CS390 Computational Game Theory and Mechanism Design	July 13, 2013

Problem Set 3

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Problem 1 (No collaborator.)
The objective cost function when x people choose the lower edge is

$$c(x) = \frac{x(\frac{x}{n})^d + (n-x)}{n},\ x \in [0, n]$$

To calculate the optimal cost, take the deriviation of $c(x)$

$$c'(x) = \frac{-1 + (d+1)\frac{x^d}{n^d}}{n}$$

Solve for $c'(x) = 0$ we have

$$x = \frac{n}{(d+1)^{1/d}}$$

so at $x = \dfrac{n}{(d + 1^{1/d})}$, $c(x)$ is optimal.
Since $(x/n)^d \leq 1$, there are still 2 equilibrium: $x = n$ and $x = n-1$. So

$$PoA = \frac{c(n)}{\min c(x)} \text{ or } \frac{c(n-1)}{\min c(x)}$$

When $n \to \infty$, $c(n) = c(n-1) = 1 + o(1)$, $PoA \to \dfrac{1}{1 - (d+1)^{-1/d} + (d+1)^{-1-1/d}}$.
When $d \to \infty$, $PoA \to +\infty$.

Problem 2 (No collaborator.)
Proof. Let the strategy profile with the optimal sum of utility be S, and the NE with the worst objective function be $S^\star$.
Consider one player i, if his utility for $S^\star$ exceed n times his utility for S, then he can change his strategy to that in S, since no matter how many people choose the same route as him, his cost on this route won't exceed n times any possible cost on this route.
This is true for any player, Then the total cost in NE won't exceed n times that of the optimal objective function. Therefore the price of anarchy won't exceed 1. □

Problem 3 (No collaborator.)
First, the pure-strategy Nash equilibria of this game are precisely the m! outcomes in which each player selects a distinct machine. Second, mixed strategy $\sigma = (\sigma_i)$ where $\sigma_i = (\frac{1}{n}, ...\frac{1}{n})$ for each player i is the unique mixed Nash equilibrium([2]. 17.2.3).
let X_{ij} denote the indicator random variable for the event that player i selects the machine j. If the first player selects machine j, then it incurs a cost of $1 + \sum_{i>1} X_{ij}$. By linearity of expectation, its expected cost on this machine is $1 + \sum_{i>1} E[X_{ij}] = 2 - \frac{1}{n}$([2]. Example 17.4).
So the objective cost at the unique mixed NE is $2 - \frac{1}{n}$ and since the optimal objective function is equal to 1, the price of anarchy is

$$PoA = 2 - \frac{1}{n}$$

References

[1] M. J. Osborne and A. Rubinstein. *A course in game theory.* MIT Press, 1994.

[2] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani (eds). *Algorithmic game theory.* Cambridge University Press, 2007. (Available at `http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf`.)