

# Biostatistics Homework 2

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```
library(ggplot2)

setwd('/Users/lumizhang/Documents/sustech/biology/classes/Biostatistics/Homework/hw2')
```

## 1 Lots of Coins

### 1.1

$$\begin{aligned}P(\text{getting a head}) &= P(\text{head}|\text{HH coin}) \cdot P(\text{HH coin}) + P(\text{head}|\text{HT coin}) \cdot P(\text{HT coin}) \\&= 1 \times \frac{1}{9} + \frac{1}{2} \times \frac{8}{9} \\&= \frac{5}{9}\end{aligned}$$

### 1.2

On the first four times of flip, we all get heads, that changes the possibility of the coin having two heads. It is much more likely than  $\frac{1}{9}$  that the coin is a fair coin. So the new probability can be calculated with:

$$\begin{aligned}P(\text{HH}|4\text{H}) &= \frac{P(4\text{H}|\text{HH}) \cdot P(\text{HH})}{P(4\text{H})} \\&= \frac{P(4\text{H}|\text{HH}) \cdot P(\text{HH})}{P(4\text{H}|\text{HT}) \cdot P(\text{HT}) + P(4\text{H}|\text{HH}) \cdot P(\text{HH})} \\&= \frac{1 \times \frac{1}{9}}{1 \times \frac{1}{9} + (\frac{1}{2})^4 \times \frac{8}{9}} = \frac{2}{3} \\P(\text{HT}|4\text{H}) &= 1 - P(\text{HH}|4\text{H}) = 1 - \frac{2}{3} = \frac{1}{3}\end{aligned}$$

Thus we can get the probability of getting another head, based on our knowledge that the first four tosses are all heads.

$$\begin{aligned}P(\text{another head}) &= P(\text{HH}|4\text{H}) \cdot P(\text{H}|\text{HH}) + P(\text{HT}|4\text{H}) \cdot P(\text{H}|\text{HT}) \\&= \frac{2}{3} \times 1 + \frac{1}{3} \times \frac{1}{2} \\&= \frac{5}{6}\end{aligned}$$

## 2 Detecting Virus with our Kit

### 2.1 & 2.2

$$P(E_1) = 0.01 \times 0.9 + 0.99 \times 0.05 = 0.0585 P(E_2) = 0.01 \times 0.95 + 0.99 \times 0.1 = 0.1085$$

## 2.3

$E_1$  and  $E_2$  are not independent, because sadly, the test positive result of any one of the kit increases the probability that the person have actually get the virus. When he do a second test, the kit is more likely to show a positive result.

$$P(E_1 \cap E_2) \neq P(E_1) \cdot P(E_2)$$

## 2.4

Under the condition that the person is or is not carrying the virus, events  $E_1$  and  $E_2$  are actually independent. So:

$$\begin{aligned} P(E_3|E_1 \cap E_2) &= \frac{P(E_1 \cap E_2|E_3) \cdot P(E_3)}{P(E_1 \cap E_2)} \\ &= \frac{P(E_1|E_3) \cdot P(E_2|E_3) \cdot P(E_3)}{P(E_1 \cap E_2|E_3) \cdot P(E_3) + P(E_1 \cap E_2|E'_3) \cdot P(E'_3)} \\ &= \frac{0.9 \times 0.95 \times 0.01}{0.9 \times 0.95 \times 0.01 + 0.05 \times 0.1 \times 0.99} \\ &= \frac{19}{30} = 0.633... \end{aligned}$$

From this problem, we can also see that:

$$P(E_1 \cap E_2) = 0.0135P(E_1) \cdot P(E_2) \neq 0.0135$$

So  $E_1$  and  $E_2$  are not independent.

## 3 Weird DIE Again

### 3.1

```
die_a <- c(1,2,3,4,5)
die_b <- c(1,2,3,4,5)
sample_space <- as.data.frame(expand.grid(die_a, die_b))
colnames(sample_space) <- c("die_a", "die_b")
sample_space
```

```
##      die_a die_b
## 1         1     1
## 2         2     1
## 3         3     1
## 4         4     1
## 5         5     1
## 6         1     2
## 7         2     2
## 8         3     2
## 9         4     2
## 10        5     2
## 11        1     3
## 12        2     3
## 13        3     3
## 14        4     3
## 15        5     3
```

```
## 16      1      4
## 17      2      4
## 18      3      4
## 19      4      4
## 20      5      4
## 21      1      5
## 22      2      5
## 23      3      5
## 24      4      5
## 25      5      5
```

```
sample_space$sum <- sample_space$die_a + sample_space$die_b
sample_space$A <- sample_space$sum == 10
sample_space$B <- sample_space$die_a == 5 | sample_space$die_b == 5
sample_space$C <- sample_space$die_a == 1 | sample_space$die_b == 1
sample_space
```

```
##      die_a die_b sum      A      B      C
## 1         1     1  2 FALSE FALSE  TRUE
## 2         2     1  3 FALSE FALSE  TRUE
## 3         3     1  4 FALSE FALSE  TRUE
## 4         4     1  5 FALSE FALSE  TRUE
## 5         5     1  6 FALSE  TRUE  TRUE
## 6         1     2  3 FALSE FALSE  TRUE
## 7         2     2  4 FALSE FALSE FALSE
## 8         3     2  5 FALSE FALSE FALSE
## 9         4     2  6 FALSE FALSE FALSE
## 10        5     2  7 FALSE  TRUE FALSE
## 11        1     3  4 FALSE FALSE  TRUE
## 12        2     3  5 FALSE FALSE FALSE
## 13        3     3  6 FALSE FALSE FALSE
## 14        4     3  7 FALSE FALSE FALSE
## 15        5     3  8 FALSE  TRUE FALSE
## 16        1     4  5 FALSE FALSE  TRUE
## 17        2     4  6 FALSE FALSE FALSE
## 18        3     4  7 FALSE FALSE FALSE
## 19        4     4  8 FALSE FALSE FALSE
## 20        5     4  9 FALSE  TRUE FALSE
## 21        1     5  6 FALSE  TRUE  TRUE
## 22        2     5  7 FALSE  TRUE FALSE
## 23        3     5  8 FALSE  TRUE FALSE
## 24        4     5  9 FALSE  TRUE FALSE
## 25        5     5 10  TRUE  TRUE FALSE
```

```
a <- sum(sample_space$A == TRUE)
b <- sum(sample_space$B == T)
c <- sum(sample_space$C == T)
ab <- sum(sample_space$A == T & sample_space$B == T)
ac <- sum(sample_space$A == T & sample_space$C == T)
sprintf("ab : %i/25, ac : %i/25", ab, ac)
```

```
## [1] "ab : 1/25, ac : 0/25"
```

We can see from the table and the printed results that

$$P(A \cap B) \neq P(A) \cdot P(B)$$

and

$$P(A \cap C) \neq P(A) \cdot P(B)$$

So  $A$  and  $B$  are not independent, and  $A$  and  $C$  are also not independent. Actually,  $A$  and  $C$  are mutually exclusive, so they are strongly dependent on each other.

### 3.2

```
sample_space$D <- sample_space$sum == 7
sample_space$E <- abs(sample_space$die_a - sample_space$die_b) == 1
sample_space$F <- sample_space$die_a < sample_space$die_b
e <- sum(sample_space$E == T)
f <- sum(sample_space$F == T)
ef <- sum(sample_space$E == T & sample_space$F == T)
sample_space[sample_space$F == T, c("E", "F")]
```

```
##           E      F
## 6    TRUE TRUE
## 11   FALSE TRUE
## 12    TRUE TRUE
## 16   FALSE TRUE
## 17   FALSE TRUE
## 18    TRUE TRUE
## 21   FALSE TRUE
## 22   FALSE TRUE
## 23   FALSE TRUE
## 24    TRUE TRUE
```

$$P(E|F) = \frac{4}{10} \neq P(E) = \frac{8}{25}$$

So  $E$  and  $F$  are not independent.

```
sample_space[sample_space$D == T, c("E", "F")]
```

```
##           E      F
## 10  FALSE FALSE
## 14    TRUE FALSE
## 18    TRUE  TRUE
## 22  FALSE  TRUE
```

$$P(E|F) = 0.5 = P(E), P(F) \neq 0$$

So they are independent under the condition that  $D$  is true.

## 4 What's a CD-ROM?

### 4.1

Number of defected old CD-ROMs are :  $500 \times 15\% = 75$

Number of defected new CD-ROMs are :  $1500 \times 5\% = 75$

So,

the probability of buying two successive broken old CD-ROMs are  $\frac{75}{500} \times \frac{74}{499} = \frac{111}{4990}$   
the probability of buying two successive broken old CD-ROMs are  $\frac{75}{1500} \times \frac{74}{1499} = \frac{37}{14990}$   
so the probability of event  $A$  that both CD-ROMs being defected is:

$$P(A) = P(\text{old}) \cdot \frac{111}{4990} + P(\text{new}) \cdot \frac{37}{14990} \approx 0.0123$$

## 4.2

$$P(\text{old}|A) = \frac{P(A|\text{old}) \cdot P(\text{old})}{P(A)} = \frac{\frac{111}{4990} \times 0.5}{0.0123} = \frac{4497}{4996} \approx 0.9$$

## 5 Activating Transcription Factor 1

First we look at the  $\text{RAS} \rightarrow \text{PI3K} \rightarrow \text{AKT} \rightarrow \text{TF1}$  pathway. The probability of this pathway succeeding is

$$0.9 \times 0.8 \times 0.9 = 0.648$$

Then we look at the  $\text{RAS} \rightarrow \text{RAF} \rightarrow \text{MEK} \rightarrow \text{ERK1 or ERK2} \rightarrow \text{TF1}$  pathway. The probability of this pathway succeeding is

$$0.8 \times 0.9 \times (1 - (1 - 0.9 \times 0.8)(1 - 0.8 \times 0.8)) = 0.647424$$

So in order for the whole  $\text{RTK} \rightarrow \text{TF1}$  to succeed, at least one of the two pathways should succeed.

$$P(\text{TF1}) = 0.9 \times (1 - (1 - 0.648)(1 - 0.647424)) \approx 0.7883$$

## 6 Weird Signals

### 6.1

The  $k$ -th signal is an independent event. So the probability of event  $A = \{\text{receive correctly}\}$  is

$$P(A) = p \cdot (1 - \epsilon_0) + (1 - p)\epsilon_1$$

### 6.2

Because receiving the signal are independent events, we know that

$$\begin{aligned} P(1011|1011) &= P(1|1) \cdot P(0|0) \cdot P(1|1) \cdot P(1|1) \\ &= (1 - \epsilon_1)(1 - \epsilon_0)(1 - \epsilon_1)(1 - \epsilon_1) \end{aligned}$$

### 6.3

Event  $A$  = three zeros. Event  $B$  = two zeros.

$$\begin{aligned} P(A) &= (1 - \epsilon_0)^3 \\ P(B) &= (1 - \epsilon_0)^2 \cdot \epsilon_0 \cdot \binom{3}{1} = 3(1 - \epsilon_0)^2 \epsilon_0 \\ P(\text{correct}) &= P(A) + P(B) = (1 - \epsilon_0)^2(1 + 2\epsilon_0) \end{aligned}$$

## 6.4

We can see the improvement from this:

$$\frac{P(\text{correct}|3\text{times})}{P(\text{correct}|1\text{time})} = \frac{(1 - \epsilon_0)^2(1 + 2\epsilon_0)}{1 - \epsilon_0} = (1 - \epsilon_0)(1 + 2\epsilon_0) = 1 + \epsilon_0 - 2\epsilon_0^2$$

Because we want an improvement, we want  $1 + \epsilon_0 - 2\epsilon_0^2$  to be larger than 1. So:

$$\begin{aligned} 1 + \epsilon_0 - 2\epsilon_0^2 &> 1 \\ \epsilon_0 &> 2\epsilon_0^2 \\ 0 < \epsilon_0 &< \frac{1}{2} \end{aligned}$$

## 6.5

$$\begin{aligned} P(0|101) &= \frac{P(101|0) \cdot P(0)}{P(101|0) \cdot P(0) + P(101|1) \cdot P(1)} \\ &= \frac{\epsilon_0^2(1 - \epsilon_0)p}{\epsilon_0^2(1 - \epsilon_0)p + (1 - \epsilon_1)^2\epsilon_1(1 - p)} \end{aligned}$$

## 7 Happy Prison

The important thing here is that out of three prisoners, two will be chosen. So no matter if our lovely prisoner of interest is chosen to be released or not, there will always be another prisoner from the other two prisoners to be chosen. In other words  $P(B|A) = P(B|A')$ . So

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')} = P(A)$$

So event  $A$  and event  $B$  are independent.

## 8 Searching in a Smarter Way

From intuition, the fact that we didn't find the object in cell  $i$  must cause a reduction of the probability that the object is in cell  $i$ , and an increase in the probability of finding the object in other cells.

We can prove this in a mathematical way (although mine might be wrong, but the results turned out to be ok). So, suppose event  $X_i$  means not finding the object in cell  $i$  and  $\Pi_i$  means the event of finding the object in the cell  $i$  (we use  $\Pi$  instead of  $\pi$  to differentiate these two).

$$\begin{aligned} P(\Pi_i|X_i) &= \frac{P(X_i|\Pi_i)P(\Pi_i)}{P(X_i)} \\ &= \frac{(1 - p_i)\pi_i}{1 - \pi_i p_i} \end{aligned}$$

We can prove that the updated information in the cell  $i$  actually leads to a decrease in probability of finding the object inside the cell. In other words  $P(\Pi_i|X_i) < \pi_i$ , since  $p_i$  doesn't change.

$$\frac{P(\Pi_i|X_i)}{\pi_i} = \frac{1 - p_i}{1 - \pi_i p_i} < 1, \text{ thus } P(\Pi_i|X_i) < \pi_i$$

Because  $\pi_i < 1$  so,  $p_i > \pi_i p_i$ ,  $1 - \pi_i p_i > 1 - p_i$ . Which fits our guess that the updated probability should be smaller.

As for other cells, the possibility should be larger,  $P(\Pi_j|X_i) > \pi_j$ .

$$P(\Pi_j|X_i) = \frac{P(X_i|\Pi_j)P(\Pi_j)}{P(X_i)} = \frac{\pi_j}{1 - \pi_i p_i}$$

Because  $1 - \pi_i p_i$  is smaller than 1 but greater than 0, we can see that  $P(\Pi_j|X_i) > \pi_j$ .

The results are consistent with our intuitions.