UNIT 12 PARTIALCORRELATION

Structure

12.1 INTRODUCTION

In Unit 11 of this block, you studied the concept of multiple correlation and multiple correlation coefficient with its properties. In this unit, you will study the partial correlation. To understand the mathematical formulation of partial correlation coefficient, you go through the Unit 10 of this block. You will learn also how to derive the multiple correlation coefficients in terms of total and partial correlation coefficients.

Section 12.2 discusses the concept of partial correlation and derivation of partial correlation coefficient formula. Multiple correlation coefficients in terms of total and partial correlation coefficients are expressed in Section 12.3.

Objectives

After reading this unit, you will be able to

- describe the concept of partial correlation;
- derive the partial correlation coefficient formula; and
- derscribe multiple correlation coefficient in terms of total and partial correlation coefficients.

12.2 COEFFICIENT OF PARTIAL CORRELATION

As we have seen in Unit 11 of this block that multiple correlation studies the joint effect of group of variables on a single variable and multiple correlation coefficient provides a degree of association between a variable and its estimate. Many times correlation between two variables is partly due to the third variable. For example correlation between height and weight is due to age. In such situations, one may be interested to know the relationship between two variables ignoring the effect of third and fourth or more other variables. Partial correlation studies this type of situations.

In fact, partial correlation is the correlation between two variables, after removing the linear effects of other variables on them.

Let us consider the case of three variables X_1, X_2 and X_3. Sometimes the correlation between two variables X_1 and X_2 may be partly due to the correlation of a third variable X_3 with both X_1 and X_2. In this type of situation one may be interested to study the correlation between X_1 and X_2 when the effect of X_3 on each of X_1 and X_2 is eliminated. This correlation is known as partial correlation. The correlation coefficient between X_1 and X_2 after eliminating the linear effect of X_3 on X_1 and X_2 is called the partial correlation coefficient.

If we consider the regression equation of X_1 on X_3 i.e.

$$X_1 = a + b_{13}X_3$$

and suppose three variables x_1, x_2 and x_3 are measured from their respective means i.e.

$$X_1 - \overline{X}_1 = x_1, X_2 - \overline{X}_2 = x_2 \text{ and } X_3 - \overline{X}_3 = x_3$$

then the regression equation of x_1 on x_3 is given by $x_1 = b_{13}x_3$.

The residual $e_{1.3}$ for x_1 can be expressed as

$$e_{1.3} = x_1 - b_{13}x_3 \qquad \ldots (1)$$

Equation (1) may be considered as a part of the dependent variable x_1 which remains when the linear effect of x_3 on x_1 is eliminated.

Similarly, the regression equation x_2 on x_3 i.e. $x_2 = b_{23}x_3$ then the residual $e_{2.3}$ is expressed as

$$e_{2.3} = x_2 - b_{23}x_3 \qquad \ldots (2)$$

which may be considered as a part of the dependent variable x_2, which remains when the linear effect of x_3 on x_2 is eliminated. Thus, the correlation between $e_{1.3}$ and $e_{2.3}$ is considered as the partial correlation coefficient.

12.2.1 Derivation of Partial Correlation Coefficient Formula

Partial correlation coefficient is the correlation coefficient between two variables after removing the linear effect of other variables on them.

If there are three variables x_1, x_2 and x_3 then partial correlation coefficient between x_1 and x_2 is denoted by $r_{12.3}$ and defined by

$$r_{12.3} = \frac{Cov(e_{1.3}, e_{2.3})}{\sqrt{V(e_{1.3})V(e_{2.3})}} \qquad \ldots (3)$$

We know that

$$Cov(e_{1.3}, e_{2.3}) = \frac{1}{N}\sum(e_{1.3} - \overline{e}_{1.3})(e_{2.3} - \overline{e}_{2.3})$$

Since x_1, x_2 and x_3 are measured from their respective means so

$\sum x_1 = \sum x_2 = \sum x_3 = 0 \Rightarrow \bar{e}_{1.3} = 0 = \bar{e}_{2.3}$ (See equations (1) and (2))

So,

$$\begin{aligned}
Cov(e_{1.3}, e_{2.3}) &= \frac{1}{N}\sum e_{1.3}e_{2.3} \\
&= \frac{1}{N}\sum (x_1 - b_{13}x_3)(x_2 - b_{23}x_3) \\
&\qquad \text{From equations (1) and (2)} \\
&= \frac{1}{N}\sum x_1x_2 - b_{13}\frac{1}{N}\sum x_2x_3 \\
&\qquad - b_{23}\frac{1}{N}\sum x_1x_3 + b_{13}b_{23}\frac{1}{N}\sum x_3^2 \\
Cov(e_{1.3}, e_{2.3}) &= r_{12}\sigma_1\sigma_2 - b_{13}r_{23}\sigma_2\sigma_3 - b_{23}r_{13}\sigma_1\sigma_3 + b_{13}b_{23}\sigma_3^2
\end{aligned}$$

Explanatory Note:

$Cov(x_1, x_2) = \frac{1}{N}\sum (x_1 - \bar{x}_1)(x_2 - \bar{x}_2)$

Since x_1, x_2 and x_3 are measured from their respective means so

$\sum x_1 = \sum x_2 = 0 \Rightarrow \bar{x}_1 = \bar{x}_2 = 0$

$Cov(x_1, x_2) = \frac{1}{N}\sum x_1x_2$

So

$\Rightarrow \frac{1}{N}\sum x_1x_2 = Cov(x_1, x_2) = r_{12}\sigma_1\sigma_2$

Similarly $\Rightarrow \frac{1}{N}\sum x_2x_3 = Cov(x_2, x_3) = r_{23}\sigma_2\sigma_3$

We know that $\sigma_3^2 = \frac{1}{N}\sum (x_3 - \bar{x}_3)^2 = \frac{1}{N}\sum x_3^2$. Similarly, other expressions can be obtained.

We know that the simple regression coefficient of x_1 on x_3 is $b_{13} = r_{13}\frac{\sigma_1}{\sigma_3}$

similarly, the regression coefficient of x_2 on x_3 is

$$b_{23} = r_{23}\frac{\sigma_2}{\sigma_3}$$

So

$$Cov(e_{1.3}, e_{2.3}) = r_{12}\sigma_1\sigma_2 - r_{13}\frac{\sigma_1}{\sigma_3}r_{23}\sigma_2\sigma_3 - r_{23}\frac{\sigma_2}{\sigma_3}r_{13}\sigma_1\sigma_3 + r_{13}\frac{\sigma_1}{\sigma_3}r_{23}\frac{\sigma_2}{\sigma_3}\sigma_3^2$$

$$Cov(e_{1.3}, e_{2.3}) = \sigma_1\sigma_2(r_{12} - r_{13}r_{23})$$

and

$$V(e_{1.3}) = \frac{1}{N}\sum (e_{1.3} - \bar{e}_{1.3})^2$$

$$V(e_{1.3}) = \frac{1}{N}\sum e_{1.3}^2 \qquad [\because \bar{e}_{1.3} = 0]$$

$$= \frac{1}{N}\sum e_{1.3}e_{1.3}$$

$$= \frac{1}{N}\sum x_1 e_{1.3} \quad \text{(By the third property of residuals)}$$

$$= \frac{1}{N}\sum x_1(x_1 - b_{13}x_3) \qquad \text{From equation (1)}$$

$$= \frac{1}{N}\sum x_1^2 - b_{13}\frac{1}{N}\sum x_1 x_3$$

$$= \sigma_1^2 - b_{13}r_{13}\sigma_1\sigma_3$$

$$= \sigma_1^2 - r_{13}\frac{\sigma_1}{\sigma_3}r_{13}\sigma_1\sigma_3$$

Since, $\quad b_{13} = r_{13}\frac{\sigma_1}{\sigma_3} = \sigma_1^2 - r_{13}^2\sigma_1^2$

$$V(e_{1.3}) = \sigma_1^2(1 - r_{13}^2)$$

Similarly $\quad V(e_{2.3}) = \sigma_2^2(1 - r_{23}^2) \qquad \ldots (4)$

Substituting the value of $Cov(e_{1.3}, e_{2.3}), V(e_{1.3})$ and $V(e_{2.3})$ in equation (3), we have

$$r_{12.3} = \frac{\sigma_1\sigma_2(r_{12} - r_{13}r_{23})}{\sqrt{\sigma_1^2(1 - r_{13}^2)\sigma_2^2(1 - r_{23}^2)}}$$

$$r_{12.3} = \frac{(r_{12} - r_{13}r_{23})}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}} \qquad \ldots (5)$$

Similarly, expression for $r_{13.2}$ may be obtained as

$$r_{13.2} = \frac{r_{13} - r_{12}r_{32}}{\sqrt{(1 - r_{12}^2)(1 - r_{32}^2)}} \qquad \ldots (6)$$

and

$$r_{23.1} = \frac{r_{23} - r_{21}r_{31}}{\sqrt{(1 - r_{21}^2)(1 - r_{31}^2)}}. \qquad \ldots (7)$$

If $r_{12.3} = 0$, i.e. partial correlation coefficient is zero but $r_{12.3} = 0 \Rightarrow r_{12} = r_{13}r_{23}$ it means correlation coefficient between X_1 and X_2 is not zero if X_3 is correlated with X_1 and X_2.

12.3 MULTIPLE CORRELATION COEFFICIENT IN TERMS OF TOTAL AND PARTIAL CORRELATION COEFFCIENTS

If three variables X_1, X_2 and X_3 are considered then multiple correlation coefficient between X_1 and joint effect of X_2 and X_3 on X_1 is

$$R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}} \qquad \ldots (8)$$

$$R^2_{1.23} = \frac{r^2_{12} + r^2_{13} - 2r_{12}r_{13}r_{23}}{1 - r^2_{23}} \qquad \ldots (9)$$

$$\Rightarrow 1 - R^2_{1.23} = 1 - \frac{r^2_{12} + r^2_{13} - 2r_{12}r_{13}r_{23}}{1 - r^2_{23}}$$

$$\Rightarrow 1 - R^2_{1.23} = \frac{1 - r^2_{23} - r^2_{12} - r^2_{13} + 2r_{12}r_{13}r_{23}}{1 - r^2_{23}} \qquad \ldots (10)$$

We know that the partial correlation coefficient between x_1 and x_3 when the effect of x_2 on each of x_1 and x_3 are eliminated is

$$r_{13.2} = \frac{(r_{13} - r_{12}r_{23})}{\sqrt{(1 - r^2_{12})(1 - r^2_{23})}} \qquad \ldots (11)$$

Squaring equation (11), we get

$$r^2_{13.2} = \frac{(r_{13} - r_{12}r_{23})^2}{(1 - r^2_{12})(1 - r^2_{23})}$$

$$\Rightarrow 1 - r^2_{13.2} = 1 - \frac{(r_{13} - r_{12}r_{23})^2}{(1 - r^2_{12})(1 - r^2_{23})}$$

$$\Rightarrow 1 - r^2_{13.2} = \frac{(1 - r^2_{12})(1 - r^2_{23}) - (r_{13} - r_{12}r_{23})^2}{(1 - r^2_{12})(1 - r^2_{23})}$$

$$\Rightarrow 1 - r^2_{13.2} = \frac{1 - r^2_{23} - r^2_{12} + r^2_{23}r^2_{12} - r^2_{13} - r^2_{23}r^2_{12} + 2r_{12}r_{13}r_{23}}{(1 - r^2_{12})(1 - r^2_{23})}$$

$$\Rightarrow 1 - r^2_{13.2} = \frac{1 - r^2_{23} - r^2_{12} - r^2_{13} + 2r_{12}r_{13}r_{23}}{(1 - r^2_{12})(1 - r^2_{23})}$$

$$\Rightarrow (1 - r^2_{13.2})(1 - r^2_{12}) = \frac{1 - r^2_{23} - r^2_{12} - r^2_{13} + 2r_{12}r_{13}r_{23}}{(1 - r^2_{23})} \qquad \ldots (12)$$

From equations (9) and (12)

$$\Rightarrow (1 - r^2_{13.2})(1 - r^2_{12}) = 1 - R^2_{1.23}$$

$$\Rightarrow (1 - r^2_{13.2})(1 - r^2_{12}) = 1 - R^2_{1.23}$$

$$\Rightarrow R_{1.23} = \sqrt{1 - (1 - r^2_{12})(1 - r^2_{13.2})}$$

It is the required formula and similarly, we may obtain

$$R_{2.13} = \sqrt{1 - (1 - r^2_{12})(1 - r^2_{23.1})}$$

and $$R_{3.12} = \sqrt{1 - (1 - r^2_{13})(1 - r^2_{32.1})}$$

Let us solve some problem on partial correlation coefficient.

Example 1: If $r_{12} = 0.60$, $r_{13} = 0.50$ and $r_{23} = 0.45$ then calculate $r_{12.3}$, $r_{13.2}$ and $r_{23.1}$.

Solution: We have

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}$$

$$= \frac{0.60 - (0.50)(0.45)}{\sqrt{\{1-(0.50)^2\}\{1-(0.45)^2\}}}$$

$$= \frac{0.60 - 0.23}{\sqrt{\{1-0.25\}\{1-0.20\}}}$$

$$= \frac{0.37}{\sqrt{0.75 \times 0.80}}$$

$$= \frac{0.37}{\sqrt{0.60}}$$

$$r_{12.3} = 0.48$$

$$r_{13.2} = \frac{r_{13} - r_{12}r_{32}}{\sqrt{(1-r_{12}^2)(1-r_{23}^2)}}$$

$$= \frac{0.50 - (0.60) \times (0.45)}{\sqrt{\{1-(0.60)^2\}\{1-(0.45)^2\}}}$$

$$= \frac{0.50 - 0.27}{\sqrt{\{1-0.36\{1-0.20\}}}$$

$$= \frac{0.23}{\sqrt{0.64 \times 0..80}}$$

$$= \frac{0.23}{\sqrt{0..51`}}$$

$$= \frac{0.23}{0.71}$$

$$r_{13.2} = 0.32$$

Now,

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1-r_{12}^2)(1-r_{13}^2)}}$$

$$= \frac{0.45 - (0.60)(0.50)}{\sqrt{\{1-(0.60)^2\}\{1-(0.50)^2\}}}$$

$$= \frac{0.45 - 0.30}{\sqrt{\{1-0.36\}\{1-0.25\}}}$$

$$= \frac{0.15}{\sqrt{0.64 \times 0.75}}$$

$$= \frac{0.15}{\sqrt{0.48}}$$

$$= \frac{0.15}{0.69}$$

$$r_{23.1} = 0.22$$

Example 2: From the following data, obtain $r_{12.3}$, $r_{13.2}$ and $r_{23.1}$.

X_1	20	15	25	26	28	40	38
X_2	12	13	16	15	23	15	28
X_3	13	15	12	16	14	18	14

Solution: To obtain partial correlation coefficients $r_{12.3}$, $r_{13.2}$ and $r_{23.1}$ we use following formulae:

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}},$$

$$r_{13.2} = \frac{r_{13} - r_{12}r_{32}}{\sqrt{(1-r_{12}^2)(1-r_{23}^2)}} \quad \text{and}$$

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1-r_{12}^2)(1-r_{13}^2)}}$$

We need r_{12} , r_{13} and r_{23} which are obtained from the following table:

S. No.	X_1	X_2	X_3	$(X_1)^2$	$(X_2)^2$	$(X_3)^2$	X_1X_2	X_1X_3	X_2X_3
1	20	12	13	400	144	169	240	260	156
2	15	13	15	225	169	225	195	225	195
3	25	16	12	625	256	144	400	300	192
4	26	15	16	676	225	256	390	416	240
5	28	23	14	784	529	196	644	392	322
6	40	15	28	1600	225	784	600	1120	420
7	38	28	14	1444	784	196	1064	532	392
Total	192	122	112	5754	2332	1970	3533	3245	1917

Now, we get the total correlation coefficient r_{12} , r_{13} and r_{23}

$$r_{12} = \frac{N(\sum X_1X_2) - (\sum X_1)(\sum X_2)}{\sqrt{\{N(\sum X_1^2) - (\sum X_1)^2\}\{N(\sum X_2^2) - (\sum X_2)^2\}}}$$

$$r_{12} = \frac{(7\times 3533) - (192)\times(122)}{\sqrt{\{(7\times 5754) - (192)\times(192)\}\{(7\times 2332) - (122)\times(122)\}}}$$

$$r_{12} = \frac{1307}{\sqrt{\{3414\}\{1440\}}} = \frac{1307}{2217.24} = 0.59$$

$$r_{13} = \frac{N(\sum X_1X_3) - (\sum X_1)(\sum X_3)}{\sqrt{\{N(\sum X_1^2) - (\sum X_1)^2\}\{N(\sum X_3^2) - (\sum X_3)^2\}}}$$

$$r_{13} = \frac{(7\times 3245) - (192)\times(112)}{\sqrt{\{(7\times 5754) - (192\times 192)\}\{(7\times 1970) - (112\times 112)\}}}$$

$$r_{13} = \frac{1211}{\sqrt{\{3414\}\{1246\}}} = \frac{1211}{2062\cdot 48} = 0.59$$

and

$$r_{23} = \frac{N(\sum X_2X_3) - (\sum X_2)(\sum X_3)}{\sqrt{\{N(\sum X_2^2) - (\sum X_2)^2\}\{N(\sum X_3^2) - (\sum X_3)^2\}}}$$

$$r_{23} = \frac{(7\times 1917) - (122)\times(112)}{\sqrt{\{7\times 2332) - (122\times 122)\}\{(7\times 1970) - (112\times 112)\}}}$$

$$r_{23} = \frac{-245}{\sqrt{1440\times 1246}} = \frac{-245}{1339.50} = -0.18$$

Now, we calculate $r_{12.3}$

We have, $r_{12} = 0.59$, $r_{13} = 0.59$ and $r_{23} = -0.18$, then

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}$$

$$= \frac{0.59 - (0.59)(-0.18)}{\sqrt{\{1-(0.59)^2\}\{1-(-0.18)^2\}}}$$

$$= \frac{0.6962}{\sqrt{0.6519\times 0.9676}}$$

$$= \frac{0.6962}{0.7942}$$

$$r_{12.3} = 0.88$$

$$r_{13.2} = \frac{r_{13} - r_{12}r_{32}}{\sqrt{(1-r_{12}^2)(1-r_{23}^2)}}$$

$$= \frac{0.59 - (0.59) \times (-0.18)}{\sqrt{\{1 - (.59)^2\}\{1 - (-0.18^2\}}}$$

$$= \frac{0.6962}{\sqrt{0.6519 \times 0.9676}}$$

$$= \frac{0.6962}{0.7942}$$

Thus,

$$r_{13.2} = 0.88$$

$$r_{23.1} = \frac{r_{23} - r_{12} r_{13}}{\sqrt{(1 - r_{12}^2)(1 - r_{13}^2)}}$$

$$= \frac{-0.18 - (0.59)(0.59)}{\sqrt{\{1 - (0.59)^2\}\{1 - (0.59)^2\}}}$$

$$= \frac{-0.5281}{\sqrt{0.6519 \times 0.6519}}$$

$$= \frac{-0.5281}{0.6519}$$

$$r_{23.1} = -0.81$$

Example 3: From the following data, obtain $r_{12.3}$, $r_{13.2}$ and $r_{23.1}$:

X_1	40	44	42	45	40	45	40	40	42	41
X_2	18	20	26	24	20	25	23	19	18	16
X_3	52	51	50	48	47	52	50	51	49	50

Solution: To obtain partial correlation coefficients $r_{12.3}$, $r_{13.2}$ and $r_{23.1}$ we use following formulae

$$r_{12.3} = \frac{r_{12} - r_{13} r_{23}}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}},$$

$$r_{13.2} = \frac{r_{13} - r_{12} r_{32}}{\sqrt{(1 - r_{12}^2)(1 - r_{23}^2)}} \quad \text{and}$$

$$r_{23.1} = \frac{r_{23} - r_{12} r_{13}}{\sqrt{(1 - r_{12}^2)(1 - r_{13}^2)}}$$

We need r_{12} , r_{13} and r_{23} which are obtained from the following table:

Here we are using shortcut method to find correlation coefficient.

Let $d_1 = X_1 - 60$, $d_2 = X_2 - 50$ and $d_3 = X_3 - 70$

S. No.	X_1	X_2	X_3	$d_1 = X_1 - 60$	$d_2 = X_2 - 50$	$d_3 = X_3 - 70$	$(d_1)^2$	$(d_2)^2$	$(d_3)^2$	d_1d_2	d_1d_3	d_2d_3
1	40	18	52	0	-2	2	0	4	4	0	0	-4
2	44	20	51	4	0	1	16	0	1	0	4	0
3	42	26	50	2	6	0	4	36	0	12	0	0
4	45	24	48	5	4	-2	25	16	4	20	-10	-8
5	40	20	47	0	0	-3	0	0	9	0	0	0
6	45	25	52	5	5	2	25	25	4	25	10	10
7	40	23	50	0	3	0	0	9	0	0	0	0
8	40	19	51	0	-1	1	0	1	1	0	0	-1
9	42	18	49	2	-2	-1	4	4	1	-4	-2	2
10	41	16	50	1	-4	0	1	16	0	-4	0	0
Total				19	9	0	75	111	24	49	2	-1

Now we get the total correlation coefficient r_{12} , r_{13} and r_{23}

$$r_{12} = \frac{N(\sum d_1 d_2) - (\sum d_1)(\sum d_2)}{\sqrt{\{N(\sum d_1^2) - (\sum d_1)^2\}\{N(\sum d_2^2) - (\sum d_2)^2\}}}$$

$$r_{12} = \frac{(10 \times 49) - (19) \times (9)}{\sqrt{\{(10 \times 75) - (19) \times (19)\}\{(10 \times 111) - (9) \times (9)\}}}$$

$$r_{12} = \frac{319}{\sqrt{\{389\}\{1029\}}} = \frac{319}{632.68} = 0.50$$

$$r_{13} = \frac{N(\sum d_1 d_3) - (\sum d_1)(\sum d_3)}{\sqrt{\{N(\sum d_1^2) - (\sum d_1)^2\}\{N(\sum d_3^2) - (\sum d_3)^2\}}}$$

$$r_{13} = \frac{(10 \times 2) - (19) \times (0)}{\sqrt{\{(10 \times 75) - (19 \times 19)\}\{(10 \times 24) - (0 \times 0)\}}}$$

$$r_{13} = \frac{20}{\sqrt{\{389\}\{240\}}} = \frac{20}{305.55} = 0.07$$

and

$$r_{23} = \frac{N(\sum d_2 d_3) - (\sum d_2)(\sum d_3)}{\sqrt{\{N(\sum d_2^2) - (\sum d_2)^2\}\{N(\sum d_3^2) - (\sum d_3)^2\}}}$$

$$r_{23} = \frac{(10 \times -1) - (9) \times (0)}{\sqrt{\{(10 \times 111) - (9 \times 9)\}\{(10 \times 24) - (0 \times 0)\}}}$$

$$r_{23} = \frac{-10}{\sqrt{\{1029\}\{240\}}} = \frac{-10}{496.95} = -\ 0.02$$

Now, we calculate $r_{12.3}$

We have, $r_{12} = 0.50$, $r_{13} = 0.07$ and $r_{23} = -0.02$, then

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}}$$

$$= \frac{0.50 - (0.07)(-0.02)}{\sqrt{\{1 - (0.07)^2\}\{1 - (-0.02)^2\}}}$$

$$= \frac{0.5014}{\sqrt{0.9951 \times 0.9996}}$$

$$= \frac{0.5014}{0.9933}$$

$$r_{12.3} = 0.50$$

$$r_{13.2} = \frac{r_{13} - r_{12}r_{32}}{\sqrt{(1 - r_{12}^2)(1 - r_{23}^2)}}$$

$$= \frac{0.20 - (0.50) \times (-0.02)}{\sqrt{\{1 - (0.50)^2\}\{1 - (-0.02)^2\}}}$$

$$= \frac{0.0800}{\sqrt{0.7500 \times 0.9996}}$$

$$= \frac{0.0800}{0.8659}$$

Now, $r_{13.2} = 0.09$

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1 - r_{12}^2)(1 - r_{13}^2)}}$$

$$= \frac{-0.02 - (0.50)(0.07)}{\sqrt{\{1 - (0.50)^2\}\{1 - (0.07)^2\}}}$$

$$= \frac{-0.0550}{\sqrt{0.7500 \times 0.9951}}$$

$$= \frac{-0.0550}{0.8639}$$

$$r_{23.1} = -0.06$$

Now let us solve some exercises

E1) If $r_{12} = 0.87$, $r_{13} = 0.82$ and $r_{23} = 0.62$, compute partial correlation coefficient $r_{12.3}$.

E2) In trivariate distribution $r_{12} = 0.8$, $r_{23} = 0.6$, $r_{13} = 0.6$

Compute (a) $r_{12.3}$ (b) $r_{23.1}$

E3) From the following data, obtain $r_{12.3}$, $r_{13.2}$ and $r_{23.1}$.

X_1	12	7	9	15	14	18	18
X_2	10	7	16	15	8	12	10
X_3	7	9	4	8	10	12	8

E4) From the following data, obtain $r_{12.3}$, $r_{13.2}$ and $r_{23.1}$ by using shortcut method

X_1	200	204	202	205	199	200	198	200	202	201
X_2	180	185	179	180	175	184	180	181	178	181
X_3	152	150	149	148	152	150	150	148	153	150

12.4 SUMMARY

In this unit, we have discussed:

1. The correlation coefficient between X_1 and X_2 after eliminating the linear effect of X_3 on X_1 and X_2 is called the partial correlation coefficient,
2. How to derive the formula of partial correlation coefficient, and
3. Multiple correlation coefficient can be expressed in terms of total and partial correlation coefficients as $R_{1.23} = \sqrt{1-(1-r_{12}^2)(1-r_{13.2}^2)}$.

12.5 SOLUTIONS /ANSWERS

E1) We have

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}$$

$$= \frac{0.87-(0.82)(0.62)}{\sqrt{\{1-(0.82)^2\}\{1-(0.62)^2\}}}$$

$$= \frac{0.87-0.51}{\sqrt{\{1-0.67\}\{1-0.38\}}}$$

$$= \frac{0.36}{\sqrt{0.33\times 0.62}} = \frac{0.36}{\sqrt{0.20}} = \frac{0.36}{0.45} = 0.80$$

E2) We have

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}$$

$$= \frac{0.8-(0.6)(0.6)}{\sqrt{\{1-(0.6)^2\}\{1-(0.6)^2\}}}$$

$$= \frac{0.44}{\sqrt{0.64 \times 0.64}} = \frac{0.44}{0.64} = 0.69$$

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1-r_{12}^2)(1-r_{13}^2)}}$$

$$= \frac{0.6-(0.8)(0.6)}{\sqrt{\{1-(0.8)^2\}\{1-(0.6)^2\}}}$$

$$= \frac{0.6-0.48}{\sqrt{0.36 \times 0.64}} = \frac{0.12}{\sqrt{0.23}} = \frac{0.12}{0.48} = 0.25$$

E3) To obtain partial correlation coefficients $r_{12.3}$, $r_{13.2}$ and $r_{23.1}$ we use following formulae:

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}},$$

$$r_{13.2} = \frac{r_{13} - r_{12}r_{32}}{\sqrt{(1-r_{12}^2)(1-r_{23}^2)}} \quad \text{and}$$

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1-r_{12}^2)(1-r_{13}^2)}}$$

We need r_{12} , r_{13} and r_{23} which are obtained from the following table:

S. No.	X_1	X_2	X_3	$(X_1)^2$	$(X_2)^2$	$(X_3)^2$	X_1X_2	X_1X_3	X_2X_3
1	12	10	7	144	100	49	120	84	70
2	7	7	9	49	49	81	49	63	63
3	9	16	4	81	256	16	144	36	64
4	15	15	8	225	225	64	225	120	120
5	14	8	10	196	64	100	112	140	80
6	18	12	12	324	144	144	216	216	144
7	18	10	8	324	100	64	180	144	80
Total	93	78	58	1343	938	518	1046	803	621

Now we get the total correlation coefficient r_{12} , r_{13} and r_{23}

$$r_{12} = \frac{N(\sum X_1X_2) - (\sum X_1)(\sum X_2)}{\sqrt{\left\{N(\sum X_1^2) - (\sum X_1)^2\right\}\left\{N(\sum X_2^2) - (\sum X_2)^2\right\}}}$$

$$r_{12} = \frac{(7 \times 1046) - (93) \times (78)}{\sqrt{\{(7 \times 1343) - (93) \times (93)\}\{(7 \times 938) - (78) \times (78)\}}}$$

$$r_{12} = \frac{68}{\sqrt{\{752\}\{482\}}} = \frac{68}{602.05} = 0.11$$

$$r_{13} = \frac{N(\sum X_1X_3) - (\sum X_1)(\sum X_3)}{\sqrt{\left\{N(\sum X_1^2) - (\sum X_1)^2\right\}\left\{N(\sum X_3^2) - (\sum X_3)^2\right\}}}$$

$$r_{13} = \frac{(7 \times 803) - (93) \times (58)}{\sqrt{\{(7 \times 1343) - (93 \times 93)\}\{(7 \times 518) - (58 \times 58)\}}}$$

$$r_{13} = \frac{227}{\sqrt{\{752\}\{262\}}} = \frac{227}{443.87} = 0.51$$

and

$$r_{23} = \frac{N(\sum X_2X_3) - (\sum X_2)(\sum X_3)}{\sqrt{\left\{N(\sum X_2^2) - (\sum X_2)^2\right\}\left\{N(\sum X_3^2) - (\sum X_3)^2\right\}}}$$

$$r_{23} = \frac{(7 \times 621) - (78) \times (58)}{\sqrt{\{7 \times 938) - (78 \times 78)\}\{(7 \times 518) - (58 \times 58)\}}}$$

$$r_{23} = \frac{-177}{\sqrt{\{482\}\{262\}}} = \frac{-177}{355.36} = -0.50$$

Now, we calculate $r_{12.3}$

We have, $r_{12} = 0.11$, $r_{13} = 0.51$ and $r_{23} = -0.50$, then

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}}$$

$$= \frac{0.11 - (0.51)(-0.50)}{\sqrt{\{1 - (0.51)^2\}\{1 - (-0.50)^2\}}}$$

$$= \frac{0.3650}{\sqrt{0.7399 \times 0.7500}} = \frac{0.3650}{0.7449}$$

$$r_{12.3} = 0.49$$

$$r_{13.2} = \frac{r_{13} - r_{12}r_{32}}{\sqrt{(1 - r_{12}^2)(1 - r_{23}^2)}}$$

$$= \frac{0.51 - (0.11) \times (-0.50)}{\sqrt{\{1 - (0.11)^2\}\{1 - (-0.50)^2\}}}$$

$$= \frac{0.5650}{\sqrt{0.9879 \times 0.7500}} = \frac{0.5650}{0.8608}$$

$$r_{13.2} = 0.66$$

Now,

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1 - r_{12}^2)(1 - r_{13}^2)}}$$

$$= \frac{-0.50 - (0.11)(0.51)}{\sqrt{\{1 - (0.11)^2\}\{1 - (0.51)^2\}}}$$

$$= \frac{-0.5561}{\sqrt{0.9879 \times 0.7399}} = \frac{-0.5561}{0.8550}$$

$$r_{23.1} = -0.65$$

E4) To obtain partial correlation coefficients $r_{12.3}$, $r_{13.2}$ and $r_{23.1}$ we use following formulae

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}},$$

$$r_{13.2} = \frac{r_{13} - r_{12}r_{32}}{\sqrt{(1 - r_{12}^2)(1 - r_{23}^2)}} \quad \text{and}$$

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1 - r_{12}^2)(1 - r_{13}^2)}}$$

We need r_{12}, r_{13} and r_{23} which are obtained from the following table:

S. No.	X_1	X_2	X_3	$d_1 = X_1 - 60$	$d_2 = X_2 - 50$	$d_3 = X_3 - 70$	$(d_1)^2$	$(d_2)^2$	$(d_3)^2$	d_1d_2	d_1d_3	d_2d_3
1	201	180	152	1	0	2	1	0	4	0	2	0
2	204	185	150	4	5	0	16	25	0	20	0	0
3	202	179	149	2	-1	-1	4	1	1	-2	-2	1
4	205	180	148	5	0	-2	25	0	4	0	-10	0
5	199	175	152	-1	-5	2	1	25	4	5	-2	-10
6	200	184	150	0	4	0	0	16	0	0	0	0
7	198	180	150	-2	0	0	4	0	0	0	0	0
8	200	181	148	0	1	-2	0	1	4	0	0	-2
9	202	178	153	2	-2	3	4	4	9	-4	6	-6
10	201	181	150	1	1	0	1	1	0	1	0	0
Total				12	3	2	56	73	26	20	-6	-17

Now we get the total correlation coefficient r_{12}, r_{13} and r_{23}

$$r_{12}=\frac{N(\sum d_1d_2)-(\sum d_1)(\sum d_2)}{\sqrt{\{N(\sum d_1^2)-(\sum d_1)^2\}\{N(\sum d_2^2)-(\sum d_2)^2\}}}$$

$$r_{12}=\frac{(10\times 20)-(12)\times(3)}{\sqrt{\{(10\times 56)-(12)\times(12)\}\{(10\times 73)-(3)\times(3)\}}}$$

$$r_{12}=\frac{164}{\sqrt{\{416\}\{721\}}}=\frac{164}{547.66}=0.30$$

$$r_{13}=\frac{N(\sum d_1d_3)-(\sum d_1)(\sum d_3)}{\sqrt{\{N(\sum d_1^2)-(\sum d_1)^2\}\{N(\sum d_3^2)-(\sum d_3)^2\}}}$$

$$r_{13}=\frac{(10\times -6)-(12)\times(3)}{\sqrt{\{(10\times 56)-(12\times 12)\}\{(10\times 26)-(3\times 3)\}}}$$

$$r_{13}=\frac{-84}{\sqrt{\{416\}\{256\}}}=\frac{-84}{326.34}=-0.26$$

and

$$r_{23}=\frac{N(\sum d_2d_3)-(\sum d_2)(\sum d_3)}{\sqrt{\{N(\sum d_2^2)-(\sum d_2)^2\}\{N(\sum d_3^2)-(\sum d_3)^2\}}}$$

$$r_{23}=\frac{(10\times -17)-(3)\times(2)}{\sqrt{\{(10\times 73)-(3\times 3)\}\{(10\times 26)-(2\times 2)\}}}$$

$$r_{23}=\frac{-176}{\sqrt{\{721\}\{256\}}}=\frac{-176}{429.62}=-0.41$$

Now, we calculate $r_{12.3}$

We have, $r_{12}=0.30$, $r_{13}=-0.26$ and $r_{23}=-0.41$, then

$$r_{12.3}=\frac{r_{12}-r_{13}r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}$$

$$=\frac{0.30-(-0.26)(-0.41)}{\sqrt{\{1-(-0.26)^2\}\{1-(-0.41)^2\}}}$$

$$=\frac{0.1934}{\sqrt{0.9324\times 0.8319}}=\frac{0.1934}{0.8807}$$

$$r_{12.3}=0.22$$

$$r_{13.2}=\frac{r_{13}-r_{12}r_{32}}{\sqrt{(1-r_{12}^2)(1-r_{23}^2)}}$$

$$=\frac{-0.26-(0.30)\times(-0.41)}{\sqrt{\{1-(0.30)^2\}\{1-(-0.41)^2\}}}$$

$$= \frac{-0.1370}{\sqrt{0.9100 \times 0.8300}} = \frac{-0.1370}{0.8701}$$

$$r_{13.2} = 0.16$$

Now,

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{(1-r_{12}^2)(1-r_{13}^2)}}$$

$$= \frac{-0.41 - (0.30)(-0.26)}{\sqrt{\{1-(0.30)^2\}\{1-(-0.26)^2\}}}$$

$$= \frac{-0.3320}{\sqrt{0.9100 \times 0.9324}} = \frac{-0.3320}{0.9211}$$

$$r_{23.1} = -0.36$$