习题 7.3(P62)

1．求下列函数的全微分.

(1) $z = xy + \frac{x}{y}$

解：$\frac{\partial z}{\partial x} = y + \frac{1}{y}$，$\frac{\partial z}{\partial y} = x - \frac{x}{y^2} = x(1 - \frac{1}{y^2})$，所以 $dz = \left(y + \frac{1}{y}\right)dx + x\left(1 - \frac{1}{y^2}\right)dy$

(2) $z = \frac{y}{\sqrt{x^2 + y^2}}$

解：$\frac{\partial z}{\partial x} = -\frac{xy}{\sqrt{(x^2 + y^2)^3}}$，$\frac{\partial z}{\partial y} = \frac{x^2}{\sqrt{(x^2 + y^2)^3}}$，

所以 $dz = -\frac{xy}{\sqrt{(x^2 + y^2)^3}}dx + \frac{x^2}{\sqrt{(x^2 + y^2)^3}}dy = \frac{-x}{\sqrt{(x^2 + y^2)^3}}(ydx - xdy)$

(3) $z = \arctan(xy)$

解：$\frac{\partial z}{\partial x} = \frac{y}{1 + x^2y^2}$，$\frac{\partial z}{\partial y} = \frac{x}{1 + x^2y^2}$，所以 $dz = \frac{1}{1 + x^2y^2}(ydx + xdy)$

(4) $u = x^{yz}$

解：$\frac{\partial u}{\partial x} = yz \cdot x^{yz-1}$，$\frac{\partial u}{\partial y} = z \cdot x^{yz} \cdot \ln x$，$\frac{\partial u}{\partial z} = y \cdot x^{yz} \cdot \ln x$，

所以 $du = yz \cdot x^{yz-1}dx + z \cdot x^{yz} \ln x dy + y \cdot x^{yz} \ln x dz$

2．设 $z = x\sin(x + y)$，求 $dz\big|_{(0,0)}$，$dz\big|_{(\frac{\pi}{4},\frac{\pi}{4})}$.

解：$\frac{\partial z}{\partial x} = \sin(x + y) + x\cos(x + y)$，$\frac{\partial z}{\partial y} = x\cos(x + y)$

$\left.\frac{\partial z}{\partial x}\right|_{(0,0)} = 0$，$\left.\frac{\partial z}{\partial y}\right|_{(0,0)} = 0$，所以 $dz\big|_{(0,0)} = 0$

$\left.\frac{\partial z}{\partial x}\right|_{(\frac{\pi}{4},\frac{\pi}{4})} = 1$，$\left.\frac{\partial z}{\partial y}\right|_{(\frac{\pi}{4},\frac{\pi}{4})} = 0$，所以 $dz\big|_{(\frac{\pi}{4},\frac{\pi}{4})} = dx$

3．求 $z = x^2y^3$ 当 $x = 2$，$y = -1$，$\Delta x = 0.02$，$\Delta y = -0.01$ 时的全增量与全微分.

解：全增量 $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = 2.02^2 \cdot (-1.01)^3 - 2^2 \cdot (-1)^3 \approx -0.204$，

$\left.\frac{\partial z}{\partial x}\right|_{(2,-1)} = 2xy^3\big|_{(2,-1)} = -4$，$\left.\frac{\partial z}{\partial y}\right|_{(2,-1)} = 3x^2y^2\big|_{(2,-1)} = 12$，所以

全微分 $dz = -4\Delta x + 12\Delta y = -0.2$

4．设 $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$，问 $f(x,y)$ 在 $(0,0)$ 是否可微？

解：当点 (x,y) 沿 $y=0$ 趋近于 $(0,0)$ 时，$\lim\limits_{\substack{x\to0\\y\to0}} f(x,y) = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{x^2\cdot 0}{x^4+0^2} = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{0}{x^4} = 0$

当点 (x,y) 沿 $y=x^2$ 趋近于 $(0,0)$ 时，$\lim\limits_{\substack{x\to0\\y\to0}} f(x,y) = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{x^2\cdot x^2}{x^4+(x^2)^2} = \lim\limits_{\substack{x\to0\\y\to0}} \dfrac{x^4}{2x^4} = \dfrac{1}{2}$

即 $\lim\limits_{\substack{x\to0\\y\to0}} f(x,y)$ 不存在，从而函数 $f(x,y)$ 在点 $(0,0)$ 处不连续，故 $f(x,y)$ 在 $(0,0)$ 不可微.

5． 求 $\sin 29^0 \tan 46^0$ 的近似值.

解：设 $f(x,y) = \sin x \tan y$，则 $f_x(x,y) = \cos x \tan y$， $f_y(x,y) = \sin x \sec^2 y$，

所求为 $f\left(\dfrac{29\pi}{180}, \dfrac{46\pi}{180}\right)$. 取 $x=\dfrac{\pi}{6}$， $y=\dfrac{\pi}{4}$， $\Delta x = -\dfrac{\pi}{180}$， $\Delta y = \dfrac{\pi}{180}$，

则 $f\left(\dfrac{\pi}{6}, \dfrac{\pi}{4}\right) = \dfrac{1}{2}$， $f_x\left(\dfrac{\pi}{6}, \dfrac{\pi}{4}\right) = \dfrac{\sqrt{3}}{2}$， $f_y\left(\dfrac{\pi}{6}, \dfrac{\pi}{4}\right) = 1$，

所以 $f\left(\dfrac{29\pi}{180}, \dfrac{46\pi}{180}\right) \approx \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\cdot\left(-\dfrac{\pi}{180}\right) + 1\cdot\dfrac{\pi}{180} \approx 0.5023$

6．矩形长 $8m$， 宽 $6m$， 当长减少 $5cm$， 宽增加 $2cm$ 时，求矩形的对角线变化的近似值.

解：设 $l = f(x,y) = \sqrt{x^2+y^2}$ ，则 $dl = \dfrac{x}{\sqrt{x^2+y^2}}\Delta x + \dfrac{y}{\sqrt{x^2+y^2}}\Delta y$，。

取 $x=8$， $y=6$， $\Delta x = -0.05$， $\Delta y = 0.02$，

则对角线增量 $\Delta l \approx dl = \frac{4}{5}\cdot\Delta x + \frac{3}{5}\cdot\Delta y = -0.028m = 2.8cm$

7. 一扇形的中心角为 60^0，半径为 $20m$，如果中心角增加 1^0，为使扇形面积保持不变，应将扇形半径减少多少（计算到小数点后 3 位）？

解：设 $S = f(r,\theta) = \frac{1}{2}r^2\theta$，则 $dS = r\theta\Delta r + \frac{1}{2}r^2\Delta\theta$，取 $r = 20$，$\theta = \frac{\pi}{3}$，$\Delta\theta = \frac{\pi}{180}$，

若要面积保持不变，应有 $dS \approx \Delta S = 0$，所以 $20\cdot\frac{\pi}{3}\Delta r + \frac{1}{2}\cdot 20^2\cdot\frac{\pi}{180} \approx 0$，

解得 $\Delta r \approx 0.167m$

8..已知圆柱体高的相对误差限为 $\varepsilon_r(h_0)$，底面直径的相对误差限为 $\varepsilon_r(d_0)$，问圆柱体体积的相对误差限是多少？

解：设 $V = f(d,h) = \frac{1}{4}\pi d^2 h$，则 $dV = \frac{1}{2}\pi dh\Delta d + \frac{1}{4}\pi d^2\Delta h$，所以圆柱体体积的相对

误差限为 $\left|\frac{dV}{V}\right| = \left|\frac{\frac{1}{2}\pi dh\Delta d + \frac{1}{4}\pi d^2\Delta h}{\frac{1}{4}\pi d^2 h}\right| \le 2\left|\frac{\Delta d}{d}\right| + \left|\frac{\Delta h}{h}\right| \le \varepsilon_r(h_0) + 2\varepsilon_r(d_0)$