习题 7.4(P72)

1．设 $z=e^{x-2y}$， $x=\sin t$， $y=t^3$，求 $\dfrac{dz}{dt}$

解：$\dfrac{dz}{dt}=\dfrac{\partial z}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dt}=e^{x-2y}\cos t-6t^2e^{x-2y}=(\cos t-6t^2)e^{\sin t-2t^3}$

2．设 $z=\arctan(xy)$， $y=e^x$，求 $\dfrac{dz}{dx}$

解：$\dfrac{dz}{dx}=\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dx}=\dfrac{y}{1+x^2y^2}+\dfrac{x}{1+x^2y^2}e^x=\dfrac{y+xe^x}{1+x^2y^2}=\dfrac{e^x+xe^x}{1+x^2e^{2x}}=\dfrac{(1+x)e^x}{1+x^2e^{2x}}$

3．设 $z=x^2\ln y$， $x=\dfrac{u}{v}$， $y=3u-2v$，求 $\dfrac{\partial z}{\partial u}$， $\dfrac{\partial z}{\partial v}$

解：$\dfrac{\partial z}{\partial u}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}=2x(\ln y)\cdot\dfrac{1}{v}+\dfrac{x^2}{y}\cdot 3=2\dfrac{u}{v^2}\ln(3u-2v)+\dfrac{3u^2}{v^2(3u-2v)}$，

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=2x(\ln y)\cdot\left(-\frac{u}{v^2}\right)+\frac{x^2}{y}\cdot(-2)=-\frac{2u^2}{v^3}\ln(3u-2v)-\frac{2u^2}{v^2(3u-2v)}$$

4．设 $u=f(\dfrac{x}{y},\dfrac{y}{z})$，其中 f 是可微函数，求 $\dfrac{\partial u}{\partial x}$， $\dfrac{\partial u}{\partial y}$， $\dfrac{\partial u}{\partial z}$

解：$\dfrac{\partial u}{\partial x}=\dfrac{1}{y}f_1'$， $\dfrac{\partial u}{\partial y}=-\dfrac{x}{y^2}f_1'+\dfrac{1}{z}f_2'$， $\dfrac{\partial u}{\partial z}=-\dfrac{y}{z^2}f_2'$

5．设 $z=f(x^2-y^2,y^2-x^2)$，其中 f 具有一阶连续偏导数，证明： $y\dfrac{\partial z}{\partial x}+x\dfrac{\partial z}{\partial y}=0$

解：$\dfrac{\partial z}{\partial x}=f_1'\cdot 2x+f_2'\cdot(-2x)$， $\dfrac{\partial z}{\partial y}=f_1'\cdot(-2y)+f_2'\cdot 2y$，

所以 $y\dfrac{\partial z}{\partial x}+x\dfrac{\partial z}{\partial y}=y\cdot\left(f_1'\cdot 2x+f_2'\cdot(-2x)\right)+x\cdot\left(f_1'\cdot(-2y)+f_2'\cdot 2y\right)=0$

6．设$u = f(x, xy, xyz)$，其中f具有连续偏导数，求$\frac{\partial u}{\partial x}$，$\frac{\partial u}{\partial y}$，$\frac{\partial u}{\partial z}$

解：$\frac{\partial u}{\partial x} = f_1' + yf_2' + yzf_3'$，$\frac{\partial u}{\partial y} = xf_2' + xzf_3'$，$\frac{\partial u}{\partial z} = xyf_3'$

7．设$z = \frac{y}{f(x^2 - y^2)}$，其中$f$是可导函数，验证$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{z}{y^2}$

证：$\frac{\partial z}{\partial x} = \frac{-yf' \cdot 2x}{f^2} = \frac{-2xyf'}{f^2}$，$\frac{\partial z}{\partial y} = \frac{f - yf' \cdot (-2y)}{f^2} = \frac{1}{f} + \frac{2y^2 f'}{f^2}$

所以$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{-2yf'}{f^2} + \frac{1}{yf} + \frac{2yf'}{f^2} = \frac{1}{yf} = \frac{\frac{y}{f}}{y^2} = \frac{z}{y^2}$

8．设$z = f(2x, \frac{x}{y})$，其中f具有二阶连续偏导数，求$\frac{\partial^2 z}{\partial x^2}$，$\frac{\partial^2 z}{\partial y^2}$.

解：$\frac{\partial z}{\partial x} = 2f_1' + \frac{1}{y}f_2'$

$$\frac{\partial^2 z}{\partial x^2} = 2\left(f_{11}'' \cdot 2 + \frac{1}{y}f_{12}''\right) + \frac{1}{y}\left(f_{21}'' \cdot 2 + \frac{1}{y}f_{22}''\right) = 4f_{11}'' + \frac{4}{y}f_{12}'' + \frac{1}{y^2}f_{22}''$$

$$\frac{\partial z}{\partial y} = -\frac{x}{y^2}f_2'$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2x}{y^3}f_2' - \frac{x}{y^2} \cdot (-\frac{x}{y^2})f_{22}'' = \frac{2x}{y^3}f_2' + \frac{x^2}{y^4}f_{22}''$$

9．已知$u = f(x, y, z)$，$y = \varphi(x)$，$z = \psi(x, y)$，其中f、φ、ψ都是可微函数，求$\frac{du}{dx}$.

解：$\frac{du}{dx} = f_x' + f_y' \cdot \frac{dy}{dx} + f_z'\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}\right) = f_x' + f_y' \cdot \varphi' + f_z'(\psi_x' + \psi_y' \cdot \varphi')$

10．设$u = f(x, ye^z, x\sin y)$，其中f是可微函数，求du.

解：$\dfrac{\partial u}{\partial x}=f_1'+\sin yf_3'$，$\dfrac{\partial u}{\partial y}=e^z f_2'+x\cos yf_3'$，$\dfrac{\partial u}{\partial z}=ye^z f_2'$，

所以 $du=\left(f_1'+\sin yf_3'\right)dx+\left(e^z f_2'+x\cos yf_3'\right)dy+ye^z f_2'dz$

11．设 $u=f(x^2+y^2+z^2)$，其中 f 是三阶可导函数，求 $\dfrac{\partial^2 u}{\partial x\partial y}$，$\dfrac{\partial^3 u}{\partial x\partial y\partial z}$

解：$\dfrac{\partial u}{\partial x}=2xf'$，$\dfrac{\partial^2 u}{\partial x\partial y}=2xf''\cdot 2y=4xyf''$，$\dfrac{\partial^3 u}{\partial x\partial y\partial z}=4xyf'''\cdot 2z=8xyzf'''$

12．设 $u=yf(\dfrac{x}{y})+xg(\dfrac{y}{x})$，其中 f、g 具有二阶连续导数，求 $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\partial y}$

解：$\dfrac{\partial u}{\partial x}=yf'\cdot\dfrac{1}{y}+g+x\cdot g'\cdot(-\dfrac{y}{x^2})=f'+g-\dfrac{y}{x}g'$

$$\frac{\partial^2 u}{\partial x^2}=f''\cdot\frac{1}{y}+g'\cdot(-\frac{y}{x^2})+\frac{y}{x^2}\cdot g'-\frac{y}{x}g''\cdot(-\frac{y}{x^2})=\frac{1}{y}f''+\frac{y^2}{x^3}g''$$

$$\frac{\partial^2 u}{\partial x\partial y}=f''\cdot(-\frac{x}{y^2})+g'\cdot\frac{1}{x}-\frac{1}{x}\cdot g'-\frac{y}{x}g''\cdot\frac{1}{x}=-\frac{x}{y^2}f''-\frac{y}{x^2}g''$$

故 $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{x}{y}f''+\dfrac{y^2}{x^2}g''-\dfrac{x}{y}f''-\dfrac{y^2}{x^2}g''=0$

13．已知 $\sin(xy)-e^{xy}-x^2y=0$，求 $\dfrac{dy}{dx}$.

解：方程两边对 x 求导，得 $\cos(xy)\left(y+x\dfrac{dy}{dx}\right)-e^{xy}\left(y+x\dfrac{dy}{dx}\right)-\left(2xy+x^2\dfrac{dy}{dx}\right)=0$，

整理方程，得 $\left(x\cos(xy)-xe^{xy}-x^2\right)\dfrac{dy}{dx}=-y\cos(xy)+ye^{xy}+2xy$，

所以 $\dfrac{dy}{dx}=-\dfrac{y[\cos(xy)-e^{xy}-2x]}{x[\cos(xy)-e^{xy}-x]}$

14．已知 $x+y+z=e^{-(x^2+y^2+z^2)}$，求 $\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$

解 1：注意到由已知方程可以确定函数 $z=z(x,y)$，对已知方程两边关于 x 求偏导：

$$1+\frac{\partial z}{\partial x}=e^{-(x^2+y^2z^2)}\left(-2x-2z\frac{\partial z}{\partial x}\right),\qquad \text{解得}\frac{\partial z}{\partial x}=-\frac{1+2xe^{-(x^2+y^2z^2)}}{1+2ze^{-(x^2+y^2z^2)}}$$

利用函数 z 关于自变量 x、y 的对称性，得 $\dfrac{\partial z}{\partial y}=-\dfrac{1+2ye^{-(x^2+y^2z^2)}}{1+2ze^{-(x^2+y^2z^2)}}$

解 2：对已知方程两边求微分：$dx+dy+dz=e^{-(x^2+y^2z^2)}\left(-2xdx-2ydy-2zdz\right)$，

整理得 $dz=\dfrac{-1-2xe^{-(x^2+y^2z^2)}}{1+2ze^{-(x^2+y^2z^2)}}dx+\dfrac{-1-2ye^{-(x^2+y^2z^2)}}{1+2ze^{-(x^2+y^2z^2)}}dy$，

所以 $\dfrac{\partial z}{\partial x}=\dfrac{-1-2xe^{-(x^2+y^2z^2)}}{1+2ze^{-(x^2+y^2z^2)}}$，$\quad\dfrac{\partial z}{\partial y}=\dfrac{-1-2ye^{-(x^2+y^2z^2)}}{1+2ze^{-(x^2+y^2z^2)}}$

15．设 $\cos^2 x+\cos^2 y+\cos^2 z=1$，求 dz．

解：方程两边求微分，得 $2\cos x(-\sin x)dx+2\cos y(-\sin y)dy+2\cos z(-\sin z)dz=0$，

整理得，$\sin 2xdx+\sin 2ydy+\sin 2zdz=0$，所以 $dz=-\dfrac{\sin 2x}{\sin 2z}dx-\dfrac{\sin 2y}{\sin 2z}dy$

16．设 $\varphi(cx-az,cy-bz)=0$，其中 f 是可微函数，证明：$a\dfrac{\partial z}{\partial x}+b\dfrac{\partial z}{\partial y}=c$

解：注意到由已知方程可以确定函数 $z=z(x,y)$，对已知方程分别对 x、y 求偏导：

$$\varphi_1'\cdot\left(c-a\frac{\partial z}{\partial x}\right)+\varphi_2'\cdot\left(-b\frac{\partial z}{\partial x}\right)=0,\qquad \varphi_1'\cdot\left(-a\frac{\partial z}{\partial y}\right)+\varphi_2'\cdot\left(c-b\frac{\partial z}{\partial y}\right)=0$$

解得：$\dfrac{\partial z}{\partial x}=\dfrac{c\varphi_1'}{a\varphi_1'+b\varphi_2'}$，$\qquad\dfrac{\partial z}{\partial y}=\dfrac{c\varphi_2'}{a\varphi_1'+b\varphi_2'}$

故 $\qquad a\dfrac{\partial z}{\partial x}+b\dfrac{\partial z}{\partial y}=\dfrac{ac\varphi_1'}{a\varphi_1'+b\varphi_2'}+\dfrac{bc\varphi_2'}{a\varphi_1'+b\varphi_2'}=\dfrac{c(a\varphi_1'+b\varphi_2')}{a\varphi_1'+b\varphi_2'}=c$

17．设 $x^2+y^2+z^2=yf(\dfrac{z}{y})$，其中 f 是可微函数，求 dz．

解：方程两边求微分，得 $2xdx+2ydy+2zdz=fdy+yf'(\frac{ydz-zdy}{y^2})$，

整理得 $2xdx+\left(2y-f+\frac{z}{y}f'\right)dy+\left(2z-f'\right)dz=0$，

所以 $dz=\frac{2x}{f'-2z}dx+\frac{2y-f+\frac{z}{y}f'}{f'-2z}dy$ $dz=\frac{2x}{f'-2z}dx+\frac{2y^2-yf+zf'}{yf'-2yz}dy$

18．设 $x+y-z=e^z$，求 $\frac{\partial^2 z}{\partial x\partial y}$

解：方程两端对 x、y 求偏导： $1-\frac{\partial z}{\partial x}=e^z\frac{\partial z}{\partial x}$， $1-\frac{\partial z}{\partial y}=e^z\frac{\partial z}{\partial y}$

解得： $\frac{\partial z}{\partial x}=\frac{1}{1+e^z}$， $\frac{\partial z}{\partial y}=\frac{1}{1+e^z}$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{-e^z\frac{\partial z}{\partial y}}{(1+e^z)^2}=-\frac{e^z}{(1+e^z)^3}$$

19．已知方程 $\frac{x}{z}=\ln\frac{z}{y}$ 定义了函数 $z=z(x,y)$，求 $\frac{\partial^2 z}{\partial x^2}$.

解：为了求导简便，将方程改写为 $x=z(\ln z-\ln y)$，此时 $z=z(x,y)$

方程两端对 x 求偏导： $1=\frac{\partial z}{\partial x}(\ln z-\ln y)+z\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}$

整理得： $\frac{\partial z}{\partial x}=\frac{1}{\ln z-\ln y+1}$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{-\frac{1}{z}\cdot\frac{\partial z}{\partial x}}{(\ln z-\ln y+1)^2}=-\frac{1}{z(\ln z-\ln y+1)^3}$$

20．设 $z+\ln z-\int_y^x e^{-t^2}dt=0$，求 $\frac{\partial^2 z}{\partial x\partial y}$

解：方程确定 $z=z(x,y)$，方程两端分别对 x 、 y 求偏导： $\frac{\partial z}{\partial x}+\frac{1}{z}\frac{\partial z}{\partial x}-e^{-x^2}=0$，

$\frac{\partial z}{\partial y}+\frac{1}{z}\frac{\partial z}{\partial y}+e^{-y^2}=0$，解得 $\frac{\partial z}{\partial x}=\frac{z}{z+1}e^{-x^2}$， $\frac{\partial z}{\partial y}=-\frac{z}{z+1}e^{-y^2}$，

所以 $$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{z}{z+1}e^{-x^2}\right)=e^{-x^2}\frac{\partial}{\partial y}\left(1-\frac{1}{z+1}\right)$$

$$=e^{-x^2}\frac{1}{(z+1)^2}\frac{\partial z}{\partial y}=-\frac{z}{(z+1)^3}e^{-(x^2+y^2)}$$

21．设 $\begin{cases} z=x^2+y^2 \\ x^2+2y^2+3z^2=20 \end{cases}$，求 $\frac{dy}{dx}$， $\frac{dz}{dx}$.

解：由方程组确定 $y=y(x)$， $z=z(x)$，方程组两端分别对 x 求导：

$\begin{cases} \frac{dz}{dx}=2x+2y\frac{dy}{dx} \\ 2x+4y\frac{dy}{dx}+6z\frac{dz}{dx}=0 \end{cases}$， 解得： $\frac{dy}{dx}=-\frac{x(6z+1)}{2y(3z+1)}$， $\frac{dz}{dx}=\frac{x}{3z+1}$

22．设 $\begin{cases} x=e^u+v \\ xy=e^u+u \end{cases}$，求 $\frac{\partial u}{\partial x}$， $\frac{\partial v}{\partial y}$

解：由方程组确定 $u=u(x,y)$， $v=v(x,y)$，方程组两端分别对 x 、 y 求偏导：

$\begin{cases} 1=e^u\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x} \\ y=e^u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x} \end{cases}$， $\begin{cases} 0=e^u\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y} \\ x=e^u\frac{\partial u}{\partial y}+\frac{\partial u}{\partial y} \end{cases}$

解得： $\frac{\partial u}{\partial x}=\frac{y}{1+e^u}$， $\frac{\partial v}{\partial x}=-\frac{xe^u}{1+e^u}$

23．设 $y=y(x)$， $z=z(x)$ 是由方程 $z=xf(x+y)$ 和 $F(x,y,z)=0$ 所确定的函数，其中 f 与 F 分别具有一阶连续导数和一阶连续偏导数，求 $\frac{dz}{dx}$

解：方程组 $z=xf(x+y),F(x,y,z)=0$ 中有三个变量，故可解得 $y=y(x)$， $z=z(x)$。

方程组两端分别对 x 求导得 $\begin{cases} \dfrac{dz}{dx} = f + xf'\left(1 + \dfrac{dy}{dx}\right) \\ F_x' + F_y'\dfrac{dy}{dx} + F_z'\dfrac{dz}{dx} = 0 \end{cases}$，

整理得 $\begin{cases} xf'\dfrac{dy}{dx} - \dfrac{dz}{dx} = -f - xf' \\ F_y'\dfrac{dy}{dx} + F_z'\dfrac{dz}{dx} = -F_x' \end{cases}$，

解得 $$\frac{dz}{dx} = \frac{\begin{vmatrix} xf' & -f - xf' \\ F_y' & -F_x' \end{vmatrix}}{\begin{vmatrix} xf' & -1 \\ F_y' & F_z' \end{vmatrix}} = \frac{-xf'F_x' + fF_y' + xf'F_y'}{xf'F_z' + F_y'}$$

24．设 $y = f(x,t)$，$F(x,y,t) = 0$，其中 f 、F 都具有一阶连续偏导数，证明：

$$\frac{dy}{dx} = \frac{f_x'F_t' - f_t'F_x'}{f_t'F_y' + F_t'}$$

分析：方程组 $y = f(x,t)$，$F(x,y,t) = 0$ 中有三个变量，故可解得 $y = y(x)$，$t = t(x)$

解：方程组两端分别对 x 求导得 $\begin{cases} y_x' - f_x' - f_t' \cdot t_x' = 0 \\ F_x' + F_y' \cdot y_x' + F_t' \cdot t_x' = 0 \end{cases}$

$$y_x' = \frac{dy}{dx} = \frac{\begin{vmatrix} f_x' & -f_t' \\ -F_x' & F_t' \end{vmatrix}}{\begin{vmatrix} 1 & -f_t' \\ F_y' & F_t' \end{vmatrix}} = \frac{f_x'F_t' - f_t'F_x'}{f_t'F_y' + F_t'}$$