习题 7.9(P102)

1．设 a、b、c 是三角形三条边的长，A、B、C 分别是此三边对应的三个角的度量，求 $\frac{\partial A}{\partial a}$，$\frac{\partial A}{\partial b}$，$\frac{\partial A}{\partial c}$.

解：由余弦定理：$2bc\cos A = b^2 + c^2 - a^2$

两端分别对 a、b、c 求偏导： $-2bc\sin A\cdot\frac{\partial A}{\partial a} = -2a$

$$2c\cos A - 2bc\sin A\cdot\frac{\partial A}{\partial b} = 2b$$

$$2b\cos A - 2bc\sin A\cdot\frac{\partial A}{\partial c} = 2c$$

整理得：$\frac{\partial A}{\partial a} = \frac{a}{bc\sin A}$，$\frac{\partial A}{\partial b} = \frac{c\cos A - b}{bc\sin A}$，$\frac{\partial A}{\partial c} = \frac{b\cos A - c}{bc\sin A}$

2．设 $z = f(y+\varphi(x-y), e^{2x})$，其中 f 具有二阶连续偏导数，φ 具有二阶偏导数，求 $\frac{\partial^2 z}{\partial x\partial y}$.

解：$\frac{\partial z}{\partial x} = f_1'\cdot\varphi' + f_2'\cdot e^{2x}\cdot 2 = f_1'\cdot\varphi' + 2e^{2x}f_2'$

$$\frac{\partial^2 z}{\partial x\partial y} = f_{11}''\cdot(1-\varphi')\varphi' + f_1'\cdot\varphi'' + 2e^{2x}f_{21}''\cdot(1-\varphi')$$

3． 证明：函数 $y(x,t) = \varphi(x+at) + \varphi(x-at) + \int_{x-at}^{x+at} f(z)dz$ 满足方程 $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

（其中 f，φ 可微）

证明：$\frac{\partial y}{\partial x} = \varphi'(x+at) + \varphi'(x-at) + f(x+at) - f(x-at)$

$$\frac{\partial^2 y}{\partial x^2} = \varphi''(x+at) + \varphi''(x-at) + f'(x+at) - f'(x-at)$$

$$\frac{\partial y}{\partial t}=a\varphi'(x+at)-a\varphi'(x-at)+af(x+at)+af(x-at)$$

$$\frac{\partial^2 y}{\partial t^2}=a^2\varphi''(x+at)+a^2\varphi''(x-at)+a^2f'(x+at)-a^2f'(x-at)$$

故 $\frac{\partial^2 y}{\partial t^2}=a^2[\varphi''(x+at)+\varphi''(x-at)+f'(x+at)-f'(x-at)]=a^2\frac{\partial^2 y}{\partial x^2}$

易出的错误： 将 $\varphi'(x+at)$ 与 $\varphi'(x-at)$ 均写成 φ'；$\varphi''(x+at)$ 与 $\varphi''(x-at)$ 均写成 φ''；$f'(x+at)$ 与 $f'(x-at)$ 均写成 f'.（注：在不发生混淆时才可以简写）

4．设函数 $f(u)$ 具有二阶连续导数，$z=f(e^x\sin y)$ 满足方程 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=e^{2x}z$，求 $f(u)$.

解：$\frac{\partial z}{\partial x}=f'\cdot e^x\sin y$，$\frac{\partial z}{\partial y}=f'\cdot e^x\cos y$

$$\frac{\partial^2 z}{\partial x^2}=f''\cdot e^x\sin y\cdot e^x\sin y+f'\cdot e^x\sin y=f''\cdot e^{2x}\sin^2 y+f'\cdot e^x\sin y$$

$$\frac{\partial^2 z}{\partial y^2}=f''\cdot e^x\cos y\cdot e^x\cos y-f'\cdot e^x\sin y=f''\cdot e^{2x}\cos^2 y-f'\cdot e^x\sin y$$

代入方程得 $f''\cdot e^{2x}=e^{2x}z=e^{2x}f$，即 $f''=f$，亦即 $f''-f=0$

其特征方程为 $r^2-1=0$，得 $r_{1,2}=\pm1$

故 $f(u)=C_1e^u+C_2e^{-u}$ （C_1、C_2 为任意常数）.

5．设 $u=u(x)$ 是由方程组 $u=f(x,y)$，$g(x,y,z)=0$，$h(x,z)=0$ 所确定的函数，其中 f、g、h 是可微函数，且 $h_z'\neq0$，$g_y'\neq0$，求 $\frac{du}{dx}$.

解：三个方程分别对 x 求导：

$$\begin{cases} \dfrac{du}{dx} = f'_x + f'_y \dfrac{dy}{dx} & (1) \\ g'_x + g'_y \cdot \dfrac{dy}{dx} + g'_z \cdot \dfrac{dz}{dx} = 0 & (2) \\ h'_x + h'_z \cdot \dfrac{dz}{dx} = 0 & (3) \end{cases}$$

由(3)式解得 $\dfrac{dz}{dx} = -\dfrac{h'_x}{h'_z}$，将其代入(2)式得

$$\frac{dy}{dx} = \frac{g'_z \cdot h'_x}{g'_y \cdot h'_z} - \frac{g'_x}{g'_y}$$，将其代入(1)式得

$$\frac{du}{dx} = f'_x + f'_y \cdot \left(\frac{g'_z \cdot h'_x}{g'_y \cdot h'_z} - \frac{g'_x}{g'_y} \right) = f'_x + \frac{f'_y \cdot g'_z \cdot h'_x}{g'_y \cdot h'_z} - \frac{f'_y \cdot g'_x}{g'_y}$$

6．设函数 $z(x, y)$ 满足 $\begin{cases} \dfrac{\partial z}{\partial x} = -\sin y + \dfrac{1}{1-xy} \\ z(1, y) = \sin y \end{cases}$，求 $z(x, y)$.

解：方程两端对 x 积分（把 y 视为常数）：$z = -x\sin y - \dfrac{1}{y}\ln|1-xy| + \varphi(y)$

由条件 $z(1, y) = \sin y$，$\varphi(y) = 2\sin y + \dfrac{1}{y}\ln|1-y|$

故 $z(x, y) = (2-x)\sin y + \dfrac{1}{y}\ln\left|\dfrac{1-y}{1-xy}\right|$

易出的错误： 利用积分公式 $\int \dfrac{1}{x}dx = \ln|x| + C$ 时，未加绝对值符号.

7． 设 $f(x, y)$ 具有一阶连续偏导数，且 $f(x, x^2) = 1$，$f'_x(x, x^2) = x$，求 $f'_y(x, x^2)$

解： 方程 $f(x, x^2) = 1$ 两端同时对 x 求导，由多元复合函数求导法得

$$f'_x(x, x^2) + f'_y(x, x^2) \cdot 2x = 0$$

故 $f'_y(x, x^2) = -\dfrac{f'_x(x, x^2)}{2x} = -\dfrac{x}{2x} = -\dfrac{1}{2}$

8．设 $f(u,v)$ 具有二阶连续偏导数，且满足 $\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}=1$，又 $g(x,y)=f(xy,\frac{1}{2}(x^2-y^2))$，求 $\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}$

解：$\frac{\partial g}{\partial x}=y\frac{\partial f}{\partial u}+x\frac{\partial f}{\partial v}$，　　$\frac{\partial g}{\partial y}=x\frac{\partial f}{\partial u}-y\frac{\partial f}{\partial v}$

$$\frac{\partial^2 g}{\partial x^2}=y(y\frac{\partial^2 f}{\partial u^2}+x\frac{\partial^2 f}{\partial u\partial v})+x(y\frac{\partial^2 f}{\partial v\partial u}+x\frac{\partial^2 f}{\partial v^2})=y^2\frac{\partial^2 f}{\partial u^2}+2xy\frac{\partial^2 f}{\partial u\partial v}+x^2\frac{\partial^2 f}{\partial v^2}$$

$$\frac{\partial^2 g}{\partial y^2}=x(x\frac{\partial^2 f}{\partial u^2}-y\frac{\partial^2 f}{\partial u\partial v})-y(x\frac{\partial^2 f}{\partial v\partial u}-y\frac{\partial^2 f}{\partial v^2})=x^2\frac{\partial^2 f}{\partial u^2}-2xy\frac{\partial^2 f}{\partial u\partial v}+y^2\frac{\partial^2 f}{\partial v^2}$$

故 $\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}=(x^2+y^2)(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2})=x^2+y^2$

9．作变换 $u=x$，$v=x^2-y^2$，求方程 $y\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$ 的解.

分析：所作的变换使得二元函数变为中间变量为 u、v 的复合函数，利用多元复合函数求导法求出 $\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$，代入方程即可得解.

解：$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\cdot 2x$，$\frac{\partial z}{\partial y}=-\frac{\partial z}{\partial v}\cdot 2y$

代入方程得　$y\left[\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\cdot 2x\right]+x\left[-2y\frac{\partial z}{\partial v}\right]=0$

整理得　$y\frac{\partial z}{\partial u}=0$，因为 $y\neq 0$（这里 $\neq$ 为不恒等于），所以 $\frac{\partial z}{\partial u}=0$

两端对 u 积分（积分时注意到 z 是 u、v 的函数，对 u 求偏导时，是把 v 作为常数）得

$z=f(v)=f(x^2-y^2)$

10．设 $z=z(x,y)$ 有二阶连续偏导数，$u=x-ay$，$v=x+ay$，变换方程 $\frac{\partial^2 z}{\partial y^2}=a^2\frac{\partial^2 z}{\partial x^2}$

解： $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}=\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=-a\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(-a\frac{\partial z}{\partial u}+a\frac{\partial z}{\partial v}\right)=-a\left(\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial y}\right)+a\left(\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial y}\right)$$

$$=-a\left((-a)\frac{\partial^2 z}{\partial u^2}\cdot+a\frac{\partial^2 z}{\partial u\partial v}\right)+a\left((-a)\frac{\partial^2 z}{\partial v\partial u}+a\frac{\partial^2 z}{\partial v^2}\right)=a^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)$$

代入方程 $\dfrac{\partial^2 z}{\partial y^2}=a^2\dfrac{\partial^2 z}{\partial x^2}$ 得： $\dfrac{\partial^2 z}{\partial u\partial v}=0$

11．设 $\vec{n}$ 是曲面 $2x^2+3y^2+z^2=6$ 在点 $P(1,1,1)$ 处的指向外侧的法向量，求函数 $u=\dfrac{\sqrt{6x^2+8y^2}}{z}$ 在点 P 处沿方向 $\vec{n}$ 的方向向量.

解：曲面在点 $P(1,1,1)$ 处的指向外侧的法向量为 $\vec{n}=\{4x,6y,2z\}\big|_P=\{4,6,2\}$

$$\vec{n}^0=\left\{\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}},\frac{1}{\sqrt{14}}\right\}$$

$$\frac{\partial u}{\partial x}=\frac{6x}{z\sqrt{6x^2+8y^2}},\quad \frac{\partial u}{\partial y}=\frac{8y}{z\sqrt{6x^2+8y^2}},\quad \frac{\partial u}{\partial z}=-\frac{\sqrt{6x^2+8y^2}}{z^2}$$

$$\left.\frac{\partial u}{\partial x}\right|_P=\frac{6}{\sqrt{14}},\quad \left.\frac{\partial u}{\partial y}\right|_P=\frac{8}{\sqrt{14}},\quad \left.\frac{\partial u}{\partial z}\right|_P=-\sqrt{14}$$

$$\left.\frac{\partial u}{\partial \vec{n}}\right|_P=\frac{6}{\sqrt{14}}\cdot\frac{2}{\sqrt{14}}+\frac{8}{\sqrt{14}}\cdot\frac{3}{\sqrt{14}}-\sqrt{14}\cdot\frac{1}{\sqrt{14}}=\frac{11}{7}$$

12． 证明：曲线 $x=e^t\cos t$， $y=e^t\sin t$， $z=e^t$ 与圆锥面 $x^2+y^2=z^2$ 所有母线以等

角相交.

分析： 曲线上的点 M 与曲面在点 M 处的母线的夹角即曲线在点 M 处的切线与曲面在点 M 处的母线的夹角。故只要证明曲线在任意一点 M 处的切向量与曲面在点 M 处母线方向向量的夹角是常数即可。注意：曲线在圆锥面上，圆锥面的顶点为原点。

证明：由于曲线方程满足圆锥面方程，故曲线在圆锥面上，设 $M(x, y, z)$ 是曲线上的任一点，则曲线在点 M 处的切向量

$$\vec{T}=\left\{\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right\}=\left\{e^t(\cos t-\sin t), e^t(\sin t+\cos t), e^t\right\}$$

圆锥面上过点 M 的母线上的向量 $\overrightarrow{OM}=\{x, y, z\}=\left\{e^t\cos t, e^t\sin t, e^t\right\}$

设 θ 为 $\vec{T}$ 与 $\overrightarrow{OM}$ 之间的夹角，则

$$\cos\theta=\frac{\vec{T}\cdot\overrightarrow{OM}}{\left|\vec{T}\right|\cdot\left|\overrightarrow{OM}\right|}=\frac{2}{\sqrt{6}}$$

θ 与 t 无关，因此 θ 与点 M 无关，故曲线与圆锥面所有母线以等角相交.

13． 证明：曲面 $z=xf\left(\dfrac{y}{x}\right)$.上的任何一点的切平面通过一定点.

证明：设 $M_0(x_0, y_0, z_0)$ 是曲面上的任一点，则曲面在点 M_0 处的法向量

$$\vec{n}=\left\{z'_x, z'_y, -1\right\}\Big|_{M_0}=\left\{f\left(\frac{y_0}{x_0}\right)-\frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right),\ f'\left(\frac{y_0}{x_0}\right),\ -1\right\}$$

又 $z_0=x_0 f\left(\dfrac{y_0}{x_0}\right)$，故曲面在点 M_0 处的切平面方程为

$$\left(f\left(\frac{y_0}{x_0}\right)-\frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right)(x-x_0)+f'\left(\frac{y_0}{x_0}\right)(y-y_0)-\left(z-x_0 f\left(\frac{y_0}{x_0}\right)\right)=0$$

整理得 $\left(f\left(\dfrac{y_0}{x_0}\right)-\dfrac{y_0}{x_0}f'\left(\dfrac{y_0}{x_0}\right)\right)x+f'\left(\dfrac{y_0}{x_0}\right)y-z=0$

原点 $(0, 0, 0)$ 满足上述方程，即曲面上的任何一点的切平面都通过原点这一定点.

14. 求曲面 $x=u+v$， $y=u^2+v^2$， $z=u^3+v^3$ 在点 $u=1$， $v=-1$ 处的切平面方程.

分析：求出法向量 $\vec{n}=\left\{z'_x,\ z'_y,\ -1\right\}$ 即可. 两种解法：(1)已知的曲面方程是参数为 u、v 的参数方程，因而 $u=u(x, y)$， $v=v(x, y)$，利用多元复合函数求导法求 z'_x、 z'_y；(2)可

以将 z 表示为 x 、 y 的函数直接求 z'_x 、 z'_y .

解：法 1：曲面方程分别对 x 、 y 求偏导：

$$\begin{cases} 1=\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial x} \\ 0=2u\dfrac{\partial u}{\partial x}+2v\dfrac{\partial v}{\partial x} \\ \dfrac{\partial z}{\partial x}=3u^2\dfrac{\partial u}{\partial x}+3v^2\dfrac{\partial v}{\partial x} \end{cases}, \qquad \begin{cases} 0=\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial y} \\ 1=2u\dfrac{\partial u}{\partial y}+2v\dfrac{\partial v}{\partial y} \\ \dfrac{\partial z}{\partial y}=3u^2\dfrac{\partial u}{\partial y}+3v^2\dfrac{\partial v}{\partial y} \end{cases}$$

将 $u=1$，$v=-1$ 代入上述方程组，解得 $\dfrac{\partial z}{\partial x}=3$， $\dfrac{\partial z}{\partial y}=0$，

故切平面的法向量 $\vec{n}=\{z'_x,\ z'_y,\ -1\}=\{3,\ 0,\ -1\}$

当 $u=1$，$v=-1$ 时， $x=0$， $y=2$， $z=0$

故切平面方程为 $3(x-0)+0(y-2)-(z-0)=0$，即 $3x-z=0$

法 2： $x^2=(u+v)^2=u^2+v^2+2uv=y+2uv$，即 $uv=\dfrac{x^2-y}{2}$

$$z=u^3+v^3=(u+v)(u^2+v^2-uv)=x(y-\frac{x^2-y}{2})=\frac{3}{2}xy-\frac{x^3}{2}$$

故 $\left.\dfrac{\partial z}{\partial x}\right|_{(0,2,0)}=\left.\dfrac{3}{2}(y-x^2)\right|_{(0,2,0)}=3$， $\left.\dfrac{\partial z}{\partial y}\right|_{(0,2,0)}=\left.\dfrac{3}{2}x\right|_{(0,2,0)}=0$

15. 已知 x 、 y 、 z 为实数，且 $e^x+y^2+|z|=3$，求证： $e^x y^2|z|\le 1$.

分析：令 $u=e^x$， $v=y^2$， $w=|z|$，只要证明在条件 $u+v+w=3\ (u>0, v\ge, w\ge 0)$

下，函数 $T=uvw$ 的最大值为 1 即可，故用拉格朗日乘数法。

也可以转化为无条件极值问题。

证明：法 1(拉格朗日乘数法): 设 $F(u,v,w)=uvw+\lambda(u+v+w-3)$，则令

$$\begin{cases} F'_u=vw+\lambda=0 \\ F'_v=uw+\lambda=0 \\ F'_w=uv+\lambda=0 \\ u+v+w=3 \end{cases},$$

解得唯一驻点 $u=v=w=1$

因为 $T(1,\dfrac{3}{2},\dfrac{1}{2})=\dfrac{3}{4}$， 而 $T(1,1,1)=1$，即 $T_{\max}=T(1,1,1)=1$，故 $uvw\le 1$，

法 2（转化为无条件极值）：由 $u+v+w=3$，得 $w=3-u-v$

故 $T=uvw=uv(3-u-v)=3uv-u^2v-uv^2$

令 $\frac{\partial T}{\partial u}=3v-2uv-v^2=0$，$\frac{\partial T}{\partial v}=3u-2uv-u^2=0$，解得唯一驻点 $u=v=1$，

又 $\left.\frac{\partial^2 T}{\partial u^2}\right|_{(1,1)}=-2v=-2=A$，$\left.\frac{\partial^2 T}{\partial u\partial v}\right|_{(1,1)}=3-2u-2v=-1=B$，

$\left.\frac{\partial^2 T}{\partial v^2}\right|_{(1,1)}=-2u=-2=C$，$AC-B^2=3>0$，$A<0$，故在点 $(1,1)$ 取得极大值，由驻点的唯一性知：点 $(1,1)$ 也是最大值点，而当 $u=v=1$ 时，$w=1$，$T_{\max}=T(1,1,1)=1$