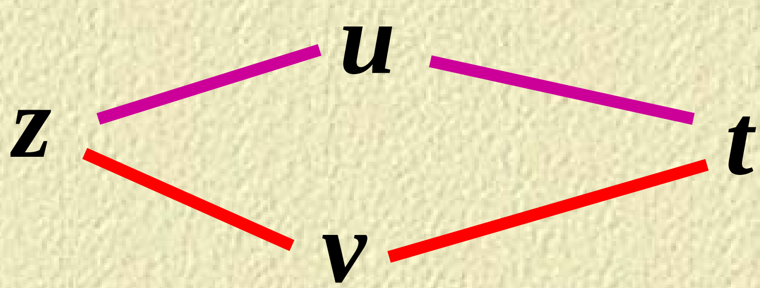
7.4 复合函数与隐函数的微分法

7.4.1 复合函数微分法（链式法则）

1. 复合函数的中间变量均是一元函数的情形

定理 1 如果函数$u=\phi(t)$及$v=\psi(t)$都在点t可导，函数$z=f(u,v)$在对应点(u,v)可微，则复合函数$z=f[\phi(t),\psi(t)]$在点t可导，且有：

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}.$$



连线相乘
分线相加

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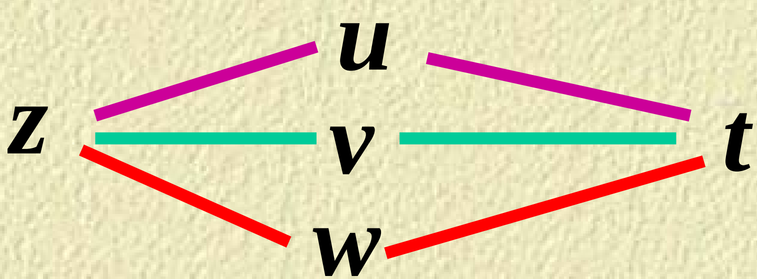
上定理的结论可推广到中间变量多于两个的情况.

如 $$\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial w}\frac{dw}{dt}$$



以上公式中的导数 $\frac{dz}{dt}$ 称为**全导数.**

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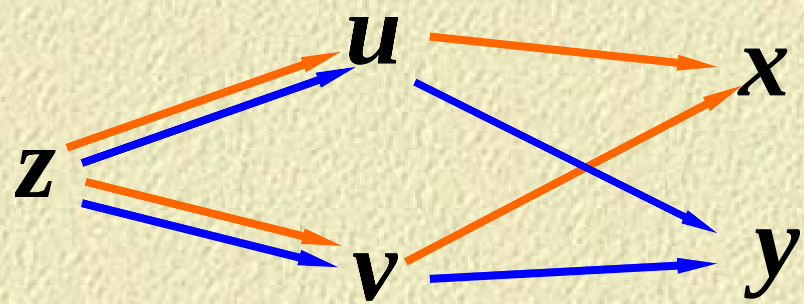
2. 复合函数的中间变量均是二(多)元函数的情形

上定理还可推广到中间变量不是一元函数而是多元函数的情况： $z = f[\phi(x,y),\psi(x,y)]$.

定理 2 如果 $u=\phi(x,y)$ 及 $v=\psi(x,y)$ 都在点 (x,y) 具有对 x 和 y 的偏导数，且函数 $z=f(u,v)$ 在对应点 (u,v) 可微，则复合函数 $z=f[\phi(x,y),\psi(x,y)]$ 在点 (x,y) 的两个偏导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

链式法则如图示



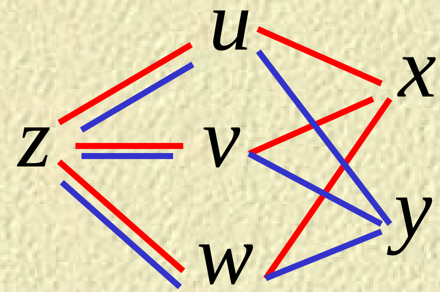
$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

求复合函数的偏导数时，关键是要分清哪些变量是中间变量，哪些变量是自变量，为了直观的看出变量间的关系，可以画出变量关系图，然后“连线相乘、分线相加”。

类似地再推广，设$u=\phi(x,y)$、$v=\psi(x,y)$、$w=w(x,y)$都在点(x,y)具有对x和y的偏导数，且函数$z=f(u,v,w)$在对应点(u,v,w)处可微，则复合函数$z=f[\phi(x,y),\psi(x,y),w(x,y)]$在点$(x,y)$的两个偏导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x},$$

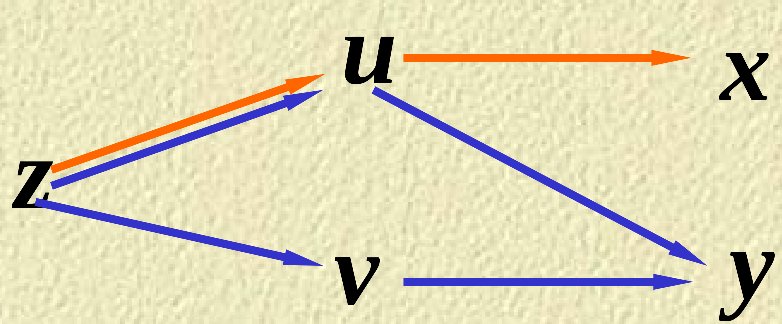


$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial y}.$$

特殊地， 复合函数的中间变量既有一元函数、又有二(多)元函数的情形

定理 3　如果$u=\phi(x,y)$在点(x,y)具有对x和y的偏导数，$v=\psi(y)$在点y可导，且函数$z=f(u,v)$在对应点(u,v)可微，则复合函数$z=f[\phi(x,y),\psi(y)]$在点(x,y)的两个偏导数存在，且有

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}, \qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{dv}{dy}.$$



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3．中间变量本身是自变量的情形

特殊地 $z = f(u, x, y)$ 其中 $u = \phi(x, y)$

即 $z = f[\phi(x, y), x, y]$, 令 $v = x$, $w = y$,

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial w}{\partial y} = 1.$$

$$\boxed{\frac{\partial z}{\partial x}} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \boxed{\frac{\partial f}{\partial x}}, \qquad \boxed{\frac{\partial z}{\partial y}} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \boxed{\frac{\partial f}{\partial y}}.$$

区别类似

两者的区别

把复合函数 $z = f[\phi(x, y), x, y]$ 中的 y 看作不变而对 x 的偏导数

把 $z = f(u, x, y)$ 中的 u 及 y 看作不变而对 x 的偏导数

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例 1 设$z=e^{u}\sin v$，而$u=xy$，$v=x+y$，

求 $\dfrac{\partial z}{\partial x}$和$\dfrac{\partial z}{\partial y}$.

解 $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=e^{u}\sin v\cdot y+e^{u}\cos v\cdot 1=e^{u}(y\sin v+\cos v),$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=e^{u}\sin v\cdot x+e^{u}\cos v\cdot 1=e^{u}(x\sin v+\cos v).$$

例 2 设$z = uv + \sin t$，而$u = e^t$，$v = \cos t$，

求全导数$\dfrac{dz}{dt}$.

解
$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial z}{\partial t}$$

$$= ve^t - u\sin t + \cos t$$

$$= e^t \cos t - e^t \sin t + \cos t$$

$$= e^t(\cos t - \sin t) + \cos t.$$

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例 3　设$w = f(x+y+z, xyz)$，f 具有二阶连续偏导数，求$\dfrac{\partial w}{\partial x}$和$\dfrac{\partial^2 w}{\partial x \partial z}$.

解　令　$u = x+y+z$，　$v = xyz$;

记　$f_1' = \dfrac{\partial f(u,v)}{\partial u}$，　$f_{12}'' = \dfrac{\partial^2 f(u,v)}{\partial u \partial v}$，

同理有　f_2'，　f_{11}''，　f_{22}''.

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = f_1' + yzf_2';$$

$$\frac{\partial^2 w}{\partial x \partial z} = \frac{\partial}{\partial z}(f_1' + yzf_2') = \frac{\partial f_1'}{\partial z} + yf_2' + yz\frac{\partial f_2'}{\partial z};$$

$$\frac{\partial f_1'}{\partial z} = \frac{\partial f_1'}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f_1'}{\partial v} \cdot \frac{\partial v}{\partial z} = f_{11}'' + xyf_{12}'';$$

$$\frac{\partial f_2'}{\partial z} = \frac{\partial f_2'}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f_2'}{\partial v} \cdot \frac{\partial v}{\partial z} = f_{21}'' + xyf_{22}'';$$

于是 $$\frac{\partial^2 w}{\partial x \partial z} = f_{11}'' + xyf_{12}'' + yf_2' + yz(f_{21}'' + xyf_{22}'')$$

$$= f_{11}'' + y(x+z)f_{12}'' + xy^2 zf_{22}'' + yf_2'.$$

7.4.2 全微分形式不变性

设函数$z = f(u,v)$具有连续偏导数，则有全微分

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv;$$

当$u = \phi(x,y)$、$v = \psi(x,y)$时，有

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

$$\Rightarrow dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

全微分形式不变形的实质：

无论 z是自变量u、v的函数或中间变量u、v的函数，它的全微分形式是一样的.

例 4 已知$e^{-xy}-2z+e^{z}=0$，求$\dfrac{\partial z}{\partial x}$和$\dfrac{\partial z}{\partial y}$.

解 $\because d(e^{-xy}-2z+e^{z})=0,$

$$\therefore e^{-xy}d(-xy)-2dz+e^{z}dz=0,$$

$$(e^{z}-2)dz=e^{-xy}(xdy+ydx)$$

$$dz=\frac{ye^{-xy}}{(e^{z}-2)}dx+\frac{xe^{-xy}}{(e^{z}-2)}dy$$

$$\frac{\partial z}{\partial x}=\frac{ye^{-xy}}{e^{z}-2},\qquad \frac{\partial z}{\partial y}=\frac{xe^{-xy}}{e^{z}-2}.$$

求偏导数的其它方法：利用全微分可求出各个偏导数。

例 5　设 $u=f(x,y)$ 有二阶连续偏导数，求 $\dfrac{\partial u}{\partial x}$ 和

$\dfrac{\partial^2 u}{\partial x^2}$ 在极坐标系中的表达式.

解　由直角坐标与极坐标的关系，得

$$u=f(\rho\cos\theta,\rho\sin\theta)=F(\rho,\theta)$$

由 $\rho=\sqrt{x^2+y^2}$， $\theta=\arctan\dfrac{y}{x}$ 或 $\theta=\pi+\arctan\dfrac{y}{x}$

得 $\dfrac{\partial\rho}{\partial x}=\dfrac{x}{\rho}=\cos\theta$， $\dfrac{\partial\rho}{\partial y}=\dfrac{y}{\rho}=\sin\theta$，

$\dfrac{\partial\theta}{\partial x}=\dfrac{-y}{\rho^2}=\dfrac{-\sin\theta}{\rho}$， $\dfrac{\partial\theta}{\partial y}=\dfrac{x}{\rho^2}=\dfrac{\cos\theta}{\rho}$

故 $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}=\frac{\partial u}{\partial \rho}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{\rho}$$

$$\frac{\partial^2 u}{\partial x^2}=\left(\frac{\partial u}{\partial \rho}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{\rho}\right)'_{\rho}\frac{\partial \rho}{\partial x}+\left(\frac{\partial u}{\partial \rho}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{\rho}\right)'_{\theta}\frac{\partial \theta}{\partial x}$$

$$=\left(\frac{\partial^2 u}{\partial \rho^2}\cos\theta-\frac{\partial^2 u}{\partial \theta\partial \rho}\frac{\sin\theta}{\rho}+\frac{\partial u}{\partial \theta}\frac{\sin\theta}{\rho^2}\right)\cos\theta$$

$$-\left(\frac{\partial^2 u}{\partial \rho\partial \theta}\cos\theta-\frac{\partial u}{\partial \rho}\sin\theta-\frac{\partial^2 u}{\partial \theta^2}\frac{\sin\theta}{\rho}-\frac{\partial u}{\partial \theta}\frac{\cos\theta}{\rho}\right)\frac{\sin\theta}{\rho}$$

$$=\frac{\partial^2 u}{\partial \rho^2}\cos^2\theta-\frac{\partial^2 u}{\partial \rho\partial \theta}\frac{\sin 2\theta}{\rho}+\frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{\rho^2}+\frac{\partial u}{\partial \theta}\frac{\sin 2\theta}{\rho^2}+\frac{\partial u}{\partial \rho}\frac{\sin^2\theta}{\rho}$$

历年研究生试题(多元复合函数求导法)

1.(87,3)设f和g为连续可微函数，$u = f(x, xy)$, $v = g(x + xy)$, 求 $\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x}$.

?

2.(88,6)设$u = yf(\frac{x}{y}) + xg(\frac{y}{x})$, 其中 f和g具有二阶连续导数，求 $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y}$.

?

3.(90,5)设$z = f(2x - y, y\sin x)$, 其中$f(u,v)$具有二阶连续偏导数，求 $\frac{\partial^2 z}{\partial x \partial y}$.

?

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4.(89,5)设$z = f(2x-y)+g(x,xy)$,其中函数$f(t)$二阶可导,$g(u,v)$具有二阶连续偏导数,求$\dfrac{\partial^2 z}{\partial x\partial y}$.

?

5.(92,5)设$z = f(e^x\sin y, x^2+y^2)$,其中$f$具有二阶连续偏导数,求$\dfrac{\partial^2 z}{\partial x\partial y}$.

?

6.(94,3)设$u = e^{-x}\sin\dfrac{x}{y}$,则$\dfrac{\partial^2 u}{\partial x\partial y}$在点$\left(2,\dfrac{1}{\pi}\right)$处的值为 ________

?

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7.(96,5)设变换 $\begin{cases} u = x - 2y \\ v = x + ay \end{cases}$ 可把方程

$$6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = 0$$ 简化为 $$\frac{\partial^2 z}{\partial u \partial v} = 0$$

求常数 a.

?

8.(98,3)设 $z = \frac{1}{x} f(xy) + y\varphi(x+y)$, f,φ 具有二阶连续导数，则 $\frac{\partial^2 z}{\partial x \partial y}$ ________.

?

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9.(00,5)设$z = f(xy, \frac{x}{y}) + g(\frac{y}{x})$,其中$f$具有二阶连续偏导数，$g$具有二阶连续导数，求$\frac{\partial^2 z}{\partial x \partial y}$.

?

10.(01,6)设$z = f(x, y)$在点$(1,1)$处可微，且

$$f(1,1) = 1, \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 2, \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = 3,$$

$\varphi(x) = f(x, f(x, x))$,求$\left.\frac{d}{dx}\varphi^3(x)\right|_{x=1}$.

?

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7.4.3 隐函数的微分法

1. 由一个方程确定的隐函数

(1). $F(x,y)=0$

隐函数存在定理 1 设函数$F(x,y)$在点$P(x_0,y_0)$的某一邻域内具有连续的偏导数，且$F(x_0,y_0)=0$，$F_y'(x_0,y_0)\neq 0$，则方程$F(x,y)=0$在点$P(x_0,y_0)$的某一邻域内恒能唯一确定一个单值连续且具有连续导数的函数$y=f(x)$，它满足条件$y_0=f(x_0)$，并有

$$\frac{dy}{dx}=-\frac{F_x'}{F_y'}$$

隐函数的求导公式

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例 5 已知 $\ln\sqrt{x^2+y^2} = \arctan\dfrac{y}{x}$，求 $\dfrac{dy}{dx}$.

解 令 $F(x,y) = \ln\sqrt{x^2+y^2} - \arctan\dfrac{y}{x}$

$$= \frac{1}{2}\ln(x^2+y^2) - \arctan\frac{y}{x}$$

则 $F'_x(x,y) = \dfrac{x+y}{x^2+y^2}$, $\quad F'_y(x,y) = \dfrac{y-x}{x^2+y^2}$,

$$\frac{dy}{dx} = -\frac{F'_x}{F'_y} = -\frac{x+y}{y-x}.$$

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(2). $F(x,y,z)=0$

隐函数存在定理 2 设函数$F(x,y,z)$在点$P(x_0,y_0,z_0)$的某一邻域内有连续的偏导数，且$F(x_0,y_0,z_0)=0$，$F_z'(x_0,y_0,z_0)\neq 0$，则方程$F(x,y,z)=0$在点$P(x_0,y_0,z_0)$的某一邻域内恒能唯一确定一个单值连续且具有连续偏导数的函数$z=f(x,y)$，它满足条件$z_0=f(x_0,y_0)$，并有

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'},\qquad \frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}.$$

例 6　设 $x^2+y^2+z^2-4z=0$，求 $\dfrac{\partial^2 z}{\partial x^2}$.

解　令　$F(x,y,z)=x^2+y^2+z^2-4z$,

则 $F'_x=2x$,　$F'_z=2z-4$, $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=\dfrac{x}{2-z}$,

$$\frac{\partial^2 z}{\partial x^2}=\frac{(2-z)+x\dfrac{\partial z}{\partial x}}{(2-z)^2}=\frac{(2-z)+x\cdot\dfrac{x}{2-z}}{(2-z)^2}$$

$$=\frac{(2-z)^2+x^2}{(2-z)^3}.$$

例 7　设$z = f(x+y+z,\ xyz)$，求$\dfrac{\partial z}{\partial x}$，$\dfrac{\partial x}{\partial y}$，$\dfrac{\partial y}{\partial z}$.

解　$u = x+y+z,\qquad v = xyz,$

则　$z = f(u,v),\qquad F = z - f(u,v)$

$$F'_x = -(F'_u \cdot 1 + F'_v \cdot yz)$$

$$F'_y = -(F'_u \cdot 1 + F'_v \cdot xz)$$

$$F'_z = 1 - (F'_u \cdot 1 + F'_v \cdot xy)$$

$$\frac{\partial z}{\partial x} = \frac{f'_u + yzf'_v}{1 - f'_u - xyf'_v},\qquad \frac{\partial x}{\partial y} = -\frac{f'_u + xzf'_v}{f'_u + yzf'_v},$$

$$\frac{\partial y}{\partial z} = \frac{1 - f'_u - xyf'_v}{f'_u + xzf'_v}.$$

2. 由方程组确定的隐函数 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$

设 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 具有连续偏导数，

并可确定 $u=u(x,y)$, $\quad v=v(x,y)$, 假定其

偏导数存在，求 $\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial x},\dfrac{\partial v}{\partial y}$.

隐函数存在定理 3　设$F(x,y,u,v)$、$G(x,y,u,v)$在点 $P(x_0,y_0,u_0,v_0)$ 的某一邻域内有对各个变量的连续偏导数，且$F(x_0,y_0,u_0,v_0)=0$，$G(x_0,y_0,u_0,v_0)=0$，且偏导数所组成的函数行列式（或称雅可比行列式）

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial v}\\ \dfrac{\partial G}{\partial u} & \dfrac{\partial G}{\partial v}\end{vmatrix}=\begin{vmatrix}F'_u & F'_v\\ G'_v & G'_v\end{vmatrix}$$

在点 $P(x_0,y_0,u_0,v_0)$ 不等于零，则方程组

$$F(x,y,u,v)=0、\quad G(x,y,u,v)=0$$

在点$P(x_0,y_0,u_0,v_0)$的某一邻域内恒能唯一确定一组单值连续且具有连续偏导数的函数

$u=u(x,y)$，$v=v(x,y)$，它们满足条件

$u_0=u(x_0,y_0)$，$v_0=v(x_0,y_0)$，并有

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{\begin{vmatrix}F'_x & F'_v\\ G'_x & G'_v\end{vmatrix}}{\begin{vmatrix}F'_u & F'_v\\ G'_u & G'_v\end{vmatrix}},$$

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$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}=-\begin{vmatrix}F'_u & F'_x\\ G'_u & G'_x\end{vmatrix}\Bigg/\begin{vmatrix}F'_u & F'_v\\ G'_u & G'_v\end{vmatrix},$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\begin{vmatrix}F'_y & F'_v\\ G'_y & G'_v\end{vmatrix}\Bigg/\begin{vmatrix}F'_u & F'_v\\ G'_u & G'_v\end{vmatrix},$$

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}=-\begin{vmatrix}F'_u & F'_y\\ G'_u & G'_y\end{vmatrix}\Bigg/\begin{vmatrix}F'_u & F'_v\\ G'_u & G'_v\end{vmatrix}.$$

特殊情形：如果方程组为 $\begin{cases} F(x,u,v)=0 \\ G(x,u,v)=0 \end{cases}$

设 $\begin{cases} F(x,u,v)=0 \\ G(x,u,v)=0 \end{cases}$ 具有连续偏导数，

并可确定隐函数 $u=u(x),\quad v=v(x)$，假定

两个隐函数导数都存在．

则可用上述方法求 $\dfrac{du}{dx},\dfrac{dv}{dx}$.

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例 8.设 $xu - yv = 0$， $yu + xv = 1$，

求 $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$ 和 $\dfrac{\partial v}{\partial y}$.

解1 直接代入公式;

解2 运用公式推导的方法，

将所给方程的两边对 x 求导并移项

$$\begin{cases} x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = -u \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} = -v \end{cases}, \qquad J = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2,$$

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在$J \neq 0$的条件下，

$$\frac{\partial u}{\partial x}=\frac{\begin{vmatrix}-u & -y\\ -v & x\end{vmatrix}}{\begin{vmatrix}x & -y\\ y & x\end{vmatrix}}=-\frac{xu+yv}{x^2+y^2},\quad \frac{\partial v}{\partial x}=\frac{\begin{vmatrix}x & -u\\ y & -v\end{vmatrix}}{\begin{vmatrix}x & -y\\ y & x\end{vmatrix}}=\frac{yu-xv}{x^2+y^2},$$

将所给方程的两边对y求导，用同样方法得

$$\frac{\partial u}{\partial y}=\frac{xv-yu}{x^2+y^2},\qquad \frac{\partial v}{\partial y}=-\frac{xu+yv}{x^2+y^2}.$$

小结

1、链式法则（分三种情况）

（特别要注意课中所讲的特殊情况——自变量　也是中间变量的情况；复合函数的高阶偏导数其结构与原复合函数相同）

2、全微分形式不变性　（理解其实质）

3、隐函数的求导法则（分以下几种情况）

(1)　$F(x,y)=0$

(2)　$F(x,y,z)=0$

(3)　$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$

思考题 设$z=f(u,v,x)$，而$u=\phi(x)$，$v=\psi(x)$，

则$$\frac{dz}{dx}=\frac{\partial f}{\partial u}\frac{du}{dx}+\frac{\partial f}{\partial v}\frac{dv}{dx}+\frac{\partial f}{\partial x},$$

试问$\frac{dz}{dx}$与$\frac{\partial f}{\partial x}$是否相同？为什么？

思考题解答 不相同.

等式左端的 z是作为一个自变量x的函数，

而等式右端最后一项 f 是作为u,v,x的三元函数，

写出来为

$$\left.\frac{dz}{dx}\right|_{x}=\left.\frac{\partial f}{\partial u}\right|_{(u,v,x)}\cdot\left.\frac{du}{dx}\right|_{x}+\left.\frac{\partial f}{\partial v}\right|_{(u,v,x)}\cdot\left.\frac{dv}{dx}\right|_{x}+\left.\frac{\partial f}{\partial x}\right|_{(u,v,x)}.$$

思考题 已知$\dfrac{x}{z}=\varphi(\dfrac{y}{z})$，其中$\varphi$为可微函数，

求$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=?$

思考题解答 记$F(x,y,z)=\dfrac{x}{z}-\varphi(\dfrac{y}{z})$， 则$F'_x=\dfrac{1}{z}$，

$F'_y=-\varphi'(\dfrac{y}{z})\cdot\dfrac{1}{z}$， $F'_z=\dfrac{-x}{z^2}-\varphi'(\dfrac{y}{z})\cdot\dfrac{(-y)}{z^2}$，

$$\frac{\partial z}{\partial x}=-\frac{F'_x}{F'_z}=\frac{z}{x-y\varphi'(\frac{y}{z})},\qquad \frac{\partial z}{\partial y}=-\frac{F'_y}{F'_z}=\frac{-z\varphi'(\frac{y}{z})}{x-y\varphi'(\frac{y}{z})},$$

于是$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z.$

历年研究生试题（隐函数求导法）

1.(91,3)由方程$xyz+\sqrt{x^2+y^2+z^2}=\sqrt{2}$所确定的函数

$z=z(x,y)$在点$(1,0,-1)$处的全微分$dz=$ ______

2.(95,5)设$u=f(x,y,z)$, $\varphi(x^2,e^y,z)=0$

$y=\sin x$,其中f,φ都具有一阶连续偏导数 ，

且$\dfrac{\partial\varphi}{\partial z}\neq 0$,求$\dfrac{du}{dx}$.

3.(99,5)设$y=y(x), z=z(x)$是由方程$z=xf(x+y)$

和$F(x,y,z)=0$所确定的函数，其中 f和F分别具有

一阶连续导数和一阶连 续偏导数,求$\dfrac{dz}{dx}$.

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1.(87,3)设f和g为连续可微函数， $u = f(x, xy)$,

$v = g(x + xy)$, 求 $\dfrac{\partial u}{\partial x} \cdot \dfrac{\partial v}{\partial x}$.

解:

$$\frac{\partial u}{\partial x} = f_1' + y f_2', \quad \frac{\partial v}{\partial x} = g'(x + xy)(1 + y)$$

则 $$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} = (f_1' + y f_2') \cdot g'(x + xy)(1 + y)$$

返回

2.(88,6)设$u = yf(\frac{x}{y}) + xg(\frac{y}{x})$,其中$f$和$g$具有二阶连续导数，求 $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y}$.

解: $$\frac{\partial u}{\partial x} = f'(\frac{x}{y}) + xg'(\frac{y}{x})\cdot(-\frac{y}{x^2}) + g(\frac{y}{x})$$

$$= f'(\frac{x}{y}) - \frac{y}{x}g'(\frac{y}{x}) + g(\frac{y}{x})$$

$$\frac{\partial^2 u}{\partial x^2} = f''(\frac{x}{y})\cdot\frac{1}{y} + \frac{y}{x^2}g'(\frac{y}{x}) + \frac{y^2}{x^3}g''(\frac{y}{x}) + g'(\frac{y}{x})(-\frac{y}{x^2})$$

$$= \frac{1}{y}f''(\frac{x}{y}) + \frac{y^2}{x^3}g''(\frac{y}{x})$$

$$\frac{\partial u}{\partial x} = f'(\frac{x}{y}) - \frac{y}{x} g'(\frac{y}{x}) + g(\frac{y}{x})$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{y} f''(\frac{x}{y}) + \frac{y^2}{x^3} g''(\frac{y}{x})$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{x}{y^2} f''(\frac{x}{y}) + \frac{1}{x} g'(\frac{y}{x}) - \frac{1}{x} g'(\frac{y}{x}) - \frac{y}{x^2} g''(\frac{y}{x})$$

$$= -\frac{x}{y^2} f''(\frac{x}{y}) - \frac{y}{x^2} g''(\frac{y}{x})$$

由此可得：$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = 0$

返回

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3.(90,5)设$z = f(2x - y, y\sin x)$,其中$f(u,v)$具有二阶连续偏导数,求$\dfrac{\partial^2 z}{\partial x \partial y}$.

返回

解: $\dfrac{\partial z}{\partial x} = 2f_1' + y\cos x f_2'$

$$\frac{\partial^2 z}{\partial x \partial y} = -2f_{11}'' + 2\sin x f_{12}'' + \cos x f_2' - y\cos x f_{21}'' + y\sin x \cos x f_{22}''$$

$$= -2f_{11}'' + (2\sin x - y\cos x) f_{12}'' + \cos x f_2' + y\sin x \cos x f_{22}''$$

4.(89,5)设$z = f(2x-y)+g(x,xy)$,其中函数$f(t)$二阶可导,$g(u,v)$具有二阶连续偏导数,求$\dfrac{\partial^2 z}{\partial x \partial y}$.

解:
$$\frac{\partial z}{\partial x} = 2f'(2x-y)+g_1'+g_2'y$$

$$\frac{\partial^2 z}{\partial x \partial y} = -2f''(2x-y)+xg_{12}''+g_2'+xyg_{22}''$$

返回

5.(92,5)设$z = f(e^x \sin y, x^2 + y^2)$,其中 f具有二阶连续偏导数，求 $\dfrac{\partial^2 z}{\partial x \partial y}$.

解: $\dfrac{\partial z}{\partial x} = e^x \sin y f_1' + 2x f_2'$

返回

$$\frac{\partial^2 z}{\partial x \partial y} = e^x \cos y f_1' + e^x \sin y[e^x \cos y f_{11}'' + 2y f_{12}'']$$

$$+ 2x[e^x \cos y f_{21}'' + 2y f_{22}'']$$

$$= e^{2x} \sin y \cos y f_{11}'' + 2e^x (y \sin y + x \cos y) f_{12}''$$

$$+ 4xy f_{22}'' + e^x \cos y f_1'$$

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6.(94,3)设 $u=e^{-x}\sin\frac{x}{y}$,则 $\frac{\partial^2 u}{\partial x\partial y}$ 在点 $\left(2,\frac{1}{\pi}\right)$ 处的值为 ________

解: $$\frac{\partial u}{\partial x}=-e^{-x}\sin\frac{x}{y}+\frac{e^{-x}}{y}\cos\frac{x}{y}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{xe^{-x}}{y^2}\cos\frac{x}{y}-\frac{e^{-x}}{y^2}\cos\frac{x}{y}+\frac{xe^{-x}}{y^3}\sin\frac{x}{y}$$

$$\left.\frac{\partial^2 u}{\partial x\partial y}\right|_{(2,\frac{1}{\pi})}=\left(\frac{\pi}{e}\right)^2$$

返回

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7.(96,5)设变换 $\begin{cases} u = x - 2y \\ v = x + ay \end{cases}$ 可把方程

$$6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = 0$$ 简化为 $$\frac{\partial^2 z}{\partial u \partial v} = 0$$

求常数 a.

分析：由题设知 $z = z(u,v)$, $u = x - 2y, v = x + ay$.

利用复合函数求导法求 出z关于x, y的二

阶偏导数，代入原方程 确定a.

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7.(96,5)设变换 $\begin{cases} u = x - 2y \\ v = x + ay \end{cases}$ 可把方程

$$6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = 0$$ 简化为 $$\frac{\partial^2 z}{\partial u \partial v} = 0$$

求常数 a.

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解: $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$, $\frac{\partial z}{\partial y} = -2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -2\frac{\partial^2 z}{\partial u^2} + (a-2)\frac{\partial^2 z}{\partial u \partial v} + a\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial y^2} = 4\frac{\partial^2 z}{\partial u^2} - 4a\frac{\partial^2 z}{\partial u \partial v} + a^2\frac{\partial^2 z}{\partial v^2}$$

返回

将上述结果代入原方程 并整理得

$$(10+5a)\frac{\partial^2 z}{\partial u \partial v} + (6+a-a^2)\frac{\partial^2 z}{\partial v^2} = 0$$

由题设知 $(10+5a) \neq 0 \quad (6+a-a^2) = 0$

解得: $a = 3$

8.(98,3)设$z=\frac{1}{x}f(xy)+y\varphi(x+y)$, f, φ具有二阶连续导数，则$\frac{\partial^2 z}{\partial x\partial y}$________.

解：$\frac{\partial z}{\partial x}=-\frac{1}{x^2}f(xy)+\frac{y}{x}f'(xy)+y\varphi'(x+y)$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{x}f'(xy)+\frac{1}{x}f'(xy)+yf''(xy)$$

$$+\varphi'(x+y)+y\varphi''(x+y)$$

$$=yf''(xy)+\varphi'(x+y)+y\varphi''(x+y)$$

9.(00,5)设$z = f(xy, \frac{x}{y}) + g(\frac{y}{x})$,其中 f具有二阶连续偏导数 ,g具有二阶连续导数 ,求 $\frac{\partial^2 z}{\partial x \partial y}$.

解: $$\frac{\partial z}{\partial x} = y f_1' + \frac{1}{y} f_2' - \frac{y}{x^2} g'$$

返回

$$\frac{\partial^2 z}{\partial x \partial y} = f_1' + y(x f_{11}'' - \frac{x}{y^2} f_{12}'') - \frac{1}{y^2} f_2'$$

$$+ \frac{1}{y}(x f_{21}'' - \frac{x}{y^2} f_{22}'') - \frac{1}{x^2} g' - \frac{y}{x^3} g''$$

$$= f_1' - \frac{1}{y^2} f_2' + yx f_{11}'' - \frac{x}{y^3} f_{22}'' - \frac{1}{x^2} g' - \frac{y}{x^3} g''$$

10.(01,6)设 $z=f(x,y)$ 在点 $(1,1)$ 处可微，且

$$f(1,1)=1,\left.\frac{\partial f}{\partial x}\right|_{(1,1)}=2,\left.\frac{\partial f}{\partial y}\right|_{(1,1)}=3,$$

$\varphi(x)=f(x,f(x,x))$，求 $\left.\dfrac{d}{dx}\varphi^3(x)\right|_{x=1}$.

解：$\varphi(1)=f(1,f(1,1))=f(1,1)=1$

返回

$$\left.\frac{d}{dx}\varphi^3(x)\right|_{x=1}=\left.\left[3\varphi^2(x)\frac{d\varphi(x)}{dx}\right]\right|_{x=1}$$

$$=3\varphi^2(x)\left.\left[f_1'(x,f(x,x))+f_2'(x,f(x,x))(f_1'(x,x)+f_2'(x,x))\right]\right|_{x=1}$$

$$=3\times1\times[2+3\times(2+3)]=51$$

1.(91,3)由方程$xyz+\sqrt{x^2+y^2+z^2}=\sqrt{2}$所确定的函数

$z=z(x,y)$在点$(1,0,-1)$处的全微分 $dz=$ ______

解1: $dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy$, 用隐函数求导法得:

$$\frac{\partial z}{\partial x}=-\frac{yz\sqrt{x^2+y^2+z^2}+x}{xy\sqrt{x^2+y^2+z^2}+z},\quad \left.\frac{\partial z}{\partial x}\right|_{(1,0,-1)}=1$$

$$\frac{\partial z}{\partial y}=-\frac{xz\sqrt{x^2+y^2+z^2}+y}{xy\sqrt{x^2+y^2+z^2}+z},\quad \left.\frac{\partial z}{\partial y}\right|_{(1,0,-1)}=-\sqrt{2}$$

$$dz=dx-\sqrt{2}dy$$

解2：方程 $xyz+\sqrt{x^2+y^2+z^2}=\sqrt{2}$两边求微分得

$$yzdx+xzdy+xydz+\frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}=0$$

将 $x=1,\quad y=0,\quad z=-1$代入上式得

$$-dy+\frac{dx-dz}{\sqrt{2}}=0$$

故 $\quad dz=dx-\sqrt{2}dy$

2.(95,5)设$u = f(x, y, z)$, $\varphi(x^2, e^y, z) = 0$

$y = \sin x$,其中f,φ都具有一阶连续偏导数 ,

且$\dfrac{\partial \varphi}{\partial z} \neq 0$,求$\dfrac{du}{dx}$.

分析：由于$y = \sin x, z = z(x)$由隐函数方程

$\varphi = 0$确定，因此， $u = f(x, y, z)$是

关于x的复合函数。

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2.(95,5)设 $u = f(x,y,z)$, $\varphi(x^2, e^y, z) = 0$

$y = \sin x$,其中 f, φ 都具有一阶连续偏导数,

且 $\dfrac{\partial \varphi}{\partial z} \neq 0$,求 $\dfrac{du}{dx}$.

解: $\dfrac{du}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dx}$, $\because \dfrac{dy}{dx} = \cos x$

$$\frac{dz}{dx} = -\frac{1}{\varphi_3'}(2x\varphi_1' + e^y \cos x \varphi_2')$$

返回

故 $$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cos x - \frac{\partial f}{\partial z}\frac{1}{\varphi_3'}(2x\varphi_1' + e^{\sin x}\cos x \varphi_2')$$

3.(99,5)设$y=y(x), z=z(x)$是由方程 $z=xf(x+y)$ 和$F(x,y,z)=0$所确定的函数，,其中 f和F分别具有一阶连续导数和一阶连 续偏导数，求 $\dfrac{dz}{dx}$.

分析：

求隐函数方程组所确定 的函数的导数，通常是在隐函数方程两边对 自变量求导。本题自变量x, y和z均是x的函数，在对两个隐函 数方程两边求导后，解出 $\dfrac{dz}{dx}$和$\dfrac{dy}{dx}$即可。

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解：等式$z = xf(x+y)$和$F(x,y,z)=0$两端

对x求导得

$$\begin{cases} \dfrac{dz}{dx} = f + x\left(1+\dfrac{dy}{dx}\right)f' \\ F'_x + F'_y\dfrac{dy}{dx} + F'_z\dfrac{dz}{dx} = 0 \end{cases}$$

消去$\dfrac{dy}{dx}$，解得

返回

$$\frac{dz}{dx} = \frac{(f+xf')F'_y - xf'F'_x}{F'_y + xf'F'_z} \qquad (F'_y + xf'F'_z \neq 0)$$

作业：

P72： 1. 至11.（共11个题）

13. 16. 17. 19. 20. 22. 23. 24.

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