Factorial modulo p in $O(p \log n)$

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In some cases it is necessary to consider complex formulas modulo p, containing factorials in both numerator and denominator. We consider the case when p is relatively small. This problem makes sense only when factorials are included in both numerator and denominator of fractions. Otherwise $p!$ and subsequent terms will reduce to zero, but in fractions all multipliers containing p can be reduced, and the resulting expression will be non-zero modulo p.

Thus, formally the task is: You want to calculate $n! \bmod p$, without taking all the multiple factors of p into account that appear in the factorial. Imaging you write down the prime factorization of $n!$, remove all factors p, and compute the product modulo p. We will denote this modified factorial with $n!_{\%p}$.

Learning how to effectively calculate this modified factorial allows us to quickly calculate the value of the various combinatorial formulae (for example, Binomial coefficients).

Algorithm

Let's write this modified factorial explicitly.

$$\begin{aligned} n!_{\%p} &= 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-2) \cdot (p-1) \cdot \underbrace{1}_{p} \cdot (p+1) \cdot (p+ \\ &\quad \cdot (2p+1) \cdot \ldots \cdot (p^2-1) \cdot \underbrace{1}_{p^2} \cdot (p^2+1) \cdot \ldots \cdot n \\ &= 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-2) \cdot (p-1) \cdot \underbrace{1}_{p} \cdot 2 \cdot \ldots \cdot (p-1 \\ &\quad \cdot \ldots \cdot (p-1) \cdot \underbrace{1}_{p^2} \cdot 1 \cdot 2 \cdot \ldots \cdot (n \bmod p) \quad (\mathrm{mo} \end{aligned}$$

It can be clearly seen that factorial is divided into several blocks of same length expect for the last one.

$$\begin{aligned} n!_{\%p} &= \underbrace{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-2) \cdot (p-1) \cdot 1}_{1\text{st}} \cdot \underbrace{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p}_{2} \\ &\quad \cdot \underbrace{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-2) \cdot (p-1) \cdot 1}_{p\text{th}} \cdot \ldots \cdot \quad \underbrace{1 \cdot 2 \cdot \ldots} \end{aligned}$$

The general part of the blocks it is easy to count — it's just $(p-1)!\ \mathrm{mod}\ p$ that you can calculate programmatically or via Wilson theorem, according to which $(p-1)! \bmod p = p-1$. To multiply these common parts of all blocks, we can raise the value to the higher power modulo p, which can be done in $O(\log n)$ operations using Binary Exponentiation; however, you may notice that the result will always be either 1 or $p-1$, depending on the parity of the index. The value of the last partial block can be calculated separately in $O(p)$. Leaving only the last elements of the blocks, we can examine that:

$$n!_{\%p} = \underbrace{\ldots \cdot 1} \cdot \underbrace{\ldots \cdot 2} \cdot \ldots \cdot \underbrace{\ldots \cdot (p-1)} \cdot \underbrace{\ldots \cdot 1} \cdot \underbrace{\ldots \cdot 1} \cdot \underbrace{\ldots}$$

And again, by removing the blocks that we already computed, we receive a "modified" factorial but with smaller dimension ($\lfloor n/p \rfloor$ blocks remain). Thus, in the calculation of "modified" the factorial $n!_{\%p}$ we did $O(p)$

operations and are left with the calculation of $(n/p)!_{\%p}$. Revealing this recursive dependence, we obtain that the recursion depth is $O(\log_p n)$, the total asymptotic behavior of the algorithm is thus $O(p \log_p n)$.

Implementation

We don't need recursion because this is a case of tail recursion and thus can be easily implemented using iteration.

```
int factmod(int n, int p) {
    int res = 1;
    while (n > 1) {
        res = (res * ((n/p) % 2 ?  p-1 : 1)) %
        for (int i = 2; i <= n%p; ++i)
            res = (res * i) % p;
        n /= p;
    }
    return res % p;
}
```

This implementation works in $O(p \log_p n)$.