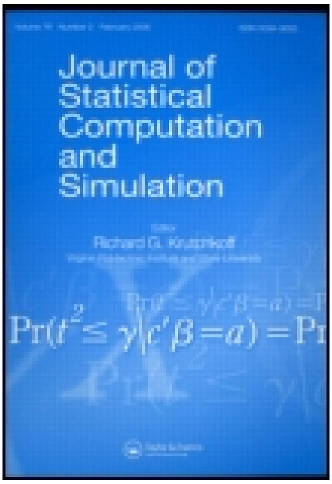

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An improved method of estimation for the parameters of the Birnbaum-Saunders distribution

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An improved method of estimation for the parameters of the Birnbaum–Saunders distribution

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The Birnbaum–Saunders (BS) distribution is a positively skewed distribution, frequently used for analysing lifetime data. In this paper, we propose a simple method of estimation for the parameters of the two-parameter BS distribution by making use of some key properties of the distribution. Compared with the maximum likelihood estimators and the modified moment estimators, the proposed method has smaller bias, but having the same mean square errors as these two estimators. We also discuss some methods of construction of confidence intervals. The performance of the estimators is then assessed by means of Monte Carlo simulations. Finally, an example is used to illustrate the method of estimation developed here.

Keywords: asymptotic confidence interval; bias; BS distribution; bootstrap confidence interval; Jackknife confidence interval; maximum likelihood estimators; mean square error; modified moment estimators

1. Introduction

The Birnbaum–Saunders (BS) distribution, proposed by Birnbaum and Saunders [1], is a flexible and useful model for the analysis of reliability data, and various inferential methods have been developed for this distribution in the literature. The cumulative distribution function (CDF) of a two-parameter BS random variable T is given by

$$F(t;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right], \quad t > 0,\ \alpha > 0,\ \beta > 0, \tag{1}$$

where $\Phi(\cdot)$ is the standard normal CDF, and β and α are the scale and shape parameters, respectively.

The BS distribution has found applications in a wide array of problems. For example, Birnbaum and Saunders [2] fitted the model to several data sets on the fatigue life of 6061-T6 aluminum coupons. Desmond [3] extended the model to failure in random environments and studied the fatigue damage at the root of a cantilever beam. Chang and Tang [4] used the distribution to model active repair times for an airborne communication transceiver. For a review of various developments on the BS distribution, interested readers may refer to Johnson et al. [5].

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The maximum likelihood estimators (MLEs) of the parameters α and β were derived originally by Birnbaum and Saunders [2] and their asymptotic distributions were obtained by Engelhardt et al. [6]. Ng et al. [7] derived the modified moment estimators (MMEs) for complete samples. Subsequently, Ng et al. [8] and Wang et al. [9] extended the MMEs to the case of type-II censored samples. Rieck [10] discussed the estimation problem based on symmetrically censored samples. Here, we propose another simple explicit estimation method and show that it has a smaller bias compared with the MMEs and the MLEs, especially in the case of small samples.

The rest of this paper proceeds as follows. In Section 2, we first describe briefly the BS distribution and some of its key properties. In Section 3, we describe the MLEs and the MMEs and the corresponding inferential results. In Section 4, we present the proposed method of estimation and show that the estimators always exist uniquely. In Section 5, we show that the proposed estimator (PE) of the shape parameter has a negative bias, and that the bias is smaller than that of the MMEs. In Section 6, we describe the interval estimation methods based on Jackknife, bootstrap and the asymptotic approaches. A Monte Carlo simulation study is carried out in Section 7 to examine the bias and mean square errors (MSEs) of the PEs, and to compare their performance with those of the MMEs and the MLEs. In Section 8, we illustrate the approach by using a real data set from the reliability literature. Finally, in Section 9, we make some concluding remarks and also point out some problems that are of interest for further study.

2. Birnbaum–Saunders distribution and some properties

The probability density function (PDF) of the BS random variable T in Equation (1) is given by

$$f(t;\alpha,\beta)=\frac{1}{2\sqrt{2\pi}\alpha\beta}\left\{\left(\frac{\beta}{t}\right)^{1/2}+\left(\frac{\beta}{t}\right)^{3/2}\right\}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{\beta}{t}+\frac{t}{\beta}-2\right)\right],\quad t>0,\ \alpha>0,\ \beta>0. \tag{2}$$

The following interesting properties of the BS distribution in Equation (1) are well-known; see, for example, Birnbaum and Saunders [1].

Property 2.1 *Suppose* $T\sim \mathrm{BS}(\alpha,\beta)$ *as defined in Equation* (1). *Then*

(1) $(1/\alpha)(\sqrt{T/\beta}-\sqrt{\beta/T})\sim N(0,1)$;
(2) $CT\sim \mathrm{BS}(\alpha, C\beta)$ *and*
(3) $1/T\sim \mathrm{BS}(\alpha,1/\beta)$.

By using Result (1) in Property 2.1, the expected value and variance of T can be readily obtained as

$$E(T)=\beta\left(1+\tfrac{1}{2}\alpha^2\right), \tag{3}$$

$$\mathrm{Var}(T)=(\alpha\beta)^2\left(1+\tfrac{5}{4}\alpha^2\right). \tag{4}$$

Similarly, by using Result (3) in Property 2.1, we readily have

$$E(T^{-1})=\beta^{-1}\left(1+\tfrac{1}{2}\alpha^2\right), \tag{5}$$

$$\mathrm{Var}(T^{-1})=\alpha^2\beta^{-2}\left(1+\tfrac{5}{4}\alpha^2\right). \tag{6}$$

These properties will be utilized in developing the new estimators later in Section 4.

3. Maximum likelihood and modified moment estimators

3.1. *Maximum likelihood estimators*

The MLEs of the parameters of the BS distribution have been discussed in detail in the literature; see, for example, Birnbaum and Saunders [2] and Ng et al. [7]. Birnbaum and Saunders [2] showed that the MLEs of α and β do exist and are unique in the case of complete samples. This result was generalized recently by Balakrishnan and Zhu [11] for different forms of censored data.

Let $(t_1, t_2, \dots, t_n)$ be a random sample of size n from the BS distribution with PDF as given in Equation (2). Then, the MLE of β (denoted by $\hat{\beta}$) can be obtained from the equation

$$\beta^2 - \beta[2r + K(\beta)] + r[s + K(\beta)] = 0, \tag{7}$$

where $s = (1/n)\sum_{i=1}^{n} t_i$, $r = [(1/n)\sum_{i=1}^{n} t_i^{-1}]^{-1}$ and $K(x) = [(1/n)\sum_{i=1}^{n}(x + t_i)^{-1}]^{-1}$ for $x \geq 0$. Since this is a non-linear equation, one may have to use either the Newton–Raphson algorithm or some other algorithm. Once $\hat{\beta}$ is obtained, the MLE of α (denoted by $\hat{\alpha}$) can be obtained explicitly as

$$\hat{\alpha} = \left[\frac{s}{\hat{\beta}} + \frac{\hat{\beta}}{r} - 2\right]^{1/2}. \tag{8}$$

Engelhardt et al. [6] showed that the asymptotic joint distribution of $\hat{\alpha}$ and $\hat{\beta}$ is bivariate normal given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \dfrac{\alpha^2}{2n} & 0 \\ 0 & \dfrac{\beta^2}{n[0.25 + \alpha^{-2} + I(\alpha)]} \end{pmatrix}\right], \tag{9}$$

where $I(\alpha) = 2\int_0^{\infty}\{[1 + g(\alpha x)]^{-1} - 0.5\}^2 \mathrm{d}\Phi(x)$ and $g(y) = 1 + y^2/2 + y(1 + y^2/4)^{1/2}$. Based on the results of an extensive Monte Carlo simulation study, Ng et al. [7] observed that

$$\mathrm{Bias}(\hat{\alpha}) \approx -\frac{\alpha}{n}, \tag{10}$$

$$\mathrm{Bias}(\hat{\beta}) \approx \frac{\alpha^2}{4n}. \tag{11}$$

3.2. *Modified moment estimators*

Ng et al. [7] proposed the MMEs from Equations (3) and (5). In this case, the unique MMEs for α and β, denoted by $\tilde{\alpha}$ and $\tilde{\beta}$, are given explicitly by

$$\tilde{\alpha} = \left\{2\left[\left(\frac{s}{r}\right)^{1/2} - 1\right]\right\}^{1/2}, \tag{12}$$

$$\tilde{\beta} = (sr)^{1/2}. \tag{13}$$

The asymptotic joint distribution of $\tilde{\alpha}$ and $\tilde{\beta}$ has been shown to be bivariate normal given by

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \dfrac{\alpha^2}{2n} & 0 \\ 0 & \dfrac{(\alpha\beta)^2}{n}\left(\dfrac{1 + (3/4)\alpha^2}{(1 + (1/2)\alpha^2)^2}\right) \end{pmatrix}\right]. \tag{14}$$

Based on the results of an extensive Monte Carlo simulation study, Ng et al. [7] also observed that the MLEs and the MMEs performed very similarly in terms of both bias and MSE, especially for small values of α. Upon inspecting the pattern of the bias of the MMEs, they found that exactly the same formulae (10) and (11) for the bias also apply to these estimators.

4. Proposed estimators

Let $T \sim \mathrm{BS}(\alpha, \beta)$ as defined in Equation (1), and $(T_1, \ldots, T_n)$ be a complete sample of size n. Then, let us define

$$Z_{ij} = T_i \frac{1}{T_j}, \quad \text{for } 1 \leq i \neq j \leq n. \tag{15}$$

It is evident that $Z_{ij} = 1/Z_{ji}$, and we thus have $\binom{n}{2}$ pairs (Z_{ij}, Z_{ji}).

By exploiting the fact that $1/T \sim \mathrm{BS}(\alpha, 1/\beta)$ (see Result 3 in Property 2.1) and the independence of T_i and T_j, we immediately find

$$\begin{aligned} E(Z_{ij}) &= E\left(T_i \frac{1}{T_j}\right) \\ &= E(T_i) E\left(\frac{1}{T_j}\right) \\ &= \beta\left(1 + \frac{\alpha^2}{2}\right) \frac{1}{\beta}\left(1 + \frac{\alpha^2}{2}\right) \\ &= \left(1 + \frac{1}{2}\alpha^2\right)^2. \end{aligned} \tag{16}$$

Then, the sample mean of z_{ij} (observed value of Z_{ij}), calculated as

$$\bar{z} = \frac{1}{2\binom{n}{2}} \sum_{1 \leq i \neq j \leq n} z_{ij}, \tag{17}$$

may be equated to $E(Z_{ij}) = (1 + (1/2)\alpha^2)^2$ and solved for α to obtain the estimator

$$\hat{\alpha}^* = [2(\sqrt{\bar{z}} - 1)]^{1/2}. \tag{18}$$

Also, since

$$E(\bar{T}) = E\left(\frac{1}{n}\sum_{i=1}^{n} T_i\right) = \beta\left(1 + \frac{1}{2}\alpha^2\right), \tag{19}$$

we can get an estimator of β (denoted by $\hat{\beta}_1^*$) as

$$\hat{\beta}_1^* = \frac{2s}{(\hat{\alpha}^*)^2 + 2} = \frac{s}{\sqrt{\bar{z}}}, \tag{20}$$

where $s = (1/n)\sum_{i=1}^{n} t_i$. Moreover, since

$$E\left[\left(\overline{\frac{1}{T}}\right)\right] = \frac{1}{n}\sum_{i=1}^{n} E\left(\frac{1}{T_i}\right) = \frac{1}{\beta}\left(1 + \frac{1}{2}\alpha^2\right), \tag{21}$$

we can also get another estimator of β (denoted by $\hat{\beta}_2^*$) as

$$\hat{\beta}_2^* = r(1 + \tfrac{1}{2}(\hat{\alpha}^*)^2) = r\sqrt{\bar{z}}, \tag{22}$$

where $r = [(1/n)\sum_{i=1}^{n} t_i^{-1}]^{-1}$ as before. Now, we can pool these two estimators of β to obtain an estimator of β as

$$\hat{\beta}^* = (\hat{\beta}_1^* \hat{\beta}_2^*)^{1/2} = \left(\frac{s}{\sqrt{\bar{z}}} r\sqrt{\bar{z}}\right)^{1/2} = (sr)^{1/2}, \tag{23}$$

which interestingly is the same as the MME $\tilde{\beta}$ given in Equation (13).

PROPERTY 4.1 *The PEs always exist uniquely.*

Proof It is equivalent to showing that $\hat{\alpha}^*$ in Equation (18) is always non-negative. For this purpose, we note that

$$\begin{aligned}
\hat{\alpha} &= [2(\sqrt{\bar{z}} - 1)]^{1/2} \\
&= \left[2\left(\sqrt{\frac{1}{2\binom{n}{2}} \sum_{1 \le i < j \le n} \left(z_{ij} + \frac{1}{z_{ij}}\right)} - 1\right)\right]^{1/2} \\
&\ge \left[2\left(\sqrt{\frac{2\binom{n}{2}}{2\binom{n}{2}}} - 1\right)\right]^{1/2} \\
&= 0,
\end{aligned}$$

as required. ∎

5. Comparison with MMEs

Ng et al. [7] observed that the performance of the MMEs is quite similar to that of the MLEs. While the MLEs are obtained by solving a non-linear equation, the MMEs have simple explicit expressions. But, they noted that both estimators are somewhat biased, and especially so in case of small sample sizes. In this section, we examine some properties of the PE $\hat{\alpha}^*$ in Equation (18) and compare it to the MME $\tilde{\alpha}$ in Equation (12).

PROPERTY 5.1 *Based on a sample $t_1, \ldots, t_n$, we have $\hat{\alpha}^* > \tilde{\alpha}$.*

Proof To prove this result is equivalent to showing that $\bar{z} \ge s/r$, where $\bar{z}$, s and r are as in Equations (17) and (7). By using the Cauchy–Schwarz inequality, we have

$$\bar{z} = \frac{1}{2\binom{n}{2}} \sum_{1 \le i \ne j \le n} t_i \frac{1}{t_j} = \frac{1}{n^2 - n} \sum_{1 \le i < j \le n} \left(t_i \frac{1}{t_j} + t_j \frac{1}{t_i}\right) \ge 1.$$

Now, by utilizing the fact that $x/y > (x + c)/(y + c)$, when $x \ge y \ge 0$ and $c > 0$, we obtain

$$\begin{aligned}
\bar{z} &= \frac{1}{n^2 - n} \sum_{1 \le i < j \le n} \left(t_i \frac{1}{t_j} + t_j \frac{1}{t_i}\right) \\
&\ge \frac{\sum_{1 \le i < j \le n} (t_i(1/t_j) + t_j(1/t_i)) + n}{n^2 - n + n} \\
&= \frac{\sum_{1 \le i \le j \le n} (t_i(1/t_j) + t_j(1/t_i))}{n^2}
\end{aligned}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} t_i \sum_{i=1}^{n} \frac{1}{t_i}$$
$$= \frac{s}{r}.$$

Hence, the result. ■

PROPERTY 5.2 $(((\hat{\alpha}^*)^2 + 2)/2)^2$ *is an unbiased estimator of* $((\alpha^2 + 2)/2)^2$.

Proof Evidently, we have

$$E\left[\left(\frac{(\hat{\alpha}^*)^2 + 2}{2}\right)^2\right] = E(\bar{Z}) = E(T_i)E\left(\frac{1}{T_j}\right) = \left(\frac{\alpha^2 + 2}{2}\right)^2.$$

■

PROPERTY 5.3 $\hat{\alpha}^*$ *is a negatively biased estimator of* α.

Proof Let $g(x) = ((x^2 + 2)/2)^2$. Then, $g(x)$ is clearly a convex function which is monotone increasing for $x \geq 0$. Therefore, by using Jensen's inequality, we immediately have

$$g[E(\hat{\alpha}^*)] \leq E[g(\hat{\alpha}^*)] = \left(\frac{\alpha^2 + 2}{2}\right)^2,$$

from which we obtain that $[E(\hat{\alpha}^*)] \leq \alpha$ by the monotonicity of $g(\cdot)$. ■

PROPERTY 5.4 *The MME* $\tilde{\alpha}$ *in Equation* (12) *is a negatively biased estimator of* α, *with its bias being greater than that of* $\hat{\alpha}^*$ *in Equation* (18), *i.e.* $\text{Bias}(\tilde{\alpha}) < \text{Bias}(\hat{\alpha}^*) < 0$.

Proof This can be readily proved by using Properties 5.1 and 5.3. ■

Property 5.4 immediately reveals that the PE of α has less bias than the MME (and the MLE) of α. This can also be seen in the simulation results presented in Table 1.

6. Interval estimation of parameters

6.1. *Jackknifing method*

From a sample of size n, after dropping the ith observation (t_i) from the sample, find the corresponding $(\hat{\alpha}^*_{(i)}, \hat{\beta}^*_{(i)})$ from the sample of remaining $(n-1)$ observations by using the formulae of the estimators $\hat{\alpha}^*$ and $\hat{\beta}^*$ in Equations (18) and (23), respectively. Then, from the set of n pairs of estimates $(\hat{\alpha}^*_{(1)}, \hat{\beta}^*_{(1)}), \ldots, (\hat{\alpha}^*_{(n)}, \hat{\beta}^*_{(n)})$, we can estimate the variances of the estimators $\hat{\alpha}^*$ and $\hat{\beta}^*$ as

$$\widehat{\text{Var}(\hat{\alpha}^*)} = \frac{n}{n-1} \sum_{i=1}^{n} (\hat{\alpha}^*_{(i)} - \bar{\hat{\alpha}})^2, \tag{24}$$

$$\widehat{\text{Var}(\hat{\beta}^*)} = \frac{n}{n-1} \sum_{i=1}^{n} (\hat{\beta}^*_{(i)} - \bar{\hat{\beta}})^2, \tag{25}$$

where $\bar{\hat{\alpha}} = (1/n) \sum_{i=1}^{n} \hat{\alpha}^*_{(i)}$ and $\bar{\hat{\beta}} = (1/n) \sum_{i=1}^{n} \hat{\beta}^*_{(i)}$.

6.2. *Bootstrap method*

From the given sample of size n, compute the estimates $(\hat{\alpha}^*, \hat{\beta}^*)$ from the formulae in Equations (18) and (23). Then, with $(\hat{\alpha}^*, \hat{\beta}^*)$ as the values of (α, β), we generate samples of size n from the BS distribution, and find the corresponding estimates $(\hat{\alpha}_1^*, \hat{\beta}_1^*), \ldots, (\hat{\alpha}_B^*, \hat{\beta}_B^*)$ from B bootstrap samples once again by using the formulas of $\hat{\alpha}^*$ and $\hat{\beta}^*$ in Equations (18) and (23), respectively. Then, from this set of B estimates, we can estimate the variances of the estimates $\hat{\alpha}^*$ and $\hat{\beta}^*$ as

$$\widehat{\mathrm{Var}(\hat{\alpha}^*)} = \frac{1}{B-1}\sum_{i=1}^{B}(\hat{\alpha}_i^* - \bar{\hat{\alpha}}^*)^2, \tag{26}$$

$$\widehat{\mathrm{Var}(\hat{\beta}^*)} = \frac{1}{B-1}\sum_{i=1}^{B}(\hat{\beta}_i^* - \bar{\hat{\beta}}^*)^2, \tag{27}$$

where $\bar{\hat{\alpha}}^* = (1/B)\sum_{i=1}^{B}\hat{\alpha}_i^*$ and $\bar{\hat{\beta}}^* = (1/B)\sum_{i=1}^{B}\hat{\beta}_i^*$.

Note that the above described Jackknife and bootstrap methods will be applied to each estimation method detailed earlier.

6.3. *Asymptotic method*

Since $\hat{\beta}^* = \tilde{\beta}$, we have the same asymptotic distribution, i.e.

$$\hat{\beta}^* \sim N\left[\beta, \frac{(\alpha\beta)^2}{n}\left(\frac{1+(3/4)\alpha^2}{(1+(1/2)\alpha^2)^2}\right)\right], \tag{28}$$

which can be used to construct an asymptotic confidence interval for β.

Next, for deriving the asymptotic distribution of $\bar{Z}$, we first of all observe

$$E(\bar{Z}) = \frac{1}{n(n-1)}\sum_{1\le i\ne j\le n} E(Z_{ij}) = \left(1+\frac{1}{2}\alpha^2\right)^2$$

and

$$\begin{aligned}
E[(\bar{Z})^2] = \frac{1}{n^2(n-1)^2}E\Bigg[&\sum_{1\le i\ne j\ne k\ne l\le n}\frac{t_i t_j}{t_k t_l} + \left(\sum_{1\le i\ne j\ne k\le n}\frac{t_i^2}{t_j t_k} + \sum_{1\le i\ne j\ne k\le n}\frac{t_j t_k}{t_i^2}\right) \\
&+2\sum_{1\le i\ne j\ne k\le n}\frac{t_i t_k}{t_k t_j} + \sum_{1\le i\ne j\le n}\frac{t_i^2}{t_j^2} + 2\binom{n}{2}\Bigg] \\
= \left(1+\frac{1}{2}\alpha^2\right)^4 &+ \frac{2\alpha^4(1+(1/2)\alpha^2)^2}{n-1} + \frac{(7/16)\alpha^8 - (5/2)\alpha^6 - 6\alpha^4 - 4\alpha^2 - 1}{n(n-1)}.
\end{aligned} \tag{29}$$

These expressions yield

$$\mathrm{Var}(\bar{Z}) = \frac{2\alpha^4(1+(1/2)\alpha^2)^2}{n-1} + \frac{(7/16)\alpha^8 - (5/2)\alpha^6 - 6\alpha^4 - 4\alpha^2 - 1}{n(n-1)}. \tag{30}$$

Thus, taking just the term of order $1/n$ in the expression of $\text{Var}(\bar{Z})$ in Equation (30), and using the central limit theorem, we have

$$\bar{Z} \underset{n\to\infty}{\longrightarrow} N\left(\left(1+\frac{1}{2}\alpha^2\right)^2, \frac{2\alpha^4(1+(1/2)\alpha^2)^2}{n}\right).$$

Now, for obtaining the asymptotic distribution of $\hat{\alpha}^*$, upon using Taylor series expansion, we obtain

$$\hat{\alpha}^* = \sqrt{2(\sqrt{\bar{z}}-1)} = g(\bar{z}) = g(a) + (\bar{z}-a)g'(a) + \frac{(\bar{z}-a)^2}{2}g''(a) + \cdots, \tag{31}$$

where $a = (1+(1/2)\alpha^2)^2$, and $g'(\cdot)$ and $g''(\cdot)$ are the first and second derivatives of the function of $g(\cdot)$. We thus get the asymptotic distribution of $\hat{\alpha}^*$ as $N(\alpha, 2\alpha^2/n)$. We also have

$$\text{Bias}(\hat{\alpha}^*) \approx -\frac{\alpha(2+3\alpha^2)}{4n(2+\alpha^2)}, \tag{32}$$

which can be used to correct the bias of $\hat{\alpha}^*$ if needs to be.

7. Simulation study

We carried out an extensive Monte Carlo simulation study for different choices of n and α by keeping $\beta = 1$, without loss of any generality. For the cases when the sample size n equals 10 and 50, and the values of α are 0.10, 0.25, 0.50, 0.75, 1.00, 1.25, 1.50 and 2.00, we have presented the empirical values of the means and MSEs of the PE in Table 1, along with the corresponding results for the MLEs and the MMEs. These empirical results were determined from 10,000 Monte Carlo simulations. The values of the shape parameter α have been chosen so as to examine the performance of the proposed estimation method under low, moderate and high skewness. The

Table 1. Simulated values of means and MSEs (within brackets) of the PE in comparison with those of MLEs and MMEs.

		PE[a]	MLEs		MMEs	
n	α	$\hat{\alpha}^*$	$\hat{\alpha}$	$\hat{\beta}$	$\tilde{\alpha}$	$\tilde{\beta}$
10	0.10	0.0976 (0.0006)	0.0926 (0.0005)	1.0003 (0.0010)	0.0926 (0.0005)	1.0003 (0.0010)
	0.25	0.2438 (0.0034)	0.2315 (0.0034)	1.0025 (0.0062)	0.2315 (0.0034)	1.0025 (0.0062)
	0.50	0.4855 (0.0136)	0.4620 (0.0137)	1.0107 (0.0242)	0.4620 (0.0137)	1.0107 (0.0242)
	0.75	0.7242 (0.0303)	0.6911 (0.0309)	1.0232 (0.0528)	0.6911 (0.0309)	1.0231 (0.0528)
	1.00	0.9600 (0.0538)	0.9188 (0.0555)	1.0384 (0.0899)	0.9186 (0.0554)	1.0384 (0.0900)
	1.25	1.1934 (0.0844)	1.1455 (0.0877)	1.0548 (0.1332)	1.1449 (0.0875)	1.0549 (0.1337)
	1.50	1.4251 (0.1224)	1.3715 (0.1280)	1.0712 (0.1805)	1.3702 (0.1274)	1.0716 (0.1820)
	2.00	1.8850 (0.2220)	1.8228 (0.2335)	1.1013 (0.2800)	1.8186 (0.2317)	1.1032 (0.2864)
50	0.10	0.0995 (0.0001)	0.0985 (0.0001)	1.0001 (0.0002)	0.0985 (0.0001)	1.0001 (0.0002)
	0.25	0.2486 (0.0006)	0.2462 (0.0006)	1.0005 (0.0012)	0.2462 (0.0006)	1.0005 (0.0012)
	0.50	0.4969 (0.0025)	0.4922 (0.0025)	1.0021 (0.0048)	0.4922 (0.0025)	1.0021 (0.0048)
	0.75	0.7445 (0.0057)	0.7378 (0.0057)	1.0044 (0.0100)	0.7378 (0.0057)	1.0045 (0.0100)
	1.00	0.9915 (0.0102)	0.9833 (0.0102)	1.0071 (0.0161)	0.9832 (0.0102)	1.0072 (0.0161)
	1.25	1.2381 (0.0158)	1.2285 (0.0160)	1.0098 (0.0223)	1.2284 (0.0160)	1.0101 (0.0225)
	1.50	1.4844 (0.0228)	1.4737 (0.0231)	1.0123 (0.0281)	1.4734 (0.0231)	1.0127 (0.0285)
	2.00	1.9764 (0.0408)	1.9641 (0.0413)	1.0161 (0.0374)	1.9631 (0.0412)	1.0172 (0.0389)

[a]The PE of β, $\hat{\beta}^*$ is identical to the MME of β, $\tilde{\beta}$.

required BS samples were generated from standard normal samples and then using the inverse relationship between the BS and normal variates in Equation (1). As observed earlier by Ng et al. [7], we also observe from our empirical results presented in Table 1 that the MLEs and MMEs are quite close both in terms of bias and MSE, and that the corresponding estimates of α are quite biased for small sample sizes especially for large values of α. However, a comparison of these results with those corresponding to the proposed method in Table 1 reveals that the proposed method of estimation of α possesses lower bias and similar MSE, especially in the case of small sample size. Moreover, we observe that the PEs of α and β have very nearly the same MSEs as the MLEs and the MMEs.

8. An illustrative example

In this section, we illustrate the results established in the preceding sections with one real data taken from the reliability literature.

Example 8.1 The data presented in Table 2, due to Birnbaum and Saunders [2], give the fatigue lifetimes of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles/s. The complete sample contains 101 observations with maximum stress per cycle as 31,000 psi.

Ng et al. [7] analysed these data by using both MLEs and MMEs. Here, we estimate the parameters by the proposed method, and then use the Jackknife, bootstrap and asymptotic methods to construct 95% CIs for α and β. All these results, presented in Table 3, are also compared with the corresponding results based on the MLEs and the MMEs. We observe in this case that all

Table 2. Data on the fatigue lifetimes of aluminum coupons, taken from Birnbaum and Saunders [2].

70	90	96	97	99	100	103	104	104	105	107	108	108	108
109	109	112	112	113	114	114	114	116	119	120	120	120	121
121	123	124	124	124	124	124	128	128	129	129	130	130	130
131	131	131	131	131	132	132	132	133	134	134	134	134	134
136	136	137	138	138	138	139	139	141	141	142	142	142	142
142	142	144	144	145	146	148	148	149	151	151	152	155	156
157	157	157	157	158	159	162	163	163	164	166	166	168	170
174	196	212											

Table 3. Estimates of the parameters (based on the PEs, MLEs and MMEs) and the corresponding 95% CIs based on the data in Table 2.

	$\hat{\alpha}$	$\hat{\beta}$
PEs	0.1712	131.8193
Jackknife 95% CI	(0.1396, 0.2028)	(127.4088, 136.2297)
Bootstrap 95% CI	(0.1473, 0.1949)	(127.4794, 136.2791)
Asymptotic 95% CI	(0.1476, 0.1948)	(127.4330, 136.2046)
MLEs	0.1704	131.8188
Jackknife 95% CI	(0.1390, 0.2018)	(127.4080, 136.2296)
Bootstrap 95% CI	(0.1480, 0.1923)	(127.5290, 135.9603)
Asymptotic 95% CI	(0.1469, 0.1939)	(127.4544, 136.1832)
MMEs	0.1704	131.8193
Jackknife 95% CI	(0.1390, 0.2018)	(127.4088, 136.2297)
Bootstrap 95% CI	(0.1480, 0.1923)	(127.4794, 136.2791)
Asymptotic 95% CI	(0.1469, 0.1939)	(127.4330, 136.2046)

Table 4. Kolmogorov–Smirnov distances and the corresponding P-values based on the PEs, MLEs and MMEs.

	PEs	MLEs	MMEs
KS-distance	0.0853	0.0850	0.0850
P-value	0.4544	0.4594	0.4595

the obtained estimates and the confidence intervals are quite similar. Finally, the Kolmogorov–Smirnov test is carried out and these results, presented in Table 4, support the BS model assumption made in our analysis.

9. Concluding remarks

In this paper, we have developed a simple and efficient method of estimating the shape and scale parameters of the BS distribution based on complete samples. We have shown that the PE of α has smaller bias compared with the MME and MLE. It would be of interest to extend the proposed method of estimation to the case when the available data are censored. Also, the extension of the method to the bivariate BS distribution [12] and the generalized BS distribution [13,14] will be of great interest. Work on these problems is currently under progress and we hope to report these findings in a future paper.

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