DISCRETE SUBGROUPS OF SL(2,$\mathbb{R}$)

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To my fellow students at the University of Puerto Rico who fight day by day for a better and accessible education for our Country and its future generations.

Abstract. The purpose of this project is to explore the close relation that exists between some functions in number theory such as the arithmetic sigma function, the sum of k squares function, the partition function and the Ramanujan tau function, with the topological group SL(2,$\mathbb{R}$) and its discrete subgroups Γ. This will be achieved through the study of modular forms its properties and its generators.

Introduction

Whenever a mathematician comes up with a conjecture he or she tries to solve it with the tools that are known to the field where the problem seems to belong. However, what does one do when they realize that the route that they are taking to solve the problem is leading them towards a dead end? They start approaching the problem from different fields of mathematics which sometimes were not thought to be closely related. Often times the results turn out to be beautiful and lead to a great advance in mathematics. In the past there have been conjectures already proven but when approach from different fields of mathematics new results have been found along with new information. Along these lines we aim to derive well known formulas in number theory using elements from the study of modular forms.

An interesting example in point is the Euler phi function. For every n that belongs to the set of natural numbers the function $\phi(n)$, gives the number positive integers not exceeding n that are relatively prime to n. This function can also be written as a modular form using the Dedekind eta function which extends the domain from the natural numbers, adding all complex numbers z with $|z| < 1$. Here is the astonishing result, taking the reciprocal of the modular function ϕ gives rise to an infinite series whose coefficients are partition functions. For m a natural number the partition function of m gives the number of ways on which m can be written as a sum of positive numbers for which the order does not matter. The Ramanujan tau function as the partition function can also be constructed with the Dedekind eta function. This tau function is closely related to the arithmetic sigma function which will be discussed and constructed later on.

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The Eisenstein series and the arithmetic sigma function are just another great example that will be discussed later on, showing just how linked modular forms and some functions in number theory are. One can start with an Eisenstein series G_k and after computing the coefficients they will get the arithmetic sigma function for the $k-1$ power. The most interesting result from this function comes from the cases where $k=1$ and $k=2$. When $k=1$, the σ function of any natural number m will give us the amount of divisors of m and when we set $k=2$ the σ function of any natural number m yields the sum of the divisors of m which is used in the search for perfect numbers.

Another modular form that will appear later on is the Jacobi theta function. With the use of the Jacobi theta function we will construct a different modular form which is a power of the θ function. This new modular form when expressed as an infinite series, will have as its coefficients the function $r_k(n)$. The function $r_k(n)$ is known as the sum of k squares function and it gives the number of ways n can be written as a sum of k squares without distinguishing the order.

1. Discrete subgroups of SL(2,$\mathbb{R}$)

Definition 1.1. A topological group G is a group that is also a topological space satisfying the T_1 axiom, such that the map of $G \times G$ into G sending (x,y) to $x \cdot y$, and the map of G into G sending x into x^{-1}, are continuous map.

Definition 1.2. Let G be a topological group, a subgroup Γ of G is a discrete subgroup if the induced topology on Γ is discrete.

Example 1.3. $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$.

Example 1.4. SL(2,$\mathbb{Z}$) is a discrete subgroup of SL(2,$\mathbb{R}$).

Proposition 1.5. *Let G be a locally compact group, and K a compact subgroup of G. Define $S = G/K$ and let h: $G \to S$ be the natural map. If A is a compact subset of S, $h^{-1}(A)$ is compact.*

Proof:

Take an open covering of G whose members have compact closures, and consider their images on S by h. Then $A \subset \cup_i h(V_i)$ with finitely many open sets V_i whose closures $\overline{V_i}$ are compact. Hence $h^{-1}(A) \subset \cup_i \overline{V_i} K$. Observe that $\overline{V_i} K$ is compact. Therefore $h^{-1}(A)$ is compact since is the closed subset of a compact set. This proof was taken out from *Arithmetic Theory of Automorphic Functions*. See [1].

Proposition 1.6. *Let G, K, S, and h be as in proposition 1.5, and Γ a subgroup of G. Then the following are equivalent:*

(1) Γ is a discrete subgroup of G.

(2) For any two compact subsets A and B of S, $\{g\in\Gamma |\ g(A)\cap B\neq\emptyset\}$ is a finite set.

Proof:

($\Rightarrow$)
Let A and B be compact subsets of S, and let $C=h^{-1}(A)$, $D=h^{-1}(B)$, $g\in\Gamma$. If $g(A)\cap B\neq\emptyset$, one has $g(C)\cap D\neq\emptyset$, hence $g\in\Gamma\cap(DC^{-1})$. By proposition 1.6 A and B are compact, hence DC^{-1} is compact. If Γ is discrete, $\Gamma\cap(DB^{-1})$ is both compact and discrete, hence must be finite.

($\Leftarrow$)
Let V be a compact neigborhood of e in G, and let $t = h(e)$. Then $\Gamma\cap V\subset\{ g\in\Gamma |\ gt\in h(V)\}$. Now let t=A and $h(V)$=B, then $\Gamma\cap V$ is a finite set. Therefore Γ is discrete. This completes the proof. This proof was taken out from *Arithmetic Theory of Automorphic Functions*. See [1].

Now that we have a general idea of the topology of a discrete subgroup Γ of SL(2,ℝ), we can define and state some of the properties of the parabolics and hyperbolics elements of SL(2,ℝ).

Definition 1.7. An element $\sigma\in$ SL(2,ℝ), $\sigma\neq\pm 1_2$ is said to be parabolic if it is conjugate to a matrix of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ for some $\lambda\in\mathbb{R}$.

Proposition 1.8. *$\sigma\in$ SL(2,ℝ) and $\sigma\neq\pm 1_2$, σ is parabolic $\Leftrightarrow$ tr(σ)= ± 2.*

Proof:

($\Rightarrow$)
Let $\sigma\in$ SL(2,ℝ) and $\sigma\neq\pm 1_2$, and assume that σ is parabolic. Consider the characteristic polynomial of σ, x^2+ tr$(\sigma)x+$det(σ)=0. Since $\sigma\in$SL(2,ℝ) the det(σ)=1. Then the characteristic polynomial becomes x^2+ tr$(\sigma)x+1$=0. We now have that 2λ=tr(σ) and λ^2=1. Then, $\lambda=\pm 1$ so $2\lambda = \pm 2 =$ tr(σ).

($\Leftarrow$)
Let $\sigma\in$ SL(2,ℝ) and $\sigma\neq\pm 1_2$, and assume that tr$(\sigma)=\pm 2$. Consider the characteristic polynomial of σ, x^2+ tr$(\sigma)x+$det(σ)=0. Since tr$(\sigma)=\pm 2$, consider the two cases:

Case I: tr(σ)=2 then $x^2+2x+1=(x+1)^2=0$, then $x = -1$ is a root of order two of this polynomial. So we have $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ which is parabolic by definition 1.7.

Case II: tr(σ)=-2 then $x^2-2x+1=(x-1)^2=0$, then $x=1$ is a root of order two of this polynomial. And we get $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ which is parabolic by definition 1.7.

This completes the proof of proposition 1.8.

Proposition 1.9. *Let $\sigma \in$ SL(2,$\mathbb{R}$), $\sigma \neq \pm 1_2$, then σ is parabolic if it has only one fixed point in $\mathbb{R} \cup \{\infty\}$.*

Proof:

For all $s \in \mathbb{R}\cup\{\infty\}$, let

$$F(s)=\{\alpha\in\mathrm{SL}(2,\mathbb{R})|\ \alpha(s)=s\},$$

$$P(s)=\{\alpha\in F(s)|\ \alpha \text{ is parabolic or } \pm 1_2\}.$$

Since SL(2,$\mathbb{R}$) acts transitively on $\mathbb{R}\cup\{\infty\}$, we can find an element σ of SL(2,$\mathbb{R}$) so that $\sigma(\infty)=s$. Then $F(s)=\sigma F(\infty)\sigma^{-1}$, $P(s)=\sigma P(\infty)\sigma^{-1}$. Now it is easy to check that

$$F(\infty)=\left\{ \begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix}|\ a\in\mathbb{R}^{\times},\ b\in\mathbb{R} \right\},$$

$$P(\infty)=\left\{ \pm\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}|\ h\in\mathbb{R} \right\} \cong\mathbb{R}\times\{\pm 1\}.$$

Therefore if an element σ of SL(2,$\mathbb{R}$) with $\sigma\neq\pm 1_2$ has only one fixed point on $\mathbb{R}\cup\{\infty\}$, then σ is parabolic. Let $\sigma\in$SL(2,$\mathbb{R}$), $\sigma\neq\pm 1_2$, and let $m\in\mathbb{Z}$ with $\sigma^m\neq\pm 1_2$. Then σ is parabolic if and only if σ^m is parabolic. This completes the proof. This proof was taken out from *Arithmetic Theory of Automorphic Functions*. See [1].

Definition 1.10. An element $\sigma\in$ SL(2,$\mathbb{R}$), $\sigma\neq\pm 1_2$ is said to be hyperbolic if it is the conjugate of a matrix of the form $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ for some $\lambda\neq\mu$, with c real and positive.

Definition 1.11. A point s of $\mathbb{R} \cup \{\infty\}$ is called a *cusp* if there exists a parabolic element τ of Γ such that $\tau(s) = s$. Also if ω is a *cusp* of Γ and $\gamma \in \Gamma$, then $\gamma(\omega)$ is also a *cusp* of Γ.

Proposition 1.12. *Let s be a cusp of Γ, and let $\Gamma_s = \{\sigma\in\Gamma|\sigma(s)=s\}$. Then Γ_s/($\Gamma\cap\{\pm 1\}$) is isomorphic to $\mathbb{Z}$. Moreover $\Gamma_s=\Gamma\cap P(s)$.*

Proof:

We know that $P(s)$ is isomorphic to $\mathbb{R}\times\{\pm1\}$. Therefore $(P(S)\cap\Gamma)/(\Gamma\cap\{\pm1\})$ is isomorphic to a non-trivial discrete subgroup of $\mathbb{R}$, Hence isomorphic to $\mathbb{Z}$. Without loss of generality assume that $s{=}\infty$, and take a generator $\sigma{=}\begin{pmatrix}\pm1 & h\\ 0 & \pm1\end{pmatrix}$ (modulo ±1) of this group. Assume that Γ_s contains a hyperbolic element $\tau = \begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix}$ with $|a|{\neq}1$, and consider τ^{-1}, if necessary assume that $|a|{<}1$. Then $\tau\sigma\tau^{-1}{=}\begin{pmatrix}\pm1 & a^2h\\ 0 & \pm1\end{pmatrix}{\in} P(s){\cap}\Gamma$. This is a contradiction, since $|a^2h|{<}|h|$. Therefore $\Gamma_s{=}P(s){\cap}\Gamma$. This completes the proof. This proof was taken out from *Arithmetic Theory of Automorphic Functions*. See [1].

2. Modular Forms

Proposition 2.1. *Let k be an integer. A $\mathbb{C}$-valued function f on $\mathcal{H}$ is called a modular form of weight k with respect to Γ, if f satisfies the following three conditions:*

(1) *f is meromorphic on $\mathcal{H}$;*

(2) *$f|_k\gamma{=}\ f$ for all $\gamma \in \Gamma$;*

(3) *f is meromorphic at every cusp of Γ.*

Notice that this last condition can be disregarded if Γ has no cusp. The precise meaning of the third condition is as it follows. Suppose Γ has a cusp s, and let $\rho{\in}\mathrm{SL}(2,\mathbb{R})$ so that $\rho(s){=}\infty$. Let $\Gamma_s{=}\{\gamma{\in}\Gamma|\ \gamma(s){=}s\}$, and we have

$$\rho\Gamma_s\rho^{-1}{\cdot}\{\pm1\}{=}\left\{\pm\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}^m\middle| m{\in}\mathbb{Z}\right\}$$

with a positive real h. By the second condition, $f|_k\rho^{-1}$ is invariant under $|_k\sigma$ for all $\sigma{\in}\rho\Gamma_s\rho^{-1}$.

Case I: k even. Since $f|_k\rho^{-1}$ is invariant under $z{\mapsto}z{+}h$, there exists a meromorphic function $\Phi(q)$ in the domain $0{<}|q|{<}r$, with a positive real r, such that

$$f|_k\rho^{-1}{=}\Phi(e^{2\pi iz/h}).$$

Then the third condition means that Φ is meromorphic at $q{=}0$.

Case II: k odd. if $-1{\in}\Gamma$ the second condition implies that $f{=}{-}f$, so that there is no automorphic form of weight k other than 0. Therefore, we assume that $-1{\notin}\Gamma$.

Then $\rho\Gamma_s\rho^{-1}$ is generated either by $\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}$, or by $-\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}$. We say that s is *regular* or *irregular*, respectively. If s is regular, the third condition is as the Case I. If s is irregular, $g(z){=}f|_k\rho^{-1}$ satisfies $g(z+h){=}{-}g(z)$, hence $g(z+2h){=}g(z)$. Then the third condition means that there exists a function Ψ meromorphic in the neighborhood of 0 such that

$$f|_k\rho^{-1}{=}\Psi(e^{\pi iz/h}).$$

The function Ψ must be an odd function. The expression $f|_k\rho^{-1}$ as a power series in $e^{2\pi iz/h}$ or in $e^{\pi iz/h}$ is called the *Fourier Expansion* of f at s; it has the following form:

$$f|_k\rho^{-1} = \sum_{n=-\infty}^{\infty} a_n e^{2\pi inz}$$

Denote by $A_k(\Gamma)$ the set of all automorphic forms of weight k with respect to Γ, and denote by $G_k(\Gamma)$ the set of all $f{\in}A_k(\Gamma)$ such that f is holomorphic on $\mathcal{H}$ and the function Φ or Ψ define in the above definition at each cusp is holomorphic at the origin; the latter condition means that the Fourier coefficients $a_n{=}0$ for $n<0$. Also denote by $S_k(\Gamma)$ the set of all $f{\in}G_k(\Gamma)$ such that the function Φ or Ψ at each cusp vanishes at the origin, i.e., the Fourier coefficients $a_n=0$ for $n\leq 0$.

Definition 2.2. An element of $S_k(\Gamma)$ is called a cusp form of weight k with respect to Γ.

This section defining modular forms was taken out from *Arithmetic Theory of Automorphic Functions*, which was the primary source for this research. See [1].

3. Jacobi Theta Function

A modular form that is strongly related to number theory is the Jacobi theta function. This function can be use to construct modular forms of half-integral weight, and it is of our deepest interest since the coefficients of this function will give rise to the sum of k squares function. But first we need to define the inverse Fourier transform and the Poisson summation formula.

Definition 3.1. Let f be an integrable function from $\mathbb{R}$ to $\mathbb{C}$, then the Fourier inverse transform of f is defined as follows

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(\omega)e^{-2\pi i\omega x}d\omega$$

for all real numbers x.

Claim 3.2. *The inverse Fourier transform of* $e^{-a\omega^2}$ *is* $\sqrt{\frac{\pi}{a}}e^{-\pi\omega^2/a}$.

Proof:

The inverse of a Fourier transform of $e^{-a\omega^2}$ is given by

$$g(x) = \int_{-\infty}^{\infty} e^{-a\omega^2} e^{-i\omega x} d\omega$$

$e^{-i\omega x}$ has period of 2π therefore we will substitute $e^{-2\pi i\omega x}$ for $e^{-i\omega x}$ in our previous equation we get that

$$g(x) = \int_{-\infty}^{\infty} e^{-a\omega^2} e^{-2\pi i\omega x} d\omega$$

Taking the derivatives of $g(x)$ we get that

$$g'(x) = \int_{-\infty}^{\infty} -2\pi i\omega e^{-a\omega^2} e^{-2\pi i\omega x} d\omega = \frac{\pi i}{a}\int_{-\infty}^{\infty} -2a\omega e^{-a\omega^2} e^{-2\pi i\omega x} d\omega$$

$$= \frac{\pi i}{a}\int_{-\infty}^{\infty} \frac{d}{d\omega}\left(e^{-a\omega^2}\right) e^{-2\pi i\omega x} d\omega = \frac{-2x\pi^2}{a}\int_{-\infty}^{\infty} e^{-a\omega^2} e^{-2\pi i\omega x} d\omega$$

The last part of the previous equality is done by the method of integration by parts. Now we have that

$$g'(x) = \tfrac{-2x\pi^2}{a}\, g(x).$$

This is an ordinary differential equation. Therefore the solution of

$$g(x) = e^{-\pi^2 x^2/a}.$$

is given by

$$g(x) = g(0)e^{-\pi^2 x^2/a}$$

with

$$g(0) = \int_{-\infty}^{\infty} e^{-a\omega^2} d\omega.$$

Let z=$\sqrt{a}\omega$, then $dz = \sqrt{a}d\omega$. We can now rewrite the previous integral as

$$\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-az^2}dz$$

.

This integral yields a well known result, which is

$$\int_{-\infty}^{\infty} e^{-az^2}dz = \sqrt{\pi},$$

then

$$g(0) = \sqrt{\tfrac{\pi}{a}}.$$

Therefore

$$g(x) = \sqrt{\tfrac{\pi}{a}}\; e^{-\pi^2 x^2/a}$$

as claimed. Elements of this proof were taken from *Applied Partial Diferential Equations*. See [2].

Proposition 3.3. *(Poisson Summation formula) Let f have a Fourier transform in $\mathbb{R}^n$ and denote its Fourier transform by $\hat{f}$, then the following equality holds*

$$\textstyle\sum_{m\in\mathbb{Z}^n} f(m) = \sum_{m\in\mathbb{Z}^n} \hat{f}(m).$$

Proof:

Let

$$g(z) = \sum_{k\in\mathbb{Z}^n} f(x+k)$$

Then g is periodic with respect to $\mathbb{Z}^n$ and C^∞. If c_m is its m-th Fourier coefficient, then

$$\sum_{m\in\mathbb{Z}^n} c_m = g(0) = \sum_{k\in\mathbb{Z}^n} f(k)$$

on the other hand, interchanging the sum and the integral, we get

$$\begin{aligned} c_m = \int_{\mathbb{R}^n/\mathbb{Z}^n} g(x)e^{-2\pi imx}dx &= \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n/\mathbb{Z}^n} f(x+k)e^{-2\pi imx}dx \\ &= \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{R}^n/\mathbb{Z}^n} f(x+k)e^{-2\pi im(x+k)}dx \\ &= \int_{\mathbb{R}^n} f(x+k)e^{-2\pi imx}dx = \hat{f}(m) \end{aligned}$$

This proves the Poisson summation formula. This proof was taken out from *Real and Functional Analysis*. See [3].

Definition 3.4. If Γ is a congruence subgroup of level N, that is to say, N is the largest integer such that $\Gamma\subset\Gamma_0(N)$, then we say that f is a modular form/function of level N,

where $\Gamma_0(N)$ is given by:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

Now we are ready to define the original theta function of Jacobi. With the use of the Fourier inverse transform and the Poisson summation formula we will prove that the original theta function of Jacobi is a modular form.

Theorem 3.5. *Let*

$$\theta(z) = \sum_{n=-\infty}^{\infty} a_n e^{i\pi m^2 z},$$

then $\theta(z)$ a modular form for $\Gamma_0(2)$ and weight $\frac{1}{2}$.

This series is absolutely convergent in $\mathcal{H}$. We only need to show that $\theta(z) = \theta(z+2)$ and $\theta(-\frac{1}{z}) = \sqrt{iz}\theta(z)$ since $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are the generators of $\Gamma_0(2)$.

The first condition of a modular form is really simple and it goes as follow:

$$\begin{aligned}\theta(z+2) &= \sum_{n=-\infty}^{\infty} a_n e^{i\pi m^2(z+2)} \\ &= \sum_{n=-\infty}^{\infty} \left(a_n e^{i\pi m^2 z} e^{i\pi m^2 2}\right) \\ &= \sum_{n=-\infty}^{\infty} a_n e^{i\pi m^2 z} \\ &= \theta(z).\end{aligned}$$

Therefore $\theta(z+2) = \theta(z)$.

The other transformation property under the map $-\frac{1}{z}$ is derived from the Poisson summation formula. Note now that the Fourier transform of e^{-ax^2} is $\sqrt{\frac{\pi}{a}}e^{-\pi^2x^2/a}$ which means that (a=πy) the transform of $e^{-\pi m^2 y}$ is $\sqrt{\frac{\pi}{a}}e^{-\pi x^2/y}$ hence

$$\begin{aligned}\theta(iy) &= \sum_{n=-\infty}^{\infty} e^{-\pi m^2 y} \\ &= \sqrt{\frac{1}{y}} \sum_{n=-\infty}^{\infty} e^{-\pi m^2/y} \\ &= \sqrt{\frac{1}{y}}\theta\big(-\frac{1}{iy}\big).\end{aligned}$$

By analytic continuation we can extend this equality to the whole complex plane and we get

$$\theta(-\tfrac{1}{z}) = \sqrt{iz}\theta(z).$$

We now have that θ is a modular function of weight $\frac{1}{2}$ and level 2. Using Poisson summation again we get that $\theta(z) = \theta(2z)$ and this function belongs to $A_{\frac{1}{2}}(4)$. There are many uses for θ, for example, if

$$\theta(z)^4 = \sum_{n=1}^{\infty} r_4(n) e^{i\pi n z},$$

where $r_k(n)$ is the sum of k squares function.

As we wanted we started with a the original theta function of Jacobi and proved that is in fact a modular form of level 2 and weight $\frac{1}{2}$, and arrive at the result that the coefficients of the summation are given by a well known function in number theory which is the sum of k squares function.

4. The Dedekind Eta Function

Another example of a modular form that is closely related to number theory is the Dedekind eta function.

Definition 4.1. $\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n \tau}\right)$

This product is absolute convergent and nonzero in $\mathcal{H}$ so $\eta(\tau)$ is analytic for $\tau \in \mathcal{H}$.

Theorem 4.2. *$\eta(\tau)$ has the following transformations properties:*

(1) $\eta(\tau + 1) = e^{\frac{\pi i \tau}{12}} \eta(\tau)$

(2) $\eta(\frac{-1}{\tau}) = (-i\tau)^{\frac{1}{2}}\ \eta(\tau)$

Proof:

Part 1.

$$\begin{aligned}
\eta(\tau + 1) &= e^{\frac{\pi i (\tau+1)}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n(\tau+1)}\right) \\
&= e^{\frac{\pi i \tau}{12}} e^{\frac{\pi i}{12}} \prod_{n=1}^{\infty} \left(1 - \left(e^{2\pi i n \tau} e^{2\pi i n}\right)\right) \\
&= e^{\frac{\pi i}{12}} e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} \left(1 - \left(e^{2\pi i n \tau} 1\right)\right. \\
&= e^{\frac{\pi i}{12}} \eta(\tau).
\end{aligned}$$

Therefore part 1 holds.

Part 2.

We will prove this equality for $\tau{=}iy$ and extend the domain by analytic continuation. The formula for $\tau{=}iy$ becomes

$$\eta(\tfrac{i}{y}) = (y)^{\frac{1}{2}} \eta(iy)$$

and if we take logarithms on both sides we get

$$\ln\eta(\tfrac{i}{y}) = \tfrac{1}{2}\ln y + \ln\eta(iy)$$

then

$$\ln\eta(iy) - \ln\eta(\tfrac{i}{y}) = -\tfrac{1}{2}\ln y$$

and

$$\begin{aligned}
\ln \eta(iy) &= -\frac{\pi y}{12} + \ln \prod_{n=1}^{\infty} \left(1 - e^{-2\pi n y}\right) \\
&= -\frac{\pi y}{12} + \sum_{n=1}^{\infty} \ln \left(1 - e^{-2\pi n y}\right) \\
\textit{taylor series} &= -\frac{\pi y}{12} - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{e^{-2\pi m n y}}{m} \\
\textit{geometric series} &= -\frac{\pi y}{12} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{e^{-2\pi m y}}{1 - e^{-2\pi m y}} \\
&= -\frac{\pi y}{12} + \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{2\pi m y}}
\end{aligned}$$

similarly

$$\ln \eta(i/y) = -\frac{\pi}{12y} + \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{2\pi m/y}},$$

Therefore we need to prove that

$$\sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{2\pi m n y}} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{1 - e^{2\pi m/y}} - \frac{\pi}{12}\left(y - \frac{1}{y}\right) = -\frac{1}{2} \ln y.$$

This will be proved with the use calculus of residues.
For fixed $y > 0$ and $n = 1, 2, \ldots$, let

$$F_n(z) = -\frac{1}{8z} \ \cot \pi i N z \ \cot \frac{\pi N z}{y},$$

where $N = \mathrm{n} + \frac{1}{2}$. Let C be the parallelogram with vertices y, i, $-y$ and $-i$. Inside C, F_n has simple poles at $z = ik/N$ and $z = ky/N$ for $k = \pm 1, \pm 2, \ldots$, $\pm n$. There is also a triple pole at $z = 0$ with residue $i(y - y^{-1})/24$. The residue at $z = ik/N$ is

$$\frac{1}{8\pi k} \cot \frac{\pi i k}{y}.$$

This is an even function of k so we have

$$\sum_{\substack{k=-n\\k\neq 0}}^{n} Res(F_n, ik/N) = 2\sum_{k=1}^{n} \frac{1}{8\pi k} \ \cot\frac{\pi i k}{y}.$$

But

$$\cot\ i\theta = \frac{\cos i\theta}{\sin i\theta} = i\ \frac{e^{-\theta}+e^{\theta}}{e^{-\theta}-e^{\theta}} = -i\ \frac{e^{2\theta}+1}{e^{2\theta}-1} = \frac{1}{i}\left(1-\frac{2}{1-e^{2\theta}}\right).$$

Now let $\theta = \pi k/y$ and we get

$$\sum_{\substack{k=-n\\k\neq 0}}^{n} Res(F_n, ik/N) = \frac{1}{4\pi i}\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{2\pi i}\sum_{k=1}^{n}\frac{1}{k}\ \frac{1}{1-e^{2\pi k/y}}.$$

Similarly

$$\sum_{\substack{k=-n\\k\neq 0}}^{n} Res(F_n, ky/N) = \frac{i}{4\pi}\sum_{k=1}^{n}\frac{1}{k} - \frac{i}{2\pi}\sum_{k=1}^{n}\frac{1}{k}\ \frac{1}{1-e^{2\pi k y}}.$$

Multiplying $2\pi i$ to the sum of all the residues of $F_n(z)$ inside C and taking the limit as $n\to\infty$ we get

$$\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{1-e^{2\pi m y}} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{1-e^{2\pi m/y}} - \frac{\pi}{12}\left(y-\frac{1}{y}\right)$$

Therefore, in order to complete the proof we only need to show that

$$\lim_{n\to\infty}\int_C F_n(z)dz = -\frac{1}{2}\ln y.$$

On the edges of C (except at the vertices) the function $zF_n(z)$ has, as $n\to\infty$, the limit $\frac{1}{8}$ on the edges connecting y, i, and $-y$, $-i$, and the limit $-\frac{1}{8}$ on the other two edges. Moreover, $F_n(z)$ is uniformly bounded on C for all n (because $N = n+\frac{1}{2}$ and $y>0$). Hence by Arzela's bounded convergence theorem we have

$$\begin{aligned}
\lim_{n\to\infty}\int_C F_n(z)dz &= \int_C \lim_{n\to\infty} zF_n(z)\frac{dz}{z}\\
&= \frac{1}{8}\Big\{-\int_{-i}^{y}+\int_{y}^{-i}-\int_{i}^{-y}+\int_{-y}^{-i}\Big\}\frac{dz}{z}\\
&= \frac{1}{4}\Big\{-\int_{-i}^{y}+\int_{y}^{i}\Big\}\frac{dz}{z}\\
&= \frac{1}{4}\Big\{-\Big(\ln y+\frac{\pi i}{2}\Big)+\Big(\frac{\pi i}{2}-\ln y\Big)\Big\} = -\frac{1}{2}\ln y.
\end{aligned}$$

This completes the proof. Notice that we only needed to show that $\eta(\tau+1) = e^{\frac{\pi i\tau}{12}}\eta(\tau)$ and $\eta(\frac{-1}{\tau}) = (-i\tau)^{\frac{1}{2}}\,\eta(\tau)$ since $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ are the generators of SL(2,$\mathbb{Z}$). The Part II of this proof was taken out from *Modular Functions and Dirichlet Series in Number Theory*. See [4]. This η function has lots of applications in number theory. For example lets consider the Δ function which is a modular form of weight 12 that is constructed from the eta function.

Proposition 4.3.

$$\Delta(x) = (2\pi)^{12}\eta^{24}(x),$$

and let

$$F(x) = (2\pi)^{-12}\Delta(x)$$

We denote $\tau(n)$ the nth coefficient of $F(x)$. Hence

$$\sum_{n=1}^{\infty}\tau(n)x^n = x\prod_{n=1}^{\infty}\big(1-x^n\big)^{24}, for|x|<1,$$

where $\tau(n)$ is the Ramanujan tau function.

This proposition was taken out from *Introduction to Elliptic Curves and Modular Forms*. See [5].

Another example is the Euler ϕ function.

Proposition 4.4.

$$\phi(q) = q^{\frac{1}{24}}\eta(q),$$

We denote p(n) the nth coefficient of $\frac{1}{\phi(q)}$. *Then*

$$\frac{1}{\phi(q)} = \sum_{n=0}^{\infty} p(n)q^n, for|q| < 1,$$

where p(n) is the partition function.

This proposition was taken out from *Introduction to Elliptic Curves and Modular Forms*. See [5].

We've shown how two functions that are constructed with the Dedekind eta function both give rise to two very important and useful formulas in number theory as the Ramanujan tau function and the partition function.

5. The Eisenstein Series

Proposition 5.1. *Let k be an even integer greater than 2. For* $z \in \mathcal{H}$ *we define the Einsenstein series as*

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(mz+n)^{2k}},$$

Then G_k *is a modular form of weight 2k for SL(2,ℤ).*

Proof:

We only need to show that $G_k(z) = G_k(z+1)$ and $G_k(-\frac{1}{z}) = (-z)^{2k}G_k(z)$ since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are the generators of SL(2,ℤ).

$$\begin{aligned} G_k(z+1) &= \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m(z+1)+n)^{2k}} \\ &= \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(mz+(m+n))^{2k}} \\ &= G_k(z) \end{aligned}$$

To prove the last part of the equality, let $m+n = r$ and note that if $(m,n) \in \mathbb{Z}^2\setminus(0,0)$ then $(m,r) \in \mathbb{Z}^2\setminus(0,0)$. Also if $(m,r) \in \mathbb{Z}^2\setminus(0,0)$ then $(m,n) \in \mathbb{Z}^2\setminus(0,0)$. Therefore both sum go through the same elements and the equality holds.

$$
\begin{aligned}
G_k\left(-\frac{1}{z}\right) &= \sum_{(m,n)\in\mathbb{Z}^2\backslash(0,0)} \frac{1}{(m(-\frac{1}{z})+n)^{2k}} \\
&= \frac{(-z)^{2k}}{(-z)^{2k}} \sum_{(m,n)\in\mathbb{Z}^2\backslash(0,0)} \frac{1}{(m(-\frac{1}{z})+n)^{2k}} \\
&= (-z)^{2k} \sum_{(m,n)\in\mathbb{Z}^2\backslash(0,0)} \frac{1}{(-z)^{2k}} \frac{1}{(m(-\frac{1}{z})+n)^{2k}} \\
&= (-z)^{2k} \sum_{(m,n)\in\mathbb{Z}^2\backslash(0,0)} \frac{1}{(-nz+m)^{2k}} \\
&= (-z)^{2k} G_k(z)
\end{aligned}
$$

Similarly to the first part, to prove the last part of the equality note that if $(m,n)\in\mathbb{Z}^2\backslash(0,0)$ then $(m,-n)\in\mathbb{Z}^2\backslash(0,0)$. Also if $(m,-n)\in\mathbb{Z}^2\backslash(0,0)$ then $(m,n)\in\mathbb{Z}^2\backslash(0,0)$. Therefore both sums go through the same elements. This completes the proof.

Now we compute the q-expansion coefficients for G_k. We will find that these coefficients are the arithmetic sigma function of n

$$
\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}
$$

Proposition 5.2. *Let k be an even integer greater than 2, and let $z\in\mathcal{H}$. Then the modular form G_k has the following q-expasion.*

$$
G_k(z) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right),
$$

where $q = e^{2\pi z}$, and the Bernoulli numbers B_k are defined by setting

$$
\frac{x}{e^x-1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}
$$

Proof:

The logarithmic derivative of the product formula for sine is

$$
\pi\cot(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty}\left(\frac{1}{a+n} + \frac{1}{a-n}\right),\ a\in\mathcal{H}.
$$

If we write the left side as $\pi i(e^{\pi ia}+e^{-\pi ia})/(e^{\pi ia}-e^{-\pi ia}) = \pi i + 2\pi i/(e^{2\pi ia}-1)$, multiply both sides by a, replace $2\pi ia$ by x, and expand both sides in powers of x, we obtain the well-known formula for $\zeta(k)$:

$$\zeta(k) = -(2\pi i)^k B_k/2k! \ \ for \ \ k > 0 \ even.$$

Next, if we successively differentiate both sides of the logarithmic derivative of the product formula for sine with respect to a and then replace a by mz, we obtain:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n m z} = -\frac{2k}{B_k}\zeta(z) \sum_{d=1}^{\infty} d^{k-1} q^{dm}.$$

Thus,

$$\begin{aligned} G_k(z) &= 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{d=1}^{\infty} d^{k-1} q^{dm}\right). \end{aligned}$$

Collecting coefficients of a fixed power q^n in the las double sum, we obtain the arithmetic sigma function as the coefficient of q^n. This completes the proof and gives us our final example of a modular form that gives rise to a number theory function. This proposition was taken out of *Introduction to Elliptic Curves and Modular Forms*. See [5].

References

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