Lecture 5 Continuous-time Fourier Transform

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Outline: **Lecture 5:** Continuous-time Fourier Transform

- Continuous-time Fourier Transform
  - Fourier Transform Representation
  - Convergence Issue

- Fourier Transform of Periodic Signal

- Examples

- Fourier Transform Properties

Outline: Lecture 5: Continuous-time Fourier Transform

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Lecture 4: 傅里叶级数

- 连续时间周期信号傅里叶级数的三种表示形式：

① 正余弦形式：

$$x(t)=A_0+\sum_{k=1}^{\infty}[A_k\cos(k\omega_0 t)+B_k\sin(k\omega_0 t)]$$

② 复指数形式：

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t},\ a_k=\begin{cases}\dfrac{A_{-k}+jB_{-k}}{2} & k<0\\ A_0 & k=0\\ \dfrac{A_k-jB_k}{2} & k>0\end{cases}$$

③ 幅度-相位形式：

$$x(t)=a_0+2\sum_{k=1}^{\infty}A'_k\cos(k\omega_0 t+\theta_k),\ a_k=A'_k e^{j\theta_k}$$

Continuous-time Fourier Transform

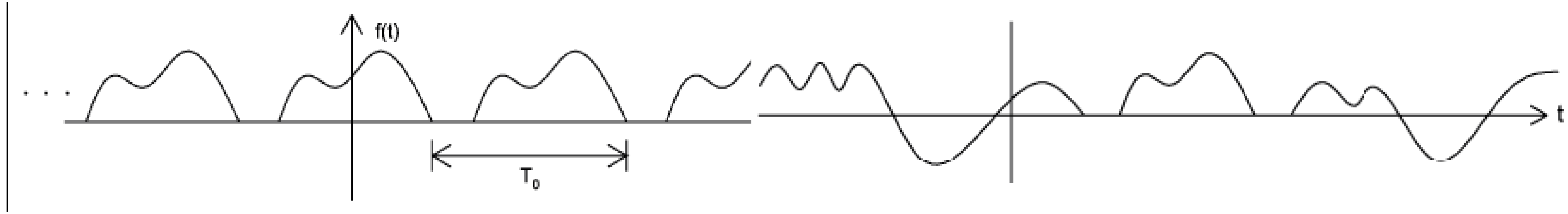
- In the previous lecture, we show a continuous-time periodic signal can be represented by a Fourier series, producing the spectral in frequency domain.

- In practice, many signals are aperiodic. For those signals, how could we evaluate their spectral?

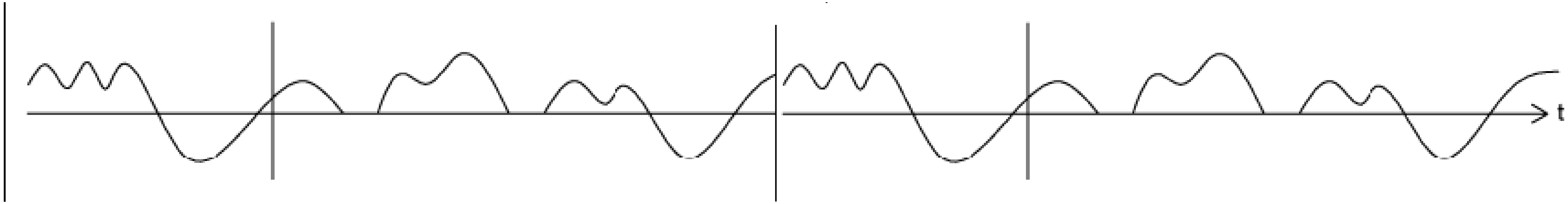
- An aperiodic signal can be considered as a periodic signal, the period of which is extremely large, *e.g.*, infinity.

Examples

- Still periodic?



- Still an aperiodic signal?

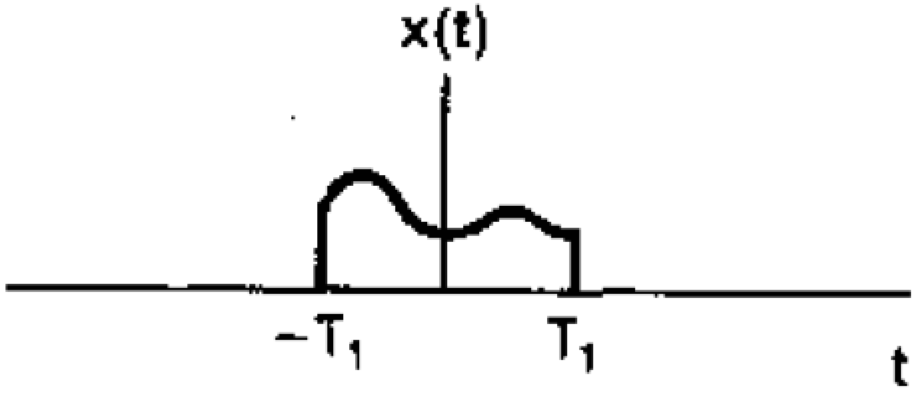


- 周期与非周期性同样具有主观性！

Fourier Transform Representation

From Fourier series to Fourier transform

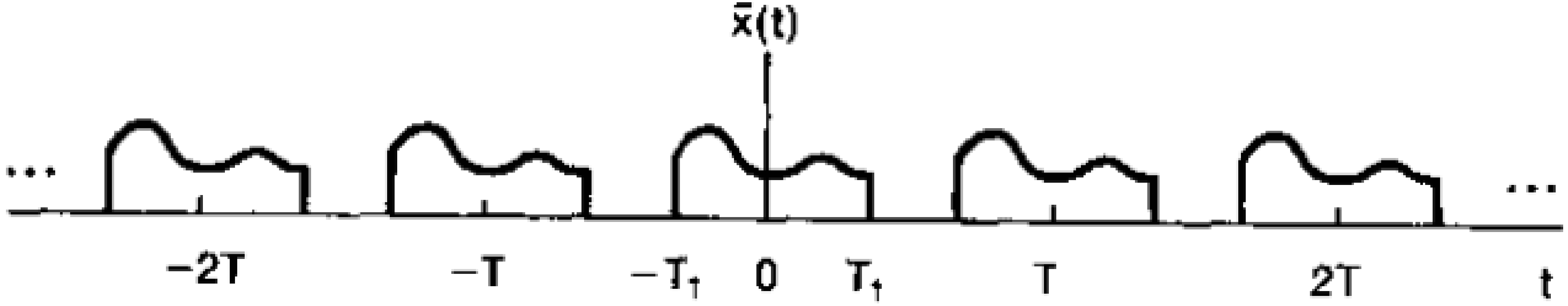
- The following signal, $x(t)$, is an aperiodic signal:



Fourier Transform Representation

From Fourier series to Fourier transform

- Assume $x(t)$ is repeated for every period of T ($T \to \infty$), as $\tilde{x}(t)$



Fourier Transform Representation

From Fourier series to Fourier transform

- Fourier series representation:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

Fourier Transform Representation

From Fourier series to Fourier transform

- In the time of $[-\frac{T}{2},\frac{T}{2}]$, $\tilde{x}(t)=x(t)$

$$a_k=\frac{1}{T}\int_{-T/2}^{T/2}\tilde{x}(t)e^{-jk\omega_0 t}dt$$

$$=\frac{1}{T}\int_{-T/2}^{T/2}x(t)e^{-jk\omega_0 t}dt=\frac{1}{T}\int_{-\infty}^{\infty}x(t)e^{-jk\omega_0 t}dt$$

$$Ta_k=\int_{-\infty}^{\infty}x(t)e^{-jk\omega_0 t}dt$$

Fourier Transform Representation

From Fourier series to Fourier transform

$$Ta_k = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt$$

- Define $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

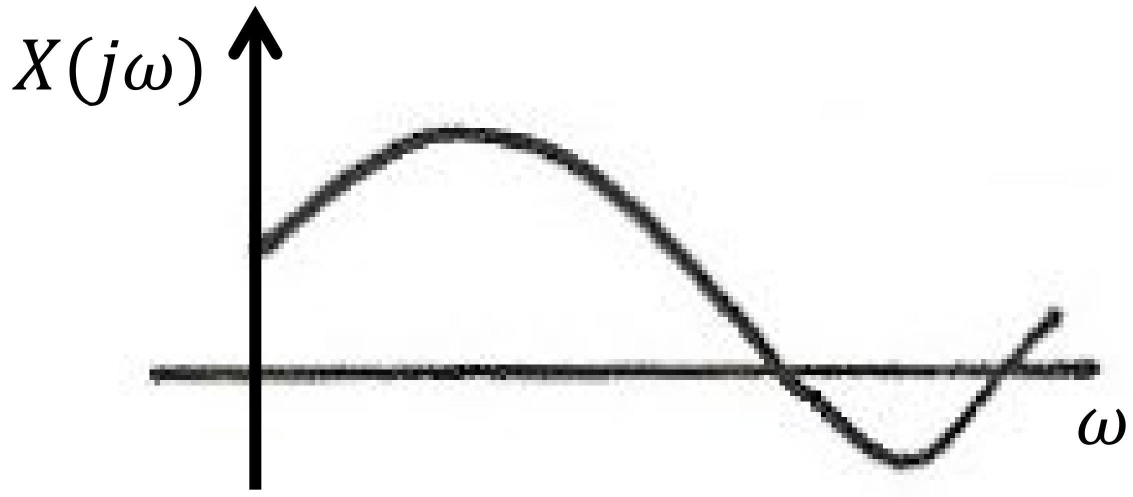
$$Ta_k = X(j\omega)|_{\omega=k\omega_0} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\,|_{\omega=k\omega_0}$$

Fourier Transform Representation

From Fourier series to Fourier transform

$$Ta_k = X(j\omega)|_{\omega=k\omega_0}$$

- $X(j\omega)$ is a continuous function related to ω：



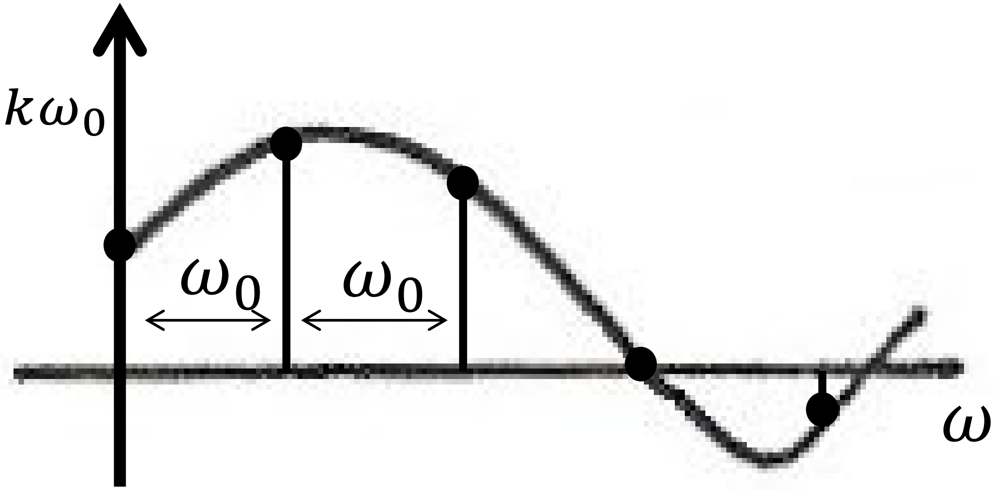
Fourier Transform Representation

From Fourier series to Fourier transform

$$Ta_k = X(j\omega)|_{\omega=k\omega_0}$$

- $X(j\omega)$ is a continuous function related to ω. In order to find discrete values, Ta_k, on $X(j\omega)$, $X(j\omega)$ is sampled for every interval of ω_0:

$$Ta_k = X(j\omega)|_{\omega=k\omega_0}$$



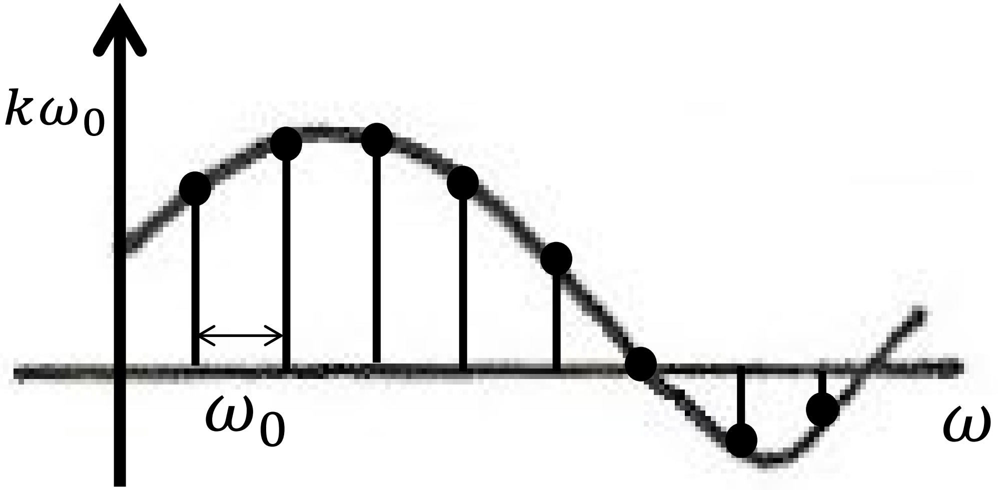
Fourier Transform Representation

From Fourier series to Fourier transform

$$Ta_k = X(j\omega)|_{\omega=k\omega_0}$$

- $T\uparrow, \omega_0\downarrow$:

$$Ta_k = X(j\omega)|_{\omega=k\omega_0}$$



Fourier Transform Representation

From Fourier series to Fourier transform

$$Ta_k = X(j\omega)|_{\omega=k\omega_0}$$

- $T \to \infty, \omega_0 \to 0$:

$$Ta_k = X(j\omega)|_{\omega=k\omega_0} = X(j\omega)$$

- $X(j\omega)$ is the envelope of Ta_k

Fourier Transform Representation

From Fourier series to Fourier transform

- For aperiodic signals, Fourier series (spectral) can be approximated by a continuous function:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Transform Representation

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \qquad T a_k = X(j\omega)|_{\omega=k\omega_0}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t}\, \omega_0$$

Fourier Transform Representation

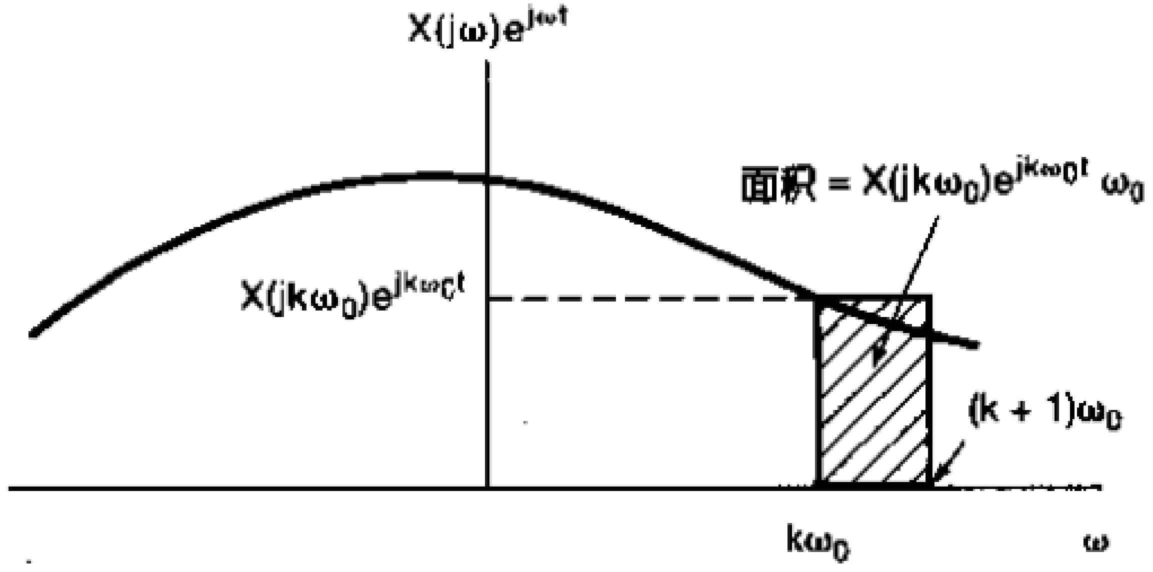
$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X\,(jk\omega_0) e^{jk\omega_0 t}\, \omega_0$$

- In the time of $[-\frac{T}{2}, \frac{T}{2}]$ ($T \to \infty$: $[-\infty, \infty]$), $\tilde{x}(t) = x(t)$

$$x(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X\,(jk\omega_0) e^{jk\omega_0 t}\, \omega_0$$

Fourier Transform Representation

$$x(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t}\, \omega_0$$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Transform Representation

Fourier transform (spectral) 傅里叶变换:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Inverse Fourier transform (signal representation) 傅里叶逆变换:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Lecture 4, 5: 傅里叶级数与傅里叶变换

- 时域连续，周期与非周期信号对应频域特点：

Periodic in time domain ↔ Discrete in frequency domain

Aperiodic in time domain ↔ Continuous in frequency domain

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- Fourier Transform of Periodic Signal

- Examples

- Fourier Transform Properties

Convergence Issue

- An aperiodic signal is considered as a periodic signal, the period of which is extremely large, *e.g.*, infinity.

- The spectral of an aperiodic signal is considered as the Fourier series with $\omega_0 \rightarrow 0$.

- Thus, convergence of Fourier transform has exactly the same requirements as Fourier series.

Convergence Issue

- A signal can be represented by Fourier transform, if it has finite energy:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

Convergence Issue

Dirichlet's condition

- A signal can be represented by Fourier transform, if

  (1) it is absolutely integrable, $\int_{-\infty}^{\infty} \left| x(t) \right| dt < \infty$

  (2) it has finite number of maxima & minima

  (3) it has finite number of discontinuities

Convergence Issue

- But there might be some exceptions …

- What about to perform Fourier transform for periodic signals?

  - Some periodic signals or their energies are not integrable in $[-\infty, \infty]$;

  - Periodic signal may have infinite number of maxima/minima and discontinuities.

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Fourier Transform of Periodic Signal

- For a periodic signal, we have the Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- Let us firstly consider the Fourier transform for a complex exponential: $e^{jk\omega_0 t}$.

Fourier Transform of Periodic Signal

- Let us firstly consider the Fourier transform for a complex exponential: $x(\mathrm{t}) = e^{jk\omega_0 t}$.

$$X(j\omega) = \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{j(k\omega_0 - \omega)t} dt$$

$$= \begin{cases} \int_{-\infty}^{\infty} 1 dt = \infty, & k\omega_0 = \omega \\ 0, & otherwise \end{cases}$$

- $x(t) = e^{jk\omega_0 t}$ is not integrable in $[-\infty, \infty]$.

Fourier Transform of Periodic Signal

Fourier transform of a impulse function with a time shift of t_0:

$$x(t) = \delta(t - t_0)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t} dt = e^{-j\omega t_0}$$

Thus,

$$\delta(t - t_0) \overset{F}{\leftrightarrow} e^{-j\omega t_0}$$

Fourier Transform of Periodic Signal

$$\delta(t-t_0) \overset{F}{\leftrightarrow} e^{-j\omega t_0}$$

- Use the inverse Fourier transform for $X(j\omega) = e^{-j\omega t_0}$:

$$x(t) = \delta(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t_0} e^{j\omega t} d\omega$$

- Let us make some tricky replacements. Replace t, t_0 and ω above by $k\omega_0, \omega$ and t:

$$\delta(k\omega_0-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jt\omega} e^{jtk\omega_0} dt$$

Fourier Transform of Periodic Signal

$$\delta(k\omega_0 - \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jt\omega} e^{jtk\omega_0} dt$$

$$2\pi\delta(\omega - k\omega_0) = \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt$$

- Compare above equation with Fourier transform: $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

$$x(t) = e^{jk\omega_0 t} \overset{F}{\leftrightarrow} X(j\omega) = 2\pi\delta(\omega - k\omega_0)$$

Fourier Transform of Periodic Signal

$$x(t) = e^{jk\omega_0 t} \overset{F}{\leftrightarrow} X(j\omega) = 2\pi\delta(\omega - k\omega_0)$$

- For periodic signal, we have the Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \overset{F}{\leftrightarrow} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$$

Fourier Transform of Periodic Signal

- Fourier transform can be performed for periodic signals by using impulse function. The spectral is the same as Fourier series.

- In such a case, we have a unified framework of Fourier transform for both periodic and aperiodic signals.

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Examples

- *Example 4.7, p.211 of textbook*

Find the spectral (Fourier transform) for the following signals: $x(t) = sin\omega_0 t$ and $x(t) = cos\omega_0 t$.

Examples

- *Example 4.7, p.211 of textbook*

Find the spectral (Fourier transform) for the following signals: $x(t) = sin\omega_0 t$ and $x(t) = cos\omega_0 t$.

$$x(t) = sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$a_k = \begin{cases} \dfrac{1}{2j}, & k = 1 \\ -\dfrac{1}{2j}, & k = -1 \\ 0, & k \neq 1 \; or \; -1 \end{cases} \Rightarrow X(j\omega) = -\frac{\pi}{j}\delta(\omega + \omega_0) + \frac{\pi}{j}\delta(\omega - \omega_0)$$

Examples

- *Example 4.7, p.211 of textbook*

Find the spectral (Fourier transform) for the following signals: $x(t) = sin\omega_0 t$ and $x(t) = cos\omega_0 t$.

$$x(t) = cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$a_k = \begin{cases} \frac{1}{2}, & k = 1 \; or \; -1 \\ 0, & k \neq 1 \; or \; -1 \end{cases} \Rightarrow X(j\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$

Fourier transform of periodic signal:

$$x(t) \overset{FS}{\leftrightarrow} a_k \rightarrow X(j\omega)$$

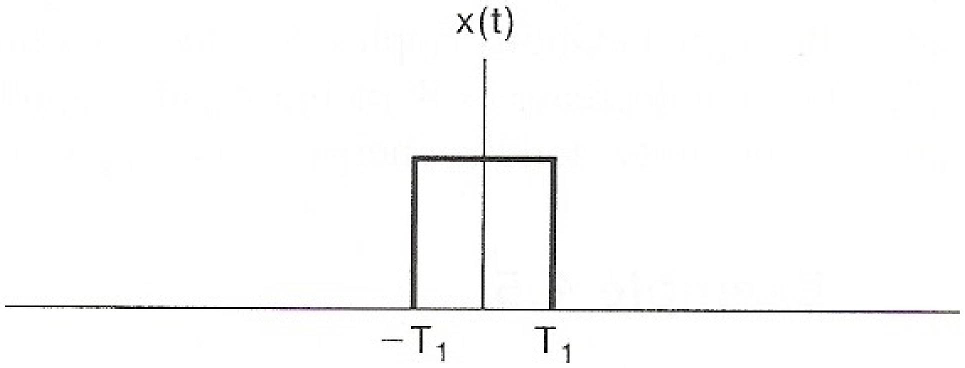
$$F$$

Examples

- *Example 4.4, p.208 of textbook*

Find the spectral (Fourier transform) for the following signal:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



Examples

- *Example 4.4, p.208 of textbook*

Find the spectral (Fourier transform) for the following signal:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-T_1}^{T_1} e^{-j\omega t} dt$$

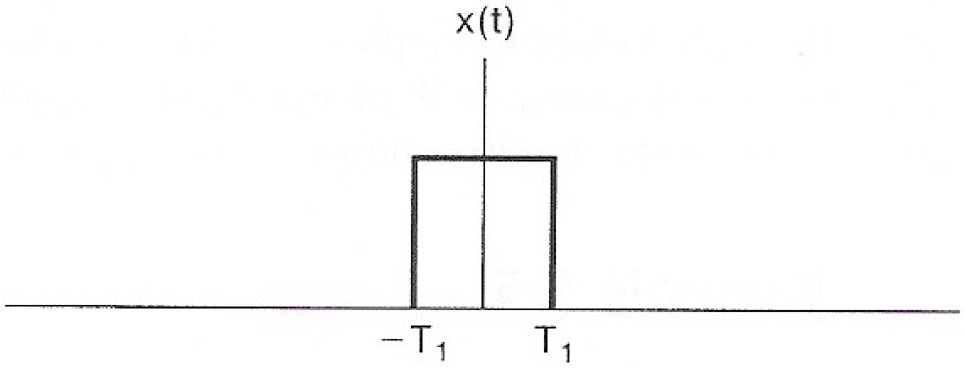
$$= -\frac{1}{j\omega} e^{-j\omega T_1} - e^{j\omega T_1} = \frac{2 sin\omega T_1}{\omega}$$

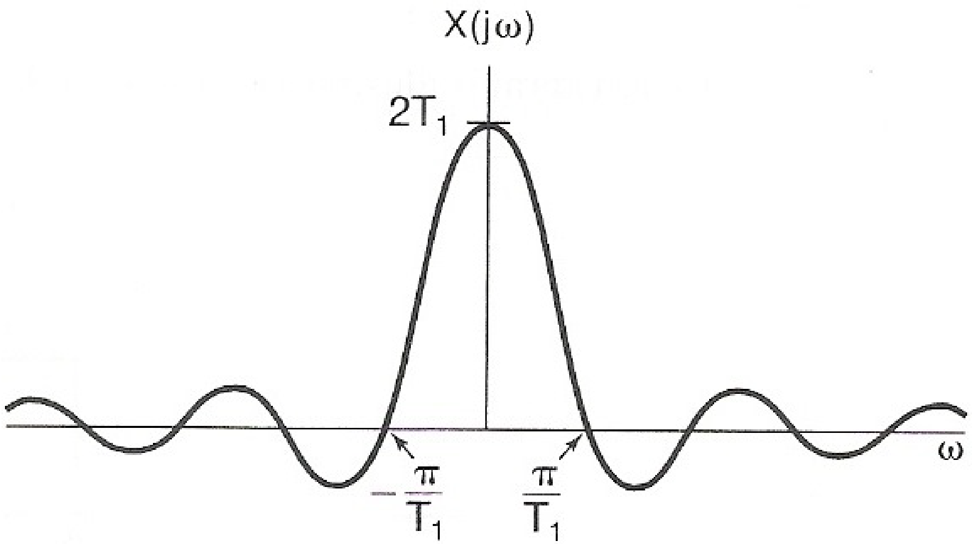
Examples

- *Example 4.4, p.208 of textbook*

Find the spectral (Fourier transform) for the following signal:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \qquad X(j\omega) = \frac{2sin\omega T_1}{\omega}$$





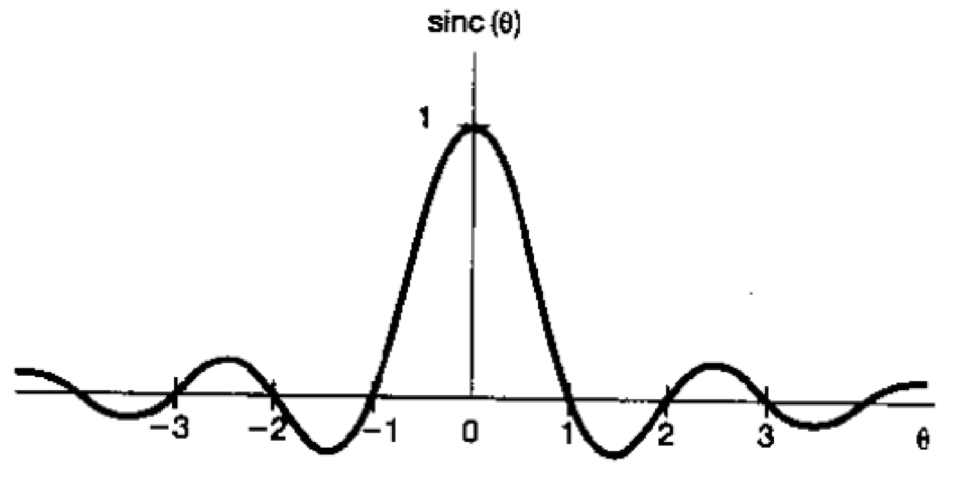
Examples

- *Example 4.4, p.208 of textbook*

Find the spectral (Fourier transform) for the following signal:

$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases}$$

- Giving the definition of *sinc* function: $sinc(\theta)=\dfrac{sin\pi\theta}{\pi\theta}$



Examples

- *Example 4.4, p.208 of textbook*

Find the spectral (Fourier transform) for the following signal:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

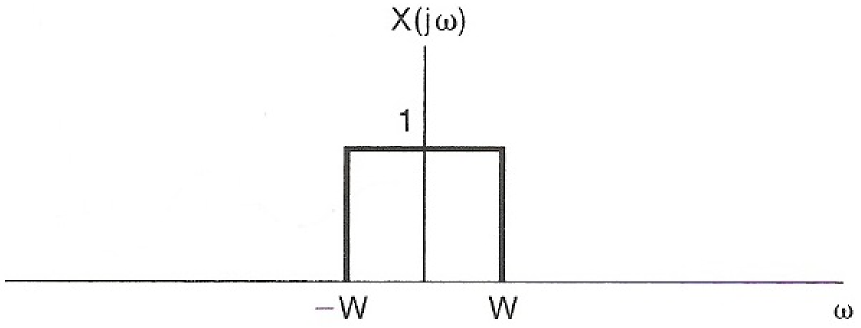
$$X(j\omega) = \frac{2sin\omega T_1}{\omega} = 2T_1 \frac{sin\pi\frac{\omega T_1}{\pi}}{\pi\frac{\omega T_1}{\pi}} = 2T_1 sinc(\frac{\omega T_1}{\pi})$$

Examples

- *Example 4.5, p.209 of textbook*

Find the signal function in time domain for the following signal:

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



Examples

- *Example 4.5, p.209 of textbook*

Find the signal function in time domain for the following signal:

$$X(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-W}^{W}e^{j\omega t}d\omega$$

$$=\frac{1}{2\pi}\frac{1}{jt}e^{jWt}-e^{-jWt}=\frac{sinWt}{\pi t}$$

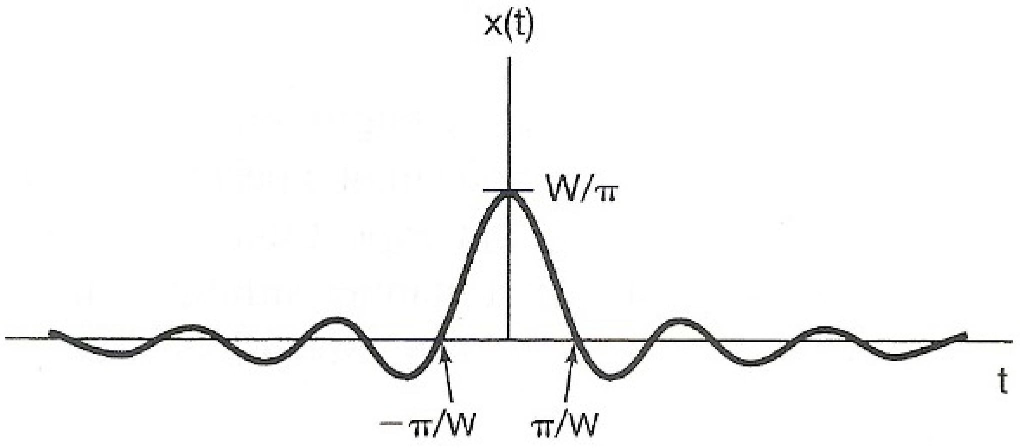
$$=\frac{W}{\pi}sinc(\frac{Wt}{\pi})$$

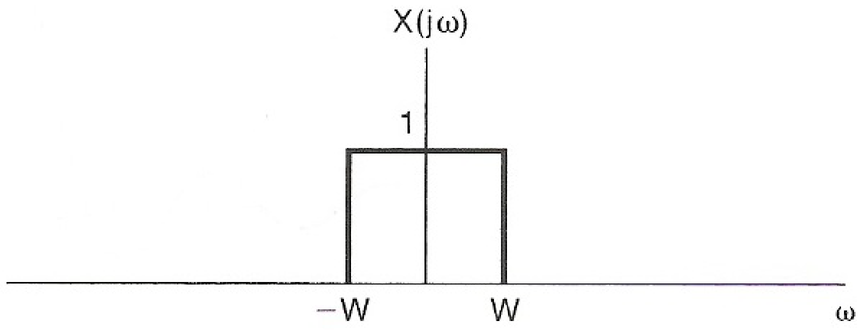
Examples

- *Example 4.5, p.209 of textbook*

Find the signal function in time domain for the following signal:

$$x(t) = \frac{\sin Wt}{\pi t} \qquad X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$





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- Fourier Transform of Periodic Signal
- Examples
- Fourier Transform Properties

Fourier Transform Properties

- The following notation is used to indicate a pair of Fourier transform:

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

Fourier Transform Properties

Linearity

Assume :

$$x(t) \overset{F}{\leftrightarrow} X(j\omega), \mathrm{y}(t) \overset{F}{\leftrightarrow} Y(j\omega),$$

Then:

$$\mathrm{ax}(t) + by(t) \overset{F}{\leftrightarrow} aX(j\omega) + bY(j\omega)$$

Fourier Transform Properties

Time shift

Assume :

$$x(t) \overset{F}{\leftrightarrow} X(j\omega), y(t) = x(t-t_0) \overset{F}{\leftrightarrow} Y(j\omega),$$

Then:

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}e^{-j\omega t_0}dt = X(j\omega)e^{-j\omega t_0}$$

- Time shift leads to unchanged amplitude and shifted phase.

Fourier Transform Properties

Conjugation

Assume:

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

Then:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)(e^{j\omega t})^* d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(-j\omega)e^{j\omega t}d\omega$$

$$x^*(t) \overset{F}{\leftrightarrow} X^*(-j\omega)$$

Fourier Transform Properties

Conjugation

Even/Odd property

$$x(t) \overset{F}{\leftrightarrow} X(j\omega), x^*(t) \overset{F}{\leftrightarrow} X^*(-j\omega)$$

- If $x(t)$ is real ($x(t) = x^*(t)$):

$$X(j\omega) = X^*(-j\omega)$$

Fourier Transform Properties

Conjugation

Even/Odd property

$$x(t) \overset{F}{\leftrightarrow} X(j\omega), x^*(t) \overset{F}{\leftrightarrow} X^*(-j\omega)$$

- If $x(t)$ is real and even ($x(t) = x^*(t), x(t) = x(-t)$):

$$X(j\omega) = X^*(-j\omega), X(j\omega) = X(-j\omega)$$

$$\Rightarrow X^*(-j\omega) = X(-j\omega)$$

$$\Rightarrow X^*(j\omega) = X(j\omega)$$

- $X(j\omega)$ is real and even

Fourier Transform Properties

Conjugation

Even/Odd property

$$x(t) \overset{F}{\leftrightarrow} X(j\omega), x^*(t) \overset{F}{\leftrightarrow} X^*(-j\omega)$$

- If $x(t)$ is real and odd ($x(t) = x^*(t), x(t) = -x(-t)$):

$$X(j\omega) = X^*(-j\omega), X(j\omega) = -X(-j\omega)$$

$$\Rightarrow X^*(-j\omega) = -X(-j\omega)$$

$$\Rightarrow X^*(j\omega) = -X(j\omega)$$

- $X(j\omega)$ is imaginary and odd

Fourier Transform Properties

Conjugation

- Any signal can be discomposed into a sum of an even and an odd

$$Ev\{x(t)\} = x_1(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$Od\{x(t)\} = x_2(t) = \frac{1}{2}[x(t) - x(-t)]$$

- If $x(t)$ is real:

$$Ev\{x(t)\} \overset{F}{\leftrightarrow} \mathbb{R}\{X(j\omega)\}$$

$$Od\{x(t)\} \overset{F}{\leftrightarrow} \mathbb{I}\{X(j\omega)\}$$

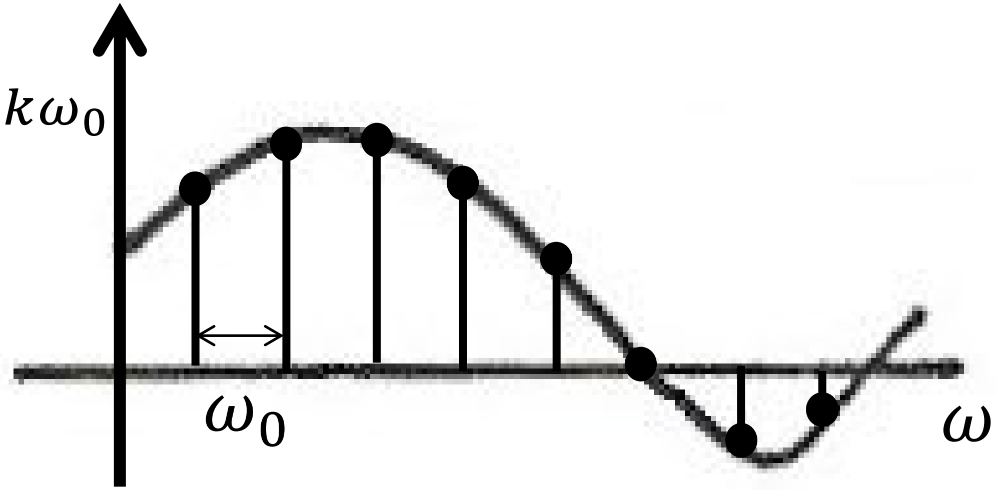
Fourier Transform Representation

From Fourier series to Fourier transform

$$Ta_k = X(j\omega)|_{\omega=k\omega_0}$$

- $T\uparrow, \omega_0\downarrow$:

$$Ta_k = X(j\omega)|_{\omega=k\omega_0}$$



Fourier Transform Properties

Differentiation

Assume:

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

Then:

$$\frac{dx(t)}{dt} = \frac{d\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right)}{dt}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega$$

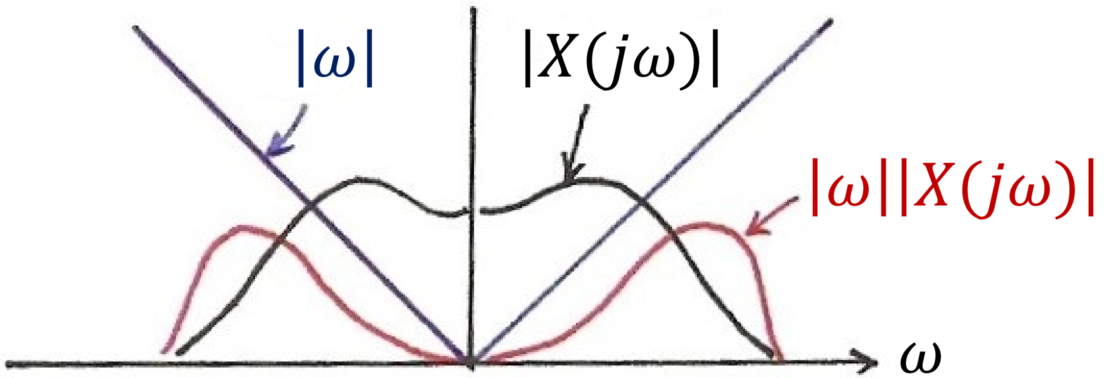
$$\frac{dx(t)}{dt} \overset{F}{\leftrightarrow} j\omega X(j\omega)$$

Fourier Transform Properties

Differentiation

$$\frac{dx(t)}{dt} \overset{F}{\leftrightarrow} j\omega X(j\omega)$$

$$|j\omega \cdot X(j\omega)| = |\omega| \cdot |X(j\omega)|$$



Enhancing higher frequencies

De-emphasizing lower frequencies

Deleting DC term (=0 for ω=0)

Integration

Assume:

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

Then:

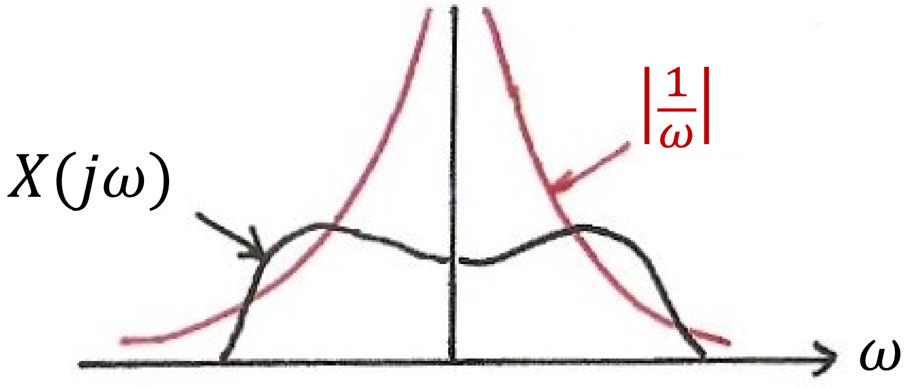
$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F}{\leftrightarrow} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Fourier Transform Properties

Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F}{\leftrightarrow} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

$$\left|\frac{1}{j\omega}\right| \cdot |X(j\omega)| = \left|\frac{1}{\omega}\right| \cdot |X(j\omega)|$$



Enhancing lower frequencies

De-emphasizing higher frequencies (smoothing effect)

Undefined for ω=0

Fourier Transform Properties

Time/frequency scaling

Assume :

$$x(t) \overset{F}{\leftrightarrow} X(j\omega), y(t) = x(at) \overset{F}{\leftrightarrow} Y(j\omega)$$

Then:

$$Y(j\omega) = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\frac{\tau}{a}}d\frac{\tau}{a}$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)e^{-j\frac{\omega}{a}\tau}d\tau = \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

$$x(at) \overset{F}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

Fourier Transform Properties

Time/frequency scaling

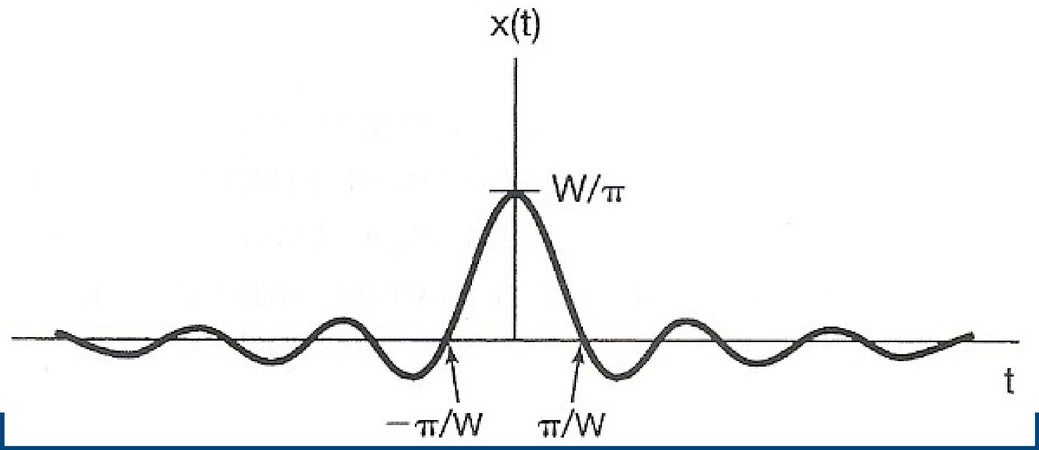
$$x(at) \overset{F}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

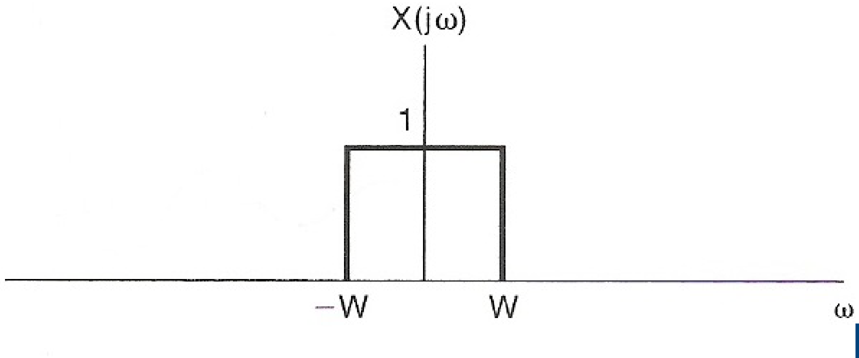
- Inverse relationship between signal "width" in time/frequency domains:

Example 4.5, p.209 of textbook

$$x(t) = \frac{sinWt}{\pi t}$$

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



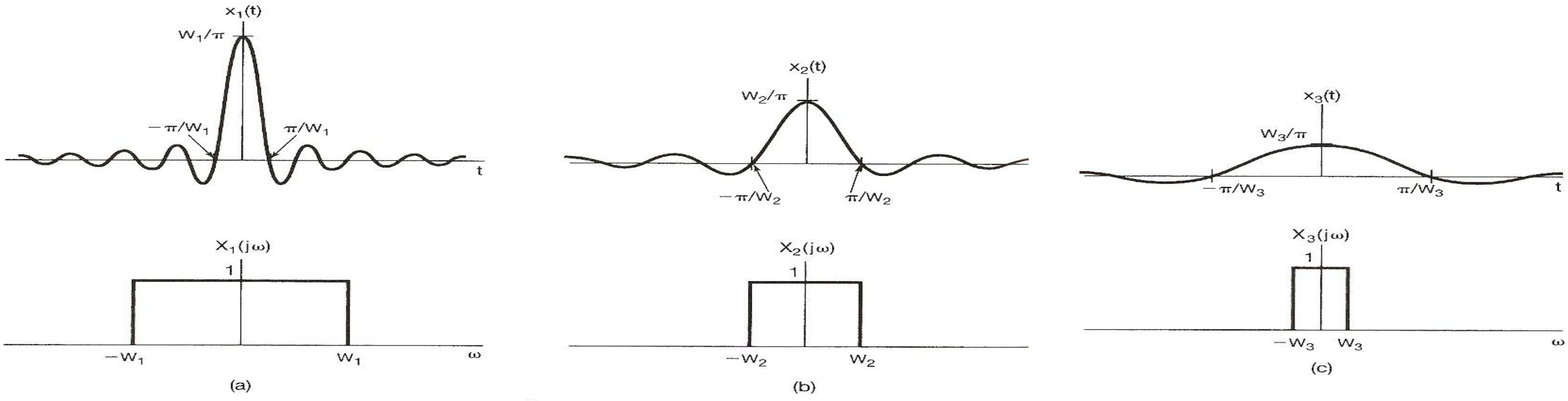


Fourier Transform Properties

Time/frequency scaling

$$x(at) \overset{F}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- Assume $W_1 > W_2 > W_3$:



Fourier Transform Properties

Duality

- Time/frequency domains are kind of "symmetric" except for a sign change (and a factor of 2π)

Fourier transform (spectral):

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Inverse Fourier transform (signal representation):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

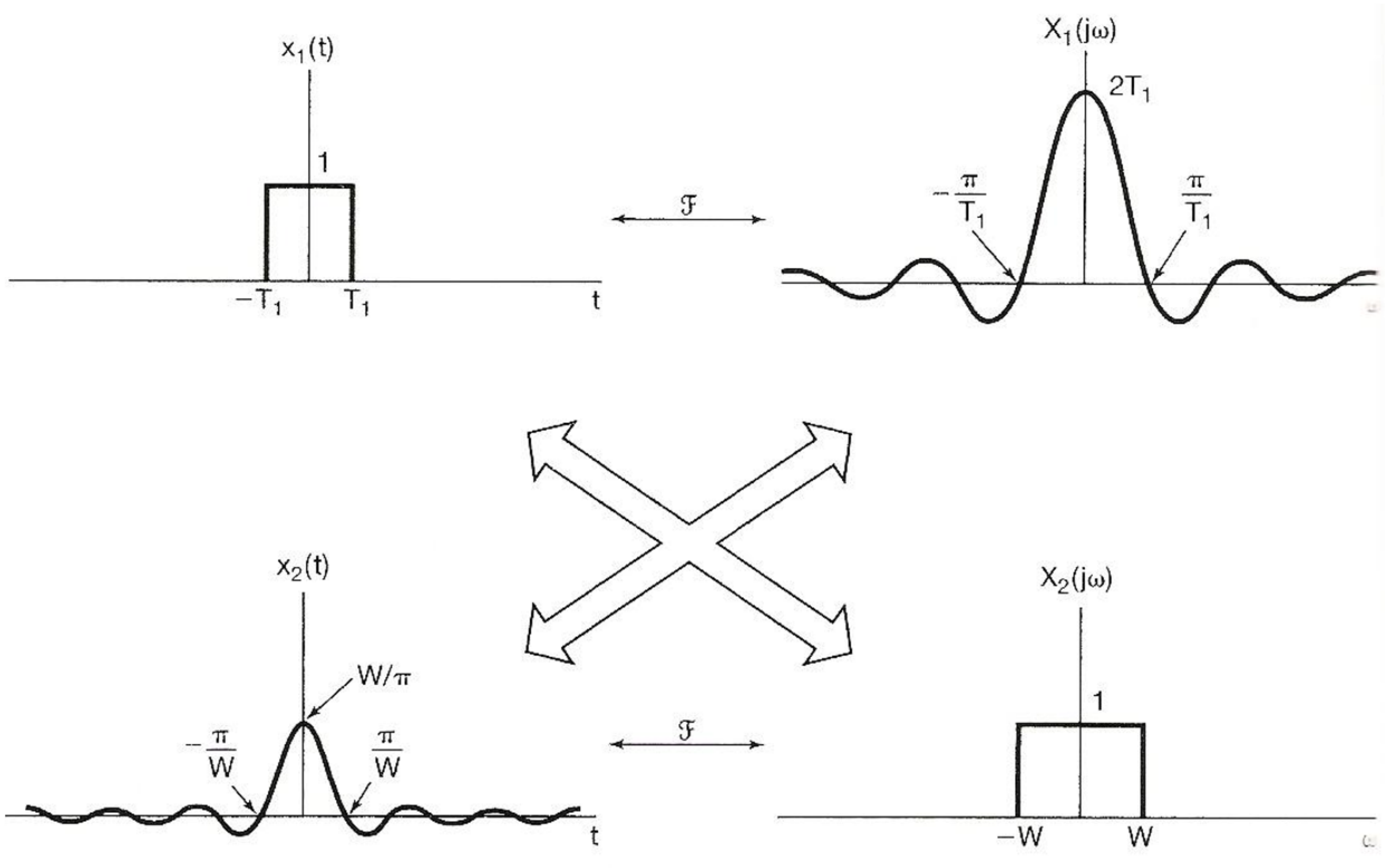
Fourier Transform Properties

Duality

- Time/frequency domains are kind of "symmetric" except for a sign change (and a factor of 2π)

Example 4.4, p.208 of textbook:

Example 4.5, p.209 of textbook:



Fourier Transform Properties

Duality

- If $x(t)$ has a Fourier Transform $X(j\omega)$, then if we form a new function of time that has the functional form of the transform, $X(t)$, it will have a Fourier Transform $x(\omega)$ that has the functional form of the original time function (but is a function of frequency).

- Mathematically, we can write:

$$x(t) \overset{F}{\leftrightarrow} X(j\omega) = X'(\omega)$$

$$X'(t) \overset{F}{\leftrightarrow} 2\pi x(-\omega)$$

Fourier Transform Properties

Duality

$$x(t) \overset{F}{\leftrightarrow} X(j\omega) = X'(\omega)$$
$$X'(t) \overset{F}{\leftrightarrow} 2\pi x(-j\omega)$$

Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X'(\omega)e^{j\omega t}d\omega$$

$$2\pi x(t) = \int_{-\infty}^{\infty} X'(\omega)e^{j\omega t}d\omega$$

Simply replace t and ω by $-\omega$ and t:

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X'(t)e^{-j\omega t}dt$$

Fourier Transform Properties

Duality

Example: Given a pair of Fourier transform: $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \overset{F}{\leftrightarrow} X(j\omega) = \frac{2sin\omega T_1}{\omega}$, find Fourier transform for $x(t) = \frac{sinWt}{\pi t}$.

Fourier Transform Properties

Duality

$$x(t) \overset{F}{\leftrightarrow} X(j\omega) = X'(\omega)$$

$$X'(t) \overset{F}{\leftrightarrow} 2\pi x(-\omega)$$

Example: Given a pair of Fourier transform: $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \overset{F}{\leftrightarrow} X(j\omega) = \frac{2\sin\omega T_1}{\omega}$, find Fourier transform for $x(t) = \frac{\sin Wt}{\pi t}$.

$$\begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \overset{F}{\leftrightarrow} \frac{2\sin\omega T_1}{\omega}$$

$$\frac{2\sin tT_1}{t} \overset{F}{\leftrightarrow} \begin{cases} 2\pi, & |\omega| < T_1 \\ 0, & |\omega| > T_1 \end{cases}$$

Parseval's relation

- The integral of the square of a function is equal to the integral of the square of its Fourier transform.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Parseval's relation

Proof:

$$\int_{-\infty}^{\infty}|x(t)|^2 dt = \int_{-\infty}^{\infty}x(t)x^*(t)dt$$

$$= \int_{-\infty}^{\infty}x(t)[\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(j\omega)e^{-j\omega t}d\omega]\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(j\omega)[\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt]\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(j\omega)X(j\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega$$

Fourier Transform Properties

Convolution

Assume :

$$x(t) \overset{F}{\leftrightarrow} X(j\omega), h(t) \overset{F}{\leftrightarrow} H(j\omega)$$

Then:

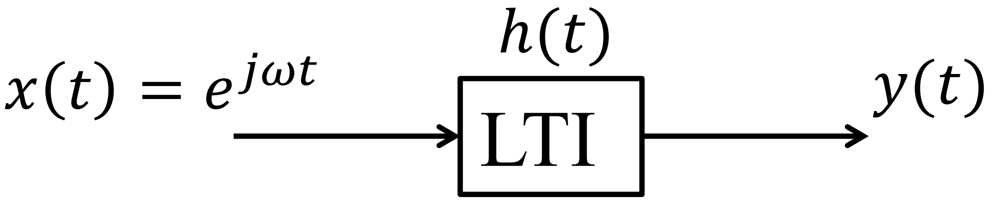
$$x(t) * h(t) \overset{F}{\leftrightarrow} X(j\omega)H(j\omega)$$

Proof: p223 of textbook

Fourier Transform Properties

Convolution

Response of LTI systems to exponential signal (Lecture 4)



$$y(t) = e^{j\omega t} * h(t) = e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau = e^{j\omega t} H(j\omega)$$

Fourier Transform Properties

Convolution

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\doteq \frac{1}{2\pi}\lim_{\omega_0 \to 0}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$= \sum_{k=-\infty}^{\infty} A_k e^{jk\omega_0 t}$$

Fourier Transform Properties

Convolution

$$x(t) = \sum_{k=-\infty}^{\infty} A_k e^{jk\omega_0 t} \longrightarrow \boxed{\text{LTI}}_{h(t)} \longrightarrow y(t)$$

$$y(t) = \sum_{k=-\infty}^{\infty} A_k e^{jk\omega_0 t} * h(t) = \sum_{k=-\infty}^{\infty} A_k e^{jk\omega_0 t} H(jk\omega_0)$$

$$= \frac{1}{2\pi} \lim_{\omega_0 \to 0} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 H(jk\omega_0)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) H(j\omega) e^{j\omega t} d\omega$$

Fourier Transform Properties

Convolution

$$x(t) * h(t) \overset{F}{\leftrightarrow} X(j\omega)H(j\omega)$$

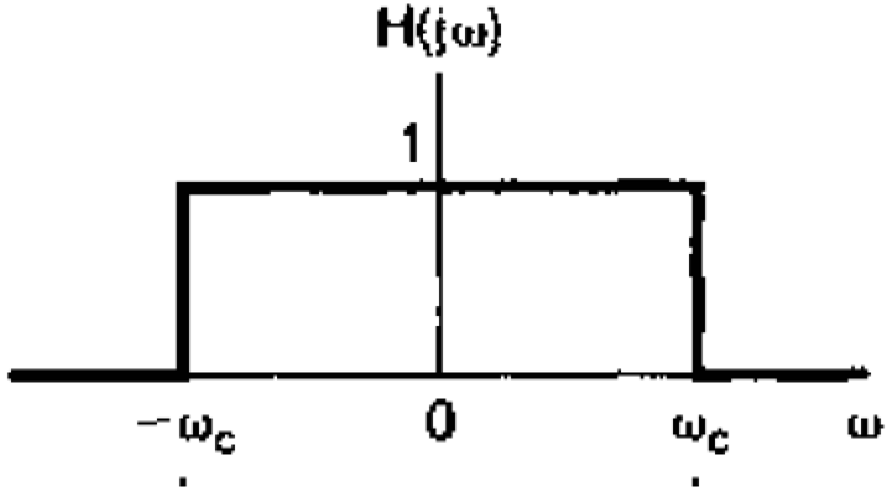
- Convolution in time domain leads to multiplication in frequency domain
- Cascade of multiple systems implies product of these frequency response, independent of the order of the cascade
- Application: filtering of signals

Fourier Transform Properties

Convolution

Ideal low-pass filter

$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



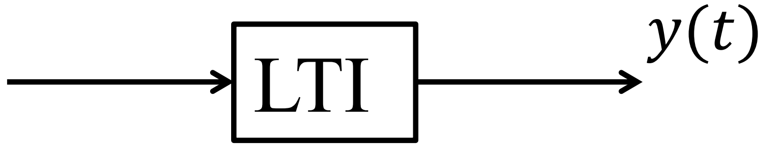
$$h(t) = \frac{\sin\omega_c t}{\pi t}$$

Fourier Transform Properties

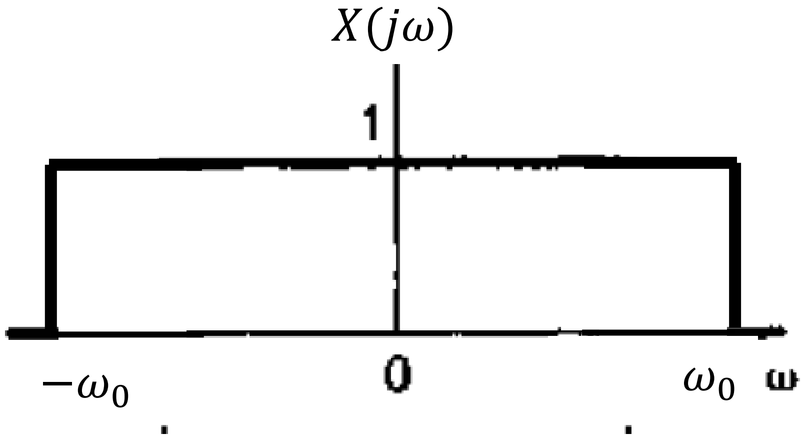
Convolution

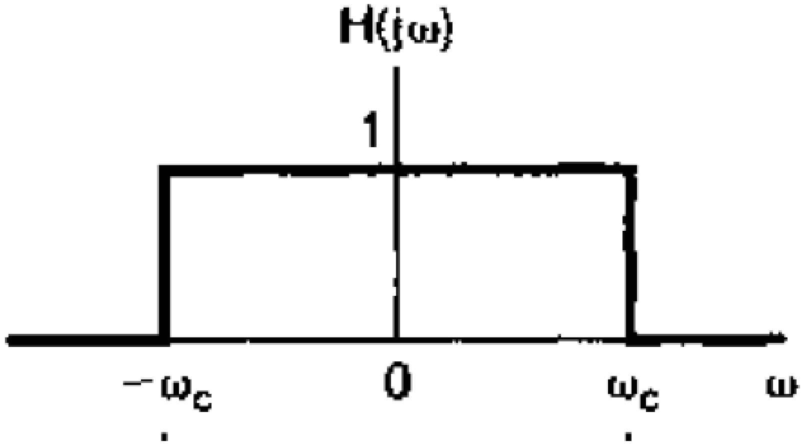
Ideal low-pass filter

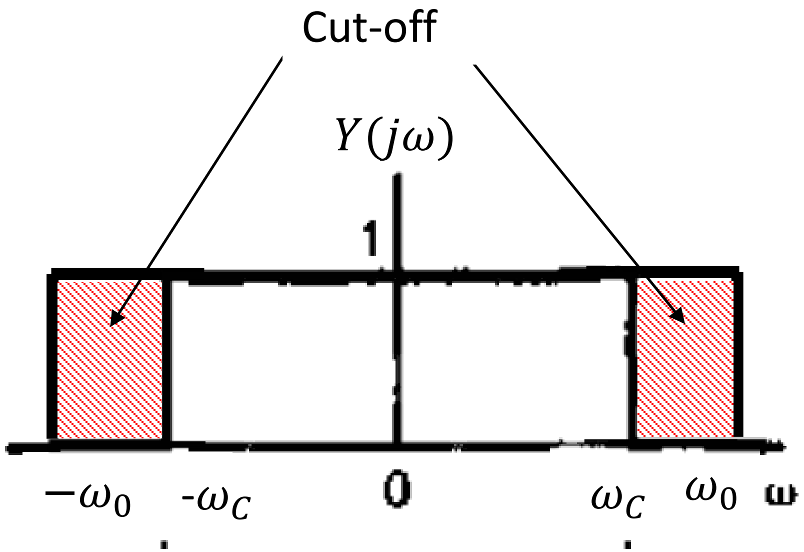
$$x(t) = \frac{sin\omega_0 t}{\pi t} \qquad h(t) = \frac{sin\omega_c t}{\pi t}$$



Assume $\omega_0 > \omega_c$:







Fourier Transform Properties

Multiplication

Assume :

$$s(t) \overset{F}{\leftrightarrow} S(j\omega), p(t) \overset{F}{\leftrightarrow} P(j\omega)$$

Then:

$$s(t)p(t) \overset{F}{\leftrightarrow} \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

- Multiplication property is a dual property of convolution property.

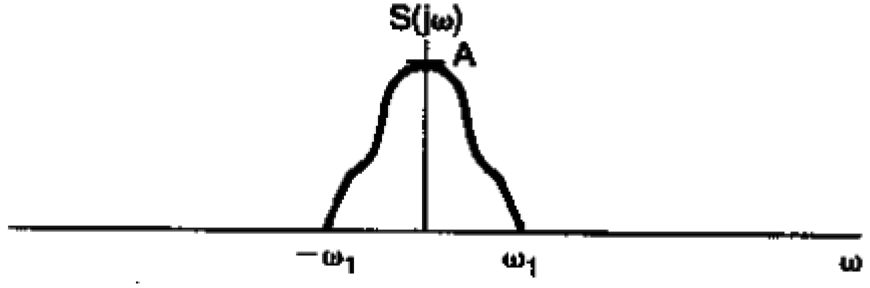
Fourier Transform Properties

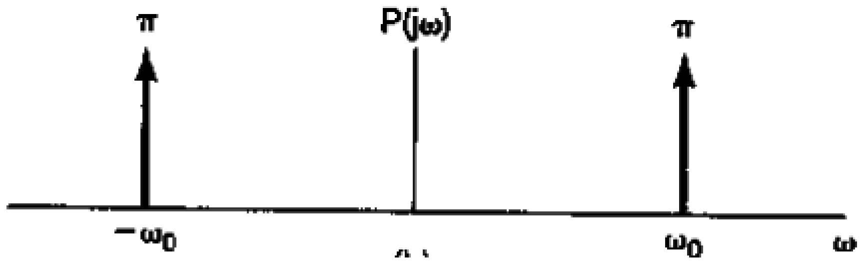
Multiplication -> Modulation

- *Example 4.21, p.229 of textbook*

$$s(t) \overset{F}{\leftrightarrow} S(j\omega)$$

$$p(t) = cos\omega_0 t \overset{F}{\leftrightarrow}$$
$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$





Fourier Transform Properties

Multiplication -> Modulation

- *Example 4.21, p.229 of textbook*

$$s(t)p(t) \overset{F}{\leftrightarrow} \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

$$\frac{1}{2\pi}[S(j\omega) * P(j\omega)] = \frac{1}{2\pi}[S(j\omega) * \pi\delta(\omega - \omega_0) + S(j\omega) * \pi\delta(\omega + \omega_0)]$$

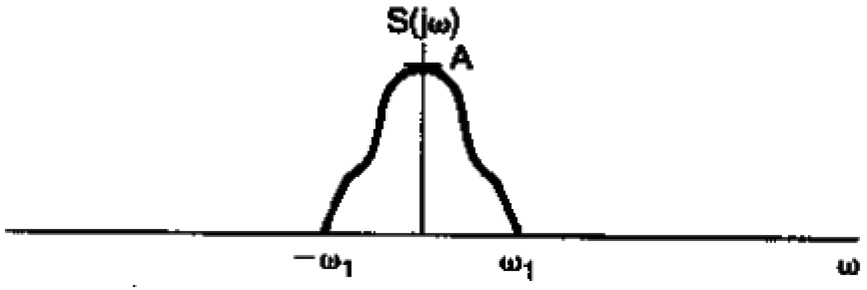
$$= \frac{1}{2\pi}[\int_{-\infty}^{\infty} S(ju)\pi\delta(u - \omega - \omega_0)du + \int_{-\infty}^{\infty} S(ju)\pi\delta(u - \omega + \omega_0)du]$$

$$= \frac{1}{2}S(j(\omega + \omega_0)) + \frac{1}{2}S(j(\omega - \omega_0))$$

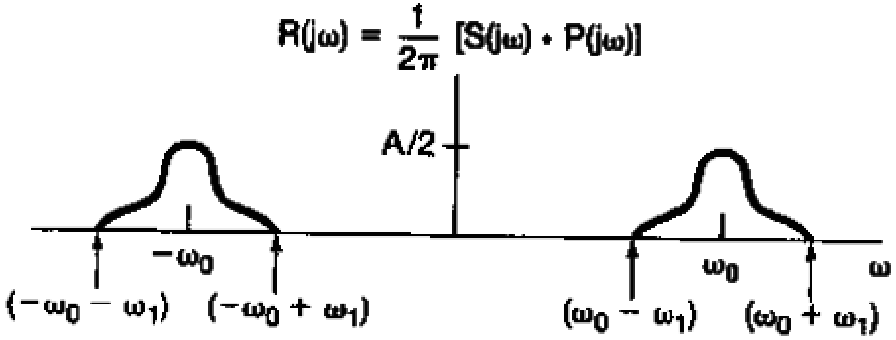
Fourier Transform Properties

Multiplication -> Modulation

- *Example 4.21, p.229 of textbook*



$\frac{1}{2\pi}[S(j\omega) * P(j\omega)]$
generally has the shape (information) of $S(j\omega)$ and the frequency of $P(j\omega)$.



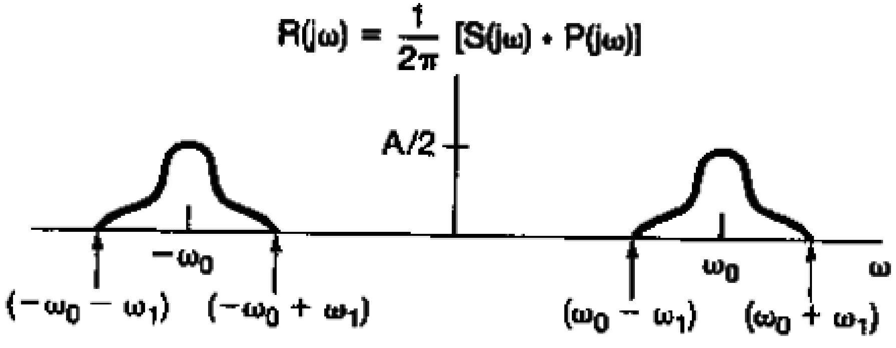
(c)

Fourier Transform Properties

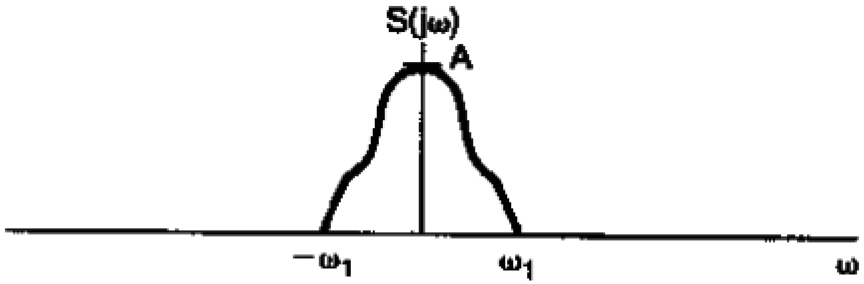
Multiplication -> Demodulation

- *Example 4.22, p.230 of textbook*

Recover $S(j\omega)$ from $R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$



(c)



Fourier Transform Properties

Multiplication -> Demodulation

- *Example 4.22, p.230 of textbook*

$$r(t)p(t) \overset{F}{\leftrightarrow} \frac{1}{2\pi}[R(j\omega) * P(j\omega)]$$

$$\frac{1}{2\pi}[R(j\omega) * P(j\omega)]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{1}{2}S\big(j(\omega+\omega_0)\big) + \frac{1}{2}S\big(j(\omega-\omega_0)\big)\right] * [\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)]\right\}$$

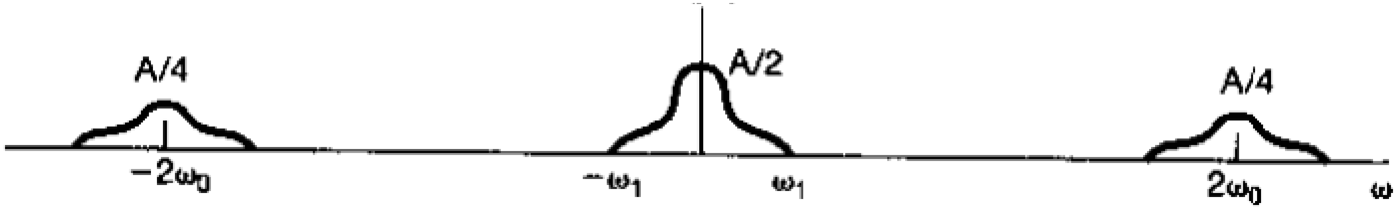
$$= \frac{1}{4}S\big(j(\omega+2\omega_0)\big) + \frac{1}{2}S(j\omega) + \frac{1}{4}S\big(j(\omega-2\omega_0)\big)$$

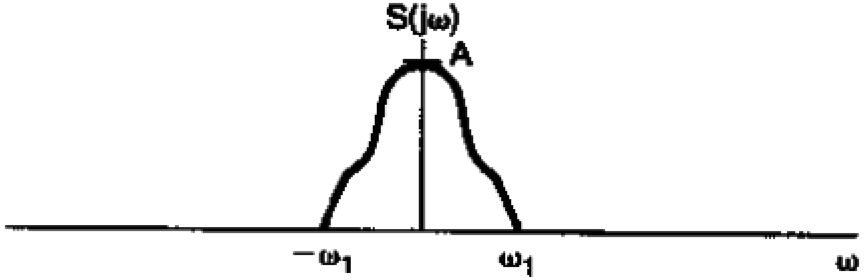
Fourier Transform Properties

Multiplication -> Demodulation

- *Example 4.22, p.230 of textbook*

$$\frac{1}{2\pi}[R(j\omega) * P(j\omega)]$$





S(jω) is recovered by putting $\frac{1}{2\pi}[R(j\omega) * P(j\omega)]$ into a low-pass filter ($\omega_c = \omega_1$)

Thank you for your listening!