CS390 Computational Game Theory and Mechanism Design July 15, 2013

Problem Set 4

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Problem 1 (No collaborator.)
obviously $\sigma_i = \frac{1}{2}H + \frac{1}{2}T$ for each player i is a Nash. In which way, each player makes the other indifferent between choosing heads or tails, so neither player has an incentive to try another strategy.
Now given positive ϵ, lets consider a strategy namely σ':

$$\sigma' = \begin{cases} (\frac{1}{2} + \frac{\epsilon}{20})H + (\frac{1}{2} - \frac{\epsilon}{20})T & i = 1, ..., 5 \\ \\ \frac{1}{2}H + \frac{1}{2}T & i = 6 \end{cases}$$

We found that

$$\begin{aligned} &u_i(\sigma) = 0 && (i = 1, ...6) \\ &|u_i(H, \sigma_6')| = |u_i(T, \sigma_6)| = 0 \le \epsilon && (i = 1, ..., 5) \\ &|u_6((\sigma_i'), H)| = |5((\frac{1}{2} - \frac{\epsilon}{20}) - (\frac{1}{2} + \frac{\epsilon}{20}))| = | - \frac{\epsilon}{2}| \le \epsilon \\ &|u_6((\sigma_i'), T)| = |5((\frac{1}{2} + \frac{\epsilon}{20}) - (\frac{1}{2} - \frac{\epsilon}{20}))| = |\frac{\epsilon}{2}| \le \epsilon \end{aligned}$$

However σ' is not a Nash since each player can always avoid receiving negative utility, so σ' yields a ϵ-Nash equillibrium.

Problem 2 (No collaborator.) W.L.G, take player 6 as the root and all other players be the children and terminated nodes.
suppose player 6 choose the strategy $\sigma = pH + qT$, $0 \le p, q \le 1$,$p + q = 1$. Player i chooses strategy (p_iH, q_iH). If $p = q$, any strategy for player i is a best response. If $p > q$, best response for other players would be $(1H, 0T)$, If $p < q$, the best response would be $(0H, 1T)$.
So the table passed from the children is

$$T_i(p_iH + q_iT, pH + qT) = \begin{cases} 1 & p = q \text{ or } p > q, p_i = 1 \text{ or } p < q, q_i = 1 \\ 0 & otherwise \end{cases}$$

We have

$$u_6(p_iH + q_iH, pH + qH) = p\sum_{i=1}^{5} q_i + q\sum_{i=1}^{5} p_i = (1-2p)(2\sum_{i=1}^{5} p_i - 5)$$

For player 6, $p \neq q$ is obvious not a best response to his children's choices since all other players can choose one pure strategy and drop player 6's utility to negative. And when $p = q$, it's a best response only if $\sum_{i=1}^{5} p_i = \frac{5}{2}$. So the table at the root is

$$T_6(pH + qT) = \begin{cases} 1 & p = q \\ 0 & \text{otherwise} \end{cases}$$

while the witness list at the root is

$$\sum_{i=1}^{5} p_i = \frac{5}{2}$$

This yields NEs with one player chooses equal probability H and T, others chooses equal sum of probability of H and T.

Problem 3 (No collaborator.)

Proof. In mixed strategy σ yielded from strategy set S, let p_i be the probability of σ_i, let $p = \sum_{i=1}^{k} \lambda_i p_i$ be the convex combination of p_i.

$\forall i, \forall b \in S_i, \forall b' \in S_i \quad T = \{s \mid s \in S,\ s_i = b\}$

$$
\begin{aligned}
\mathbb{E}_{s\sim\sigma|_{s_i=b}} u_i(s) &= \sum_{s\in T} \frac{p(s)}{\displaystyle\sum_{s'\in S} p(s')} u_i(s) \\
&= \sum_{s\in T} \frac{\displaystyle\sum_{j=1}^{k} \lambda_j p_j(s)}{\displaystyle\sum_{s'\in S} p(s')} u_i(s) \\
&= \sum_{j=1}^{k} \lambda_j \frac{\displaystyle\sum_{s'\in S} p_j(s')}{\displaystyle\sum_{s'\in S} p(s')} \sum_{s\in T} \frac{p_j(s)}{\displaystyle\sum_{s'\in S} p_j(s')} u_i(s) \\
&= \sum_{j=1}^{k} \lambda_j \frac{\displaystyle\sum_{s'\in S} p_j(s')}{\displaystyle\sum_{s'\in S} p(s')} \mathbb{E}_{s\sim\sigma_j|_{s_i=b}} u_i(s) \\
&\geq \sum_{j=1}^{k} \lambda_j \frac{\displaystyle\sum_{s'\in S} p_j(s')}{\displaystyle\sum_{s'\in S} p(s')} \mathbb{E}_{s\sim\sigma_j|_{s_i=b}} u_i(b', s_{-i}) \\
&= \mathbb{E}_{s\sim\sigma|_{s_i=b}} u_i(b', s_{-i}) \text{ (by symmetry)}
\end{aligned}
$$

So

$$\mathbb{E}_{s\sim\sigma|_{s_i=b}} u_i(s) \geq \mathbb{E}_{s\sim\sigma|_{s_i=b}} u_i(b', s)$$

So for any finite game G, any finite convex combination of CEs is still a CE of G. $\square$

References

[1] M. J. Osborne and A. Rubinstein. *A course in game theory.* MIT Press, 1994.

[2] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani (eds). *Algorithmic game theory.* Cambridge University Press, 2007. (Available at `http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf`.)