Finding Power of Factorial Divisor

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You are given two numbers n and k. Find the largest power of k x such that $n!$ is divisible by k^x.

Prime k

Let's first consider the case of prime k. The explicit expression for factorial

$$n! = 1 \cdot 2 \cdot 3 \dots (n-1) \cdot n$$

Note that every k-th element of the product is divisible by k, i.e. adds $+1$ to the answer; the number of such elements is $\Bigl\lfloor \frac{n}{k} \Bigr\rfloor$.

Next, every k^2-th element is divisible by k^2, i.e. adds another $+1$ to the answer (the first power of k has already been counted in the previous paragraph). The number of such elements is $\left\lfloor \dfrac{n}{k^2} \right\rfloor$.

And so on, for every i each k^i-th element adds another $+1$ to the answer, and there are $\left\lfloor \dfrac{n}{k^i} \right\rfloor$ such elements.

The final answer is

$$\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n}{k^2} \right\rfloor + \ldots + \left\lfloor \frac{n}{k^i} \right\rfloor + \ldots$$

The sum is of course finite, since only approximately the first $\log_k n$ elements are not zeros. Thus, the runtime of this algorithm is $O(\log_k n)$.

Implementation

```
int fact_pow (int n, int k) {
    int res = 0;
    while (n) {
        n /= k;
        res += n;
    }
    return res;
}
```

Composite k

The same idea can't be applied directly. Instead we can factor k, representing it as $k = k_1^{p_1} \cdot \ldots \cdot k_m^{p_m}$. For each k_i, we find the number of times it is present in $n!$ using the algorithm described above - let's call this value a_i. The answer for composite k will be

$$\min_{i=1 \ldots m} \frac{a_i}{p_i}$$