

2007-2008 学年第二学期期中试题(A 卷)参考解答及评分标准

2008 年 4 月 18 日

一、填空题 (每小题 4 分, 共 24 分)

$$1. \quad 3x^2 + 2y^2 + 2z^2 = 13, \quad \vec{n} = \pm\{3, -2, 4\}, \text{ or } \vec{n} = \pm\{6, -4, 8\}$$

$$2. \quad \vec{b} = \left\{ \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0 \right\}; \quad 3. \quad \lambda = 2, \quad d = \frac{29}{\sqrt{30}};$$

$$4. \quad dz(1,0) = 2dx - 3dy; \quad 5. \quad \text{最大值为 } 5; \quad 6. \quad I = \int_{-2}^1 dx \int_{x^2}^{2-x} f(x,y) dy.$$

二、(10 分)

$$\text{解: } \frac{\partial z}{\partial x} = 2xyf'_1 + \frac{1}{y}f'_2; \quad \dots\dots\dots 3 \text{ 分}$$

$$\frac{\partial z}{\partial y} = x^2f'_1 - \frac{x}{y^2}f'_2; \quad \dots\dots\dots 6 \text{ 分}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2xf'_1 - \frac{1}{y^2}f'_2 + 2x^3yf''_{11} - \frac{x^2}{y}f''_{12} - \frac{x}{y^3}f''_{22}. \quad \dots\dots\dots 10 \text{ 分}$$

三、(10 分)

$$\text{解: } \begin{cases} \frac{\partial f}{\partial x} = 6x^2 + y - 2x = 0 \\ \frac{\partial f}{\partial y} = x - 2y = 0 \end{cases}, \text{ 得驻点: } (0,0), \quad \left(\frac{1}{4}, \frac{1}{8}\right). \text{ 又}$$

$$\frac{\partial^2 f}{\partial x^2} = 12x - 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 1, \quad \frac{\partial^2 f}{\partial y^2} = -2. \quad \dots\dots\dots 4 \text{ 分}$$

$$\text{在点 } (0,0) \text{ 处: } A = \frac{\partial^2 f}{\partial x^2} = -2, \quad B = \frac{\partial^2 f}{\partial x \partial y} = 1, \quad C = \frac{\partial^2 f}{\partial y^2} = -2. \text{ 且}$$

$$B^2 - AC = -3 < 0, \quad \text{又 } A = -2 < 0, \text{ 所以}$$

$$f(x,y) \text{ 在 } (0,0) \text{ 处取得极大值, } f_{\max} = 0, (0,0) \text{ 为极大值点; } \dots\dots\dots 7 \text{ 分}$$

$$\text{同理在点 } \left(\frac{1}{4}, \frac{1}{8}\right) \text{ 处: } A = \frac{\partial^2 f}{\partial x^2} = 1, \quad B = \frac{\partial^2 f}{\partial x \partial y} = 1, \quad C = \frac{\partial^2 f}{\partial y^2} = -2. \text{ 且}$$

$$B^2 - AC = 3 > 0, \text{ 所以 } f(x,y) \text{ 在 } \left(\frac{1}{4}, \frac{1}{8}\right) \text{ 处不取得极值 } \dots\dots\dots 10 \text{ 分}$$

四、(12 分) 解: (1) 切线 L 的方程:

$$\begin{cases} 2x + 2y \frac{dy}{dx} + 4z \frac{dz}{dx} = 0 \\ 2 + \frac{dy}{dx} + \frac{dz}{dx} = 0 \end{cases}, \text{ 在点 } M(1, -2, 1) \text{ 处解得: } \begin{cases} \frac{dy}{dx} = -\frac{3}{4} \\ \frac{dz}{dx} = -\frac{5}{4} \end{cases}$$

可得切向量为 $\vec{\tau} = \{4, -3, -5\}$, 所以切线 L 的方程为: $\frac{x-1}{4} = \frac{y+2}{-3} = \frac{z-1}{-5}$.

(注: 此处求切向量还可用第八题的方法)5 分

(2) 切平面 π 的方程:

法向量 $\vec{n} = \{4x, -2y, 2\}|_M = 2\{2, 2, 1\}$, 所以切平面 π 的方程为:

$$2x + 2y + z + 1 = 0. \quad \dots\dots\dots 9 \text{ 分}$$

(3) 夹角:

$$\sin \varphi = \frac{|\vec{\tau} \cdot \vec{n}|}{|\vec{\tau}| |\vec{n}|} = \frac{1}{5\sqrt{2}}, \text{ 所以夹角 } \varphi = \arcsin \frac{1}{5\sqrt{2}}. \dots\dots\dots 12 \text{ 分}$$

五、(10 分) 解: $z + x \frac{\partial z}{\partial x} + e^z \frac{\partial z}{\partial x} - e^{x^2} = 0, \Rightarrow \frac{\partial z}{\partial x} = \frac{e^{x^2} - z}{x + e^z}. \dots\dots\dots 3 \text{ 分}$

$$x \frac{\partial z}{\partial y} + e^z \frac{\partial z}{\partial y} + 2e^{4y^2} = 0, \Rightarrow \frac{\partial z}{\partial y} = -\frac{2e^{4y^2}}{x + e^z}. \dots\dots\dots 6 \text{ 分}$$

$$\frac{\partial z}{\partial y} + x \frac{\partial^2 z}{\partial x \partial y} + e^z \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} + e^z \frac{\partial^2 z}{\partial x \partial y} = 0, \quad \dots\dots\dots 8 \text{ 分}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = -\frac{\cancel{\frac{\partial z}{\partial y}} + e^z \cancel{\frac{\partial z}{\partial y}} \cancel{\frac{\partial z}{\partial x}}}{x + e^z} = \frac{2e^{4y^2}(x + e^z) + 2e^{z+4y^2}(e^{x^2} - z)}{(x + e^z)^3}. \dots\dots\dots 10 \text{ 分}$$

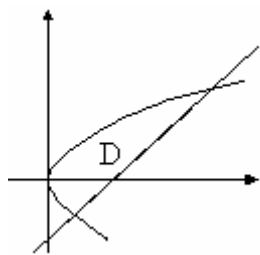
六、(12 分) 解: 求交点: $\begin{cases} y = \sqrt{2-x^2} \\ x = y^2 \end{cases}$, 得 (1,1), 其极坐标为 $(\sqrt{2}, \frac{\pi}{4})$,

又曲线 $x = y^2$ 的极坐标方程为: $\rho = \frac{\cos \theta}{\sin^2 \theta}, \dots\dots\dots 3 \text{ 分}$

$$\text{故 } I = \int_0^{\frac{\pi}{4}} d\theta \int_0^{\sqrt{2}} f(\rho \cos \theta, \rho \sin \theta) \rho d\rho + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{\frac{\cos \theta}{\sin^2 \theta}} f(\rho \cos \theta, \rho \sin \theta) \rho d\rho \dots\dots\dots 7 \text{ 分}$$

$$\begin{aligned}
 I &= \iint_D \sqrt{x^2 + y^2} dx dy = \int_0^{\frac{\pi}{4}} d\theta \int_0^{\sqrt{2}} \rho^2 d\rho + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{\frac{\cos\theta}{\sin^2\theta}} \rho^2 d\rho \\
 &= \frac{\sqrt{2}}{6} \pi + \frac{2}{45} (1 + \sqrt{2}). \quad \dots\dots\dots 12 \text{ 分}
 \end{aligned}$$

七、(10 分) 解: $V = \iint_D |2 - x + y| dx dy \quad \dots\dots\dots 3 \text{ 分}$



$$\begin{aligned}
 &= \iint_D (2 - x + y) dx dy \\
 &= \int_{-1}^2 dy \int_{y^2}^{2+y} (2 - x + y) dx \quad \dots\dots\dots 6 \text{ 分} \\
 &= \frac{81}{20} \quad \dots\dots\dots 10 \text{ 分}
 \end{aligned}$$

八、(12 分)

解: 构造拉格朗日函数:

$$\begin{aligned}
 &F(x, y, z) = x^2 + y^2 + z^2 + \lambda(z - x^2 - y^2) + \mu(4 - x - y - z) \\
 &\begin{cases} F'_x = 2x - 2\lambda x - \mu = 0 \\ F'_y = 2y - 2\lambda y - \mu = 0 \\ F'_z = 2z + \lambda - \mu = 0 \\ z = x^2 + y^2 \\ x + y + z = 4 \end{cases} \quad \text{, 得驻点: } (-2, -2, 8), (1, 1, 2), \dots\dots\dots 5 \text{ 分}
 \end{aligned}$$

又 $u(-2, -2, 8) = 72$, $u(1, 1, 2) = 6$,

所以最大值为 $u_{\max} = 72$, 最大值点为: $M(-2, -2, 8)$. $\dots\dots\dots 8 \text{ 分}$

Γ 在上述最大值点 M 处的切向量为:

$$\vec{\tau} = \{2x, 2y, -1\}|_M \times \{1, 1, 1\} = \{-4, -4, -1\} \times \{1, 1, 1\} = \{-3, 3, 0\}$$

又点 M 的向径为: $\overrightarrow{OM} = \{-2, -2, 8\}$,

$\vec{\tau} \cdot \overrightarrow{OM} = \{-3, 3, 0\} \cdot \{-2, -2, 8\} = 0$, 所以 $\vec{\tau} \perp \overrightarrow{OM}$, 即

曲线 Γ 在上述取得最大值点处的切向量与最大值点的向径正交.

$\dots\dots\dots 12 \text{ 分}$