7.7 二元函数的泰勒公式

一元函数的泰勒公式：

$$f(x)=f(x_0)+f'(x_0)(x-x_0)$$
$$+\frac{f''(x_0)}{2}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$
$$+\frac{f^{(n+1)}\big(x_0+\theta(x-x_0)\big)}{(n+1)!}(x-x_0)^{n+1}\qquad(0<\theta<1).$$

意义：可用n次多项式来近似表达函数$f(x)$，且误差是当$x\to x_0$时比$(x-x_0)^n$高阶的无穷小.

一元函数的泰勒公式(令 $x=x_0+h$):

$$f(x_0+h)=f(x_0)+f'(x_0)h+\frac{f''(x_0)}{2}h^2+\cdots$$

$$+\frac{f^{(n)}(x_0)}{n!}h^n+\frac{f^{(n+1)}\left(x_0+\theta h\right)}{(n+1)!}h^{n+1}\quad(0<\theta<1).$$

一元函数的麦克劳林公式:

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n$$

$$+\frac{f^{(n+1)}\left(\theta x\right)}{(n+1)!}x^{n+1}\quad(0<\theta<1).$$

问题 能否用两个变量的多项式来近似表达一个给定的二元函数，并能具体地估算出误差的大小.

即 设$z=f(x,y)$在点$P(x_0,y_0)$的某一邻域内连续且有直到$n+1$阶的连续偏导数，$P'(x_0+h,y_0+k)$为此邻域内任一点，能否把函数$f(x_0+h,y_0+k)$近似地表达为$h=x-x_0,k=y-y_0$的 n次多项式，且误差是当$\rho=\sqrt{h^2+k^2}\to 0$时比ρ^n高阶的无穷小.

定理 设$z=f(x,y)$在点(x_0,y_0)的某一邻域内连续且有直到$n+1$阶的连续偏导数，(x_0+h,y_0+k)为此邻域内任一点，则有

$$f(x_0+h,y_0+k)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)$$

$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k),\qquad (0<\theta<1)$$

其中记号$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)\stackrel{\Delta}{=}hf_x'(x_0,y_0)+kf_y'(x_0,y_0)$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)\stackrel{\Delta}{=}h^2f_{xx}''(x_0,y_0)+2hkf_{xy}''(x_0,y_0)+k^2f_{yy}''(x_0,y_0)$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{m} f(x_0,y_0) \triangleq \sum_{p=0}^{m} C_m^p h^p k^{m-p} \frac{\partial^m f}{\partial x^p \partial y^{m-p}}\bigg|_{(x_0,y_0)}$$

上述公式中

$$R_n=\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k), \quad (0<\theta<1)$$

称为拉格朗日型余项

故上述公式称为二元函数 $f(x,y)$ 在点 (x_0,y_0) 处的 n 阶带拉格朗日型余项的泰勒公式。

证 设$F(t)=f(x_0+ht,y_0+kt)$, $(0\le t\le 1)$.

显然 $F(0)=f(x_0,y_0)$, $F(1)=f(x_0+h,y_0+k)$.

由$F(t)$的定义及多元复合函数的求导法则,可得

$$F'(t)=hf'_x(x_0+ht,y_0+kt)+kf'_y(x_0+ht,y_0+kt)$$

$$=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0+ht,y_0+kt),$$

$$F''(t)=h^2f''_{xx}(x_0+ht,y_0+kt)$$

$$+2hkf''_{xy}(x_0+ht,y_0+kt)+k^2f''_{yy}(x_0+ht,y_0+kt)$$

…… …… …… ……

$$F^{(n+1)}(t)=\sum_{p=0}^{n+1} C_{n+1}^{p} h^{p} k^{n+1-p} \left.\frac{\partial^{n+1} f}{\partial x^{p} \partial y^{n+1-p}}\right|_{(x_0+ht,\,y_0+kt)}$$

$$=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+ht,\,y_0+kt).$$

利用一元函数的麦克劳林公式，得

$$F(1)=F(0)+F'(0)+\frac{1}{2!}F''(0)+\cdots$$

$$+\frac{1}{n!}F^{(n)}(0)+\frac{1}{(n+1)!}F^{(n+1)}(\theta),\quad (0<\theta<1)$$

将 $F(0)=f(x_0,y_0)$，$F(1)=f(x_0+h,y_0+k)$ 及上面求得的 $F(t)$ 直到 n 阶导数在 $t=0$ 的值，以及 $F^{(n+1)}(t)$ 在 $t=\theta$ 的值代入上式. 即得

$$f(x_0+h,y_0+k)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)+R_n$$

其中

$$R_n=\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k)\quad(0<\theta<1)$$

在n阶泰勒公式中, 如果取$x_0=0, y_0=0$, 则称为n阶麦克劳林公式.

$$f(x,y)=f(0,0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)$$

$$+\frac{1}{2!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0,0)+\cdots+\frac{1}{n!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^n f(0,0)$$

$$+\frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{n+1} f(\theta x,\theta y),$$

$$(0<\theta<1)$$

由二元函数的泰勒公式知:

余项R_n中的各阶偏导数连续,故它们的绝对值在点(x_0, y_0)的某一邻域内都不超过某一正的常数M.

于是, 有下面的误差估计式:

$$\left|R_n\right| \le \frac{M}{(n+1)!}\left(|h| + |k|\right)^{n+1}$$

$$= \frac{M}{(n+1)!}\rho^{n+1}\left(|\cos\alpha| + |\sin\alpha|\right)^{n+1} \quad (\rho = \sqrt{h^2 + k^2})$$

$$\le \frac{2^{n+1}}{(n+1)!}M\rho^{n+1},$$

由上式得, 误差$\left|R_n\right|$是当$\rho \to 0$时比ρ^n高阶无穷小.

则泰勒公式的余项 $R_n = o(\rho^n)$

这种形式的余项称为皮亚诺型余项.

带拉格朗日余项的一阶泰勒公式为

$$f(x_0+h, y_0+k)$$

$$= f(x_0, y_0) + f_x'(x_0, y_0)h + f_y'(x_0, y_0)k + R_1$$

其中

$$R_1 = \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0+\theta h, y_0+\theta k) \quad (0<\theta<1)$$

带皮亚诺余项时, $R_1 = o(\rho)$ $\quad (\rho = \sqrt{h^2+k^2})$

带拉格朗日余项的二阶泰勒公式为

$$f(x_0+h, y_0+k)$$

$$= f(x_0, y_0) + f'_x(x_0, y_0)h + f'_y(x_0, y_0)k$$

$$+ \frac{1}{2!}[f''_{x^2}(x_0, y_0)h^2 + 2f''_{xy}(x_0, y_0)hk + f''_{y^2}(x_0, y_0)k^2]$$

$$+ R_2$$

其中

$$R_2 = \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0+\theta h, y_0+\theta k) \quad (0<\theta<1)$$

带皮亚诺余项时，$R_2 = o(\rho^2) \quad (\rho = \sqrt{h^2+k^2})$

带拉格朗日余项的二阶麦克劳林公式为

$$f(x,y)$$

$$= f(0,0) + f'_x(0,0)x + f'_y(0,0)y$$

$$+ \frac{1}{2!}[f''_{x^2}(0,0)x^2 + 2f''_{xy}(0,0)xy + f''_{y^2}(0,0)y^2]$$

$$+ R_2$$

其中 $R_2 = \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(\theta x, \theta y) \quad (0 < \theta < 1)$

带皮亚诺余项时，$R_2 = o(\rho^2) \quad (\rho = \sqrt{x^2 + y^2})$

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例 1（书中例1）求函数 $f(x,y)=\ln(1+x+y)$ 的二阶麦克劳林公式（带拉格朗日余项）.

解 $f(0,0)=0$

$$f_x'(x,y)=\frac{1}{1+x+y}=f_y'(x,y)$$

$$f_{xx}''(x,y)=-\frac{1}{(1+x+y)^2}=f_{xy}''(x,y)=f_{yy}''(x,y)$$

$$\frac{\partial^3 f}{\partial x^p\partial y^{3-p}}=\frac{2!}{(1+x+y)^3},\qquad (p=0,1,2,3)$$

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)=xf_x(0,0)+yf_y(0,0)=x+y$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right)^2 f(0,0)$$

$$= x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0) = -(x+y)^2$$

故

$$\ln(1 + x + y) = x + y - \frac{1}{2}(x+y)^2 + R_2$$

其中 $R_2 = \dfrac{1}{3!}\left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} \right)^3 f(\theta x, \theta y)$

$$= \frac{1}{3} \cdot \frac{(x+y)^3}{(1 + \theta x + \theta y)^3}, \qquad (0 < \theta < 1)$$

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小结

1、二元函数的泰勒公式

$$f(x_0+h,y_0+k)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)+R_n$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m f(x_0,y_0)\triangleq\sum_{p=0}^{m}C_m^p h^p k^{m-p}\frac{\partial^m f}{\partial x^p\partial y^{m-p}}\bigg|_{(x_0,y_0)}$$

$$R_n=\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k),\quad (0<\theta<1)$$

或 $R_n=o(\rho^n)$

2、 n 阶麦克劳林公式

$$f(x,y)=f(0,0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)$$

$$+\frac{1}{2!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0,0)+\cdots+\frac{1}{n!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^n f(0,0)+R_n$$

其中 $\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^m f(0,0)\overset{\Delta}{=}\sum_{p=0}^{m}C_m^p x^p y^{m-p}\frac{\partial^m f}{\partial x^p\partial y^{m-p}}\Big|_{(0,0)}$

$$R_n=\frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{n+1} f(\theta x,\theta y),\quad (0<\theta<1)$$

或 $R_n=o(\rho^n)$

作业：

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