1 Multi-dimensional state space and value function iteration

1.1 Motivation

In economic models, we often have to solve multi-dimensional optimization problems numerically. In many cases, we can solve for the optimal policy functions by finding the root of a system of non-linear equations. For example, consider the simple case of a two-period OLG model where the household works l_t^1 hours in the first period receiving the gross labor income $w_t l_t^1$ and retires in the second period receiving a pension p_{t+1}. He consumes c_t^1 and c_{t+1}^2 in young and old age so that his intertemporal budget constraint is presented by

$$c_t^1 + \frac{c_{t+1}^2}{1+r_{t+1}} = (1-\tau)w_t l_t^1 + \frac{p_{t+1}}{1+r_{t+1}}, \tag{1}$$

where w_t, r_t and τ denote the wage rate, the interest rate r_t and proportional wage tax rate.
He maximizes his life-time utility

$$U(c_t^1, c_{t+1}^2, l_t^1) = u(c_t^1, l_t^1) + \beta u(c_{t+1}^2, 0), \tag{2}$$

where instantaneous utility $u(c,l)$ is a function of consumption c and labor l. Utility in the second period of the life when labor supply is zero is discounted by the factor β.
Maximizing (2) subject to his budget constraint (1) for given exogenous results in the two first-order conditions

$$u_c(c_t^1, l_t^1) = \beta(1+r_{t+1})u_c(c_{t+1}^2, 0) \tag{3a}$$
$$u_c(c_t^1, l_t^1) = -(1-\tau)w_t u_l(c_t^1, l_t^1), \tag{3b}$$

where $u_x(.)$ denotes the partial derivative of the instantaneous utility function with respect to the argument $x \in \{c, l\}$.
In this simple example, it is straightforward to compute the solution to this optimization problem by solving two non-linear equations (3). We only need to choose functional forms and parameters for the utility function and apply standard methods like the Newton-Rhapson algorithm.
The same method can be applied in this simple two-period model if pensions $p_{t+1}(x_t)$ depend on the contributions to the social security system of the individual at young age, $x_t = \tau w_t l_t^1$. In this case, the first-order condition with respect to labor, (3b), has to be adjusted as follows:

$$u_c(c_t^1, l_t^1) + \beta \frac{u_c(c_{t+1}^2, 0)p'_{t+1}(x_t)\tau w_t}{1+r_{t+1}} = -(1-\tau)w_t u_l(c_t^1, l_t^1). \tag{4}$$

Again, the solution to the optimization problem can be found with the help of a standard algorithm for non-linear equations.
The problem, however, becomes significantly more complicated once the problem is extended from a two-period OLG problem to a 70-period OLG problem where periods correspond to years like in the standard model of **?**.
Adding uncertainty with respect to individual productivity to this model further complicates the computation due to the curse of dimensionality. The life-time contributions of a household depends on the time path of exogenous shocks to his individual productivity so that the problem cannot be formulated in terms of a finite set of non-linear equations. Instead, we have to resort to the solution of the value function problem and end up solving an OLG model with multi-dimensional state space.
In the following, we compare different methods to compute a standard 70-period model with heterogenous agents that have continuous state variables of higher dimension. We will focus on a particular example with two continuous state variables in the form of the individual assets and cumulated earnings and solve it with the help of value function iteration (without interpolation between grid points). The computational time for the GAUSS and PYTHON programs (Windows 10, 64 BIT system, 32 MB RAM, Intel Xeon 2.90 GHz processor):

- Gauss: 3h 58m
- Python: 25h 23m

1.2 The model

1.2.1 Demographics

Households live a maximum of $T = 70$ periods. Periods are equal to one year. Households are born at age 1 (corresponding to real life-time age 20). All agents of age s in period t survive until age $s+1$ with probability ϕ_t^s, with $\phi_t^{70} = 0$. Let N_t denote the number of households at period t. We assume that the population grows at a constant rate, $\frac{N_{t+1}}{N_t} = 1 + g_n$.
The first $n_w = 45$ periods, the agents are working, the last $n_r = 25$ periods, they are retired and receive pensions that depend on the social-security contributions while young.

1.2.2 Preferences

Households maximize expected intertemporal utility at the beginning of age 1 in period t

$$\max \sum_{s=1}^{J} \beta^{s-1} \left(\Pi_{j=1}^{s} \phi_{t+j-1}^{j-1} \right) \left[u(c_{t+s-1}^s, l_{t+s-1}^s) + v(g_{t+s-1}) \right], \tag{5}$$

where $\beta > 0$ denotes the discount factor. Instantaneous utility $u(c,l)$ is specified as a function of consumption c and labor l

$$u(c,l) = \frac{c^{1-\sigma} - 1}{1-\sigma} - \nu_0 \frac{l^{1+\frac{1}{\nu_1}}}{1 + \frac{1}{\mu_1}}. \tag{6}$$

ν_1 denotes the Frisch labor supply elasticity. The utility from government consumption is additive so that it does not have any effect on the optimizing behavior of the household.
The total time endowment is equal to one and allocated between leisure $1-l$ and work l. We will impose a constraint on the maximum working hours, $l_t^s \leq l^{max}$.
If the s-year aged worker is employed in period t, his total gross labor income, $\epsilon(\eta, s) l_t^s w_t$, consists of the product of his working time l_t^s, his idiosyncratic productivity $\epsilon(z, s)$, and the wage per efficiency unit w_t. The household's labor productivity $\epsilon(\eta, s)$ is stochastic and also depends on his age s according to $\epsilon(\eta, s) = \eta e^{\bar{y}^s}$. In addition, all persons receive transfers tr_t from the government. The worker pays labor income taxes τ^w and social security contribution τ^b proportional to his labor income, $\epsilon(\eta, s) l_t^s w_t$. The average accumulated contributions of the $s+1$-year old household at the beginning of period $t+1$ are summarized by the accounting variable at age $s+1$, x_{t+1}^{s+1} as follows:

$$x_{t+1}^{s+1} = \begin{cases} \frac{(s-1)x_t^s + \tau^b \epsilon(\eta,s) l_t^s w_t}{s} & s = 1, \ldots, n_w \\ x_t^s & s = n_w + 1, \ldots, T, \end{cases} \tag{7}$$

with initial cumulated earnings equal to zero at the beginning of the life, $x_t^1 = 0$. Notice that the workers do not accrue interest on their social security payments. In old age, the retired worker receives a pension $pen(x_t^s)$ that depends on his average contributions over the life-cycle. In particular, we assume that the pension system redistributes between those with little contributions to those with high contributions. The pension system will be specified in more detail in the next section.
Households accumulate assets a_t^s with initial wealth $a_t^1 = 0$. Workers can save at the interest rate r_t in period t. Interest income is taxed at rate τ^r. However, workers are credit-constrained so that assets cannot be negative, $a_t^s \geq 0$. Accordingly, the budget constraint of the household is presented by

$$a_{t+1}^{s+1} = \begin{cases} (1 + (1-\tau^r) r_t) a_t^s + (1 - \tau^w - \tau^b) \epsilon(\eta, s) l_t^s w_t + tr_t - c_t^s & s = 1, \ldots, n_w, \\ (1 + (1-\tau^r) r_t) a_t^s + pen(x_t^s) - c_t^s & s = n_w + 1, \ldots, T. \end{cases} \tag{8}$$

1.2.3 Technology

Output is produced with the help of capital K_t and effective labor L_t according to the standard Cobb-Douglas function:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}. \tag{9}$$

Capital K_t also depreciates at rate δ. Productivity A_t grows at the exogenous rate g.

Firms are competitive and maximize profits $\Pi_t = Y_t - r_t K_t - w_t L_t - \delta K_{c,t}$ such that factor prices are given by:

$$w_t = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}, \tag{10a}$$
$$r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - \delta. \tag{10b}$$

1.2.4 Government and Social Security

The government levies income taxes τ^w and τ^r on labor and capital income. The entrepreneur's profit income is taxed at rate τ_y. In addition, the government confiscates all accidental bequests Beq_t. It pays total pensions Pen_t and transfers Tr_t, provides a certain level G_t of total public expenditures, and pays interest on the accumulated debt D_t. In each period, the government budget is financed by issuing government debt:

$$Pen_t + Tr_t + G_t + r_t D_t - Tax_t - Beq_t = D_{t+1} - D_t \tag{11}$$

1.2.5 Households' Optimization Problem

In the following, we formulate the problem $V(z)$ denotes the value function of the household. For $s \leq 45$, the young household decides if he becomes an entrepreneur or an employed worker:

$$V(z) = \max\left\{V_e(z), V_w(z)\right\}, \tag{12}$$

where $V_e(z)$ and $V_w(z)$ denote the value functions of the household who become an entrepreneur and a worker during the current period. Let $\eta(z) \in \{0,1\}$ with $\eta = 1$ ($\eta = 0$) denote the policy function whether the household chooses to become an entrepreneur or not. During retirement, households do neither work nor become entrepreneur so that $\eta = 0$ for $s > 45$.
In the following, we will describe the optimization problem for the working household, $s \leq 45$ in the stationary equilibrium. Let $z = (a, \eta, s)$ denote the individual state vector.[1]
For the working agent,

$$V_w(a,x,\theta,\epsilon,s) = \max_{c,n,a'}\left\{u(c,1-n) + \beta\phi_s \sum_{\theta'} prob(\theta'|\theta)\, V(a',x',\theta',\epsilon,s+1)\right\}, \tag{13}$$

subject to (7) and:

$$a' = a + (1-\tau_w-\tau_b) n_t e(\epsilon,s) w_t + (1-\tau_r) ra + tr - c \tag{14a}$$
$$a \geq 0. \tag{14b}$$

The old's problem. During retirement, the household is neither working nor becoming an entrepreneur. His entrepreneurial ability is set equal to $\theta = 0$. Instead, the person receives pensions depending on his contributions x. For $s \geq 46$, His value function is given by:

$$V(a,x,\theta,\epsilon,s) = \max_{c,a'}\left\{u(c,1) + \beta\phi_s \sum_{\theta'} prob(\theta'|\theta)\, V(a',x,\theta',\epsilon,s+1)\right\}, \tag{15}$$

subject to

$$a' = a + (1-\tau_w) b(x) + (1-\tau_r) ra + tr - c \tag{16a}$$
$$a \geq 0. \tag{16b}$$

Notice that the state variable x (average contributions to the pension system) remains constant. In addition, the retired worker pays labor income taxes on his pensions.

[1] In the following, we drop the index for the period t whenever it does not imply ambiguity.

1.3 Stationary Equilibrium

In order to express the equilibrium in terms of stationary variables, we have to divide aggregate variables by the mass of the total population N_t. Therefore, we define the following stationary variables, $\tilde{X}_t \equiv \frac{X_t}{N_t}$, for the aggregate variables $X \in \{Pen, Tr, G, D, Beq, Tax, Y, Y_c, Y_e, K, K_c, K_e, L, L_c, L_e, C, A, \}$, where K_e and L_e denote total capital and employment in the entrepreneurial sector. Let m denote the invariant distribution of z in the stationary equilibrium where the sum of all individuals is normalized to one.
In the equilibrium, the following conditions hold:

1. Households maximize their intertemporal utility as described in Section 1.2.5 implying the policy functions $c(.)$ and $a'(.)$, and labor supply $n(.)$ of the entrepreneurs and the investment $k(.)$ of the entrepreneurs.

2. In a factor market equilibrium, factors are rewarded with their marginal product presented by (10).

3. Total production $\tilde{Y}_t$ is equal to the sum of production in both sectors, $\tilde{Y}_t^c$ and $\tilde{Y}_t^e$:
$$\tilde{Y} = \tilde{Y}_c + \tilde{Y}_e \tag{17}$$

4. The government budget (11) is financed by debt in every period t:
$$\tilde{P}en_t + \tilde{T}r_t + \tilde{G}_t + r_t\tilde{D}_t = \tilde{T}ax_t + (1+g_N)\tilde{D}_{t+1} - \tilde{D}_t + \tilde{B}eq_t \tag{18}$$

5. Individual and aggregate behavior are consistent:
$$\tilde{Y}_e = \sum_{s=1}^{45}\sum_{\theta}\sum_{\epsilon}\int_a\int_x \theta\left(k(a,x,\theta,e,s)\right)^{\nu}(\epsilon\bar{n})^{1-\nu}\,\eta(a,x,\theta,\epsilon)\;m(a,x,\theta,\epsilon,s)\;da\;dx, \tag{19a}$$
$$\tilde{P}en = \sum_{s=46}^{70}\sum_{\theta}\sum_{\epsilon}\int_a\int_x b(x)m(a,x,\theta,\epsilon,s)\;da\;dx, \tag{19b}$$
$$\tilde{B}eq' = \sum_{s=1}^{70}\sum_{\theta}\sum_{\epsilon}\int_a\int_x (1-\phi_s)\left(1+(1-\tau_r)r\right)a'(a,x,\theta,\epsilon,s)\;m(a,x,\theta,\epsilon,s)\;da\;dx, \tag{19c}$$
$$\tilde{T}r_t = tr_t, \tag{19d}$$
$$\tilde{A} = \sum_{s=1}^{70}\sum_{\theta}\sum_{\epsilon}\int_a\int_x a\;m(a,x,\theta,\epsilon,s)\;da\;dx, \tag{19e}$$
$$\tilde{A}_e = \sum_{s=1}^{45}\sum_{\theta}\sum_{\epsilon}\int_a\int_x a\,\eta(a,x,\theta,\epsilon)\;m(a,x,\theta,\epsilon,s)\;da\;dx, \tag{19f}$$
$$\tilde{C} = \sum_{s=1}^{70}\sum_{\theta}\sum_{\epsilon}\int_a\int_x c(a,x,\theta,\epsilon,s)\;m(a,x,\theta,\epsilon,s)\;da\;dx, \tag{19g}$$
$$\tilde{L}_c = \sum_{s=1}^{45}\sum_{\theta}\sum_{\epsilon}\int_a\int_x I_{\eta(a,x,\theta,\epsilon,s)=0}\cdot n(a,x,\theta,\epsilon,s)\;m(a,x,\theta,\epsilon,s)\;da\;dx, \tag{19h}$$
$$\tilde{L}_e = \sum_{s=1}^{45}\sum_{\theta}\sum_{\epsilon}\int_a\int_x \bar{n}\,\eta(a,x,\theta,\epsilon,s)\;m(a,x,\theta,\epsilon,s)\;da\;dx, \tag{19i}$$
$$\tilde{K}_e = \sum_{s=1}^{45}\sum_{\theta}\sum_{\epsilon}\int_a\int_x k(a,x,\theta,e,s)\;\eta(a,x,\theta,\epsilon,s)m(a,x,\theta,\epsilon,s)\;da\;dx, \tag{19j}$$
$$\tilde{K} = \tilde{K}_e + \tilde{K}_c, \tag{19k}$$
$$\tilde{L} = \tilde{L}_e + \tilde{L}_c, \tag{19l}$$
$$\tilde{\Pi} = \tilde{Y}_e - \delta\tilde{K}_e - r\left(\tilde{K}_e - \tilde{A}_e\right) \tag{19m}$$
$$\tilde{T}ax = \tau_r r\left(\tilde{A} - \tilde{A}_e\right) + \tau_w wL_c + \tau_y\tilde{\Pi} + \tau_w\tilde{P}en. \tag{19n}$$

where $I_{\eta=0}$ denotes an index function that takes the value of one if $\eta = 0$ and zero otherwise and $\tilde{A}^e$ and $\tilde{\Pi}$ are total assets and total profits of the entrepreneurs.

6. The capital market clears: Total household savings are equal to the sum of the total capital employed in the non-entrepreneurial and the entrepreneurial sector plus government debt:

$$\tilde{A} = \tilde{K}_c + \tilde{K}_e + \tilde{D} \tag{20}$$

7. The labor market clears such that $\tilde{L}_c$ and $\tilde{L}_e$ are given by (19h) and (19i), respectively.

8. Profits in the credit sector are zero.

9. The goods market clears such that:

$$\tilde{Y} = \tilde{C} + \tilde{G} + (g_N + \delta)\tilde{K}. \tag{21}$$

2 Calibration

Our model is calibrated for the US economy. Our survival probabilities ϕ_s are taken from the United Nations (2002) world population projections.[2] The average life-time of the households amounts to 77.4 years in 2013.
For the discount factor, we choose the parameter values $\beta = 0.1011$ in accordance with the empirical estimates of Hurd (1989), who explicitly accounts for mortality risk[3] In addition, we choose the utility parameters $\sigma = 2.0$ and $\gamma = 0.28$ and the production parameters $\alpha = 0.36$, $A = 1.0$, and $\delta = 0.08$ that are standard in the business cycle literature.[4] The value of $\gamma = 0.28$, which measures the relative share of consumption and leisure in utility, implies an average labor supply of the worker approximately equal to 0.3. In accordance with the workers' average labor supply, we set the inelastic labor supply of the entrepreneurs equal to $\bar{n} = 0.3$. The model parameters are presented in Table 1.

Table 1: Benchmark calibration

Population	g_N=1.1%			
Preferences	β=0.99	γ=0.28	σ=2.0	$\bar{n}$=0.3
Individual abilities	$\{\epsilon_1, \epsilon_2\}$=$\{0.57, 1.43\}$	$\{\theta_1, \theta_2\}$=$\{0, 0.6\}$		
Production	α=0.36	δ=0.08	A=1	ν=0.88
Credit market	f=0.75			
Government	τ_b=9.59% G/Y=0.195	τ_w=0.248 D/Y=0.75	τ_r=0.429	τ_y=0.20

Following Krueger and Ludwig (2006), we choose the permanent efficiency types of the workers $\{\epsilon_1, \epsilon_2\} = \{0.57, 1.43\}$ implying a Gini cofficient of wage income at the amount 0.332, in good accordance with empirical observations from the US economy as presented by Budría Rodríguez, Díaz-Giménez, Quadrini, and Ríos-Rull (2002). The age-efficiency $\bar{y}_s$ profile is taken from Hansen (1993).[5]
In accordance with the US pension system, pensions $b(x)$ are modeled as a piecewise linear function of average past earnings following Huggett and Parra (2010):

$$b(x) = \begin{cases} b^{min} + 0.9x & \text{if } x \leq 0.2\bar{x} \\ b^{min} + 0.9(0.2\bar{x}) + 0.32(x - 0.2\bar{x}) & \text{if } 0.2\bar{x} < x \leq 1.24\bar{x} \\ b^{min} + 0.9(0.2\bar{x}) + 0.32(1.24\bar{x} - 0.2\bar{x}) + 0.15(x - 1.24\bar{x}) & \text{if } x > 1.24\bar{x} \end{cases} \tag{22}$$

[2]The survival probabilities for the year 2013 and 2050 are presented in Fig. **??** in the Appendix.
[3]Related research that uses such a value for β includes İmrohoroğlu et al. (1995) and Huggett (1996).
[4]See, for example, Heer and Maussner (2009).
[5]In the Appendix, we graph the age-efficiency profile in Fig. **??**.

where $\bar{x}$ denotes the average value of accumulated pension contributions x among the retired in the economy. Following Huggett and Ventura (2000), we set the lump-sum benefit b^{min} equal to 12.42% of GDP per capita in the model economy. Depending on which earnings bracket the retired agent's average contributions x were situated, he received 90% of the first 20 $\bar{x}$, 32% of the next 104% of $\bar{x}$, and 15% of the remaining earnings $(x - 1.24\bar{x})$ in 1994. Therefore, the marginal benefit rate declines with average earnings. The social security contribution rate τ_b is calibrated so that the budget of the social security balances.
The remaining parameters of the government policy that we need to calibrate are the two tax rates τ_r and τ_w, government expenditures G, and the debt level D. The two tax rates $\tau_r = 42.9\%$ and $\tau_w = 24.8\%$ are computed as the average values of the effective US tax rates over the time period 1965-1988 that are reported by Mendoza et al. (1994). The profit tax for entrepreneurs is set equal to $\tau_y = 0.20$. The share of government consumption in GDP is $G/Y = 19.5\%$, which is equal to the average ratio of G/Y in the US during 1959-1993 according to the Economic Report of the President (1994). The debt-to-equity ratio is set equal to 75%. The transfers-to-GDP ratio is determined endogenously so that the government budget balances implies a constant government debt D.

3 Computation

3.1 The Algorithm

The computation of the stationary equilibrium in our large-scale OLG model with heterogeneity and stochastic income is described in the following Algorithm 3.1.

Algorithm 3.1: Computation of the Model in Section 2
Find the solution to a large-scale OLG model

1. Parameterize the utility, production and population parameters.

2. Discretize individual stochastic productivity $\eta \in \mathscr{E}$ with n_η grid points. Estimate the Markov transition matrix for the dynamics of η.

3. Initialize the aggregate variables $\{\tilde{K}, \tilde{L}, \widetilde{Tr}, \tau\}$.

4. Choose an asset grid

   $$\mathscr{A} = \{a_1, a_2, \dots, a_{n_a}\},\ a_i < a_j,\ i < j = 1, 2, \dots n_a.$$

   and cumulated earnings grid

   $$\mathscr{X} = \{x_1, x_2, \dots, x_{n_x}\},\ x_i < x_j,\ i < j = 1, 2, \dots n_x.$$

   In case of discrete value function iteration, also choose a grid over labor supply

   $$\mathscr{L} = \{l_1, k_2, \dots, l_{n_l}\},\ l_i < l_j,\ i < j = 1, 2, \dots n_l.$$

5. Compute the optimal policy functions $c(z)$, $a'(z)$ and $l(z)$.

6. Compute the distribution of assets and aggregate variables.

7. Update the aggregate variables and return to step 5 until convergence.

8. Check if the grid over $\mathscr{A}$ and $\mathscr{E}$ was chosen large enough. If not, return to step 4.

In Step 4, we use value function iteration without interpolation. In the programs *OLG_simple_VI.py* and *OLG_simple_VI.gss*, we compute the value function

$$\max_{e'} u(c, l) + \beta \mathbb{E} v(a', e', \eta', s+1).$$

over a grid of $n_a \times n_x \times n_l = 100 \times 20 \times 100 = 200,000$ points and allocate the maximum (using the commands *'maxc()'* and *'np.where(bellman == np.max(bellman))'* in GAUSS and PYTHON, respectively.
The maximum value is saved as $v(a, a', e, \eta, s)$.