习题 4.4(P247)

1. 计算下列定积分.

(1) $\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x dx$

解: $\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x dx = \int_0^{\frac{\pi}{2}} \cos^5 x dx - \int_0^{\frac{\pi}{2}} \cos^7 x dx$

$= \frac{4}{5} \cdot \frac{2}{3} - \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{1}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{105}$

(2) $\int_1^{e^2} \frac{dx}{x\sqrt{1+\ln x}}$

解: $\int_1^{e^2} \frac{dx}{x\sqrt{1+\ln x}} = \int_1^{e^2} \frac{d(1+\ln x)}{\sqrt{1+\ln x}} = 2\sqrt{1+\ln x}\Big|_1^{e^2} = 2(\sqrt{3}-1)$

(3) $\int_{\ln 2}^{2\ln 2} \frac{dx}{e^x - 1}$

解: $\int_{\ln 2}^{2\ln 2} \frac{dx}{e^x - 1} = \int_{\ln 2}^{2\ln 2} \frac{1 - e^x + e^x}{e^x - 1} = \int_{\ln 2}^{2\ln 2} \frac{d(e^x - 1)}{e^x - 1} - \int_{\ln 2}^{2\ln 2} dx$

$= [\ln|e^x - 1| - x]_{\ln 2}^{2\ln 2} = \ln \frac{3}{2}$

(4) $\int_3^8 \frac{x}{\sqrt{1+x}} dx$

解: $\int_3^8 \frac{x}{\sqrt{1+x}} dx = \int_3^8 (\sqrt{1+x} - \frac{1}{\sqrt{1+x}}) d(1+x) = \left[\frac{2}{3}(1+x)^{\frac{3}{2}} - 2(1+x)^{\frac{1}{2}}\right]_3^8 = 10\frac{2}{3}$

(5) $\int_1^2 \frac{\sqrt{x^2-1}}{x} dx$

解: $\int_1^2 \frac{\sqrt{x^2-1}}{x} dx \xlongequal{x=\sec t} \int_0^{\frac{\pi}{3}} \frac{\tan t}{\sec t} \cdot \sec t \tan t dt = \int_0^{\frac{\pi}{3}} \tan^2 t dt = \int_0^{\frac{\pi}{3}} (\sec^2 t - 1) dt$

$= (\tan t - t)\Big|_0^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3}$

(6) $\int_0^1 \sqrt{(1-x^2)^3}dx$

解: $\int_0^1 \sqrt{(1-x^2)^3}dx = \int_0^{\frac{\pi}{2}} \sqrt{(1-\sin^2 t)^3}\cos t dt = \int_0^{\frac{\pi}{2}} \cos^4 t dt = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{16}$

(7) $\int_1^3 \frac{dx}{x\sqrt{x^2+5x+1}}$

解: $\int_1^3 \frac{dx}{x\sqrt{x^2+5x+1}} \overset{x=\frac{1}{t}}{=\!=\!=} -\int_1^{\frac{1}{3}} \frac{dt}{\sqrt{t^2+5t+1}} = -\int_1^{\frac{1}{3}} \frac{d(t+\frac{5}{2})}{\sqrt{(t+\frac{5}{2})^2-\frac{21}{4}}}$

$= -\ln\left|t+\frac{5}{2}+\sqrt{t^2+5t+1}\right|_1^{\frac{1}{3}} = \ln(\frac{7}{2}+\sqrt{7}) - \ln\frac{9}{2} = \ln\frac{7+2\sqrt{7}}{9}$

(8) $\int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x}dx$

解: $\int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x}dx = \int_0^{\pi} \sqrt{\sin^3 x \cdot |\cos x|}dx$

$$= \int_0^{\frac{\pi}{2}} (\sin x)^{\frac{3}{2}} \cos x dx - \int_{\frac{\pi}{2}}^{\pi} (\sin x)^{\frac{3}{2}} \cos x dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin x)^{\frac{3}{2}} d(\sin x) - \int_{\frac{\pi}{2}}^{\pi} (\sin x)^{\frac{3}{2}} d(\sin x)$$

$$= \frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_0^{\frac{\pi}{2}} - \frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_{\frac{\pi}{2}}^{\pi} = \frac{4}{5}$$

(9) $\int_0^{-\ln 2} \sqrt{1-e^{2x}}dx$

解: $\int_0^{-\ln 2} \sqrt{1-e^{2x}}dx \overset{\sqrt{1-e^{2x}}=t}{\underset{x=\frac{1}{2}\ln(1-t^2)}{=\!=\!=\!=}} \int_0^{\frac{\sqrt{3}}{2}} \frac{-t^2}{1-t^2}dt = \int_0^{\frac{\sqrt{3}}{2}} (1-\frac{1}{1-t^2})dt$

$= \int_0^{\frac{\sqrt{3}}{2}} (1-\frac{1}{2}\cdot\frac{1}{1-t}-\frac{1}{2}\cdot\frac{1}{1+t})dt = (t+\frac{1}{2}\ln\left|\frac{1-t}{1+t}\right|)\Big|_0^{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2}+\frac{1}{2}\ln(\frac{2-\sqrt{3}}{2+\sqrt{3}})$

$= \frac{\sqrt{3}}{2}+\ln(2-\sqrt{3})$

2. 计算下列定积分.

(1) $\int_1^e x^2 \ln x dx$

解: $\int_1^e x^2 \ln x dx = \frac{1}{3}\int_1^e \ln x dx^3 = \frac{1}{3}\left[x^3 \ln x\Big|_1^e - \int_1^e x^3 \frac{1}{x} dx \right]$

$= \frac{1}{3}(e^3 - \frac{1}{3}e^3 + \frac{1}{3}) = \frac{2}{9}e^3 + \frac{1}{9}$

(2) $\int_0^{\sqrt{3}} x \arctan x dx$

解: $\int_0^{\sqrt{3}} x \arctan x dx = \int_0^{\sqrt{3}} \arctan x d(\frac{x^2}{2}) = \frac{x^2}{2}\arctan x\Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x^2}{2(1+x^2)} dx$

$= \frac{\pi}{2} - \frac{1}{2}\int_0^{\sqrt{3}} (1 - \frac{1}{1+x^2}) dx = \frac{\pi}{2} - \frac{1}{2}(x - \arctan x)\Big|_0^{\sqrt{3}} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

(3) $\int_0^{\frac{\pi}{2}} e^{2x} \cos x dx$

解: $I_1 = \int_0^{\frac{\pi}{2}} e^{2x} \cos x dx = \int_0^{\frac{\pi}{2}} e^{2x} d(\sin x) = e^{2x} \sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2e^{2x} \sin x dx$

$= e^{\pi} + 2\int_0^{\frac{\pi}{2}} e^{2x} d(\cos x) = e^{\pi} + 2\left[e^{2x} \cos x\Big|_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} e^{2x} \cos x dx \right] = e^{\pi} - 2 - 4I_1$

$\therefore I_1 = \frac{1}{5}(e^{\pi} - 2)$

(4) $\int_0^3 \arcsin\sqrt{\frac{x}{1+x}} dx$

解: $\int_0^3 \arcsin\sqrt{\frac{x}{1+x}} dx \xlongequal[1+x=\frac{1}{1-t^2}]{\sqrt{\frac{x}{1+x}}=t} \int_0^{\frac{\sqrt{3}}{2}} \arcsin t d(\frac{1}{1-t^2})$

$= \frac{\arcsin t}{1-t^2}\Big|_0^{\frac{\sqrt{3}}{2}} - \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{(1-t^2)^{\frac{3}{2}}} dt \xlongequal{t=\sin y} \frac{4}{3}\pi - \int_0^{\frac{\pi}{3}} \frac{1}{\cos^2 y} dy$

$= \frac{4}{3}\pi - \tan y\Big|_0^{\frac{\pi}{3}} = \frac{4}{3}\pi - \sqrt{3}$

(5) $\int_0^{\frac{\pi}{2}} \cos^7 x dx$

解: 直接用公式$\int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots\cdots \frac{2}{3}$ (n为奇数)

$$\int_0^{\frac{\pi}{2}} \cos^7 x dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{16}{35}$$

(6) $\int_0^{\pi} \sin^8 \frac{x}{2} dx$

解: $\int_0^{\pi} \sin^8 \frac{x}{2} dx \xlongequal{\frac{x}{2}=t} 2\int_0^{\frac{\pi}{2}} \sin^8 t dt = 2 \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{35}{128}\pi$

(7) $\int_{-\pi}^{\pi} x \cos x dx$

解: 被积函数是奇函数，故 $\int_{-\pi}^{\pi} x \cos x dx = 0$

(8) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x} dx$

解: $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{s\sin^2 x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} x d(-\cot x) = -x\cot x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot x dx$

$$= (\frac{1}{4} - \frac{\sqrt{3}}{9})\pi + \ln|\sin x|\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = (\frac{1}{4} - \frac{\sqrt{3}}{9})\pi + \frac{1}{2}\ln\frac{3}{2}$$

(9) $\int_1^e \sin(\ln x) dx$

解: $I = \int_1^e \sin(\ln x) dx = x\sin(\ln x)\Big|_1^e - \int_1^e \cos(\ln x) dx$

$$= e\sin 1 - \left[x\cos(\ln x)\Big|_1^e + \int_1^e \sin(\ln x) dx \right] = e\sin 1 - e\cos 1 + 1 - I$$

$\therefore \quad I = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}$

3. 计算下列定积分.

(1) $\int_{-5}^{5}\frac{x^3\sin^2 x}{1+x^2+x^4}dx$

解: 被积函数$\frac{x^3\sin^2 x}{1+x^2+x^4}$为奇函数，故$\int_{-5}^{5}\frac{x^3\sin^2 x}{1+x^2+x^4}dx=0$

(2) $\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x\arcsin x}{\sqrt{1-x^2}}dx$

解: $\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x\arcsin x}{\sqrt{1-x^2}}dx=2\int_0^{\frac{1}{2}}\frac{x\arcsin x}{\sqrt{1-x^2}}dx=2\int_0^{\frac{1}{2}}\arcsin x\,d(-\sqrt{1-x^2})$

$$=2(-\sqrt{1-x^2}\arcsin x\Big|_0^{\frac{1}{2}}+\int_0^{\frac{1}{2}}dx)=2(-\frac{\sqrt{3}}{12}\pi+\frac{1}{2})=1-\frac{\sqrt{3}}{6}\pi$$

(3) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}\,dx$

解: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}\,dx=2\int_0^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}\,dx=2\int_0^{\frac{\pi}{2}}\sqrt{\cos x}\sin x\,dx$

$$=-2\int_0^{\frac{\pi}{2}}\sqrt{\cos x}\,d(\cos x)=-2\cdot\frac{2}{3}(\cos x)^{\frac{3}{2}}\Big|_0^{\frac{\pi}{2}}=\frac{4}{3}$$

4. 证明: $\int_x^1\frac{dt}{1+t^2}=\int_1^{\frac{1}{x}}\frac{dt}{1+t^2}$

证明: $\int_x^1\frac{dt}{1+t^2}\overset{t=\frac{1}{s}}{=\!=\!=}\int_{\frac{1}{x}}^1\frac{-\frac{1}{s^2}ds}{1+(\frac{1}{s})^2}=\int_1^{\frac{1}{x}}\frac{ds}{1+s^2}=\int_1^{\frac{1}{x}}\frac{dt}{1+t^2}$

5. 证明: $\int_0^1 x^m(1-x)^n dx=\int_0^1 x^n(1-x)^m dx$，其中$m$、$n$为正整数.

证明: $\int_0^1 x^m(1-x)^n dx\overset{x=1-t}{=\!=\!=}-\int_1^0 t^n(1-t)^m dx$

$$=\int_0^1 t^n(1-t)^m dx=\int_0^1 x^n(1-x)^m dx$$

推论：$\int_0^a x^m(a-x)^n dx=\int_0^a x^n(a-x)^m dx$

6. 设 $I_n=\int_0^{\frac{\pi}{4}}\tan^n xdx$，其中 n 为大于 1 的整数，证明：$I_n=\frac{1}{n-1}-I_{n-2}$，并利用此递推公式计算 $\int_0^{\frac{\pi}{4}}\tan^5 xdx$

证明：$I_n=\int_0^{\frac{\pi}{4}}\tan^n xdx=\int_0^{\frac{\pi}{4}}\tan^{n-2}x(\sec^2 x-1)dx=\int_0^{\frac{\pi}{4}}\tan^{n-2}xd(\tan x)-I_{n-2}$

$$=\left.\frac{\tan^{n-1}x}{n-1}\right|_0^{\frac{\pi}{4}}-I_{n-2}=\frac{1}{n-1}-I_{n-2}$$

$$\int_0^{\frac{\pi}{4}}\tan^5 xdx=I_5=\frac{1}{4}-I_3=\frac{1}{4}-(\frac{1}{2}-I_1)=-\frac{1}{4}+\int_0^{\frac{\pi}{4}}\tan xdx=-\frac{1}{4}-\ln\left.|\cos x|\right|_0^{\frac{\pi}{4}}$$

$$=\frac{1}{2}\ln 2-\frac{1}{4}$$

7. 设 $f(x)=\begin{cases}1+x^2 & 0\le x\le 1\\ 2-x & 1<x<2\end{cases}$，计算 $\int_0^2 f(x)e^x dx$

解：$\int_0^2 f(x)e^x dx=\int_0^1(1+x^2)e^x dx+\int_1^2(2-x)e^x dx=\int_0^1(1+x^2)de^x+\int_1^2(2-x)de^x$

$$=\left.(1+x^2)e^x\right|_0^1-\int_0^1 2xe^x dx+\left.(2-x)e^x\right|_1^2+\int_1^2 e^x dx$$

$$=2e-1-\int_0^1 2xde^x-e+\left.e^x\right|_1^2=e^2-1-\left.(2xe^x-2e^x)\right|_0^1=e^2-3$$