Computing Quasi-normal Modes of Black Holes using Pseudo-Spectral Method

Numerical Methods: Term Paper

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Abstract

In this report, quasi-normal modes of Schwarzschild black holes are computed for spin 0, 1 and 2 fields numerically by pseudo-spectral method using Chebyshev basis. The obtained quasi-normal frequencies are compared with the results from WKB approximation available in the literature.

1 Introduction

When a physical system is subjected to small perturbations, it vibrates with a set of natural frequencies, which depends only on the intrinsic properties of the system. If the medium is dissipative, these oscillations will eventually die out and the system relaxes to its original state. These modes of oscillations are called quasi-normal modes. The resulting frequencies are complex, with the imaginary part representing the exponential decay of amplitude with time. These characteristic frequencies do not depend on the initial condition but only on the physical parameters of the system under consideration. A real black hole is not an isolated system. It is surrounded by matter distribution such as accretion disks, stars, planets, magnetic fields, etc. Black holes formed during supernovae or remnant black hole from a merger event, are all in a perturbed state. When a black hole is perturbed, it emits gravitational waves, which includes quasi-normal modes(QNMs). Identifying the quasi-normal frequencies from the gravitational waves helps in estimating the parameters of the black hole under study. Stability of the black hole to external perturbations can also be analysed.

2 Perturbation Equations

The metric of asymptotically flat, spherically symmetric black hole background is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1}$$

where M is the mass of the Schwarzschild black hole and $r \in [0, \infty)$ is the radial coordinate. $r = 2M$ represents event horizon and there is a physical singularity at $r = 0$. To compute quasi-normal modes, the region of interest is $r \in (2M, \infty)$.

A black hole background can be perturbed by introducing probe fields like scalar, electromagnetic fields or by adding perturbation to the metric itself. In the linear approximation, the problem can described as fields propagating in the unperturbed black hole spacetime and we just need to solve their respective equations of motion. For example, Klein-Gordon equation for scalar fields, Maxwell's equation for electromagnetic fields and Einstein's equation for gravitational perturbations. The resulting partial differential equations are solved by decomposing the solution into Fourier modes($e^{-i\omega t}$) to remove the time derivatives. In case of Schwarzschild black hole, the metric is spherically symmetric and angular part can be expanded in terms of spherical harmonics. This will result in a second-order Schrodinger like differential equation given by

$$\frac{d^2\Phi_s(r)}{dr_*^2} + \left[\omega^2 - V(r)\right] \Phi_s(r) = 0. \tag{2}$$

where

$$V(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right]. \tag{3}$$

where $s = 0, 1, 2$ for massless scalar, electromagnetic, and gravitational(axial) perturbations respectively. The tortoise coordinate is defined as $\frac{dr}{dr_*} = \left(1 - \frac{2M}{r}\right)$.

The boundary conditions can be imposed as follows:

- Nothing comes out of the black hole interior. Thus, there are only ingoing waves near the horizon. $\Phi_s(r,t) \sim e^{-i\omega(t+r_*)}$ as $r_* \to -\infty$.
- Nothing can enter from outside the spacetime, which would mean there are only outgoing waves at spatial infinity. $\Phi_s(r,t) \sim e^{-i\omega(t-r_*)}$ as $r_* \to \infty$.

In order to apply pseudo-spectral method, the perturbation equation should be expressed in Eddington-Finkelstein coordinates [3][6]. In terms of these coordinates, equation (2) can be written as

$$(1-u)u^3\Psi_s^{"} - u(u^2 - 2i\omega)\Psi_s^{'} + \left(s^2u^2 - l(l+1)u - 2i\omega\right)\Psi_s(u) = 0 \tag{4}$$

where $u = 1/r$ and $r_s = 2M = 1$. Boundary conditions can be incorporated in the above equation by calculating the asymptotic solutions near the horizon and spatial infinity. Substitute the ansatz $\Psi_s(u) = (1-u)^\alpha$ [6] for solutions near horizon in equation (4). Two solutions, $\alpha = 0$ (interpreted as in-going waves) and $\alpha = 2i\omega$ (outgoing) is obtained and the second solution has to be neglected. For spatial infinity, use the ansatz $\Psi_s = u^\beta$ which would lead to the solution $\Psi_s(u) = ae^{2i\omega/u}u^{-2i\omega} + bu$. For divergent solution, b is set to 0. Applying the transformation,

$$\Psi_s(u) = e^{2i\omega/u}u^{-2i\omega}\psi_s(u) \tag{5}$$

where ψ_s is assumed to be a analytic function in the interval $u \in [0,1]$ ensures the boundary conditions are satisfied. Make a final transformation $u = (x+1)/2$ to change the domain of interest to $x \in [-1,1]$. The final differential equation is:

$$\begin{aligned}
&-2\,(1-x)\left(\frac{x+1}{2}\right)^3\phi_s^{"}(x) + 2\left[\left(\frac{x+1}{2}\right)^3 - i\,(x+1)\left(x^2+2x-1\right)\lambda\right]\phi_s^{'}(x) \\
&+\left[l(l+1)\frac{x+1}{2} - s^2\frac{(1+x)^2}{4} - 4i\lambda - 4(1+x)(3+x)\lambda^2\right]\phi_s(x) = 0
\end{aligned} \tag{6}$$

where $\lambda = M\omega$ and the problem reduces to a quadratic eigenvalue problem in the domain $x \in [-1,1]$ and the boundary conditions are implicitly incorporated in the differential equation. This equation can now be solved using pseudo-spectral method.

3 Pseudo-Spectral Method

The differential equation can be written in the form

$$a_2(x,\lambda,\lambda^2)\phi_s^{"}(x) + a_1(x,\lambda,\lambda^2)\phi_s^{'}(x) + a_0(x,\lambda,\lambda^2)\phi_s(x) = 0 \tag{7}$$

where $a_i(x,\lambda,\lambda^2) = a_{i,0}(x) + \lambda a_{i,1}(x) + \lambda^2 a_{i,2}(x)$. Expand the solution using Chebyshev basis

$$\phi_s(x) = \sum_{k=0}^{N-1} c_k T_k(x) \tag{8}$$

Calculate the differential equation on the collocation points $x_i = \cos\left(\frac{\pi i}{N-1}\right)$ where $i = 0, \cdots, N-1$. The equation can be written in matrix form as,

$$(M_0 + \lambda M_1 + \lambda^2 M_2)\phi = 0 \tag{9}$$

The matrices M_0, M_1, M_2 are given by,

$$\begin{aligned}(M_0)_{ij} &= a_{0,0}(x_j)\delta_{ij} + a_{1,0}(x_j)D_{ij} + a_{2,0}(x_j)D^2_{ij}\\(M_1)_{ij} &= a_{0,1}(x_j)\delta_{ij} + a_{1,1}(x_j)D_{ij} + a_{2,1}(x_j)D^2_{ij}\\(M_2)_{ij} &= a_{0,2}(x_j)\delta_{ij} + a_{1,2}(x_j)D_{ij} + a_{2,2}(x_j)D^2_{ij}\end{aligned} \tag{10}$$

where D and D^2 are derivative matrices obtained using Chebyshev basis. The quadratic eigenvalue equation (9) can be converted to a linear eigenvalue problem by redefining the matrices.

$$(\tilde{M_0} + \lambda\tilde{M_1}).\vec{\phi} = 0 \tag{11}$$

where $\tilde{M_0} = \begin{pmatrix} M_0 & M_1 \\ \mathbb{0} & \mathbb{1} \end{pmatrix}$, $\tilde{M_1} = \begin{pmatrix} \mathbb{0} & M_2 \\ -\mathbb{1} & \mathbb{0} \end{pmatrix}$, and $\vec{\phi} = \begin{pmatrix} \phi \\ \lambda\phi \end{pmatrix}$. The generalized linear eigenvalue equation $\tilde{M_0}\vec{\phi} = -\lambda\tilde{M_1}\vec{\phi}$ can be solved using the scipy inbuilt function `scipy.linalg.eig` . In addition to the QNM frequencies, this method also generates several false eigenvalues. The actual eigenvalues have to be picked by using the fact that the QNM frequencies should not depend on the number of collocation points used. We can compute the eigenvalues for different values of N and compare them to find the common eigenvalues.

4 Results

The values of quasinormal frequencies obtained for $s = 0, 1, 2$ are given in table 1. The results from pseudo spectral method using 40 polynomials is given in first column. The second column contains quasinormal frequencies calculated using 6th order WKB approximation found in the literature.

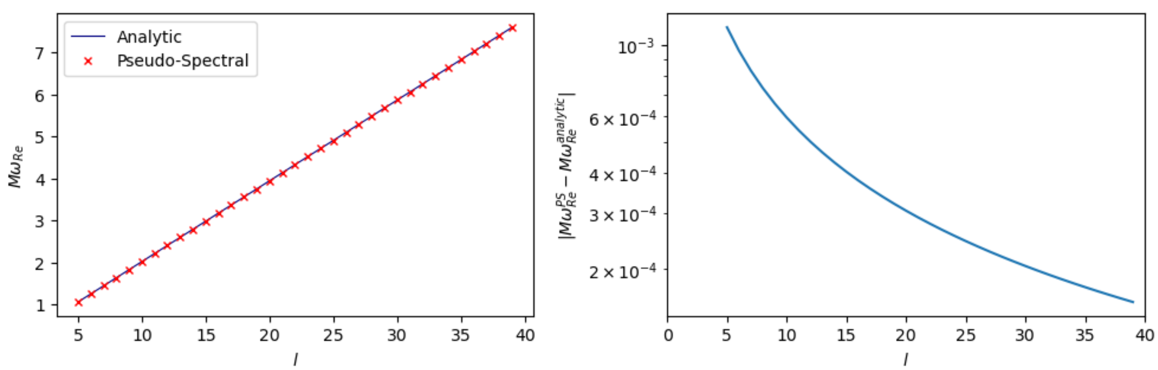
$s = 0$	Pseudo-Spectral($N = 40$)	6th order WKB(Ref. [4])
$l = 0, n = 0$	±0.110455-0.104895i	±0.1105 - 0.1008i
$l = 1, n = 0$	±0.292936-0.097660i	±0.2929 - 0.0978i
$l = 1, n = 1$	±0.264449-0.306257i	±0.2645 - 0.3065i
$l = 2, n = 0$	±0.483644-0.096759i	±0.4836 - 0.0968i
$l = 2, n = 1$	±0.463851-0.295604i	±0.4638 - 0.2956i
$l = 2, n = 2$	±0.430544-0.508558i	±0.4304 - 0.5087i
$l = 3, n = 0$	±0.675366-0.096500i	-
$l = 3, n = 1$	±0.660671-0.292285i	-
$l = 3, n = 2$	±0.633626-0.496008i	-
$l = 3, n = 3$	±0.598773-0.711222i	-
$s = 1$	Pseudo-Spectral($N = 40$)	6th order WKB(Ref. [4])
$l = 1, n = 0$	±0.248263-0.092488i	± 0.2482 - 0.0926i
$l = 1, n = 1$	±0.214515-0.293668i	± 0.2143 - 0.2941i
$l = 2, n = 0$	±0.457596-0.095004i	± 0.4576 - 0.0950i
$l = 2, n = 1$	±0.436542-0.290710i	± 0.4365 - 0.2907i
$l = 2, n = 2$	±0.401187-0.501587i	± 0.4009 - 0.5017i
$l = 3, n = 0$	±0.656899-0.095616i	± 0.6569 - 0.0956i
$l = 3, n = 1$	±0.641737-0.289728i	± 0.6417 - 0.2897i
$l = 3, n = 2$	±0.613832-0.492066i	± 0.6138 - 0.4921i
$l = 3, n = 3$	±0.577918-0.706331i	± 0.5775 - 0.7065i
$s = 2$	Pseudo-Spectral($N = 40$)	6th order WKB(Ref. [4])
$l = 2, n = 0$	±0.373672-0.088962i	±0.3736 - 0.0890i
$l = 2, n = 1$	±0.346711-0.273915i	±0.3463 - 0.2735i
$l = 2, n = 2$	±0.301054-0.478276i	±0.2985 - 0.4776i
$l = 3, n = 0$	±0.599443-0.092703i	±0.5994 - 0.0927i
$l = 3, n = 1$	±0.582644-0.281298i	±0.5826 - 0.2813i
$l = 3, n = 2$	±0.551685-0.479093i	±0.5516 - 0.4790i
$l = 3, n = 3$	±0.511963-0.690336i	±0.5111 - 0.6905i

Table 1: Quasi-normal frequencies $M\omega$ for $s = 0, 1, 2$ field perturbations compared with Ref. [4]

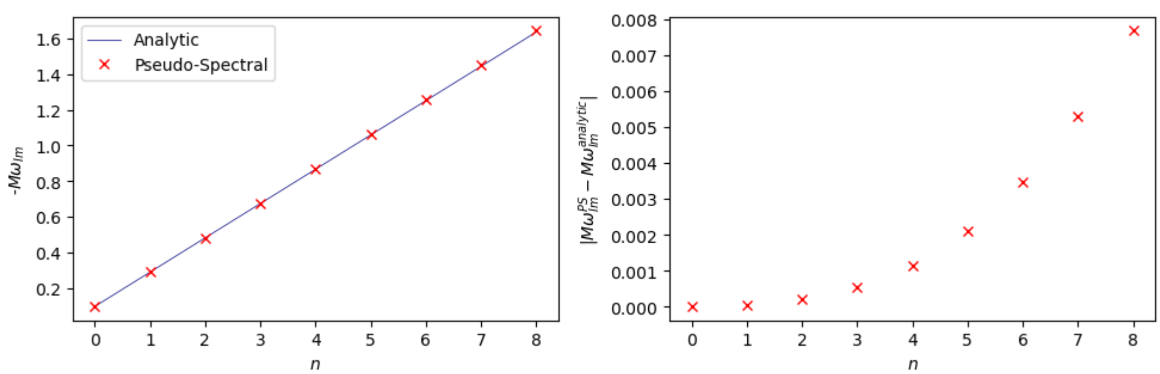
- QNMs for $l >> 1$: Analytical solutions were obtained in Ref. [2]

$$
\begin{aligned}
M\omega_{Re} &= \frac{1}{3\sqrt{3}}\left(l + \frac{1}{2}\right) \\
M\omega_{Im} &= -\frac{1}{3\sqrt{3}}\left(n + \frac{1}{2}\right)
\end{aligned}
\tag{12}
$$

Plots comparing the numerical results from pseudo-spectral method and analytical results is given in figure (1)

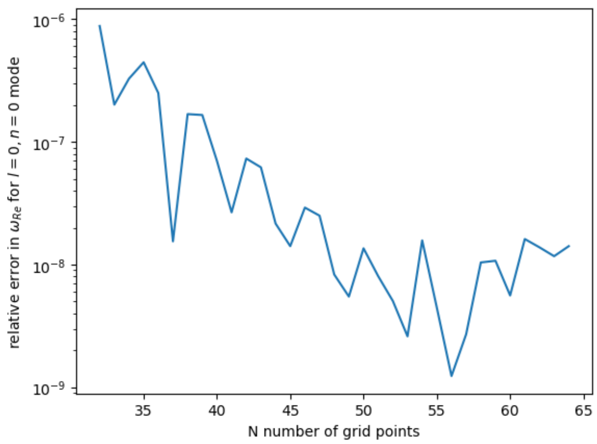


(a)

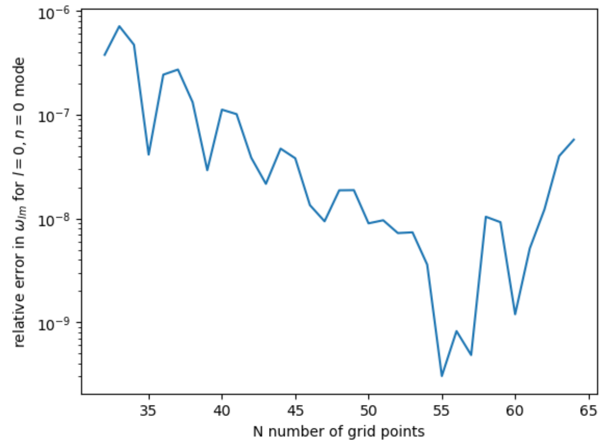


(b)

Figure 1: (a) Real parts of frequencies as a function of l for $s = 0$, $n = 0$(left) and the difference between analytical and pseudo-spectral methods (right). (b) Imaginary parts of frequencies as a function of n (left) and the difference between analytical and pseudo-spectral methods (right), for $s = 0$, $l = 30$.



(a) Real part



(b) Imaginary part

Figure 2: Relative error in ω for $l = 0, s = 0$ as a function of grid points N

Figure (2) shows the dependence of relative error in real and imaginary parts of the quasi-normal frequency for $l = 0, n = 0$ for $s = 0$ with the number of grid points N. As it can be seen, the error overall decreases till $N = 55$, then starts increasing. It is not quite clear why this is could be occurring, but it is suspected to be due to the use of `scipy.linalg.eigvals` function. The discrepancy could be caused by the loss of numerical accuracy in the eigenvalue calculation due to the limitation of hardware double-precision (16- digits). Higher working precision might be required to get more accurate results.

5 Conclusions

Quasi-normal frequencies for different l and s values were calculated for Schwarzschild black hole using Pseudo-Spectral method with Chebyshev polynomials as cardinal functions. The imaginary part of the QNM frequencies were found to be negative for massless scalar, electromagnetic and axial gravitational perturbations indicating damping and we can conclude that Schwarzschild black holes are stable towards such perturbations.

Calculating QN frequencies using pseudo-spectral method does not require any initial guess. This may be considered as an advantage compared to methods like Leaver's continued fraction method [5] available in literature. The error associated with this method is $O(\frac{1}{N^N})$ and if the eigenfunctions are analytic, it can be approximated with exponential convergence. The only assumption made in this method is the analyticity of the eigenfunctions. And the only approximation is the grid size. One of the problems with this method is the occurrence of extra eigenvalues.

References

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