COMPUTATIONS IN THE RELATIVE SKEIN ALGEBRA OF A LOCAL ANNULUS

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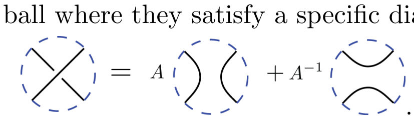
Abstract. Let M be a compact, oriented, 3-manifold of the form $F \times [0,1]$ where F is an oriented surface of genus g and b boundary components. $K_A(M)$ is the skein algebra of M for A an N-th root of unity. The purpose of this paper is to develop tools to simplify a skein in $K_A(M)$ by finding a local annulus in F and simplifying the skein in such annulus.

Introduction

The purpose of this paper is to develop local tools for the study of skein algebras of compact, oriented, 3-manifolds of the form $F \times [0,1]$ where F is an oriented surface for A an N-th root of unity. The main idea behind it being that if we can find an annulus around F where the skein behaves in a certain way, then we can apply these tools to simplify it in this annulus, hence simplifying it in F. For example, if the skein comes inside annulus and wraps around k-times for some large value of k, then we would like to be able to write this as a linear combination of skeins with a smaller winding number and possibly some central elements. This in parts is an extension, or rather a generalization of The Spiral Lemma introduced by Bloomquist and Frohman in [5]. Proofs throughout this article will be broken into even and odd cases and/or positive and negative cases. The results will turn out to be symmetric hence we will only provide the proof of one of them and state the formula for the other case as a corollary.

1. Skein Modules

Consider a 3-manifold M. "The Kauffman bracket skein module of M is an algebraic invariant $K(M)$ built from the set of all framed links in M." [1] A framed link is an "embedded collection of annuli considered up to isotopy in M. [1] We call $\mathcal{L}_M$, the set which consists of framed links union the empty link $\emptyset$. Now that we have the set $\mathcal{L}_M$ we are interested in establishing relations among links. We say three links L, L_0 and L_∞ are *Kauffman skein related* if we can embedded them identically except in a ball where they satisfy a specific diagrammatic property



$L \sqcup \bigcirc$ denotes L union with an unlinked 0-framed unknot.

Using the property described above we can then proceed to define the Kauffman bracket skein algebra. Let $A = e^{\pi i/N}$, where N is odd. Let R denote the ring of Laurent polynomials $\mathbb{Z}[A, A^{-1}]$ and $R\mathcal{L}_M$ the free R-module on the basis $\mathcal{L}_M$. We

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already had the definition of what it means pictorially for three links to be Kauffman skein related hence now we want to assign algebraic relations to these pictures. If L, L_0 and L_∞ are Kauffman skein related then $L - AL_0 - A^{-1}L_\infty$ is what we call a *skein relation*. We also have a $framing\ relation$ given by $L \sqcup \bigcirc + A^2L + A^{-2}L$ for any $L \in \mathcal{L}_M$. Now we can construct the $Kauffman\ bracket\ skein\ algebra$ $K(M)$. Define $K(M) = R\mathcal{L}_M/S(M)$ where $S(M)$ is the smallest submodule of $R\mathcal{L}_M$ containing all possible skein and framing relations.

We are interested in the study of 3-manifolds of the form $M = F \times [0,1]$, for F an oriented surfaces of genus g and b boundary components. Let $\Sigma_{n,m}$ to be the surface of genus n and m boundary components. Even though $\Sigma_{0,3}$ and $\Sigma_{1,1}$ are not homeomorphic $\Sigma_{0,3} \times [0,1]$ and $\Sigma_{1,1} \times [0,1]$ are. This tells us that different skein algebras can have the same underlying skein module. Notice also that the simple closed curves in F act as a basis for the links in the algebra. Since once you apply your skein relation and get rid of all crossing all you are left is with a collection of links, but if we project these links orthogonally onto F they look like simple closed curves in F. It makes sense now then to refer to $K_A(M)$ by $K_A(F)$ since this will make clear the underlying multiplication with which the algebra is equipped.

2. K_A(Ann) and the relevance of T_N to the study of skeins

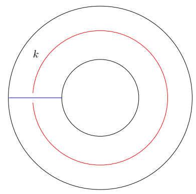
The skein algebra of an annulus at A an N-th root of unity, by definition is the collection of all framed links in the annulus including the empty link, mod out by the skein relation and the framing relation. That is to say $K_A(Ann) = \mathbb{C}\mathcal{L}_F / < L - AL_0 - A^{-1}L_\infty,\ L \sqcup \bigcirc + A^2L + A^{-2}L >$. However notice that the only nontrivial simple diagrams in the annulus are just linear combinations of powers of the the one that wraps around it, call it ω. Hence you don't need to worry about the skein relation because there won't be any crossings, since if you have two copies of ω you can always move then around a little so that they become disjoint. The framing relation will reduce a disjoint union with an unknot to a scalar multiplication. When we put all of this together it tells us that the skein algebra of an annulus is precisely the polynomial ring in one variable. That is to say, $K_A(Ann) = \mathbb{C}[\omega]$.

The Tchebychev polynomials of the first kind are defined in the following recurrent way, $T_0 = 2$, $T_1 = \omega$, $T_k = \omega T_{k-1} - T_{k-2}$. Notice that if $\omega = 2\cos(z)$ for $z \in \mathbb{C}$, then $T_k = 2\cos(kz)$. Hence this polynomials are related to the cosine function. These polynomials were introduced by Frohman and Gelca into the study of skein algebras in [2] to compute the skein algebra of the noncommutative torus.

The used of the Tchevychev polynomials of the first kind to study skein algebras is not limited to the noncummutative torus. A map $\tau : K_{-1}(M) \to K_N(M)$ is said to be a *threading map* if $\tau(L) = T_N(L)$ for some fixed N and for all $L \in K_{-1}(M)$. Bonahon and Wong showed in [4] the existence of such $\mathbb{C}$-linear maps and that $\tau(L)$ is central for all $L \in K_{-1}(M)$. They also proved that when $M = F \times [0,1]$, the map $\tau : K_{-1}(F) \to K_N(F)$ is a homomorphism of algebras. Another important consequence is that for an oriented 3-manifold M, $K_{-1}(M)$ is the universal character ring of $\pi_1(M)$, proved by Bullock in [3]. That is to say, "[...] if $X(\pi_1(M))$ is the ring of $SL_2\mathbb{C}$-characters of $\pi_1(M)$ and $\sqrt{0}$ is the nilradical of $K_{-1}(M)$ then the map, $\Theta : K_{-1}(M)/\sqrt{0} \to X(\pi_1(M))$ [...] is an isomorphism" [3]. The following result is from Bloomquist and Frohman in [5]:

Lemma 2.1.

$$1 * T_k(\omega) =$$



$$= A^k C_k + A^{-k} C_{-k}$$

and

$$T_k(\omega) * 1 = A^{-k} C_k + A^k C_{-k}.$$

See figure 1 for diagram of C_k. This provides a method to simplify a skein in $K_A(F)$ for A an N-th root of unity, given that we can find an annulus in F where the skein comes in, spirals around and then leaves. Their main focus being that if it spirals around more than N times, then one can simplify this skein locally as $T_N(\omega)$ times some single strand crossing along, minus a spiral of winding number less than N. The latter since when you plug in the negative spiral it will cancel some of the times that it loops around the annulus. This leaves us with a linear combination, locally much easier to understand because the $T_N(\omega)$s are central [4] and the other term is a spiral of smaller winding degree.

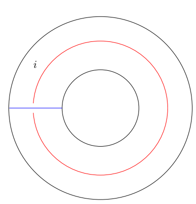
3. S_n in a local Annulus.

The Tchebychev polynomials of the second kind are defined by the same recurrence relation as of the first kind but with a different value at $k = 0$. Let $S_0 = 1$, $S_1 = \omega$, $S_k = \omega S_{k-1} - S_{k-2}$. They are related to T_ks in the following way, $T_k = S_k - S_{k-2}$. For computational purposes we will extend the definition to include $k = -1$ and define $S_{-1} := 0$. The S_ks played an important roll in the construction of quantum invariants. For references see [6], [7]. The advantage of threading with the S_ks instead of the T_ks is that if we look at the skein algebra as a Module of $\mathbb{Z}[A^{\pm 1}]$ then the S_ks produce a basis for $K_A(Ann)$ and the T_ks don't since $T_0(\omega)$ equals two times the empty link.

From now on we will assume that we are working inside an annulus in F of $M = F \times [0,1]$. Our goal for this section and the paper is to find a formula in terms of simple diagrams for $1_l * S_i(\omega) * 1_r$, where the l and r are there to specify which one strand we are multiplying first. When dealing with only one strand we will drop the subscripts.

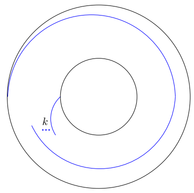
Proposition 3.1. *Let $i \in \mathbb{Z}$, $i \geq 0$, then*

$$1 * S_i(\omega) =$$



$$= \sum_{k=-i,\ by\ 2}^{i} A^k C_k$$

Where C_k spirals around k-times clockwise, when $k > 0$ and counterclockwise when $k < 0$.

$C^k =$ 

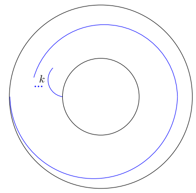
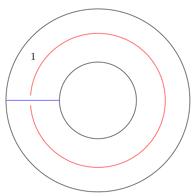
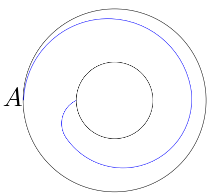
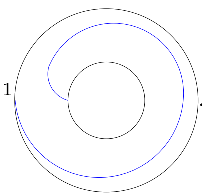
$, C^{-k} =$ 

Figure 1

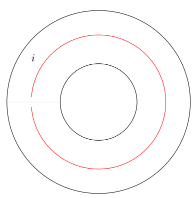
Proof. We will proceed with a proof by induction. Case $i = 0$ is trivial.

Case $i = 1$.



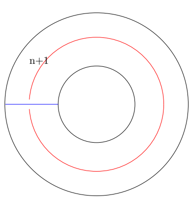
$= A$  $+ A^{-1}$  .

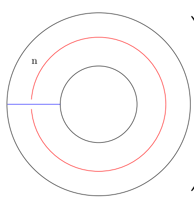
Suppose it is true for $i = n$.

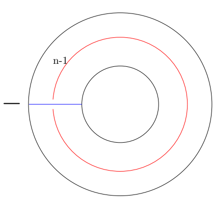


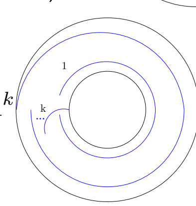
$$= \sum_{k=-n,\ by\ 2}^{n} A^k C_k.$$

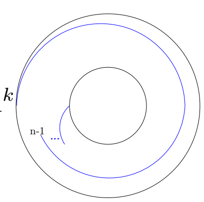
Case of $i = n + 1$

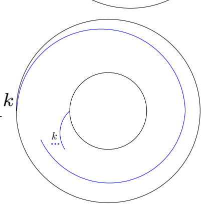


$= \Bigg(\omega *$ 

$\Bigg) -$ 

$= \Bigg(\sum_{k=-n,\ by\ 2}^{n} \omega * A^k$ 

$\Bigg) - \sum_{k=-(n-1),\ by\ 2}^{n-1} A^k$ 

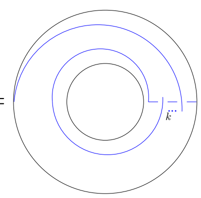
$= \sum_{k=-(n+1),\ by\ 2}^{n+1} A^k$ 

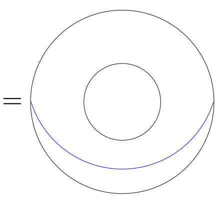
Notice that $\omega * C_k = AC_{k+1} + A^{-1}C_{k-1}$ which makes the last equality hold. $\square$

Corollary 3.2. *Let $i < 0$, then* $S_i * 1 = \sum_{k=-i,\ by\ 2}^{i} A^{-k}C_k$.

Proof. Proof similar to that of proposition 3.1. $\square$

We now know that we can express $1 * S_i(\omega)$ as a linear combination of C_ks. The next step to understanding $1_l * S_i(\omega)\ 1_r$ is to understand $C_k * 1$. Multiplying $C_k * 1$ will give rise to three new types of diagrams η_k for $k \in \mathbb{Z} \cup \{0\}$, and τ and $\rho(0)$. η_k satisfies a recurrent relation and that is the focus of the following proposition and corollary.

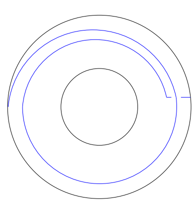
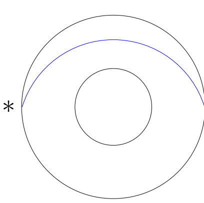
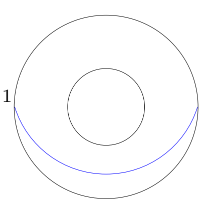
Let $\eta_k =$ 

$, \tau =$ 

Where η_k spirals around k times. $\rho(0)$ will be defined later on when it becomes necessary.

Proposition 3.3. *Let $k > 0$, then* $\eta_k = A^k S_k(\omega) * \eta_0 + A^{k-2}S_{k-1}(\omega) * \tau$.

Proof. Case $k = 1$

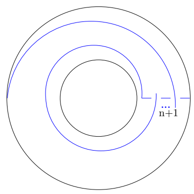
 $= \Bigg(A\omega *$  $\Bigg) + A^{-1}$ 

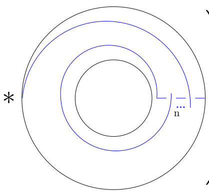
$$= A^1 S_1(\omega) * \eta_0 + A^{1-2} S_0(w) * \tau$$

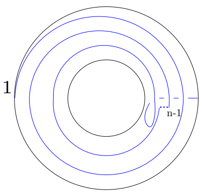
Suppose it is true for $k = n$

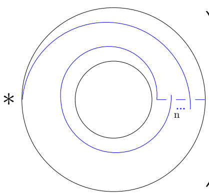
$$\eta_k = A^n S_n(\omega) * \eta_0 + A^{n-2} S_{n-1}(\omega) * \tau.$$

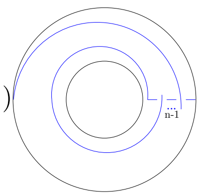
Case for $k = n + 1$
We will start smoothing from the innermost crossing towards the outermost.

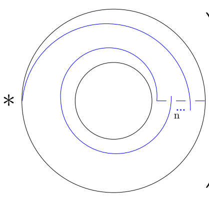


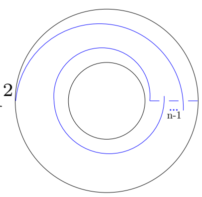
$= \left(A\omega * \right.$ 

$\left. \right) + A^{-1}$ 

$= \left(A\omega * \right.$ 

$\left. \right) + A^{-1}(-A^{3})$ 

$= \left(A\omega * \right.$ 

$\left. \right) - A^{2}$ 

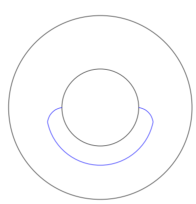
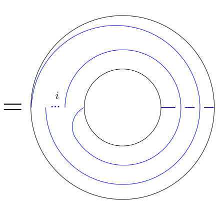
$$
\begin{aligned}
&= [(A\omega)(A^{n}S_{n}(\omega) * \eta_0 + A^{n-2}S_{n-1}(\omega) * \tau)] \\
&\quad - A^{2}(A^{n-1}S_{n-1}(\omega) * \eta_0 + A^{(n-1)-2}S_{(n-1)-1}(\omega) * \tau)] \\
&= [A^{n+1}(\omega S_{n}(\omega) - S_{n-1}(\omega)) * \eta_0] + [A^{n-1}(\omega S_{n-1}(\omega) - S_{n-2}(\omega)) * \tau] \\
&= [A^{n+1}S_{n+1}(\omega)] * \eta_0 + [A^{n-1}S_{n}(\omega)] * \tau \\
&= [A^{n+1}S_{n+1}(\omega)] * \eta_0 + [A^{(n+1)-2}S_{(n+1)-1}(\omega)] * \tau
\end{aligned}
$$

□

Corollary 3.4. *Let $k > 0$, then $\eta_{-k} = A^{2-k}S_{k-1}(\omega) * \eta_0 + A^{-k}S_{k}(\omega) * \tau$.*

Proof. Proof similar to that of proposition 3.3. □

Now that we've develop a formula for η_k let's go back to $C_k * 1$. Smoothing $C_k * 1$ will result in a sum of two elements. $A\eta_{k-1} * \rho(0)$ and $A^{-1}h(C_{k-1} * 1)$ where $h : Ann \to Ann$ is a $\mathbb{C}$-linear diffeomorphism that fixes the outside boundary but twists the inside boundary by 180° clockwise and everything in between rotates smoothly. We proceed to define $\rho(0)$ and for a matter of simplicity we will rename $C_k * 1$.

Let $\rho(0) =$ 
, and $\gamma(i, 0) = C_i * 1 =$ 

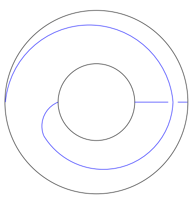
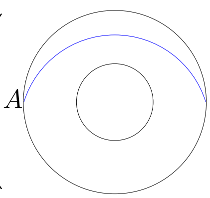
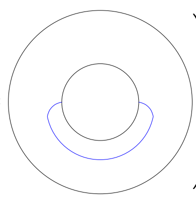
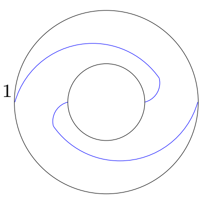
where i indicates how many times it spirals around.

In general $\gamma\left(k, \frac{l}{2}\right) = h^{l}\gamma(k, 0)$. Notice that h fixes ω, η_0 and τ and it acts on $\rho(0)$ by rotating it so we can define $\rho\left(\frac{l}{2}\right)$.

Proposition 3.5. *Let $k > 0$, then*

$$\begin{aligned}\gamma(k,0) = &\sum_{i=0}^{k-1} A^{2(i+1)-k} S_i(\omega) * \eta_0 * \rho\left(\frac{(k-1)-i}{2}\right)\\ &+ \sum_{i=0}^{k-2} A^{2(i+1)-k} S_i(\omega) * \tau * \rho\left(\frac{(k-2)-i}{2}\right)\\ &+ A^{-k}\gamma\left(0, \frac{k}{2}\right).\end{aligned}$$

Proof. We will proceed by induction.
Case $i = 1$

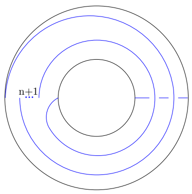
 $=$ $\Bigg(A$  $*$  $\Bigg) + A^{-1}$ 

$$= A^1 S_0(\omega) * \eta_0 * \rho(0) + A^1 S_{-1}(w) * \tau + \gamma\left(0, \frac{1}{2}\right)$$

Suppose it is true for $i = n$

$$\begin{aligned}\gamma(n,0) = &\sum_{i=0}^{n-1} A^{2(i+1)-n} S_i(\omega) * \eta_0 * \rho\left(\frac{(n-1)-i}{2}\right)\\ &+ \sum_{i=0}^{n-2} A^{2(i+1)-n} S_i(\omega) * \tau * \rho\left(\frac{(n-2)-i}{2}\right) + A^{-n}\gamma\left(0, \frac{n}{2}\right).\end{aligned}$$

Case $i = n+1$ As in the case with η_k, we will start smoothing from the innermost crossing towards the outermost.



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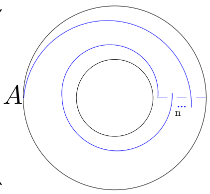

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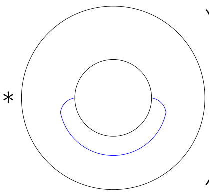
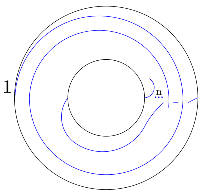


$\Bigg)$ $+ A^{-1}$



$$
\begin{aligned}
&= A\eta_n * \rho(0) + A^{-1}\gamma\left(n, \frac{1}{2}\right) \\
&= A\eta_n * \rho(0) + A^{-1}h(\gamma(n,0)) \\
&= A[A^n S_n(\omega) * \eta_0 + A^{n-2}S_{n-1}(\omega) * \tau] * \rho(0) \\
\text{h is linear} \quad &\quad + A^{-1}h\left(\sum_{i=0}^{n-1} A^{2(i+1)-n} S_i(\omega) * \eta_0 * \rho\left(\frac{(n-1)-i}{2}\right)\right) \\
&\quad + A^{-1}h\left(\sum_{i=0}^{n-2} A^{2(i+1)-n} S_i(\omega) * \tau * \rho\left(\frac{(n-2)-i}{2}\right)\right) + A^{-1}h\left(A^{-n}\gamma\left(0, \frac{n}{2}\right)\right) \\
&= A^{n+1}S_n(\omega) * \eta_0 * \rho(0) + A^{n-1}S_{n-1}(\omega) * \tau * \rho(0) \\
&\quad + \sum_{i=0}^{n-1} A^{2(i+1)-n-1} S_i(\omega) * \eta_0 * h\left(\rho\left(\frac{(n-1)-i}{2}\right)\right) \\
&\quad + \sum_{i=0}^{n-2} A^{2(i+1)-n-1} S_i(\omega) * \tau * h\left(\rho\left(\frac{(n-2)-i}{2}\right)\right) + \ A^{-n-1}h\left(\gamma\left(0, \frac{n}{2}\right)\right) \\
&= A^{n+1}S_n(\omega) * \eta_0 * \rho(0) + A^{n-1}S_{n-1}(\omega) * \tau * \rho(0) \\
&\quad + \sum_{i=0}^{n-1} A^{2(i+1)-(n+1)} S_i(\omega) * \eta_0 * \rho\left(\frac{n-i}{2}\right) \\
&\quad + \sum_{i=0}^{n-2} A^{2(i+1)-(n+1)} S_i(\omega) * \tau * \rho\left(\frac{(n-1)-i}{2}\right) + \ A^{-(n+1)}\gamma\left(0, \frac{n}{2} + \frac{1}{2}\right) \\
\text{collecting terms} &= \sum_{i=0}^{n} A^{2(i+1)-(n+1)} S_i(\omega) * \eta_0 * \rho\left(\frac{n-i}{2}\right) \\
&\quad + \sum_{i=0}^{n-1} A^{2(i+1)-(n+1)} S_i(\omega) * \tau * \rho\left(\frac{(n-1)-i}{2}\right) + \ A^{-(n+1)}\gamma\left(0, \frac{n+1}{2}\right)
\end{aligned}
$$

□

Corollary 3.6. *Let $k > 0$ and $l \in \mathbb{Z}$, then*

$$
\begin{aligned}
\gamma\left(k,\frac{l}{2}\right) = &\sum_{i=0}^{k-1} A^{2(i+1)-k} S_i(\omega) * \eta_0 * \rho\left(\frac{(k-1)-i+l}{2}\right) \\
&+ \sum_{i=0}^{k-2} A^{2(i+1)-k} S_i(\omega) * \tau * \rho\left(\frac{(k-2)-i+l}{2}\right) \\
&+ A^{-k}\gamma\left(0,\frac{k+l}{2}\right).
\end{aligned}
$$

Proof. Proof similar to that of proposition 3.5 combined with the definition of h. □

Corollary 3.7. *Let $k > 0$, then*

$$
\begin{aligned}
\gamma(-k,0) = &\sum_{i=0}^{k-1} A^{-2(i+1)-k} S_i(\omega) * \tau * \rho\left(\frac{-k+i}{2}\right) \\
&+ \sum_{i=0}^{k-2} A^{-2(i+1)-k} S_i(\omega) * \eta_0 * \rho\left(\frac{-(k-1)-i}{2}\right) \\
&+ A^{-k}\gamma\left(0,\frac{-k}{2}\right).
\end{aligned}
$$

Proof. Proof similar to that of proposition 3.5. □

Corollary 3.8. *Let $k > 0$, then*

$$
\begin{aligned}
\gamma(-k,\frac{l}{2}) = &\sum_{i=0}^{k-1} A^{-2(i+1)-k} S_i(\omega) * \tau * \rho\left(\frac{-k+i+l}{2}\right) \\
&+ \sum_{i=0}^{k-2} A^{-2(i+1)-k} S_i(\omega) * \eta_0 * \rho\left(\frac{-(k-1)-i+l}{2}\right) \\
&+ A^{-k}\gamma\left(0,\frac{-k+l}{2}\right).
\end{aligned}
$$

Proof. Proof similar to that of proposition 3.5 combined with the definition of h. □

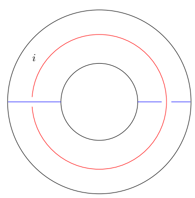
The following two lemmas are simple results but they will become very useful for the prove of the main theorem.

Lemma 3.9. *Let $k > 0$, then*

$$
\begin{aligned}
A^k\gamma(k,0)+A^{-k}\gamma(-k,0) = {} & A^{2k}S_{k-1}(\omega)*\eta_0*\rho(0)+A^{-2k}S_{k-1}(\omega)*\tau*\rho\left(\frac{-1}{2}\right)\\
&+\sum_{i=0}^{k-2}\left(A^{2(i+1)}+A^{-2(i+1)}\right)S_i(\omega)*\left[\eta_0*\rho\left(\frac{k-1-i}{2}\right)+\tau*\rho\left(\frac{-k+i}{2}\right)\right]\\
&+\gamma\left(0,\frac{k}{2}\right)+\gamma\left(0,\frac{-k}{2}\right).
\end{aligned}
$$

Theorem 3.10. *Main Theorem*
Let $i>0$ and even, then

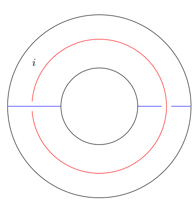
$$1_l * S_i(\omega) * 1_r =$$



$$
\begin{aligned}
&=\sum_{k=-i\ by\ 2}^{i}\gamma\left(0,\frac{k}{2}\right)+\sum_{k=2\ by\ 2}^{i}A^{2k}S_{k-1}(\omega)*\eta_0*\rho(0)+A^{-2k}S_{k-1}(\omega)*\tau*\rho\left(\frac{1}{2}\right)\\
&+\sum_{n=0,\ by\ 2}^{i-2}\left(\frac{i-l}{2}\right)\left(A^{2(n+1)}+A^{-2(n+1)}\right)S_n(\omega)*\left[\eta_0*\rho\left(\frac{1}{2}\right)+\tau*\rho(0)\right]\\
&+\sum_{n=1,\ by\ 2}^{i-3}\left(\frac{i-(n+1)}{2}\right)\left(A^{2(n+1)}+A^{-2(n+1)}\right)S_n(\omega)*\left[\eta_0*\rho(0)+\tau*\rho\left(\frac{1}{2}\right)\right].
\end{aligned}
$$

Corollary 3.11. *Let $i>0$ and odd, then*

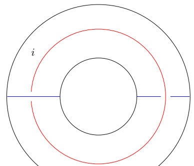
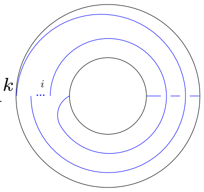
$$1_l * S_i(\omega) * 1_r =$$



$$
\begin{aligned}
&=\sum_{k=-i\ by\ 2}^{i}\gamma\left(0,\frac{k}{2}\right)+\sum_{k=1\ by\ 2}^{i}A^{2k}S_{k-1}(\omega)*\eta_0*\rho(0)+A^{-2k}S_{k-1}(\omega)*\tau*\rho\left(\frac{1}{2}\right)\\
&+\sum_{m=1,\ by\ 2}^{i-2}\left(\frac{i-m}{2}\right)\left(A^{2(m+1)}+A^{-2(m+1)}\right)S_m(\omega)*\left[\eta_0*\rho\left(\frac{1}{2}\right)+\tau*\rho(0)\right]\\
&+\sum_{m=0,\ by\ 2}^{i-3}\left(\frac{i-(m+1)}{2}\right)\left(A^{2(m+1)}+A^{-2(m+1)}\right)S_m(\omega)*\left[\eta_0*\rho(0)+\tau*\rho\left(\frac{1}{2}\right)\right].
\end{aligned}
$$

We will prove theorem 3.11. The proofs of 3.12 and 3.13 follow in a similar fashion.

Proof.

$$1_l * S_i(\omega) * 1_r =$$  $$= \sum_{k=-i,\ by\ 2}^{i} A^k$$ 

$$= \gamma(0,0) + \sum_{k=2\ by\ 2}^{i} A^k\gamma(k,0) + A^{-k}\gamma(-k,0)$$

$$\text{by lemma 1.9} = \sum_{k=-i\ by\ 2}^{i} A^k\gamma\left(0,\frac{k}{2}\right) + \sum_{k=2\ by\ 2}^{i} A^{2k}S_{k-1}(\omega)*\eta_0*\rho(0) + A^{-2k}S_{k-1}(\omega)*\tau*\rho\left(\frac{-1}{2}\right)$$
$$+ \sum_{k=2\ by\ 2}^{i}\sum_{n=0}^{k-2}\left(A^{2(n+1)} + A^{-2(n+1)}\right)S_n(\omega)*\left[\eta_0*\rho\left(\frac{k-1-n}{2}\right) + \tau*\rho\left(\frac{-k+n}{2}\right)\right]$$

split the sum over n even and odd we get

$$= \sum_{k=-i\ by\ 2}^{i} A^k\gamma\left(0,\frac{k}{2}\right) + \sum_{k=2\ by\ 2}^{i} A^{2k}S_{k-1}(\omega)*\eta_0*\rho(0) + A^{-2k}S_{k-1}(\omega)*\tau*\rho\left(\frac{1}{2}\right)$$
$$+ \sum_{k=2\ by\ 2}^{i}\sum_{n=0\ by2}^{k-2}\left(A^{2(n+1)} + A^{-2(n+1)}\right)S_n(\omega)*\left[\eta_0*\rho\left(\frac{k-1-n}{2}\right) + \tau*\rho\left(\frac{-k+n}{2}\right)\right]$$
$$+ \sum_{k=2\ by\ 2}^{i}\sum_{n=0\ by1}^{k-3}\left(A^{2(n+1)} + A^{-2(n+1)}\right)S_n(\omega)*\left[\eta_0*\rho\left(\frac{k-1-n}{2}\right) + \tau*\rho\left(\frac{-k+n}{2}\right)\right]$$

Notice that we can simplify the ρ modulo 2 in each individual sum

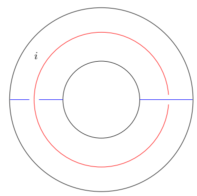
$$= \sum_{k=-i\ by\ 2}^{i} A^k\gamma\left(0,\frac{k}{2}\right) + \sum_{k=2\ by\ 2}^{i} A^{2k}S_{k-1}(\omega)*\eta_0*\rho(0) + A^{-2k}S_{k-1}(\omega)*\tau*\rho\left(\frac{1}{2}\right)$$
$$+ \sum_{k=2\ by\ 2}^{i}\sum_{n=0\ by\ 2}^{k-2}\left(A^{2(n+1)} + A^{-2(n+1)}\right)S_n(\omega)*\left[\eta_0*\rho\left(\frac{1}{2}\right) + \tau*\rho\left(0\right)\right]$$
$$+ \sum_{k=2\ by\ 2}^{i}\sum_{n=1\ by\ 2}^{k-3}\left(A^{2(n+1)} + A^{-2(n+1)}\right)S_n(\omega)*\left[\eta_0*\rho\left(0\right) + \tau*\rho\left(\frac{1}{2}\right)\right]$$

collecting terms we get

$$= \sum_{k=-i\ by\ 2}^{i} A^k\gamma\left(0,\frac{k}{2}\right) + \sum_{k=2\ by\ 2}^{i} A^{2k}S_{k-1}(\omega)*\eta_0*\rho(0) + A^{-2k}S_{k-1}(\omega)*\tau*\rho\left(\frac{1}{2}\right)$$
$$+ \sum_{m=0\ by\ 2}^{i-2}\left(\frac{i-m}{2}\right)\left(A^{2(m+1)} + A^{-2(m+1)}\right)S_m(\omega)*\left[\eta_0*\rho\left(\frac{1}{2}\right) + \tau*\rho\left(0\right)\right]$$
$$+ \sum_{m=1\ by\ 2}^{i-3}\left(\frac{i-(m+1)}{2}\right)\left(A^{2(m+1)} + A^{-2(m+1)}\right)S_m(\omega)*\left[\eta_0*\rho\left(0\right) + \tau*\rho\left(\frac{1}{2}\right)\right]$$

□

Remark 3.12. Notice that

$$1_r * S_i(\omega) * 1_l =$$ 

is the same picture as in 1.11 or 1.12 depending if i is even or odd rotated by 180 degrees. Instead of smoothing the left crossing first, start with the right one it will yield the same formula with every basis diagram rotated 180 degrees. All of these calculations can also be done for T_k in a similar and easier fashion.

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