CS 740 Execise 1

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1. Show: $2^n \in O(n!)$

n	2^n	$n!$	$(2^n < n!)?$
1	2	1	✗
2	4	2	✗
3	8	6	✗
4	16	24	✓
5	32	120	✓

Table 1: Enumerating function values for $n \in [1, 4]$

With the enumeration in Table 1, let's assume $C = 1$ and $n_0 = 4$, s.t.
$2^n \leq C.n! = n!$ for all $n \geq n_0 = 4$.

Proof. (With mathematical induction)
① When $n = 4$, we have $2^4 = 16$ and $4! = 24$, so $2^n \leq n!$ holds.
② Assume $2^n \leq n!$ holds for $n = k$ where $k > 4$.
Then, when $n = k + 1$, we have $2^{k+1} = 2.2^k \leq 2.k! \leq (k + 1).k! = (k + 1)!$,
which means $2^{k+1} \leq (k + 1)!$
$\therefore 2^n \leq n!$ for all $n \geq 4$
$\therefore 2^n \in O(n!)$ $\square$

2. Show: $n! \notin O(2^n)$

Assume $n! \in O(2^n)$, we have $\exists C_0$ and n_0 s.t. $n! \leq C_0.2^n$ for all $n \geq n_0$.
And according to Execise 1, we have $\exists C_1$ and n_1 s.t. $2^n \leq C_1.n!$ for all $n \geq n_1$.
Let $C_1 = C_0$, we can compute $n_0 \leq n_1 < +\infty$ such that $2^n \leq C_1.n!$ for all $n \geq n_1$, which controdicts the assumption.
$\therefore n! \notin O(2^n)$

3. Show: if $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$

$\because f \in O(g)$
$\therefore \exists\ C$ and n_0 s.t. $f(n) \leq C.g(n)$ for all $n \geq n_0$
And $\because g \in O(h)$
$\therefore \exists\ C'$ and n_0' s.t. $g(n) \leq C'h(n)$ for all $n \geq n_0'$.
$\Rightarrow C.g(n) \leq C.C'.h(n)$ for all $n \geq n_0'$.

Let $C_1 = C.C'$ and $n_1 = \max\{n_0, n_0'\}$, we have $f(n) \leq C_1.h(n)$ for all $n \geq n_1$.
$\therefore f \in O(h)$