P102EX5 设函数 $u(x)$ 由方程组 $\begin{cases} u = f(x, y), \\ g(x, y, z) = 0 \text{所确定,} \\ h(x, z) = 0. \end{cases}$

其中f、g、h是可微函数，且 $\dfrac{\partial g}{\partial y} \neq 0, \dfrac{\partial h}{\partial z} \neq 0,$ 试求 $\dfrac{du}{dx}$.

解 将方程组的变元 u 以及 y, z 都看成是 x 的函数.

方程组各方程两边对 x 求导,得

$$\begin{cases} \dfrac{du}{dx} = f'_x + f'_y \dfrac{dy}{dx}, & (1) \\ g'_x + g'_y \cdot \dfrac{dy}{dx} + g'_z \cdot \dfrac{dz}{dx} = 0, & (2) \\ h'_x + h'_z \cdot \dfrac{dz}{dx} = 0. & (3) \end{cases}$$

由(3)得 $\dfrac{dz}{dx}=-\dfrac{h_x'}{h_z'}$， 代入(2)得 $\dfrac{dy}{dx}=\dfrac{g_z'\cdot h_x'}{g_y'\cdot h_z'}-\dfrac{g_x'}{g_y'}$，

代入(1)得 $\dfrac{du}{dx}=f_x'-\dfrac{f_y'\cdot g_x'}{g_y'}+\dfrac{f_y'\cdot g_z'\cdot h_x'}{g_y'\cdot h_z'}$.

设$z = z(x, y)$是由方程 $f(y-x, yz) = 0$所确定的隐函数,其中函数f具有二阶连续的偏导数，求 $\frac{\partial^2 z}{\partial x^2}$.

解 方程 $f(y-x, yz) = 0$两端对x求偏导

$$-f_1' + yf_2' \frac{\partial z}{\partial x} = 0 \qquad (1) \qquad\qquad \frac{\partial z}{\partial x} = \frac{f_1'}{yf_2'} \qquad (2)$$

对(2)式求会比较繁！！！

(1)式两端再对 x求偏导

$$f_{11}'' - yf_{12}'' \frac{\partial z}{\partial x} - yf_{21}'' \frac{\partial z}{\partial x} + y^2 f_{22}'' \left(\frac{\partial z}{\partial x}\right)^2 + yf_2' \frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{1}{yf_2'}\left[-f_{11}''+y(f_{12}''+f_{21}'')\frac{\partial z}{\partial x}-y^2 f_{22}''(\frac{\partial z}{\partial x})^2\right]$$

$$=\frac{1}{yf_2'}\left[-f_{11}''+2yf_{12}''\frac{\partial z}{\partial x}-y^2 f_{22}''(\frac{\partial z}{\partial x})^2\right]$$

$$=\frac{1}{yf_2'}\left[-f_{11}''+2yf_{12}''\cdot\frac{f_1'}{yf_2'}-y^2 f_{22}''(\frac{f_1'}{yf_2'})^2\right]$$

$$=\frac{1}{y(f_2')^3}\left[-(f_2')^2 f_{11}''+2f_1'\cdot f_2'\cdot f_{12}''-(f_1')^2 f_{22}''\right]$$