Lecture 8 Discrete-time Fourier Transform

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Outline: **Lecture 8:** Discrete-time Fourier Transform

- Discrete-time Fourier Transform
    - Fourier Transform Representation
    - Fourier Transform of Periodic Signal
- Fourier Transform Properties
- Examples
- Duality

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Discrete-time Fourier Transform

Periodicity

Continuity	**Periodic**	**Aperiodic**
Continuous	Fourier series $x(t) \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x(t) \overset{F}{\leftrightarrow} X(j\omega)$
Discrete	Fourier series $x[n] \overset{FS}{\leftrightarrow} a_k$	???

Discrete-time Fourier Transform

Periodicity

Continuity		**Periodic**	**Aperiodic**
	Continuous	Fourier series $x(t) \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x(t) \overset{F}{\leftrightarrow} X(j\omega)$
	Discrete	Fourier series $x[n] \overset{FS}{\leftrightarrow} a_k$	**???**

Periodic time-domain signal↔Discrete frequency-domain signal

Discrete-time Fourier Transform

Periodicity

Continuity	**Periodic**	**Aperiodic**
Continuous	Fourier series $x(t) \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x(t) \overset{F}{\leftrightarrow} X(j\omega)$
Discrete	Fourier series $x[n] \overset{FS}{\leftrightarrow} a_k$	???

Continuous time-domain signal↔Aperiodic frequency-domain signal

Discrete-time Fourier Transform

Periodicity

	Periodic	**Aperiodic**
Continuity: Continuous	Fourier series $x(t) \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x(t) \overset{F}{\leftrightarrow} X(j\omega)$
Continuity: Discrete	Fourier series $x[n] \overset{FS}{\leftrightarrow} a_k$	Continuous and periodic frequency-domain signal?

Discrete-time Fourier Transform

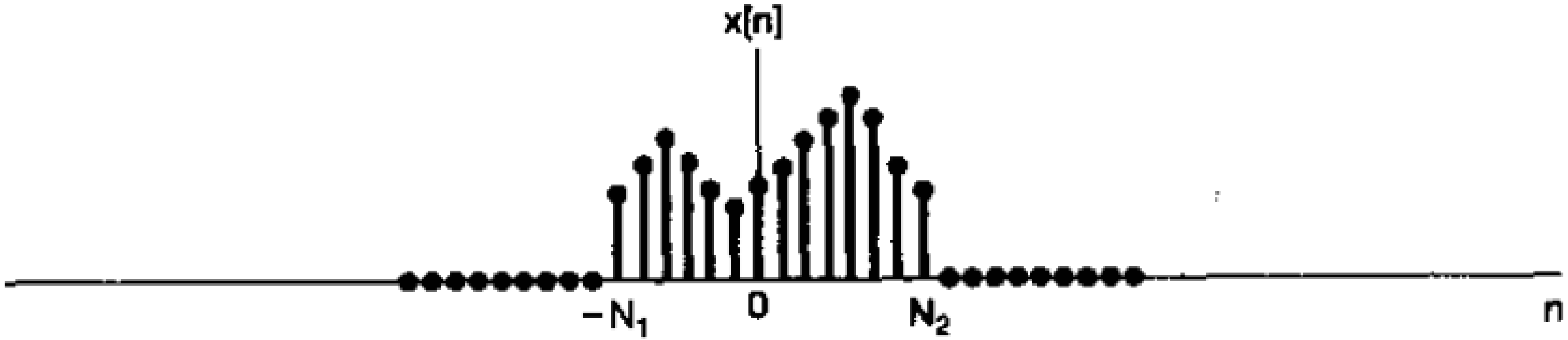
- In the previous lecture, we show a discrete-time periodic signal can be represented by a Fourier series, producing the spectral in frequency domain.

- An aperiodic signal can be considered as a periodic signal, the period of which is extremely large, *e.g.*, infinity.

Fourier Transform Representation

From Fourier series to Fourier transform

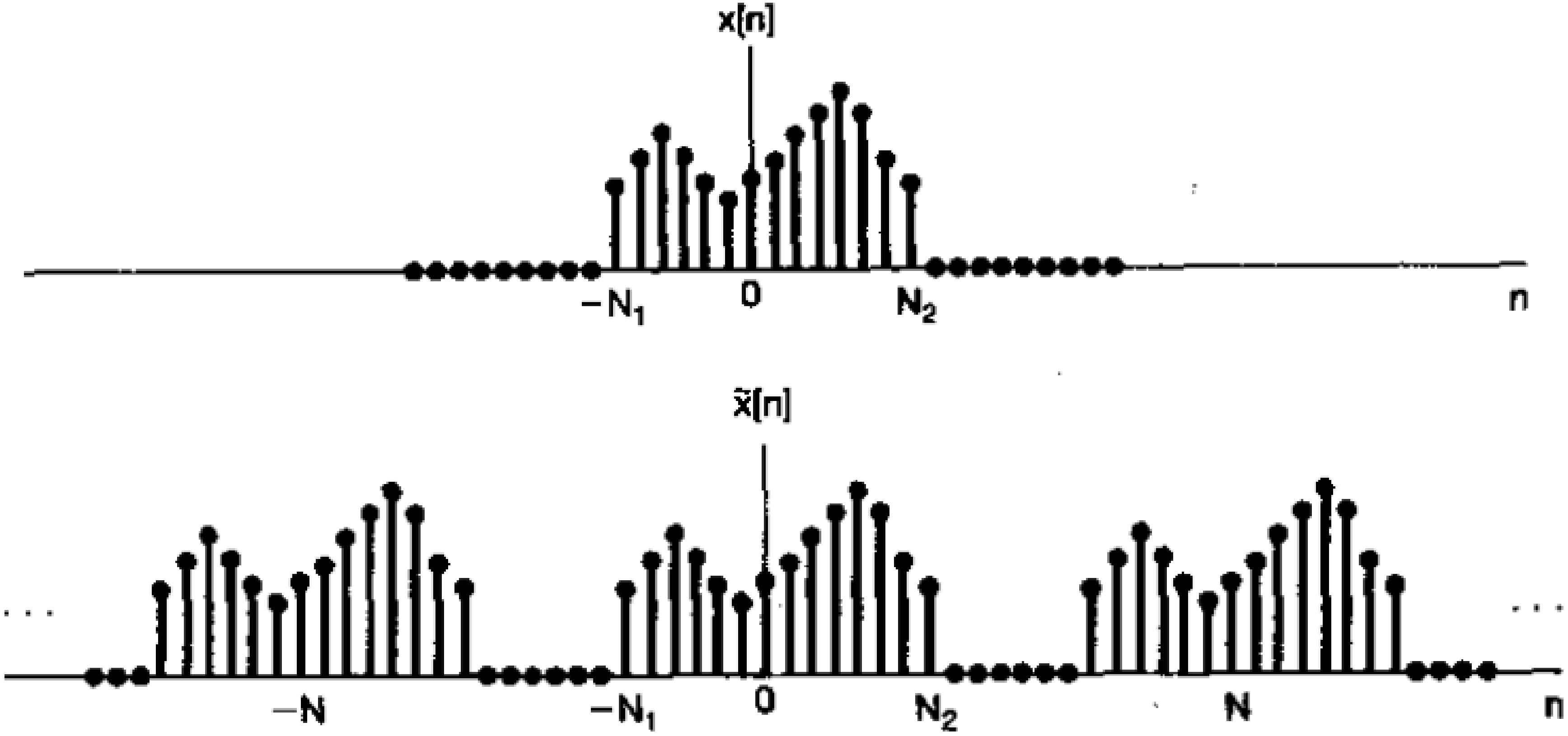
- The following signal, $x[n]$, is an aperiodic signal:



Fourier Transform Representation

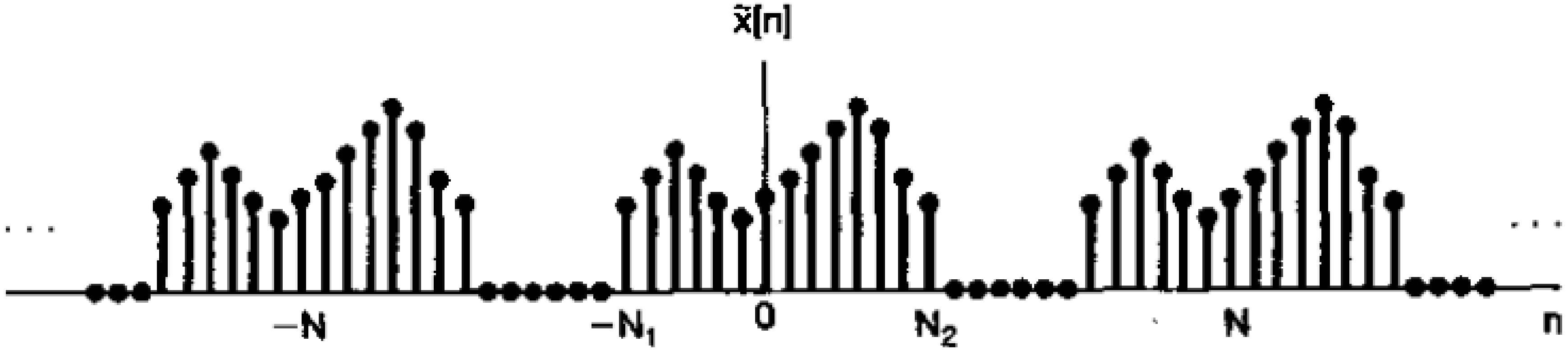
From Fourier series to Fourier transform

- Assume $x[n]$ is repeated for every period of N ($\mathrm{N} \rightarrow \infty$), as $\tilde{x}[n]$



Fourier Transform Representation

From Fourier series to Fourier transform



- Fourier series representation:

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jk\omega_0 n}$$

Fourier Transform Representation

From Fourier series to Fourier transform

- In the time of $[-N_1, N_2]$, $\tilde{x}[n] = x[n]$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$$

$$N a_k = \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$$

Fourier Transform Representation

From Fourier series to Fourier transform

$$Na_k = \sum_{n=-\infty}^{\infty} x\,[n]e^{-jk\omega_0 n}$$

- Define $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x\,[n]e^{-j\omega n}$

$$Na_k = X(e^{j\omega})|_{\omega=k\omega_0} = \sum_{n=-\infty}^{\infty} x\,[n]e^{-j\omega n}\,|_{\omega=k\omega_0}$$

Fourier Transform Representation

From Fourier series to Fourier transform

$$Na_k = X(e^{j\omega})|_{\omega=k\omega_0}$$

- $N \to \infty, \omega_0 \to 0$:

$$Na_k = X(e^{j\omega})|_{\omega=k\omega_0} = X(e^{j\omega})$$

- For aperiodic signals, Fourier series (spectral) can be approximated by a continuous function:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Fourier Transform Representation

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}, \qquad N a_k = X\left(e^{j\omega}\right)\Big|_{\omega=k\omega_0}$$

$$\tilde{x}(t) = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

Fourier Transform Representation

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

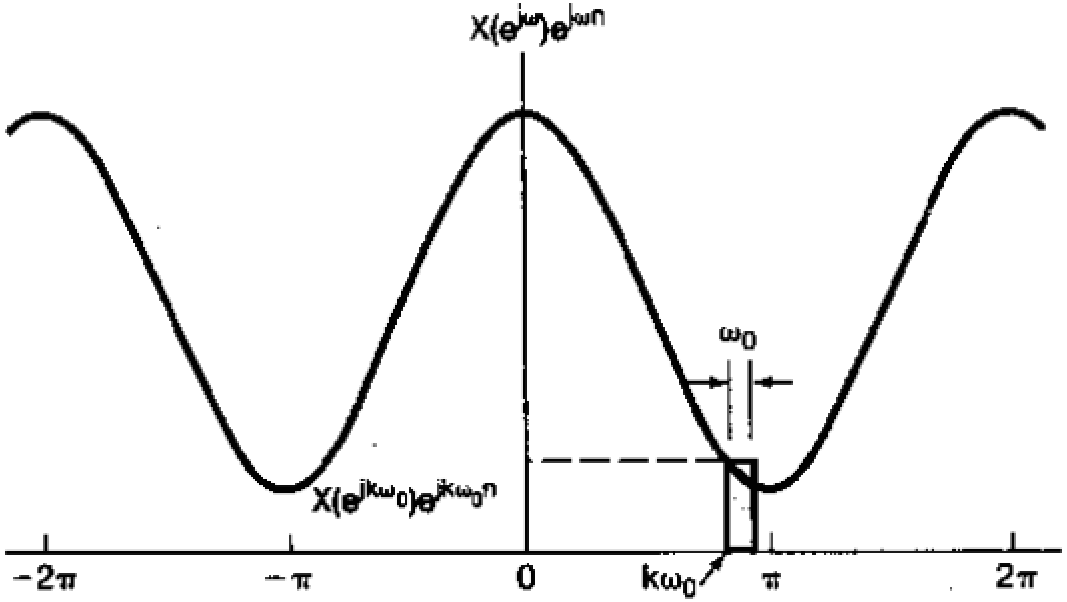
- In the time of $[0, N-1]$ ($N \to \infty$: $[0, \infty]$), $\tilde{x}[n] = x[n]$

$$x[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

Fourier Transform Representation

$$x[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

$X\left(e^{j\omega}\right)$ is periodic with period 2π



$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

Fourier Transform Representation

Fourier transform (spectral):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Inverse Fourier transform (signal representation):

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

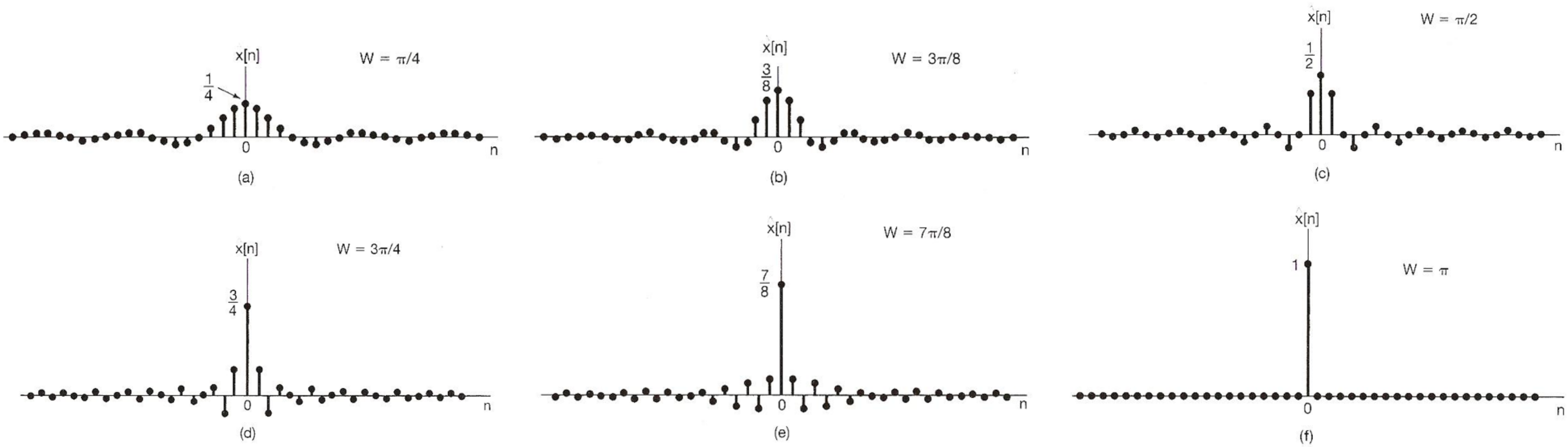
There is no convergence issue for discrete-time Fourier transform!

Fourier Transform Representation

Example 5.4, p.260 of textbook

$$x[n] = \delta[n], X(e^{j\omega}) = 1$$

- Assume $\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n} d\omega = \frac{sinWn}{\pi n}$, and increase W from 0 to π:



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Fourier Transform of Periodic Signal

- For a periodic signal, we have the Fourier series representation:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

- Let us firstly consider the Fourier transform for a complex exponential: $e^{jk\omega_0 n}$.

Fourier Transform of Periodic Signal

- For a continuous-time signal, $e^{j\omega_0 t}$, the Fourier transform is:

$$e^{j\omega_0 t} \overset{F}{\leftrightarrow} 2\pi\delta(\omega - \omega_0)$$

- Is the above transform pair valid for discrete-time signal? But it is not periodic:

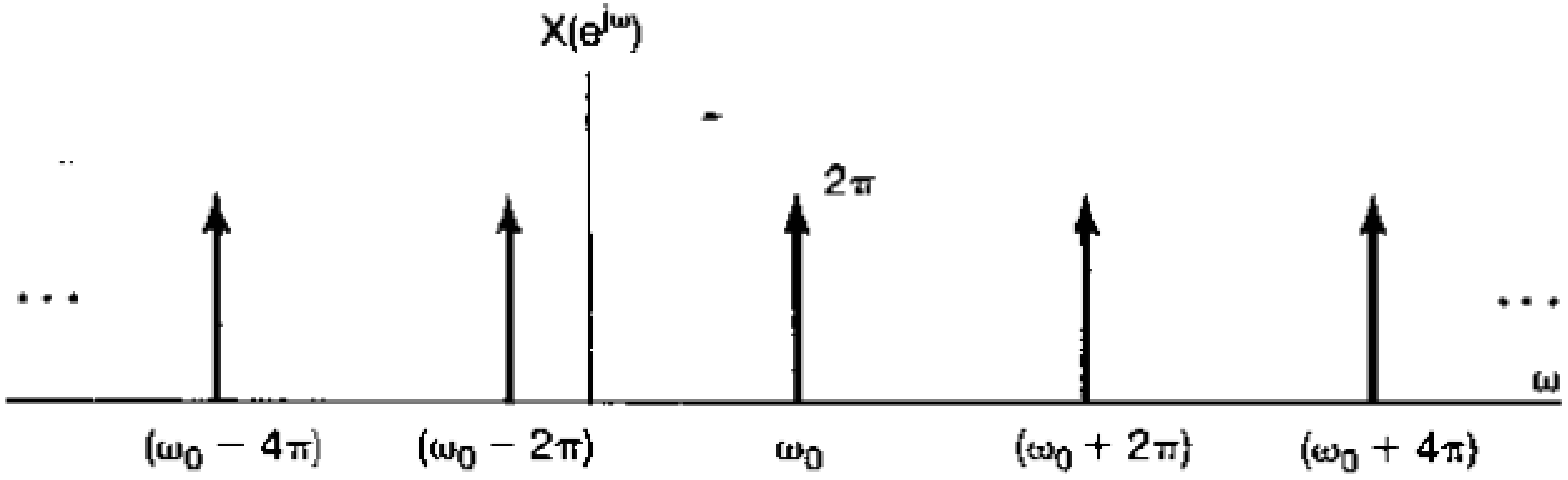
$$e^{j\omega_0 n} \overset{F}{\leftrightarrow} 2\pi\delta(\omega - \omega_0)$$

- Put in some terms to force it having a period of 2π:

$$e^{j\omega_0 n} \overset{F}{\leftrightarrow} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

Fourier Transform of Periodic Signal

$$x[n] = e^{j\omega_0 n} \overset{F}{\leftrightarrow} X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$



Fourier Transform of Periodic Signal

$$x[n] = e^{j\omega_0 n} \overset{F}{\leftrightarrow} X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

Proof:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)\, e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} 2\pi\delta(\omega - \omega_0)e^{j\omega n}d\omega = e^{j\omega_0 n}$$

Fourier Transform of Periodic Signal

$$e^{j\omega_0 n} \overset{F}{\leftrightarrow} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

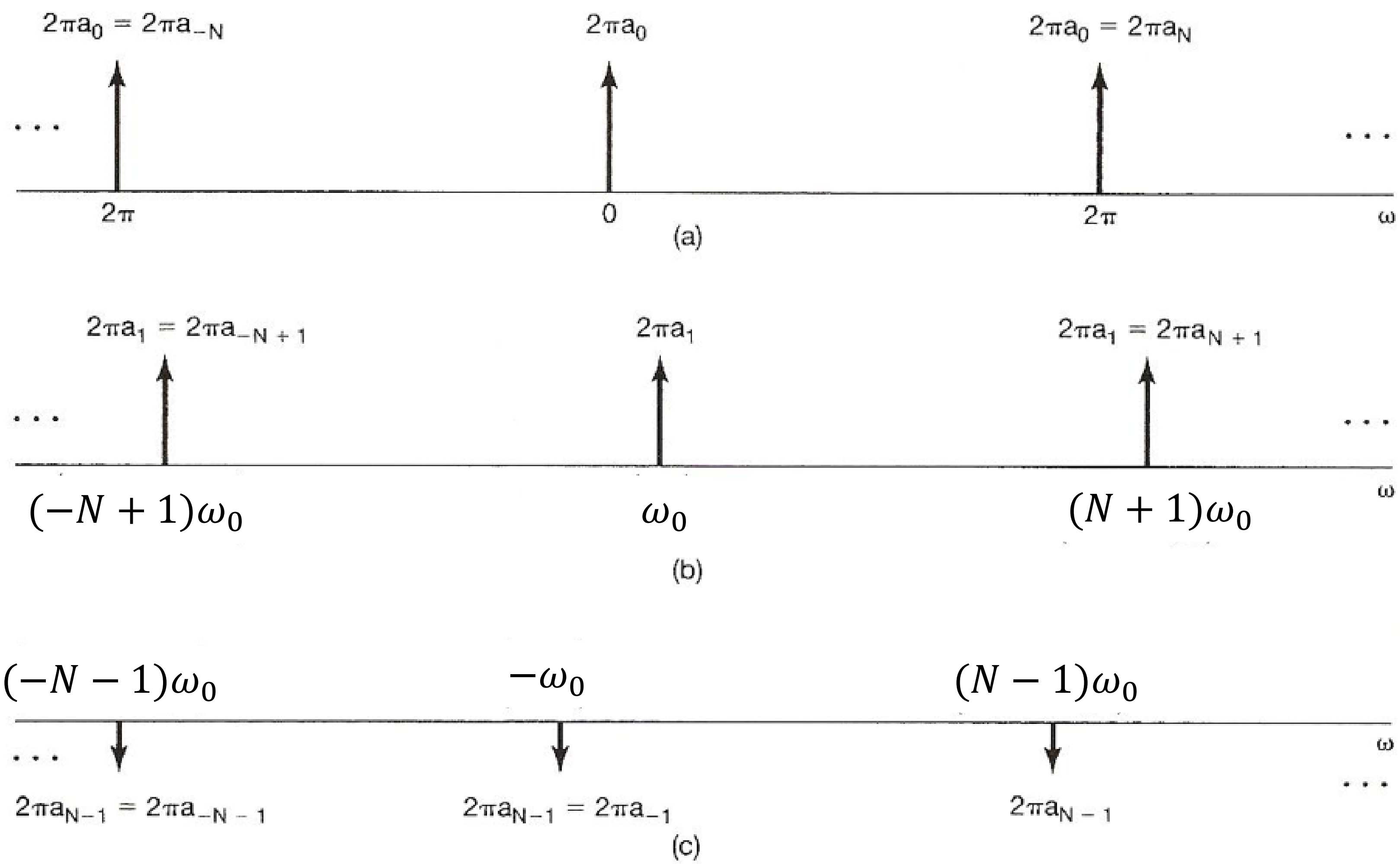
- For periodic signal, we have the Fourier series representation:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

$$X\left(e^{j\omega}\right) = \sum_{k=\langle N\rangle} a_k \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0 - 2\pi l)$$

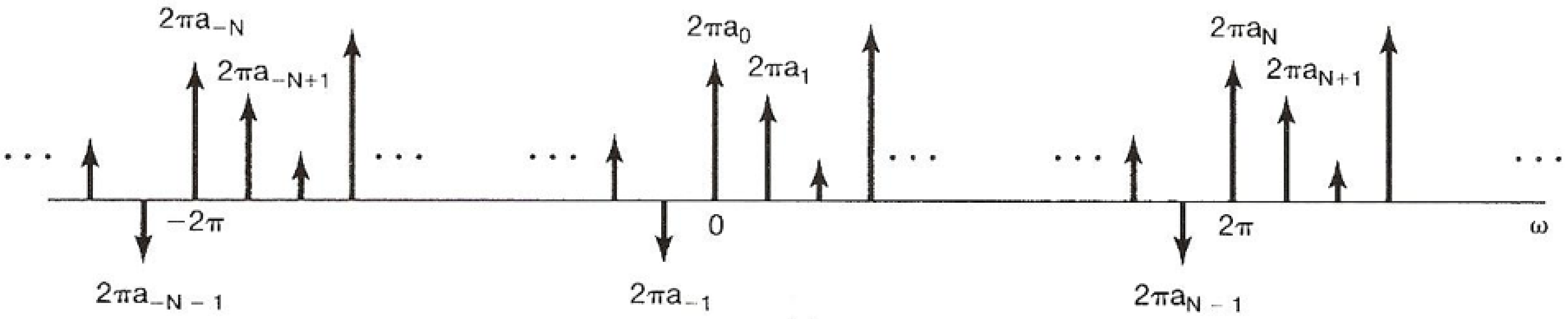
Fourier Transform of Periodic Signal

$$X(e^{j\omega}) = \sum_{k=\langle N\rangle} a_k \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0 - 2\pi l)$$



Fourier Transform of Periodic Signal

$$X(e^{j\omega}) = \sum_{k=\langle N\rangle} a_k \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0 - 2\pi l)$$



(d)

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Fourier Transform of Periodic Signal

- Fourier series representation of periodic signal:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

- Fourier transform representation of periodic signal:

$$x(t) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi}\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)\, e^{j\omega n}d\omega$$

$$= \sum_{k=\langle N\rangle} a_k \delta(\omega - k\omega_0)\, e^{j\omega n} = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

Fourier Transform of Periodic Signal

- Fourier transform can be performed for periodic signals by using impulse function. The spectral is the same as Fourier series.

- In such a case, we have a unified framework of Fourier transform for both periodic and aperiodic signals.

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Fourier Transform Properties

- The following notation is used to indicate a pair of Fourier transform:

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

Fourier Transform Properties

Linearity

Assume :

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega}), \mathrm{y}[n] \overset{F}{\leftrightarrow} Y(e^{j\omega})$$

Then:

$$\mathrm{ax}[n] + by[n] \overset{F}{\leftrightarrow} aX(e^{j\omega}) + bY(e^{j\omega})$$

Fourier Transform Properties

Time shift

Assume :

$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\omega}\right), \mathrm{y}[n] = x[n-n_0] \overset{F}{\leftrightarrow} Y(e^{j\omega})$$

Then:

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x\,[n-n_0]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x\,[n]e^{-j\omega n}e^{-j\omega n_0} = X\left(e^{j\omega}\right)e^{-j\omega n_0}$$

- Time shift leads to unchanged amplitude and shifted phase.

Fourier Transform Properties

Frequency shift

Assume :

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega}), \mathrm{y}[n] \overset{F}{\leftrightarrow} Y(e^{j\omega}) = X(e^{j(\omega-\omega_0)})$$

Then:

$$y[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j(\omega-\omega_0)})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega\, e^{j\omega_0 n} = x[n]e^{j\omega_0 n}$$

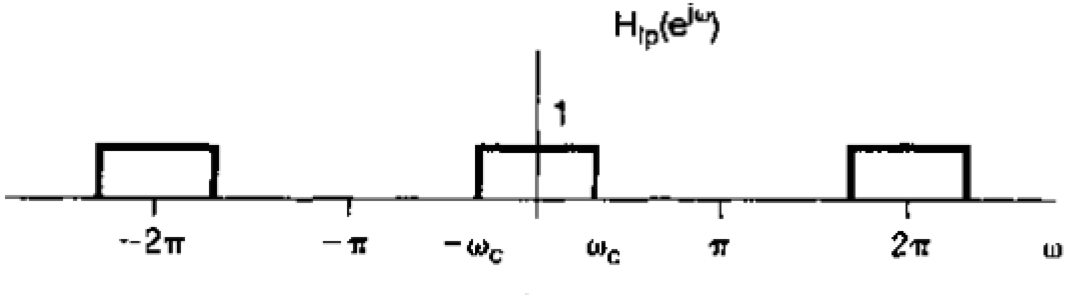
- Frequency shift leads to unchanged amplitude and shifted phase.

Fourier Transform Properties

Frequency shift

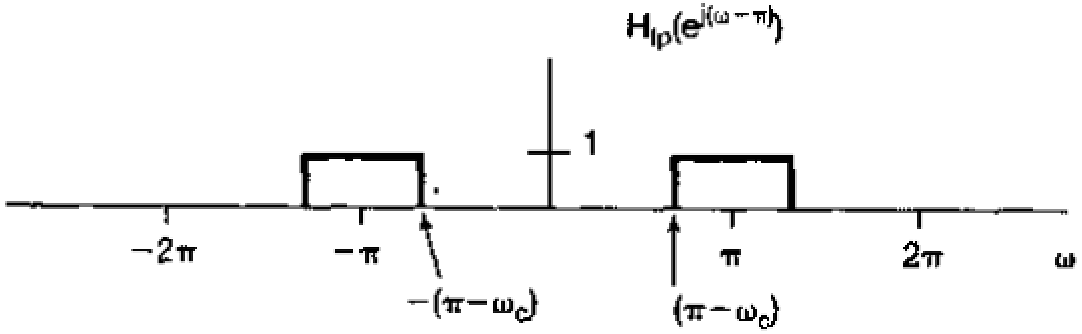
- Relationship between low-pass and high-pass filter:

Low-pass filter:



(a)

High-pass filter:



(b)

$$H_{hp}\left(e^{j\omega}\right) = H_{lp}\left(e^{j(\omega-\pi)}\right)$$

$$h_{hp}[n] = h_{lp}[n]e^{j\pi n} = (-1)^n h_{lp}[n]$$

Fourier Transform Properties

Conjugation

Assume:

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

Then:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$x^*[n] = \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega})(e^{j\omega n})^*d\omega = \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega})e^{-j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{-j\omega})e^{j\omega n}d\omega$$

$$x^*[n] \overset{F}{\leftrightarrow} X^*(e^{-j\omega})$$

Fourier Transform Properties

Conjugation

Even/Odd property

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega}), x^*[n] \overset{F}{\leftrightarrow} X^*(e^{-j\omega})$$

- If $x[n]$ is real ($x[n] = x^*[n]$):

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

Fourier Transform Properties

Conjugation

Even/Odd property

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega}), x^*[n] \overset{F}{\leftrightarrow} X^*(e^{-j\omega})$$

- If $x[n]$ is real and even ($x[n] = x^*[n], x[n] = x[-n]$):

$$X(e^{j\omega}) = X^*(e^{-j\omega}), X(e^{j\omega}) = X(e^{-j\omega})$$

$$\Rightarrow X^*(e^{-j\omega}) = X(e^{-j\omega})$$

$$\Rightarrow X^*(e^{j\omega}) = X(e^{j\omega})$$

- $X(e^{j\omega})$ is real and even

Fourier Transform Properties

Conjugation

Even/Odd property

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega}), x^*[n] \overset{F}{\leftrightarrow} X^*(e^{-j\omega})$$

- If $x[n]$ is real and odd ($x[n] = x^*[n], x[n] = -x[-n]$):

$$X(e^{j\omega}) = X^*(e^{-j\omega}), X(e^{j\omega}) = -X(e^{-j\omega})$$

$$\Rightarrow X^*(e^{-j\omega}) = -X(e^{-j\omega})$$

$$\Rightarrow X^*(e^{j\omega}) = -X(e^{j\omega})$$

- $X(e^{j\omega})$ is imaginary and odd

Fourier Transform Properties

Conjugation

- Any signal can be discomposed into a sum of an even and an odd

$$Ev\{x[n]\} = x_1[n] = \frac{1}{2}[x[n] + x[-n]]$$

$$Od\{x[n]\} = x_2[n] = \frac{1}{2}[x[n] - x[-n]]$$

- If $x[n]$ is real:

$$Ev\{x[n]\} \overset{F}{\leftrightarrow} \mathbb{R}\{X(e^{j\omega})\}$$

$$Od\{x[n]\} \overset{F}{\leftrightarrow} \mathbb{I}\{X(e^{j\omega})\}$$

Fourier Transform Properties

Differencing

Assume:

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

Then:

$$x[n] - x[n-1] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega - \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega(n-1)}d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})(e^{j\omega n} - e^{j\omega n}e^{-j\omega})d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})(1 - e^{-j\omega})e^{j\omega n}d\omega$$

Fourier Transform Properties

Accumulation

Assume:

$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\omega}\right)$$

Then:

$$\sum_{m=-\infty}^{n} x[m] \overset{F}{\leftrightarrow} \frac{1}{1-e^{-j\omega}} X\left(e^{j\omega}\right) + \pi X\left(e^{j0}\right) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Fourier Transform Properties

Time reversal

Assume :

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

Then:

$$x[-n] \overset{F}{\leftrightarrow} X(e^{-j\omega})$$

Fourier Transform Properties

Time expansion

Assume :

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega}), x_{(k)}[n] \overset{F}{\leftrightarrow} X_{(k)}(e^{j\omega}),$$

$$x_{(k)}[n] = \begin{cases} x[n/k], if\ n\ is\ a\ multiple\ of\ k \\ 0, \qquad\qquad otherwise \end{cases}$$

Then:

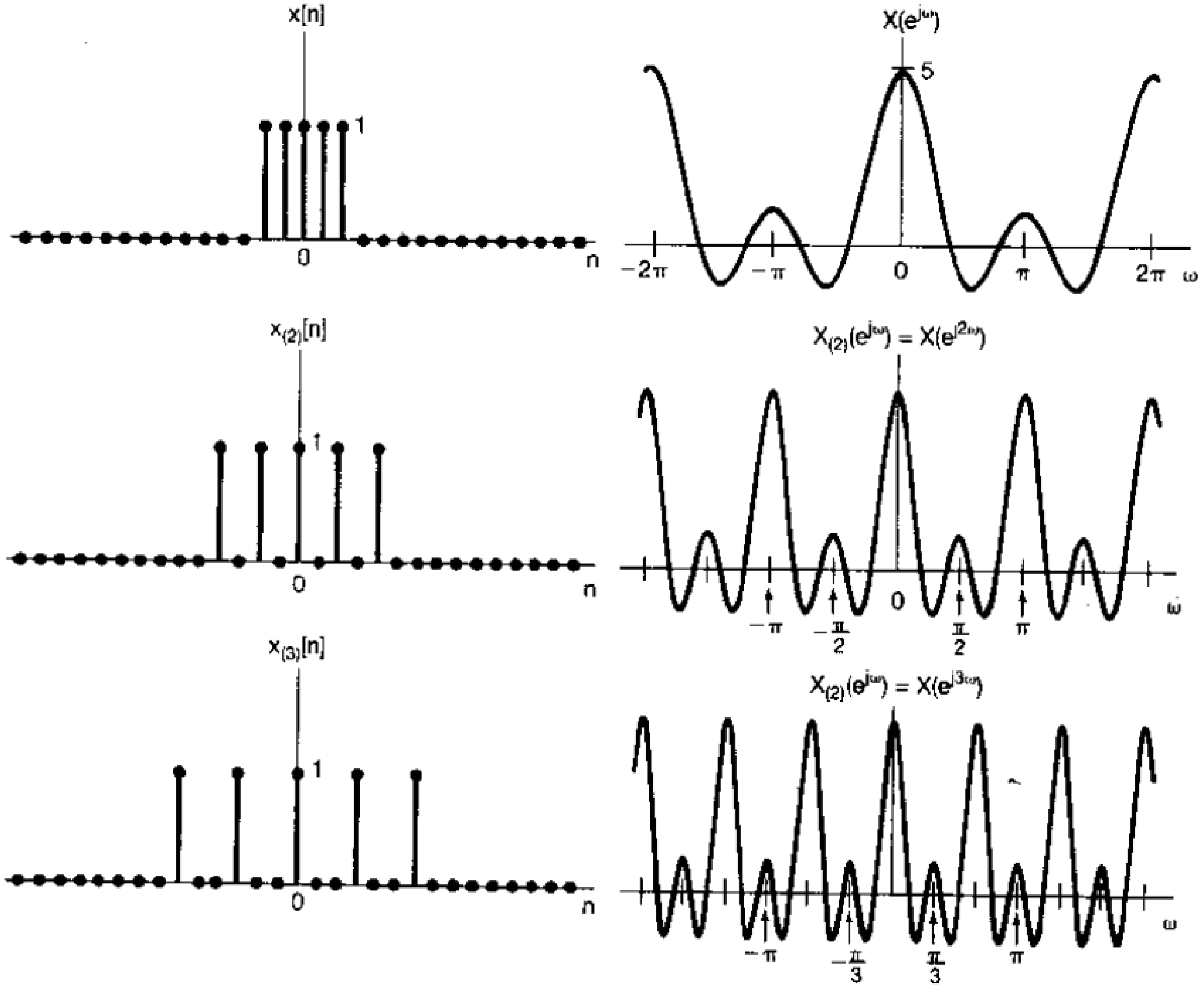
$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{n/k=-\infty}^{\infty} x\left[\frac{n}{k}\right]e^{-j\omega\frac{n}{k}k}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-jk\omega n} = X(e^{jk\omega})$$

Fourier Transform Properties

Time expansion

As k increases, $x_{(k)}[n]$ spreads out, while its transform is compressed.



Fourier Transform Properties

Differentiation in frequency domain

Assume :

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

Then:

$$\frac{dX(e^{j\omega})}{d\omega} = \frac{d\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}}{d\omega} = \sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\omega n}$$

$$-jnx[n] \overset{F}{\leftrightarrow} \frac{dX(e^{j\omega})}{d\omega}, nx[n] \overset{F}{\leftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Fourier Transform Properties

Parseval's relation

- The sum of the square of a function is equal to the integral of the square of its Fourier transform in a period of 2π.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Fourier Transform Properties

Parseval's relation

Proof:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]x^*[n]$$

$$= \sum_{n=-\infty}^{\infty} x[n]\frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega})e^{-j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega})[\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}]\, d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega})X(e^{j\omega})d\omega = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Fourier Transform Properties

Convolution

Assume :

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega}), h[n] \overset{F}{\leftrightarrow} H(e^{j\omega})$$

Then:

$$x[n] * h[n] \overset{F}{\leftrightarrow} X(e^{j\omega})H(e^{j\omega})$$

Proof: similar to continuous-time transformation, p223 of textbook

Fourier Transform Properties

Multiplication

Assume :

$$x_1[n] \overset{F}{\leftrightarrow} X_1(e^{j\omega}), x_2[n] \overset{F}{\leftrightarrow} X_2(e^{j\omega})$$

Then:

$$x_1[n]x_2[n] \overset{F}{\leftrightarrow} \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

Periodic convolution

Fourier Transform Properties

Multiplication

$$x_1[n]x_2[n] \overset{F}{\leftrightarrow} \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

Proof:

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_2[n][\frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right)e^{j\theta n}d\theta]e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\theta}\right)\sum_{n=-\infty}^{\infty} x_2[n]e^{-j(\omega-\theta)n}\,d\theta = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

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Examples

- *Example 5.2, p.257 of textbook*

Find the spectral (Fourier transform) for the following signals: $x[n] = a^{|n|}, |a| < 1$

Examples

- *Example 5.2, p.257 of textbook*

Find the spectral (Fourier transform) for the following signal: $x[n] = a^{|n|}, |a| < 1$

$x[n] = a^{|n|}$ is the even part of $\mathrm{y}[n] = \begin{cases} 1, & n = 0 \\ 2a^n u[n], & \mathrm{n} \neq 0 \end{cases}$

According to conjugation property:

$$\mathrm{y}[n] = \begin{cases} 1, & n = 0 \\ 2a^n u[n], & \mathrm{n} \neq 0 \end{cases} \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$x[n] = a^{|n|} \overset{F}{\leftrightarrow} \mathbb{R}\{X(e^{j\omega})\}$$

Examples

- *Example 5.2, p.257 of textbook*

Find the spectral (Fourier transform) for the following signal: $x[n] = a^{|n|}, |a| < 1$

$$X(e^{j\omega}) = y[0]e^{-j\omega 0} + \sum_{n=1}^{\infty} y[n]e^{-j\omega n} = 1 + \sum_{n=1}^{\infty} 2a^n u[n]e^{-j\omega n}$$

$$= 1 + 2\sum_{n=1}^{\infty} (ae^{-j\omega})^n = 1 + 2\frac{ae^{-j\omega}}{1 - ae^{-j\omega}} = \frac{1 + ae^{-j\omega}}{1 - ae^{-j\omega}}$$

Examples

- *Example 5.2, p.257 of textbook*

Find the spectral (Fourier transform) for the following signal: $x[n] = a^{|n|}, |a| < 1$

$$X(e^{j\omega}) = \frac{1 + ae^{-j\omega}}{1 - ae^{-j\omega}} = \frac{1 - a^2 - 2jasin\omega}{1 + a^2 - 2cos\omega}$$

$$x[n] = a^{|n|} \overset{F}{\leftrightarrow} \mathbb{R}\{X(e^{j\omega})\} = \frac{1 - a^2}{1 + a^2 - 2cos\omega}$$

Examples

- *Example 5.13, p.273 of textbook*

Compute the time-domain signal for $y[n] = \alpha^n u[n] * \alpha^n u[n]$

Examples

- *Example 5.13, p.273 of textbook*

Compute the time-domain signal for y$[n] = \alpha^n u[n] * \alpha^n u[n]$

$$\alpha^n u[n] \overset{F}{\leftrightarrow} \frac{1}{1 - \alpha e^{-j\omega}}$$

Based on the convolution property:

$$y[n] = \alpha^n u[n] * \alpha^n u[n] \overset{F}{\leftrightarrow} (\frac{1}{1 - \alpha e^{-j\omega}})^2$$

Examples

- *Example 5.13, p.273 of textbook*

Compute the time-domain signal for y$[n] = \alpha^n u[n] * \alpha^n u[n]$

$$\alpha^n u[n] \overset{F}{\leftrightarrow} \frac{1}{1-\alpha e^{-j\omega}}$$

$$\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = -j.-\alpha e^{-j\omega}.-1.\left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2$$

$$\Rightarrow \quad \left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2 = \frac{j}{a} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

Examples

- *Example 5.13, p.273 of textbook*

Compute the time-domain signal for $y[n] = \alpha^n u[n] * \alpha^n u[n]$

$$\alpha^n u[n] \overset{F}{\leftrightarrow} \frac{1}{1-\alpha e^{-j\omega}}$$

According to the property of differencing in frequency domain:

$$n\alpha^n u[n] \overset{F}{\leftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

Examples

- *Example 5.13, p.273 of textbook*

Compute the time-domain signal for $\mathrm{y}[n] = \alpha^n u[n] * \alpha^n u[n]$

$$n\alpha^n u[n] \overset{F}{\leftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$\left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2 = \frac{j}{a}e^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{1}{a}e^{j\omega}\sum_{n=-\infty}^{\infty} n\alpha^n u[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} n\alpha^{n-1}u[n]\, e^{-j\omega(n-1)} = \sum_{n=-\infty}^{\infty}(n+1)\alpha^n u[n+1]\, e^{-j\omega n}$$

Examples

- *Example 5.13, p.273 of textbook*

Compute the time-domain signal for $y[n] = \alpha^n u[n] * \alpha^n u[n]$

$$y[n] = \alpha^n u[n] * \alpha^n u[n] \overset{F}{\leftrightarrow} \left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2$$

$$\left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2 = \sum_{n=-\infty}^{\infty} (n+1)\alpha^n u[n+1]\, e^{-j\omega n}$$

$$\Rightarrow \quad y[n] = \alpha^n u[n] * \alpha^n u[n] \overset{F}{\leftrightarrow} (n+1)\alpha^n u[n+1]$$

Outline: **Lecture 8:** Discrete-time Fourier Transform

- Discrete-time Fourier Transform
  - Fourier Transform Representation
  - Fourier Transform of Periodic Signal
- Fourier Transform Properties
- Examples
- Duality

Duality

Fourier transform (spectral):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x\,[n]e^{-j\omega n}$$

Inverse Fourier transform (signal representation):

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

There is no duality between Fourier transform and inverse transform for discrete-time aperiodic signal.

Duality

Discrete-time Fourier series representation:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

Discrete-time Fourier series coefficient:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

But there is a duality between Fourier series and time-domain signal for discrete-time periodic signal.

Duality

Discrete-time Fourier series duality

Assume

$$f[m] = \frac{1}{N} \sum_{r=\langle N \rangle} g[r] e^{-jm\omega_0 r}$$

Replace m, r by k, n respectively:

$$f[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk\omega_0 n}$$

$$g[n] \overset{FS}{\leftrightarrow} f[k]$$

Duality

Discrete-time Fourier series duality

Assume

$$f[m] = \frac{1}{N} \sum_{r=\langle N \rangle} g[r] e^{-jm\omega_0 r}$$

Replace m, r by n, $-k$ respectively:

$$f[n] = \frac{1}{N} \sum_{k=\langle N \rangle} g[-k] e^{jk\omega_0 n}$$

$$f[n] \overset{FS}{\leftrightarrow} \frac{1}{N} g[-k]$$

Duality

Discrete-time Fourier series duality

- If $g[n]$ has a Fourier series f[k], then if we form a new function of time that has the functional form of the series, *f[n],* it will have a Fourier series g[k] that has the functional form of the original time function (but is a function of frequency).

- Mathematically, we can write:

$$g[n] \overset{FS}{\leftrightarrow} f[k]$$

$$f[n] \overset{FS}{\leftrightarrow} \frac{1}{N} g[-k]$$

Duality

Discrete-time Fourier series duality

Example: Given the following pair of Fourier series representation:

$$x[n] = \begin{cases} 1, & |n| \le 2 \\ 0, & 2 < |n| \le 4 \end{cases} \overset{FS}{\leftrightarrow} a_k = \begin{cases} \frac{1}{9}\frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \text{ is mulptiple of } 9 \\ \frac{5}{9}, & k \text{ is not multiple of } 9 \end{cases}$$

Find Fourier series coefficients for $x[n] = \begin{cases} \frac{1}{9}\frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & k \text{ is mulptiple of } 9 \\ \frac{5}{9}, & k \text{ is not multiple of } 9 \end{cases}$

Duality

Discrete-time Fourier series duality

$$x[n] = \begin{cases} \dfrac{1}{9}\dfrac{\sin(5\pi n/9)}{\sin(\pi n/9)}, k \text{ is mulptiple of } 9 \\ \dfrac{5}{9}, \quad\quad k \text{ is not multiple of } 9 \end{cases} \overset{FS}{\leftrightarrow} a_k = \begin{cases} \dfrac{1}{9}, & |k| \le 2 \\ 0, & 2 < |k| \le 4 \end{cases}$$

Duality

Discrete-time Fourier transform

Fourier transform (spectral):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Inverse Fourier transform (signal representation):

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Continuous-time Fourier series

Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Fourier series coefficient:

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

Duality between discrete-time Fourier transform and continuous-time Fourier series

Duality

Discrete-time Fourier series duality

Example: Given the following pair of Fourier series representation:

$$x(t)=\begin{cases}1, & |t|\le T_1\\ 0, & T_1<|t|\le \pi\end{cases} \overset{FS}{\leftrightarrow} a_k=\frac{\sin(kT_1)}{k\pi}$$

Find Fourier transform for $x[n]=\dfrac{\sin(\pi n/2)}{\pi n}$

Duality

Discrete-time Fourier series duality

Fourier series coefficient:

$$\frac{\sin(kT_1)}{k\pi} = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt,$$

$$x(t) = \begin{cases} 1, & |t| \le T_1 \\ 0, & T_1 < |t| \le \pi \end{cases}, \qquad T = 2\pi$$

Inverse Fourier transform (signal representation):

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\Rightarrow \quad X(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \pi/2 \\ 0, & \pi/2 < |\omega| \le \pi \end{cases}$$

Duality

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time			Discrete time		
	Time domain		**Frequency domain**	**Time domain**		**Frequency domain**
Fourier Series	$x(t) =$ (3.38) $\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	<C>	$a_k =$ (3.39) $\frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency duality	$x[n] =$ (3.94) $\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	<B> duality	$a_k =$ (3.95) $\frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) =$ (4.8) $\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	<A> duality	$X(j\omega) =$ (4.9) $\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] =$ (5.8) $\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ discrete time aperiodic in time	<D>	$X(e^{j\omega}) =$ (5.9) $\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

Summary

Periodicity

Continuity	**Periodic**	**Aperiodic**
Continuous	Fourier series $x(t) \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x(t) \overset{F}{\leftrightarrow} X(j\omega)$
Discrete	Fourier series $x[n] \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$

Periodic in time domain

$\updownarrow$

Discrete in frequency domain

Summary

Periodicity

Continuity	**Periodic**	**Aperiodic**
Continuous	Fourier series $x(t) \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x(t) \overset{F}{\leftrightarrow} X(j\omega)$
Discrete	Fourier series $x[n] \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$

Aperiodic in time domain

$\updownarrow$

Continuous in frequency domain

Summary

Periodicity

Continuity

	Periodic	**Aperiodic**
Continuous	Fourier series $x(t) \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x(t) \overset{F}{\leftrightarrow} X(j\omega)$
Discrete	Fourier series $x[n] \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$

Continuous in time domain

$\updownarrow$

Aperiodic in frequency domain

Summary

Continuity	**Periodicity**	**Periodic**	**Aperiodic**
	Continuous	Fourier series $x(t) \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x(t) \overset{F}{\leftrightarrow} X(j\omega)$
	Discrete	Fourier series $x[n] \overset{FS}{\leftrightarrow} a_k$	Fourier transform $x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$

Discrete in time domain

$\updownarrow$

Periodic in frequency domain

Summary

Periodic in time domain $\leftrightarrow$ Discrete in frequency domain

Aperiodic in time domain $\leftrightarrow$ Continuous in frequency domain

Continuous in time domain $\leftrightarrow$ Aperiodic in frequency domain

Discrete in time domain $\leftrightarrow$ Periodic in frequency domain

Thank you for your listening!