

# Gaussian Mixture Models

## General Principles

To discover group structures or clusters in data, we can use a **Gaussian Mixture Model (GMM)**. This is a parametric clustering method. A GMM assumes that the data is generated from a mixture of a **pre-specified number ( $K$ )** of different Gaussian distributions. The model's goal is to figure out:

1. **The properties of each of the  $K$  clusters:** For each of the  $K$  clusters, it estimates its center (mean  $\mu$ ) and its shape/spread (covariance  $\Sigma$ ).
2. **The mixture weights:** It estimates the proportion of the data that belongs to each cluster.
3. **The assignment of each data point:** It determines the probability of each data point belonging to each of the  $K$  clusters.

## Considerations

### Caution

- A GMM is a Bayesian model that considers uncertainty in all its parameters, *except for the number of clusters,  $K$* , which must be fixed in advance.
- The key parameters and their priors are:
  - **Number of Clusters  $K$ :** This is a **fixed hyperparameter** that you must choose before running the model. Choosing the right  $K$  often involves running the model multiple times and using model comparison criteria (like cross-validation, AIC, or BIC).
  - **Cluster Weights  $w$ :** These are the probabilities of drawing a data point from any given cluster. Since there are a fixed number  $K$  of them and they must sum to 1, they are typically given a **Dirichlet** prior. A symmetric **Dirichlet** prior (e.g., `Dirichlet(1, 1, ..., 1)`) represents an initial belief that all clusters are equally likely.

- **Cluster Parameters ( $\mu, \Sigma$ ):** Each of the  $K$  clusters has a mean  $\mu$  and a covariance matrix  $\Sigma$ . We place priors on these to define our beliefs about their plausible values.
- Like the DPMM, the model is often implemented in its marginalized form . Instead of explicitly assigning each data point to a cluster, we integrate out this choice. This creates a smoother probability surface for the inference algorithm to explore, leading to much more efficient computation.
- To increase accuracy we run a k-means algorithm to initialize the cluster mean priors.

## Example

Below is an example of a GMM implemented in BI. The goal is to cluster a synthetic dataset into a pre-specified  $K=4$  groups.

## Python

```
from BI import bi
from sklearn.datasets import make_blobs

# Generate synthetic data
data, true_labels = make_blobs(
    n_samples=500, centers=8, cluster_std=0.8,
    center_box=(-10,10), random_state=101
)

# The model
def gmm(data, K, initial_means): # Here K is the *exact* number of clusters
    D = data.shape[1] # Number of features
    alpha_prior = 0.5 * jnp.ones(K)
    w = dist.dirichlet(concentration=alpha_prior, name='weights')

    with dist.plate("components", K): # Use fixed K
        mu = dist.multivariatenormal(loc=initial_means, covariance_matrix=0.1*jnp.eye(D), name='mu')
        sigma = dist.halfcauchy(1, shape=(D,), event=1, name='sigma')
        Lcorr = dist.lkjcholesky(dimension=D, concentration=1.0, name='Lcorr')
```

```

scale_tril = sigma[..., None] * Lcorr

dist.mixturestamefamily(
    mixing_distribution=dist.categorical(probs=w, create_obj=True),
    component_distribution=dist.multivariatenormal(loc=mu, scale_tril=scale_tril, create_obj=True),
    name="obs",
    obs=data
)

m.data_on_model = {"data": data, "K": 4 }
m.fit(gmm) # Optimize model parameters through MCMC sampling
m.plot(X=data, sampler=m.sampler) # Prebuild plot function for GMM

```

## Mathematical Details

This section describes the generative process for a GMM.

$$\begin{pmatrix} Y_{i,1} \\ \vdots \\ Y_{i,D} \end{pmatrix} \sim \text{MVN} \left( \begin{pmatrix} \mu_{z_i,1} \\ \vdots \\ \mu_{z_i,D} \end{pmatrix}, \Sigma_{z_i} \right)$$

$$\begin{pmatrix} \mu_{k,1} \\ \vdots \\ \mu_{k,D} \end{pmatrix} \sim \text{MVN} \left( \begin{pmatrix} A_{k,1} \\ \vdots \\ A_{k,D} \end{pmatrix}, B \right)$$

$$\Sigma_k = \sigma_k \Omega_k \sigma_k$$

$$\sigma_k \sim \text{HalfCauchy}(1)$$

$$\Omega_k \sim \text{LKJ}(2)$$

$$z_i \sim \text{Categorical}(\pi)$$

$$\pi \sim \text{Dirichlet}(0.5, \dots, 0.5)$$

Where :

- $\begin{pmatrix} Y_{[i,1]} \\ \vdots \\ Y_{[i,D]} \end{pmatrix}$  is the  $i$ -th observation of a D-dimensional data array.
- $\begin{pmatrix} \mu_{[k,1]} \\ \vdots \\ \mu_{[k,D]} \end{pmatrix}$  is the  $k$ -th parameter vector of dimension D.
- $\begin{pmatrix} A_{[k,1]} \\ \vdots \\ A_{[k,D]} \end{pmatrix}$  is a prior for the  $k$ -th mean vector as derived by a *KMEANS* clustering algorithm.
- $B$  is the prior covariance of the cluster means, and is setup as a diagonal matrix with 0.1 along the diagonal.
- $\Sigma_k$  is the DxD covariance matrix of the  $k$ -th cluster (it is composed from  $\sigma_k$  and  $\Omega_k$ ).
- $\sigma_k$  is a diagonal matrix of standard deviations for the  $k$ -th cluster.
- $\Omega_k$  is a correlation matrix for the  $k$ -th cluster.
- $z_i$  is a latent variable that maps observation  $i$  to cluster  $k$ .
- $\pi$  is a vector of  $K$  cluster weights.

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## Notes

### Note

The primary challenge of the GMM compared to the DPMM is the need to **manually specify the number of clusters  $K$** . If the chosen  $K$  is too small, the model may merge distinct clusters. If  $K$  is too large, it may split natural clusters into meaningless subgroups. Therefore, applying a GMM often involves an outer loop of model selection where one fits the model for a range of  $K$  values and uses a scoring metric to select the best one.

## Reference(s)

C. M. Bishop (2006). *Pattern Recognition and Machine Learning*. Springer. (Chapter 9)