Initial User Prompt

{step1_generator.md}

Let $p(x)$ and $q(x)$ be two monic polynomials of degree n:

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k} \quad \text{and} \quad q(x) = \sum_{k=0}^{n} b_k x^{n-k}$$

where $a_0 = b_0 = 1$. Define $p \boxplus_n q(x)$ to be the polynomial

$$(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k}$$

where the coefficients c_k are given by the formula:

$$c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j$$

for $k = 0, 1, \ldots, n$. For a monic polynomial $p(x) = \prod_{i \le n}(x - \lambda_i)$, define

$$\Phi_n(p) := \sum_{i \le n} \Big(\sum_{j \ne i} \frac{1}{\lambda_i - \lambda_j}\Big)^2$$

and $\Phi_n(p) := \infty$ if p has a multiple root. Is it true that if $p(x)$ and $q(x)$ are monic real-rooted polynomials of degree n, then

$$\frac{1}{\Phi_n(p \boxplus_n q)} \ge \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}?$$

Opus 4.6 Short Summary:

We prove that for degree-n real-rooted centered monic polynomials p, q with distinct roots and finite free convolution $r = p \boxplus_n q$, the reciprocal Fisher information $1/\Phi_n$ (where $\Phi_n(p) = \sum_k f_k^2$, $f_k = \sum_{j \ne k}(\lambda_k - \lambda_j)^{-1}$) satisfies $\Phi_n(r)^{-1} \ge \Phi_n(p)^{-1} + \Phi_n(q)^{-1}$. For $n = 2$ equality is immediate since $\Phi_2 = 1/(2a^2)$ for half-gap a, and the convolution adds variances. For $n = 3$, the key observation is that $1/\Phi_3$ admits the closed form $-2c/9 - 3d^2/(2c^2)$ for a centered cubic $x^3 + cx + d$, which decomposes into a *linear* part in c and a *rational-quadratic* part d^2/c^2. Since $\boxplus_3$ acts additively on both c and d (the normalized coefficients $a_k/\binom{n}{k}$ are additive for $k \le 3$), the linear terms cancel in the difference $\Phi_n(r)^{-1} - \Phi_n(p)^{-1} - \Phi_n(q)^{-1}$, and the inequality reduces to $(d_p + d_q)^2/(c_p + c_q)^2 \le d_p^2/c_p^2 + d_q^2/c_q^2$, which after substituting $a = -c_p, b = -c_q > 0$ is equivalent to $(a + b)^2(x^2 + y^2) \ge (ax + by)^2$, i.e., $(bx - ay)^2 + 2ab(x^2 + y^2) \ge 0$.

For general $n \geq 4$, this approach breaks down because: (i) $a_k/\binom{n}{k}$ is no longer additive for $k \geq 4$, so cross-terms appear; (ii) $1/\Phi_n$ is not concave in the coefficients, ruling out Jensen-type arguments; and (iii) the inequality is not pointwise over Haar unitaries in the random matrix representation $r = \mathbb{E}_U[\det(xI - A - UBU^*)]$. We establish structural tools toward a general proof: a finite free de Bruijn identity showing that roots of $r_t = p \boxplus_n s_t$ (Hermite heat flow) evolve by the Dyson equation $\dot{\gamma}_k = f_k$, yielding $(1/\Phi_n)' = 2S/\Phi_n^2 \geq 0$ where $S = \sum_{i<j}(f_i - f_j)^2/(\gamma_i - \gamma_j)^2$ is a Dirichlet form; an algebraic identity $\sum_{i<j}(f_i - f_j)/(\lambda_i - \lambda_j) = \Phi_n$; and finite free subordination $G_r(z) = G_p(\omega_1(z))$ with $\omega_1 + \omega_2 = z + 1/G_r(z)$. The inequality is verified numerically (180,000 tests, $n \leq 10$, zero violations), and the most promising path to a general proof is establishing the Dirichlet form bound $S \geq \Phi_n^2/[n(n-1)]$ along the heat flow, which numerical evidence supports with minimum ratio ≈ 1.