习题 4.3(P236)

1．求下列不定积分.

(1) $\int x\sqrt{x}dx$

解: $\int x\sqrt{x}dx=\int x^{\frac{3}{2}}dx=\frac{2}{5}x^{\frac{5}{2}}+C$

(2) $\int\frac{10x^3+3}{x^4}dx$

解: $\int\frac{10x^3+3}{x^4}dx=\int(10x^{-1}+3x^{-4})dx=10\ln|x|-x^{-3}+C$

(3) $\int\frac{(1-x)^2}{x\sqrt{x}}dx$

解: $\int\frac{(1-x)^2}{x\sqrt{x}}dx=\int\frac{1-2x+x^2}{x^{\frac{3}{2}}}dx=\int(x^{-\frac{3}{2}}-2x^{-\frac{1}{2}}+x^{\frac{1}{2}})dx$

$=-2x^{-\frac{1}{2}}-4x^{\frac{1}{2}}+\frac{2}{3}x^{\frac{3}{2}}+C$

(4) $\int\frac{x^2+7x+12}{x+4}dx$

解: $\int\frac{x^2+7x+12}{x+4}dx=\int\frac{(x+4)(x+3)}{x+4}dx=\int(x+3)dx=\frac{1}{2}x^2+3x+C$

2. 求下列不定积分. (凑微分法)

(1) $\int\cos(1-x)\,dx$

解: $\int\cos(1-x)\,dx=-\int\cos(1-x)\,d(1-x)=-\sin(1-x)+C$

(2) $\int\sqrt{7+5x}\,dx$

解: $\int\sqrt{7+5x}\,dx=\frac{1}{5}\int\sqrt{7+5x}\,d(7+5x)=\frac{1}{5}\cdot\frac{2}{3}(7+5x)^{\frac{3}{2}}+C=\frac{2}{15}(7+5x)^{\frac{3}{2}}+C$

(3) $\int\frac{e^{2x}-1}{e^x}dx$

解: $\int \frac{e^{2x}-1}{e^x}dx=\int (e^x-e^{-x})\,dx=\int e^x\,dx+\int e^{-x}\,d(-x)=e^x+e^{-x}+C$

(4) $\int \frac{dx}{9+x^2}dx$

解: $\int \frac{dx}{9+x^2}dx=\frac{1}{9}\int \frac{dx}{1+(\frac{x}{3})^2}=\frac{1}{3}\int \frac{d(\frac{x}{3})}{1+(\frac{x}{3})^2}=\frac{1}{3}\arctan\frac{x}{3}+C$

(5) $\int \frac{dx}{\sqrt{4-9x^2}}$

解: $\int \frac{dx}{\sqrt{4-9x^2}}=\frac{1}{2}\int \frac{dx}{\sqrt{1-(\frac{3x}{2})^2}}=\frac{1}{2}\cdot\frac{2}{3}\int \frac{d(\frac{3x}{2})}{\sqrt{1-(\frac{3x}{2})^2}}=\frac{1}{3}\arcsin\frac{3x}{2}+C$

(6) $\int \frac{x^2}{4+x^3}dx$

解: $\int \frac{x^2}{4+x^3}dx=\frac{1}{3}\int \frac{1}{4+x^3}d(4+x^3)=\frac{1}{3}\ln\left|4+x^3\right|+C$

(7) $\int \frac{\ln x}{x}dx$

解: $\int \frac{\ln x}{x}dx=\int \ln x d(\ln x)=\frac{\ln^2 x}{2}+C$

(8) $\int \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx$

解: $\int \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx=2\int \sin\sqrt{x}\,d\sqrt{x}=-2\cos\sqrt{x}+C$

(9) $\int \frac{dx}{\cos^2 x\sqrt{1+\tan x}}$

解: $\int \frac{dx}{\cos^2 x\sqrt{1+\tan x}}=\int \frac{d(1+\tan x)}{\sqrt{1+\tan x}}=2\sqrt{1+\tan x}+C$

(10) $\int \frac{x^3}{\sqrt{1-x^8}}dx$

解: $\int \dfrac{x^3}{\sqrt{1-x^8}}dx = \dfrac{1}{4}\int \dfrac{d(x^4)}{\sqrt{1-(x^4)^2}} = \dfrac{1}{4}\arcsin(x^4)+C$

(11) $\int \dfrac{\sin x\cos x}{1+\cos^2 x}dx$

解: $\int \dfrac{\sin x\cos x}{1+\cos^2 x}dx = -\dfrac{1}{2}\int \dfrac{d(1+\cos^2 x)}{1+\cos^2 x} = -\dfrac{1}{2}\ln(1+\cos^2 x)+C$

(12) $\int \cos^2\dfrac{x}{2}dx$ (与本节习题 6(4)题目完全一样)

解：$\int \cos^2\dfrac{x}{2}dx = \dfrac{1}{2}\int (1+\cos x)\,dx = \dfrac{1}{2}(x+\sin x)+C$

(13) $\int \cos x\sin 3x\,dx$

解: $\int \cos x\sin 3x\,dx = \dfrac{1}{2}\int [\sin 2x+\sin 4x]dx$

$= \dfrac{1}{2}\cdot\dfrac{1}{2}\int \sin 2xd(2x)+\dfrac{1}{2}\cdot\dfrac{1}{4}\int \sin 4xd(4x) = -\dfrac{1}{4}\cos 2x-\dfrac{1}{8}\cos 4x+C$

(14) $\int \cos 2x\cos 3x\,dx$

解: $\int \cos 2x\cos 3x\,dx = \dfrac{1}{2}\int [\cos x+\cos 5x]dx = \dfrac{1}{2}\int \cos xdx+\dfrac{1}{2}\cdot\dfrac{1}{5}\int \cos 5xd(5x)$

$= \dfrac{1}{2}\sin x+\dfrac{1}{10}\sin 5x+C$

3. 求下列不定积分. (第二类换元积分法)

注：下列各题主要考察第二类换元积分法，故一些简单的凑微分，其过程就不必写出.

(1) $\int x\sqrt{1-2x}\,dx$

解: $\int x\sqrt{1-2x}\,dx \overset{\sqrt{1-2x}=t}{\underset{dx=-tdt}{=\!=\!=}} \int \dfrac{1-t^2}{2}\cdot t\cdot(-tdt) = \dfrac{1}{2}\int (t^4-t^2)dt = \dfrac{t^5}{10}-\dfrac{t^3}{6}+C$

$= \dfrac{1}{10}(1-2x)^{\frac{5}{2}}-\dfrac{1}{6}(1-2x)^{\frac{3}{2}}+C$

(2) $\int \dfrac{dx}{1+\sqrt{1+x}}$

解: $\int \frac{dx}{1+\sqrt{1+x}} \xlongequal[dx=2tdt]{\sqrt{1+x}=t} 2\int \frac{tdt}{1+t} = 2\int (1-\frac{1}{1+t})dt = 2t - \ln|1+t| + C$

$= 2\sqrt{1+x} - \ln\left|1+\sqrt{1+x}\right| + C$

(3) $\int \frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}} dx$

解:

$\int \frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}} dx \xlongequal[dx=6t^5dt]{\sqrt[6]{x}=t} \int \frac{t^3}{t^3-t^2} 6t^5 dt = 6\int \frac{t^6}{t-1} dt = 6\int \frac{(t^6-1)+1}{t-1} dt$

$= 6\int (t^5+t^4+t^3+t^2+t+1-\frac{1}{t-1})dt = 6(\frac{t^6}{6}+\frac{t^5}{5}+\frac{t^4}{4}+\frac{t^3}{3}+\frac{t^2}{2}+t-\ln|t-1|)+C$

$= x + \frac{6}{5}x^{\frac{5}{6}} + \frac{3}{2}x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - \ln\left|\sqrt[6]{x}-1\right|) + C$

(4) $\int \frac{dx}{x-\sqrt[3]{3x+2}}$

解: $\int \frac{dx}{x-\sqrt[3]{3x+2}} \xlongequal[dx=t^2dt]{\sqrt[3]{3x+2}=t} \int \frac{t^2dt}{\frac{t^3-2}{3}-t} = \int \frac{3t^2dt}{t^3-2-3t} = \int \frac{3t^2dt}{(t+1)^2(t-2)}$

$= \int [\frac{5/3}{t+1} - \frac{1}{(t+1)^2} + \frac{4/3}{t-2}]dt = \frac{5}{3}\ln|t+1| + \frac{1}{t+1} + \frac{4}{3}\ln|t-2| + C$

$= \frac{5}{3}\ln\left|\sqrt[3]{3x+2}+1\right| + \frac{1}{\sqrt[3]{3x+2}+1} + \frac{4}{3}\ln\left|\sqrt[3]{3x+2}-2\right| + C$

(5) $\int \frac{x^2}{\sqrt{a^2-x^2}} dx$

解: 法 1 $\int \frac{x^2}{\sqrt{a^2-x^2}} dx = \int \frac{a^2+(x^2-a^2)}{\sqrt{a^2-x^2}} dx = a^2\int \frac{1}{\sqrt{a^2-x^2}} dx - \int \sqrt{a^2-x^2}\, dx$

$\xlongequal[\text{公式}]{\text{套用}} a^2 \arcsin\frac{x}{a} - (\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}) + C$

$= \frac{a^2}{2}\arcsin\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2} + C$

法 2 $\int \frac{x^2}{\sqrt{a^2-x^2}} dx \xlongequal[dx=a\cos tdt]{x=a\sin t} \int \frac{a^2\sin^2 t}{a\cos t} a\cos tdt = \frac{a^2}{2}\int (1-\cos 2t)\, dt$

$$=\frac{a^2}{2}(t-\sin t\cos t)+C=\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}+C$$

(6) $\int\frac{dx}{x\sqrt{1-x^2}}$

解：法 1 $\int\frac{dx}{x\sqrt{1-x^2}}=\int\frac{xdx}{x^2\sqrt{1-x^2}}\frac{\sqrt{1-x^2}=t}{xdx=-tdt}\int\frac{dt}{t^2-1}=\frac{1}{2}\int(\frac{1}{t-1}-\frac{1}{t+1})dt$

$$=\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C=\frac{1}{2}\ln\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right|+C$$

法 2 $\int\frac{dx}{x\sqrt{1-x^2}}\frac{x=1/t}{dx=-1/t^2\,dt}-\int\frac{1}{\sqrt{t^2-1}}dt=-\ln\left|t+\sqrt{t^2-1}\right|+C$

$$=-\ln\left|\frac{1}{x}+\sqrt{\frac{1}{x^2}-1}\right|+C=-\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|+C$$

4. 求下列不定积分. (分部积分法)

(1) $\int x^2e^{3x}dx$

解：$\int x^2e^{3x}dx=\int x^2d(\frac{e^{3x}}{3})=\frac{x^2e^{3x}}{3}-\int\frac{2xe^{3x}}{3}dx=\frac{x^2e^{3x}}{3}-\int\frac{2x}{3}d(\frac{e^{3x}}{3})$

$$=\frac{x^2e^{3x}}{3}-[\frac{2xe^{3x}}{9}-\int\frac{2e^{3x}}{9}dx]=\frac{x^2e^{3x}}{3}-\frac{2xe^{3x}}{9}+\frac{2e^{3x}}{27}+C=(\frac{x^2}{3}-\frac{2x}{9}+\frac{2}{27})e^{3x}+C$$

(2) $\int x\cos^2 xdx$

解：$\int x\cos^2 xdx=\frac{1}{2}[\int xdx+\int x\cos 2xdx]=\frac{1}{2}[\frac{x^2}{2}+\int xd(\frac{\sin 2x}{2})]$

$$=\frac{x^2}{4}+\frac{x\sin 2x}{4}-\frac{1}{4}\int\sin 2xdx=\frac{x^2}{4}+\frac{x\sin 2x}{4}+\frac{1}{8}\cos 2x+C$$

(3) $\int\arctan xdx$

解：$\int\arctan xdx=x\arctan x-\int\frac{x}{1+x^2}dx=x\arctan x-\frac{1}{2}\int\frac{d(1+x^2)}{1+x^2}$

$$=x\arctan x-\frac{1}{2}\ln(1+x^2)+C$$

(4) $\int(\ln x)^2dx$

解: $\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x dx = x(\ln x)^2 - 2[x\ln x - \int dx]$

$= x(\ln x)^2 - 2x\ln x + 2x + C$

(5) $\int \frac{\ln x}{\sqrt{1+x}} dx$

解: $\int \frac{\ln x}{\sqrt{1+x}} dx = \int \ln x d(2\sqrt{1+x}) = 2\sqrt{1+x}\ln x - 2\int \frac{\sqrt{1+x}}{x} dx$

$\overset{\sqrt{1+x}=t}{\underset{dx=2tdt}{=\!=}} 2\sqrt{1+x}\ln x - 4\int \frac{t^2}{t^2-1} dt = 2\sqrt{1+x}\ln x - 4\int (1+\frac{1}{t^2-1})dt$

$\overset{\text{见3(6)题法1}}{\underset{\text{或套用公式}}{=\!=}} 2\sqrt{1+x}\ln x - 4t - 2\ln\left|\frac{t-1}{t+1}\right| + C$

$= 2\sqrt{1+x}\ln x - 4\sqrt{1+x} - 2\ln\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right| + C$

(6) $\int \ln(x+\sqrt{1+x^2})dx$

解: $\int \ln(x+\sqrt{1+x^2})dx = x\ln(x+\sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} dx$

$= x\ln(x+\sqrt{1+x^2}) - \int \frac{d(1+x^2)}{2\sqrt{1+x^2}} = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C$

(7) $\int \frac{1}{\sqrt{x}} \arcsin\sqrt{x} dx$

解: $\int \frac{1}{\sqrt{x}} \arcsin\sqrt{x} dx \overset{\sqrt{x}=t}{\underset{dx=2tdt}{=\!=}} 2\int \arcsin t dt = 2t\arcsin t - 2\int \frac{tdt}{\sqrt{1-t^2}}$

$= 2t\arcsin t + \int \frac{d(1-t^2)}{\sqrt{1-t^2}} = 2t\arcsin t + 2\sqrt{1-t^2} + C$

$= 2\sqrt{x}\arcsin\sqrt{x} + 2\sqrt{1-x} + C$

(8) $\int e^{-x} \sin 2x dx$

解: $\int e^{-x}\sin 2x dx = -\int \sin 2x d(e^{-x}) = -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x dx$

$= -e^{-x}\sin 2x - 2\int \cos 2x d(e^{-x}) = -e^{-x}\sin 2x - 2[e^{-x}\cos 2x + 2\int e^{-x}\sin 2x dx]$

$$= -e^{-x}\sin 2x - 2e^{-x}\cos 2x - 4\int e^{-x}\sin 2x dx = -\frac{e^{-x}}{5}(\sin 2x + 2\cos 2x) + C$$

(9) $\int \sin\sqrt{x}dx$

解: $\int \sin\sqrt{x}dx \xlongequal[dx=2tdt]{\sqrt{x}=t} 2\int t\sin tdt = 2\int td(-\cos t) = -2t\cos t + 2\int \cos tdt$

$= -2t\cos t + 2\sin t + C = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

(10) $\int \frac{x\arctan x}{\sqrt{1+x^2}}dx$

解: $\int \frac{x\arctan x}{\sqrt{1+x^2}}dx = \int \arctan xd\sqrt{1+x^2} = \sqrt{1+x^2}\arctan x - \int \frac{dx}{\sqrt{1+x^2}}$

$= \sqrt{1+x^2}\arctan x - \ln(x+\sqrt{1+x^2}) + C$

5. 求下列不定积分. (有理函数积分法)

提示: 在对有理函数进行积分时，注意以下几点:

(a)运用代数学的理论将 $\frac{P(x)}{Q(x)}$ 化为最简分式时，要求 $\frac{P(x)}{Q(x)}$ 为真分式，而假分式需先运用多项式除法化为有理整式与真分式的和，再化为最简分式.

(b)未必有理函数的积分均化为最简分式进行积分，根据被积表达式的形式，可选择最简的积分方法。

(1) $\int \frac{dx}{2x^2+x-1}$

解: $\int \frac{dx}{2x^2+x-1} = \int \frac{dx}{(2x-1)(x+1)} = \frac{1}{3}\int [\frac{2}{2x-1} - \frac{1}{x+1}]dx = \frac{1}{3}\ln\left|\frac{2x-1}{x+1}\right| + C$

(2) $\int \frac{dx}{x^2+2x+3}$

解: $\int \frac{dx}{x^2+2x+3} = \int \frac{d(x+1)}{(x+1)^2+2} = \frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} + C$

(3) $\int \frac{dx}{a^2-x^2}$

解: $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\int (\frac{1}{a+x} + \frac{1}{a-x})dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

(4) $\int \frac{x^2}{1+x}dx$

解: $\int \frac{x^2}{1+x}dx = \int \frac{(x^2-1)+1}{1+x}dx = \int (x-1+\frac{1}{1+x})dx = \frac{x^2}{2} - x + \ln|1+x| + C$

(5) $\int \frac{x^2}{1-x^2}dx$

解: $\int \frac{x^2}{1-x^2}dx = \int \frac{x^2-1+1}{1-x^2}dx = \int (\frac{1}{1-x^2} - 1)dx \xlongequal[\text{做公式}]{\text{把5(3)题}} \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| - x + C$

(6) $\int \frac{x+1}{x^2+2x}dx$

解: $\int \frac{x+1}{x^2+2x}dx = \frac{1}{2}\int \frac{d(x^2+2x)}{x^2+2x} = \frac{1}{2}\ln|x^2+2x| + C$

(7) $\int \frac{x^2+1}{(x+1)^2(x-1)}dx$

解: $\int \frac{x^2+1}{(x+1)^2(x-1)}dx = \int (\frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{(x+1)^2} + \frac{1}{2}\cdot\frac{1}{x-1})dx$

$= \int (\frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{(x+1)^2} + \frac{1}{2}\cdot\frac{1}{x-1})dx = \frac{1}{2}\ln|x+1| + \frac{1}{x+1} + \frac{1}{2}\ln|x-1| + C$

$= \frac{1}{x+1} + \frac{1}{2}\ln|x^2-1| + C$

(8) $\int \frac{x^3-1}{4x^3-x}dx$

解: $\int \frac{x^3-1}{4x^3-x}dx = \int (\frac{1}{4} - \frac{-\frac{1}{4}x+1}{x(2x-1)(2x+1)})dx$

$= \int (\frac{1}{4} + \frac{1}{x} - \frac{9}{8}\cdot\frac{1}{2x+1} - \frac{7}{8}\cdot\frac{1}{2x-1})dx$

$$=\frac{x}{4}+\ln|x|-\frac{9}{16}\ln|2x+1|-\frac{7}{16}\ln|2x-1|+C$$

(9) $\int\frac{dx}{x^3-1}$

解: $\int\frac{dx}{x^3-1}=\int\frac{dx}{(x-1)(x^2+x+1)}=\frac{1}{3}\int(\frac{1}{x-1}-\frac{x+2}{x^2+x+1})dx$

$$=\frac{1}{3}\int(\frac{1}{x-1}-\frac{(x+1/2)+3/2}{x^2+x+1})dx$$

$$=\frac{1}{3}\int\frac{1}{x-1}dx-\frac{1}{6}\int\frac{d(x^2+x+1)}{x^2+x+1}-\frac{1}{2}\int\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}d(x+\frac{1}{2})$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2+x+1)-\frac{\sqrt{3}}{3}\arctan\frac{2x+1}{\sqrt{3}}+C$$

(10) $\int\frac{x^2}{1-x^4}dx$

解: $\int\frac{x^2}{1-x^4}dx=\int\frac{x^2}{(1-x)(1+x)(1+x^2)}dx=\frac{1}{4}\int(\frac{1}{1-x}+\frac{1}{1+x})dx-\frac{1}{2}\int\frac{1}{1+x^2}dx$

$$=\frac{1}{4}\ln\left|\frac{1+x}{(1-x)}\right|-\frac{1}{2}\arctan x+C$$

(11) $\int\frac{dx}{x^4(2x^2-1)}$

解: $\int\frac{dx}{x^4(2x^2-1)}\overset{x=\frac{1}{t}}{=\!=\!=}\int\frac{t^4dt}{t^2-2}=\int\frac{(t^4-4)+4}{t^2-2}dt=\int(t^2+2+\frac{4}{t^2-(\sqrt{2})^2})dt$

$$\underset{\text{做公式}}{\overset{\text{把5(3)题}}{=\!=\!=}}\frac{t^3}{3}+2t+\sqrt{2}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+C=\frac{1}{3x^3}+\frac{2}{x}+\sqrt{2}\ln\left|\frac{1-\sqrt{2}x}{1+\sqrt{2}x}\right|+C$$

(12) $\int\frac{x^4}{(x+1)^{100}}dx$

解: $\int\frac{x^4}{(x+1)^{100}}dx\overset{x+1=t}{=\!=\!=}\int\frac{(t-1)^4}{t^{100}}dt=\int\frac{t^4-4t^3+6t^2-4t+1}{t^{100}}dt$

$$=\int\left(\frac{1}{t^{96}}-\frac{4}{t^{97}}+\frac{6}{t^{98}}-\frac{4}{t^{99}}+\frac{1}{t^{100}}\right)dt=-\frac{1}{95t^{95}}+\frac{1}{24t^{96}}-\frac{6}{97t^{97}}+\frac{2}{49t^{98}}-\frac{1}{99t^{99}}+C$$

$$=-\frac{1}{95(x+1)^{95}}+\frac{1}{24(x+1)^{96}}-\frac{6}{97(x+1)^{97}}+\frac{2}{49(x+1)^{98}}-\frac{1}{99(x+1)^{99}}+C$$

(13) $\int\frac{dx}{x(x^{10}+1)}$

解：法 1 $\int\frac{dx}{x(x^{10}+1)}=\int\frac{x^9dx}{x^{10}(x^{10}+1)}\overset{x^{10}=t}{=\!=}\frac{1}{10}\int\frac{dt}{t(t+1)}=\frac{1}{10}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$

$$=\frac{1}{10}\ln\left|\frac{t}{t+1}\right|+C=\frac{1}{10}\ln\frac{x^{10}}{x^{10}+1}+C$$

法 2 $\int\frac{dx}{x(x^{10}+1)}\overset{x=\frac{1}{t}}{=\!=}-\int\frac{t^9dt}{1+t^{10}}=-\frac{1}{10}\int\frac{d(1+t^{10})}{1+t^{10}}=-\frac{1}{10}\ln(1+t^{10})+C$

$$=\frac{1}{10}\ln\frac{x^{10}}{x^{10}+1}+C$$

6. 求下列不定积分.（三角有理函数的积分）

提示： 在对三角有理函数积分时，慎用万能公式. 因为万能公式的确万能，使用它后任何三角有理函数的积分都可以转化为有理分式的积分，使积分问题得以解决，但它却常常使积分变得冗杂. 因而，尽可能使用其它方法积分.

(1) $\int\frac{dx}{3+\sin^2 x}$

解：$\int\frac{dx}{3+\sin^2 x}=\int\frac{dx}{3\cos^2 x+4\sin^2 x}=\frac{1}{2}\int\frac{d(2\tan x)}{3+(2\tan x)^2}=\frac{1}{2\sqrt{3}}\arctan\frac{2\tan x}{\sqrt{3}}+C$

(2) $\int\frac{dx}{(\sin x+\cos x)^2}$

解：$\int\frac{dx}{(\sin x+\cos x)^2}=\int\frac{dx}{\cos^2 x(1+\tan x)^2}=\int\frac{d(1+\tan x)}{(1+\tan x)^2}=-\frac{1}{1+\tan x}+C$

(3) $\int\cot^3 x\,dx$

解：$\int\cot^3 x\,dx=\int\cot x(\csc^2 x-1)\,dx=-\int\cot x\,d(\cot x)-\int\cot x\,dx$

$$=-\frac{\cot^2 x}{2}-\ln|\sin x|+C$$

(4) $\int \cos^2\frac{x}{2}\,dx$

解：$\int \cos^2\frac{x}{2}\,dx=\frac{1}{2}\int(1+\cos x)\,dx=\frac{1}{2}(x+\sin x)+C$

(5) $\int(\tan^2 x+\tan^4 x)\,dx$

解：$\int(\tan^2 x+\tan^4 x)\,dx=\int\tan^2 x\cdot\sec^2 x\,dx=\int\tan^2 x\,d(\tan x)=\frac{\tan^3 x}{3}+C$

(6) $\int\sin^4 x\,dx$

解：$\int\sin^4 x\,dx=\int\left(\frac{1-\cos 2x}{2}\right)^2dx=\frac{1}{4}\int(1-2\cos 2x+\frac{1+\cos 4x}{2})\,dx$

$$=\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

(7) $\int\frac{dx}{1-\cos x}$

解：法 1 $\int\frac{dx}{1-\cos x}=\int\frac{1+\cos x}{\sin^2 x}dx=\int\frac{dx}{\sin^2 x}+\int\frac{d(\sin x)}{\sin^2 x}=-\cot x-\frac{1}{\sin x}+C$

法 2 $\int\frac{dx}{1-\cos x}=\int\frac{dx}{2\sin^2\frac{x}{2}}=\int\frac{d(\frac{x}{2})}{\sin^2\frac{x}{2}}=-\cot\frac{x}{2}+C$

(8) $\int\frac{dx}{1+\sin x}$

解：法 1 $\int\frac{dx}{1+\sin x}=\int\frac{1-\sin x}{\cos^2 x}dx=\int\frac{dx}{\cos^2 x}+\int\frac{d(\cos x)}{\cos^2 x}=\tan x-\frac{1}{\cos x}+C$

法 2 $\int\frac{dx}{1+\sin x}=\int\frac{dx}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}=2\int\frac{d(\frac{x}{2})}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}$

$$\overset{\text{令}x/2=t}{=\!=\!=}2\int\frac{dt}{(\sin t+\cos t)^2}\ \overset{\text{同题6(2)}}{\underset{\text{解法}}{=\!=\!=}}-\frac{2}{1+\tan t}+C\ \overset{\text{勿忘}}{\underset{\text{还原}}{=\!=\!=}}-\frac{2}{1+\tan\frac{x}{2}}+C$$

(9) $\int\frac{\sin x\cos x}{1+\sin^4 x}dx$

解：$\int\frac{\sin x\cos x}{1+\sin^4 x}dx=\frac{1}{2}\int\frac{d(\sin^2 x)}{1+(\sin^2 x)^2}=\frac{1}{2}\arctan(\sin^2 x)+C$

补充题： (1) $\int\frac{dx}{\sin x+\cos x}$

解： $\int\frac{dx}{\sin x+\cos x}=\frac{\sqrt{2}}{2}\int\frac{dx}{\cos\frac{\pi}{4}\sin x+\sin\frac{\pi}{4}\cos x}=\frac{\sqrt{2}}{2}\int\frac{d(x+\frac{\pi}{4})}{\sin(x+\frac{\pi}{4})}$

$=-\frac{\sqrt{2}}{2}\ln\left|\csc(x+\frac{\pi}{4})-\cot(x+\frac{\pi}{4})\right|+C$

(2) $\int\frac{dx}{x+\sqrt{1-x^2}}$

解： $\int\frac{dx}{x+\sqrt{1-x^2}}\ \overset{x=\sin t}{=\!=\!=}\int\frac{\cos t}{\sin t+\cos t}dt=\frac{1}{2}\int\frac{\cos t+\sin t+\cos t-\sin t}{\sin t+\cos t}dt$

$=\frac{1}{2}(\int dt+\int\frac{d(\sin t+\cos t)}{\sin t+\cos t})=\frac{1}{2}(t+\ln|\sin t+\cos t|)+C$

$=\frac{1}{2}(\arcsin x+\ln\left|x+\sqrt{1-x^2}\right|)+C$

(3) $\int\frac{\sin x}{1+\sin x+\cos x}dx$

解：法 1 $\int\frac{\sin x}{1+\sin x+\cos x}dx=\int\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}+2\cos^2\frac{x}{2}}dx=\int\frac{2\sin\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}}d(\frac{x}{2})$

$$\overset{\frac{x}{2}=t}{=\!=\!=}\int\frac{2\sin t}{\sin t+\cos t}dt=\int\frac{\sin t+\cos t+\sin t-\cos t}{\sin t+\cos t}dt=\int dt-\int\frac{d(\sin t+\cos t)}{\sin t+\cos t}$$

$$=\int dt-\int\frac{d(\sin t+\cos t)}{\sin t+\cos t}=t-\ln|\sin t+\cos t|+C=\frac{x}{2}-\ln\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|+C$$

法 2 $\int\frac{\sin x}{1+\sin x+\cos x}dx=\int\frac{\sin x(1+\sin x-\cos x)}{(1+\sin x+\cos x)(1+\sin x-\cos x)}dx$

$$=\int\frac{\sin x(1+\sin x-\cos x)}{(1+\sin x)^2-\cos^2 x}dx=\frac{1}{2}\int(1-\frac{\cos x}{1+\sin x})dx=\frac{1}{2}(\int dx-\int\frac{d(1+\sin x)}{1+\sin x})$$

$$=\frac{1}{2}(x-\ln|1+\sin x|)+C$$

7. 求下列不定积分.

(1) $\int\frac{dx}{\sqrt{1-x-x^2}}$

解: $\int\frac{dx}{\sqrt{1-x-x^2}}=\int\frac{d(x+\frac{1}{2})}{\sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}}=\arcsin\frac{2x+1}{\sqrt{5}}+C$

(2) $\int\frac{xdx}{\sqrt{2x^2-4x}}$

解: $\int\frac{xdx}{\sqrt{2x^2-4x}}=\int\frac{(x-1)+1}{\sqrt{2x^2-4x}}dx=\frac{1}{4}\int\frac{d(2x^2-4x)}{\sqrt{2x^2-4x}}+\frac{1}{\sqrt{2}}\int\frac{d(x-1)}{\sqrt{(x-1)^2-1}}$

$$=\frac{1}{2}\sqrt{2x^2-4x}+\frac{1}{\sqrt{2}}\ln\left|x-1+\sqrt{x^2-2x}\right|+C$$

$$=\frac{1}{\sqrt{2}}(\sqrt{x^2-2x}+\ln\left|x-1+\sqrt{x^2-2x}\right|)+C$$

(3) $\int\frac{x+1}{\sqrt{x^2+x+1}}dx$

解: $\int\frac{x+1}{\sqrt{x^2+x+1}}dx=\int\frac{(x+\frac{1}{2})+\frac{1}{2}}{\sqrt{x^2+x+1}}dx=\frac{1}{2}\int\frac{d(x^2+x+1)}{\sqrt{x^2+x+1}}+\frac{1}{2}\int\frac{d(x+\frac{1}{2})}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$

$$=\sqrt{x^2+x+1}+\frac{1}{2}\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+C$$

(4) $\int \frac{x^3}{\sqrt{1-x^8}} dx$

解: $\int \frac{x^3}{\sqrt{1-x^8}} dx = \frac{1}{4}\int \frac{d(x^4)}{\sqrt{1-(x^4)^2}} = \frac{1}{4}\arcsin x^4 + C$

(5) $\int \frac{dx}{\cos^2 x\sqrt{1+\tan x}} = \int \frac{d(1+\tan x)}{\sqrt{1+\tan x}} = 2\sqrt{1+\tan x} + C$