User Prompt

Let $p(x)$ and $q(x)$ be two monic polynomials of degree n:

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k} \quad \text{and} \quad q(x) = \sum_{k=0}^{n} b_k x^{n-k}$$

where $a_0 = b_0 = 1$. Define $p \boxplus_n q(x)$ to be the polynomial

$$(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k}$$

where the coefficients c_k are given by the formula:

$$c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j$$

for $k = 0, 1, \ldots, n$. For a monic polynomial $p(x) = \prod_{i \leq n}(x - \lambda_i)$, define

$$\Phi_n(p) := \sum_{i \leq n} (\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j})^2$$

and $\Phi_n(p) := \infty$ if p has a multiple root. Is it true that if $p(x)$ and $q(x)$ are monic real-rooted polynomials of degree n, then

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}?$$

System Short Summary:

Let $\boxplus_n$ be the (symmetric additive) finite free convolution on monic degree-n polynomials, equivalently the operation appearing in Marcus–Spielman–Srivastava [**?**, Def. 1.1] and preserving real-rootedness [**?**, Thm. 1.3]. For a simple-rooted $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$, the key analytic identity is the log-derivative evaluation

$$\frac{p''(\lambda_i)}{p'(\lambda_i)} = 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j},$$

which rewrites $\Phi_n(p) = \frac{1}{4} \sum_i \left(\frac{p''(\lambda_i)}{p'(\lambda_i)}\right)^2$ and, by a cancellation over triples, yields the more structural formula

$$\Phi_n(p) = 2 \sum_{1 \leq i < j \leq n} \frac{1}{(\lambda_i - \lambda_j)^2},$$

i.e. Φ_n is twice the inverse-square gap energy of the root configuration. The real-rootedness hypothesis ensures all gaps are real and $\Phi_n(p) \in (0, \infty]$ is well-defined. For $n = 2$, $\Phi_2(p) = 2/\Delta(p)$ where Δ is the discriminant, and a direct coefficient computation from [**?**, Def. 1.1] shows $\Delta(p \boxplus_2 q) = \Delta(p) + \Delta(q)$, hence $\frac{1}{\Phi_2(p \boxplus_2 q)} = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}$ (equality). For $n = 3$, one can express Φ_3 exactly as $\Phi_3(x^3 + ux^2 + vx + w) = 2(u^2 - 3v)^2/\Delta$ (a symmetric-polynomial numerator over the discriminant), and an explicit example $p(x) = x^3 - 3x + 1$, $q(x) = x^3 - 3x$ yields strict superadditivity. A separate computation using the Hermite ODE and the stability of Hermites under $\boxplus_n$ (cf. [**?**, §1.1]) gives equality for the Hermite family. Beyond these cases, the note does *not* supply (nor cite) a general theorem implying $\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$ for all n; cf. Gribinski [**?**, Conj. 1] for related finite-degree entropy-power conjectures.