Lecture 7 Fourier Series of Discrete-time Signals

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Outline: **Lecture 7:** Fourier Series of Discrete-time Signals

- Discrete Time Fourier Series
  - Fourier Series Representation
  - Determination of Fourier Series Coefficient
  - Fourier Series Properties

- Application Example
  - System Characterization
  - Filtering

Outline: **Lecture 7:** Fourier Series of Discrete-time Signals

- Discrete Time Fourier Series

  Fourier Series Representation

  Determination of Fourier Series Coefficient

  Fourier Series Properties

- Application Example

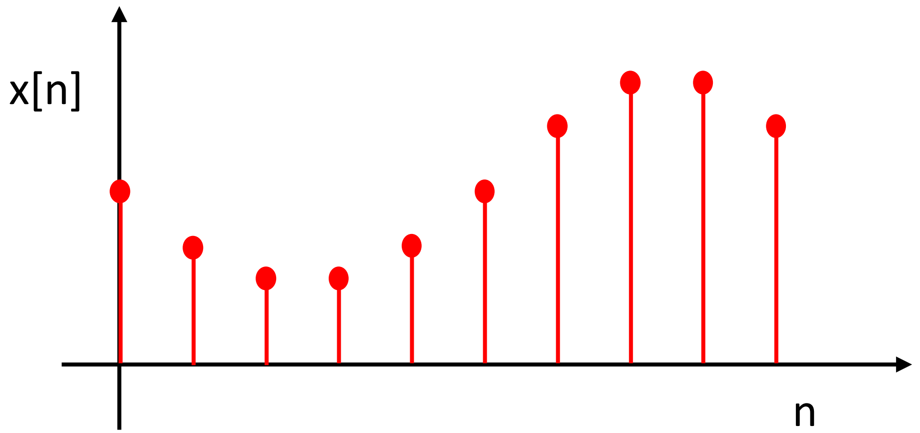
  System Characterization

  Filtering

Discrete Time Fourier Series Representation

Discrete-time signal

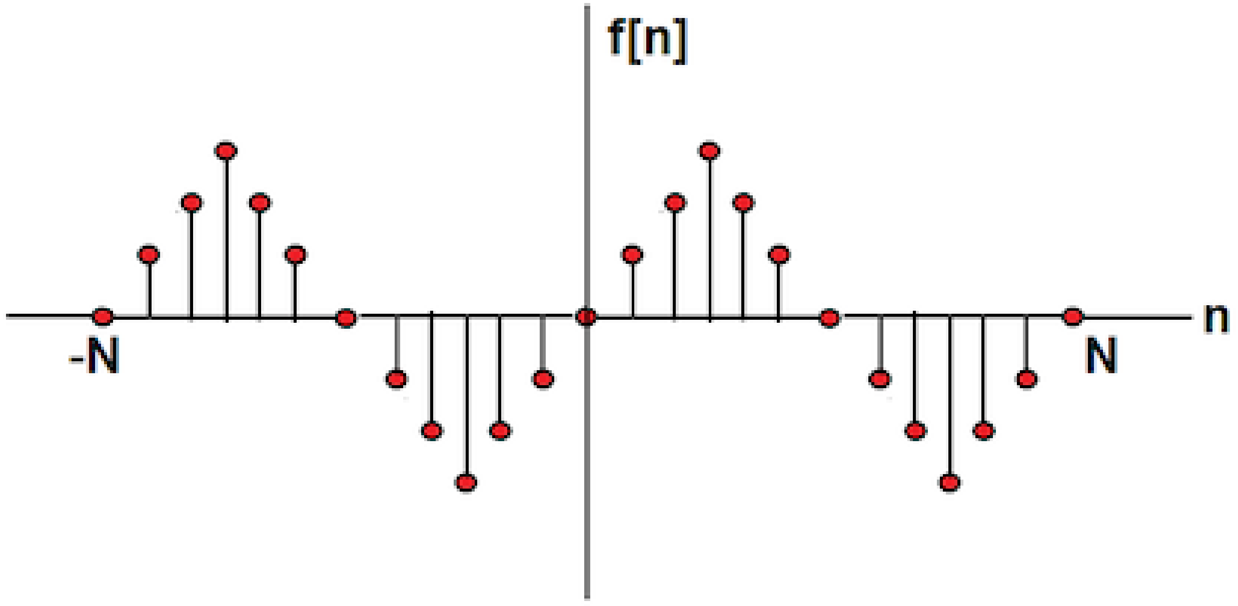
- A discrete-time signal is a time series consisting of a sequence of quantities, denoted by $x[n]$, where n is an integer value that varies discretely.



Discrete Time Fourier Series Representation

Periodic discrete-time signal

- A discrete-time periodic signal satisfies $x[n] = x[n+N]$, where N is the period. Thus, the frequency $\omega_0 = \frac{2\Pi}{N}$.



Discrete Time Fourier Series Representation

Fourier series

- Any arbitrary periodic function can be represented by a harmonically related trigonometric series.

- For continuous-time signal:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2, \ldots$$

- This should also be correct for discrete-time signal.

Discrete Time Fourier Series Representation

Fourier series

- For continuous-time signal:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2, \ldots$$

- For discrete-time signal:

$$x[n] = \sum_{k} a_k e^{jk\omega_0 n}, k = 0, \pm 1, \pm 2, \ldots$$

Is k still located in the range of $[-\infty, \infty]$?

Discrete Time Fourier Series Representation

- Important Differences Between Continuous-time and Discrete-time Exponential/Sinusoidal Signals

  – For discrete-time, signals with frequencies ω_0 and $\omega_0 + m \cdot 2\pi$ are identical. This is Not true for continuous-time.

重要！

Discrete-time:

$$e^{j(\omega_0 + m \cdot 2\pi)n} = e^{j\omega_0 n}$$

Continuous-time:

$$e^{j(\omega_0 + m \cdot 2\pi)t} \neq e^{j\omega_0 t}$$

Continuous/Discrete Sinusoidals

Discrete-time:

$$e^{j(\omega_0+m\cdot 2\pi)n} = \cos(\omega_0 n + m\cdot 2\pi\, n) + j\sin(\omega_0 n + m\cdot 2\pi\, n)$$

$$= \cos(\omega_0 n) + j\sin(\omega_0 n) \ \ (\text{as } m.n \text{ is an integer})$$

$$= e^{j\omega_0 n}$$

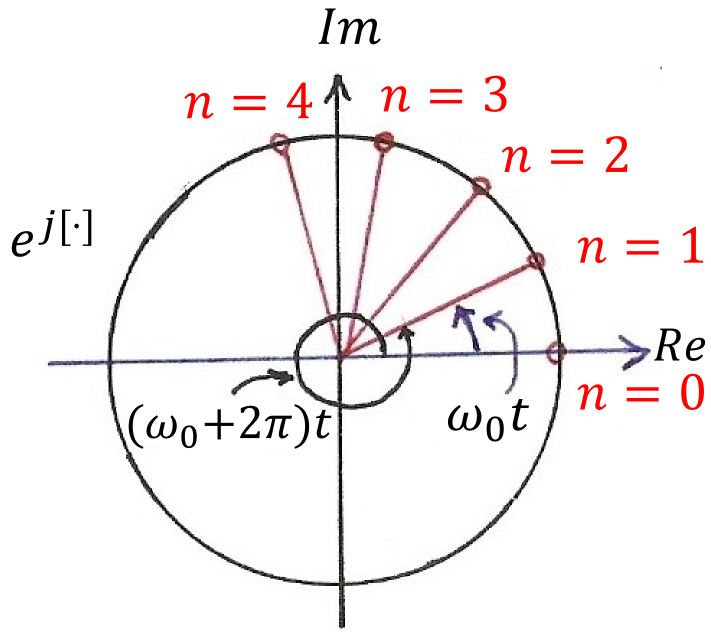
重要！

Continuous-time:

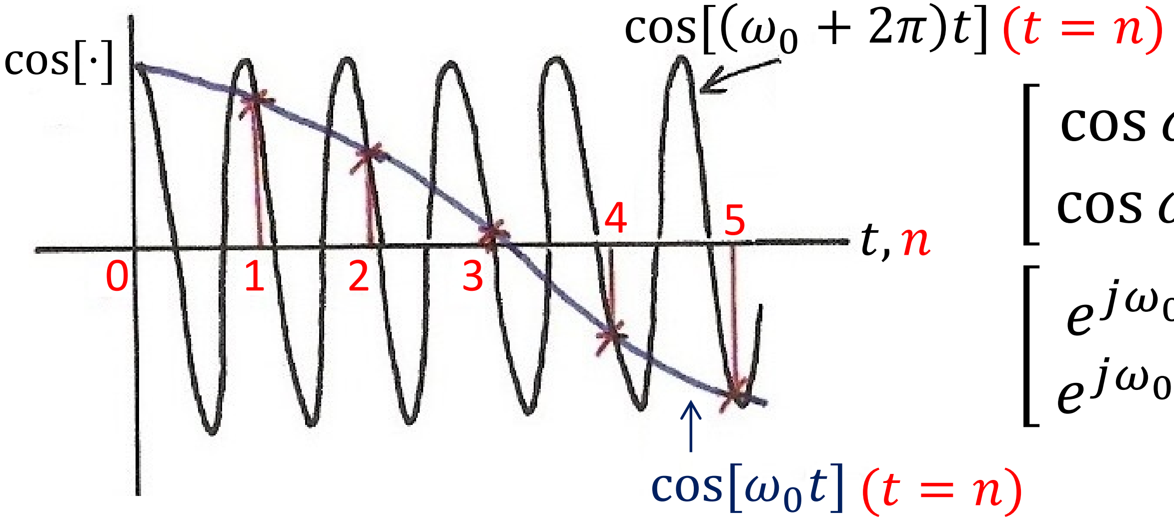
$$e^{j(\omega_0+m\cdot 2\pi)t} = \cos(\omega_0 t + m\cdot 2\pi\, t) + j\sin(\omega_0 t + m\cdot 2\pi\, t)$$

$$\neq \cos(\omega_0 t) + j\sin(\omega_0 t) \ \ (\text{as } m.t \text{ may not be an integer})$$

$$= e^{j\omega_0 t}$$

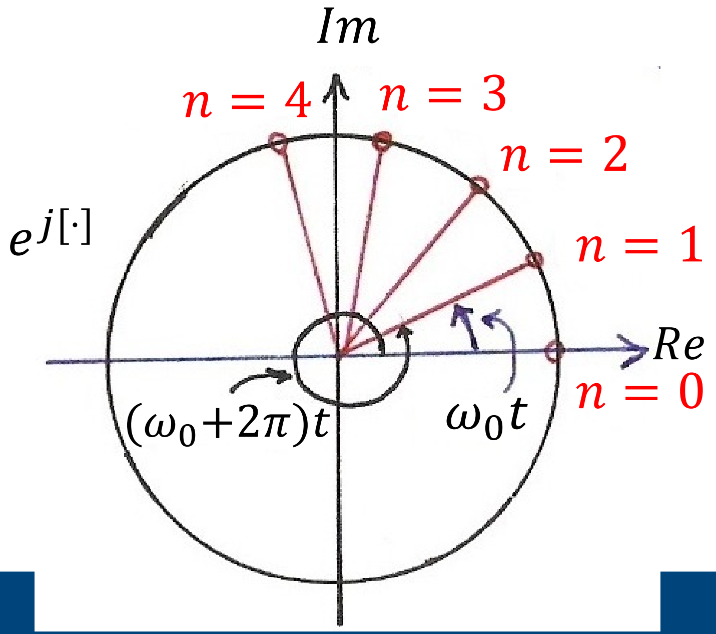


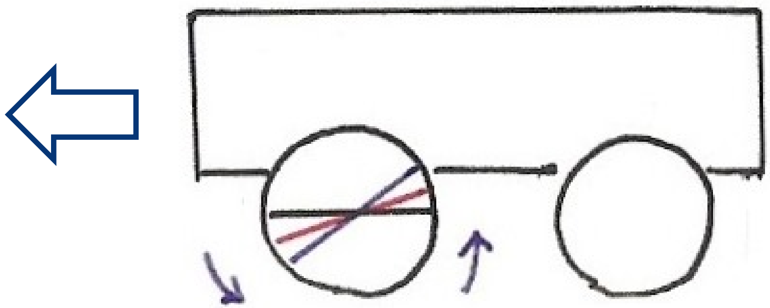
Discrete Time Fourier Series Representation



$$\begin{cases} \cos \omega_0 t \neq \cos(\omega_0 + 2\pi)t \\ \cos \omega_0 n = \cos(\omega_0 + 2\pi)n \end{cases}$$

$$\begin{cases} e^{j\omega_0 t} \neq e^{j(\omega_0+2\pi)t} \\ e^{j\omega_0 n} = e^{j(\omega_0+2\pi)n} \end{cases}$$





Discrete Time Fourier Series Representation

Fourier series

- For discrete-time signal, consider the frequency $\Phi_{\mathrm{k+N}}[n]$ for k+N:

$$\Phi_{\mathrm{k+N}}[n] = e^{j(k+N)\omega_0 n}$$

$$= e^{jk\omega_0 n} . e^{jN\omega_0 n}$$

$$= e^{jk\omega_0 n} . e^{jN\frac{2\Pi}{N}n}$$

$$= e^{jk\omega_0 n} = \Phi_{\mathrm{k}}[n]$$

Discrete Time Fourier Series Representation

Fourier series

- There are only N distinct frequencies for discrete-time signal. Those are:

$$\Phi_0[n], \Phi_1[n], \dots \Phi_{N-1}[n]$$

or

$$\Phi_1[n], \Phi_2[n], \dots \Phi_N[n]$$

or

……

Discrete Time Fourier Series Representation

Discrete-time Fourier series representation

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

$\langle N \rangle$ indicates N consecutive integers, *e.g.*, [0, N-1], [1, N], [2, N+1], *etc.*

Outline: Lecture 7: Fourier Series of Discrete-time Signals

- Discrete Time Fourier Series
  - Fourier Series Representation
  - Determination of Fourier Series Coefficient
  - Fourier Series Properties

- Application Example
  - System Characterization
  - Filtering

Determination of Fourier Series Coefficient

- Continuous-time Fourier series coefficient:

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

- Discrete-time Fourier series coefficient:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n}$$

Determination of Fourier Series Coefficient

Determination of discrete-time Fourier series coefficient

Proof:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

$$\sum_{n=\langle N \rangle} x[n]\, e^{-jr\omega_0 n} = \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}\, e^{-jr\omega_0 n}$$

$$\sum_{n=\langle N \rangle} x[n]\, e^{-jr\omega_0 n} = \sum_{k=\langle N \rangle} a_k \sum_{n=\langle N \rangle} e^{j(k-r)\omega_0 n}$$

Determination of Fourier Series Coefficient

Determination of discrete-time Fourier series coefficient

Proof:

- For k=r:

$$\sum_{n=\langle N\rangle} e^{j(k-r)\omega_0 n} = \sum_{n=\langle N\rangle} 1 = N$$

- For k≠ r:

$$\sum_{n=\langle N\rangle} e^{j(k-r)\omega_0 n} = \sum_{n=\langle N\rangle} \cos[\omega_0(k-r)n] + j\sum_{n=\langle N\rangle} \sin[\omega_0(k-r)n] = 0$$

Determination of Fourier Series Coefficient

Determination of discrete-time Fourier series coefficient

Proof:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

$$\sum_{n=\langle N \rangle} x[n]\, e^{-jr\omega_0 n} = \sum_{k=\langle N \rangle} a_k \sum_{n=\langle N \rangle} e^{j(k-r)\omega_0 n}$$

$$\sum_{n=\langle N \rangle} x[n]\, e^{-jr\omega_0 n} = a_k N$$

Determination of Fourier Series Coefficient

Discrete-time Fourier series representation:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

Discrete-time Fourier series coefficient:

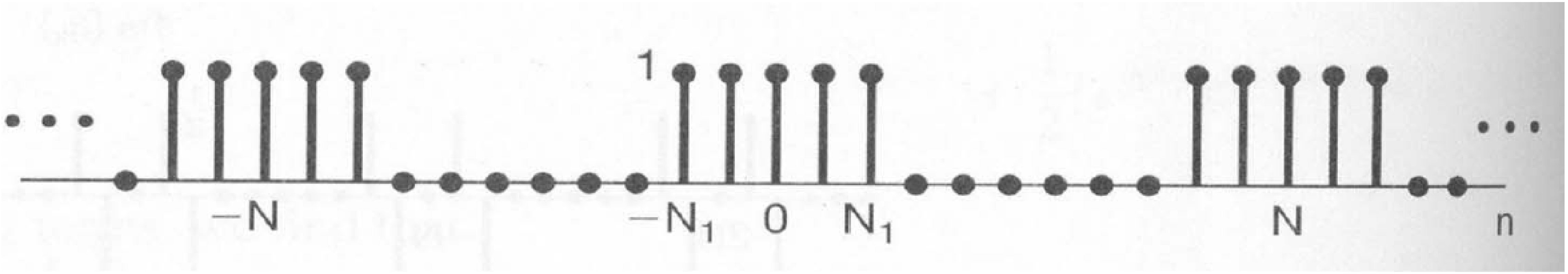
$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

There is no convergence issue for discrete-time Fourier series!

Determination of Fourier Series Coefficient

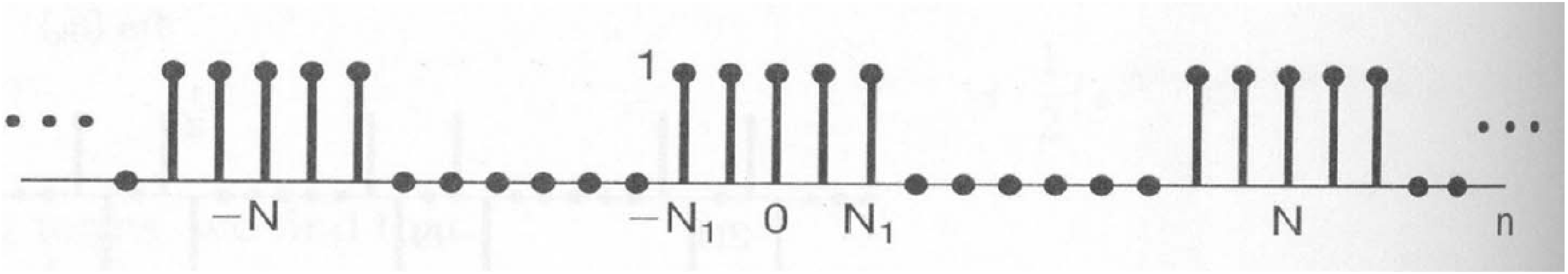
Example

Find a_k for the following discrete-time periodic signal



Determination of Fourier Series Coefficient

Solution



$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n}$$

$$= \frac{1}{N}\frac{e^{jk\omega_0 N_1} - e^{-jk\omega_0(N_1+1)}}{1 - e^{-jk\omega_0}} = \frac{1}{N}\frac{e^{-jk\frac{\omega_0}{2}}}{e^{-jk\frac{\omega_0}{2}}}\frac{e^{jk\omega_0(N_1+\frac{1}{2})} - e^{-jk\omega_0(N_1+\frac{1}{2})}}{e^{jk\frac{\omega_0}{2}} - e^{-jk\frac{\omega_0}{2}}}$$

Determination of Fourier Series Coefficient

Solution

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n}$$

$$= \frac{1}{N}\frac{e^{jk\omega_0 N_1} - e^{-jk\omega_0(N_1+1)}}{1-e^{-jk\omega_0}} = \frac{1}{N}\frac{e^{-jk\frac{\omega_0}{2}}}{e^{-jk\frac{\omega_0}{2}}}\frac{e^{jk\omega_0(N_1+\frac{1}{2})} - e^{-jk\omega_0(N_1+\frac{1}{2})}}{e^{jk\frac{\omega_0}{2}} - e^{-jk\frac{\omega_0}{2}}}$$

$$= \frac{1}{N}\frac{\sin[k\omega_0(N_1+\frac{1}{2})]}{\sin(\frac{k\omega_0}{2})}, \qquad k \neq 0, \pm N, \pm 2N, \ldots$$

Determination of Fourier Series Coefficient

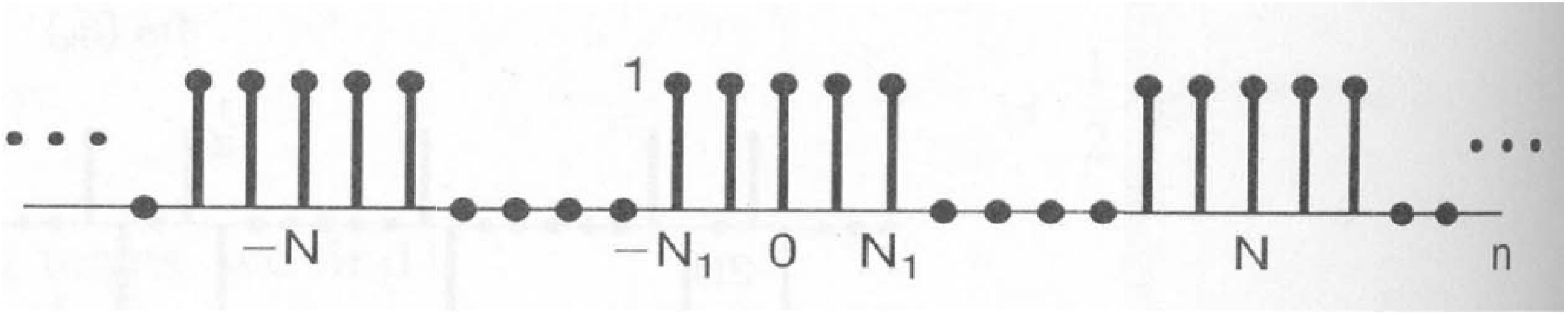
Solution

For $k = 0, \pm N, \pm 2N, \ldots$:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} = \frac{2N_1 + 1}{N}$$

Determination of Fourier Series Coefficient

Assume $\mathrm{N} = 9, N_1 = 2$:

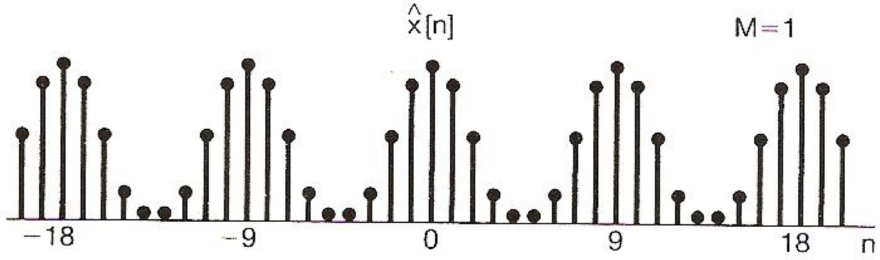


$$a_k = \begin{cases} \dfrac{2N_1+1}{N} = \dfrac{5}{9}, & k = 0, \pm 9, \pm 18, \ldots \\ \dfrac{1}{N}\dfrac{\sin[k\omega_0(N_1+\frac{1}{2})]}{\sin(\frac{k\omega_0}{2})} = \dfrac{\sin(\frac{5\pi}{9}k)}{9\sin(\frac{4\pi}{9}k)}, & k \neq 0, \pm 9, \pm 18, \ldots \end{cases}$$

Determination of Fourier Series Coefficient

Compute the partial sum $\hat{x}[n]$ for $N = 9, N_1 = 2$ as follows:

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk\omega_0 n}$$



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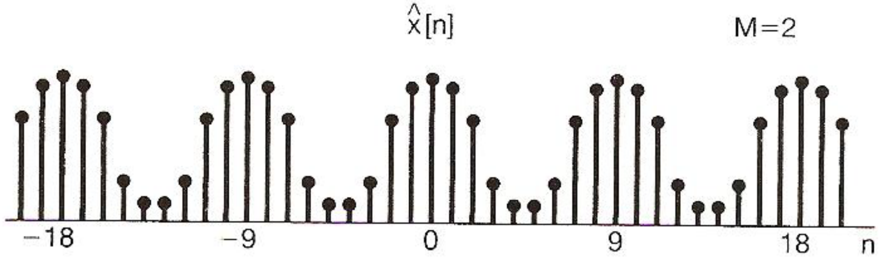

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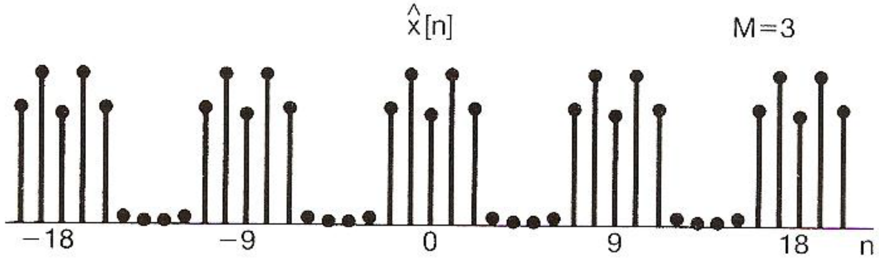

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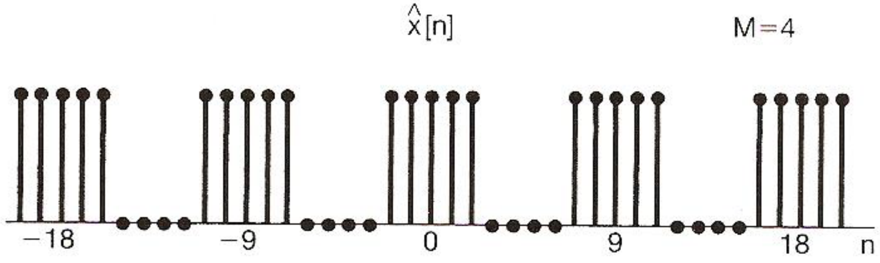

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- Discrete Time Fourier Series
  - Fourier Series Representation
  - Determination of Fourier Series Coefficient
  - Fourier Series Properties

- Application Example
  - System Characterization
  - Filtering

Fourier Series Properties

- The following notation is used to indicate a signal, $x[n]$, can be represented by Fourier series with the coefficients, a_k.

$$x[n] \overset{FS}{\leftrightarrow} a_k$$

Fourier Series Properties

Time scaling

Assume :

$$x[n] \overset{FS}{\leftrightarrow} a_k, x_{(m)}[n] \overset{FS}{\leftrightarrow} b_k, x_{(m)}[n] = \begin{cases} x[n/m], if\ n\ is\ a\ multiple\ of\ m \\ 0, \qquad\qquad otherwise \end{cases}$$

Then:

The period for $x_{(m)}[n]$ is mN

$$b_k = \frac{1}{mN} \sum_{n/m=\langle N \rangle} x[n/m] e^{-jk\omega_0 \frac{n}{m}}$$

$$= \frac{1}{m} \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{m} a_k$$

Fourier Series Properties

Multiplication

Assume :

$$x[n] \overset{FS}{\leftrightarrow} a_k, \mathrm{y}[n] \overset{FS}{\leftrightarrow} b_k,$$

Then:

$$\mathrm{x}(t)y(t) \overset{FS}{\leftrightarrow} d_k = \sum_{l=\langle N\rangle} a_l b_{k-l}$$

Fourier Series Properties

First difference

$$x[n] \overset{FS}{\leftrightarrow} a_k, \mathrm{x}[n] - x[n-1] \overset{FS}{\leftrightarrow} d_k$$

Based on time shift property:

$$x[n-1] \overset{FS}{\leftrightarrow} a_k e^{-jk\omega_0}$$

Based on linearity property:

$$\mathrm{x}[n] - x[n-1] \overset{FS}{\leftrightarrow} d_k = a_k - a_k e^{-jk\omega_0}$$

Fourier Series Properties

Parseval's relation

- The average power of a function in a period is equal to the sum of the square (power) of its Fourier series coefficients.

$$\frac{1}{N}\sum_{k=\langle N\rangle}|x[n]|^2=\sum_{k=\langle N\rangle}|a_k|^2$$

Outline: **Lecture 8:** Fourier Series of Discrete-time Signals

- Discrete Time Fourier Series
  - Fourier Series Representation
  - Determination of Fourier Series Coefficient
  - Fourier Series Properties

- Application Example
  - System Characterization
  - Filtering

System Characterization

$$x(t), x[n] \longrightarrow \overset{h(t), h[n]}{\boxed{\text{LTI}}} \longrightarrow y(t), y[n]$$

Assume $x(t) = e^{j\omega t}, x[n] = e^{j\omega n}$

$$y(t) = e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = e^{j\omega t}\mathrm{H}(\mathrm{j}\omega) = \mathrm{x}(\mathrm{t}).\mathrm{H}(\mathrm{j}\omega)$$

$$y[n] = e^{j\omega n}\sum_{l=-\infty}^{\infty} h[l]e^{-j\omega l} = e^{j\omega n}\mathrm{H}\left(e^{j\omega}\right) = \mathrm{x}[\mathrm{n}].\mathrm{H}\left(e^{j\omega}\right)$$

System Characterization

For discrete-time signal $x[n] = e^{j\omega n}$ and LTI system h$[n]$:

$$y[n] = \sum_{l=-\infty}^{\infty} h[l]x[n-l]$$

$$= \sum_{l=-\infty}^{\infty} h[l]e^{j\omega\,(n-l)} = \sum_{l=-\infty}^{\infty} e^{j\omega n}h[l]e^{-j\omega l}$$

$$= e^{j\omega n} \sum_{l=-\infty}^{\infty} h[l]e^{-j\omega l}$$

$$= \mathrm{x}[\mathrm{n}].\,\mathrm{H}\left(e^{j\omega}\right)$$

System Characterization

$$x(t), x[n] \longrightarrow \overset{h(t), h[n]}{\boxed{\text{LTI}}} \longrightarrow y(t), y[n]$$

For arbitrary periodic signal $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\, \mathrm{H}(\mathrm{j}\omega) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\, \mathrm{H}(jk\omega_0)$$

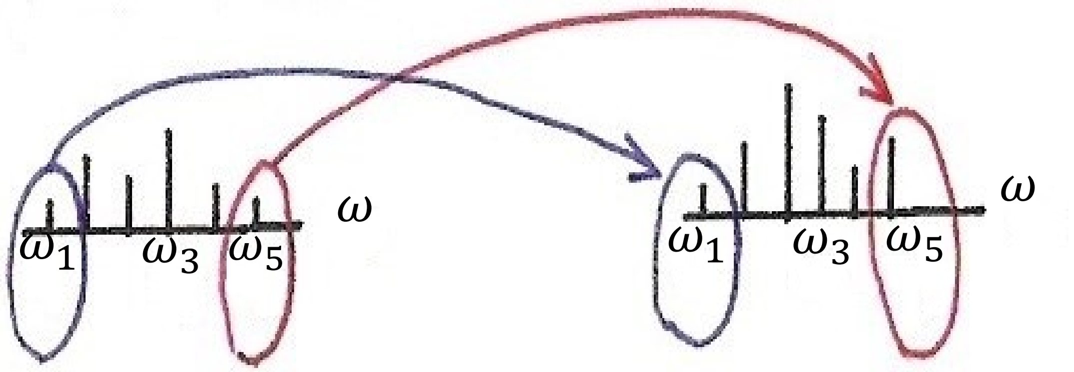
$$y[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}\, \mathrm{H}\left(e^{j\omega}\right) = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}\, \mathrm{H}\left(e^{jk\omega_0}\right)$$

System Characterization

- $H(j\omega)$ and $H(e^{j\omega})$ are frequency-related functions, known as frequency response.

- If the input has a frequency component, the output will exactly have the same frequency component, except scaled by a constant.

- Frequency domain:



Outline: **Lecture 7:** Fourier Series of Discrete-time Signals

- Discrete Time Fourier Series
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Filtering

- Modifying the amplitude/phase of the different frequency components in a signal, including eliminating some frequency components entirely.

  frequency-shaping filter （频率成形滤波器）

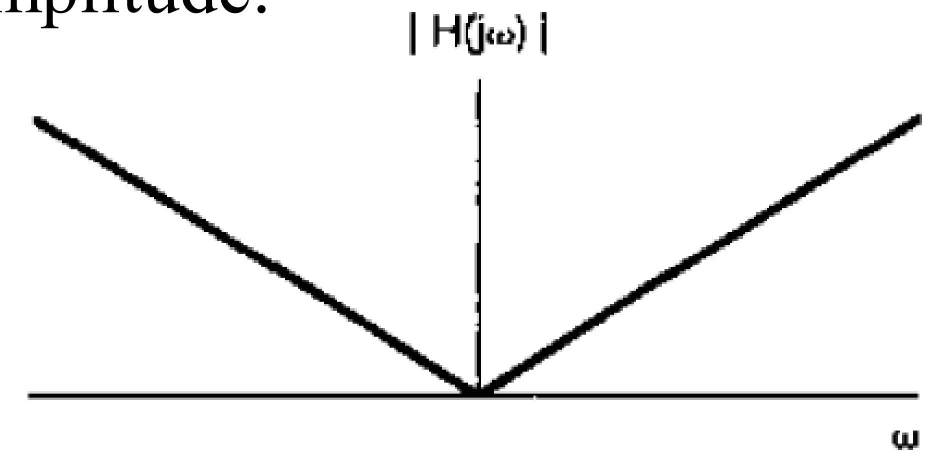
  frequency-selective filter （频率选择滤波器）

Filtering

Example: frequency-shaping filter

$$H(j\omega) = j\omega$$

Amplitude:



- Amplify high frequency components

Filtering

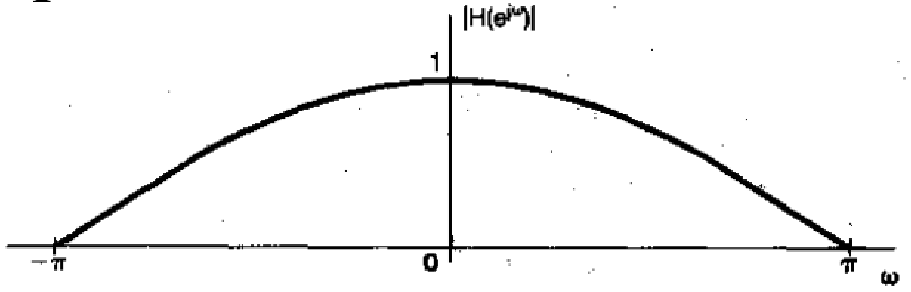
Example: frequency-shaping filter

$$h[n] = \frac{1}{2}[\delta[n] + \delta[n-1]]$$

$$\mathrm{H}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \frac{1}{2}[\delta[n] + \delta[n-1]]e^{-j\omega n} = \frac{1}{2}[1 + e^{-j\omega}]$$

$$= e^{-j\omega/2}\cos(\frac{\omega}{2})$$

Amplitude:



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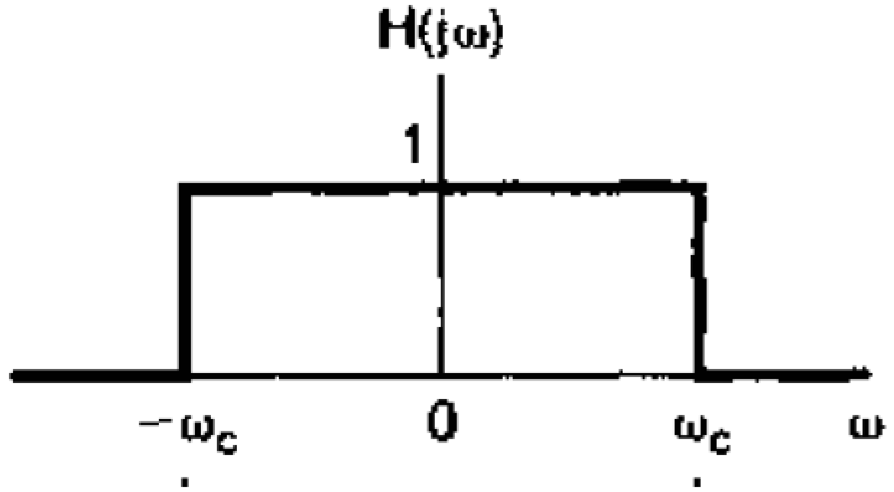
- Compress high frequency components

Filtering

Example: frequency-selective filter

- Ideal low-pass filter

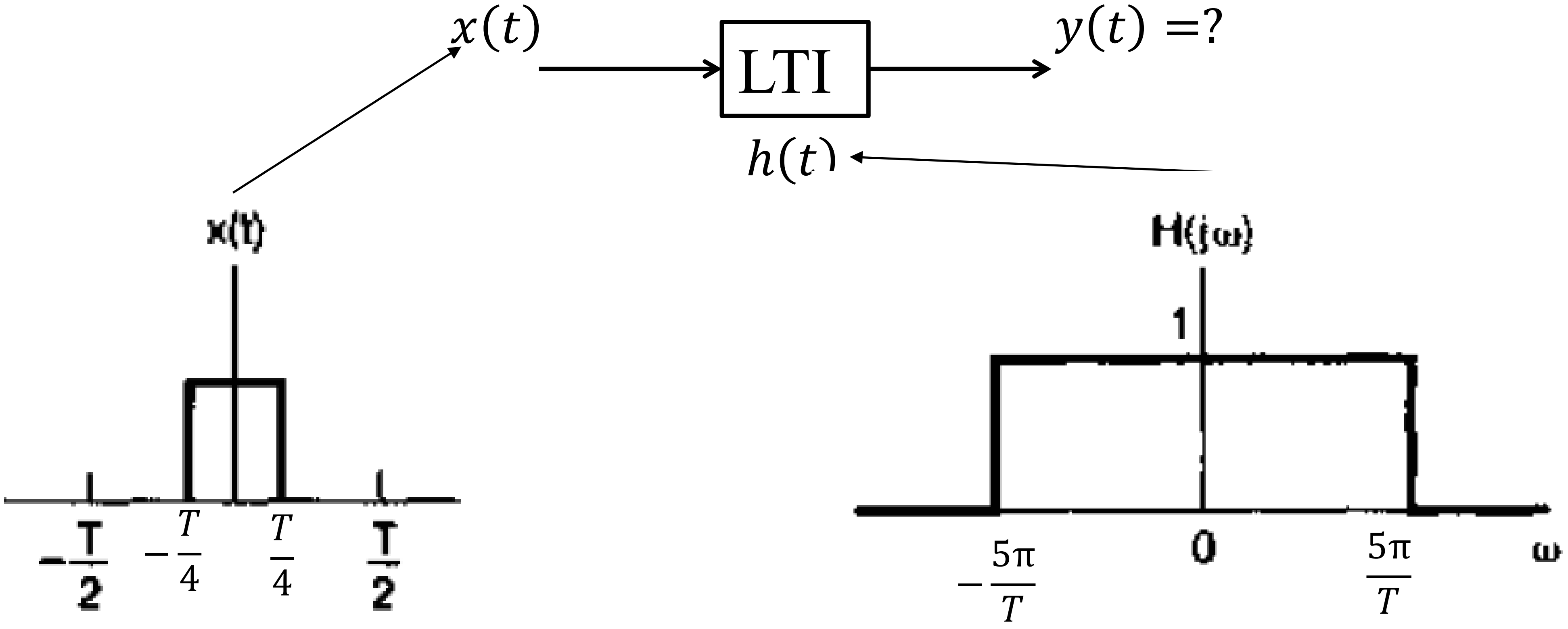
$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



Filtering

Example: frequency-selective filter

- Ideal low-pass filter



Filtering

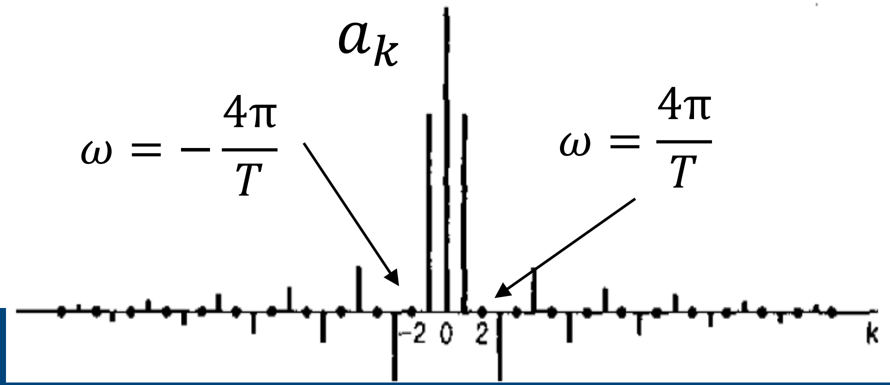
Example: frequency-selective filter

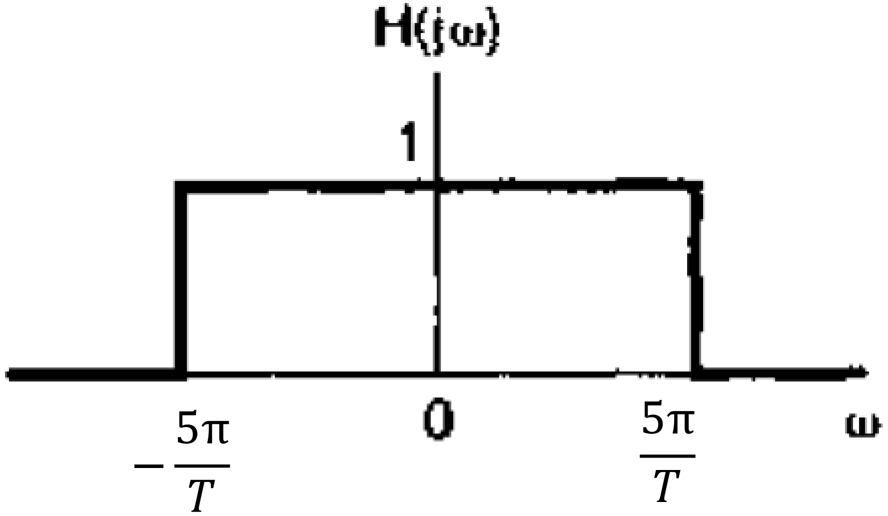
- Ideal low-pass filter

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = ?$$

$$h(t)$$

$$a_k = \begin{cases} \dfrac{\sin(k\frac{\pi}{2})}{k\pi}, & k \neq 0 \\ \dfrac{1}{2}, & k = 0 \end{cases}$$





Filtering

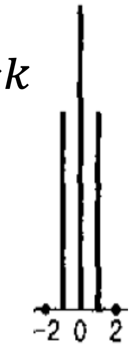
Example: frequency-selective filter

- Ideal low-pass filter

$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$

$h(t)$

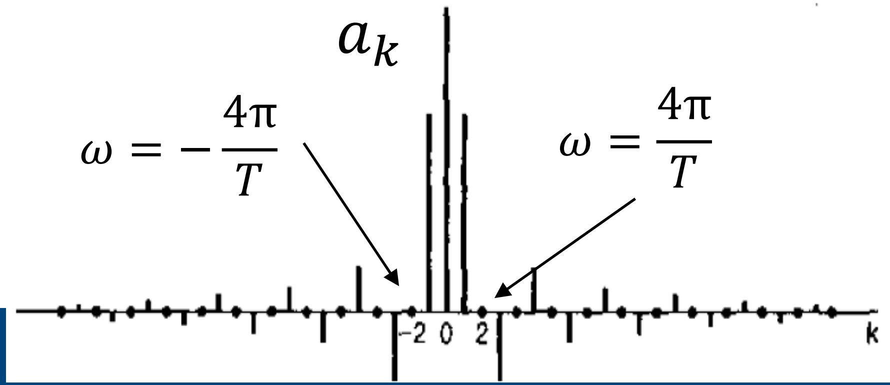
a_k

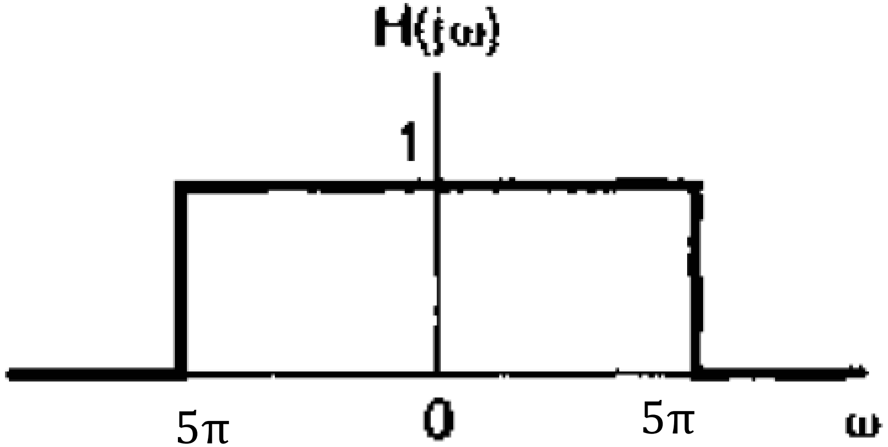


$$a_k = \begin{cases} \dfrac{\sin(k\frac{\pi}{2})}{k\pi}, & k \neq 0 \\ \dfrac{1}{2}, & k = 0 \end{cases}$$

a_k

$\omega = -\dfrac{4\pi}{T}$ $\omega = \dfrac{4\pi}{T}$





$-\dfrac{5\pi}{T}$ $\dfrac{5\pi}{T}$

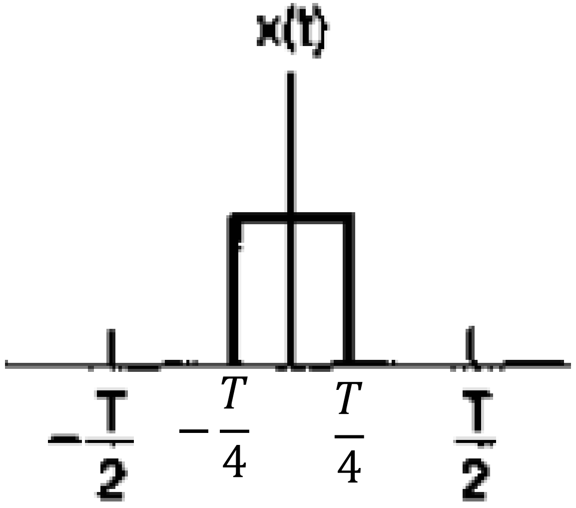
Filtering

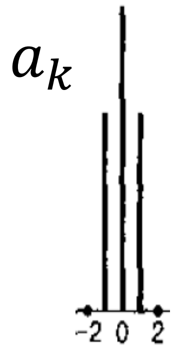
Example: frequency-selective filter

- Ideal low-pass filter

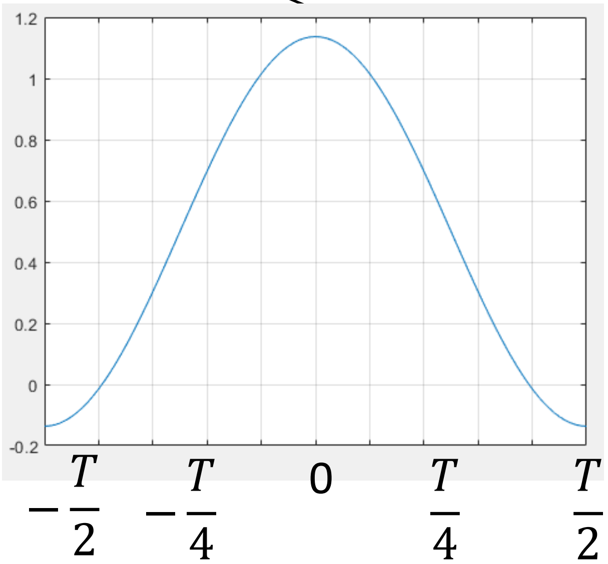
$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t)$

$h(t)$





$$a_{-2} = 0$$
$$a_{-1} = 1/\pi$$
$$a_0 = 1/2$$
$$a_1 = 1/\pi$$
$$a_2 = 0$$



Filtering

Example: frequency-selective filter

- Ideal filters (continuous-time signal)

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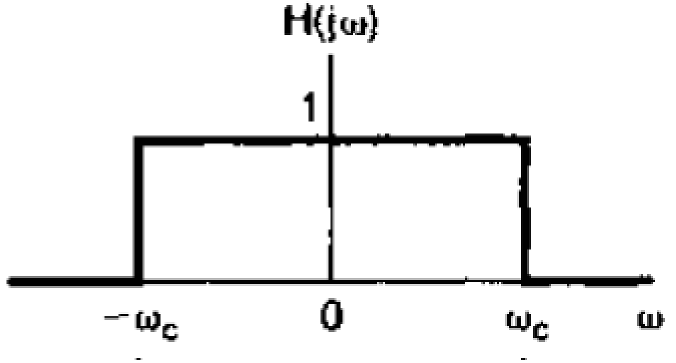
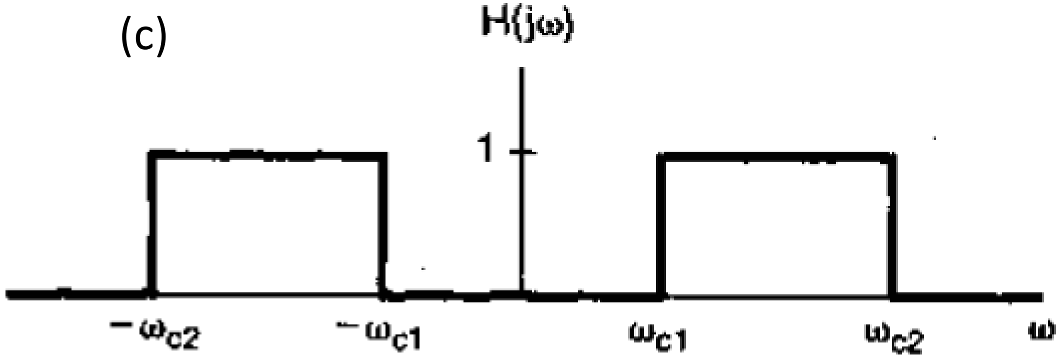
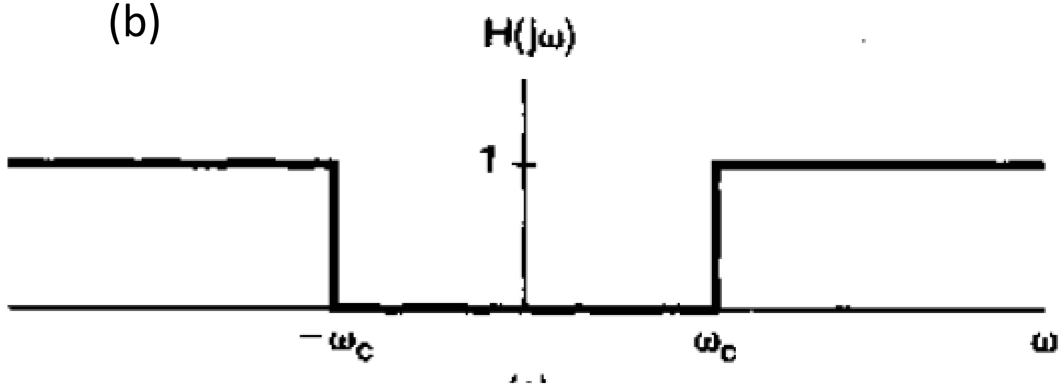


(c)



(b)



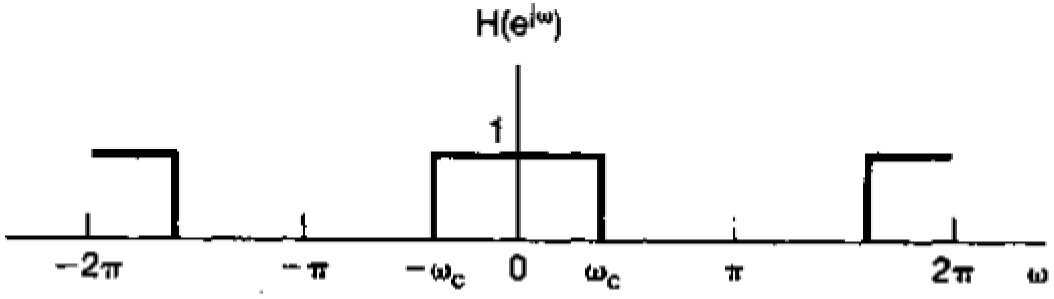
(a) Low-pass filter
(b) High-pass filter
(c) Band-pass filter

Filtering

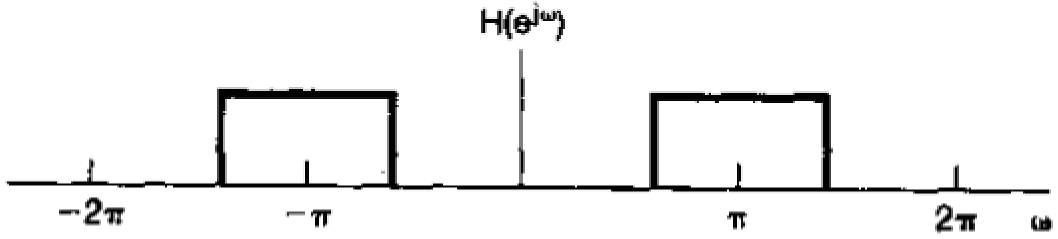
Example: frequency-selective filter

- Ideal filters (discrete-time signal)

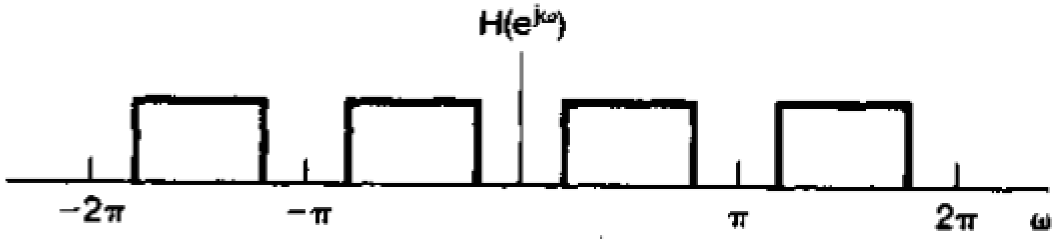
(a)



(b)



(c)



(a) Low-pass filter
(b) High-pass filter
(c) Band-pass filter

Thank you for your listening!