Lecture 2 Complex Numbers & Signal Property

Complex Numbers – the basics

The notes cover the basic definitions and properties of complex numbers (Boas 2.1-2.5). The story starts from finding solutions for the simple algebraic equation $x^2+1=0$. There is no real solution to the equation. But, if we introduce the notion of imaginary numbers,

$$j \equiv \sqrt{-1}, \qquad \text{with } j^2 = -1$$

one can write down the solutions $x = \pm j$. Going beyond the pure imaginary numbers, one can introduce the complex number with real and imaginary parts,

$$z = x + jy$$

where x and y are real numbers, representing the real and imaginary parts respectively.

Complex land looks different

For complex numbers, an amazing identity arises

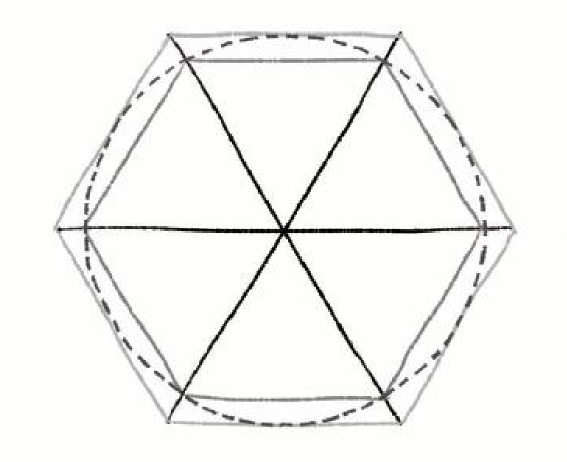
$$e^{j\pi} = -1$$

It is quite remarkable that two irrational numbers e and π can be related by the magic number j!

圆周率π与自然对数e

- 圆周率π：表示圆直径与周长之间的比例，对直径为1的圆，周长为π

- 圆周率的确定：对直径为1的圆，画出与其内接与外切的多边形，当边数趋于无穷大，内接与外切多边形周长趋于相等，周长为π



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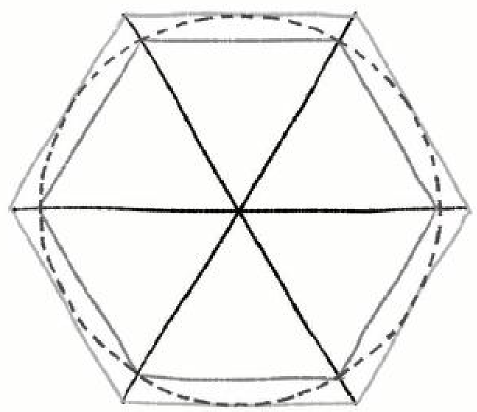


- 自然对数e：

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n$$

- 圆周率与自然对数反映了在恒定自然规律约束下，最终收敛。

$$e^{j\pi} = -1$$

Complex plane

It is convenient to plot $z = x + jy$ in the two-dimensional complex plane, using the real part x and the imaginary part y as Cartesian coordinates. Clearly, one can also write the same complex number in polar form,

$$z = x + jy = r\cos\theta + jr\sin\theta$$

Here $r = \sqrt{x^2 + y^2} = |z|$ is the absolute value of z (distance to the origin) and $\theta = \tan^{-1}(y/x)$ is the corresponding angle. Complex conjugate is defined as mirror mapping to the $x-axis$, i.e.

$$\bar{z} = x - jy = r(\cos\theta - j\sin\theta) = r[\cos(-\theta) + j\sin(-\theta)]$$

Since we can plot a complex number in the two dimensional plane, a natural question pops out: Is a complex number a two-dimensional vector?

Complex Numbers – the basics

Compute the following product

$$\bar{z}_1 z_2 = (x_1 - jy_1)(x_2 + jy_2) = (x_1 x_2 + y_1 y_2) + j(x_1 y_2 - x_2 y_1)$$

You can convince yourself that the real part is just the inner product and the imaginary part is the outer product for two-dimensional vectors.

Complex equations

Since a complex number contain real and imaginary parts, a complex equation amount to two real equations. For instance, $z^2 = 2j$ can be decomposed into two equations,

$$x^2 - y^2 = 0$$
$$2xy = 2$$

Note that both x and y are real. Thus, two solutions $x = y = 1$ and $x = y = -1$ are found for the complex equation $z^2 = 2j$.

Taylor expansions

Now I would like to establish an important identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

The easiest way to prove the above identity is through Taylor expansions,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Choose the variable $x = j\theta$ in the Taylor expansion for the exponential function and one can check that it equals the sum of the Taylor series for the sinusoidal functions.

泰勒展开式

- 泰勒展开：很多复杂的函数可以用多项式的形式表达

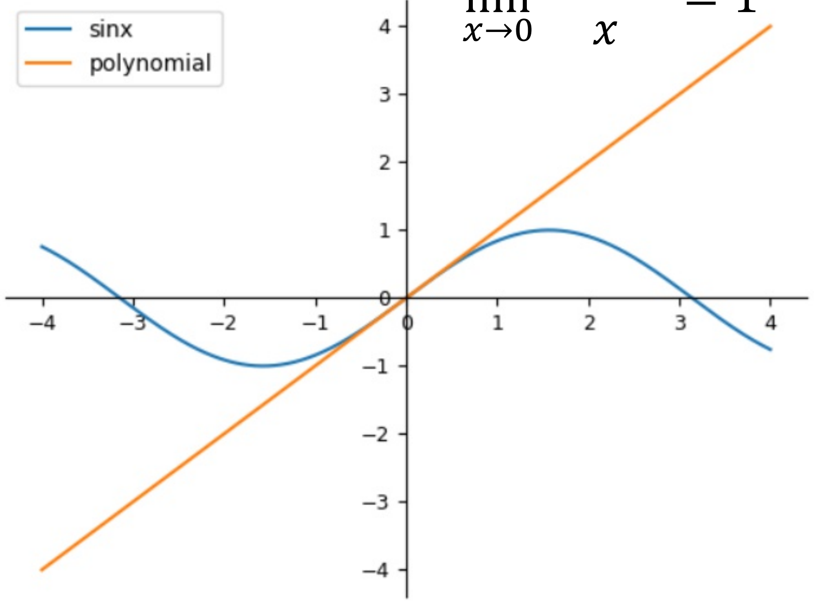
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

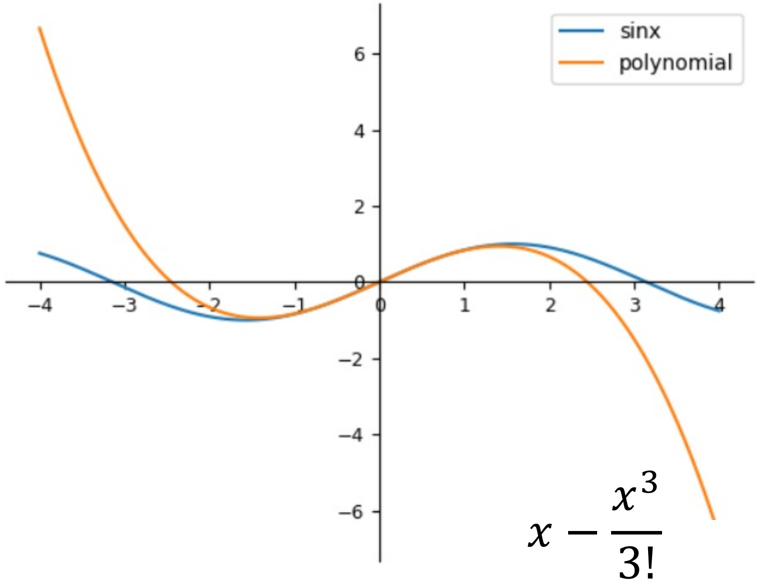
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

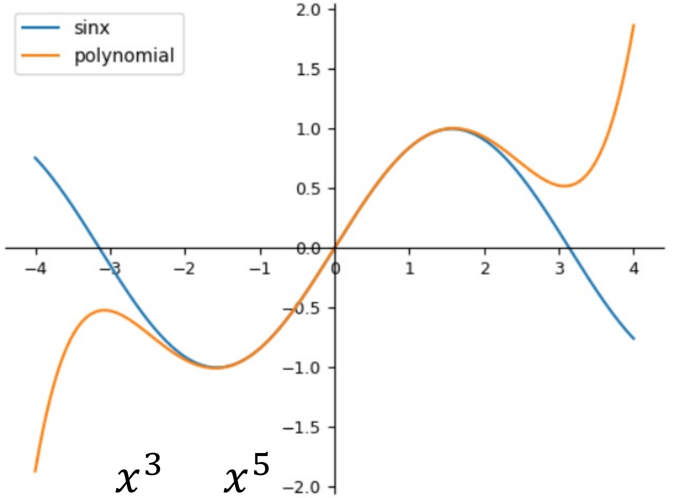
- 对 *f(x)=sin(x)*:

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$





$$x - \frac{x^3}{3!}$$

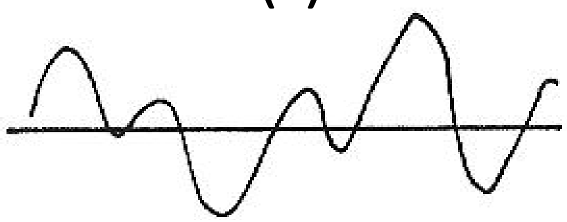


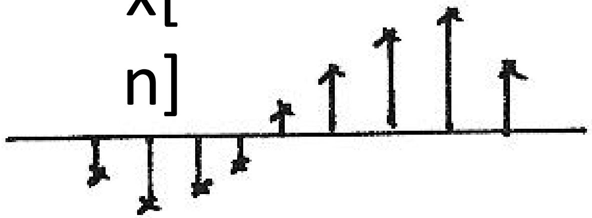
$$x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Signal Property

Continuous/Discrete-time Signals

$x(t), x[n]$





Signal Energy/Power

$$E = \int_{t_1}^{t_2} | x(t) |^2 \, dt \ , \quad E = \sum_{n=n_1}^{n_2} | x[n] |^2$$

$$P = E / (t_2 - t_1) \quad , \quad P = E / (n_2 - n_1 + 1)$$

"Electrical" Signal Energy & Power

- It is often useful to characterise signals by measures such as **energy** and **power**

For example, the **instantaneous power** of a resistor is:

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

- and the **total energy** expanded over the interval $[t_1, t_2]$ is:

$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

- and the **average energy** is:

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

- How are these concepts defined for any continuous or discrete time signal?

Generic Signal Energy and Power

- **Total energy** of a continuous signal $x(t)$ over $[t_1, t_2]$ is:

$$E = \int_{t_1}^{t_2} \left| x(t) \right|^2 dt$$

- where |.| denote the magnitude of the (complex) number.

Similarly for a discrete time signal $x[n]$ over $[n_1, n_2]$:

$$E = \sum_{n=n_1}^{n_2} \left| x[n] \right|^2$$

- By dividing the quantities by $(t_2\text{-}t_1)$ and $(n_2\text{-}n_1+1)$, respectively, gives the **average power**, P

- Note that these are similar to the electrical analogies (voltage), but they are different, both value and dimension.

Energy and Power over Infinite Time

- For many signals, we're interested in examining the power and energy over an infinite time interval (-∞, ∞). These quantities are therefore defined by:

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- If the sums or integrals do not converge, the energy of such a signal is infinite

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- Two important (sub)classes of signals
  1. Finite total energy (and therefore zero average power)
  2. Finite average power (and therefore infinite total energy)
- Signal analysis over infinite time, all depends on the "tails" (limiting behaviour)

Transformation of A Signal

- Time Shift

$$x(t) \to x(t - t_0) \quad , \quad x[n] \to x[n - n_0]$$

- Time Reversal

$$x(t) \to x(-t) \quad , \quad x[n] \to x[-n]$$

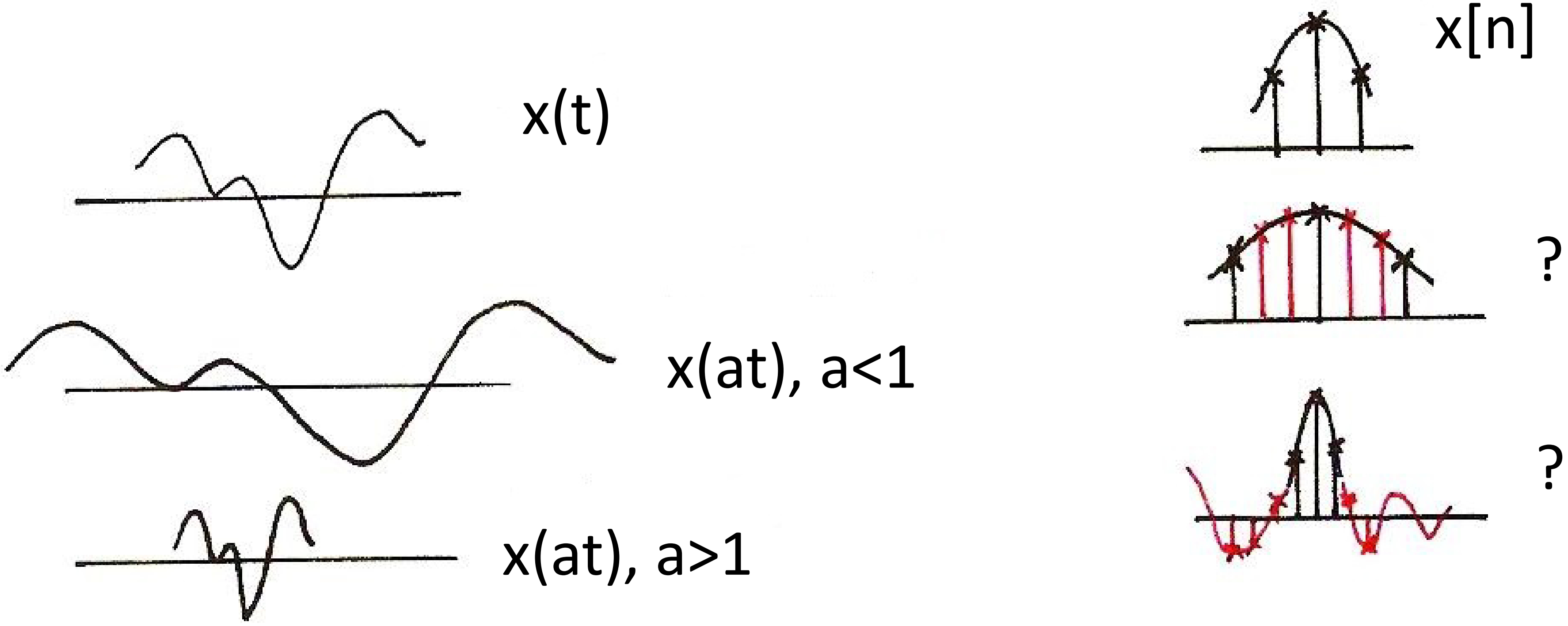
- Time Scaling

$$x(t) \to x(at) \quad , \quad x[n] \to ?$$

- Combination

$$x(t) \to x(at + b) \quad , \quad x[n] \to ?$$

Time Scaling



Periodic Signal

$$x(t) = x(t+T) \quad , \quad T : period$$

$$x(t) = x(t+mT) \quad , \quad m : \text{integer}$$

T_0 : Fundamental period : the smallest positive value of T

aperiodic : NOT periodic

$$x[n] = x[n+N] = x[n+mN] \quad , \quad N_0$$

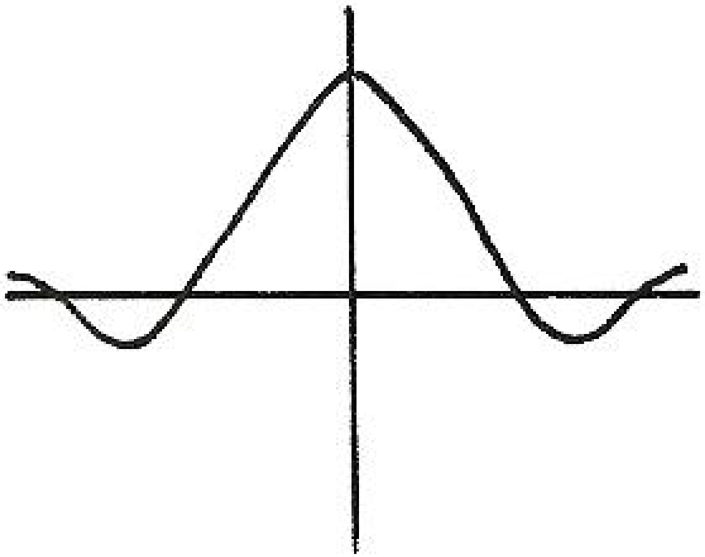
Even/Odd Signals

- Even $x(-t) = x(t)$, $x[-n] = x[n]$
- Odd $x(-t) = -x(t)$, $x[-n] = -x[n]$
- Any signal can be discomposed into a sum of an even and an odd

$$x_1(t) = \frac{1}{2}[x(t) + x(-t)], \quad x_2(t) = \frac{1}{2}[x(t) - x(-t)]$$

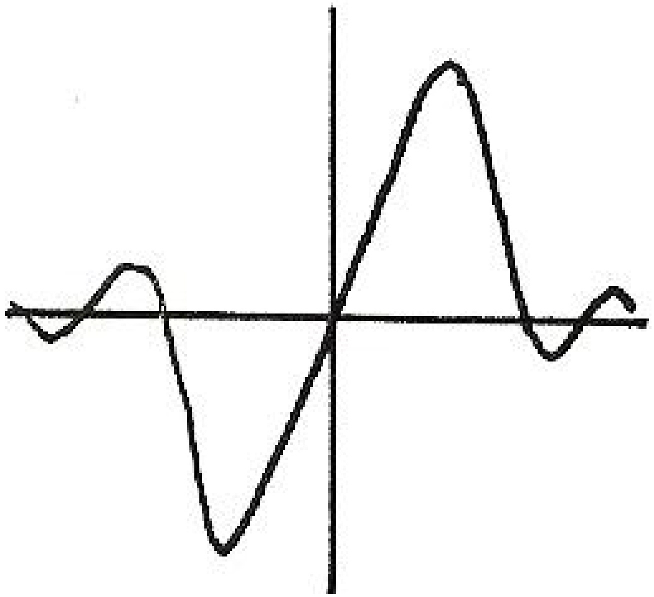
Even/Odd

Even



$x(-t)=x(t)$

Odd



$x(-t)=-x(t)$

Exponential/Sinusoidal Signals

● Basic Building Blocks from which one can construct many different signals and define frameworks for analyzing many different signals efficiently

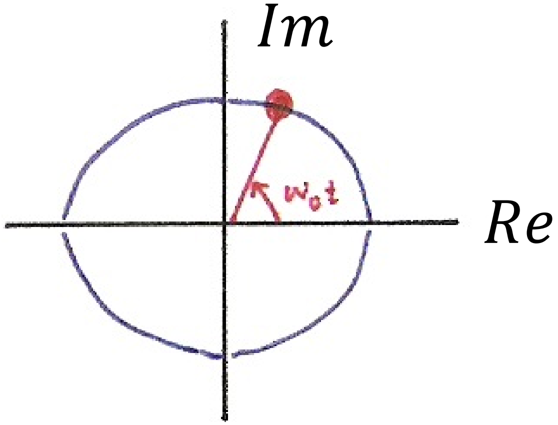
$$x(t) = e^{j\omega_0 t}, \quad \text{fundamental period} \quad T_0 = \frac{2\pi}{|\omega_0|}$$

$$\text{fundamental frequency} \quad \omega_0 = \frac{2\pi}{T_0}$$

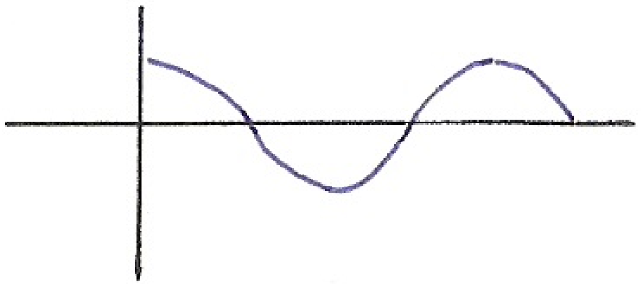
$$\omega_0 : rad / \sec$$

Exponential/Sinusoidal Signals

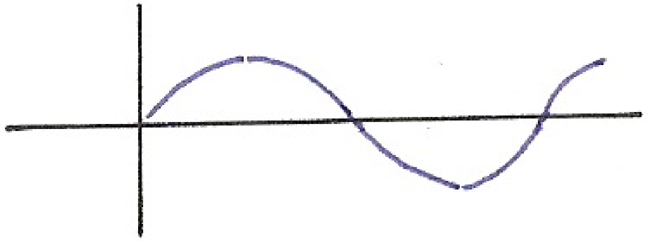
$x(t) = e^{j\omega_0 t}$



$Re\{e^{j\omega_0 t}\} = \cos \omega_0 t$



$Im\{e^{j\omega_0 t}\} = \sin \omega_0 t$



$e^{jx} = \cos x + j \sin x$

Exponential/Sinusoidal Signals

- Harmonically related signal sets

$$\{\phi_k(t) = e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2, \ldots\}$$

fundamental period $T_1 = \dfrac{2\pi}{|\omega_0|}$

Harmonical period $T_k = \dfrac{2\pi}{|k\omega_0|}$

fundamental frequency $|\omega_0|$

Harmonical frequency $|k\omega_0|$

Exponential/Sinusoidal Signals

- Sinusoidal signal

$$x(t) = A\cos(\omega_0 t + \phi) = \mathrm{Re}\{Ae^{j(\omega_0 t + \phi)}\}$$

- General format

$$x(t) = Ce^{at} = |C|e^{j\theta} \cdot e^{(r+j\omega_0)t} = |C|e^{rt} \cdot e^{j(\omega_0 t + \theta)}$$

- Discrete-Time

$$x[n] = e^{j\omega_0 n}, \omega_0 : rad$$

$$x[n] = A\cos(\omega_0 n + \phi)$$

$$x[n] = Ce^{\beta n}$$

Exponential/Sinusoidal Signals

- Important Differences Between Continuous-time and Discrete-time Exponential/Sinusoidal Signals

  重要！

  – For discrete-time, signals with frequencies ω_0 and $\omega_0 + m \cdot 2\pi$ are identical, where m is an integer. This is not true for continuous-time signal.

Discrete-time:

$$e^{j(\omega_0 + m \cdot 2\pi)n} = e^{j\omega_0 n}$$

Continuous-time:

$$e^{j(\omega_0 + m \cdot 2\pi)t} \neq e^{j\omega_0 t}$$

Continuous/Discrete Sinusoidals

重要！

Discrete-time:

$$e^{j(\omega_0+m\cdot 2\pi)n} = \cos(\omega_0 n + m\cdot 2\pi\, n) + j\sin(\omega_0 n + m\cdot 2\pi\, n)$$

$$= \cos(\omega_0 n) + j\sin(\omega_0 n) \ \ \text{(as } m.n \text{ is an integer)}$$

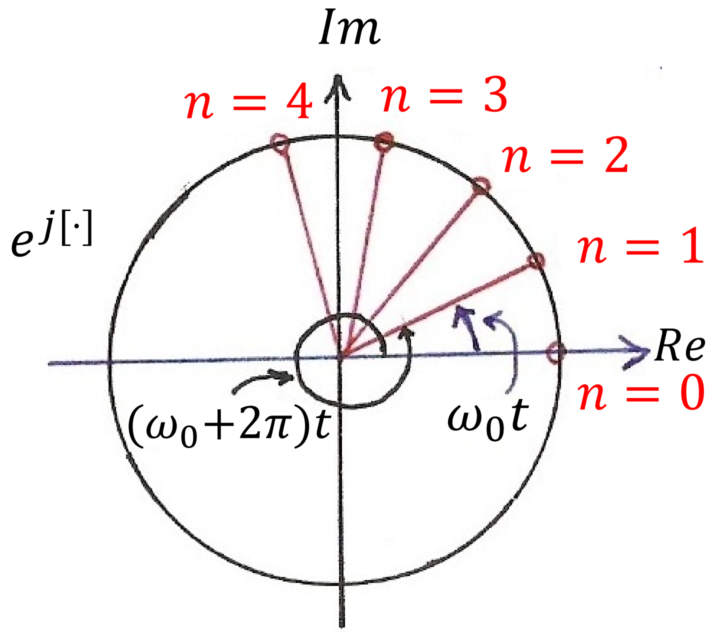
$$= e^{j\omega_0 n}$$

Continuous-time:

$$e^{j(\omega_0+m\cdot 2\pi)t} = \cos(\omega_0 t + m\cdot 2\pi\, t) + j\sin(\omega_0 t + m\cdot 2\pi\, t)$$

$$\neq \cos(\omega_0 t) + j\sin(\omega_0 t) \ \ \text{(as } m.t \text{ may not be an integer)}$$

$$= e^{j\omega_0 t}$$



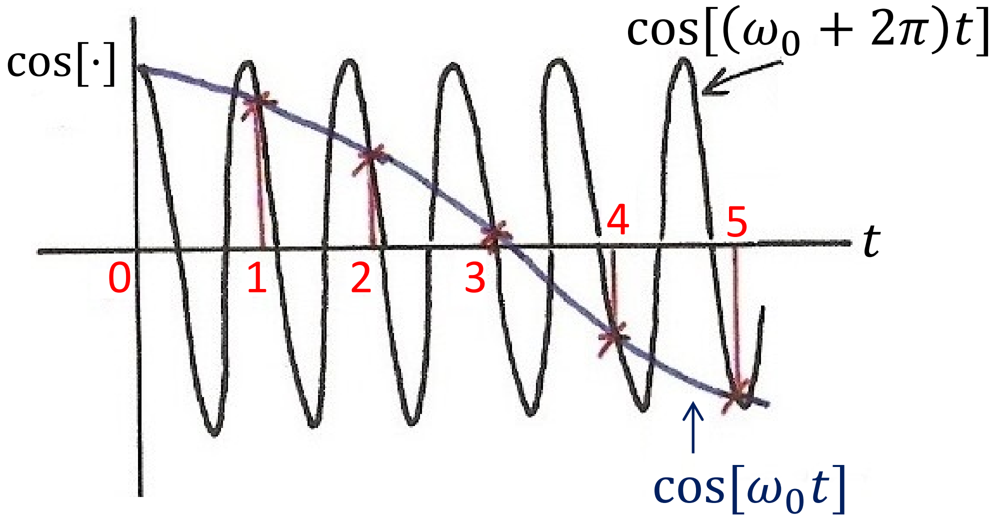
Continuous/Discrete Sinusoidals

Continuous-time:

$$e^{j(\omega_0+m\cdot 2\pi)t} = \cos(\omega_0 t + m\cdot 2\pi\, t) + j\sin(\omega_0 t + m\cdot 2\pi\, t)$$

$$\neq \cos(\omega_0 t) + j\sin(\omega_0 t) \ \ \text{(as } m.t \text{ may not be an integer)}$$

$$= e^{j\omega_0 t}$$



$$\cos\omega_0 t \neq \cos(\omega_0+2\pi)t$$

$$e^{j\omega_0 t} \neq e^{j(\omega_0+2\pi)t}$$

Continuous/Discrete Sinusoidals

Discrete-time:

$$e^{j(\omega_0+m\cdot 2\pi)n}$$

$$= \cos(\omega_0 n + m\cdot 2\pi\, n)$$
$$+ j\sin(\omega_0 n + m\cdot 2\pi\, n)$$

$$= \cos(\omega_0 n) + j\sin(\omega_0 n)$$

$$= e^{j\omega_0 n}$$

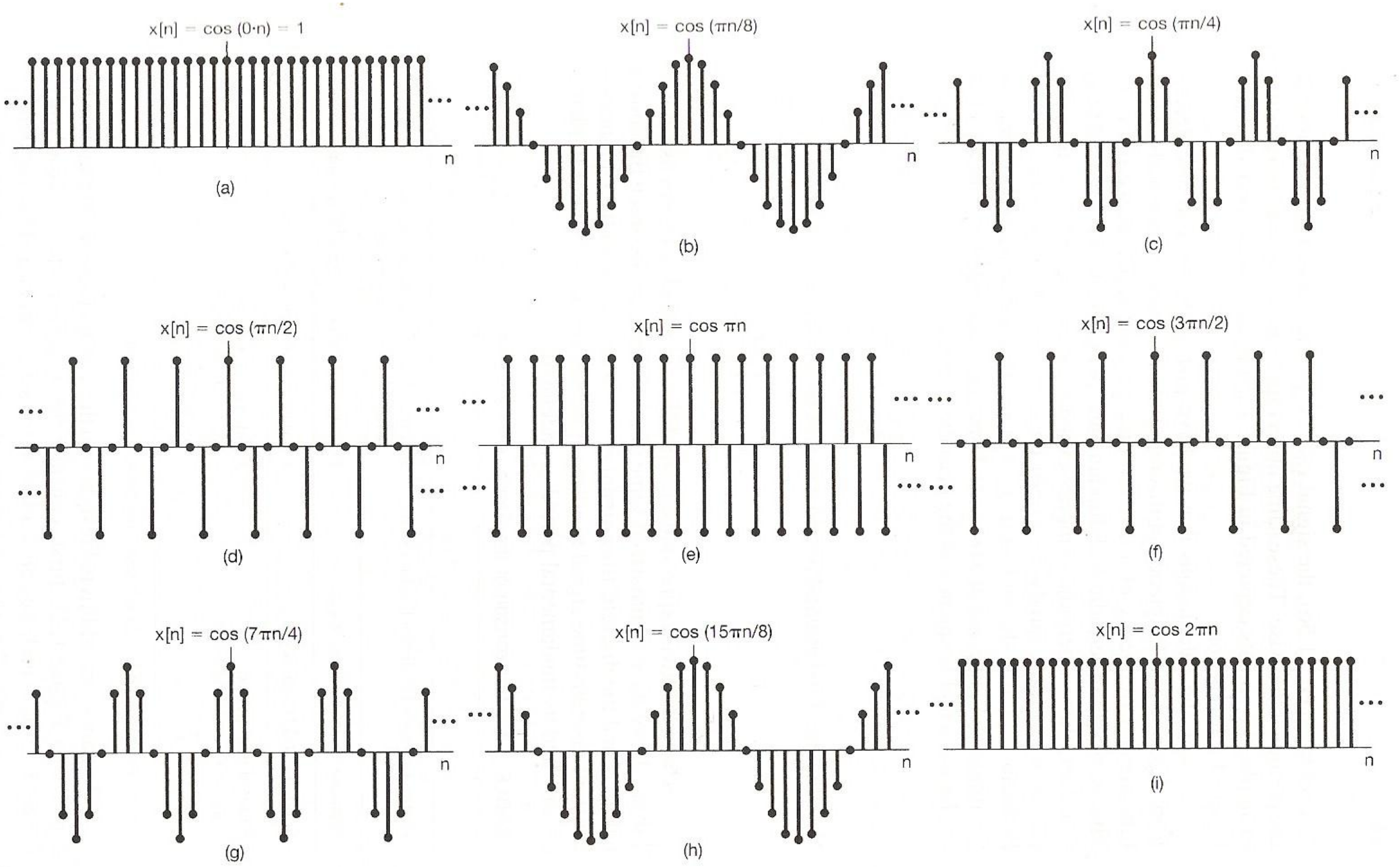


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

Exponential/Sinusoidal Signals

- Important Differences Between Continuous-time and Discrete-time Exponential/Sinusoidal Signals
  - For discrete-time, ω_0 is usually defined only for $[-\pi, \pi]$ or $[0, 2\pi]$. For continuous-time, ω_0 is defined for $(-\infty, \infty)$
  - For discrete-time, the signal is periodic only when

$$\omega_0 N = 2\pi m, \quad \omega_0 = \left(\frac{2\pi}{N}\right)m = 2\pi\left(\frac{m}{N}\right)$$

Exponential/Sinusoidal Signals

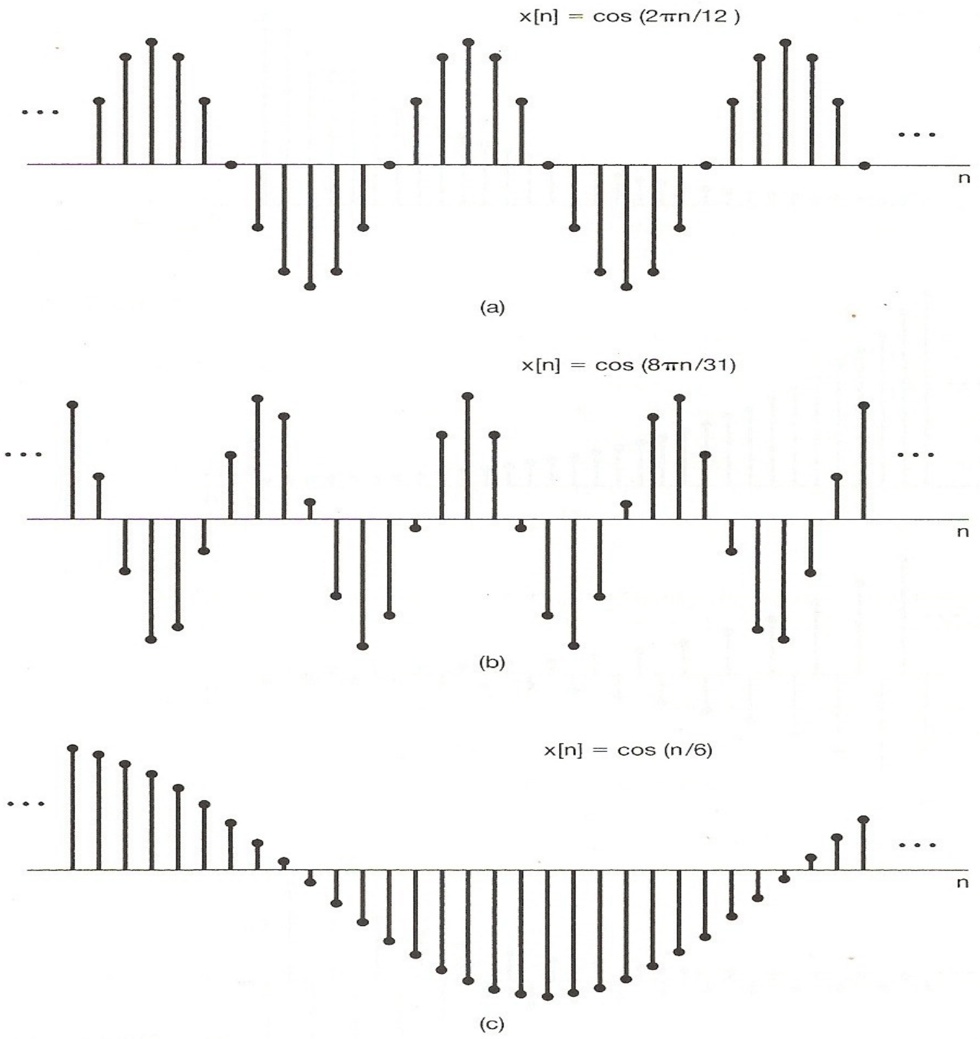


Figure 1.25 Discrete-time sinusoidal signals.

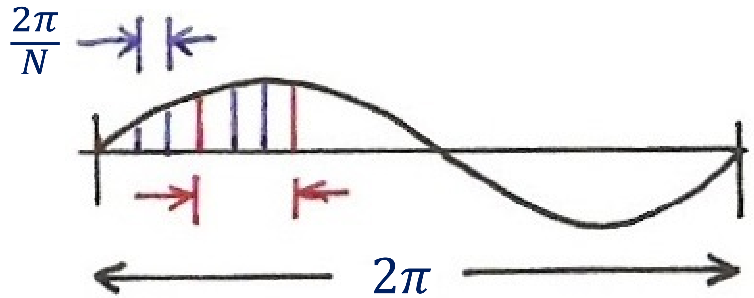
Harmonically Related Signal Sets

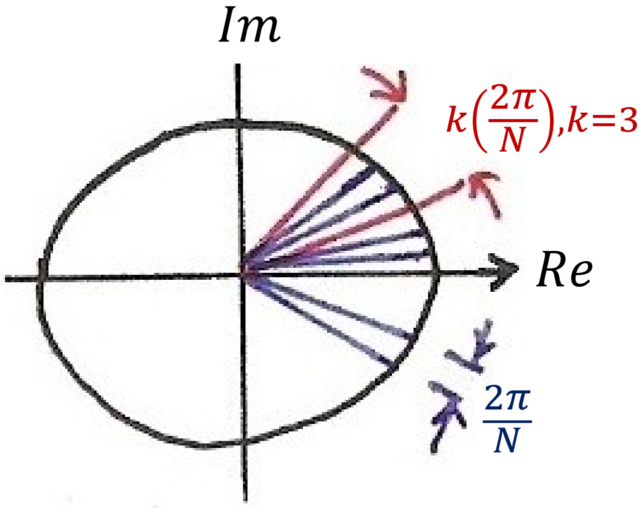
For being periodic

real period in cycles

$$\omega_k \cdot \underline{\underline{N}} = 2\pi \cdot \overline{\overline{k}} \quad \Longrightarrow \quad \omega_k = k\left(\frac{2\pi}{N}\right)$$

real period in *n*





$$\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n} \quad \Longrightarrow \quad \phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n} = \phi_k[n]$$

Exponential/Sinusoidal Signals

- Harmonically related discrete-time signal sets

$$\{\phi_k[n] = e^{jk(\frac{2\pi}{N})n} \quad , \qquad \mathrm{k} = 0, \pm 1, \pm 2, \ldots\ldots\}$$

all with common period N

$$\phi_{k+N}[n] = \phi_k[n]$$

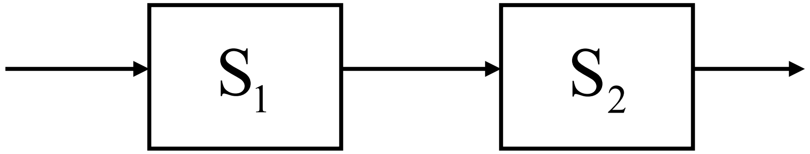
This is different from continuous case. Only N distinct signals in this set.

Fundamental Concepts (Systems)

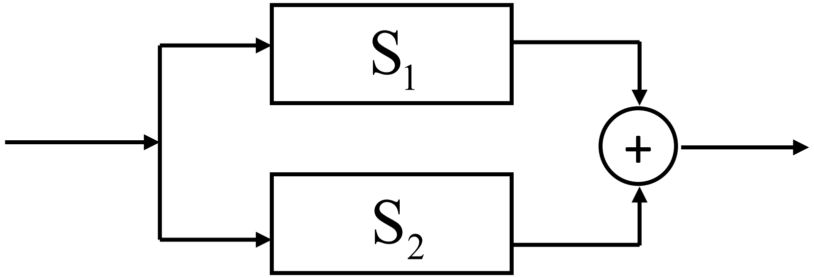
- Continuous/Discrete-time Systems

- Interconnections of Systems

  – Series

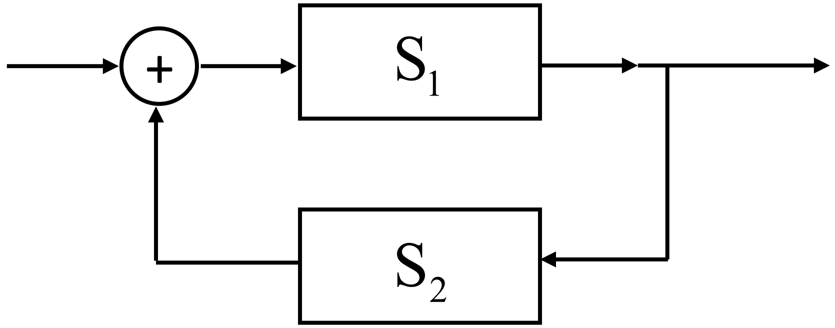


  – Parallel



Fundamental Concepts (Systems)

- Interconnections of Systems
  - –Feedback



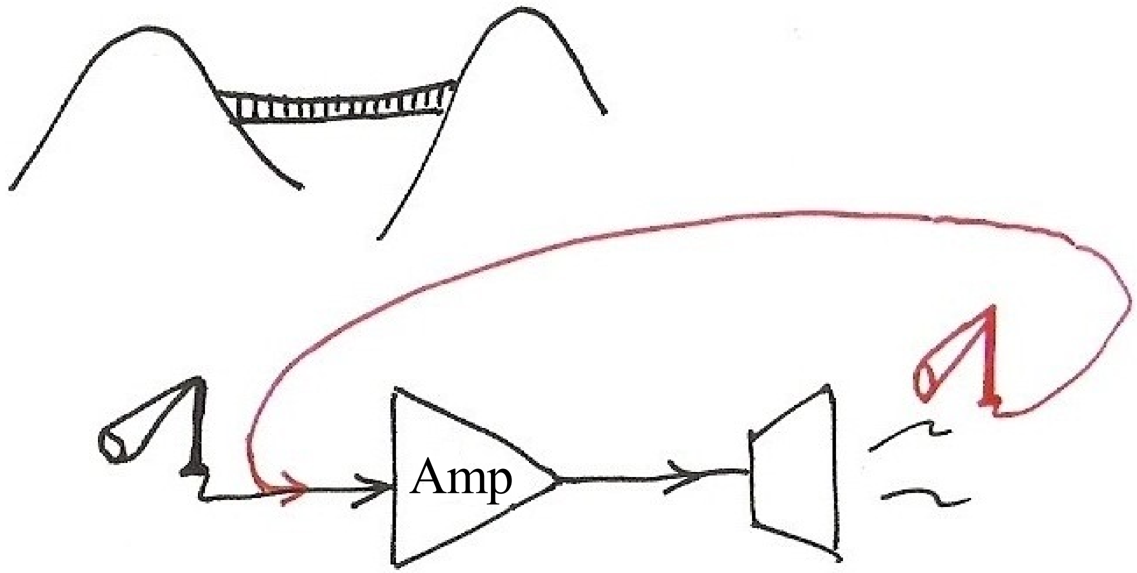
  - –Combinations

Fundamental Concepts (Systems)

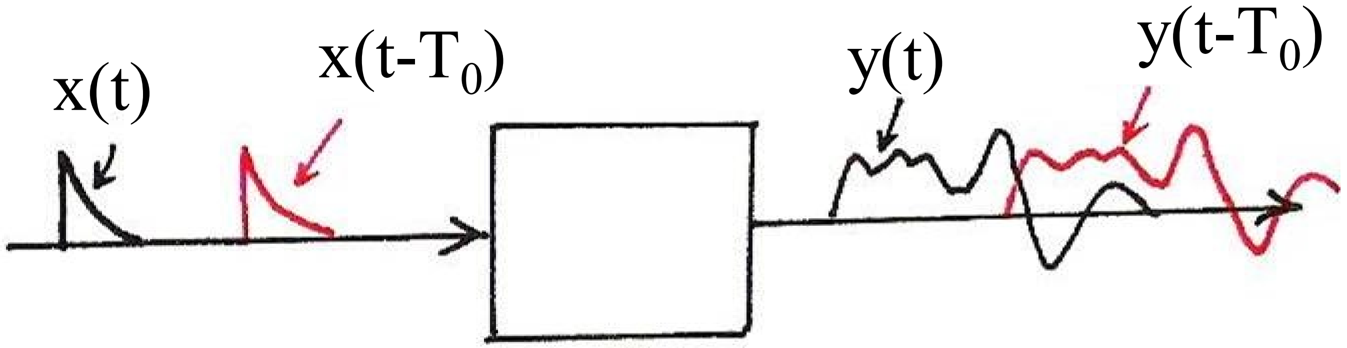
- Stability
  - stable : bounded inputs lead to bounded outputs
- Time Invariance
  - time invariant : behavior and characteristic of the system are fixed over time

Stability

Examples of unstable systems



Time Invariance



Stability

- Linearity
  - linear : superposition property
  - scaling or homogeneity property
  - additive property

$$x_k[n] \to y_k[n]$$

$$\sum_k a_k x_k[n] \to \sum_k a_k y_k[n]$$

$$x[n] \to y[n]$$

$$ax[n] \to ay[n]$$

$$x_i[n] \to y_i[n]$$

$$x_1[n] + x_2[n] \to y_1[n] + y_2[n]$$

Stability

- Memoryless/With Memory
  - Memoryless : output at a given time depends only on the input at the same time

    eg. $$y[n] = (ax[n] - x^2[n])^2$$

  - With Memory

    eg. $$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Stability

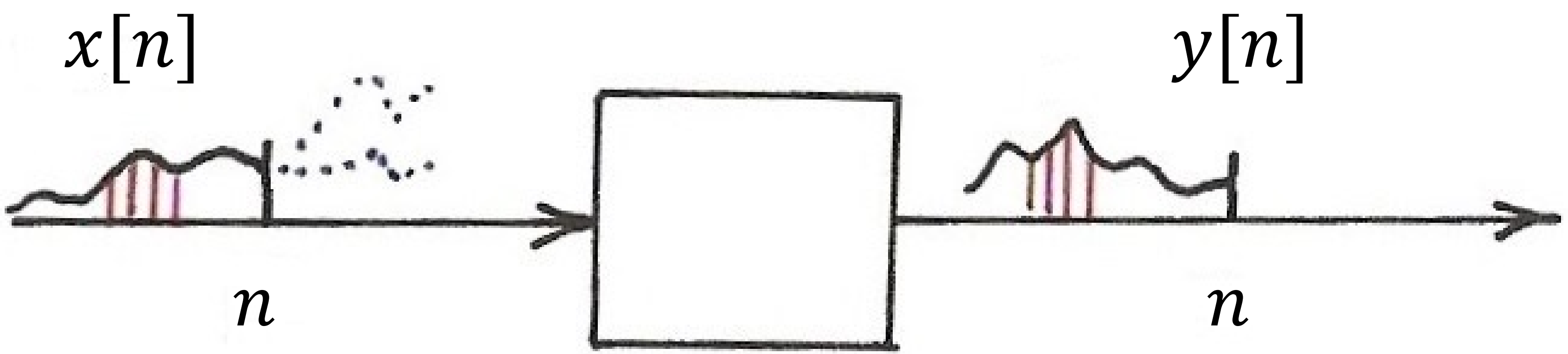
- Invertibility
  - invertible : distinct inputs lead to distinct outputs, i.e. an inverse system exits

    eg.
$$y[n] = \sum_{k=-\infty}^{n} x[k]$$
$$z[n] = y[n] - y[n-1]$$

- Causality
  - causal : output at any time depends on input at the same time and in the past

    eg.
$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Causality



$$y[n] = \sum_{k=-\infty}^{n+m} x[k]$$

本章部分关键词汇中英文对照表

复数	Complex number
实数	Real number
虚数	Imaginary number
直角坐标系	Cartesian coordinate
极坐标系	Polar coordinate
欧拉公式	Euler’s formula
泰勒展开式	Taylor expansion
基波信号	Fundamental signal
谐波信号	Harmonic signal

Thank you for your listening!