User Prompt

For a graph $G = (V, E)$, let $G_S = (V, E(S, S))$ denote the graph with the same vertex set, but only the edges between vertices in S. Let L be the Laplacian matrix of G and let L_S be the Laplacian of G_S. I say that a set of vertices S is ϵ-light if the matrix $\epsilon L - L_S$ is positive semidefinite. Does there exist a constant $c > 0$ so that for every graph G and every ϵ between 0 and 1, V contains an ϵ-light subset S of size at least $c\epsilon|V|$?

System Short Summary:

Fix a connected graph $G = (V, E)$ with Laplacian L and $\varepsilon \in (0, 1)$. The candidate approach starts by classifying edges by effective resistance $R_{\text{eff}}(e) = b_e^\top L^\dagger b_e$. By Foster's theorem $\sum_{e \in E} R_{\text{eff}}(e) = |V| - 1$, hence the set $E_{\text{heavy}} := \{e : R_{\text{eff}}(e) > \varepsilon\}$ has size $|E_{\text{heavy}}| < |V|/\varepsilon$. Viewing $H = (V, E_{\text{heavy}})$, a standard degree-pruning plus random-sampling/greedy-filtering argument yields an independent set S in H with $\mathbb{E}|S| = \Omega(\varepsilon|V|)$, and therefore (for some realization) $|S| \geq \varepsilon|V|/(8e)$ while $G[S]$ contains no heavy edges (i.e. every edge in $E(S)$ has $R_{\text{eff}}(e) \leq \varepsilon$). This is the only place where Foster's theorem and the hypothesis $\varepsilon \in (0, 1)$ are used: they bound the heavy-edge density so that a large independent set exists.

The intended spectral conclusion is $L_S \preceq \varepsilon L$. A useful linearization is $L_S \preceq \frac{1}{2} \sum_{u \in S} L_u$ where $L_u = \sum_{v \sim u} b_{uv} b_{uv}^\top$ is the star Laplacian at u; thus it would suffice to find S with $\sum_{u \in S} L_u \preceq 2\varepsilon L$. However, "no heavy edges" does not imply ε-lightness: from the variational characterization of effective resistance one only gets the per-edge domination $b_e b_e^\top \preceq R_{\text{eff}}(e)\, L$, so summing over $e \in E(S)$ yields $L_S \preceq (\sum_{e \in E(S)} R_{\text{eff}}(e))\, L$, which requires control of the *total* resistance mass of $E(S)$, not just $\max_{e \in E(S)} R_{\text{eff}}(e)$. In dense graphs (e.g. K_n) all edges have $R_{\text{eff}}(e) = 2/n \ll \varepsilon$ while $L_S \preceq \varepsilon L$ forces $|S| \leq \varepsilon n$, so the candidate proof's final step needs an additional, genuinely nontrivial matrix-selection/rounding lemma beyond Foster/independent-set considerations.