Target search with stochastic resetting

Optimizing the average time of search for RTP using resetting as a Poisson process

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1 Introduction

We find search problems everywhere in our lives , some notable examples include proteins searching for a site to bind and bacteria searching for food. We might not notice it, but even in our lives we face plenty of search problems, for example did you ever thought how do we look for a friend among a crowd of people?, its kind of a stochastic process, you see how this happens is that your eye will randomly point to some spot in crowd and you will start searching for a familiar face in the neighborhood of that spot, after looking for a few seconds your eyes will focus on a different spot and the same process continues.

Even in more academic setting you will encounter these kind of problems for example you might have to numerically find the ground state of a system, in these kind of problems what we do is we change the state of a system and see how our energy changes, if it decreases the energy we accept the new state and that's how system like Ising models evolve computationally in Monte Carlo simulation, although this is not the case for ising model but in some cases you might find the state to be in local minima, than these kind of method to find the ground state might not work. That's where resetting helps us.

These were just some examples where resetting might be useful, but in this report we will focus on target search for a special type of motion called Random Tumble Particle(RTP) motion and will see how resetting helps in optimising search time

2 Example of Search time without resetting

We look at a simple example of 1D Brownian motion, by solving Fokker Plank equation it can be analytically calculated that the distribution of first passage time or the time it takes for the particle to find the target withour resetting is give by:

$$P(t_f|x_0) = \frac{x_0}{\sqrt{2\pi}} \frac{exp(-x_0^2/2t_f)}{t_f^{3/2}} \quad [1]$$

Hence, the average first passage time:

$$\langle t_f \rangle = \int_0^\infty t_f \frac{x_0}{\sqrt{2\pi}} \frac{exp(-x_0^2/2t_f)}{t_f^{3/2}} dt_f \to \infty$$

This is not just the case of Brownian motion, as we will see in simulations, RTPs also show this kind of divergence of average search time, and we will see that how stochastic resetting curb this divergence

3 RTP motion

It consists of alternating sequence of "Run" and "Tumble", during "Run" phase particle will move in a direction with constant speed and during "Tumble" phase it will reorient itself in a random direction and than again the "Run" phase starts and it will start moving with constant speed in that new direction
The "Tumble" phase will we Poisson process, that means, that particle will tumble in a new direction with Poisson parameter γ
In 1d case tumbling will means that particle will just move in opposite direction after tumbling.
Just like any other stochastic motion RTP motion can be decribed using a Lnagevin equation

4 Relevant Langevin equations

We will focus on Over damped Langevin equations, this means that any kind of acceleration of particle dies of very quickly as compared to the time scale of motion.
This happens to particles moving in a very high viscous medium, that means in the equation of motion we can just ignore the acceleration term, and hence for 1d RTP, the over damped Langevin equation is:

$$\frac{dx}{dt} = \sigma(t)v \tag{1}$$

Here, σ gives the direction of speed and will randomly change between $+1$ and -1, with Poisson rate γ, simulating this equation is straight forward, to simulate the tumbling, we know that Poisson process can be simulated by calculating time of arrivals using exponential distribution given by:

$$P(t) = \gamma exp(-\gamma t)$$

We will take out a time from this distribution and will add it to time of particle and that will be the time at which next tumbling will take place, rest of simulating is just discretizing the time into smaller time steps and calculating new position for the particle.
Second, Langevin equation we will simulate is RTP with diffusion term and is given by:

$$\frac{dx}{dt} = \sigma(t)v + \sqrt{2D}\eta(t) \tag{2}$$

Here, D is the diffusion constant and $\eta(t)$ is delta correlated Gaussian white noise, that means that:

$$\langle \eta(t) \rangle = 0 \quad \langle \eta(t)\eta(t') \rangle = \delta_{tt'}$$

Integrating the above langevin equation for small time step Δt, if we define integral of the $\eta(t)$ as $\xi_{\Delta t}(t)$, we will see that $\xi_{\Delta_t}(t)$, has zero mean and Δt variance and due to CLT it will have Gaussian distribution, defining $\xi(t)$ as Gaussian noise with zero mean and 1 variance, the above Langevin equation for small time step Δt becomes:

$$\Delta x = \sigma(t)v\Delta t + \sqrt{2D\Delta t}\xi(t) \tag{3}$$

This is the equation we will simulate for this case

Up next is RTPs confined in Harmonic Potential $V = \frac{\alpha}{2}x^2$, that means force is $f(x) = -\alpha x$, the Langevin equation is this given by:

$$\frac{dx}{dt} = -\alpha x + \sigma(t)v \tag{4}$$

The simulation of this equation will take place the same way it did for free RTP case.

Now, we move onto some 2D cases that we will simulate, first up is 2D free RTP, whose over damped Langevin equation is:

$$\frac{dx}{dt} = vcos\left(\theta(t)\right) \quad \frac{dy}{dt} = vsin\left(\theta(t)\right) \tag{5}$$

Here, $\theta(t)$ takes a new value from $[0, 2\pi]$ randomly with Poisson rate γ, also in this case particle is confined in the 8×8 box.
The Poisson process in this case will be simulated the same way as done in first case by calculating arrival times from exponential distribution

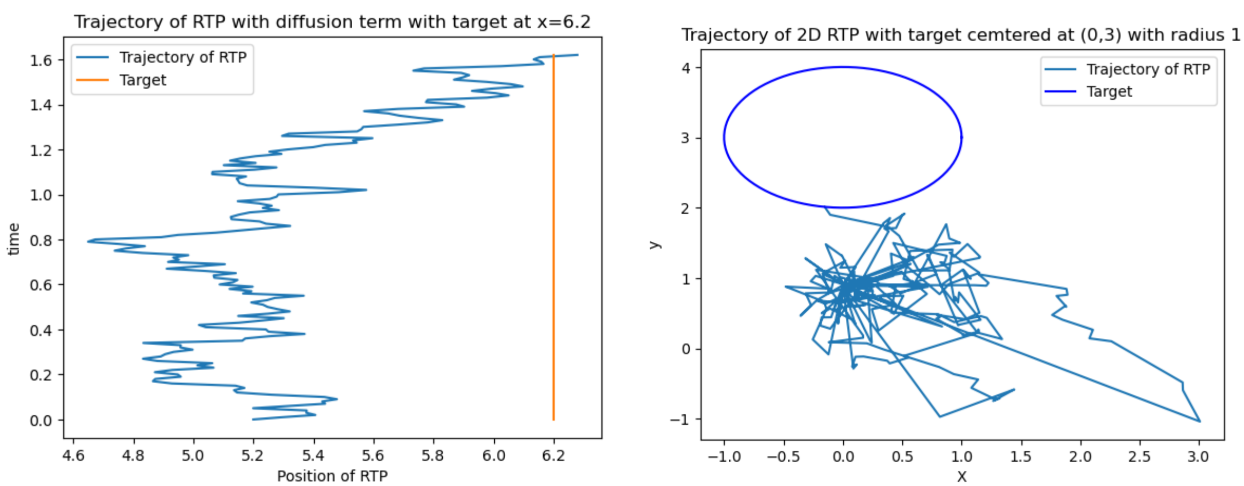
Now we have our last case in which $2D$ RTP is confined in $V = \frac{\alpha}{2}(x^2 + y^2)$, hence the langevin equation is:

$$\frac{dx}{dt} = vcos\,(\theta(t)) - \alpha x \quad \frac{dy}{dt} = vsin\,(\theta(t)) - \alpha y \tag{6}$$

In all of the cases discussed above the resetting has been done as a Poisson process with Poisson rate r, this has been simulated the same way we simulated tumbling by calculating tumbling times, for resetting at each resetting time calculated form the exponential distribution, both position and direction of the RTP has been resetted to the initial position and direction of the particle

5 Simulations and results

First we will see how trajectory of these RTPs particle looks like with stochastic resetting:

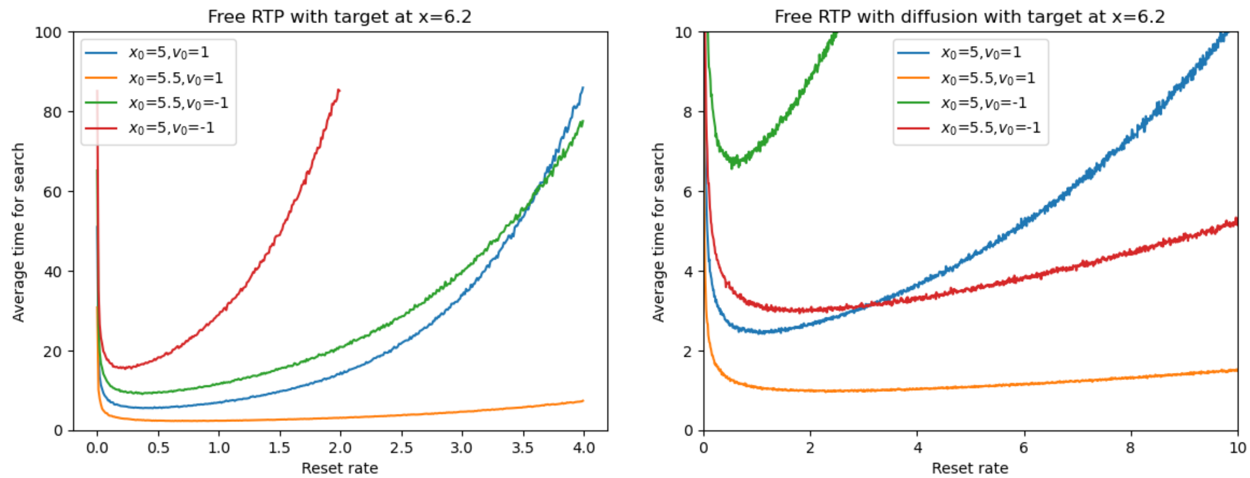


(a) RTP with diffusion for $D = 0.5$, reset rate=1, $\gamma = 1$

(b) 2D RTP with reset rate=0.5, $\gamma = 5$

Figure 1: Trajectory of free RTP with diffusion (a) described by equation (2,3) with initial starting position $x_0 = 5.2$ and initial velocity $\vec{v_0} = 1$ with target at $x = 6.2$, (b) 2D RTP described by equation 5, starting at initial position $(0, 0.9)$ with initial direction given by $\theta = 0$ with circle representing target region

Now we move onto the results of average search time for each of the case discussed above:



(a) RTP with $\gamma = 1$

(b) RTP plus diffusion with $\gamma = 1,\ D = 0.5$

Figure 2: Average Search time for case of free RTP (a) described by equation 1 and RTP with diffusion (b) described by equation (2,3), here x_0, v_0 gives the initial position and direction of particle with target at $x = 6.2$

As mentioned in the section 2 we can see that zero resetting rate leads to diverging average search time even for RTPs

We can also see that if particle is initial closer to the target than the minima of average time of search falls farther away on the x-axis, and if particle is initially away from target than, the the Resetting rate for which minima occurs is much lower.

This can be easily explained because if the particle is initially farther away from the target than particle will need some time to reach the target even in the ideal case, hence higher resetting rate will not give particle enough time to reach the target before being resetted to original position.

Hence we expect that as the distance between the target and initial position of particle increases the position of minimum search time should tend to zero.

Also for starting velocity opposite to the direction of target, resetting rate for minimum search time is again closer to zero as compared to same scenario with starting velocity towards the target, this can easily explained by the fact that with initial velocity opposite to the direction of particle, particle will move away from target until its direction changes, hence the same effect of being farther away from target will move the minima towards zero

We also observe that with diffusion term added to Langevin equation, particle can have much larger resetting rate and much lower search time.

We will see same type of observations in other simulations too, and the explanation for these observation will be same as this one in those cases too.

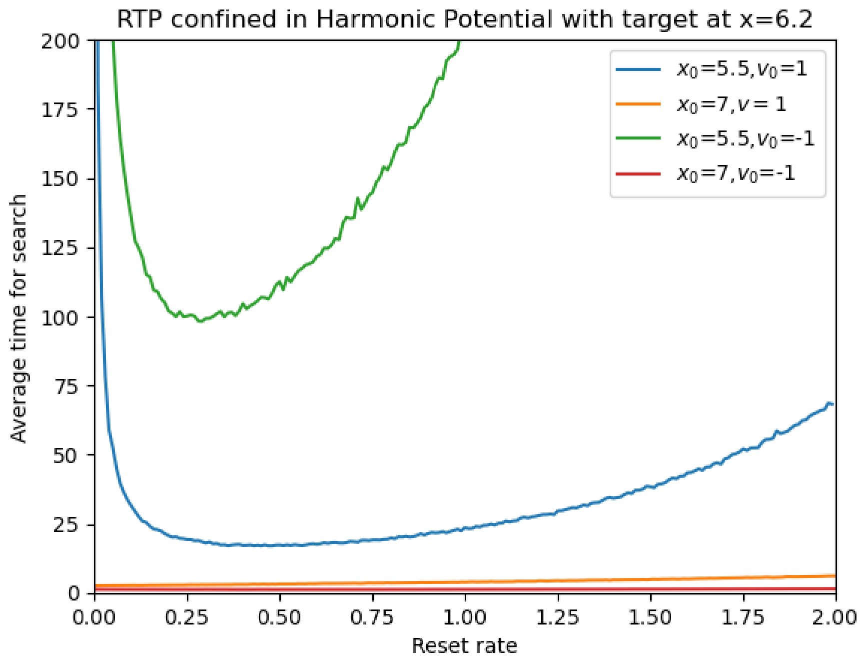
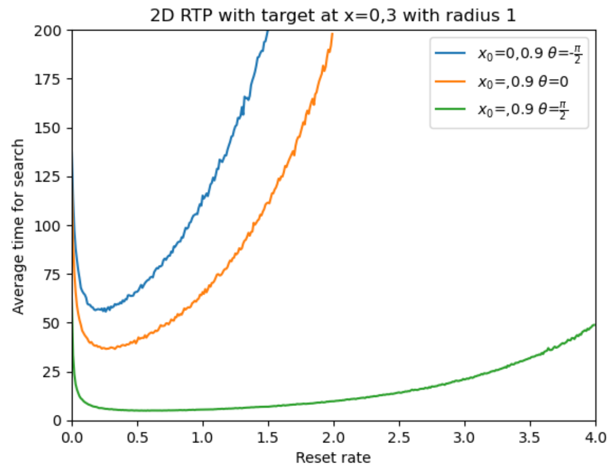


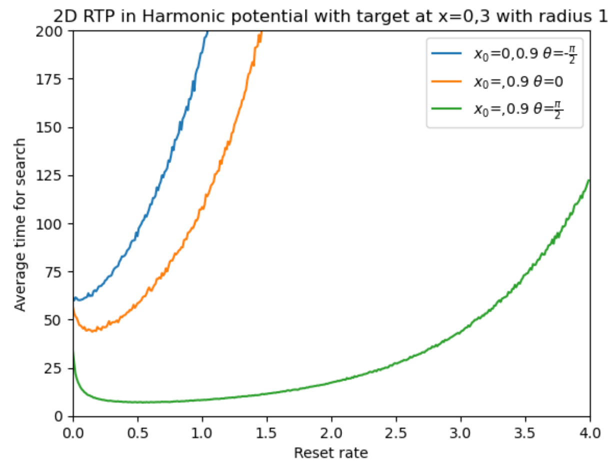
Figure 3: Average Search time for case of RTP inside harmonic potential described by equation 4 with $\gamma = 1$, and potential is $V = (\alpha/2)x^2$, with $\alpha = 0.1$, here x_0, v_0 gives the initial position and direction of particle with target at $x = 6.2$

One thing to keep in mind in this case is that due to harmonic potential, particle experiences a force towards the origin, although this force might not be enough to overcome the "Run" velocity of particle

We observe that if particle's starting position is to the left of target for target being on positive x axis, the resetting is useful and we do see the divergence for zero resetting rate, although, this system prefers much lower resetting rate as compared to previous cases, but if particle is starting to the right of target, than resetting isn't useful as we can see from the slope of those curves, it might even be increasing the average search time, this is because there's always a force on the particle towards the origin, hence the particle that is to the right of target, will eventually be forced to move towards the target, hence resetting might reset whatever progress has been made due to that force.



(a) 2D RTP confined in 8×8 square, with initial position $(0, 0.9)$ and $\gamma = 1$



(b) 2D RTP confined in potential of the form $V = (\alpha/2)(x^2 + y^2)$,with $\alpha = 0.1$, initial position $(0, 0.9)$ and $\gamma = 1$

Figure 4: Average Search time for case of 2D free RTP (a) confined in a box described by equation 5 and RTP confined in harmonic potential (b) described by equation 6, here x_0, θ gives the initial position and direction of particle with target at $(0, 3)$ of radius 1, since target is at $y - axis$ and particle starts at $y - axis$, so $\theta = \pi/2$ simulates the case particle with initial velocity towards the target, $\theta = 0$ gives the case of tangential initial velocity and $\theta = -\pi/2$ simulates the case of particle moving away from the target initially

Average time of search is much larger in this case, we also observe that at zero resetting rate, the average search time might not be diverging in these cases, but still there exit a minimum average search time for finite reset rate, hence in-spite of no divergence in search time, resetting is still beneficial.
We also observe that minimum of search time occurs for much larger resetting rate if particle have some component of velocity in the direction of target, but it occurs for smaller resetting rate if particle have no component in the direction of target, to explain this we can extend the explanation of 1D RTP case

6 Conclusion

- There exist a divergence of average search time for zero resetting rate in 1D RTPs studied in this report
- Stochastic resetting removes that divergence of average search time in all the cases studied in this report
- There exist a non zero finite resetting rate for which the average search time is minimum
- Even in the absence of any divergence, a low resetting rate is always beneficial for search processes

References

[1] Brownian Functionals in Physics and Computer Science SATYA N. MAJUMDAR

[2] Stochastic Resetting: A (Very) Brief Review Shamik Gupta, Arun M. Jayannavar