Compressed Sensing Reconstruction using ADMM

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22MAT230(Mathematics For Computing 4)

Motivation

- Traditionalsampling follows Nyquist rate
- High-dimensional signals often have sparse representations
- Compressed Sensing (CS) allows recovery from fewer measurements

Measurement model:

$$b=Ax \ (m<<n)$$

Where:

x: sparse signal

A: sensing matrix

b: measurement vector

Sparsity Assumption

Sparsity Concept

- Signal contains very few non-zero entries
- Let sparsity level be sss

$$\|x\|_0 \leqslant s$$

- ℓ_0-norm counts number of non-zero elements
- Leads to non-convex optimization

Compressed Sensing Problem

Equation formulation:

$$\min_{x} \|x\|_0 \qquad s.t.\, Ax = b$$

- ℓ_0 norm counts non-zero elements
- Direct solution is NP-hard

$$\min_{x} \|Ax - b\|_2^2 \qquad s.t\, \|x\|_0 \leqslant s$$

Variable Splitting

$$y = Ax + b$$

Augmented Lagrangian:

$$L_p\left(x, y, z\right) = f\left(y\right) - z^T\left(Ax + b - y\right) + \frac{\rho}{2}\|Ax + b - y\|^2$$

Where:
z: Lagrange multiplier
ρ: penalty parameter

ADMM Iteration – x update

$$x^{k+1} = argmin \|Ax - b_k\|^2 \qquad s.t. \|x\|_0 \leqslant s$$

Solved using:
- Gradient step
- Limited shrinkage thresholding
- Hard thresholding to keep top-ss entries

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ADMM Iteration – y update

$$y^{k+1} = argminf(y) + \frac{\rho}{2}\|Ax^{k+1} + b - y\|^2$$

- Typically a least-squares problem
- Has closed-form solution

ADMM Iteration – z update

Multiplier update:

$$z^{k+1} = z^k + \rho \left(y^{k+1} - Ax^{k+1} - b \right)$$

- Enforces consistency of constraints
- Ensures convergence

Gradient step in x-update

Gradient Descent Step:

$$x - \frac{1}{v} A^T (Ax - b_k)$$

- Moves solution toward data consistency
- Requires thresholding to enforce sparsity

LST and Hard Thresholding

Why LST?

- Hard thresholding → unstable
- Soft thresholding → bias

LST Properties:

- Suppresses small coefficients
- Preserves large coefficients
- Improves stability

HardThresholding Operator

Keeps top-s largest magnitude values
Sets remaining entries to zero
Ensures sparsity constraint

Restricted Isometry Property

RIP provides a condition on the sensing matrix that guarantees sparse signals are not distorted.

$$(1 - \delta_s)\|x\|_2^2 \leqslant \|Ax\|_2^2 \leqslant (1 + \delta_s)\|x\|_2^2$$

δ_s : RIP constant

$\|Ax\|_2^2$: Energy after measurement

$\|x\|_2^2$ Energy of sparse signal

Hard Thresholding

Hard thresholding operator $H_s(.)$ keeps only the top-sss largest magnitude entries and sets the rest to zero.

$$H_s(x)_i = \begin{cases} x_i, & \text{if } |x|_i \text{ is among top s values} \\ 0, & \text{otherwise} \end{cases}$$

Properties:

- Enforces exact sparsity
- Simple and fast
- Can be unstable when used alone

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Limited Shrinkage Thresholding (LST)

LSTisageneralized thresholdingoperatorthat:
- Suppresses small coefficients
- Preserves large coefficients
- Limits the amount of shrinkage applied

$$
LT_\lambda = \begin{cases} 0, & |t| \leqslant k\left(\lambda\right) \\ \text{t-limited shrinkage}, & |t| > k\left(\lambda\right) \end{cases}
$$

Where:

λ : threshold parameter

$k\left(\lambda\right)$: shrinkage bound

Combined Use in ADMM

In the x-update step:

$$x^{k+1} \in H_s . LT_\lambda \left(x^k - \frac{1}{v} A^T \left(Ax^k - b_k \right) \right)$$

Interpretation:

1. Gradient step → data consistency
2. LST → suppress noise, control shrinkage
3. Hard thresholding → enforce sparsity

Experimental Results

Compressed Sensing using ADMM + LST

Legend: Original, Reconstructed

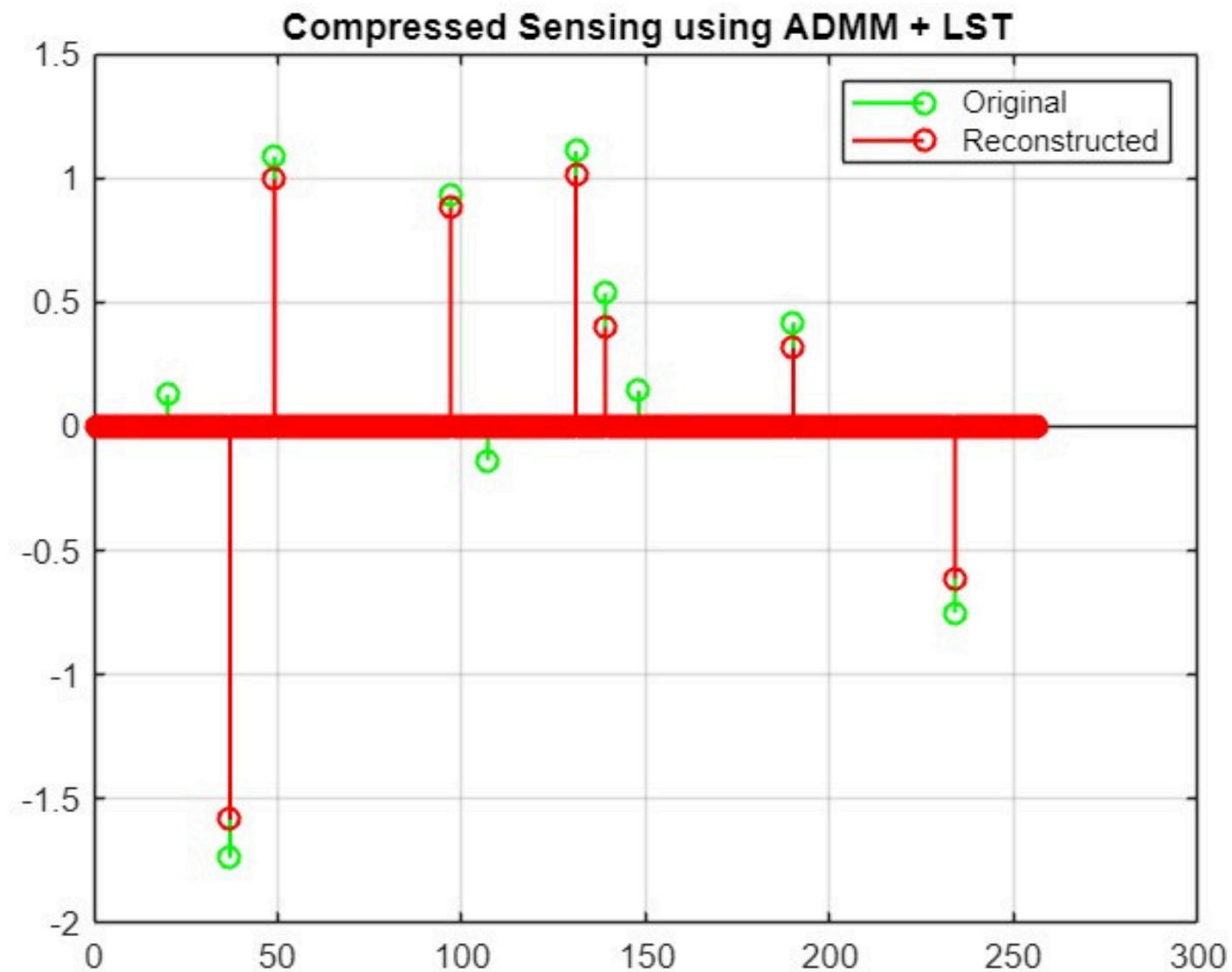Figure Description:

- Green stems: Original sparse signal Xtrue
- Red stems: Reconstructed signal using ADMM + LST
- Signal length n=256, sparsity level s=10
- Number of measurements m≪n

Obser vations

- Most non-zero locations of the original signal are correctly identified
- Reconstructed signal maintains strong sparsity
- Small amplitude mismatch is observed in some coefficients
- No numerical divergence or instability

Conclusion and Future Work

This project implementedcompressed sensing reconstruction using an ADMM framework combined with Limited Shrinkage Thresholding to solve an ℓ_0-constrained sparse recovery problem. Variable splitting enabled efficient alternating minimization, while LST improved numerical stability and reduced bias compared to classical thresholding methods. The MATLAB implementation follows the theoretical formulation of the base paper, and the results demonstrate stable convergence and effective sparse signal recovery under the Restricted Isometry Property assumption. Future work includes extending the method using inertial ADMM for faster convergence, comparing performance with classical algorithms such as Iterative Hard Thresholding, analyzing robustness under noisy measurements, and applying the framework to higher-dimensional problems such as image reconstruction.