CS390 Computational Game Theory and Mechanism Design July 22, 2013

Problem Set 6

2012 ACM Class 5120309027 Huang Zen

Problem 1 (No collaborator.)
Proof.

1. From $\begin{cases} u_1(\theta, a) > u_1(\theta, c) \\ u_1(\theta', c) > u_1(\theta', a) \end{cases}$ we have $L(\theta, a) \supset L(\theta', a)$. By symmetry we also have $L(\theta', b) \supset L(\theta, b)$. So f is monotone.

2. $c \notin f(\theta), c \notin f(\theta')$, so f do not satisfy NVP.

3. Suppose there exists a mechanism M which fully Nash-implements f. From $o(\theta, pNE(M)) = f(\theta)$, we have there exist a column that the outcomes are all a. and from $o(\theta', pNE(M)) = f(\theta')$, then there exist a row that the outcomes are all b. That leads to $a = b$, what is contradiction.

□

Problem 2 (No collaborator.)

Claim 1

Proof.

Let $A^{OPT} = \underset{A'}{argmax} \sum_i v_i(A'_i)$, suppose $A_i^{OPT} = \emptyset$. Then

$$P_i = \max_{A'} \sum_{j \neq i} v_j(A'_j) - \sum_{j \neq i} v_j(A_j^{OPT}) = 0$$

So that we have claim 1.

□

Claim 2

Proof.

Let $A^{OPT} = \underset{A'}{argmax} \sum_i v_i(A'_i)$, we have

$$P_i = \max_{A'} \sum_{j \neq i} v_j(A'_j) - \sum_{j \neq i} v_j(A_j^{OPT})$$

$\forall i \in N, \theta_i \in \Theta_i$, and $v_{-i} \in \Theta_{-i}$, where $v_i = \theta_i$, we have

$$\begin{aligned} u_i(\theta_i, VCG(\theta_i, v_{-i})) &= \theta_i(A_i) - P_i \\ &= v_i(A_i) - P_i \\ &= v_i(A_i) + \sum_{j \neq i} v_j(A_j^{OPT}) - \max_{A'} \sum_{j \neq i} v_j(A'_j) \\ &= \sum_j v_j(A_j^{OPT}) - \max_{A'} \sum_{j \neq i} v_j(A'_j) \\ &\geq 0 \end{aligned}$$

$\square$

Problem 3 (No collaborator.)

(a) *Proof.* First we show that M implements F.

$$M(\theta) = \begin{cases} (0, p) & \sum_i \theta_i < c \\ (1, p) & \sum_i \theta_i \geq c \end{cases} \subseteq F(\theta)$$

So M implements F.
Next we show that M is DST, which means $\forall \theta, i$, θ_i is a dominant strategy. $\forall v_{-i}$

(1) If $\sum_{j \neq i} v_j + \theta_i < c$

i. If $\sum_j v_j < c$, then $u_i(\theta_i, M(\theta_i, v_{-i})) = u_i(\theta_i, M(v_i, v_{-i})) = 0$.

ii. If $\sum_j v_j \geq c$, then $\begin{cases} u_i(\theta_i, M(\theta_i, v_{-i})) = 0 \\ u_i(\theta_i, M(v_i, v_{-i})) = \sum_{j \neq i} v_j + \theta_i - c \leq 0 \end{cases}$.

(2) If $\sum_{j \neq i} v_j + \theta_i \geq c$

i. If $\sum_j v_j < c$, then $\begin{cases} u_i(\theta_i, M(\theta_i, v_{-i})) = \sum_{j \neq i} v_j + \theta_i - c \geq 0 \\ u_i(\theta_i, M(v_i, v_{-i})) = 0 \end{cases}$.

ii. If $\sum_j v_j \geq c$, then $u_i(\theta_i, M(\theta_i, v_{-i})) = u_i(\theta_i, M(v_i, v_{-i})) = \sum_{j \neq i} v_j + \theta_i - c$.

To all the above cases, $u_i(\theta_i, M(\theta_i, v_{-i})) \geq u_i(\theta_i, M(v_i, v_{-i}))$, so that $\forall \theta, i$, θ_i is a dominant strategy.
Thus we have Groves mechanism DST-implements F. $\square$

(b) $N = 2, c = 1, \theta_1 = \theta_2 = 10$, the revenue is -18.

References

[1] M. J. Osborne and A. Rubinstein. *A course in game theory.* MIT Press, 1994.

[2] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani (eds). *Algorithmic game theory.* Cambridge University Press, 2007. (Available at `http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf`.)