Lecture 4 Fourier Series of Continuous-time Signals

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Outline: **Lecture 4:** Fourier Series of Continuous-time Signals

- Time/Frequency Domain
- Response of LTI Systems to Exponential Signal
- Fourier Series
    - Fourier Series Representation
    - Convergence Issue
    - Fourier Series Properties

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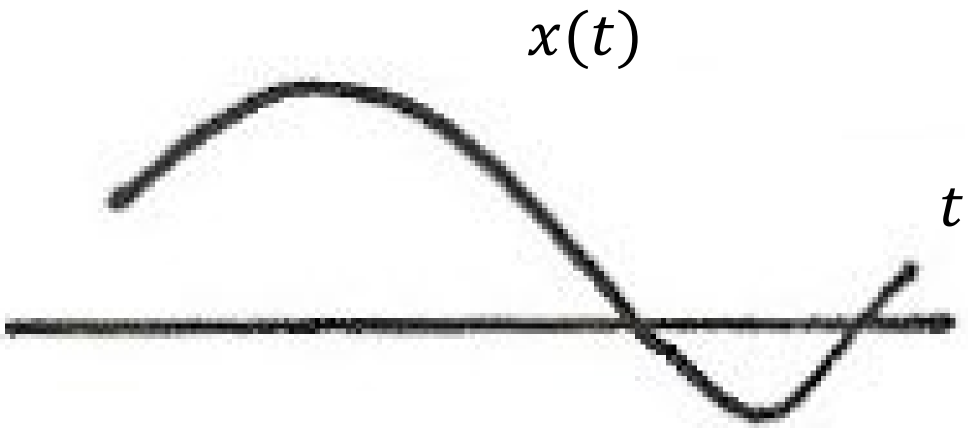
  - Fourier Series Properties

Time/Frequency Domain

Time domain signal

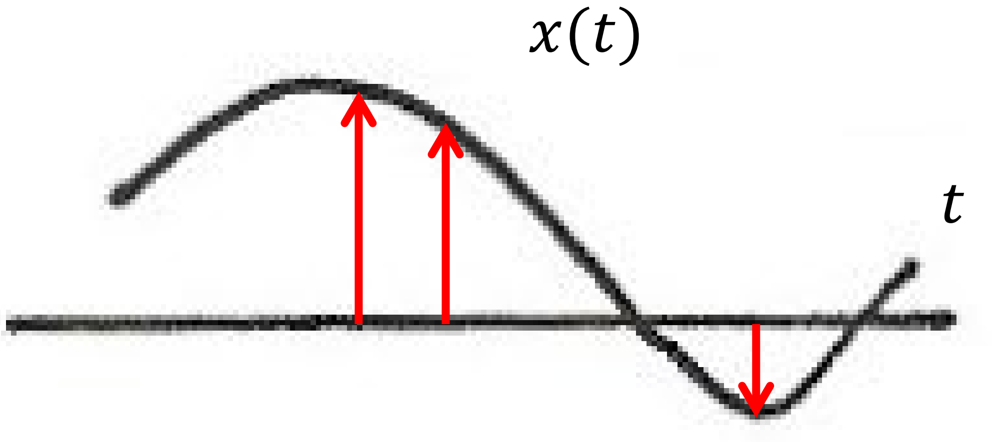
- A *signal* is a function, in the mathematical sense, normally a function of time.



Time/Frequency Domain

Time domain signal

- Time domain analysis examines the amplitude vs. time characteristics of a measuring signal.

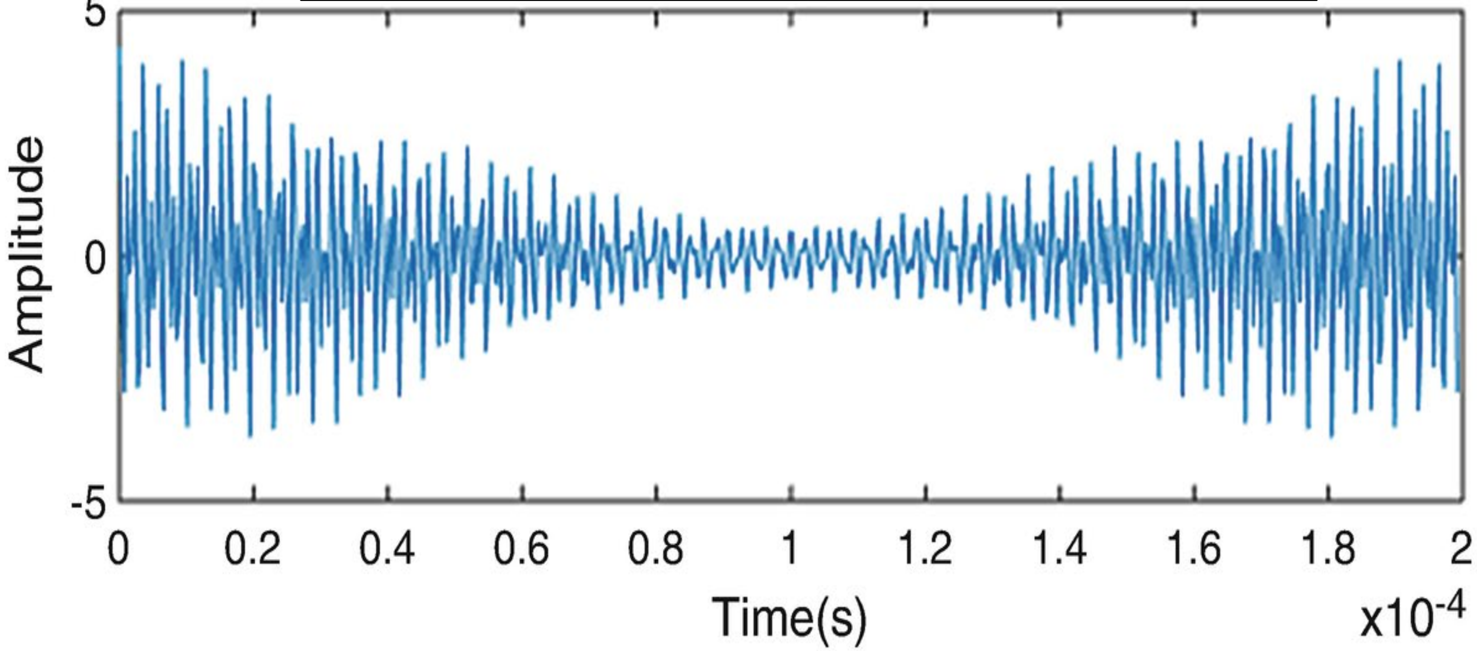


- Time domain is how people tend to think that things work, but in many situations, signals in time domain can be complicated.

Time/Frequency Domain

Time domain signal

An audio signal in time domain



- We can easily observe the loudness of a sound, but what else?
  - Whose voice is this?
  - How much is the noise?

Time/Frequency Domain

Frequency domain

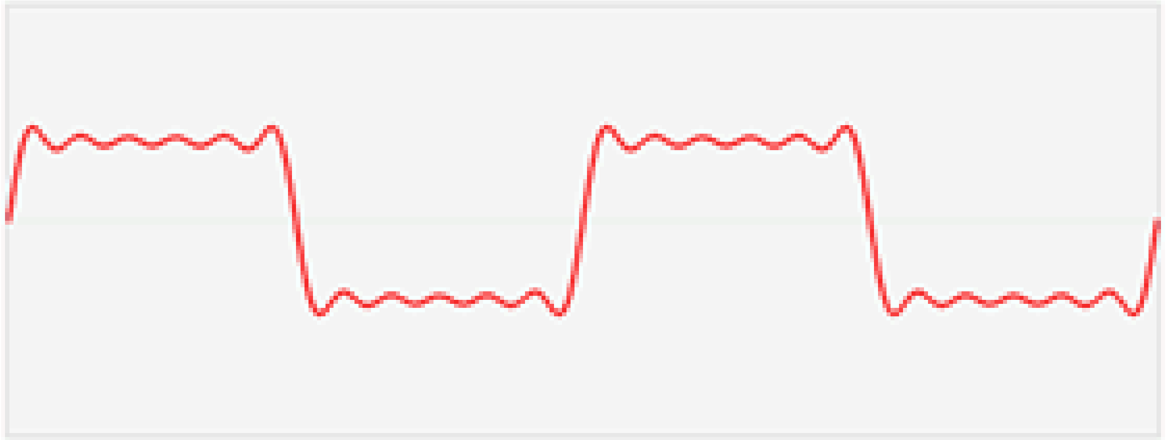
- In 1807, J.B.J Fourier proposed to use trigonometric functions and series to solve heat equation in a metal plate.

- Fourier claimed ***any arbitrary periodic function can be represented by a harmonically related trigonometric series.***

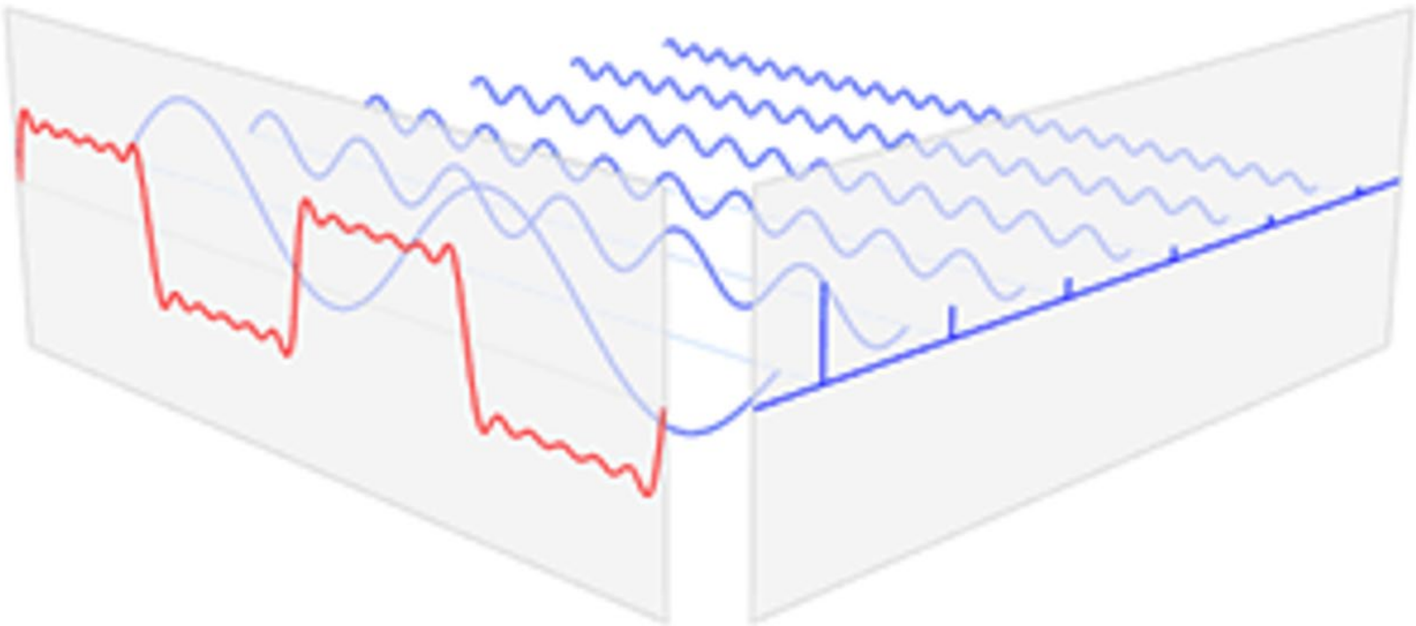
- Fourier series representation:

$$x(t) = A_0 + \sum_{k=1}^{\infty} [A_k \cos(k\omega_0 t) + B_k \sin(k\omega_0 t)]$$

Time/Frequency Domain



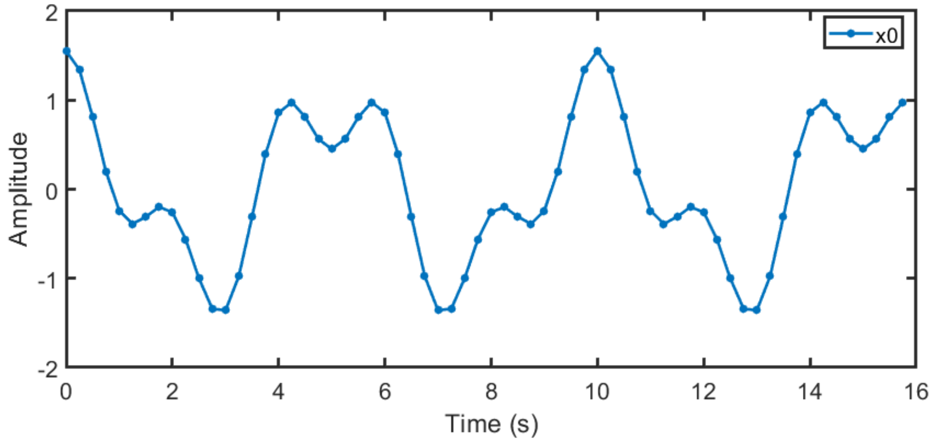
Time/Frequency Domain



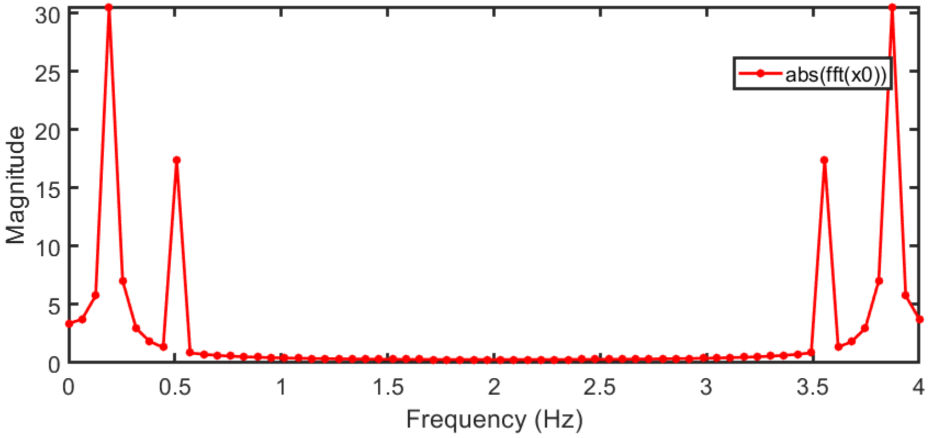
Time/Frequency Domain

Frequency domain

- Frequency domain shows how much of the signal lies within each given frequency band over a range of frequencies.



Time domain signal

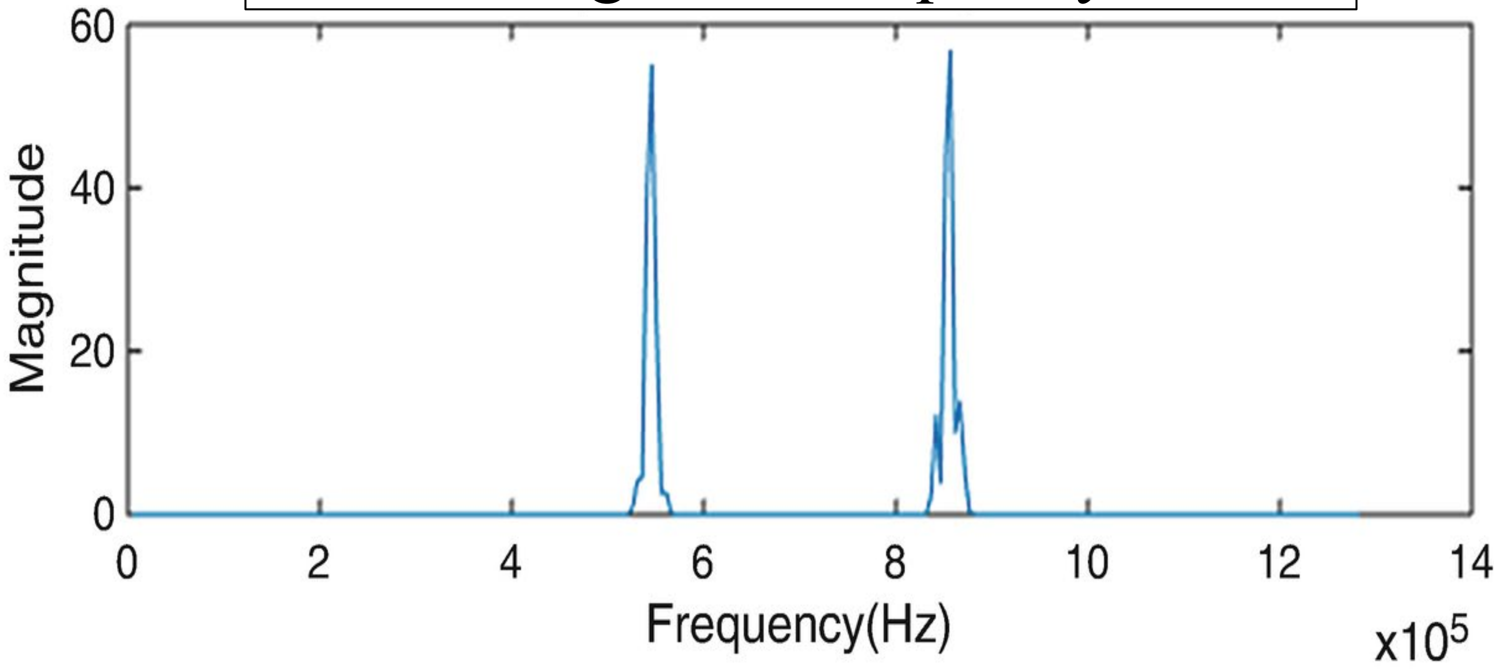


Frequency domain signal

Time/Frequency Domain

Why frequency domain is important

An audio signal in frequency domain



- We can collect even more information in frequency domain, *e.g.*:
  - Speaker recognition based on frequency-based coefficients
  - Noise identification based on noise frequency characteristics

Time/Frequency Domain

Why frequency domain is important

- Frequency domain is a different perspective on signal, which providing us more useful information.

- Generally,

  - Time domain shows how we tend to think that things work;
  - Frequency domain can often provide many detailed performance specifications.

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    - Fourier Series Representation
    - Convergence Issue
    - Fourier Series Properties

Response of LTI Systems to Exponential Signal

Fourier series sine-cosine form

$$x(t) = A_0 + \sum_{k=1}^{\infty} [A_k \cos(k\omega_0 t) + B_k \sin(k\omega_0 t)]$$

- Sinusoids can be represented by complex exponential function

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Response of LTI Systems to Exponential Signal

Fourier series exponential form

$$x(t) = A_0 + \sum_{k=1}^{\infty} [A_k \cos(k\omega_0 t) + B_k \sin(k\omega_0 t)]$$

$$= A_0 + \sum_{k=1}^{\infty} \left[\frac{A_k}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) + \frac{B_k}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)\right]$$

$$= A_0 + \sum_{k=1}^{\infty} \frac{A_k - jB_k}{2} e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} \frac{A_{-k} + jB_{-k}}{2} e^{jk\omega_0 t}$$

Response of LTI Systems to Exponential Signal

Fourier series exponential form

$$x(t) = A_0 + \sum_{k=1}^{\infty}[A_k\cos(k\omega_0 t) + B_k \sin(k\omega_0 t)]$$

$$= A_0 + \sum_{k=1}^{\infty}\frac{A_k - jB_k}{2}e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1}\frac{A_{-k} + jB_{-k}}{2}e^{jk\omega_0 t}$$

Assume

$$a_k = \begin{cases} \frac{A_{-k}+jB_{-k}}{2} & k < 0 \\ A_0 & k = 0 \\ \frac{A_k-jB_k}{2} & k > 0 \end{cases}$$

Response of LTI Systems to Exponential Signal

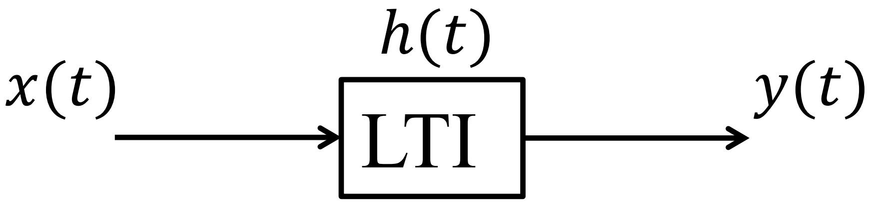
Fourier series exponential form

$$x(t) = A_0 + \sum_{k=1}^{\infty} [A_k \cos(k\omega_0 t) + B_k \sin(k\omega_0 t)]$$

$$= A_0 + \sum_{k=1}^{\infty} \frac{A_k - jB_k}{2} e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} \frac{A_{-k} + jB_{-k}}{2} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

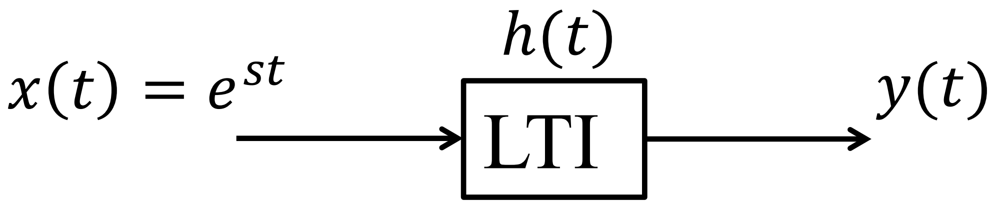
Response of LTI Systems to Exponential Signal



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Assume $x(t) = e^{st}$

Response of LTI Systems to Exponential Signal

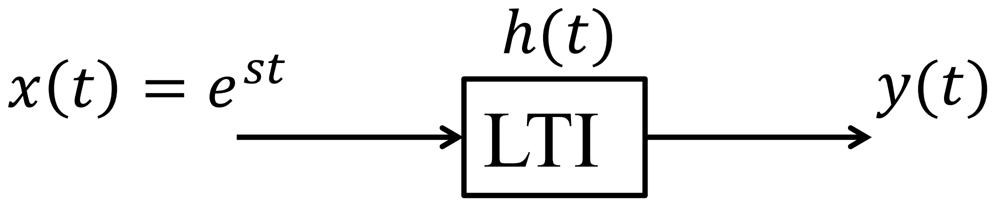


$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{s\,(t-\tau)}d\tau = \int_{-\infty}^{\infty} e^{st}h(\tau)e^{-s\tau}d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

Response of LTI Systems to Exponential Signal



$$y(t) = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

$\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$ is a constant, thus we define $\mathrm{H(s)} = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$:

$$y(t) = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = e^{st}\mathrm{H(s)} = \mathrm{x(t).H(s)}$$

Response of LTI Systems to Exponential Signal

$$y(t) = e^{st}\mathrm{H(s)} = \mathrm{x(t).H(s)}$$

- Response of a LTI system to an exponential signal is this exponential signal multiplied with a constant.

- We call an exponential function, e^{st}, as the eigenfunction of any LTI system.

- We call the constant, $\mathrm{H(s)}$, as the eigenvalue associated with the eigenfunction e^{st}.

Response of LTI Systems to Exponential Signal

Why do we specifically study LTI response to exponential signal?

- Fourier series can be considered as a composition of exponential signals:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Response of LTI Systems to Exponential Signal

Why do we specifically study LTI response to exponential signal?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \xrightarrow{\quad} \boxed{\text{LTI}}_{\,h(t)} \xrightarrow{\quad} y(t)$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} H(jk\omega_0)$$

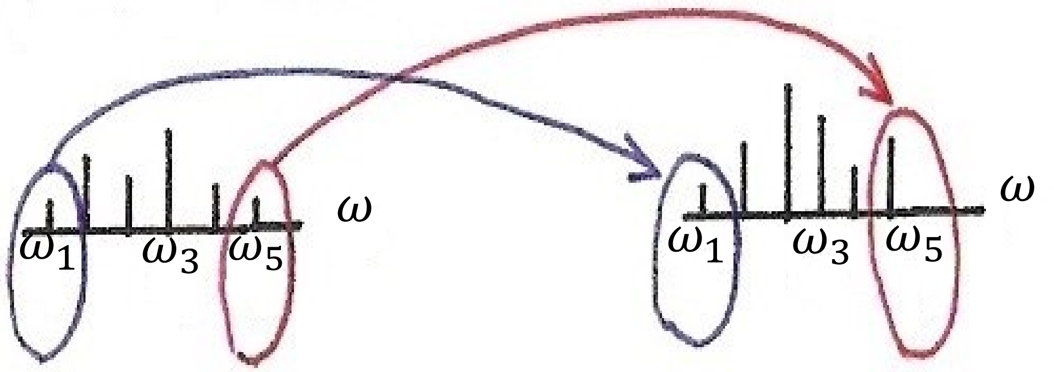
- LTI response to a periodic signal represented by Fourier series is still a periodic signal represented by Fourier series.

Response of LTI Systems to Exponential Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \rightarrow \quad y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} H(jk\omega_0)$$

- If the input has a frequency component, the output will exactly have the same frequency component, except scaled by a constant.

- Frequency domain:



Response of LTI Systems to Exponential Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \rightarrow \quad y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} H(jk\omega_0)$$

- Each frequency component never split to other frequency components, no convolution involved, realizing fast and convenient signal processing.

Outline: **Lecture 4:** Fourier Series of Continuous-time Signals

- Time/Frequency Domain

- Response of LTI Systems to Exponential Signal

- Fourier Series
    - Fourier Series Representation
    - Convergence Issue
    - Fourier Series Properties

Fourier Series Representation

Harmonically related complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

$$k = 0, \pm 1, \pm 2, \ldots$$

- A harmonic of such a wave is a wave with a frequency that is a positive integer multiple of the frequency of the original wave, known as the fundamental frequency.

- Fundamental frequency: ω_0 $(k = \pm 1)$
- 2nd harmonic: $2\omega_0$ $(k = \pm 2)$
- 3rd harmonic: $3\omega_0$ $(k = \pm 3)$ …

Fourier Series Representation

Determination of Fourier series coefficient - a_k

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt$$

Fourier Series Representation

Determination of Fourier series coefficient - a_k

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt$$

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j\omega_0 t(k-n)} dt$$

Fourier Series Representation

Determination of Fourier series coefficient - a_k

- For k=n:

$$\int_0^T e^{j\omega_0 t(k-n)}dt = \int_0^T 1dt = T$$

- For k≠n:

$$\int_0^T e^{j\omega_0 t(k-n)}dt = \int_0^T \cos[\omega_0 t(k-n)]dt + j\int_0^T sin[\omega_0 t(k-n)]dt$$

$$= 0 \text{ (since k} - \text{n is an integer)}$$

Fourier Series Representation

Determination of Fourier series coefficient - a_k

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j\omega_0 t(k-n)}dt$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = Ta_n$$

Fourier Series Representation

Fourier series representation （傅里叶级数表示）：

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Fourier series coefficient（傅里叶级数系数）：

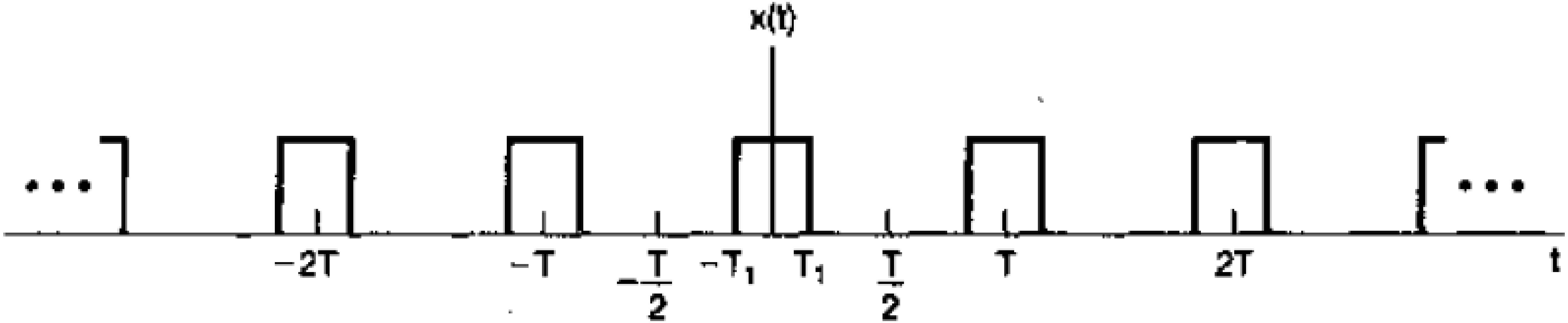
$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

Fourier Series Representation

Example:

Find Fourier series coefficients for the following periodic signal.

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



Fourier Series Representation

Solutions:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < {}^{T}/_{2} \end{cases}$$

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt$$

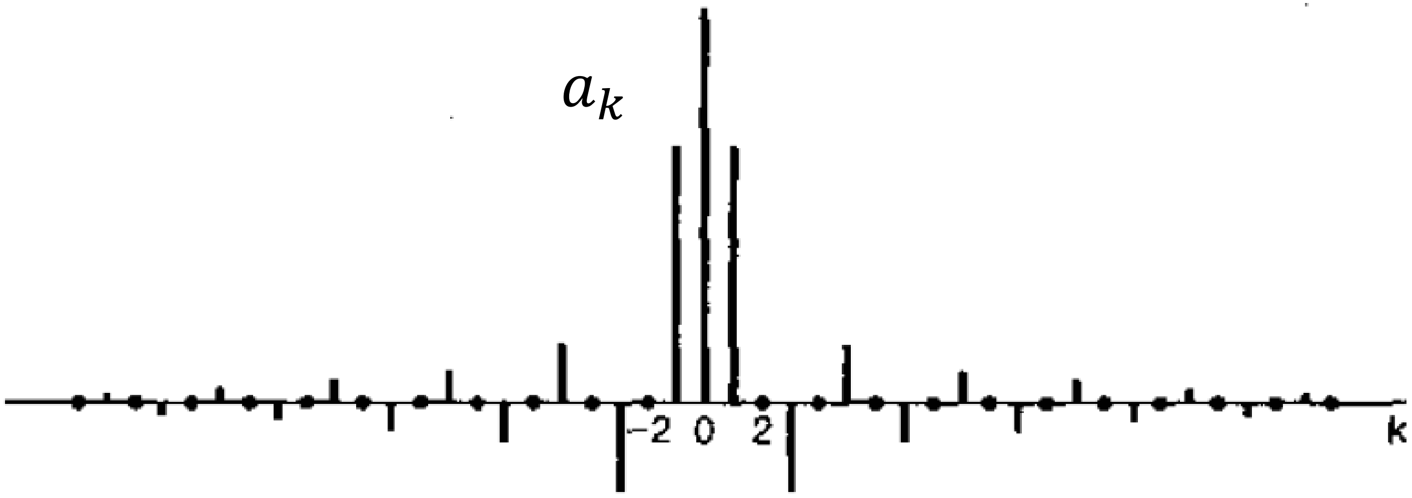
$$= \begin{cases} \dfrac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{jk\omega_0 \dfrac{2\pi}{\omega_0}} = \dfrac{\sin(k\omega_0 T_1)}{k\pi}, & k \neq 0 \\ \dfrac{1}{T}\displaystyle\int_{-T_1}^{T_1} dt = \dfrac{2T_1}{T}, & k = 0 \end{cases}$$

Fourier Series Representation

Assume $T_1 = {}^{T}/_{4}$:

$$a_k = \begin{cases} \dfrac{\sin(k\omega_0 T_1)}{k\pi}, & k \neq 0 \\ \dfrac{1}{2}, & k = 0 \end{cases}$$

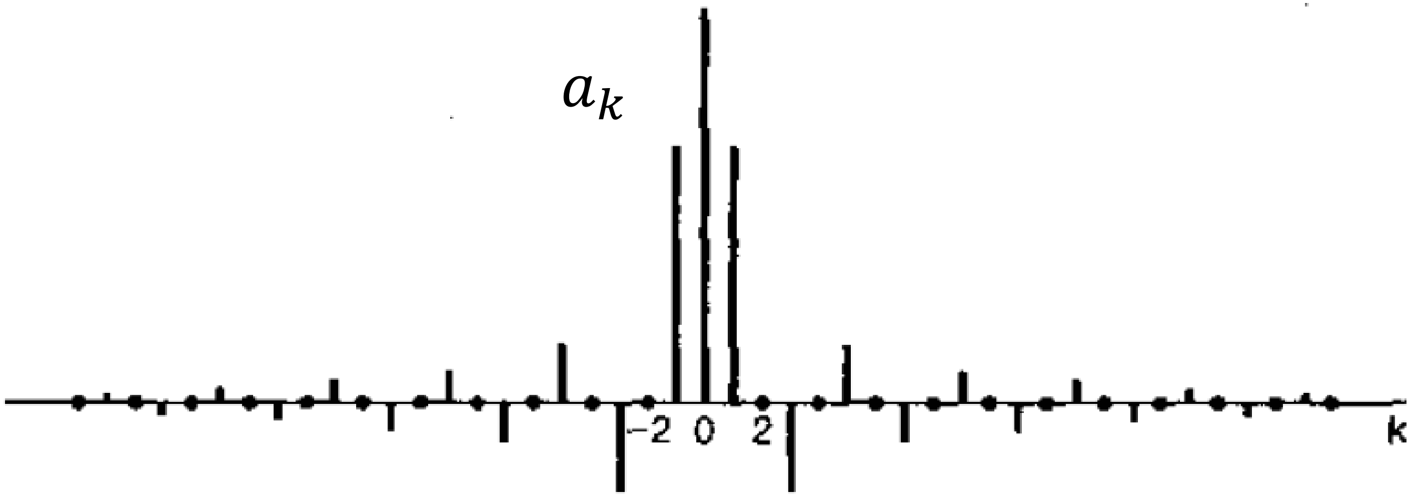
Spectral:



Fourier Series Representation

Assume $T_1 = {}^{T}/_{4}$:

$$a_k = \begin{cases} \dfrac{\sin(k\omega_0 T_1)}{k\pi}, & k \neq 0 \\ \dfrac{1}{2}, & k = 0 \end{cases}$$



But is this correct?

Fourier Series Representation

Fourier series sine-cosine form

$$x(t) = A_0 + \sum_{k=1}^{\infty} [A_k \cos(k\omega_0 t) + B_k \sin(k\omega_0 t)],$$

$$k = 0, \pm 1, \pm 2, \dots$$

Fourier series amplitude-phase form

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} A'_k \cos(k\omega_0 t + \theta_k),$$

$$k = 0, \pm 1, \pm 2, \dots$$

Fourier Series Representation

Amplitude-phase form derived from exponential form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= a_0 + \sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} a_k e^{jk\omega_0 t}$$

$$= a_0 + \sum_{k=1}^{\infty} (a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t})$$

Fourier Series Representation

Amplitude-phase form derived from exponential form

- Remember in slide 16, we have assumed: $a_k = \begin{cases} \frac{A_{-k}+jB_{-k}}{2} & k < 0 \\ A_0 & k = 0 \\ \frac{A_k - jB_k}{2} & k > 0 \end{cases}$, thus

a_k and a_{-k} (for $k \in [1, \infty]$) are a pair of complex conjugates.

- $e^{jk\omega_0 t}$ and $e^{-jk\omega_0 t}$ (for $k \in [1, \infty]$) are a pair of complex conjugates.

- $a_k e^{jk\omega_0 t}$ and $a_{-k} e^{-jk\omega_0 t}$ (for $k \in [1, \infty]$) are also a pair of complex conjugates.

Fourier Series Representation

Amplitude-phase form derived from exponential form

- Remember in slide 16, we have assumed: $a_k = \begin{cases} \frac{A_{-k}+jB_{-k}}{2} & k < 0 \\ A_0 & k = 0 \\ \frac{A_k-jB_k}{2} & k > 0 \end{cases}$, thus

a_k and a_{-k} (for $k \in [1, \infty]$) are a pair of complex conjugates.

- Note that the above hypothesis is true, only if A_k and B_k are real, and thus the signal is a real signal.

Fourier Series Representation

Amplitude-phase form derived from exponential form

- Remember in slide 16, we have assumed: $a_k = \begin{cases} \frac{A_{-k}+jB_{-k}}{2} & k < 0 \\ A_0 & k = 0 \\ \frac{A_k - jB_k}{2} & k > 0 \end{cases}$, thus

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- $e^{jk\omega_0 t}$ and $e^{-jk\omega_0 t}$ (for $k \in [1, \infty]$) are a pair of complex conjugates.

- $a_k e^{jk\omega_0 t}$ and $a_{-k} e^{-jk\omega_0 t}$ (for $k \in [1, \infty]$) are also a pair of complex conjugates.

Fourier Series Representation

Amplitude-phase form derived from exponential form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= a_0 + \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right)$$

$$= a_0 + \sum_{k=1}^{\infty} 2\mathbb{R}\{a_k e^{jk\omega_0 t}\}$$

Fourier Series Representation

Amplitude-phase form derived from exponential form

- Use polar coordinate system to represent a_k as: $a_k = A'_k e^{j\theta_k}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= a_0 + \sum_{k=1}^{\infty} 2\mathbb{R}\{a_k e^{jk\omega_0 t}\} = a_0 + \sum_{k=1}^{\infty} 2A'_k \mathbb{R}\{e^{j(k\omega_0 t + \theta_k)}\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} A'_k \cos(k\omega_0 t + \theta_k)$$

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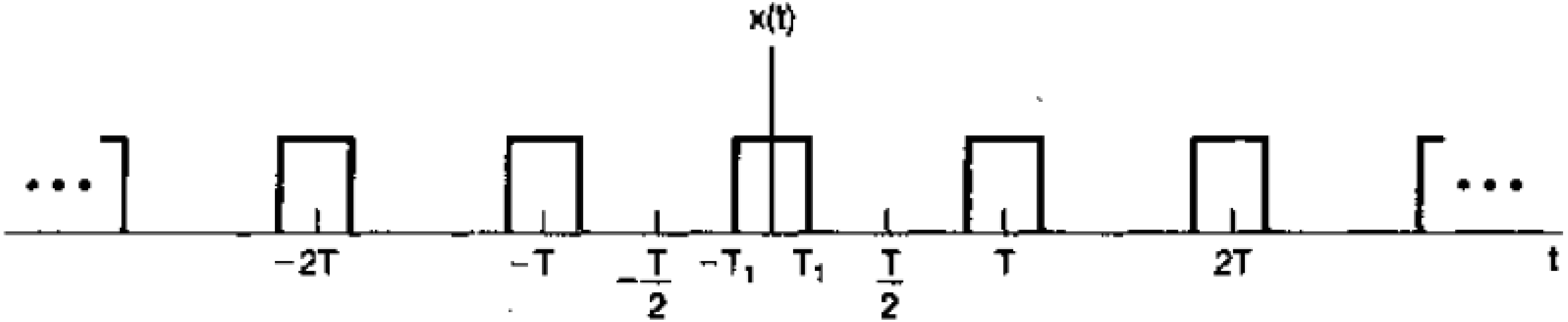
  - Fourier Series Representation
  - Convergence Issue
  - Fourier Series Properties

Convergence Issue

Question:

- In time domain, this signal is not continuous with discontinuity.

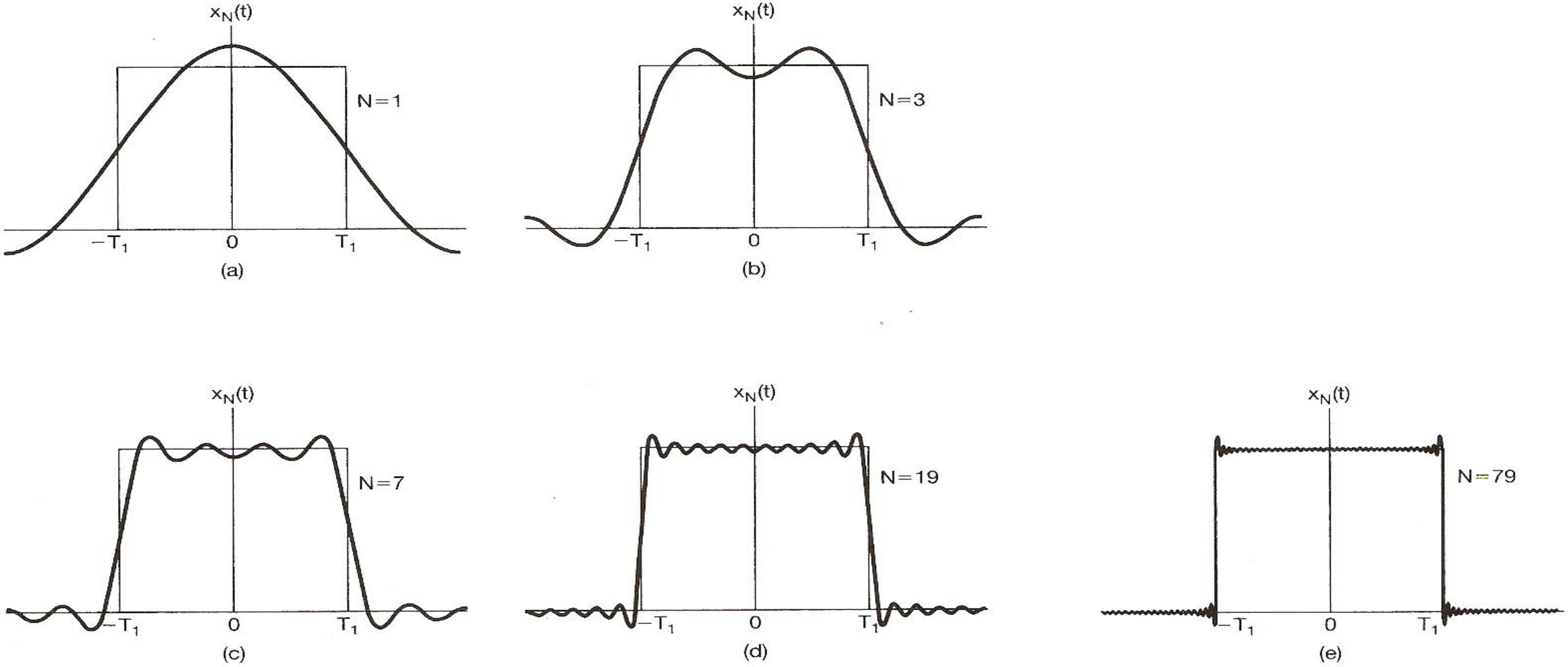


- In frequency domain, this signal is represented by a trigonometric series, each of which is continuous.

- How is the discontinuity represented?

Convergence Issue

Gibbs phenomenon（吉伯斯现象）



Convergence Issue

Gibbs phenomenon

- The partial sum of the Fourier series has large oscillations near the jump, which might cause a fixed overshoot above the function itself.

- As the number of terms rises, the error of the approximation is reduced in width and energy.

- At the jump discontinuities, the limit will converge to the average of the values of the function on either side of the jump. (Dirichlet Theorem 狄利赫里定理)

Convergence Issue

- A period signal can be represented by Fourier series, if it has finite energy:

$$\int_T |x(t)|^2 dt < \infty$$

Convergence Issue

Dirichlet’s condition（狄利赫里条件）

- A signal can be represented by Fourier series expansion, if

  (1) it is absolutely integrable, $\int_T |x(t)|\, dt < \infty$

  (2) it has finite number of maxima & minima in a period

  (3) it has finite number of discontinuities in a period

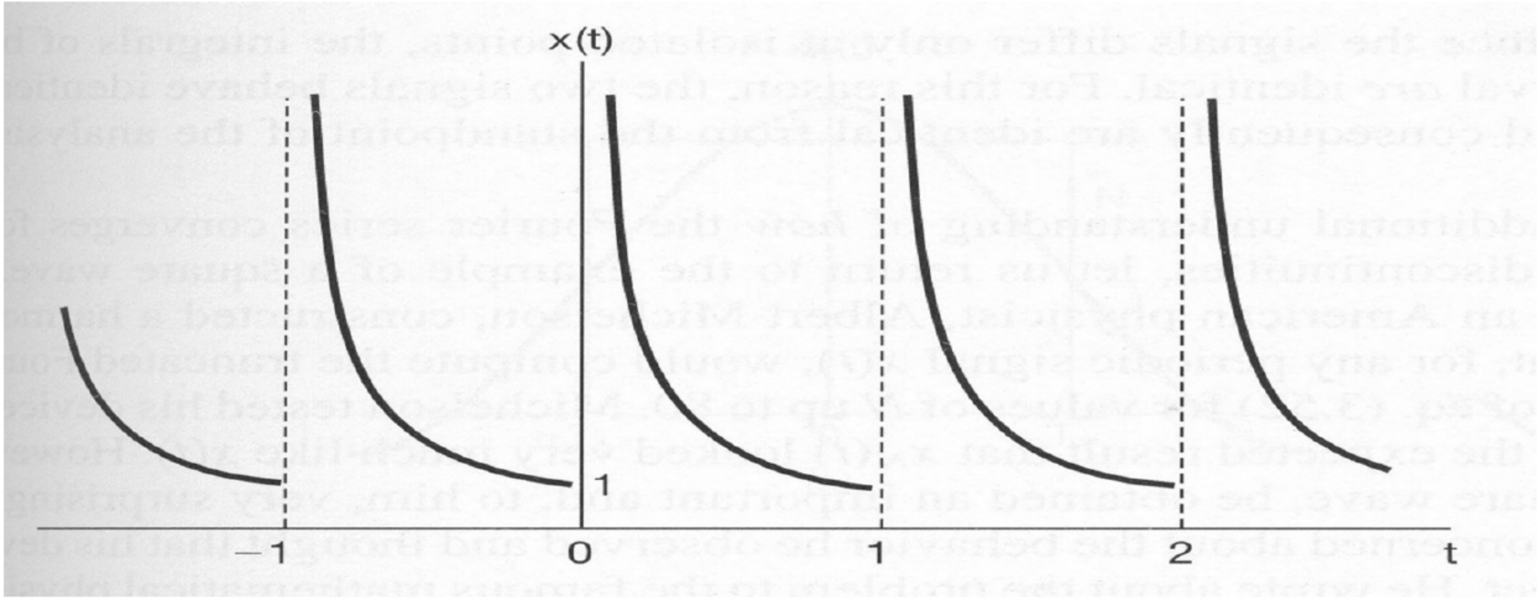
Convergence Issue

Dirichlet's condition

- A signal can be represented by Fourier series expansion, if
  (1) it is absolutely integrable, $\int_T |x(t)|\, dt < \infty$

- The signal below can not be represented by Fourier series expansion:

$$x(t) = \frac{1}{t}, 0 < t \leq 1$$



Convergence Issue

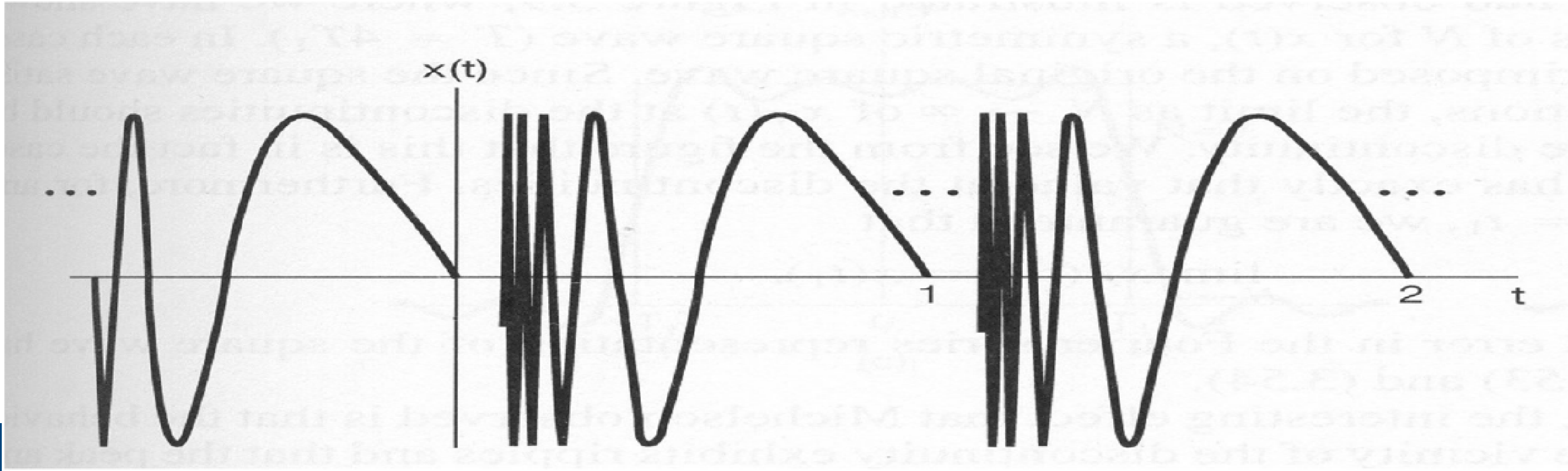
Dirichlet's condition

- A signal can be represented by Fourier series expansion, if

  (2) it has finite number of maxima & minima in a period

- The signal below can not be represented by Fourier series expansion:

$$x(t) = \sin(\frac{2\pi}{t}), 0 < t \leq 1$$



Convergence Issue

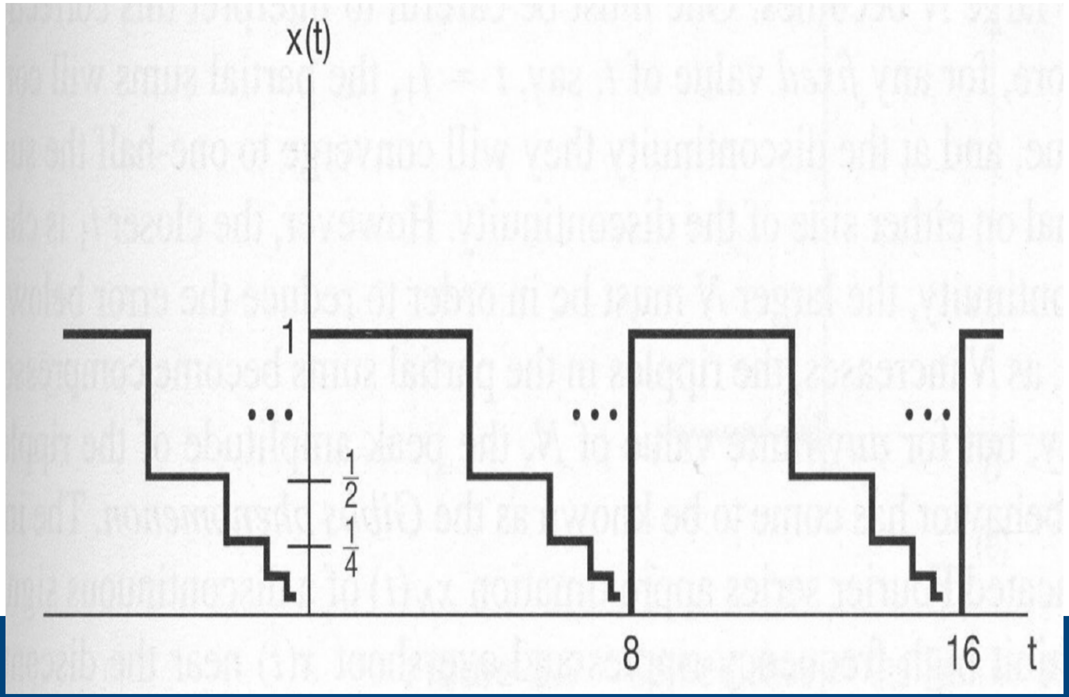
Dirichlet's condition

- A signal can be represented by Fourier series expansion, if

  (3) it has finite number of discontinuities in a period

- The signal below can not be represented by Fourier series expansion:

$$x(t)=\begin{cases}1, & 0\le t<4\\ 1/2, & 4\le t<6\\ 1/4, & 6\le t<7\\ & \vdots\end{cases}$$



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Fourier Series Properties

- The following notation is used to indicate a signal, $x(t)$, can be represented by Fourier series with the coefficients, a_k.

$$x(t) \overset{FS}{\leftrightarrow} a_k$$

Fourier Series Properties

Linearity

Assume :

$$x(t) \overset{FS}{\leftrightarrow} a_k, \mathrm{y}(t) \overset{FS}{\leftrightarrow} b_k,$$

Then:

$$\mathrm{Ax}(t) + By(t) \overset{FS}{\leftrightarrow} Aa_k + Bb_k$$

Fourier Series Properties

Time shift

Assume :

$$x(t) \overset{FS}{\leftrightarrow} a_k, y(t) = x(t - t_0) \overset{FS}{\leftrightarrow} b_k$$

Then:

$$b_k = \frac{1}{T}\int_0^T x(t - t_0)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}e^{-jk\omega_0 t_0}dt = a_k e^{-jk\omega_0 t_0}$$

- Time shift leads to unchanged amplitude and shifted phase.

Fourier Series Properties

Time reversal

Assume :

$$x(t) \overset{FS}{\leftrightarrow} a_k, y(t) = x(-t) \overset{FS}{\leftrightarrow} b_k$$

Then:

$$b_k = \frac{1}{T}\int_0^T x(-t)e^{jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_0^T x(t)e^{j(-k)\omega_0 t}dt = a_{-k}$$

$$x(-t) \overset{FS}{\leftrightarrow} a_{-k}$$

Fourier Series Properties

Time scaling

Assume :

$$x(t) \overset{FS}{\leftrightarrow} a_k$$

Then:

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 \alpha t}$$

$$x(\alpha t) \overset{FS}{\leftrightarrow} a_k$$

- Note that though Fourier series coefficients, a_k, is unchanged, Fourier series representation is different, because each harmonic component is different: for $x(t)$, the harmonic frequency is kω_0; for $x(\alpha t)$, it is k$\omega_0 \alpha$.

Fourier Series Properties

Multiplication

Assume :

$$x(t) \overset{FS}{\leftrightarrow} a_k, y(t) \overset{FS}{\leftrightarrow} b_k,$$

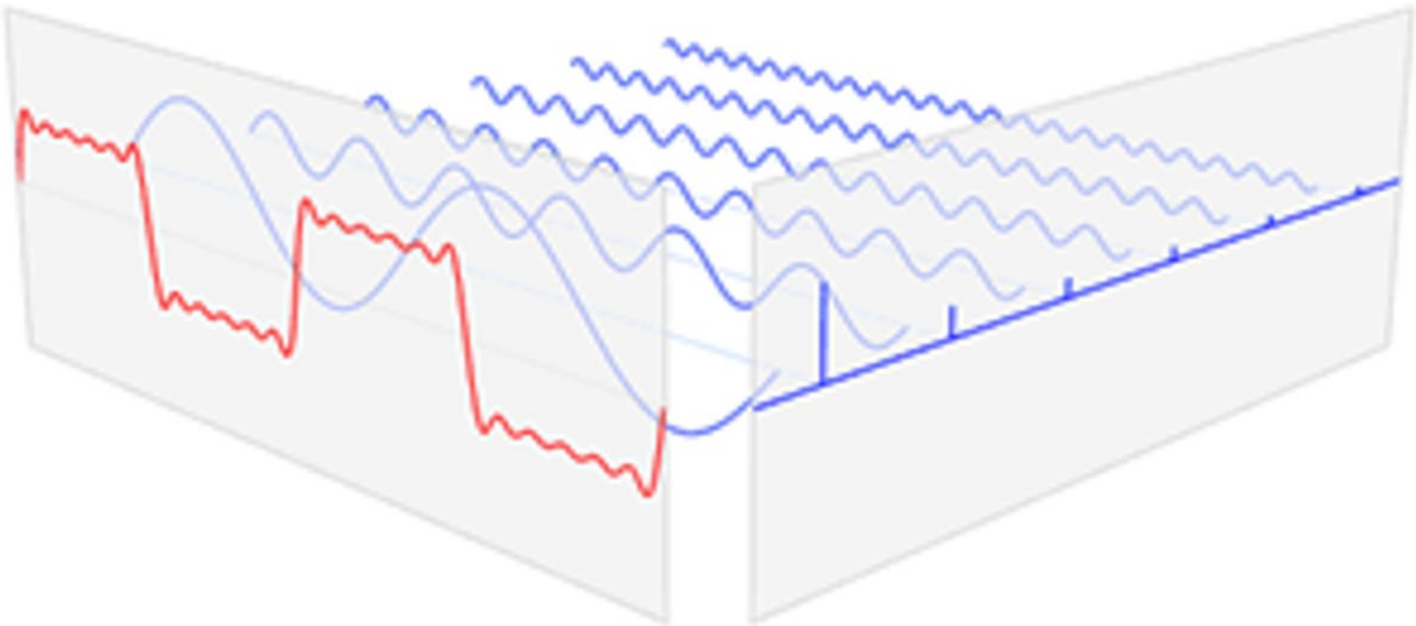
Then:

$$d_k e^{jk\omega_0 t} = \sum_{k_1} \sum_{k_2} a_{k_1} b_{k_2} e^{j(k_1+k_2)\omega_0 t}, k_1 + k_2 = k$$

Let $k_1 = l, k_2 = k - l$:

$$\mathrm{x}(t)y(t) \overset{FS}{\leftrightarrow} d_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k * b_k$$

Time/Frequency Domain



不同傅里叶级数表示形式的相互转换

解：首先回忆傅里叶级数表示的三种形式：

① 正余弦形式：

$$x(t) = A_0 + \sum_{k=1}^{\infty}[A_k\cos(k\omega_0 t) + B_k\sin(k\omega_0 t)]$$

② 复指数形式：

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, a_k = \begin{cases} \dfrac{A_{-k} + jB_{-k}}{2} & k < 0 \\ A_0 & k = 0 \\ \dfrac{A_k - jB_k}{2} & k > 0 \end{cases}$$

③ 幅度-相位形式：

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A'_k\cos(k\omega_0 t + \theta_k), a_k = A'_k e^{j\theta_k}$$

Fourier Series Properties

Conjugation

Assume :

$$x(t) \overset{FS}{\leftrightarrow} a_k$$

Then:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* (e^{jk\omega_0 t})^* = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

$$x^*(t) \overset{FS}{\leftrightarrow} a_{-k}^*$$

Fourier Series Properties

Conjugation

Even/Odd property

$$x(t) \overset{FS}{\leftrightarrow} a_k, x^*(t) \overset{FS}{\leftrightarrow} a^*_{-k}$$

- If $x(t)$ is real ($x(t) = x^*(t)$):

$$a_k = a^*_{-k}$$

Fourier Series Properties

Conjugation

Even/Odd property

$$x(t) \overset{FS}{\leftrightarrow} a_k, x^*(t) \overset{FS}{\leftrightarrow} a_{-k}^*$$

- If $x(t)$ is real and even $(x(t) = x^*(t), x(t) = x(-t))$:

$$a_k = a_{-k}^*, a_k = a_{-k}$$

$$\Rightarrow a_{-k}^* = a_{-k}$$

$$\Rightarrow a_k^* = a_k$$

- a_k is real and even

Fourier Series Properties

Conjugation

Even/Odd property

$$x(t) \overset{FS}{\leftrightarrow} a_k, x^*(t) \overset{FS}{\leftrightarrow} a^*_{-k}$$

- If $x(t)$ is real and odd $(x(t) = x^*(t), x(t) = -x(-t))$:

$$a_k = a^*_{-k}, a_k = -a_{-k}$$

$$\Rightarrow a^*_{-k} = -a_{-k}$$

$$\Rightarrow a^*_k = -a_k$$

- a_k is imaginary and odd

Fourier Series Properties

Parseval's relation（帕斯瓦尔关系）

- The average power of a function in a period is equal to the sum of the square (power) of its Fourier series coefficients.

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Fourier Series Properties

Parseval's relation

Proof:

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = \sum_{k=-\infty}^{\infty} a_k\, a_k^*$$

$$= \sum_{k=-\infty}^{\infty} a_k \,[\frac{1}{T}\int_0^T x^*(t)e^{jk\omega_0 t}dt] = \frac{1}{T}\int_0^T x^*(t)[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}]dt$$

$$= \frac{1}{T}\int_0^T x^*(t)x(t)dt = \frac{1}{T}\int_0^T |x(t)|^2\, dt$$

Thank you for your listening!