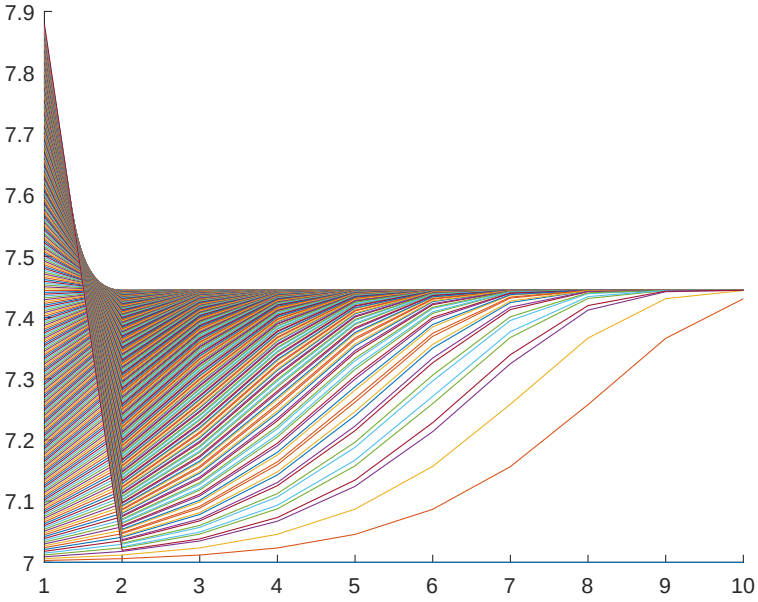
表 2: 迭代 5 次过程

$x^{(0)}$	7.4	7.2	7.01	7.80	7.88
$x^{(1)}$	7.44	7.31	7.019775	7.16	7.0176
$x^{(2)}$	7.4444	7.403775	7.038670136	7.2624	7.03450304
$x^{(3)}$	7.44444444	7.440722936	7.073975668	7.36987904	7.066327546
$x^{(4)}$	7.444444444	7.444413283	7.135638438	7.431934445	7.122756569
$x^{(5)}$	7.444444444	7.444444442	7.229881858	7.444092319	7.211607493
$f(x^{(5)})$	70.24372086	70.24372086	70.94969577	70.24372212	71.11655611

表 3: 大范围迭代结果

初始点	迭代终止点	初始点	迭代终止点
5	-Inf	9	-Inf
5.25	-Inf	9.25	-Inf
5.5	-Inf	9.5	-Inf
5.75	-Inf	9.75	-Inf
6	-Inf	10	-Inf
6.25	-Inf	10.25	-Inf
6.5	-Inf	10.5	-Inf
6.75	-Inf	10.75	-Inf
7	7	11	-Inf
7.25	7.44444444444445	11.25	-Inf
7.5	7.44444444444445	11.5	-Inf
7.75	7.44444444444445	11.75	-Inf
8	-Inf	12	-Inf
8.25	-Inf	12.25	-Inf
8.5	-Inf	12.5	-Inf
8.75	-Inf	12.75	-Inf


7.9
7.8
7.7
7.6
7.5
7.4
7.3
7.2
7.1
7
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6


7.9
7.8
7.7
7.6
7.5
7.4
7.3
7.2
7.1
7
1
2
3
4
5
6
7
8
9
10

3 Problem 5.8

3.1 重要参数

$$f(x_1,x_2) = -9x_1 - 10x_2 - \mu\left[\ln(-x_1 - x_2 + 100) + \ln(-x_1 + x_2 + 50) + \ln(x_1) + \ln(x_2)\right]$$

$$\begin{aligned}\boldsymbol{g}(x_1,x_2) &= \nabla f(x_1,x_2)\\ &= \begin{bmatrix} -9 - \dfrac{\mu}{x_1} + \dfrac{\mu}{100 - x_1 - x_2} + \dfrac{\mu}{50 - x_1 + x_2} \\ -10 - \dfrac{\mu}{x_2} + \dfrac{\mu}{100 - x_1 - x_2} - \dfrac{\mu}{50 - x_1 + x_2} \end{bmatrix}\end{aligned}$$

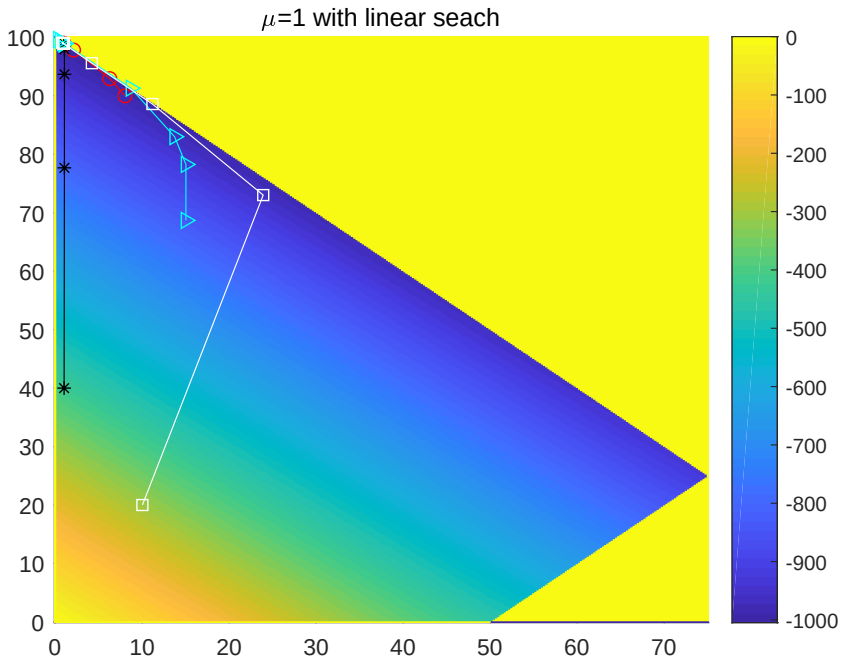
$$\begin{aligned}&\boldsymbol{G}[x_1,x_2]\\ =&\nabla \boldsymbol{g}(x_1,x_2)\\ =&\begin{bmatrix} \dfrac{1}{{x_1}^2} + \dfrac{1}{(100 - x_1 - x_2)^2} + \dfrac{1}{(50 - x_1 + x_2)^2} & \dfrac{1}{(100 - x_1 - x_2)^2} - \dfrac{1}{(50 - x_1 + x_2)^2} \\ \dfrac{1}{(100 - x_1 - x_2)^2} - \dfrac{1}{(50 - x_1 + x_2)^2} & \dfrac{1}{{x_2}^2} + \dfrac{1}{(100 - x_1 - x_2)^2} + \dfrac{1}{(50 - x_1 + x_2)^2} \end{bmatrix}\end{aligned}$$

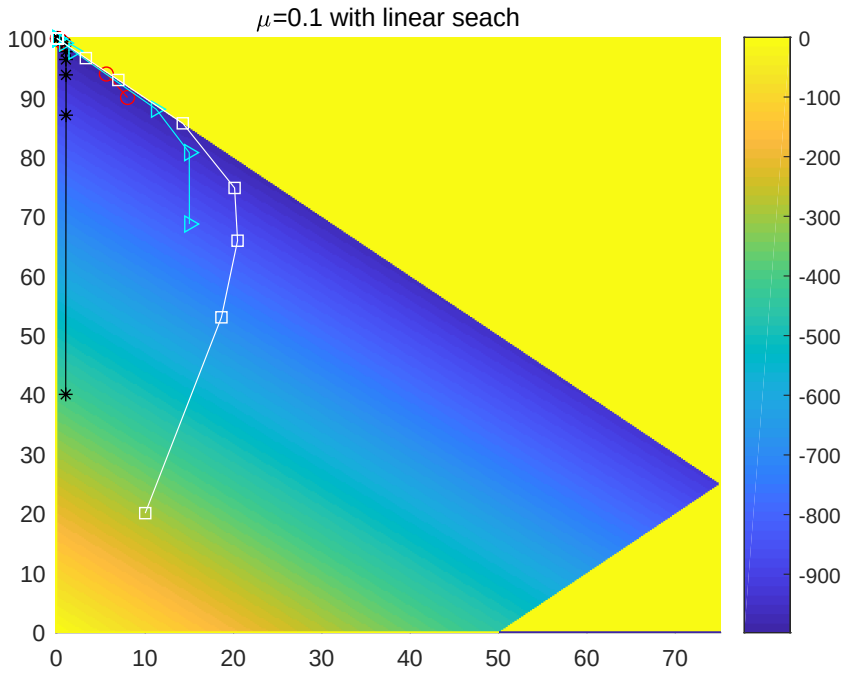
3.2 算法伪代码

Algorithm 3 Newton-Armijo method for problem(5.8)

1: Given $\boldsymbol{x}^{(0)}$ and compute $\boldsymbol{g}(\boldsymbol{x}) = \nabla f(\boldsymbol{x})$
2: Compute $\boldsymbol{G}(\boldsymbol{x}) = \nabla \boldsymbol{g}(\boldsymbol{x})^T$
3: Set $\boldsymbol{g}^{(0)} = \boldsymbol{g}(\boldsymbol{x}^{(0)}), k = 0$
4: **while** $\|\boldsymbol{g}^{(k)}\|_2 > \epsilon$ **do**
5: Set $\boldsymbol{s}^{(k)} = -{\boldsymbol{G}^{(k)}}^{-1}\boldsymbol{g}^{(k)}$
6: Compute α_k by Backtracking-Armijo Line Search
7: Set $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha_k \boldsymbol{s}^{(k)}$
8: Set k=k+1
9: **end while**

3.3 计算结果展示





3.4 总结分析

没加线搜索和越界判定之前，牛顿法总是一两次就冲到了定义域外，然后无法收敛。

然后我加了个越界判定: 若下一步的迭代点不在定义域内，则将步长 α 缩小一半，牛顿法很好地收敛到了全局最优点。

```
%Check函数检查是否越界，以缩短步长ak
while(Check(x+ak*double(p')))
    ak=0.5*ak;
    xk=x+ak*double(p');
end
```

而后在越界判定的基础上加了个线搜索：

```
%采用Armijo法则计算近似步长ak
while(F(xk(1),xk(2)) > (F(x(1),x(2))+0.01*double(p'*g)*ak)
||Check(x+ak*double(p')))%Check函数检查是否越界
    ak=0.5*ak;
    xk=x+ak*double(p');
```

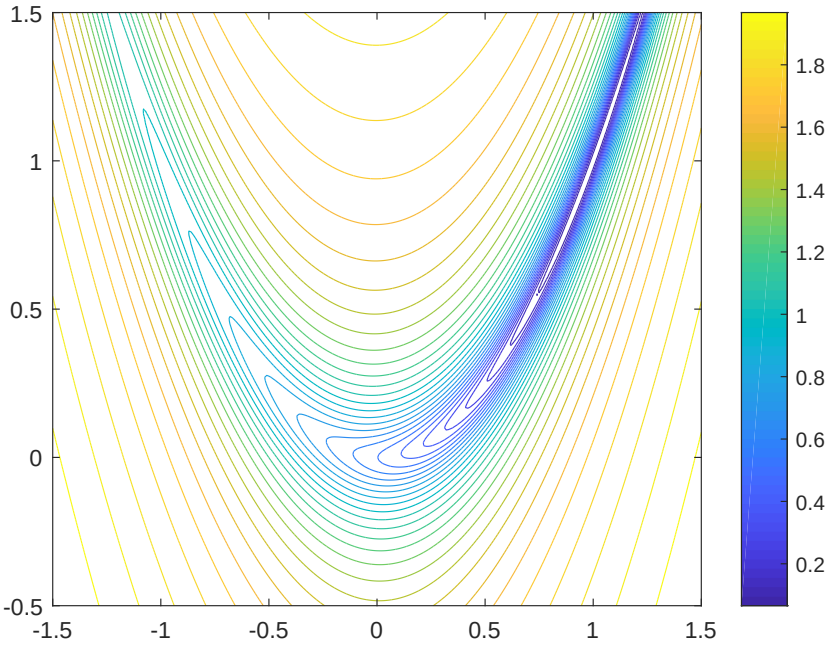
```
6 end
```

然而得到的结果和没加线搜索一样，可见在迭代的前期，越界判定起到了一定类似于线搜索缩短步长的效果，而到了迭代的后期，已经牛顿法的基本步长已经满足 Armijo 法则，还原成了基本牛顿法。

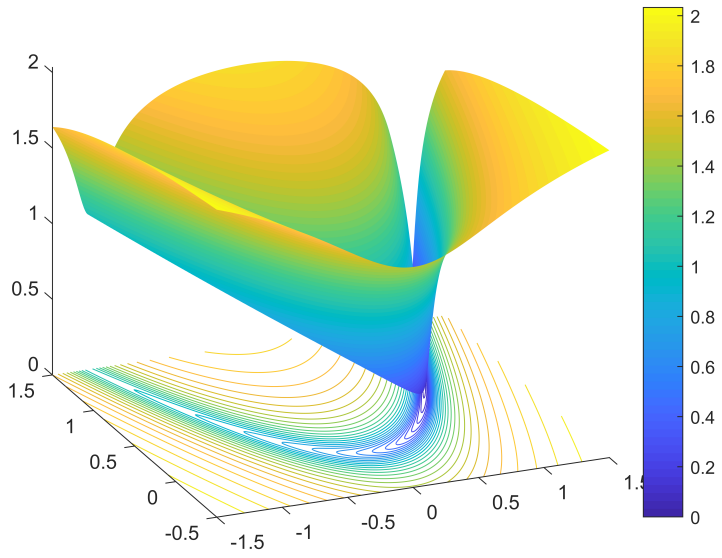
4 Problem5.9

4.1 Rosenbrock 函数图像

首先画出 Rosenbrock 函数的图像及等值线如下[1]:



[1]由于 Rosenbrock 函数过于陡峭，因此对原函数进行了取对数处理，以便于观察其特点。



4.2 算法伪代码

Algorithm 4 Backtracking-Armijo Line Search

Choose $\bar{\alpha} > 0, \quad \gamma, \rho \in (0, 1)$
Set $\alpha = \bar{\alpha}$
while $\phi(\alpha) > \phi(0) + \rho\phi'(0)\alpha$ **do**
    Set $\alpha = \gamma\alpha$
end while
return α as α_k

4.3 计算结果展示

本题中 Armijo 线搜索的参数为 $\gamma = 0.5, \rho = 0.01$, 并设置最大搜索步长为 200.

然后分别以梯度下降法和牛顿法迭代，并画出等高线、运动轨迹、迭代值如下:

Algorithm 5 Steepest-denscent-Armijo method for problem(5.9)

Given $\boldsymbol{x}^{(0)}$ and G
Set $\boldsymbol{p}^{(0)} = -\boldsymbol{g}^{(0)}, k = 0$
while $\|\boldsymbol{g}^{(k)}\| > \epsilon$ **do**
    Compute α_k by Backtracking-Armijo Line Search
    Set $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha_k \boldsymbol{p}^{(k)}$
    Set $\boldsymbol{g}^{(k+1)} = g(\boldsymbol{x}^{(k+1)})$
    Set $\boldsymbol{p}^{(k)} = -\boldsymbol{g}^{(k)}$
    k=k+1
end while

Algorithm 6 Newton-Armijo method for problem(5.9)

Given $\boldsymbol{x}^{(0)}$ and compute $\boldsymbol{g}(\boldsymbol{x}) = \nabla f(\boldsymbol{x})$
Compute $\boldsymbol{G}(\boldsymbol{x}) = \nabla \boldsymbol{g}(\boldsymbol{x})^T$
Set $\boldsymbol{g}^{(0)} = \boldsymbol{g}(\boldsymbol{x}^{(0)}), k = 0$
while $\|\boldsymbol{g}^{(k)}\|_2 > \epsilon$ **do**
    Set $\boldsymbol{s}^{(k)} = -{\boldsymbol{G}^{(k)}}^{-1} \boldsymbol{g}^{(k)}$
    Compute α_k by Backtracking-Armijo Line Search
    Set $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha_k \boldsymbol{s}^{(k)}$
    Set k=k+1
end while

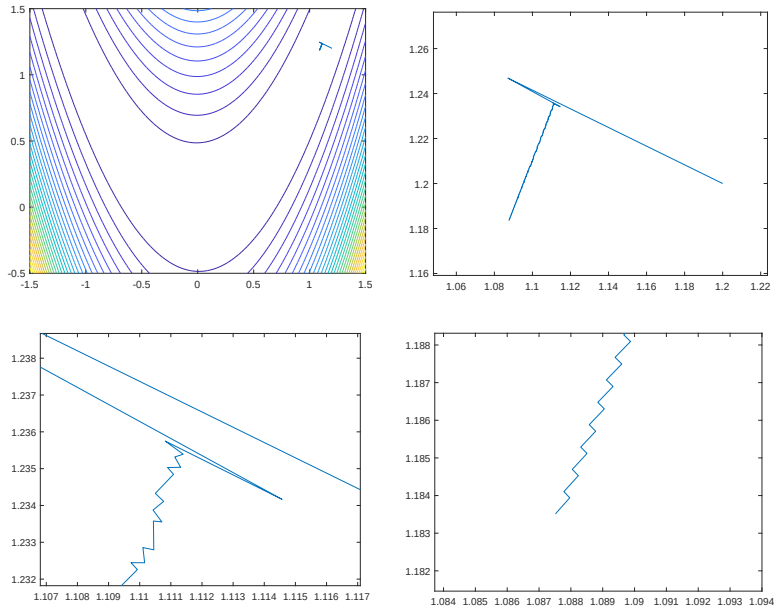


图 5: Steepest-denscent in (1.2,1.2)

Step[190]: x=[1.088846 1.186477] optim_fx=0.007973
Step[191]: x=[1.089052 1.186303] optim_fx=0.007937
Step[192]: x=[1.088577 1.185882] optim_fx=0.007924
Step[193]: x=[1.088779 1.185710] optim_fx=0.007889
Step[194]: x=[1.088310 1.185288] optim_fx=0.007874
Step[195]: x=[1.088507 1.185118] optim_fx=0.007841
Step[196]: x=[1.088044 1.184696] optim_fx=0.007825
Step[197]: x=[1.088236 1.184529] optim_fx=0.007793
Step[198]: x=[1.087780 1.184104] optim_fx=0.007776
Step[199]: x=[1.087965 1.183941] optim_fx=0.007745
Step[200]: x=[1.087517 1.183515] optim_fx=0.007727
最速下降法,共迭代 200 步
结果：
x=[1.087517e+00 1.183515e+00] optim_fx=0.007727

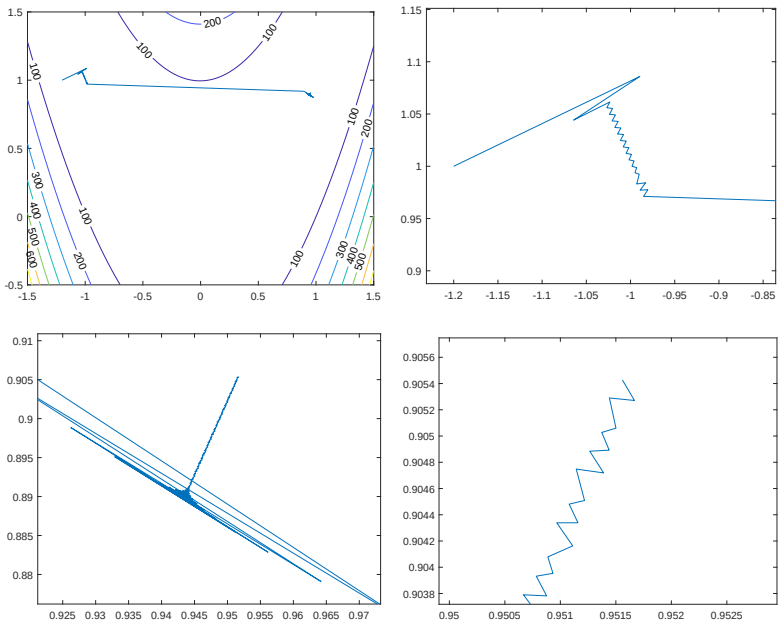


图 6: Steepest-denscent in (-1.2,1)

Step[190]: x=[0.951079 0.904481] optim_fx=0.002394
Step[191]: x=[0.951218 0.904509] optim_fx=0.002389
Step[192]: x=[0.951143 0.904748] optim_fx=0.002388
Step[193]: x=[0.951390 0.904719] optim_fx=0.002381
Step[194]: x=[0.951265 0.904884] optim_fx=0.002375
Step[195]: x=[0.951440 0.904892] optim_fx=0.002370
Step[196]: x=[0.951373 0.905027] optim_fx=0.002365
Step[197]: x=[0.951501 0.905060] optim_fx=0.002361
Step[198]: x=[0.951442 0.905290] optim_fx=0.002358
Step[199]: x=[0.951668 0.905271] optim_fx=0.002352
Step[200]: x=[0.951559 0.905427] optim_fx=0.002347
最速下降法,共迭代 200 步
结果：
x=[9.515588e-01 9.054274e-01] optim_fx=0.002347

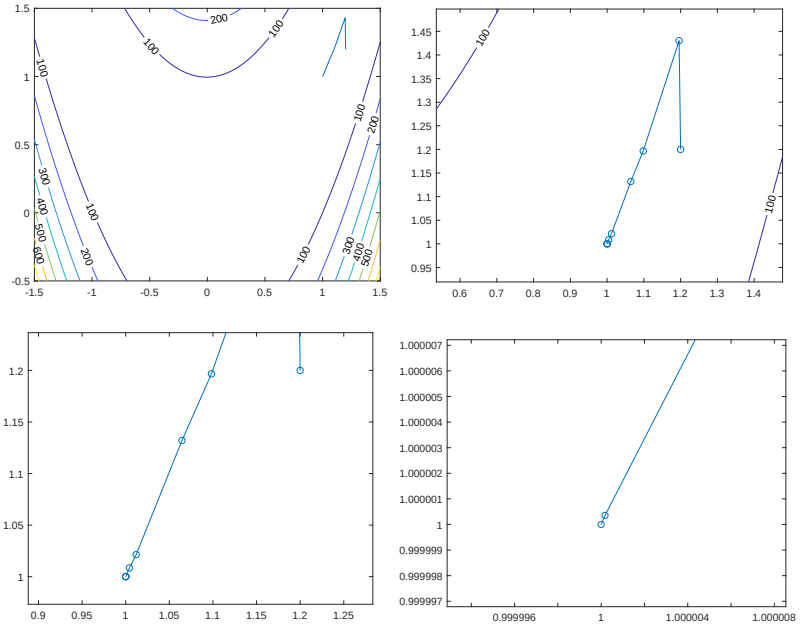


图 7: Newton-Armijo in (1.2,1.2)

Step[1]: x=[1.200000 1.200000] optim_fx=5.800000
Step[2]: x=[1.195918 1.430204] optim_fx=0.038384
Step[3]: x=[1.098284 1.196688] optim_fx=0.018762
Step[4]: x=[1.064488 1.131993] optim_fx=0.004289
Step[5]: x=[1.011992 1.021372] optim_fx=0.000903
Step[6]: x=[1.004261 1.008481] optim_fx=0.000019
Step[7]: x=[1.000050 1.000083] optim_fx=0.000000
Step[8]: x=[1.000000 1.000000] optim_fx=0.000000
Step[9]: x=[1.000000 1.000000] optim_fx=0.000000

牛顿 Armijo 回溯法,,共迭代 9 步
结果：
x=[1.000000e+00 1.000000e+00] optim_fx=0.000000

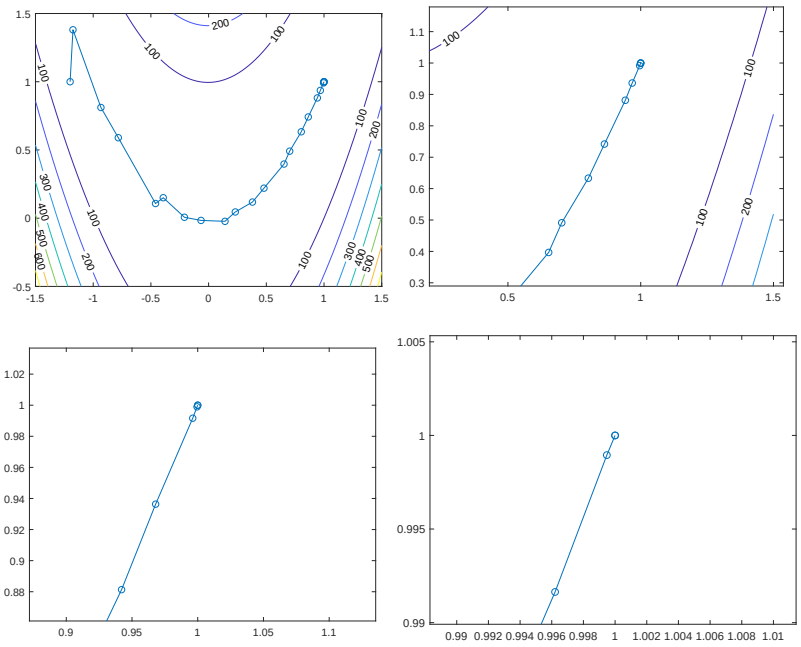


图 8: Newton-Armijo in (-1.2,1)

Step[15]: x=[0.802786 0.633221] optim_fx=0.051535
Step[16]: x=[0.863491 0.741931] optim_fx=0.019993
Step[17]: x=[0.942079 0.881336] optim_fx=0.007169
Step[18]: x=[0.967992 0.936337] optim_fx=0.001070
Step[19]: x=[0.996210 0.991639] optim_fx=0.000078
Step[20]: x=[0.999479 0.998948] optim_fx=0.000000
Step[21]: x=[0.999999 0.999998] optim_fx=0.000000
Step[22]: x=[1.000000 1.000000] optim_fx=0.000000
牛顿 Armijo 回溯法,,共迭代 22 步
结果：
x=[1.000000e+00 1.000000e+00] optim_fx=0.000000

5 Problem 5.19

5.1 算法伪代码

Algorithm 7 Conjugate gradient method method for problem(5.19)

1: Given $\boldsymbol{x}^{(0)}$ and G
2: Compute $\boldsymbol{g}^{(0)} = \boldsymbol{G}\boldsymbol{x}^{(0)} + b$
3: Set $\boldsymbol{p}^{(0)} = -\boldsymbol{g}^{(0)}, k = 0$
4: **while** $\|\boldsymbol{g}^{(k)}\| > \epsilon$ **do**
5: Compute α_k by Backtracking-Armijo Line Search
6: Set $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha_k \boldsymbol{p}^{(k)}$
7: Set $\boldsymbol{g}^{(k+1)} = g(\boldsymbol{x}^{(k+1)})$
8: Set $\boldsymbol{p}^{(k)} = -\boldsymbol{g}^{(k)}$
9: k=k+1
10: **end while**

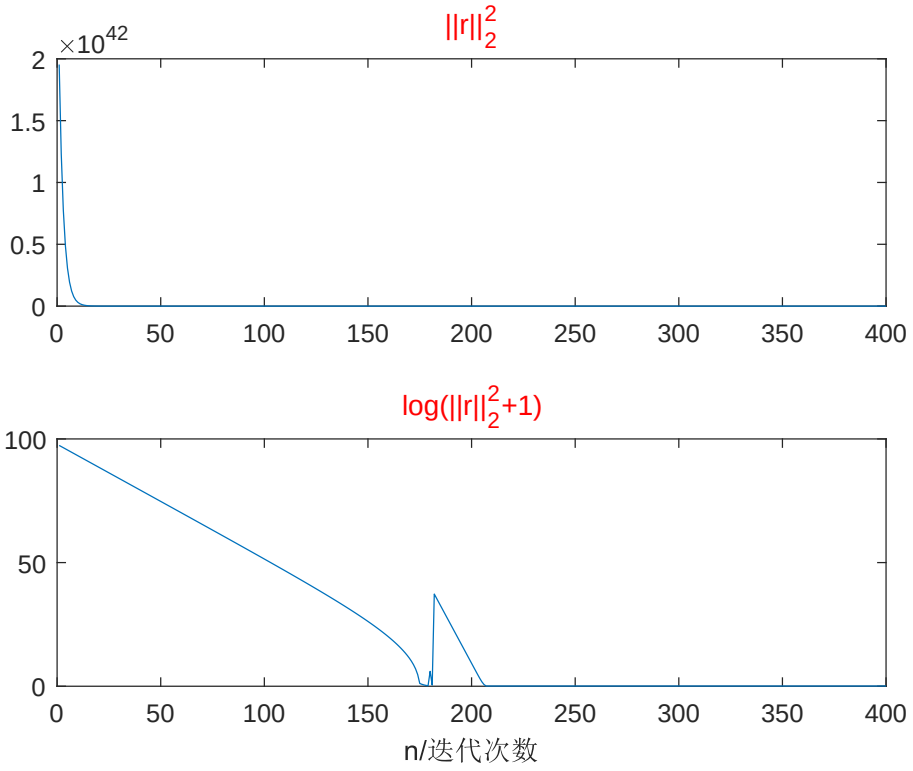
5.2 计算结果展示

迭代次数和求解的值如下:

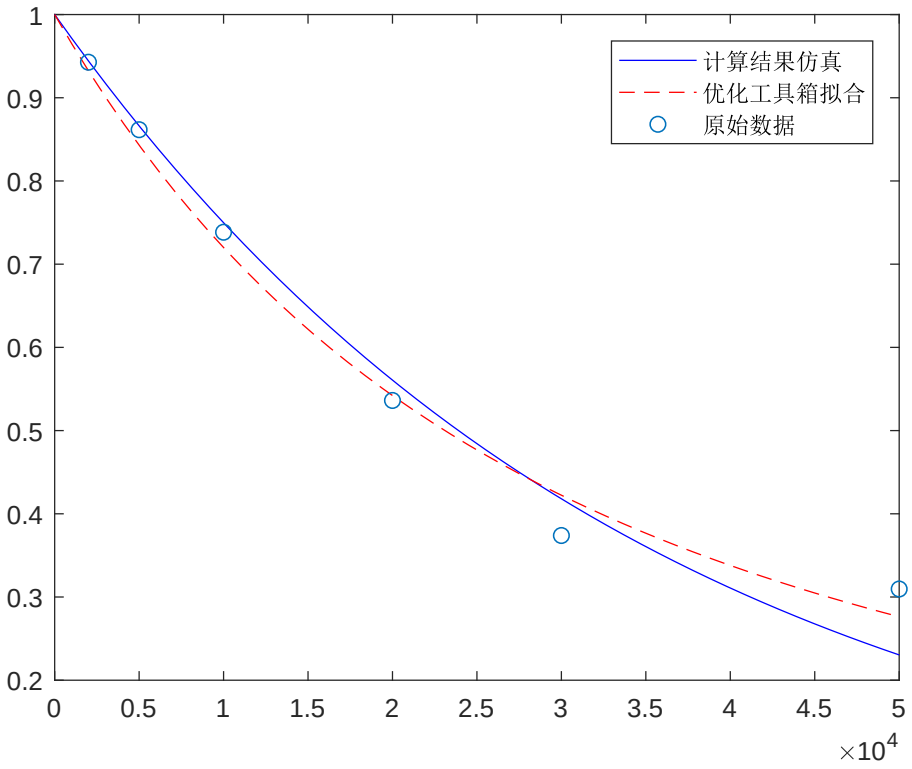
n=5	**n=8**	**n=12**	**n=20**
k=6	k=19	k=35	k=66
x	x	x	x
5.00E+00	5.90E-11	-9.61E+00	-1.10E+01
-1.20E+02	-6.97E-11	8.15E+02	1.05E+03
6.30E+02	-5.16E-10	-1.65E+04	-2.40E+04
-1.12E+03	1.12E-09	1.36E+05	2.20E+05
6.30E+02	3.17E-10	-5.36E+05	-9.65E+05
	-6.55E-10	1.03E+06	1.99E+06
	-6.57E-10	-6.43E+05	-1.25E+06
	4.59E-10	-6.58E+05	-1.34E+06
		8.04E+05	8.83E+05
		6.63E+05	1.69E+06
		-1.24E+06	3.88E+05
		4.66E+05	-1.31E+06
			-1.71E+06
			-5.28E+05
			1.21E+06
			2.00E+06
			9.45E+05
			-1.43E+06
			-2.65E+06
			1.89E+06

6 Problem 5.27

此题采用线搜索确定步长时，得到的结果误差极大，因为 $\phi'(0)$ 在离稳定点较远时数量级高达 10^{40} 量级，导致线搜索得到的步长极小，几乎为 0，无法收敛, 经反复调整参数都没能取得好的结果，最后只好手动确定步长 $\alpha_k = 0.05$，此时效果良好，残量的 2-范数随迭代次数的下降情况如下：（由于数量级巨大，为了更好的显示残差的波动，将原图中将残差取对数处理后并排参考）



又采用 MATLAB 中优化工具箱中的 lsqnonlin 函数进行拟合，得到的结果比我的程序算出来的略好，将两者进行比较，比较结果如下：



6.1 计算结果展示

表 4: 结果比较

	程序计算	工具箱拟合
stv	0.125639950119876	0.104420208306470
x_1	3.323336983976929e-04	-0.009615612533368
x_2	3.516673367231929e+02	-19.446505801429495

6.2 算法伪代码

Algorithm 8 Gauss-Newton method for problem(5.27)

1: Given $\boldsymbol{x}^{(0)}$ and G
2: Set $\boldsymbol{p}^{(0)} = -\boldsymbol{g}^{(0)}, k = 0$
3: **while** $\|\boldsymbol{g}^{(k)}\| > \epsilon$ **do**
4: Set $\alpha_k = -\dfrac{{\boldsymbol{p}^{(k)}}^T \boldsymbol{g}^{(k)}}{{\boldsymbol{p}^{(k)}}^T \boldsymbol{G} \boldsymbol{p}^{(k)}}$
5: Set $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha_k \boldsymbol{p}^{(k)}$
6: Set $\boldsymbol{g}^{(k+1)} = g(\boldsymbol{x}^{(k+1)})$
7: Set $\boldsymbol{p}^{(k)} = -\boldsymbol{g}^{(k)}$
8: k=k+1
9: **end while**