习题 8.4(P141)

1．试求下列曲面面积.

(1)平面 $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$ 被三坐标面所截部分.

解：平面 $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$ 在 xOy 面上的投影为 $D_{xy}: 0\le y\le 2, 0\le x\le 1-\frac{y}{2}$

$$S=\iint\limits_{D_{xy}}\sqrt{1+(z'_x)^2+(z'_y)^2}\,d\sigma=\iint\limits_{D_{xy}}\sqrt{1+3^2+(\frac{3}{2})^2}\,d\sigma$$

$$=\frac{7}{2}\iint\limits_{D_{xy}}d\sigma=\frac{7}{2}\times D_{xy}\text{的面积}=\frac{7}{2}\times\frac{1}{2}\times 1\times 2=\frac{7}{2}$$

(2) 球面 $x^2+y^2+z^2=R^2$ 被平面 $z=\frac{R}{4}$ 和 $z=\frac{R}{2}$ 所夹部分.

解：球面 $z=\sqrt{R^2-x^2-y^2}$ 在 xOy 面上的投影为 $D_{xy}: \frac{3}{4}R^2\le x^2+y^2\le\frac{15}{16}R^2$

$$S=\iint\limits_{D_{xy}}\sqrt{1+(z'_x)^2+(z'_y)^2}\,d\sigma=\iint\limits_{D_{xy}}\frac{R}{\sqrt{R^2-x^2-y^2}}\,d\sigma$$

$$=R\int_0^{2\pi}d\theta\int_{\frac{\sqrt{3}}{2}R}^{\frac{\sqrt{15}}{4}R}\frac{\rho}{\sqrt{R^2-\rho^2}}\,d\rho=2\pi R\cdot(-\frac{1}{2})\int_{\frac{\sqrt{3}}{2}R}^{\frac{\sqrt{15}}{4}R}\frac{d(R^2-\rho^2)}{\sqrt{R^2-\rho^2}}=\frac{\pi}{2}R^2$$

(3) 球面 $x^2+y^2+z^2=3\ (z\ge 0)$ 和抛物面 $x^2+y^2=2z$ 所围区域的边界曲面.

解：两个曲面的交线为 $\begin{cases}x^2+y^2+z^2=3\\ x^2+y^2=2z\end{cases}$ 即 $\begin{cases}x^2+y^2=2\\ z=1\end{cases}$

在 xOy 平面上的投影 $D_{xy}: x^2+y^2\le 2$，

$$S=S_1+S_2=\iint\limits_{D_{xy}}\frac{\sqrt{3}}{\sqrt{3-x^2-y^2}}\,d\sigma+\iint\limits_{D_{xy}}\sqrt{1+x^2+y^2}\,d\sigma$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\left(\frac{\sqrt{3}}{\sqrt{3-\rho^2}}+\sqrt{1+\rho^2}\right)\rho\,d\rho=2\pi\int_0^{\sqrt{2}}\left(\frac{\sqrt{3}}{\sqrt{3-\rho^2}}+\sqrt{1+\rho^2}\right)\rho\,d\rho=\frac{16}{3}\pi$$

(4) 曲面 $z=\sqrt{9-y^2}$ 被柱面 $|x|+|y|=1$ 截下的部分.

解：曲面 $z=\sqrt{9-y^2}$ 被柱面 $|x|+|y|=1$ 截下的第一卦限部分在 xOy 平面上的投影

$D_1: x \ge 0, y \ge 0, x+y \le 1$

$z'_x = 0$，$z'_y = -\dfrac{y}{\sqrt{9-y^2}}$ $\quad dS = \sqrt{1+(z'_x)^2+(z'_y)^2}\,d\sigma = \dfrac{3}{\sqrt{9-y^2}}$

由对称性得 $S = 4\iint\limits_{D_1} \dfrac{3}{\sqrt{9-y^2}} d\sigma = 12\int_0^1 dy \int_0^{1-y} \dfrac{1}{\sqrt{3^2-y^2}} dx$

$$= 12\int_0^1 \frac{1-y}{\sqrt{9-y^2}} dy = 12[\arcsin\frac{y}{3} + \sqrt{9-y^2}]_0^1$$

$$= 12[\arcsin\frac{1}{3} + 2\sqrt{2} - 3]$$

(5)锥面 $z = \sqrt{x^2+y^2}$ 被柱面 $z^2 = 2x$ 截下的部分.

解：联立 $\begin{cases} z = \sqrt{x^2+y^2} \\ z^2 = 2x \end{cases}$ 并消去 z ，得 $x^2+y^2 = 2x$

所以此曲面在 xOy 平面上的投影 $D_{xy}: x^2+y^2 \le 2x$，

$$S = \iint\limits_{D_{xy}} \sqrt{1+(z'_x)^2+(z'_y)^2}\,d\sigma = \iint\limits_{D_{xy}} \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,d\sigma$$

$$= \iint\limits_{D_{xy}} \sqrt{2}\,d\sigma = \sqrt{2} \times D_{xy}\text{的面积} = \sqrt{2}\pi$$

2．试求下列物体的质心坐标.

(1)均匀薄板 $D: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1\ (x \ge 0, y \ge 0)$.

解：$\bar{x} = \dfrac{\iint\limits_D x\,d\sigma}{\iint\limits_D d\sigma} = \dfrac{\iint\limits_D x\,d\sigma}{\frac{1}{4}\pi ab} = \dfrac{4}{\pi ab}\int_0^b dy \int_0^{a\sqrt{1-\frac{y^2}{b^2}}} x\,dx = \dfrac{2}{\pi ab}\int_0^b a^2\left(1-\dfrac{y^2}{b^2}\right)dy = \dfrac{4a}{3\pi}$

同理可得 $\bar{y} = \dfrac{4b}{3\pi}$，故质心坐标为 $\left(\dfrac{4a}{3\pi}, \dfrac{4b}{3\pi}\right)$

(2) 球体 $V: x^2+y^2+z^2 \le 2az\ (a>0)$ 中各点的密度与该点到原点的距离成正比.

解：由题意：$\rho_V(x,y,z) = k\sqrt{x^2+y^2+z^2}$

由对称性知：$\bar{x} = \bar{y} = 0$

$$m=\iiint\limits_V \rho_V(x,y,z)dV=k\iiint\limits_V \sqrt{x^2+y^2+z^2}dV=k\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^{2a\cos\varphi}r\cdot r^2\sin\varphi dr$$

$$=2\pi k\int_0^{\frac{\pi}{2}}d\varphi\int_0^{2a\cos\varphi}r^3\sin\varphi dr=2\pi k\int_0^{\frac{\pi}{2}}-4a^4\cos^4\varphi d(\cos\varphi)=\frac{8}{5}\pi a^4k$$

$$\bar{z}=\frac{1}{m}\iiint\limits_V z\rho_V(x,y,z)dV=\frac{k}{m}\iiint\limits_V z\sqrt{x^2+y^2+z^2}dV$$

$$=\frac{k}{m}\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^{2a\cos\varphi}r\cos\varphi\cdot r\cdot r^2\sin\varphi dr=\frac{2\pi k}{m}\int_0^{\frac{\pi}{2}}d\varphi\int_0^{2a\cos\varphi}r^4\cos\varphi\sin\varphi dr$$

$$=\frac{2\pi k}{m}\int_0^{\frac{\pi}{2}}-\frac{32}{5}a^5\cos^6\varphi d(\cos\varphi)=\frac{2\pi k}{m}\cdot\frac{32}{35}a^5=\frac{64}{35m}\pi a^5k=\frac{8}{7}a$$

故质心坐标为$\left(0,0,\ \frac{8}{7}a\right)$

(3) 物体$V:\sqrt{x^2+y^2}\le z\le h$，密度$\rho=1+x^2+y^2$

解：由对称性知：$\bar{x}=\bar{y}=0$

$$m=\iiint\limits_V \rho(x,y,z)dV=\iiint\limits_V(1+x^2+y^2)dV=\int_0^{2\pi}d\theta\int_0^h d\rho\int_\rho^h(1+\rho^2)\rho dz$$

$$=2\pi\int_0^h(1+\rho^2)\rho(h-\rho)d\rho=\frac{1}{30}(10+3h^2)\pi h^3$$

$$\bar{z}=\frac{1}{m}\iiint\limits_V z\rho(x,y,z)dV=\frac{1}{m}\iiint\limits_V z(1+x^2+y^2)dV$$

$$=\frac{1}{m}\int_0^{2\pi}d\theta\int_0^h d\rho\int_\rho^h z(1+\rho^2)\rho dz=\frac{\pi}{m}\int_0^h(1+\rho^2)\rho(h^2-\rho^2)d\rho$$

$$=\frac{(3+h^2)}{12m}\pi h^4=\frac{5h(3+h^2)}{2(10+3h^2)}$$

故质心坐标为$\left(0,0,\ \frac{5h(3+h^2)}{2(10+3h^2)}\right)$

(4) 均匀物体V由曲面$z=x^2+y^2$与$z=\sqrt{x^2+y^2}$围成.

解：由对称性知：$\bar{x}=\bar{y}=0$

$$\bar{z}=\frac{\iiint\limits_V zdV}{\iiint\limits_V dV}=\frac{\int_0^{2\pi}d\theta\int_0^1\rho d\rho\int_{\rho^2}^{\rho}zdz}{\int_0^{2\pi}d\theta\int_0^1\rho d\rho\int_{\rho^2}^{\rho}dz}=\frac{2\pi\int_0^1\frac{1}{2}\rho(\rho^2-\rho^4)d\rho}{2\pi\int_0^1\rho(\rho-\rho^2)d\rho}=\frac{1/24}{1/12}=\frac{1}{2}$$

故质心坐标为 $\left(0,\ 0,\ \frac{1}{2}\right)$

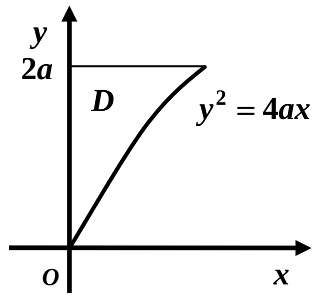
3．设面密度为 ρ_A 的均匀薄板 $D=\left\{(x,\ y)\middle|\ x\ge 0,\ 0\le y\le 2a,\ y\ge\sqrt{4ax}\right\}$

求薄板对 x 轴的转动惯量.

解：$J_x=\iint\limits_D \rho_A\, y^2 d\sigma=\int_0^{2a}dy\int_0^{\frac{1}{4a}y^2}\rho_A\, y^2dx$

$=\rho_A\int_0^{2a}\frac{1}{4a}y^4dy=\frac{\rho_A}{4a}\cdot\frac{(2a)^5}{5}$

$=\frac{8a^4}{5}\rho_A$



4．质量为 m 的圆锥形陀螺，底半径为 R，高为 h，试求陀螺绕其对称轴的转动惯量.

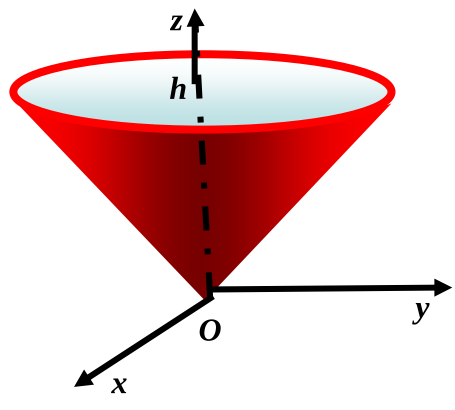
解：如图建立坐标系，圆锥的体积 $V=\frac{1}{3}\pi R^2h$，密度为 $\rho_v=\frac{m}{V}=\frac{3m}{\pi R^2h}$，则

$J_z=\iiint\limits_V (x^2+y^2)\rho_v dV$

$=\iint\limits_{D_{xy}}(x^2+y^2)\rho_v dxdy\int_{\frac{h}{R}\sqrt{x^2+y^2}}^{h}dz$

$=\int_0^{2\pi}d\theta\int_0^R\rho_v\rho^2(h-\frac{h}{R}\rho)\rho d\rho$

$=2\pi h(\frac{R^4}{4}-\frac{R^4}{5})\rho_v=\frac{3mR^2}{10}$



5．求密度为 ρ 的均匀上半球体 $0\le z\le\sqrt{R^2-x^2-y^2}$ 对 y 轴的转动惯量.

解 1：（用柱坐标变换）$J_y=\rho\iiint\limits_V(x^2+z^2)dV=4\rho\iiint\limits_{V_1}(x^2+z^2)dV$

$=4\rho\iint\limits_{D_{xz}}(x^2+z^2)dxdz\int_0^{\sqrt{R^2-x^2-y^2}}dy=4\rho\int_0^{\frac{\pi}{2}}d\theta\int_0^R r^2\sqrt{R^2-r^2}\,rdr$

$$= 2\pi\rho\int_0^R r^3\sqrt{R^2 - r^2}\,dr = 2\pi\rho R^5\int_0^{\frac{\pi}{2}} \sin^3 t(1-\sin^2 t)dt$$

$$\underline{\underline{r = R\sin t}}\ 2\pi\rho\int_0^{\frac{\pi}{2}} R^5\sin^3 t\cos^2 t dt = 2\pi\rho R^5\int_0^{\frac{\pi}{2}} \sin^3 t(1-\sin^2 t)dt$$

$$= 2\pi\rho R^5\int_0^{\frac{\pi}{2}} (\sin^3 t - \sin^5 t)dt = 2\pi\rho R^5(\frac{2}{3} - \frac{4}{5}\cdot\frac{2}{3}) = \frac{4}{15}\pi\rho R^5$$

解 2：（用球坐标变换）

$$J_y = \rho\iiint_V (x^2+z^2)dV = \rho\int_0^{2\pi} d\theta\int_0^{\frac{\pi}{2}} d\varphi\int_0^R \left(r^2\sin^2\varphi\cos^2\theta + r^2\cos^2\varphi\right)r^2\sin\varphi dr$$

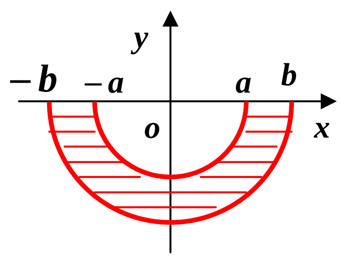
$$= \frac{\rho R^5}{5}\int_0^{2\pi} d\theta\int_0^{\frac{\pi}{2}} \left(\sin^3\varphi\cos^2\theta + \cos^2\varphi\sin\varphi\right)d\varphi = \frac{\rho R^5}{5}\int_0^{2\pi}\left(\frac{2}{3}\cos^2\theta + \frac{1}{3}\right)d\theta$$

$$= \frac{4}{15}\pi\rho R^5$$

6．设半圆环 $a^2 \le x^2 + y^2 \le b^2\ (y \le 0)$ 薄板的密度 $\rho(x, y) = y$，求薄板对原点处质量为 m 的质点的引力.

解：如图，由对称性知 $\overrightarrow{F_x} = 0$

$$\overrightarrow{F_y} = \iint_D \frac{km\ y\cdot y}{(x^2+y^2)^{\frac{3}{2}}}dxdy$$



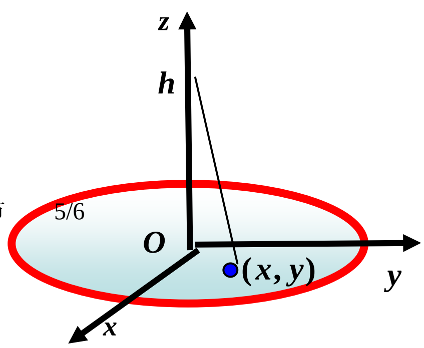
$$= \int_\pi^{2\pi} d\theta\int_a^b \frac{km\rho^2\sin^2\theta}{\rho^3}\rho\,d\rho = km(b-a)\int_\pi^{2\pi}\sin^2\theta\,d\theta$$

$$= km(b-a)\cdot\frac{\pi}{2} = \frac{\pi km}{2}(b-a)$$

7．设有一半径为 R、面密度为 ρ_A 的均匀圆板，在板的中心垂线上距圆心 h 处有一单位质点 P，求圆板对质点 P 的引力

解：如图建立坐标系，由对称性得

$$\overrightarrow{F_x} = \overrightarrow{F_y} = 0$$



$$\overrightarrow{F_z}=\iint\limits_D \frac{-k\rho_A h d\sigma}{(x^2+y^2+h^2)^{\frac{3}{2}}}$$

$$=\int_0^{2\pi}d\theta\int_0^R \frac{-k\rho_A h\rho}{(\rho^2+h^2)^{\frac{3}{2}}}d\rho$$

$$=2\pi k\rho_A h(\frac{1}{\sqrt{R^2+h^2}}-\frac{1}{h})=-2\pi k\rho_A(1-\frac{h}{\sqrt{R^2+h^2}})$$

8．设有质量为 M 的均匀柱体 $V: x^2+y^2\le R^2,\ 0\le z\le h$

在点 $A(0,0,b)\ (b>h)$ 处有一质量为 m 的质点，求柱体对质点的引力.

解：由对称性得

$$\overrightarrow{F_x}=\overrightarrow{F_y}=0$$

$$\overrightarrow{F_z}=\iiint\limits_V \frac{km\rho_v(z-b)}{[x^2+y^2+(z-b)^2]^{\frac{3}{2}}}dV=\iint\limits_{D_{xy}}dxdy\int_0^h \frac{km\rho_v(z-b)}{[x^2+y^2+(z-b)^2]^{\frac{3}{2}}}dz$$

$$=km\rho_v\iint\limits_{D_{xy}}\left(\frac{1}{[x^2+y^2+b^2]^{\frac{1}{2}}}-\frac{1}{[x^2+y^2+(h-b)^2]^{\frac{1}{2}}}\right)dxdy$$

$$=km\rho_v\int_0^{2\pi}d\theta\int_0^R\left(\frac{1}{[\rho^2+b^2]^{\frac{1}{2}}}-\frac{1}{[\rho^2+(h-b)^2]^{\frac{1}{2}}}\right)\rho d\rho$$

$$=2\pi km\rho_v[\sqrt{R^2+b^2}-\sqrt{R^2+(h-b)^2}-h]$$

$$=2\pi km\frac{M}{\pi R^2 h}[\sqrt{R^2+b^2}-\sqrt{R^2+(h-b)^2}-h]$$

$$=\frac{2kmM}{R^2 h}[\sqrt{R^2+b^2}-\sqrt{R^2+(h-b)^2}-h]$$