User Prompt

For a graph $G = (V, E)$, let $G_S = (V, E(S, S))$ denote the graph with the same vertex set, but only the edges between vertices in S. Let L be the Laplacian matrix of G and let L_S be the Laplacian of G_S. I say that a set of vertices S is ϵ-light if the matrix $\epsilon L - L_S$ is positive semidefinite. Does there exist a constant $c > 0$ so that for every graph G and every ϵ between 0 and 1, V contains an ϵ-light subset S of size at least $c\epsilon|V|$?

System Result

Theorem 1 (What the "heavy-edge" approach proves, and where it stops)**.** *Let $G = (V, E)$ be a finite, connected, simple graph on $n := |V|$ vertices, with (combinatorial) Laplacian*

$$L \;=\; \sum_{\{u,v\}\in E} b_{\{u,v\}} b_{\{u,v\}}^{\top}, \qquad b_{\{u,v\}} := \mathbf{e}_u - \mathbf{e}_v.$$

For $S \subseteq V$, let L_S be the Laplacian of the induced subgraph $G[S]$, embedded as an $n \times n$ matrix. Fix $\varepsilon \in (0, 1)$.

(A) (Combinatorial reduction.) *There exists $S \subseteq V$ with $|S| \geq \frac{\varepsilon n}{8\mathrm{e}}$ such that $G[S]$ contains no edge e whose effective resistance $R_{\text{eff}}(e)$ (in G) exceeds ε.*

(B) (The missing spectral step.) *The conclusion in (A)* does not *imply that S is ε-light (i.e. that $L_S \preceq \varepsilon L$). In particular, even if every edge of $G[S]$ satisfies $R_{\text{eff}}(e) \leq \varepsilon$, it may still happen that $L_S \not\preceq \varepsilon L$.*

(C) (An exact spectral reformulation.) *For each $u \in V$ define the star Laplacian*

$$L_u \;:=\; \sum_{v\colon\ \{u,v\}\in E} b_{\{u,v\}} b_{\{u,v\}}^{\top}.$$

Then for every $S \subseteq V$ one has

$$L_S \;\preceq\; \frac{1}{2} \sum_{u\in S} L_u.$$

Consequently, to prove that S is ε-light it suffices to show

$$\sum_{u\in S} L_u \;\preceq\; 2\varepsilon\, L.$$

The candidate solution implicitly requires a "log-free" matrix-selection/rounding statement of this type; no such statement is proved there.

Proof. **Effective resistance preliminaries.** Let $L^{\dagger}$ denote the Moore–Penrose pseudoinverse. For an edge $e = \{u, v\}$, define

$$R_{\text{eff}}(e) \;:=\; b_e^{\top} L^{\dagger} b_e.$$

Foster's theorem states that

$$\sum_{e\in E} R_{\mathrm{eff}}(e) \;=\; n-1,$$

see Theorem 5.1 of Ge [1, Theorem 5.1] (an arXiv preprint which quotes Foster's original result and provides a proof).

Proof of (A). Define the set of "heavy" edges

$$E_{\mathrm{heavy}} \;:=\; \{e\in E:\; R_{\mathrm{eff}}(e) > \varepsilon\}.$$

By Foster's theorem,

$$|E_{\mathrm{heavy}}|\,\varepsilon \;<\; \sum_{e\in E_{\mathrm{heavy}}} R_{\mathrm{eff}}(e) \;\le\; \sum_{e\in E} R_{\mathrm{eff}}(e) \;=\; n-1,$$

so $|E_{\mathrm{heavy}}| < n/\varepsilon$.
Let $H = (V, E_{\mathrm{heavy}})$ and write $d_H(v)$ for the degree of v in H. Then

$$\sum_{v\in V} d_H(v) \;=\; 2|E_{\mathrm{heavy}}| \;<\; \frac{2n}{\varepsilon}.$$

Let $V_{\mathrm{high}} := \{v\in V:\; d_H(v) > \frac{4}{\varepsilon}\}$ and $V_{\mathrm{low}} := V\setminus V_{\mathrm{high}}$. By Markov's inequality,

$$|V_{\mathrm{high}}|\cdot\frac{4}{\varepsilon} \;<\; \frac{2n}{\varepsilon} \quad\Rightarrow\quad |V_{\mathrm{high}}| < \frac{n}{2} \quad\Rightarrow\quad |V_{\mathrm{low}}| \ge \frac{n}{2}.$$

Let $H_{\mathrm{low}} := H[V_{\mathrm{low}}]$; then $\Delta(H_{\mathrm{low}}) \le \lfloor 4/\varepsilon\rfloor$.
Now sample $S_{\mathrm{raw}} \subseteq V_{\mathrm{low}}$ by including each vertex independently with probability $p := \varepsilon/4$. To obtain an independent set in H_{low}, fix an arbitrary total order on V_{low} and keep a vertex $v\in S_{\mathrm{raw}}$ iff no larger neighbor of v (in H_{low}) lies in S_{raw}; call the resulting set S. Then S is an independent set in H_{low} (hence in H), so $G[S]$ contains no edge from E_{heavy}.
Moreover, for each $v\in V_{\mathrm{low}}$,

$$\mathbb{P}(v\in S) \;\ge\; p\,(1-p)^{d_{H_{\mathrm{low}}}(v)} \;\ge\; p\,(1-p)^{4/\varepsilon},$$

and with $p = \varepsilon/4$ one has $(1-p)^{4/\varepsilon} \ge \mathrm{e}^{-1}$ (since $(1-x)^{1/x} \ge \mathrm{e}^{-1}$ for $x\in(0,1)$). Therefore,

$$\mathbb{E}|S| \;=\; \sum_{v\in V_{\mathrm{low}}} \mathbb{P}(v\in S) \;\ge\; |V_{\mathrm{low}}|\cdot\frac{\varepsilon}{4}\cdot\mathrm{e}^{-1} \;\ge\; \frac{n}{2}\cdot\frac{\varepsilon}{4}\cdot\mathrm{e}^{-1} \;=\; \frac{\varepsilon n}{8\mathrm{e}}.$$

In particular, there exists a realization with $|S| \ge \varepsilon n/(8\mathrm{e})$, and by construction $E(S)\cap E_{\mathrm{heavy}} = \varnothing$. This completes (A).

Proof of (B). We first record a standard inequality relating effective resistance to quadratic forms. For any edge $e = \{u,v\}$ and any $x\in\mathbb{R}^V$ with $x_u \ne x_v$, set

$y := x/(x_u - x_v)$ so that $y_u - y_v = 1$. By the variational characterization of effective resistance,

$$R_{\mathrm{eff}}(e) \;=\; \min\{z^\top Lz : \; z_u - z_v = 1\} \;\le\; y^\top Ly \;=\; \frac{x^\top Lx}{(x_u - x_v)^2}.$$

Equivalently,

$$(x_u - x_v)^2 \;\le\; R_{\mathrm{eff}}(e)\, x^\top Lx \qquad \text{for all } x \in \mathbb{R}^V, \tag{1}$$

which is the matrix inequality $b_e b_e^\top \preceq R_{\mathrm{eff}}(e)\, L$.

Summing (1) over $e \in E(S)$ gives the sufficient (but generally weak) bound

$$x^\top L_S x \;=\; \sum_{e\in E(S)} (x_u - x_v)^2 \;\le\; \Big(\sum_{e\in E(S)} R_{\mathrm{eff}}(e)\Big)\, x^\top Lx \quad \Rightarrow \quad L_S \;\preceq\; \Big(\sum_{e\in E(S)} R_{\mathrm{eff}}(e)\Big)\, L.$$

Thus, controlling *each* $R_{\mathrm{eff}}(e)$ by ε does not force $L_S \preceq \varepsilon L$ unless one also controls the *total* sum of resistances over edges in $E(S)$, which can be large.

A concrete example is the complete graph $G = K_n$. For K_n, every edge has effective resistance $R_{\mathrm{eff}}(e) = 2/n$ (this is a standard computation; it also follows from symmetry and Foster's theorem), so for any fixed $\varepsilon > 0$ and all sufficiently large n one has $R_{\mathrm{eff}}(e) \le \varepsilon$ for *every* edge. Nevertheless, if $S \subseteq V$ has size $|S| = s$, then (on $\mathbf{1}^\perp$) the Laplacian eigenvalues satisfy $\lambda_{\max}(L) = n$ and $\lambda_{\max}(L_S) = s$ (since $L(K_m)$ has eigenvalue m with multiplicity $m-1$), so $L_S \preceq \varepsilon L$ would force $s \le \varepsilon n$. Taking, say, $s = \lfloor n/2 \rfloor$ and $\varepsilon = 0.1$ shows that "all edges light" does not imply ε-lightness of S. This proves (B).

Proof of (C). Expand $\sum_{u\in S} L_u$ as a sum over edges. Each induced edge $e = \{u, v\} \in E(S)$ appears *twice* in $\sum_{u\in S} L_u$ (once in L_u and once in L_v), while each edge crossing S and $V \setminus S$ appears *once*, and all other edges do not appear. Therefore,

$$\frac{1}{2}\sum_{u\in S} L_u \;=\; \sum_{e\in E(S)} b_e b_e^\top \;+\; \frac{1}{2}\sum_{e\in\partial S} b_e b_e^\top \;\succeq\; \sum_{e\in E(S)} b_e b_e^\top \;=\; L_S,$$

where ∂S denotes edges with exactly one endpoint in S. Hence $L_S \preceq \frac{1}{2}\sum_{u\in S} L_u$, and the stated sufficient condition follows immediately. This completes (C). □

Remark 1 (Why interlacing-polynomial citations do *not* complete the candidate proof)**.** The candidate solution asserts that logarithmic losses from matrix concentration can be removed "by interlacing polynomials". One precise interlacing-polynomial existence result is Theorem 1.4 of Marcus–Spielman–Srivastava [3, Theorem 1.4] (arXiv:1306.3969), which guarantees a norm bound for a sum $\sum_i v_i v_i^*$ of independent random rank-one matrices under the hypotheses $\sum_i \mathbb{E}[v_i v_i^*] = I$ and $\mathbb{E}\|v_i\|^2 \le \epsilon$. However, applying [3, Theorem 1.4] would require a representation of the *vertex-induced* Laplacian L_S (or of $\sum_{u\in S} L_u$) as a sum of independent rank-one matrices whose individual contributions are uniformly small in the relevant normalization. The candidate proof does not provide such a representation, nor does it establish a matrix-selection/rounding principle of the form in Theorem 1(C). Thus the interlacing-polynomial citation, as used there, does not presently fill the gap.

References

[1] J. Ge, *Effective Resistances of Nearly Complete Bipartite Graphs*, arXiv:1904.07766. (Foster's theorem stated as Theorem 5.1.)

[2] J. A. Tropp, *User-Friendly Tail Bounds for Sums of Random Matrices*, arXiv:1004.4389. (Matrix Chernoff bounds stated as Theorem 1.1.)

[3] A. W. Marcus, D. A. Spielman, and N. Srivastava, *Interlacing Families II: Mixed Characteristic Polynomials and the Kadison–Singer Problem*, arXiv:1306.3969. (Interlacing existence bound stated as Theorem 1.4.)