Using Brownian Bridge for Fast Simulation of Jump-Diffusion Processes and Barrier Options

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Barrier options are one of the most popular derivatives in the financial markets. The authors present a fast and unbiased Monte Carlo approach to pricing barrier options when the underlying security follows a simple jump-diffusion process with constant parameters and a continuously monitored barrier. Two algorithms are based on the Brownian bridge concept. The first one is based on a sampling approach to evaluate an integral that results from application of the Brownian bridge. The second approach approximates that integral using a Taylor series expansion.

Both methods significantly reduce bias and speed convergence compared to the standard Monte Carlo simulation approach. For example, the first method achieves zero bias. In addition, it is about 100 times faster than the conventional Monte Carlo method that achieves acceptable bias. In developing the second algorithm, the authors derive a novel approach for obtaining a first-passage time density integral using a Taylor series expansion. This approach is potentially useful in other applications, where the expectation of some function over the first-passage time distribution needs to be derived.

It is widely acknowledged that the standard assumption of lognormal stock price diffusion with constant volatility (as in the Black-Scholes framework) is inadequate to characterize financial markets. Empirical studies examining market returns and derivatives valuations all point to excess kurtosis and skewness.

To address these empirical observations, two dominant approaches have been researched: the stochastic volatility approach, and the jump-diffusion approach. In this article we consider the jump-diffusion approach.

Jump-diffusion processes were originally proposed by Merton [1976]. He derives an infinite-sum formula for a plain vanilla option, assuming normally distributed jump sizes (see also Trautmann and Beinert [1995] for an extension). Since then, many research articles have studied jump-diffusion processes. The problem is that for many exotic options analytical expressions have been very difficult to come by in a jump-diffusion framework.

We consider barrier options, one of the most popular exotic options. Barrier options can have a variety of possible features, but the general concept is that the payoff depends on whether the underlying asset price hits a specific barrier level.

There are generally two types of barrier options: knock-out and knock-in. For the knock-out, the option is valid only as long as the barrier is never touched during the life of the option. For the knock-in, the option becomes valid whenever the barrier is touched during the life of the option.

Because of a parity relationship between knock-out and knock-in options, it is generally sufficient to study one of the two types. In our case, we consider knock-out options. We also consider a constant rebate payment to be paid to the optionholder at the first time the barrier is hit, if there is any barrier crossing during the life of the option.

Valuation of barrier options calls for solution of the first-passage time problem, or boundary-crossing problem, which has been a widely studied topic in mathematics for the last 50 years (see Karatzas and Shreve [1991]). A closed-form expression for the case of Brownian motion has long been well-known (the so-called inverse Gaussian density). Based on this, an analytical formula for barrier options under the standard lognormal process has been derived (Hull [1997], Wilmott [1998]).

Closed-form expressions for the case of jump-diffusion processes are, however, very few. Kou and Wang [2000] derive an expression for the first-passage time when jump-sizes follow a double-exponential distribution (see also Boyarchenko and Levendorskii [2000], Mordecki [2000]).

Because of the difficulty in obtaining general analytical expressions for barrier options under the jump-diffusion framework, much of the work has focused on numerical or Monte Carlo valuation methods. We consider only the latter.

The typical Monte Carlo approach would be to examine the equivalent risk-neutral process, simulate jump-diffusion paths according to this process, and compute the expected discounted payoff by averaging over the many simulated paths. There is, however, a problem with this approach. Time discretization of the jump-diffusion paths introduces some bias in the pricing estimate.

This bias should not be underestimated. In our simulations we get biases as high as 10% (for a time discretization of one point per trading day). To reduce the bias, one should make the discretization fine enough, but as a result the computation becomes very slow.

Our work takes account of these two issues: the estimation bias, and the computation speed. Specifically, we propose two Monte Carlo techniques to value barrier options for jump-diffusion processes. The first method eliminates the bias completely, while the second method reduces it significantly. In addition, both methods are computationally orders of magnitude more efficient than the conventional Monte Carlo method.

The new algorithms are based on the Brownian bridge concept (see Karatzas and Shreve [1991]). This concept is used to obtain the boundary-crossing probability density, given that we observe the two end-points of the process.

The Brownian bridge concept has been used before in other related problems. Duffie and Lando [2001] use the concept to obtain the conditional distribution of a firm's assets at time *t,* given no default before *t*. Beaglehole, Dybvig, and Zhou [1997] and El Babsiri and Noel [1998] use the same concept to value options whose payoffs depend on extreme values, using pure diffusion processes for the underlying assets (see also Andersen and Brotherton-Ratcliffe [1996]). Atiya [2000] considers jump-diffusion processes, and uses the Brownian bridge concept to compute the boundary-crossing probability more efficiently.

We focus on a continuously monitored barrier, although a discretely monitored barrier is more common in practice. A number of authors take discrete monitoring into account. For example, Broadie, Glasserman, and Kou [1997] derive a correction to the barrier option price obtained under a continuously monitored barrier in order to obtain an approximation of the price under the discretely monitored barrier. Also, Nahum [1999] prices look-back options using jump-diffusion processes and approximates the discrete monitoring of the extreme value using Brownian meander.

For the two proposed approaches, we start by generating the jump-instants of the process, as well as the asset value immediately before and immediately after the jump-instant. In between these generated points, we have a pure diffusion with known end-points, hence a Brownian bridge. The two methods compute the expected payoff of the rebate portion of an interjump interval differently. The first proposed method, the uniform sampling approach, is based on sampling from a uniform distribution in a fashion similar to the rejection method. The second approach, is the series expansion approach. As the name suggests, it evaluates the payoff integral corresponding to the interjump interval using a series expansion.

I. MATHEMATICAL MODEL

We assume the underlying security follows a simple jump-diffusion process that consists of two parts, a continuous and a discontinuous part. The continuous part is a geometric Brownian motion with constant instantaneous drift μ and volatility σ. The discontinuous part represents the change in the security value upon arrival of the rare event. Rare events include major disasters or political changes, or the release of unexpected firm or economic news. The process is:

$$\frac{dS_t}{S_{t-}} = \mu \, dt + \sigma \, dZ_t + V_t dq_t \tag{1}$$

where the first two terms on the right-hand side represent the continuous part, the third term represents the discontinuous part, and $q = (q_t)_{t \geq 0}$ is a homogeneous Poisson process with a non-negative intensity parameter λ (annual frequency of jumps).

Let T_1, T_2, …, denote the arrival times of the jumps. Moreover, let $V_{T_i} = (S_{T_i^+}/S_{T_i^-}) - 1$, where V_{T_1}, V_{T_2}, …, are i.i.d. random variables representing the successive percentage changes in the security value at the jump-events. Note that $(S_{T_i^+}/S_{T_i^-}) \geq 0$ since the security price is assumed to be non-negative at all times. $Z = (Z_t)_{t \geq 0}$ is a standard Brownian motion $Z_T \sim N(0, T)$. We also assume that the processes Z, q, and V_{T_i} are jointly independent.

Let A_i be the logarithm of the ratio of the security value after and before the jump. We assume it is normally distributed and state-independent (we could assume other distributions as well):

$$A_i = \ln S_{T_i^+} - \ln S_{T_i^-} = \ln(V_{T_i} + 1)$$

and

$$f(A_i) \sim N(\mu_A, \sigma_A^2)$$

Let $J = \sum_{i=1}^{m} A_i$ be the sum of the log jump-sizes in the interval $[0, T]$. Under this assumption, J is also normally distributed:

$$J \sim N(\mu_A \lambda\, T,\ [\mu_A^2 + \sigma_A^2]\lambda\, T)$$

Also let

$$k = E^P(V_{T_i}) = \exp\left(\mu_A + \frac{\sigma_A^2}{2}\right) - 1$$

where P is the real-world measure.

Applying the Doléans-Dade stochastic exponential formula for semimartingales, we get a unique solution to the stochastic differential equation (see Metivier [1982], Jacod and Shiryayev [1987], and Duffie [2001]):

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma\, Z_T^Q + J^Q - \lambda^Q\, E^Q(V_{T_i})T\right] \tag{2}$$

where Q is the equivalent martingale measure that makes the security process a local martingale.

Equation (1) can also be written as

$$\frac{dS_t}{S_{t-}} = \mu\, dt + \sigma\, dZ_t + \int_{\Re} V_t\, \tilde{\nu}(dt, dA)$$

where $\tilde{\nu}$ is the P-compensated Poisson random measure, and the jump-size space is the real line $\Re$(Last and Brandt [1995]).

Pham [1997] characterizes the equivalent martingale measures by their Radon-Nykodim density with respect to P:

$$\frac{dQ}{dP} = \varepsilon\left(-\int_0^T \theta\, dZ_t + \int_0^T \int_{\Re} (p(A) - 1)\, \tilde{\nu}(dt, dA)\right)$$

where $\varepsilon(.)$ is the exponential semimartingale of Doléans-Dade, θ is the market price of the diffusion risk, and $p(A)$ is the market price of the jump-risk.

θ and $p(A)$ are related as follows:

$$\mu - r = \theta\, \sigma + \lambda \int_{\Re} V_{T_i}[1 - p(A)] f(A) dA \tag{3}$$

which is used to get a unique θ for a choice of $p(A)$.

Under the Q measure:

$$\lambda^Q = \lambda \int_{\Re} p(A) f(A) dA$$

$$f^Q(A) = \frac{p(A) f(A)}{\int_{\Re} p(A) f(A) dA}$$

$$Z_T^Q = Z_T + \theta\, T$$

where Z_T^Q is a Q-Brownian motion

$$J^Q = \sum_{i=1}^{m^Q} A_i$$

here m^Q: the number of jumps in the interval $[0, T]$, where the Poisson process q now has intensity of λ^Q.

$$E^Q(V_{T_i}) = \int_{\Re} V_{T_i} f^Q(A)\, dA$$

This gives us many equivalent martingale measures, which leads us into an incomplete market. Each measure has a different combination of the three parameters: the drift, the jump-intensity, and the jump-size distribution. In such an incomplete market, perfect hedging is not possible.

Pham [1999] reviews three different hedging criteria in an incomplete market: superhedging, mean-variance hedging, and shortfall risk minimization. Because we focus on the computational technique, we consider only the simplest measure and do not price the jump-risk (as considered by Merton [1976]). Our proposed computational approach does apply with minor modifications to the general case where the jump-risk is priced.

This means that $p(A) = 1$ (the real-world and risk-neutral distributions for the jump component are the same). From (3) we get

$$\theta = \frac{(\mu - r)}{\sigma}$$

which implies:

$$\lambda^Q = \lambda,\ f^Q(A) = f(A),\ E^Q(V_{T_i}) = k$$

Substituting this in (2) we get:

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma Z_T^Q + J - \lambda kT\right]$$

Let $x(t) = \ln S_t$

Then:

$$x(T) - x(0) = cT + \sigma Z_T^Q + J$$
$$\sim N([r - \frac{\sigma^2}{2} - \lambda k]T + \mu_A \lambda T,\ \sigma^2 T + [\mu_A^2 + \sigma_A^2]\lambda T) \quad (4)$$

where $c = r - [\sigma^2/2] - \lambda k$. The proposed algorithms consider mainly the transformed process $x(t)$.

In this article we apply our approach to down-and-out barrier call options, for which we use the shorthand DOC. For these options, a payoff of $\max(S_T - X, 0)$ is paid at maturity T if the barrier is never crossed during the life of the option. If a crossing occurs before maturity, a rebate is paid at the time of the first crossing, and there is no subsequent payoff.

The option value for a continuously monitored barrier is

$$DOC = \exp(-rT) \int_{\ln X}^{\infty} \max(\exp(x(T)) - X, 0) \times P(x(T) \in dx, \inf_{0 \le s \le T} x(s) > \ln(H))dx + R\int_0^T \exp(-rt)h(t)dt \quad (5)$$

where $\exp(-rt)$ is the discounting term, X is the strike value, H is the barrier level, R is the constant rebate value, and $h(t)$ is the density of the first-passage time. The term $P[x(T) \in dx, \inf_{0 \le s \le T} x(s) > \ln(H)]$ represents the joint distribution of the random variable $x(T)$ and the event that the asset price stays above the barrier during the life of the option (Harrison [1985]).

Note that the first-passage time density for jump-diffusion processes does not have a closed-form solution, except in two cases: when the jumps are doubly exponential or exponentially distributed (Kou and Wang [2000], Mordecki [2000]), or when the jumps can have only non-negative values (Blake and Lindsey [1973]). Our proposed algorithm represents efficient ways to evaluate the integrals in the valuation formula.

II. SUMMARY OF THE METHODS AND PRELIMINARIES

Description of the Proposed Methods

First, we generate the jump-instants from the inter-jump density function. Then, we generate the asset value immediately before the first jump using a normal distribution. Next, the value immediately after the jump is generated, using the jump-size distribution. Then, the value immediately before the second jump is generated using a normal distribution. We continue in this manner until we reach the expiration time.

We use a normal distribution because in between any two jumps the process follows a pure diffusion, and hence, given the previous jump value, the ending value is normally distributed.

If any of the generated values are below the barrier, we know that at some time the process has crossed the barrier. Yet there is also a chance that, previous to that, in between any two jumps the path has crossed the barrier. To tackle this situation, we consider the probability density of the barrier crossing-time, given the end-point values (which are already generated and are hence known). The derivation of such a density is based on the Brownian bridge concept described below.

In the first proposed method, we randomly generate a value for the discounted payoff corresponding to the particular interjump interval using that density (that is, the conditional barrier crossing-time density). Because there is also a non-zero probability that no crossing has occurred in that interval, we use the concept of rejection sampling, where "rejection" of a generated point means no crossing occurred in the interval.

This procedure is performed sequentially starting from the interval $t = 0$ until the first jump, then the interval from the first jump until the second jump, and so on until expiration time. If still no crossing has happened at that point, then the option has not been knocked out. We then generate the terminal asset value and compute the corresponding discounted payoff.

In the second approach, rather than generating the discounted payoff corresponding to any interjump interval randomly, we evaluate the expectation of the discounted payoff corresponding to that interval (again given the two end-points). That expectation, however, is an integral that is hard to evaluate in closed form. We approximate the integral using a series expansion. This method also proceeds in a sequential manner until reaching the option expiration time.

Brownian Bridge Concept

Let the jump-instants be $T_1, \ldots, T_K$. As mentioned, these are the first variables to be generated. Let $x(T_i^-)$ be the process value immediately before the *i-th* jump and $x(T_i^+)$ the process value immediately after the *i-th* jump. These values are generated sequentially, that is, first $x(T_1^-)$, then $x(T_1^+)$, then $x(T_2^-)$, then $x(T_2^+)$, and so on.

The process follows a pure Brownian motion in between any two jumps. Since we know the initial and terminal values, however, the Brownian motion is actually a tied-down Brownian motion or a Brownian bridge (Karatzas and Shreve [1991], Revuz and Yor [1994]).

Let B_s be a Brownian bridge in the interval $[T_{i-1}, T_i]$ with $B_{T_{i-1}^+} = x(T_{i-1}^+)$, $B_{T_i^-} = x(T_i^-)$, and $\tau = (T_i - T_{i-1})$ where $dx_t = cdt + \sigma dz_t^Q$. From Karatzas and Shreve [1991], we can obtain the probability that the minimum of B_s is always above the barrier in the interval τ:

$$P_i = P(\inf_{T_{i-1} \le s \le T_i} B_s > \ln H \mid B_{T_{i-1}^+} = x(T_{i-1}^+), B_{T_i^-} = x(T_i^-))$$
$$= \begin{cases} 1 - \exp\left(-\dfrac{2\left[\ln H - x(T_{i-1}^+)\right]\left[\ln H - x(T_i^-)\right]}{\tau\sigma^2}\right) & \textit{if} \quad x(T_i^-) > \ln H \\ 0 & \textit{otherwise} \end{cases} \tag{6}$$

where $\ln H$ is the barrier level.

We are also interested in obtaining the density function of the first-passage time t, given the two end-point values. First, define $C(t)$ as the event the process crosses the barrier for the first time in the interval $[t, t + dt]$. The conditional first-passage density is defined as:

$$g_i(t) = p(C(t) \in dt \mid x(T_{i-1}^+), x(T_i^-))$$

From Feller [1968]:

$$\begin{aligned} g_i(t) &= P(C(t) \in dt \mid x(T_{i-1}^+), x(T_i^-)) \\ &= \frac{P(C(t) \in dt, x(T_i^-) \in dx \mid x(T_{i-1}^+))}{P(x(T_i^-) \in dx \mid x(T_{i-1}^+))} \\ &= \frac{P(C(t) \in dt \mid x(T_{i-1}^+)) * P(x(T_i^-) \in dx \mid C(t), x(T_{i-1}^+))}{P(x(T_i^-) \in dx \mid x(T_{i-1}^+))} \\ &= \frac{P(C(t) \in dt \mid x(T_{i-1}^+)) * P(x(T_i^-) \in dx \mid x(t) = \ln H)}{P(x(T_i^-) \in dx \mid x(T_{i-1}^+))} \end{aligned} \tag{7}$$

From Rogers and Williams [1994], we have the first-passage time (inverse Gaussian) density:

$$P(C(t) \in dt \mid x(T_{i-1}^+)) = \frac{x(T_{i-1}^+) - \ln H}{\sqrt{2\pi}\sigma}(t - T_{i-1})^{-\frac{3}{2}} \times \exp\left(-\frac{[x(T_{i-1}^+) - \ln H + c(t - T_{i-1})]^2}{2(t - T_{i-1})\sigma^2}\right) \tag{8}$$

The other two components of the formula are the normal density functions (Feller [1968]):

$$P(x(T_i^-) \in dx \mid x(T_{i-1}^+)) = \frac{1}{\sqrt{2\pi(T_i - T_{i-1})}\sigma} \times \exp\left(-\frac{[x(T_i^-) - x(T_{i-1}^+) - c(T_i - T_{i-1})]^2}{2(T_i - T_{i-1})\sigma^2}\right) \tag{9}$$

$$P(x(T_i^-) \in dx \mid x(t) = \ln H) = \frac{1}{\sqrt{2\pi(T_i - t)}\sigma} \times \exp\left(-\frac{[x(T_i^-) - \ln H - c(T_i - t)]^2}{2(T_i - t)\sigma^2}\right) \quad (10)$$

Substituting (8), (9), and (10) in (7), after some algebraic grouping, we get:

$$g_i(t) = \frac{x(T_{i-1}^+) - \ln H}{2\gamma\pi\sigma^2}(t - T_{i-1})^{-\frac{3}{2}}(T_i - t)^{-\frac{1}{2}} \times \exp\left(-\left[\frac{(x(T_i^-) - \ln H - c(T_i - t))^2}{2(T_i - t)\sigma^2} + \frac{(x(T_{i-1}^+) - \ln H + c(t - T_{i-1}))^2}{2(t - T_{i-1})\sigma^2}\right]\right) \quad (11)$$

where

$$\gamma = \frac{1}{\sqrt{2\pi\tau}\sigma} \exp\left(-\frac{[x(T_{i-1}^+) - x(T_i^-) + c\tau]^2}{2\sigma^2\tau}\right) \quad (12)$$

This conditional first-passage density is used in the rebate payoff portion corresponding to the interval $[T_{i-1}, T_i]$.

III. THE PROPOSED SIMULATION ALGORITHMS

For ease of notation, let $T_0 = 0$ and $T_{K+1} = T$. Also let I represent the index of the first jump, if any, that crosses the barrier during the life of the option:

$$I = \min(i{:}\, x(T_j^-) > \ln H\ ;\ j = 1, \ldots, i,\ x(T_j^+) > \ln H$$
$$j = 1, \ldots, i-1,\ \ x(T_i^+) \le \ln H) \quad (13)$$

If no such I exists, then $I = 0$. Let $M(s) = M$ denote the index of the interjump period in which the time s falls. This means that $s \in [T_{M-1}, T_M]$. Then:

$$P(C(s) \in ds \mid x(T_{i-1}^+), x(T_i^-), i = 1, \ldots, K+1) = \begin{cases} g_M(s)\prod_{j=1}^{M-1} P_j & \text{if} \quad M < I \quad or \quad I = 0 \\ g_M(s)\prod_{j=1}^{M-1} P_j + \prod_{j=1}^{M} P_j \delta(s - T_I) & \text{if} \quad M = I \\ 0 & \text{if} \quad M > I \end{cases} \quad (14)$$

where δ is the Dirac's delta function.

The price of the down-and-out barrier option can then be computed as the expectation over $\{T_i, x(T_{i-1}^+), x(T_i^-){:}\ i = 1, \ldots, K+1\}$ of Equation (15), which represents the discounted expectation (with respect to the other random components) of the payoff:

$$DOC = \sum_{i=1}^{U} R \prod_{j=1}^{i-1} P_j \int_{T_{i-1}}^{T_i} \exp(-rs) g_i(s) ds + A_I \exp(-rT_I) R \prod_{j=1}^{I} P_j + (1 - A_I)\exp(-rT)\max(\exp(x(T)) - X, 0) \prod_{j=1}^{K+1} P_j \quad (15)$$

where

$$A_I = \begin{cases} 1 & \text{if} \quad I \ne 0 \\ 0 & \text{if} \quad I = 0 \end{cases}$$

and

$$U = \begin{cases} I & \text{if} \quad I \ne 0 \\ K+1 & \text{if} \quad I = 0 \end{cases}$$

The integral in Equation (15) cannot be evaluated in closed form. We have derived a series expansion method to evaluate it. By retaining up to the second-order term of the expansion, we can get a very accurate approximation. The detailed formulas are described in the appendix. We use this approximation in the second proposed approach.

Uniform Sampling

In the uniform sampling method we evaluate the integral in Equation (15) by sequentially generating a value from a distribution in the appropriate range, and then evaluating the integrand at the generated value of s. Specifically, the idea of the algorithm is to consider the interjump periods sequentially.

Consider, for example, the period $[T_{i-1}, T_i]$. We generate a variable from a distribution uniform in an interval starting from T_{i-1}, but extending beyond T_i by a factor of $1/(1 - P_i)$ (steps 4b and 4c in the algorithm below). This adjusts for the fact that the total probability of crossing the barrier any time within period i is $(1 - P_i)$.

If the generated point s falls in the interjump interval $[T_{i-1}, T_i]$, a barrier crossing has occurred at that generated point. We then sample the integrand function $g_i(s)$ at that generated point. Or, if the generated point s falls outside the interval $[T_{i-1}, T_i]$ (which happens with probability P_i), that point is rejected. This means no barrier crossing has occurred in the interval, and we proceed to the next interval and repeat the whole process again. The steps of the algorithm are as follows:

1. For $n = 1$ to N perform Monte Carlo runs as follows (steps 2–5):
2. Generate jump-instants T_i by generating the interjump times $(T_i - T_{i-1})$ according to the given density (e.g., exponential).
3. For $i = 1$ to K(K is the number of jumps that occur during the life of the option):
   a. Generate $x(T_i^-)$ from a Gaussian distribution of mean $x(T_{i-1}^+) + c(T_i - T_{i-1})$ and standard deviation $\sigma\sqrt{T_i - T_{i-1}}$. The initial state is $x(0) = x(T_0^+)$.
   b. Generate the size of jump i, A_i, according to the given jump-size distribution.
   c. Compute the post-jump value: $x(T_i^+) = x(T_i^-) + A_i$.
4. For intervals $i = 1$ to $K + 1$:
   a. Compute the intraperiod probability of no barrier crossing P_i according to Equation (6).
   b. Let $b = (T_i - T_{i-1})/(1 - P_i)$
   c. Generate s from a distribution uniform in the interval $[T_{i-1}, T_{i-1} + b]$.
   d. If $s \in [T_{i-1}, T_i]$, then the first-passage time to the barrier occurred in this interval $[T_{i-1}, T_i]$. In this case, we evaluate $g_i(s)$ by substituting the generated s into Equation (11). Then:
      - DiscPayoff(n) $= Rbg_i(s)\exp(-rs)$.
      - Exit loop, and perform another Monte Carlo cycle (steps 2–5).
   e. If $s \notin [T_{i-1}, T_i]$, then the first-passage time has not yet occurred.
   f. If $x(T_i^+) \leq \ln H$, then the jump crossed the barrier (index I as defined in Equation (13) equals i). The payoff becomes:
      - DiscPayoff(n) $= R\exp(-rT_I)$.
      - Exit loop, and perform another Monte Carlo cycle (steps 2–5).
   g. If $x(T_i^+) > \ln H$, then examine the next interval, that is, increment i, and perform another iteration of step 4.
5. No crossing occurred during the life of the option. The payoff is given by:
   - DiscPayoff(n) $= R\exp(-rT)\max(\exp(x(T)) - X, 0)$.
   - Perform another Monte Carlo cycle (steps 2–5).
6. If $n = N$, i.e., we have completed all cycles of the Monte Carlo simulation, obtain the estimate for the option price:

$$DOC = \frac{1}{N}\sum_{n=1}^{N} DiscPayoff(n)$$

Much of the complexity in the first algorithm comes from simulating the exact point within the interval at which the barrier is crossed. We also test a much simpler version of the uniform sampling approach. To save on computational effort, rather than evaluating the lengthy formula of $g_i(s)$ in step 4d, we simply assume that if interval $[T_{i-1}, T_i]$ is accepted as a crossing interval, the barrier crossing occurs at the midpoint: $(T_{i-1} + T_i)/2$. This is only an approximation, but if r is small, a small misestimation of the exact barrier crossing location will have little effect on the discounted value of the rebate.*

Series Expansion

In the second approach we focus on approximating the integral appearing in Equation (15), rather than evaluating it using a Monte Carlo sampling approach. The integral cannot be evaluated in closed form, unless $r = 0$, in which case it equals $1 - P_i$. Since r is usually small, a Taylor series expansion in r, retaining a few terms, would be fairly accurate (see the appendix for details of the series expansion).

We again proceed in a sequential way as in the uniform sampling approach. Here are the details of the algorithm:

1. For $n = 1$ to N, perform Monte Carlo runs as follows (steps 2–5):
2. Initialize DiscPayoff(n) = 0. Generate T_i by generating the interjump times $(T_i - T_{i-1})$ according to the given density (e.g., exponential).
3. For $i = 1$ to K(K is the number of jumps that occur during the life of the option):
   a. Generate $x(T_i^-)$ from a Gaussian distribution of mean $x(T_{i-1}^+) + c(T_i - T_{i-1})$ and standard deviation $\sigma\sqrt{T_i - T_{i-1}}$. The initial state is $x(0) = x(T_0^+)$.
   b. Generate A_i according to the given jump-size distribution.

c. Compute the post-jump value: $x(T_i^+) = x(T_i^-) + A_i$.

4. For $i = 1$ to $K + 1$:

a. Compute P_i according to Equation (6).

b. Compute the integral

$$J = \int_{T_{i-1}}^{T_i} \exp(-rs) g_i(s) ds$$

according to Equations (A-1) or (A-2) in the appendix.

c. DiscPayoff(n) = DiscPayoff(n) + $RJ\prod_{j=1}^{i-1} P_j$.

d. If $x(T_i^-) \leq \ln H$, then exit loop and perform another Monte Carlo cycle (steps 2-5). Note that in such a case, $P_j = 0$ for $j \geq i$, so the discounted payoff for subsequent ranges will be zero.

e. If $x(T_i^+) \leq \ln H$, then the jump crossed the barrier (index I as defined in Equation (13) equals i). The payoff becomes:

- DiscPayoff(n) = DiscPayoff(n) + $R \exp(-rT_I)\prod_{j=1}^{i} P_j$.
- Exit loop, and perform another Monte Carlo cycle (steps 2-5).

f. If $x(T_i^+) > \ln H$, then examine the next interval, that is, increment i, and perform another iteration of step 4.

5. No crossing occurred during the life of the option. This means that all values of $x(T_i^+)$; $i = 1, \ldots, K$ and $x(T_i^-)$; $i = 1, \ldots, K + 1$ are above the barrier level. The payoff is given by:

- DiscPayoff(n) = DiscPayoff(n) + $\exp(-rT) \prod_{j=1}^{K+1} P_j \max(\exp(x(T)) - X, 0)$.
- Perform another Monte Carlo cycle (steps 2-5).

6. If $n = N$, i.e., we have completed all cycles of the Monte Carlo simulation, obtain the estimate for the option price:

$$DOC = \frac{1}{N} \sum_{n=1}^{N} DiscPayoff(n)$$

A Note on Estimation Bias

The uniform sampling method provides an unbiased estimate of the option price. The reason is that DiscPayoff(n) is drawn from the distribution

$$P(DiscPayoff \in dx \,|\, x(T_{i-1}^+), x(T_i^-), i = 1, \ldots, K+1)$$

where DiscPayoff represents the true discounted payoff (note that we generate s, and s along with $x(T)$ uniquely determines the discounted payoff). Because T_i, $x(T_i^-)$, and $x(T_i^+)$ are generated according to their joint-distribution, we have

$$E(Discpayoff(n)) = DiscPayoff$$

and hence it is an unbiased estimate of the true option price. On the other hand, the midpoint approximation of the uniform sampling approach introduces some bias. Also, the Taylor series approach produces some bias because of series truncation. As we will see in the simulation experiments, for these two approaches the bias is very small.

IV. SIMULATION RESULTS

We test the proposed algorithms on a number of examples. To obtain an idea of the comparable advantage of the proposed method, we also implement the standard Monte Carlo procedure on these same examples.

In the standard Monte Carlo approach, we discretize time finely (let Δ be the discretization interval). Starting from zero time, we sequentially generate values of the risk-neutralized asset price by simulating the stochastic differential equation forward, and in the process we generate the Wiener increments from a normal distribution. We simulate the standard method using a variety of possible time discretization sizes Δ.

We run MATLAB programs on a Pentium III 733 MHz computer: 1 million Monte Carlo iterations for each method, except for the uniform sampling method we run 10 million iterations.

The reason for the difference is that we know the uniform sampling method has zero bias. We therefore use its simulation result to obtain the real value of the option price. Hence we use more runs to obtain a more accurate estimate. We measure the bias and the variance per iteration and the CPU time per million iterations for each method. We also compute a measure that combines speed and accuracy, by multiplying mean square (MS) error by the CPU time per iteration.

Example 1

In Example 1 we use parameter values as follows: $S_0 = 50$, $X = 55$, $H = 45$, $R = 1$, $r = 0.05$, $\sigma = 0.3$, $\lambda = 8$, $\sigma_A = 0.05$, $\mu_A = 0$, and $T = 1$ year.

Results are shown in Exhibit 1.

EXHIBIT 1
Example 1 Comparisons

Method	Bias (absolute)	Bias (%)	Std Dev (per iteration)	CPU Time (per million iterations)	MS Error* × CPU (per iteration)
Standard Monte Carlo Δ = 0.0002	0.05	1.18	9.9	115,923	11.362
Standard Monte Carlo Δ = 0.001	0.084	1.86	9.9	23,645	2.317
Standard Monte Carlo Δ = 0.004	0.174	3.86	9.9	6,774	0.664
Uniform Sampling	0	0	10.9	1,432	0.170
Uniform Sampling— Midpoint Approximation	–0.014	–0.31	9.8	1289	0.124
Series Expansion	–0.008	–0.18	9.28	10,647	0.916

*Standard deviation is per iteration, while the CPU time is per million iterations. The true value of the barrier option is 4.513. *Mean square error.*

EXHIBIT 2
Example 2 Comparisons

Method	Bias (absolute)	Bias (%)	Std Dev (per iteration)	CPU Time (per million iterations)	MS Error* × CPU (per iteration)
Standard Monte Carlo Δ = 0.0002	0.12	2.34	14.6	76,849	16.382
Standard Monte Carlo Δ = 0.001	0.28	5.22	14.9	16,621	3.691
Standard Monte Carlo Δ = 0.004	0.50	9.49	15.1	4,186	0.955
Uniform Sampling	0	0	14.7	529	0.114
Uniform Sampling— Midpoint Approximation	–0.005	–0.08	14.5	517	0.109
Series Expansion	–0.01	–0.2	11.9	3,249	0.460

*Standard deviation is per iteration, while the CPU time is per million iterations. The true value of the barrier option is 5.303. *Mean square error.*

EXHIBIT 3
Example 3 Comparisons

Method	Bias (absolute)	Bias (%)	Std Dev (per iteration)	CPU Time (per million iterations)	MS Error* × CPU (per iteration)
Standard Monte Carlo Δ = 0.0002	0.013	0.14	18.0	178,230	57.746
Standard Monte Carlo Δ = 0.001	0.054	0.60	18.1	37,973	12.440
Standard Monte Carlo Δ = 0.004	0.091	1.01	18.1	9,142	2.955
Uniform Sampling	0	0	18.1	666	0.218
Uniform Sampling— Midpoint Approximation	–0.008	–0.09	18.0	563	0.182
Series Expansion	–0.009	–0.10	17.63	3,737	1.162

*Standard deviation is per iteration, while the CPU time is per million iterations. The true value of the barrier option is 9.013. *Mean square error.*

Example 2

In Example 2, we use parameter values as follows: $S_0 = 100$, $X = 110$, $H = 95$, $R = 1$, $r = 0.05$, $\sigma = 0.25$, $\lambda = 2$, $\sigma_A = 0.1$, $\mu_A = 0$, and $T = 1$ year.

Results are shown in Exhibit 2.

Example 3

In Example 3, we use parameter values as follows: $S_0 = 100$, $X = 110$, $H = 85$, $R = 1$, $r = 0.05$, $\sigma = 0.25$, $\lambda = 2$, $\sigma_A = 0.1$, $\mu_A = 0$, and $T = 1$ year

Results are shown in Exhibit 3.

The standard deviation numbers are per iteration, so for M iterations we would divide the standard deviation by $\sqrt{M}$. One can see that the standard Monte Carlo method produces significant bias even as we shorten the time-step size at the expense of the CPU time. Also note that the uniform sampling method, in addition to producing zero bias, is significantly faster than the other two methods.

For acceptable bias for the standard Monte Carlo method, it seems that one should choose $\Delta = 0.0002$. For such a choice, the uniform sampling method is typically more than 100 times faster. The midpoint version of the uniform sampling approach is even a little faster, although at the expense of a little more bias.

The series expansion approach is also consistently better than the standard Monte Carlo method in terms of both bias and CPU time. The bias in the series expansion approach is due to approximation of the integral using only up to the second term in r in the Taylor series.

A surprising observation here is that the standard deviation per iteration does not differ much across methods. The uniform sampling method produces only one point per Monte Carlo path, while the series expansion method goes through great pains to compute expectations. Its improvement in standard deviation is not as great as we would have expected.

V. CONCLUSION

Obtaining efficient and accurate valuation of exotic options has been a major goal for academicians and financial institutions. Because of the lack of analytical solutions, improving the computational methodology has been of significant importance. As a step in this direction, we have presented a fast and low-bias approach to pricing down-and-out barrier options when the underlying security follows a jump-diffusion process and the barrier is continuously monitored. The use of the Brownian bridge significantly reduces bias and speeds convergence compared to the standard Monte Carlo approach.

Another approach for approximating the first-passage time density integral uses Taylor series expansion. We show that the series expansion approach is still a significantly better approach than the standard Monte Carlo approach. In follow-up work we plan to examine the issue of discrete monitoring of the barrier crossing, and incorporate stochastic volatility of the jump-diffusion process.

APPENDIX
Integral of the First-Passage Density

The integral

$$\int_{T_{i-1}}^{T_i} \exp(-rs)\, g_i(s)\, ds$$

appearing in Equation (15) cannot be evaluated in closed form, unless $r = 0$, in which case it will equal $1 - P_i$. Taking off from the fact that r is usually small (for example $r = 0.05$), we propose an approximation of the integral by a Taylor series expansion in r.

Specifically, we obtain the Laplace transform of the integral. Then we take the Taylor series expansion of the Laplace transform in terms of r. We retain up to the second-order terms of r. For the case of $r = 0.05$, for example, the third-order term is 0.000125, which is negligible. We then take the inverse Laplace transform of the truncated series.

For simplicity, let us shift the time axis by subtracting T_{i-1}. The integral becomes

$$J(\tau) = \exp(-rT_{i-1}) \int_0^{\tau} \exp(-rt)\, \hat{g}_i(t)\, dt$$

where $\tau = (T_i - T_{i-1})$, and $\hat{g}_i(t)$ is the time-shifted version of $g_i(t)$. From previous derivation we can use Equation (11) to get:

$$J(\tau) = \frac{\exp(-rT_{i-1})}{\gamma} \int_0^{\tau} f(t) h(\tau - t)\, dt$$

$$f(t) = \frac{\exp(-rt)\left[x(T_{i-1}^{+}) - \ln H\right]}{\sqrt{2\pi\sigma^2}}\, t^{-\frac{3}{2}} \exp\left(-\frac{(x(T_{i-1}^{+}) - \ln H + ct)^2}{2t\sigma^2} \right)$$

$$h(t) = \frac{1}{\sqrt{2\pi\sigma^2}}\, t^{-\frac{1}{2}} \exp\left(-\frac{(x(T_i^{-}) - \ln H - ct)^2}{2t\sigma^2} \right)$$

From Gradshteyn and Ryzhik [2000], we have the Laplace transforms:

$$\Im\left[at^{-\frac{3}{2}}\exp\left(-\frac{a^2}{4t}\right)\right] = 2\sqrt{\pi}\exp(-a\sqrt{s}) \qquad a > 0$$

and

$$\Im\left[t^{-\frac{1}{2}}\exp\left(-\frac{a^2}{4t}\right)\right] = \sqrt{\pi}\,\frac{\exp(-|a|\sqrt{s})}{\sqrt{s}}$$

Using these, together with the shift and convolution theorems, we get the Laplace transform of $J(\tau)$:

$$J(s) = \exp(-rT_{i-1}) \times \frac{\exp\left(-\frac{c}{\sigma^2}\left[x(T_{i-1}^{+}) - x(T_i^{-}) + \left[x(T_{i-1}^{+}) - \ln H\right]\sqrt{\frac{2\sigma^2 s}{c^2}+1}\right]\right)}{\sigma\sqrt{2}\gamma} \times \frac{\exp\left(-\left|\frac{\left[x(T_i^{-}) - \ln H\right]}{\sigma}\right|\sqrt{2}\sqrt{s+\frac{c^2}{2\sigma^2}-r}\right)}{\sqrt{s+\frac{c^2}{2\sigma^2}-r}}$$

We take the Taylor series expansion of $J(s)$ in terms of r, and keep up to the second-order term of r. Then we take the inverse Laplace transform of the resulting series. After some algebra and simplification, the final expression for J turns out to be:

For the case that $x(T_i^-) > \ln H$:

$$J(\tau) \approx \exp(-rT_{i-1})\exp\left(-\frac{2\left[x(T_{i-1}^{+}) - \ln H\right]\left[x(T_i^{-}) - \ln H\right]}{\tau\sigma^2}\right) + \frac{r\left[x(T_{i-1}^{+}) - \ln H\right]\exp(rT_i - 2rT_{i-1})}{8\sigma}\left(A_1 + C_1 B\right) \qquad \text{(A-1)}$$

For the case that $x(T_i^-) \leq \ln H$:

$$J(\tau) \approx \exp(-rT_{i-1}) + \frac{r\left[x(T_{i-1}^{+}) - \ln H\right]\exp(rT_i - 2rT_{i-1})}{8\sigma}\left(A_2 + C_2 B\right) \qquad \text{(A-2)}$$

where

$$A_1 = \frac{2r}{\sigma}\tau\left[x(T_{i-1}^{+}) - x(T_i^{-})\right]\exp\left(-\frac{2\left[x(T_{i-1}^{+}) - \ln H\right]\left[x(T_i^{-}) - \ln H\right]}{\tau\sigma^2}\right)$$

$$C_1 = \sqrt{2\pi\tau}\;\exp\left(\frac{\left[x(T_{i-1}^{+}) - x(T_i^{-})\right]^2}{2\tau\sigma^2}\right)\left[\Phi\left(\frac{x(T_{i-1}^{+}) + x(T_i^{-}) - 2\ln H}{\sqrt{2\tau\sigma^2}}\right) - 1\right]$$

$$B = 8 - 2r\tau + \frac{2r}{\sigma^2}\left[x(T_{i-1}^{+}) - x(T_i^{-})\right]\left[x(T_{i-1}^{+}) + x(T_i^{-}) - 2\ln H\right]$$

$$A_2 = \frac{2r\tau}{\sigma}\left[x(T_{i-1}^{+}) + x(T_i^{-}) - 2\ln H\right]$$

$$C_2 = \sqrt{2\pi\tau}\;\exp\left(\frac{\left[x(T_{i-1}^{+}) - x(T_i^{-})\right]^2}{2\tau\sigma^2}\right)\left[\Phi\left(\frac{x(T_{i-1}^{+}) - x(T_i^{-})}{\sqrt{2\tau\sigma^2}}\right) - 1\right]$$

where $\Phi(z)$ is the cumulative normal distribution:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}\exp\left(-\frac{y^2}{2}\right)dy$$

The accuracy of the approximation has been verified using a number of numerical examples.

ENDNOTES

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