Solutions

1) (a) $u = e^{x^2-y^2}$ $\quad v = e^{x^2+y^2}$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^{x^2-y^2}\cdot 2x & e^{x^2-y^2}\cdot(-2y) \\ e^{x^2+y^2}\cdot 2x & e^{x^2+y^2}\cdot 2y \end{vmatrix}$$

$$= e^{2x^2}\cdot 4xy + e^{2x^2} 4xy$$

$$= 8xy\, e^{2x^2}$$

(b) $u = e^x \cos y$ $\quad v = e^x \sin y$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

$$= e^{2x} \cos^2 y + e^{2x} \sin^2 y$$

$$= e^{2x}.$$

(c) $x = \frac{u}{v}$, $\quad y = 4u^2 - 4v^2$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 2u & -8v \end{vmatrix}$$

$$= -8 + \frac{2u^2}{v^2}$$

(d) $x = \rho \sin\phi \cos\theta$, $\quad y = \rho \sin\phi\, \sin\theta$, $\quad z = \rho\cos\phi$

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\,\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2 \Big[\sin\phi \cos\theta \left(\sin^2\phi \cos\theta \right) - \cos\phi \cos\theta \left(-\sin\phi \cos\phi \cos\theta \right) - \sin\phi \sin\theta \left(-\sin^2\phi \sin\theta - \cos^2\phi \sin\theta \right) \Big]$$

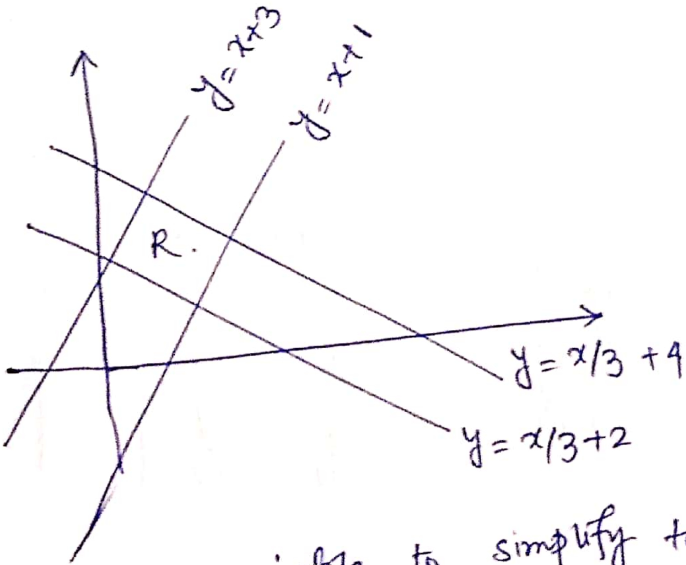
$$= \rho^2 \left[\sin^3\phi \cos^2\theta + \sin\phi \cos^2\phi \cos^2\theta + \sin\phi \sin^2\theta \right]$$

$$= \rho^2 \left[\sin\phi \cos^2\theta \left(\sin^2\phi + \cos^2\phi \right) + \sin\phi \sin^2\theta \right]$$

$$= \rho^2 \left[\sin\phi \cos^2\theta + \sin\phi \sin^2\theta \right]$$

$$= \rho^2 \sin\phi$$

2) (a) The Region R is sketched in fig 1.



We use the change the variable to simplify the integral

Let, $u = y - x$, $\quad v = y + x/3$

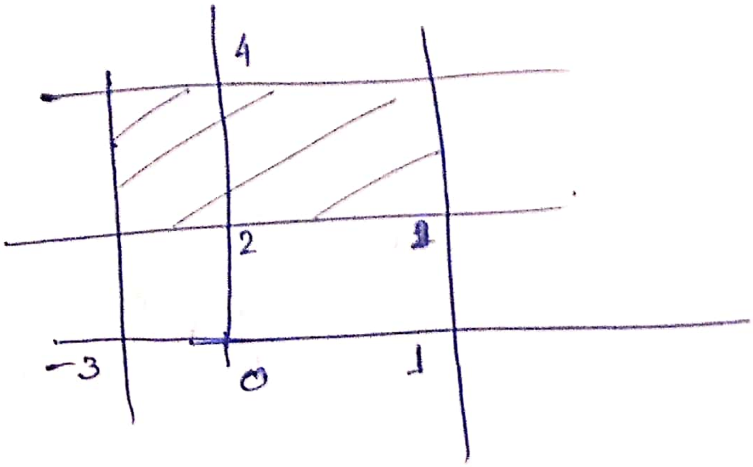
We have, $y = x+1 \Rightarrow y - x = 1 \Rightarrow u = 1$

$y = x - 3 \Rightarrow y - x = -3 \Rightarrow u = -3$

$y = -\frac{x}{3} + 2 \Rightarrow y + \frac{x}{3} = 2 \Rightarrow v = 2$

$y = -\frac{x}{3} + 4 \Rightarrow y + \frac{x}{3} = 4 \Rightarrow v = 4$

The pull back S of the Region R is rectangle show in fig 2



Calculate Jacobian –

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ \frac{1}{3} & 1 \end{vmatrix} = -\frac{4}{3}.$$

Then absoulate value of Jacobian is –

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{3}{4}$$

Hence differential is – $dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{3}{4} du\,dv$

$$\therefore \iint_R (y-x)\,dx\,dy$$

$$= \iint_S u \cdot \frac{3}{4}\,du\,dv = \frac{3}{4}\int_{-3}^{1} u\,du \int_{2}^{4} dv = \frac{3}{4}\left[\frac{u^2}{2}\right]_{-3}^{1}\left[v\right]_2^4$$

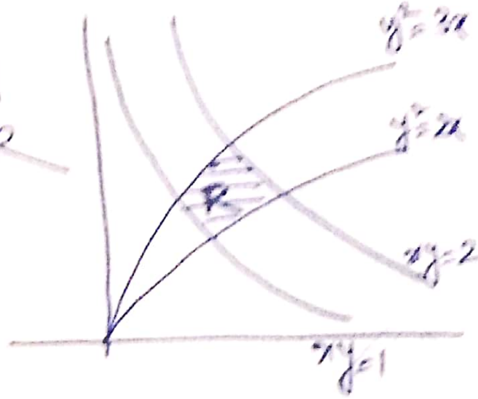
$$= \frac{3}{4}\left(\frac{1}{2}-\frac{9}{2}\right)(2)$$

$$= -6.$$

(b)

The Region R is sketched in fig—.

We apply the following substitution of variable to simplify the region R



$$\left.\begin{array}{l} u = \dfrac{y^2}{x} \\ v = xy \end{array}\right\}$$

The pullback S of region R given by

$y^2 = 2x \Rightarrow \dfrac{y^2}{x} = 2 \Rightarrow u = 2$

$y^2 = 3x \Rightarrow \dfrac{y^2}{x} = 3 \Rightarrow u = 3.$

$xy = 1 \Rightarrow v = 1$

$xy = 2 \Rightarrow v = 2$

To find the Jacobian transformation we express the variable x, y in term u, v.

$u = y^2/x, \Rightarrow x = \dfrac{y^2}{u}$, $\quad v = xy = \dfrac{y^2}{u}\cdot y$

$\therefore uv = y^3$

$y = (uv)^{1/3}$

$x = \dfrac{(uv)^{2/3}}{u} = u^{-\frac{1}{3}} v^{\frac{2}{3}}$

Now, $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

$= \begin{vmatrix} v^{2/3}\cdot\left(-\frac{1}{3}u^{-4/3}\right) & u^{-\frac{1}{3}}\frac{2}{3}v^{-\frac{1}{3}} \\ \frac{1}{3}u^{-2/3}\cdot v^{\frac{1}{3}} & \frac{1}{3}\cdot u^{+\frac{1}{3}} v^{-\frac{2}{3}} \end{vmatrix}$

$= -\frac{1}{9}u^{-1}v^{0} - \frac{2}{9}u^{-1} = -\frac{1}{3}u^{-1} = -\frac{1}{3u}$

$\therefore dx\,dy = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \dfrac{1}{3u}\,du\,dv.$

$\therefore I = \iint_R dx\,dy = \iint_S \dfrac{du\,dv}{3u} = \int_2^3 \dfrac{du}{3u}\int_1^2 dv = \frac{1}{3}\log\frac{3}{2}\cdot(2-1)$

$= \frac{1}{3}\log\frac{3}{2}$

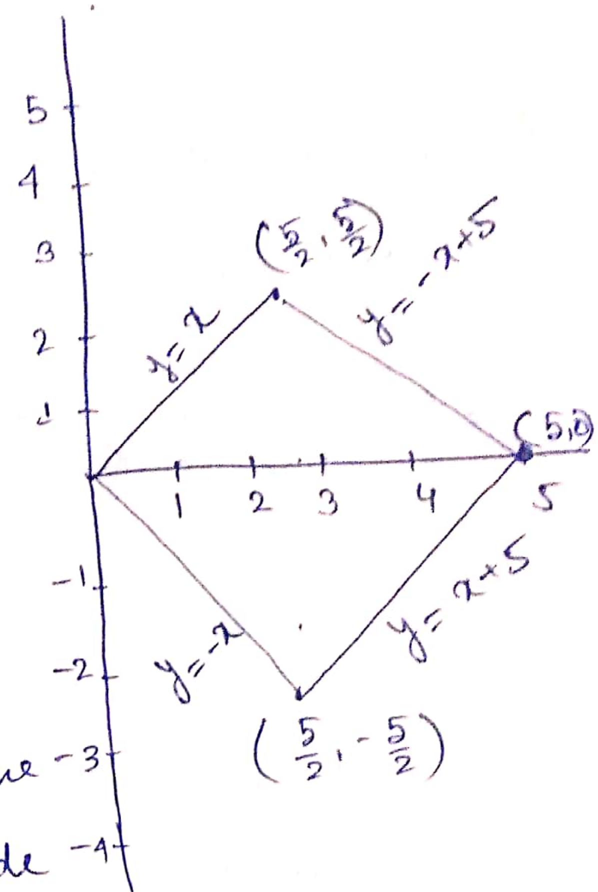
(c) Plugging in transformation gives —

$y = x \Rightarrow v = 0.$

$y = -x \Rightarrow u = 0$

$y = -x+5 \Rightarrow u = 5/4$

$y = x - 5 \Rightarrow v = 5/6$



Therefore the region S in u-v plane is then a rectangle whose side are given by $u = 0,\ v = 0,\ u = 5/4,\ v = 5/6.$

Jacobian —

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -12$$

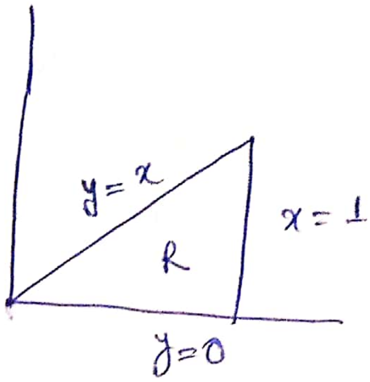
$$I = \iint_R (x+y)\,dA = \int_{v=0}^{5/6}\int_{u=0}^{5/4} \left[(2u+3v)+(2u-3v)\right] |-12|\,du\,dv$$

$$= \int_{v=0}^{5/6}\int_{u=0}^{5/4} 48u\,du\,dv$$

$$= 48 \cdot \left[\frac{u^2}{2}\right]_0^{5/4} \cdot \left[v\right]_0^{5/6}$$

$$= \cancel{48}^{\,3} \times \frac{25}{\cancel{32}_{2}} \times \frac{5}{\cancel{6}_{2}} = \frac{125}{24}$$

d) Here, R is the region bounded by $y=x$, $y=0$ and $x=1$



then, the given integral

$$I = \int_0^1 dx \int_0^x \sqrt{x^2+y^2}\, dx\, dy$$

$$= \iint_R \sqrt{x^2+y^2}\, dx\, dy \text{ where.}$$

Let, $x = r\cos\theta$, $y = r\sin\theta$.

$\therefore$ Jacobian. $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

and $I = \iint_{R'} r\cdot r\, dr\, d\theta$ where, R is mapped into R'

where, $r\sin\theta = 0$, $r\sin\theta = r\cos\theta$, $r\cos\theta = 1$

suggest that θ changes 0 to $\pi/4$, and $\therefore$ $r = \sec\theta$.

$$\therefore I = \int_{\theta=0}^{\pi/4}\left\{\int_{r=0}^{\sec\theta} r^2\, dr\right\} d\theta = \frac{1}{3}\int_{\theta=0}^{\pi/4} \sec^3\theta\, d\theta$$

$$= \frac{1}{3}\int_{\theta=0}^{\pi/4} \sqrt{1+\tan^2\theta}\; d(\tan\theta)$$

$$= \frac{1}{3}\int_{z=0}^{1} \sqrt{1+z^2}\, dz$$

$$= \frac{1}{3}\left[\frac{z}{2}\left(\sqrt{1+z^2}\right) + \frac{1}{2}\log\left[z+\sqrt{1+z^2}\right]\right]_0^1$$

$$= \frac{1}{3}\left[\frac{1}{\sqrt{2}} + \frac{1}{2}\log\left(1+\sqrt{2}\right)\right]$$

$$= \frac{1}{6}\left(\sqrt{2} + \log\left(1+\sqrt{2}\right)\right)$$

Scanned with CamScanner

(e)

Change to polar coordinates.

$x = r\cos\theta, \quad y = r\sin\theta.$

$\therefore$ Jacobian $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r.$

Then given circle $x^2 + y^2 - 2ax = 0$ becomes

$\therefore r^2 - 2ar\cos\theta = 0.$

$\Rightarrow r = 2a\cos\theta$

$\therefore I = \iint_{R'} r\sqrt{4a^2 - r^2}\, dr\, d\theta$, R'; upperhalf of $r = 2a\cos\theta$.

$= \int_{\theta=0}^{\pi/2} \left\{ \int_{r=0}^{2a\cos\theta} r\sqrt{4a^2 - r^2}\, dr \right\} d\theta.$

$= \frac{8}{3}a^3 \int_{\theta=0}^{\pi/2} (1 - \sin^3\theta)\, d\theta.$

$= \left[[\theta]_0^{\pi/2} + \int_{\theta=0}^{\pi/2} (1 - \cos^2\theta)\, d(\cos\theta) \right] \frac{8}{3}a^3$

$= \frac{8}{3}\frac{\pi}{2}a^3 + \frac{8}{3}a^3 \left[\cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi/2}$

$= \frac{8}{3}a^3\left[\pi/2 + \left(-1 + \frac{1}{3}\right)\right]$

$= a^3\frac{8}{3}\left(\frac{\pi}{2} + \frac{2}{3}\right) = \frac{4}{9}a^3 (3\pi - 4)$

[Ans]

$\int_{r=0}^{2a\cos\theta} r\sqrt{4a^2 - r^2}\, dr$

$4a^2 - r^2 = z^2$

$\int_{2a}^{2a\sin\theta} z \cdot (-2)\, dz$

$= -\frac{1}{3}(8a^3\sin^3\theta - 8a^3)$

$= \frac{8a^3}{3}(1 - \sin^3\theta)$

(f) Here, G in xyz space is bounded by the planes

$x = \frac{y}{2}$, $x = \frac{y}{2} + 1$; $y = 0$, $y = 4$; $z = 0$, $z = 4$;

We have to use

$u = \frac{2x - y}{2}$, $v = \frac{y}{2}$ and $w = z/3$. — (1)

We need to solve for x, y and z, ~~and~~

we get,

$$\left.\begin{aligned} u + v &= x \\ 2v &= y \\ 3w &= z \end{aligned}\right\} \quad \text{— (2)}$$

we can find corresponding surface for region G and the limits of integration in uvw space.

Equation in xyz for the region D.	Corresponding equation in uvw space	Limit for the integration in uvw space.
$x = y/2$	$u + v = v$	$u = 0$.
$x = y/2 + 1$	$u + v = v + 1$	$u = 1$
$y = 0$,	$2v = 0$	$v = 0$.
$y = 4$	$2v = 4$	$v = 2$
$z = 0$.	$3w = 0$	$w = 0$.
$z = 3$	$3w = 3$	$w = 1$.

Now, we calculate the Jacobian.

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6$$

Now, $I = \int_0^3 \int_0^4 \int_{y/2}^{y/2+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz$

$= \int_{w=0}^{1} \int_{v=0}^{2} \int_{u=0}^{1} \left(u + v + \frac{3w}{3}\right) |J|\, du\, dv\, dw$

$= 6 \int_{w=0}^{1} \int_{v=0}^{2} \int_{u=0}^{1} (u+v+w)\, du\, dv\, dw$

$= 6 \int_{w=0}^{1} \int_{v=0}^{2} \left[\frac{u^2}{2} + uv + uw\right]_{u=0}^{1} dv\, dw$

$= 6 \int_{w=0}^{1} \int_{v=0}^{2} \frac{1}{2} + v + w \quad dv\, dw$

$= 6 \int_{w=0}^{1} \left[\frac{v}{2} + \frac{v^2}{2} + vw\right]_{v=0}^{2} dw$

$= 6 \int_{w=0}^{1} 3 + 2w \quad dw$

$= 6 \left[3w + w^2\right]_{w=0}^{1}$

$= 6 \times 4 = 24$

(g) The ball is centered at the origin. Hence the region of integration U in spherical coordinates is described by

$$x = \rho \sin\theta \cos\phi \qquad y = \rho \sin\theta \sin\phi \qquad z = \rho \cos\theta$$

where, $0 \le \rho \le 1$, $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$

Here, $J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin\theta$

Writting the co-ordinates in spherical co-ordinates we have,

$$I = \iiint_U e^{(x^2+y^2+z^2)^{3/2}} dx\,dy\,dz = \iiint_{U'} e^{\rho^{2 \cdot 3/2}} \cdot \rho^2 \sin\theta \, d\rho\, d\theta\, d\phi$$

$$= \iiint_{U'} e^{\rho^3} \cdot \rho^2 \sin\theta \, d\rho\, d\theta\, d\phi$$

$$= \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} e^{\rho^3} \cdot \rho^2 \sin\theta \, d\rho\, d\phi\, d\theta$$

$$= \frac{1}{3} \int_{\rho=0}^{1} e^{\rho^3} \cdot d(\rho^3) \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta$$

$$= \frac{1}{3} \cdot \left[e^{\rho^3}\right]_0^1 \left[\phi\right]_0^{2\pi} \left[-\cos\theta\right]_0^{\pi}$$

$$= \frac{1}{3} \cdot (e-1) \cdot 2\pi \cdot (1+1)$$

$$= \frac{4\pi}{3}(e-1) \quad \text{(Ans)}$$

(3)

(a) The surface area of the graph $z = f(x, y)$ over a domain D in xy plane is

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\,dy$$

$\therefore$ ~~$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$~~

Now, $z = \sqrt{a^2 - x^2 - y^2}$

$\therefore \quad x^2 + y^2 + z^2 = a^2$

$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{x^2 + y^2 + z^2}{z^2} = \frac{a^2}{z^2}$$

$$= \frac{a^2}{a^2 - x^2 - y^2} \quad \text{--- (1)}$$

put $x = r\cos\theta$, $y = r\sin\theta$,

(1) becomes $= \dfrac{a^2}{a^2 - r^2}$,

and $x^2 + y^2 = ay$

$\Rightarrow r^2 = ra\sin\theta$.

$\Rightarrow r = a\sin\theta$.

$$\therefore S = \iint_{x^2+y^2 \le ay} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\,dy$$

$$= 2\int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sin\theta} \frac{a}{\sqrt{a^2 - r^2}}\, r\,dr\,d\theta.$$

$$= 2a \int_0^{\pi} \left[-\sqrt{a^2 - r^2}\right]_0^{a\sin\theta} d\theta$$

$$= 2a \int_0^{\pi} \left(-a|\cos\theta| + a\right) d\theta$$

$$= 2a \left(-2a + a\pi\right)$$

$$= 2a^2(\pi - 2)$$

$$\int_0^{\pi} |\cos\theta|\, d\theta.$$

$$= \int_0^{\pi/2} \cos\theta\, d\theta + \int_{\pi/2}^{\pi} -\cos\theta\, d\theta$$

$$= \left[\sin\theta\right]_0^{\frac{\pi}{2}} - \left[\sin\theta\right]_{\frac{\pi}{2}}^{\pi}$$

$$= 1 + 1$$

$$= 2$$

(b) Given. $D = \quad y = 3x^2 + 3z^2$ — (1)

$y = 6$ — (2)

Let, D be the circle/disk we get by equatting (1) and (2)

$6 = 3x^2 + 3z^2$

or, $x^2 + z^2 = 2$

So, D will be disk $x^2 + z^2 \leq 2$

In this case the surface is in the form $= 3x^2 + 3y^2$

$y = g(x, z)$

We will use the formula for surface integral

$\iint_S f(x, y, z)\, ds = \iint_D f(x, g(x,z), z) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}\, dA$

$\iint_S 40y\, ds \quad = \iint_D 40(3x^2 + 3z^2) \sqrt{1 + (6x)^2 + (6z)^2}\, dA$

$= 120 \iint_D (x^2 + z^2) \sqrt{36(x^2+z^2)+1}\, dA.$

$\therefore x^2 + z^2 = r^2$

Let, $x = r\cos\theta$, $z = r\sin\theta$,

because D is the disk $x^2 + z^2 \leq 2$,

$\therefore$ then, $0 \leq \theta \leq 2\pi$,

$0 \leq r \leq \sqrt{2}$

$\therefore \iint_S 40y\, ds = 120 \int_0^{2\pi} \int_0^{\sqrt{2}} r^2 \sqrt{36r^2+1}\; r\, dr\, d\theta$

$= 120 \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{36r^2+1}\; r^3\, dr\, d\theta.$

Let, $36r^2+1 = u$ $\qquad r^2 = \frac{1}{36}(u-1)$

$\Rightarrow 72r\,dr = du$

~~$\Rightarrow 36r\,dr = du$~~

$$\iint_S 40y\,ds = 120\int_0^{2\pi}\int_1^{73}\left(\frac{1}{72}\right)\left(\frac{1}{36}\right)(u-1)\,u^{\frac{1}{2}}\,du\,d\theta$$

$$= 120\times\frac{1}{72\times 36}\int_0^{2\pi}\int_1^{73} u^{3/2}-u^{1/2}\,du\,d\theta$$

$$= \frac{5}{108}\int_0^{2\pi}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)_1^{73}d\theta$$

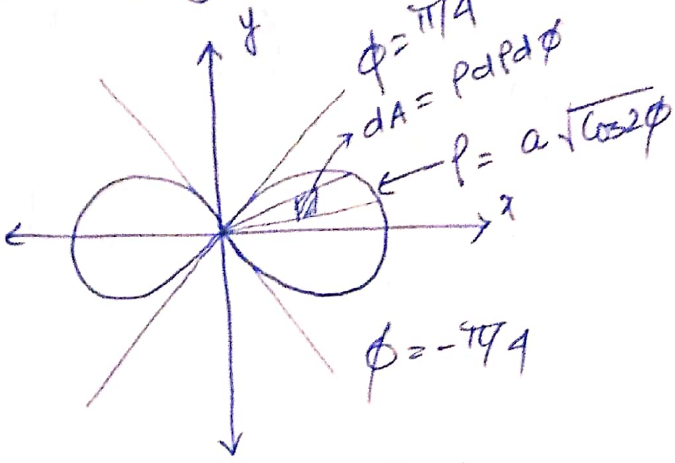
$$= \frac{5}{108}\left[\left(\frac{2}{5}(73)^{5/2}-\frac{2}{3}(73)^{3/2}\right)-\left(-\frac{4}{15}\right)\right](2\pi)$$

$$= \frac{5\pi}{64}\left[\frac{2}{5}(73)^{5/2}-\frac{2}{3}(73)^{3/2}+\frac{4}{15}\right]$$

$$= 5176.8958$$

(c) $\rho^2 = a^2 \cos 2\phi$

Here the curve is given directly in polar form (ρ, ϕ)



The required Area — (Grander)

$$4 \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{a\sqrt{\cos 2\phi}} \rho \, d\rho \, d\phi$$

$$= 4 \int_{\phi=0}^{\pi/4} \left[\frac{\rho^2}{2}\right]_0^{a\sqrt{\cos 2\phi}} d\phi$$

$$= \frac{4}{2} \int_{\phi=0}^{\pi/4} a^2 \cos 2\phi \, d\phi$$

$$= 2a^2 \left[\frac{\sin 2\phi}{2}\right]_0^{\pi/4}$$

$$= \underline{a^2}$$

#
(d)

given inequalities

$0 \le 2x - 3y + z \le 5$

$1 \le x + 2y \le 4$

$-3 \le x - z \le 6$

Let, $u = 2x - 3y + z$ $\quad v = x + 2y$ $\quad w = x - z$

Calculate Jacobian —

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= -9$$

$$\therefore |J(u,v,w)| = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \frac{1}{9}$$

Volume of the solid is

$$V = \iiint |J(u,v,w)| \, du \, dv \, dw$$

$$= \int_{u=0}^{5} \int_{v=1}^{4} \int_{w=-3}^{6} \frac{1}{9} \, du \, dv \, dw$$

$$= \frac{1}{9} \times 5 \times 3 \times 9 = 15$$

(e) The surface S lies in the plane $2x+3y+6z=60$

$\therefore\ 2+6\frac{\partial z}{\partial x}=0$

$\frac{\partial z}{\partial x}=-\frac{1}{3}$

and $\frac{\partial z}{\partial y}=-\frac{1}{2}$

The area of S is found by calculating the suffae integral over S for $f(x,y,z)=1$

The projection of surface S onto xy plane is given by $D=\{(x,y)\mid x^2-2x+y^2=(x-1)^2+y^2\leq 1\}$

Hence the Area of S is given by

$$\iint_S 1\,dS=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

$$=\iint_D\sqrt{1+\frac{1}{9}+\frac{1}{9}}\,dx\,dy$$

$$=\frac{7}{6}\iint_D dx\,dy$$

$$=\frac{7}{6}\times\text{Area of }D$$

$$=\frac{7}{6}\times\pi=\frac{7\pi}{6}.$$

$$=\frac{7}{6}\times$$

(f) Here, $x^2+y^2+z^2=1$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\therefore ds = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{\frac{x^2+z^2+y^2}{z^2}}$$

$$= \frac{1}{z}$$

~~Then the integral becomes.~~

Let, D be the projection of surface S onto x-y plane i.e $D = \{(x,y) \mid x^2+y^2 \le 1\}$

$$\iint_S z^2 ds = \iint_D z^2 \cdot \frac{1}{z}\, dx\, dy$$

$$= \iint_D \sqrt{1-x^2-y^2}\, dx\, dy$$

Let, $x = r\cos\theta$, $y = r\sin\theta$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \sqrt{1-r^2}\; r\, dr\, d\theta$$

$$= -\int_0^{2\pi} d\theta \int_1^0 \frac{1}{2}\sqrt{u}\, du$$

$$= 2\pi \times \frac{1}{2} \times \frac{2}{3} = \frac{2\pi}{3}.$$

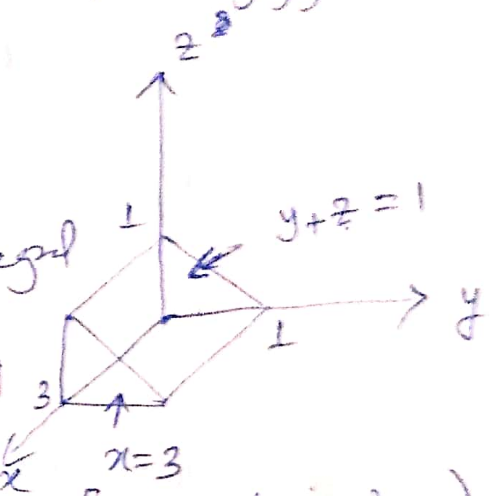
(8)

The volume is given by the tripple integral = $\iiint dv$

putting z on the outer integrals,

y on the intermediate and

x on to the ~~integ~~ inner integral



Here, the limits of z are $z=0$ to $z=1$

for each, value of z, y varies $y=0$ to $y=1-z$ (on the sloping face)

For each combination of y and z, x varies from. $x=0$ to $x=3$

Thus the value of the figure is —

$$\iiint dv$$

$$= \int_{z=0}^{1} \int_{y=0}^{1-z} \int_{x=0}^{3} dx\,dy\,dz$$

$$= \int_{z=0}^{1} \int_{y=0}^{1-z} [x]_0^3\,dy\,dz = \int_{z=0}^{1} \int_{y=0}^{1-z} 3\,dy\,dz$$

$$= 3\int_{z=0}^{1} [y]_0^{1-z}\,dz$$

$$= 3\int_{z=0}^{1} 1-z\,dz$$

$$= 3\left[z-\frac{z^2}{2}\right]_0^1$$

$$= 3\left[1-\frac{1}{2}\right] = \frac{3}{2}$$

(h)

Let, $x = r\cos\theta$,

$y = r\sin\theta$,

$z = z$.

~~Then Now~~ The given region D is defined by

$y=0$, $z=0$, $y=x$ and $z = 4-x^2-y^2$

$\therefore$ Therefore r is varies from ~~a~~ 0 to 2.

θ is varies from 0 to $\pi/4$

and z varies between 0 to $4-r^2$

$\therefore$ ~~Be The transformation $(r,\theta,z) \to (r\cos\theta, r\sin\theta, z)$~~

~~is bijection between~~

Consider $B = \{(r,\theta,z) : 0 \le r \le 2,\ 0 \le \theta \le \pi/4,\ 0 \le z \le 4-r^2\}$

$\therefore$ Now,
$$\iiint_D e^{x^2+y^2}\,dx\,dy\,dz$$

$$= \iiint_B e^{r^2}.\ r\,dz\,d\theta\,dr \quad [\because J(r,\theta,z) = r]$$

$$= \int_{r=0}^{2}\int_{\theta=0}^{\pi/4}\int_{z=0}^{4-r^2} e^{r^2}.r\,dz\,d\theta\,dr$$

$$= \int_{r=0}^{2}\int_{\theta=0}^{\pi/4} e^{r^2}.r(4-r^2)\,dr\,d\theta$$

$$= \frac{\pi}{4}\int_{r=0}^{2} .r(4-r^2)\,e^{r^2}\,dr.$$

Let, $r^2 = u$ $\quad \frac{du}{2} = r\,dr$

$$\therefore \frac{\pi}{4} \cdot \frac{1}{2} \int_0^4 e^u (4-u)\, du$$

$$= \frac{\pi}{8} \left[4 \int_0^4 e^u\, du - \int_0^4 u e^u\, du \right]$$

$$= \frac{\pi}{8} \left[4 (e^4 - 1) - (u e^u - e^u)_0^4 \right]$$

$$= \frac{\pi}{8} \left[4 (e^4 - 1) - (4e^4 - e^4 + 1) \right]$$

$$= \frac{\pi}{8} \left[4e^4 - 4 - 4e^4 + e^4 - 1 \right]$$

$$= \frac{\pi}{8} \left[e^4 - 5 \right]$$