习题 7.2(P57)

1．求下列函数的偏导数.

(1) $z=\sin(xy)+\cos^2(xy)$

解：将 y 看作常量，得 $\dfrac{\partial z}{\partial x}=y\cos(xy)-2y\cos(xy)\sin(xy)$

$$=y[\cos(xy)-\sin(2xy)]$$

将 x 看作常量（或利用函数关于自变量的对称性由 $\dfrac{\partial z}{\partial x}$），得

$$\frac{\partial z}{\partial y}=x\cos(xy)-2x\cos(xy)\sin(xy)$$

$$=x[\cos(xy)-\sin(2xy)]$$

(2) $z=\sqrt{\ln(xy)}$

解：将 y 看作常量，得 $\dfrac{\partial z}{\partial x}=\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{\ln(xy)}}\cdot\dfrac{1}{xy}\cdot y=\dfrac{1}{2x\sqrt{\ln(xy)}}$；

将 x 看作常量（或利用函数关于自变量的对称性由 $\dfrac{\partial z}{\partial x}$），得 $\dfrac{\partial z}{\partial y}=\dfrac{1}{2y\sqrt{\ln(xy)}}$

(3) $z=\ln(\tan\dfrac{x}{y})$

解：将 y 看作常量，得 $\dfrac{\partial z}{\partial x}=\dfrac{1}{\tan\dfrac{x}{y}}\cdot\sec^2\dfrac{x}{y}\cdot\dfrac{1}{y}=\dfrac{2}{y}\csc\dfrac{2x}{y}$

将 x 看作常量，得 $\dfrac{\partial z}{\partial y}=\dfrac{1}{\tan\dfrac{x}{y}}\cdot\sec^2\dfrac{x}{y}\cdot\left(-\dfrac{x}{y^2}\right)=-\dfrac{2x}{y^2}\csc\dfrac{2x}{y}$

(4) $z=\sqrt{x}\arctan y$

解：将 y 看作常量，得 $\dfrac{\partial z}{\partial x}=\dfrac{1}{2\sqrt{x}}\cdot\arctan y$；

将 x 看作常量，得 $\dfrac{\partial z}{\partial y}=\dfrac{\sqrt{x}}{1+y^2}$

(5) $z=\ln(x+\sqrt{x^2+y^2})$

解：将 y 看作常量，得 $\dfrac{\partial z}{\partial x}=\dfrac{1}{x+\sqrt{x^2+y^2}}\cdot\left(1+\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{x^2+y^2}}\cdot 2x\right)=\dfrac{1}{\sqrt{x^2+y^2}}$；

将 x 看作常量，得 $\dfrac{\partial z}{\partial y}=\dfrac{1}{x+\sqrt{x^2+y^2}}\cdot\dfrac{1}{2}\cdot\dfrac{2y}{\sqrt{x^2+y^2}}=\dfrac{y}{x^2+y^2+x\sqrt{x^2+y^2}}$

(6) $u=x^{\frac{y}{z}}$

解：$\dfrac{\partial u}{\partial x}=\dfrac{y}{z}\cdot x^{\frac{y}{z}-1}$ ；

$$\frac{\partial u}{\partial y}=\frac{\partial(x^{\frac{1}{z}})^y}{\partial y}=x^{\frac{y}{z}}\ln x^{\frac{1}{z}}=\frac{1}{z}x^{\frac{y}{z}}\ln x\text{；}$$

$$\frac{\partial u}{\partial z}=\frac{\partial\left(x^y\right)^{\frac{1}{z}}}{\partial z}=x^{\frac{y}{z}}\ln x^y\left(-\frac{1}{z^2}\right)=-\frac{y}{z^2}x^{\frac{y}{z}}\ln x$$

(7) $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$

解：$\dfrac{\partial u}{\partial x}=-\dfrac{1}{2}\cdot\dfrac{2x}{\sqrt{\left(x^2+y^2+z^2\right)^3}}=-\dfrac{x}{\sqrt{\left(x^2+y^2+z^2\right)^3}}$；

由函数关于自变量的对称性及 $\dfrac{\partial u}{\partial x}$，得

$$\frac{\partial u}{\partial y}=-\frac{y}{\sqrt{\left(x^2+y^2+z^2\right)^3}}\text{；}$$

$$\frac{\partial u}{\partial z}=-\frac{z}{\sqrt{\left(x^2+y^2+z^2\right)^3}}$$

2．设 $f(x,y)=e^{3x}\ln(2y)$，求 $f'_x(0,1)$, $f'_y(0,e^{-1})$.

解 1：$f'_x(x,y)=3e^{3x}\ln(2y)$，所以 $f'_x(0,1)=3\ln 2$；

$f'_y(x,y)=e^{3x}\dfrac{1}{y}$，所以 $f'_y(0,e^{-1})=e$

解 2：$f(x,1)=e^{3x}\ln 2$，所以 $f'_x(x,1)=3e^{3x}\ln 2$，因此 $f'_x(0,1)=3\ln 2$；

$f(0,y)=\ln(2y)$，所以 $f_y'(0,y)=\dfrac{1}{y}$，因此 $f_y'(0,e^{-1})=e$

3．设 $z=(1+xy)^y$，求 $\left.\dfrac{\partial z}{\partial x}\right|_{(1,1)}$，$\left.\dfrac{\partial z}{\partial y}\right|_{(1,1)}$

解：$\left.\dfrac{\partial z}{\partial x}\right|_{(1,1)}=y^2(1+xy)^{y-1}\Big|_{(1,1)}=1$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1)}=\left.\frac{\partial(e^{y\ln(1+xy)})}{\partial y}\right|_{(1,1)}=(1+xy)^y(\ln(1+xy)+\frac{xy}{1+xy})\Big|_{(1,1)}=1+2\ln 2$$

易出的错： 对 y 求偏导时，运用指数函数求导法则：$\dfrac{\partial z}{\partial y}=(1+xy)^y\ln(1+xy)\cdot x$

4．曲线 $\begin{cases} z=\dfrac{x^2+y^2}{4} \\ y=4 \end{cases}$ 在点 $(2,4,5)$ 处的切线与 x 轴正向所成的夹角是多少？

解：$\left.\dfrac{\partial z}{\partial x}\right|_{(2,4)}=\dfrac{1}{2}x\Big|_{(2,4)}=1$，根据偏导数的几何意义知所求夹角为 $\dfrac{\pi}{4}$

5．曲面 $z=x^2+\dfrac{y^2}{6}$ 和 $z=\dfrac{x^2+y^2}{3}$ 被平面 $y=2$ 所截，得两条平面曲线，求这两条曲线交点处切线的夹角。

解：将 $y=2$ 分别代入两个曲面方程得两条平面曲线方程 $z=x^2+\dfrac{2}{3}$，$z=\dfrac{x^2+4}{3}$，联立上述两个方程得 xoz 坐标面上的两条平面曲线的交点 $(1,\dfrac{5}{3})$，$(-1,\dfrac{5}{3})$

$$k_{11}=(x^2+\frac{2}{3})'\Big|_{(1,\frac{5}{3})}=2x\Big|_{(1,\frac{5}{3})}=2 \qquad k_{12}=(x^2+\frac{2}{3})'\Big|_{(-1,\frac{5}{3})}=2x\Big|_{(-1,\frac{5}{3})}=-2$$

$$k_{21}=(\frac{x^2+4}{3})'\Big|_{(1,\frac{5}{3})}=\frac{2x}{3}\Big|_{(1,\frac{5}{3})}=\frac{2}{3} \qquad k_{22}=(\frac{x^2+4}{3})'\Big|_{(-1,\frac{5}{3})}=\frac{2x}{3}\Big|_{(-1,\frac{5}{3})}=-\frac{2}{3}$$

由 $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$ 及直线夹角的定义（$0\le\theta\le\dfrac{\pi}{2}$）得

$\tan\theta_1 = \left|\dfrac{k_{11}-k_{21}}{1+k_{11}\cdot k_{21}}\right| = \dfrac{4}{7}$　　　　$\tan\theta_2 = \left|\dfrac{k_{12}-k_{22}}{1+k_{12}\cdot k_{22}}\right| = \dfrac{4}{7}$

故 $\theta = \arctan\dfrac{4}{7}$

6．设 $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$，求 $f'_x(0, 0)$，$f'_y(0, 0)$．

解：$f'_x(0, 0) = \lim\limits_{x\to 0}\dfrac{f(x, 0) - f(0, 0)}{x} = \lim\limits_{x\to 0}\dfrac{0-0}{x} = 0$

$f'_y(0, 0) = \lim\limits_{y\to 0}\dfrac{f(0, y) - f(0, 0)}{y} = \lim\limits_{x\to 0}\dfrac{0-0}{y} = 0$

7．求下列函数的各二阶偏导数.

(1)　$z = \dfrac{x+y}{x-y}$

解：$\dfrac{\partial z}{\partial x} = \dfrac{-2y}{(x-y)^2}$，　　$\dfrac{\partial z}{\partial y} = \dfrac{2x}{(x-y)^2}$，

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{4y}{(x-y)^3}$，　　$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{4x}{(x-y)^3}$，　　$\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{\partial^2 z}{\partial y\partial x} = \dfrac{-2(x+y)}{(x-y)^3}$

(2) $z = x^{2y}$

解：$\dfrac{\partial z}{\partial x} = 2y\cdot x^{2y-1}$，　　$\dfrac{\partial z}{\partial y} = x^{2y}\cdot\ln x^2 = 2(\ln x)x^{2y}$，

$\dfrac{\partial^2 z}{\partial x^2} = 2y(2y-1)x^{2y-2}$，　　$\dfrac{\partial^2 z}{\partial y^2} = 4x^{2y}\ln^2 x$，

$\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{\partial^2 z}{\partial y\partial x} = 2x^{2y-1} + 4yx^{2y-1}\ln x$

(3) $z = \arctan\dfrac{y}{x}$

解：$\frac{\partial z}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)=\frac{-y}{x^2+y^2}$， $\frac{\partial z}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}$，

$$\frac{\partial^2 z}{\partial x^2}=\frac{2xy}{\left(x^2+y^2\right)^2},\quad \frac{\partial^2 z}{\partial y^2}=-\frac{2xy}{\left(x^2+y^2\right)^2},\quad \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

8. 设 $u=\frac{1}{\sqrt{x^2+y^2+z^2}}$，求 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$.

解：设 $r=\sqrt{x^2+y^2+z^2}$，则由第 1 题第（7）小题，得 $\frac{\partial u}{\partial x}=-\frac{x}{r^3}$，所以

$$\frac{\partial^2 u}{\partial x^2}=-\frac{1}{r^3}+\frac{3x}{r^4}\cdot\frac{\partial r}{\partial x}=-\frac{1}{r^3}+\frac{3x^2}{r^5}.$$

由函数关于自变量的对称性，得

$\frac{\partial^2 u}{\partial y^2}=-\frac{1}{r^3}+\frac{3y^2}{r^5}$，$\frac{\partial^2 u}{\partial z^2}=-\frac{1}{r^3}+\frac{3z^2}{r^5}$，所以

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{1}{r^3}+\frac{3x^2}{r^5}-\frac{1}{r^3}+\frac{3y^2}{r^5}-\frac{1}{r^3}+\frac{3z^2}{r^5}=0$$

9. 设 $z=\ln(e^x+e^y)$，证明：$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-(\frac{\partial^2 z}{\partial x\partial y})^2=0$

解：$\frac{\partial z}{\partial x}=\frac{e^x}{e^x+e^y}$， $\frac{\partial z}{\partial y}=\frac{e^y}{e^x+e^y}$， $\frac{\partial^2 z}{\partial x^2}=\frac{e^x\cdot e^y}{\left(e^x+e^y\right)^2}$， $\frac{\partial^2 z}{\partial y^2}=\frac{e^x\cdot e^y}{\left(e^x+e^y\right)^2}$，

$\frac{\partial^2 z}{\partial x\partial y}=\frac{-e^x\cdot e^y}{\left(e^x+e^y\right)^2}$，将上述三个二阶偏导数代入即可证出.