# Computations On The Relative Skein Algebra of a Local Annulus

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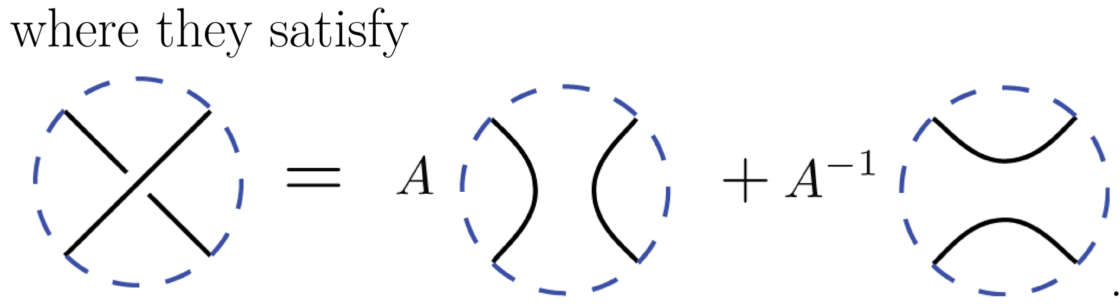
## Abstract

Let $M$ be a compact, oriented, 3-manifold of the form $F \times [0,1]$ where $F$ is an oriented surface of genus $g$ and $b$ boundary components. $K_A(M)$ is the skein algebra of $M$ for A an $N$-th root of unity. The purpose of this project is to develop tools to simplify a skein in $K_A(M)$ by finding a local annulus in $F$ and simplifying the skein in such annulus.

## Background

Consider a 3-manifold $M$. The Kauffman bracket skein module of $M$ is an algebraic invariant $K(M)$ built from the set of all framed links in $M$. A framed link is an embedded collection of annuli considered up to isotopy in $M$. We call $\mathcal{L}_M$, the set which consists of framed links union the empty link $\emptyset$. Three links $L, L_0$ and $L_\infty$ are *Kauffman skein related* if we can embedded them identically except in a ball where they satisfy




$L \sqcup \bigcirc$ denotes $L$ union with an unlinked 0-framed unknot.

Let $A = e^{\pi i/N}$, where $N$ is odd. Let $R$ denote the ring of Laurent polynomials $\mathbb{Z}[A, A^{-1}]$ and $R\mathcal{L}_M$ the free $R$-module on the basis $\mathcal{L}_M$. We already had the definition of what it means pictorially for three links to be Kauffman skein related so lets assign an algebraic relation to these pictures. If $L, L_0$ and $L_\infty$ are Kauffman skein related then $L - AL_0 - A^{-1}L_\infty$ represents the *skein relation*. We also have a *framing relation* given by $L \sqcup \bigcirc + A^2L + A^{-2}L$ for any $L \in \mathcal{L}_M$. The *Kauffman bracket skein algebra* $K(M)$ is then defined by $K(M) = R\mathcal{L}_M/S(M)$ where $S(M)$ is the smallest submodule of $R\mathcal{L}_M$ containing all possible skein and framing relations.

The Tchebychev polynomials of the second kind are defined in the following recurrent way: $S_0 = 1$, $S_1 = \omega$, $S_k = \omega S_{k-1} - S_{k-2}$. For computational purposes we will extend the definition to include $k = -1$ and define $S_{-1} := 0$.

## $K_A(Ann)$

The skein algebra of an annulus at $A$ an $N$-th root of unity, by definition is the collection of all framed links in the annulus including the empty link, mod out by the skein relation and the framing relation. That is to say $K_A(Ann) = \mathbb{C}\mathcal{L}_{Ann}/ < L - AL_0 - A^{-1}L_\infty,\ L \sqcup \bigcirc + A^2L + A^{-2}L >$. Notice that the only nontrivial simple diagrams on the annulus are just linear combinations of powers of the the one that wraps around it, call it $\omega$. Hence we don't need to worry about the skein relation because there won't be any crossings, since if you have two copies of $\omega$ you can always move then around a little so that they become disjoint. The framing relation will reduce a disjoint union with an unknot to a scalar multiplication. When we put all of this together it tells us that the skein algebra of an annulus is precisely the polynomial ring in one variable. That is to say, $K_A(Ann) = \mathbb{C}[\omega]$.

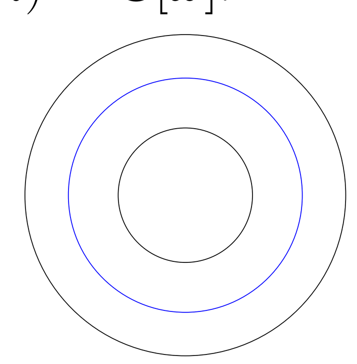


Figure 1: $\omega$ going around the annulus

## Basic Diagrams

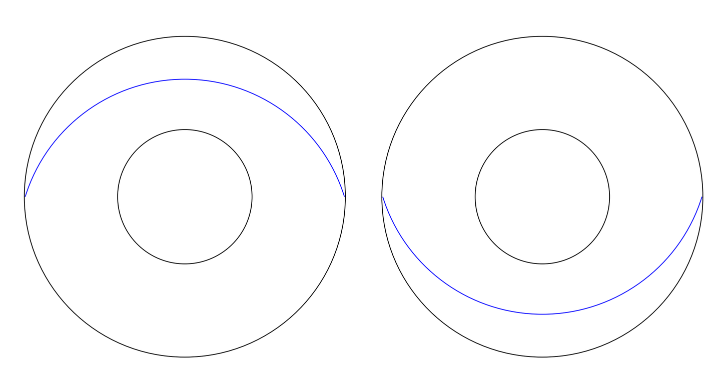


Figure 2: $\eta(0)$ and $\tau$

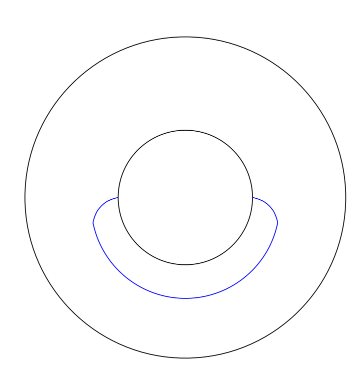


Figure 3: $\rho(0)$

## More diagrams

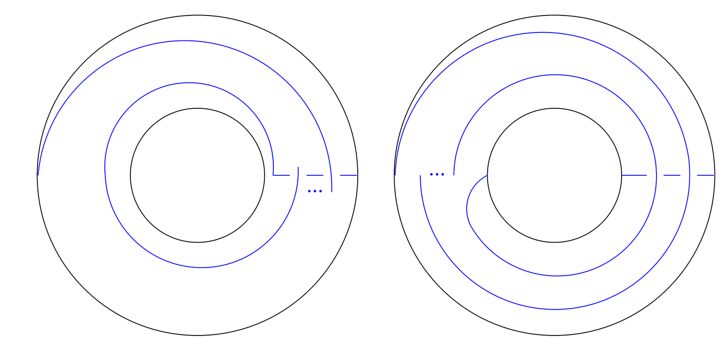
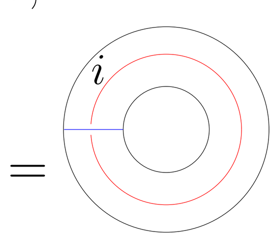


Figure 4: $\eta(k)$ and $\gamma(k,0)$

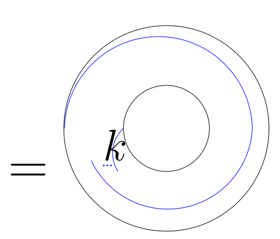
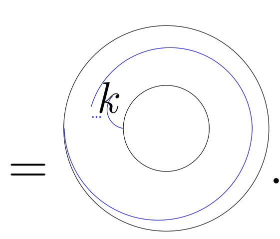
Lemma 1:

Let $i \in \mathbb{Z}$, $i \geq 0$, then

$1 * S_i(\omega) =$  $= \sum_{k=-i,\ by\ 2}^{i} A^k C_k.$



Where $C_k$ spirals around $k$-times clockwise, when $k > 0$ and counterclockwise when $k < 0$.

$C^k =$  $, C^{-k} =$  .



Lemma 2:

Let $k > 0$, then
$\eta_k = A^k S_k(\omega) * \eta_0 + A^{k-2} S_{k-1}(\omega) * \tau.$

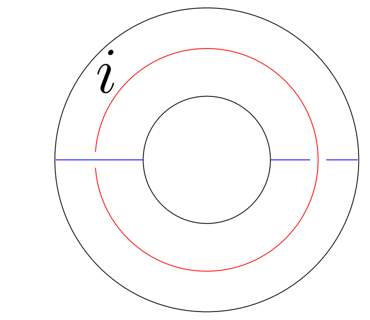
Lemma 3:

$$\begin{aligned}\gamma(k,0) = &\sum_{i=0}^{k-1} A^{2(i+1)-k} S_i(\omega) * \eta_0 * \rho\left(\frac{(k-1)-i}{2}\right) \\ &+ \sum_{i=0}^{k-2} A^{2(i+1)-k} S_i(\omega) * \tau * \rho\left(\frac{(k-2)-i}{2}\right) \\ &+ A^{-k}\gamma\left(0, \frac{k}{2}\right).\end{aligned}$$

Let $h : Ann \to Ann$ be a homeomorphism that fixes the outside boundary and half twists the inside one. The second argument of the $\gamma$s corresponds to the number of times you apply such a homeomorphism, $h$ to $\gamma(0,0)$, that is to say $\gamma(0, \frac{k}{2}) = h^k(\gamma(0,0))$.

## Main Theorem

Let $i \geq 2$ and even, then

$1_l * S_i(\omega) * 1_r =$ 



$$\sum_{k=-i\ by\ 2}^{i} \gamma\left(0, \tfrac{k}{2}\right) + \sum_{k=2\ by\ 2}^{i} A^{2k} S_{k-1}(\omega) * \eta_0 * \rho(0) + A^{-2k} S_{k-1}(\omega) * \tau * \rho\left(\tfrac{1}{2}\right)$$

$$+ \sum_{n=0,\ by\ 2}^{i-2} \left(\tfrac{i-l}{2}\right)\left(A^{2(n+1)} + A^{-2(n+1)}\right) S_n(\omega) * \left[\eta_0 * \rho\left(\tfrac{1}{2}\right) + \tau * \rho(0)\right]$$

$$+ \sum_{n=1,\ by\ 2}^{i-3} \left(\tfrac{i-(n+1)}{2}\right)\left(A^{2(n+1)} + A^{-2(n+1)}\right) S_n(\omega) * \left[\eta_0 * \rho(0) + \tau * \rho\left(\tfrac{1}{2}\right)\right].$$

## Future Work

We are currently using these results to study the Kauffman Bracket Skein Algebra of the once-punctured torus. Once we understand the once punctured torus case we will proceed to use these techniques in the study of surfaces of arbitrary genus and boundary components.

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