习题 1.4(P50)

1 求下列函数的极限.

(1). $\lim\limits_{x\to 0}\dfrac{\tan kx}{x}$ (k为常数)

解： $\lim\limits_{x\to 0}\dfrac{\tan kx}{x}=\lim\limits_{x\to 0}\dfrac{k}{\cos kx}\cdot\dfrac{\sin kx}{kx}=k$

(2). $\lim\limits_{x\to 0^+}\dfrac{x}{\sqrt{1-\cos x}}$

解： $\lim\limits_{x\to 0^+}\dfrac{x}{\sqrt{1-\cos x}}=\lim\limits_{x\to 0^+}\dfrac{x}{\sqrt{2\sin^2\dfrac{x}{2}}}=\lim\limits_{x\to 0^+}\dfrac{x}{\sqrt{2}\sin\dfrac{x}{2}}$

$=\lim\limits_{x\to 0^+}\dfrac{2}{\sqrt{2}}\cdot\dfrac{\dfrac{x}{2}}{\sin\dfrac{x}{2}}=\dfrac{2}{\sqrt{2}}=\sqrt{2}$

(3). $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$

解： $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin x}{\cos x}(1-\cos x)}{x^3}=\lim\limits_{x\to 0}\dfrac{1}{\cos x}\cdot\dfrac{\sin x}{x}\cdot\dfrac{2\sin^2(x/2)}{x^2}$

$=\lim\limits_{x\to 0}\dfrac{1}{\cos x}\cdot\dfrac{\sin x}{x}\cdot\dfrac{2\sin^2(x/2)}{4(x/2)^2}=\dfrac{1}{2}$

(4). $\lim\limits_{x\to \pi}\dfrac{\sin 2x}{\sin 3x}$

解： $\lim\limits_{x\to \pi}\dfrac{\sin 2x}{\sin 3x}\overset{令t=x-\pi}{=\!=\!=\!=}\lim\limits_{t\to 0}\dfrac{\sin(2\pi+2t)}{\sin(3\pi+3t)}=\lim\limits_{t\to 0}\dfrac{\sin 2t}{-\sin 3t}$

$=-\lim\limits_{t\to 0}\dfrac{\sin 2t}{2t}\cdot\dfrac{3t}{\sin 3t}\cdot\dfrac{2}{3}=-\dfrac{2}{3}$

(5). $\lim\limits_{x\to 0^+}\dfrac{\cos x-1}{x^{\frac{3}{2}}}$

解： $\lim\limits_{x\to0^+}\dfrac{\cos x-1}{x^{\frac{3}{2}}}=\lim\limits_{x\to0^+}\dfrac{-2\sin^2\frac{x}{2}}{x^{\frac{3}{2}}}=-\lim\limits_{x\to0^+}\dfrac{2x^{\frac{1}{2}}}{4}\cdot\dfrac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}=0$

(6). $\lim\limits_{x\to1}(1-x)\tan\dfrac{\pi x}{2}$

解： $\lim\limits_{x\to1}(1-x)\tan\dfrac{\pi x}{2}\overset{令t=1-x}{=\!=\!=\!=}\lim\limits_{t\to0}t\cdot\tan(\dfrac{\pi}{2}-\dfrac{\pi t}{2})=\lim\limits_{t\to0}t\cdot\cot(\dfrac{\pi t}{2})$

$$=\frac{2}{\pi}\lim_{t\to0}\frac{\frac{\pi t}{2}}{\sin(\frac{\pi t}{2})}\cdot\cos(\frac{\pi t}{2})=\frac{2}{\pi}$$

(7). $\lim\limits_{x\to0}x\cot2x$

解： $\lim\limits_{x\to0}x\cot2x=\lim\limits_{x\to0}\dfrac{2x}{\sin2x}\cdot\dfrac{\cos2x}{2}=\dfrac{1}{2}$

（8）. $\lim\limits_{x\to0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2x}$

解： $\lim\limits_{x\to0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2x}=\lim\limits_{x\to0}\dfrac{1-\cos x}{\sin^2x\,(\sqrt{2}+\sqrt{1+\cos x})}=\dfrac{1}{2\sqrt{2}}\lim\limits_{x\to0}\dfrac{2\sin^2(x/2)}{\sin^2x}$

$$=\frac{1}{2\sqrt{2}}\lim_{x\to0}\frac{\sin^2(x/2)}{2(x/2)^2}\cdot\frac{x^2}{\sin^2x}=\frac{1}{4\sqrt{2}}\lim_{x\to0}\left(\frac{\sin(x/2)}{(x/2)}\right)^2\cdot\left(\frac{x}{\sin x}\right)^2=\frac{\sqrt{2}}{8}$$

（9）. $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}$

解： $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}\overset{令t=x-a}{=\!=\!=\!=}\lim\limits_{t\to0}\dfrac{\sin(a+t)-\sin a}{t}=\lim\limits_{t\to0}\dfrac{2\cos(a+\frac{t}{2})\sin\frac{t}{2}}{t}$

$$=\lim_{t\to0}\cos(a+\frac{t}{2})\cdot\frac{\sin t/2}{t/2}=\cos a$$

（10）. $\lim\limits_{x\to\infty}x\arcsin\dfrac{n}{x}\quad(n\in N^+)$

解： $\lim\limits_{x\to\infty} x\arcsin\frac{n}{x} \overset{\text{令}t=\arcsin\frac{n}{x}}{\underset{x=\frac{n}{\sin t}}{=\!=\!=}} \lim\limits_{t\to0}\frac{n}{\sin t}\cdot t = n$

（11）. $\lim\limits_{x\to0}\frac{\sqrt{2+\tan x}-\sqrt{2+\sin x}}{x^3}$

解： $\lim\limits_{x\to0}\frac{\sqrt{2+\tan x}-\sqrt{2+\sin x}}{x^3} = \lim\limits_{x\to0}\frac{\tan x-\sin x}{x^3(\sqrt{2+\tan x}+\sqrt{2+\sin x})} \overset{\text{由题(2)}}{\underset{\text{即得}}{=\!=}} \frac{\sqrt{2}}{8}$

（12）. $\lim\limits_{x\to\frac{\pi}{6}}\tan 3x\cdot\tan\left(\frac{\pi}{6}-x\right)$

解： $\lim\limits_{x\to\frac{\pi}{6}}\tan 3x\cdot\tan\left(\frac{\pi}{6}-x\right) \overset{\text{令t}=\frac{\pi}{6}-x}{=\!=\!=} \lim\limits_{t\to0}\tan 3\left(\frac{\pi}{6}-t\right)\cdot\tan t$

$=\lim\limits_{t\to0}\tan\left(\frac{\pi}{2}-3t\right)\cdot\tan t = \lim\limits_{t\to0}\cot 3t\cdot\tan t = \frac{1}{3}\lim\limits_{t\to0}\frac{\cos 3t}{\cos t}\cdot\frac{3t}{\sin 3t}\cdot\frac{\sin t}{t} = \frac{1}{3}$

（13）. $\lim\limits_{x\to\frac{\pi}{3}}\frac{1-2\cos x}{\sin\left(x-\frac{\pi}{3}\right)}$

解： $\lim\limits_{x\to\frac{\pi}{3}}\frac{1-2\cos x}{\sin\left(x-\frac{\pi}{3}\right)} \overset{\text{令t}=x-\frac{\pi}{3}}{=\!=\!=} \lim\limits_{t\to0}\frac{1-2\cos(t+\frac{\pi}{3})}{\sin t} = \lim\limits_{t\to0}\frac{1-2[\cos t\cos\frac{\pi}{3}-\sin t\sin\frac{\pi}{3}]}{\sin t}$

$=\lim\limits_{t\to0}\frac{1-\cos t+\sqrt{3}\sin t}{\sin t} = \lim\limits_{t\to0}\frac{2\sin^2(t/2)}{\sin t}+\sqrt{3} = \sqrt{3}$

（14）. $\lim\limits_{x\to0}(1-x)^{\frac{1}{x}}$

解法 1： $\lim\limits_{x\to0}(1-x)^{\frac{1}{x}} = \lim\limits_{x\to0}\left\{(1-x)^{-\frac{1}{x}}\right\}^{-1} = e^{-1}$

解法 2：因为当 $x\to0$ 时， $f(x)=-x\to0$， $g(x)=\frac{1}{x}\to\infty$， $\lim\limits_{x\to0}f(x)g(x)=-1$

故 $\lim\limits_{x\to0}(1-x)^{\frac{1}{x}} = e^{-1}$

（15）． $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^{x}$

解法 1： $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^{x}=\lim\limits_{x\to\infty}\left(1-\dfrac{1}{1+x}\right)^{x}$

$$=\lim_{x\to\infty}\left\{\left(1-\frac{1}{1+x}\right)^{-(x+1)}\right\}^{-1}\cdot\left(1-\frac{1}{1+x}\right)^{-1}=e^{-1}$$

解法 2：因为当 $x\to\infty$ 时，$f(x)=-\dfrac{1}{1+x}\to 0$，$g(x)=x\to\infty$，$\lim\limits_{x\to 0}f(x)g(x)=-1$

故 $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^{x}=e^{-1}$

（16）． $\lim\limits_{x\to\infty}\left(\dfrac{3-2x}{2-2x}\right)^{x}$

解法 1： $\lim\limits_{x\to\infty}\left(\dfrac{3-2x}{2-2x}\right)^{x}=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{2-2x}\right)^{x}$

$$=\lim_{x\to\infty}\left\{\left(1+\frac{1}{2-2x}\right)^{2-2x}\right\}^{-\frac{1}{2}}\cdot\left(1+\frac{1}{2-2x}\right)=e^{-\frac{1}{2}}$$

解法 2：因为当 $x\to\infty$ 时，$f(x)=\dfrac{1}{2-2x}\to 0$，$g(x)=x\to\infty$，$\lim\limits_{x\to\infty}f(x)g(x)=-\dfrac{1}{2}$

故 $\lim\limits_{x\to\infty}\left(\dfrac{3-2x}{2-2x}\right)^{x}=e^{-\frac{1}{2}}$

（17）． $\lim\limits_{x\to 0}\left(1+\dfrac{x}{2}\right)^{\frac{x-1}{x}}$

解法 1： $\lim\limits_{x\to 0}\left(1+\dfrac{x}{2}\right)^{\frac{x-1}{x}}=\lim\limits_{x\to 0}\left(1+\dfrac{x}{2}\right)^{1-\frac{1}{x}}=\lim\limits_{x\to 0}\left\{\left(1+\dfrac{x}{2}\right)^{\frac{2}{x}}\right\}^{-\frac{1}{2}}\cdot\left(1+\dfrac{x}{2}\right)=e^{-\frac{1}{2}}$

解法 2：因为当 $x\to 0$ 时， $f(x)=\dfrac{x}{2}\to 0$， $g(x)=\dfrac{x-1}{x}\to\infty$， $\lim\limits_{x\to 0}f(x)g(x)=-\dfrac{1}{2}$

故 $\lim\limits_{x\to 0}\left(1+\dfrac{x}{2}\right)^{\frac{x-1}{x}}=e^{-\frac{1}{2}}$

（18）. $\lim\limits_{x\to\infty}\left(\dfrac{x^2}{x^2-1}\right)^x$

解法 1： $\lim\limits_{x\to\infty}\left(\dfrac{x^2}{x^2-1}\right)^x=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x^2-1}\right)^x=\lim\limits_{x\to\infty}\left\{\left(1+\dfrac{1}{x^2-1}\right)^{x^2}\right\}^{\frac{1}{x}}$

$$=\lim_{x\to\infty}\left\{\left(1+\frac{1}{x^2-1}\right)^{x^2-1}\right\}^{\frac{1}{x}}\cdot\left(1+\frac{1}{x^2-1}\right)^{\frac{1}{x}}=e^0\times1^0=1$$

解法 2：因为当 $x\to\infty$ 时， $f(x)=\dfrac{1}{x^2-1}\to 0$， $g(x)=x\to\infty$， $\lim\limits_{x\to 0}f(x)g(x)=0$

$$\text{故}\lim_{x\to\infty}\left(\frac{x^2}{x^2-1}\right)^x=e^0=1$$

（19）. $\lim\limits_{x\to\infty}\left(\dfrac{x^2-1}{x^2+1}\right)^{x^2}$

解法 1： $\lim\limits_{x\to\infty}\left(\dfrac{x^2-1}{x^2+1}\right)^{x^2}=\lim\limits_{x\to\infty}\left(1-\dfrac{2}{x^2+1}\right)^{x^2}$

$$=\lim_{x\to\infty}\left\{\left(1-\frac{2}{x^2+1}\right)^{-\frac{x^2+1}{2}}\right\}^{-2}\cdot\left(1-\frac{2}{x^2+1}\right)^{-1}=e^{-2}$$

解法 2:因为当 $x\to\infty$ 时，$f(x)=-\dfrac{2}{x^2+1}\to 0$，$g(x)=x^2\to\infty$，$\lim\limits_{x\to 0}f(x)g(x)=-2$

$$\text{故}\lim_{x\to\infty}\left(\frac{x^2-1}{x^2+1}\right)^{x^2}=e^{-2}$$

（20）. $\lim\limits_{x\to 0}\dfrac{\arcsin x}{x}$

解： $\lim\limits_{x\to 0}\dfrac{\arcsin x}{x}\overset{\text{令}t=\arcsin x}{=\!=\!=\!=}\lim\limits_{t\to 0}\dfrac{t}{\sin t}=\lim\limits_{t\to 0}\dfrac{1}{\sin t/t}=1$

2. 已知 $\lim\limits_{x\to\infty}\left(\dfrac{x-2}{x}\right)^{kx}=\dfrac{1}{e}$， 求常数 k .

解：$\lim\limits_{x\to\infty}\left(\dfrac{x-2}{x}\right)^{kx}=\lim\limits_{x\to\infty}\left(1-\dfrac{2}{x}\right)^{kx}$

因为当 $x\to\infty$ 时，$f(x)=-\dfrac{2}{x}\to 0$，$g(x)=kx\to\infty$，$\lim\limits_{x\to 0}f(x)g(x)=-2k$

故 $\lim\limits_{x\to\infty}\left(\dfrac{x-2}{x}\right)^{kx}=e^{-2k}=\dfrac{1}{e}$，即 $-2k=-1$，得 $k=\dfrac{1}{2}$

3. 讨论函数 $f(x)=\begin{cases}\dfrac{\sin x}{x} & x<0\\ (1+x)^{\frac{1}{x}} & x>0\end{cases}$，当 $x\to 0$ 时，极限是否存在.

解：$f(0-0)=\lim\limits_{x\to 0^-}\dfrac{\sin x}{x}=1$，$f(0+0)=\lim\limits_{x\to 0^+}(1+x)^{\frac{1}{x}}=e$

$f(0-0)\neq f(0+0)$，故当 $x\to 0$ 时，极限不存在.

4. 计算 $\lim\limits_{n\to\infty}\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{2^2}\cdots\cdots\cos\dfrac{\theta}{2^n}$，$\theta$ 为任意非零常数.

解：$\lim\limits_{n\to\infty}\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{2^2}\cdots\cdots\cos\dfrac{\theta}{2^n}=\lim\limits_{n\to\infty}\dfrac{\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{2^2}\cdots\cdots\cos\dfrac{\theta}{2^n}\cdot 2^n\sin\dfrac{\theta}{2^n}}{2^n\sin\dfrac{\theta}{2^n}}$

$$=\lim_{n\to\infty}\frac{\sin\theta}{2^n\sin\dfrac{\theta}{2^n}}=\lim_{n\to\infty}\frac{\sin\theta}{\theta}=\lim_{n\to\infty}\frac{\dfrac{\theta}{2^n}}{\sin\dfrac{\theta}{2^n}}=\frac{\sin\theta}{\theta}$$