CS390 Computational Game Theory and Mechanism Design	July 1, 2013

Lecture 1, Part 2

Scribed by Yiqing Hua(5120309062) Xiaoyang Lin(5120309438)

The first part is arranged by Xiaoyang Lin(5120309438),revised by Yiqing Hua(5120309062)

1 Zero-Sum Games

Definition 1 *A two players Z-S game ({1,2},S,u) s.t*

$$\forall s \in \mathbf{S}, u_1(s) + u_2(s) = 0 \tag{1}$$

2 Example

	$Morality$	$Tax-cuts$
$Economy$	$3 \quad -3$	$-1 \quad 1$
$Society$	$-2 \quad 2$	$1 \quad -1$

3 Process

if player 1(P1) acts first $\sigma_1 \in \Delta(\{Economy, Society\})$,player 2 σ_2 will max u_2,then P1 gets $min_{\sigma_2} u_1(\sigma_1, \sigma_2)$ using σ_1
That is P1 choose σ_1 s.t $\sigma_1 \in arg \quad max_{\sigma_1} min_{\sigma_2} u_1(\sigma_1, \sigma_2)$
So the $u_1 = max_{\sigma_1} min_{\sigma_2} u_1(\sigma_1, \sigma_2)$

if player 2(P2) acts first $\sigma_2 \in \Delta(\{Morality, Tax-cuts\})$,player 1 σ_1 will max u_1,then P2 gets $min_{\sigma_1} u_2(\sigma_2, \sigma_1)$ using σ_2
That is P2 choose σ_2 s.t $\sigma_2 \in arg \quad max_{\sigma_2} min_{\sigma_1} u_2(\sigma_2, \sigma_1)$
So the $u_2 = max_{\sigma_2} min_{\sigma_1} u_2(\sigma_2, \sigma_1) = -min_{\sigma_2} max_{\sigma_1} u_1(\sigma_1, \sigma_2) = -u_1(\sigma_1, \sigma_2)$

Theorem 1 $\forall$ *finite Z-S game,*$max_{\sigma_1} min_{\sigma_2} u_1(\sigma_1, \sigma_2) = min_{\sigma_2} max_{\sigma_1} u_1(\sigma_1, \sigma_2)$

Proof 1 *the Game is finite* $\Rightarrow \exists NE(\widehat{\sigma_1}, \widehat{\sigma_2})$
$\Rightarrow u_2(\widehat{\sigma_1}, \widehat{\sigma_2}) \geq u_2(\widehat{\sigma_1}, \sigma_2) \quad \forall \sigma_2$

$\Rightarrow u_1(\widehat{\sigma_1}, \widehat{\sigma_2}) \leq u_1(\widehat{\sigma_1}, \sigma_2) \quad \forall \sigma_2$
$\Rightarrow u_1(\widehat{\sigma_1}, \widehat{\sigma_2}) = min_{\sigma_2} u_1(\widehat{\sigma_1}, \sigma_2) \leq max_{\sigma_1} min_{\sigma_2} u_1(\sigma_1, \sigma_2)$
By symmetry $,u_1(\widehat{\sigma_1}, \widehat{\sigma_2}) \geq u_1(\sigma_1, \widehat{\sigma_2}) \quad \forall \sigma_1$
$\Rightarrow u_1(\widehat{\sigma_1}, \widehat{\sigma_2}) \geq min_{\sigma_2} u_1(\sigma_1, \sigma_2) \quad \forall \sigma_1$
$\Rightarrow u_1(\widehat{\sigma_1}, \widehat{\sigma_2}) \geq max_{\sigma_1} min_{\sigma_2} u_1(\sigma_1, \sigma_2)$

From the analysis ,we can get $u_1(\widehat{\sigma_1}, \widehat{\sigma_2}) = max_{\sigma_1} min_{\sigma_2} u_1(\widehat{\sigma_1}, \sigma_2)$
By symmetry, $u_2(\widehat{\sigma_1}, \widehat{\sigma_2}) = max_{\sigma_2} min_{\sigma_1} u_2(\sigma_1, \sigma_2) = -min_{\sigma_2} max_{\sigma_1} u_1(\sigma_1, \sigma_2)$

So we can conclude

$$max_{\sigma_1} min_{\sigma_2} u_1(\sigma_1, \sigma_2) = min_{\sigma_2} max_{\sigma_1} u_1(\sigma_1, \sigma_2)$$

Theorem 2 *If* $\sigma_1^* \in arg \quad max_{\sigma_1} min_{\sigma_2} u_1(\sigma_1, \sigma_2)$ *and* $\sigma_2^* \in arg \quad min_{\sigma_2} max_{\sigma_1} u_1(\sigma_1, \sigma_2)$ *then* (σ_1^*, σ_2^*) *is a NE*

Proof 2 *let* $v^* = max_{\sigma_1} min_{\sigma_2} u_1(\sigma_1, \sigma_2) = min_{\sigma_2} max_{\sigma_1} u_1(\sigma_1, \sigma_2)$
According to Theorem 1, $-v^* = max_{\sigma_2} min_{\sigma_1} u_2(\sigma_1, \sigma_2)$

$$\left.\begin{array}{r} u_1(\sigma_1^*, \sigma_2) \geq v^* \quad \forall \sigma_2 \\ u_2(\sigma_1, \sigma_2^*) \geq -v^* \quad \forall \sigma_1 \end{array}\right\} \text{let} \quad \sigma_1 = \sigma_1^* \quad \text{and} \quad \sigma_2 = \sigma_2^* \text{in these two inequalities we obtain}$$

$u_1(\sigma_1^*, \sigma_2^*) = v^*$

and using $u_2 = -u_1$*,we conclude* (σ_1^*, σ_2^*) *is a NE.*

4 Iterated elimination

The second part is arranged by Yiqing Hua(5120309062),revised by Xiaoyang Lin(5120309438)

In this class we talked about iterated elimination. In a strategic game, if one pure s-trategy strictly dominates the others then it is obvious the strategy we seek for. We'll talk about the notion of strictly dominence, and the process to eliminate the strategies strictly dominated by the others so that none of the players may choose them.

Definition
Iterated elimination are defined as follows.

Definition 2 *In a strategic game* $\langle N, S, u \rangle$*, a set* $X \subseteq S$ *survives iterated elimination of strictly dominated strategy of* $X = X_1 \times X_2 \cdots \times X_n$ *if* $\exists$ *a finite sequence* $S^0, S^1 \ldots S^K$ *s.t*

- $S^0 = S$
- $S^k = S_1^k \times S_2^k \dots S_n^k$
- $S^K = X$
- $\forall 0 \leq k \leq K, S_i^{k+1} \subseteq S_i^k$ *and* $S^k \backslash S^{k+1} \neq \emptyset$
- $\forall i,\ s_i \in S_i^k \backslash S_i^{k+1},$ *if* $\exists \sigma_i \in \delta(S_i^k),$ *s.t* $u_i(\sigma_i, S_{-i}) > u_i(s_i, S_{-i})\ \forall S_{-i} \in S_{-i}^k$. *For* $G^k = \langle N, S^k, u \rangle$, *we say that* s_i *is strictly dominated by* σ_i *in* G^k.
- $\forall i,\ S_i^K$ *contains no strategy that is strictly dominated over* S^K.

Remark. Note that σ_i can be either pure strategy or mixed strategy while $s_i \in S_i^k$ are all pure strategies.

5 Example

Given the game $\langle \{1, 2\}, \{T, M, B\} \times \{L, R\}, u \rangle$. The utility is given as the following table. The iterated elimination acts like follows.

	L	R
T	(3, 0)	(0, 1)
M	(0, 0)	(3, 1)
B	(1, 1)	(1, 0)

- First we have $S^0 = S = \{T, M, B\} \times \{L, R\}$
  Since no pure strategy can dominate any strategy of each player, we should look for some mixed strategy to do our elimination.
  We choose $\sigma_1 = \frac{1}{2}M + \frac{1}{2}T$. Then we have
  $$u_1(\sigma_1, L) = 1.5 > 1 = u_1(B, L)$$
  $$u_1(\sigma_1, R) = 1.5 > 1 = u_1(B, R)$$
  Therefore, B is strictly dominated by σ_1. We can get S^1.
- $S^1 = \{T, M\} \times \{L, R\}$
  Consider player 2, we have
  $$u_2(T, R) = 1 > 0 = u_2(T, L)$$
  $$u_2(M, R) = 1 > 0 = u_2(M, L)$$
  Therefore, L is strictly dominated by R. We now have S^2.

- $S^2 = \{T, M\} \times \{R\}$
  Since $u_1(M, R) = 3 > 0 = u_1(T, R)$, we can now eliminate strategy T.

- And now we have the final state $S^3 = \{M\} \times \{R\}$.

References

[1] M. J. Osborne and A. Rubinstnnein. *A course in game theory.* MIT Press, 1994.

[2] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani (eds). *Algorithmic game theory.* Cambridge University Press, 2007. (Available at `http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf`.)