Lecture 6 Sampling

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实验二 信号表示

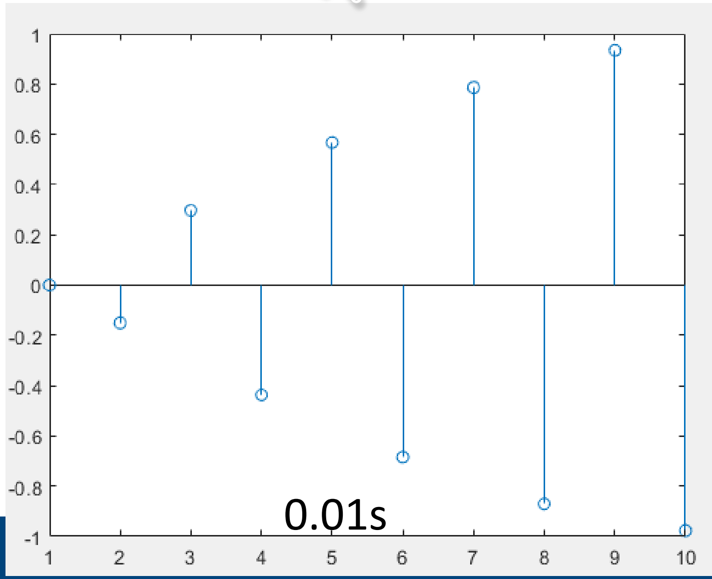
在实验二教学PPT中，我们已学会如何根据音调的频率构建其对应的信号，信号用如下正弦波形式表示：

$$x(t) = A\,sin(2\pi f t + \phi)$$

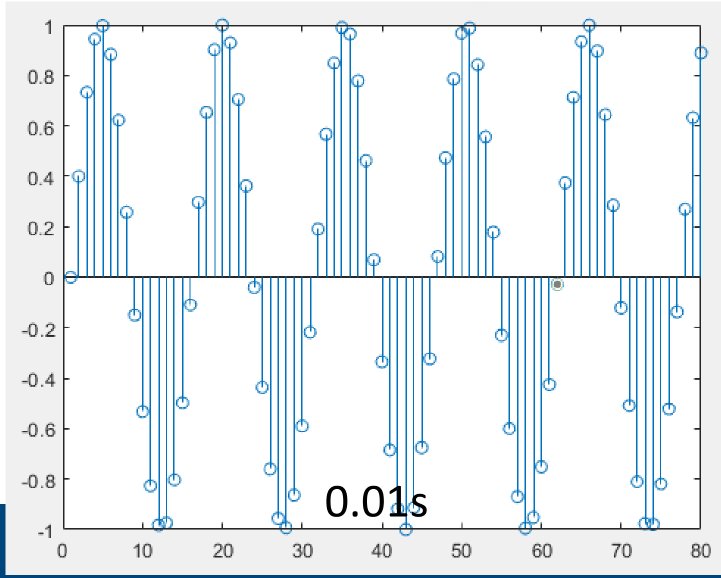
3. 1）Fs=1000；2）Fs=8000；3）Fs=80000。在上述三种采样频率下，所产生的信号与声音是否有所不同？为什么？

高音C（哆）：f=524Hz

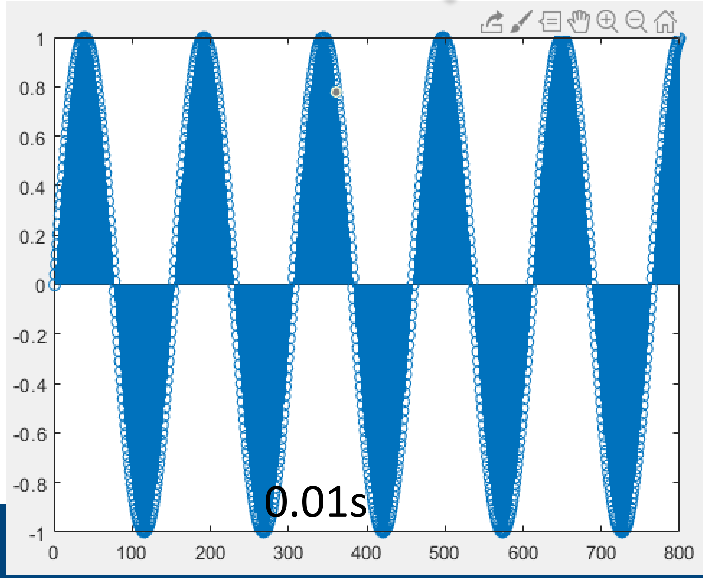
Fs=1000Hz



Fs=8000Hz



Fs=80000Hz



实验二 信号表示

在实验二教学PPT中，我们已学会如何根据音调的频率构建其对应的信号，信号用如下正弦波形式表示：

$$x(t) = A\,sin(\,2\pi f t + \phi)$$

3. 1）Fs=1000；2）Fs=8000；3）Fs=80000。在上述三种采样频率下，所产生的信号与声音是否有所不同？为什么？

取决于给定信号的频率f：
上述结果表明采样频率Fs对信号的影响与信号本身的频率f相关。实际上，根据奈奎斯特采样定理，当Fs>2f时，任何采集到的离散时间信号都具有原始信号的全部特征，因此不同Fs无任何区别（当f取值小于500时）；Fs≤2f时，采集到的离散时间信号不具有原始信号的全部特征，则会导致声音信号失真。

若f取值较低（低于500），三种采样频率的声音无任何区别；
若500<f<4000，则Fs=1000：音调低；Fs=8000或Fs=80000，音调高且相同；
若f>4000，则Fs=1000：音调最低；Fs=8000：音调较低；Fs=80000，音调最高。（注意，f不要太取太大，否则声音太刺耳）

Outline: Lecture 6: Sampling

- Introduction

- The Sampling Theorem
    - Impulse train sampling
    - Sampling with a zero-order hold

- Interpolation

- Aliasing

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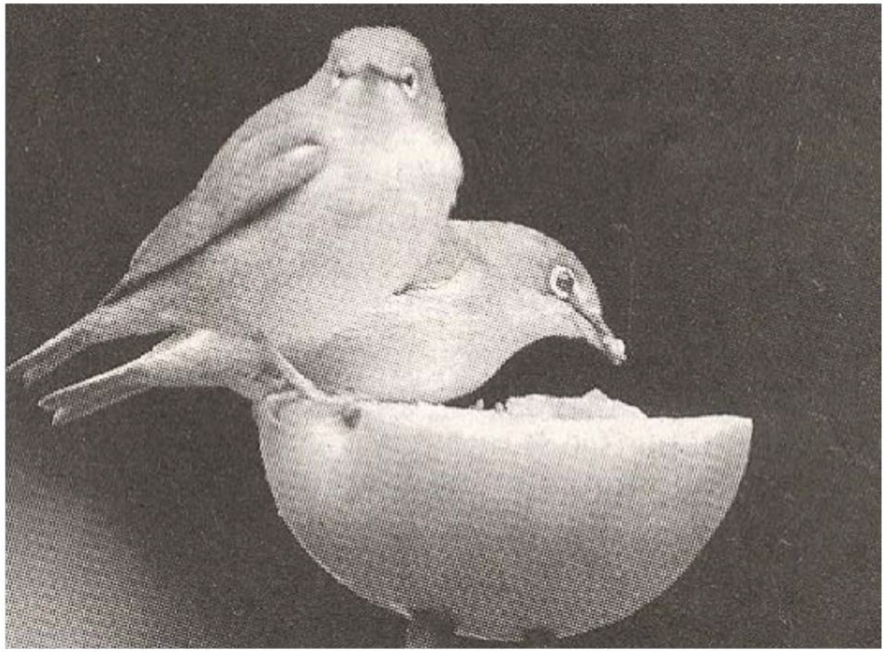
Introduction

- Continuous-time signal is often represented by discrete-time signal:
  - Image transmission in fax machine;
  - Signals in TV display: 60Hz or beyond refresh rates;
  - Film: 12 or beyond frames per second.



Zoom in

Introduction

- Motivation: a discrete-time represented continuous-time signal can be digitally handled, using computers:

  - Accurate, programmable, flexible, reproducible, powerful;

  - Compatible to digital networks and relevant technologies;

  - All signals look the same when digitized, except at different rates, and thus can be supported by the same network.

- Question: under which conditions can a continuous-time signal be uniquely specified by its discrete-time samples?

Introduction

- Question: under which conditions can a continuous-time signal be uniquely specified by its discrete-time samples?

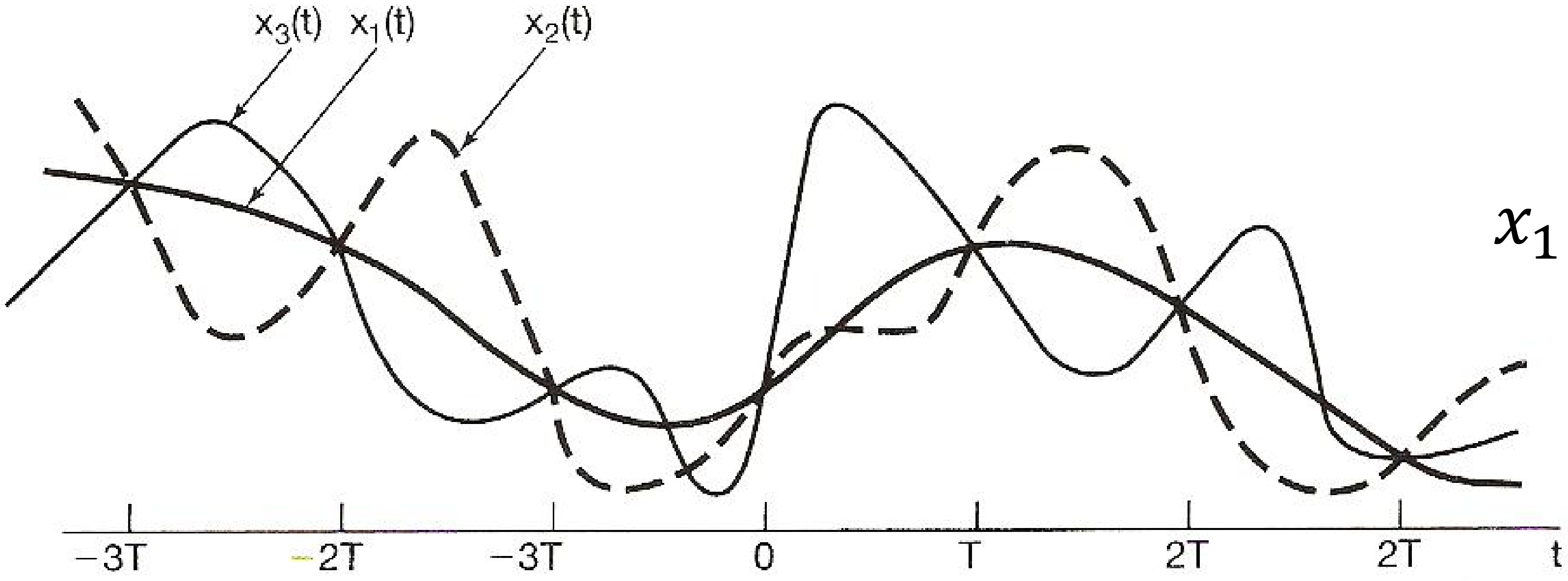


Figure 7.1 Three continuous-time signals with identical values at integer multiples of T.

$$x_1(kT) = x_2(kT) = x_3(kT)$$

Introduction

- Question: under which conditions can a continuous-time signal be uniquely specified by its discrete-time samples?

- For a given set of samples, it might be produced by different continuous-time signals.

- But if a continuous-time signal is sampled in a relatively high frequency, can the samples be used to represent the signal? How high should the sampling frequency be?

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The Sampling Theorem

Impulse train sampling

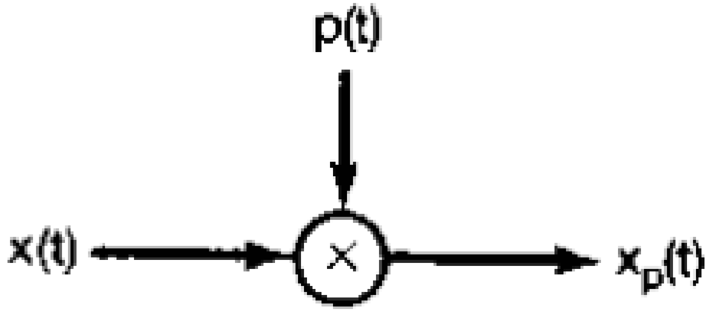
- Mathematical model:

  Time domain:

$$x_p(t) = x(t)p(t)$$

  Frequency domain:

$$X_p(j\omega) = \frac{1}{2\pi}[X(j\omega) * P(j\omega)]$$



The Sampling Theorem

Impulse train sampling（冲激串采样）

- Mathematical model:

  Time domain:

$$x_p(t) = x(t)p(t)$$

  where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta\,(t - nT)$$

T: sampling period

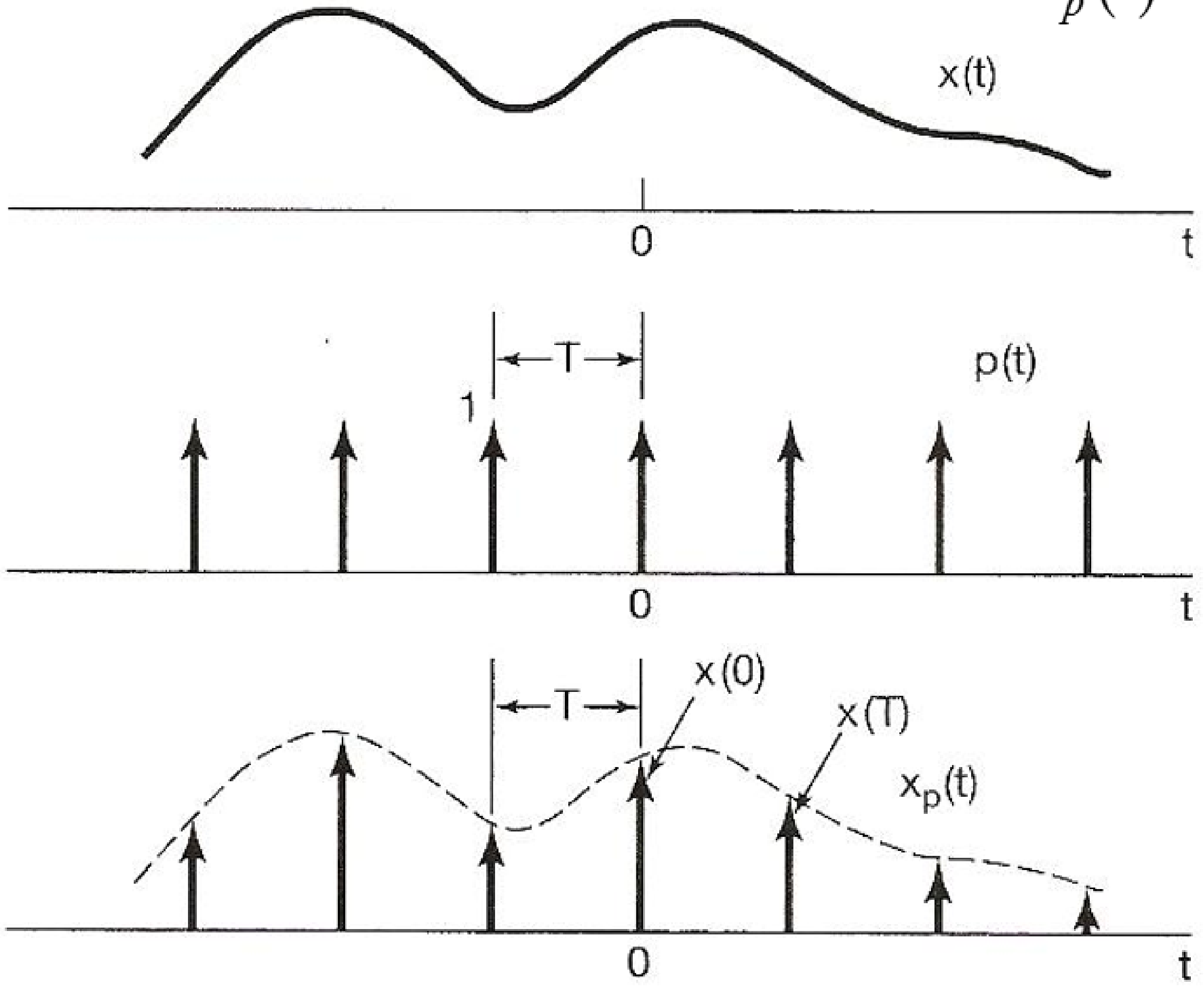
$\omega_s = \frac{2\pi}{T}$: sampling frequency

The Sampling Theorem

Impulse train sampling（冲激串采样）

Time domain:

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta\,(t - nT)$$



The Sampling Theorem

Impulse train sampling

Frequency domain:

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\omega_s\right)$$

$$X_p(j\omega) = \frac{1}{2\pi}[X(j\omega) * P(j\omega)] = \frac{1}{T}\sum_{k=-\infty}^{\infty}X\left(j(\omega - k\omega_s)\right)$$

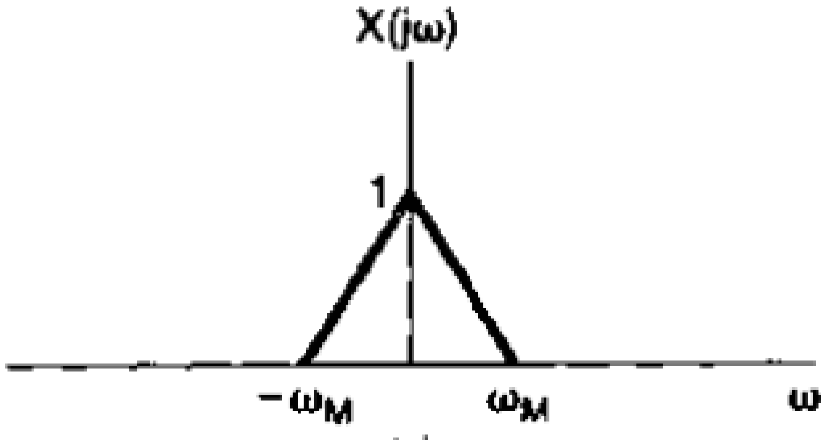
The Sampling Theorem

Impulse train sampling

Frequency domain:

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j(\omega - k\omega_s)\right)$$

Assume $X(j\omega)$ is a band-limited signal with its frequency in the range of $\pm\omega_M$:

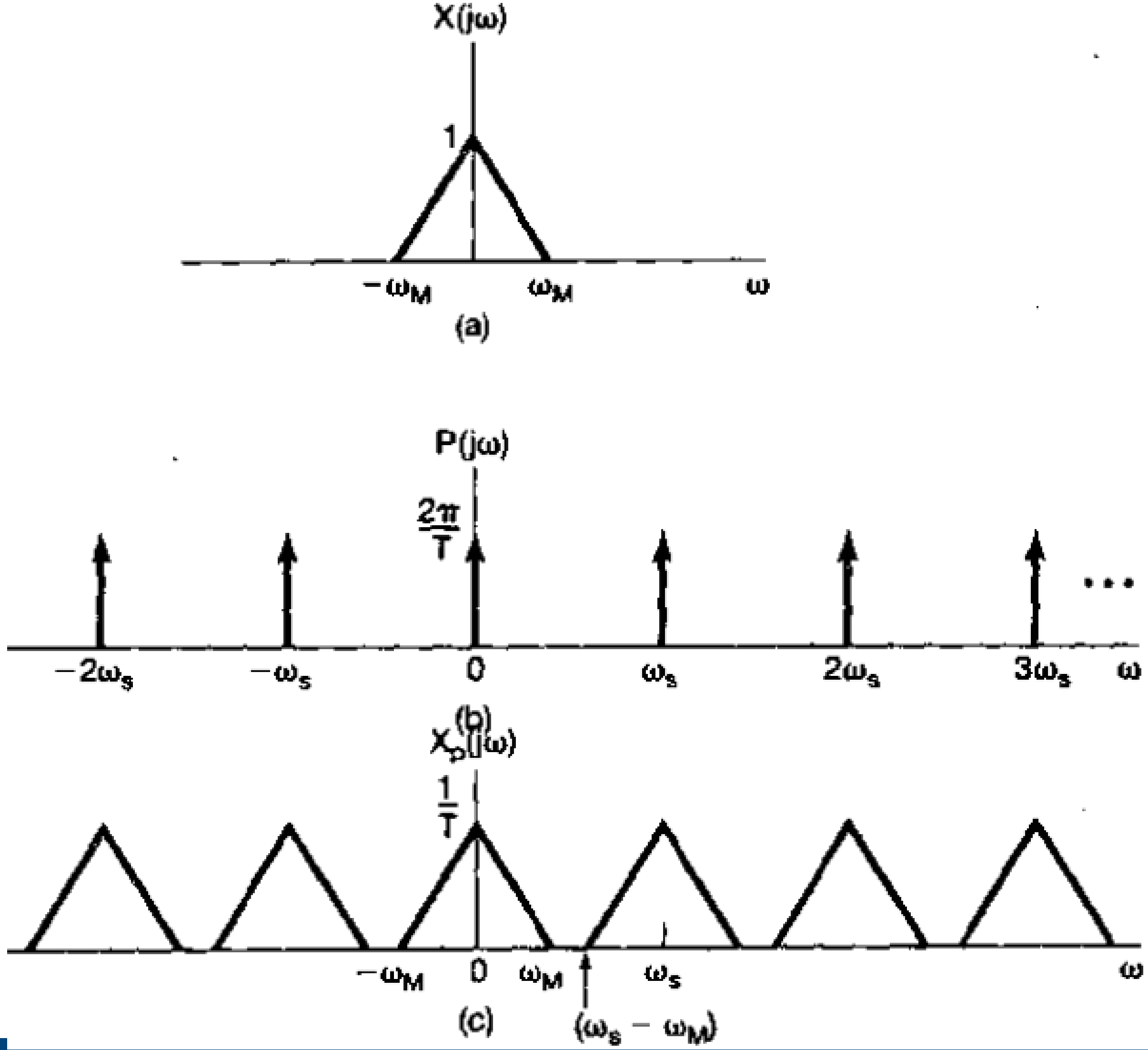


The Sampling Theorem

Impulse train sampling

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j(\omega - k\omega_s)\right)$$

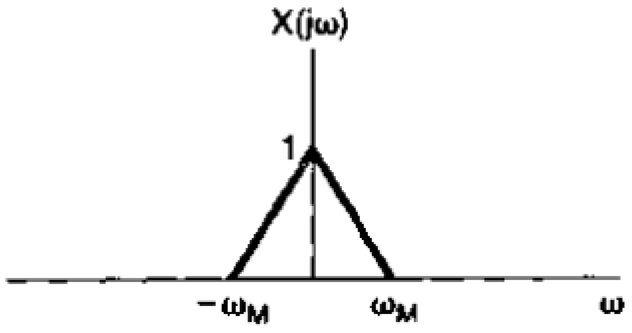
For $\omega_s - \omega_M > \omega_M \Rightarrow \omega_s > 2\omega_M$:



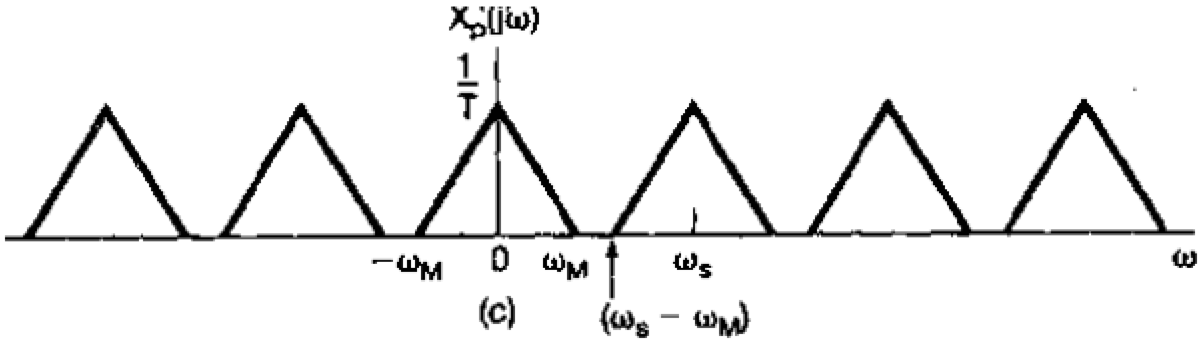
The Sampling Theorem

Impulse train sampling

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j(\omega - k\omega_s)\right)$$



For $\omega_s - \omega_M > \omega_M \Rightarrow \omega_s > 2\omega_M$:



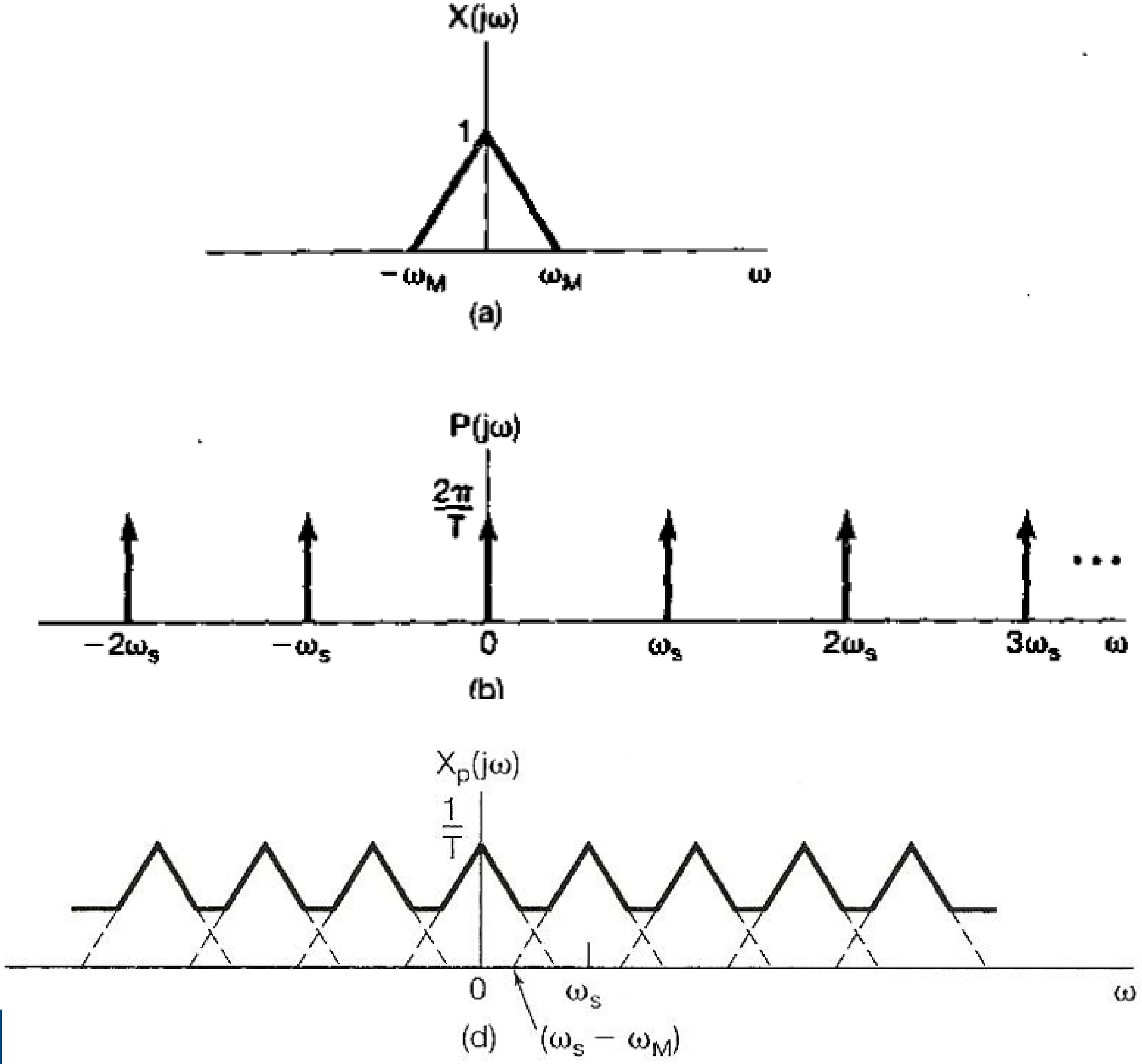
- In such a case, $x(t)$ can be precisely reconstructed by feeding $x_p(t)$ into an ideal low-pass filter with gain T and cut-off frequency $\omega_c \in (\omega_M, \omega_s - \omega_M)$

The Sampling Theorem

Impulse train sampling

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j(\omega - k\omega_s)\right)$$

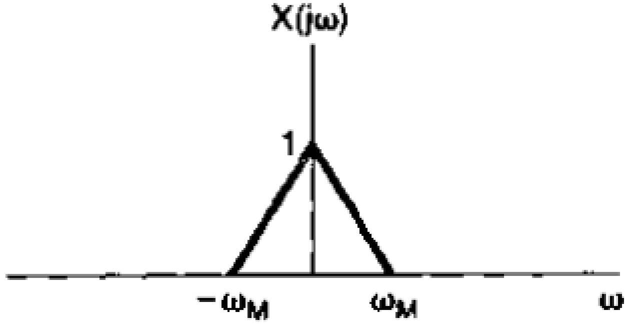
For $\omega_s - \omega_M < \omega_M \Rightarrow \omega_s < 2\omega_M$:



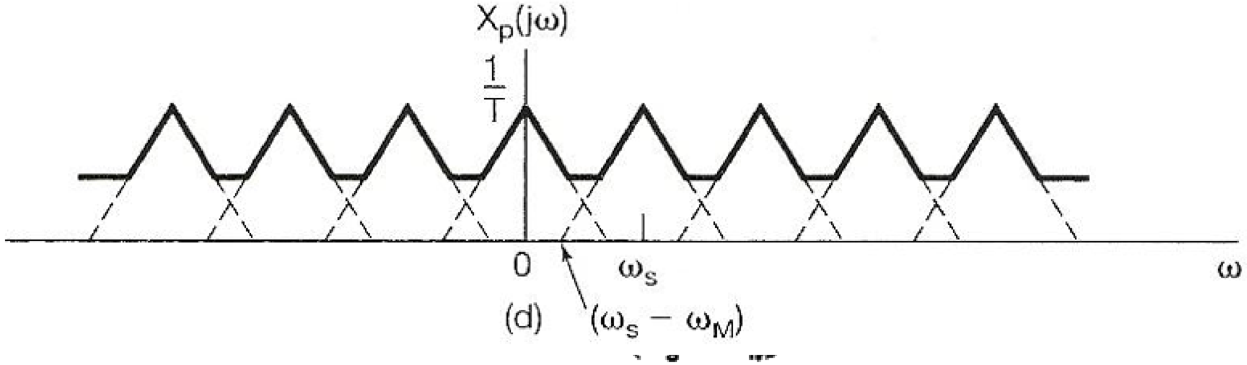
The Sampling Theorem

Impulse train sampling

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j(\omega - k\omega_s)\right)$$



For $\omega_s - \omega_M < \omega_M \Rightarrow \omega_s < 2\omega_M$:



- Spectrum overlapped, frequency components confused: aliasing effect （混叠现象）

=> can’t be reconstructed by low-pass filtering

The Sampling Theorem

Nyquist sampling theorem（奈奎斯特采样定理）

- A band-limited （带限） continuous-time signal can be sampled and perfectly reconstructed from its samples if the waveform is sampled over twice as fast as it's highest frequency component.

The highest frequency of $x(t)$: ω_M

The highest sampling frequency that may cause aliasing effect (Nyquist rate奈奎斯特率):

$$\omega_s = 2\omega_M$$

The Sampling Theorem

Practical issues

- Application of non-ideal low-pass filters for signal reconstruction:

  - non-ideal low-pass filters need be accurate enough determined by acceptable level of distortion

- In practice, signals might not be band-limited:

  - pre-filtering

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  Impulse train sampling

  Sampling with a zero-order hold

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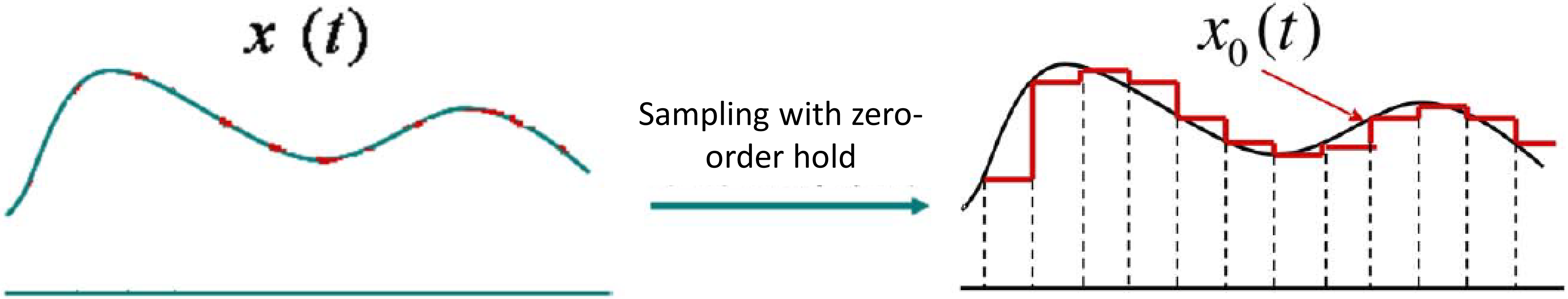
- Aliasing

The Sampling Theorem

Sampling with a zero-order hold（零阶保持采样）

- Zero-order hold: holding the sampled value until the next sample taken

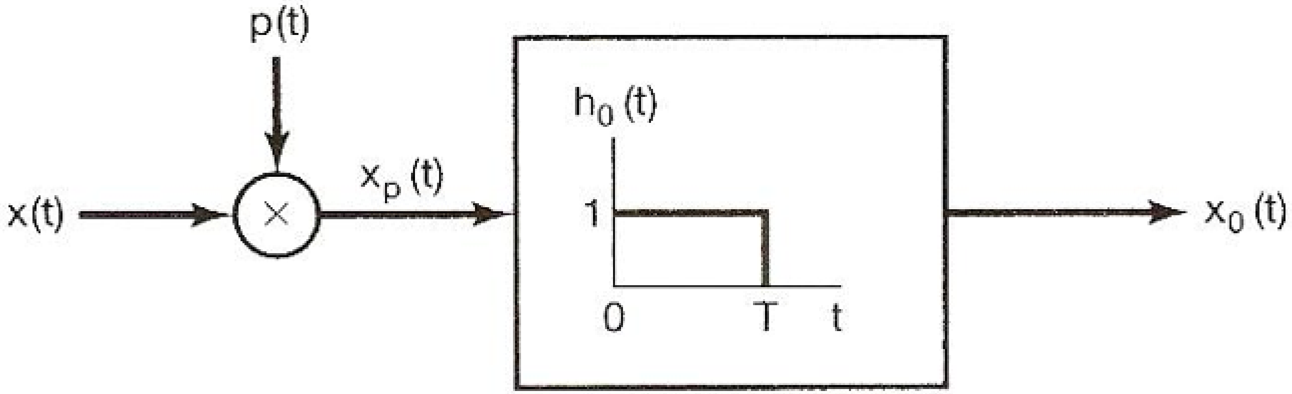


- The output signal of zero-order hold sampling, $x_0(t)$, is a staircase signal.

The Sampling Theorem

Sampling with a zero-order hold

- Zero-order hold: modeled by an impulse train sampler followed by a system with rectangular impulse response

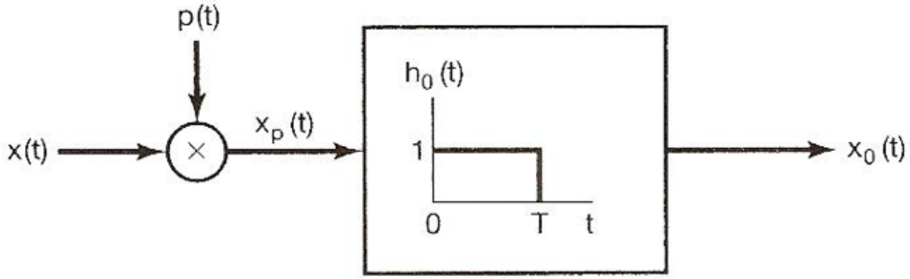


$$H_0(j\omega) = e^{-j\omega T/2}[\frac{2\sin(\omega T/2)}{\omega}]$$

The Sampling Theorem

Sampling with a zero-order hold

- Zero-order hold: modeled by an impulse train sampler followed by a system with rectangular impulse response



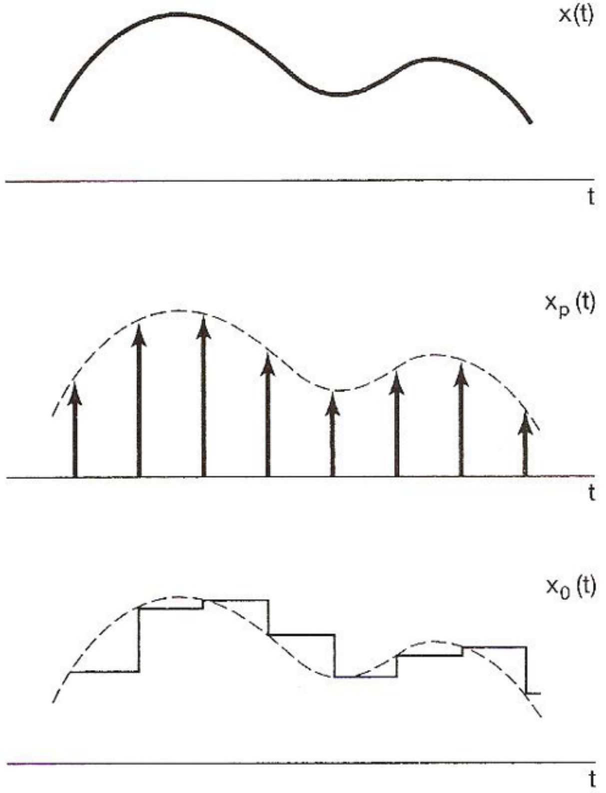
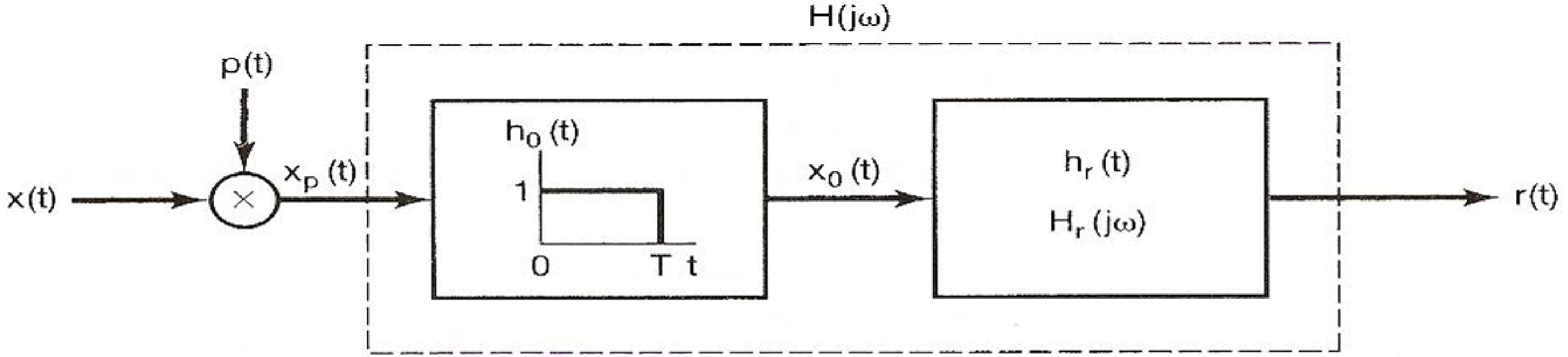


Figure 7.6 Zero-order hold as impulse-train sampling followed by an LTI system with a rectangular impulse response.

The Sampling Theorem

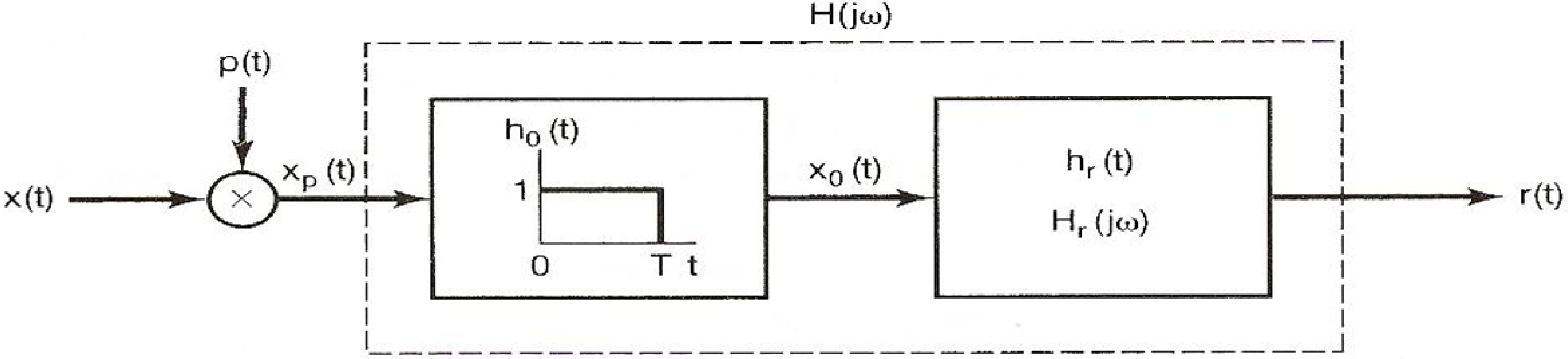
Sampling with a zero-order hold

- Cascaded systems are required to reconstruct $x(t)$ from $x_0(t)$:



The Sampling Theorem

Sampling with a zero-order hold

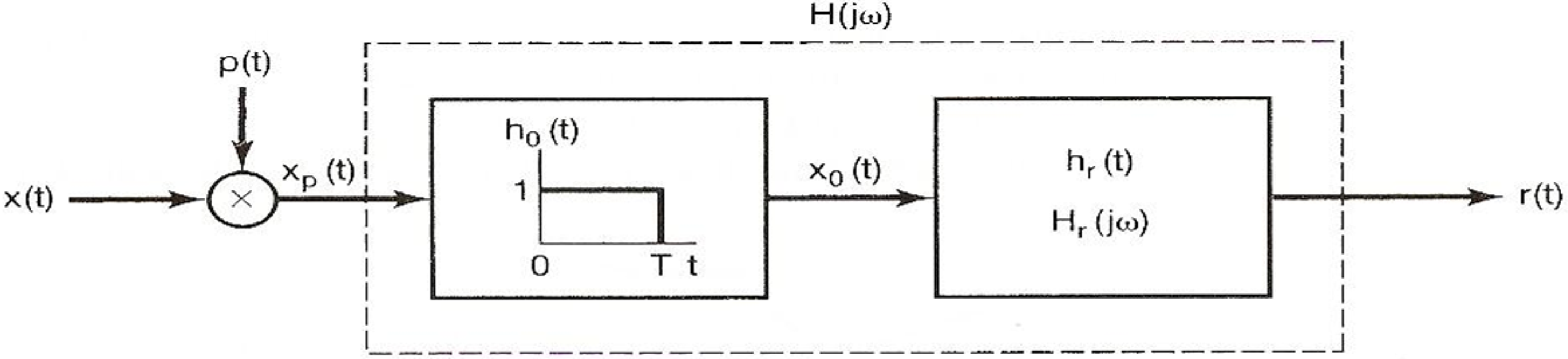


- Remember that continuous-time signal reconstruction is realized by putting the samples into an ideal low-pass filter:

=> In fact, $H(j\omega)$ is the frequency response of an ideal low-pass filter

The Sampling Theorem

Sampling with a zero-order hold



$$H(j\omega) = \begin{cases} T, |\omega| < \omega_c \\ 0, |\omega| > \omega_c \end{cases} \qquad \omega_M < \omega_c < \omega_s - \omega_M$$

The Sampling Theorem

Sampling with a zero-order hold

$$H(j\omega) = \begin{cases} T, |\omega| < \omega_c \\ 0, |\omega| > \omega_c \end{cases} \qquad \omega_M < \omega_c < \omega_s - \omega_M$$

$$H_0(j\omega) = e^{-j\omega T/2}[\frac{2\sin(\omega T/2)}{\omega}]$$

$$\Rightarrow H_r(j\omega) = \frac{H(j\omega)}{H_0(j\omega)} = \frac{e^{\frac{j\omega T}{2}} H(j\omega)}{\frac{2\sin(\omega T/2)}{\omega}}$$

The Sampling Theorem

Practical issues

- $H(j\omega)$ is an ideal low-pass filter, which is impractical, and thus the system, $H_r(j\omega)$, is impractical as well.

- In many situations, the signal sampled by a zero-order hold, $x_0(t)$, is considered as an approximated reconstruction of $\mathrm{x}(t)$ (with a very coarse interpolation), and can be used for further signal processing.

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Interpolation（内插）

- Interpolation is the process to reconstruct a signal from its samples

- Continuous-time signal reconstruction is realized by putting the samples into an ideal low-pass filter.

- Ideal interpolation is to use the impulse response of an ideal low-pass filter as the interpolation function:

$$x_r(t) = x_p(t) * h(t)$$

where

$$h(t) = T\frac{sin\omega_c t}{\pi t}$$

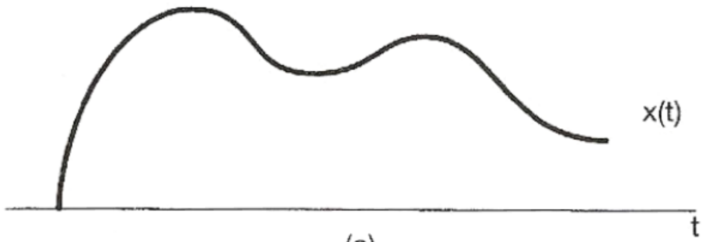
Interpolation

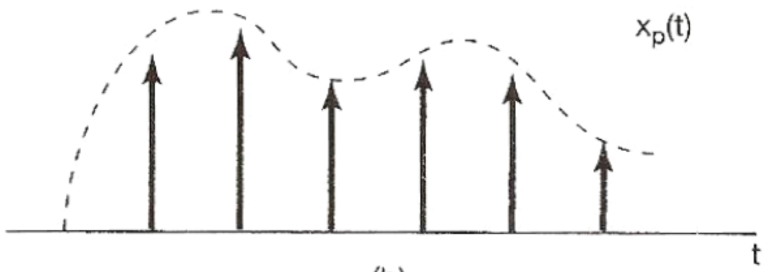
- Ideal interpolation is to use the impulse response of an ideal low-pa as the interpolation function:

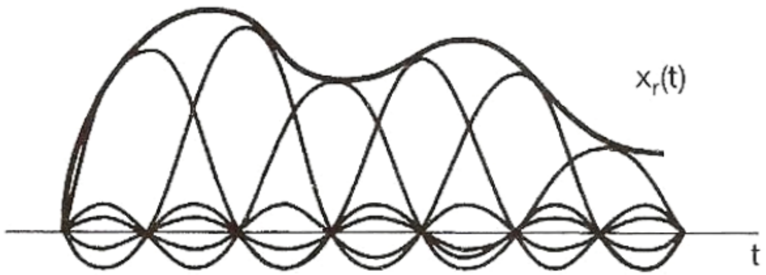
$$x_r(t) = x_p(t) * h(t)$$

where

$$h(t) = T\frac{sin\omega_c t}{\pi t}$$







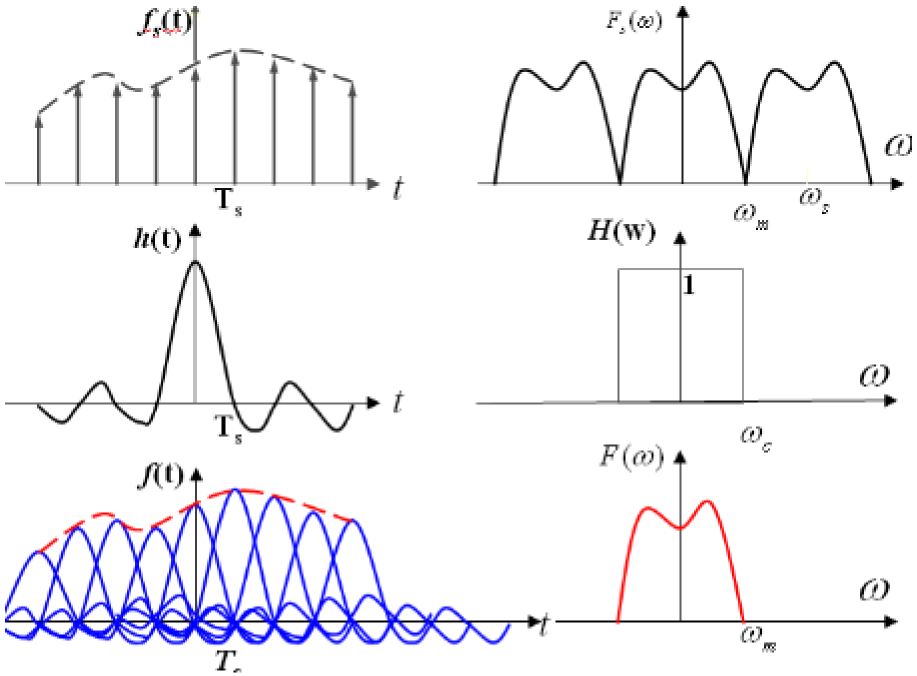
Lecture1 - 系统的因果性（Causality）

- **Causal**: a system is causal if the output at a time, only depends on input values up to that time. （系统的输出只取决于现在的输入及过去的输入）

- 具有因果性的系统才是物理可实现的系统

- 因果性的意义：判断所设计的系统是否是可以被实现的

- 例：理想低通滤波器不具有因果性，是不可能被实现的



Lecture3 - 系统的因果性（Causality）

- Causality
  - continuous-time

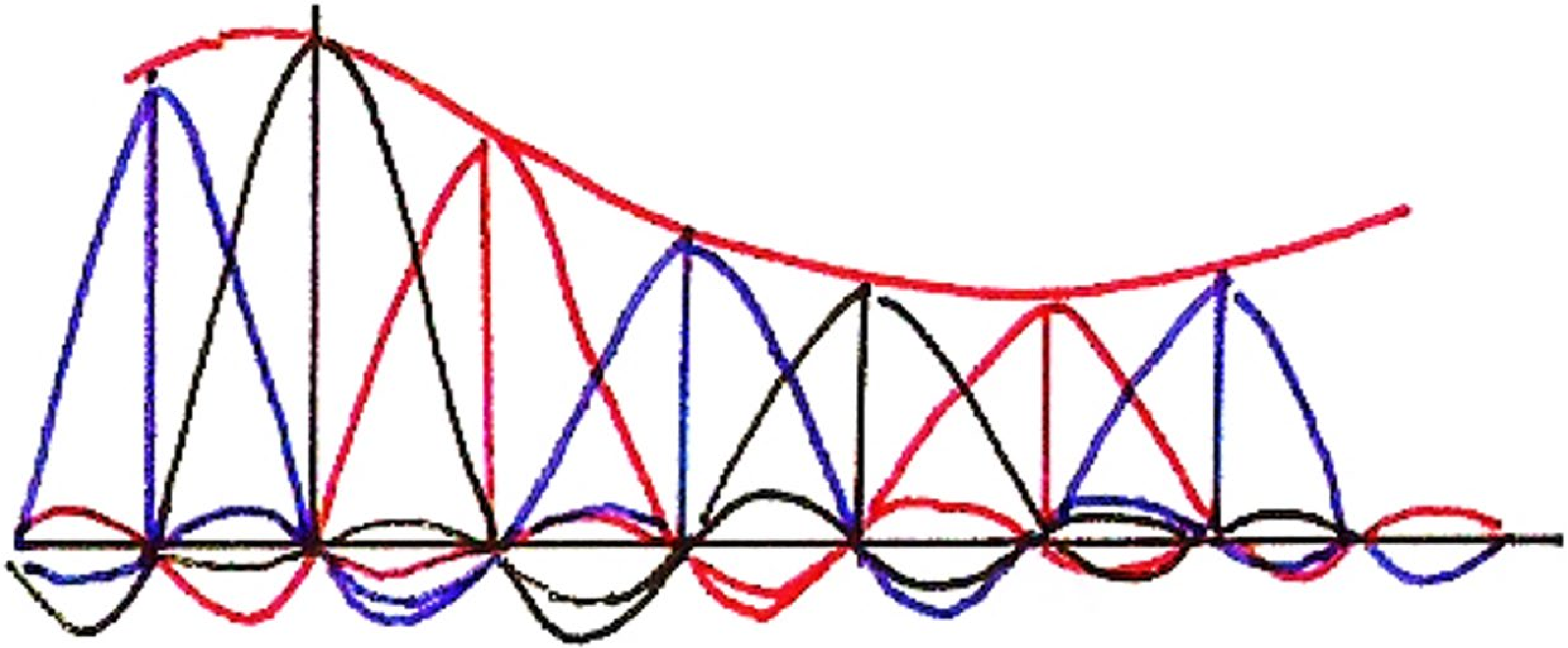
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

  - Causal iff h(t)=0,t<0

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

Interpolation

- Ideal interpolation:

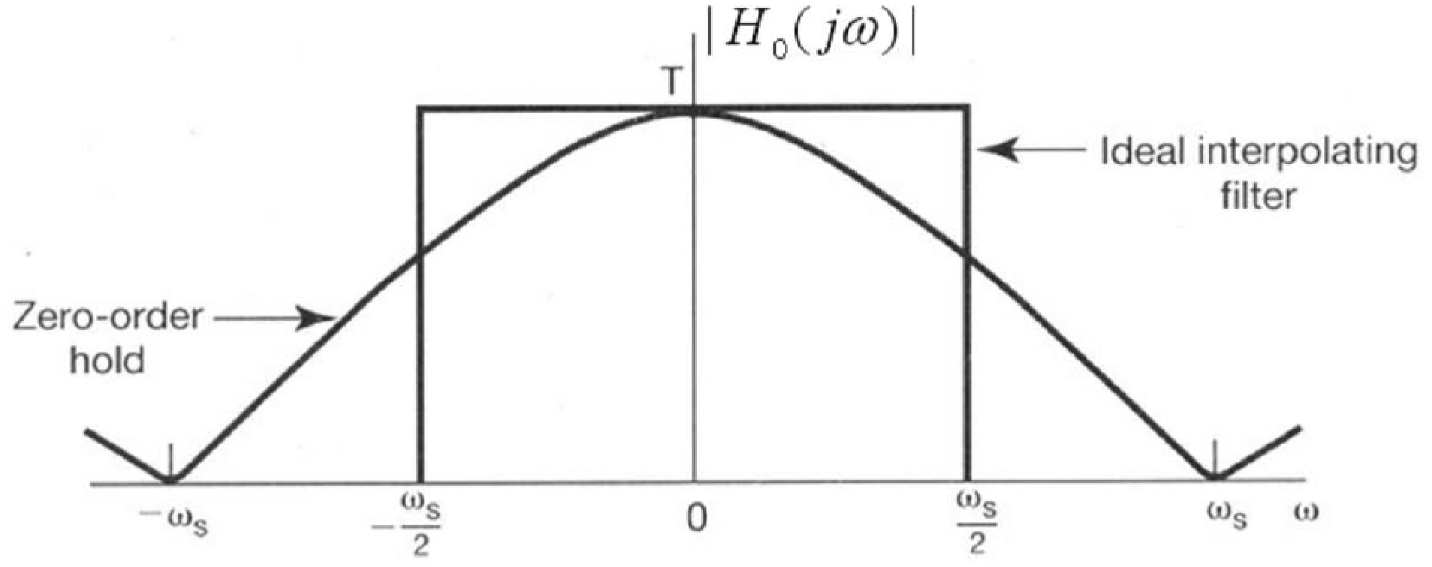


Interpolation

- Zero-order hold can be considered as a “coarse” interpolation

- Transfer function of zero-order hold, $h_0(t)$, is a rough approximation of an ideal low-pass filter:

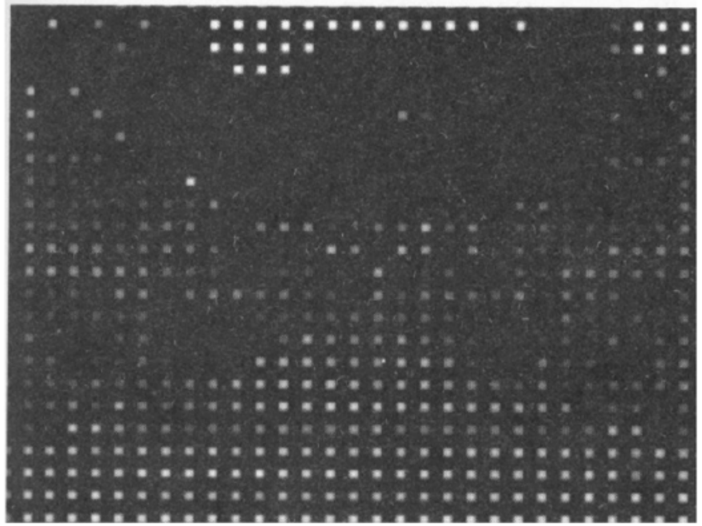
$$H_0(j\omega) = e^{-j\omega T/2}[\frac{2\sin(\omega T/2)}{\omega}]$$

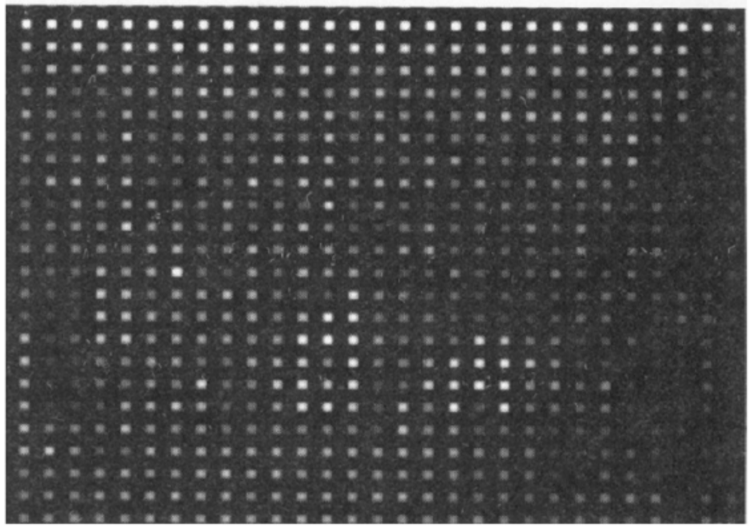


Interpolation

Original pictures:

Impulse train sampled pictures:

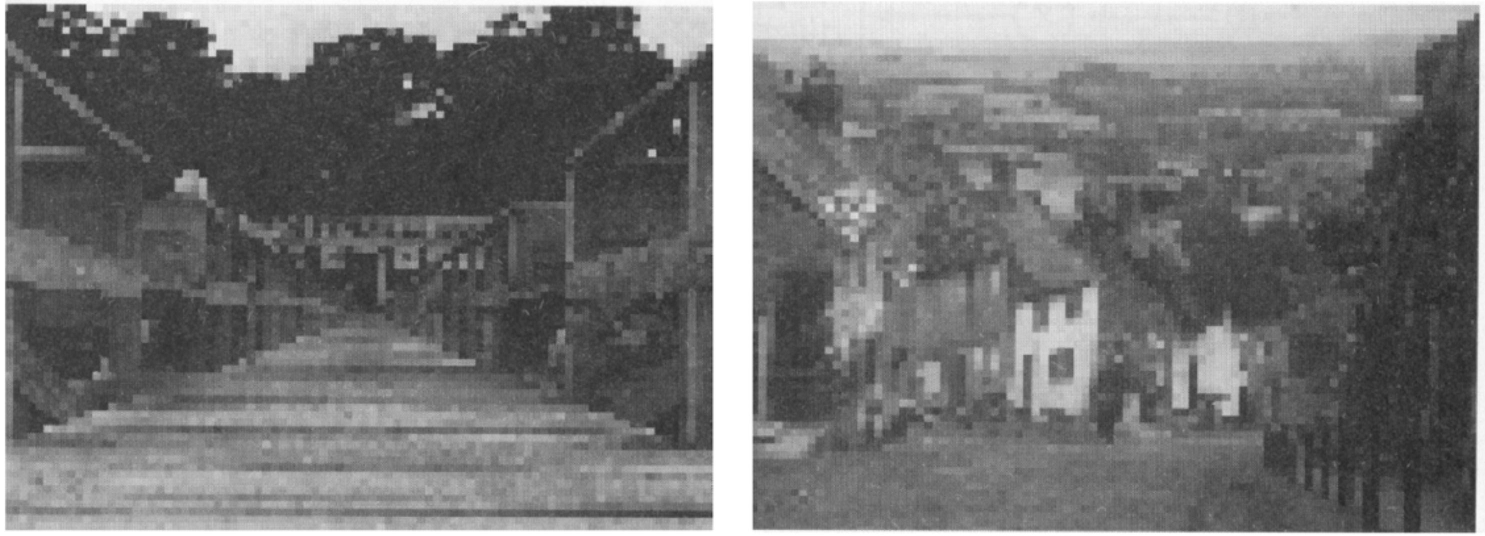




Interpolation

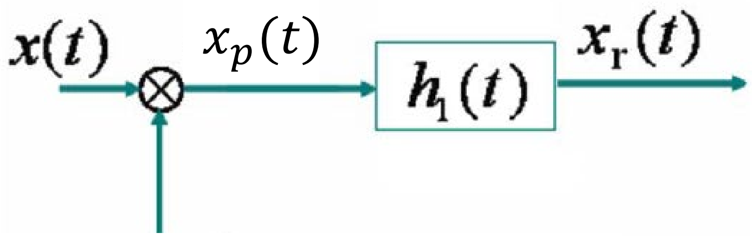
Original pictures:

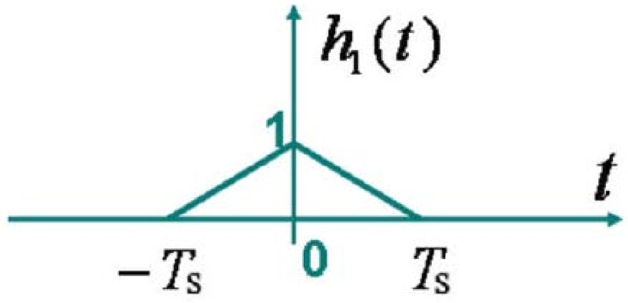
Zero-order hold sampled pictures:



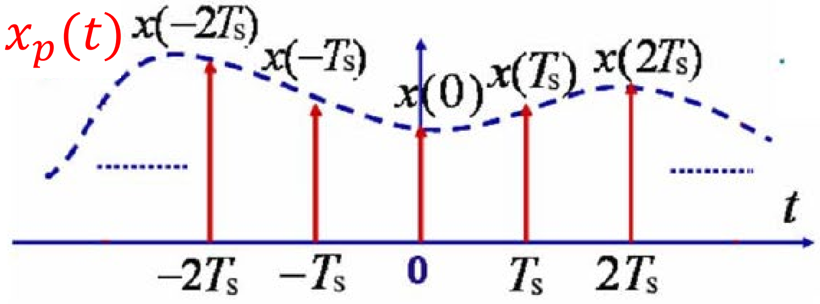
Interpolation

- For smoother interpolations, nth-order hold ($n \geq 1$) might be used, e.g., first-order hold:

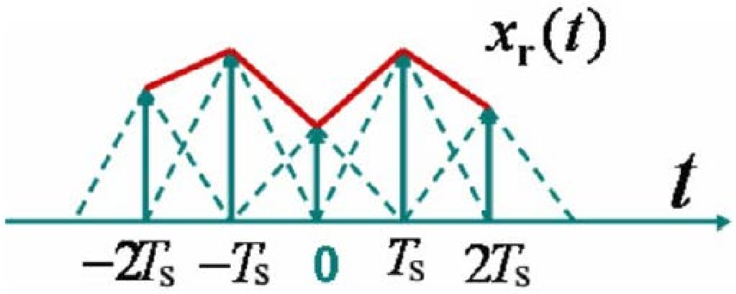




$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



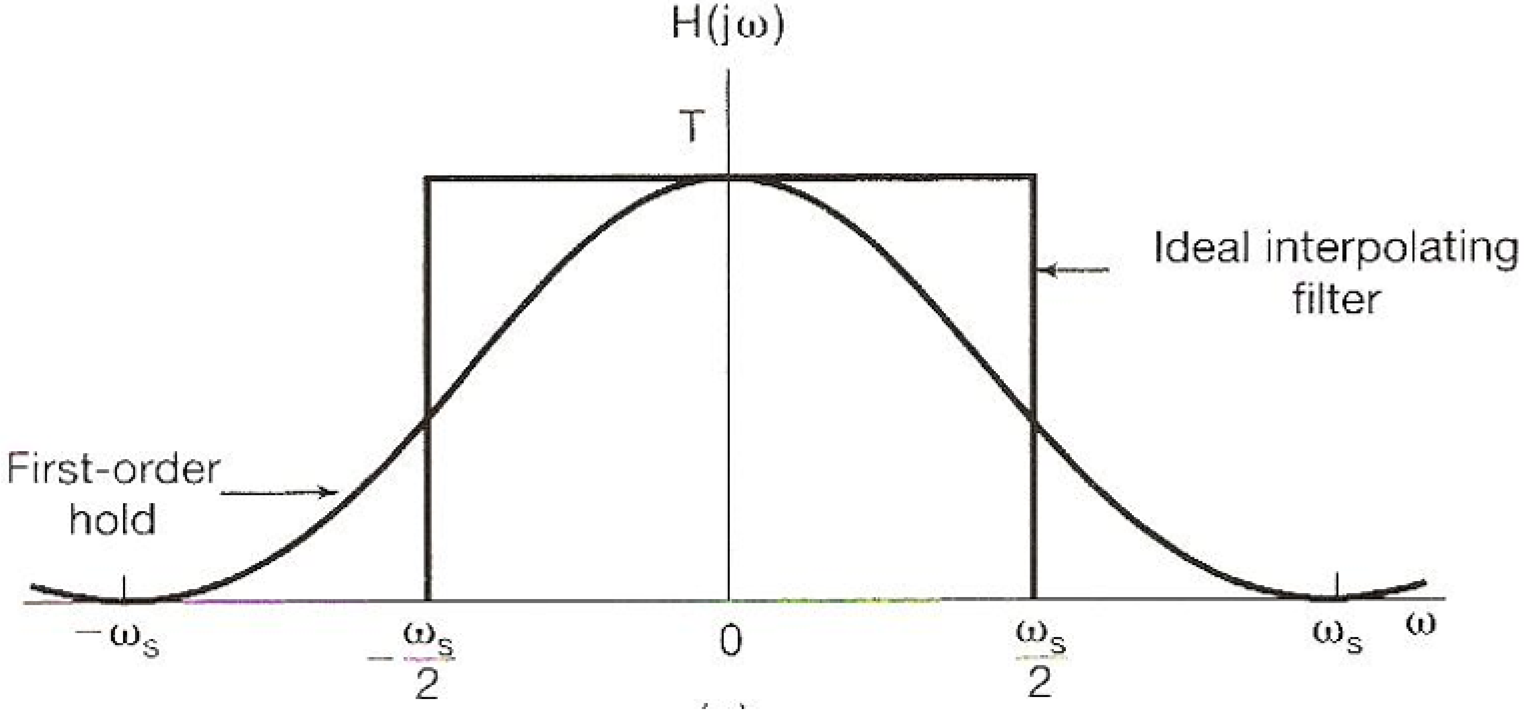
Interpolation function, $h_1(t)$, is a triangular function.



Interpolation

- Transfer function of first-order hold, $h_0(t)$, is a rough approximatic ideal low-pass filter:

$$H(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$



Interpolation

Zero-order hold sampled pictures:

First-order hold sampled pictures:

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- Aliasing

Aliasing（混叠现象）

- When $\omega_s < 2\omega_M$, spectrum overlapped, frequency components confused, resulting in aliasing effect, such that the sampled signal can’t be reconstructed by low-pass filtering.

Aliasing

- Consider a signal $x(t) = cos\omega_0 t$:

  - sampled at sampling frequency $\omega_s = \frac{2\pi}{T}$;

  - reconstructed by an ideal low-pass filter with $\omega_c = \frac{\omega_s}{2}$;

  - reconstructed signal: $x_r(t)$

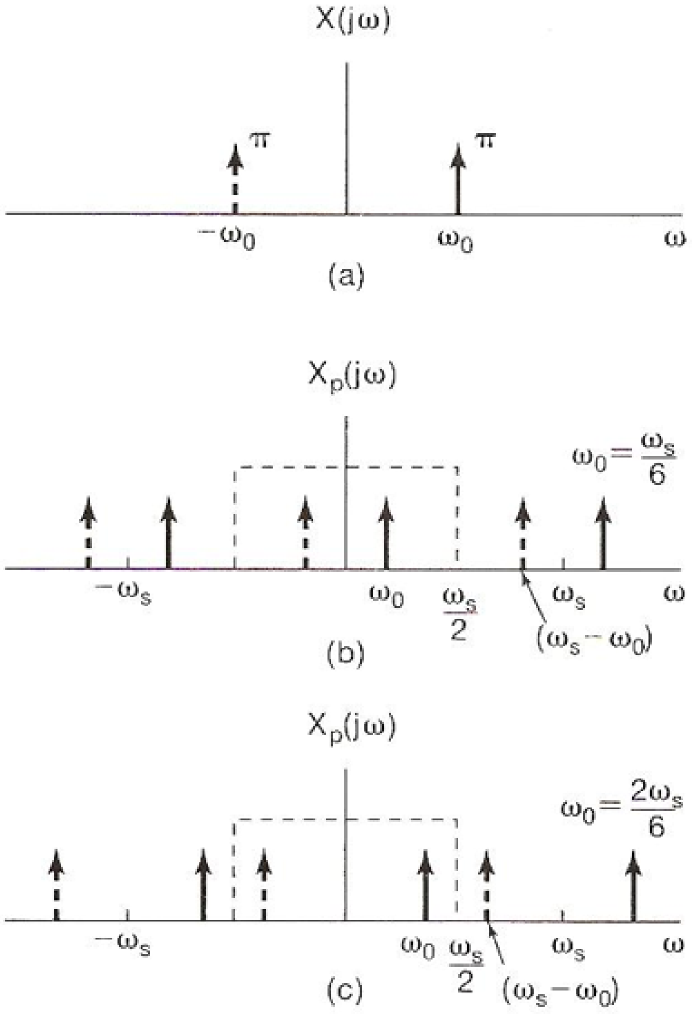
- According to Nyquist sampling theorem, $x_r(t) = x(t)$ if $\omega_s > 2\omega_0$

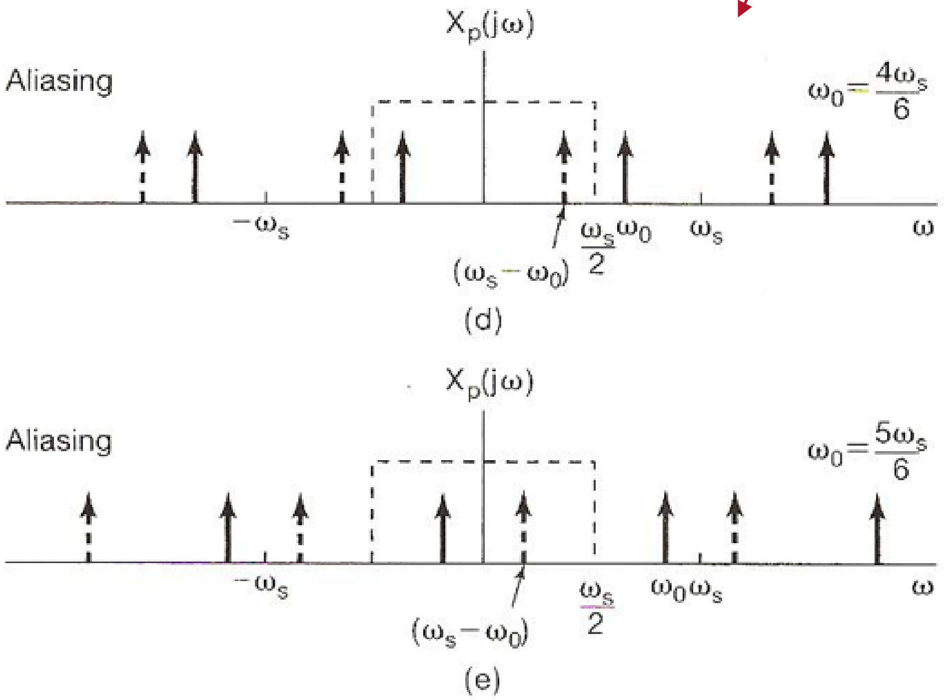
Aliasing

- Assume a fixed ω_s, and vary ω_0:

$$x(t) = cos\omega_0 t$$

$$x_r(t) = \cos(\omega_s - \omega_0)t$$





Aliasing

- Assume $x(t) = \cos(\omega_0 t + \varphi)$

$$x(t) = \cos(\omega_0 t + \varphi) = \frac{1}{2}(e^{j(\omega_0 t+\varphi)} + e^{-j(\omega_0 t+\varphi)})$$

- Using the Fourier transform:

$$e^{j\omega_0 t} \overset{F}{\leftrightarrow} 2\pi\delta(\omega - \omega_0)$$

$$e^{-j\omega_0 t} \overset{F}{\leftrightarrow} 2\pi\delta(\omega + \omega_0)$$

And the time shift property: $x(t - t_0) \overset{F}{\leftrightarrow} e^{-j\omega t_0} X(j\omega)$

Aliasing

- Using the Fourier transform:

$$e^{j(\omega_0 t+\varphi)+} \overset{F}{\leftrightarrow} 2\pi\delta(\omega-\omega_0)e^{j\varphi}$$

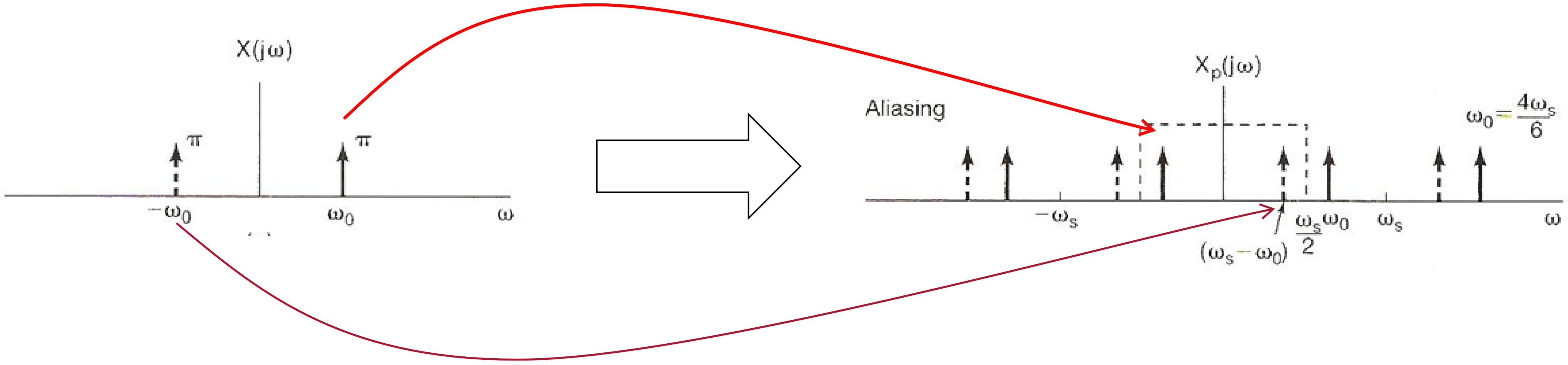
$$e^{-j(\omega_0 t+\varphi)} \overset{F}{\leftrightarrow} 2\pi\delta(\omega+\omega_0)e^{-j\varphi}$$

$$\cos(\omega_0 t+\varphi) \overset{F}{\leftrightarrow} \pi e^{j\varphi}\delta(\omega-\omega_0)+\pi e^{-j\varphi}\delta(\omega+\omega_0)$$

Aliasing

$$x(t) = \cos(\omega_0 t + \varphi) \overset{F}{\leftrightarrow} X(j\omega) = \pi e^{j\varphi}\delta(\omega - \omega_0) + \pi e^{-j\varphi}\delta(\omega + \omega_0)$$

- After feeding the samples into low-pass filter, ω_0 and $-\omega_0$ components become $\omega_0 - \omega_s$ and $\omega_s - \omega_0$ components, respectively.



Aliasing

$$x(t) = \cos(\omega_0 t + \varphi) \overset{F}{\leftrightarrow} X(j\omega) = \pi e^{j\varphi}\delta(\omega - \omega_0) + \pi e^{-j\varphi}\delta(\omega + \omega_0)$$

$$X_r(j\omega) = \pi e^{j\varphi}\delta(\omega - (\omega_0 - \omega_s)) + \pi e^{-j\varphi}\delta(\omega + (\omega_0 - \omega_s))$$

$$\Rightarrow\ x_r(t) = \cos((\omega_0 - \omega_s)t + \varphi)$$

Aliasing

$$x(t) = \cos(\omega_0 t + \varphi), \qquad x_r(t) = \cos((\omega_0 - \omega_s)t + \varphi)$$

- Even if $\omega_s = 2\omega_0$, phase is changed (reversed):

$$x_r(t) = \cos(-\omega_0 t + \varphi) = \cos(\omega_0 t - \varphi)$$

- The original signal cannot be reconstructed even if the sampling frequency is exactly twice of the highest frequency of the signal.

- Nyquist sampling theorem: in order to precisely reconstruct a continuous-time signal, the sampling frequency has to be larger than twice of the highest frequency of the signal.

Thank you for your listening!