Lecture 3

Convolution and Linear Time-invariant Systems

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content

Convolution

- **For discrete signal, convolution sum:**

$$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$

- **For continuous signal, convolution integral:**

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$

2.1 Discrete-time Systems: the Convolution Sum

- Representing an arbitrary signal as a sequence of unit impulses

$$x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

an unit impulse located at n = k on the index n

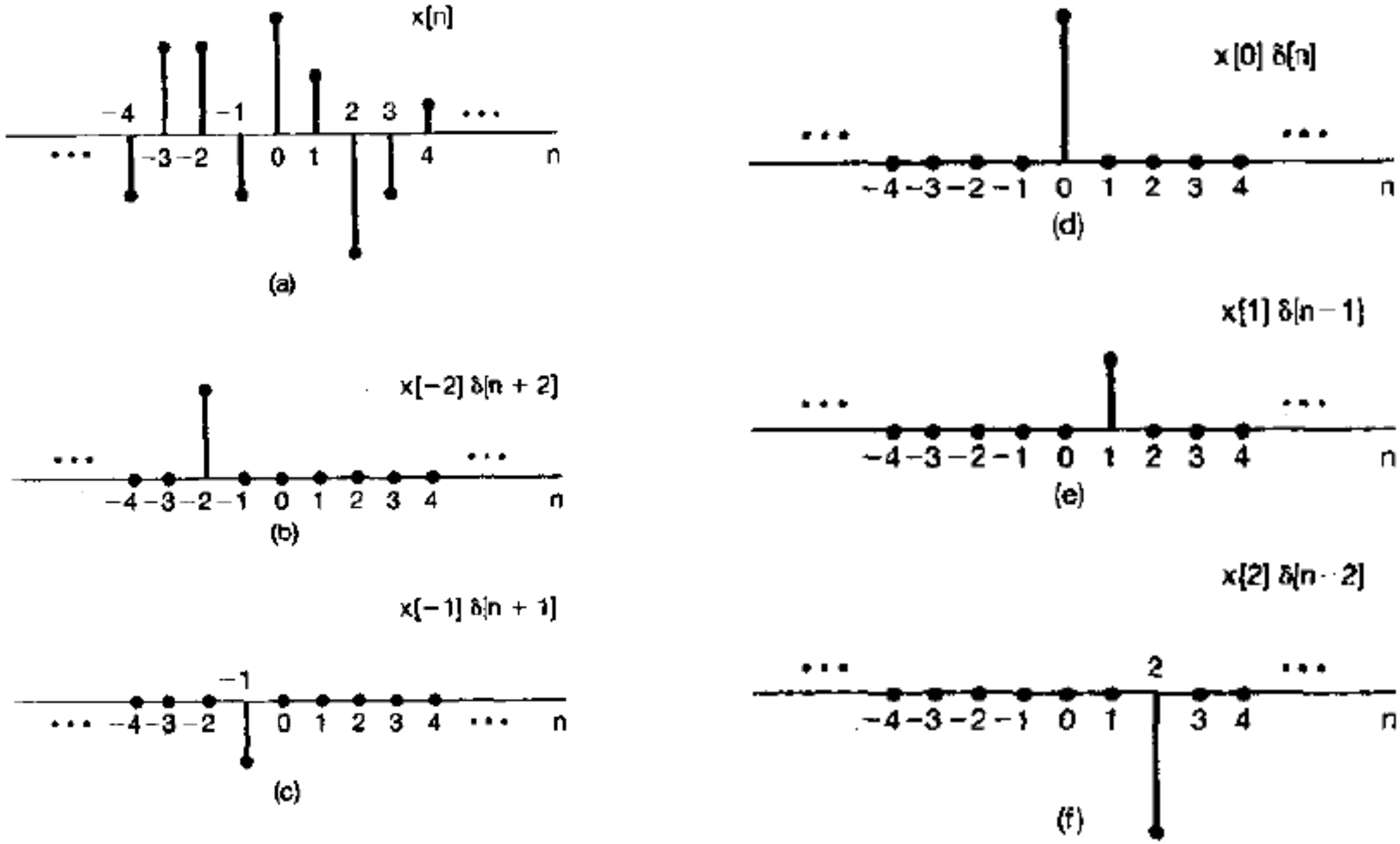


图 2.1 一个离散时间信号分解为一组加权的移位脉冲之和

2.1 Discrete-time Systems: the Convolution Sum

- Representing an arbitrary signal as a sequence of unit impulses

$$x[n] = x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

an unit impulse located at n = k on the index n

See Fig. 2.1, p.76 of text

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \quad \text{a special case}$$

2.1 Discrete-time Systems: the Convolution Sum

- Convolution sum can be used to represent an arbitrary signal as a sequence of unit impulses

$$x[n] = x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- For discrete signal, convolution sum:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- A Different Way to visualize the convolution sum
  - looked at on the index k

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Contribution to the output signal at time n

input signal

reflected-over version of $h[k]$ located at $k = n$

  - on the dummy index k, $h[k]$ is reflected over and shifted to k=n, weighted by $x[k]$ and summed to produce an output sample $y[n]$ at time n

See Figs 2.5, 2.6, 2.7, pp. 83-85 of text

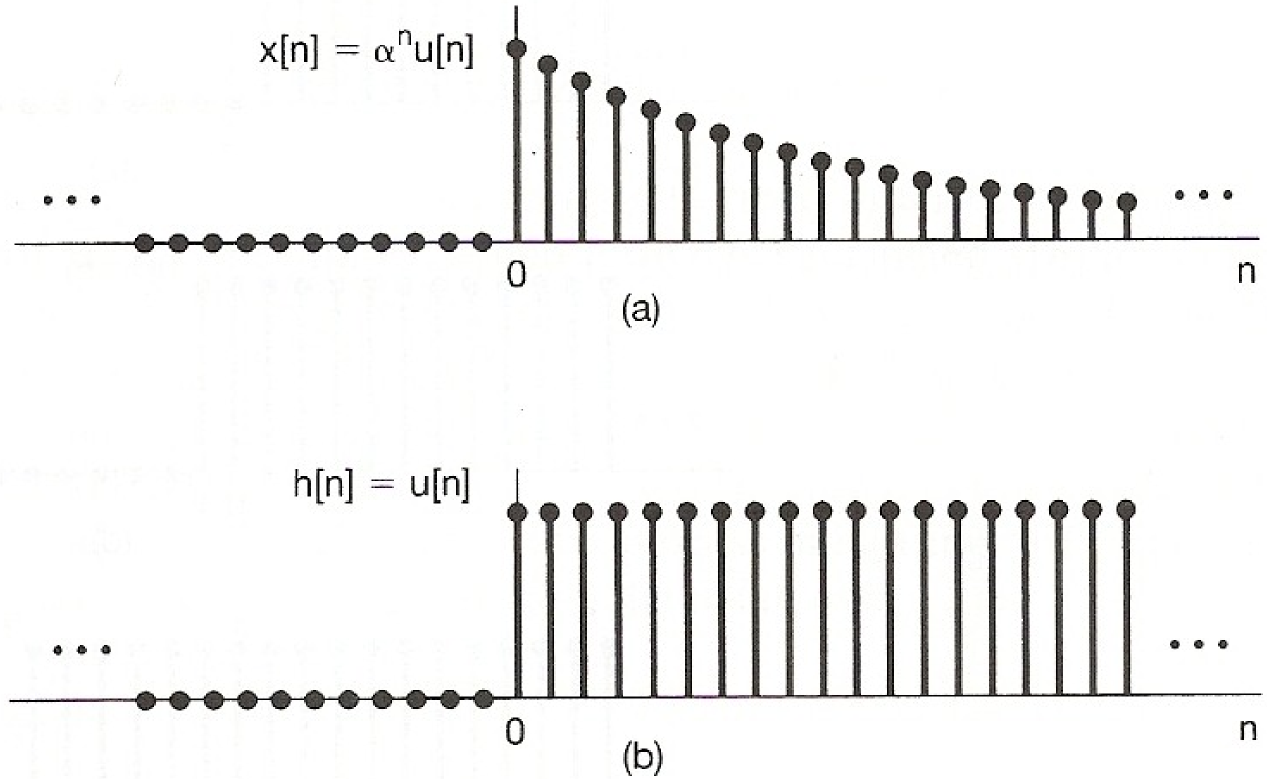


Figure 2.5 The signals $x[n]$ and $h[n]$ in Example 2.3.

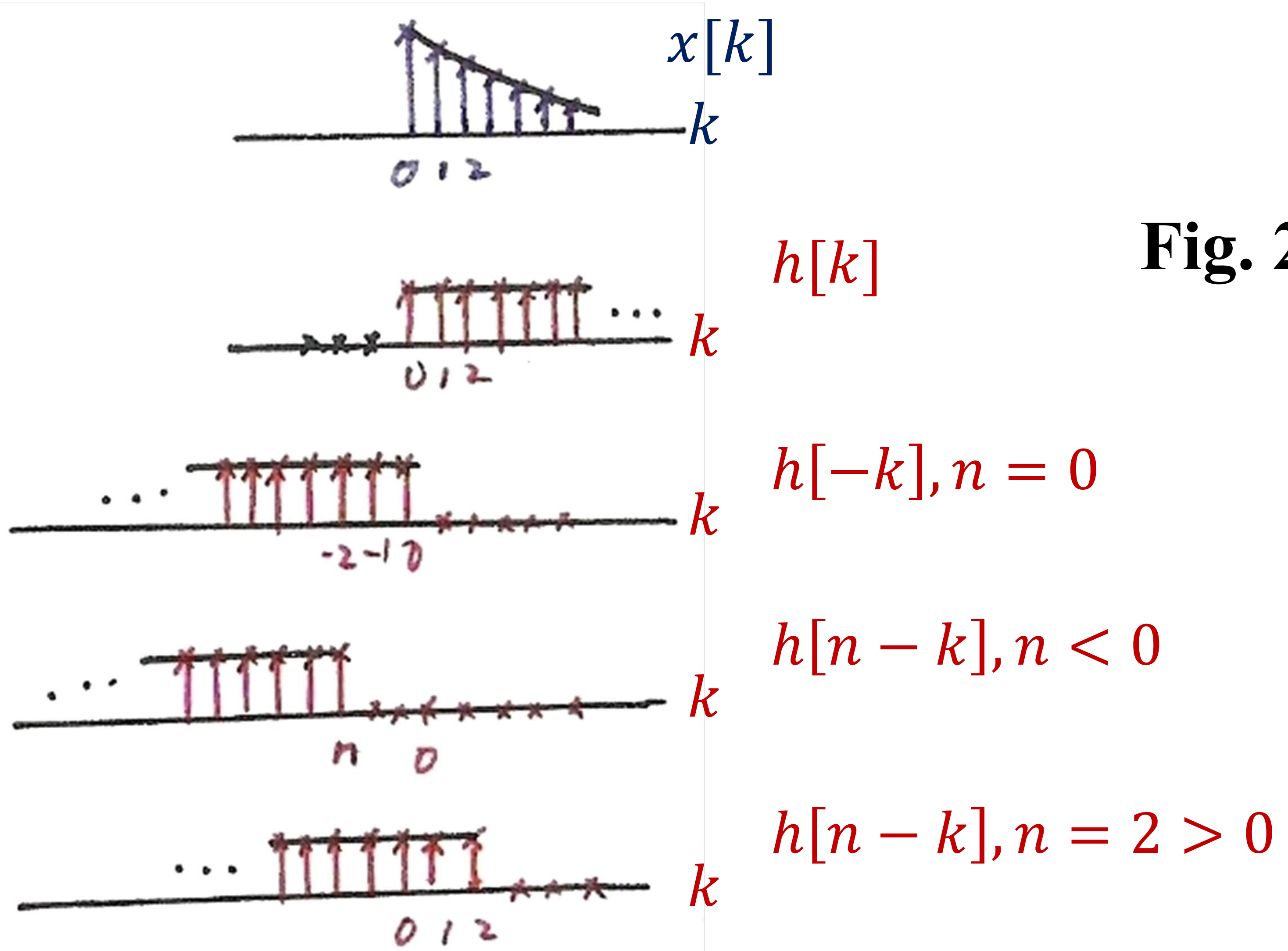


Fig. 2.6

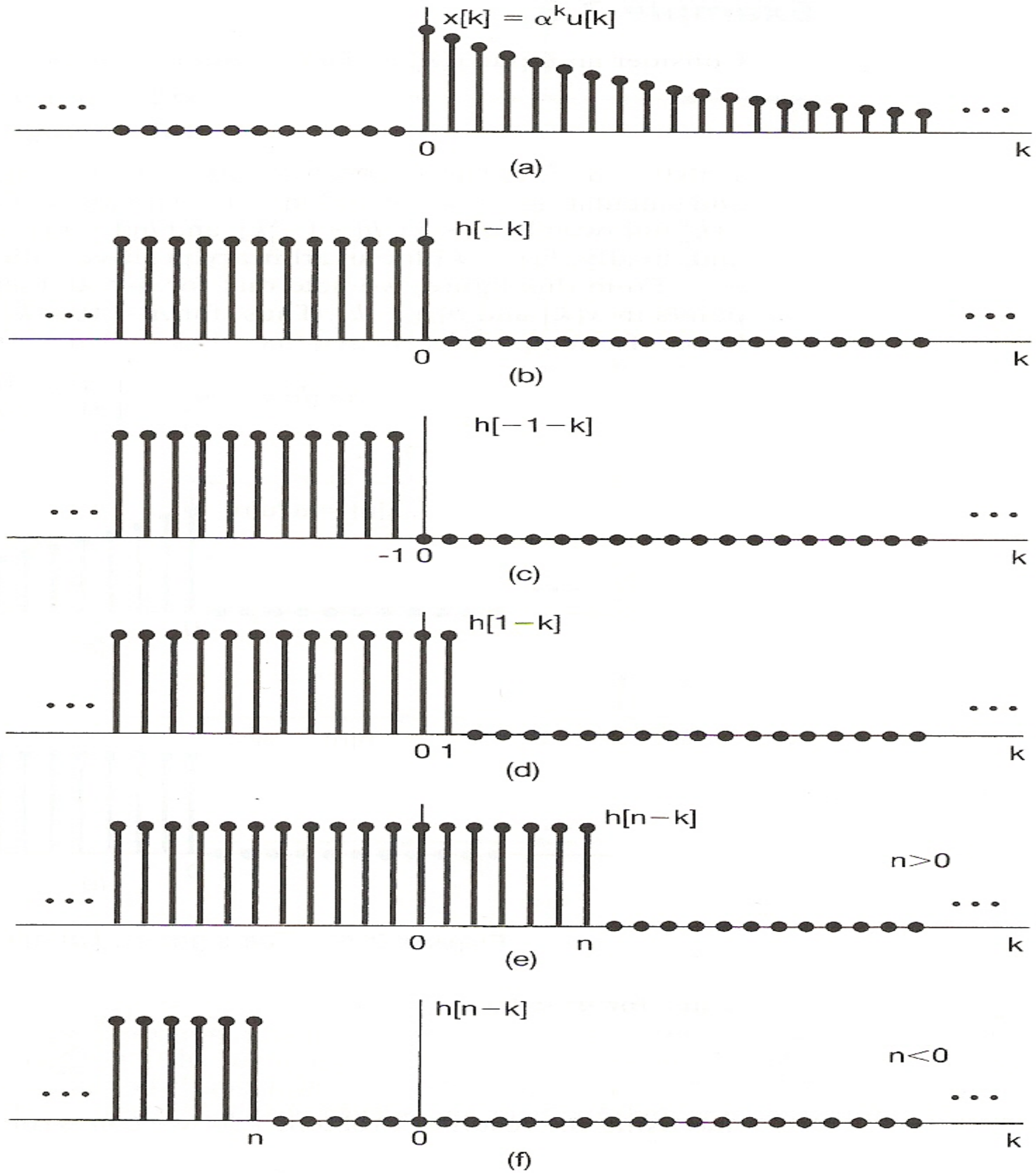


Figure 2.6 Graphical interpretation of the calculation of the convolution sum for Example 2.3.

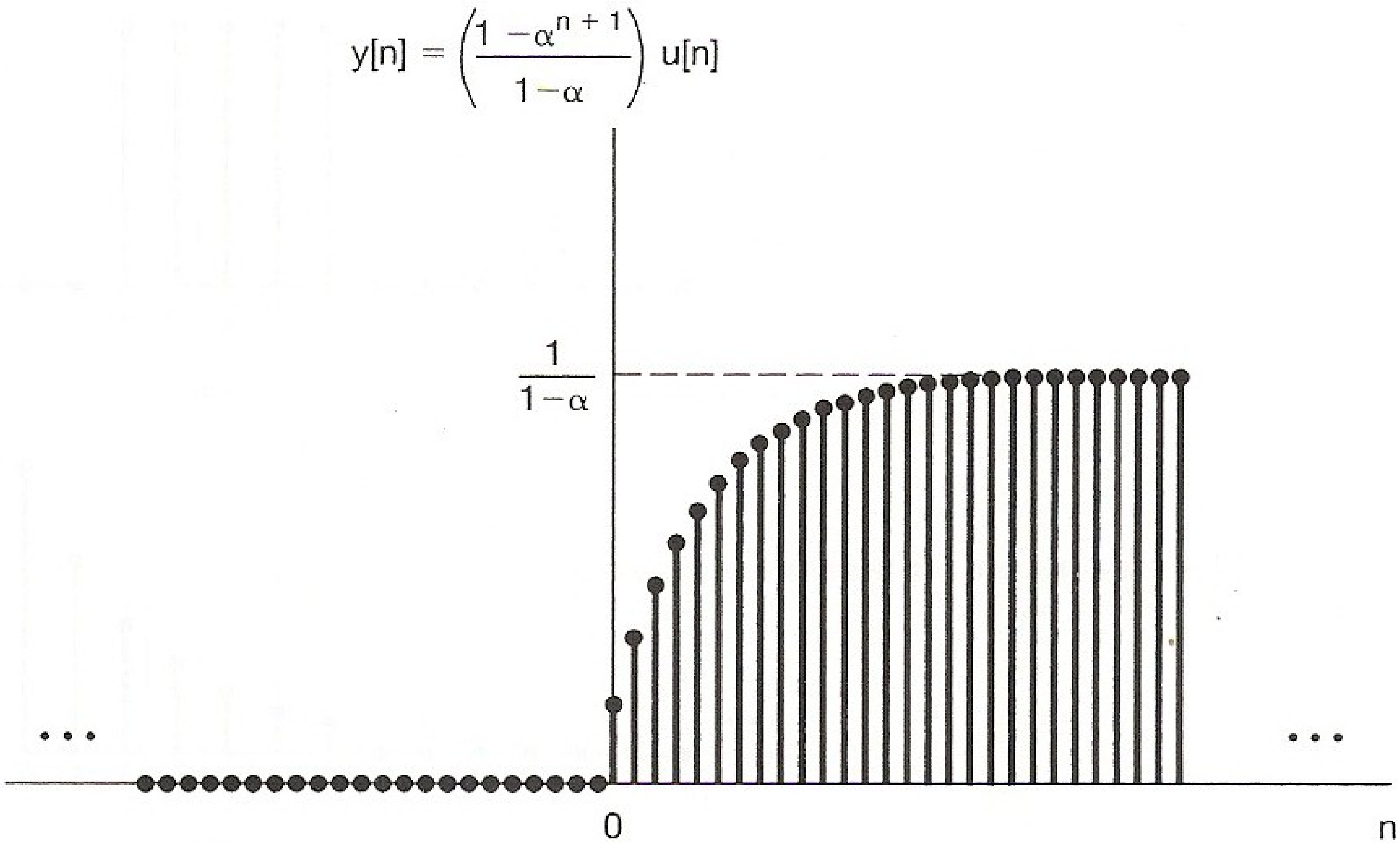


Figure 2.7 Output for Example 2.3.

2.2 Continuous-time System : the Convolution Integral

- Representing an arbitrary signal as an integral of impulses

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

See Fig 2.12 , p.91 of text

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

an impulse located at $t = \tau$ whose value is $x(\tau)$ (合成)

$$u(t) = \int_{0}^{\infty} \delta(t - \tau)d\tau$$ a special case

- A Different Way to visualize the convolution integral
  - Look on the index τ

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

output signal at time t

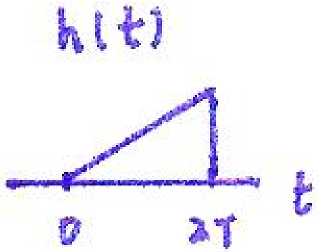
input signal

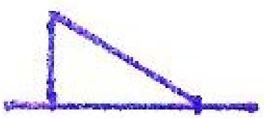
reflected-over version of $h(t)$ located at $\tau = t$

  - On the dummy index τ, $h(t)$ is reflected over and shifted to $\tau = t$, weighted by $x(t)$ and integrated to produce the output value at time t, $y(t)$

    see Figs.2.19,2.20,2.21,pp.100-101 of text

- A linear time-invariant system is completely characterized by its unit impulse response





$-2T$ $\quad$ 0

$h(-t)$

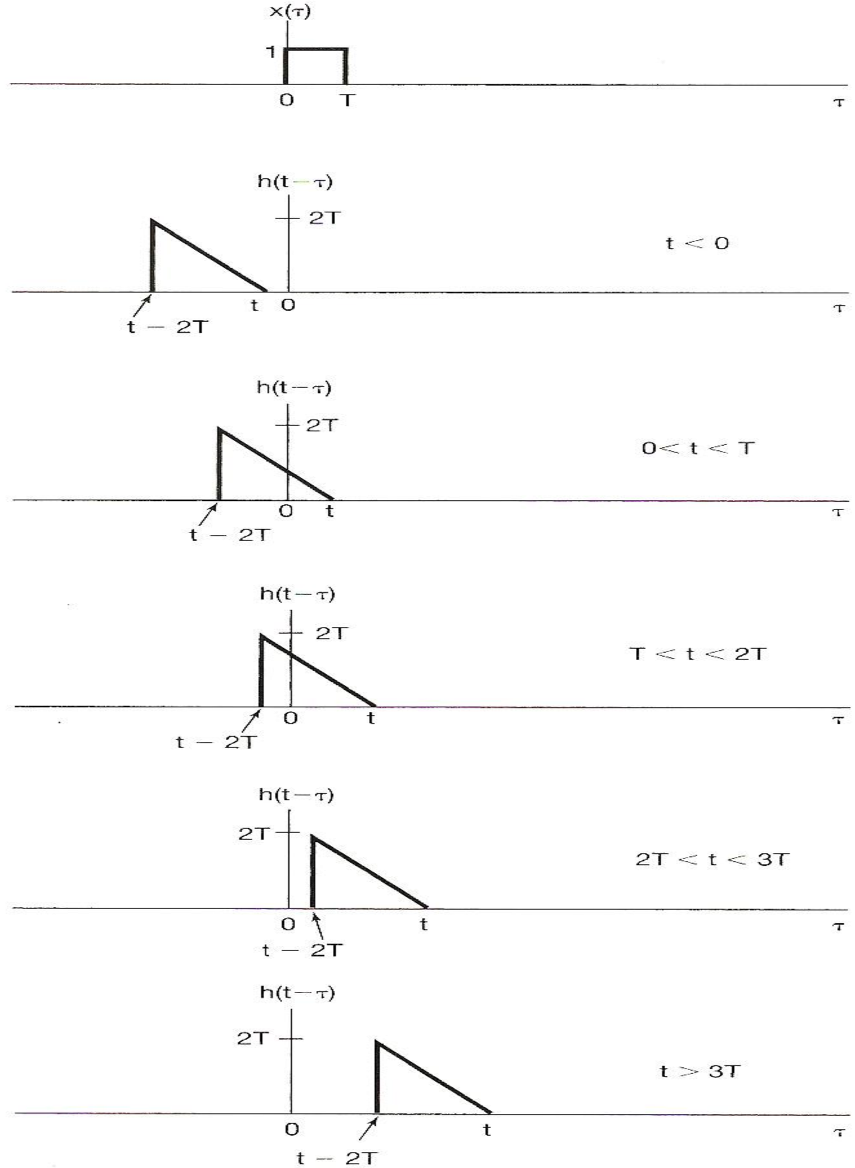
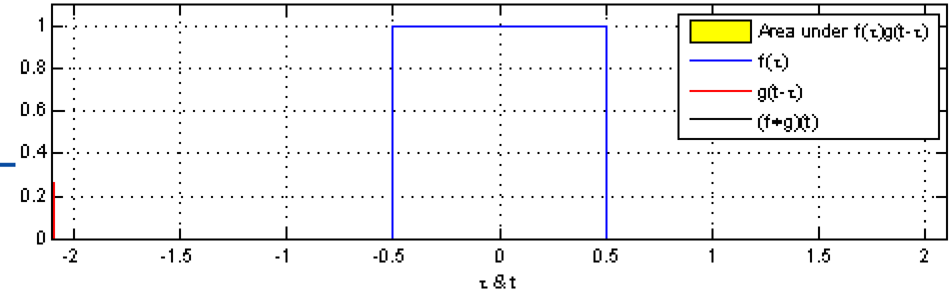
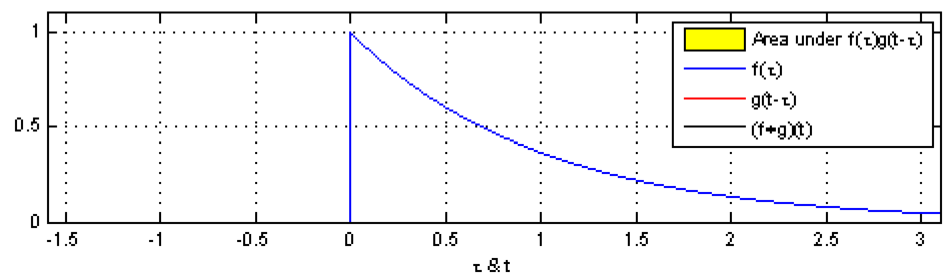


Figure 2.19 Signals $x(\tau)$ and $h(t-\tau)$ for different values of t for Example 2.7.



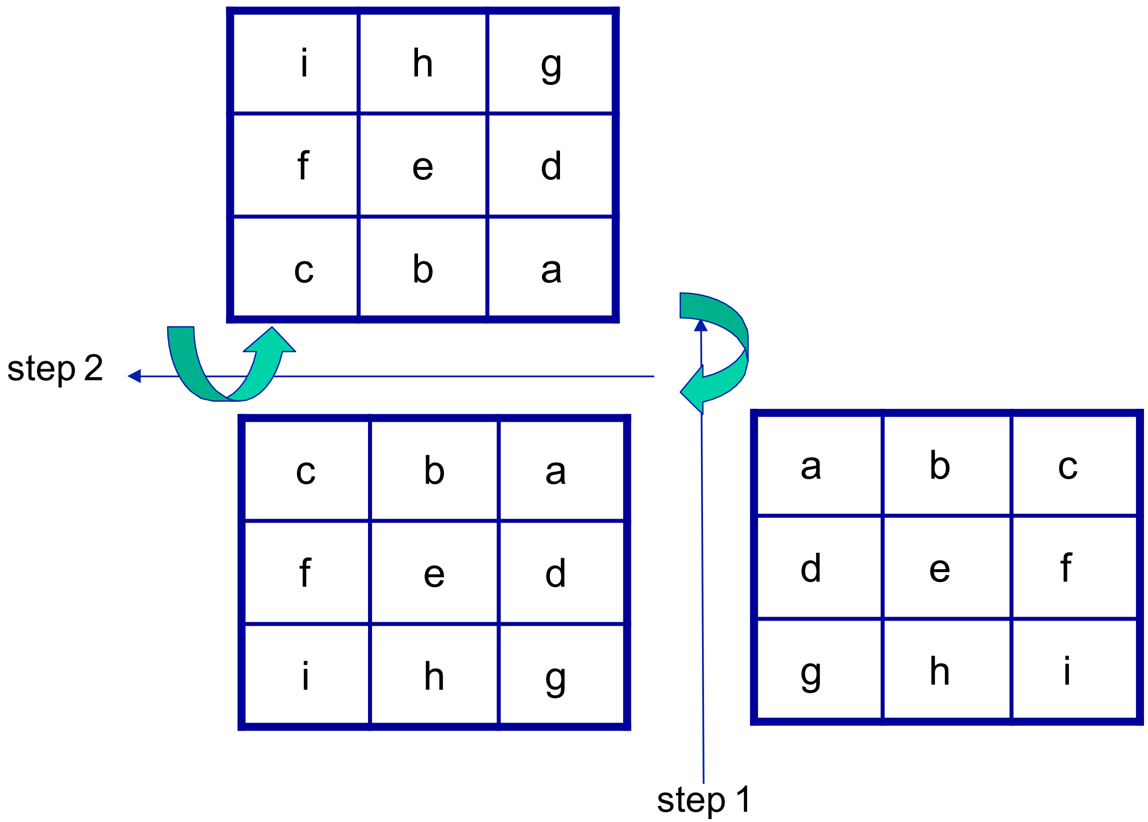
In this example, the red-colored "pulse", $g(\tau)$, is an even function ($g(-\tau) = g(\tau)$), so convolution is equivalent to correlation. A snapshot of this "movie" shows functions $g(t-\tau)$ and $f(\tau)$ (in blue) for some value of parameter t, which is arbitrarily defined as the distance from the $\tau = 0$ axis to the center of the red pulse. The amount of yellow is the area of the product $f(\tau) \cdot g(t-\tau)$, computed by the convolution/correlation integral. The movie is created by continuously changing t and recomputing the integral. The result (shown in black) is a function of t, but is plotted on the same axis as τ, for convenience and comparison.



In this depiction, $f(\tau)$ could represent the response of an RC circuit to a narrow pulse that occurs at $\tau = 0$. In other words, if $g(\tau) = \delta(\tau)$, the result of convolution is just $f(t)$. But when $g(\tau)$ is the wider pulse (in red), the response is a "smeared" version of $f(t)$. It begins at $t = -0.5$, because we defined t as the distance from the $\tau = 0$ axis to the center of the wide pulse (instead of the leading edge).

Matrix perspective

$$f[n_1,n_2]**h[n_1,n_2]=\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}f[m_1,m_2]h[n_1-m_1,n_2-m_2]$$



Convolution Example

$$f[n_1,n_2]**h[n_1,n_2]=\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty} f[m_1,m_2]h[n_1-m_1,n_2-m_2]$$

h

1	-1	-1
1	2	-1
1	1	1

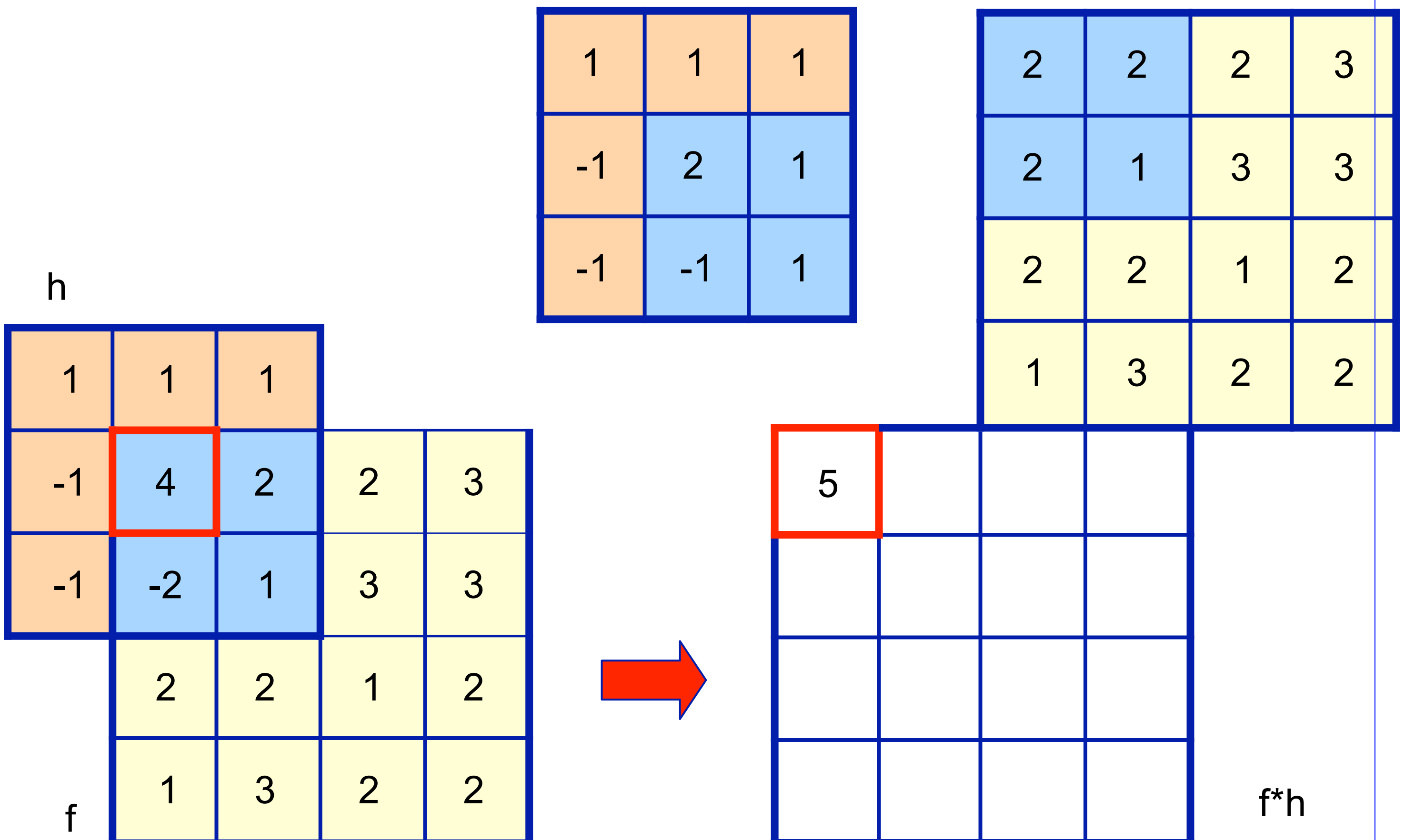
Rotate

1	1	1
-1	2	1
-1	-1	1

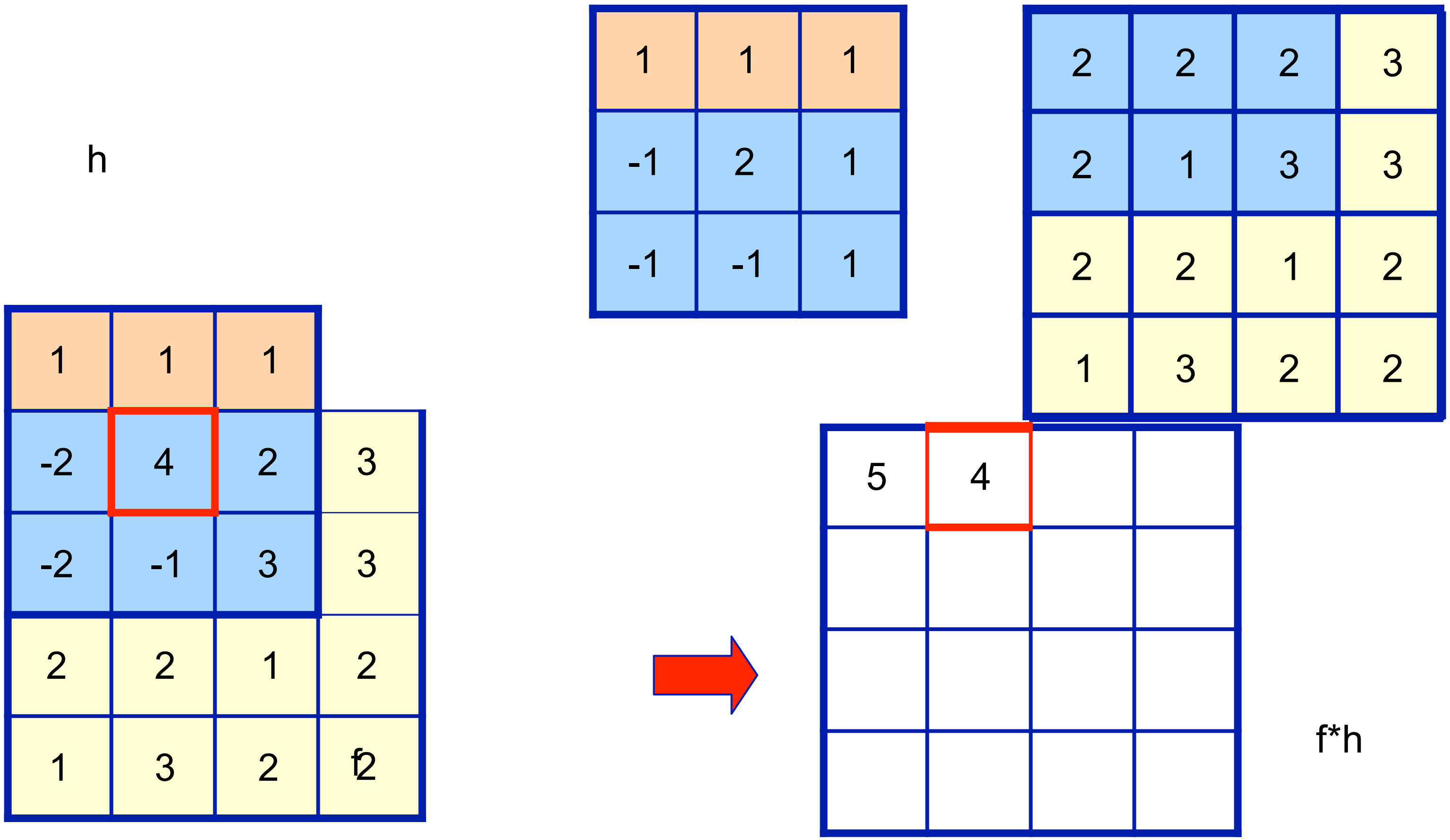
f

2	2	2	3
2	1	3	3
2	2	1	2
1	3	2	2

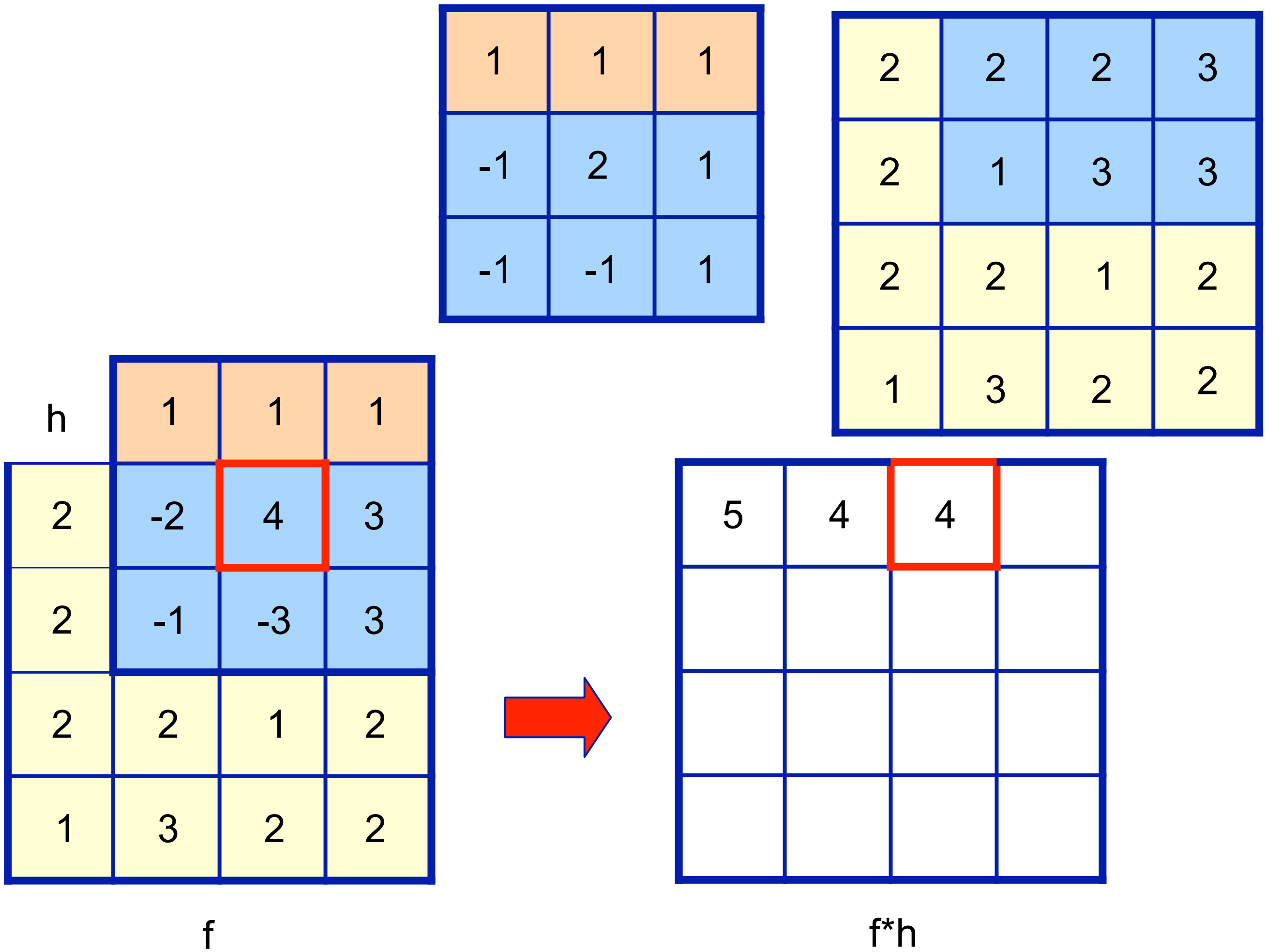
Convolution Example



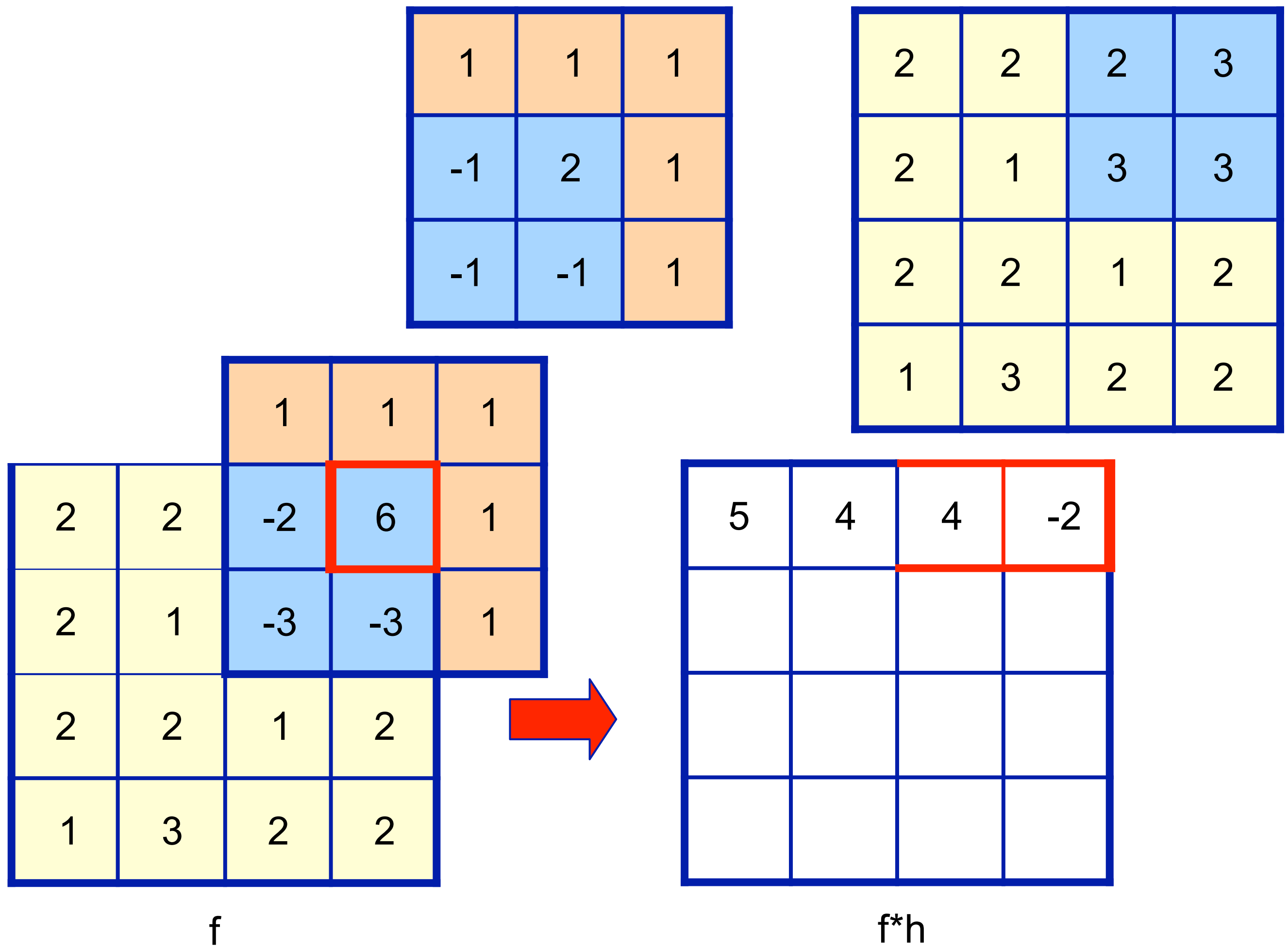
Convolution Example



Convolution Example



Convolution Example



Convolution Example

1	1	1
-1	2	1
-1	-1	1

2	2	2	3
2	1	3	3
2	2	1	2
1	3	2	2

h

1	2	2	2	3
-1	4	1	3	3
-1	-2	2	1	2
	1	3	2	2

f

5	4	4	-2
9			

f*h

Convolution Example

1	1	1
-1	2	1
-1	-1	1

2	2	2	3
2	1	3	3
2	2	1	2
1	3	2	2

h

2	2	2	3
-2	2	3	3
-2	-2	1	2
1	3	2	2

f

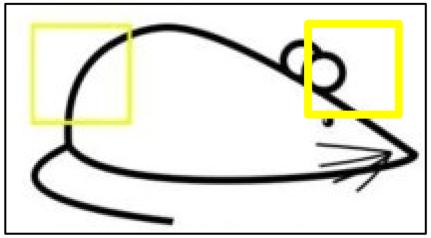
5	4	4	-2
9	6		

f*h

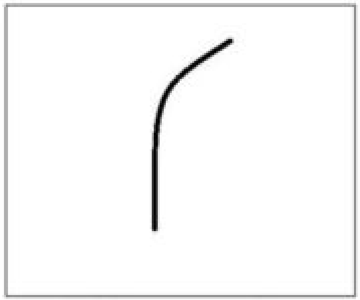
卷积运算的物理意义

- 对图像来说，卷积运算被用来判断输入图像与特征图像的匹配程度。
- 例如，对以下输入图片，找到符合特征图像的部分：

输入图像



特征图像



卷积运算的物理意义

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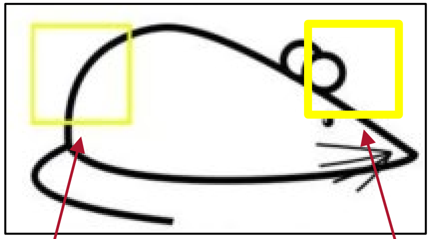
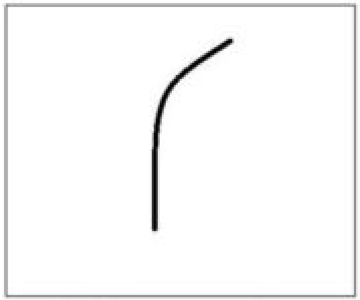


特征图像



- 用像素值表示：

0	0	0	0	0	30	0
0	0	0	0	30	0	0
0	0	0	30	0	0	0
0	0	0	30	0	0	0
0	0	0	30	0	0	0
0	0	0	30	0	0	0
0	0	0	0	0	0	0

0	0	0	0	0	0	0
0	40	0	0	0	0	0
40	0	40	0	0	0	0
40	20	0	0	0	0	0
0	50	0	0	0	0	0
0	0	50	0	0	0	0
25	25	0	50	0	0	0

0	0	0	0	0	0	30
0	0	0	0	50	50	50
0	0	0	20	50	0	0
0	0	0	50	50	0	0
0	0	0	50	50	0	0
0	0	0	50	50	0	0
0	0	0	50	50	0	0

卷积运算的物理意义

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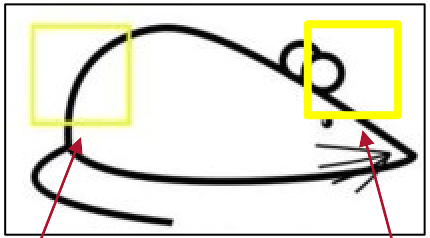
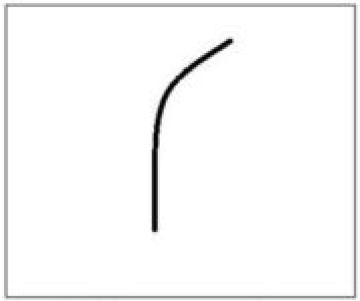


特征图像



0	0	0	0	0	30	0
0	0	0	0	30	0	0
0	0	0	30	0	0	0
0	0	0	30	0	0	0
0	0	0	30	0	0	0
0	0	0	30	0	0	0
0	0	0	0	0	0	0

0	0	0	0	0	0	0
0	40	0	0	0	0	0
40	0	40	0	0	0	0
40	20	0	0	0	0	0
0	50	0	0	0	0	0
0	0	50	0	0	0	0
25	25	0	50	0	0	0

0	0	0	0	0	0	30
0	0	0	0	50	50	50
0	0	0	20	50	0	0
0	0	0	50	50	0	0
0	0	0	50	50	0	0
0	0	0	50	50	0	0
0	0	0	50	50	0	0

内积

6600

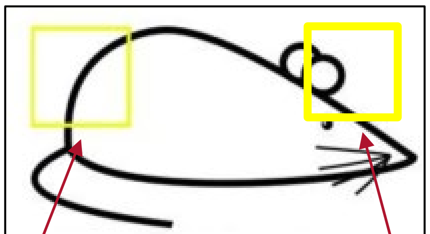
内积

2500

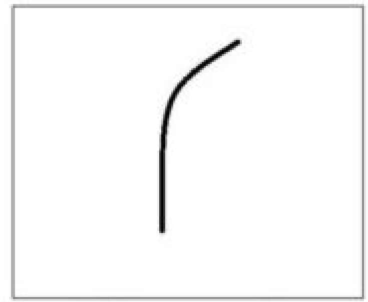
卷积运算的物理意义

- 例如，对以下输入图片，找到符合特征图像的部分：

输入图像



特征图像



0	0	0	0	0	30	0
0	0	0	0	30	0	0
0	0	0	30	0	0	0
0	0	0	30	0	0	0
0	0	0	30	0	0	0
0	0	0	30	0	0	0
0	0	0	0	0	0	0

0	0	0	0	0	0	0
0	40	0	0	0	0	0
40	0	40	0	0	0	0
40	20	0	0	0	0	0
0	50	0	0	0	0	0
0	0	50	0	0	0	0
25	25	0	50	0	0	0

0	0	0	0	0	0	30
0	0	0	0	50	50	50
0	0	0	20	50	0	0
0	0	0	50	50	0	0
0	0	0	50	50	0	0
0	0	0	50	50	0	0
0	0	0	50	50	0	0

内积

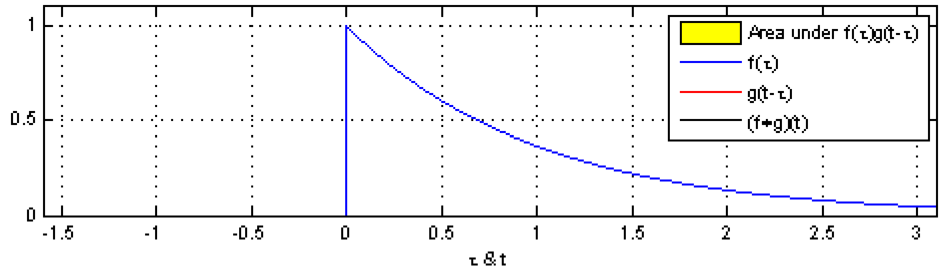
内积

更符合所需特征 ← 6600

2500

卷积运算的物理意义

- 信号处理中，卷积运算的本质与上述图像卷积运算类似：



- 卷积运算的结果表示一维曲线f(t)与特征曲线g(t)的匹配程度，可用来滤除我们所不希望的特征：滤波

Linear Time-invariant Systems

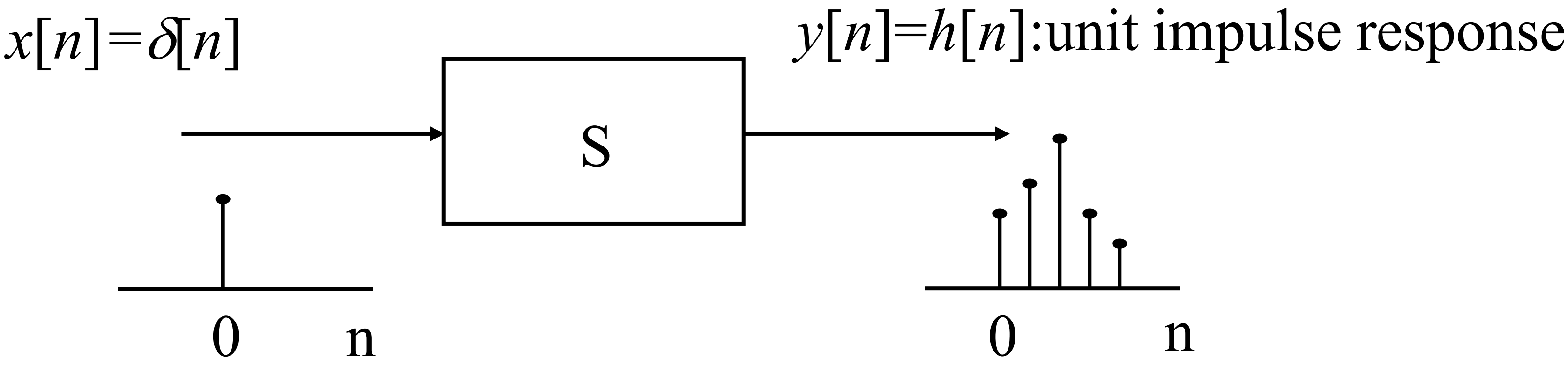
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\,\tau$$

But what is $h[n]$ or $h(t)$?

Linear Time-invariant Systems

- Defining the output for an unit impulse input as the Unit Impulse Response, $h[n]$

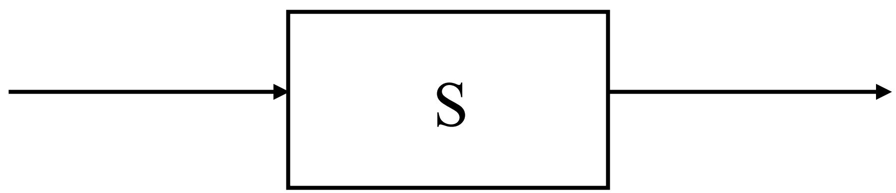


Linear Time-invariant Systems

- Defining the output for an unit impulse input as the Unit Impulse Response, $h[n]$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



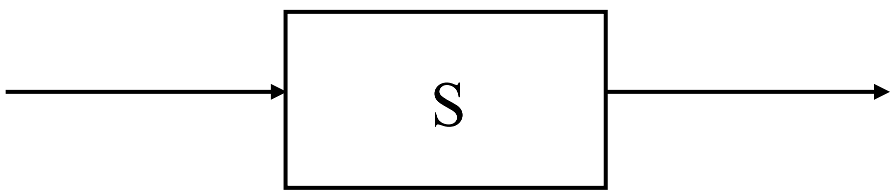
- Theoretically, for any $x[n]$, which can be constructed by $\delta[n]$, the system output $y[n]$ can be computed based on $h[n]$.

Linear Time-invariant Systems

- Defining the output for an unit impulse input as the Unit Impulse Response, $h[n]$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



- Theoretically, for any $x[n]$, which can be constructed by $\delta[n]$, the system output $y[n]$ can be computed based on $h[n]$, only if the system is a linear time-invariant system.

线性（Linear）系统

- **Linear**: a system is linear if the output of the scaled sum of two input signals is the equivalent scaled sum of outputs （如果输入信号是两个信号的加权和，那么输出信号也是这两个输入信号对应输出的加权和）

- 例：如果系统对输入信号$x(t)$的输出是$y(t)$，则系统对输入信号$ax_1(t)+bx_2(t)$的输出为$ay_1(t)+by_2(t)$

- 线性系统对信号在频域表示下的处理具有重要意义。

时不变（Time-invariant）系统

- **Time-invariance**: a system is time invariant if the system’s output is the same, given the same input signal, regardless of time.（系统特性不随时间发生变化）

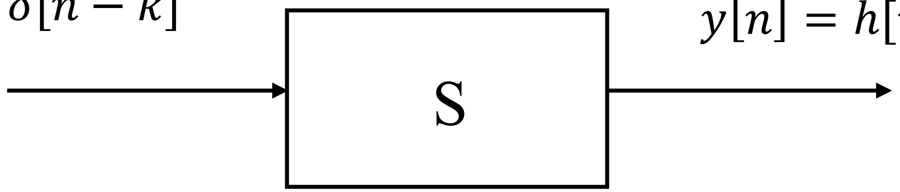
- 例：如果系统对输入信号$x(t)$的输出是$y(t)$，则系统对输入信号$x(t-t_0)$的输出为$y(t-t_0)$

- 时不变系统对信号在频域表示下的处理具有重要意义。

Linear Time-invariant Systems

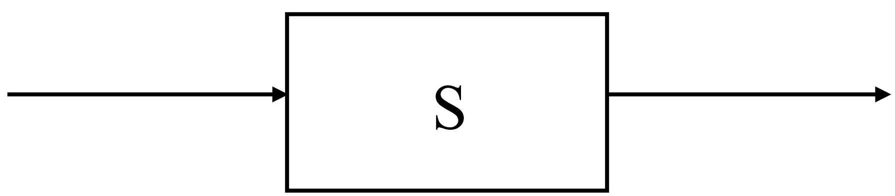
- Time-invariant: for any k, response of $\delta[n-k]$ is always $h[n-k]$

$x[n] = \delta[n-k]$ $\qquad$ $y[n] = h[n-k]$



- Linearity: response can always be represented by the scaled sum of $h[n-k]$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



Properties of Linear Time-invariant Systems

- Commutative Property （交换律）

$$x[n]*h[n]=h[n]*x[n]$$

$$x(t)*h(t)=h(t)*x(t)$$

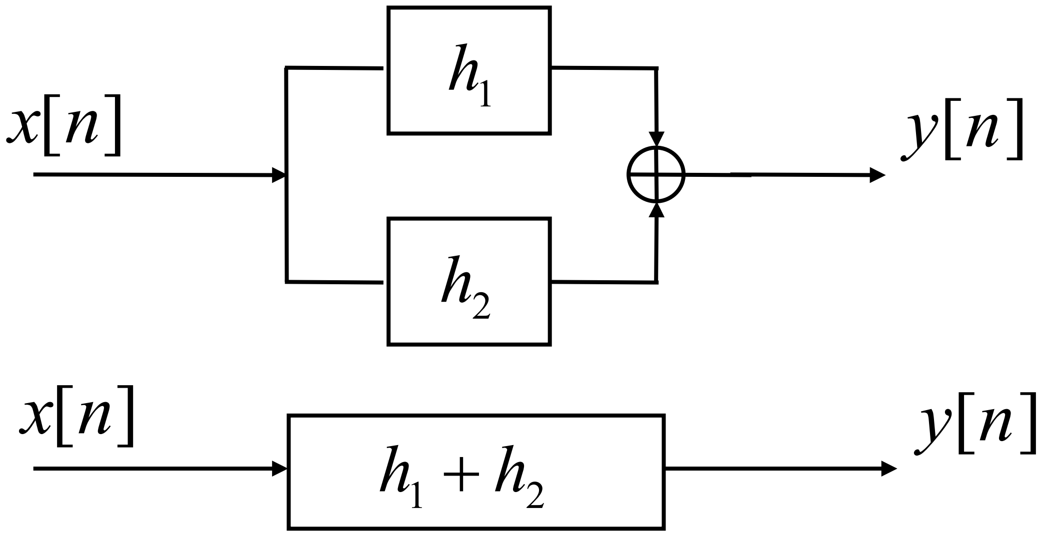
  – the role of input signal and unit impulse response is interchangeable, giving the same output signal

  – In evaluating the convolution sum or integral, the input signal can be reflected over and weighted by the unit impulse response

- Distributive Property (分配率)

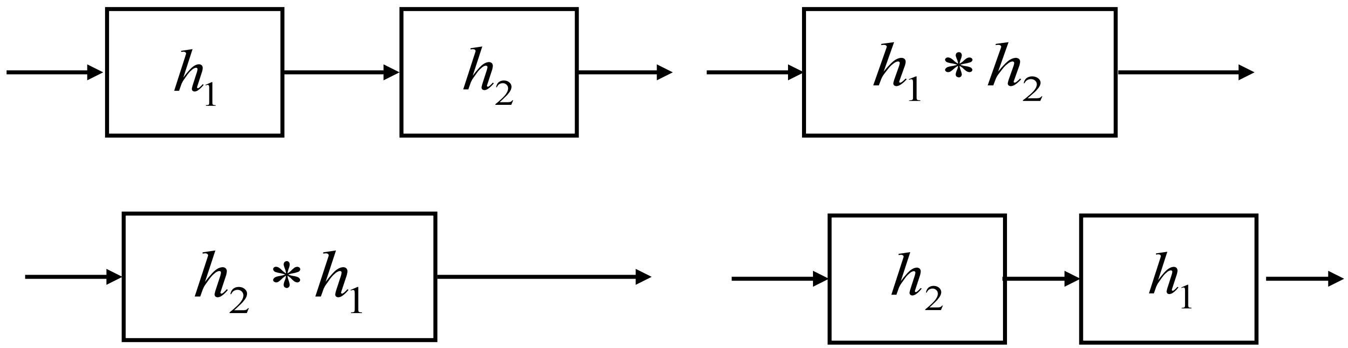
$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$$

$$x(t)*[h_1(t)+h_2(t)] = x(t)*h_1(t)+x(t)*h_2(t)$$



- Associative Property（结合律）

$$x[n]*(h_1[n]*h_2[n])=(x[n]*h_1[n])*h_2[n]$$



  - Cascade of two systems gives an unit impulse response which is the convolution of the unit impulse responses of the two individual systems
  - The behavior of a cascade of two systems is independent of the order in which the two systems are cascaded

- Causality
  - causal if $y[n]$ dose not depend on $x[k]$ for $k > n$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  - Causal iff $h[n]$=0, $n < 0$

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

- Causality
  - continuous-time

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

  - Causal iff h(t)=0,t<0

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Unit step response

output for an unit step function input

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]$$ Running sum

$$h[n] = s[n] - s[n-1]$$ First difference

similarly

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$ Running integral

$$h(t) = \frac{ds(t)}{dt}$$ First derivative

- Unit step response

output for an unit step function input

$$s[n] = u[n] * h[n] = \sum_{k=0}^{\infty} h[n-k]$$

令$k' = n - k$:

$$s[n] = u[n] * h[n] = \sum_{k=0}^{\infty} h[n-k] = \sum_{k'=n}^{-\infty} h[k'] = \sum_{k=n}^{-\infty} h[k]$$

$$\Rightarrow \ s[n-1] = \sum_{k=n-1}^{-\infty} h[k]$$

因此：

$$s[n] - s[n-1] = \sum_{k=n}^{-\infty} h[k] - \sum_{k=n-1}^{-\infty} h[k] = h[n]$$

本章部分关键词汇中英文对照表

卷积	Convolution
卷积和	Convolution sum
卷积积分	Convolution integral
单位冲激响应	Unit impulse response
单位阶跃响应	Unit step response
线性	Linearity
时不变	Time-invariant