习题 4.5(P254)

1. 求下列广义积分.

(1) $\int_0^{+\infty} e^{-x}dx$

解：$\int_0^{+\infty} e^{-x}dx = -e^{x}\Big|_0^{+\infty} = -(0-e^0) = 1$

(2) $\int_1^{+\infty} \frac{dx}{x(x+1)}$

解：$\int_1^{+\infty} \frac{dx}{x(x+1)} = \int_1^{+\infty}(\frac{1}{x}-\frac{1}{x+1})dx = \ln\left|\frac{x}{x+1}\right|\Big|_1^{+\infty} = -\ln\frac{1}{2} = \ln 2$

(3) $\int_{-\infty}^{-1} \frac{dx}{x^2(x^2+1)}$

解：$\int_{-\infty}^{-1} \frac{dx}{x^2(x^2+1)} = \int_{-\infty}^{-1}(\frac{1}{x^2}-\frac{1}{x^2+1})dx = -\frac{1}{x}\Big|_{-\infty}^{-1} - \arctan x\Big|_{-\infty}^{-1} = 1-\frac{\pi}{4}$

(4) $\int_0^{+\infty} xe^{-x^2}dx$

解：$\int_0^{+\infty} xe^{-x^2}dx = -\frac{1}{2}\int_0^{+\infty} e^{-x^2}d(-x^2) = -\frac{1}{2}e^{-x^2}\Big|_0^{+\infty} = \frac{1}{2}$

(5) $\int_1^{+\infty} \frac{\arctan x}{x^2}dx$

解：$\int_1^{+\infty} \frac{\arctan x}{x^2}dx = -\int_1^{+\infty}\arctan x d(\frac{1}{x}) = -\left(\frac{1}{x}\arctan x\Big|_1^{+\infty} - \int_1^{+\infty}\frac{1}{x}\cdot\frac{1}{1+x^2}dx\right)$

$= \frac{\pi}{4} + \int_1^{+\infty}(\frac{1}{x}-\frac{x}{1+x^2})dx = \frac{\pi}{4} + \ln\frac{x}{\sqrt{1+x^2}}\Big|_1^{+\infty} = \frac{\pi}{4}+\frac{1}{2}\ln 2$

(6) $\int_0^{+\infty} e^{-ax}\cos bxdx \quad (a>0)$

解：$\int_0^{+\infty} e^{-ax}\cos bxdx = -\frac{1}{a}\int_0^{+\infty}\cos bxd(e^{-ax})$

$= -\frac{1}{a}\left(e^{-ax}\cos bx\Big|_0^{+\infty} - \int_0^{+\infty} e^{-ax}b(-\sin bx)dx\right)$

$= \frac{1}{a} - \frac{b}{a}\int_0^{+\infty} e^{-ax}\sin bxdx = \frac{1}{a}+\frac{b}{a^2}\int_0^{+\infty}\sin bxd(e^{-ax})$

$= \frac{1}{a}+\frac{b}{a^2}\left(e^{-ax}\sin bx\Big|_0^{+\infty} - b\int_0^{+\infty} e^{-ax}\cos bxdx\right) = \frac{1}{a}-\frac{b^2}{a^2}\int_0^{+\infty} e^{-ax}\cos bxdx$

$\therefore \int_0^{+\infty} e^{-ax}\cos bx dx = \dfrac{a}{a^2+b^2}$

(7) $\int_0^1 \dfrac{dx}{\sqrt{x}}$

解：$\int_0^1 \dfrac{dx}{\sqrt{x}} = \lim\limits_{s\to 0^+} \int_s^1 \dfrac{1}{\sqrt{x}}dx = \lim\limits_{s\to 0^+} 2\sqrt{x}\Big|_s^1 = \lim\limits_{s\to 0^+}[2-\sqrt{s}\,]= 2$

(8) $\int_0^1 \ln x dx$

解：$\int_0^1 \ln x dx = \lim\limits_{s\to 0^+} \int_s^1 \ln x dx = \lim\limits_{s\to 0^+}[x\ln x\big|_s^1 - \int_s^1 dx] = \lim\limits_{s\to 0^+}[x\ln x - x]\big|_s^1$

$= -1 + \lim\limits_{s\to 0^+} \dfrac{\ln s - 1}{\frac{1}{s}} \xlongequal[\text{法则}]{\text{洛必达}} -1 + \lim\limits_{s\to 0^+} -s = -1$

(9) $\int_0^1 \dfrac{x}{\sqrt{1-x^2}}dx$

解：$\int_0^1 \dfrac{x}{\sqrt{1-x^2}}dx = \lim\limits_{s\to 0^+} -\dfrac{1}{2}\int_s^1 \dfrac{d(1-x^2)}{\sqrt{1-x^2}} = -\dfrac{1}{2}\lim\limits_{s\to 0^+} 2\sqrt{1-x^2}\Big|_s^1 = -\lim\limits_{s\to 0^+}[0-\sqrt{1-s^2}\,] = 1$

(10) $\int_a^{2a} \dfrac{dx}{(x-a)^{\frac{3}{2}}}$

解：$\int_a^{2a} \dfrac{dx}{(x-a)^{\frac{3}{2}}} = \lim\limits_{s\to a^+} \int_s^{2a} \dfrac{d(x-a)}{(x-a)^{\frac{3}{2}}} = \lim\limits_{s\to a^+} \dfrac{-2}{(x-a)^{\frac{1}{2}}}\Bigg|_s^{2a} = \dfrac{-2}{a^{\frac{1}{2}}} + \lim\limits_{s\to a^+} \dfrac{2}{(s-a)^{\frac{1}{2}}} = +\infty$

故该广义积分发散.

2. 求曲线 $y = xe^{-\frac{x^2}{2}}$ 与其渐近线之间的面积

解：因为 $\lim\limits_{x\to\infty} y = xe^{-\frac{x^2}{2}} = 0$，故曲线 $y = xe^{-\frac{x^2}{2}}$ 的渐近线为 $y = 0$，故曲线 $y = xe^{-\frac{x^2}{2}}$ 与该渐近线之间的面积

$$A = \int_{-\infty}^{+\infty} \left| xe^{-\frac{x^2}{2}} \right| dx = \int_{-\infty}^{0} -xe^{-\frac{x^2}{2}}dx + \int_0^{+\infty} xe^{-\frac{x^2}{2}}dx$$

$$= \int_{-\infty}^{0} e^{-\frac{x^2}{2}} d(-\frac{x^2}{2}) - \int_0^{+\infty} e^{-\frac{x^2}{2}} d(-\frac{x^2}{2}) = e^{-\frac{x^2}{2}}\Bigg|_{-\infty}^{0} - e^{-\frac{x^2}{2}}\Bigg|_0^{+\infty} = 1+1 = 2$$