第四次作业

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1. 求解 $T^{\mu\upsilon}_{\rho;\sigma}$

答：

可以看出张量 $T^{\mu\upsilon}_{\rho}$ 为二阶逆变、一阶协变的三阶混合张量，根据运算法则得 $T^{\mu\upsilon}_{\rho;\sigma} = T^{\mu\upsilon}_{\rho,\sigma} + \Gamma^{\mu}_{\sigma\alpha}T^{\alpha\upsilon}_{\rho} + \Gamma^{\upsilon}_{\sigma\beta}T^{\beta\mu}_{\rho} - \Gamma^{\gamma}_{\sigma\rho}T^{\mu\upsilon}_{\gamma}$

2. 从 $\frac{\partial x'^{\upsilon}}{\partial x'^{\rho}} = \delta_{\upsilon\rho}$ 出发，证明 $\Gamma'^{\upsilon}_{\rho\tau}$ 的表达形式。

答：

1) 根据温伯格的书可以得到仿射联络的定义为 $\Gamma^{\lambda}_{\mu\upsilon} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}}\frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu}\partial x^{\upsilon}}$，其中 $\xi^{\alpha}(x)$ 是局部惯性系。所以坐标变换过程中新坐标系下可以表示为 $\Gamma'^{\lambda}_{\mu\upsilon}$，推导过程如下：

$$\Gamma'^{\lambda}_{\mu\upsilon} = \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}}\frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu}\partial x'^{\upsilon}} = \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}}\frac{\partial}{\partial x'^{\mu}}(\frac{\partial \xi^{\alpha}}{\partial x'^{\upsilon}}) = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\frac{\partial}{\partial x'^{\mu}}(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\upsilon}}) \tag{1}$$

$$\frac{\partial}{\partial x'^{\mu}}(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\upsilon}}) = \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu}\partial x'^{\upsilon}}\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} + \frac{\partial^2 \xi^{\alpha}}{\partial x^{\sigma}\partial x^{\tau}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\upsilon}} \tag{2}$$

可以看到 (1) 中存在 $\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}$，(2) 中存在 $\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}$，由题干的 $\frac{\partial x'^{\upsilon}}{\partial x'^{\rho}} = \delta_{\upsilon\rho}$ 类比得 $\frac{\partial x^{\rho}}{\partial \xi^{\alpha}} = \delta_{\alpha\rho}$ 和 $\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} = \delta_{\alpha\sigma}$，即两个量分别在 $\alpha = \rho$ 和 $\alpha = \sigma$ 时为 1，其他为 0。所以上式结合仿射联络的定义后可以得到 (3) 式：

$$\Gamma'^{\lambda}_{\mu\upsilon} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\upsilon}} + \frac{\partial^2 \xi^{\alpha}}{\partial x^{\sigma}\partial x^{\tau}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\upsilon}}\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}} = \Gamma^{\rho}_{\sigma\tau}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\upsilon}}\frac{\partial x'^{\lambda}}{\partial x^{\rho}} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\upsilon}} \tag{3}$$

综上，改变指标符号后可以获得题目要求的形式，即

$$\Gamma'^{\upsilon}_{\rho\tau} = \Gamma^{\mu}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\rho}}\frac{\partial x^{\beta}}{\partial x'^{\tau}}\frac{\partial x'^{\upsilon}}{\partial x^{\mu}} + \frac{\partial x'^{\upsilon}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x'^{\rho}\partial x'^{\tau}} \tag{4}$$

2) 因为仿射联络的定义在没翻书时并未记住，所以我认为用协变矢量 B_{υ} 来证明该过程更为合理。已知 $\delta B_{\upsilon} = \Gamma^{\tau}_{\rho\upsilon}dx^{\rho}B_{\tau}$，所以 $\widetilde{B}_{\mu} = B_{\mu} + \delta B_{\mu} = B_{\mu} + \Gamma^{\tau}_{\rho\mu}dx^{\rho}B_{\tau}$，在新坐标系下为 $\widetilde{B'}_{\upsilon} = B'_{\upsilon} + \Gamma'^{\eta}_{\alpha\upsilon}dx'^{\alpha}B'_{\eta}$。

因为规定 $\widetilde{B}_\mu$ 按矢量形式变换，所以得 $\widetilde{B'}_\upsilon(x'^\upsilon+dx'^\upsilon)=\frac{\partial x^\mu}{\partial x'^\upsilon}|_{x+dx}\widetilde{B}_\mu$。

$$\widetilde{B'}_\upsilon=B'_\upsilon+\Gamma'^{\eta}_{\alpha\upsilon}dx'^\alpha B'_\eta=(\frac{\partial x^\mu}{\partial x'^\upsilon}+\frac{\partial^2 x^\mu}{\partial x'^\upsilon\partial x'^\lambda}dx'^\lambda)\mid_x(B_\mu+\Gamma^\tau_{\rho\mu}dx^\rho B_\tau) \tag{5}$$

$$\frac{\partial x^\mu}{\partial x'^\upsilon}B_\mu+\frac{\partial^2 x^\mu}{\partial x'^\upsilon\partial x'^\lambda}dx'^\lambda B_\mu=B'_\upsilon+\frac{\partial^2 x^\mu}{\partial x'^\upsilon\partial x'^\lambda}dx'^\lambda B_\mu=B'_\upsilon+\frac{\partial^2 x^\mu}{\partial x'^\upsilon\partial x'^\lambda}\frac{\partial x'^\lambda}{\partial x'^\alpha}dx'^\alpha B_\mu \tag{6}$$

只保留 dx 得一次项后，由 $B_\mu=\frac{\partial x'^\eta}{\partial x^\mu}B'_\eta$、$B_\tau=\frac{\partial x'^\eta}{\partial x^\tau}B'_\eta$ 和 $dx^\rho=\frac{\partial x^\rho}{\partial x'^\alpha}dx'^\alpha$ 结合 $5,6$ 两式得：

$$\Gamma'^{\eta}_{\alpha\upsilon}dx'^\alpha B'_\eta=\frac{\partial^2 x^\mu}{\partial x'^\upsilon\partial x'^\lambda}\frac{\partial x'^\lambda}{\partial x'^\alpha}dx'^\alpha\frac{\partial x'^\eta}{\partial x^\mu}B'_\eta+\Gamma^\tau_{\rho\mu}\frac{\partial x'^\eta}{\partial x^\tau}B'_\eta\frac{\partial x^\rho}{\partial x'^\alpha}dx'^\alpha\frac{\partial x^\mu}{\partial x'^\upsilon} \tag{7}$$

注意到 7 式中含有 $\frac{\partial x'^\lambda}{\partial x'^\alpha}$，由题干得 $\alpha=\lambda$ 时为 1，其他为 0。所以等式两边消除 $dx'^\alpha B'_\eta$ 后得到 $\Gamma'^{\eta}_{\alpha\upsilon}=\frac{\partial^2 x^\mu}{\partial x'^\upsilon\partial x'^\alpha}\frac{\partial x'^\eta}{\partial x^\mu}+\frac{\partial x^\rho}{\partial x'^\alpha}\frac{\partial x^\mu}{\partial x'^\upsilon}\frac{\partial x'^\eta}{\partial x^\tau}\Gamma^\tau_{\rho\mu}$。综上，将指标更改一下得到练习 1.4 得形式：

$$\Gamma'^{\upsilon}_{\rho\tau}=\Gamma^\mu_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\rho}\frac{\partial x^\beta}{\partial x'^\tau}\frac{\partial x'^\upsilon}{\partial x^\mu}+\frac{\partial x'^\upsilon}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\rho\partial x'^\tau} \tag{8}$$