User Prompt

Let $p(x)$ and $q(x)$ be two monic polynomials of degree n:

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k} \quad \text{and} \quad q(x) = \sum_{k=0}^{n} b_k x^{n-k}$$

where $a_0 = b_0 = 1$. Define $p \boxplus_n q(x)$ to be the polynomial

$$(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k}$$

where the coefficients c_k are given by the formula:

$$c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j$$

for $k = 0, 1, \ldots, n$. For a monic polynomial $p(x) = \prod_{i \le n}(x - \lambda_i)$, define

$$\Phi_n(p) := \sum_{i \le n} (\sum_{j \ne i} \frac{1}{\lambda_i - \lambda_j})^2$$

and $\Phi_n(p) := \infty$ if p has a multiple root. Is it true that if $p(x)$ and $q(x)$ are monic real-rooted polynomials of degree n, then

$$\frac{1}{\Phi_n(p \boxplus_n q)} \ge \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}?$$

System Result:

Theorem 1 (Verified special cases; status of the general inequality)**.** *Fix $n \ge 1$. For monic polynomials*

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k}, \qquad q(x) = \sum_{k=0}^{n} b_k x^{n-k},$$

define $p \boxplus_n q(x) = \sum_{k=0}^{n} c_k x^{n-k}$ by

$$c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j \qquad (0 \le k \le n).$$

For $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$, set

$$\Phi_n(p) := \sum_{i=1}^{n} \Big(\sum_{j \ne i} \frac{1}{\lambda_i - \lambda_j} \Big)^2, \qquad \text{and } \ \Phi_n(p) := \infty \text{ if } p \text{ has a multiple root.}$$

Consider the inequality

$$\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)} \tag{1}$$

for monic real-rooted degree-n polynomials p, q (interpreting $1/\infty = 0$).

(i) ($n = 2$) The inequality (1) holds with equality for all monic real-rooted quadratics.

(ii) (A strict example at $n = 3$) There exist monic real-rooted cubics p, q for which (1) holds strictly.

(iii) (Hermite family) For the probabilists' Hermite polynomials H_n and their natural scalings, (1) holds with equality for every n.

(iv) (General n) The full inequality (1) is not established by the arguments in the candidate solution; in particular, the appeal to a general "finite free Stam inequality" for this specific functional Φ_n is not substantiated by the cited literature. The status of (1) for general n remains open based on the material presented here.

Proof. **A. Identification with symmetric additive convolution.** The operation $\boxplus_n$ is the same (up to the standard sign convention for coefficients) as the *symmetric additive convolution* $+_n$ introduced by Marcus–Spielman–Srivastava; see [1, Definition 1.1] for the defining coefficient formula (written there with the alternating-sign coefficient convention). In particular, if p and q are real-rooted, then so is $p \boxplus_n q$ by [1, Theorem 1.3]. (This real-rootedness preservation is conceptually relevant but not needed for the explicit computations in parts (i)–(iii) below.)

B. A convenient identity for Φ_n. Assume $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$ has simple roots. Then for each i,

$$\frac{p''(\lambda_i)}{p'(\lambda_i)} \;=\; 2\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}. \tag{2}$$

Indeed, writing $p'(x) = \sum_i \prod_{j \neq i}(x - \lambda_j)$ and evaluating at $x = \lambda_i$ gives $p'(\lambda_i) = \prod_{j \neq i}(\lambda_i - \lambda_j)$, while differentiating again and evaluating at $x = \lambda_i$ yields $p''(\lambda_i) = 2\sum_{j \neq i}\prod_{k \neq i,j}(\lambda_i - \lambda_k) = 2p'(\lambda_i)\sum_{j \neq i}\frac{1}{\lambda_i - \lambda_j}$. Thus

$$\Phi_n(p) = \frac{1}{4}\sum_{i=1}^{n}\Big(\frac{p''(\lambda_i)}{p'(\lambda_i)}\Big)^2. \tag{3}$$

If p has a multiple root then at least one denominator in (2) vanishes and $\Phi_n(p) = \infty$ by definition.

C. Proof of (i): the case $n = 2$ (equality). Let $p(x) = (x - \lambda_1)(x - \lambda_2) = x^2 + a_1 x + a_2$. Then

$$\Phi_2(p) = \Big(\frac{1}{\lambda_1 - \lambda_2}\Big)^2 + \Big(\frac{1}{\lambda_2 - \lambda_1}\Big)^2 = \frac{2}{(\lambda_1 - \lambda_2)^2}.$$

Writing $\Delta(p) := (\lambda_1 - \lambda_2)^2 = a_1^2 - 4a_2$ (the discriminant), we have $\Phi_2(p) = 2/\Delta(p)$ and

$$\frac{1}{\Phi_2(p)} = \frac{\Delta(p)}{2}. \tag{4}$$

Now let $q(x) = x^2 + b_1 x + b_2$ and set $r := p \boxplus_2 q = x^2 + c_1 x + c_2$. By the defining formula,

$$c_1 = a_1 + b_1, \qquad c_2 = a_2 + \frac{1}{2} a_1 b_1 + b_2.$$

Hence

$$\Delta(r) = c_1^2 - 4c_2 = (a_1 + b_1)^2 - 4\Big(a_2 + \frac{1}{2} a_1 b_1 + b_2\Big) = (a_1^2 - 4a_2) + (b_1^2 - 4b_2) = \Delta(p) + \Delta(q).$$

Combining with (4) gives

$$\frac{1}{\Phi_2(p \boxplus_2 q)} = \frac{\Delta(r)}{2} = \frac{\Delta(p)}{2} + \frac{\Delta(q)}{2} = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}.$$

Thus (1) holds with equality for $n = 2$.

D. A closed form for Φ_3 and a strict example (ii). Let $p(x) = \prod_{i=1}^{3}(x - \lambda_i) = x^3 + Ax^2 + Bx + C$ have three distinct real roots and discriminant

$$\Delta(p) := \prod_{1 \le i < j \le 3} (\lambda_i - \lambda_j)^2.$$

Set $s_1 := \lambda_1 + \lambda_2 + \lambda_3 = -A$ and $s_2 := \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = B$. A direct algebraic simplification starting from

$$\Phi_3(p) = \sum_{i=1}^{3} \Big(\frac{1}{\lambda_i - \lambda_j} + \frac{1}{\lambda_i - \lambda_k}\Big)^2 \quad (\{i, j, k\} = \{1, 2, 3\})$$

shows

$$\Phi_3(p) = \frac{2(s_1^2 - 3s_2)^2}{\Delta(p)} = \frac{2(A^2 - 3B)^2}{\Delta(p)}. \tag{5}$$

(One convenient route is to rewrite each term over the common denominator $\Delta(p)$ and then express the symmetric numerator in the basis $\{s_1^4, s_1^2 s_2, s_2^2, s_1 s_3\}$, noting that the $s_1 s_3$ coefficient cancels.)
Now take

$$p(x) = x^3 - 3x + 1 \quad (A = 0,\ B = -3), \qquad q(x) = x^3 - 3x \quad (A = 0,\ B = -3).$$

Their discriminants are those of depressed cubics:

$$\Delta(x^3 + Bx + C) = -4B^3 - 27C^2,$$

so $\Delta(p) = -4(-3)^3 - 27 \cdot 1^2 = 81$ and $\Delta(q) = -4(-3)^3 - 27 \cdot 0^2 = 108$. By (5),

$$\Phi_3(p) = \frac{2(0 - 3(-3))^2}{81} = 2, \qquad \Phi_3(q) = \frac{2(0 - 3(-3))^2}{108} = \frac{3}{2}.$$

Next compute $r := p \boxplus_3 q$. Using the defining coefficient formula with $n = 3$ and noting $a_1 = b_1 = 0$,

$$r(x) = x^3 + 0 \cdot x^2 + (a_2 + b_2)\, x + (a_3 + b_3) = x^3 - 6x + 1,$$

so $A = 0$, $B = -6$, $C = 1$, and $\Delta(r) = -4(-6)^3 - 27 \cdot 1^2 = 837$. Again by (5),

$$\Phi_3(r) = \frac{2(0 - 3(-6))^2}{837} = \frac{648}{837} = \frac{24}{31}.$$

Therefore

$$\frac{1}{\Phi_3(p \boxplus_3 q)} = \frac{31}{24} \quad \text{while} \quad \frac{1}{\Phi_3(p)} + \frac{1}{\Phi_3(q)} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6} = \frac{28}{24},$$

so (1) holds strictly in this example.

E. Equality for the Hermite family (iii). Let H_n denote the probabilists' Hermite polynomial, characterized by the differential equation

$$H_n''(x) - xH_n'(x) + nH_n(x) = 0, \tag{6}$$

and normalized to be monic of degree n (equivalently, $H_n(x) = e^{-D^2/2}x^n$ as in [1, §1.1]). If λ is a root of H_n, then $H_n(\lambda) = 0$ and (6) gives

$$\frac{H_n''(\lambda)}{H_n'(\lambda)} = \lambda.$$

Combining with (2) yields the "electrostatic" identity

$$\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \frac{\lambda_i}{2} \qquad (\lambda_i \text{ a root of } H_n),$$

and hence

$$\Phi_n(H_n) = \sum_{i=1}^{n} \left(\frac{\lambda_i}{2}\right)^2 = \frac{1}{4} \sum_{i=1}^{n} \lambda_i^2. \tag{7}$$

For monic H_n, the coefficient of x^{n-2} is $-\binom{n}{2}$, so $\sum_i \lambda_i^2 = n(n-1)$ (Newton sums), and

$$\Phi_n(H_n) = \frac{n(n-1)}{4}. \tag{8}$$

For $a > 0$, define the scaling $H_{n,a}(x) := a^n H_n(x/a)$ (still monic). Its roots are $a\lambda_i$, and from the scaling in the score $\sum_{j \neq i} 1/(a\lambda_i - a\lambda_j) = (1/a) \sum_{j \neq i} 1/(\lambda_i - \lambda_j)$ we get

$$\Phi_n(H_{n,a}) = \frac{n(n-1)}{4a^2} \quad \Longrightarrow \quad \frac{1}{\Phi_n(H_{n,a})} = \frac{4a^2}{n(n-1)}.$$

Marcus–Spielman–Srivastava note (via [1, Theorem 1.2] or directly from the coefficient formula [1, Definition 1.1]) that

$$H_{n,a} \boxplus_n H_{n,b} = H_{n,c} \qquad \text{where } c = \sqrt{a^2 + b^2} \tag{9}$$

(see the displayed identity in [1, §1.1] for Hermite polynomials). Consequently,

$$\frac{1}{\Phi_n(H_{n,a} \boxplus_n H_{n,b})} = \frac{1}{\Phi_n(H_{n,c})} = \frac{4(a^2+b^2)}{n(n-1)} = \frac{1}{\Phi_n(H_{n,a})} + \frac{1}{\Phi_n(H_{n,b})},$$

so (1) holds with equality on this family.

F. On the general inequality (iv). The candidate solution asserts that (1) is a known general theorem (a "finite free Stam inequality") for the functional Φ_n above. However, the standard (infinite-dimensional) free Stam inequality is a statement about Voiculescu's free Fisher information Φ^* under *free additive convolution* (for freely independent noncommutative random variables); see [3, §3, Φ^*.4]. In contrast, a general Stam-type inequality for the *finite-degree* functional Φ_n defined here and the finite convolution $\boxplus_n$ is not derived by the computations in parts (i)–(iii), and it does not follow from the cited finite-free convolution theory alone. Related finite-degree entropy-power conjectures for roots/discriminants are discussed by Gribinski [2, Conjecture 1], who explicitly distinguishes such conjectures from a (future) development of a Fisher-information analogue. Thus, based on the presented material, the general validity of (1) for all n remains open. □

References

[1] A. Marcus, D. Spielman, and N. Srivastava, *Finite free convolutions of polynomials*, arXiv:1504.00350v2.

[2] E. Gribinski, *A notion of entropy on the roots of polynomials*, arXiv:1907.12826v1.

[3] D. Voiculescu, *Free entropy*, arXiv:math/0103168v1.