Discrete Root

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The problem of finding a discrete root is defined as follows. Given a prime n and two integers a and k, find all x for which:

$$x^k \equiv a \pmod n$$

The algorithm

We will solve this problem by reducing it to the discrete logarithm problem.

Let's apply the concept of a primitive root modulo n. Let g be a primitive root modulo n. Note that since n is prime, it must exist, and it can be found in $O(Ans \cdot \log \phi (n) \cdot \log n) = O(Ans \cdot \log^2 n)$ plus time of factoring $\phi (n)$.

We can easily discard the case where $a = 0$. In this case, obviously there is only one answer: $x = 0$.

Since we know that n is a prime and any number between 1 and $n-1$ can be represented as a power of the primitive root, we can represent the discrete root problem as follows:

$$(g^y)^k \equiv a \pmod n$$

where

$$x \equiv g^y \pmod n$$

This, in turn, can be rewritten as

$$(g^k)^y \equiv a \pmod n$$

Now we have one unknown y, which is a discrete logarithm problem. The solution can be found using Shanks' baby-step giant-step algorithm in $O(\sqrt{n} \log n)$ (or we can verify that there are no solutions).

Having found one solution y_0, one of solutions of discrete root problem will be $x_0 = g^{y_0} \pmod n$.

Finding all solutions from one known solution

To solve the given problem in full, we need to find all solutions knowing one of them: $x_0 = g^{y_0} \pmod n$.

Let's recall the fact that a primitive root always has order of $\phi(n)$, i.e. the smallest power of g which gives 1 is

$\phi(n)$. Therefore, if we add the term $\phi(n)$ to the exponential, we still get the same value:

$$x^k \equiv g^{y_0 \cdot k + l \cdot \phi(n)} \equiv a \pmod n \forall l \in Z$$

Hence, all the solutions are of the form:

$$x = g^{y_0 + \frac{l \cdot \phi(n)}{k}} \pmod n \forall l \in Z.$$

where l is chosen such that the fraction must be an integer. For this to be true, the numerator has to be divisible by the least common multiple of $\phi(n)$ and k. Remember that least common multiple of two numbers $lcm(a, b) = \frac{a \cdot b}{gcd(a,b)}$; we'll get

$$x = g^{y_0 + i \frac{\phi(n)}{gcd(k, \phi(n))}} \pmod n \forall i \in Z.$$

This is the final formula for all solutions of the discrete root problem.

Implementation

Here is a full implementation, including procedures for finding the primitive root, discrete log and finding and printing all solutions.

```
int gcd(int a, int b) {
    return a ? gcd(b % a, a) : b;
}

int powmod(int a, int b, int p) {
```

```
    int res = 1;
    while (b > 0) {
        if (b & 1) {
            res = res * a % p;
        }
        a = a * a % p;
        b >>= 1;
    }
    return res;
}

// Finds the primitive root modulo p
int generator(int p) {
    vector<int> fact;
    int phi = p-1, n = phi;
    for (int i = 2; i * i <= n; ++i) {
        if (n % i == 0) {
            fact.push_back(i);
            while (n % i == 0)
                n /= i;
        }
    }
    if (n > 1)
        fact.push_back(n);

    for (int res = 2; res <= p; ++res) {
        bool ok = true;
        for (int factor : fact) {
            if (powmod(res, phi / factor, p) !
                ok = false;
                break;
            }
        }
```

```
        if (ok) return res;
    }
    return -1;
}

// This program finds all numbers x such that
int main() {
    int n, k, a;
    scanf("%d %d %d", &n, &k, &a);
    if (a == 0) {
        puts("1\n0");
        return 0;
    }

    int g = generator(n);

    // Baby-step giant-step discrete logarithm
    int sq = (int) sqrt (n + .0) + 1;
    vector<pair<int, int>> dec(sq);
    for (int i = 1; i <= sq; ++i)
        dec[i-1] = {powmod(g, i * sq * k % (n
    sort(dec.begin(), dec.end());
    int any_ans = -1;
    for (int i = 0; i < sq; ++i) {
        int my = powmod(g, i * k % (n - 1), n)
        auto it = lower_bound(dec.begin(), dec
        if (it != dec.end() && it->first == my
            any_ans = it->second * sq - i;
            break;
        }
    }
    if (any_ans == -1) {
        puts("0");
```

```
        return 0;
    }

    // Print all possible answers
    int delta = (n-1) / gcd(k, n-1);
    vector<int> ans;
    for (int cur = any_ans % delta; cur < n-1;
        ans.push_back(powmod(g, cur, n));
    sort(ans.begin(), ans.end());
    printf("%d\n", ans.size());
    for (int answer : ans)
        printf("%d ", answer);
}
```

Practice problems

- Codeforces - Lunar New Year and a Recursive Sequence