The cases x = 2 to 5 are listed in A001157 – A001160, x = 6 to 24 are listed in A013954 – A013972.

Properties

Formulas at prime powers

For a prime number p,

$$\begin{aligned}\sigma_0(p) &= 2\\ \sigma_0(p^n) &= n+1\\ \sigma_1(p) &= p+1\end{aligned}$$

because by definition, the factors of a prime number are 1 and itself. Also, where $p_n\#$ denotes the primorial,

$$\sigma_0(p_n\#) = 2^n$$

since n prime factors allow a sequence of binary selection (p_i or 1) from n terms for each proper divisor formed.

Clearly, $1 < \sigma_0(n) < n$ and $\sigma(n) > n$ for all $n > 2$.

The divisor function is multiplicative, but not completely multiplicative. The consequence of this is that, if we write

$$n = \prod_{i=1}^{r} p_i^{a_i}$$

where $r = \omega(n)$ is the number of distinct prime factors of n, p_i is the ith prime factor, and a_i is the maximum power of p_i by which n is divisible, then we have

$$\sigma_x(n) = \prod_{i=1}^{r} \frac{p_i^{(a_i+1)x} - 1}{p_i^x - 1}$$

which is equivalent to the useful formula:

$$\sigma_x(n) = \prod_{i=1}^{r} \sum_{j=0}^{a_i} p_i^{jx} = \prod_{i=1}^{r} (1 + p_i^x + p_i^{2x} + \cdots + p_i^{a_i x}).$$

It follows (by setting x = 0) that $d(n)$ is:

$$\sigma_0(n) = \prod_{i=1}^{r} (a_i + 1).$$

For example, if n is 24, there are two prime factors (p_1 is 2; p_2 is 3); noting that 24 is the product of $2^3 \times 3^1$, a_1 is 3 and a_2 is 1. Thus we can calculate $\sigma_0(24)$ as so:

$$\begin{aligned}\sigma_0(24) &= \prod_{i=1}^{2} (a_i + 1)\\ &= (3+1)(1+1) = 4 \cdot 2 = 8.\end{aligned}$$

The eight divisors counted by this formula are 1, 2, 4, 8, 3, 6, 12, and 24.