Proofs About Expected Value and Variance

𝕃umi

March 8, 2023

We would like to prove three equations mentioned on Biostatistics course on March the 7th.

1 Pre-requirements

Expected values are defined as follows.

$$E(X) = \sum_x xP_X(x) \tag{1}$$

Variances are defined as follows.

$$\mathrm{Var}(X) = E([X - E(X)]^2) \tag{2}$$

2 Random Variables Built on Random Variables

Suppose we have random variables X and Y. Y and X are related through the funciton $y = f(x)$. We would like to prove that

$$E(Y) = \sum_x g(x)P_X(x) \tag{3}$$

We start from the basic definition of expected values.

$$\begin{aligned} E(Y) &= \sum_y yP_Y(y) \\ &= \sum_x g(x)P_{g(X)}(g(x)) \end{aligned} \tag{4}$$

Thus, what we need to prove is $P_{g(X)}(g(x)) = P_X(x)$.

$$\begin{aligned} P_{g(x)}(g(x)) &= \sum P(g(X) = g(x)) \\ &= \sum P(X = g^{-1}[g(x)]) \\ &= \sum P(X = x) \\ &= P_X(x) \end{aligned} \tag{5}$$

3 Nudging X in a Linear Way

The second equation to prove is when we nudge X a bit and see what happens to the expected value of X.

$$E(\alpha X + \beta) = \alpha E(X) + \beta \tag{6}$$

This is quite easy to prove with the basic definition.

$$\begin{aligned} E(\alpha X + \beta) &= \sum_x (\alpha x + \beta) P(\alpha x + \beta) \\ &= \sum_x \alpha x P(\alpha x + \beta) + \sum_x \beta P(\alpha x + \beta) \end{aligned} \tag{7}$$

We know that,

$$\sum_x P(\alpha x + \beta) = 1, P(\alpha x + \beta) = P(x) \tag{8}$$

So

$$\begin{aligned} E(\alpha x + \beta) &= \alpha \sum_x x P(x) + \beta \\ &= \alpha E(X) + \beta \end{aligned} \tag{9}$$