

The EM algorithm II: theory and inference

Julia Wrobel

Overview

Today, we cover:

- EM theory: why does it work?
- Inference for EM estimates

Announcements

- HW3 posted and due 3/4 at 10:00AM
 - Update: Extra credit if you implement SEM or Louis' method
- Where to find lab solutions

Readings:

- Chapter 4: The EM Algorithm, in Peng
- Givens and Hoeting Chapter 4

EM: notation

- Y : observed data vector
- Z : vector of data that are missing
- θ : vector of parameters we want to estimate
- $p(y, z|\theta)$: complete data density
- $p(y|\theta) = \int_z p(y, z|\theta) dz$: observed data density
 - $l(\theta|y) = \log f(y|\theta)$: observed data likelihood
- $p(z|y, \theta)$: conditional density of missing data given observed data

EM: review

(1) **E-Step:** Let θ_0 be the current estimate of θ . Define

$$Q(\theta|\theta_0) = E_z [\log p(y, z|\theta)|y, \theta_0]$$

(2) **M-Step:** Maximize $Q(\theta|\theta_0)$ with respect to θ to get next value of θ

(3) Iterate between E and M steps until convergence.

Note: E-step expectation taken WRT missing data density,

$$p(z|y, \theta) = \frac{p(y, z|\theta)}{p(y|\theta)}$$

EM Issues

1. Local vs. global max: may be multiple modes, EM may converge to a saddle point
 - **Solution:** try multiple starting values
2. Bad initialization can be a problem
 - Use information from the context
 - Use a crude method to find initial values (such as method of moments, grid search)

EM: intuition

Idea: In order to estimate θ via MLE *using only the observed data*, need to be able to maximize $l(\theta|y) = \log f(y|\theta) = \int_z p(y, z|\theta) dz$

- BUT $l(\theta|y)$ difficult to maximize because of the integral
- INSTEAD: assuming $p(y, z|\theta)$ has some nice form (like EF)
 - If we have estimate of missing data Z , can easily evaluate $p(y, z|\theta)$

To do this, we construct surrogate function (called Q function)

- Q is expected value of log likelihood for $p(y, z|\theta)$ *with respect to conditional distribution of missing given observed data*, $p(z|y, \theta)$, for current estimate of parameters, θ_0
- **M-Step** maximizes this surrogate function
 - Akin to filling in the missing data then taking the MLE for θ

EM: Why does it work?

$Q(\theta|\theta_0)$ function serves as a lower bound to the observed data density $p(y|\theta)$.

- The EM is a **minorization** approach. Instead of directly maximizing the log-likelihood, which is hard, the algorithm constructs a minorizing function and optimizes that function instead.

A function g *minorizes* f over \mathcal{X} at y if:

1. $g(x) \leq f(x)$ for all $x \in \mathcal{X}$
2. $g(y) = f(y)$

EM: Why does it work?

Because $Q(\theta|\theta_0)$ minorizes $l(\theta|y)$, maximizing it is guaranteed to increase (or at least not decrease) $l(\theta|y)$.

- This is because if θ_n is our current estimate of θ and $Q(\theta|\theta_n)$ minorizes $l(\theta|y)$ at θ_n , then we have

$$l(\theta_{n+1}|y) \geq Q(\theta_{n+1}|\theta_n) \geq Q(\theta_n|\theta_n) = l(\theta_n|y)$$

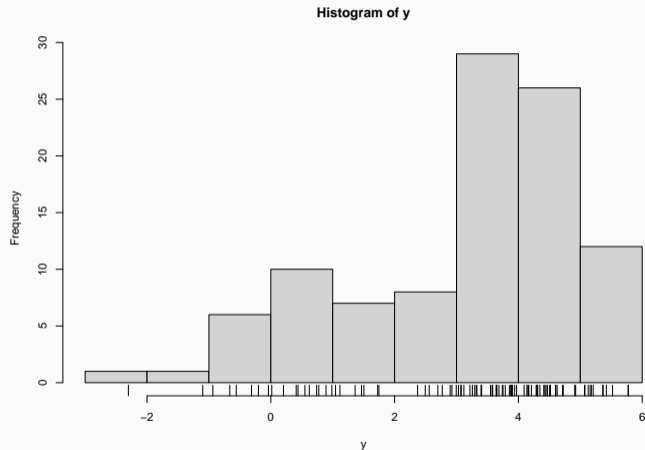
Example: minorization in Two-part Gaussian mixture model

Suppose we have data y_1, \dots, y_n that are sampled independently from a two-part mixture of Normals with density

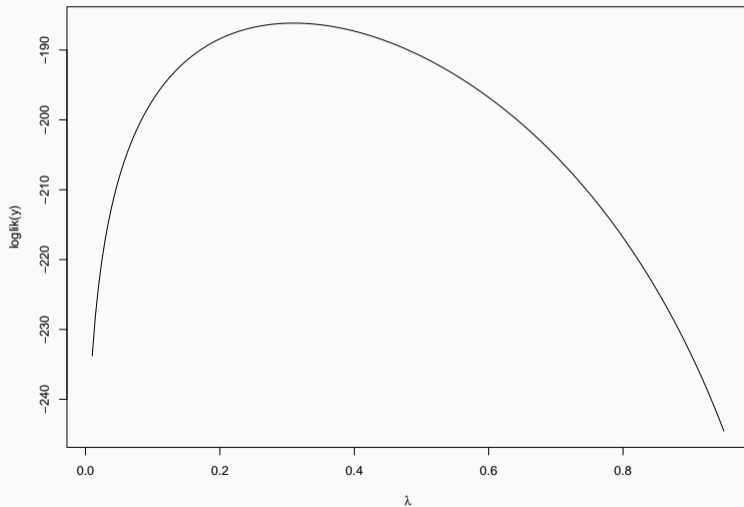
$$p(y|\theta) = \lambda \mathcal{N}(y|\mu_1, \sigma_1^2) + (1 - \lambda) \mathcal{N}(y|\mu_2, \sigma_2^2).$$

We can simulate some data from this model:

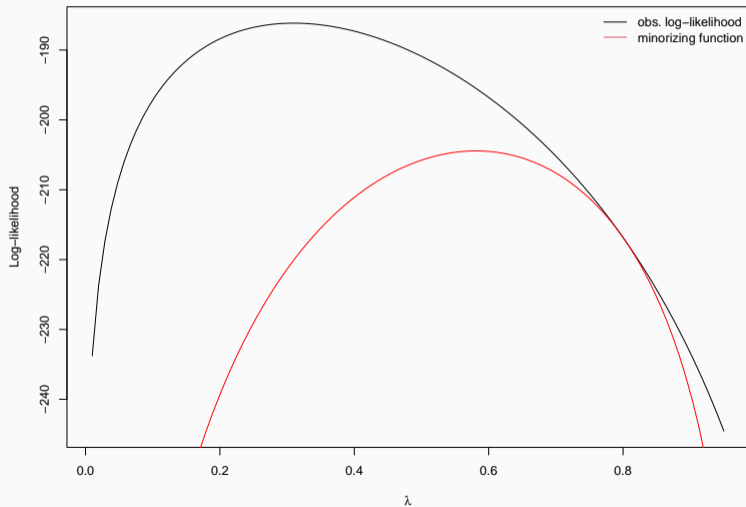
Two-part Gaussian mixture model



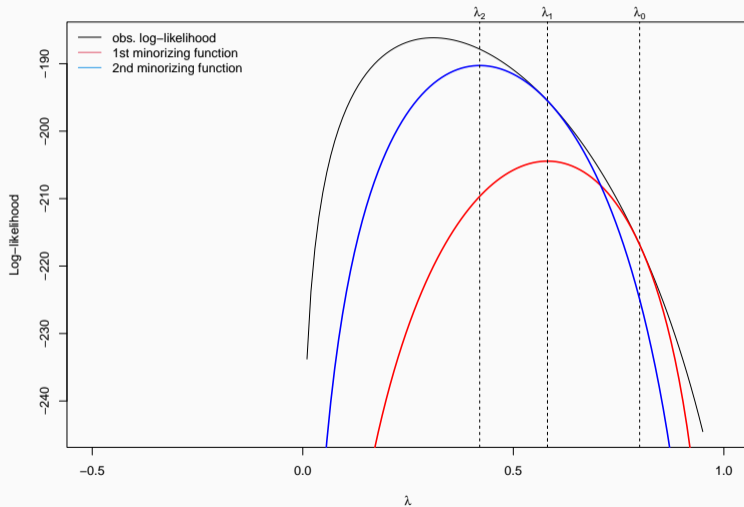
Two-part Gaussian mixture model



Two-part Gaussian mixture model



Two-part Gaussian mixture model



EM: proof of ascent property

First, some definitions

- **Bayes rule:** $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$
- **Kullback-Leibler divergence** (aka “relative entropy”):

$$\int \log \frac{p(x)}{q(x)} p(x) dx \geq 0 \text{ for densities } p(x), q(x)$$

- KL divergence is non-negative
- Attains its minimum 0 when $p(x)$ and $q(x)$ are equal

EM: proof of ascent property

EM: proof of ascent property

EM: proof of ascent property

EM: proof of ascent property

Theorem: at each iteration of the EM algorithm,

$$\log p(y|\theta^{t+1}) \geq \log p(y|\theta^t),$$

and equality holds if and only if $\theta^{t+1} = \theta^t$.

Proof: The definition of θ^{t+1} gives

$$Q(\theta^{t+1}|\theta^t) \geq Q(\theta^t|\theta^t) \implies$$

$$E_z [\log p(y, z|\theta^{t+1})|y, \theta^t] \geq E_z [\log p(y, z|\theta^t)|y, \theta^t]$$

Using Bayes rule and the law of conditional probability, this can be expanded to...

EM: proof of ascent property

$$E [\log p(z|y, \theta^{t+1})|y, \theta^t] + \log p(y|\theta^{t+1}) \geq E [\log p(z|y, \theta^t)|y, \theta^t] + \log p(y|\theta^t) \quad (1)$$

Also, by non-negativity of KL divergence,

$$\int_z \log \frac{p(z|y, \theta^t)}{p(z|y, \theta^{t+1})} p(z|y, \theta^t) dz = E \left[\log \frac{p(z|y, \theta^t)}{p(z|y, \theta^{t+1})} |y, \theta^t \right] \geq 0 \quad (2)$$

Combining (1) and (2) yields

$$\log p(y|\theta^{t+1}) \geq \log p(y|\theta^t)$$

EM: proof of ascent property

Combining (3) and (4), we have

$$\log p(y, z|\theta^{t+1}) = \log p(y, z|\theta^t).$$

The uniqueness of θ leads to $\theta^{t+1} = \theta^t$

EM Inference

Original EM paper did not discuss how to obtain any measures of uncertainty, such as standard errors.

- *Information matrix from observed data log-likelihood* would provide inference
 - Like observed data log-likelihood, often difficult to compute because of the missing data
- Louis' method
- Supplemented EM (SEM)
- Bootstrap

EM Inference

$$p(y|\theta) = \frac{p(y, z|\theta)}{p(z|y, \theta)}$$

$$-\log p(y|\theta) = -\log p(y, z|\theta) - [-\log p(z|y, \theta)]$$

$$E[-\nabla^2 \log p(y|\theta)] = E[-\nabla^2 \log p(y, z|\theta)] - E[-\nabla^2 \log p(z|y, \theta)]$$

$$I_Y(\theta) = I_{Y,Z}(\theta) - I_{Z|Y}(\theta)$$

Louis's method

$$I_Y(\theta) = I_{Y,Z}(\theta) - I_{Z|Y}(\theta)$$

- Presumably, $I_{Y,Z}(\theta)$ is reasonable to compute because based on complete data
- What is $I_{Z|Y}(\theta)$?
- $S(y|\theta) = \nabla \log p(y|\theta)$: observed data score function
- $S(y, z|\theta) = \nabla \log p(y, z|\theta)$: complete data score function

Louis's method

$$I_Y(\theta) = I_{Y,Z}(\theta) - I_{Z|Y}(\theta)$$

- $S(y|\theta) = \nabla \log p(y|\theta)$: observed score function
- $S(y, z|\theta) = \nabla \log p(y, z|\theta)$: complete data score function

Louis (1982) showed that

$$I_{Z|Y}(\theta) = E[S(y, z|\theta)S(y, z|\theta)^T] - S(y|\theta)S(y|\theta)^T$$

Where expectation is taken WRT missing data density $p(z|y, \theta)$.

Louis's method

$$I_Y(\theta) = I_{Y,Z}(\theta) - I_{Z|Y}(\theta)$$

- $I_{Z|Y}(\theta) = E[S(y, z|\theta)S(y, z|\theta)^T] - S(y|\theta)S(y|\theta)^T$
 - At MLE $\hat{\theta}$, $S(y|\hat{\theta}) = 0$

$$I_Y(\hat{\theta}) = I_{Y,Z}(\hat{\theta}) - E[S(y, z|\theta)S(y, z|\theta)^T]$$

Louis's method

$$I_Y(\hat{\theta}) = I_{Y,Z}(\hat{\theta}) - E[S(y, z|\theta)S(y, z|\theta)^T] \quad (1)$$

$$= -E[\nabla^2 \log p(y, z|\theta)|\hat{\theta}, y] - E[S(y, z|\theta)S(y, z|\theta)^T] \quad (2)$$

$$= -Q''(\hat{\theta}|\hat{\theta}) - E[S(y, z|\theta)S(y, z|\theta)^T] \quad (3)$$

Louis's estimator should be evaluated at last iteration of EM algorithm.

Supplemented EM (SEM)

Meng & Rubin, 1991: *Using EM to obtain asymptotic variance-covariance matrices: The SEM algorithm.*

Background: EM defines a mapping $\Psi : \theta^{t+1} = \Psi(\theta^t)$

- $\Psi(\theta) = (\Psi_1(\theta), \dots, \Psi_p(\theta))$ and $\theta = (\theta_1, \dots, \theta_p)$
- When EM converges, it converges to a fixed point of this mapping, so $\hat{\theta} = \Psi(\hat{\theta})$
- $\Psi'(\theta)$ is the Jacobian matrix where $[\Psi'(\theta)]_{i,j} = \frac{\partial \Psi_i(\theta)}{\partial \theta_j}$

Supplemented EM (SEM)

Dempster et al showed that

$$\Psi'(\hat{\theta})^T = I_{Z|Y}(\hat{\theta}) I_{Y,Z}(\hat{\theta})^{-1} \quad (1)$$

The missing information principle says that

$$\begin{aligned} I_Y(\hat{\theta}) &= I_{Y,Z}(\hat{\theta}) - I_{Z|Y}(\hat{\theta}) \\ &= \left[\mathcal{I} - I_{Z|Y}(\hat{\theta}) I_{Y,Z}(\hat{\theta})^{-1} \right] I_{Y,Z}(\hat{\theta}) \end{aligned}$$

Then, substituting (1) and inverting gives

$$\widehat{Var}(\hat{\theta}) = I_Y(\hat{\theta})^{-1} = I_{Y,Z}(\hat{\theta})^{-1} \left[\mathcal{I} - \Psi'(\hat{\theta}^T) \right]^{-1}. \quad (2)$$

Supplemented EM (SEM)

$$\widehat{Var}(\hat{\theta}) = I_Y(\hat{\theta})^{-1} = I_{Y,Z}(\hat{\theta})^{-1} \left[\mathcal{J} - \Psi'(\hat{\theta}^T) \right]^{-1} \quad (2)$$

- (2) means that the observed-data asymptotic variance can be obtained by inflating the complete-data asymptotic variance by the factor $\left[\mathcal{J} - \Psi'(\hat{\theta}^T) \right]^{-1}$
- Smaller missingness \implies smaller Ψ' \implies less variance inflation and faster convergence.

SEM Algorithm

SEM consists of three steps:

1. The evaluation of $I_{Y,Z}(\hat{\theta})$
2. The evaluation of $\Psi'(\hat{\theta})$
3. The evaluation of $\widehat{Var}(\hat{\theta})$

Evaluation of $I_{Y,Z}(\hat{\theta})$

- For exponential family, $I_{Y,Z}(\hat{\theta}) = -E \left[\nabla^2 \log p(y, z | \theta) | \hat{\theta}, y \right]$ should be easy to obtain.
- This is the second derivative of the Q function evaluated at $\hat{\theta}$

SEM Algorithm

Estimation of $\Psi'(\hat{\theta})$

1. Run EM algorithm to convergence to obtain MLE $\hat{\theta}$.
2. Pick a new starting point, θ^0 . θ^0 should be some small distance from $\hat{\theta}$ but not equal to $\hat{\theta}$ in any component.

SEM Algorithm

Estimation of $\Psi'(\hat{\theta})$

1. Run EM algorithm to convergence to obtain MLE $\hat{\theta}$.
2. Pick a new starting point, θ^0 .
3. Repeat the following until r_{ij}^k is stable:
 - Calculate $\theta^k = \Psi(\theta^{k-1})$ using one step of EM
 - For each $i = 1, \dots, p$:
 - Let $\theta^k(i) = (\hat{\theta}_1, \dots, \hat{\theta}_{i-1}, \hat{\theta}_i^k, \hat{\theta}_{i+1}, \dots, \hat{\theta}_p)$, i.e., replace i^{th} element of $\hat{\theta}$ with the i^{th} element of θ^k .
 - Perform one step of EM on $\theta^k(i)$ to obtain $\Psi[\theta^k(i)]$
 - Obtain $r_{ij}^k = \{\Psi[\theta^k(i)] - \hat{\theta}\} / \{\theta_i^k - \hat{\theta}_i\}$ for $j = 1, \dots, p$

SEM Algorithm

- The MLE $\hat{\theta}$ should be obtained at a very low tolerance (e.g. $\epsilon = 10^{-12}$)

The final r_{ij} is taken to be the first value of r_{ij}^k satisfying $|r_{ij}^k - r_{ij}^{k-1}| \leq \epsilon$, where k can be different for different (i, j) .

Bootstrapping

Goal is to obtain estimate of covariance matrix for EM parameters. To do a simple nonparametric bootstrap given an *iid* sample of observed data y_1, \dots, y_n , do the following:

1. Calculate $\hat{\theta}_{EM}$
2. Sample data y_1, \dots, y_n with replacement, and for each sample y_b^* , calculate a bootstrap estimate $\hat{\theta}_b^*$
3. Repeat step 2 B times to obtain $\theta_1^*, \dots, \theta_B^*$ bootstrap parameter estimates.
4. Sample covariance matrix of θ^* can be used as covariance of $\hat{\theta}_{EM}$.

Comparing EM inference approaches

- Louis's Method
 - Requires calculation of the conditional expectation of the square of the complete-data score function, which is specific to each problem
- SEM
 - Obtains covariance matrix by using only the code for computing the complete-data covariance matrix, the code for EM itself, and code for standard matrix operations.
- Bootstrapping
 - Conceptually simple
 - May be prohibitively slow if your EM algorithm is slow to converge

Speeding up EM

Sometimes convergence of EM can be very slow. Some methods to help with this:

- Louis's Acceleration
- SQUAREM

References

- Louis's method original paper
 - Finding observed information using the EM algorithm, JRSSB 1982
- SEM original paper (Meng & Rubin)
 - Using EM to obtain asymptotic variance-covariance matrices: The SEM algorithm, JASA 1991

Exercise

Try doing some inference for the two-part GMM.