Assignment 5

PUNDI BINDUSREE
CS21BTECH11048
Papoulis Chapter6 Ex-6.25

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Outline

Question

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Let **x** be the lifetime of a certain electric bulb, and y that of its replacement after the failure of the first bulb. Suppose **x** and **y** are independent with common exponential density function with parameter λ.Find the probability that the combined lifetime exceeds 2λ. What is the probability that the replacement outlasts the original component by λ?

solution

Given

- the lifetime of a certain electric bulb $= \mathbf{x}$
- and lifetime of its replacement after the failure of the first bulb $= \mathbf{y}$
- and also given that $\mathbf{x}$ and $\mathbf{y}$ are independent with common exponential density function with parameter λ .

solution

- pdf of an exponential distribution of X with parameter $\lambda = \frac{1}{\lambda}e^{-\frac{x}{\lambda}}u(x)$
- pdf of an exponential distribution of Y with parameter $\lambda = \frac{1}{\lambda}e^{-\frac{y}{\lambda}}u(y)$

solution

Given that X and Y are independent therefore

1. $f_{XY}(x,y) = f_X(x).f_Y(y)$
2. Let Z=X+Y

solution

3.

$$\text{then } f_Z(z) = \int_{x=0}^{z} f_{XY}(x,y)dx.dyz > 0 \quad (1)$$

$$= \int_{x=0}^{z} f_X(x)f_Y(y)dxdy \quad (2)$$

$$= \int_{x=0}^{z} f_X(x)f_Y(z-x)dx \quad (3)$$

solution

$$= \int_{x=0}^{z} (\frac{1}{\lambda})^2 e^{-\frac{x}{\lambda}} e^{-\frac{z-x}{\lambda}} dx \tag{4}$$

$$= \frac{1}{\lambda}^2 \int_{x=0}^{z} e^{-\frac{z}{\lambda}} \int_{x=0}^{z} dx \tag{5}$$

$$= \frac{z}{\lambda^2} e^{-\frac{z}{\lambda}} \tag{6}$$

solution

Required to find the probability that the combined lifetime exceeds 2λ

4. so here $Z > 2\lambda$

$$P(Z > 2\lambda) = \int_{2\lambda}^{\infty} \frac{z}{\lambda^2} e^{-\frac{z}{\lambda}} dz \tag{7}$$
$$= \int_{2}^{\infty} a e^{-a} da \tag{8}$$
$$= 3e^{-2} \tag{9}$$

solution

- so the required probability is $3e^{-2} = 0.407$ (approximately by rounding off)

solution

5. Now need to find the probability that the replacement outlasts the original component by λ
   - Let W=Y-X
   - then Y=W+X

solution

1. similarly from above process we get the required probability as

   -
$$f_W(w) = \int_{x=0}^{z} f_X(x) f_Y(w+2x)dx \tag{10}$$
$$= \int_{x=0}^{z} (\frac{1}{\lambda})^2 e^{-\frac{x}{\lambda}} e^{-\frac{w+2x}{\lambda}} dx \tag{11}$$
$$= \frac{1}{2\lambda} e^{-\frac{w}{\lambda}}, w > 0 \tag{12}$$

solution

6. now here $w > \lambda$

$$F_W(w) = \int_{\lambda}^{\infty} \frac{1}{2\lambda} e^{-\frac{w}{\lambda}} dw \tag{13}$$
$$= \frac{1}{2e} \tag{14}$$

solution

- Therefore the required probability that replacement outlasts the original component by λ is $= \frac{1}{2e}$