# 习题 4.2(P217)

1．求下列导数.

(1) $\frac{d}{dx}\int_x^1 \frac{\sin t}{t}dt$

解：$\frac{d}{dx}\int_x^1 \frac{\sin t}{t}dt = -\frac{d}{dx}\int_1^x \frac{\sin t}{t}dt = -\frac{\sin x}{x}$

(2) $\frac{d}{dx}\int_0^{x^2}\sqrt{1+t^2}dt$

解：$\frac{d}{dx}\int_0^{x^2}\sqrt{1+t^2}dt = \sqrt{1+(x^2)^2}\cdot(x^2)' = 2x\sqrt{1+x^4}$

(3) $\frac{d}{dx}\int_{\sin x}^2 e^{t^2}dt$

解：$\frac{d}{dx}\int_{\sin x}^2 e^{t^2}dt = -e^{\sin^2 x}(\sin x)' = -e^{\sin^2 x}\cos x$

(4) $\frac{d}{dx}\int_{x^2}^{e^x}\frac{\ln t}{t}dt$

解：$\frac{d}{dx}\int_{x^2}^{e^x}\frac{\ln t}{t}dt = \frac{\ln e^x}{e^x}(e^x)' - \frac{\ln x^2}{x^2}(x^2)' = x - \frac{4\ln x}{x}$

2. 设$\int_0^y e^t dt + 3\int_0^x \cos t dt = 0$，求$\frac{dy}{dx}$.

解：两端同时对 $x$ 求导，得$e^y\cdot\frac{dy}{dx} + 3\cos x = 0$，所以$\frac{dy}{dx} = -\frac{3\cos x}{e^y}$

3．设$\begin{cases} x = \int_1^t u\ln u du \\ y = \int_t^1 u^2\ln u du \end{cases}$，求$\frac{dy}{dx}$.

解：$\frac{dy}{dt} = -t^2\ln t$，$\frac{dx}{dt} = t\ln t$，所以$\frac{dy}{dx} = \frac{-t^2\ln t}{t\ln t} = -t$

4．求下列极限.

(1) $\lim\limits_{x\to0}\dfrac{\int_0^x\cos t^2dt}{x}$

解：$\lim\limits_{x\to0}\dfrac{\int_0^x\cos t^2dt}{x}=\lim\limits_{x\to0}\dfrac{\cos x^2}{1}=1$

(2) $\lim\limits_{x\to0}\dfrac{\int_{\cos x}^1 e^{-t^2}dt}{x^2}$

解：$\lim\limits_{x\to0}\dfrac{\int_{\cos x}^1 e^{-t^2}dt}{x^2}=\lim\limits_{x\to0}\dfrac{-e^{-\cos^2 x}(-\sin x)}{2x}=\dfrac{e^{-1}}{2}=\dfrac{1}{2e}$

(3) $\lim\limits_{x\to0}\dfrac{\int_0^{\sin x}\sqrt{\tan t}dt}{\int_0^{\tan x}\sqrt{\sin t}dt}$

解：$\lim\limits_{x\to0}\dfrac{\int_0^{\sin x}\sqrt{\tan t}dt}{\int_0^{\tan x}\sqrt{\sin t}dt}=\lim\limits_{x\to0}\dfrac{\sqrt{\tan(\sin x)}\cdot\cos x}{\sqrt{\sin(\tan x)}\sec^2 x}=\lim\limits_{x\to0}\dfrac{\sqrt{\tan(\sin x)}}{\sqrt{\sin(\tan x)}}$

$=\lim\limits_{x\to0}\dfrac{x\sqrt{\tan(\sin x)}}{x\sqrt{\sin(\tan x)}}\overset{\text{无穷小}}{\underset{\text{代换}}{=\!=}}\lim\limits_{x\to0}\dfrac{\tan x}{\sqrt{\sin(\tan x)}}\cdot\dfrac{\sqrt{\tan(\sin x)}}{\sin x}$

$=\lim\limits_{x\to0}\dfrac{\tan x}{\sqrt{\sin(\tan x)}}\cdot\lim\limits_{x\to0}\dfrac{\sqrt{\tan(\sin x)}}{\sin x}=\lim\limits_{t\to0}\dfrac{t}{\sqrt{\sin t}}\cdot\lim\limits_{t\to0}\dfrac{\sqrt{\tan t}}{t}$

$=\lim\limits_{t\to0}\dfrac{\sqrt{\tan t}}{\sqrt{\sin t}}=\sqrt{\lim\limits_{x\to0}\dfrac{\tan t}{\sin t}}\overset{\text{无穷小}}{\underset{\text{代换}}{=\!=}}\sqrt{\lim\limits_{x\to0}\dfrac{t}{t}}=1$

(4) $\lim\limits_{x\to0}\dfrac{\int_0^{x^2}e^{t^2}dt}{\int_0^x te^{2t^2}dt}$ 或 $\lim\limits_{x\to0}\dfrac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x te^{2t^2}dt}$ （书中习题有错误）

解：$\lim\limits_{x\to0}\dfrac{\int_0^{x^2}e^{t^2}dt}{\int_0^x te^{2t^2}dt}=\lim\limits_{x\to0}\dfrac{2xe^{x^4}}{xe^{2x^2}}=2$

或 $\lim\limits_{x\to0}\dfrac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x te^{2t^2}dt}=\lim\limits_{x\to0}\dfrac{2\left(\int_0^x e^{t^2}dt\right)e^{x^2}}{xe^{2x^2}}=\lim\limits_{x\to0}\dfrac{2\left(\int_0^x e^{t^2}dt\right)}{x}=\lim\limits_{x\to0}\dfrac{2e^{x^2}}{1}=2$

5．设 $F(x)=\int_0^x(t^2-x^2)f'(t)dt$，求 $F'(x)$

解：$F(x)=\int_0^x t^2 f'(t)dt - x^2\int_0^x f'(t)dt$，

则 $F'(x)=x^2 f'(x)-2x\int_0^x f'(t)dt - x^2 f'(x)$

$=-2x\int_0^x f'(t)dt=-2x[f(x)\big|_0^x]=-2x[f(x)-f(0)]$

6. 求 $F(x)=\int_0^x te^{-t^2}dt$ 的极值.

解：令 $F'(x)=xe^{-x^2}=0$，得驻点 $x=0$，且当 $x\in(-\delta,0)$ 时，$F'(x)<0$，当 $x\in(0,\delta)$ 时，$F'(x)>0$，故 $x=0$ 为极小值点，极小值为 $F(0)=0$

7．设 $f(x)$ 为连续正值函数，证明：当 $x>0$ 时，$F(x)=\dfrac{\int_0^x tf(t)dt}{\int_0^x f(t)dt}$ 为单调增加函数.

解：$F'(x)=\dfrac{xf(x)\int_0^x f(t)dt - f(x)\int_0^x tf(t)dt}{\left(\int_0^x f(t)dt\right)^2}=\dfrac{f(x)\int_0^x (x-t)f(t)dt}{\left(\int_0^x f(t)dt\right)^2}$，

由假设，当 $0<t<x$ 时，$f(t)>0$，$(x-t)f(t)>0$，所以 $\int_0^x (x-t)f(t)dt>0$，所以 $F'(x)>0(x>0)$，从而当 $x>0$ 时，$F(x)$ 为单调增加函数

8．计算下列定积分.

(1) $\int_1^3 x^3dx$

解：$\int_1^3 x^3dx=\dfrac{1}{4}x^4\Big|_1^3=\dfrac{1}{4}(3^4-1)=20$

(2) $\int_4^9 \sqrt{x}(1+\sqrt{x})dx$

解：$\int_4^9 \sqrt{x}(1+\sqrt{x})dx=\int_4^9 (x^{\frac{1}{2}}+x)dx=(\dfrac{2}{3}x^{\frac{3}{2}}+\dfrac{1}{2}x^2)\Big|_4^9=\dfrac{271}{6}=45\dfrac{1}{6}$

(3) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec x\tan xdx$

解：被积函数是奇函数，且积分区间关于原点对称，所以 $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec x \tan x dx = 0$

(4) $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}}$

解：$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin x\Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{\pi}{6}$

(5) $\int_0^2 |1-x| dx$

解：$\int_0^2 |1-x| dx = \int_0^1 (1-x)dx + \int_1^2 (x-1)dx = (x - \frac{1}{2}x^2)\Big|_0^1 + (\frac{1}{2}x^2 - x)\Big|_0^1 = \frac{1}{2} + \frac{1}{2} = 1$

(6) $\int_2^3 (x+1)e^x dx$

解：$\int_2^3 (x+1)e^x dx = +\int_2^3 (e^x + xe^x)dx = xe^x\Big|_2^3 = 3e^3 - 2e^2$

9. 设 $f(x) = \begin{cases} x^2 & x \in [0,1) \\ x & x \in [1,2] \end{cases}$，求 $\Phi(x) = \int_0^x f(t)dt$ 在 $[0,2]$ 上的表达式，并讨论 $\Phi(x)$ 在 $(0,2)$ 内的连续性.

解：当 $x \in [0,1)$ 时，$\Phi(x) = \int_0^x t^2 dt = \frac{x^3}{3}$，

当 $x \in [1,2]$ 时，$\Phi(x) = \int_0^1 t^2 dt + \int_1^x t dt = \frac{1}{3} + \frac{t^2}{2}\Big|_1^x = \frac{x^2}{2} - \frac{1}{6}$

故 $\Phi(x) = \begin{cases} \frac{x^3}{3} & x \in [0,1) \\ \frac{x^2}{2} - \frac{1}{6} & x \in [1,2] \end{cases}$，由于 $\Phi(1) = \frac{1}{3}$，$\lim\limits_{x \to 1^-} \Phi(x) = \lim\limits_{x \to 1^-} \frac{x^3}{3} = \frac{1}{3} = \Phi(1)$，

$\lim\limits_{x \to 1^+} \Phi(x) = \lim\limits_{x \to 1^+} (\frac{x^2}{2} - \frac{1}{6}) = \frac{1}{3} = \Phi(1)$，故 $\Phi(x)$ 在 $(0,2)$ 内的连续.