Introduction to Machine Learning (67577)

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Exercise 1

2. Theoretical Part

2.1 Mathematical Background

2.1.1 Linear Algebra

1

Let l_2 be an Euclidean norm: $\forall v \in \mathbb{R}^d : \|v\|_2 = \sqrt{\sum v_i^2}$
Let A be a corresponding matrix of $T : V \to W$
If A is orthogonal, then:

$$\forall x \in V \ \ \|Ax\|_2^2 = (Ax)^T(Ax) = x^T A^T Ax =$$

$$\overbrace{=}^{A\,is\,orthogonal} \quad x^T I x = x^T x = \|x\|_2^2$$

Since each norm holds: $\|\cdot\| \geq 0$, we get:

$$\forall x \in V \ \ \|Ax\|_2^2 = \|x\|_2^2$$

$$\iff$$

$$\forall x \in V \ \ \|Ax\|_2 = \|x\|_2$$

Any two norms on a finite-dimensional vector space are equivalent, so:

$$\forall x \in V \ \ \|Ax\| = \|x\| \Rightarrow T\ is\ isometric\ tranformation$$

2

Conversely, given the SVD of $A = U\Sigma V^T$ we can recover the EVD of $A^T A$ or of AA^T:

$$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

It holds:

$$AA^T = \left(U\Sigma V^T\right)\left(U\Sigma V^T\right)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T - EVD\ of\ AA^T$$

$$A^T A = \left(U\Sigma V^T\right)^T \left(U\Sigma V^T\right) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T\Sigma V^T - EVD\ of\ A^T A$$

Let compute the eigenvalues of the $AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$:

$$det(AA^T - \lambda I) = (2 - \lambda) \cdot (6 - \lambda) = 0$$

$$\iff$$

$$\lambda_1 = 2, \lambda_2 = 6$$

For $\lambda = 2$ the eigenvector is a vector:

$$(AA^T - 2I)(v_1) = 0$$

$$\iff$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \cdot v_1 = 0$$

$$\iff$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda = 6$ the eigenvector is a vector:

$$(AA^T - 6I)(v_1) = 0$$

$$\iff$$

$$\begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} \cdot v_2 = 0$$

$$\iff$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore, $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let compute the Σ matrix:

$$AA^T = U\Sigma\Sigma^T U^T = U\Sigma^2 U^T =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\iff$$

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{6} \end{bmatrix}$$

Given that $A = U\Sigma V^T$ we can deduce that $\Sigma^{-1}U^T A = V^T$:

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

We got:

$$U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{6} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} = A$$

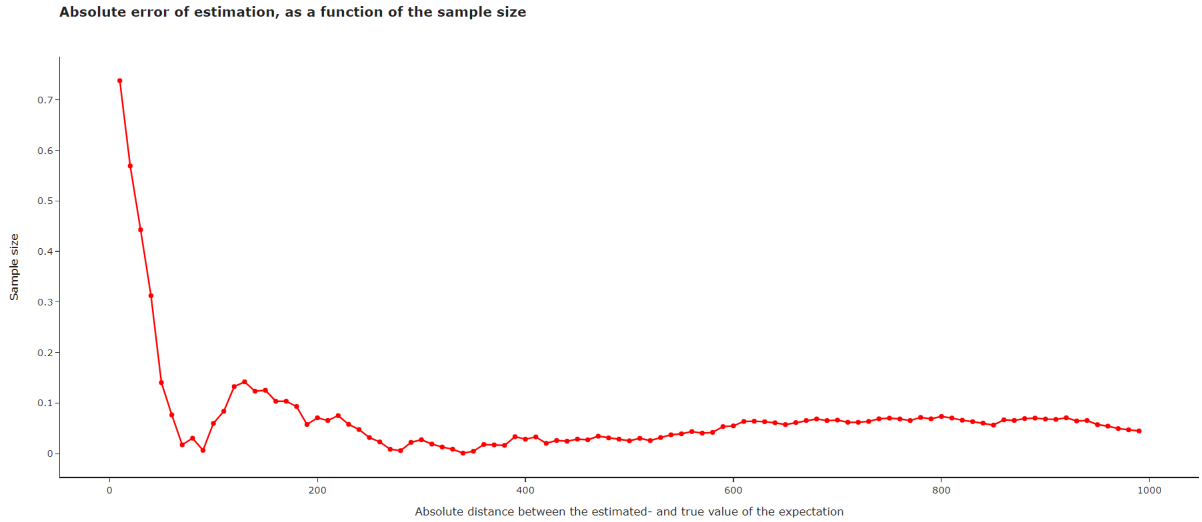
3. Practical Part

3.1 Univariate Gaussian Estimation

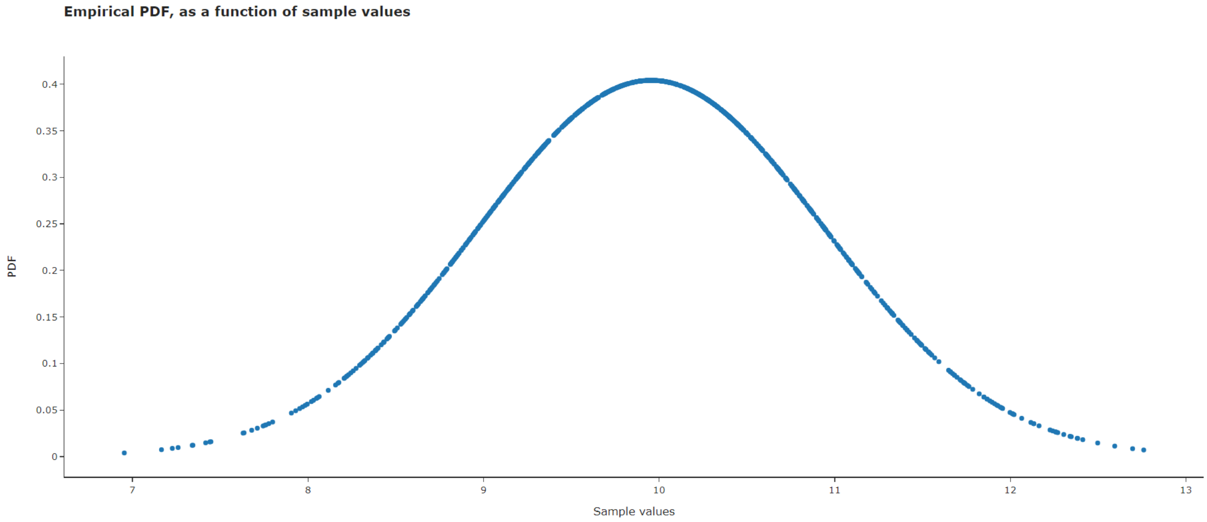
1

```
(9.954743292509804, 0.9742344563121542)
```

2



3



I would expect to see a graph depicting a distribution very similar to a normal distribution with 10 and variance of 1

3.1 Multivariate Gaussian Estimation

4

```
[-0.02282878 -0.04313959  3.9932571  -0.02038981]
[[ 0.91667608  0.16634444 -0.03027563  0.46288271]
 [ 0.16634444  1.9741828  -0.00587789  0.04557631]
 [-0.03027563 -0.00587789  0.97960271 -0.02036686]
 [ 0.46288271  0.04557631 -0.02036686  0.9725373 ]]
```

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