CS390 Computational Game Theory and Mechanism Design July 5, 2013

Problem Set 1

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Problem 1 (No collaborator.)

(a) There is particular no pure Nash equilibrium. However there is a unique mixed Nash equilibrium where the two players both choose their three actions in equal probability.

Proof.

First we show that the mixed strategy profile $\big((\frac{1}{3},\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{1}{3},\frac{1}{3})\big)$ is a NE. Given one's strategy with $\delta_1 = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$, then the utility of another player would be $u_2 = \sum(\frac{1}{3}p_i - \frac{1}{3}p_i) = 0$, so any mixed strategy for player2 is a best response. Then particularly the one with $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ is a best mixed response, for both the players.

Then we show that there's only one such mixed NE. Since the R-P-C game is Zero-Sum, when one player's strategy is not $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$, the other player would be able to choose his strategy so that $u_2 > 0$, for instance put all probability in the action which beats the other's most likely action. And while the same for player1 can choose $u_1 > 0$, since $u_1 + u_2 = 0$, there can't be another mixed NE. $\square$

(b) A mixed NE is $\big((\frac{2}{3},\frac{1}{3}),(\frac{1}{3},\frac{2}{3})\big)$.

Proof.

Given player1's strategy with $\sigma_1 = (\frac{2}{3},\frac{1}{3})$. Let σ_2 be (p,q). Then $u_2 = 1\frac{2}{3}p + 2\frac{1}{3}q = \frac{2}{3}(p+q) = \frac{2}{3}$, so any strategy of player 2 is a best mixed response, particularly the one with $\sigma_2 = (\frac{1}{3},\frac{2}{3})$ is. By symmetry, this is true for player 2 either. $\square$

(c) Obviously there are two pure NE in BoS with both B or both S. As (b) shows there's another mixed NE. We prove this is the only mixed NE so that we have in total 3 equilibrium.

Proof.

Suppose player1 chooses some strategy $\sigma_1 = (p,q)$. Suppose $p > \frac{2}{3}$. Then the best response for player2 would be $\sigma_2 = (1,0)$. In that case the best response for player1 is $\sigma_1 = (1,0)$, which is a pure NE counted. By symmetry, this is true for $p < \frac{2}{3}$. So $\big((\frac{2}{3},\frac{1}{3}),(\frac{1}{3},\frac{2}{3})\big)$ is the only mixed NE. $\square$

(d) Two players each pick a positive number. The utility for each would be the number they choose. Obviously there's no NE since each can always find a better strategy by changing to a bigger number.

Problem 2 (No collaborator.)

(a) Formulate a first price auction as a normal form game.

$$
\begin{aligned}
N &= \{1, 2, 3, \cdots, n\} \\
S &= N_+^n \\
u_i &= \begin{cases} v_i - s_{max} & s_i = s_{max} \text{ and } i < j \text{ if } s_j = s_{max} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Then if there is a NE, player 1 must obtains the object.

Proof.

If not, let player p($p > 1$) be the one obtains the object. Then we have $s_1 < s_k \leq v_k < v_1$. Then player can change his bid to at least v_k to win the auction with non-zero utility. □

Then we find all the NE. As is shown before, any NE must have $s_{max} = s_1$. Also s_{max} can't be lower than v_2, otherwise player 2 can change his bit to v_2 to rise the s_{max}. So any NE $S = (s_1, s_2, \cdots, s_n)$ follows:

- $v_1 \geq s_1 \geq v_2$
- $\forall j, s_j \leq s_1$

(b) Formulate weak dominance:

A strategy s_i weakly dominate s_i' if

$$\forall s_{-i}, u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

and a strategy is weakly dominant if it weakly dominate any other strategies.

In a second price auction the bid v_i of any player i is a weakly dominant strategy.

Proof.

We will show for player i, bidding v_i is a weak dominance. Let s_i' be another bid.In the case $\max\limits_{j \neq i} s_j \geq v_i$, then with s_i', player i either loses the object or get non-positive utility. In another case $\max\limits_{j \neq i} s_j < v_i$, player i either loses the object or wins with the same utility as s_i does. □

Finally, let's consider a equilibriam in which the winner is not player 1. Let $N = 3$.

$$v_1 = 3, v_2 = 2, v_3 = 1$$

$$s_1 = 1, s_2 = 100, s_3 = 2$$

It is indeed a NE and in this case player 2 wins the object.

Problem 3 (No collaborator.) In the first iteration, all numbers in $[34, 100]$ will be eliminated, since they cannot be $\frac{1}{3}$ of the avarage number. Similarly, the following sequence of elimination will be $[12, 33]$, $[5, 12]$, $[3, 5]$, $[2, 3]$ and what is left is 1.

References

[1] M. J. Osborne and A. Rubinstein. *A course in game theory.* MIT Press, 1994.

[2] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani (eds). *Algorithmic game theory.* Cambridge University Press, 2007. (Available at `http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf`.)