习题 7.5(76)

1．求 $z=3x^4+xy+y^3$ 在点 $(1,2)$ 处与 x 轴正向成 135^0 角、与 y 轴正向成 45^0 角的方向上的方向导数.

解：$gradz(1,\ 2)=\left\{\frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y}\right\}\Big|_{(1,\ 2)}=\{12x^3+y,\ \ x+3y^2\}\big|_{(1,\ 2)}=\{14,\ 13\}$，

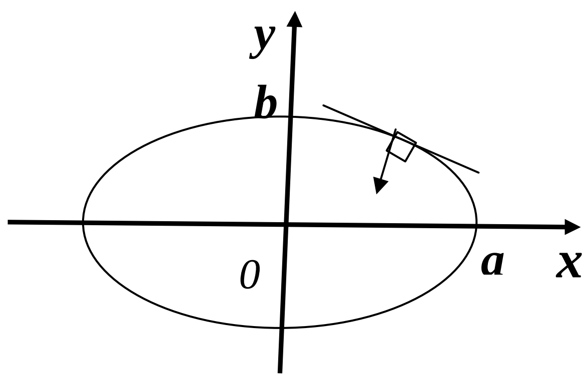
所给方向 $\vec{l}$ 的单位向量为 $\vec{l^0}=\{\cos(135^0),\cos(45^0)\}=\left\{-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right\}$，

所求方向导数为 $\frac{\partial z}{\partial \vec{l}}\Big|_{(1,\ 2)}=gradz(1,2)\cdot\vec{l^0}=-\frac{\sqrt{2}}{2}$

2． 求 $z=1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$，在点 $\left(\frac{a}{\sqrt{2}},\ \frac{b}{\sqrt{2}}\right)$ 处沿曲线 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 在此点内法线方向上的方向导数.

解：曲线 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 如图，设

$F(x,\ y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$，则曲线在点 $\left(\frac{a}{\sqrt{2}},\ \frac{b}{\sqrt{2}}\right)$ 处的法向量



$\vec{n}=\{F'_x,\ F'_y\}\big|_{(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}})}$

$\left\{\frac{2x}{a^2},\ \frac{2y}{b^2}\right\}\Big|_{(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}})}=\left\{\frac{\sqrt{2}}{a},\frac{\sqrt{2}}{b}\right\}$

曲线在该点处的内法线的单位向量为 $\vec{e}=\vec{n^0}=\left\{-\frac{b}{\sqrt{a^2+b^2}},-\frac{a}{\sqrt{a^2+b^2}}\right\}$

$$gradz(\frac{a}{\sqrt{2}},\ \frac{b}{\sqrt{2}})=\left\{\frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y}\right\}\Big|_{(\frac{a}{\sqrt{2}},\ \frac{b}{\sqrt{2}})}=\left\{-\frac{2x}{a^2},\ -\frac{2y}{b^2}\right\}\Big|_{(\frac{a}{\sqrt{2}},\ \frac{b}{\sqrt{2}})}=\left\{-\frac{\sqrt{2}}{a},\ -\frac{\sqrt{2}}{b}\right\}$$

故 $\frac{\partial z}{\partial \vec{e}}\Big|_{(\frac{a}{\sqrt{2}},\ \frac{b}{\sqrt{2}})}=gradz(\frac{a}{\sqrt{2}},\ \frac{b}{\sqrt{2}})\cdot\vec{e}$

$$= \left\{-\frac{\sqrt{2}}{a},\ -\frac{\sqrt{2}}{b}\right\} \cdot \left\{-\frac{b}{\sqrt{a^2+b^2}},\ -\frac{a}{\sqrt{a^2+b^2}}\right\} = \frac{\sqrt{2(a^2+b^2)}}{ab}$$

3．求 $u = xyz$ 在点 $(5, 1, 2)$ 处到的点 $(9, 4, 14)$ 的方向导数.

解：方向 $\vec{l} = \{9-5, 4-1, 14-2\} = \{4, 3, 12\}$，与它同向单位向量为 $\vec{l^0} = \left\{\frac{4}{13}, \frac{3}{13}, \frac{12}{13}\right\}$

$$grad u(5, 1, 2) = \left\{\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right\}\Bigg|_{(5,1,2)} = \{yz,\ xz,\ xy\}\big|_{(5,1,2)} = \{2, 10, 5\},$$

所以 $\left.\frac{\partial z}{\partial \vec{l}}\right|_{(5,1,2)} = grad u(5,1,2) \cdot \vec{l^0} = 2 \cdot \frac{4}{13} + 10 \cdot \frac{3}{13} + 5 \cdot \frac{12}{13} + = \frac{98}{13}$

4．设 $z = f(x, y)$，其中 f 具有一阶连续偏导数，已知四点 $A(1, 3)$、$B(3, 3)$、$C(1, 7)$、$D(6, 15)$，如果 $f(x, y)$ 在点 A 处沿 $\overrightarrow{AB}$ 方向的方向导数等于 3，沿 $\overrightarrow{AC}$ 方向的方向导数等于 26，求 $f(x, y)$ 在 A 处沿 $\overrightarrow{AD}$ 方向的方向导数.

解：$\overrightarrow{AB} = \{2,0\}$，$\overrightarrow{AC} = \{0,4\}$，所以与 $\overrightarrow{AB}$、$\overrightarrow{AC}$ 同方向的单位向量分别是 x 轴正向单位向量、y 轴正向单位向量，根据偏导数与方向导数之间的关系：$\left.\frac{\partial z}{\partial \vec{x}}\right|_A = 3$，$\left.\frac{\partial z}{\partial \vec{y}}\right|_A = 26$，

$\overrightarrow{AD} = \{5,12\}$，与 $\overrightarrow{AD}$ 同向的单位向量是 $\vec{l^0} = \left\{\frac{5}{13}, \frac{12}{13}\right\}$，

所以所求方向导数为 $\left.\frac{\partial z}{\partial \vec{l}}\right|_A = 3 \cdot \frac{5}{13} + 26 \cdot \frac{12}{13} + = \frac{327}{13}$

5．设 $f(x, y, z) = x^2 + 3y^2 + 3z^2 + xy + 3x - 2y - 6z$，求 $grad f(0, 0, 0)$，

$grad f(1, 1, 1)$

解：$\frac{\partial f}{\partial x} = 2x + y + 3$，$\frac{\partial f}{\partial y} = 6y + x - 2$，$\frac{\partial f}{\partial z} = 6z - 6$

$grad f(0, 0, 0) = \{3, -2, -6\}$，　　$grad f(1, 1, 1) = \{6, 5, 0\}$

6．求数量场 $u=\dfrac{x}{x^2+y^2+z^2}$ 在点 $A(1,2,2)$ 及点 $B(-3,1,0)$ 处的梯度之间的夹角.

解：$\dfrac{\partial u}{\partial x}=\dfrac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$，$\dfrac{\partial u}{\partial y}=\dfrac{-2xy}{(x^2+y^2+z^2)^2}$，$\dfrac{\partial u}{\partial z}=\dfrac{-2xz}{(x^2+y^2+z^2)^2}$

在点 A 处：$\dfrac{\partial u}{\partial x}=\dfrac{7}{81}$，$\dfrac{\partial u}{\partial y}=-\dfrac{4}{81}$，$\dfrac{\partial u}{\partial z}=-\dfrac{4}{81}$

在点 B 处：$\dfrac{\partial u}{\partial x}=-\dfrac{4}{50}$，$\dfrac{\partial u}{\partial y}=\dfrac{3}{50}$，$\dfrac{\partial u}{\partial z}=0$

$$gradu\big|_A=\left\{\frac{7}{81},-\frac{4}{81},-\frac{4}{81}\right\}\triangleq\vec{a}\,,\qquad gradu\big|_B=\left\{-\frac{4}{50},\frac{5}{50},0\right\}\triangleq\vec{b}$$

$$\cos<\vec{a},\vec{b}>=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}=-\frac{8}{9}\,,\qquad 故<\vec{a},\vec{b}>=\arccos\left(-\frac{8}{9}\right)$$

7． 设数量场 $z=\dfrac{1}{2}\ln(x^2+y^2)$，求 $gradz$，并证明此数量场的等值线上任一点 $(x,\ y)$ 处的切线与 $gradz$ 垂直.

证明：$gradz=\left\{\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}\right\}=\left\{\dfrac{x}{x^2+y^2},\dfrac{y}{x^2+y^2}\right\}\triangleq\vec{g}$

数量场的等值线为 $\dfrac{1}{2}\ln(x^2+y^2)=C$，即 $x^2+y^2=R^2$

上式两端对 x 求导得 $2x+2y\dfrac{dy}{dx}=0$，故得 $\dfrac{dy}{dx}=-\dfrac{x}{y}$

而等值线上任一点 $(x,\ y)$ 处切线的切向量 $\vec{T}=\left\{1,\dfrac{dy}{dx}\right\}=\left\{1,\ -\dfrac{x}{y}\right\}$

由于 $\vec{g}\cdot\vec{T}=\left\{\dfrac{x}{x^2+y^2},\dfrac{y}{x^2+y^2}\right\}\cdot\left\{1,\ -\dfrac{x}{y}\right\}=0$，即 $\vec{g}\perp\vec{T}$，

故等值线上任一点 $(x,\ y)$ 处的切线与 $gradz$ 垂直.