习题 1.3(P44)

求下列极限.

1. $\lim\limits_{x\to 1}\dfrac{3x^2-1}{x^2+2x+4}$

解：$\lim\limits_{x\to 1}\dfrac{3x^2-1}{x^2+2x+4}=\dfrac{3(\lim\limits_{x\to 1}x)^2-1}{(\lim\limits_{x\to 1}x)^2+2\lim\limits_{x\to 1}x+4}=\dfrac{2}{7}$

2. $\lim\limits_{x\to 3}\dfrac{2x^2-7x+3}{x^2+4x-21}$

解：$\lim\limits_{x\to 3}\dfrac{2x^2-7x+3}{x^2+4x-21}=\lim\limits_{x\to 3}\dfrac{(2x-1)(x-3)}{(x+7)(x-3)}=\lim\limits_{x\to 3}\dfrac{(2x-1)}{(x+7)}=\dfrac{5}{10}=\dfrac{1}{2}$

3. $\lim\limits_{x\to\infty}\dfrac{3x^2-1}{x^2-2x+3}$

解：$\lim\limits_{x\to\infty}\dfrac{3x^2-1}{x^2-2x+3}=\lim\limits_{x\to\infty}\dfrac{3-\dfrac{1}{x^2}}{1-\dfrac{2}{x}+\dfrac{3}{x^2}}=3$

4. $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2+1}}{x+1}$

解：$\lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2+1}}{x+1}\overset{\text{分子分母}}{\underset{\text{同除}x}{=\!=\!=}}\lim\limits_{x\to+\infty}\dfrac{\sqrt{1+\dfrac{1}{x^2}}}{1+\dfrac{1}{x}}=1$

5. $\lim\limits_{x\to 0}\dfrac{4x^3+x}{5x^2+2x}$

解：$\lim\limits_{x\to 0}\dfrac{4x^3+x}{5x^2+2x}=\lim\limits_{x\to 0}\dfrac{4x^2+1}{5x+2}=\dfrac{1}{2}$

6. $\lim\limits_{x\to 0^+}\dfrac{x-\sqrt{x}}{\sqrt{x}}$

解：$\lim\limits_{x\to 0^+}\dfrac{x-\sqrt{x}}{\sqrt{x}}\overset{\text{分子分母}}{\underset{\text{同除}\sqrt{x}}{=\!=\!=}}\lim\limits_{x\to 0^+}\dfrac{\sqrt{x}-1}{1}=-1$

7. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+9}-3}{x^2}$

解：$\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+9}-3}{x^2}\overset{\text{分子有}}{\underset{\text{理化}}{=\!=}}\lim\limits_{x\to 0}\dfrac{1}{(\sqrt{x^2+9}+3)}=\dfrac{1}{6}$

8. $\lim\limits_{x\to 2}\dfrac{\sqrt{x+2}-2}{\sqrt{x+7}-3}$

解：$\lim\limits_{x\to 2}\dfrac{\sqrt{x+2}-2}{\sqrt{x+7}-3}\overset{\text{分子分母}}{\underset{\text{都有理化}}{=\!=}}\lim\limits_{x\to 2}\dfrac{(x-2)(\sqrt{x+7}+3)}{(x-2)(\sqrt{x+2}+2)}=\lim\limits_{x\to 2}\dfrac{(\sqrt{x+7}+3)}{(\sqrt{x+2}+2)}=\dfrac{6}{4}=\dfrac{3}{2}$

9. $\lim\limits_{x\to 1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^3}\right)$

解：$\lim\limits_{x\to 1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^3}\right)=\lim\limits_{x\to 1}\left(\dfrac{1+x+x^2-3}{1-x^3}\right)$

$=\lim\limits_{x\to 1}\dfrac{(x-1)(x+2)}{(1-x)(1+x+x^2)}=\lim\limits_{x\to 1}\dfrac{-(x+2)}{(1+x+x^2)}=-1$

10. $\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$

解：$\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}=\lim\limits_{h\to 0}\dfrac{h(2x+h)}{h}=\lim\limits_{h\to 0}(2x+h)=2x$

11. $\lim\limits_{x\to +\infty}\sqrt{x}(\sqrt{x+1}-\sqrt{x})$

解：$\lim\limits_{x\to +\infty}\sqrt{x}(\sqrt{x+1}-\sqrt{x})\overset{\text{分子有}}{\underset{\text{理化}}{=\!=}}\lim\limits_{x\to +\infty}\dfrac{\sqrt{x}}{(\sqrt{x+1}+\sqrt{x})}$

$\overset{\text{分子分母}}{\underset{\text{同除}\sqrt{x}}{=\!=}}\lim\limits_{x\to +\infty}\dfrac{1}{(\sqrt{1+\frac{1}{x}}+1)}=\dfrac{1}{2}$

12. $\lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1}$ $(m\neq n\text{为正整数})$

解：法 1 利用公式：$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1})$

$\lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1}=\lim\limits_{x\to 1}\dfrac{(x-1)(x^{m-1}+x^{m-2}+\cdots+1)}{(x-1)(x^{n-1}+x^{n-2}+\cdots+1)}$

$$=\lim_{x\to 1}\frac{(x^{m-1}+x^{m-2}+\cdots+1)}{(x^{n-1}+x^{n-2}+\cdots+1)}=\frac{m}{n}$$

法 2（利用微分中值定理）设 $f(x)=x^m$， $g(x)=x^n$，

则由柯西中值得 $\dfrac{x^m-1}{x^n-1}=\dfrac{m\xi^{m-1}}{n\xi^{n-1}}$， 即 $\dfrac{x^m-1}{x^n-1}=\dfrac{m}{n}\xi^{m-n}$ （ξ 介于 x 和 1 之间）

故 $\displaystyle\lim_{x\to 1}\frac{x^m-1}{x^n-1}=\lim_{\xi\to 1}\frac{m}{n}\xi^{m-n}=\frac{m}{n}$

13. $\displaystyle\lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{2x+1}}$

解： $$\lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{2x+1}}\xlongequal[\text{同除}\sqrt{x}]{\text{分子分母}}\lim_{x\to+\infty}\frac{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}}{\sqrt{2+\frac{1}{x}}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

14. $\displaystyle\lim_{x\to\infty}\frac{x+\cos x}{x-\cos x}$

解： $$\lim_{x\to\infty}\frac{x+\cos x}{x-\cos x}=\lim_{x\to\infty}\frac{1+\frac{\cos x}{x}}{1-\frac{\cos x}{x}}=1$$

(因为 $\displaystyle\lim_{x\to\infty}\left|\frac{\cos x}{x}\right|\le\lim_{x\to\infty}\frac{1}{|x|}=0,$ 由夹逼定理得 $\displaystyle\lim_{x\to\infty}\frac{\cos x}{x}=0$)

(或因为无穷小乘以有界变量还是无穷小，故 $\displaystyle\lim_{x\to\infty}\frac{\cos x}{x}=0$)

15. $\displaystyle\lim_{x\to 1}\frac{x+x^2+\cdots+x^n-n}{x-1}$

解： $$\lim_{x\to 1}\frac{x+x^2+\cdots+x^n-n}{x-1}=\lim_{x\to 1}\frac{(x-1)+(x^2-1)+\cdots+(x^n-1)}{x-1}$$

$$=\lim_{x\to 1}[1+(x+1)+(x^2+x+1)+\cdots+(x^{n-1}+x^{n-2}+\cdots+1)]$$

$$=1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

16． $\lim\limits_{x\to0}\dfrac{(1+mx)^n-(1+nx)^m}{x^2}$ $(m\neq n$为正整数)

解： $\lim\limits_{x\to0}\dfrac{(1+mx)^n-(1+nx)^m}{x^2}\overline{\overline{\text{利用二项式}}}{\text{定理展开}}$

$$=\lim_{x\to0}\frac{(1+nmx+C_n^2m^2x^2+\cdots+m^nx^n)-(1+mnx+C_m^2n^2x^2+\cdots+n^mx^m)}{x^2}$$

$$=\lim_{x\to0}\frac{(C_n^2m^2x^2+\cdots+m^nx^n)-(C_m^2n^2x^2+\cdots+n^mx^m)}{x^2}$$

$$=\lim_{x\to0}[(C_n^2m^2+C_n^3m^3x+\cdots+m^nx^{n-2})-(C_m^2n^2+C_m^3n^3x+\cdots+n^mx^{m-2})]$$

$$=C_n^2m^2-C_m^2n^2=\frac{n(n-1)}{2}m^2-\frac{m(m-1)}{2}n^2=\frac{mn(n-m)}{2}$$

17． $\lim\limits_{n\to\infty}\dfrac{2^{n+1}+3^{n+1}}{2^n+3^n}$

解： $\lim\limits_{n\to\infty}\dfrac{2^{n+1}+3^{n+1}}{2^n+3^n}\ \dfrac{\text{分子分母}}{\text{同除}3^{n+1}}\ \lim\limits_{n\to\infty}\dfrac{\left(\dfrac{2}{3}\right)^{n+1}+1}{\dfrac{2}{3}\cdot\left(\dfrac{2}{3}\right)^n+\dfrac{1}{3}}=\dfrac{1}{\dfrac{1}{3}}=3$

18． $\lim\limits_{n\to\infty}\dfrac{1+2+3+\cdots+(n-1)}{n^2}$

解： $\lim\limits_{n\to\infty}\dfrac{1+2+3+\cdots+(n-1)}{n^2}=\lim\limits_{n\to\infty}\dfrac{(n-1)n/2}{n^2}=\dfrac{1}{2}$

19． $\lim\limits_{n\to\infty}\dfrac{(n+1)(n+2)(n+3)}{3n^3}$

解： $\lim\limits_{n\to\infty}\dfrac{(n+1)(n+2)(n+3)}{3n^3}=\lim\limits_{n\to\infty}\dfrac{1}{3}(1+\dfrac{1}{n})(1+\dfrac{2}{n})(1+\dfrac{3}{n})=\dfrac{1}{3}$

20． $\lim\limits_{n\to\infty}\dfrac{(\sqrt{n^2+1}+n)^2}{\sqrt[3]{n^6+1}}$

解： $\lim\limits_{n\to\infty}\dfrac{(\sqrt{n^2+1}+n)^2}{\sqrt[3]{n^6+1}}\ \dfrac{\text{分子分母}}{\text{同除}n^2}\ \lim\limits_{n\to\infty}\dfrac{(\sqrt{1+\dfrac{1}{n^2}}+1)^2}{\sqrt[3]{1+\dfrac{1}{n^6}}}=4$

21. $\lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^n}\right)$

解：$\lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^n}\right)=\lim\limits_{n\to\infty}\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}=2$

22. $\lim\limits_{n\to\infty}\left(\frac{1}{2!}+\frac{2}{3!}+\cdots+\frac{n}{(n+1)!}\right)$

解：$\lim\limits_{n\to\infty}\left(\frac{1}{2!}+\frac{2}{3!}+\cdots+\frac{n}{(n+1)!}\right)=\lim\limits_{n\to\infty}\left(\frac{1}{2}+\frac{3-1}{3!}+\cdots+\frac{(n+1)-1}{(n+1)!}\right)$

$=\lim\limits_{n\to\infty}\left(\frac{1}{2}+(\frac{1}{2!}-\frac{1}{3!})+(\frac{1}{3!}-\frac{1}{4!})+\cdots+(\frac{1}{n!}-\frac{1}{(n+1)!})\right)=\lim\limits_{n\to\infty}\left(1-\frac{1}{(n+1)!}\right)=1$

23. $\lim\limits_{n\to\infty}n\left(\frac{1}{n^2+\pi}+\frac{1}{n^2+2\pi}+\cdots+\frac{1}{n^2+n\pi}\right)$

解：因为 $n\cdot\frac{n}{n^2+n\pi}\le n\left(\frac{1}{n^2+\pi}+\frac{1}{n^2+2\pi}+\cdots+\frac{1}{n^2+n\pi}\right)\le n\cdot\frac{n}{n^2+\pi}$

而 $\lim\limits_{n\to\infty}\frac{n^2}{n^2+n\pi}=1$，　$\lim\limits_{n\to\infty}\frac{n^2}{n^2+\pi}=1$

由夹逼定理得: $\lim\limits_{n\to\infty}n\left(\frac{1}{n^2+\pi}+\frac{1}{n^2+2\pi}+\cdots+\frac{1}{n^2+n\pi}\right)=1$