BRAIDS AND KNOTS IN MATHEMATICS

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Abstract. We resume the content of a conference titled "Braids and Knots in Mathematics" given in the Garden of Science. In particular, we define in parallel braids and knots in space and explain how projections and Reidemeister moves can help reduce our study to the plane. Then we put in evidence a structure on the braids and explain how an operation on these braids allows us to associate them with words on these, which then classifies them. In the case of knots, we explain how the study of invariants allow the distinction of some knots even if the complete classification of knots still remains an open problem.

Acknowledgements

CIEL and the Garden of Science have given us the opportunity to make a conference entitled "Braids and Knots in Mathematics". This conference was followed by a series of questions that have prompted us to extend our work. Ms. Bopp, director of the IREM in Strasbourg, allowed us to use the course from l'Ouvert to draft this article. We thank all those who encouraged us to engage in these projects.

Introduction

Braids and knots have the privilege to occupy regularly a place in research journals as well as popular magazines. This enthusiasm reflects the great diversity of areas subsequently assembled with braids and knots. A quick search on Arxiv.org gives more than 300 responses to the word braids or knots, revealing the great vitality of mathematical research in these areas. But this enthusiasm is also due to the variety of applications of these theories [8,9]: the cryptography of quantum computers through the modeling of particle trajectories or genetics...

We present braids and knots starting from the intuition that we have of these objects in everyday life and then define these mathematical objects. Specifically, knots for mathematicians are pieces of string closed in space and intertwined, while the braids are pieces rope attached to the top and bottom and intertwined. A major difference with common knots is that the knots and braids in mathematics do not unravel at all!

One of the first steps to simplify objects and work better with knots and braids is to find a way to represent them on the plane. we define knots and braids on the plane, by projecting these spatial objects onto a plane in a general way. It is then appropriate to consider the Reidemeister moves that describe how these drawings our knots or braids on the plane behave when we change projection. These movements enable us to establish that knots and braids in space can be represented by their drawings on the plane. After these definitions in parallel for knots and braids, we explain how the closure of a braid makes a knot.

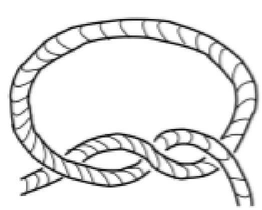
The problem of the definition has being solved, we would like to distinguish two different knots or braids. This classification problem, a natural one for mathematicians, is different for knots and braids; Indeed, a group structure, similar to the structure of the integers with addition can be placed on the braids. A composition can be defined by pasting one braid below the other and this operation is sufficiently rich in properties to allow us to cut out a simple braid and reduce the problem of classification of braids to a much simpler one (classification of words whose letters are simple braids).

For knots, we can still join two knots but this operation does not have the previous structure and we do not get the classification in this manner. Mathematicians have then define invariants to answer the question "When are two knots similar". An invariant is a mathematical object associated to a knot, such as its minimum number of crossing, which allows us to say that two knots with different invariants are different. The problem is that two different knots can have the same invariant. Invariants more and more subtle and sophisticated can be assign to knots but for now, no invariant can distinguish all knots.

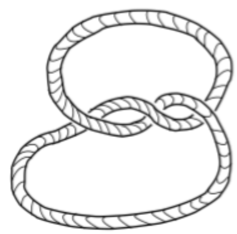
1. Parallel definition of Braids and Knots

1.1. **Knots and crossings.**

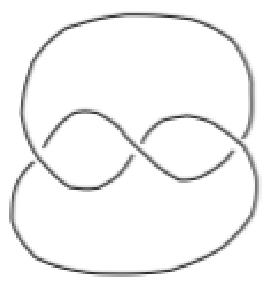
When speaking about knots, we can not imagine talking about mathematics; rather we associate this term to other areas such as sailing or climbing. In a simpler way we might think about shoelaces, in other words the knots that occur in everyday life, those that are made with pieces of string. Let's look more closely at simplest one, the square knot:



This knot is typical of those that can be made, in particular it is easy to untie. If one makes a complicated knot, we can always untie it, although as everyone has experienced, that can sometimes be too difficult and take very long. So in a sense, all knots are the same because we can unravel them. To diversify, mathematicians, change this intuitive definition by gluing the two ends dangling from the end of string, resulting in the case of the following picture:



For the moment, as a matter of example, we have considered knots with thickness (like a piece of string); but we will continue considering knots with no thickness, which gives the following diagram:



Note that the drawing indicates whether a strand passes above or below another strand.

To draw our knots we have been forced to put them flat on a sheet of paper, but in reality what we look at it is these same knots but in the ambient space (3 dimensional). We can now give a definition of a knot.

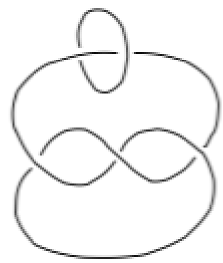
Definition 1.1. A knot is a closed curve with no repated points in space, it is the image of a continuous function $\gamma : [0,1] \to \mathbb{R}^3$ such that $\gamma(0) = \gamma(1)$ and $\gamma(t) \neq \gamma(t')$ for two distinct elements t and t'.

We will also consider the objects a little more general: links. In everything we have done previously we only had one piece of string; we now perform the same operations with several pieces of string.

Definition 1.2. A link is a disjoint union of closed curves with no repeated points in space.

Here's an example of a link:

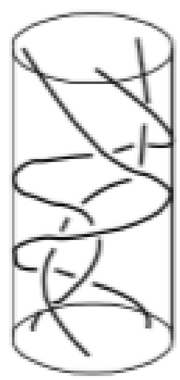
Each component will be a closed curve (which corresponds to a of the pieces of string). By using this vocabulary, a knot is an link that has only one component. We note that in this example one of the components (one of the pieces of string) is a circle. This circle is a knot (even if in real life you would not call it a knot), the



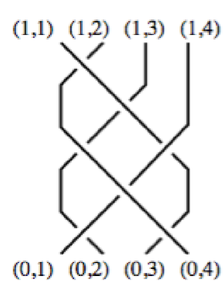
simplest knot there is, it's called the unknot or trivial knot (trivial meaning simplest in mathematics).

1.2. **Braids of n strands.**

In the same way as for the knots, with braids we consider the pieces of string in space attached at the top and bottom of a cylinder as shown in the drawing below and intertwined.



To draw, we project the braids on the plane. We can always find a projection that allows us to draw the braids as strands tied to the top and bottom as shown in the following figure. Moreover, deforming the pieces of string, it is always possible to consider that the crossings are only between two strands.



We can then now give a mathematical definition of a braid has n strands.

Definition 1.3. A geometric braid consists of n open curves (shoelaces) attached at the coordinates $(1,1),...(1,n)$ at the top and $(0,1),...,(0,n)$ at the bottom, which always go down such that the only points of intersection between the open curves are double points so that we know what strand passes over the other.

We will represent the top strand with a solid line and the one below with a discontinuous line as shown in the previous example.

1.3. **Projections and Reidemeister moves.**

Knot projections

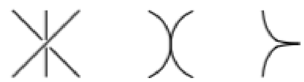
In the previous part, knots and links were defined as objects that live in space, but a flat representation allows us to understand these objects more easily. The operation carried out is a projection (we take our piece of string and crush it against a wall). And set up a drawing convention to recognize in each crossing the relative position of the two pieces. The convention is the same as for the braids and is given by the following diagrams:

For example, if we consider the following diagram and start at the $\times$ mark following the arrow, it passes successively above, below, above, below, above, below and returns to the initial point.

More generally, we always assign an orientation to a knot or a component of a link. The orientation will be represented on the diagram by an arrow. A knot (link) with an orientation will be referred as an oriented knot (link).

The drawing of a knot on the plane is called a *knot diagram*. We will now consider how these diagrams represent objects in space. For this we want a general projection. General means that from the graph we can recover the knot or link. Otherwise the projection is disastrous as in the drawings below:

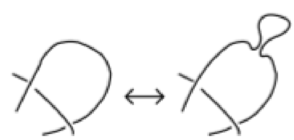
If the projection locally looks like one of these designs, the knot is not perfectly defined in space (there are several knots that have this projection). Fortunately, we



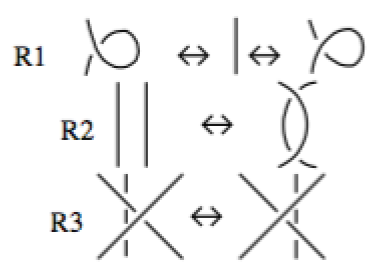
can show that there is still a general projection. For a knot or link diagram, we understand in fact a projection of a general knot or link.

Reidemeister moves for knots

A classic problems in Mathematics is the problem of classification of the objects we've studied. In this context, we are interested in the classification of knots and links. We will therefore establish a classification criteria. For example, if you want to store your library, you must first decide whether you choose a thematic classification, alphabetical, by author or title. In our case it is first necessary to decide when to consider two knots as the same. For this lets go back to the strings for a moment. Take a piece of string, tie some knots and glue the ends together. We would like then that all manipulations that can done with our piece of string in space except cutting, to always give the same knot: two knots are the same if we can transform one into the other by continuous manipulation in space. The term basically means continuous operation without cutting the piece of string. These permitted manipulations being spelled, allow for the study of knot diagrams. There are basically four types and in the following diagrams we show the pictures locally. Deformations between two crossings, as in the following example are permitted,



and the three types of deformation involving crossings are the so-called Reidemeister moves, named after the German mathematician who discover them. These manipulations happen on both side of the arrow signifying that we can move from one diagram to another, in one direction or the other.



It may be noted that to move from side of the arrows to the other, one goes through a disastrous situation, like the ones before. For example, in the case of R2 to move the right case to the one on the left, pulling the two ends one to the right the other to the left, just before having two parallel ends we get one of those disastrous situation.

It remains to see if all the manipulations of a knot in space can be seen on the knot diagram through Reidemeister moves as well as manipulation between the crossings (trivial manipulations). Reidemeister has given a proof [5].

Theorem 1.4 (Reidemeister 1927)**.** *If we can turn a knot into another knot by continuous manipulation in space, the same result can be achieved by manipulating the projection with Reidemeister moves and trivial manipulations of the knot diagram on the plane.*

This theorem means that to study the classification of knots in space, it suffices to study their diagrams using trivial manipulations and Reidemeister moves. So we reduced the problem from a space to a plane, from three dimensions to two. In the same way, this result remains true for links and for oriented knots and links. Reidemeister moves for knots and links should be possible with any orientation, but we can show that they can be obtained from the two following movements.

Reidemeister moves for braids

As the case of knots, general projections are used to obtain the geometric definition of a braid on the plane. So we must also take into account possible movements in space and their impact on drawing on the plane. All trivial manipulations (without touching the crossings) are authorized as well as the Reidemeister moves. However, these movements are slightly different from those regarded for knots, for example the Reidemeister R1 move bringing the strand back is excluded. We in fact have the R2 move (taking into account the orientation from the top down) and which is represented in the following figure and the Reidemeister R3 move that itself remains unchanged.

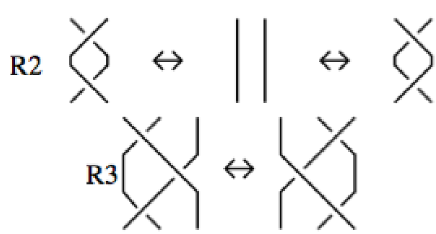


FIGURE 1

We can then define the braids on the plane by allowing these movements, moreover they match the intuition as stated by the following theorem (which follows from the Reidemeister theorem).

Theorem 1.5. *By using the trivial manipulation without touching the crossings, as well as the Reidemeister R2 and R3 moves, the geometric braids correspond to braids in space.*

1.4. **The closure of a braid is a knot.**

The braids and knots come from pieces of string intertwined in space, so it is natural to try to connect the two mathematical notions we just define to represent these pieces of string. This is possible by performing the following construction. Given a braid, we can close it by connecting the ends together without adding crossings as shown in the following figure.

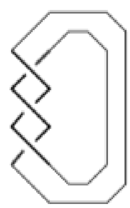
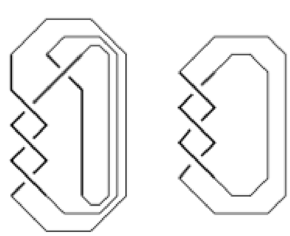


FIGURE 2

We then get a knot or link. Alexander's theorem then asserts that all knots and links can be obtained by closing a braid.

Theorem 1.6 (Alexander 1923)**.** *Every knot or link is the closure of a braid.*

This result is a result of existence as a natural question arises: being given a knot, how to find a braid whose closure in this knot. Vogel [11] gives an algorithm for "closing" a knot. We can then "carry" the results by closing braids into knots. However if we get some results this way, the problem of the classification of knots can not be derived from the classification of braids. In fact, we will see that the classification of braids comes from a structure on the set of braids, but this same structure can not be put on the knots. Note also that the closure of very different braids can give the same knots as shown in the example below.



2. A structure for braids

In this section, we show how to put a structure on braids that allows us to reduce the study of braids into a study of words and deduce a classification.

2.1. **General remarks.**

Let's start by looking at these sets of braids and their connection with numbers or permutations.

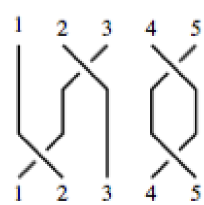
Connection with numbers

Notice that $\mathcal{B}_2$ consists of all braids with two strands. We have two types of elementary crossings, we say positive or negative as shown respectively in the following figures.

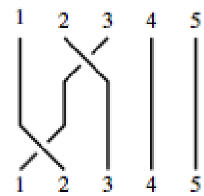
For the braids with two strands, we can count the number of crossings with their signs and get an integer associated to the braid. The Reidemeister R2 move (Fig. 1) corresponds exactly with operations $1 - 1 = 0 = -1 + 1$ and the braid of Figure (2), for example, corresponds to the number -3. We can then identify two strand braids with integers! For integers, we have an operation, namely addition, that gives us a structure. We will build the corresponding structure for braids in general and illustrate its properties thanks to the example of the two strand braids. But for now we continue to look at the connection with another set, the permutations.

Connection with permutations

Let us consider the braid with 5 strands shown below



Number the strands and follow them. The first strand ends in position 2 and the second in 3, the third strand ends in position 1. Moreover the strands 4 and 5 remain in position 4 and 5. So we have a permutation associated to the positions of this braid which we denote $1 \to 2 \to 3$, or (123). In the same way, let us consider a braid with n number of strands. Follow the strands and make note of the final position of each strand. We get a permutation of $1, ..., N$ (that is to say another way to arrange the numbers 1 to n) associated to the braid. Notice that this permutation is invariant of the Reidemeister moves and then we can define an application of all n strands braids (denoted by $\mathcal{B}_n$) to the set of permutations of $1, ..., N$ (denoted $\mathcal{S}_n$). This method does not allow us to identify the braids and permutations as shown in the following figure or the permutation (123) obtained from another braid.



In a sense braids represent permutations with the operations carried!

A braid in general

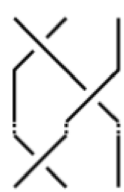
So far, we have always considered braids with n strands. How to define a braid in a general way? Take a 3 strand braid and juxtapose it with a braid with two trivial strands (that is to say not intertwined). We get a 5 strand braid (see the previous example). In a general way, we can always juxtapose trivial braids beside the strands of another braid to get the number of strands that we want. In this way we can now consider a general braid. We are then ready to look at the structure of these braids.

2.2. **Operations and structures on braids.**

To learn more about braids, we would like to know how to get a braid from others. For that we will look at how to create a braid from two other and by analogy with numbers (corresponding to the braids with two strands), what properties, and what structure can we hoped for braids, we quickly summarize the structure in a table (Fig. 3).

How do we build a braid from two others?

From two braids, we can eventually add strands to get the two braids to have the same size and then paste the second braid below the first and connect the strands as shown in the following figure.



We obtain an operation on the set of braids is that we called composition of two braids. In the example of the braids with two strands we can identify them with the number of crossings. The composition of two braids then its identified with the addition of integers.

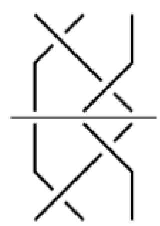
What properties are associated with this operation?

By analogy with numbers, see what properties should be checked by this composition.

The composition of braids like addition of numbers is associative. For braids, this amounts to saying that if we take three braids and stack them one below the other, the result of the composition is the same as first connecting the first to the second from the top and then connecting the resulting braid obtained with the one below or the opposite: connecting the two braids underneath and then the one on top.

The composition as the addition possesses an identity element. Consider the braid made from strands not intertwined; we will call this the trivial braid. Composition with this braid does not change and therefore the trivial braid is an identity element.

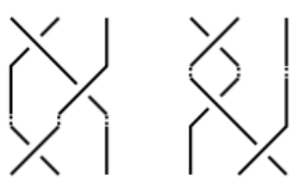
Each braid admits an inverse braid. Given a braid, we need to find another braid that, when we compose them, gives a result which is identified with the trivial braid (equivalent to zero). This is possible by considering the mirror image of the braid. By composing a braid with its mirror image, we get a braid that reduces into the trivial braid by Reidemeister moves as shown in the following figure.



A set with an operation that verifies the preceding properties (associativity and existence of an identity element and inverse for all elements) is called a group. We now have a theorem.

Theorem 2.1. *The composition endows the collection of braids with a group structure.*

Note that we do not have all the properties of addition of numbers. For example addition is commutative. For braids, this is only true if the number of strands is greater than or equal to three. The following figure shows the example of two 3 strand braids that result in two different braids depending on the order of operation (To see this, it suffices to remark that the two associated permutations are different).

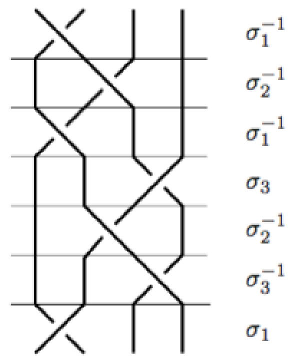


The advantage of this structure

The advantage of this structure is that allow us reduce the braids to the composition of simpler braids. Keeping our analogy with numbers, we can write $5 = 1+1+1+1+1$ and thus reduce the number 5 to the addition of simpler numbers, here all equal to 1.

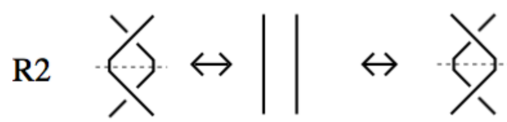
We can do the same for braids. For this we have to explain what "simple" braids are. Consider the braids with a single positive crossing between the i and $i+1$ strand. We denote this braid σ_i. Similarly we denote σ_i^{-1} the braid made from only one negative crossing between the strands i and $i+1$.

Consider now the braid with 4 strands on the following figure.

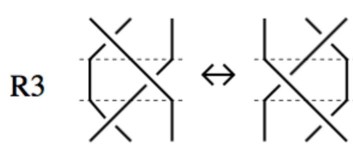


The braid is stretched to isolate each individual crossing and then we get what we call a word made from σ_i letters. The corresponding word is $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_1$.

Note that we can get more words from a braid that is otherwise stretched by performing Reidemeister moves on it. Fortunately, we know the relationships that these manipulations impose on the words. Specifically, when we can get two different words by stretching the braid differently, this requires a relationship of form $\sigma_i\sigma_j = \sigma_j\sigma_i$ and it is possible that the strands $i, i+1, j, j+1$ are all distinct that is to say $|i-j| \geq 2$. The Reidemeister R2 moves is written as $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = T$, where T denotes the trivial braid.



The Reidemeister R3 moves is written as $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.



We can then show that these are the only relationships involved and therefore the braid group identifies with the words made from the letters $\sigma_i^{\pm 1}$ and are subject to the preceding relations, this is a theorem of E. Artin [2].

Theorem 2.2 (Artin 1947)**.** *The braid group of n strands is represented by the group generated by $(\sigma_i \pm 1)_{i=1,\dots,n-1}$ with relations*

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{for all } i,j \text{ such that } |i-j| \geq 2 \\ \sigma_i\sigma_i^{-1} &= \sigma_i^{-1}\sigma_i = 1 && \text{for all } i = 1,\dots,n-1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for all } i = 1,\dots,n-2 \end{aligned}$$

This structure and this way of expressing them as the product of simple braids (generators) allows for the classification of braids by the use of words and comes with an algorithm that determines whether two braids are identical.

Operation	Placing the second braid below the first	Adding the crossings using the numbers
Associativity	Yes	(2+3)+5 = 2+(3+5)
Identity element	The trivial braid	0+2 = 2+0
Inverse	Mirror image	2-2=-2+2=0
Commutability	No	2+3=3+2
Generator	$(\sigma_i)_{i=1,\dots,n-1}$	1

FIGURE 3. Review of the braid structure

2.3. Recent work on the structures: total ordering.

Recently, a new structure was built for the braids. Dehornoy [3] shows how to define a total ordering on braids, that is to say given two braids, decide which one is the largest. For braids to be called positive, means is that we can associate a word that contains only the letters σ_i with positive powers, this ordering corresponds to the lexicographical ordering where the letters are ordered $\sigma_1 < \sigma_2 < \dots < \sigma_{n-1}$. For example $\sigma_1\sigma_2\sigma_4\sigma_2\sigma_1^2\sigma_8 < \sigma_1\sigma_4\sigma_2^3$ because of the first letter that differs we have that $\sigma_2 < \sigma_4$. Although for other braids, the definition is a little more complicated, this order allows you to say that two braids α and β are equal if and only if $\alpha \leq \beta$ and $\beta \leq \alpha$. Recent work also improved the speed of the differentiation algorithm for braids and others are on the way to create cryptographic algorithms based on calculations on the braids.

3. Invariants

Returning to the knots and their classification.

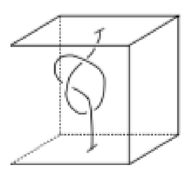
3.1. Using the Reidemeister theorem.

One might think that a priori the Reidemeister theorem provides the classification of knots. Indeed, if one takes two knot diagrams, it is enough to know if they are the same by applying a Reidemeister move on one of the diagrams, then compare and so on. If the two diagrams represent the same knot the process will terminate, otherwise it would be endless. The problem is that we can not conclude if the process does not stop. In the case that the two diagrams represent two different knots, we might be in trouble or it might be that needs more work. To force this process to end, we can try to apply the Reidemeister moves only if it makes the number of crossings decrease. Unfortunately, this is not enough because sometimes we have to make things more complicated (increase the number of crossings) in order to reduce the knot. For example, let us consider the following diagram:

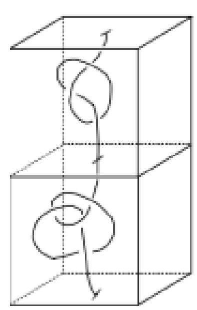
In fact we can see that this diagram is equivalent to that of the unknot but to transform it to a circle by Reidemeister moves, it is necessary to increase the number of crossings.

3.2. **A structure for knots?**

What happens when trying to put a structure on the set of knots? The first thing is how to put a composition on all knots. Let's try to define it as we did for braids: we cut the knot and then we put it in a box as shown in the diagram below (the box serves just as visual support).

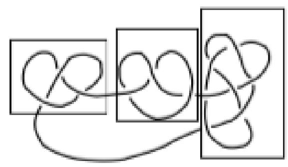


To compose knots put one box under the other:



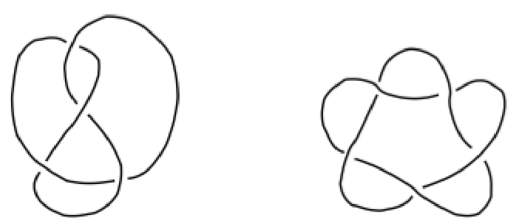
The you can glue the two ends to obtain a knot.

Now consider the properties similar to those we obtained for braids. The identity element of this composition is the unknot. This knot product is associative and commutative. By tightening for example the first knot can pass through the second along the string and get the composition in the other direction. The problem is that a knot has no inverse for this operation. Nevertheless by such process one can obtained a decomposition of knots into simpler knots, that we can not decompose. There are then, as for integers, arithmetic knots. Schubert has studied the properties of this arithmetic for 50 years. For example, the next knot has been broken down into three simpler knots.

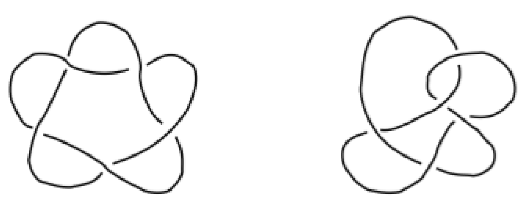


3.3. **Construction of invariants.** Neither the study of a structure on the knots nor the Reidemeister theorem, have brought closure to the classification of knots. Nevertheless this Reidemeister theorem is the key to the construction of invariants. To understand what an invariant is we must first specify invariant with respect to what? If one considers the knot, the invariance must be compared to the continuous manipulation of space, but since we can reduce a knot diagram, the invariance is trivial compared to manipulations and Reidemeister moves.

To understand this idea of invariant, let us consider a "characteristic" element of the knot like the number of crossings. A diagram can be associated with its number of crossings. But if we make a Reidemeister R1 move, for example, the number of crossing changes. This is not something that depends only on the knot but also its representation. On the other side, by considering the minimum number of crossings for a knot diagrams, we get an invariant. The following knots



do not have the same minimum number of crossings, so they are different, while the knots below



have the same number of minimum crossings but they are still different. Notice that it is easy to convince yourself by looking at the drawings that these knots are presented with their minimal number of crossings, but finding the minimum number of crossings of a knot is in general a very difficult problem.

In a general manner an invariant is the association of a knot diagram with some algebraic object (a number, a polynomial ...). For this association to be an invariant, it must be invariant under trivial manipulations and Reidemeister moves. As seen on a particular case, an invariant is generally an answer in the negation of the question: do these two diagrams represent the same knot? An invariant which is different for every knot will be called *complete*. There are many invariants such as the Alexander[1] polynomial (built in the year 1920) and Jones[2] (1984) which are, contrary to the minimum number of crossings, easily calculable. We know that the majority of invariants are not complete, nevertheless, we conjecture that the Vassiliev invariants $(1989-90)$ are. Meanwhile, the classification of knots remains an open problem.

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[1] We can read the article "One Hundred Years of algebraic topology" of C. Kassel that appeared in l'Ouvert 106 for a definition of the Alexander invariant.

[2] We can read the article "Topology of Knots" of V. Touraev that appeared in l'Ouvert 66 for a definition of the Jones invariant.